Sage Reference Manual: Discrete Valuations and Discrete Pseudo-Valuations

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The Sage Development Team

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Valuations can be defined conveniently on some Sage rings such as p-adic rings and function fields.

1.1 p-adic valuations

Valuations on number fields can be easily specified if they uniquely extend the valuation of a rational prime:

\begin{verbatim}
sage: v = QQ.valuation(2)
sage: v(1024)
10
\end{verbatim}

They are normalized such that the rational prime has valuation 1:

\begin{verbatim}
sage: K.<a> = NumberField(x^2 + x + 1)
sage: v = K.valuation(2)
sage: v(1024)
10
\end{verbatim}

If there are multiple valuations over a prime, they can be obtained by extending a valuation from a smaller ring:

\begin{verbatim}
sage: K.<a> = NumberField(x^2 + x + 1)  
sage: v = K.valuation(7)  
Traceback (most recent call last):
...  
ValueError: The valuation Gauss valuation induced by 7-adic valuation does not approximate a unique extension of 7-adic valuation with respect to x^2 + x + 1  
sage: w,ww = QQ.valuation(7).extensions(K)  
sage: w(a + 3), ww(a + 3)
(1, 0)  
sage: w(a + 5), ww(a + 5)
(0, 1)
\end{verbatim}

1.2 Valuations on Function Fields

Similarly, valuations can be defined on function fields:

\begin{verbatim}
sage: K.<x> = FunctionField(QQ)  
sage: v = K.valuation(x)  
sage: v(1/x)
-1
\end{verbatim}
On extensions of function fields, valuations can be created by providing a prime on the underlying rational function field when the extension is unique:

```python
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)
sage: v = L.valuation(x)
sage: v(x)
1
```

Valuations can also be extended from smaller function fields:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(x - 4)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)
sage: v.extensions(L)
[[ (x - 4)-adic valuation, v(y + 2) = 1 ]-adic valuation,
 [ (x - 4)-adic valuation, v(y - 2) = 1 ]-adic valuation]
```
2.1 Mac Lane valuations

Internally, all the above is backed by the algorithms described in [Mac1936I] and [Mac1936II]. Let us consider the extensions of $\text{K.valuation}(x - 4)$ to the field $L$ above to outline how this works internally.

First, the valuation on $K$ is induced by a valuation on $\mathbb{Q}[x]$. To construct this valuation, we start from the trivial valuation on $\mathbb{Q}$ and consider its induced Gauss valuation on $\mathbb{Q}[x]$, i.e., the valuation that assigns to a polynomial the minimum of the coefficient valuations:

```sage
R.<x> = QQ[]
v = GaussValuation(R, valuations.TrivialValuation(QQ))
```

The Gauss valuation can be augmented by specifying that $x - 4$ has valuation 1:

```sage
v = v.augmentation(x - 4, 1); v
```

This valuation then extends uniquely to the fraction field:

```sage
K.<x> = FunctionField(QQ)
v = v.extension(K); v
```

Over the function field we repeat the above process, i.e., we define the Gauss valuation induced by it and augment it to approximate an extension to $L$:

```sage
R.<y> = K[]
w = GaussValuation(R, v)
w = w.augmentation(y - 2, 1); w
```

```sage
L.<y> = K.extension(y^2 - x)
ww = w.extension(L); ww
```

2.2 Limit valuations

In the previous example the final valuation $ww$ is not merely given by evaluating $w$ on the ring $K[y]$: 
Instead \( w \) is given by a limit, i.e., an infinite sequence of augmentations of valuations:

\[
\text{sage: } w._\text{base_valuation}
\]

[ Gauss valuation induced by \((x - 4)\)-adic valuation, \(v(y - 2) = 1\), \(\ldots\) ]

The terms of this infinite sequence are computed on demand:

\[
\text{sage: } w._\text{base_valuation._approximation}
\]

[ Gauss valuation induced by \((x - 4)\)-adic valuation, \(v(y - 2) = 1\) ]

\[
\text{sage: } w(y - 1/4*x - 1)
\]

2

\[
\text{sage: } w._\text{base_valuation._approximation}
\]

[ Gauss valuation induced by \((x - 4)\)-adic valuation, \(v(y + 1/64*x^2 - 3/8*x - 3/4) = \ldots\) ]

-3

### 2.3 Non-classical valuations

Using the low-level interface we are not limited to classical valuations on function fields that correspond to points on the corresponding projective curves. Instead we can start with a non-trivial valuation on the field of constants:

\[
\text{sage: } v = \text{QQ}\text{.valuation}(2)
\]

\[
\text{sage: } R.<x> = \text{QQ}\[]
\]

\[
\text{sage: } w = \text{GaussValuation}(R, v) \quad \# \text{ v is not trivial}
\]

\[
\text{sage: } K.<x> = \text{FunctionField(QQ)}
\]

\[
\text{sage: } w = w\text{.extension(K)}
\]

\[
\text{sage: } w\text{.residue_field()}
\]

Rational function field in x over Finite Field of size 2
CHAPTER THREE

MAC LANE APPROXIMANTS

The main tool underlying this package is an algorithm by Mac Lane to compute, starting from a Gauss valuation on a polynomial ring and a monic squarefree polynomial $G$, approximations to the limit valuation which send $G$ to infinity:

```sage
v = QQ.valuation(2)
R.<x> = QQ[]
f = x^5 + 3*x^4 + 5*x^3 + 8*x^2 + 6*x + 12
v.mac_lane_approximants(f) # random output (order may vary)
```

From these approximants one can already see the residual degrees and ramification indices of the corresponding extensions. The approximants can be pushed to arbitrary precision, corresponding to a factorization of $f$:

```sage
v.mac_lane_approximants(f, required_precision=10) # random output
```

```python
[[ Gauss valuation induced by 2-adic valuation, v(x^2 + x + 1) = 3 ],
  [ Gauss valuation induced by 2-adic valuation, v(x) = 1/2 ],
  [ Gauss valuation induced by 2-adic valuation, v(x) = 1 ]]
```

```python
[[ Gauss valuation induced by 2-adic valuation, v(x^2 + 193*x + 13/21) = 10 ],
  [ Gauss valuation induced by 2-adic valuation, v(x + 86) = 10 ],
  [ Gauss valuation induced by 2-adic valuation, v(x) = 1/2, v(x^2 + 36/11*x + 2/17) = -11 ]]
```
The theory was originally described in [Mac1936I] and [Mac1936II]. A summary and some algorithmic details can also be found in Chapter 4 of [Rüt2014].
5.1 Value groups of discrete valuations

This file defines additive sub(semi-)groups of $\mathbb{Q}$ and related structures.

AUTHORS:

- Julian Rüth (2013-09-06): initial version

EXAMPLES:

```sage
v = ZZ.valuation(2)
sage: v.value_group()
Additive Abelian Group generated by 1
sage: v.value_semigroup()
Additive Abelian Semigroup generated by 1
```

```python
class sage.rings.valuation.value_group.DiscreteValuationCodomain
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

    The codomain of discrete valuations, the rational numbers extended by $\pm\infty$.
    
    EXAMPLES:

```sage
class sage.rings.valuation.value_group.DiscreteValueGroup(generator)
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

    The value group of a discrete valuation, an additive subgroup of $\mathbb{Q}$ generated by generator.
    
    INPUT:
    
    - generator -- a rational number
```

Note: We do not rely on the functionality provided by additive abelian groups in Sage since these require the underlying set to be the integers. Therefore, we roll our own $\mathbb{Z}$-module here. We could have used `AdditiveAbelianGroupWrapper` here, but it seems to be somewhat outdated. In particular, generic group functionality should now come from the category and not from the super-class. A facade of $\mathbb{Q}$ appeared to be the better approach.
EXAMPLES:

```python
sage: from sage.rings.valuation.value_group import DiscreteValueGroup
sage: D1 = DiscreteValueGroup(0); D1
Trivial Additive Abelian Group
sage: D2 = DiscreteValueGroup(4/3); D2
Additive Abelian Group generated by 4/3
sage: D3 = DiscreteValueGroup(-1/3); D3
Additive Abelian Group generated by 1/3
```

denominator()

Return the denominator of a generator of this group.

EXAMPLES:

```python
sage: from sage.rings.valuation.value_group import DiscreteValueGroup
sage: DiscreteValueGroup(3/8).denominator()
8
```

```python
sage: from sage.rings.valuation.value_group import DiscreteValueGroup
sage: DiscreteValueGroup(-3/8).gen()
3/8
```

index(other)

Return the index of other in this group.

INPUT:

- other – a subgroup of this group

EXAMPLES:

```python
sage: from sage.rings.valuation.value_group import DiscreteValueGroup
sage: DiscreteValueGroup(3/8).index(DiscreteValueGroup(3))
8
sage: DiscreteValueGroup(3).index(DiscreteValueGroup(3/8))
Traceback (most recent call last):
  ... ValueError: other must be a subgroup of this group
sage: DiscreteValueGroup(3).index(DiscreteValueGroup(0))
Traceback (most recent call last):
  ... ValueError: other must have finite index in this group
sage: DiscreteValueGroup(0).index(DiscreteValueGroup(3))
1
sage: DiscreteValueGroup(0).index(DiscreteValueGroup(0))
1
```

is_trivial()

Return whether this is the trivial additive abelian group.

EXAMPLES:

```python
from sage.rings.valuation.value_group import DiscreteValueGroup
sage: DiscreteValueGroup(-3/8).is_trivial()
False
sage: DiscreteValueGroup(0).is_trivial()
True
```

**numerator()**

Return the numerator of a generator of this group.

**EXAMPLES:**

```python
from sage.rings.valuation.value_group import DiscreteValueGroup
sage: DiscreteValueGroup(3/8).numerator()
3
```

**some_elements()**

Return some typical elements in this group.

**EXAMPLES:**

```python
from sage.rings.valuation.value_group import DiscreteValueGroup
sage: DiscreteValueGroup(-3/8).some_elements()
[3/8, -3/8, 0, 42, 3/2, -3/2, 9/8, -9/8]
```

```python
class sage.rings.valuation.value_group.DiscreteValueSemigroup(generators)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

The value semigroup of a discrete valuation, an additive subsemigroup of \( \mathbb{Q} \) generated by \( \text{generators} \).

**INPUT:**

- \( \text{generators} - \text{rational numbers} \)

**EXAMPLES:**

```python
from sage.rings.valuation.value_group import DiscreteValueSemigroup
sage: D1 = DiscreteValueSemigroup(0); D1
Trivial Additive Abelian Semigroup
sage: D2 = DiscreteValueSemigroup(4/3); D2
Additive Abelian Semigroup generated by 4/3
sage: D3 = DiscreteValueSemigroup([-1/3, 1/2]); D3
Additive Abelian Semigroup generated by -1/3, 1/2
```

**gens()**

Return the generators of this semigroup.

**EXAMPLES:**

```python
from sage.rings.valuation.value_group import DiscreteValueSemigroup
sage: D1 = DiscreteValueSemigroup(0); D1
Trivial Additive Abelian Semigroup
sage: D2 = DiscreteValueSemigroup(4/3); D2
Additive Abelian Semigroup generated by 4/3
sage: D3 = DiscreteValueSemigroup([-1/3, 1/2]); D3
Additive Abelian Semigroup generated by -1/3, 1/2
```

**is_group()**

Return whether this semigroup is a group.

**EXAMPLES:**

```python
```
```python
sage: from sage.rings.valuation.value_group import DiscreteValueSemigroup
sage: DiscreteValueSemigroup(1).is_group()
False
sage: D = DiscreteValueSemigroup([-1, 1])
sage: D.is_group()
True
```

Invoking this method also changes the category of this semigroup if it is a group:

```python
sage: D in AdditiveMagmas().AdditiveAssociative().AdditiveUnital().
   AdditiveInverse()
True
```

### is_trivial()

Return whether this is the trivial additive abelian semigroup.

**EXAMPLES:**

```python
sage: from sage.rings.valuation.value_group import DiscreteValueSemigroup
sage: DiscreteValueSemigroup(-3/8).is_trivial()
False
sage: DiscreteValueSemigroup([]).is_trivial()
True
```

### some_elements()

Return some typical elements in this semigroup.

**EXAMPLES:**

```python
sage: from sage.rings.valuation.value_group import DiscreteValueSemigroup
sage: list(DiscreteValueSemigroup([-3/8,1/2]).some_elements())
[0, -3/8, 1/2, ...]
```

## 5.2 Discrete valuations

This file defines abstract base classes for discrete (pseudo-)valuations.

**AUTHORS:**

- Julian Rüth (2013-03-16): initial version

**EXAMPLES:**

Discrete valuations can be created on a variety of rings:

```python
sage: ZZ.valuation(2)
2-adic valuation
sage: GaussianIntegers().valuation(3)
3-adic valuation
sage: QQ.valuation(5)
5-adic valuation
sage: Zp(7).valuation()
7-adic valuation
```

```python
sage: K.<x> = FunctionField(QQ)
sage: K.valuation(x)
```

(continues on next page)
(x)-adic valuation
sage: K.valuation(x^2 + 1)
(x^2 + 1)-adic valuation
sage: K.valuation(1/x)

Valuation at the infinite place
sage: R.<x> = QQ[]
sage: v = QQ.valuation(2)
sage: w = GaussValuation(R, v)
sage: w.augmentation(x, 3)
[ Gauss valuation induced by 2-adic valuation, v(x) = 3 ]

We can also define discrete pseudo-valuations, i.e., discrete valuations that send more than just zero to infinity:

sage: w.augmentation(x, infinity)
[ Gauss valuation induced by 2-adic valuation, v(x) = +Infinity ]

class sage.rings.valuation.valuation.DiscretePseudoValuation(parent)
    Bases: sage.categories.morphism.Morphism

Abstract base class for discrete pseudo-valuations, i.e., discrete valuations which might send more that just zero to infinity.

INPUT:
  • domain – an integral domain

EXAMPLES:

sage: v = ZZ.valuation(2); v
2-adic valuation
sage: v.is_equivalent(2, 1)
False
sage: v.is_equivalent(2, -2)
True
sage: v.is_equivalent(2, 0)
False
sage: v.is_equivalent(0, 0)
True

class sage.rings.valuation.valuation.DiscreteValuation(parent)
    Bases: sage.rings.valuation.valuation.DiscretePseudoValuation

Abstract base class for discrete valuations.

EXAMPLES:

sage: v = QQ.valuation(2)
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, v)
sage: w = v.augmentation(x, 1337); w
[ Gauss valuation induced by 2-adic valuation, v(x) = 1337 ]
\textbf{is\_discrete\_valuation()} \\
Return whether this valuation is a discrete valuation.

\textbf{EXAMPLES:}

```
sage: v = valuations.TrivialValuation(ZZ)
sage: v.is_discretevaluation()
True
```

\textbf{mac\_lane\_approximant}(G, valuation, approximants=None) \\
Return the approximant from \texttt{mac\_lane\_approximants()} for \texttt{G} which is approximated by or approximates valuation.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{G} – a monic squarefree integral polynomial in a univariate polynomial ring over the domain of this valuation
  \item \texttt{valuation} – a valuation on the parent of \texttt{G}
  \item \texttt{approximants} – the output of \texttt{mac\_lane\_approximants()}. If not given, it is computed.
\end{itemize}

\textbf{EXAMPLES:}

```
sage: v = QQ.valuation(2)
sage: R.<x> = QQ[]
sage: G = x^2 + 1

We can select an approximant by approximating it:

```
sage: w = GaussValuation(R, v).augmentation(x + 1, 1/2)
sage: v.mac_lane_approximant(G, w)
[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2 ]
```

As long as this is the only matching approximant, the approximation can be very coarse:

```
sage: w = GaussValuation(R, v)
sage: v.mac_lane_approximant(G, w)
[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2 ]
```

Or it can be very specific:

```
sage: w = GaussValuation(R, v).augmentation(x + 1, 1/2).augmentation(G, \rightarrow \infty)
sage: v.macLane_approximant(G, w)
[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2 ]
```

But it must be an approximation of an approximant:

```
sage: w = GaussValuation(R, v).augmentation(x, 1/2)
sage: v.mac_lane_approximant(G, w)
Traceback (most recent call last):
...
ValueError: The valuation [ Gauss valuation induced by 2-adic valuation, v(x) \rightarrow 1/2 ] is not an approximant for a valuation which extends 2-adic valuation with respect to x^2 + 1 since the valuation of x^2 + 1 does not increase in every step
```

The valuation must single out one approximant:
sage: G = x^2 - 1
sage: w = GaussValuation(R, v)
sage: v.mac_lane_approximant(G, w)
Traceback (most recent call last):
...
ValueError: The valuation Gauss valuation induced by 2-adic valuation does
\[\rightarrow\]not approximate a unique extension of 2-adic valuation with respect to \(x^2 - 1\)

sage: w = GaussValuation(R, v).augmentation(x + 1, 1)
sage: v.mac_lane_approximant(G, w)
Traceback (most recent call last):
...
ValueError: The valuation [ Gauss valuation induced by 2-adic valuation, \(v(x + 1) = 1\) ] does not approximate a unique extension of 2-adic valuation with
\[\rightarrow\]respect to \(x^2 - 1\)

sage: w = GaussValuation(R, v).augmentation(x + 1, 2)
sage: v.mac_lane_approximant(G, w)
[ Gauss valuation induced by 2-adic valuation, \(v(x + 1) = +\text{Infinity}\] )

sage: w = GaussValuation(R, v).augmentation(x + 3, 2)
sage: v.mac_lane_approximant(G, w)
[ Gauss valuation induced by 2-adic valuation, \(v(x + 1) = 1\] )

mac_lane_approximants 
\(G, \text{ assume_squarefree=}\text{False}, \text{ require_final_EF=}\text{True}, \text{ required_precision=}\text{-1}, \text{ require_incomparability=}\text{False}, \text{ require_maximal_degree=}\text{False}, \text{ algorithm=}\text{‘serial’}\) 
Return approximants on \(K[x]\) for the extensions of this valuation to \(L = K[x]/(G)\). If \(G\) is an irreducible polynomial, then this corresponds to extensions of this valuation to the completion of \(L\).

INPUT:

- \(G\) – a monic squarefree integral polynomial in a univariate polynomial ring over the domain of this valuation

- \text{ assume_squarefree} – a boolean (default: \text{False}), whether to assume that \(G\) is squarefree. If \text{True}, the squafreeness of \(G\) is not verified though it is necessary when \text{require_final_EF} is set for the algorithm to terminate.

- \text{ require_final_EF} – a boolean (default: \text{True}); whether to require the returned key polynomials to be in one-to-one correspondance to the extensions of this valuation to \(L\) and require them to have the ramification index and residue degree of the valuations they correspond to.

- \text{ required_precision} – a number or infinity (default: -1); whether to require the last key polynomial of the returned valuations to have at least that valuation.

- \text{ require_incomparability} – a boolean (default: \text{False}); whether to require require the returned valuations to be incomparable (with respect to the partial order on valuations defined by comparing them pointwise.)

- \text{ require_maximal_degree} – a boolean (default: \text{False}); whether to require the last key polynomial of the returned valuation to have maximal degree. This is most relevant when using this algorithm to compute approximate factorizations of \(G\), when set to \text{True}, the last key polynomial has the same degree as the corresponding factor.

- \text{ algorithm} – one of \text{“serial”} or \text{“parallel”} (default: \text{“serial”}); whether or not to parallelize the algorithm.

5.2. Discrete valuations
EXAMPLES:

```python
sage: v = QQ.valuation(2)
sage: R.<x> = QQ[]
sage: v.mac_lane_approximants(x^2 + 1)
[[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2 ]]
sage: v.mac_lane_approximants(x^2 + 1, required_precision=Infinity)
[[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2, v(x^2 + 1) = +Infinity ]]
sage: v.mac_lane_approximants(x^2 + x + 1)
[[ Gauss valuation induced by 2-adic valuation, v(x^2 + x + 1) = +Infinity ]]
```

Note that \( G \) does not need to be irreducible. Here, we detect a factor \( x + 1 \) and an approximate factor \( x + 1 \) (which is an approximation to \( x - 1 \)):

```python
sage: v.mac_lane_approximants(x^2 - 1)
[[ Gauss valuation induced by 2-adic valuation, v(x + 1) = +Infinity ],
  [ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1 ]]
```

However, it needs to be squarefree:

```python
sage: v.mac_lane_approximants(x^2)
Traceback (most recent call last):
  ...
ValueError: G must be squarefree
```

A case that triggered an assertion at some point:

```python
sage: v = QQ.valuation(3)
sage: R.<x> = QQ[]
sage: f = x^36 + 60552000*x^33 + 268157412*x^30 + 173881701*x^27 +
    266324841*x^24 + 83125683*x^21 + 111803814*x^18 + 31925826*x^15 +
    205726716*x^12 + 17990262*x^9 + 351459648*x^6 + 127014399*x^3 + 359254116
sage: v.mac_lane_approximants(f)
[[ Gauss valuation induced by 3-adic valuation, v(x^12 - 3*x^9 + 54*x^6 + 27/2*x^3 + 405/2) = 13/2,
  v(x^36 + 60552000*x^33 + 268157412*x^30 + 173881701*x^27 +
  266324841*x^24 + 83125683*x^21 + 111803814*x^18 + 31925826*x^15 +
  205726716*x^12 + 17990262*x^9 + 351459648*x^6 + 127014399*x^3 + 359254116) = +Infinity ]]
```

```
```
montes_factorization(G, assume_squarefree=False, required_precision=None)
Factor \( G \) over the completion of the domain of this valuation.

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INPUT:
• \( G \) – a monic polynomial over the domain of this valuation
• \texttt{assume\_squarefree} – a boolean (default: \texttt{False}), whether to assume \( G \) to be squarefree
• \texttt{required\_precision} – a number or infinity (default: infinity); if infinity, the returned polynomials are actual factors of \( G \), otherwise they are only factors with precision at least \texttt{required\_precision}.

ALGORITHM:

We compute \texttt{mac\_lane\_approximants()} with \texttt{required\_precision}. The key polynomials approximate factors of \( G \). This can be very slow unless \texttt{required\_precision} is set to zero. Single factor lifting could improve this significantly.

EXAMPLES:

```python
sage: k=Qp(5,4)
sage: v = k.valuation()
sage: R.<x>=k[]
sage: G = x^2 + 1
sage: v.montes_factorization(G)
((1 + O(5^4))*x + 2 + 5 + 2*5^2 + 5^3 + O(5^4)) * ((1 + O(5^4))*x + 3 + 3*5 +...
˓→2*5^2 + 3*5^3 + O(5^4))
```

The computation might not terminate over incomplete fields (in particular because the factors can not be represented there):

```python
sage: R.<x> = QQ[]
sage: v = QQ.valuation(2)
sage: v.montes_factorization(x^6 - 1)
(x - 1) * (x + 1) * (x^2 - x + 1) * (x^2 + x + 1)
```

REFERENCES:

The underlying algorithm is described in [Mac1936II] and thoroughly analyzed in [GMN2008].

```python
class sage.rings.valuation.valuation.InfiniteDiscretePseudoValuation(parent)

Bases: sage.rings.valuation.valuation.DiscretePseudoValuation

Abstract base class for infinite discrete pseudo-valuations, i.e., discrete pseudo-valuations which are not discrete valuations.

EXAMPLES:

```python
sage: v = QQ.valuation(2)
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, v)
sage: w = v.augmentation(x, infinity); w # indirect doctest
[ Gauss valuation induced by 2-adic valuation, v(x) = +Infinity ]
```

```python
is_discrete_valuation()

Return whether this valuation is a discrete valuation.

EXAMPLES:
```
sage: v = QQ.valuation(2)
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, v)
sage: v.is_discrete_valuation()
True
sage: w = v.augmentation(x, infinity)
sage: w.is_discrete_valuation()
False

class sage.rings.valuation.valuation.MacLaneApproximantNode(valuation, parent, ef, principal_part_bound, coefficients, valuations)

Bases: object

A node in the tree computed by `DiscreteValuation.mac_lane_approximants()`

Leaves in the computation of the tree of approximants `mac_lane_approximants()`. Each vertex consists of a tuple \((v, ef, p, coeffs, vals)\) where \(v\) is an approximant, i.e., a valuation, \(ef\) is a boolean, \(p\) is the parent of this vertex, and \(coeffs\) and \(vals\) are cached values. (Only \(v\) and \(ef\) are relevant, everything else are caches/debug info.) The boolean \(ef\) denotes whether \(v\) already has the final ramification index \(E\) and residue degree \(F\) of this approximant. An edge \(V – P\) represents the relation \(P.v \leq V.v\) (pointwise on the polynomial ring \(K[x]\)) between the valuations.

class sage.rings.valuation.valuation.NegativeInfiniteDiscretePseudoValuation(parent)

Bases: sage.rings.valuation.valuation.InfiniteDiscretePseudoValuation

Abstract base class for pseudo-valuations which attain the value \(\infty\) and \(-\infty\), i.e., whose domain contains an element of valuation \(\infty\) and its inverse.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ)).augmentation(x, -infinity)
sage: K.<x> = FunctionField(QQ)
sage: w = K.valuation(v)
sage: w.is_negative_pseudo_valuation()
True
```

is_negative_pseudo_valuation()

Return whether this valuation attains the value \(-\infty\).

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ)).augmentation(x, -infinity)
sage: v.is_negative_pseudo_valuation()
False
sage: K.<x> = FunctionField(QQ)
sage: w = K.valuation(v)
sage: w.is_negative_pseudo_valuation()
True
```
5.3 Spaces of valuations

This module provides spaces of exponential pseudo-valuations on integral domains. It currently only provides support for such valuations if they are discrete, i.e., their image is a discrete additive subgroup of the rational numbers extended by \( \infty \).

AUTHORS:

- Julian Rüth (2016-10-14): initial version

EXAMPLES:

```python
sage: QQ.valuation(2).parent()
Discrete pseudo-valuations on Rational Field
```

Note: Note that many tests not only in this module do not create instances of valuations directly since this gives the wrong inheritance structure on the resulting objects:

```python
sage: from sage.rings.valuation.valuation_space import DiscretePseudoValuationSpace
sage: from sage.rings.valuation.trivial_valuation import TrivialDiscretePseudoValuation
sage: H = DiscretePseudoValuationSpace(QQ)
sage: v = TrivialDiscretePseudoValuation(H)
sage: v._test_category()
Traceback (most recent call last):
  ...:
AssertionError: False is not true
```

Instead, the valuations need to be created through the `__make_element_class__` of the containing space:

```python
sage: from sage.rings.valuation.trivial_valuation import TrivialDiscretePseudoValuation
sage: v = H.__make_element_class__(TrivialDiscretePseudoValuation)(H)
sage: v._test_category()
```

The factories such as `TrivialPseudoValuation` provide the right inheritance structure:

```python
sage: v = valuations.TrivialPseudoValuation(QQ)
sage: v._test_category()
```

class `sage.rings.valuation.valuation_space.DiscretePseudoValuationSpace`(*domain*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.categories.homset.Homset`

The space of discrete pseudo-valuations on `domain`.

EXAMPLES:

```python
sage: from sage.rings.valuation.valuation_space import DiscretePseudoValuationSpace
sage: H = DiscretePseudoValuationSpace(QQ)
sage: QQ.valuation(2) in H
True
```

Note: We do not distinguish between the space of discrete valuations and the space of discrete pseudo-valuations. This is entirely for practical reasons: We would like to model the fact that every discrete valuation
is also a discrete pseudo-valuation. At first, it seems to be sufficient to make sure that the \texttt{in} operator works which can essentially be achieved by overriding \_\_element\_constructor\_ of the space of discrete pseudo-valuations to accept discrete valuations by just returning them. Currently, however, if one does not change the parent of an element in \_\_element\_constructor\_ to \texttt{self}, then one can not register that conversion as a coercion. Consequently, the operators \texttt{\leq} and \texttt{\geq} can not be made to work between discrete valuations and discrete pseudo-valuations on the same domain (because the implementation only calls \_\_richcmp\_ if both operands have the same parent.) Of course, we could override \_\_ge\_ and \_\_le\_ but then we would likely run into other surprises. So in the end, we went for a single homspace for all discrete valuations (pseudo or not) as this makes the implementation much easier.

\textbf{Todo:} The comparison problem might be fixed by trac ticket \#22029 or similar.

\textbf{class ElementMethods}

Provides methods for discrete pseudo-valuations that are added automatically to valuations in this space.

\textbf{EXAMPLES:}

Here is an example of a method that is automagically added to a discrete valuation:

\begin{verbatim}
sage: \texttt{from sage.rings.valuation.valuation_space import DiscretePseudoValuationSpace}
sage: H = DiscretePseudoValuationSpace(QQ)
sage: QQ.valuation(2).is_discrete_pseudo_valuation() # indirect doctest
True
\end{verbatim}

The methods will be provided even if the concrete type is not created with \_\_make\_element\_class\_:\ procedure:

\begin{verbatim}
sage: \texttt{from sage.rings.valuation.valuation import DiscretePseudoValuation}
sage: m = DiscretePseudoValuation(H)
sage: m.parent() \texttt{is} H
True
sage: m.is_discrete_pseudo_valuation()
True
\end{verbatim}

However, the category framework advises you to use inheritance:

\begin{verbatim}
sage: m._test_category()
Traceback (most recent call last):
...
AssertionError: False is not true
\end{verbatim}

Using \_\_make\_element\_class\_, makes your concrete valuation inherit from this class:

\begin{verbatim}
sage: m = H._\_make\_element\_class\_(DiscretePseudoValuation)(H)
sage: m._test_category()
\end{verbatim}

\textbf{change\_domain}(\texttt{ring})

Return this valuation over \texttt{ring}.

Unlike \texttt{extension()} or \texttt{restriction()}, this might not be completely sane mathematically. It is essentially a conversion of this valuation into another space of valuations.

\textbf{EXAMPLES:}
sage: v = QQ.valuation(3)
sage: v.change_domain(ZZ)
3-adic valuation

**element_with_valuation**(s)
Return an element in the domain of this valuation with valuation s.

**EXAMPLES:**

sage: v = ZZ.valuation(2)
sage: v.element_with_valuation(10)
1024

**extension**(ring)
Return the unique extension of this valuation to ring.

**EXAMPLES:**

sage: v = ZZ.valuation(2)
sage: w = v.extension(QQ)
sage: w.domain()
Rational Field

**extensions**(ring)
Return the extensions of this valuation to ring.

**EXAMPLES:**

sage: v = ZZ.valuation(2)
sage: v.extensions(QQ)

[2-adic valuation]

**inverse**(x, precision)
Return an approximate inverse of x.

The element returned is such that the product differs from 1 by an element of valuation at least precision.

**INPUT:**
  * x – an element in the domain of this valuation
  * precision – a rational or infinity

**EXAMPLES:**

sage: v = ZZ.valuation(2)
sage: x = 3
sage: y = v.inverse(3, 2); y
3
sage: x*y - 1
8

This might not be possible for elements of positive valuation:

sage: v.inverse(2, 2)
Traceback (most recent call last):
  ...
ValueError: element has no approximate inverse in this ring

Of course this always works over fields:

5.3. Spaces of valuations
sage: v = QQ.valuation(2)
sage: v.inverse(2, 2)
1/2

is_discrete_pseudovaluation()
Return whether this valuation is a discrete pseudo-valuation.

EXAMPLES:

sage: QQ.valuation(2).is_discrete_pseudovaluation()
True

is_discretevaluation()
Return whether this valuation is a discrete valuation, i.e., whether it is a discrete pseudo valuation that only sends zero to ∞.

EXAMPLES:

sage: QQ.valuation(2).is_discrete_valuation()
True

is_negative_pseudovaluation()
Return whether this valuation is a discrete pseudo-valuation that does attain −∞, i.e., it is non-trivial and its domain contains an element with valuation ∞ that has an inverse.

EXAMPLES:

sage: QQ.valuation(2).is_negative_pseudovaluation()
False

is_trivial()
Return whether this valuation is trivial, i.e., whether it is constant ∞ or constant zero for everything but the zero element.

Subclasses need to override this method if they do not implement uniformizer().

EXAMPLES:

sage: QQ.valuation(7).is_trivial()
False

lift(X)
Return a lift of X in the domain which reduces down to X again via reduce().

EXAMPLES:

sage: v = QQ.valuation(2)
sage: v.lift(v.residue_ring().one())
1

lower_bound(x)
Return a lower bound of this valuation at x.

Use this method to get an approximation of the valuation of x when speed is more important than accuracy.

EXAMPLES:
reduce(x)
Return the image of x in the residue_ring() of this valuation.

EXAMPLES:

```python
sage: v = QQ.valuation(2)
sage: v.reduce(2)
0
sage: v.reduce(1)
1
sage: v.reduce(1/3)
1
Traceback (most recent call last):
  ...  
ValueError: reduction is only defined for elements of non-negative valuation
```

residue_field()
Return the residue field of this valuation, i.e., the field of fractions of the residue_ring(), the elements of non-negative valuation modulo the elements of positive valuation.

EXAMPLES:

```python
sage: QQ.valuation(2).residue_field()
Finite Field of size 2
sage: valuations.TrivialValuation(QQ).residue_field()
Rational Field
sage: valuations.TrivialValuation(ZZ).residue_field()
Rational Field
sage: GaussValuation(ZZ['x'], ZZ.valuation(2)).residue_field()
Rational function field in x over Finite Field of size 2
```

residue_ring()
Return the residue ring of this valuation, i.e., the elements of non-negative valuation modulo the elements of positive valuation. EXAMPLES:

```python
sage: QQ.valuation(2).residue_ring()
Finite Field of size 2
sage: valuations.TrivialValuation(QQ).residue_ring()
Rational Field
sage: valuations.TrivialValuation(ZZ).residue_ring()
Integer Ring
sage: GaussValuation(ZZ['x'], ZZ.valuation(2)).residue_ring()
Univariate Polynomial Ring in x over Finite Field of size 2
```

Note that a residue ring always exists, even when a residue field may not:

```python
sage: valuations.TrivialPseudoValuation(QQ).residue_ring()
Quotient of Rational Field by the ideal (1)
sage: valuations.TrivialValuation(ZZ).residue_ring()
Integer Ring
sage: GaussValuation(ZZ['x'], ZZ.valuation(2)).residue_ring()
Univariate Polynomial Ring in x over Finite Field of size 2
```

restriction(ring)
Return the restriction of this valuation to ring.

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EXAMPLES:

```python
sage: v = QQ.valuation(2)
sage: w = v.restriction(ZZ)
sage: w.domain()
Integer Ring
```

`scale(scalar)`

Return this valuation scaled by `scalar`.

**INPUT:**
- `scalar` – a non-negative rational number or infinity

**EXAMPLES:**

```python
sage: v = ZZ.valuation(3)
sage: w = v.scale(3)
sage: w(3)
3
```

Scaling can also be done through multiplication with a scalar:

```python
sage: w/3 == v
True
```

Multiplication by zero produces the trivial discrete valuation:

```python
sage: w = 0*v
sage: w(3)
0
sage: w(0)
+Infinity
```

Multiplication by infinity produces the trivial discrete pseudo-valuation:

```python
sage: w = infinity*v
sage: w(3)
+Infinity
sage: w(0)
+Infinity
```

`separating_element(others)`

Return an element in the domain of this valuation which has positive valuation with respect to this valuation but negative valuation with respect to the valuations in `others`.

**EXAMPLES:**

```python
sage: v2 = QQ.valuation(2)
sage: v3 = QQ.valuation(3)
sage: v5 = QQ.valuation(5)
sage: v2.separating_element([v3,v5])
4/15
```

`shift(x,s)`

Shift `x` in its expansion with respect to `uniformizer()` by `s` “digits”.

For non-negative `s`, this just returns `x` multiplied by a power of the uniformizer `\( \pi \)`.

For negative `s`, it does the same but when not over a field, it drops coefficients in the `\( \pi \)`-adic expansion which have negative valuation.
EXAMPLES:

```python
sage: v = ZZ.valuation(2)
sage: v.shift(1, 10)
1024
sage: v.shift(11, -1)
5
```

For some rings, there is no clear \(\pi\)-adic expansion. In this case, this method performs negative shifts by iterated division by the uniformizer and substraction of a lift of the reduction:

```python
sage: R.<x> = ZZ[]
sage: v = ZZ.valuation(2)
sage: w = GaussValuation(R, v)
sage: w.shift(x, 1)
2*x
sage: w.shift(2*x, -1)
x
sage: w.shift(x + 2*x^2, -1)
x^2
```

`simplify(x, error=None, force=False)`

Return a simplified version of \(x\).

Produce an element which differs from \(x\) by an element of valuation strictly greater than the valuation of \(x\) (or strictly greater than \(error\) if set.)

If \(force\) is not set, then expensive simplifications may be avoided.

EXAMPLES:

```python
sage: v = ZZ.valuation(2)
sage: v.simplify(6, force=True)
2
sage: v.simplify(6, error=0, force=True)
0
```

`uniformizer()`

Return an element in the domain which has positive valuation and generates the value group of this valuation.

EXAMPLES:

```python
sage: QQ.valuation(11).uniformizer()
11
```

Trivial valuations have no uniformizer:

```python
sage: from sage.rings.valuation.valuation_space import
         DiscretePseudoValuationSpace
sage: v = DiscretePseudoValuationSpace(QQ).an_element()
sage: v.is_trivial()
True
sage: v.uniformizer()
Traceback (most recent call last):
  ...
ValueError: Trivial valuations do not define a uniformizing element
```

`upper_bound(x)`

Return an upper bound of this valuation at \(x\).
Use this method to get an approximation of the valuation of $x$ when speed is more important than accuracy.

**EXAMPLES:**

```python
sage: v = ZZ.valuation(2)
sage: v.upper_bound(2^10)
10
```

**value_group()**

Return the value group of this discrete pseudo-valuation, the discrete additive subgroup of the rational numbers which is generated by the valuation of the `uniformizer()`.

**EXAMPLES:**

```python
sage: QQ.valuation(2).value_group()
Additive Abelian Group generated by 1
```

A pseudo-valuation that is $\infty$ everywhere, does not have a value group:

```python
sage: from sage.rings.valuation.valuation_space import...
˓→DiscretePseudoValuationSpace
sage: v = DiscretePseudoValuationSpace(QQ).an_element()
Traceback (most recent call last):
  ...
ValueError: The trivial pseudo-valuation that is infinity everywhere does...
  → not have a value group.
```

**value_semigroup()**

Return the value semigroup of this discrete pseudo-valuation, the additive subsemigroup of the rational numbers which is generated by the valuations of the elements in the domain.

**EXAMPLES:**

Most commonly, in particular over fields, the semigroup is the group generated by the valuation of the uniformizer:

```python
sage: G = QQ.valuation(2).value_semigroup(); G
Additive Abelian Semigroup generated by -1, 1
sage: G in AdditiveMagmas().AdditiveAssociative().AdditiveUnital().AdditiveInverse()
True
```

If the domain is a discrete valuation ring, then the semigroup consists of the positive elements of the `value_group()`:

```python
sage: Zp(2).valuation().value_semigroup()
Additive Abelian Semigroup generated by 1
```

The semigroup can have a more complicated structure when the uniformizer is not in the domain:

```python
sage: v = ZZ.valuation(2)
sage: R.<x> = ZZ[]
sage: w = GaussValuation(R, v)
sage: u = w.augmentation(x, 5/3)
sage: u.value_semigroup()
Additive Abelian Semigroup generated by 1, 5/3
```
class sage.rings.valuation.valuation_space.ScaleAction
    Bases: sage.categories.action.Action

    Action of integers, rationals and the infinity ring on valuations by scaling it.

    EXAMPLES:

    sage: v = QQ.valuation(5)
sage: from operator import mul
sage: v.parent().get_action(ZZ, mul, self_on_left=False)
Left action by Integer Ring on Discrete pseudo-valuations on Rational Field

5.4 Trivial valuations

AUTHORS:

• Julian Rüth (2016-10-14): initial version

EXAMPLES:

sage: v = valuations.TrivialValuation(QQ); v
Trivial valuation on Rational Field
sage: v(1)
0

class sage.rings.valuation.trivial_valuation.TrivialDiscretePseudoValuation(parent)
    Bases: sage.rings.valuation.trivial_valuation.TrivialDiscretePseudoValuation_base,
           sage.rings.valuation.valuation.InfiniteDiscretePseudoValuation

    The trivial pseudo-valuation that is $\infty$ everywhere.

    EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(QQ); v
Trivial pseudo-valuation on Rational Field

lift(X)
    Return a lift of $X$ to the domain of this valuation.

    EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(QQ)
sage: v.lift(v.residue_ring().zero())
0

reduce(x)
    Reduce $x$ modulo the positive elements of this valuation.

    EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(QQ)
sage: v.reduce(1)
0

residue_ring()
    Return the residue ring of this valuation.

    EXAMPLES:
valuations.TrivialPseudoValuation(QQ).residue_ring()
Quotient of Rational Field by the ideal (1)

value_group()
Return the value group of this valuation.

EXAMPLES:
A trivial discrete pseudo-valuation has no value group:

sage: v = valuations.TrivialPseudoValuation(QQ)
sage: v.value_group()
Traceback (most recent call last):
...  
ValueError: The trivial pseudo-valuation that is infinity everywhere does not have a value group.

class sage.rings.valuation.trivial_valuation.TrivialDiscretePseudoValuation_base (parent)
Bases: sage.rings.valuation.valuation.DiscretePseudoValuation

Base class for code shared by trivial valuations.

EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(ZZ); v
Trivial pseudo-valuation on Integer Ring

is_negative_pseudo_valuation()
Return whether this valuation attains the value $-\infty$.

EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(QQ)
sage: v.is_negative_pseudovaluation()
False

is_trivial()
Return whether this valuation is trivial.

EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(QQ)
sage: v.is_trivial()
True

uniformizer()
Return a uniformizing element for this valuation.

EXAMPLES:

sage: v = valuations.TrivialPseudoValuation(ZZ)
sage: v.uniformizer()
Traceback (most recent call last):
...  
ValueError: Trivial valuations do not define a uniformizing element

class sage.rings.valuation.trivial_valuation.TrivialDiscreteValuation (parent)
The trivial valuation that is zero on non-zero elements.

**EXAMPLES:**

```python
sage: v = valuations.TrivialValuation(QQ); v
Trivial valuation on Rational Field
```

**extensions** *(ring)*

Return the unique extension of this valuation to *ring*.

**EXAMPLES:**

```python
sage: v = valuations.TrivialValuation(ZZ)
sage: v.extensions(QQ)
[Trivial valuation on Rational Field]
```

**lift** *(X)*

Return a lift of *X* to the domain of this valuation.

**EXAMPLES:**

```python
sage: v = valuations.TrivialValuation(QQ)
sage: v.lift(v.residue_ring().zero())
0
```

**reduce** *(x)*

Reduce *x* modulo the positive elements of this valuation.

**EXAMPLES:**

```python
sage: v = valuations.TrivialValuation(QQ)
sage: v.reduce(1)
1
```

**residue_ring** *

Return the residue ring of this valuation.

**EXAMPLES:**

```python
sage: valuations.TrivialValuation(QQ).residue_ring()
Rational Field
```

**value_group** *

Return the value group of this valuation.

**EXAMPLES:**

```python
A trivial discrete valuation has a trivial value group:
```

```python
sage: v = valuations.TrivialValuation(QQ)
sage: v.value_group()
Trivial Additive Abelian Group
```

```python
class sage.rings.valuation.trivial_valuation.TrivialValuationFactory(clazz, parent, *args, **kwargs)

Bases: sage.structure.factory.UniqueFactory

Create a trivial valuation on domain.
```

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EXAMPLES:

```
sage: v = valuations.TrivialValuation(QQ); v
Trivial valuation on Rational Field
sage: v(1)
0
```

**create_key**(domain)
Create a key that identifies this valuation.

EXAMPLES:

```
sage: valuations.TrivialValuation(QQ) is valuations.TrivialValuation(QQ)  # indirect doctest
True
```

**create_object**(version, key, **extra_args)
Create a trivial valuation from key.

EXAMPLES:

```
sage: valuations.TrivialValuation(QQ)  # indirect doctest
Trivial valuation on Rational Field
```

## 5.5 Gauss valuations on polynomial rings

This file implements Gauss valuations for polynomial rings, i.e. discrete valuations which assign to a polynomial the minimal valuation of its coefficients.

AUTHORS:


EXAMPLES:

A Gauss valuation maps a polynomial to the minimal valuation of any of its coefficients:

```
sage: R.<x> = QQ[]
sage: v0 = QQ.valuation(2)
sage: v = GaussValuation(R, v0); v
Gauss valuation induced by 2-adic valuation
sage: v(2*x + 2)
1
```

Gauss valuations can also be defined iteratively based on valuations over polynomial rings:

```
sage: v = v.augmentation(x, 1/4); v
[ Gauss valuation induced by 2-adic valuation, v(x) = 1/4 ]
sage: v = v.augmentation(x^4+2*x^3+2*x^2+2*x+2, 4/3); v
[ Gauss valuation induced by 2-adic valuation, v(x) = 1/4, v(x^4 + 2*x^3 + 2*x^2 +
 2*x + 2) = 4/3 ]
sage: S.<T> = R[]
sage: w = GaussValuation(S, v); w
Gauss valuation induced by [ Gauss valuation induced by 2-adic valuation, v(x) = 1/4,
  v(x^4 + 2*x^3 + 2*x^2 + 2*x + 2) = 4/3 ]
sage: w(2*T + 1)
0
```
class sage.rings.valuation.gauss_valuation.GaussValuationFactory

Create a Gauss valuation on domain.

INPUT:

• domain – a univariate polynomial ring

• v – a valuation on the base ring of domain, the underlying valuation on the constants of the polynomial ring (if unspecified take the natural valuation on the valued ring domain.)

EXAMPLES:

The Gauss valuation is the minimum of the valuation of the coefficients:

```
sage: v = QQ.valuation(2)
sage: R.<x> = QQ[]
sage: w = GaussValuation(R, v)
sage: w(2)
1
sage: w(x)
0
sage: w(x + 2)
0
```

create_key(domain, v=None)

Normalize and check the parameters to create a Gauss valuation.

create_object(version, key, **extra_args)

Create a Gauss valuation from normalized parameters.

class sage.rings.valuation.gauss_valuation.GaussValuation_generic(parent, v)

A Gauss valuation on a polynomial ring domain.

INPUT:

• domain – a univariate polynomial ring over a valued ring \( R \)

• v – a discrete valuation on \( R \)

EXAMPLES:

```
sage: R = Zp(3,5)
sage: S.<x> = R[]
sage: v0 = R.valuation()
sage: v = GaussValuation(S, v0); v
Gauss valuation induced by 3-adic valuation
sage: S.<x> = QQ[]
sage: v = GaussValuation(S, QQ.valuation(5)); v
Gauss valuation induced by 5-adic valuation
```

E()

Return the ramification index of this valuation over its underlying Gauss valuation, i.e., 1.

EXAMPLES:

```
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
```

(continues on next page)
\begin{verbatim}
sage: v = GaussValuation($)
sage: v.E()
1
\end{verbatim}

\textbf{F()}

Return the degree of the residue field extension of this valuation over the Gauss valuation, i.e., 1.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<u> = QQ(4,5)
sage: S.<x> = R[

sage: v = GaussValuation($)
sage: v.F()
1
\end{verbatim}

\textbf{augmentation_chain()}

Return a list with the chain of augmentations down to the underlying Gauss valuation.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<u> = QQ(4,5)
sage: S.<x> = R[

sage: v = GaussValuation($)
sage: v.augmentation_chain()
[Gauss valuation induced by 2-adic valuation]
\end{verbatim}

\textbf{change_domain(ring)}

Return this valuation as a valuation over \texttt{ring}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: v = ZZ.valuation(2)
sage: R.<x> = ZZ[

sage: w = GaussValuation(R, v)
sage: w.change_domain(QQ['x'])
Gauss valuation induced by 2-adic valuation
\end{verbatim}

\textbf{element_with_valuation(s)}

Return a polynomial of minimal degree with valuation \texttt{s}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: v.element_with_valuation(-2)
1/4
\end{verbatim}

\textbf{equivalence_unit(s, reciprocal=False)}

Return an equivalence unit of valuation \texttt{s}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{s} – an element of the \texttt{value_group()}
  \item \texttt{reciprocal} – a boolean (default: False); whether or not to return the equivalence unit as the \texttt{equivalence_reciprocal()} of the equivalence unit of valuation \texttt{-s}
\end{itemize}

\textbf{EXAMPLES:}
```python
sage: S.<x> = Qp(3,5)[]
sage: v = GaussValuation(S)
sage: v.equivalence_unit(2)
3^2 + O(3^7)
sage: v.equivalence_unit(-2)
3^-2 + O(3^3)
```

**extensions** *(ring)*

Return the extensions of this valuation to *ring*.

**EXAMPLES:**

```python
sage: v = ZZ.valuation(2)
sage: R.<x> = ZZ[]
sage: w = GaussValuation(R, v)
sage: w.extensions(GaussianIntegers()['x'])
[Gauss valuation induced by 2-adic valuation]
```

**is_gauss_valuation()**

Return whether this valuation is a Gauss valuation.

**EXAMPLES:**

```python
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.is_gauss_valuation()
True
```

**is_trivial()**

Return whether this is a trivial valuation (sending everything but zero to zero.)

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: v.is_trivial()
True
```

**lift** *(F)*

Return a lift of *F*.

**INPUT:**

- *F* – a polynomial over the *residue_ring()* of this valuation

**OUTPUT:**

a (possibly non-monic) polynomial in the domain of this valuation which reduces to *F*

**EXAMPLES:**

```python
sage: S.<x> = Qp(3,5)[]
sage: v = GaussValuation(S)
sage: f = x^2 + 2*x + 16
sage: F = v.reduce(f); F
x^2 + 2*x + 1
sage: g = v.lift(F); g
(1 + O(3^5))*x^2 + (2 + O(3^5))*x + 1 + O(3^5)
sage: v.is_equivalent(f,g)
```

(continues on next page)
True
sage: g.parent() is v.domain()
True

See also:
reduce()

\texttt{lift\_to\_key}(F)

Lift the irreducible polynomial $F$ from the residue_ring() to a key polynomial over this valuation.

**INPUT:**

- $F$ – an irreducible non-constant monic polynomial in residue_ring() of this valuation

**OUTPUT:**

A polynomial $f$ in the domain of this valuation which is a key polynomial for this valuation and which, for a suitable equivalence unit $R$, satisfies that the reduction of $Rf$ is $F$

**EXAMPLES:**

```
sage: R.<u> = QQ
sage: S.<x> = R[]
sage: v = GaussValuation(S, QQ.valuation(2))
sage: y = v.residue_ring().gen()
sage: f = v.lift_to_key(y^2 + y + 1); f
x^2 + x + 1
```

\texttt{lower\_bound}(f)

Return an lower bound of this valuation at $f$.

Use this method to get an approximation of the valuation of $f$ when speed is more important than accuracy.

**EXAMPLES:**

```
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.lower_bound(1024*x + 2)
1
sage: v(1024*x + 2)
1
```

\texttt{monic\_integral\_model}(G)

Return a monic integral irreducible polynomial which defines the same extension of the base ring of the domain as the irreducible polynomial $G$ together with maps between the old and the new polynomial.

**EXAMPLES:**

```
sage: R.<x> = Qp(2, 5)[]
sage: v = GaussValuation(R)
sage: v.monic_integral_model(5*x^2 + 1/2*x + 1/4)
(Ring endomorphism of Univariate Polynomial Ring in x over 2-adic Field with capped relative precision 5
   Defn: (1 + O(2^5))*x |--> (2^-1 + O(2^4))*x,
Ring endomorphism of Univariate Polynomial Ring in x over 2-adic Field with capped relative precision 5
   Defn: (1 + O(2^5))*x |--> (2 + O(2^6))*x,
(1 + O(2^5))*x^2 + (1 + 2^2 + 2^3 + O(2^5))*x + 1 + 2^2 + 2^3 + O(2^5))
```
\texttt{reduce}(f, \text{check=True}, \text{degree\_bound=None}, \text{coefficients=None}, \text{valuations=None})

Return the reduction of \(f\) modulo this valuation.

INPUT:

\begin{itemize}
\item \(f\) – an integral element of the domain of this valuation
\item \texttt{check} – whether or not to check whether \(f\) has non-negative valuation (default: True)
\item \texttt{degree\_bound} – an a-priori known bound on the degree of the result which can speed up the computation (default: not set)
\item \texttt{coefficients} – the coefficients of \(f\) as produced by \texttt{coefficients()} or None (default: None); ignored
\item \texttt{valuations} – the valuations of \texttt{coefficients} or None (default: None); ignored
\end{itemize}

OUTPUT:

A polynomial in the \texttt{residue\_ring()} of this valuation.

EXAMPLES:

\begin{verbatim}
sage: S.<x> = Qp(2,5)[]
sage: v = GaussValuation(S)
sage: f = x^2 + 2*x + 16
sage: v.reduce(f)
x^2
sage: v.reduce(f).parent() is v.residue_ring()
True
\end{verbatim}

The reduction is only defined for integral elements:

\begin{verbatim}
sage: f = x^2/2
sage: v.reduce(f)
Traceback (most recent call last):
  ...
ValueError: reduction not defined for non-integral elements and (2^-1 + O(2^˓→4))*x^2 is not integral over Gauss valuation induced by 2-adic valuation
\end{verbatim}

See also:

\texttt{lift()}

\texttt{residue\_ring()}

Return the residue ring of this valuation, i.e., the elements of valuation zero modulo the elements of positive valuation.

EXAMPLES:

\begin{verbatim}
sage: S.<x> = Qp(2,5)[]
sage: v = GaussValuation(S)
sage: v.residue_ring()
Univariate Polynomial Ring in x over Finite Field of size 2 (using ...)
\end{verbatim}

\texttt{restriction}(\texttt{ring})

Return the restriction of this valuation to \texttt{ring}.

EXAMPLES:
sage: v = ZZ.valuation(2)
sage: R.<x> = ZZ[]
sage: w = GaussValuation(R, v)
sage: w.restriction(ZZ)
2-adic valuation

scale(scalar)
Return this valuation scaled by scalar.

EXAMPLES:

sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: 3*v # indirect doctest
Gauss valuation induced by 3 * 2-adic valuation

simplify(f, error=None, force=False, size_heuristic_bound=32, effective_degree=None, phiadic=True)
Return a simplified version of \( f \).

Produce an element which differs from \( f \) by an element of valuation strictly greater than the valuation of \( f \) (or strictly greater than \( \text{error} \) if set.)

INPUT:

• \( f \) – an element in the domain of this valuation
• \( \text{error} \) – a rational, infinity, or None (default: None), the error allowed to introduce through the simplification
• \( \text{force} \) – whether or not to simplify \( f \) even if there is heuristically no change in the coefficient size of \( f \) expected (default: False)
• \( \text{effective\_degree} \) – when set, assume that coefficients beyond \( \text{effective\_degree} \) can be safely dropped (default: None)
• \( \text{size\_heuristic\_bound} \) – when \( \text{force} \) is not set, the expected factor by which the coefficients need to shrink to perform an actual simplification (default: 32)
• \( \text{phiadic} \) – whether to simplify in the \( x \)-adic expansion; the parameter is ignored as no other simplification is implemented

EXAMPLES:

sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: f = x^10/2 + 1
sage: v.simplify(f)
(2^-1 + O(2^4))*x^10 + 1 + O(2^5)

uniformizer()
Return a uniformizer of this valuation, i.e., a uniformizer of the valuation of the base ring.

EXAMPLES:

sage: S.<x> = QQ[]
sage: v = GaussValuation(S, QQ.valuation(5))
sage: v.uniformizer()
5

(continues on next page)
sage: v.uniformizer().parent() is S
True

\textbf{upper\_bound}(f)

Return an upper bound of this valuation at \(f\).

Use this method to get an approximation of the valuation of \(f\) when speed is more important than accuracy.

\textbf{EXAMPLES:}

```
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.upper_bound(1024*x + 1)
10
sage: v(1024*x + 1)
0
```

\textbf{valuations}(f, \text{coefficients=None, call\_error=False})

Return the valuations of the \(f_i \phi^i\) in the expansion \(f = \sum f_i \phi^i\).

\textbf{INPUT:}

- \(f\) – a polynomial in the domain of this valuation
- \text{coefficients} – the coefficients of \(f\) as produced by \text{coefficients}() or \text{None} (default: \text{None}); this can be used to speed up the computation when the expansion of \(f\) is already known from a previous computation.
- \text{call\_error} – whether or not to speed up the computation by assuming that the result is only used to compute the valuation of \(f\) (default: \text{False})

\textbf{OUTPUT:}

A list, each entry a rational numbers or infinity, the valuations of \(f_0, f_1 \phi, \ldots\)

\textbf{EXAMPLES:}

```
sage: R = ZZ
sage: S.<x> = R[]
sage: v = GaussValuation(S, R.valuation(2))
sage: f = x^2 + 2*x + 16
sage: list(v.valuations(f))
[4, 1, 0]
```

\textbf{value\_group}()

Return the value group of this valuation.

\textbf{EXAMPLES:}

```
sage: S.<x> = QQ[]
sage: v = GaussValuation(S, QQ.valuation(5))
sage: v.value_group()
Additive Abelian Group generated by 1
```

\textbf{value\_semigroup}()

Return the value semigroup of this valuation.

\textbf{EXAMPLES:}

```
sage: S.<x> = QQ[]
sage: v = GaussValuation(S, QQ.valuation(5))
sage: v.value_semigroup()
Additive Abelian Group generated by 1
```
5.6 Valuations on polynomial rings based on \(\phi\)-adic expansions

This file implements a base class for discrete valuations on polynomial rings, defined by a \(\phi\)-adic expansion.

AUTHORS:


EXAMPLES:

The Gauss valuation is a simple example of a valuation that relies on \(\phi\)-adic expansions:

```sage
eval = QQ.valuation(5)
S.<x> = QQ[]
v = GaussValuation(S, eval)
v.value_semigroup()
```

Additive Abelian Semigroup generated by -1, 1

In this case, \(\phi = x\), so the expansion simply lists the coefficients of the polynomial:

```sage
f = x^2 + 2*x + 2
list(v.coefficients(f))
```

\([2, 2, 1]\)

Often only the first few coefficients are necessary in computations, so for performance reasons, coefficients are computed lazily:

```sage
v.coefficients(f)
```

<generator object ...coefficients at 0x...>

Another example of a DevelopingValuation is an augmented valuation:

```sage
w = v.augmentation(x^2 + x + 1, 3)
```

Here, the expansion lists the remainders of repeated division by \(x^2 + x + 1\):

```sage
list(w.coefficients(f))
```

\([x + 1, 1]\)

class \texttt{sage.rings.valuation.developing_valuation.DevelopingValuation}(parent, \phi)

Bases: \texttt{sage.rings.valuation.valuation.DiscretePseudoValuation}

Abstract base class for a discrete valuation of polynomials defined over the polynomial ring domain by the \(\phi\)-adic development.

EXAMPLES:

```sage
R.<x> = QQ[]
v = GaussValuation(R, QQ.valuation(7))
```

\texttt{coefficients}(f)

Return the \(\phi\)-adic expansion of \(f\).

INPUT:
• \( f \) – a monic polynomial in the domain of this valuation

**OUTPUT:**

An iterator \( f_0, f_1, \ldots, f_n \) of polynomials in the domain of this valuation such that \( f = \sum_i f_i \phi^i \)

**EXAMPLES:**

```python
sage: R = Qp(2,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: f = x^2 + 2*x + 3
sage: list(v.coefficients(f))  # note that these constants are in the polynomial ring
[1 + 2 + O(2^5), 2 + O(2^6), 1 + O(2^5)]
sage: v = v.augmentation( x^2 + x + 1, 1)
sage: list(v.coefficients(f))
[(1 + O(2^5))*x + 2 + O(2^5), 1 + O(2^5)]
```

**effective_degree** \((f, \text{valuations=}\text{None})\)

Return the effective degree of \( f \) with respect to this valuation.

The effective degree of \( f \) is the largest \( i \) such that the valuation of \( f \) and the valuation of \( f_i \phi^i \) in the development \( f = \sum_j f_j \phi^j \) coincide (see [Mac1936II] p.497.)

**INPUT:**

• \( f \) – a non-zero polynomial in the domain of this valuation

**EXAMPLES:**

```python
sage: R = Zp(2,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.effective_degree(x)
1
sage: v.effective_degree(2*x + 1)
0
```

**newton_polygon** \((f, \text{valuations=}\text{None})\)

Return the newton polygon of the \( \phi \)-adic development of \( f \).

**INPUT:**

• \( f \) – a polynomial in the domain of this valuation

**EXAMPLES:**

```python
sage: R = Qp(2,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: f = x^2 + 2*x + 3
sage: v.newton_polygon(f)
Finite Newton polygon with 2 vertices: (0, 0), (2, 0)
sage: v = v.augmentation( x^2 + x + 1, 1)
sage: v.newton_polygon(f)
Finite Newton polygon with 2 vertices: (0, 0), (1, 1)
sage: v.newton_polygon( f * v.phi()^3 )
Finite Newton polygon with 2 vertices: (3, 3), (4, 4)
```
\textbf{\texttt{phi}()}

Return the polynomial \( \phi \), the key polynomial of this valuation.

\textbf{EXAMPLES:}

```
sage: R = Zp(2, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.phi()
(1 + O(2^5))*x
```

\textbf{valuations} \((f)\)

Return the valuations of the \( f_i \phi^i \) in the expansion \( f = \sum f_i \phi^i \).

\textbf{INPUT:}

- \( f \) – a polynomial in the domain of this valuation

\textbf{OUTPUT:}

A list, each entry a rational numbers or infinity, the valuations of \( f_0, f_1 \phi, \ldots \)

\textbf{EXAMPLES:}

```
sage: R = Qp(2, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S, R.valuation())
sage: f = x^2 + 2*x + 16
sage: list(v.valuations(f))
[4, 1, 0]
```

\section{5.7 Inductive valuations on polynomial rings}

This module provides functionality for inductive valuations, i.e., finite chains of \textit{augmented valuations} on top of a \textit{Gauss valuation}.

\textbf{AUTHORS:}

- Julian Rüth (2016-11-01): initial version

\textbf{EXAMPLES:}

A \textit{Gauss valuation} is an example of an inductive valuation:

```
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
```

Generally, an inductive valuation is an augmentation of an inductive valuation, i.e., a valuation that was created from a Gauss valuation in a finite number of augmentation steps:

```
sage: w = v.augmentation(x, 1)
sage: w = w.augmentation(x^2 + 2*x + 4, 3)
```

\textbf{REFERENCES:}

Inductive valuations are originally discussed in [Mac1936I] and [Mac1936II]. An introduction is also given in Chapter 4 of [Rüt2014].
class sage.rings.valuation.inductive_valuation.FinalInductiveValuation(parent, phi)

Bases: sage.rings.valuation.inductive_valuation.InductiveValuation

Abstract base class for an inductive valuation which can not be augmented further.

class sage.rings.valuation.inductive_valuation.FiniteInductiveValuation(parent, phi)


Abstract base class for iterated augmented valuations on top of a Gauss valuation which is a discrete valuation, i.e., the last key polynomial has finite valuation.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
```

extensions (other)

Return the extensions of this valuation to other.

EXAMPLES:

```python
sage: R.<x> = ZZ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(ZZ))
sage: K.<x> = FunctionField(QQ)
sage: v.extensions(K)
[Trivial valuation on Rational Field]
```

class sage.rings.valuation.inductive_valuation.InductiveValuation(parent, phi)

Bases: sage.rings.valuation.developing_valuation.DevelopingValuation

Abstract base class for iterated augmented valuations on top of a Gauss valuation.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(5))
```

E()

Return the ramification index of this valuation over its underlying Gauss valuation.

EXAMPLES:

```python
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.E()
1
```

F()

Return the residual degree of this valuation over its Gauss extension.

EXAMPLES:

```python
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
```

(continues on next page)
augmentation_chain()

Return a list with the chain of augmentations down to the underlying Gauss valuation.

EXAMPLES:

```python
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.augmentation_chain()
[Gauss valuation induced by 2-adic valuation]
```

element_with_valuation(s)

Return a polynomial of minimal degree with valuation s.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: v.element_with_valuation(-2)
1/4
```

Depending on the base ring, an element of valuation s might not exist:

```python
sage: R.<x> = ZZ[]
sage: v = GaussValuation(R, ZZ.valuation(2))
sage: v.element_with_valuation(-2)
Traceback (most recent call last):
... ValueError: s must be in the value semigroup of this valuation but -2 is not in Additive Abelian Semigroup generated by 1
```

equivalence_reciprocal (f, coefficients=None, valuations=None, check=True)

Return an equivalence reciprocal of f.

An equivalence reciprocal of f is a polynomial h such that f \cdot h is equivalent to 1 modulo this valuation (see [Mac1936II] p.497.)

INPUT:

- f – a polynomial in the domain of this valuation which is an equivalence_unit()
- coefficients – the coefficients of f in the \phi()-adic expansion if known (default: None)
- valuations – the valuations of coefficients if known (default: None)
- check – whether or not to check the validity of f (default: True)

Warning: This method may not work over \(p\)-adic rings due to problems with the xgcd implementation there.

EXAMPLES:

```python
sage: R = Zp(3,5)
sage: S.<x> = R[]
```
In an extended valuation over an extension field:

```python
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v = v.augmentation(x^2 + x + u, 1)
sage: f = 2*x + u
sage: h = v.equivalence_reciprocal(f); h
(u + 1) + O(2^5)
sage: v.is_equivalent(f*h, 1)
True
```

Extending the valuation once more:

```python
sage: v = v.augmentation((x^2 + x + u)^2 + 2*x*(x^2 + x + u) + 4*x, 3)
sage: h = v.equivalence_reciprocal(f); h
(u + 1) + O(2^5)
sage: v.is_equivalent(f*h, 1)
True
```

equivalence_unit (s, reciprocal=False)
Return an equivalence unit of valuation s.

INPUT:

- s – an element of the value_group()
- reciprocal – a boolean (default: False); whether or not to return the equivalence unit as the equivalence_reciprocal() of the equivalence unit of valuation -s.

EXAMPLES:

```python
sage: S.<x> = Qp(3,5)[]
sage: v = GaussValuation(S)
sage: v.equivalence_unit(2)
3^2 + O(3^7)
sage: v.equivalence_unit(-2)
3^-2 + O(3^3)
```

Note that this might fail for negative s if the domain is not defined over a field:

```python
sage: v = ZZ.valuation(2)
sage: R.<x> = ZZ[]
sage: w = GaussValuation(R, v)
sage: w.equivalence_unit(1)
2
sage: w.equivalence_unit(-1)
Traceback (most recent call last):
  ...
ValueError: s must be in the value semigroup of this valuation but -1 is not in Additive Abelian Semigroup generated by 1
```
**is_equivalence_unit** *(f, valuations=None)*
Return whether the polynomial \( f \) is an equivalence unit, i.e., an element of \( \text{effective_degree}() \) zero (see [Mac1936II] p.497.)

**INPUT:**
- \( f \) – a polynomial in the domain of this valuation

**EXAMPLES:**
```
sage: R = Zp(2,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.is_equivalence_unit(x)
False
sage: v.is_equivalence_unit(S.zero())
False
sage: v.is_equivalence_unit(2*x + 1)
True
```

**is_gauss_valuation()**
Return whether this valuation is a Gauss valuation over the domain.

**EXAMPLES:**
```
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.is_gauss_valuation()
True
```

**monic_integral_model** *(G)*
Return a monic integral irreducible polynomial which defines the same extension of the base ring of the domain as the irreducible polynomial \( G \) together with maps between the old and the new polynomial.

**EXAMPLES:**
```
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: v.monic_integral_model(5*x^2 + 1/2*x + 1/4)
(Ring endomorphism of Univariate Polynomial Ring in x over Rational Field
 Defn: x |--> 1/2*x,
 Ring endomorphism of Univariate Polynomial Ring in x over Rational Field
 Defn: x |--> 2*x,
 x^2 + 1/5*x + 1/5)
```

**mu()**
Return the valuation of \( \phi() \).

**EXAMPLES:**
```
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: v.mu()
0
```

**class** `sage.rings.valuation.inductive_valuation.InfiniteInductiveValuation` *(parent, base_valuation)*
**Bases:** `sage.rings.valuation.inductive_valuation.FinalInductiveValuation`, `sage.rings.valuation.valuation.InfiniteDiscretePseudoValuation`
Abstract base class for an inductive valuation which is not discrete, i.e., which assigns infinite valuation to its last key polynomial.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]

sage: v = GaussValuation(R, QQ.valuation(2))

sage: w = v.augmentation(x^2 + x + 1, infinity)
```

**change_domain**(ring)

Return this valuation over ring.

**EXAMPLES:**

We can turn an infinite valuation into a valuation on the quotient:

```python
sage: R.<x> = QQ[]

sage: v = GaussValuation(R, QQ.valuation(2))

sage: w = v.augmentation(x^2 + x + 1, infinity)

sage: w.change_domain(R.quo(x^2 + x + 1))
```

5.7. Inductive valuations on polynomial rings

```
(continues on next page)
```
See also:

* augmented_valuation

**equivalence_decomposition** *(f, assume_not_equivalence_unit=False, coefficients=None, valuations=None, compute_unit=True, degree_bound=None)*

Return an equivalence decomposition of \( f \), i.e., a polynomial \( g(x) = e(x) \prod_i \phi_i(x) \) with \( e(x) \) an equivalence unit and the \( \phi_i \) key polynomials such that \( f \) is equivalent to \( g \).

**INPUT:**

- \( f \) – a non-zero polynomial in the domain of this valuation
- \( \text{assume_not_equivalence_unit} \) – whether or not to assume that \( f \) is not an equivalence unit (default: False)
- \( \text{coefficients} \) – the coefficients of \( f \) in the \( \phi_i \)-adic expansion if known (default: None)
- \( \text{valuations} \) – the valuations of coefficients if known (default: None)
- \( \text{compute_unit} \) – whether or not to compute the unit part of the decomposition (default: True)
- \( \text{degree_bound} \) – a bound on the degree of the \_equivalence_reduction() of \( f \) (default: None)

**ALGORITHM:**

We use the algorithm described in Theorem 4.4 of [Mac1936II]. After removing all factors \( \phi \) from a polynomial \( f \), there is an equivalence unit \( R \) such that \( RF \) has valuation zero. Now \( RF \) can be factored as \( \prod_i \alpha_i \) over the residue_field(). Lifting all \( \alpha_i \) to key polynomials \( \phi_i \) gives \( RF = \prod_i R_i f_i \) for suitable equivalence units \( R_i \) (see lift_to_key()). Taking \( R' \) an equivalence_reciprocal() of \( R \), we have \( f \) equivalent to \( (R' \prod_i R_i) \prod_i \phi_i \).

**EXAMPLES:**

```python
sage: R.<u> = Qq(4,10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.equivalence_decomposition(S.zero())
Traceback (most recent call last):
... ValueError: equivalence decomposition of zero is not defined
sage: v.equivalence_decomposition(S.one())
1 + O(2^10)
sage: v.equivalence_decomposition(x^2+2)
((1 + O(2^10))*x)^2
sage: v.equivalence_decomposition(x^2+1)
((1 + O(2^10))*x + 1 + O(2^10))^2
```

A polynomial that is an equivalence unit, is returned as the unit part of a Factorization, leading to a unit non-minimal degree:

```python
sage: w = v.augmentation(x, 1)
sage: F = w.equivalence_decomposition(x^2+1); F
(1 + O(2^10))*x^2 + 1 + O(2^10)
sage: F.unit()
(1 + O(2^10))*x^2 + 1 + O(2^10)
```
However, if the polynomial has a non-unit factor, then the unit might be replaced by a factor of lower degree:

```python
sage: f = x * (x^2 + 1)
sage: F = w.equivalence_decomposition(f); F
(1 + O(2^10))*x
sage: F.unit()
1 + O(2^10)
```

Examples over an iterated unramified extension:

```python
sage: v = v.augmentation(x^2 + x + u, 1)
sage: v = v.augmentation((x^2 + x + u)^2 + 2*x*(x^2 + x + u) + 4*x, 3)
sage: v.equivalence_decomposition(x)
(1 + O(2^10))*x
sage: F = v.equivalence_decomposition( v.phi() )
sage: len(F)
1
sage: F = v.equivalence_decomposition( v.phi() * (x^4 + 4*x^3 + (7 + 2*u)*x^2 + (8 + 4*u)*x + 1023 + 3*u) )
sage: len(F)
2
```

`is_equivalence_irreducible(f, coefficients=None, valuations=None)`

Return whether the polynomial `f` is equivalence-irreducible, i.e., whether its `equivalence_decomposition()` is trivial.

ALGORITHM:

We use the same algorithm as in `equivalence_decomposition()` we just do not lift the result to key polynomials.

INPUT:

- `f` – a non-constant polynomial in the domain of this valuation

EXAMPLES:

```python
sage: R.<u> = Qq(4,5)
sage: S.<x> = R[

```
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.is_key(x)
True
sage: v.is_key(2*x, explain = True)
(False, 'phi must be monic')
sage: v.is_key(x^2, explain = True)
(False, 'phi must be equivalence irreducible')
sage: w = v.augmentation(x, 1)
sage: w.is_key(x + 1, explain = True)
(False, 'phi must be minimal')

is_minimal (f, assume_equivalence_irreducible=False)
Return whether the polynomial \( f \) is minimal with respect to this valuation.

A polynomial \( f \) is minimal with respect to \( v \) if it is not a constant and any non-zero polynomial \( h \) which is \( v \)-divisible by \( f \) has at least the degree of \( f \).

A polynomial \( h \) is \( v \)-divisible by \( f \) if there is a polynomial \( c \) such that \( f c \) is_equivalent() to \( h \).

ALGORITHM:
Based on Theorem 9.4 of [Mac1936II].

EXAMPLES:

sage: R.<u> = Qq(4, 10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: y = v.residue_ring().gen()
sage: u0 = v.residue_ring().base_ring().gen()
mac_lane_step\( (G, \text{ principal_part_bound}=\text{None}, \text{ assume_squarefree}=\text{False}, \text{ assume_equivalence_irreducible}=\text{False}, \text{ report_degree_bounds_and_caches}=\text{False}, \text{ coefficients}=\text{None}, \text{ valuations}=\text{None}, \text{ check}=\text{True}, \text{ allow_equivalent_key}=\text{True}) \)

Perform an approximation step towards the squarefree monic non-constant integral polynomial \( G \) which is not an equivalence unit.

This performs the individual steps that are used in \textit{mac_lane_approximants}().

**INPUT:**

- \( G \) – a squarefree monic non-constant integral polynomial \( G \) which is not an equivalence unit
- \textit{principal_part_bound} – an integer or None (default: None), a bound on the length of the principal part, i.e., the section of negative slope, of the Newton polygon of \( G \)
- \textit{assume_squarefree} – whether or not to assume that \( G \) is squarefree (default: False)
- \textit{assume_equivalence_irreducible} – whether or not to assume that \( G \) is equivalence irreducible (default: False)
- \textit{report_degree_bounds_and_caches} – whether or not to include internal state with the returned value (used by \textit{mac_lane_approximants}() to speed up sequential calls)
- \textit{coefficients} – the coefficients of \( G \) in the \( \phi() \)-adic expansion if known (default: None)
- \textit{valuations} – the valuations of coefficients if known (default: None)
- \textit{check} – whether to check that \( G \) is a squarefree monic non-constant integral polynomial and not an equivalence unit (default: True)
- \textit{allow_equivalent_key} – whether to return valuations which end in essentially the same key polynomial as this valuation but have a higher valuation assigned to that key polynomial (default: True)

**EXAMPLES:**

We can use this method to perform the individual steps of \textit{mac_lane_approximants}():

```
sage: R.<x> = QQ[]
sage: v = QQ.valuation(2)
sage: f = x^36 + 1160/81*x^31 + 9920/27*x^30 + 1040/81*x^26 + 52480/81*x^25 +
    220160/81*x^24 - 5120/81*x^21 - 143360/81*x^19 + 12451840/81*x^18
    - 20316160/567*x^16 - 198737920/189*x^15 - 1129840640/81*x^14 -
    8192/81*x^11 + 553650/81*x^10 +
    5242880/21*x^9 + 2118123520/567*x^8 + 15460204544/567*x^7
    + 6509559808/81*x^6 - 16777216/567*x^2 - 268435456/567*x - 1073741824/567
sage: v.mac_lane_approximants(f)
[[ Gauss valuation induced by 2-adic valuation, v(x + 2056) = 23/2 ],
 [ Gauss valuation induced by 2-adic valuation, v(x) = 11/9 ],
 [ Gauss valuation induced by 2-adic valuation, v(x) = 2/5, v(x^5 + 4) = 7/2 ]]
```

Starting from the Gauss valuation, a MacLane step branches off with some linear key polynomials in the above example:
The computation of MacLane approximants would now perform a MacLane step on each of these branches, note however, that a direct call to this method might produce some unexpected results:

```
sage: V1[0].mac_lane_step(f)
[[ Gauss valuation induced by 2-adic valuation, v(x) = 3/5, v(x^5 + 8) = 5 ],
[ Gauss valuation induced by 2-adic valuation, v(x) = 3/5, v(x^10 + 8*x^5 + \cdots - 64) = 7 ],
[ Gauss valuation induced by 2-adic valuation, v(x) = 3 ],
[ Gauss valuation induced by 2-adic valuation, v(x) = 11/9 ]]
```

Note how this detected the two augmentations of V1[0] but also two other valuations that we had seen in the previous step and that are greater than V1[0]. To ignore such trivial augmentations, we can set `allow_equivalent_key`:

```
sage: V1[0].mac_lane_step(f, allow_equivalent_key=False)
[[ Gauss valuation induced by 2-adic valuation, v(x) = 3/5, v(x^5 + 8) = 5 ],
[ Gauss valuation induced by 2-adic valuation, v(x) = 3/5, v(x^10 + 8*x^5 + \cdots - 64) = 7 ]]
```

**minimal_representative** *(f)*

Return a minimal representative for *f*, i.e., a pair *e*, *a* such that *f* is_equivalent() to *ea*, *e* is an equivalence unit, and *a* is_minimal() and monic.

**INPUT:**

- *f* – a non-zero polynomial which is not an equivalence unit

**OUTPUT:**

A factorization which has *e* as its unit and *a* as its unique factor.

**ALGORITHM:**

We use the algorithm described in the proof of Lemma 4.1 of [Mac1936II]. In the expansion \( f = \sum_i f_i \phi^i \) take \( e = f_i \) for the largest *i* with \( f_i \phi^i \) minimal (see `effective_degree()`). Let \( h \) be the equivalence_reciprocal() of *e* and take *a* given by the terms of minimal valuation in the expansion of *ef*.

**EXAMPLES:**

```
sage: R.<u> = Qq(4,10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.minimal_representative(x + 2)
(1 + O(2^10))*x
sage: v = v.augmentation(x, 1)
sage: v.minimal_representative(x + 2)
(1 + O(2^10))*x + 2 + O(2^11)
sage: f = x^3 + 6*x + 4
sage: F = v.minimal_representative(f); F
(2 + 2^2 + O(2^11)) * ((1 + O(2^10))*x + 2 + O(2^11))
```
5.8 Augmented valuations on polynomial rings

Implements augmentations of (inductive) valuations.

AUTHORS:
• Julian Rüth (2013-04-15): initial version

EXAMPLES:
Starting from a Gauss valuation, we can create augmented valuations on polynomial rings:

```sage
code
R.<x> = QQ[]
v = GaussValuation(R, QQ.valuation(2))
w = v.augmentation(x, 1); w
[ Gauss valuation induced by 2-adic valuation, v(x) = 1 ]
sage: w(x)
1
```

This also works for polynomial rings over base rings which are not fields. However, much of the functionality is only available over fields:

```sage
code
R.<x> = ZZ[]
v = GaussValuation(R, ZZ.valuation(2))
w = v.augmentation(x, 1); w
[ Gauss valuation induced by 2-adic valuation, v(x) = 1 ]
sage: w(x)
1
```

REFERENCES:
Augmentations are described originally in [Mac1936I] and [Mac1936II]. An overview can also be found in Chapter 4 of [Rüt2014].

```python
code
class AugmentedValuationFactory
Bases: sage.structure.factory.UniqueFactory

Factory for augmented valuations.

EXAMPLES:
This factory is not meant to be called directly. Instead, augmentation() of a valuation should be called:

```sage
code
R.<x> = QQ[]
v = GaussValuation(R, QQ.valuation(2))
w = v.augmentation(x, 1) # indirect doctest
```

Note that trivial parts of the augmented valuation might be dropped, so you should not rely on _basevaluation to be the valuation you started with:

```python
sage: ww = w.augmentation(x, 2)
sage: ww._base_valuation is v
True
```

**create_key** *(base_valuation, phi, mu, check=True)*

Create a key which uniquely identifies the valuation over `base_valuation` which sends `phi` to `mu`.

**Note:** The uniqueness that this factory provides is not why we chose to use a factory. However, it makes pickling and equality checks much easier. At the same time, going through a factory makes it easier to enforce that all instances correctly inherit methods from the parent Hom space.

```python
create_object (version, key)
```

Create the augmented valuation represented by `key`.

**class** `sage.rings.valuation.augmented_valuation.AugmentedValuation_base

An augmented valuation is a discrete valuation on a polynomial ring. It extends another discrete valuation `v` by setting the valuation of a polynomial `f` to the minumum of `v(f[i])mu` when writing `f = sum_i f[i]phi[i]`.

**INPUT:**

- `v` -- an InductiveValuation on a polynomial ring
- `phi` -- a key polynomial over `v`
- `mu` -- a rational number such that `mu > v(phi)` or infinity

**EXAMPLES:**

```python
sage: K.<u> = CyclotomicField(5)
sage: R.<x> = K[

sage: v = GaussValuation(R, K.valuation(2))
sage: w = v.augmentation(x, 1/2); w  
# indirect doctest

[ Gauss valuation induced by 2-adic valuation, v(x) = 1/2 ]

sage: ww = w.augmentation(x^4 + 2*x^2 + 4*u, 3); ww

[ Gauss valuation induced by 2-adic valuation, v(x) = 1/2, v(x^4 + 2*x^2 + 4*u) = 3 ]
```

**E()**

Return the ramification index of this valuation over its underlying Gauss valuation.

**EXAMPLES:**

```python
sage: K.<u> = Qq(4, 5)
sage: S.<x> = R[

sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1)
sage: w.E()

1

sage: w = v.augmentation(x, 1/2)
sage: w.E()

2
```
\textbf{F()}

Return the degree of the residue field extension of this valuation over the underlying Gauss valuation.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1)
sage: w.F()
2
sage: w = v.augmentation(x, 1/2)
sage: w.F()
1
\end{verbatim}

\textbf{augmentation_chain()}

Return a list with the chain of augmentations down to the underlying Gauss valuation.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x, 1)
sage: w.augmentation_chain()
[[ Gauss valuation induced by 2-adic valuation, v(x) = 1 ],
  Gauss valuation induced by 2-adic valuation]
\end{verbatim}

For performance reasons, (and to simplify the underlying implementation,) trivial augmentations might get dropped. You should not rely on \texttt{augmentation\_chain()} to contain all the steps that you specified to create the current valuation:

\begin{verbatim}
sage: ww = w.augmentation(x, 2)
sage: ww.augmentation_chain()
[[ Gauss valuation induced by 2-adic valuation, v(x) = 2 ],
  Gauss valuation induced by 2-adic valuation]
\end{verbatim}

\textbf{change\_domain(ring)}

Return this valuation over \texttt{ring}.

\textbf{EXAMPLES:}

We can change the domain of an augmented valuation even if there is no coercion between rings:

\begin{verbatim}
sage: R.<x> = GaussianIntegers()[]
sage: v = GaussValuation(R, GaussianIntegers().valuation(2))
sage: v = v.augmentation(x, 1)
sage: v.change_domain(QQ['x'])
[ Gauss valuation induced by 2-adic valuation, v(x) = 1 ]
\end{verbatim}

\textbf{element\_with\_valuation(s)}

Create an element of minimal degree and of valuation \texttt{s}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{s} – a rational number in the value group of this valuation
\end{itemize}

\textbf{OUTPUT:}

An element in the domain of this valuation
EXAMPLES:

```python
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[

sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.element_with_valuation(0)
1 + O(2^5)
sage: w.element_with_valuation(1/2)
(1 + O(2^5))*x^2 + (1 + O(2^5))*x + u + O(2^5)
sage: w.element_with_valuation(1)
2 + O(2^6)
sage: c = w.element_with_valuation(-1/2); c
(2^-1 + O(2^4))*x^2 + (2^-1 + O(2^4))*x + u*2^-1 + O(2^4)
sage: w(c)
-1/2
sage: w.element_with_valuation(1/3)
Traceback (most recent call last):
... ValueError: s must be in the value group of the valuation but 1/3 is not in...
```

```
→ Additive Abelian Group generated by 1/2.
```

equivalence_unit \((s, \text{reciprocal=}False)\)

Return an equivalence unit of minimal degree and valuation \(s\).

**INPUT:**

- \(s\) – a rational number
- \(\text{reciprocal}\) – a boolean (default: False); whether or not to return the equivalence unit as the \(\text{equivalence_reciprocal()}\) of the equivalence unit of valuation \(-s\).

**OUTPUT:**

A polynomial in the domain of this valuation which \(\text{is_equivalence_unit()}\) for this valuation.

**EXAMPLES:**

```
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[

sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1)
sage: w.equivalence_unit(0)
1 + O(2^5)
sage: w.equivalence_unit(-4)
2^-4 + O(2)
sage: c = w.element_with_valuation(-1/2); c
(2^-1 + O(2^4))*x^2 + (2^-1 + O(2^4))*x + u*2^-1 + O(2^4)
sage: w(c)
-1/2
sage: w.element_with_valuation(1/3)
Traceback (most recent call last):
... ValueError: s must be in the value group of the valuation but 1/3 is not in...
```

Since an equivalence unit is of effective degree zero, \(\phi\) must not divide it. Therefore, its valuation is in the value group of the base valuation:

```
sage: w = v.augmentation(x, 1/2)
sage: w.equivalence_unit(3/2)
Traceback (most recent call last):
... ValueError: 3/2 is not in the value semigroup of 2-adic valuation
sage: w.equivalence_unit(1)
2 + O(2^6)
```

An equivalence unit might not be integral, even if \(s \geq 0\):
extensions(ring)

Return the extensions of this valuation to ring.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: w.extensions(GaussianIntegers().fraction_field()['x'])
[[ Gauss valuation induced by 2-adic valuation, v(x^2 + x + 1) = 1 ]]
```

is_gauss_valuation()

Return whether this valuation is a Gauss valuation.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: w.is_gauss_valuation()  # False
```

is_negative_pseudo_valuation()

Return whether this valuation attains $-\infty$.

EXAMPLES:

No element in the domain of an augmented valuation can have valuation $-\infty$, so this method always returns False:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: w = v.augmentation(x, infinity)
sage: w.is_negative_pseudo_valuation()  # False
```

is_trivial()

Return whether this valuation is trivial, i.e., zero outside of zero.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: w.is_trivial()  # False
```

monic_integral_model(G)

Return a monic integral irreducible polynomial which defines the same extension of the base ring of the domain as the irreducible polynomial $G$ together with maps between the old and the new polynomial.

EXAMPLES:
psi()  
Return the minimal polynomial of the residue field extension of this valuation.

OUTPUT:
A polynomial in the residue ring of the base valuation

EXAMPLES:

```sage
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.psi()
x^2 + x + u0
```

restriction(ring)  
Return the restriction of this valuation to ring.

EXAMPLES:

```sage:sage: K = GaussianIntegers().fraction_field()
sage: R.<x> = K[]
sage: v = GaussValuation(R, K.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: w.restriction(QQ['x'])
[ Gauss valuation induced by 2-adic valuation, v(x^2 + x + 1) = 1 ]
```

scale(scalar)  
Return this valuation scaled by scalar.

EXAMPLES:

```sage:sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: 3*w # indirect doctest
[ Gauss valuation induced by 3 * 2-adic valuation, v(x^2 + x + 1) = 3 ]
```

uniformizer()  
Return a uniformizing element for this valuation.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: w.uniformizer()
2
```

```python
class sage.rings.valuation.augmented_valuation.FinalAugmentedValuation(
    parent, v, phi, mu)

Bases: sage.rings.valuation.augmented_valuation.AugmentedValuation_base,
       sage.rings.valuation.inductive_valuation.FinalInductiveValuation

An augmented valuation which can not be augmented anymore, either because it augments a trivial valuation or because it is infinite.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: w = v.augmentation(x, 1)

lift (F)

Return a polynomial which reduces to F.

INPUT:

- F – an element of the residue_ring()

ALGORITHM:

We simply undo the steps performed in reduce().

OUTPUT:

A polynomial in the domain of the valuation with reduction F

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: w = v.augmentation(x, 1)
sage: w.lift(1/2)
1/2
sage: w = v.augmentation(x^2 + x + u, infinity)
sage: w.lift(w.residue_ring().gen())
x
```

A case with non-trivial base valuation:

```python
sage: R.<u> = Qq(4, 10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, infinity)
sage: w.lift(w.residue_ring().gen())
(1 + O(2^10))*x
```

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**reduce** *(f, check=True, degree_bound=None, coefficients=None, valuations=None)*

Reduction of an element with respect to this valuation.

**INPUT:**
- *f* – an element in the domain of this valuation
- *check* – whether or not to check whether *f* has non-negative valuation (default: True)
- *degree_bound* – an a-priori known bound on the degree of the result which can speed up the computation (default: not set)
- *coefficients* – the coefficients of *f* as produced by *coefficients()* or None (default: None); this can be used to speed up the computation when the expansion of *f* is already known from a previous computation.
- *valuations* – the valuations of *coefficients* or None (default: None); ignored

**OUTPUT:**

An element of the *residue_ring()* of this valuation, the reduction modulo the ideal of elements of positive valuation.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: w = v.augmentation(x, 1)
sage: w.reduce(x^2 + x + 1)
1
sage: w = v.augmentation(x^2 + x + 1, infinity)
sage: w.reduce(x)
ul
```

**residue_ring()**

Return the residue ring of this valuation, i.e., the elements of non-negative valuation modulo the elements of positive valuation.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: w = v.augmentation(x, 1)
sage: w.residue_ring()
Rational Field
sage: w = v.augmentation(x^2 + x + 1, infinity)
sage: w.residue_ring()
Number Field in ul with defining polynomial x^2 + x + 1
```

An example with a non-trivial base valuation:

```python
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, infinity)
sage: w.residue_ring()
Finite Field in ul of size 2^2
```

Since trivial extensions of finite fields are not implemented, the resulting ring might be identical to the residue ring of the underlying valuation:
**class** sage.rings.valuation.augmented_valuation.FinalFiniteAugmentedValuation(\(parent, v, phi, mu\))

An augmented valuation which is discrete, i.e., which assigns a finite valuation to its last key polynomial, but which can not be further augmented.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, valuations.TrivialValuation(QQ))
sage: w = v.augmentation(x, 1)
```

**lower_bound(\(f\))**

Return a lower bound of this valuation at \(f\).

Use this method to get an approximation of the valuation of \(f\) when speed is more important than accuracy.

**ALGORITHM:**

The main cost of evaluation is the computation of the `coefficients()` of the \(phi()\)-adic expansion of \(f\) (which often leads to coefficient bloat.) So unless \(phi()\) is trivial, we fall back to valuation which this valuation augments since it is guaranteed to be smaller everywhere.

**EXAMPLES:**

```python
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.lower_bound(x^2 + x + u)
0
```

**simplify(\(f\), error=None, force=False, effective_degree=None, size_heuristic_bound=32, phiadic=False)**

Return a simplified version of \(f\).
Produce an element which differs from $f$ by an element of valuation strictly greater than the valuation of $f$ (or strictly greater than error if set.)

**INPUT:**

- $f$ – an element in the domain of this valuation
- error – a rational, infinity, or None (default: None), the error allowed to introduce through the simplification
- force – whether or not to simplify $f$ even if there is heuristically no change in the coefficient size of $f$ expected (default: False)
- effective_degree – when set, assume that coefficients beyond effective_degree in the $\phi()$-adic development can be safely dropped (default: None)
- size_heuristic_bound – when force is not set, the expected factor by which the coefficients need to shrink to perform an actual simplification (default: 32)
- phiadic – whether to simplify the coefficients in the $\phi$-adic expansion recursively. This often times leads to huge coefficients in the $x$-adic expansion (default: False, i.e., use an $x$-adic expansion.)

**EXAMPLES:**

```python
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.simplify(x^10/2 + 1, force=True)
(u + 1)*2^-1 + O(2^4)
```

Check that trac ticket #25607 has been resolved, i.e., the coefficients in the following example are small::

```python
sage: R.<x> = QQ[]
sage: K.<a> = NumberField(x^3 + 6)
sage: R.<x> = K[]
sage: v = GaussValuation(R, K.valuation(2))
sage: v = v.augmentation(x, 3/2)
sage: v = v.augmentation(x^4 + 16*x^2 + 32*x + 64, 20/3)
sage: F.<x> = FunctionField(K)
sage: S.<y> = F[]
sage: v = F.valuation(v)
sage: G = y^2 - 2*x^5 + 8*x^3 + 80*x^2 + 128*x + 192
sage: v.mac_lane_approximants(G)  # [ Gauss valuation induced by Valuation on rational function field induced by [ Gauss valuation induced by 2-adic valuation, v(x) = 3/2, v(x^2 + 8) = 13/4, v(x^4 + 16*x^2 + 32*x + 64) = 20/3 ] }, v(y + 4*x + 8) = 31/8 ]
```

**upper_bound** ($f$)

Return an upper bound of this valuation at $f$.

Use this method to get an approximation of the valuation of $f$ when speed is more important than accuracy.

**ALGORITHM:**

Any entry of `valuations()` serves as an upper bound. However, computation of the $\phi()$-adic expansion of $f$ is quite costly. Therefore, we produce an upper bound on the last entry of `valuations()`, namely the valuation of the leading coefficient of $f$ plus the valuation of the appropriate power of $\phi()$.

**EXAMPLES:**

```python
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.upper_bound(x^2 + x + u)
1/2
```
valuations \((f, \text{coefficients}=\text{None}, \text{call_error}=\text{False})\)

Return the valuations of the \(f_i \phi^i\) in the expansion \(f = \sum_i f_i \phi^i\).

INPUT:

- \(f\) – a polynomial in the domain of this valuation
- \(\text{coefficients}\) – the coefficients of \(f\) as produced by \(\text{coefficients()}\) or None (default: None); this can be used to speed up the computation when the expansion of \(f\) is already known from a previous computation.
- \(\text{call_error}\) – whether or not to speed up the computation by assuming that the result is only used to compute the valuation of \(f\) (default: False)

OUTPUT:

An iterator over rational numbers (or infinity) \([v(f_0), v(f_1 \phi), \ldots]\)

EXAMPLES:

```python
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: list(w.valuations( x^2 + 1 ))
[0, 1/2]
sage: ww = w.augmentation((x^2 + x + u)^2 + 2, 5/3)
sage: list(ww.valuations( ((x^2 + x + u)^2 + 2)^3 ))
[+Infinity, +Infinity, +Infinity, 5]
```

value_group()

Return the value group of this valuation.

EXAMPLES:

```python
sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.value_group()
Additive Abelian Group generated by 1/2
sage: ww = w.augmentation((x^2 + x + u)^2 + 2, 5/3)
sage: ww.value_group()
Additive Abelian Group generated by 1/6
```

value_semigroup()

Return the value semigroup of this valuation.

EXAMPLES:

```python
sage: R.<u> = Zq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.value_semigroup()
Additive Abelian Semigroup generated by 1/2
sage: ww = w.augmentation((x^2 + x + u)^2 + 2, 5/3)
sage: ww.value_semigroup()
Additive Abelian Semigroup generated by 1/6
```

(continues on next page)
sage: w = w.augmentation((x^2 + x + u)^2 + 2, 5/3)
sage: ww.value_semigroup()
Additive Abelian Semigroup generated by 1/2, 5/3

class sage.rings.valuation.augmented_valuation.InfiniteAugmentedValuation (parent, v, phi, mu):

Bases:
    sage.rings.valuation.augmented_valuation.FinalAugmentedValuation, 
sage.rings.valuation.inductive_valuation.InfiniteInductiveValuation

An augmented valuation which is infinite, i.e., which assigns valuation infinity to its last key polynomial (and which can therefore not be augmented further.)

EXAMPLES:

sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x, infinity)

lower_bound(f)

Return a lower bound of this valuation at f.

Use this method to get an approximation of the valuation of f when speed is more important than accuracy.

EXAMPLES:

sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, infinity)

simplify(f, error=None, force=False, effective_degree=None)

Return a simplified version of f.

Produce an element which differs from f by an element of valuation strictly greater than the valuation of f (or strictly greater than error if set.)

INPUT:

• f – an element in the domain of this valuation
• error – a rational, infinity, or None (default: None), the error allowed to introduce through the simplification
• force – whether or not to simplify f even if there is heuristically no change in the coefficient size of f expected (default: False)
• effective_degree – ignored; for compatibility with other simplify methods

EXAMPLES:

sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, infinity)
sage: w.simplify(x^10/2 + 1, force=True)
(u + 1)*2^-1 + O(2^4)

upper_bound(f)
Return an upper bound of this valuation at f.
Use this method to get an approximation of the valuation of f when speed is more important than accuracy.

EXAMPLES:

sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[
]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, infinity)
sage: w.upper_bound(x^2 + x + u)
+Infinity

valuations(f, coefficients=None, call_error=False)
Return the valuations of the $f_\phi^i$ in the expansion $f = \sum_i f_i \phi^i$.

INPUT:
• f – a polynomial in the domain of this valuation
• coefficients – the coefficients of f as produced by coefficients() or None (default: None); this can be used to speed up the computation when the expansion of f is already known from a previous computation.
• call_error – whether or not to speed up the computation by assuming that the result is only used to compute the valuation of f (default: False)

OUTPUT:
An iterator over rational numbers (or infinity) $[v(f_0), v(f_1 \phi), \ldots]$.

EXAMPLES:

sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[
]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x, infinity)
sage: list(w.valuations(x^2 + 1))
[0, +Infinity, +Infinity]

value_group()
Return the value group of this valuation.

EXAMPLES:

sage: R.<u> = Qq(4, 5)
sage: S.<x> = R[
]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x, infinity)
sage: w.value_group()
Additive Abelian Group generated by 1

value_semigroup()
Return the value semigroup of this valuation.

EXAMPLES:
```python
sage: R.<u> = Zq(4, 5)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x, infinity)
sage: w.value_semigroup()
Additive Abelian Semigroup generated by 1
```

**class** `sage.rings.valuation.augmented_valuation.NonFinalAugmentedValuation`


An augmented valuation which can be augmented further.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
```

**lift** *(F, report_coefficients=False)*

Return a polynomial which reduces to F.

**INPUT:**

- F – an element of the `residue_ring()`
- report_coefficients – whether to return the coefficients of the `phi()`-adic expansion or the actual polynomial (default: False, i.e., return the polynomial)

**OUTPUT:**

A polynomial in the domain of the valuation with reduction F, monic if F is monic.

**ALGORITHM:**

Since this is the inverse of `reduce()`, we only have to go backwards through the algorithm described there.

**EXAMPLES:**

```python
sage: R.<u> = Qq(4, 10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: y = w.residue_ring().gen()
sage: ul = w.residue_ring().base().gen()

sage: w.lift(1)
1 + O(2^10)
sage: w.lift(0)
0
sage: w.lift(ul)
(1 + O(2^10))*x
sage: w.reduce(w.lift(y)) == y
True
sage: w.reduce(w.lift(y + ul + 1)) == y + ul + 1
```

(continues on next page)
sage: w = w.augmentation((x^2 + x + u)^2 + 2, 5/3)
sage: y = ww.residue_ring().gen()
sage: u2 = ww.residue_ring().base().gen()

sage: ww.reduce(ww.lift(y)) == y
True
sage: ww.reduce(ww.lift(1)) == 1
True
sage: ww.reduce(ww.lift(y + 1)) == y + 1
True

A more complicated example:

sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: ww = w.augmentation((x^2 + x + u)^2 + 2*x*(x^2 + x + u) + 4*x, 3)
sage: u = ww.residue_ring().base().gen()

sage: F = ww.residue_ring()(u); F
u2
sage: f = ww.lift(F); f
(2^-1 + O(2^9))*x^2 + (2^-1 + O(2^9))*x + u*2^-1 + O(2^9)

sage: F == ww.reduce(f)
True

lift_to_key(F, check=True)
Lift the irreducible polynomial $F$ to a key polynomial.

INPUT:

- $F$ – an irreducible non-constant polynomial in the residue_ring() of this valuation
- check – whether or not to check correctness of $F$ (default: True)

OUTPUT:

A polynomial $f$ in the domain of this valuation which is a key polynomial for this valuation and which, for a suitable equivalence unit $R$, satisfies that the reduction of $Rf$ is $F$

ALGORITHM:

We follow the algorithm described in Theorem 13.1 [Mac1936I] which, after a lift() of $F$, essentially shifts the valuations of all terms in the $\phi$-adic expansion up and then kills the leading coefficient.

EXAMPLES:

sage: R.<u> = Qq(4, 10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)

sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: y = w.residue_ring().gen()
sage: f = w.lift_to_key(y + 1); f
(1 + O(2^10))*x^4 + (2 + O(2^11))*x^3 + (1 + u*2 + O(2^10))*x^2 + (u*2 + O(2^10))*x + (u + 1) + u*2 + O(2^10)

sage: w.is_key(f)
True

5.8. Augmented valuations on polynomial rings

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A more complicated example:

```python
sage: v = GaussValuation(S)
sage: w = v.augmentation(x^2 + x + u, 1)
sage: ww = w.augmentation((x^2 + x + u)^2 + 2*x*(x^2 + x + u) + 4*x, 3)

sage: u = ww.residue_ring().base().gen()
sage: y = ww.residue_ring().gen()
sage: f = ww.lift_to_key(y^3+y+u)
sage: f.degree()
12
sage: ww.is_key(f)
True
```

**reduce** *(f, check=True, degree_bound=None, coefficients=None, valuations=None)*

Reduce *f* modulo this valuation.

**INPUT:**

- *f* – an element in the domain of this valuation
- *check* – whether or not to check whether *f* has non-negative valuation (default: *True*)
- *degree_bound* – an a-priori known bound on the degree of the result which can speed up the computation (default: not set)
- *coefficients* – the coefficients of *f* as produced by *coefficients()* or None (default: None); this can be used to speed up the computation when the expansion of *f* is already known from a previous computation.
- *valuations* – the valuations of coefficients or None (default: None)

**OUTPUT:**

an element of the *residue_ring()* of this valuation, the reduction modulo the ideal of elements of positive valuation

**ALGORITHM:**

We follow the algorithm given in the proof of Theorem 12.1 of [Mac1936I]: If *f* has positive valuation, the reduction is simply zero. Otherwise, let *f* = \(\sum f_i \phi^i\) be the expansion of *f*, as computed by *coefficients()*). Since the valuation is zero, the exponents *i* must all be multiples of \(\tau\), the index the value group of the base valuation in the value group of this valuation. Hence, there is an equivalence_unit() *Q* with the same valuation as \(\phi^\tau\). Let *Q*’ be its equivalence_reciprocal(). Now, rewrite each term \(f_i \phi^i = (f_i Q')(\phi^\tau Q^{-1})\); it turns out that the second factor in this expression is a lift of the generator of the *residue_field()*). The reduction of the first factor can be computed recursively.

**EXAMPLES:**

```python
sage: R.<u> = Qq(4, 10)
sage: S.<x> = R[]
sage: v = GaussValuation(S)
sage: v.reduce(x)
x
sage: v.reduce(S(u))
u0
sage: w = v.augmentation(x^2 + x + u, 1/2)
sage: w.reduce(S.one())
1
```

(continues on next page)
sage: w.reduce(S(2))
0
sage: w.reduce(S(u))
u0
sage: w.reduce(x)  # this gives the generator of the residue field extension of w over v
u1
sage: f = (x^2 + x + u)^2 / 2
sage: w.reduce(f)
x
sage: w.reduce(f + x + 1)
x + u1 + 1
sage: ww = w.augmentation((x^2 + x + u)^2 + 2, 5/3)
sage: g = ((x^2 + x + u)^2 + 2)^3 / 2^5
sage: ww.reduce(g)
x
sage: ww.reduce(f)
1
sage: ww.is_equivalent(f, 1)
True
sage: ww.reduce(f + g)
x
sage: ww.reduce(f * g)
x
residue_ring()
Return the residue ring of this valuation, i.e., the elements of non-negative valuation modulo the elements of positive valuation.

EXAMPLES:

sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = v.augmentation(x^2 + x + 1, 1)
sage: w.residue_ring()
Univariate Polynomial Ring in x over Finite Field in u1 of size 2^2

Since trivial valuations of finite fields are not implemented, the resulting ring might be identical to the residue ring of the underlying valuation:

sage: w = v.augmentation(x, 1)
sage: w.residue_ring()
Univariate Polynomial Ring in x over Finite Field of size 2 (using ...)

class sage.rings.valuation.augmented_valuation.NonFinalFiniteAugmentedValuation


An augmented valuation which is discrete, i.e., which assigns a finite valuation to its last key polynomial, and which can be augmented further.

EXAMPLES:
5.9 Valuations which are defined as limits of valuations.

The discrete valuation of a complete field extends uniquely to a finite field extension. This is not the case anymore for fields which are not complete with respect to their discrete valuation. In this case, the extensions essentially correspond to the factors of the defining polynomial of the extension over the completion. However, these factors only exist over the completion and this makes it difficult to write down such valuations with a representation of finite length.

More specifically, let \( v \) be a discrete valuation on \( K \) and let \( L = K[x]/(G) \) a finite extension thereof. An extension of \( v \) to \( L \) can be represented as a discrete pseudo-valuation \( w' \) on \( K[x] \) which sends \( G \) to infinity. However, such \( w' \) might not be described by an augmented valuation over a Gauss valuation anymore. Instead, we may need to write is as a limit of augmented valuations.

The classes in this module provide the means of writing down such limits and resulting valuations on quotients.

AUTHORS:

• Julian Rüth (2016-10-19): initial version

EXAMPLES:

In this function field, the unique place of \( K \) which corresponds to the zero point has two extensions to \( L \). The valuations corresponding to these extensions can only be approximated:

\[
\begin{align*}
\text{sage: } & K.<x> = FunctionField(QQ) \\
\text{sage: } & R.<y> = K[] \\
\text{sage: } & L.<y> = K.extension(y^2 - x) \\
\text{sage: } & v = K.valuation(1) \\
\text{sage: } & w = v.extensions(L); w \\
& \quad [\text{(x - 1)-adic valuation, } v(y + 1) = 1 \text{-adic valuation}, \\
& \quad (x - 1)-adic valuation, v(y - 1) = 1 \text{-adic valuation}]
\end{align*}
\]

The same phenomenon can be observed for valuations on number fields:

\[
\begin{align*}
\text{sage: } & K = QQ \\
\text{sage: } & R.<t> = K[] \\
\text{sage: } & L.<t> = K.extension(t^2 + 1) \\
\text{sage: } & v = QQ.valuation(5) \\
\text{sage: } & w = v.extensions(L); w \\
& \quad [\text{5-adic valuation, } v(t + 2) = 1 \text{-adic valuation}, \\
& \quad 5-adic valuation, v(t + 3) = 1 \text{-adic valuation}]
\end{align*}
\]

Note: We often rely on approximations of valuations even if we could represent the valuation without using a limit. This is done to improve performance as many computations already can be done correctly with an approximation:

\[
\begin{align*}
\text{sage: } & K.<x> = FunctionField(QQ) \\
\text{sage: } & R.<y> = K[] \\
\text{sage: } & L.<y> = K.extension(y^2 - x) \\
\text{sage: } & v = K.valuation(1/x) \\
\text{sage: } & w = v.extension(L); w \\
\end{align*}
\]

(continues on next page)
Valuation at the infinite place

sage: w._base_valuation._base_valuation._improve_approximation()
sage: w._base_valuation._base_valuation._approximation

[ Gauss valuation induced by Valuation at the infinite place, v(y) = 1/2, v(y^2 - 1/x) = +Infinity ]

REFERENCES:

Limits of inductive valuations are discussed in [Mac1936I] and [Mac1936II]. An overview can also be found in Section 4.6 of [Rüt2014].

class sage.rings.valuation.limit_valuation.LimitValuationFactory

Bases: sage.structure.factory.UniqueFactory

Return a limit valuation which sends the polynomial G to infinity and is greater than or equal than base_valuation.

INPUT:

- base_valuation – a discrete (pseudo-)valuation on a polynomial ring which is a discrete valuation on the coefficient ring which can be uniquely augmented (possibly only in the limit) to a pseudo-valuation that sends G to infinity.
- G – a squarefree polynomial in the domain of base_valuation.

EXAMPLES:

sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = valuations.LimitValuation(v, x)
sage: w(x)
+Infinity

create_key(base_valuation, G)

Create a key from the parameters of this valuation.

EXAMPLES:

Note that this does not normalize base_valuation in any way. It is easily possible to create the same limit in two different ways:

sage: R.<x> = QQ[]
sage: v = GaussValuation(R, QQ.valuation(2))
sage: w = valuations.LimitValuation(v, x) # indirect doctest
sage: v = v.augmentation(x, infinity)
sage: u = valuations.LimitValuation(v, x)
sage: u == w
False

The point here is that this is not meant to be invoked from user code. But mostly from other factories which have made sure that the parameters are normalized already.

create_object(version, key)

Create an object from key.

EXAMPLES:

5.9. Valuations which are defined as limits of valuations.
sage: R.<x> = QQ[]  
sage: v = GaussValuation(R, QQ.valuation(2))  
sage: w = valuations.LimitValuation(v, x^2 + 1)  
# indirect doctest

class sage.rings.valuation.limit_valuation.LimitValuation_generic (parent,  
               approximation)
Bases: sage.rings.valuation.valuation.DiscretePseudoValuation

Base class for limit valuations.

A limit valuation is realized as an approximation of a valuation and means to improve that approximation when necessary.

EXAMPLES:

sage: K.<x> = FunctionField(QQ)  
sage: R.<y> = K[]  
sage: L.<y> = K.extension(y^2 - x)  
sage: v = K.valuation(0)  
sage: w = v.extension(L)  
sage: w._base_valuation  
[ Gauss valuation induced by (x)-adic valuation, v(y) = 1/2 , ... ]

The currently used approximation can be found in the _approximation field:

sage: w._base_valuation._approximation  
[ Gauss valuation induced by (x)-adic valuation, v(y) = 1/2 ]

reduce (f, check=True)

Return the reduction of \( f \) as an element of the \( \text{residue\_ring()} \).

INPUT:

- \( f \) – an element in the domain of this valuation of non-negative valuation
- \( \text{check} \) – whether or not to check that \( f \) has non-negative valuation (default: True)

EXAMPLES:

sage: K.<x> = FunctionField(QQ)  
sage: R.<y> = K[]  
sage: L.<y> = K.extension(y^2 - (x - 1))  
sage: v = K.valuation(0)  
sage: w = v.extension(L)  
sage: w.reduce(y)  
# indirect doctest

ul

class sage.rings.valuation.limit_valuation.MacLaneLimitValuation (parent,  
               approximation,  
               G)
rings.valuation.valuation.InfiniteDiscretePseudoValuation

A limit valuation that is a pseudo-valuation on polynomial ring \( K[x] \) which sends a square-free polynomial \( G \) to infinity.

This uses the MacLane algorithm to compute the next element in the limit.
It starts from a first valuation approximation which has a unique augmentation that sends \( G \) to infinity and whose uniformizer must be a uniformizer of the limit and whose residue field must contain the residue field of the limit.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: K.<i> = QQ.extension(x^2 + 1)
sage: v = K.valuation(2)
sage: u = v._base_valuation; u
[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2 , ... ]
```

**element_with_valuation\((s)\)**

Return an element with valuation \( s \).

**extensions\((\text{ring})\)**

Return the extensions of this valuation to \( \text{ring} \).

**EXAMPLES:**

```python
sage: v = GaussianIntegers().valuation(2)
sage: u = v._base_valuation
sage: u.extensions(QQ['x'])
[[ Gauss valuation induced by 2-adic valuation, v(x + 1) = 1/2 , ... ]]
```

**is_negative_pseudo_valuation()**

Return whether this valuation attains \(-\infty\).

**EXAMPLES:**

For a Mac Lane limit valuation, this is never the case, so this method always returns True:

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = QQ.valuation(2)
sage: u = v.extension(L)
sage: u.is_negative_pseudo_valuation()
False
```

**lift\((F)\)**

Return a lift of \( F \) from the residue_ring() to the domain of this valuation.

**EXAMPLES:**

```python
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^4 - x^2 - 2*x - 1)
sage: v = K.valuation(1)
sage: w = v.extensions(L)[1]; w
[ (x - 1)-adic valuation, v(y^2 - 2) = 1 ]-adic valuation
sage: s = w.reduce(y); s
u1
sage: w.lift(s) # indirect doctest
y
```

**lower_bound\((f)\)**

Return a lower bound of this valuation at \( x \).
Use this method to get an approximation of the valuation of \( x \) when speed is more important than accuracy.

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = QQ.valuation(2)
sage: u = v.extension(L)
sage: u.lower_bound(1024*t + 1024)
10
sage: u(1024*t + 1024)
21/2
```

**residue_ring()**

Return the residue ring of this valuation, which is always a field.

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = QQ.valuation(2)
sage: w = v.extension(L)
sage: w.residue_ring()
Finite Field of size 2
```

**restriction(ring)**

Return the restriction of this valuation to \( \text{ring} \).

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = QQ.valuation(2)
sage: w = v.extension(L)
sage: w._base_valuation.restriction(K)
2-adic valuation
```

**simplify** (\( f, \text{error=None, force=False} \))

Return a simplified version of \( f \).

Produce an element which differs from \( f \) by an element of valuation strictly greater than the valuation of \( f \) (or strictly greater than \( \text{error} \) if set.)

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = QQ.valuation(2)
sage: u = v.extension(L)
sage: u.simplify(t + 1024, force=True)
t
```

**uniformizer()**

Return a uniformizing element for this valuation.

**EXAMPLES:**

```python
```
upper_bound(f)
Return an upper bound of this valuation at \( x \).
Use this method to get an approximation of the valuation of \( x \) when speed is more important than accuracy.

EXAMPLES:

```sage
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = QQ.valuation(2)
sage: u = v.extension(L)
sage: u.upper_bound(1024*t + 1024)
21/2
sage: u(1024*t + 1024)
21/2
```

value_semigroup()
Return the value semigroup of this valuation.

5.10 Valuations which are implemented through a map to another valuation

EXAMPLES:

Extensions of valuations over finite field extensions \( L = K[x]/(G) \) are realized through an infinite valuation on \( K[x] \) which maps \( G \) to infinity:

```sage
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)

sage: v = K.valuation(0)
sage: w = v.extension(L)

sage: w._base_valuation
(Gauss valuation induced by (x)-adic valuation, v(y) = 1/2 , ... )
```

AUTHORS:
- Julian Rüth (2016-11-10): initial version

```sage
class sage.rings.valuation.mapped_valuation.FiniteExtensionFromInfiniteValuation(parent, base_valuation)
```

5.10. Valuations which are implemented through a map to another valuation 73
A valuation on a quotient of the form $L = K[x]/(G)$ with an irreducible $G$ which is internally backed by a pseudo-valuations on $K[x]$ which sends $G$ to infinity.

**INPUT:**

- **parent** – the containing valuation space (usually the space of discrete valuations on $L$)
- **basevaluation** – an infinite valuation on $K[x]$ which takes $G$ to infinity

**EXAMPLES:**

```python
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)
sage: v = K.valuation(0)
sage: w = v.extension(L); w
(x)-adic valuation
```

**lower_bound**($x$)

Return an lower bound of this valuation at $x$.

Use this method to get an approximation of the valuation of $x$ when speed is more important than accuracy.

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 5)
sage: u,uu = v.extensions(L)
sage: u.lower_bound(t + 2)
0
sage: u(t + 2)
1
```

**restriction**($ring$)

Return the restriction of this valuation to $ring$.

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 2)
sage: w = v.extension(L)
sage: w.restriction(K) is v
True
```

**simplify**(x, **error=None, force=False**)

Return a simplified version of $x$.

Produce an element which differs from $x$ by an element of valuation strictly greater than the valuation of $x$ (or strictly greater than $error$ if set.)

**EXAMPLES:**

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 5)
```
upper_bound(x)

Return an upper bound of this valuation at x.

Use this method to get an approximation of the valuation of x when speed is more important than accuracy.

EXAMPLES:

```python
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 5)
sage: u, uu = v.extensions(L)
sage: u.upper_bound(t + 2) >= 1
True
sage: u(t + 2)
1
```

5.10. Valuations which are implemented through a map to another valuation

```python
class sage.rings.valuation.mapped_valuation.MappedValuation_base (parent, base_valuation)

Bases: sage.rings.valuation.valuation.DiscretePseudoValuation

A valuation which is implemented through another proxy "base" valuation.

EXAMPLES:

```
sage: v = K.valuation(0)
sage: w = v.extension(L); w
\[(x)-adic valuation\]

\textbf{element\_with\_valuation}(s)

Return an element with valuation \(s\).

\textbf{EXAMPLES:}

\begin{verbatim}
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 5)
sage: u,uu = v.extensions(L)
sage: u.element_with_valuation(1)
5
\end{verbatim}

\textbf{lift}(F)

Lift \(F\) from the \texttt{residue\_field()} of this valuation into its domain.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)
sage: v = K.valuation(2)
sage: w = v.extension(L)
sage: w.lift(w.residue_field().gen())
y
\end{verbatim}

\textbf{reduce}(f)

Return the reduction of \(f\) in the \texttt{residue\_field()} of this valuation.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - (x - 2))
sage: v = K.valuation(0)
sage: w = v.extension(L)
sage: w.reduce(y)
u1
\end{verbatim}

\textbf{residue\_ring()}

Return the residue ring of this valuation.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 2)
sage: v.extension(L).residue_ring()
Finite Field of size 2
\end{verbatim}
\textbf{simplify}(x, error=None, force=False)

Return a simplified version of \(x\).

Produce an element which differs from \(x\) by an element of valuation strictly greater than the valuation of \(x\) (or strictly greater than \(error\) if set.)

If \(force\) is not set, then expensive simplifications may be avoided.

\textbf{EXAMPLES:}

```sage
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)
sage: v = K.valuation(0)
sage: w = v.extensions(L)[0]
```

As \_relative\_size() misses the bloated term \(x^{32}\), the following term does not get simplified:

```sage
sage: w.simplify(y + x^32)
y + x^{32}
```

In this case the simplification can be forced but this should not happen as a default as the recursive simplification can be quite costly:

```sage
sage: w.simplify(y + x^32, force=True)
y
```

\textbf{uniformizer}()

Return a uniformizing element of this valuation.

\textbf{EXAMPLES:}

```sage
sage: K = QQ
sage: R.<t> = K[]
sage: L.<t> = K.extension(t^2 + 1)
sage: v = valuations.pAdicValuation(QQ, 2)
sage: v.extension(L).uniformizer()
t + 1
```

\section*{5.11 Valuations which are scaled versions of another valuation}

\textbf{EXAMPLES:}

```sage
sage: 3*ZZ.valuation(3)
3 * 3-adic valuation
```

\textbf{AUTHORS:}

- Julian Rüth (2016-11-10): initial version

class \texttt{sage.rings.valuation.scaled\_valuation.ScaledValuationFactory}

Bases: \texttt{sage.structure.factory.UniqueFactory}

Return a valuation which scales the valuation \texttt{base} by the factor \texttt{s}.

\textbf{EXAMPLES:}
create_key(base, s)
Create a key which uniquely identifies a valuation.

create_object(version, key)
Create a valuation from key.

class sage.rings.valuation.scaled_valuation.ScaledValuation_generic(parent, base_valuation, s)
Bases: sage.rings.valuation.valuation.DiscreteValuation
A valuation which scales another base_valuation by a finite positive factor s.

EXAMPLES:

```
sage: v = 3*ZZ.valuation(3); v
3 * 3-adic valuation
```

extensions(ring)
Return the extensions of this valuation to ring.

EXAMPLES:

```
sage: v = 3*ZZ.valuation(5)
sage: v.extensions(GaussianIntegers().fraction_field())
[3 * [ 5-adic valuation, v(x + 2) = 1 ]-adic valuation, 3 * [ 5-adic valuation, v(x + 3) = 1 ]-adic valuation]
```

lift(F)
Lift F from the residue_field() of this valuation into its domain.

EXAMPLES:

```
sage: v = 3*ZZ.valuation(2)
sage: v.lift(1)
1
```

reduce(f)
Return the reduction of f in the residue_field() of this valuation.

EXAMPLES:

```
sage: v = 3*ZZ.valuation(2)
sage: v.reduce(1)
1
```

residue_ring()
Return the residue field of this valuation.

EXAMPLES:

```
sage: v = 3*ZZ.valuation(2)
sage: v.residue_ring()
Finite Field of size 2
```

restriction(ring)
Return the restriction of this valuation to ring.
EXAMPLES:

```python
sage: v = 3*QQ.valuation(5)
sage: v.restriction(ZZ)
3 * 5-adic valuation
```

`uniformizer()`
Return a uniformizing element of this valuation.

EXAMPLES:

```python
sage: v = 3*ZZ.valuation(2)
sage: v.uniformizer()
2
```

`value_semigroup()`
Return the value semigroup of this valuation.

EXAMPLES:

```python
sage: v2 = QQ.valuation(2)
sage: (2*v2).value_semigroup()
Additive Abelian Semigroup generated by -2, 2
```

## 5.12 Discrete valuations on function fields

AUTHORS:

- Julian Rüth (2016-10-16): initial version

EXAMPLES:

We can create classical valuations that correspond to finite and infinite places on a rational function field:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(1); v
(x - 1)-adic valuation
sage: v = K.valuation(x^2 + 1); v
(x^2 + 1)-adic valuation
sage: v = K.valuation(1/x); v
Valuation at the infinite place
```

Note that we can also specify valuations which do not correspond to a place of the function field:

```python
sage: R.<x> = QQ[]
sage: w = valuations.GaussValuation(R, QQ.valuation(2))
sage: v = K.valuation(w); v
2-adic valuation
```

Valuations on a rational function field can then be extended to finite extensions:

```python
sage: v = K.valuation(x - 1); v
(x - 1)-adic valuation
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x)
sage: w = v.extensions(L); w
```

(continues on next page)
REFERENCES:
An overview of some computational tools relating to valuations on function fields can be found in Section 4.6 of [Rüt2014]. Most of this was originally developed for number fields in [Mac1936I] and [Mac1936II].

```python
class sage.rings.function_field.function_field_valuation.ClassicalFunctionFieldValuation_base
Bases:
    sage.rings.function_field.function_field_valuation.
    DiscreteFunctionFieldValuation_base

Base class for discrete valuations on rational function fields that come from points on the projective line.

class sage.rings.function_field.function_field_valuation.DiscreteFunctionFieldValuation_base
Bases:
    sage.rings.valuation.valuation.DiscreteValuation

Base class for discrete valuations on function fields.

extensions(L)
Return the extensions of this valuation to L.

EXAMPLES:

```sage```
K.<x> = FunctionField(QQ)
sage: v = K.valuation(x + 1); v  # indirect doctest
(x + 1)-adic valuation
```

A finite place with residual degree:

```sage```
w = K.valuation(x^2 + 1); w
(x^2 + 1)-adic valuation
```

A finite place with ramification:

```sage```
K.<t> = FunctionField(GF(3))
sage: u = L.valuation(x^3 - t); u
(x^3 + 2*t)-adic valuation
```

A finite place with residual degree and ramification:
```python
sage: q = L.valuation(x^6 - t); q
(x^6 + 2*t)-adic valuation
```

```python
class sage.rings.function_field.function_field_valuation.FunctionFieldExtensionMappedValuation

Bases: sage.rings.function_field.function_field_valuation.
FunctionFieldMappedValuationRelative_base

A valuation on a finite extensions of function fields \( L = K[y]/(G) \) where \( K \) is another function field which redirects to another base_valuation on an isomorphism function field \( M = K[y]/(H) \).

The isomorphisms must be trivial on \( K \).

EXAMPLES:
```
sage: K.<x> = FunctionField(GF(2))
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 + y + x^3)
sage: v = K.valuation(1/x)
sage: w = v.extension(L)
sage: w(x)
-1
sage: w(y)
-3/2
sage: w.uniformizer()
1/x^2*y
```

```python
restriction(ring)

Return the restriction of this valuation to ring.

EXAMPLES:
```
sage: K.<x> = FunctionField(GF(2))
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 + y + x^3)
sage: v = K.valuation(1/x)
sage: w = v.extension(L)
sage: w.restriction(K)
is v True
```
```
class sage.rings.function_field.function_field_valuation.FunctionFieldFromLimitValuation


A valuation on a finite extensions of function fields \( L = K[y]/(G) \) where \( K \) is another function field.

EXAMPLES:
```
```
```
```
```
scale(\texttt{scalar})

Return this valuation scaled by \texttt{scalar}.

EXAMPLES:

```
sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - (x^2 + x + 1))
sage: v = K.valuation(x - 1) \# indirect doctest
sage: w = v.extension(L); w
(x - 1)-adic valuation
sage: 3*w
3 * (x - 1)-adic valuation
```
sage: K.<x> = FunctionField(GF(2))
sage: v = K.valuation(1/x); v
Valuation at the infinite place

is_discrete_valuation()
Return whether this is a discrete valuation.

EXAMPLES:

sage: K.<x> = FunctionField(QQ)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 - x^4 - 1)
sage: v = K.valuation(1/x)
sage: w0,w1 = v.extensions(L)
sage: w0.is_discrete_valuation()
True

scale(scalar)
Return this valuation scaled by scalar.

EXAMPLES:

sage: K.<x> = FunctionField(GF(2))
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 + y + x^3)
sage: v = K.valuation(1/x)
sage: w = v.extension(L)
sage: 3*w
3 * (x)-adic valuation (in Rational function field in x over Finite Field of size 2 after x |--> 1/x)

class sage.rings.function_field.function_field_valuation.FunctionFieldValuationFactory
Bases: sage.structure.factory.UniqueFactory
Create a valuation on domain corresponding to prime.

INPUT:

• domain – a function field

• prime – a place of the function field, a valuation on a subring, or a valuation on another function field together with information for isomorphisms to and from that function field

EXAMPLES:

sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(1); v # indirect doctest
(x - 1)-adic valuation
sage: v(x)
0
sage: v(x - 1)
1

See sage.rings.function_field.function_field.FunctionField.valuation() for further examples.

create_key_and_extra_args(domain, prime)
Create a unique key which identifies the valuation given by prime on domain.

create_key_and_extra_args_from_place(domain, generator)
Create a unique key which identifies the valuation at the place specified by generator.

5.12. Discrete valuations on function fields
create_key_and_extra_args_from_valuation(domain, valuation)
Create a unique key which identifies the valuation which extends valuation.

create_key_and_extra_args_from_valuation_on_isomorphic_field(domain, valuation, to_valuation_domain, from_valuation_domain)
Create a unique key which identifies the valuation which is valuation after mapping through to_valuation_domain.

create_object(version, key, **extra_args)
Create the valuation specified by key.

EXAMPLES:

```
sage: K.<x> = FunctionField(QQ)
sage: R.<x> = QQ[]
sage: w = valuations.GaussValuation(R, QQ.valuation(2))
sage: v = K.valuation(w); v
# indirect doctest
2-adic valuation
```

class sage.rings.function_field.function_field_valuation.FunctionFieldValuation_base(parent)
Bases: sage.rings.valuation.valuation.DiscretePseudoValuation
Abstract base class for any discrete (pseudo-)valuation on a function field.

class sage.rings.function_field.function_field_valuation.InducedRationalFunctionFieldValuation_base(parent, base_valuation)
Bases: sage.rings.function_field.function_field_valuation.FunctionFieldValuation_base
Base class for function field valuation induced by a valuation on the underlying polynomial ring.

extensions(L)
Return all extensions of this valuation to L which has a larger constant field than the domain of this valuation.

EXAMPLES:

```
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(x^2 + 1)
sage: L.<x> = FunctionField(GaussianIntegers().fraction_field())
sage: v.extensions(L) # indirect doctest
[(x - I)-adic valuation, (x + I)-adic valuation]
```

lift(F)
Return a lift of F to the domain of this valuation such that reduce() returns the original element.

EXAMPLES:

```
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(x)
sage: v.lift(0)
0
sage: v.lift(1)
1
```

reduce(f)
Return the reduction of f in residue_ring().

EXAMPLES:
residue_ring()

Return the residue field of this valuation.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(x^2 + 1)
sage: v.reduce(x)
ul

restriction(ring)

Return the restriction of this valuation to ring.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: K.valuation(x).restriction(QQ)
Trivial valuation on Rational Field
```

simplify(f, error=None, force=False)

Return a simplified version of f.

Produce an element which differs from f by an element of valuation strictly greater than the valuation of f (or strictly greater than error if set.)

If force is not set, then expensive simplifications may be avoided.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(2)
sage: f = (x + 1)/(x - 1)
As the coefficients of this fraction are small, we do not simplify as this could be very costly in some cases:
sage: v.simplify(f)
(x + 1)/(x - 1)
```

However, simplification can be forced:

```python
sage: v.simplify(f, force=True)
3
```

uniformizer()

Return a uniformizing element for this valuation.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: K.valuation(x).uniformizer()
x
```

value_group()

Return the value group of this valuation.

EXAMPLES:
```python
sage: K.<x> = FunctionField(QQ)
sage: K.valuation(x).value_group()
Additive Abelian Group generated by 1
```

class sage.rings.function_field.function_field_valuation.InfiniteRationalFunctionFieldValuation

Bases:
  sage.rings.function_field.function_field_valuation.FunctionFieldMappedValuationRelative_base,
  sage.rings.function_field.function_field_valuation.RationalFunctionFieldValuation_base

Valuation of the infinite place of a function field.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = K.valuation(1/x)  # indirect doctest
```

class sage.rings.function_field.function_field_valuation.NonClassicalRationalFunctionFieldValuation

Bases:
  sage.rings.function_field.function_field_valuation.InducedRationalFunctionFieldValuation_base,
  sage.rings.function_field.function_field_valuation.RationalFunctionFieldValuation_base

Valuation induced by a valuation on the underlying polynomial ring which is non-classical.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = GaussValuation(QQ['x'], QQ.valuation(2))
sage: w = K.valuation(v); w  # indirect doctest
2-adic valuation
```

residue_ring()

Return the residue field of this valuation.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: v = valuations.GaussValuation(QQ['x'], QQ.valuation(2))
sage: w = K.valuation(v)
sage: w.residue_ring()  # indirect doctest
Rational function field in x over Finite Field of size 2
```

class sage.rings.function_field.function_field_valuation.RationalFunctionFieldMappedValuation

Bases:
  sage.rings.function_field.function_field_valuation.

Valuation of a function field.

EXAMPLES:

```python
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 + 2*x)
sage: w.extension(L).residue_ring()
Function field in u2 defined by u2^2 + x
```
Valuation on a rational function field that is implemented after a map to an isomorphic rational function field.

EXAMPLES:

```python
sage: K.<x> = FunctionField(QQ)
sage: R.<x> = QQ[]
sage: w = GaussValuation(R, QQ.valuation(2)).augmentation(x, 1)
sage: w = K.valuation(w)
sage: v = K.valuation((w, K.hom([~K.gen()]), K.hom([~K.gen()]))); v
Valuation on rational function field induced by [ Gauss valuation induced by 2-adic valuation, v(x) = 1 ] (in Rational function field in x over Rational Field after x |--> 1/x)
```

class `sage.rings.function_field.function_field_valuation.RationalFunctionFieldValuation_base`

Bases: `sage.rings.function_field.function_field_valuation.FunctionFieldValuation_base`

Base class for valuations on rational function fields.

**element_with_valuation**(s)

Return an element with valuation s.

EXAMPLES:

```python
sage: K.<a> = NumberField(x^3+6)
sage: v = K.valuation(2)
sage: R.<x> = K[]
sage: w = GaussValuation(R, v).augmentation(x, 1/123)
sage: w = w.extension(K)
sage: w.element_with_valuation(122/123)
sage: w.element_with_valuation(1)
2/x
```

5.13 \( p \)-adic valuations on number fields and their subrings and completions

EXAMPLES:

```python
sage: ZZ.valuation(2)
2-adic valuation
sage: QQ.valuation(3)
3-adic valuation
sage: CyclotomicField(5).valuation(5)
5-adic valuation
sage: GaussianIntegers().valuation(7)
7-adic valuation
sage: Zp(11).valuation()
11-adic valuation
```

These valuations can then, e.g., be used to compute approximate factorizations in the completion of a ring:

5.13. \( p \)-adic valuations on number fields and their subrings and completions
AUTHORS:

- Julian Rüth (2013-03-16): initial version

REFERENCES:

The theory used here was originally developed in [Mac1936I] and [Mac1936II]. An overview can also be found in Chapter 4 of [Rüt2014].

```python
sage: v = ZZ.valuation(2)
sage: R.<x> = ZZ[]
sage: f = x^5 + x^4 + x^3 + x^2 + x - 1
sage: v.montes_factorization(f, required_precision=20)
(x + 676027) * (x^4 + 372550*x^3 + 464863*x^2 + 385052*x + 297869)
```

```
AUTHORS:

- Julian Rüth (2013-03-16): initial version

REFERENCES:

The theory used here was originally developed in [Mac1936I] and [Mac1936II]. An overview can also be found in Chapter 4 of [Rüt2014].

```python
class sage.rings.padics.padic_valuation.PadicValuationFactory
    Bases: sage.structure.factory.UniqueFactory

    Create a prime-adic valuation on R.

    INPUT:

    - R – a subring of a number field or a subring of a local field in characteristic zero
    - prime – a prime that does not split, a discrete (pseudo-)valuation, a fractional ideal, or None (default: None)

    EXAMPLES:

    For integers and rational numbers, prime is just a prime of the integers:

    ```
sage: valuations.pAdicValuation(ZZ, 3)
3-adic valuation
sage: valuations.pAdicValuation(QQ, 3)
3-adic valuation
```

prime may be None for local rings:

```python
sage: valuations.pAdicValuation(Qp(2))
2-adic valuation
sage: valuations.pAdicValuation(Zp(2))
2-adic valuation
```

But it must be specified in all other cases:

```python
sage: valuations.pAdicValuation(ZZ)
Traceback (most recent call last):
 ... ValueError: prime must be specified for this ring
```

It can sometimes be beneficial to define a number field extension as a quotient of a polynomial ring (since number field extensions always compute an absolute polynomial defining the extension which can be very costly):

```python
sage: R.<x> = QQ[]
sage: K.<a> = NumberField(x^2 + 1)
sage: R.<x> = K[]
sage: L.<b> = R.quo(x^2 + a)
```

(continues on next page)
sage: valuations.pAdicValuation(L, 2)
2-adic valuation

See also:
NumberField_generic.valueation(), Order.valueation(), pAdicGeneric.valueation(), RationalField.valueation(), IntegerRing_class.valueation().

create_key_and_extra_args (R, prime=None, approximants=None)
Create a unique key identifying the valuation of R with respect to prime.

EXAMPLES:
sage: QQ.valueation(2) # indirect doctest
2-adic valuation

create_key_and_extra_args_for_number_field (R, prime, approximants)
Create a unique key identifying the valuation of R with respect to prime.

EXAMPLES:
sage: GaussianIntegers().valueation(2) # indirect doctest
2-adic valuation

create_key_and_extra_args_for_number_field_from_ideal (R, I, prime)
Create a unique key identifying the valuation of R with respect to I.

Note: prime, the original parameter that was passed to create_key_and_extra_args(), is only used to provide more meaningful error messages

EXAMPLES:
sage: GaussianIntegers().valueation(GaussianIntegers().ideal(2)) # indirect doctest
2-adic valuation

create_key_and_extra_args_for_number_field_from_valuation (R, v, prime, approximants)
Create a unique key identifying the valuation of R with respect to v.

Note: prime, the original parameter that was passed to create_key_and_extra_args(), is only used to provide more meaningful error messages

EXAMPLES:
sage: GaussianIntegers().valueation(ZZ.valueation(2)) # indirect doctest
2-adic valuation

create_key_for_integers (R, prime)
Create a unique key identifying the valuation of R with respect to prime.

EXAMPLES:
sage: QQ.valueation(2) # indirect doctest
2-adic valuation

5.13. $p$-adic valuations on number fields and their subrings and completions 89
create_key_for_local_ring \( (R, \text{prime}) \)
Create a unique key identifying the valuation of \( R \) with respect to \( \text{prime} \).

EXAMPLES:
```
sage: Qp(2).valuation() # indirect doctest
2-adic valuation
```

create_object \( (\text{version}, \text{key}, **\text{extra}_\text{args}) \)
Create a \( p \)-adic valuation from \( \text{key} \).

EXAMPLES:
```
sage: ZZ.valuation(5) # indirect doctest
5-adic valuation
```

class sage.rings.padics.padic_valuation.pAdicFromLimitValuation
\( \text{parent, approximant, } G, \text{approximants} \)
A \( p \)-adic valuation on a number field or a subring thereof, i.e., a valuation that extends the \( p \)-adic valuation on the integers.

EXAMPLES:
```
sage: v = GaussianIntegers().valuation(3); v
3-adic valuation
```

extensions \( (\text{ring}) \)
Return the extensions of this valuation to \( \text{ring} \).

EXAMPLES:
```
sage: v = GaussianIntegers().valuation(3)
sage: v.extensions(v.domain().fraction_field())
[3-adic valuation]
```

class sage.rings.padics.padic_valuation.pAdicValuation_base
\( \text{parent, } p \)
Abstract base class for \( p \)-adic valuations.

INPUT:

- \text{ring} – an integral domain
- \text{p} – a rational prime over which this valuation lies, not necessarily a uniformizer for the valuation

EXAMPLES:
```
sage: ZZ.valuation(3)
3-adic valuation
sage: QQ.valuation(5)
5-adic valuation
```
For \`p\'-adic rings, \`p\' has to match the \`p\' of the ring.
```
sage: v = valuations.pAdicValuation(Zp(3), 2); v
```
Traceback (most recent call last):
...
ValueError: prime must be an element of positive valuation

change_domain(ring)
Change the domain of this valuation to ring if possible.

EXAMPLES:

```
sage: v = ZZ.valuation(2)
sage: v.change_domain(QQ).domain()
Rational Field
```

extensions(ring)
Return the extensions of this valuation to ring.

EXAMPLES:

```
sage: v = ZZ.valuation(2)
sage: v.extensions(GaussianIntegers())
[2-adic valuation]
```

is_totally_ramified(G, include_steps=False, assume_squarefree=False)
Return whether G defines a single totally ramified extension of the completion of the domain of this valuation.

INPUT:

- G – a monic squarefree polynomial over the domain of this valuation
- include_steps – a boolean (default: False); where to include the valuations produced during the process of checking whether G is totally ramified in the return value
- assume_squarefree – a boolean (default: False); whether to assume that G is square-free over the completion of the domain of this valuation. Setting this to True can significantly improve the performance.

ALGORITHM:

This is a simplified version of sage.rings.valuation.valuation.DiscreteValuation.mac_lane_approximants().

EXAMPLES:

```
sage: k = Qp(5,4)
sage: v = k.valuation()
sage: R.<x> = k[]
sage: G = x^2 + 1
sage: v.is_totally_ramified(G)
False
sage: G = x + 1
sage: v.is_totally_ramified(G)
True
sage: G = x^2 + 2
sage: v.is_totally_ramified(G)
False
sage: G = x^2 + 5
sage: v.is_totally_ramified(G)
True
```
We consider an extension as totally ramified if its ramification index matches the degree. Hence, a trivial extension is totally ramified:

```python
sage: R.<x> = QQ[]
sage: v = QQ.valuation(2)
sage: v.is_totally_ramified(x)
True
```

```
(True, [Gauss valuation induced by 5-adic valuation, [ Gauss valuation, \( \rightarrow \) induced by 5-adic valuation, \( v((1 + O(5^4)) \cdot x) = 1/2 \) ]])
```

**is_unramified** \((G, \text{include_steps}=False, \text{assume_squarefree}=False)\)

Return whether \(G\) defines a single unramified extension of the completion of the domain of this valuation.

**INPUT:**

- \(G\) – a monic squarefree polynomial over the domain of this valuation
- \(\text{include_steps}\) – a boolean (default: False); whether to include the approximate valuations that were used to determine the result in the return value.
- \(\text{assume_squarefree}\) – a boolean (default: False); whether to assume that \(G\) is square-free over the completion of the domain of this valuation. Setting this to True can significantly improve the performance.

**EXAMPLES:**

We consider an extension as unramified if its ramification index is 1. Hence, a trivial extension is unramified:

```python
sage: R.<x> = QQ[]
sage: v = QQ.valuation(2)
sage: v.is_unramified(x)
True
```

If \(G\) remains irreducible in reduction, then it defines an unramified extension:

```python
sage: v.is_unramified(x^2 + x + 1)
True
```

However, even if \(G\) factors, it might define an unramified extension:

```python
sage: v.is_unramified(x^2 + 2*x + 4)
True
```

**lift** \((x)\)

Lift \(x\) from the residue field to the domain of this valuation.

**INPUT:**

- \(x\) – an element of the \(\text{residue_field}()\)

**EXAMPLES:**

```python
sage: v = ZZ.valuation(3)
sage: xbar = v.reduce(4)
sage: v.lift(xbar)
1
```
p()

Return the $p$ of this $p$-adic valuation.

EXAMPLES:

```
sage: GaussianIntegers().valuation(2).p()
sage: 2
```

reduce(x)

Reduce $x$ modulo the ideal of elements of positive valuation.

INPUT:

• $x$ – an element in the domain of this valuation

OUTPUT:

An element of the `residue_field()`.

EXAMPLES:

```
sage: v = ZZ.valuation(3)
sage: v.reduce(4)
sage: 1
```

restriction(ring)

Return the restriction of this valuation to `ring`.

EXAMPLES:

```
sage: v = GaussianIntegers().valuation(2)
sage: v.restriction(ZZ)
sage: 2-adic valuation
```

value_semigroup()

Return the value semigroup of this valuation.

EXAMPLES:

```
sage: v = GaussianIntegers().valuation(2)
sage: v.value_semigroup()
sage: Additive Abelian Semigroup generated by 1/2
```

class sage.rings.padics.padic_valuation.pAdicValuation_int (parent, p)

A $p$-adic valuation on the integers or the rationals.

EXAMPLES:

```
sage: v = ZZ.valuation(3); v
sage: 3-adic valuation
```

inverse(x, precision)

Return an approximate inverse of $x$.

The element returned is such that the product differs from 1 by an element of valuation at least precision.

INPUT:

• $x$ – an element in the domain of this valuation

5.13. $p$-adic valuations on number fields and their subrings and completions
• precision – a rational or infinity

EXAMPES:

```python
sage: v = ZZ.valuation(2)
sage: x = 3
sage: y = v.inverse(3, 2); y
3
sage: x*y - 1
8
```

This might not be possible for elements of positive valuation:

```python
sage: v.inverse(2, 2)
Traceback (most recent call last):
  ...  
ValueError: element has no approximate inverse in this ring
```

Unless the precision is very small:

```python
sage: v.inverse(2, 0)
1
```

`residue_ring()`

Return the residue field of this valuation.

EXAMPES:

```python
sage: v = ZZ.valuation(3)
sage: v.residue_ring()
Finite Field of size 3
```

`simplify(x, error=None, force=False, size_heuristic_bound=32)`

Return a simplified version of `x`.

Produce an element which differs from `x` by an element of valuation strictly greater than the valuation of `x` (or strictly greater than `error` if set.)

INPUT:

• `x` – an element in the domain of this valuation

• `error` – a rational, infinity, or None (default: None), the error allowed to introduce through the simplification

• `force` – ignored

• `size_heuristic_bound` – when `force` is not set, the expected factor by which the `x` need to shrink to perform an actual simplification (default: 32)

EXAMPES:

```python
sage: v = ZZ.valuation(2)
sage: v.simplify(6, force=True)
2
sage: v.simplify(6, error=0, force=True)
0
```

In this example, the usual rational reconstruction misses a good answer for some moduli (because the absolute value of the numerator is not bounded by the square root of the modulus):
uniformizer()

Return a uniformizer of this \( p \)-adic valuation, i.e., \( p \) as an element of the domain.

EXAMPLES:

```python
sage: v = ZZ.valuation(3)
sage: v.uniformizer()
3
```

class sage.rings.padics.padic_valuation.pAdicValuation_padic(parent)

Bases: sage.rings.padics.padic_valuation.pAdicValuation_base

The \( p \)-adic valuation of a complete \( p \)-adic ring.

INPUT:

- \( R \) – a \( p \)-adic ring

EXAMPLES:

```python
sage: v = Qp(2).valuation(); v
2-adic valuation

sage: R = Zp(3)
sage: v = R.valuation()
sage: v.element_with_valuation(3)
3^3 + O(3^23)

sage: K = Qp(3)
sage: R.<y> = K[]
sage: L.<y> = K.extension(y^2 + 3*y + 3)
sage: L.valuation().element_with_valuation(3/2)
y^3 + O(y^43)
```

lift(x)

Lift \( x \) from the \( \text{residue_field()} \) to the domain of this valuation.

INPUT:

- \( x \) – an element of the \( \text{residue_field()} \) of this valuation

EXAMPLES:
reduce(x)
Reduce x modulo the ideal of elements of positive valuation.

INPUT:
- x – an element of the domain of this valuation

OUTPUT:
An element of the residue_field().

EXAMPLES:
```python
sage: R = Zp(3)
sage: v = R.valuation()
sage: xbar = v.reduce(R(4))
sage: v.lift(xbar)
1 + O(3^20)
```

residue_ring()
Return the residue field of this valuation.

EXAMPLES:
```python
sage: Qq(9, names='a').valuation().residue_ring()
Finite Field in a0 of size 3^2
```

shift(x, s)
Shift x in its expansion with respect to uniformizer() by s “digits”.
For non-negative s, this just returns x multiplied by a power of the uniformizer π.
For negative s, it does the same but when not over a field, it drops coefficients in the π-adic expansion which have negative valuation.

EXAMPLES:
```python
sage: R = ZpCA(2)
sage: v = R.valuation()
sage: v.shift(R.one(), 1)
2 + O(2^20)
sage: v.shift(R.one(), -1)
O(2^19)
sage: S.<y> = R[]
sage: S.<y> = R.extension(y^3 - 2)
sage: v = S.valuation()
sage: v.shift(1, 5)
y^5 + O(y^60)
```

simplify(x, error=None, force=False)
Return a simplified version of x.
Produce an element which differs from x by an element of valuation strictly greater than the valuation of x (or strictly greater than error if set.)

INPUT:
• x – an element in the domain of this valuation

• error – a rational, infinity, or None (default: None), the error allowed to introduce through the
  simplification

• force – ignored

EXAMPLES:

```
sage: R = Zp(2)
sage: v = R.valuation()
sage: v.simplify(6)
2 + O(2^21)
sage: v.simplify(6, error=0)
0
```

uniformizer()

Return a uniformizer of this valuation.

EXAMPLES:

```
sage: v = Zp(3).valuation()
sage: v.uniformizer()
3 + O(3^21)
```
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