Coercion

Release 10.2

The Sage Development Team

Dec 06, 2023
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1.1 What is coercion all about?

The primary goal of coercion is to be able to transparently do arithmetic, comparisons, etc. between elements of distinct sets.

As a concrete example, when one writes $1 + 1/2$ one wants to perform arithmetic on the operands as rational numbers, despite the left being an integer. This makes sense given the obvious and natural inclusion of the integers into the rational numbers. The goal of the coercion system is to facilitate this (and more complicated arithmetic) without having to explicitly map everything over into the same domain, and at the same time being strict enough to not resolve ambiguity or accept nonsense. Here are some examples:

```
sage: 1 + 1/2
3/2
sage: R.<x,y> = ZZ[]
sage: R
Multivariate Polynomial Ring in x, y over Integer Ring
sage: parent(x)
Multivariate Polynomial Ring in x, y over Integer Ring
sage: parent(1/3)
Rational Field
sage: x+1/3
x + 1/3
sage: parent(x+1/3)
Multivariate Polynomial Ring in x, y over Rational Field

sage: GF(5)(1) + CC(I)
Traceback (most recent call last):
  ... TypeError: unsupported operand parent(s) for +: 'Finite Field of size 5' and 'Complex Field with 53 bits of precision'
```
1.2 Parents and Elements

Parents are objects in concrete categories, and Elements are their members. Parents are first-class objects. Most things in Sage are either parents or have a parent. Typically whenever one sees the word Parent one can think Set. Here are some examples:

```
sage: parent(1)
Integer Ring
sage: parent(1) is ZZ
True
sage: ZZ
Integer Ring
sage: parent(1.50000000000000000000000000000000000)
Real Field with 120 bits of precision
sage: parent(x)
Symbolic Ring
sage: x^sin(x)
x^sin(x)
sage: R.<t> = Qp(5)[]
sage: f = t^3-5; f
(1 + O(5^20))*t^3 + 4*5 + 4*5^2 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8 + 4*5^9 +
   4*5^10 + 4*5^11 + 4*5^12 + 4*5^13 + 4*5^14 + 4*5^15 + 4*5^16 + 4*5^17 + 4*5^18 + 4*5^19 +
   4*5^20 + 0(5^21)
sage: parent(f)
Univariate Polynomial Ring in t over 5-adic Field with capped relative precision 20
```

There is an important distinction between Parents and types:

```
sage: a = GF(5).random_element()
sage: b = GF(7).random_element()
sage: type(a)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: type(b)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: type(a) == type(b)
True
sage: parent(a)
Finite Field of size 5
sage: parent(a) == parent(b)
False
```

However, non-Sage objects do not really have parents, but we still want to be able to reason with them, so their type is used instead:
In fact, under the hood, a special kind of parent “The set of all Python objects of class T” is used in these cases. Note that parents are not always as tight as possible.

1.3 Maps between Parents

Many parents come with maps to and from other parents.

Sage makes a distinction between being able to convert between various parents, and coerce between them. Conversion is explicit and tries to make sense of an object in the target domain if at all possible. It is invoked by calling:

```sage
sage: ZZ(5)
5
sage: ZZ(10/5)
2
sage: QQ(10)
10
sage: parent(QQ(10))
Rational Field
sage: a = GF(5)(2); a
2
sage: parent(a)
Finite Field of size 5
sage: parent(ZZ(a))
Integer Ring
sage: GF(71)(1/5)
57
sage: ZZ(1/2)
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

Conversions need not be canonical (they may for example involve a choice of lift) or even make sense mathematically (e.g. constructions of some kind).

```sage
sage: ZZ("123")
123
sage: ZZ(GF(5)(14))
4
sage: ZZ["x"]([4,3,2,1])
x^3 + 2*x^2 + 3*x + 4
sage: a = Qp(5, 10)(1/3); a
2 + 3*5 + 5*2 + 3*5*3 + 5*4 + 3*5*5 + 5*6 + 3*5*7 + 5*8 + 3*5*9 + 0(5*10)
```
On the other hand, Sage has the notion of a **coercion**, which is a canonical morphism (occasionally up to a conventional choice made by developers) between parents. A coercion from one parent to another **must** be defined on the whole domain, and always succeeds. As it may be invoked implicitly, it should be obvious and natural (in both the mathematically rigorous and colloquial sense of the word). Up to inescapable rounding issues that arise with inexact representations, these coercion morphisms should all commute. In particular, if there are coercion maps $A \rightarrow B$ and $B \rightarrow A$, then their composites must be the identity maps.

Coercions can be discovered via the `Parent.has_coerce_map_from()` method, and if needed explicitly invoked with the `Parent.coerce()` method:

```python
sage: QQ.has_coerce_map_from(ZZ)
True
sage: QQ.has_coerce_map_from(RR)
False
sage: ZZ['x'].has_coerce_map_from(QQ)
False
sage: ZZ['x'].has_coerce_map_from(ZZ)
True
sage: ZZ['x'].coerce(5)
5
sage: ZZ['x'].coerce(5).parent()
Univariate Polynomial Ring in x over Integer Ring
sage: ZZ['x'].coerce(5/1)
Traceback (most recent call last):
  ... TypeError: no canonical coercion from Rational Field to Univariate Polynomial Ring in x over Integer Ring
```
CHAPTER TWO

BASIC ARITHMETIC RULES

Suppose we want to add two element, \( a \) and \( b \), whose parents are \( A \) and \( B \) respectively. When we type \( a+b \) then

1. If \( A \) is \( B \), call \( a._add_(b) \)
2. If there is a coercion \( \phi : B \to A \), call \( a._add_((\phi)(b)) \)
3. If there is a coercion \( \phi : A \to B \), call \( (\phi)(a)._add_(b) \)
4. Look for \( Z \) such that there is a coercion \( \phi_A : A \to Z \) and \( \phi_B : B \to Z \), call \( \phi_A(a)._add_((\phi_B)(b)) \)

These rules are evaluated in order; therefore if there are coercions in both directions, then the parent of \( a._add_b \) is \( A \) – the parent of the left-hand operand is used in such cases.

The same rules are used for subtraction, multiplication, and division. This logic is embedded in a coercion model object, which can be obtained and queried.

```python
sage: parent(1 + 1/2)
Rational Field
sage: cm = coercion_model; cm
<...>
sage: cm.explain(ZZ, QQ)
Coercion on left operand via
  Natural morphism:
    From: Integer Ring
    To: Rational Field
Arithmetic performed after coercions.
Result lives in Rational Field
Rational Field

sage: cm.explain(ZZ[['x','y']], QQ['x'])
Coercion on left operand via
  Coercion map:
    From: Multivariate Polynomial Ring in x, y over Integer Ring
    To: Multivariate Polynomial Ring in x, y over Rational Field
Coercion on right operand via
  Coercion map:
    From: Univariate Polynomial Ring in x over Rational Field
    To: Multivariate Polynomial Ring in x, y over Rational Field
Arithmetic performed after coercions.
Result lives in Multivariate Polynomial Ring in x, y over Rational Field
Multivariate Polynomial Ring in x, y over Rational Field
```

The coercion model can be used directly for any binary operation (callable taking two arguments).
There are also actions in the sense that a field $K$ acts on a module over $K$, or a permutation group acts on a set. These are discovered between steps 1 and 2 above.

```
sage: cm.bin_op(77, 9, gcd)
1
```

Note that by github issue #14711 Sage’s coercion system uses maps with weak references to the domain. Such maps should only be used internally, and so a copy should be used instead (unless one knows what one is doing):

```
sage: QQ._internal_coerce_map_from(int)
(map internal to coercion system -- copy before use)
Native morphism:
  From: Set of Python objects of class 'int'
  To:    Rational Field
sage: copy(QQ._internal_coerce_map_from(int))
Native morphism:
  From: Set of Python objects of class 'int'
  To:    Rational Field
```

Note that the user-visible method (without underscore) automates this copy:

```
sage: copy(QQ.coerce_map_from(int))
Native morphism:
  From: Set of Python objects of class 'int'
  To:    Rational Field
```

```
sage: QQ.has_coerce_map_from(RR)
False
sage: QQ['x'].get_action(QQ)
Right scalar multiplication by Rational Field on Univariate Polynomial Ring in x over...
  -- Rational Field
sage: QQ2 = QQ^2
```

(continues on next page)
sage: (QQ2).get_action(QQ)
Right scalar multiplication by Rational Field on Vector space of dimension 2 over Rational Field

sage: QQ['x'].get_action(RR)
Right scalar multiplication by Real Field with 53 bits of precision on Univariate Polynomial Ring in x over Rational Field
3.1 Methods to implement

- Arithmetic on Elements: _add_, _sub_, _mul_, _div_
  This is where the binary arithmetic operators should be implemented. Unlike Python's __add__, both operands are guaranteed to have the same Parent at this point.

- Coercion for Parents: _coerce_map_from_
  Given two parents R and S, R._coerce_map_from_(S) is called to determine if there is a coercion \( \phi : S \rightarrow R \). Note that the function is called on the potential codomain. To indicate that there is no coercion from S to R (self), return False or None. This is the default behavior. If there is a coercion, return True (in which case an morphism using R._element_constructor_ will be created) or an actual Morphism object with S as the domain and R as the codomain.

- Actions for Parents: _get_action_ or _rmul_, _lmul_, _act_on_, _acted_upon_
  Suppose one wants R to act on S. Some examples of this could be \( R = \mathbb{Q}, S = \mathbb{Q}[x] \) or \( R = \text{Gal}(S/\mathbb{Q}) \) where S is a number field. There are several ways to implement this:
  - If R is the base of S (as in the first example), simply implement _rmul_ and/or _lmul_ on the Elements of S. In this case \( r \cdot s \) gets handled as \( s \cdot \text{rmul}(r) \) and \( s \cdot r \) as \( s \cdot \text{lmul}(r) \). The argument to _rmul_ and _lmul_ are guaranteed to be Elements of the base of S (with coercion happening beforehand if necessary).
  - If R acts on S, one can define the methods _act_on_ on Elements of R or _acted_upon_ on Elements of S. In this case \( r \cdot s \) gets handled as \( r \cdot \text{act_on}(s, \text{True}) \) or \( s \cdot \text{acted_upon}(r, \text{False}) \) and \( s \cdot r \) as \( r \cdot \text{act_on}(s, \text{False}) \) or \( s \cdot \text{acted_upon}(r, \text{True}) \). There is no constraint on the type or parents of objects passed to these methods; raise a TypeError or ValueError if the wrong kind of object is passed in to indicate the action is not appropriate here.
  - If either R acts on S or S acts on R, one may implement R._get_action_ to return an actual Action object to be used. This is how non-multiplicative actions must be implemented, and is the most powerful and complete way to do things.

It should be noted that for the first way to work, elements of S are required to be ModuleElements. This requirement is likely to be lifted in the future.

- Element conversion/construction for Parents: use _element_constructor_ not __call__
  The Parent.__call__() method dispatches to _element_constructor_. When someone writes \( R(x, \ldots) \), this is the method that eventually gets called in most cases. See the documentation on the __call__ method below.

Parents may also call the self._populate_coercion_lists_ method in their __init__ functions to pass any callable for use instead of _element_constructor_, provide a list of Parents with coercions to self (as an alternative
to implementing \_\_coerce_map\_\_from\_\_, provide special construction methods (like \_\_integer\_ for \texttt{ZZ}), etc. This also allows one to specify a single coercion embedding \textit{out} of self (whereas the rest of the coercion functions all specify maps \textit{into} self). There is extensive documentation in the docstring of the \_\_populate\_\_coercion\_\_lists\_\_ method.

### 3.2 Example

Sometimes a simple example is worth a thousand words. Here is a minimal example of setting up a simple Ring that handles coercion. (It is easy to imagine much more sophisticated and powerful localizations, but that would obscure the main points being made here.)

```python
class Localization(Ring):
    def __init__(self, primes):
        ""
        Localization of \texttt{\ZZ} away from primes.
        ""
        Ring.__init__(self, base=ZZ)
        self._primes = primes
        self._populate_coercion_lists_()
        
    def _repr_(self):
        ""
        How to print self.
        ""
        return "%s localized at %s" % (self.base(), self._primes)
        
    def _element_constructor_(self, x):
        ""
        Make sure x is a valid member of self, and return the constructed element.
        ""
        if isinstance(x, LocalizationElement):
            x = x._value
        else:
            x = QQ(x)
        for p, e in x.denominator().factor():
            if p not in self._primes:
                raise ValueError("Not integral at %s" % p)
        return LocalizationElement(self, x)
        
    def _coerce_map_from_(self, S):
        ""
        The only things that coerce into this ring are:
        - the integer ring
        - other localizations away from fewer primes
        ""
        if S is ZZ:
            return True
        elif isinstance(S, Localization):
            return all(p in self._primes for p in S._primes)
```

(continues on next page)
class LocalizationElement(RingElement):

    def __init__(self, parent, x):
        RingElement.__init__(self, parent)
        self._value = x

    # We're just printing out this way to make it easy to see what's going on in the examples.
    def _repr_(self):
        return "LocalElt(%s)" % self._value

    # Now define addition, subtraction, and multiplication of elements.
    # Note that left and right always have the same parent.
    def __add__(left, right):
        return LocalizationElement(left.parent(), left._value + right._value)
    def __sub__(left, right):
        return LocalizationElement(left.parent(), left._value - right._value)
    def __mul__(left, right):
        return LocalizationElement(left.parent(), left._value * right._value)

    # The basering was set to ZZ, so c is guaranteed to be in ZZ
    def __rmul__(self, c):
        return LocalizationElement(self.parent(), c * self._value)
    def __lmul__(self, c):
        return LocalizationElement(self.parent(), self._value * c)

That's all there is to it. Now we can test it out:

```
sage: R = Localization([2]); R
Integer Ring localized at [2]
sage: R(1)
LocalElt(1)
sage: R(1/2)
LocalElt(1/2)
sage: R(1/3)
Traceback (most recent call last):
  ... ValueError: Not integral at 3
sage: R.coerce(1)
LocalElt(1)
sage: R.coerce(1/4)
Traceback (click to the left for traceback)
  ... TypeError: no canonical coercion from Rational Field to Integer Ring localized at [2]
```
sage: R(1/2) + R(3/4)
LocalElt(5/4)
sage: R(1/2) + 5
LocalElt(11/2)
sage: 5 + R(1/2)
LocalElt(11/2)
sage: R(1/2) + 1/7
Traceback (most recent call last):
...  
TypeError: unsupported operand parent(s) for +: 'Integer Ring localized at [2]' and 'Rational Field'
sage: R(3/4) * 7
LocalElt(21/4)
sage: R.get_action(ZZ)
Right scalar multiplication by Integer Ring on Integer Ring localized at [2]
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.explain(R, ZZ, operator.add)
Coercion on right operand via
  Conversion map:
    From: Integer Ring
    To: Integer Ring localized at [2]
Arithmetic performed after coercions.
Result lives in Integer Ring localized at [2]
Integer Ring localized at [2]
sage: cm.explain(R, ZZ, operator.mul)
Action discovered.
  Right scalar multiplication by Integer Ring on Integer Ring localized at [2]
Result lives in Integer Ring localized at [2]
Integer Ring localized at [2]
sage: R6 = Localization([2,3]); R6
Integer Ring localized at [2, 3]
sage: R6(1/3) - R(1/2)
LocalElt(-1/6)
sage: parent(R6(1/3) - R(1/2))
Integer Ring localized at [2, 3]
sage: R.has_coerce_map_from(ZZ)
True
sage: R.coerce_map_from(ZZ)
Conversion map:
  From: Integer Ring
  To: Integer Ring localized at [2]
sage: R6.coerce_map_from(R)
Conversion map:
  From: Integer Ring localized at [2]
  To: Integer Ring localized at [2, 3]
sage: R6.coerce(R(1/2))
LocalElt(1/2)

sage: cm.explain(R, R6, operator.mul)
Coercion on left operand via
    Conversion map:
        From: Integer Ring localized at [2]
        To:   Integer Ring localized at [2, 3]
Arithmetic performed after coercions.
Result lives in Integer Ring localized at [2, 3]

3.3 Provided Methods

• __call__

This provides a consistent interface for element construction. In particular, it makes sure that conversion always gives the same result as coercion, if a coercion exists. (This used to be violated for some Rings in Sage as the code for conversion and coercion got edited separately.) Let \( R \) be a Parent and assume the user types \( R(x) \), where \( x \) has parent \( X \). Roughly speaking, the following occurs:

1. If \( X \) is \( R \), return \( x \) (*)
2. If there is a coercion \( f : X \to R \), return \( f(x) \)
3. If there is a coercion \( f : R \to X \), try to return \( f^{-1}(x) \)
4. Return \( R._element_constructor_(x) \) (**)

Keywords and extra arguments are passed on. The result of all this logic is cached.

(*) Unless there is a “copy” keyword like \( R(x, \text{copy}=\text{False}) \)

(**) Technically, a generic morphism is created from \( X \) to \( R \), which may use magic methods like \_integer\_ or other data provided by \_populate_coercion_lists\_.

• coerce

Coerces elements into self, raising a type error if there is no coercion map.

• coerce_map_from, convert_map_from

Returns an actual Morphism object to coerce/convert from another Parent to self. Barring direct construction of elements of \( R \), \( R\text{.convert_map_from}(S) \) will provide a callable Python object which is the fastest way to convert elements of \( S \) to elements of \( R \). From Cython, it can be invoked via the cdef \_call\_ method.

• has_coerce_map_from

Returns True or False depending on whether or not there is a coercion. \( R\text{.has_coerce_map_from}(S) \) is shorthand for \( R\text{.coerce_map_from}(S) \) is not None

• get_action

This will unwind all the \_get_action\_, \_rmul\_, \_lmul\_, \_act_on\_, \_acted_upon\_, ... methods to provide an actual Action object, if one exists.
New parents are discovered using an algorithm in sage/category/pushout.py. The fundamental idea is that most Parents in Sage are constructed from simpler objects via various functors. These are accessed via the `construction()` method, which returns a (simpler) Parent along with a functor with which one can create self.

```
sage: CC.construction()
(AlgebraicClosureFunctor, Real Field with 53 bits of precision)
sage: RR.construction()
(Completion[+Infinity, prec=53], Rational Field)
sage: QQ.construction()
(FractionField, Integer Ring)
sage: ZZ.construction()  # None
sage: Zp(5).construction()
(Completion[5, prec=20], Integer Ring)
sage: QQ.completion(5, 100, {})
5-adic Field with capped relative precision 100
sage: c, R = RR.construction()
sage: a = CC.construction()[0]
sage: a.commutes(c)
False
sage: RR == c(QQ)
True
```

```
sage: sage.categories.pushout.construction_tower(Frac(CDF['x']))
[(None, Fraction Field of Univariate Polynomial Ring in x over Complex Double Field),
 (FractionField, Univariate Polynomial Ring in x over Complex Double Field),
 (Poly[x], Complex Double Field),
 (AlgebraicClosureFunctor, Real Double Field),
 (Completion[+Infinity, prec=53], Rational Field),
 (FractionField, Integer Ring)]
```

Given Parents R and S, such that there is no coercion either from R to S or from S to R, one can find a common Z with coercions $R \to Z$ and $S \to Z$ by considering the sequence of construction functors to get from a common ancestor to both R and S. We then use a heuristic algorithm to interleave these constructors in an attempt to arrive at a suitable Z (if one exists). For example:

```
sage: ZZ['x'].construction()
(Poly[x], Integer Ring)
sage: QQ.construction()
(FractionField, Integer Ring)
```

(continues on next page)
The common ancestor is \( \mathbb{Z} \) and our options for \( \mathbb{Z} \) are \( \text{Frac}(\mathbb{Z}[x]) \) or \( \text{Frac}(\mathbb{Z})[x] \). In Sage we choose the later, treating the fraction field functor as binding “more tightly” than the polynomial functor, as most people agree that \( \mathbb{Q}[x] \) is the more natural choice. The same procedure is applied to more complicated Parents, returning a new Parent if one can be unambiguously determined.
5.1 The coercion model

The coercion model manages how elements of one parent get related to elements of another. For example, the integer 2 can canonically be viewed as an element of the rational numbers. (The parent of a non-element is its Python type.)

```
sage: ZZ(2).parent()
Integer Ring
sage: QQ(2).parent()
Rational Field
```

The most prominent role of the coercion model is to make sense of binary operations between elements that have distinct parents. It does this by finding a parent where both elements make sense, and doing the operation there. For example:

```
sage: a = 1/2; a.parent()
Rational Field
sage: b = ZZ['x'].gen(); b.parent()
Univariate Polynomial Ring in x over Integer Ring
sage: a + b
x + 1/2
sage: (a + b).parent()
Univariate Polynomial Ring in x over Rational Field
```

If there is a coercion (see below) from one of the parents to the other, the operation is always performed in the codomain of that coercion. Otherwise a reasonable attempt to create a new parent with coercion maps from both original parents is made. The results of these discoveries are cached. On failure, a `TypeError` is always raised.

Some arithmetic operations (such as multiplication) can indicate an action rather than arithmetic in a common parent. For example:

```
sage: E = EllipticCurve(37a)  # needs sage.schemes
sage: P = E(0,0)  # needs sage.schemes
sage: 5*P  # needs sage.schemes
(1/4 : -5/8 : 1)
```

where there is action of $\mathbb{Z}$ on the points of $E$ given by the additive group law. Parents can specify how they act on or are acted upon by other parents.

There are two kinds of ways to get from one parent to another, coercions and conversions.
Coercions are canonical (possibly modulo a finite number of deterministic choices) morphisms, and the set of all coercions between all parents forms a commuting diagram (modulo possibly rounding issues). \( \mathbb{Z} \rightarrow \mathbb{Q} \) is an example of a coercion. These are invoked implicitly by the coercion model.

Conversions try to construct an element out of their input if at all possible. Examples include sections of coercions, creating an element from a string or list, etc. and may fail on some inputs of a given type while succeeding on others (i.e. they may not be defined on the whole domain). Conversions are always explicitly invoked, and never used by the coercion model to resolve binary operations.

For more information on how to specify coercions, conversions, and actions, see the documentation for `Parent`.

### class `sage.structure.coerce.CoercionModel`

Bases: `object`

See also `sage.categories.pushout`

**EXAMPLES:**

```python
sage: f = ZZ['t', 'x'].0 + QQ['x'].0 + CyclotomicField(13).gen(); f
#- needs sage.rings.number_field
t + x + zeta13
sage: f.parent()
#- needs sage.rings.number_field
Multivariate Polynomial Ring in t, x
over Cyclotomic Field of order 13 and degree 12
sage: ZZ['x', 'y'].0 + ~Frac(QQ['y']).0
(x*y + 1)/y
sage: MatrixSpace(ZZ['x'], 2, 2)(2) + ~Frac(QQ['x']).0
#- needs sage.modules
[(2*x + 1)/x 0]
[ 0 (2*x + 1)/x]
sage: f = ZZ['x,y,z'].0 + QQ['w,x,z,a'].0; f
w + x
sage: f.parent()
Multivariate Polynomial Ring in w, x, y, z, a over Rational Field
sage: ZZ['x,y,z'].0 + ZZ['w,x,z,a'].1
2*x
```

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**analyse** (`xp, yp, op='mul'`)

Emulate the process of doing arithmetic between `xp` and `yp`, returning a list of steps and the parent that the result will live in.

The `explain()` method is easier to use, but if one wants access to the actual morphism and action objects (rather than their string representations), then this is the function to use.

**EXAMPLES:**

```python
sage: cm = sage.structure.element.get_coercion_model()
sage: GF7 = GF(7)
sage: steps, res = cm.analyse(GF7, ZZ)
sage: steps
['Coercion on right operand via',
Natural morphism:
From: Integer Ring
(continues on next page)```
To: Finite Field of size 7, 'Arithmetic performed after coercions.'][

```
sage: res
Finite Field of size 7
sage: f = steps[1]; type(f)
<class 'sage.rings.finite_rings.integer_mod.Integer_to_IntegerMod'>
sage: f(100)
2
```

### bin_op(x, y, op)

Execute the operation op on x and y.

It first looks for an action corresponding to op, and failing that, it tries to coerce x and y into a common parent and calls op on them.

If it cannot make sense of the operation, a TypeError is raised.

**INPUT:**

- x – the left operand
- y – the right operand
- op – a python function taking 2 arguments

**Note:** op is often an arithmetic operation, but need not be so.

**EXAMPLES:**

```
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.bin_op(1/2, 5, operator.mul)
5/2
```

The operator can be any callable:

```
sage: R.<x> = ZZ['x']
sage: cm.bin_op(x^2 - 1, x + 1, gcd)
x + 1
```

Actions are detected and performed:

```
sage: M = matrix(ZZ, 2, 2, range(4))  # needs sage.modules
sage: V = vector(ZZ, [5, 7])  # needs sage.modules
sage: cm.bin_op(M, V, operator.mul)  # needs sage.modules
(7, 31)
```

### canonical_coercion(x, y)

Given two elements x and y, with parents S and R respectively, find a common parent Z such that there are coercions f : S → Z and g : R → Z and return f(x), g(y), which will have the same parent.

Raises a type error if no such Z can be found.

**EXAMPLES:**
Coercion, Release 10.2

```python
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.canonical_coercion(mod(2, 10), 17)
(2, 7)
sage: # needs sage.modules
sage: x, y = cm.canonical_coercion(1/2, matrix(ZZ, 2, 2, range(4)))
sage: x
[1/2 0]
[ 0 1/2]
sage: y
[0 1]
[2 3]
sage: parent(x) is parent(y)
True
```

There is some support for non-Sage datatypes as well:

```python
sage: x, y = cm.canonical_coercion(int(5), 10)
sage: type(x), type(y)
(<class 'sage.rings.integer.Integer'>, <class 'sage.rings.integer.Integer'>)
sage: x, y = cm.canonical_coercion(int(5), complex(3))
sage: type(x), type(y)
(<class 'complex'>, <class 'complex'>)
sage: class MyClass:
    ....:     def _sage_(self):
    ....:         return 13
sage: a, b = cm.canonical_coercion(MyClass(), 1/3)
sage: a, b
(13, 1/3)
sage: type(a)
<class 'sage.rings.rational.Rational'>
```

We also make an exception for 0, even if \( Z \) does not map in:

```python
sage: canonical_coercion(vector([1, 2, 3]), 0) # needs sage.modules
((1, 2, 3), (0, 0, 0))
sage: canonical_coercion(GF(5)(0), float(0))
(0, 0)
```

**coercion_maps** \((R, S)\)

Give two parents \( R \) and \( S \), return a pair of coercion maps \( f : R \to Z \) and \( g : S \to Z \), if such a \( Z \) can be found.

In the (common) case that \( R = Z \) or \( S = Z \) then `None` is returned for \( f \) or \( g \) respectively rather than constructing (and subsequently calling) the identity morphism.

If no suitable \( f, g \) can be found, a single `None` is returned. This result is cached.

**Note:** By [github issue #14711](https://github.com/sagemath/sage/issues/14711), coerce maps should be copied when using them outside of the coercion system, because they may become defunct by garbage collection.
EXAMPLES:

```python
sage: cm = sage.structure.element.get_coercion_model()
sage: f, g = cm.coercion_maps(ZZ, QQ)
sage: print(copy(f))
Natural morphism:
  From: Integer Ring
  To:  Rational Field
sage: print(g)
None

sage: ZZx = ZZ['x']
sage: f, g = cm.coercion_maps(ZZx, QQ)
sage: print(f)
(map internal to coercion system -- copy before use)
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:  Univariate Polynomial Ring in x over Rational Field
sage: print(g)
(map internal to coercion system -- copy before use)
Polynomial base injection morphism:
  From: Rational Field
  To:  Univariate Polynomial Ring in x over Rational Field

sage: K = GF(7)
sage: cm.coercion_maps(QQ, K) is None
True
```

Note that to break symmetry, if there is a coercion map in both directions, the parent on the left is used:

```python
sage: # needs sage.modules
sage: V = QQ^3
sage: W = V.__class__(QQ, 3)
sage: V == W
True
sage: V is W
False
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.coercion_maps(V, W)
(None, (map internal to coercion system -- copy before use)
Coercion map:
  From: Vector space of dimension 3 over Rational Field
  To:  Vector space of dimension 3 over Rational Field)
sage: cm.coercion_maps(W, V)
(None, (map internal to coercion system -- copy before use)
Coercion map:
  From: Vector space of dimension 3 over Rational Field
  To:  Vector space of dimension 3 over Rational Field)
sage: v = V([1,2,3])
sage: w = W([1,2,3])
sage: parent(v + w) is V
True
```

(continues on next page)
Coercion, Release 10.2

sage: parent(w + v) is W
True

common_parent(*args)
Compute a common parent for all the inputs.

It’s essentially an \(n\)-ary canonical coercion except it can operate on parents rather than just elements.

INPUT:

- args – a set of elements and/or parents

OUTPUT:

A Parent into which each input should coerce, or raises a TypeError if no such Parent can be found.

EXAMPLES:

sage: cm = sage.structure.element.get_coercion_model()
sage: cm.common_parent(ZZ, QQ)
Rational Field

\[\texttt{\#} \text{needs sage.rings.real_mpfr}\]
Real Field with 53 bits of precision

sage: ZZT = ZZ[[ 'T']]
sage: QQT = QQ[[ 'T']]
sage: cm.common_parent(ZZT, QQT, RDF)
Power Series Ring in T over Real Double Field

sage: cm.common_parent(4r, 5r)
<class 'int'>
sage: cm.common_parent(int, float, ZZ)
<class 'float'>

\[\texttt{\#} \text{needs sage.rings.real_mpfr}\]
Real Field with 10 bits of precision

There are some cases where the ordering does matter, but if a parent can be found it is always the same:

sage: QOxy = QQ[['x,y']]
sage: QQyz = QQ[['y,z']]
sage: cm.common_parent(QOxy, QQyz) == cm.common_parent(QQyz, QOxy)
True

sage: QQzt = QQ[['z,t']]
sage: cm.common_parent(QQxy, QQyz, QQzt)
Multivariate Polynomial Ring in x, y, z, t over Rational Field

\[\texttt{\#} \text{needs sage.rings.real_mpfr}\]
sage: cm.common_parent(QQxy, QQzt, QQyz)
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents:
'Multivariate Polynomial Ring in x, y over Rational Field' and
'Multivariate Polynomial Ring in z, t over Rational Field'

discover_action(R, S, op, r=None, s=None)

INPUT:
• $R$ – the left Parent (or type)
• $S$ – the right Parent (or type)
• $op$ – the operand, typically an element of the operator module
• $r$ – (optional) element of $R$
• $s$ – (optional) element of $S$.

OUTPUT:
• An action $A$ such that $s \, op \, r$ is given by $A(s, r)$.

The steps taken are illustrated below.

EXAMPLES:

```sage
sage: P, x = ZZ['x']
sage: P.get_action(ZZ)
Right scalar multiplication by Integer Ring on
  Univariate Polynomial Ring in x over Integer Ring
sage: ZZ.get_action(P) is None
True
sage: cm = sage.structure.element.get_coercion_model()
```

If $R$ or $S$ is a Parent, ask it for an action by/on $R$:

```sage
sage: cm.discover_action(ZZ, P, operator.mul)
Left scalar multiplication by Integer Ring on
  Univariate Polynomial Ring in x over Integer Ring
```

If $R$ or $S$ a type, recursively call `get_action()` with the Sage versions of $R$ and/or $S$:

```sage
sage: cm.discover_action(P, int, operator.mul)
Right scalar multiplication by Integer Ring on
  Univariate Polynomial Ring in x over Integer Ring
  with precomposition on right by Native morphism:
  From: Set of Python objects of class 'int'
  To: Integer Ring
```

If $op$ is division, look for action on right by inverse:

```sage
sage: cm.discover_action(P, ZZ, operator.truediv)
Right inverse action by Rational Field on
  Univariate Polynomial Ring in x over Integer Ring
  with precomposition on right by Natural morphism:
  From: Integer Ring
  To: Rational Field
```

Check that github issue #17740 is fixed:

```sage
sage: R = GF(5)['x']
sage: cm.discover_action(R, ZZ, operator.truediv)
Right inverse action by Finite Field of size 5
  on Univariate Polynomial Ring in x over Finite Field of size 5
  with precomposition on right by Natural morphism:
  From: Integer Ring
```

(continues on next page)
To: Finite Field of size 5

```
sage: cm.bin_op(R.gen(), 7, operator.truediv).parent()
Univariate Polynomial Ring in x over Finite Field of size 5
```

Check that [github issue #18221](https://github.com/sagemath/sage/issues/18221) is fixed:

```
sage: # needs sage.combinat sage.modules
sage: F.<x> = FreeAlgebra(QQ)
sage: x / 2
1/2*x
sage: cm.discover_action(F, ZZ, operator.truediv)
Right inverse action by Rational Field on
  Free Algebra on 1 generators (x,) over Rational Field
  with precomposition on right by Natural morphism:
    From: Integer Ring
    To: Rational Field
```

discover_coercion($R, S$)

This actually implements the finding of coercion maps as described in the `coercion_maps()` method.

**EXAMPLES:**

```
sage: cm = sage.structure.element.get_coercion_model()
```

If $R$ is $S$, then two identity morphisms suffice:

```
sage: cm.discover_coercion(SR, SR)  # needs sage.symbolic
     (None, None)
```

If there is a coercion map either direction, use that:

```
sage: cm.discover_coercion(ZZ, QQ)  # needs sage.rings.real_mpfr
    ((map internal to coercion system -- copy before use)
     Natural morphism:
       From: Integer Ring
       To: Rational Field, None)
sage: cm.discover_coercion(RR, QQ)  # needs sage.rings.real_mpfr
    (None, (map internal to coercion system -- copy before use)
     Generic map:
       From: Rational Field
       To: Real Field with 53 bits of precision)
```

Otherwise, try and compute an appropriate cover:

```
sage: ZZxy = ZZ['x','y']
sage: cm.discover_coercion(ZZxy, RDF)
    ((map internal to coercion system -- copy before use)
     Call morphism:
       From: Multivariate Polynomial Ring in x, y over Integer Ring
       To: Multivariate Polynomial Ring in x, y over Real Double Field,
          (map internal to coercion system -- copy before use))
```

(continues on next page)
Polynomial base injection morphism:
From: Real Double Field
To: Multivariate Polynomial Ring in x, y over Real Double Field)

Sometimes there is a reasonable “cover,” but no canonical coercion:

```
sage: sage.categories.pushout.pushout(QQ, QQ^3)
Vector space of dimension 3 over Rational Field

sage: print(cm.discover_coercion(QQ, QQ^3))
None
```

```
Polynomial base injection morphism:
From: Real Double Field
To: Multivariate Polynomial Ring in x, y over Real Double Field

Sometimes there is a reasonable “cover,” but no canonical coercion:

```
sage: sage.categories.pushout.pushout(QQ, QQ^3)
Vector space of dimension 3 over Rational Field

sage: print(cm.discover_coercion(QQ, QQ^3))
None
```

```
division_parent(P)
Deduces where the result of division in P lies by calculating the inverse of P.one() or P.an_element().

The result is cached.

EXAMPLES:
```
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.division_parent(ZZ)
Rational Field
sage: cm.division_parent(QQ)
Rational Field
sage: ZZx = ZZ['x']
sage: cm.division_parent(ZZx)
Fraction Field of Univariate Polynomial Ring in x over Integer Ring
sage: K = GF(41)
sage: cm.division_parent(K)
Finite Field of size 41
sage: Zmod100 = Integers(100)
sage: cm.division_parent(Zmod100)
Ring of integers modulo 100
sage: S5 = SymmetricGroup(5)
sage: cm.division_parent(S5)
Symmetric group of order 5! as a permutation group
```

```
exception_stack()
Returns the list of exceptions that were caught in the course of executing the last binary operation. Useful
for diagnosis when user-defined maps or actions raise exceptions that are caught in the course of coercion
detection.

If all went well, this should be the empty list. If things aren’t happening as you expect, this is a good place
to check. See also coercion_traceback().

EXAMPLES:
```
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.record_exceptions()
sage: 1/2 + 2
5/2
```
```
sage: cm.exception_stack()
[]
sage: 1/2 + GF(3)(2)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +:
'Rational Field' and 'Finite Field of size 3'

Now see what the actual problem was:

sage: import traceback
sage: cm.exception_stack()
['Traceback (most recent call last):...', 'Traceback (most recent call last):... →'
]  
sage: print(cm.exception_stack()[-1])
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents:
'Rational Field' and 'Finite Field of size 3'

This is typically accessed via the coercion_traceback() function.

sage: coercion_traceback()
Traceback (most recent call last):
...
TypeError: no common canonical parent for objects with parents:
'Rational Field' and 'Finite Field of size 3'

\texttt{explain}(xp, yp, op='mul', verbosity=2)

This function can be used to understand what coercions will happen for an arithmetic operation between xp and yp (which may be either elements or parents). If the parent of the result can be determined then it will be returned.

EXAMPLES:

sage: cm = sage.structure.element.get_coercion_model()
sage: cm.explain(ZZ, ZZ)
Identical parents, arithmetic performed immediately.
Result lives in Integer Ring
Integer Ring

sage: cm.explain(QQ, int)
Coercion on right operand via
Native morphism:
  From: Set of Python objects of class 'int'
  To: Rational Field
Arithmetic performed after coercions.
Result lives in Rational Field
Rational Field

sage: R = ZZ['x']
sage: cm.explain(R, QQ)
Action discovered.
   Right scalar multiplication by Rational Field
   on Univariate Polynomial Ring in x over Integer Ring
Result lives in Univariate Polynomial Ring in x over Rational Field
Univariate Polynomial Ring in x over Rational Field

```
sage: cm.explain(ZZ['x'], QQ, operator.add)
Coercion on left operand via
   Ring morphism:
       From: Univariate Polynomial Ring in x over Integer Ring
       To:     Univariate Polynomial Ring in x over Rational Field
       Defn: Induced from base ring by
              Natural morphism:
                  From: Integer Ring
                  To:     Rational Field
Coercion on right operand via
   Polynomial base injection morphism:
       From: Rational Field
       To:     Univariate Polynomial Ring in x over Rational Field
Arithmetic performed after coercions.
Result lives in Univariate Polynomial Ring in x over Rational Field
```

Sometimes with non-sage types there is not enough information to deduce what will actually happen:

```
sage: R100 = RealField(100)          # needs sage.rings.real_mpfr
sage: cm.explain(R100, float, operator.add)    # needs sage.rings.real_mpfr
Right operand is numeric, will attempt coercion in both directions.
Unknown result parent.
sage: parent(R100(1) + float(1))        # needs sage.rings.real_mpfr
<class 'float'>
sage: cm.explain(QQ, float, operator.add)
Right operand is numeric, will attempt coercion in both directions.
Unknown result parent.
sage: parent(QQ(1) + float(1))
<class 'float'>
```

Special care is taken to deal with division:

```
sage: cm.explain(ZZ, ZZ, operator.truediv)
Identical parents, arithmetic performed immediately.
Result lives in Rational Field
Rational Field
sage: ZZx = ZZ['x']
sage: QQx = QQ['x']
sage: cm.explain(ZZx, QQx, operator.truediv)
Coercion on left operand via
   Ring morphism:
```
From: Univariate Polynomial Ring in x over Integer Ring
To: Univariate Polynomial Ring in x over Rational Field
Defn: Induced from base ring by
    Natural morphism:
        From: Integer Ring
        To: Rational Field
Arithmetic performed after coercions.
Result lives in Fraction Field of Univariate Polynomial Ring in x over Rational Field
Fractions Field of Univariate Polynomial Ring in x over Rational Field

sage: cm.explain(int, ZZ, operator.truediv)
Coercion on left operand via
    Native morphism:
        From: Set of Python objects of class 'int'
        To: Integer Ring
Arithmetic performed after coercions.
Result lives in Rational Field
Rational Field

sage: cm.explain(ZZx, ZZ, operator.truediv)
Action discovered.
    Right inverse action by Rational Field
    on Univariate Polynomial Ring in x over Integer Ring
    with precomposition on right by Natural morphism:
        From: Integer Ring
        To: Rational Field
Result lives in Univariate Polynomial Ring in x over Rational Field
Univariate Polynomial Ring in x over Rational Field

Note: This function is accurate only in so far as analyse() is kept in sync with the bin_op() and canonical_coercion() which are kept separate for maximal efficiency.

get_action(R, S, op='mul', r=None, s=None)
Get the action of R on S or S on R associated to the operation op.

EXAMPLES:

sage: cm = sage.structure.element.get_coercion_model()
sage: ZZx = ZZ['x']
sage: cm.get_action(ZZx, ZZ, operator.mul)
Right scalar multiplication by Integer Ring
on Univariate Polynomial Ring in x over Integer Ring
sage: cm.get_action(ZZx, QQ, operator.mul)
Right scalar multiplication by Rational Field
on Univariate Polynomial Ring in x over Integer Ring
sage: QQx = QQ['x']
sage: cm.get_action(QQx, int, operator.mul)
Right scalar multiplication by Integer Ring
on Univariate Polynomial Ring in x over Rational Field
with precomposition on right by Native morphism:
From: Set of Python objects of class 'int'
To:   Integer Ring

generate: A = cm.get_action(QQx, ZZ, operator.truediv); A
Right inverse action by Rational Field
on Univariate Polynomial Ring in x over Rational Field
with precomposition on right by Natural morphism:
  From: Integer Ring
  To:   Rational Field

generate: x = QQx.gen()
generate: A(x+10, 5)
1/5*x + 2

\textbf{get\_cache()}

This returns the current cache of coercion maps and actions, primarily useful for debugging and introspection.

\textbf{EXAMPLES:}

\begin{verbatim}
 sage: cm = sage.structure.element.get_coercion_model()
 sage: cm.canonical_coercion(1, 2/3)
 (1, 2/3)
 sage: maps, actions = cm.get_cache()
\end{verbatim}

Now let us see what happens when we do a binary operations with an integer and a rational:

\begin{verbatim}
 sage: left_morphism_ref, right_morphism_ref = maps[ZZ, QQ]
\end{verbatim}

Note that by \url{github issue #14058} the coercion model only stores a weak reference to the coercion maps in this case:

\begin{verbatim}
 sage: left_morphism_ref
 <weakref at ...; to 'sage.rings.rational.Z_to_Q' at ...>
\end{verbatim}

Moreover, the weakly referenced coercion map uses only a weak reference to the codomain:

\begin{verbatim}
 sage: left_morphism_ref()
 (map internal to coercion system -- copy before use)
 Natural morphism:
   From: Integer Ring
   To:   Rational Field
\end{verbatim}

To get an actual valid map, we simply copy the weakly referenced coercion map:

\begin{verbatim}
 sage: print(copy(left_morphism_ref()))
 Natural morphism:
   From: Integer Ring
   To:   Rational Field
 sage: print(right_morphism_ref)
 None
\end{verbatim}

We can see that it coerces the left operand from an integer to a rational, and doesn’t do anything to the right.

Now for some actions:
sage: R.<x> = ZZ['x']
sage: 1/2 * x
1/2*x
sage: maps, actions = cm.get_cache()
sage: act = actions[QQ, R, operator.mul]; act
Left scalar multiplication by Rational Field
on Univariate Polynomial Ring in x over Integer Ring
sage: act.actor()
Rational Field
sage: act.domain()
Univariate Polynomial Ring in x over Integer Ring
sage: act.codomain()
Univariate Polynomial Ring in x over Rational Field
sage: act(1/5, x+10)
1/5*x + 2

record_exceptions(value=True)

Enables (or disables) recording of the exceptions suppressed during arithmetic.
Each time that record_exceptions is called (either enabling or disabling the record), the exception_stack is cleared.

reset_cache()

Clear the coercion cache.
This should have no impact on the result of arithmetic operations, as the exact same coercions and actions will be re-discovered when needed.
It may be useful for debugging, and may also free some memory.
EXAMPLES:

```python
sage: cm = sage.structure.element.get_coercion_model()
sage: len(cm.get_cache()[0])  # random
42
sage: cm.reset_cache()
sage: cm.get_cache()
({}, {})
```

richcmp(x, y, op)

Given two arbitrary objects x and y, coerce them to a common parent and compare them using rich comparison operator op.
EXAMPLES:

```python
sage: from sage.structure.element import get_coercion_model
sage: from sage.structure.richcmp import op_LT, op_LE, op_EQ, op_NE, op_GT, op_GE
sage: richcmp = get_coercion_model().richcmp
sage: richcmp(None, None, op_EQ)
True
sage: richcmp(None, 1, op_LT)
True
sage: richcmp("hello", None, op_LE)
False
```
sage: richcmp(-1, 1, op_GE)
False
sage: richcmp(int(1), float(2), op_GE)
False

If there is no coercion, we only support == and !=:

sage: x = QQ.one(); y = GF(2).one()
sage: richcmp(x, y, op_EQ)
False
sage: richcmp(x, y, op_NE)
True
sage: richcmp(x, y, op_GT)
Traceback (most recent call last):
  ...TypeError: unsupported operand parent(s) for >:
'Rational Field' and 'Finite Field of size 2'

We support non-Sage types with the usual Python convention:

sage: class AlwaysEqual():
    ...:     def __eq__(self, other):
    ...:         return True
sage: x = AlwaysEqual()
sage: x == 1
True
sage: 1 == x
True

verify_action(action, R, S, op, fix=True)
Verify that action takes an element of R on the left and S on the right, raising an error if not.

This is used for consistency checking in the coercion model.

EXAMPLES:

sage: R.<x> = ZZ['x']
sage: cm = sage.structure.element.get_coercion_model()
sage: cm.verify_action(R.get_action(QQ), R, QQ, operator.mul)
Right scalar multiplication by Rational Field
on Univariate Polynomial Ring in x over Integer Ring
sage: cm.verify_action(R.get_action(QQ), RDF, R, operator.mul)
Traceback (most recent call last):
  ...RuntimeError: There is a BUG in the coercion model:
    Action found for R <built-in function mul> S does not have the correct
    domains
    R = Real Double Field
    S = Univariate Polynomial Ring in x over Integer Ring
    (should be Univariate Polynomial Ring in x over Integer Ring, Rational
    Field)
    action = Right scalar multiplication by Rational Field
    on Univariate Polynomial Ring in x over Integer Ring
    (<class 'sage.structure.coerce_actions.RightModuleAction'>)
verify_coercion_maps(R, S, homs, fix=False)

Make sure this is a valid pair of homomorphisms from $R$ and $S$ to a common parent. This function is used to protect the user against buggy parents.

EXAMPLES:

```
sage: cm = sage.structure.element.get_coercion_model()
sage: homs = QQ.coerce_map_from(ZZ), None
sage: cm.verify_coercion_maps(ZZ, QQ, homs) == homs
True
sage: homs = QQ.coerce_map_from(ZZ), RR.coerce_map_from(QQ)
sage: cm.verify_coercion_maps(ZZ, QQ, homs) == homs
Traceback (most recent call last):
...  
RuntimeError: ('BUG in coercion model, codomains must be identical',
  Natural morphism:
  From: Integer Ring
  To:   Rational Field,
  Generic map:
  From: Rational Field
  To:   Real Field with 53 bits of precision)
```

sage.structure.coerce.is_mpmath_type(t)

Check whether the type $t$ is a type whose name starts with either mpmath. or sage.libs.mpmath.

EXAMPLES:

```
sage: from sage.structure.coerce import is_mpmath_type
sage: is_mpmath_type(int)
False
sage: import mpmath
sage: is_mpmath_type(mpmath.mpc(2))
False
sage: is_mpmath_type(type(mpmath.mpc(2)))
True
sage: is_mpmath_type(type(mpmath.mpf(2)))
True
```

sage.structure.coerce.is_numpy_type(t)

Return True if and only if $t$ is a type whose name starts with numpy.

EXAMPLES:

```
sage: from sage.structure.coerce import is_numpy_type
sage: # needs numpy
sage: import numpy
sage: is_numpy_type(numpy.int16)
True
sage: is_numpy_type(numpy.floating)
True
sage: is_numpy_type(numpy.ndarray)
True
sage: is_numpy_type(numpy.matrix)
```

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sage: is_numpy_type(int)
False
sage: is_numpy_type(Integer)
False
sage: is_numpy_type(Sudoku)
False

\texttt{sage.structure.coerce.parent_is_integers}(P)

Check whether the type or parent represents the ring of integers.

\textbf{EXAMPLES:}

```python
sage: from sage.structure.coerce import parent_is_integers
sage: parent_is_integers(int)
True
sage: parent_is_integers(float)
False
sage: parent_is_integers(bool)
True
sage: parent_is_integers(dict)
False

sage: import numpy
sage: parent_is_integers(numpy.int16)
True
sage: parent_is_integers(numpy.uint64)
True
sage: parent_is_integers(float)
False

sage: import gmpy2
sage: parent_is_integers(gmpy2.mpz)
True
sage: parent_is_integers(gmpy2.mpq)
False
```

Ensure (\texttt{github issue #27893}) is fixed:

```python
sage: K.<f> = QQ[]
sage: gmpy2.mpz(2) * f
2*f
```

\texttt{sage.structure.coerce.parent_is_numerical}(P)

Test if elements of the parent or type \( P \) can be numerically evaluated as complex numbers (in a canonical way).

\textbf{EXAMPLES:}

```python
```
sage: from sage.structure.coerce import parent_is_numerical
sage: import gmpy2
sage: [parent_is_numerical(R) for R in [QQ, int, complex, gmpy2.mpc]]  #→ needs sage.rings.complex_double
[True, True, True, True]
sage: [parent_is_numerical(R) for R in [RR, CC]]  #→ needs sage.rings.real_mpfr
[True, True]
sage: parent_is_numerical(QuadraticField(-1))  #→ needs sage.rings.number_field
True
sage: import numpy; parent_is_numerical(numpy.complexfloating)  #→ needs numpy
True
sage: parent_is_numerical(SR)  #→ needs sage.symbolic
False
sage: [parent_is_numerical(R) for R in [QQ['x'], QQ[['x']], str]]  #→ needs sage.libs.flint
[False, False, False]

sage.structure.coerce.parent_is_real_numerical(P)
Test if elements of the parent or type P can be numerically evaluated as real numbers (in a canonical way).

EXAMPLES:

sage: from sage.structure.coerce import parent_is_real_numerical
sage: import gmpy2
sage: [parent_is_real_numerical(R) for R in [RR, QQ, ZZ, RLF, int, float, gmpy2.mpq]]  #→ needs sage.rings.number_field
[True, True, True, True, True, True, True]
sage: parent_is_real_numerical(QuadraticField(2))  #→ needs sage.rings.number_field
True
sage: import numpy; parent_is_real_numerical(numpy.integer)  #→ needs numpy
True
sage: parent_is_real_numerical(QuadraticField(-1))  #→ needs sage.rings.number_field
False
sage: [parent_is_real_numerical(R) for R in [CC, complex, gmpy2.mpc, numpy.complexfloating]]  #→ needs sage.libs.flint
[False, False, False, False]
(continues on next page)
 sage.structure.coerce.py_scalar_parent(py_type)

Returns the Sage equivalent of the given python type, if one exists. If there is no equivalent, return None.

EXAMPLES:

```python
sage: from sage.structure.coerce import py_scalar_parent
sage: py_scalar_parent(int)
Integer Ring
sage: py_scalar_parent(float)
Real Double Field
# needs sage.rings.complex_double
Complex Double Field
sage: py_scalar_parent(complex)
Complex Double Field
sage: py_scalar_parent(bool)
Integer Ring
sage: py_scalar_parent(dict),
(Non),

sage: import fractions
sage: py_scalar_parent(fractions.Fraction)
Rational Field

sage: # needs numpy
sage: import numpy
sage: py_scalar_parent(numpy.int16)
Integer Ring
sage: py_scalar_parent(numpy.int32)
Integer Ring
sage: py_scalar_parent(numpy.uint64)
Integer Ring
sage: py_scalar_parent(numpy.double)
Real Double Field

sage: import gmpy2
sage: py_scalar_parent(gmpy2.mpz)
Integer Ring
sage: py_scalar_parent(gmpy2.mpq)
Rational Field
sage: py_scalar_parent(gmpy2.mpfr)
Real Double Field
# needs sage.rings.complex_double
Complex Double Field
```

sage.structure.coerce.py_scalar_to_element(x)

Convert x to a Sage Element if possible.

If x was already an Element or if there is no obvious conversion possible, just return x itself.

EXAMPLES:
Coercion, Release 10.2

```python
sage: from sage.structure.coerce import py_scalar_to_element
sage: x = py_scalar_to_element(42)
sage: x, parent(x)
(42, Integer Ring)
sage: x = py_scalar_to_element(int(42))
sage: x, parent(x)
(42, Integer Ring)
sage: x = py_scalar_to_element(float(42))
sage: x, parent(x)
(42.0, Real Double Field)
sage: x = py_scalar_to_element(complex(42))
# needs sage.rings.complex_double
sage: x, parent(x)
(42.0000000000000 + 0.00000000000000*I, Complex Double Field)
sage: py_scalar_to_element('hello')
'hello'

sage: from fractions import Fraction
sage: f = Fraction(int(2^100), int(3^100))
```

Note that bools are converted to 0 or 1:

```python
sage: py_scalar_to_element(False), py_scalar_to_element(True)
(0, 1)
```

Test gmpy2’s types:

```python
sage: import gmpy2
sage: x = py_scalar_to_element(gmpy2.mpz(42))
sage: x, parent(x)
(42, Integer Ring)
sage: x = py_scalar_to_element(gmpy2.mpq('3/4'))
sage: x, parent(x)
(3/4, Rational Field)
sage: x = py_scalar_to_element(gmpy2.mpfr(42.57))
sage: x, parent(x)
(42.5700000000000, Real Double Field)
sage: x = py_scalar_to_element(gmpy2.mpc(int(42), int(42)))
# needs sage.rings.complex_double
```

Test compatibility with `py_scalar_parent()`:

```python
sage: from sage.structure.coerce import py_scalar_parent
sage: elt = [True, int(42), float(42), complex(42)]
sage: for x in elt:
    # needs sage.rings.complex_double
    ....:    assert py_scalar_parent(type(x)) == py_scalar_to_element(x).parent()
```

(continues on next page)
5.2 Coerce actions

class sage.structure.coerce_actions.ActOnAction
    Bases: GenericAction

    Class for actions defined via the _act_on_ method.

class sage.structure.coerce_actions.ActedUponAction
    Bases: GenericAction

    Class for actions defined via the _acted_upon_ method.

class sage.structure.coerce_actions.GenericAction
    Bases: Action

    codomain()

    Returns the “codomain” of this action, i.e. the Parent in which the result elements live. Typically, this
    should be the same as the acted upon set.

    EXAMPLES:

    Note that coerce actions should only be used inside of the coercion model. For this test, we need to strongly
    reference the domains, for otherwise they could be garbage collected, giving rise to random errors (see
    github issue #18157).

Set $\mathbb{P}^1(\mathbb{Q})$ of all cusps

```python
sage: # needs sage.groups
sage: S3 = SymmetricGroup(3)
sage: QQxyz = QQ['x,y,z']
Sage: A = sage.structure.coerce_actions.ActOnAction(S3, QQxyz, False)
sage: A.codomain()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

### class sage.structure.coerce_actions.IntegerAction

**Bases:** Action

Abstract base class representing some action by integers on something. Here, “integer” is defined loosely in the “duck typing” sense.

**INPUT:**

- $Z$ – a type or parent representing integers

For the other arguments, see Action.

**Note:** This class is used internally in Sage’s coercion model. Outside of the coercion model, special precautions are needed to prevent domains of the action from being garbage collected.

### class sage.structure.coerce_actions.IntegerMulAction

**Bases:** IntegerAction

Implement the action $\mathbb{n \cdot a} = a + a + ... + a$ via repeated doubling.

Both addition and negation must be defined on the set $M$.

**INPUT:**

- $Z$ – a type or parent representing integers
- $M$ – a $\mathbb{ZZ}$-module
- $m$ – (optional) an element of $M$

**EXAMPLES:**

```python
sage: from sage.structure.coerce_actions import IntegerMulAction
sage: R.<x> = QQ['x']
sage: act = IntegerMulAction(ZZ, R)
sage: act(5, x)
5*x
sage: act(0, x)
0
sage: act(-3, x-1)
-3*x + 3
```

### class sage.structure.coerce_actions.IntegerPowAction

**Bases:** IntegerAction

The right action $\mathbb{a}\mathbb^n = a * a * ... * a$ where $n$ is an integer.

The action is implemented using the `_pow_int` method on elements.
INPUT:

- \( Z \) – a type or parent representing integers
- \( M \) – a parent whose elements implement \_pow\_int
- \( m \) – (optional) an element of \( M \)

EXAMPLES:

```
sage: from sage.structure.coerce_actions import IntegerPowAction
sage: R.<x> = LaurentSeriesRing(QQ)
sage: act = IntegerPowAction(ZZ, R)
sage: act(x, 5)
x^5
sage: act(x, -2)
x^-2
sage: act(x, int(5))
x^5
```

```python
class sage.structure.coerce_actions.LeftModuleAction
Bases: ModuleAction
class sage.structure.coerce_actions.ModuleAction
Bases: Action
```

Module action.

See also:

This is an abstract class, one must actually instantiate a \texttt{LeftModuleAction} or a \texttt{RightModuleAction}.

INPUT:

- \( G \) – the actor, an instance of \texttt{Parent}.
- \( S \) – the object that is acted upon.
- \( g \) – optional, an element of \( G \).
- \( a \) – optional, an element of \( S \).
- \( \text{check} \) – if True (default), then there will be no consistency tests performed on sample elements.

NOTE:

By default, the sample elements of \( S \) and \( G \) are obtained from \texttt{an\_element()}, which relies on the implementation of an \_an\_element\_() method. This is not always available. But usually, the action is only needed when one already has two elements. Hence, by github issue \#14249, the coercion model will pass these two elements to the \texttt{ModuleAction} constructor.

The actual action is implemented by the \_rmul\_ or \_lmu1\_ function on its elements. We must, however, be very particular about what we feed into these functions, because they operate under the assumption that the inputs lie exactly in the base ring and may segfault otherwise. Thus we handle all possible base extensions manually here.

```
codomain()
```

The codomain of self, which may or may not be equal to the domain.

EXAMPLES:

Note that coerce actions should only be used inside of the coercion model. For this test, we need to strongly reference the domains, for otherwise they could be garbage collected, giving rise to random errors (see github issue \#18157).
Coercion, Release 10.2

```
sage: from sage.structure.coerce_actions import LeftModuleAction
sage: ZZxyz = ZZ['x,y,z']
sage: A = LeftModuleAction(QQ, ZZxyz)
sage: A.codomain()
Multivariate Polynomial Ring in x, y, z over Rational Field
```

domain()

The domain of self, which is the module that is being acted on.

EXAMPLES:

Note that coerce actions should only be used inside of the coercion model. For this test, we need to strongly reference the domains, for otherwise they could be garbage collected, giving rise to random errors (see github issue #18157).

```
sage: from sage.structure.coerce_actions import LeftModuleAction
sage: ZZxyz = ZZ['x,y,z']
sage: A = LeftModuleAction(QQ, ZZxyz)
sage: A.domain()
Multivariate Polynomial Ring in x, y, z over Integer Ring
```

class sage.structure.coerce_actions.PyScalarAction

Bases: Action

class sage.structure.coerce_actions.RightModuleAction

Bases: ModuleAction

```
sage: from sage.structure.coerce_actions import detect_element_action
sage: ZZx = ZZ['x']
sage: M = MatrixSpace(ZZ, 2)
# needs sage.modules
sage: detect_element_action(ZZx, ZZ, False)
Left scalar multiplication by Integer Ring on Univariate Polynomial Ring in x over Integer Ring
sage: detect_element_action(ZZx, QQ, True)
Right scalar multiplication by Rational Field on Univariate Polynomial Ring in x over Integer Ring
sage: detect_element_action(Cusps, M, False)
# needs sage.modular sage.modules
Left action by Full MatrixSpace of 2 by 2 dense matrices over Integer Ring on Set P^1(QQ) of all cusps
sage: detect_element_action(Cusps, M, True),
# needs sage.modular sage.modules
(None,)
sage: detect_element_action(ZZ, QQ, True),
(None,)
```

40 Chapter 5. Modules
5.3 Coerce maps

class sage.structure.coerce_maps.CCallableConvertMap_class

Bases: Map

class sage.structure.coerce_maps.CallableConvertMap

Bases: Map

This lets one easily create maps from any callable object.

This is especially useful to create maps from bound methods.

EXAMPLES:

```
sage: from sage.structure.coerce_maps import CallableConvertMap
sage: def foo(P, x): return x/2
sage: f = CallableConvertMap(ZZ, QQ, foo)
sage: f(3)
3/2
sage:
Conversion via foo map:
  From: Integer Ring
  To:  Rational Field
```

Create a homomorphism from $\mathbb{R}$ to $\mathbb{R}^+$ viewed as additive groups.

```
sage: # needs sage.symbolic
sage: f = CallableConvertMap(RR, RR, exp, parent_as_first_arg=False)
sage: f(0)
1.00000000000000
sage: f(1)
2.71828182845905
sage: f(-3)
0.0497870683678639
```

class sage.structure.coerce_maps.DefaultConvertMap

Bases: Map

This morphism simply calls the codomain’s element_constructor method, passing in the codomain as the first argument.

EXAMPLES:

```
sage: QQ[['x']].coerce_map_from(QQ)
Coercion map:
  From: Rational Field
  To:  Power Series Ring in x over Rational Field
```

class sage.structure.coerce_maps.DefaultConvertMap_unique

Bases: DefaultConvertMap

This morphism simply defers action to the codomain’s element_constructor method, WITHOUT passing in the codomain as the first argument.

This is used for creating elements that don’t take a parent as the first argument to their __init__ method, for example, Integers, Rationals, Algebraic Reals... all have a unique parent. It is also used when the element_constructor is a bound method (whose self argument is assumed to be bound to the codomain).
class sage.structure.coerce_maps.ListMorphism
    Bases: Map
class sage.structure.coerce_maps.NamedConvertMap
    Bases: Map
    This is used for creating elements via the _xxx_ methods.
    For example, many elements implement an _integer_ method to convert to ZZ, or a _rational_ method to convert to QQ.

method_name
class sage.structure.coerce_maps.TryMap
    Bases: Map
sage.structure.coerce_maps.test_CCallableConvertMap(domain, name=None)
    For testing CCallableConvertMap_class.

5.4 Coercion via construction functors

class sage.categories.pushout.AlgebraicClosureFunctor
    Bases: ConstructionFunctor
    Algebraic Closure.
    EXAMPLES:
    
    sage: # needs sage.rings.complex_double sage.rings.number_field
    sage: F = CDF.construction()[0]
sage: F(QQ)
    Algebraic Field
    sage: F(RR) # needs sage.rings.real_mpfr
    Complex Field with 53 bits of precision
    sage: F(F(QQ)) is F(QQ)
    True

merge(other)
    Mathematically, Algebraic Closure subsumes Algebraic Extension. However, it seems that people do want to work with algebraic extensions of RR. Therefore, we do not merge with algebraic extension.

rank = 3

class sage.categories.pushout.AlgebraicExtensionFunctor(polys, names, embeddings=None, structures=None, cyclotomic=None, precs=None, implementations=None, *, residue=None, latex_names=None, **kwds)
    Bases: ConstructionFunctor
    Algebraic extension (univariate polynomial ring modulo principal ideal).
    EXAMPLES:
Coercion, Release 10.2

```
sage: x = polygen(QQ, 'x')
sage: K.<a> = NumberField(x^3 + x^2 + 1)  # needs sage.rings.number_field
sage: F = K.construction()[0]  # needs sage.rings.number_field
sage: F(ZZ['t'])  # needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
over Univariate Polynomial Ring in t over Integer Ring
with modulus a^3 + a^2 + 1
```

Note that, even if a field is algebraically closed, the algebraic extension will be constructed as the quotient of a univariate polynomial ring:

```
sage: F(CC)  # needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
over Complex Field with 53 bits of precision
with modulus a^3 + a^2 + 1.00000000000000
sage: F(RR)  # needs sage.rings.number_field
Univariate Quotient Polynomial Ring in a
over Real Field with 53 bits of precision
with modulus a^3 + a^2 + 1.00000000000000
```

Note that the construction functor of a number field applied to the integers returns an order (not necessarily maximal) of that field, similar to the behaviour of `ZZ.extension(...)`:

```
sage: F(ZZ)  # needs sage.rings.number_field
Order in Number Field in a with defining polynomial x^3 + x^2 + 1
```

This also holds for non-absolute number fields:

```
sage: # needs sage.rings.number_field
sage: x = polygen(QQ, 'x')
sage: K.<a,b> = NumberField([x^3 + x^2 + 1, x^2 + x + 1])
sage: F = K.construction()[0]
sage: O = F(ZZ); O
Relative Order in Number Field in a
with defining polynomial x^3 + x^2 + 1 over its base field
sage: O.ambient() is K
True
```

Special cases are made for cyclotomic fields and residue fields:

```
sage: # needs sage.rings.number_field
sage: C = CyclotomicField(8)
sage: F, R = C.construction()
sage: F
AlgebraicExtensionFunctor
sage: R
Rational Field
```

(continues on next page)
Coercion, Release 10.2

sage: F(R)
Cyclotomic Field of order 8 and degree 4
sage: F(ZZ)
Maximal Order in Cyclotomic Field of order 8 and degree 4

sage: # needs sage.rings.number_field
sage: K.<z> = CyclotomicField(7)
sage: P = K.factor(17)[0][0]
sage: k = K.residue_field(P)
sage: F, R = k.construction()
sage: F
AlgebraicExtensionFunctor
sage: R
Cyclotomic Field of order 7 and degree 6
sage: F(R) is k
True
sage: F(ZZ)
Residue field of Integers modulo 17
sage: F(CyclotomicField(49))
Residue field in zbar of Fractional ideal (17)

expand()

Decompose the functor \( F \) into sub-functors, whose product returns \( F \).

EXAMPLES:

sage: # needs sage.rings.number_field
sage: P.<x> = QQ[]
sage: K.<a> = NumberField(x^3 - 5, embedding=0)
sage: L.<b> = K.extension(x^2 + a)
sage: F, R = L.construction()
sage: prod(F.expand())(R) == L
True
sage: K = NumberField([x^2 - 2, x^2 - 3], 'a')
sage: F, R = K.construction()
sage: F
AlgebraicExtensionFunctor
sage: L = F.expand(); L
[AlgebraicExtensionFunctor, AlgebraicExtensionFunctor]
sage: L[-1](QQ)
Number Field in a1 with defining polynomial x^2 - 3

merge(other)

Merging with another \texttt{AlgebraicExtensionFunctor}.

INPUT:

other -- Construction Functor.

OUTPUT:

- If \texttt{self==other}, \texttt{self} is returned.
- If \texttt{self} and \texttt{other} are simple extensions and both provide an embedding, then it is tested whether one of the number fields provided by the functors coerces into the other; the functor associated with
the target of the coercion is returned. Otherwise, the construction functor associated with the pushout of the codomains of the two embeddings is returned, provided that it is a number field.

- If these two extensions are defined by Conway polynomials over finite fields, merges them into a single extension of degree the \( \text{lcm} \) of the two degrees.
- Otherwise, \text{None} is returned.

REMARK:
Algebraic extension with embeddings currently only works when applied to the rational field. This is why we use the admittedly strange rule above for merging.

EXAMPLES:
The following demonstrate coercions for finite fields using Conway or pseudo-Conway polynomials:

```python
sage: k = GF(3^2, prefix='z'); a = k.gen()  # needs sage.rings.finite_rings
sage: l = GF(3^3, prefix='z'); b = l.gen()  # needs sage.rings.finite_rings
sage: a + b  # indirect doctest
˓→ z6^5 + 2*z6^4 + 2*z6^3 + z6^2 + 2*z6 + 1

Note that embeddings are compatible in lattices of such finite fields:

```python
sage: m = GF(3^5, prefix='z'); c = m.gen()
sage: (a + b) + c == a + (b + c)  # indirect doctest
True
```

Coercion is also available for number fields:

```python
sage: P.<x> = QQ[]
sage: L.<b> = NumberField(x^8 - x^4 + 1, embedding=CDF.0)
sage: M1.<c1> = NumberField(x^2 + x + 1, embedding=b^4 - 1)
sage: M2.<c2> = NumberField(x^2 + 1, embedding=-b^6)
sage: M1.coerce_map_from(M2)
sage: M2.coerce_map_from(M1)
sage: c1 + c2; parent(c1 + c2)  # indirect doctest
˓→ -b^6 + b^4 - 1

Number Field in b with defining polynomial x^8 - x^4 + 1 with b = -0.2581904510252087? + 0.965925826890683?*I
```

```python
sage: pushout(M1['x'], M2['x'])  # needs sage.rings.finite_rings
˓→ Univariate Polynomial Ring in x over Number Field in b with defining polynomial x^8 - x^4 + 1 with b = -0.2581904510252087? + 0.965925826890683?*I
```
In the previous example, the number field $L$ becomes the pushout of $M_1$ and $M_2$ since both are provided with an embedding into $L$, and since $L$ is a number field. If two number fields are embedded into a field that is not a numberfield, no merging occurs:

```python
sage: # needs sage.rings.complex_double sage.rings.number_field
sage: cbrt2 = CDF(2)^(1/3)
sage: zeta3 = CDF.zeta(3)
sage: K.<a> = NumberField(x^3 - 2, embedding=cbrt2 * zeta3)
sage: L.<b> = NumberField(x^6 - 2, embedding=1.1)
sage: K.coerce_map_from(L)
sage: L.coerce_map_from(K)
sage: pushout(K, L)  # needs sage.rings.finite_rings
```

```
Traceback (most recent call last):
... CoercionException: ('Ambiguous Base Extension', Number Field in a with defining polynomial x^3 - 2 with a = -0.6299605249474365? + 1.091123635971722?*I,
Number Field in b with defining polynomial x^6 - 2 with b = 1.122462048309373?)
```

```
rank = 3
```

class sage.categories.pushout.BlackBoxConstructionFunctor(box)

Bases: ConstructionFunctor

Construction functor obtained from any callable object.

EXAMPLES:

```python
sage: from sage.categories.pushout import BlackBoxConstructionFunctor
sage: FG = BlackBoxConstructionFunctor(gap)
sage: FS = BlackBoxConstructionFunctor(singular)
sage: FG
BlackBoxConstructionFunctor
sage: FG(ZZ)
# needs sage.libs.gap
Integers
sage: FG(ZZ).parent()
# needs sage.libs.gap
Gap
sage: FS(QQ['t'])
# needs sage.libs.singular
polynomial ring, over a field, global ordering
// coefficients: QQ
// number of vars : 1
//   block 1 : ordering lp
//     : names   t
//   block 2 : ordering C
sage: FG == FS
# needs sage.libs.gap sage.libs.singular
False
sage: FG == loads(dumps(FG))
# needs sage.libs.gap
True
```
class sage.categories.pushout.CompletionFunctor(p, prec, extras=None)

Bases: ConstructionFunctor

Completion of a ring with respect to a given prime (including infinity).

EXAMPLES:

```sage
# needs sage.rings.padics
sage: R = Zp(5)
sage: R
5-adic Ring with capped relative precision 20

sage: F1 = R.construction()[0]
sage: F1
Completion[5, prec=20]

sage: F1(ZZ) is R
True

sage: F1(QQ)
5-adic Field with capped relative precision 20

sage: F2 = RR.construction()[0]
sage: F2
Completion[+Infinity, prec=53]

sage: F2(QQ) is RR
True

sage: P.<x> = ZZ[]
sage: Px = P.completion(x)  # currently the only implemented completion of P
sage: Px
Power Series Ring in x over Integer Ring

sage: F3 = Px.construction()[0]
sage: F3(GF(3)['x'])
Power Series Ring in x over Finite Field of size 3
```

commutes(other)

Completion commutes with fraction fields.

EXAMPLES:

```sage
sage: F1 = Zp(5).construction()[0]  # needs sage.rings.padics
sage: F2 = QQ.construction()[0]
sage: F1.commutes(F2)  # needs sage.rings.padics
True
```

merge(other)

Two Completion functors are merged, if they are equal. If the precisions of both functors coincide, then a Completion functor is returned that results from updating the extras dictionary of self by otherextras. Otherwise, if the completion is at infinity then merging does not increase the set precision, and if the completion is at a finite prime, merging does not decrease the capped precision.

EXAMPLES:
```python
sage: # needs sage.rings.padics
sage: R1.<a> = Zp(5, prec=20)

sage: R2 = Qp(5, prec=40)

sage: R2(1) + a  # indirect doctest
(1 + O(5^20))*a + 1 + O(5^40)

sage: R3 = RealField(30)

sage: R4 = RealField(50)

sage: R3(1) + R4(1)  # indirect doctest
2.0000000

sage: (R3(1) + R4(1)).parent()
Real Field with 30 bits of precision
```

```
rank = 4

class sage.categories.pushout.CompositeConstructionFunctor(*args)
    Bases: ConstructionFunctor

    A Construction Functor composed by other Construction Functors.

    INPUT:

    F1, F2,...: A list of Construction Functors. The result is the composition
                F1 followed by F2 followed by ...

    EXAMPLES:

    sage: from sage.categories.pushout import CompositeConstructionFunctor
    sage: F = CompositeConstructionFunctor(QQ.construction()[0], ZZ['x'].constriction()[0], ...
                                                QQ.construction()[0], ZZ['y'].constriction()[0])
    sage: F
    Poly[y](FractionField(Poly[x](FractionField(...))))
    True
    sage: F == CompositeConstructionFunctor(*F.all)
    True
    sage: F(GF(2)['t'])
    Univariate Polynomial Ring in y
    over Fraction Field of Univariate Polynomial Ring in x
    over Fraction Field of Univariate Polynomial Ring in t
    over Finite Field of size 2 (using GF2X)

    expand()

    Return expansion of a CompositeConstructionFunctor.

    Note: The product over the list of components, as returned by the expand() method, is equal to self.

    EXAMPLES:

    sage: from sage.categories.pushout import CompositeConstructionFunctor
    sage: F = CompositeConstructionFunctor(QQ.construction()[0], ...
                                        ZZ['x'].constriction()[0], ...
                                        QQ.construction()[0], ...
    sage: F
```

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(continued from previous page)

....:

```python
sage: F
Poly[y](FractionField(Poly[x](FractionField(...))))
```

```python
sage: prod(F.expand()) == F
True
```

```python
class sage.categories.pushout.ConstructionFunctor

Bases: Functor

Base class for construction functors.

A construction functor is a functorial algebraic construction, such as the construction of a matrix ring over a given ring or the fraction field of a given ring.

In addition to the class `Functor`, construction functors provide rules for combining and merging constructions. This is an important part of Sage’s coercion model, namely the pushout of two constructions: When a polynomial \( p \) in a variable \( x \) with integer coefficients is added to a rational number \( q \), then Sage finds that the parents \( \mathbb{Z}[x] \) and \( \mathbb{Q} \) are obtained from \( \mathbb{Z} \) by applying a polynomial ring construction respectively the fraction field construction. Each construction functor has an attribute `rank`, and the rank of the polynomial ring construction is higher than the rank of the fraction field construction. This means that the pushout of \( \mathbb{Q} \) and \( \mathbb{Z}[x] \), and thus a common parent in which \( p \) and \( q \) can be added, is \( \mathbb{Q}(x) \), since the construction functor with a lower rank is applied first.

```python
sage: F1, R = QQ.construction()
sage: F1
FractionField
sage: R
Integer Ring
sage: F2, R = (ZZ['x']).construction()
sage: F2
Poly[x]
sage: R
Integer Ring
sage: F3 = F2.pushout(F1)
sage: F3
Poly[x](FractionField(...))
sage: F3(R)
Univariate Polynomial Ring in x over Rational Field
```

When composing two construction functors, they are sometimes merged into one, as is the case in the Quotient construction:

```python
sage: Q15, R = (ZZ.quo(15*ZZ)).construction()
sage: Q15
QuotientFunctor
sage: Q35, R = (ZZ.quo(35*ZZ)).construction()
sage: Q35
QuotientFunctor
```

(continues on next page)
Functors can not only be applied to objects, but also to morphisms in the respective categories. For example:

```python
sage: P.<x,y> = ZZ[]
sage: F = P.construction()[0]; F
MPoly[x,y]
sage: A.<a,b> = GF(5)[]
sage: f = A.hom([a + b, a - b], A)
sage: F(A)
Multivariate Polynomial Ring in x, y
   over Multivariate Polynomial Ring in a, b over Finite Field of size 5
sage: F(f)
Ring endomorphism of Multivariate Polynomial Ring in x, y
   over Multivariate Polynomial Ring in a, b over Finite Field of size 5
   Defn: Induced from base ring by
       Ring endomorphism of Multivariate Polynomial Ring in a, b
       over Finite Field of size 5
       Defn: a |--> a + b
       b |--> a - b
sage: F(f)(F(A)(x)*a)
(a + b)*x
```

```python
coercion_reversed = False
```

```python
common_base(other_functor, self_bases, other_bases)
```
This function is called by `pushout()` when no common parent is found in the construction tower.

**Note:** The main use is for multivariate construction functors, which use this function to implement recursion for `pushout()`.

**INPUT:**
- `other_functor` – a construction functor.
- `self_bases` – the arguments passed to this functor.
- `other_bases` – the arguments passed to the functor `other_functor`.

**OUTPUT:**
Nothing, since a `CoercionException` is raised.

**Note:** Overload this function in derived class, see e.g. `MultivariateConstructionFunctor`.

```python
commutes(other)
```
Determine whether `self` commutes with another construction functor.

**Note:** By default, `False` is returned in all cases (even if the two functors are the same, since in this case `merge()` will apply anyway). So far there is no construction functor that overloads this method. Anyway,
this method only becomes relevant if two construction functors have the same rank.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: P = ZZ['t'].construction()[0]
sage: F.commutes(P)
False
sage: P.commutes(F)
False
sage: F.commutes(F)
False
```

`expand()`
Decompose `self` into a list of construction functors.

**Note:** The default is to return the list only containing `self`.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: F.expand()
[FractionField]
sage: Q = ZZ.quo(2).construction()[0]
sage: Q.expand()
[QuotientFunctor]
sage: P = ZZ['t'].construction()[0]
sage: FP = F*P
sage: FP.expand()
[FractionField, Poly[t]]
```

`merge(other)`
Merge `self` with another construction functor, or return `None`.

**Note:** The default is to merge only if the two functors coincide. But this may be overloaded for subclasses, such as the quotient functor.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: P = ZZ['t'].construction()[0]
sage: F.merge(F)
FractionField
sage: F.merge(P)
```

`pushout(other)`
Composition of two construction functors, ordered by their ranks.
Note:

- This method seems not to be used in the coercion model.
- By default, the functor with smaller rank is applied first.

```python
class sage.categories.pushout.EquivariantSubobjectConstructionFunctor(S, action=<built-in function mul>,
    side='left',
    other_action=None,
    other_side='left')
```

**Bases:** ConstructionFunctor

Constructor for subobjects invariant or equivariant under given semigroup actions.

Let $S$ be a semigroup that acts on a parent $X$ as $s \cdot x$ (action, side='left') or acts on $X$ as $x \cdot s$ (action, side='right'), and (possibly trivially) acts on $X$ as $s \ast x$ (other_action, other_side='left') or acts on $X$ as $x \ast s$ (other_action, other_side='right).

The $S$-equivariant subobject is the subobject

$$X^S := \{x \in X : s \cdot x = s \ast x, \forall s \in S\}$$

when side = other_side = 'left' and mutatis mutandis for the other values of side and other_side.

When other_action is trivial, $X^S$ is called the $S$-invariant subobject.

**EXAMPLES:**

Monoterm symmetries of a tensor, here only for matrices: row (index 0), column (index 1); the order of the extra element 2 in a permutation determines whether it is a symmetry or an antisymmetry:

```python
sage: # needs sage.groups sage.modules
sage: GSym01 = PermutationGroup([[0,1,2,3]]); GSym01
Permutation Group with generators [(0,1)]

sage: GASym01 = PermutationGroup([[0,1,2,3]]); GASym01
Permutation Group with generators [(0,1,2,3)]

sage: from sage.categories.action import Action
sage: from sage.structure.element import Matrix

sage: class TensorIndexAction(Action):
...    def _act_(self, g, x):
...        if isinstance(x, Matrix):
...            if g(0) == 1:
...                if g(2) == 2:
...                    return x.transpose()
...                else:
...                    return -x.transpose()
...            else:
...                return x
...        raise NotImplementedError

sage: M = matrix([[1, 2], [3, 4]]); M
[1 2]
[3 4]

sage: GSym01_action = TensorIndexAction(GSym01, M.parent())

sage: GSym01_action.act(GSym01.0, M)
```

(continues on next page)
sage: GASym01_action.act(GASym01.0, M)
[-1 -3]
[-2 -4]
sage: Sym01 = M.parent().invariant_module(GSym01, action=GSym01_action); Sym01
(Permutation Group with generators [(0,1)])-invariant submodule
of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: list(Sym01.basis())
[B[0], B[1], B[2]]
sage: list(Sym01.basis().map(Sym01.lift))
[[1 0] [0 1] [0 0]]
[[0 0], [1 0], [0 1]]
sage: ASym01 = M.parent().invariant_module(GASym01, action=GASym01_action)
sage: ASym01
(Permutation Group with generators [(0,1)(2,3)])-invariant submodule
of Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: list(ASym01.basis())
[B[0]]
sage: list(ASym01.basis().map(ASym01.lift))
[[0 1], [-1 0]]

sage: from sage.categories.pushout import pushout
sage: pushout(Sym01, QQ)
(Permutation Group with generators [(0,1)])-invariant submodule
of Full MatrixSpace of 2 by 2 dense matrices over Rational Field

class sage.categories.pushout.FractionField
    Bases: ConstructionFunctor

    Construction functor for fraction fields.

    EXAMPLES:

sage: F = QQ.construction()[0]
sage: F
FractionField
sage: F.domain() Category of integral domains
sage: F.codomain() Category of fields
sage: F(GF(5)) is GF(5) True
sage: F(ZZ['t']) Fraction Field of Univariate Polynomial Ring in t over Integer Ring

5.4. Coercion via construction functors
Fraction Field of Multivariate Polynomial Ring in x, y over Rational Field
Defn: x |--> x + 2*y
       y |--> 3*x - y
sage: F(f)(1/x)
1/(x + 2*y)
sage: F == loads(dumps(F))
True

rank = 5

class sage.categories.pushout.IdentityConstructionFunctor
    Bases: ConstructionFunctor

    A construction functor that is the identity functor.

    rank = -100

class sage.categories.pushout.InfinitePolynomialFunctor(gens, order, implementation)
    Bases: ConstructionFunctor

    A Construction Functor for Infinite Polynomial Rings (see
    infinite_polynomial_ring).

    AUTHOR:
    - Simon King

    This construction functor is used to provide uniqueness of
    infinite polynomial rings as parent structures. As usual, the
    construction functor allows for constructing pushouts.

    Another purpose is to avoid name conflicts of variables of the to-be-constructed
    infinite polynomial ring with variables of the base ring, and moreover to keep the
    internal structure of an Infinite Polynomial Ring as simple as possible: If
    variables \(v_1, \ldots, v_n\) of the given base ring generate an ordered
    sub-monoid of the monomials of the ambient Infinite Polynomial Ring, then they
    are removed from the base ring and merged with the generators of the ambient ring.
    However, if the orders don’t match, an error is raised, since there was a name conflict
    without merging.

    EXAMPLES:

    sage: A.<a,b> = InfinitePolynomialRing(ZZ['t'])
sage: A.construction()[(InfPoly{[a,b], "lex", "dense"},
                      Univariate Polynomial Ring in t over Integer Ring)]
sage: type(_[0])
<class 'sage.categories.pushout.InfinitePolynomialFunctor'>
sage: B.<x,y,a_3,a_1> = PolynomialRing(QQ, order='lex')
sage: B.construction()[(MPoly[x,y,a_3,a_1], Rational Field)]
sage: A.construction()[0] * B.construction()[0]
InfPoly{[a,b], "lex", "dense"}(MPoly[x,y](...))

Apparently the variables \(a_1, a_3\) of the polynomial ring are merged with the variables \(a_0, a_1, a_2, \ldots\) of the
infinite polynomial ring; indeed, they form an ordered sub-structure. However, if the polynomial ring was given a
different ordering, merging would not be allowed, resulting in a name conflict:

sage: R = PolynomialRing(QQ, names=['x','y','a_3','a_1'])
sage: A.construction()[0] * R.construction()[0]
Traceback (most recent call last):
...
CoercionException: Incompatible term orders lex, degrevlex

In an infinite polynomial ring with generator $a_*$, the variable $a_3$ will always be greater than the variable $a_1$. Hence, the orders are incompatible in the next example as well:

```
sage: R = PolynomialRing(QQ, names=['x','y','a_1','a_3'], order='lex')
sage: A.construction()[0] * R.construction()[0]
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('a', 'b'),['a_1', 'a_3']) are incompatible
```

Another requirement is that after merging the order of the remaining variables must be unique. This is not the case in the following example, since it is not clear whether the variables $x, y$ should be greater or smaller than the variables $b_*$:

```
sage: R = PolynomialRing(QQ, names=['a_3','a_1','x','y'], order='lex')
sage: A.construction()[0] * R.construction()[0]
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('a', 'b'),['a_3', 'a_1']) are incompatible
```

Since the construction functors are actually used to construct infinite polynomial rings, the following result is no surprise:

```
sage: C.<a,b> = InfinitePolynomialRing(B); C
Infinite polynomial ring in a, b
over Multivariate Polynomial Ring in x, y over Rational Field
```

There is also an overlap in the next example:

```
sage: X.<w,x,y> = InfinitePolynomialRing(ZZ)
sage: Y.<x,y,z> = InfinitePolynomialRing(QQ)
```

$X$ and $Y$ have an overlapping generators $x_*, y_*$. Since the default lexicographic order is used in both rings, it gives rise to isomorphic sub-monoids in both $X$ and $Y$. They are merged in the pushout, which also yields a common parent for doing arithmetic:

```
sage: P = sage.categories.pushout.pushout(Y,X); P
Infinite polynomial ring in w, x, y, z over Rational Field
sage: w^2+z^3
w_2 + z_3
sage: _.parent() is P
True
```

**expand()**

Decompose the functor $F$ into sub-functors, whose product returns $F$.

**EXAMPLES:**
merge(other)

Merge two construction functors of infinite polynomial rings, regardless of monomial order and implementation.

The purpose is to have a pushout (and thus, arithmetic) even in cases when the parents are isomorphic as rings, but not as ordered rings.

EXAMPLES:

```python
sage: X.<x,y> = InfinitePolynomialRing(QQ, implementation='sparse')
sage: Y.<x,y> = InfinitePolynomialRing(QQ, order='degrevlex')
sage: X.construction()
[InfPoly([x,y], "lex", "sparse"), Rational Field]
sage: Y.construction()
[InfPoly([x,y], "degrevlex", "dense"), Rational Field]
sage: Y.construction()[0].merge(Y.construction()[0])
InfPoly([x,y], "degrevlex", "dense")
sage: y[3] + X(x[2])
x_2 + y_3
sage: _.parent().construction()
[InfPoly([x,y], "degrevlex", "dense"), Rational Field]
```

rank = 9.5

class sage.categories.pushout.LaurentPolynomialFunctor(var, multi_variate=False)

Bases: ConstructionFunctor

Construction functor for Laurent polynomial rings.

EXAMPLES:

```python
sage: L.<t> = LaurentPolynomialRing(ZZ)
sage: F = L.construction()[0]
sage: F
LaurentPolynomialFunctor
sage: F(QQ)
Univariate Laurent Polynomial Ring in t over Rational Field
sage: K.<x> = LaurentPolynomialRing(ZZ)
sage: F(K)
Univariate Laurent Polynomial Ring in t
```
over Univariate Laurent Polynomial Ring in \( x \) over Integer Ring

```python
sage: P.<x,y> = ZZ[]
sage: f = P.hom([x + 2*y, 3*x - y], P)
sage: F(f)
```

Ring endomorphism of Univariate Laurent Polynomial Ring in \( t \) over Multivariate Polynomial Ring in \( x, y \) over Integer Ring

```
Defn: Induced from base ring by
Ring endomorphism of Multivariate Polynomial Ring in \( x, y \) over Integer Ring
Defn: \( x \mapsto x + 2*y \)
\( y \mapsto 3*x - y \)
```

```python
sage: F(f)(x*F(P).gen()^-2 + y*F(P).gen()^3)
```

\((x + 2*y)*t^{-2} + (3*x - y)*t^3\)

merge(other)

Two Laurent polynomial construction functors merge if the variable names coincide.

The result is multivariate if one of the arguments is multivariate.

EXAMPLES:

```python
sage: from sage.categories.pushout import LaurentPolynomialFunctor
sage: F1 = LaurentPolynomialFunctor('t')
sage: F2 = LaurentPolynomialFunctor('t', multi_variate=True)
sage: F1.merge(F2)
LaurentPolynomialFunctor
```

```python
sage: F1.merge(F2)(LaurentPolynomialRing(GF(2), 'a'))
```

```
# needs sage.modules
Multivariate Laurent Polynomial Ring in \( a, t \) over Finite Field of size 2
```

```python
sage: F1.merge(F1)(LaurentPolynomialRing(GF(2), 'a'))
```

```
# needs sage.modules
Univariate Laurent Polynomial Ring in \( t \) over
Univariate Laurent Polynomial Ring in \( a \) over Finite Field of size 2
```

rank = 9

class sage.categories.pushout.MatrixFunctor(nrows, ncols, is_sparse=False)

Bases: ConstructionFunctor

A construction functor for matrices over rings.

EXAMPLES:

```python
sage: MS = MatrixSpace(ZZ, 2, 3)
```

```python
sage: F = MS.construction()[0]; F
MatrixFunctor
```

```python
sage: MS = MatrixSpace(ZZ, 2)
```

```python
sage: F = MS.construction()[0]; F
MatrixFunctor
```

```python
sage: P.<x,y> = QQ[]
```

```python
sage: R = F(P); R
Full MatrixSpace of 2 by 2 dense matrices
over Multivariate Polynomial Ring in \( x, y \) over Rational Field
```

```python
sage: f = P.hom([x+y, x-y], P); F(f)
```

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Coercion, Release 10.2

Ring endomorphism
of Full MatrixSpace of 2 by 2 dense matrices
over Multivariate Polynomial Ring in x, y over Rational Field
Defn: Induced from base ring by
  Ring endomorphism
  of Multivariate Polynomial Ring in x, y over Rational Field
  Defn: x |--> x + y
        y |--> x - y
sage: M = R([x, y, x*y, x + y])
sage: F(f)(M)
\[
\begin{pmatrix}
  x + y & x - y \\
  x^2 - y^2 & 2x
\end{pmatrix}
\]

merge(other)

Merging is only happening if both functors are matrix functors of the same dimension.
The result is sparse if and only if both given functors are sparse.

EXAMPLES:

sage: # needs sage.modules
sage: F1 = MatrixSpace(ZZ, 2, 2).construction()[0]
sage: F2 = MatrixSpace(ZZ, 2, 3).construction()[0]
sage: F3 = MatrixSpace(ZZ, 2, 2, sparse=True).construction()[0]
sage: F1.merge(F2)
MatrixFunctor
sage: F1.merge(F3)
MatrixFunctor
sage: F13 = F1.merge(F3)
sage: F13.is_sparse
False
sage: F1.is_sparse
False
sage: F3.is_sparse
True
sage: F3.merge(F3).is_sparse
True

rank = 10

class sage.categories.pushout.MultiPolynomialFunctor(vars, term_order)

Bases: ConstructionFunctor

A constructor for multivariate polynomial rings.

EXAMPLES:

sage: P.<x,y> = ZZ[]
sage: F = P.construction()[0]; F
MPoly[x,y]
sage: A.<a,b> = GF(5)[]
sage: F(A)
Multivariate Polynomial Ring in x, y
over Multivariate Polynomial Ring in a, b over Finite Field of size 5
sage: f = A.hom([a+b, a-b], A)
Coercion, Release 10.2

sage: F(f)
Ring endomorphism of Multivariate Polynomial Ring in x, y
over Multivariate Polynomial Ring in a, b over Finite Field of size 5
  Defn: Induced from base ring by
    Ring endomorphism of Multivariate Polynomial Ring in a, b over Finite Field
      of size 5
      Defn: a |--> a + b
            b |--> a - b
sage: F(f)(F(A)(x)*a)
(a + b)*x

expand()

Decompose self into a list of construction functors.

EXAMPLES:

sage: F = QQ['x,y,z,t'].construction()[0]; F
MPoly[x,y,z,t]
sage: F.expand()
[MPoly[t], MPoly[z], MPoly[y], MPoly[x]]

Now an actual use case:

sage: R.<x,y,z> = ZZ[]
sage: S.<z,t> = QQ[]
sage: x+t
x + t
sage: parent(x+t)
Multivariate Polynomial Ring in x, y, z over Rational Field
sage: T.<y,s> = QQ[]
sage: x + s
Traceback (most recent call last):
  ...
TypeError: unsupported operand parent(s) for +:
  'Multivariate Polynomial Ring in x, y, z over Integer Ring' and
  'Multivariate Polynomial Ring in y, s over Rational Field'
sage: R = PolynomialRing(ZZ, 'x', 50)
sage: S = PolynomialRing(GF(5), 'x', 20)
sage: R.gen(0) + S.gen(0)
2*x0

merge(other)

Merge self with another construction functor, or return None.

EXAMPLES:

sage: F = sage.categories.pushout.MultiPolynomialFunctor(['x', 'y'], None)
sage: G = sage.categories.pushout.MultiPolynomialFunctor(['t'], None)
sage: F.merge(G) is None
True
sage: F.merge(F)
MPoly[x,y]

rank = 9

5.4. Coercion via construction functors
class sage.categories.pushout.MultivariateConstructionFunctor
Bases: ConstructionFunctor

An abstract base class for functors that take multiple inputs (e.g. Cartesian products).

common_base(other_functor, self_bases, other_bases)
This function is called by pushout() when no common parent is found in the construction tower.

INPUT:
• other_functor – a construction functor.
• self_bases – the arguments passed to this functor.
• other_bases – the arguments passed to the functor other_functor.

OUTPUT:
A parent.

If no common base is found a sage.structure.coerce_exceptions.CoercionException is raised.

Note: Overload this function in derived class, see e.g. MultivariateConstructionFunctor.

class sage.categories.pushout.PermutationGroupFunctor(gens, domain)
Bases: ConstructionFunctor

EXAMPLES:

sage: from sage.categories.pushout import PermutationGroupFunctor
sage: PF = PermutationGroupFunctor([PermutationGroupElement([(1,2)])], [1,2]); PF
PermutationGroupFunctor[(1,2)]

gens()

EXAMPLES:

sage: P1 = PermutationGroup([(1,2)]) # needs sage.groups
sage: PF, P = P1.construction()
# needs sage.groups
sage: PF.gens()
# needs sage.groups
((1,2),)

merge(other)
Merge self with another construction functor, or return None.

EXAMPLES:

sage: # needs sage.groups
sage: P1 = PermutationGroup([(1,2)])
sage: PF1, P = P1.construction()
sage: P2 = PermutationGroup([(1,3)])
sage: PF2, P = P2.construction()
sage: PF1.merge(PF2)
PermutationGroupFunctor[(1,2), (1,3)]
class sage.categories.pushout.PolynomialFunctor(var, multi_variate=False, sparse=False, implementation=None)

Bases: ConstructionFunctor

Construction functor for univariate polynomial rings.

EXAMPLES:

```python
sage: P = ZZ['t'].construction()[0]
sage: P(GF(3))
Univariate Polynomial Ring in t over Finite Field of size 3
sage: P == loads(dumps(P))
True
sage: R.<x,y> = GF(5)[]
sage: f = R.hom([x + 2*y, 3*x - y], R)
sage: P(f)((x+y) * P(R).0)
(-x + y)*t
```

By github issue #9944, the construction functor distinguishes sparse and dense polynomial rings. Before, the following example failed:

```python
sage: R.<x> = PolynomialRing(GF(5), sparse=True)
sage: F, B = R.construction()
sage: F(B) is R
True
sage: S.<x> = PolynomialRing(ZZ)
sage: R.has_coerce_map_from(S)
False
sage: S.has_coerce_map_from(R)
False
sage: S.0 + R.0
2*x
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Finite Field of size 5
sage: (S.0 + R.0).parent().is_sparse()
False
```

merge(other)

Merge self with another construction functor, or return None.

Note: Internally, the merging is delegated to the merging of multipolynomial construction functors. But in effect, this does the same as the default implementation, that returns None unless the to-be-merged functors coincide.

EXAMPLES:

```python
sage: P = ZZ['x'].construction()[0]
sage: Q = ZZ['y','x'].construction()[0]
sage: P.merge(Q)
sage: P.merge(P) is P
True
```
class sage.categories.pushout.QuotientFunctor(I, names=None, as_field=False, domain=None, codomain=None, **kwds)

Bases: ConstructionFunctor

Construction functor for quotient rings.

**Note:** The functor keeps track of variable names. Optionally, it may keep track of additional properties of the quotient, such as its category or its implementation.

**EXAMPLES:**

```python
sage: P.<x,y> = ZZ[

sage: Q = P.quo([x^2 + y^2] * P)
sage: F = Q.construction()[0]
sage: F(QQ['x','y'])
Quotient of Multivariate Polynomial Ring in x, y over Rational Field
by the ideal (x^2 + y^2)
sage: F(QQ['x','y']) == QQ['x','y'].quo([x^2 + y^2] * QQ['x','y'])
True
sage: F(QQ['x','y','z'])
Traceback (most recent call last):
... CoercionException: Cannot apply this quotient functor to
Multivariate Polynomial Ring in x, y, z over Rational Field
```

```python
sage: merge(other)

Two quotient functors with coinciding names are merged by taking the gcd of their moduli, the meet of their domains, and the join of their codomains.

In particular, if one of the functors being merged knows that the quotient is going to be a field, then the merged functor will return fields as well.

**EXAMPLES:**

```python
sage: # needs sage.libs.pari
sage: P.<x> = QQ[]
sage: Q1 = P.quo([[x^2+1]*2*(x^2-3)])
sage: Q2 = P.quo([[x^2+1]*2^(x^5+3)])
sage: from sage.categories.pushout import pushout
sage: pushout(Q1,Q2)     # indirect doctest
Univariate Quotient Polynomial Ring in xbar over Rational Field
with modulus x^4 + 2*x^2 + 1
```

The following was fixed in github issue #8800:
Coercion, Release 10.2

```python
sage: pushout(GF(5), Integers(5))
```

```
# needs sage.libs.pari
Finite Field of size 5
```

**rank** = 4.5

```python
class sage.categories.pushout.SubspaceFunctor(basis)
```

**Bases:** ConstructionFunctor

Constructing a subspace of an ambient free module, given by a basis.

**Note:** This construction functor keeps track of the basis. It can only be applied to free modules into which this basis coerces.

**EXAMPLES:**

```python
sage: # needs sage.modules
sage: M = ZZ^3
sage: S = M.submodule([(1,2,3), (4,5,6)]); S
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
sage: F = S.construction()[0]
sage: F(GF(2)^3)
Vector space of degree 3 and dimension 2 over Finite Field of size 2
User basis matrix:
[1 0 1]
[0 1 0]
```

**coercion_reversed** = True

**merge**(other)

Two Subspace Functors are merged into a construction functor of the sum of two subspaces.

**EXAMPLES:**

```python
sage: # needs sage.modules
sage: M = GF(5)^3
sage: S1 = M.submodule([(1,2,3),(4,5,6)])
sage: S2 = M.submodule([(2,2,3)])
sage: F1 = S1.construction()[0]
sage: F2 = S2.construction()[0]
sage: F1.merge(F2)
SubspaceFunctor
sage: F1.merge(F2)(GF(5)^3) == S1 + S2
True
```

5.4. Coercion via construction functors
rank = 11

class sage.categories.pushout.VectorFunctor(n=None, is_sparse=False, inner_product_matrix=None, *,
with_basis='standard', basis_keys=None,
name_mapping=None, latex_name_mapping=None)

Bases: ConstructionFunctor

A construction functor for free modules over commutative rings.

EXAMPLES:

```python
sage: # needs sage.modules
sage: F = (ZZ^3).construction()[0]
sage: F
VectorFunctor
sage: F(GF(2) '['t']') #˓→ needs sage.libs.ntl
Ambient free module of rank 3
over the principal ideal domain Univariate Polynomial Ring in t
over Finite Field of size 2 (using GF2X)
```

merge(other)

Two constructors of free modules merge, if the module ranks and the inner products coincide. If both have
explicitly given inner product matrices, they must coincide as well.

EXAMPLES:

Two modules without explicitly given inner product allow coercion:

```python
sage: M1 = QQ^3  #˓→ needs sage.modules
sage: P.<t> = ZZ[]
sage: M2 = FreeModule(P, 3)  #˓→ needs sage.modules
sage: M1([1,1/2,1/3]) + M2([t,t^2+t,3])  # indirect doctest  #˓→ needs sage.modules
(t + 1, t^2 + t + 1/2, 10/3)
```

If only one summand has an explicit inner product, the result will be provided with it:

```python
sage: M3 = FreeModule(P, 3, inner_product_matrix=Matrix(3, 3, range(9)))  #˓→ needs sage.modules
sage: M1([1,1/2,1/3]) + M3([t,t^2+t,3])  #˓→ needs sage.modules
(t + 1, t^2 + t + 1/2, 10/3)
sage: (M1([1,1/2,1/3]) + M3([t,t^2+t,3])).parent().inner_product_matrix()  #˓→ needs sage.modules
[0 1 2]
[3 4 5]
[6 7 8]
```

If both summands have an explicit inner product (even if it is the standard inner product), then the products
must coincide. The only difference between M1 and M4 in the following example is the fact that the default
inner product was explicitly requested for M4. It is therefore not possible to coerce with a different inner
product:
sage: # needs sage.modules
sage: M4 = FreeModule(QQ, 3, inner_product_matrix=Matrix(3, 3, 1))
sage: M4 == M1
True
sage: M4.inner_product_matrix() == M1.inner_product_matrix()
True
sage: M4([1,1/2,1/3]) + M3([t,t^2+t,3])  # indirect doctest
Traceback (most recent call last):
  ...TypeError: unsupported operand parent(s) for +:
'Ambient quadratic space of dimension 3 over Rational Field
Inner product matrix:
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and
'Ambient free quadratic module of rank 3 over the integral domain
Univariate Polynomial Ring in t over Integer Ring
Inner product matrix:
\[
\begin{bmatrix}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8
\end{bmatrix}
\]
Names are removed when they conflict:

sage: # needs sage.modules
sage: from sage.categories.pushout import VectorFunctor, pushout
sage: M_ZZx = FreeModule(ZZ['x'], 4, with_basis=None, name='M_ZZx')
sage: N_ZZx = FreeModule(ZZ['x'], 4, with_basis=None, name='N_ZZx')
sage: pushout(M_ZZx, QQ)
Rank-4 free module M_ZZx_base_ext
over the Univariate Polynomial Ring in x over Rational Field
sage: pushout(M_ZZx, N_ZZx)
Rank-4 free module
over the Univariate Polynomial Ring in x over Integer Ring
sage: pushout(pushout(M_ZZx, N_ZZx), QQ)
Rank-4 free module
over the Univariate Polynomial Ring in x over Rational Field

rank = 10

sage.categories.pushout.construction_tower(R)

An auxiliary function that is used in pushout() and pushout_lattice().

INPUT:

An object

OUTPUT:

A constructive description of the object from scratch, by a list of pairs of a construction functor and an object to which the construction functor is to be applied. The first pair is formed by None and the given object.

EXAMPLES:

sage: from sage.categories.pushout import construction_tower
sage: construction_tower(MatrixSpace(FractionField(QQ['t']), 2))  #...
sage.categories.pushout.expand_tower(tower)

An auxiliary function that is used in pushout().

INPUT:
A construction tower as returned by construction_tower().

OUTPUT:
A new construction tower with all the construction functors expanded.

EXAMPLES:

```python
sage: from sage.categories.pushout import construction_tower, expand_tower
sage: construction_tower(QQ['x,y,z'])
[(None, Multivariate Polynomial Ring in x, y, z over Rational Field),
 (MPoly[x,y,z], Rational Field),
 (FractionField, Integer Ring)]

sage: expand_tower(construction_tower(QQ['x,y,z']))
[(None, Multivariate Polynomial Ring in x, y, z over Rational Field),
 (MPoly[z], Univariate Polynomial Ring in y
 over Univariate Polynomial Ring in x over Rational Field),
 (MPoly[y], Univariate Polynomial Ring in x over Rational Field),
 (MPoly[x], Rational Field),
 (FractionField, Integer Ring)]
```

sage.categories.pushout.pushout(R, S)

Given a pair of objects $R$ and $S$, try to construct a reasonable object $Y$ and return maps such that canonically $R \leftarrow Y \rightarrow S$.

ALGORITHM:
This incorporates the idea of functors discussed at Sage Days 4. Every object $R$ can be viewed as an initial object and a series of functors (e.g. polynomial, quotient, extension, completion, vector/matrix, etc.). Call the series of increasingly simple objects (with the associated functors) the “tower” of $R$. The construction method is used to create the tower.

Given two objects $R$ and $S$, try to find a common initial object $Z$. If the towers of $R$ and $S$ meet, let $Z$ be their join. Otherwise, see if the top of one coerces naturally into the other.

Now we have an initial object and two ordered lists of functors to apply. We wish to merge these in an unambiguous order, popping elements off the top of one or the other tower as we apply them to $Z$.

- If the functors are of distinct types, there is an absolute ordering given by the rank attribute. Use this.
- Otherwise:
  - If the tops are equal, we (try to) merge them.
  - If exactly one occurs lower in the other tower, we may unambiguously apply the other (hoping for a later merge).
Coercion, Release 10.2

- If the tops commute, we can apply either first.
- Otherwise fail due to ambiguity.

The algorithm assumes by default that when a construction \( F \) is applied to an object \( X \), the object \( F(X) \) admits a coercion map from \( X \). However, the algorithm can also handle the case where \( F(X) \) has a coercion map to \( X \) instead. In this case, the attribute coercion_reversed of the class implementing \( F \) should be set to True.

**EXAMPLES:**

Here our “towers” are \( R = \text{Complete}_7(\text{Frac}(\mathbb{Z})) \) and \( \text{Frac}(\text{Poly}_x(\mathbb{Z})) \), which give us \( \text{Frac}(\text{Poly}_x(\text{Complete}_7(\text{Frac}(\mathbb{Z})))) \):

```
sage: from sage.categories.pushout import pushout
sage: pushout(Qp(7), Frac(ZZ['x']))
needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
over 7-adic Field with capped relative precision 20
```

Note we get the same thing with

```
sage: pushout(Zp(7), Frac(QQ['x']))
needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
over 7-adic Field with capped relative precision 20
```

Note that polynomial variable ordering must be unambiguously determined.

```
sage: pushout(ZZ['x,y,z'], QQ['w,z,t'])
Traceback (most recent call last):
  ... CoercionException: ('Ambiguous Base Extension',
Multivariate Polynomial Ring in x, y, z over Integer Ring,
Multivariate Polynomial Ring in w, z, t over Rational Field)
sage: pushout(ZZ['x,y,z'], QQ['w,x,z,t'])
Multivariate Polynomial Ring in w, x, y, z, t over Rational Field
```

Some other examples:

```
sage: pushout(Zp(7)['y'], Frac(QQ['t'])['x,y,z'])
needs sage.rings.padics
Multivariate Polynomial Ring in x, y, z
over Fraction Field of Univariate Polynomial Ring in t
over 7-adic Field with capped relative precision 20
```

```
sage: pushout(ZZ['x,y,z'], Frac(ZZ['x'])['y'])
Multivariate Polynomial Ring in y, z
over Fraction Field of Univariate Polynomial Ring in x
over Integer Ring
```

```
sage: pushout(MatrixSpace(RDF, 2, 2), Frac(ZZ['x']))
needs sage.modules
Full MatrixSpace of 2 by 2 dense matrices
over Fraction Field of Univariate Polynomial Ring in x over Real Double Field
```

(continues on next page)
Coercion, Release 10.2

A construction with coercion_reversed=True (currently only the SubspaceFunctor construction) is only applied if it leads to a valid coercion:

```
sage: # needs sage.modules
sage: A = ZZ^2
sage: V = span([[1, 2]], QQ)
sage: P = sage.categories.pushout.pushout(A, V)
sage: P.has_coerce_map_from(A)
True
```

Some more tests with coercion_reversed=True:

```
sage: from sage.categories.pushout import ConstructionFunctor
sage: class EvenPolynomialRing(type(QQ['x'])):
    ....:     def __init__(self, base, var):
    ....:         super().__init__(base, var)
    ....:         self.register_embedding(base[var])
    ....:     def __repr__(self):
    ....:         return "Even Power " + super().__repr__()
    ....:     def construction(self):
    ....:         return EvenPolynomialFunctor(), self.base()[self.variable_name()]
    ....:     def _coerce_map_from_(self, R):
    ....:         return self.base().has_coerce_map_from(R)

sage: class EvenPolynomialFunctor(ConstructionFunctor):
    ....:     rank = 10
    ....:     coercion_reversed = True
    ....:     def __init__(self):
    ....:         ConstructionFunctor.__init__(self, Rings(), Rings())
    ....:     def _apply_functor(self, R):
```

(continues on next page)
Some more tests related to univariate/multivariate constructions. We consider a generalization of polynomial rings, where in addition to the coefficient ring $C$ we also specify an additive monoid $E$ for the exponents of the indeterminate. In particular, the elements of such a parent are given by

$$\sum_{i=0}^{l} c_i x^{e_i},$$

with $c_i \in C$ and $e_i \in E$. We define

```python
sage: class GPolynomialRing(Parent):
    ....:     def __init__(self, coefficients, var, exponents):
    ....:         self.coefficients = coefficients
    ....:         self.var = var
    ....:         self.exponents = exponents
    ....:         super().__init__(category=Rings())
    ....:     def _repr_(self):
    ....:         return 'Generalized Polynomial Ring in %s(%s) over %s' % (self.var, self.exponents, self.coefficients)
```

(continues on next page)
....:     def construction(self):
....:         return GPolynomialFunctor(self.var, self.exponents), self.coefficients
....:     def _coerce_map_from_(self, R):
....:         return self.coefficients.has_coerce_map_from(R)

and

sage: class GPolynomialFunctor(ConstructionFunctor):
     rank = 10
     def __init__(self, var, exponents):
         self.var = var
         self.exponents = exponents
         ConstructionFunctor.__init__(self, Rings(), Rings())
     def _repr_(self):
         return 'GPoly[%s^{%s}]' % (self.var, self.exponents)
     def _apply_functor(self, coefficients):
         return GPolynomialRing(coefficients, self.var, self.exponents)
     def merge(self, other):
         if isinstance(other, GPolynomialFunctor) and self.var == other.var:
             exponents = pushout(self.exponents, other.exponents)
         return GPolynomialFunctor(self.var, exponents)

We can construct a parent now in two different ways:

sage: GPolynomialRing(QQ, 'X', ZZ)
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
sage: GP_ZZ = GPolynomialFunctor('X', ZZ); GP_ZZ
GPoly[X^(Integer Ring)]
sage: GP_ZZ(QQ)
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field

Since the construction

sage: GP_ZZ(QQ).construction()
(GPoly[X^(Integer Ring)], Rational Field)

uses the coefficient ring, we have the usual coercion with respect to this parameter:

sage: pushout(GP_ZZ(ZZ), GP_ZZ(QQ))
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
sage: pushout(GP_ZZ(ZZ['t']), GP_ZZ(QQ))
Generalized Polynomial Ring in X^(Integer Ring)
    over Univariate Polynomial Ring in t over Rational Field
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(ZZ['b,c']))
Generalized Polynomial Ring in X^(Integer Ring)
    over Multivariate Polynomial Ring in a, b over Integer Ring
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(QQ['b,c']))
Generalized Polynomial Ring in X^(Integer Ring)
    over Multivariate Polynomial Ring in a, b over Integer Ring
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(ZZ['c,d']))
Traceback (most recent call last):
...    CoercionException: ('Ambiguous Base Extension', ...)
sage: GP_QQ = GPolynomialFunctor('X', QQ)
sage: pushout(GP_ZZ(ZZ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Integer Ring
sage: pushout(GP_QQ(ZZ), GP_ZZ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Integer Ring

sage: GP_ZZt = GPolynomialFunctor('X', ZZ[t])
sage: pushout(GP_ZZt(ZZ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t over Rational Field) over Integer Ring
sage: pushout(GP_QQ(ZZ), GP_ZZ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Rational Field
sage: pushout(GP_ZZt(QQ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t over Rational Field) over Rational Field
sage: pushout(GP_ZZt(ZZ[\text{a,b}]), GP_QQ(ZZ[\text{c,d}]))
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', ...)
sage: pushout(GP_ZZt(ZZ[\text{a,b}]), GP_QQ(ZZ[\text{b,c}]))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t over Rational Field)
over Multivariate Polynomial Ring in a, b, c over Integer Ring

Some tests with Cartesian products:

sage: from sage.sets.cartesian_product import CartesianProduct
sage: A = CartesianProduct((ZZ[\text{x}], QQ[\text{y}], QQ[z]),
....: Sets().CartesianProducts())
....: B = CartesianProduct((ZZ[\text{x}], ZZ[\text{y}], ZZ[t][\text{z}]),
....: Sets().CartesianProducts())

sage: A.construction()
(\text{The cartesian_product} functorial construction,
(\text{Univariate Polynomial Ring in x over Integer Ring,}
 \text{Univariate Polynomial Ring in y over Rational Field,}
 \text{Univariate Polynomial Ring in z over Rational Field}))
sage: pushout(A, B)
The Cartesian product of
(\text{Univariate Polynomial Ring in x over Integer Ring,}
 \text{Univariate Polynomial Ring in y over Rational Field,}
 \text{Univariate Polynomial Ring in z over}
 \text{Univariate Polynomial Ring in t over Rational Field})

sage: pushout(ZZ, cartesian_product([ZZ, QQ]))
Traceback (most recent call last):
...
CoercionException: 'NoneType' object is not iterable

5.4. Coercion via construction functors
sage: from sage.categories.pushout import PolynomialFunctor
sage: from sage.sets.cartesian_product import CartesianProduct
sage: class CartesianProductPoly(CartesianProduct):
    ....: def __init__(self, polynomial_rings):
    ....:     sort = sorted(polynomial_rings,
    ....:                    key=lambda P: P.variable_name())
    ....:     super().__init__(sort, Sets().CartesianProducts())
    ....: def vars(self):
    ....:     return tuple(P.variable_name()
    ....:                   for P in self.cartesian_factors())
    ....: def _pushout_(self, other):
    ....:     if isinstance(other, CartesianProductPoly):
    ....:         s_vars = self.vars()
    ....:         o_vars = other.vars()
    ....:         if s_vars == o_vars:
    ....:             return
    ....:         return pushout(CartesianProductPoly(
    ....:             self.cartesian_factors() +
    ....:             tuple(f for f in other.cartesian_factors()
    ....:                  if f.variable_name() not in s_vars),
    ....:             CartesianProductPoly(
    ....:             other.cartesian_factors() +
    ....:             tuple(f for f in self.cartesian_factors()
    ....:                  if f.variable_name() not in o_vars))
    ....:     C = other.construction()
    ....:     if C is None:
    ....:         return
    ....:     elif isinstance(C[0], PolynomialFunctor):
    ....:         return pushout(self, CartesianProductPoly((other,)))

sage: pushout(CartesianProductPoly((ZZ['x'],)),)
    CartesianProductPoly((ZZ['y'],)))
The Cartesian product of
(Univariate Polynomial Ring in x over Integer Ring,
 Univariate Polynomial Ring in y over Integer Ring)
sage: pushout(CartesianProductPoly((ZZ['x'], ZZ['y'])),)
    CartesianProductPoly((ZZ['x'], ZZ['z'])))
The Cartesian product of
(Univariate Polynomial Ring in x over Integer Ring,
 Univariate Polynomial Ring in y over Integer Ring,
 Univariate Polynomial Ring in z over Integer Ring)
sage: pushout(CartesianProductPoly((QQ['a,b'][['x'], QQ['y']]),),
    # needs sage.symbolic
    CartesianProductPoly((ZZ['b,c'][['x'], SR['z']])))
The Cartesian product of
(Multivariate Polynomial Ring in a, b, c over Rational Field,
 Univariate Polynomial Ring in y over Rational Field,
 Univariate Polynomial Ring in z over Symbolic Ring)

sage: pushout(CartesianProductPoly((ZZ['x'],)), ZZ['y']))
The Cartesian product of
(continues on next page)
(Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Integer Ring)
sage: pushout(QQ['b,c']['y'], CartesianProductPoly((ZZ['a,b']['x'],)))
The Cartesian product of
  (Univariate Polynomial Ring in x over
    Multivariate Polynomial Ring in a, b over Integer Ring,
  Univariate Polynomial Ring in y over
    Multivariate Polynomial Ring in b, c over Rational Field)
sage: pushout(CartesianProductPoly((ZZ['x'],)), ZZ)
Traceback (most recent call last):
  ... CoercionException: No common base ("join") found for
The cartesian_product functorial construction(...) and None(Integer Ring):
(Multivariate) functors are incompatible.

AUTHORS:
  • Robert Bradshaw
  • Peter Bruin
  • Simon King
  • Daniel Krenn
  • David Roe

sage.categories.pushout.pushout_lattice(R, S)
Given a pair of objects \( R \) and \( S \), try to construct a reasonable object \( Y \) and return maps such that canonically \( R \leftarrow Y \rightarrow S \).

ALGORITHM:
This is based on the model that arose from much discussion at Sage Days 4. Going up the tower of constructions of \( R \) and \( S \) (e.g. the reals come from the rationals come from the integers), try to find a common parent, and then try to fill in a lattice with these two towers as sides with the top as the common ancestor and the bottom will be the desired ring.

See the code for a specific worked-out example.

EXAMPLES:

```python
sage: from sage.categories.pushout import pushout_lattice
sage: A, B = pushout_lattice(Qp(7), Frac(ZZ['x']))
# needs sage.rings.padics
sage: A.codomain()
# needs sage.rings.padics
Fraction Field of Univariate Polynomial Ring in x
  over 7-adic Field with capped relative precision 20
sage: A.codomain() == B.codomain()
# needs sage.rings.padics
True
sage: A, B = pushout_lattice(ZZ, MatrixSpace(ZZ[['x']], 3, 3))
# needs sage.modules
sage: B
# needs sage.modules
```

(continues on next page)
Identity endomorphism of Full MatrixSpace of 3 by 3 dense matrices over Power Series Ring in x over Integer Ring

AUTHOR:
  • Robert Bradshaw

sage.categories.pushout.type_to_parent(P)

An auxiliary function that is used in pushout().

INPUT:
A type

OUTPUT:
A Sage parent structure corresponding to the given type

5.5 Group, ring, etc. actions on objects

The terminology and notation used is suggestive of groups acting on sets, but this framework can be used for modules, algebras, etc.

A group action \( G \times S \to S \) is a functor from \( G \) to Sets.

Warning: An Action object only keeps a weak reference to the underlying set which is acted upon. This decision was made in github issue #715 in order to allow garbage collection within the coercion framework (this is where actions are mainly used) and avoid memory leaks.

```sage
def from sage.categories.action import Action
class P: pass
A = Action(P(),P())
import gc
_ = gc.collect()
A
<repr(<sage.categories.action.Action at 0x...>) failed:
RuntimeError: This action acted on a set that became garbage collected>
```

To avoid garbage collection of the underlying set, it is sufficient to create a strong reference to it before the action is created.

```sage
_ = gc.collect()
from sage.categories.action import Action
class P: pass
q = P()
A = Action(P(),q)
gc.collect()
0
A
Left action by __main__.P ... at ... on __main__.P ... at ...
```

AUTHOR:
  • Robert Bradshaw: initial version
class sage.categories.action.Action
    Bases: Functor
    
The action of $G$ on $S$.
    
    INPUT:
    
    • $G$ – a parent or Python type
    • $S$ – a parent or Python type
    • is_left – (boolean, default: True) whether elements of $G$ are on the left
    • op – (default: None) operation. This is not used by $Action$ itself, but other classes may use it
    
    $G$
    
    act($g$, $x$)
    
    This is a consistent interface for acting on $x$ by $g$, regardless of whether it’s a left or right action.
    
    If needed, $g$ and $x$ are converted to the correct parent.
    
    EXAMPLES:
    
    sage: R.<x> = ZZ []
sage: from sage.structure.coerce_actions import IntegerMulAction
sage: A = IntegerMulAction(ZZ, R, True)  # Left action
sage: A.act(5, x)
5*x
sage: A.act(int(5), x)
5*x
sage: A = IntegerMulAction(ZZ, R, False)  # Right action
sage: A.act(5, x)
5*x
sage: A.act(int(5), x)
5*x

actor()
codomain()domain()is_left()left_domain()opoperation()right_domain()

class sage.categories.action.ActionEndomorphism
    Bases: Morphism
    
The endomorphism defined by the action of one element.
    
    EXAMPLES:
Coercion, Release 10.2

```python
sage: A = ZZ['x'].get_action(QQ, self_on_left=False, op=operator.mul)
sage: A
Left scalar multiplication by Rational Field on Univariate Polynomial Ring in x over Integer Ring
sage: A(1/2)
Action of 1/2 on Univariate Polynomial Ring in x over Integer Ring under Left scalar multiplication by Rational Field on Univariate Polynomial Ring in x over Integer Ring.
```

class sage.categories.action.InverseAction

Bases: Action

An action that acts as the inverse of the given action.

EXAMPLES:

```python
sage: V = QQ^3
    # needs sage.modules
sage: v = V((1, 2, 3))
    # needs sage.modules
sage: cm = get_coercion_model()
sage: # needs sage.modules
sage: a = cm.get_action(V, QQ, operator.mul)
sage: a
Right scalar multiplication by Rational Field on Vector space of dimension 3 over Rational Field
sage: ~a
Right inverse action by Rational Field on Vector space of dimension 3 over Rational Field
sage: (~a)(v, 1/3)
(3, 6, 9)

sage: # needs sage.modules
sage: b = cm.get_action(QQ, V, operator.mul)
sage: b
Left scalar multiplication by Rational Field on Vector space of dimension 3 over Rational Field
sage: ~b
Left inverse action by Rational Field on Vector space of dimension 3 over Rational Field
sage: (~b)(1/3, v)
(3, 6, 9)

sage: c = cm.get_action(ZZ, list, operator.mul)
sage: c
Left action by Integer Ring on 'list'
sage: ~c
Traceback (most recent call last):
...
TypeError: no inverse defined for Left action by Integer Ring on 'list'
```
class sage.categories.action.PrecomposedAction

    Bases: Action

    A precomposed action first applies given maps, and then applying an action to the return values of the maps.

    EXAMPLES:

    We demonstrate that an example discussed on github issue #14711 did not become a problem:

    sage: # needs sage.modular
    sage: E = ModularSymbols(11).2
    sage: s = E.modular_symbol_rep()
    sage: del E, s
    sage: import gc
    sage: _ = gc.collect()
    sage: E = ModularSymbols(11).2
    sage: v = E.manin_symbol_rep()
    sage: c,x = v[0]
    sage: y = x.modular_symbol_rep()
    sage: coercion_model.get_action(QQ, parent(y), op=operator.mul)
    Left scalar multiplication by Rational Field
    on Abelian Group of all Formal Finite Sums over Rational Field
    with precomposition on right by Coercion map:
    From: Abelian Group of all Formal Finite Sums over Integer Ring
    To: Abelian Group of all Formal Finite Sums over Rational Field

    codomain()

    domain()

    left_precomposition
    The left map to precompose with, or None if there is no left precomposition map.

    right_precomposition
    The right map to precompose with, or None if there is no right precomposition map.

5.6 Containers for storing coercion data

This module provides TripleDict and MonoDict. These are structures similar to WeakKeyDictionary in Python’s weakref module, and are optimized for lookup speed. The keys for TripleDict consist of triples (k1,k2,k3) and are looked up by identity rather than equality. The keys are stored by weakrefs if possible. If any one of the components k1, k2, k3 gets garbage collected, then the entry is removed from the TripleDict. Key components that do not allow for weakrefs are stored via a normal refcounted reference. That means that any entry stored using a triple (k1,k2,k3) so that none of the k1,k2,k3 allows a weak reference behaves as an entry in a normal dictionary: Its existence in TripleDict prevents it from being garbage collected.

That container currently is used to store coercion and conversion maps between two parents (github issue #715) and to store homsets of pairs of objects of a category (github issue #11521). In both cases, it is essential that the parent structures remain garbage collectable, it is essential that the data access is faster than with a usual WeakKeyDictionary, and we enforce the “unique parent condition” in Sage (parent structures should be identical if they are equal).

MonoDict behaves similarly, but it takes a single item as a key. It is used for caching the parents which allow a coercion map into a fixed other parent (github issue #12313).

By github issue #14159, MonoDict and TripleDict can be optionally used with weak references on the values.
Note that this kind of dictionary is also used for caching actions and coerce maps. In previous versions of Sage, the cache was by strong references and resulted in a memory leak in the following example. However, this leak was fixed by github issue #715, using weak references:

```python
sage: # needs sage.combinat sage.modules sage.rings.finite_rings
sage: K.<t> = GF(2^55)
sage: for i in range(20):
    ....:     a = K.random_element()
    ....:     E = EllipticCurve(j=a)
    ....:     P = E.random_point()
    ....:     Q = 2*P
sage: L = [Partitions(n) for n in range(200)]  # purge strong cache in...
˓→CachedRepresentation
sage: import gc
sage: n = gc.collect()
```

```python
class sage.structure.coerce_dict.MonoDict
    Bases: object

    This is a hashtable specifically designed for (read) speed in the coercion model.

    It differs from a python WeakKeyDictionary in the following important ways:
    • Comparison is done using the ‘is’ rather than ‘==’ operator.
    • Only weak references to the keys are stored if at all possible. Keys that do not allow for weak
      references are stored with a normal refcounted reference.
    • The callback of the weak references is safe against recursion, see below.

    There are special cdef set/get methods for faster access. It is bare-bones in the sense that not all
    dictionary methods are implemented.

    IMPLEMENTATION:

    It is implemented as a hash table with open addressing, similar to python’s dict.

    INPUT:
    • data – optional iterable defining initial data, as dict or iterable of (key, value) pairs.
    • weak_values – optional bool (default False). If it is true, weak references to the values in this
      dictionary will be used, when possible.

    EXAMPLES:

    ```python
    sage: from sage.structure.coerce_dict import MonoDict
    sage: L = MonoDict()
    sage: a = 'a'; b = 'ab'; c = '-15'
    sage: L[a] = 1
    sage: L[b] = 2
    sage: L[c] = 3
    ```

    The key is expected to be a unique object. Hence, the item stored for c cannot be obtained by providing
    another equal string:
Coercion, Release 10.2

sage: L[a]
1
sage: L[b]
2
sage: L[c]
3
sage: L['-15']
Traceback (most recent call last):
  ...                      
KeyError: '-15'

Not all features of Python dictionaries are available, but iteration over the dictionary items is possible:

sage: sorted(L.items())
[('-15', 3), ('a', 1), ('ab', 2)]
sage: del L[c]
sage: sorted(L.items())
[('a', 1), ('ab', 2)]
sage: len(L)
2
sage: for i in range(1000):
    ....: L[i] = i
sage: len(L)
1002
sage: L['a']
1
sage: L['c']
Traceback (most recent call last):
  ...                      
KeyError: 'c'

Note that `MW` also accepts values that do not allow for weak references:

sage: MW[k] = int(5)
sage: MW[k]
5

The following demonstrates that :class:`MonoDict` is safer than :class:`~weakref.WeakKeyDictionary` against recursions created by nested callbacks; compare :trac:`15069` (the mechanism used now is different, though):

sage: M = MonoDict()
sage: class A: pass
sage: a = A()
sage: prev = a
sage: for i in range(1000):
    ....:    newA = A()
    ....:    M[prev] = newA
    ....:    prev = newA
sage: len(M)
1000
sage: del a
sage: len(M)

(continues on next page)

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The corresponding example with a Python :class:`weakref.WeakKeyDictionary`
would result in a too deep recursion during deletion of the dictionary
items::

    sage: import weakref
    sage: M = weakref.WeakKeyDictionary()
    sage: a = A()
    sage: prev = a
    sage: for i in range(1000):
    ....:     newA = A()
    ....:     M[prev] = newA
    ....:     prev = newA
    sage: len(M)
    1000

Check that also in the presence of circular references, :class:`MonoDict`
gets properly collected::

    sage: import gc
    sage: def count_type(T):
    ....:     return len([c for c in gc.get_objects() if isinstance(c,T)])
    sage: gc.freeze() # so that gc.collect() only deals with our trash
    sage: N = count_type(MonoDict)
    sage: for i in range(100):
    ....:     V = [MonoDict({"id":j+100*i}) for j in range(100)]
    ....:     n = len(V)
    ....:     for i in range(n): V[i][V[(i+1)%n]] = (i+1)%n
    ....:     del V
    ....:     _ = gc.collect()
    ....:     assert count_type(MonoDict) == N
    True
    sage: gc.unfreeze()

AUTHORS:

- Simon King (2012-01)
- Nils Bruin (2012-08)
- Simon King (2013-02)
- Nils Bruin (2013-11)

**copy()**

Return a copy of this :class:`MonoDict` as Python dict.

**EXAMPLES:**

    sage: from sage.structure.coerce_dict import MonoDict
    sage: L = MonoDict()
    sage: L[1] = 42
    sage: L.copy()
    {1: 42}
items()
Iterate over the (key, value) pairs of this `MonoDict`.

EXAMPLES:

```python
sage: from sage.structure.coerce_dict import MonoDict
sage: L = MonoDict()
sage: L[1] = None
sage: L.items()
<...generator object at ...>
sage: sorted(L.items())
[(1, None), (2, True)]
```

class sage.structure.coerce_dict.MonoDictEraser
Bases: object

Erase items from a `MonoDict` when a weak reference becomes invalid.

This is of internal use only. Instances of this class will be passed as a callback function when creating a weak reference.

EXAMPLES:

```python
sage: from sage.structure.coerce_dict import MonoDict
sage: class A:
...     pass
sage: a = A()
sage: M = MonoDict()
sage: M[a] = 1
sage: len(M)
1
sage: del a
sage: import gc
sage: n = gc.collect()
sage: len(M)  # indirect doctest
0
```

AUTHOR:

- Simon King (2012-01)
- Nils Bruin (2013-11)

class sage.structure.coerce_dict.ObjectWrapper
Bases: object

A simple fast wrapper around a Python object. This is like a 1-element tuple except that it does not keep a reference to the wrapped object.

class sage.structure.coerce_dict.TripleDict
Bases: object

This is a hashtable specifically designed for (read) speed in the coercion model.

It differs from a python dict in the following important ways:

- All keys must be sequence of exactly three elements. All sequence types (tuple, list, etc.) map to the same item.
• Any of the three key components that support weak-refs are stored via a weakref. If any of these components gets garbage collected then the entire entry is removed. In that sense, this structure behaves like a nested WeakKeyDictionary.

• Comparison is done using the ‘is’ rather than ‘==' operator.

There are special cdef set/get methods for faster access. It is bare-bones in the sense that not all dictionary methods are implemented.

INPUT:

• data – optional iterable defining initial data, as dict or iterable of (key, value) pairs.
• weak_values – optional bool (default False). If it is true, weak references to the values in this dictionary will be used, when possible.

IMPLEMENTATION:

It is implemented as a hash table with open addressing, similar to python’s dict.

EXAMPLES:

```python
sage: from sage.structure.coerce_dict import TripleDict
sage: L = TripleDict()
sage: a = 'a'; b = 'b'; c = 'c'
sage: L[a,b,c] = 1
sage: L[a,b,c]
1
sage: L[c,b,a] = -1
sage: sorted(L.items())
[(('a', 'b', 'c'), 1), (('c', 'b', 'a'), -1)]
sage: del L[a,b,c]
sage: list(L.items())
[('c', 'b', 'a'), -1]
sage: len(L)
1
sage: for i in range(1000):
    ....:     L[i,i,i] = i
sage: len(L)
1001
sage: L = TripleDict(L)
sage: L[c,b,a]
-1
sage: L[a,b,c]
Traceback (most recent call last):
  ...
  KeyError: ('a', 'b', 'c')
sage: L[a]
Traceback (most recent call last):
  ...
  KeyError: 'a'
sage: L[a] = 1
Traceback (most recent call last):
  ...
  KeyError: 'a'
```

AUTHORS:

• Robert Bradshaw, 2007-08
copy()

Return a copy of this \textit{TripleDict} as Python dict.

EXAMPLES:

\begin{lstlisting}[language=Python]
sage: from sage.structure.coerce_dict import TripleDict
dsage: L = TripleDict()
dsage: L[1,2,3] = 42
dsage: L.copy()
{(1, 2, 3): 42}
\end{lstlisting}

items()

Iterate over the (key, value) pairs of this \textit{TripleDict}.

EXAMPLES:

\begin{lstlisting}[language=Python]
sage: from sage.structure.coerce_dict import TripleDict
dsage: L = TripleDict()
dsage: L[1,2,3] = None
dsage: L.items()
<...generator object at ...>
dsage: list(L.items())
[[(1, 2, 3), None]]
\end{lstlisting}

\textbf{class} \texttt{sage.structure.coerce_dict.TripleDictEraser}

\texttt{Bases: object}

Erases items from a \textit{TripleDict} when a weak reference becomes invalid.

This is of internal use only. Instances of this class will be passed as a callback function when creating a weak reference.

EXAMPLES:

\begin{lstlisting}[language=Python]
sage: from sage.structure.coerce_dict import TripleDict
dsage: class A: pass
dsage: a = A()
dsage: T = TripleDict()
dsage: T[a,ZZ,None] = 1
dsage: T[ZZ,a,1] = 2
dsage: T[a,a,ZZ] = 3
dsage: len(T)
3
dsage: del a
dsage: import gc
dsage: n = gc.collect()
dsage: len(T) # indirect doctest
0
\end{lstlisting}

AUTHOR:
5.7 Exceptions raised by the coercion model

```python
exception sage.structure.coerce_exceptions.CoercionException
   Bases: TypeError
```

This is the baseclass of exceptions that the coercion model raises when trying to discover coercions. We do not use standard Python exceptions to avoid inadvertently catching and suppressing real errors.

Usually one raises this to indicate the attempted action is not implemented/appropriate, but if there are other things to try not to immediately abort to the user.
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