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INTRODUCTORY MATERIAL

- Combinatorics quickref
- Introduction to combinatorics in Sage
CHAPTER
TWO

THEMATIC INDEXES

• Algebraic combinatorics
  – Combinatorial Hopf algebras
  – Cluster algebras and quivers
  – Crystals
  – Root Systems
  – Symmetric Functions
  – FullyCommutativeElements

• Counting
• Enumerated sets and combinatorial objects
• Enumerated sets of partitions, tableaux, …
• Finite state machines, automata, transducers
• Combinatorial species
• Combinatorial designs and incidence structures
• Posets
• Combinatorics on words
• A bijectionist’s toolkit
• Output functions
• Rankers
• Combinatorial maps
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CHAPTER
FOUR

RELATED TOPICS

• Coding Theory
• Discrete dynamics
• Graph Theory
5.1 Comprehensive Module List

Note: This list is currently sorted in alphabetical order w.r.t. the module names. It can be updated semi-automatically by running in src/sage/combinat:

```
find -name "*.py*" | sed 's|\..pyx\?$||; s|\./| sage/combinat/|' | LANG=en_US.UTF-8 LC_COLLATE=C sort > /tmp/module_list.rst
```

and copy pasting the result back there.

Todo: See github issue #17421 for desirable improvements.

5.1.1 Abstract Recursive Trees

The purpose of this class is to help implement trees with a specific structure on the children of each node. For instance, one could want to define a tree in which each node sees its children as linearly (see the Ordered Trees module) or cyclically ordered.

Tree structures

Conceptually, one can define a tree structure from any object that can contain others. Indeed, a list can contain lists which contain lists which contain lists, and thus define a tree... The same can be done with sets, or any kind of iterable objects.

While any iterable is sufficient to encode trees, it can prove useful to have other methods available like isomorphism tests (see next section), conversions to DiGraphs objects (see `as_digraph()`) or computation of the number of automorphisms constrained by the structure on children. Providing such methods is the whole purpose of the `AbstractTree` class.

As a result, the `AbstractTree` class is not meant to be instantiated, but extended. It is expected that classes extending this one may also inherit from classes representing iterables, for instance `ClonableArray` or `ClonableList`

Constrained Trees

The tree built from a specific container will reflect the properties of the container. Indeed, if A is an iterable class whose elements are linearly ordered, a class B extending both of `AbstractTree` and A will be such that the children of a node will be linearly ordered. If A behaves like a set (i.e. if there is no order on the elements it contains), then two trees will be considered as equal if one can be obtained from the other through permutations between the children of a same node (see next section).
Combinatorics, Release 10.0

Paths and ID

It is expected that each element of a set of children should be identified by its index in the container. This way, any node of the tree can be identified by a word describing a path from the root node.

Canonical labellings

Equality between instances of classes extending both AbstractTree and A is entirely defined by the equality defined on the elements of A. A canonical labelling of such a tree, however, should be such that two trees a and b satisfying a == b have the same canonical labellings. On the other hand, the canonical labellings of trees a and b satisfying a != b are expected to be different.

For this reason, the values returned by the canonical_labelling method heavily depend on the data structure used for a node’s children and should be overridden by most of the classes extending AbstractTree if it is incoherent with the data structure.

Authors

- Florent Hivert (2010-2011): initial revision
- Frédéric Chapoton (2011): contributed some methods

class sage.combinat.abstract_tree.AbstractClonableTree

Bases: AbstractTree

Abstract Clonable Tree.

An abstract class for trees with clone protocol (see list_clone). It is expected that classes extending this one may also inherit from classes like ClonableArray or ClonableList depending whether one wants to build trees where adding a child is allowed.

Note: Due to the limitation of Cython inheritance, one cannot inherit here from ClonableElement, because it would prevent us from later inheriting from ClonableArray or ClonableList.

How should this class be extended?

A class extending AbstractClonableTree should satisfy the following assumptions:

- An instantiable class extending AbstractClonableTree should also extend the ClonableElement class or one of its subclasses generally, at least ClonableArray.
- To respect the Clone protocol, the AbstractClonableTree.check() method should be overridden by the new class.

See also the assumptions in AbstractTree.

check()

Check that self is a correct tree.

This method does nothing. It is implemented here because many extensions of AbstractClonableTree also extend sage.structure.list_clone.ClonableElement, which requires it.

It should be overridden in subclasses in order to check that the characterizing property of the respective kind of tree holds (eg: two children for binary trees).

EXAMPLES:

sage: OrderedTree([[],[]]).check()
sage: BinaryTree([[],[],[]]).check()
class sage.combinat.abstract_tree.AbstractLabelledClonableTree(parent, children, label=None, check=True)

Bases: AbstractLabelledTree, AbstractClonableTree

Abstract Labelled Clonable Tree

This class takes care of modification for the label by the clone protocol.

**Note:** Due to the limitation of Cython inheritance, one cannot inherit here from ClonableArray, because it would prevent us to inherit later from ClonableList.

**map_labels(f)**

Apply the function \( f \) to the labels of \( self \)

This method returns a copy of \( self \) on which the function \( f \) has been applied on all labels (a label \( x \) is replaced by \( f(x) \)).

**EXAMPLES:**

```sage
sage: LT = LabelledOrderedTree
sage: t = LT([LT([],label=1),LT([],label=7)],label=3); t
3[1[], 7[]]
sage: t.map_labels(lambda z:z+1)
4[2[], 8[]]
sage: LBT = LabelledBinaryTree
sage: bt = LBT([LBT([],label=1),LBT([],label=4)],label=2); bt
2[1[., .], 4[., .]]
sage: bt.map_labels(lambda z:z+1)
3[2[., .], 5[., .]]
```

**set_label(path, label)**

Change the label of subtree indexed by \( path \) to \( label \).

**INPUT:**

- \( path \) – None (default) or a path (list or tuple of children index in the tree)
- \( label \) – any sage object

**OUTPUT:** Nothing, \( self \) is modified in place

**Note:** \( self \) must be in a mutable state. See sage.structure.list_clone for more details about mutability.

**EXAMPLES:**

```sage
t = LabelledOrderedTree([[[],[]],[]])
sage: t.set_label((0,), 4)
Traceback (most recent call last):
... ValueError: object is immutable; please change a copy instead.
sage: with t.clone() as t:
....:   t.set_label((0,), 4)
(continues on next page)```
sage: t
None[4[], None[None[], None[]]]
sage: with t.clone() as t:
    ....: t.set_label((1,0), label = 42)
sage: t
None[4[], None[42[], None[]]]

Todo: Do we want to implement the following syntactic sugar:

```python
with t.clone() as tt:
    tt.labels[1,2] = 3
```

**set_root_label**(*label*)

Set the label of the root of self.

**INPUT:** label – any Sage object

**OUTPUT:** None, self is modified in place

**Note:** self must be in a mutable state. See `sage.structure.list_clone` for more details about mutability.

**EXAMPLES:**

```python
sage: t = LabelledOrderedTree([[], [[], []]])
sage: t.set_root_label(3)
Traceback (most recent call last):
    ...:
ValueError: object is immutable; please change a copy instead.
sage: with t.clone() as t:
    ....: t.set_root_label(3)
sage: t.label()
3
sage: t
3[None[], None[None[], None[]]]
```

This also works for binary trees:

```python
sage: bt = LabelledBinaryTree([[], []])
sage: bt.set_root_label(3)
Traceback (most recent call last):
    ...:
ValueError: object is immutable; please change a copy instead.
sage: with bt.clone() as bt:
    ....: bt.set_root_label(3)
sage: bt.label()
3
sage: bt
3[None[., .], None[., .]]
```
class sage.combinat.abstract_tree.AbstractLabelledTree(parent, children, label=None, check=True)

Bases: AbstractTree

Abstract Labelled Tree.

Typically a class for labelled trees is constructed by inheriting from a class for unlabelled trees and AbstractLabelledTree.

**How should this class be extended?**

A class extending AbstractLabelledTree should respect the following assumptions:

- For a labelled tree \( T \) the call \( T.parent().unlabelled_trees() \) should return a parent for unlabelled trees of the same kind: for example,
  - if \( T \) is a binary labelled tree, \( T.parent() \) is LabelledBinaryTrees() and \( T.parent().unlabelled_trees() \) is BinaryTrees
  - if \( T \) is an ordered labelled tree, \( T.parent() \) is LabelledOrderedTrees() and \( T.parent().unlabelled_trees() \) is OrderedTrees
- In the same vein, the class of \( T \) should contain an attribute \( \_UnLabelled \) which should be the class for the corresponding unlabelled trees.

See also the assumptions in AbstractTree.

See also:

AbstractTree

\texttt{as\_digraph()}

Return a directed graph version of \texttt{self}.

\begin{center}
\textbf{Warning:} At this time, the output makes sense only if \texttt{self} is a labelled binary tree with no repeated labels and no None labels.
\end{center}

**EXAMPLES:**

\begin{verbatim}
sage: LT = LabelledOrderedTrees()
sage: t1 = LT([LT([],label=6),LT([],label=1)],label=9)
sage: t1.as_digraph()
Digraph on 3 vertices
\end{verbatim}

\begin{verbatim}
sage: t = BinaryTree([[None, None],[[]],None])
sage: lt = t.canonical_labelling()
sage: lt.as_digraph()
Digraph on 4 vertices
\end{verbatim}

\texttt{label(path=None)}

Return the label of \texttt{self}.

INPUT:

- \texttt{path} -- None (default) or a path (list or tuple of children index in the tree)

OUTPUT: the label of the subtree indexed by \texttt{path}

**EXAMPLES:**
sage: t = LabelledOrderedTree([[], []], label = 3)
sage: t.label()
3
sage: t[0].label()

sage: t = LabelledOrderedTree([LabelledOrderedTree([], 5), []], label = 3)
sage: t.label()
3
sage: t[0].label()
5
sage: t[1].label()
sage: t.label([0])
5

labels()

Return the list of labels of self.

EXAMPLES:

sage: LT = LabelledOrderedTree
sage: t = LT([LT([], label='b'), LT([], label='c')], label='a')
sage: t.labels()
['a', 'b', 'c']
sage: LBT = LabelledBinaryTree
sage: LBT([LBT([], label=1), LBT([], label=4)], label=2).labels()
[2, 1, 4]

leaf_labels()

Return the list of labels of the leaves of self.

In case of a labelled binary tree, these “leaves” are not actually the leaves of the binary trees, but the nodes whose both children are leaves!

EXAMPLES:

sage: LT = LabelledOrderedTree
sage: t = LT([LT([], label='b'), LT([], label='c')], label='a')
sage: t.leaf_labels()
['b', 'c']
sage: LBT = LabelledBinaryTree
sage: bt = LBT([LBT([], label='b'), LBT([], label='c')], label='a')
sage: bt.leaf_labels()
['b', 'c']
sage: LBT([], label='1').leaf_labels()
['1']
sage: LBT(None).leaf_labels()
[]

shape()

Return the unlabelled tree associated to self.

EXAMPLES:
```python
sage: t = LabelledOrderedTree([], [()], label = 25).shape(); t
[[[], []]]
sage: LabelledBinaryTree([], [([], [])], label = 25).shape()
[[., .], [[., .], [., .]]]
sage: LRT = LabelledRootedTree
sage: tb = LRT([], label='b')
sage: LRT([tb, tb], label='a').shape()
[[], []]
```

```python
class sage.combinat.abstract_tree.AbstractTree
    Bases: object
    
    Abstract Tree.
    
    There is no data structure defined here, as this class is meant to be extended, not instantiated.
    
    How should this class be extended?
    
    A class extending `AbstractTree` should respect several assumptions:
    
    • For a tree `T`, the call `iter(T)` should return an iterator on the children of the root `T`.
    
    • The `canonical_labelling` method should return the same value for trees that are considered equal (see the “canonical labellings” section in the documentation of the `AbstractTree` class).
    
    • For a tree `T` the call `T.parent().labelled_trees()` should return a parent for labelled trees of the same kind: for example,
      - if `T` is a binary tree, `T.parent()` is `BinaryTrees()` and `T.parent().labelled_trees()` is `LabelledBinaryTrees()`
      - if `T` is an ordered tree, `T.parent()` is `OrderedTrees()` and `T.parent().labelled_trees()` is `LabelledOrderedTrees()`

    `breadth_first_order_traversal(action=None)`
    
    Run the breadth-first post-order traversal algorithm and subject every node encountered to some procedure `action`. The algorithm is:
    
    ```python
    queue <- [ root ];
    while the queue is not empty:
        node <- pop( queue );
        manipulate the node;
        prepend to the queue the list of all subtrees of the node (from the rightmost to the leftmost).
    ```
    
    INPUT:
    
    • `action` – (optional) a function which takes a node as input, and does something during the exploration
    
    OUTPUT:
    
    None. (This is not an iterator.)
    
    EXAMPLES:
    
    For example, on the following binary tree `b`:
the breadth-first order traversal algorithm explores \( b \) in the following order of nodes: 3, 1, 7, 2, 5, 8, 4, 6.

**canonical_labelling** (*shift=1*)

Return a labelled version of \( self \).

The actual canonical labelling is currently unspecified. However, it is guaranteed to have labels in 1...\( n \) where \( n \) is the number of nodes of the tree. Moreover, two (unlabelled) trees compare as equal if and only if their canonical labelled trees compare as equal.

**EXAMPLES:**

```python
sage: t = OrderedTree([[[], [], [], [], [], []], [], []])
sage: t.canonical_labelling()
1[2[., .], 3[4[., .], 5[., .]], 7[., .]], 8[9[., .], 10[., .]], 11[12[., .], 13[., .]]
```

**depth()**

Return the depth of \( self \).

**EXAMPLES:**

```python
sage: OrderedTree().depth()
1
sage: OrderedTree([]).depth()
1
sage: OrderedTree([[[]]]).depth()
2
sage: OrderedTree([[[], []]]).depth()
3
sage: OrderedTree([[[]], [[], []], [[], []], [[], []], [[], []]]).depth()
4
```

**iterative_post_order_traversal** (*action=None*)

Run the depth-first post-order traversal algorithm (iterative implementation) and subject every node encountered to some procedure \( action \). The algorithm is:

explore each subtree (by the algorithm) from the leftmost one to the rightmost one;
then manipulate the root with function `action` (in the case of a binary tree, only if the root is not a leaf).

INPUT:
- `action` – (optional) a function which takes a node as input, and does something during the exploration

OUTPUT:
None. (This is not an iterator.)

See also:
- `post_order_traversal_iter()`

**iterative_pre_order_traversal** *(action=None)*
Run the depth-first pre-order traversal algorithm (iterative implementation) and subject every node encountered to some procedure `action`. The algorithm is:

manipulate the root with function `action` (in the case of a binary tree, only if the root is not a leaf);
then explore each subtree (by the algorithm) from the leftmost one to the rightmost one.

INPUT:
- `action` – (optional) a function which takes a node as input, and does something during the exploration

OUTPUT:
None. (This is not an iterator.)

See also:
- `pre_order_traversal_iter()`
- `pre_order_traversal()`

**node_number()**
Return the number of nodes of `self`.

See also:
`node_number_at_depth()`, `node_number_to_the_right()`

EXAMPLES:

```sage
sage: OrderedTree().node_number()
1
sage: OrderedTree([]).node_number()
1
sage: OrderedTree([[],[[]]]).node_number()
3
sage: OrderedTree([[],[[]]]).node_number()
4
sage: OrderedTree([[[], [[], [[], [[], [[], [[], []]]]]]]).node_number()
13
```

EXAMPLES:
node_number_at_depth(depth)
Return the number of nodes at a given depth.
This counts all nodes that are at the given depth.
Here the root is considered to have depth 0.

INPUT:
• depth – an integer

See also:
node_number(), node_number_to_the_right(), paths_at_depth()

EXAMPLES:

```python
sage: T = OrderedTree([[[], [[]]], [[[], [[]]], []]])
sage: ascii_art(T)
   ___o____
   /  /  /
 o  o  o  o
 /  /  /  /
 o  o  o
 |  |
 o  o
 |  |
 o

sage: [T.node_number_at_depth(i) for i in range(6)]
[1, 3, 4, 2, 1, 0]
```

node_number_to_the_right(path)
Return the number of nodes at the same depth and to the right of the node identified by path.
This counts the nodes that are at the same depth as the given one, and strictly to its right.

See also:
node_number(), node_number_at_depth(), paths_to_the_right()

EXAMPLES:

```python
sage: T = OrderedTree([[[], [[]]], [[[], [[]]], []]])
sage: ascii_art(T)
   ___o____
   /  /  /
 o  o  o  o
 /  /  /  /
 o  o  o
 |  |
 o  o
 |  |
 o

(continues on next page)```
Combinatorics, Release 10.0

| | | | | |
| | | | | |
| o | o | |
| | o |
sage: T.node_number_to_the_right(())
0
sage: T.node_number_to_the_right((0,))
2
sage: T.node_number_to_the_right((0,1))
2
sage: T.node_number_to_the_right((0,1,0))
1

sage: T = OrderedTree([[]])
sage: T.node_number_to_the_right(())
0

paths()
Return a generator for all paths to nodes of self.

OUTPUT:
This method returns a list of sequences of integers. Each of these sequences represents a path from the root node to some node. For instance, (1,3,2,5,0,3) represents the node obtained by choosing the 1st child of the root node (in the ordering returned by iter), then the 3rd child of its child, then the 2nd child of the latter, etc. (where the labelling of the children is zero-based).

The root element is represented by the empty tuple ()

See also:
paths_at_depth(), paths_to_the_right()

EXAMPLES:

sage: list(OrderedTree([]).paths())
[()]
sage: list(OrderedTree([[],[]]).paths())
[(), (0,), (1,), (1, 0)]
sage: list(BinaryTree([[],[],[]]).paths())
[(), (0,), (1,), (1, 0), (1, 1)]

paths_at_depth(depth, path=[])
Return a generator for all paths at a fixed depth.

This iterates over all paths for nodes that are at the given depth.
Here the root is considered to have depth 0.

INPUT:

* depth – an integer
* path – optional given path (as a list) used in the recursion
**Warning:** The path option should not be used directly.

See also: 
paths(), paths_to_the_right(), node_number_at_depth()

EXAMPLES:

```sage
T = OrderedTree([[[], [[]], []], [[], [[[]]], []], []])
sage: ascii_art(T)
____o_______
 / / / / /
_o__ o o o o
 / / |
o o_ o_ o_
 / / / /
o o o o |
| | o

sage: list(T.paths_at_depth(0))
[()]
sage: list(T.paths_at_depth(2))
[(0, 0), (0, 1), (2, 0)]
sage: list(T.paths_at_depth(4))
[(0, 1, 1, 0)]
sage: list(T.paths_at_depth(5))
[]

sage: T2 = OrderedTree([[]])
sage: list(T2.paths_at_depth(0))
[()]
```

**paths_to_the_right**(*path*)

Return a generator of paths for all nodes at the same depth and to the right of the node identified by path. This iterates over the paths for nodes that are at the same depth as the given one, and strictly to its right.

INPUT:

* path – any path in the tree

See also: 
paths(), paths_at_depth(), node_number_to_the_right()

EXAMPLES:

```sage
sage: T = OrderedTree([[[], [[]], []], [[], [[[]]], []], []])
sage: ascii_art(T)
____o_______
 / / / / /
_o__ o o o o
 / / |
o o_ o_ o_
 / / / /
o o o o |
| | o

sage: list(T.paths_at_depth(0))
[()]
sage: list(T.paths_at_depth(2))
[(0, 0), (0, 1), (2, 0)]
sage: list(T.paths_at_depth(4))
[(0, 1, 1, 0)]
sage: list(T.paths_at_depth(5))
[]

sage: T2 = OrderedTree([[]])
sage: list(T2.paths_at_depth(0))
[()]
```

(continues on next page)
```
sage: g = T.paths_to_the_right(())
sage: list(g)
[]

sage: g = T.paths_to_the_right((0,))
sage: list(g)
[(1,), (2,)]

sage: g = T.paths_to_the_right((0,1))
sage: list(g)
[(1, 0), (1, 1)]

sage: g = T.paths_to_the_right((0,1,0))
sage: list(g)
[(1, 1, 0)]

sage: g = T.paths_to_the_right((1,2))
sage: list(g)
[]
```

**post_order_traversal**(action=None)

Run the depth-first post-order traversal algorithm (recursive implementation) and subject every node encountered to some procedure action. The algorithm is:

explore each subtree (by the algorithm) from the leftmost one to the rightmost one;
then manipulate the root with function `action` (in the case of a binary tree, only if the root is not a leaf).

**INPUT:**

* action – (optional) a function which takes a node as input, and does something during the exploration

**OUTPUT:**

None. (This is not an iterator.)

**See also:**

* post_order_traversal_iter()
* iterative_post_order_traversal()

**post_order_traversal_iter()**

The depth-first post-order traversal iterator.

This method iters each node following the depth-first post-order traversal algorithm (recursive implementation). The algorithm is:

explore each subtree (by the algorithm) from the leftmost one to the rightmost one;
then yield the root (in the case of binary trees, only if it is not a leaf).
EXAMPLES:

For example on the following binary tree $b$:

```
  ___3___
 /     \
1    _7_
 \
 2   5 8
 / \
4   6
```

(only the nodes are shown), the depth-first post-order traversal algorithm explores $b$ in the following order of nodes: 2, 1, 4, 6, 5, 8, 7, 3.

For another example, consider the labelled tree:

```
  __1___
 / / / |
2 6 8_ |
 | | | / / |
3_ 7 9 10 |
 / / |
4 5 |
```

The algorithm explores this tree in the following order: 4, 5, 3, 2, 7, 6, 9, 10, 8, 1.

**pre_order_traversal**($\text{action=}\text{None}$)

Run the depth-first pre-order traversal algorithm (recursive implementation) and subject every node encountered to some procedure $\text{action}$. The algorithm is:

- manipulate the root with function `action` (in the case of a binary tree, only if the root is not a leaf);
- then explore each subtree (by the algorithm) from the leftmost one to the rightmost one.

**INPUT:**

- $\text{action}$ – (optional) a function which takes a node as input, and does something during the exploration

**OUTPUT:**

None. (This is \textit{not} an iterator.)

**EXAMPLES:**

For example, on the following binary tree $b$:

```
  ___3___
 /     \
1    _7_
 \
 2   5 8
 / \
4   6
```

the depth-first pre-order traversal algorithm explores $b$ in the following order of nodes: 3, 1, 2, 7, 5, 4, 6, 8.

Another example:
The algorithm explores this tree in the following order: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

See also:

- `pre_order_traversal_iter()`
- `iterative_pre_order_traversal()`

`pre_order_traversal_iter()`

The depth-first pre-order traversal iterator.

This method iterates each node following the depth-first pre-order traversal algorithm (recursive implementation). The algorithm is:

```
yield the root (in the case of binary trees, if it is not a leaf);
then explore each subtree (by the algorithm) from the leftmost one to the rightmost one.
```

EXAMPLES:

For example, on the following binary tree b:

```
| ___3___ |
| /     \ |
| 1     _7_ |
| \     /  |
| 2     5  |
|     /   |
|     4   |
```

(only the nodes shown), the depth-first pre-order traversal algorithm explores b in the following order of nodes: 3, 1, 2, 7, 5, 4, 6, 8.

Another example:

```
| ___4___ |
| /     / |
| 2     6  |
|     /   |
|     3   |
|     /   |
|     4   |
```

The algorithm explores this labelled tree in the following order: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

`subtrees()`

Return a generator for all nonempty subtrees of `self`.
The number of nonempty subtrees of a tree is its number of nodes. (The word “nonempty” makes a difference only in the case of binary trees. For ordered trees, for example, all trees are nonempty.)

EXAMPLES:

```python
sage: list(OrderedTree([]).subtrees())
[[]]
sage: list(OrderedTree([[]],[[]]).subtrees())
[[], [[]], [][[]], [[]][[]]]
sage: list(OrderedTree([[]],[[]]).canonical_labelling().subtrees())
[1[2[], 3[4[]]], 2[], 3[4[]], 4[]]
sage: list(BinaryTree([[],[]],[[],[]]).subtrees())
[[[., .], [[[., .], [., .]], [., .], [[., .], [., .]], [., .], [., .]]],
sage: v = BinaryTree([[],[]])
sage: list(v.canonical_labelling().subtrees())
[2[1[., .], 3[., .]], 1[., .], 3[., .]]
```

**to_hexacode()**

Transform a tree into an hexadecimal string.

The definition of the hexacode is recursive. The first letter is the valence of the root as an hexadecimal (up to 15), followed by the concatenation of the hexacodes of the subtrees.

This method only works for trees where every vertex has valency at most 15.

See **from_hexacode()** for the reverse transformation.

EXAMPLES:

```python
sage: from sage.combinat.abstract_tree import from_hexacode
sage: LT = LabelledOrderedTrees()
sage: from_hexacode('2010', LT).to_hexacode()
'2010'
sage: LT.an_element().to_hexacode()
'3020010'
sage: t = from_hexacode('a0000000000000000', LT)
sage: t.to_hexacode()
'a0000000000'
sage: OrderedTrees(6).an_element().to_hexacode()
'500000'
```

**tree_factorial()**

Return the tree-factorial of self.

Definition:

The tree-factorial $T!$ of a tree $T$ is the product $\prod_{v \in T} \#\text{children}(v)$.

EXAMPLES:

```python
sage: LT = LabelledOrderedTrees()
sage: t = LT([LT([],label=6),LT([],label=1)],label=9)
sage: t.tree_factorial()
(continues on next page)
```
sage.combinat.abstract_tree.from_hexacode(ch, parent=None, label='@')

Transform an hexadecimal string into a tree.

**INPUT:**

- ch – an hexadecimal string
- parent – kind of trees to be produced. If None, this will be LabelledOrderedTrees
- label – a label (default: '@') to be used for every vertex of the tree

See *AbstractTree.to_hexacode()* for the description of the encoding
See _from_hexacode_aux() for the actual code

**EXAMPLES:**

```python
sage: from sage.combinat.abstract_tree import from_hexacode
sage: from_hexacode('12000', LabelledOrderedTrees())
@[@[@[], @[]]]
sage: from_hexacode('12000')
@[@[@[], @[]]]
sage: from_hexacode('1200', LabelledOrderedTrees())
@[@[@[], @[]]]
```

It can happen that only a prefix of the word is used:

```python
sage: from_hexacode('a' + '14*0', LabelledOrderedTrees())
@[@[], @[], @[], @[], @[], @[], @[], @[], @[], @[], @[]]
```

One can choose the label:

```python
sage: from_hexacode('1200', LabelledOrderedTrees(), label='o')
o[@[@[], @[]]]
```

One can also create other kinds of trees:

```python
sage: from_hexacode('1200', OrderedTrees())
[[[][], [][]]]
```

### 5.1.2 Affine Permutations

```python
class sage.combinat.affine_permutation.AffinePermutation(parent, lst, check=True)
```

**Bases:** ClonableArray

An affine permutation, represented in the window notation, and considered as a bijection from $\mathbb{Z}$ to $\mathbb{Z}$.

**EXAMPLES:**
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p
Type A affine permutation with window [3, -1, 0, 6, 5, 4, 10, 9]

apply_simple_reflection(i, side='right')

Apply a simple reflection.

INPUT:

• i – an integer
• side – (default: 'right') determines whether to apply the reflection on the 'right' or 'left'

EXAMPLES:

sage: p = AffinePermutationGroup(['A',7,1])([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.apply_simple_reflection(3)
Type A affine permutation with window [3, -1, 6, 0, 5, 4, 10, 9]
sage: p.apply_simple_reflection(11)
Type A affine permutation with window [3, -1, 6, 0, 5, 4, 10, 9]
sage: p.apply_simple_reflection(3, 'left')
Type A affine permutation with window [4, -1, 0, 6, 5, 3, 10, 9]
sage: p.apply_simple_reflection(11, 'left')
Type A affine permutation with window [4, -1, 0, 6, 5, 3, 10, 9]

grassmannian_quotient(i=0, side='right')

Return the Grassmannian quotient.

Factors self into a unique product of a Grassmannian and a finite-type element. Returns a tuple containing
the Grassmannian and finite elements, in order according to side.

INPUT:

• i – (default: 0) an element of the index set; the descent checked for

EXAMPLES:

sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: gq=p.grassmannian_quotient()
sage: gq
(Type A affine permutation with window [-1, 0, 3, 4, 5, 6, 9, 10],
Type A affine permutation with window [3, 1, 2, 6, 5, 4, 8, 7])
sage: gq[0].is_i_grassmannian()
True
sage: 0 not in gq[1].reduced_word()
True
sage: prod(gq)==p
True
sage: gqLeft=p.grassmannian_quotient(side='left')
sage: 0 not in gqLeft[0].reduced_word()
True
sage: gqLeft[1].is_i_grassmannian(side='left')
True

(continues on next page)
sage: prod(gqLeft)==p
True

index_set()

Index set of the affine permutation group.

EXAMPLES:

sage: A = AffinePermutationGroup(['A',7,1])
sage: A.index_set()
(0, 1, 2, 3, 4, 5, 6, 7)

is_i_grassmannian(i=0, side='right')

Test whether self is i-grassmannian, i.e., either is the identity or has i as the sole descent.

INPUT:

• i – an element of the index set
• side – determines the side on which to check the descents

EXAMPLES:

sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.is_i_grassmannian()
False
sage: q=A.from_word([3,2,1,0])
sage: q.is_i_grassmannian()
True
sage: q=A.from_word([2,3,4,5])
sage: q.is_i_grassmannian(5)
True
sage: q.is_i_grassmannian(2, side='left')
True

is_one()

Tests whether the affine permutation is the identity.

EXAMPLES:

sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.is_one()
False
sage: q=A.one()
sage: q.is_one()
True

lower_covers(side='right')

Return lower covers of self.

The set of affine permutations of one less length related by multiplication by a simple transposition on the indicated side. These are the elements that self covers in weak order.

EXAMPLES:
Combinatorics, Release 10.0

\begin{verbatim}
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.lower_covers()
[Type A affine permutation with window [-1, 3, 0, 6, 5, 4, 10, 9],
 Type A affine permutation with window [3, -1, 0, 6, 5, 4, 10, 9],
 Type A affine permutation with window [3, -1, 0, 6, 4, 5, 10, 9],
 Type A affine permutation with window [3, -1, 0, 6, 5, 4, 9, 10]]

reduced_word()
Returns a reduced word for the affine permutation.

EXAMPLES:

\begin{verbatim}
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.reduced_word()
[0, 7, 4, 1, 0, 7, 5, 4, 2, 1]
\end{verbatim}

signature()
Signature of the affine permutation, \((-1)^l\), where \(l\) is the length of the permutation.

EXAMPLES:

\begin{verbatim}
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.signature()
1
\end{verbatim}

to_weyl_group_element()
The affine Weyl group element corresponding to the affine permutation.

EXAMPLES:

\begin{verbatim}
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.to_weyl_group_element()
[ 0 -1 0 1 0 0 1 0]
[ 1 -1 0 1 0 0 1 -1]
[ 1 -1 0 1 0 0 0 0]
[ 0 0 0 1 0 0 0 0]
[ 0 0 0 1 0 -1 1 0]
[ 0 0 0 1 -1 0 1 0]
[ 0 0 0 0 0 0 1 0]
[ 0 -1 1 0 0 0 1 0]
\end{verbatim}

sage.combinat.affine_permutation.AffinePermutationGroup(cartan_type)
Wrapper function for specific affine permutation groups.

These are combinatorial implementations of the affine Weyl groups of types \(A, B, C, D,\) and \(G\) as permutations of the set of all integers. the basic algorithms are derived from [BB2005] and [Eri1995].

EXAMPLES:

\begin{verbatim}
sage: ct = CartanType(['A',7,1])
sage: A = AffinePermutationGroup(ct)
\end{verbatim}
\end{verbatim}
The group of affine permutations of type ['A', 7, 1]

We define an element of $A$:

```python
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p
Type A affine permutation with window [3, -1, 0, 6, 5, 4, 10, 9]
```

We find the value $p(1)$, considering $p$ as a bijection on the integers. This is the same as calling the `value()` method:

```python
sage: p.value(1)
3
sage: p(1) == p.value(1)
True
```

We can also find the position of the integer 3 in $p$ considered as a sequence, equivalent to finding $p^{-1}(3)$:

```python
sage: p.position(3)
1
sage: (p^-1)(3)
1
```

Since the affine permutation group is a group, we demonstrate its group properties:

```python
sage: A.one()
Type A affine permutation with window [1, 2, 3, 4, 5, 6, 7, 8]
sage: q = A([0, 2, 3, 4, 5, 6, 7, 9])
sage: p * q
Type A affine permutation with window [1, -1, 0, 6, 5, 4, 10, 11]
sage: q * p
Type A affine permutation with window [3, -1, 1, 6, 5, 4, 10, 8]
sage: p^-1
Type A affine permutation with window [0, -1, 1, 6, 5, 4, 10, 11]
sage: p^-1 * p == A.one()
True
sage: p * p^-1 == A.one()
True
```

If we decide we prefer the Weyl Group implementation of the affine Weyl group, we can easily get it:

```python
sage: p.to_weyl_group_element()
[ 0 -1 0 0 0 0 0 0]
[ 1 0 0 0 0 0 0 -1]
[ 1 -1 0 0 0 0 0 0]
[ 0 0 0 0 0 0 0 0]
[ 0 0 0 1 0 -1 1 0]
[ 0 0 0 1 -1 0 1 0]
[ 0 0 0 0 0 0 0 0]
[ 0 -1 1 0 0 0 0 0]
```
We can find a reduced word and do all of the other things one expects in a Coxeter group:

```
sage: p.has_right_descent(1)
True
sage: p.apply_simple_reflection(1)
Type A affine permutation with window [-1, 3, 0, 6, 5, 4, 10, 9]
sage: p.apply_simple_reflection(0)
Type A affine permutation with window [1, -1, 0, 6, 5, 4, 10, 11]
sage: p.reduced_word()
[0, 7, 4, 1, 0, 7, 5, 4, 2, 1]
sage: p.length()
10
```

The following methods are particular to type $A$. We can check if the element is fully commutative:

```
sage: p.is_fully_commutative()
False
sage: q.is_fully_commutative()
True
```

We can also compute the affine Lehmer code of the permutation, a weak composition with $k + 1$ entries:

```
sage: p.to_lehmer_code()
[0, 3, 3, 0, 1, 2, 0, 1]
```

Once we have the Lehmer code, we can obtain a $k$-bounded partition by sorting the Lehmer code, and then reading the row lengths. There is a unique 0-Grassmanian (dominant) affine permutation associated to this $k$-bounded partition, and a $k$-core as well.

```
sage: p.to_bounded_partition()
[5, 3, 2]
sage: p.to_dominant()
Type A affine permutation with window [-2, -1, 1, 3, 4, 8, 10, 13]
sage: p.to_core()
[5, 3, 2]
```

Finally, we can take a reduced word for $p$ and insert it to find a standard composition tableau associated uniquely to that word:

```
sage: p.tableau_of_word(p.reduced_word())
[[], [1, 6, 9], [2, 7, 10], [], [3], [4, 8], [], [5]]
```

We can also form affine permutation groups in types $B$, $C$, $D$, and $G$:

```
sage: B = AffinePermutationGroup(['B',4,1])
sage: B.an_element()
Type B affine permutation with window [-1, 3, 4, 11]
sage: C = AffinePermutationGroup(['C',4,1])
sage: C.an_element()
Type C affine permutation with window [2, 3, 4, 10]
sage: D = AffinePermutationGroup(['D',4,1])
sage: D.an_element()
```

(continues on next page)
Type D affine permutation with window [-1, 3, 11, 5]

```
sage: G = AffinePermutationGroup(['G',2,1])
sage: G.an_element()
Type G affine permutation with window [0, 4, -1, 8, 3, 7]
```

```python
class sage.combinat.affine_permutation.AffinePermutationGroupGeneric(cartan_type)
    Bases: UniqueRepresentation, Parent

    The generic affine permutation group class, in which we define all type-free methods for the specific affine permutation groups.

    cartan_matrix()
        Returns the Cartan matrix of self.
        EXAMPLES:
        ```
sage: AffinePermutationGroup(['A',7,1]).cartan_matrix()
[ 2 -1 0 0 0 0 0 -1]
[-1 2 -1 0 0 0 0 0]
[ 0 -1 2 -1 0 0 0 0]
[ 0 0 -1 2 -1 0 0 0]
[ 0 0 0 -1 2 -1 0 0]
[ 0 0 0 0 -1 2 -1 0]
[ 0 0 0 0 0 -1 2 -1]
[-1 0 0 0 0 0 -1 2]
```

cartan_type()
    Returns the Cartan type of self.
    EXAMPLES:
    ```
sage: AffinePermutationGroup(['A',7,1]).cartan_type()
['A', 7, 1]
```

classical()
    Returns the finite permutation group.
    EXAMPLES:
    ```
sage: A = AffinePermutationGroup(['A',7,1])
sage: A.classical()
Symmetric group of order 8! as a permutation group
```

from_word(w)
    Builds an affine permutation from a given word. Note: Already in category as from_reduced_word, but this is less typing!
    EXAMPLES:
    ```
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: A.from_word([0, 7, 4, 1, 0, 7, 5, 4, 2, 1])
Type A affine permutation with window [3, -1, 0, 6, 5, 4, 10, 9]
```
**index_set()**

EXAMPLES:

```
sage: AffinePermutationGroup(['A',7,1]).index_set()
(0, 1, 2, 3, 4, 5, 6, 7)
```

**is_crystallographic()**

Tells whether the affine permutation group is crystallographic.

EXAMPLES:

```
sage: AffinePermutationGroup(['A',7,1]).is_crystallographic()
True
```

**random_element(n=None)**

Return a random affine permutation of length $n$.

If $n$ is not specified, then $n$ is chosen as a random non-negative integer in $[0, 1000]$.

Starts at the identity, then chooses an upper cover at random. Not very uniform: actually constructs a uniformly random reduced word of length $n$. Thus we most likely get elements with lots of reduced words!

For the actual code, see `sage.categories.coxeter_group.random_element_of_length()`.

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A',7,1])
sage: A.random_element()  # random
Type A affine permutation with window [-12, 16, 19, -1, -2, 10, -3, 9]
sage: p = A.random_element(10)
sage: p.length() == 10
True
```

**rank()**

Rank of the affine permutation group, equal to $k + 1$.

EXAMPLES:

```
sage: AffinePermutationGroup(['A',7,1]).rank()
8
```

**reflection_index_set()**

EXAMPLES:

```
sage: AffinePermutationGroup(['A',7,1]).reflection_index_set()
(0, 1, 2, 3, 4, 5, 6, 7)
```

**weyl_group()**

Returns the Weyl Group of the same type as `self`.

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A',7,1])
sage: A.weyl_group()
Weyl Group of type ['A', 7, 1] (as a matrix group acting on the root space)
```
class sage.combinat.affine_permutation.AffinePermutationGroupTypeA(cartan_type)
Bases: AffinePermutationGroupGeneric

Element
alias of AffinePermutationTypeA

from_lehmer_code(C, typ='decreasing', side='right')
Return the affine permutation with the supplied Lehmer code (a weak composition with \(k+1\) parts, at least one of which is 0).

INPUT:
• typ – 'increasing' or 'decreasing' (default: 'decreasing'); type of product
• side – 'right' or 'left' (default: 'right'); whether the decomposition is from the right or left

EXAMPLES:

```python
sage: import itertools
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.to_lehmer_code()
[0, 3, 3, 0, 1, 2, 0, 1]
sage: A.from_lehmer_code(p.to_lehmer_code()) == p
True
sage: orders = ('increasing','decreasing')
sage: sides = ('left','right')
sage: all(A.from_lehmer_code(p.to_lehmer_code(o,s),o,s) == p
.....:     for o,s in itertools.product(orders,sides))
True
```

one()
Return the identity element.

EXAMPLES:

```python
sage: AffinePermutationGroup(['A',7,1]).one()
Type A affine permutation with window [1, 2, 3, 4, 5, 6, 7, 8]
```

class sage.combinat.affine_permutation.AffinePermutationGroupTypeB(cartan_type)
Bases: AffinePermutationGroupTypeC

Element
alias of AffinePermutationTypeB

class sage.combinat.affine_permutation.AffinePermutationGroupTypeC(cartan_type)
Bases: AffinePermutationGroupGeneric

Element
alias of AffinePermutationTypeC

one()
Return the identity element.

EXAMPLES:
sage: ct=CartanType(['C',4,1])
sage: C = AffinePermutationGroup(ct)
sage: C.one()
Type C affine permutation with window [1, 2, 3, 4]
sage: C.one()*C.one()==C.one()
True

class sage.combinat.affine_permutation.AffinePermutationGroupTypeD(cartan_type)
    Bases: AffinePermutationGroupTypeC
    Element
        alias of AffinePermutationTypeD

class sage.combinat.affine_permutation.AffinePermutationGroupTypeG(cartan_type)
    Bases: AffinePermutationGroupGeneric
    Element
        alias of AffinePermutationTypeG

    one()
        Return the identity element.
        EXAMPLES:
            sage: AffinePermutationGroup(['G',2,1]).one()
            Type G affine permutation with window [1, 2, 3, 4, 5, 6]

class sage.combinat.affine_permutation.AffinePermutationGroupTypeA(parent, lst, check=True)
    Bases: AffinePermutation

    apply_simple_reflection_left(i)
        Apply the simple reflection to the values $i, i+1$.
        EXAMPLES:
            sage: p = AffinePermutationGroup(['A',7,1])([3, -1, 0, 6, 5, 4, 10, 9])
            sage: p.apply_simple_reflection_left(3)
            Type A affine permutation with window [4, -1, 0, 6, 5, 3, 10, 9]
            sage: p.apply_simple_reflection_left(11)
            Type A affine permutation with window [4, -1, 0, 6, 5, 3, 10, 9]

    apply_simple_reflection_right(i)
        Apply the simple reflection to positions $i, i+1$.
        INPUT:
            * i – an integer
        EXAMPLES:
            sage: p = AffinePermutationGroup(['A',7,1])([3, -1, 0, 6, 5, 4, 10, 9])
            sage: p.apply_simple_reflection_right(3)
            Type A affine permutation with window [3, -1, 6, 0, 5, 4, 10, 9]
            sage: p.apply_simple_reflection_right(11)
            Type A affine permutation with window [3, -1, 6, 0, 5, 4, 10, 9]
check()

Check that self is an affine permutation.

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p
Type A affine permutation with window [3, -1, 0, 6, 5, 4, 10, 9]
sage: q = A([1,2,3])  # indirect doctest
Traceback (most recent call last):
... ValueError: length of list must be k+1=8
sage: q = A([1,2,3,4,5,6,7,9])  # indirect doctest
Traceback (most recent call last):
... ValueError: window does not sum to 36
sage: q = A([1,1,3,4,5,6,7,9])  # indirect doctest
Traceback (most recent call last):
... ValueError: entries must have distinct residues
```

flip_automorphism()

The Dynkin diagram automorphism which fixes \( s_0 \) and reverses all other indices.

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.flip_automorphism()
Type A affine permutation with window [0, -1, 5, 4, 3, 9, 10, 6]
```

has_left_descent(i)

Determine whether there is a descent at \( i \).

INPUT:

- \( i \) – an integer

EXAMPLES:

```
sage: p = AffinePermutationGroup(['A',7,1])([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.has_left_descent(1)
True
sage: p.has_left_descent(9)
True
sage: p.has_left_descent(0)
True
```

has_right_descent(i)

Determine whether there is a descent at \( i \).

INPUT:

- \( i \) – an integer

EXAMPLES:
is_fully_commutative()
Determine whether self is fully commutative, i.e., has no reduced words with a braid.

EXAMPLES:

```python
sage: A = AffinePermutationGroup(['A', 7, 1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.is_fully_commutative()
False
sage: q = A([-3, -2, 0, 7, 9, 2, 11, 12])
sage: q.is_fully_commutative()
True
```

maximal_cyclic_decomposition(typ='decreasing', side='right', verbose=False)
Find the unique maximal decomposition of self into cyclically decreasing/increasing elements.

INPUT:

- **typ** – 'increasing' or 'decreasing' (default: 'decreasing'); chooses whether to find increasing or decreasing sets
- **side** – 'right' or 'left' (default: 'right') chooses whether to find maximal sets starting from the left or the right
- **verbose** – (default: False) print extra information while finding the decomposition

EXAMPLES:

```python
sage: p = AffinePermutationGroup(['A', 7, 1])([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.maximal_cyclic_decomposition()
[[0, 7], [4, 1, 0], [7, 5, 4, 2, 1]]
sage: p.maximal_cyclic_decomposition(side='left')
[[1, 0, 7, 5, 4], [1, 0, 5], [2, 1]]
sage: p.maximal_cyclic_decomposition(typ='increasing', side='right')
[[1], [5, 0, 1, 2], [4, 5, 7, 0, 1]]
sage: p.maximal_cyclic_decomposition(typ='increasing', side='left')
[[0, 1, 2, 4, 5], [4, 7, 0, 1], [7]]
```

maximal_cyclic_factor(typ='decreasing', side='right', verbose=False)
For an affine permutation \( x \), find the unique maximal subset \( A \) of the index set such that \( x = yd_A \) is a reduced product.

INPUT:

- **typ** – 'increasing' or 'decreasing' (default: 'decreasing'); chooses whether to find increasing or decreasing sets
- **side** – 'right' or 'left' (default: 'right') chooses whether to find maximal sets starting from the left or the right
• **verbose** – True or False. If True, outputs information about how the cyclically increasing element was found.

**EXAMPLES:**

```python
sage: p = AffinePermutationGroup(['A',7,1])([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.maximal_cyclic_factor()
[7, 5, 4, 2, 1]
sage: p.maximal_cyclic_factor(side='left')
[1, 0, 7, 5, 4]
sage: p.maximal_cyclic_factor('increasing','right')
[4, 5, 7, 0, 1]
sage: p.maximal_cyclic_factor('increasing','left')
[0, 1, 2, 4, 5]
```

**position**(i)

Find the position j such the `self.value(j) == i`.

**EXAMPLES:**

```python
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.position(3)
1
sage: p.position(11)
9
```

**promotion**()

The Dynkin diagram automorphism which sends $s_i$ to $s_{i+1}$.

**EXAMPLES:**

```python
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.promotion()
Type A affine permutation with window [2, 4, 0, 1, 7, 6, 5, 11]
```

**tableau_of_word**(w, typ='increasing', side='right', alpha=None)

Finds a tableau on the Lehmer code of `self` corresponding to the given reduced word.

For a full description of this algorithm, see [Den2012].

**INPUT:**

- w – a reduced word for `self`
- typ – 'increasing' or 'decreasing'; the type of Lehmer code used
- side – 'right' or 'left'
- alpha – a content vector; w should be of type alpha; specifying alpha produces semistandard tableaux

**EXAMPLES:**

```python
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.tableau_of_word(p.reduced_word())
(continues on next page)
```
Combinatorics, Release 10.0

(continued from previous page)

\[
[[], [1, 6, 9], [2, 7, 10], [], [3], [4, 8], [], [5]]
\]

sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: w=p.reduced_word()
sage: w
[0, 7, 4, 1, 0, 7, 5, 4, 2, 1]
sage: alpha=[5,3,2]
sage: p.tableau_of_word(p.reduced_word(), alpha=alpha)
[[], [1, 2, 3], [1, 2, 3], [], [1], [1, 2], [], [1]]
sage: p.tableau_of_word(p.reduced_word(), side='left')
[[1, 4, 9], [6], [], [], [3, 7], [8], [], [2, 5, 10]]
sage: p.tableau_of_word(p.reduced_word(), typ='increasing', side='right')
[[9, 10], [1, 2], [], [], [3, 4], [8], [], [5, 6, 7]]
sage: p.tableau_of_word(p.reduced_word(), typ='increasing', side='left')
[[1, 2], [4, 5, 6], [9, 10], [], [3], [7, 8], [], []]

to_bounded_partition(typ='decreasing', side='right')

Return the \(k\)-bounded partition associated to the dominant element obtained by sorting the Lehmer code.

INPUT:

- typ – ‘increasing’ or ‘decreasing’ (default: ‘decreasing’.) Chooses whether to find increasing or decreasing sets.
- side – ‘right’ or ‘left’ (default: ‘right’.) Chooses whether to find maximal sets starting from the left or the right.

EXAMPLES:

sage: A = AffinePermutationGroup(['A',2,1])
sage: p=A.from_lehmer_code([4,1,0])
sage: p.to_bounded_partition()
[2, 1, 1, 1]

to_core(typ='decreasing', side='right')

Returns the core associated to the dominant element obtained by sorting the Lehmer code.

INPUT:

- typ – ‘increasing’ or ‘decreasing’ (default: ‘decreasing’.)
- side – ‘right’ or ‘left’ (default: ‘right’.) Chooses whether to find maximal sets starting from the left or the right.

EXAMPLES:

sage: A = AffinePermutationGroup(['A',2,1])
sage: p=A.from_lehmer_code([4,1,0])
sage: p.to_bounded_partition()
[2, 1, 1, 1]
sage: p.to_core()
[4, 2, 1, 1]

to_dominant(typ='decreasing', side='right')

Finds the Lehmer code and then sorts it. Returns the affine permutation with the given sorted Lehmer code; this element is 0-dominant.
INPUT:

- **typ** – 'increasing' or 'decreasing' (default: 'decreasing') chooses whether to find increasing or decreasing sets
- **side** – 'right' or 'left' (default: 'right') chooses whether to find maximal sets starting from the left or the right

EXAMPLES:

```python
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.to_dominant()
Type A affine permutation with window [-2, -1, 1, 3, 4, 8, 10, 13]
sage: p.to_dominant(typ='increasing', side='left')
Type A affine permutation with window [3, 4, -1, 5, 0, 9, 6, 10]
```

to_lehmer_code(**typ='decreasing', side='right'**)

Return the affine Lehmer code.

There are four such codes; the options typ and side determine which code is generated. The codes generated are the shape of the maximal cyclic decompositions of self according to the given typ and side options.

INPUT:

- **typ** – 'increasing' or 'decreasing' (default: 'decreasing'); chooses whether to find increasing or decreasing sets
- **side** – 'right' or 'left' (default: 'right') chooses whether to find maximal sets starting from the left or the right

EXAMPLES:

```python
sage: import itertools
sage: A = AffinePermutationGroup(['A',7,1])
sage: p=A([3, -1, 0, 6, 5, 4, 10, 9])
sage: orders = ('increasing', 'decreasing')
sage: sides = ('left', 'right')
sage: for o, s in itertools.product(orders, sides):
    ....:     p.to_lehmer_code(o, s)
[2, 3, 2, 0, 1, 2, 0, 0]
[2, 2, 0, 0, 2, 1, 0, 3]
[3, 1, 0, 2, 1, 0, 3]
[0, 3, 3, 0, 1, 2, 0, 1]
sage: for a in itertools.product(orders, sides):
    ....:     A.from_lehmer_code(p.to_lehmer_code(a[0], a[1]), a[0], a[1]) == p
True
True
True
True
```

to_type_a()

Return an embedding of self into the affine permutation group of type A. (For type A, just returns self.)

EXAMPLES:
\begin{verbatim}
sage: p = AffinePermutationGroup(['A',7,1])([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.to_type_a() is p
True
\end{verbatim}

\textbf{value}(i, base_window=False)

Return the image of the integer \(i\) under this permutation.

\textbf{INPUT:}

\begin{itemize}
  \item base_window – boolean; indicating whether \(i\) is in the base window; if True, will run a bit faster, but the method will screw up if \(i\) is not actually in the index set
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = AffinePermutationGroup(['A',7,1])
sage: p = A([3, -1, 0, 6, 5, 4, 10, 9])
sage: p.value(1)
3
sage: p.value(9)
11
\end{verbatim}

\begin{class}
\textbf{sage.combinat.affine_permutation.AffinePermutationTypeB}(parent, lst, check=True)
Bases: \textit{AffinePermutationTypeC}

\textbf{apply_simple_reflection_left}(i)

Apply the simple reflection indexed by \(i\) on values.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = AffinePermutationGroup(['B',4,1])
sage: p=B([-5,1,6,-2])
sage: p.apply_simple_reflection_left(0)
Type B affine permutation with window [-5, -2, 6, 1]
sage: p.apply_simple_reflection_left(2)
Type B affine permutation with window [-5, 1, 7, -3]
sage: p.apply_simple_reflection_left(4)
Type B affine permutation with window [-4, 1, 6, -2]
\end{verbatim}

\textbf{apply_simple_reflection_right}(i)

Apply the simple reflection indexed by \(i\) on positions.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = AffinePermutationGroup(['B',4,1])
sage: p=B([-5,1,6,-2])
sage: p.apply_simple_reflection_right(1)
Type B affine permutation with window [1, -5, 6, -2]
sage: p.apply_simple_reflection_right(0)
Type B affine permutation with window [-1, 5, 6, -2]
sage: p.apply_simple_reflection_right(4)
Type B affine permutation with window [-5, 1, 6, 11]
\end{verbatim}

\textbf{check}()

Check that \texttt{self} is an affine permutation.

\textbf{EXAMPLES:}
sage: B = AffinePermutationGroup(['B',4,1])
sage: x = B([-5,1,6,-2])
sage: x
Type B affine permutation with window [-5, 1, 6, -2]

has_left_descent(i)
Determines whether there is a descent at i.
INPUT:
• i – an integer
EXAMPLES:
sage: B = AffinePermutationGroup(['B',4,1])
sage: p = B([-5,1,6,-2])
sage: [p.has_left_descent(i) for i in B.index_set()]
[True, True, False, False, True]

has_right_descent(i)
Determines whether there is a descent at index i.
INPUT:
• i – an integer
EXAMPLES:
sage: B = AffinePermutationGroup(['B',4,1])
sage: p = B([-5,1,6,-2])
sage: [p.has_right_descent(i) for i in B.index_set()]
[True, False, False, True, False]

class sage.combinat.affine_permutation.AffinePermutationTypeC(parent, lst, check=True)
Bases: AffinePermutation

apply_simple_reflection_left(i)
Apply the simple reflection indexed by i on values.
EXAMPLES:
sage: C = AffinePermutationGroup(['C',4,1])
sage: x = C([-1,5,3,7])
sage: for i in C.index_set(): x.apply_simple_reflection_left(i)
Type C affine permutation with window [1, 5, 3, 7]
Type C affine permutation with window [-2, 5, 3, 8]
Type C affine permutation with window [-1, 5, 2, 6]
Type C affine permutation with window [-1, 6, 4, 7]
Type C affine permutation with window [-1, 4, 3, 7]

apply_simple_reflection_right(i)
Apply the simple reflection indexed by i on positions.
EXAMPLES:
```python
sage: C = AffinePermutationGroup(['C', 4, 1])
sage: x = C([-1, 5, 3, 7])
sage: for i in C.index_set(): x.apply_simple_reflection_right(i)
Type C affine permutation with window [1, 5, 3, 7]
Type C affine permutation with window [5, -1, 3, 7]
Type C affine permutation with window [-1, 3, 5, 7]
Type C affine permutation with window [-1, 5, 7, 3]
Type C affine permutation with window [-1, 5, 3, 2]
```

**check()**

Check that `self` is an affine permutation.

**EXAMPLES:**

```python
sage: C = AffinePermutationGroup(['C', 4, 1])
sage: x = C([-1, 5, 3, 7])
sage: x
Type C affine permutation with window [-1, 5, 3, 7]
```

**has_left_descent(i)**

Determine whether there is a descent at `i`.

**INPUT:**

- `i` – an integer

**EXAMPLES:**

```python
sage: C = AffinePermutationGroup(['C', 4, 1])
sage: x = C([-1, 5, 3, 7])
sage: for i in C.index_set(): x.has_left_descent(i)
True
False
True
False
True
```

**has_right_descent(i)**

Determine whether there is a descent at index `i`.

**INPUT:**

- `i` – an integer

**EXAMPLES:**

```python
sage: C = AffinePermutationGroup(['C', 4, 1])
sage: x = C([-1, 5, 3, 7])
sage: for i in C.index_set(): x.has_right_descent(i)
True
False
True
False
True
```
position(i)
Find the position j such the self.value(j)=i

EXAMPLES:

```python
sage: C = AffinePermutationGroup(['C',4,1])
sage: x = C.one()
sage: [x.position(i) for i in range(-10,10)] == list(range(-10,10))
True
```

to_type_a()
Return an embedding of self into the affine permutation group of type A.

EXAMPLES:

```python
sage: C = AffinePermutationGroup(['C',4,1])
sage: x = C([-1,5,3,7])
sage: x.to_type_a()  # Type A affine permutation with window [-1, 5, 3, 7, 2, 6, 4, 10, 9]
```

value(i)
Return the image of the integer i under this permutation.

EXAMPLES:

```python
sage: C = AffinePermutationGroup(['C',4,1])
sage: x = C.one()
sage: [x.value(i) for i in range(-10,10)] == list(range(-10,10))
True
```

class sage.combinat.affine_permutation.AffinePermutationTypeD(parent, lst, check=True)
Bases: AffinePermutationTypeC

apply_simple_reflection_left(i)
Apply simple reflection indexed by i on values.

EXAMPLES:

```python
sage: D = AffinePermutationGroup(['D',4,1])
sage: p=D([-1,6,5,-2])
sage: p.apply_simple_reflection_left(0)
Type D affine permutation with window [-2, -6, 5, 1]
sage: p.apply_simple_reflection_left(1)
Type D affine permutation with window [2, -6, 5, -1]
sage: p.apply_simple_reflection_left(4)
Type D affine permutation with window [1, -4, 3, -2]
```

apply_simple_reflection_right(i)
Apply the simple reflection indexed by i on positions.

EXAMPLES:

```python
sage: D = AffinePermutationGroup(['D',4,1])
sage: p=D([-1,6,5,-2])
sage: p.apply_simple_reflection_right(0)
Type D affine permutation with window [6, -1, 5, -2]
```
sage: p.apply_simple_reflection_right(1)
Type D affine permutation with window [-6, 1, 5, -2]
sage: p.apply_simple_reflection_right(4)
Type D affine permutation with window [1, -6, 11, 4]

check()
Check that self is an affine permutation.

EXAMPLES:

sage: D = AffinePermutationGroup(['D',4,1])
sage: p = D([1,-6,5,-2])
sage: p
Type D affine permutation with window [1, -6, 5, -2]

has_left_descent(i)
Determine whether there is a descent at i.

INPUT:
  • i – an integer

EXAMPLES:

sage: D = AffinePermutationGroup(['D',4,1])
sage: p=D([1,-6,5,-2])
sage: [p.has_left_descent(i) for i in D.index_set()]
[True, True, False, True, True]

has_right_descent(i)
Determine whether there is a descent at index i.

INPUT:
  • i – an integer

EXAMPLES:

sage: D = AffinePermutationGroup(['D',4,1])
sage: p=D([1,-6,5,-2])
sage: [p.has_right_descent(i) for i in D.index_set()]
[True, True, False, True, False]

class sage.combinat.affine_permutation.AffinePermutationTypeG(parent, lst, check=True)
Bases: AffinePermutation

apply_simple_reflection_left(i)
Apply simple reflection indexed by i on values.

EXAMPLES:

sage: G = AffinePermutationGroup(['G',2,1])
sage: p=G([2, 10, -5, 12, -3, 5])
sage: p.apply_simple_reflection_left(0)
Type G affine permutation with window [0, 10, -7, 14, -3, 7]
sage: p.apply_simple_reflection_left(1)
apply_simple_reflection_right(i)

Apply the simple reflection indexed by i on positions.

EXAMPLES:

```
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: p.apply_simple_reflection_right(0)
Type G affine permutation with window [-9, -1, -5, 12, 8, 16]
sage: p.apply_simple_reflection_right(1)
Type G affine permutation with window [10, 2, 12, -5, 5, -3]
sage: p.apply_simple_reflection_right(2)
Type G affine permutation with window [2, -5, 10, -3, 12, 5]
```

check()

Check that self is an affine permutation.

EXAMPLES:

```
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: p
Type G affine permutation with window [2, 10, -5, 12, -3, 5]
```

has_left_descent(i)

Determines whether there is a descent at i.

INPUT:

• i – an integer

EXAMPLES:

```
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: [p.has_left_descent(i) for i in G.index_set()]
[False, True, False]
```

has_right_descent(i)

Determines whether there is a descent at index i.

INPUT:

• i – an integer

EXAMPLES:

```
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: [p.has_right_descent(i) for i in G.index_set()]
[False, False, True]
```
position($i$)

Find the position $j$ such the $\text{self.value}(j) = i$.

EXAMPLES:

\begin{verbatim}
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: [p.position(i) for i in p]
[1, 2, 3, 4, 5, 6]
\end{verbatim}

to_type_a()

Return an embedding of $\text{self}$ into the affine permutation group of type A.

EXAMPLES:

\begin{verbatim}
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: p.to_type_a()
Type A affine permutation with window [2, 10, -5, 12, -3, 5]
\end{verbatim}

value($i$, base_window=False)

Return the image of the integer $i$ under this permutation.

INPUT:

- base_window – boolean indicating whether $i$ is between 1 and $k+1$; if True, will run a bit faster, but the method will screw up if $i$ is not actually in the index set

EXAMPLES:

\begin{verbatim}
sage: G = AffinePermutationGroup(['G',2,1])
sage: p = G([2, 10, -5, 12, -3, 5])
sage: [p.value(i) for i in [1..12]]
[2, 10, -5, 12, -3, 5, 8, 16, 1, 18, 3, 11]
\end{verbatim}

5.1.3 Algebraic combinatorics

Thematic tutorials

- Algebraic Combinatorics in Sage
- Lie Methods and Related Combinatorics in Sage
- Linear Programming (Mixed Integer)

Enumerated sets of combinatorial objects

- Enumerated sets of partitions, tableaux, …
- GelfandTsetlinPattern, GelfandTsetlinPatterns
- KnutsonTaoPuzzleSolver
Groups and Algebras

- Catalog of algebras
- Groups
  - SymmetricGroup, CoxeterGroup, WeylGroup
  - PartitionAlgebra
  - IwahoriHeckeAlgebra
  - SymmetricGroupAlgebra
  - NilCoxeterAlgebra
  - AffineNilTemperleyLiebTypeA
- Descent Algebras
- Diagram and Partition Algebras
- Blob Algebras

Combinatorial Representation Theory

- Root Systems
- Crystals
- Rigged configurations
- Cluster algebras and quivers
- KazhdanLusztigPolynomial
- SymmetricGroupRepresentation
- SpechtModule
- Yang-Baxter Graphs
- Hall Polynomials
- Key polynomials

Operads and their algebras

- Free Dendriform Algebras
- Free Pre-Lie Algebras
- Free Zinbiel Algebras
5.1.4 Combinatorics

Introductory material

- Combinatorics quickref
- Introduction to combinatorics in Sage

Thematic indexes

- Algebraic combinatorics
  - Combinatorial Hopf algebras
  - Cluster algebras and quivers
  - Crystals
  - Root Systems
  - Symmetric Functions
  - FullyCommutativeElements
- Counting
- Enumerated sets and combinatorial objects
- Enumerated sets of partitions, tableaux, …
- Finite state machines, automata, transducers
- Combinatorial species
- Combinatorial designs and incidence structures
- Posets
- Combinatorics on words
- A bijectionist’s toolkit

Utilities

- Output functions
- Rankers
- Combinatorial maps
- Miscellaneous
5.1.5 Alternating Sign Matrices

AUTHORS:

• Mike Hansen (2007): Initial version
• Pierre Cange, Luis Serrano (2012): Added monotone triangles
• Travis Scrimshaw (2013-28-03): Added element class for ASM’s and made MonotoneTriangles inherit from GelfandTsetlinPatterns
• Jessica Striker (2013): Added additional methods
• Vincent Delecroix (2017): cleaning

class sage.combinat.alternating_sign_matrix.AlternatingSignMatrices(n)
    Bases: UniqueRepresentation, Parent
    Class of all \( n \times n \) alternating sign matrices.
    An alternating sign matrix of size \( n \) is an \( n \times n \) matrix of '0's, '1's and '-1's such that the sum of each row and column is 1 and the non-zero entries in each row and column alternate in sign.
    Alternating sign matrices of size \( n \) are in bijection with monotone triangles with \( n \) rows.
    INPUT:
    • \( n \) – an integer, the size of the matrices.
    EXAMPLES:
    This will create an instance to manipulate the alternating sign matrices of size 3:

    sage: A = AlternatingSignMatrices(3)
sage: A
    Alternating sign matrices of size 3
    sage: A.cardinality()
    7

    Notably, this implementation allows to make a lattice of it:

    sage: L = A.lattice()
sage: L
    Finite lattice containing 7 elements
    sage: L.category()
    Category of facade finite enumerated lattice posets

    Element
    alias of AlternatingSignMatrix
cardinality()

Return the cardinality of self.

The number of $n \times n$ alternating sign matrices is equal to

$$\prod_{k=0}^{n-1} \frac{(3k + 1)!}{(n + k)!} = \frac{1!4!7!10! \cdots (3n - 2)!}{n!(n + 1)!(n + 2)!(n + 3)! \cdots (2n - 1)!}$$

EXAMPLES:

```python
sage: [AlternatingSignMatrices(n).cardinality() for n in range(11)]
[1, 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700]
```

cover_relations()

Iterate on the cover relations between the alternating sign matrices.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: for (a,b) in A.cover_relations():
    ....:     eval('a, b')

(\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\])
(\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
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\begin{bmatrix}
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0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
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\begin{bmatrix}
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\end{bmatrix}
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\begin{bmatrix}
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\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\])
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\begin{bmatrix}
0 & 1 & 0 \\
-1 & 1 & 0 \\
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\end{bmatrix},
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0 & 1 & 0 \\
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0 & 1 & 0
\end{bmatrix}
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\begin{bmatrix}
0 & 0 & 1 \\
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\end{bmatrix},
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0 & 0 & 1 \\
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0 & 1 & 0
\end{bmatrix}
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\begin{bmatrix}
0 & 1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
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0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\])
```
(continues on next page)
first()

Return the first alternating sign matrix.

EXAMPLES:

```
sage: AlternatingSignMatrices(5).first()
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
```

from_contre_tableau(comps)

Return an alternating sign matrix from a contre-tableau.

EXAMPLES:

```
sage: ASM = AlternatingSignMatrices(3)
sage: ASM.from_contre_tableau([[1, 2, 3], [1, 2], [1]])
[0 0 1]
[0 1 0]
[1 0 0]
sage: ASM.from_contre_tableau([[1, 2, 3], [2, 3], [3]])
[1 0 0]
[0 1 0]
[0 0 1]
```

from_corner_sum(corner)

Return an alternating sign matrix from a corner sum matrix.

EXAMPLES:

```
sage: A = AlternatingSignMatrices(3)
sage: A.from_corner_sum(matrix([[0,0,0],[0,1,1],[0,1,2],[0,1,2,3]]))
[1 0 0]
[0 1 0]
[0 0 1]
sage: A.from_corner_sum(matrix([[0,0,0],[0,0,1],[0,1,2],[0,1,2,3]]))
[0 1 0]
[1 -1 1]
[0 1 0]
```

from_height_function(height)

Return an alternating sign matrix from a height function.

EXAMPLES:
from sage.combinat.alternating_sign_matrix import AlternatingSignMatrices

```python
sage: A = AlternatingSignMatrices(3)
sage: A.from_height_function(matrix([[0,1,2,3],[1,2,1,2],[2,3,2,1],[3,2,1,0]]))
[0 0 1]
[1 0 0]
[0 1 0]
sage: A.from_height_function(matrix([[0,1,2,3],[1,2,1,2],[2,1,2,1],[3,2,1,0]]))
[ 0 1 0]
[ 1 -1 1]
[ 0 1 0]
```

**from_monotone_triangle**(triangle, check=True)

Return an alternating sign matrix from a monotone triangle.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: A.from_monotone_triangle([[3, 2, 1], [2, 1], [1]])
[1 0 0]
[0 1 0]
[0 0 1]
sage: A.from_monotone_triangle([[3, 2, 1], [3, 2], [3]])
[0 0 1]
[0 1 0]
[1 0 0]
sage: A.from_monotone_triangle([[3, 2, 1], [2, 2], [1]])
Traceback (most recent call last):
... ValueError: not a valid triangle
```

**gyration_orbit_sizes()**

Return the sizes of gyration orbits of self.

EXAMPLES:

```python
sage: AlternatingSignMatrices(3).gyration_orbit_sizes()
[3, 2]
sage: AlternatingSignMatrices(4).gyration_orbit_sizes()
[4, 8, 2, 8, 8, 8, 2, 2]
sage: A = AlternatingSignMatrices(5)
 sage: li = [5,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,4,4,5,4,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,2,2,2,2]
 sage: A.gyration_orbit_sizes() == li
 True
```

**gyration_orbits()**

Return the list of gyration orbits of self.

EXAMPLES:
```
sage: AlternatingSignMatrices(3).gyration_orbits()
(([[1 0 0] [0 0 1] [ 0 1 0]], [[0 1 0] [0 1 0] [ 1 -1 1]], [[0 0 1], [1 0 0], [ 0 1 0]]),
 ([0 1 0] [1 0 0]
 [1 0 0] [0 0 1]
 [0 0 1], [0 1 0]),
 ([0 0 1] [0 1 0]
 [1 0 0] [0 0 1]
 [0 1 0], [1 0 0]));
```

**last()**

Return the last alternating sign matrix.

**EXAMPLES:**

```
sage: AlternatingSignMatrices(5).last()
[[0 0 0 0 1]
 [0 0 0 1 0]
 [0 0 1 0 0]
 [0 1 0 0 0]
 [1 0 0 0 0]]
```

**lattice()**

Return the lattice of the alternating sign matrices of size $n$, created by `LatticePoset`.

**EXAMPLES:**

```
sage: A = AlternatingSignMatrices(3)
sage: L = A.lattice()
sage: L
Finite lattice containing 7 elements
```

**matrix_space()**

Return the underlying matrix space.

**EXAMPLES:**

```
sage: A = AlternatingSignMatrices(3)
sage: A.matrix_space()
sage: A.matrix_space()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
```

**random_element()**

Return a uniformly random alternating sign matrix.

**EXAMPLES:**
sage: AlternatingSignMatrices(7).random_element()  # random
[ 0 0 0 0 1 0 0]
[ 0 0 1 0 -1 0 1]
[ 0 0 0 0 1 0 0]
[ 0 1 -1 0 0 1 0]
[ 1 -1 1 0 0 0 0]
[ 0 0 0 1 0 0 0]
[ 0 1 0 0 0 0 0]
sage: a = AlternatingSignMatrices(5).random_element()
sage: bool(a.number_negative_ones()) or a.is_permutation()
True

This is done using a modified version of Propp and Wilson’s “coupling from the past” algorithm. It creates a uniformly random Gelfand-Tsetlin triangle with top row \([n, n−1, \ldots, 2, 1]\), and then converts it to an alternating sign matrix.

**size()**

Return the size of the matrices in self.

class sage.combinat.alternating_sign_matrix.AlternatingSignMatrix(parent, asm)

An alternating sign matrix.

An alternating sign matrix is a square matrix of 0’s, 1’s and −1’s such that the sum of each row and column is 1 and the non-zero entries in each row and column alternate in sign.

These were introduced in [MRR1983].

**ASM_compatible(B)**

Return True if self and B are compatible alternating sign matrices in the sense of [EKLP1992]. (If self is of size \(n\), B must be of size \(n+1\).)

In [EKLP1992], there is a notion of a pair of ASM’s with sizes differing by 1 being compatible, in the sense that they can be combined to encode a tiling of the Aztec Diamond.

**EXAMPLES:**

```python
sage: A = AlternatingSignMatrix(matrix([[0,0,1,0],[0,1,-1,1],[1,0,0,0],[0,0,1,-0]]))
sage: B = AlternatingSignMatrix(matrix([[0,0,1,0,0],[0,0,0,1,0],[1,0,0,-1,1],[0,-1,0,0,0],[0,0,0,1,0]]))
sage: AASM_compatible(B)
True
sage: A = AlternatingSignMatrix(matrix([[0,1,0],[0,1,0]])
sage: B = AlternatingSignMatrix(matrix([[0,0,1,0],[1,0,0,0],[0,0,1,0],[0,1,0,-0]]))
sage: AASM_compatible(B)
False
```

**ASM_compatible_bigger()**

Return all ASM’s compatible with self that are of size one greater than self.

Given an \(n \times n\) alternating sign matrix \(A\), there are as many ASM’s of size \(n + 1\) compatible with \(A\) as 2 raised to the power of the number of 1’s in \(A\) [EKLP1992].

**EXAMPLES:**
sage: A = AlternatingSignMatrix([[1,0],[0,1]])
sage: A.ASM_compatible_bigger()

sage: B = AlternatingSignMatrix([[0,1],[1,0]])
sage: B.ASM_compatible_bigger()

sage: B = AlternatingSignMatrix([[0,1,0],[1,-1,1],[0,1,0]])
sage: len(B.ASM_compatible_bigger()) == 2**4
True

ASM_compatible_smaller()
Return the list of all ASMs compatible with self that are of size one smaller than self.

Given an alternating sign matrix \( A \) of size \( n \), there are as many ASM’s of size \( n - 1 \) compatible with it as 2 raised to the power of the number of \(-1\)’s in \( A \) [EKLP1992].

EXAMPLES:

sage: A = AlternatingSignMatrix(matrix([[0,0,1,0],[0,1,-1,1],[1,0,0,0],[0,0,1,0]]))
sage: A.ASM_compatible_smaller()

sage: B = AlternatingSignMatrix(matrix([[1,0,0],[0,0,1],[0,1,0]]))
sage: B.ASM_compatible_smaller()

corner_sum_matrix()
Return the corner sum matrix of self.

EXAMPLES:

sage: A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).corner_sum_matrix()

(continues on next page)
gyration()

Return the alternating sign matrix obtained by applying gyration to the height function in bijection with self.

Gyration acts on height functions as follows. Go through the entries of the matrix, first those for which the sum of the row and column indices is even, then for those for which it is odd, and increment or decrement the squares by 2 wherever possible such that the resulting matrix is still a height function. Gyration was first defined in [Wie2000] as an action on fully-packed loops.

EXAMPLES:

```sage
A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).gyration()
[0 0 1]
[0 1 0]
[1 0 0]
sage: A([[0, 1, 0],[1, -1, 1],[0, 1, 0]]).gyration()
[1 0 0]
[0 1 0]
[0 0 1]
sage: A([[0, 0, 1],[1, 0, 0],[0, 1, 0]]).gyration()
[0 1 0]
[0 0 1]
[1 0 0]
sage: A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).gyration().gyration()
[ 0 1 0]
[ 1 -1 1]
[ 0 1 0]
sage: A([[0, 0, 1],[1, 0, 0],[0, 0, 1]]).gyration().gyration().gyration()
[1 0 0]
[0 1 0]
[0 0 1]
```

```sage
A = AlternatingSignMatrices(4)
sage: M = A([[0,0,1,0],[1,0,0,0],[0,1,-1,1],[0,0,1,0]])
sage: for i in range(5):
    .....
    M = M.gyration()
sage: M
```
**gyration_orbit()**

Return the gyration orbit of self (including self).

**EXAMPLES:**

```python
sage: AlternatingSignMatrix([[0, 1, 0], [1, -1, 1], [0, 1, 0]]).gyration_orbit() 
[[ 0 1 0] 
 [ 1 -1 1] 
 [ 0 1 0] 
] 

sage: AlternatingSignMatrix([[0, 1, 0, 0], [1, -1, 1, 0], [0, 1, -1, 1], [0, 0, 1, 0]]).gyration_orbit() 
[[ 0 1 0 0] 
 [ 1 -1 1 0] 
 [ 0 1 -1 1] 
 [ 0 0 1 0] 
] 

sage: len(AlternatingSignMatrix([[0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, -1, 0, 0, 0, 1], 
[0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0]]).gyration_orbit()) 
12
```

**height_function()**

Return the height function from self.

A height function corresponding to an \( n \times n \) ASM is an \( (n+1) \times (n+1) \) matrix such that the first row is 0, 1, ..., \( n \), the last row is \( n, n-1, \ldots, 1, 0 \), and the difference between adjacent entries is 1.

**EXAMPLES:**

```python
sage: A = AlternatingSignMatrices(3) 
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).height_function() 
[0 1 2 3] 
[1 0 1 2] 
[2 1 0 1] 
[3 2 1 0] 

sage: asm = A([[0, 1, 0],[1, -1, 1],[0, 1, 0]]) 
sage: asm.height_function() 
[0 1 2 3]
```
Combinatorics, Release 10.0

| 1 2 1 2 |
| 2 1 2 1 |
| 3 2 1 0 |

\[
\text{sage: asm} = A([[\text{0, 0, 1}, [1, \text{0, 0}, [\text{0, 1, 0}]]])
\]

\[
\text{sage: asm.height}() \\
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 1 & 2 \\
2 & 3 & 2 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}
\]

\[
\text{sage: A} = \text{AlternatingSignMatrices}(4)
\]

\[
\text{sage: all(A.from_height_function(a.height_function()) == a for a in A)}
\]

True

\section*{inversion_number()}

Return the inversion number of \text{self}.

If we denote the entries of the alternating sign matrix as \(a_{i,j}\), the inversion number is defined as \(\sum_{i>j} \sum_{j<k} a_{i,j}a_{k,l}\). When restricted to permutation matrices, this gives the usual inversion number of the permutation.

This definition is equivalent to the one given in [MRR1983].

\subsection*{EXAMPLES:}

\[
\text{sage: A} = \text{AlternatingSignMatrices}(3)
\text{sage: A([[1, 0, 0], [0, 1, 0], [0, 0, 1]])}.\text{inversion_number()}
\]

0

\[
\text{sage: asm} = A([[\text{0, 0, 1}, [1, \text{0, 0}, [\text{0, 1, 0}]]])
\text{sage: asm.inversion_number()}
\]

2

\[
\text{sage: asm} = A([[\text{0, 1, 0}, [1, -1, 1], [0, 1, 0]])
\text{sage: asm.inversion_number()}
\]

2

\[
\text{sage: P} = \text{Permutations}(5)
\text{sage: all(p.number_of_inversions()==AlternatingSignMatrix(p.to_matrix()).\text{\rightarrow inversion_number()}}
\]

True

\section*{is_permutation()}

Return True if \text{self} is a permutation matrix and False otherwise.

\subsection*{EXAMPLES:}

\[
\text{sage: A} = \text{AlternatingSignMatrices}(3)
\text{sage: asm} = A([[\text{0,1,0}, [1,\text{0,0}, [\text{0,0,1}]))
\text{sage: asm.is_permutation()}
\]

True

\[
\text{sage: asm} = A([[\text{0,1,0}, [1,-1,1], [0,1,0]])
\text{sage: asm.is_permutation()}
\]

False

\section*{left_key()}

Return the left key of the alternating sign matrix \text{self}.
The left key of an alternating sign matrix was defined by Lascoux in [Lasc] and is obtained by successively removing all the $-1$’s until what remains is a permutation matrix. This notion corresponds to the notion of left key for semistandard tableaux. So our algorithm proceeds as follows: we map `self` to its corresponding monotone triangle, view that monotone triangle as a semistandard tableau, take its left key, and then map back through monotone triangles to the permutation matrix which is the left key.

See also [Ava2007].

**EXAMPLES:**

```python
sage: A = AlternatingSignMatrices(3)
sage: A([[0,0,1],[1,0,0],[0,1,0]]).left_key()
[0 0 1]
[1 0 0]
[0 1 0]
sage: t = A([[0,1,0],[1,-1,1],[0,1,0]]).left_key(); t
[1 0 0]
[0 0 1]
[0 1 0]
sage: parent(t)
Alternating sign matrices of size 3
```

**left_key_as_permutation()**

Return the permutation of the left key of `self`.

See also:

- `left_key()`

**EXAMPLES:**

```python
sage: A = AlternatingSignMatrices(3)
sage: A([[0,0,1],[1,0,0],[0,1,0]]).left_key_as_permutation()
[3, 1, 2]
sage: t = A([[0,1,0],[1,-1,1],[0,1,0]]).left_key_as_permutation(); t
[1, 3, 2]
sage: parent(t)
Standard permutations
```

**link_pattern()**

Return the link pattern corresponding to the fully packed loop corresponding to `self`.

**EXAMPLES:**

We can extract the underlying link pattern (a non-crossing partition) from a fully packed loop:

```python
sage: A = AlternatingSignMatrix([[0, 1, 0], [1, -1, 1], [0, 1, 0]])
sage: A.link_pattern()
[(1, 2), (3, 6), (4, 5)]
sage: B = AlternatingSignMatrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
sage: B.link_pattern()
[(1, 6), (2, 5), (3, 4)]
```

**number_negative_ones()**

Return the number of entries in `self` equal to -1.
EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: asm = A([[0,1,0],[1,0,0],[0,0,1]])
sage: asm.number_negative_ones()
0
sage: asm = A([[0,1,0],[1,-1,1],[0,1,0]])
sage: asm.number_negative_ones()
1
```

**rotate_ccw()**

Return the counterclockwise quarter turn rotation of `self`.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).rotate_ccw()
[0 0 1]
[0 1 0]
[1 0 0]
sage: asm = A([[0, 0, 1],[1, 0, 0],[0, 1, 0]])
sage: asm.rotate_ccw()
[1 0 0]
[0 0 1]
[0 1 0]
```

**rotate_cw()**

Return the clockwise quarter turn rotation of `self`.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).rotate_cw()
[0 0 1]
[0 1 0]
[1 0 0]
sage: asm = A([[0, 0, 1],[1, 0, 0],[0, 1, 0]])
sage: asm.rotate_cw()
[0 1 0]
[1 0 0]
[0 0 1]
```

**to_dyck_word(algorithm)**

Return a Dyck word determined by the specified algorithm.

The algorithm `last_diagonal` uses the last diagonal of the monotone triangle corresponding to `self`. The algorithm `link_pattern` returns the Dyck word in bijection with the link pattern of the fully packed loop. Note that these two algorithms in general yield different Dyck words for a given alternating sign matrix.

**INPUT:**

- `algorithm` - either `last_diagonal` or `link_pattern`

**EXAMPLES:**
sage: A = AlternatingSignMatrices(3)
sage: A([[0,1,0],[1,0,0],[0,0,1]]).to_dyck_word(algorithm = 'last_diagonal')
[1, 1, 0, 0, 1, 0]
sage: d = A([[0,1,0],[1,-1,1],[0,1,0]]).to_dyck_word(algorithm = 'last_diagonal →'); d
[1, 1, 0, 1, 0, 0]
sage: parent(d)
Complete Dyck words
sage: A = AlternatingSignMatrices(3)
sage: asm = A([[0,1,0],[1,0,0],[0,0,1]])
sage: asm.to_dyck_word(algorithm = 'link_pattern')
[1, 0, 1, 0, 1, 0]
sage: asm = A([[0,1,0],[1,-1,1],[0,1,0]])
sage: asm.to_dyck_word(algorithm = 'link_pattern')
[1, 0, 1, 1, 0, 0]
sage: A = AlternatingSignMatrices(4)
sage: asm = A([[0,0,1,0],[1,0,0,0],[0,1,-1,1],[0,0,1,0]])
sage: asm.to_dyck_word(algorithm = 'link_pattern')
[1, 1, 1, 0, 1, 0, 0, 0]
sage: asm.to_dyck_word()  # Error
Traceback (most recent call last):
  ...
TypeError: ...to_dyck_word() ...argument...
sage: asm.to_dyck_word(algorithm = 'notamethod')
Traceback (most recent call last):
  ...
ValueError: unknown algorithm 'notamethod'

\textbf{to\_fully\_packed\_loop()}

Return the fully packed loop configuration from self.

\textbf{See also:}
\textit{FullyPackedLoop}

\textbf{EXAMPLES:}

```python
sage: asm = AlternatingSignMatrix([[1,0,0],[0,1,0],[0,0,1]])
sage: fpl = asm.to\_fully\_packed\_loop()
sage: fpl
```

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</table>

\textbf{to\_matrix()}

Return self as a regular matrix.
EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: asm = A([[1, 0, 0],[0, 1, 0],[0, 0, 1]])
sage: m = asm.to_matrix(); m
[1 0 0]
[0 1 0]
[0 0 1]
sage: m.parent()
Full MatrixSpace of 3 by 3 dense matrices over Integer Ring
```

### to_monotone_triangle()

Return a monotone triangle from self.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).to_monotone_triangle()
[[3, 2, 1], [2, 1], [1]]
sage: asm = A([[0, 1, 0],[1, -1, 1],[0, 1, 0]])
sage: asm.to_monotone_triangle()
[[3, 2, 1], [3, 1], [2]]
sage: asm = A([[0, 0, 1],[1, 0, 0],[0, 1, 0]])
sage: asm.to_monotone_triangle()
[[3, 2, 1], [3, 1], [3]]
sage: asm = A([[1, 0, 0],[0, 1, 0],[0, 0, 1]])
sage: asm.to_monotone_triangle() == asm
True
```

### to_permutation()

Return the corresponding permutation if self is a permutation matrix.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: asm = A([[0,1,0],[1,0,0],[0,0,1]])
sage: p = asm.to_permutation(); p
[2, 1, 3]
sage: parent(p)
Standard permutations
sage: asm = A([[0,1,0],[1,-1,1],[0,1,0]])
sage: asm.to_permutation()
Traceback (most recent call last):
...
ValueError: not a permutation matrix
```

### to_semistandard_tableau()

Return the semistandard tableau corresponding the monotone triangle corresponding to self.

EXAMPLES:

```python
sage: A = AlternatingSignMatrices(3)
sage: t = A([[0,1,0],[1,-1,1],[0,1,0]]).to_semistandard_tableau(); t
[[1, 1, 2], [2, 3], [3]]
```

(continues on next page)
Semistandard tableaux

\textbf{to\_six\_vertex\_model()}

Return the six vertex model configuration from self.

This method calls \texttt{sage.combinat.six_vertex_model.from\_alternating\_sign\_matrix}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: asm = AlternatingSignMatrix([[0,1,0],[1,-1,1],[0,1,0]])
sage: asm.to_six_vertex_model()
\end{verbatim}

\begin{verbatim}
^ ^ ^
| | |  --> # -> # <- # <--
| ^ |
|--|--|--|--|--<--
| | |  ^ | ^ |
| V |
|--|--|--|--|--<--
| | |  V | V |
|--|--|--|--|--<--
| | |  V V V
\end{verbatim}

\textbf{transpose()}

Return self transposed.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = AlternatingSignMatrices(3)
sage: A([[1, 0, 0],[0, 1, 0],[0, 0, 1]]).transpose()
[1 0 0]
[0 1 0]
[0 0 1]
sage: asm = A([[0, 0, 1],[1, 0, 0],[0, 1, 0]])
sage: asm.transpose()
[0 1 0]
[0 0 1]
[1 0 0]
\end{verbatim}

\textbf{class \texttt{sage.combinat.alternating\_sign\_matrix}\_ContreTableaux}

\texttt{Parent}

Factory class for the combinatorial class of contre tableaux of size \(n\).

\textbf{EXAMPLES:}

\begin{verbatim}
sage: ct4 = ContreTableaux(4); ct4
Contre tableaux of size 4
sage: ct4.cardinality()
42
\end{verbatim}
cardinality()
EXAMPLES:

```sage
sage: [ContreTableaux(n).cardinality() for n in range(11)]
[1, 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700]
```

class sage.combinat.alternating_sign_matrix.MonotoneTriangles(n)

Bases: GelfandTsetlinPatternsTopRow

Monotone triangles with \( n \) rows.

A monotone triangle is a number triangle \((a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq i}\) on \( \{1, \ldots, n\} \) such that:

- \( a_{i,j} < a_{i,j+1} \)
- \( a_{i+1,j} < a_{i,j} \leq a_{i+1,j+1} \)

This notably requires that the bottom column is \([1, \ldots, n] \).

Alternatively a monotone triangle is a strict Gelfand-Tsetlin pattern with top row \((n, \ldots, 2, 1)\).

INPUT:

- \( n \) – The number of rows in the monotone triangles

EXAMPLES:

This represents the monotone triangles with base \([3,2,1]\):

```sage
sage: M = MonotoneTriangles(3)
sage: M
Monotone triangles with 3 rows
sage: M.cardinality()
7
```

The monotone triangles are a lattice:

```sage
sage: M.lattice()
Finite lattice containing 7 elements
```

Monotone triangles can be converted to alternating sign matrices and back:

```sage
sage: M = MonotoneTriangles(5)
sage: A = AlternatingSignMatrices(5)
sage: all(A.from_monotone_triangle(m).to_monotone_triangle() == m for m in M)
True
```

cardinality()

Cardinality of self.

The number of monotone triangles with \( n \) rows is equal to

\[
\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \frac{1!4!7!10! \cdots (3n-2)!}{n!(n+1)!(n+2)!(n+3)! \cdots (2n-1)!}
\]

EXAMPLES:

```sage
sage: M = MonotoneTriangles(4)
sage: M.cardinality()
42
```
**cover_relations()**

Iterate on the cover relations in the set of monotone triangles with \( n \) rows.

EXAMPLES:

```python
sage: M = MonotoneTriangles(3)
sage: for (a, b) in M.cover_relations():
    ....:     eval('a, b')
([[[3, 2, 1], [2, 1], [1]], [[3, 2, 1], [2, 1], [2]]],
 [[[3, 2, 1], [2, 1], [1]], [[3, 2, 1], [3, 1], [1]]],
 [[[3, 2, 1], [2, 1], [2]], [[3, 2, 1], [3, 1], [2]]],
 [[[3, 2, 1], [3, 1], [1]], [[3, 2, 1], [3, 1], [2]]],
 [[[3, 2, 1], [3, 1], [2]], [[3, 2, 1], [3, 1], [3]]],
 [[[3, 2, 1], [3, 1], [3]], [[3, 2, 1], [3, 2], [3]]],
 [[[3, 2, 1], [3, 2], [2]], [[3, 2, 1], [3, 2], [3]]])
```

**lattice()**

Return the lattice of the monotone triangles with \( n \) rows.

EXAMPLES:

```python
sage: M = MonotoneTriangles(3)
sage: P = M.lattice()
sage: P
Finite lattice containing 7 elements
```

class sage.combinat.alternating_sign_matrix.TruncatedStaircases

Bases: Parent

Factory class for the combinatorial class of truncated staircases of size \( n \) with last column \( \text{last\_column} \).

EXAMPLES:

```python
sage: t4 = TruncatedStaircases(4, [2,3]); t4
Truncated staircases of size 4 with last column [2, 3]
sage: t4.cardinality()
4
```

class sage.combinat.alternating_sign_matrix.TruncatedStaircases_nlastcolumn

Bases: TruncatedStaircases

cardinality()

EXAMPLES:

```python
sage: T = TruncatedStaircases(4, [2,3])
sage: T.cardinality()
4
```
5.1.6 Backtracking

This library contains a generic tool for constructing large sets whose elements can be enumerated by exploring a search space with a (lazy) tree or graph structure.

- **GenericBacktracker**: Depth first search through a tree described by a `children` function, with branch pruning, etc.

This module has mostly been superseded by `RecursivelyEnumeratedSet`.

```python
class sage.combinat.backtrack.GenericBacktracker(initial_data, initial_state):
    Bases: object
    A generic backtrack tool for exploring a search space organized as a tree, with branch pruning, etc.
    See also `RecursivelyEnumeratedSet_forest` for handling simple special cases.
```

```python
class sage.combinat.backtrack.PositiveIntegerSemigroup:
    Bases: UniqueRepresentation, RecursivelyEnumeratedSet_forest
    The commutative additive semigroup of positive integers.
    This class provides an example of algebraic structure which inherits from `RecursivelyEnumeratedSet_forest`. It builds the positive integers a la Peano, and endows it with its natural commutative additive semigroup structure.

    EXAMPLES:
    sage: from sage.combinat.backtrack import PositiveIntegerSemigroup
    sage: PP = PositiveIntegerSemigroup()
    sage: PP.category()
    Join of Category of monoids and Category of commutative additive semigroups and...
    Category of infinite enumerated sets and Category of facade sets
    sage: PP.cardinality()
    +Infinity
    sage: PP.one()
    1
    sage: PP.an_element()
    1
    sage: some_elements = list(PP.some_elements()); some_elements
    [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23,...
    24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43,...
    44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63,...
    64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83,...
    84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]
```

```python
children(x)
    Return the single child x+1 of the integer x

    EXAMPLES:
    sage: from sage.combinat.backtrack import PositiveIntegerSemigroup
    sage: PP = PositiveIntegerSemigroup()
    sage: list(PP.children(1))
    [2]
    sage: list(PP.children(42))
    [43]
```
one()
Return the unit of self.

EXAMPLES:

```
sage: from sage.combinat.backtrack import PositiveIntegerSemigroup
sage: PP = PositiveIntegerSemigroup()
sage: PP.one()
1
```

roots()
Return the single root of self.

EXAMPLES:

```
sage: from sage.combinat.backtrack import PositiveIntegerSemigroup
sage: PP = PositiveIntegerSemigroup()
sage: list(PP.roots())
[1]
```

5.1.7 Baxter permutations
class sage.combinat.baxter_permutations.BaxterPermutations
Bases: UniqueRepresentation, Parent
The combinatorial class of Baxter permutations.

A Baxter permutation is a permutation avoiding the generalized permutation patterns 2−413−3 and 3−142−2. In other words, a permutation $\sigma$ is a Baxter permutation if for any subword $u := u_1u_2u_3u_4$ of $\sigma$ such that the letters $u_2$ and $u_3$ are adjacent in $\sigma$, the standardized version of $u$ is neither 2413 nor 3142.


INPUT:

• $n$ – (default: None) a nonnegative integer, the size of the permutations.

OUTPUT:

Return the combinatorial class of the Baxter permutations of size $n$ if $n$ is not None. Otherwise, return the combinatorial class of all Baxter permutations.

EXAMPLES:

```
sage: BaxterPermutations(5)
Baxter permutations of size 5
sage: BaxterPermutations()
Baxter permutations
```

class sage.combinat.baxter_permutations.BaxterPermutations_all($n$=None)
Bases: DisjointUnionEnumeratedSets, BaxterPermutations
The enumerated set of all Baxter permutations.

See BaxterPermutations for the definition of Baxter permutations.

EXAMPLES:
```python
from sage.combinat.baxter_permutations import BaxterPermutations_all
BaxterPermutations_all()
```

**to_pair_of_twin_binary_trees(p)**

Apply a bijection between Baxter permutations of size `self._n` and the set of pairs of twin binary trees with `self._n` nodes.

**INPUT:**
- `p` – a Baxter permutation.

**OUTPUT:**

The pair of twin binary trees \((T_L, T_R)\) where \(T_L\) (resp. \(T_R\)) is obtained by inserting the letters of `p` from left to right (resp. right to left) following the binary search tree insertion algorithm. This is called the **Baxter P-symbol** in [Gir2012] Definition 4.1.

**Note:** This method only works when `p` is a permutation. For words with repeated letters, it would return two “right binary search trees” (in the terminology of [Gir2012]), which conflicts with the definition in [Gir2012].

**EXAMPLES:**

```python
BaxterPermutations().to_pair_of_twin_binary_trees(Permutation([]))
(., .)
BaxterPermutations().to_pair_of_twin_binary_trees(Permutation([1, 2, 3]))
(1[., 2[., 3[., .]], 3[2[1[., .], .], .]])
BaxterPermutations().to_pair_of_twin_binary_trees(Permutation([3, 4, 1, 2]))
(3[1[., 2[., .]], 4[., .]], 2[1[., .], 4[3[., .], .]])
```

```python
class sage.combinat.baxter_permutations.BaxterPermutations_size(n)
Bases: BaxterPermutations

The enumerated set of Baxter permutations of a given size.

See **BaxterPermutations** for the definition of Baxter permutations.

**EXAMPLES:**

```python
from sage.combinat.baxter_permutations import BaxterPermutations_size
BaxterPermutations_size(5)
Baxter permutations of size 5
```

**cardinality()**

Return the number of Baxter permutations of size `self._n`.

For any positive integer \(n\), the number of Baxter permutations of size \(n\) equals

\[
\sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+1} \binom{n+1}{k+1} \binom{n+1}{k+1} .
\]

This is OEIS sequence A001181.

**EXAMPLES:**
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```
sage: [BaxterPermutations(n).cardinality() for n in range(13)]
[1, 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, 1882960, 11140560]
sage: BaxterPermutations(3r).cardinality()
6
sage: parent(_)
Integer Ring
```

5.1.8 A bijectionist’s toolkit

AUTHORS:

Quick reference

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>set_statistics()</td>
<td>Declare statistics that are preserved by the bijection.</td>
</tr>
<tr>
<td>set_value_restrictions()</td>
<td>Restrict the values of the statistic on an element.</td>
</tr>
<tr>
<td>set_constant_blocks()</td>
<td>Declare that the statistic is constant on some sets.</td>
</tr>
<tr>
<td>set_distributions()</td>
<td>Restrict the distribution of values of the statistic on some elements.</td>
</tr>
<tr>
<td>set_intertwining_relations()</td>
<td>Declare that the statistic intertwines with other maps.</td>
</tr>
<tr>
<td>set_quadratic_relation()</td>
<td>Declare that the statistic satisfies a certain relation.</td>
</tr>
<tr>
<td>set_homomesic()</td>
<td>Declare that the statistic is homomesic with respect to a given set partition.</td>
</tr>
<tr>
<td>statistics_table()</td>
<td>Print a table collecting information on the given statistics.</td>
</tr>
<tr>
<td>statistics_fibers()</td>
<td>Collect elements with the same statistics.</td>
</tr>
<tr>
<td>constant_blocks()</td>
<td>Return the blocks on which the statistic is constant.</td>
</tr>
<tr>
<td>solutions_iterator()</td>
<td>Iterate over all possible solutions.</td>
</tr>
<tr>
<td>possible_values()</td>
<td>Return all possible values for a given element.</td>
</tr>
<tr>
<td>minimal_subdistributions_iterator()</td>
<td>Iterate over the minimal subdistributions.</td>
</tr>
</tbody>
</table>

A guided tour

Consider the following combinatorial statistics on a permutation:

- \( wex \), the number of weak excedences,
- \( fix \), the number of fixed points,
- \( des \), the number of descents (after appending 0),
- \( adj \), the number of adjacencies (after appending 0), and
- \( llis \), the length of a longest increasing subsequence.

Moreover, let \( rot \) be action of rotation on a permutation, i.e., the conjugation with the long cycle.

It is known that there must exist a statistic \( s \) on permutations, which is equidistributed with \( llis \) but additionally invariant under \( rot \). However, at least very small cases do not contradict the possibility that one can even find a statistic \( s \), invariant under \( rot \) and such that \( (s, wex, fix) \sim (llis, des, adj) \). Let us check this for permutations of size at most 3:
```python
data = N = 3
data = A = B = [pi for n in range(N+1) for pi in Permutations(n)]
data = def alpha1(p): return len(p.weak_excedences())
data = def alpha2(p): return len(p.fixed_points())
data = def beta1(p): return len(p.descents(final_descent=True)) if p else 0
data = def beta2(p): return len([e for (e, f) in zip(p, p[1:]+[0]) if e == f+1])
data = tau = Permutation.longest_increasing_subsequence_length
data = def rotate_permutation(p):
    ....: cycle = Permutation(tuple(range(1, len(p)+1)))
    ....: return Permutation([cycle.inverse()(p(cycle(i))) for i in range(1, len(p)+1)])
data = bij = Bijectionist(A, B, tau)
data = bij.set_statistics((len, len), (alpha1, beta1), (alpha2, beta2))
data = a, b = bij.statistics_table()
data = table(a, header_row=True, frame=True)
```

<table>
<thead>
<tr>
<th>a</th>
<th>α_1(a)</th>
<th>α_2(a)</th>
<th>α_3(a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[]</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[1]</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>[1, 2]</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>[2, 1]</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[1, 2, 3]</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>[1, 3, 2]</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>[2, 1, 3]</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>[2, 3, 1]</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>[3, 1, 2]</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[3, 2, 1]</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b</th>
<th>τ</th>
<th>β_1(b)</th>
<th>β_2(b)</th>
<th>β_3(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[]</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[1]</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>[1, 2]</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[2, 1]</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>[1, 2, 3]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[1, 3, 2]</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

(continues on next page)
On the other hand, we can check that there is no rotation invariant statistic on non-crossing set partitions which is equidistributed with the Strahler number on ordered trees:

```python
sage: N = 8
sage: A = [SetPartition(d.to_noncrossing_partition()) for n in range(N) for d in DyckWords(n)]
sage: B = [t for n in range(1, N+1) for t in OrderedTrees(n)]
sage: def theta(m):
    return SetPartition([[i % m.size() + 1 for i in b] for b in m])
```

Code for the Strahler number can be obtained from FindStat. The following code is equivalent to $\tau = \text{findstat}(397)$:

```python
sage: def tau(T):
    if len(T) == 0:
        return 1
    else:
        l = [tau(S) for S in T]
        m = max(l)
        if l.count(m) == 1:
            return m
        else:
            return m+1
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((lambda a: a.size(), lambda b: b.node_number()-1))
```

(continues on next page)

5.1. Comprehensive Module List
Next we demonstrate how to search for a bijection, instead of An example identifying $s$ and $S$:

```python
sage: N = 4
sage: A = [dyck_word for n in range(1, N) for dyck_word in DyckWords(n)]
sage: B = [binary_tree for n in range(1, N) for binary_tree in BinaryTrees(n)]
sage: concat_path = lambda D1, D2: DyckWord(list(D1) + list(D2))
sage: concat_tree = lambda B1, B2: concat_path(B1.to_dyck_word(), B2.to_dyck_word()).to_binary_tree()
sage: bij = Bijectionist(A, B)
sage: bij.set_intertwining_relations((2, concat_path, concat_tree))
sage: bij.set_statistics((lambda d: d.semilength(), lambda t: t.node_number()))

sage: for D in sorted(bij.minimal_subdistributions_iterator(), key=lambda x: x[0][0], x):
            ascii_art(D)
...
```

The output is in a form suitable for FindStat:

```python
sage: findmap(list(bij.minimal_subdistributions_iterator()))
# optional --
```

```bash
0: Mp00034 (quality [100])
1: Mp00061oMp00023 (quality [100])
2: Mp00018oMp00140 (quality [100])
```

```python
class sage.combinat.bijectionist.Bijectionist(A, B, tau=None, alpha_beta=(), P=None, pi_rho=(), phi_psi=(), Q=None, elements_distributions=(), value_restrictions=(), solver=None, key=None)
```
**Bases:** SageObject

A toolbox to list all possible bijections between two finite sets under various constraints.

**INPUT:**
- \(A, B\) – sets of equal size, given as a list
- \(\tau\) – (optional) a function from \(B\) to \(Z\), in case of None, the identity map \(\lambda x: x\) is used
- \(\alpha, \beta\) – (optional) a list of pairs of statistics \(\alpha\) from \(A\) to \(W\) and \(\beta\) from \(B\) to \(W\)
- \(P\) – (optional) a partition of \(A\)
- \(\pi, \rho\) – (optional) a list of triples \((k, \pi, \rho)\), where
  - \(\pi\) – a \(k\)-ary operation composing objects in \(A\) and
  - \(\rho\) – a \(k\)-ary function composing statistic values in \(Z\)
- \(\text{elements\_distributions}\) – (optional) a list of pairs \((tA, tZ)\), specifying the distributions of \(tA\)
- \(\text{value\_restrictions}\) – (optional) a list of pairs \((a, tZ)\), restricting the possible values of \(a\)
- \(\text{solver}\) – (optional) the backend used to solve the mixed integer linear programs

\(W\) and \(Z\) can be arbitrary sets. As a natural example we may think of the natural numbers or tuples of integers.

We are looking for a statistic \(s : A \rightarrow Z\) and a bijection \(S : A \rightarrow B\) such that
- \(s = \tau \circ S\): the statistics \(s\) and \(\tau\) are equidistributed and \(S\) is an intertwining bijection.
- \(\alpha = \beta \circ S\): the statistics \(\alpha\) and \(\beta\) are equidistributed and \(S\) is an intertwining bijection.
- \(s\) is constant on the blocks of \(P\).
- \(s(\pi(a_1, \ldots, a_k)) = \rho(s(a_1), \ldots, s(a_k))\).

Additionally, we may require that
- \(s(a) \in Z_a\) for specified sets \(Z_a \subseteq Z\), and
- \(\pi\) has a specified distribution for specified sets \(\tilde{A} \subset A\).

If \(\tau\) is the identity, the two unknown functions \(s\) and \(S\) coincide. Although we do not exclude other bijective choices for \(\tau\), they probably do not make sense.

If we want that \(S\) is graded, i.e. if elements of \(A\) and \(B\) have a notion of size and \(S\) should preserve this size, we can add grading statistics as \(\alpha\) and \(\beta\). Since \(\alpha\) and \(\beta\) will be equidistributed with \(S\) as an intertwining bijection, \(S\) will then also be graded.

In summary, we have the following two commutative diagrams, where \(s\) and \(S\) are unknown functions.

\[
\begin{array}{ccc}
A & \xleftarrow{\alpha} & A^k \\
\downarrow{S} & \swarrow{s} & \downarrow{\pi} \\
B & \xrightarrow{\beta} & Z
\end{array}
\]

\[
\begin{array}{ccc}
A & \xleftarrow{\alpha} & A^k \\
\downarrow{\pi} & \swarrow{s} & \downarrow{\phi} \\
Z & \xrightarrow{\beta} & \tilde{Z}
\end{array}
\]

**Note:** If \(\tau\) is the identity map, the partition \(P\) of \(A\) necessarily consists only of singletons.

**Note:** The order of invocation of the methods with prefix set, i.e., \(\text{set\_statistics()}\), \(\text{set\_intertwining\_relations()}\), \(\text{set\_constant\_blocks()}\), etc., is irrelevant. Calling any of these methods a second time overrides the previous specification.
**constant_blocks** *(singletons=False, optimal=False)*

Return the set partition \( P \) of \( A \) such that \( s : A \to Z \) is known to be constant on the blocks of \( P \).

**INPUT:**

• **singletons** – (optional, default: False) whether or not to include singleton blocks in the output

• **optimal** – (optional, default: False) whether or not to compute the coarsest possible partition

**Note:** computing the coarsest possible partition may be computationally expensive, but may speed up generating solutions.

**EXAMPLES:**

```sage
sage: A = B = ["a", "b", "c"]
sage: bij = Bijectionist(A, B, lambda x: 0)
sage: bij.set_constant_blocks(["a", "b"])
sage: bij.constant_blocks()  
{{'a', 'b'}}

sage: bij.constant_blocks(singletons=True)  
{{'a', 'b'}}, {{'c'}}
```

**minimal_subdistributions_blocks_iterator()**

Return all representatives of minimal subsets \( \tilde{P} \) of \( P \) together with submultisets \( \tilde{Z} \) with \( s(\tilde{P}) = \tilde{Z} \) as multisets.

**Warning:** If there are several solutions with the same support (i.e., the sets of block representatives are the same), only one of these will be found, even if the distributions are different, see the doctest below. To find all solutions, use `minimal_subdistributions_iterator()`, which is, however, computationally more expensive.

**EXAMPLES:**

```sage
sage: A = B = [permutation for n in range(3) for permutation in Permutations(n)]
sage: bij = Bijectionist(A, B, len)
sage: bij.set_statistics((len, len))
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d:tuple(sorted(d.items()))):
   print(solution)
   
   {[], [0], [1], [1, 2], [2], [2, 1], [2]}

Another example:

```sage
sage: N = 2; A = B = [dyck_word for n in range(N+1) for dyck_word in DyckWords(n)]
sage: def tau(D): return D.number_of_touch_points()
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((lambda d: d.semilength(), lambda d: d.semilength()))
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d:
   tuple(sorted(d.items()))):
```

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(continued from previous page)

....: print(solution)
{[]} : 0, [1, :] : 1, [1, 0, 1, 0] : 1, [1, 1, 0, 0] : 2
sage: for subdistribution in bij.minimal_subdistributions_blocks_iterator():
....: print(subdistribution)
([[]], [0])
([1, 0], [1])
([[[1, 0, 1, 0], [1, 1, 0, 0]], [1, 2]])

An example with two elements of the same block in a subdistribution:

sage: A = B = ["a", "b", "c", "d", "e"]

An example with overlapping minimal subdistributions:

Fedor Petrov’s example from https://mathoverflow.net/q/424187:

(continues on next page)
The following solution is not found, because it happens to have the same support as the other:

```python
sage: D = set(A).difference(['b7', 'b8', 'd'])
sage: sorted(a.replace("b", "a") for a in D)
[['a1', 'a2', 'a3', 'a4', 'a5', 'a6', 'a7', 'a8', 'a9', 'a10'], [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1])
```

But it is, by design, included here:

```python
sage: sorted(D) in [d for d in bij.minimal_subdistributions_iterator()]
True
```

### minimal_subdistributions_iterator()

Return all minimal subsets \( \tilde{A} \) of \( A \) together with submultisets \( \tilde{Z} \) with \( s(\tilde{A}) = \tilde{Z} \) as multisets.

#### EXAMPLES:

```python
sage: A = B = [permutation for n in range(3) for permutation in Permutations(n)]
sage: bij = Bijectionist(A, B, len)
sage: bij.set_statistics((len, len))
sage: for sol in bij.solutions_iterator():
    print(sol)
{[]} 0
{[1]} 1
{[1, 2]} 2
{[2, 1]} 2
```

Another example:

```python
sage: N = 2; A = B = [dyck_word for n in range(N+1) for dyck_word in DyckWords(n)]
sage: def tau(D):
    return D.number_of_touch_points()

sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((lambda d: d.semilength(), lambda d: d.semilength()))
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
    print(solution)
{[]} 0
{[1, 0]} 1
{[1, 0, 1, 0]} 2
{[1, 1, 0, 0]} 1
```

print(subdistribution)
([[[]], [0]]
([[[], [1], [1]]
([[[], [1, 0], [1, 1, 0, 0], [1, 2]])

An example with two elements of the same block in a subdistribution:

```
sage: A = B = ['a', 'b', 'c', 'd', 'e']
sage: tau = {'a': 1, 'b': 1, 'c': 2, 'd': 2, 'e': 3}.get
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_constant_blocks([["a", "b"]])
sage: bij.set_value_restrictions("a", [1, 2])
sage: bij.constant_blocks(optimal=True)
{'a': 'b'}
sage: list(bij.minimal_subdistributions_iterator())
[[['a', 'b', 'c', 'd', 'e'], [1, 1, 2, 2, 3]]
```
**Warning:** Any restriction imposed by a previous invocation of `set_constant_blocks()` will be overwritten, including restrictions discovered by `set_intertwining_relations()` and `solutions_iterator()`!

A common example is to use the orbits of a bijection acting on \( A \). This can be achieved using the function `orbit_decomposition()`.

**INPUT:**

- \( P \) – a set partition of \( A \), singletons may be omitted

**EXAMPLES:**

Initially the partitions are set to singleton blocks. The current partition can be reviewed using `constant_blocks()`:

```sage
A = B = 'abcd'
sage: bij = Bijectionist(A, B, lambda x: B.index(x) % 2)
sage: bij.constant_blocks()
{}
sage: bij.set_constant_blocks([[a', 'c']])
sage: bij.constant_blocks()
{[a', 'c']}
```

We now add a map that combines some blocks:

```sage
def pi(p1, p2): return 'abcdefgh'[A.index(p1) + A.index(p2)]
def rho(s1, s2): return (s1 + s2) % 2
sage: bij.set_intertwining_relations((2, pi, rho))
sage: list(bij.solutions_iterator())
[{
'a': 0,
'b': 1,
'c': 0,
'd': 1
}]
sage: bij.constant_blocks()
{[a', 'c'], [b', 'd']}
```

Setting constant blocks overrides any previous assignment:

```sage
sage: bij.set_constant_blocks([[a', 'b']])
sage: bij.constant_blocks()
{[a', 'b']}
```

If there is no solution, and the coarsest partition is requested, an error is raised:

```sage
sage: bij.constant_blocks(optimal=True)
Traceback (most recent call last):
... StopIteration
```

**set_distributions(**`elements_distributions`)**

Specify the distribution of \( s \) for a subset of elements.

**Warning:** Any restriction imposed by a previous invocation of `set_distributions()` will be overwritten!
INPUT:

- one or more pairs of \((\tilde{A}, Z)\), where \(\tilde{A} \subseteq A\) and \(Z\) is a list of values in \(Z\) of the same size as \(\tilde{A}\).

This method specifies that \(\{s(a) | a \in \tilde{A}\}\) equals \(Z\) as a multiset for each of the pairs.

When specifying several distributions, the subsets of \(A\) do not have to be disjoint.

ALGORITHM:

We add

\[
\sum_{a \in \tilde{A}} x_{p(a), z} t^z = \sum_{z \in Z} t^z,
\]

where \(p(a)\) is the block containing \(a\), for each given distribution as a vector equation to the MILP.

EXAMPLES:

```
sage: A = B = [permutation for n in range(4) for permutation in Permutations(n)]
sage: tau = Permutation.longest_increasing_subsequence_length
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((len, len))
sage: bij.set_distributions(([Permutation([1, 2, 3]), Permutation([1, 3, 2])],
    ...  [1, 3]))
sage: for sol in sorted(list(bij.solutions_iterator()), key=lambda d:
    ...    tuple(sorted(d.items()))):
    ....:    print(sol)
    {[], \{\}\}: 0, \{1\}: 1, \{1, 2\}: 1, \{2, 1\}: 2, \{1, 2, 3\}: 3, \{1, 3, 2\}: 1, \{2, 1, 3\}: 2,
    ...
    → \{2, 3, 1\}: 2, \{3, 2, 1\}: 2}
    {[], \{1\}: 1, \{1, 2\}: 1, \{2, 1\}: 2, \{1, 2, 3\}: 3, \{1, 3, 2\}: 1, \{2, 1, 3\}: 2,
    ...
    → \{2, 3, 1\}: 2, \{3, 1, 2\}: 2, \{3, 2, 1\}: 2}
    {[], \{1\}: 1, \{1, 2\}: 1, \{2, 1\}: 2, \{1, 2, 3\}: 3, \{1, 3, 2\}: 1, \{2, 1, 3\}: 2,
    ...
    → \{2, 3, 1\}: 2, \{3, 1, 2\}: 2, \{3, 2, 1\}: 2}
    {[], \{1\}: 1, \{1, 2\}: 2, \{2, 1\}: 1, \{1, 2, 3\}: 3, \{1, 3, 2\}: 1, \{2, 1, 3\}: 2,
    ...
    → \{2, 3, 1\}: 2, \{3, 1, 2\}: 2, \{3, 2, 1\}: 2}

sage: bij.constant_blocks(optimal=True)
{{[2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]}}

sage: sorted(bij.minimal_subdistributions_blocks_iterator(), key=lambda d:
    ...    (len(d[0]), d[0]))

[(([]), [0]),
  ([[]], [1]),
  ([[]], [1, 3]),
  ([1, 2], [2, 1]),
  ([1, 2], [1, 2]),
  ([1, 3], [1, 3], [1, 3]])
```

We may also specify multiple, possibly overlapping distributions:

```
sage: bij.set_distributions(([Permutation([1, 2, 3]), Permutation([1, 3, 2])],
    ...  [1, 3]),
    ...
    ([Permutation([1, 3, 2]), Permutation([3, 2, 1]),
      Permutation([2, 1, 3])], [1, 2, 2]))
sage: for sol in sorted(list(bij.solutions_iterator()), key=lambda d:
    ...    tuple(sorted(d.items()))):
    ....:    print(sol)
    {[], \{\}\}: 0, \{1\}: 1, \{1, 2\}: 1, \{2, 1\}: 2, \{1, 2, 3\}: 3, \{1, 3, 2\}: 1, \{2, 1, 3\}: 2,
    ...
    → \{2, 3, 1\}: 2, \{3, 1, 2\}: 2, \{3, 2, 1\}: 2}
```

(continues on next page)
→ [2, 3, 1]: 2, [3, 1, 2]: 2, [3, 2, 1]: 2

sage: bij.constant_blocks(optimal=True)

{{[1], [1, 3, 2]}, {[2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]}}

sage: sorted(bij.minimal_subdistributions_blocks_iterator(), key=lambda d:

→ (len(d[0]), d[0]))

[([[[]]], [0]), ([[1]], [1]),

→ ([[1, 2, 3]], [3]),

→ ([[2, 3, 1]], [2]),

→ ([[1, 2], [2, 1]], [1, 2])]

set_homomesic(Q)

Assert that the average of $s$ on each block of $Q$ is constant.

INPUT:

• $Q$ – a set partition of $A$

EXAMPLES:

sage: A = B = [1, 2, 3]
sage: bij = Bijectionist(A, B, lambda b: b % 3)
sage: bij.set_homomesic([[1, 2], [3]])
sage: list(bij.solutions_iterator())

[{{1: 2, 2: 0, 3: 1}, {1: 0, 2: 2, 3: 1}}]

set_intertwining_relations(*pi_rho)

Add restrictions of the form $s(\pi(a_1, \ldots, a_k)) = \rho(s(a_1), \ldots, s(a_k))$.

Warning: Any restriction imposed by a previous invocation of set_intertwining_relations() will be overwritten!

INPUT:

• $pi\_rho$ – one or more tuples $(k, \pi: A^k \to A, \rho: Z^k \to Z, \tilde{A})$ where $\tilde{A}$ (optional) is a $k$-ary function that returns true if and only if a $k$-tuple of objects in $A$ is in the domain of $\pi$

ALGORITHM:

The relation

$$s(\pi(a_1, \ldots, a_k)) = \rho(s(a_1), \ldots, s(a_k))$$

for each pair $(\pi, \rho)$ implies immediately that $s(\pi(a_1, \ldots, a_k))$ only depends on the blocks of $a_1, \ldots, a_k$.

The MILP formulation is as follows. Let $a_1, \ldots, a_k \in A$ and let $a = \pi(a_1, \ldots, a_k)$. Let $z_1, \ldots, z_k \in Z$ and let $z = \rho(z_1, \ldots, z_k)$. Suppose that $a_i \in p_i$ for all $i$ and that $a \in p$.

We then want to model the implication

$$x_{p_1, z_1} = 1, \ldots, x_{p_k, z_k} = 1 \Rightarrow x_{p, z} = 1.$$
We achieve this by requiring

\[ x_{p,z} \geq 1 - k + \sum_{i=1}^{k} x_{p_i,z_i}. \]

Note that \( z \) must be a possible value of \( p \) and each \( z_i \) must be a possible value of \( p_i \).

EXAMPLES:

We can concatenate two permutations by increasing the values of the second permutation by the length of the first permutation:

```python
sage: def concat(p1, p2): return Permutation(p1 + [i + len(p1) for i in p2])
```

We may be interested in statistics on permutations which are equidistributed with the number of fixed points, such that concatenating permutations corresponds to adding statistic values:

```python
sage: A = B = [permutation for n in range(4) for permutation in Permutations(n)]
sage: bij = Bijectionist(A, B, Permutation.number_of_fixed_points)
sage: bij.set_statistics((len, len))
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
    ....: print(solution)
```

The domain of the composition may be restricted. E.g., if we concatenate only permutations starting with a 1, we obtain fewer forced elements:

```python
sage: in_domain = lambda p1, p2: (not p1 or p1(1) == 1) and (not p2 or p2(1) == 1)
sage: bij.set_intertwining_relations((2, concat, lambda x, y: x + y))
```
We can also restrict according to several composition functions. For example, we may additionally concatenate permutations by incrementing the elements of the first:

```
sage: skew_concat = lambda p1, p2: Permutation([i + len(p2) for i in p1] + list(p2))
sage: bij.set_intertwining_relations((2, skew_concat, lambda x, y: x + y))
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
...:     print(solution)
```

However, this yields no solution:

```
sage: bij.set_intertwining_relations((2, concat, lambda x, y: x + y), (2, skew_concat, lambda x, y: x + y))
sage: list(bij.solutions_iterator())
[]
```

```
set_quadratic_relation(*phi_psi)
```

Add restrictions of the form $s \circ \psi \circ s = \phi$.

**INPUT:**

- phi_psi – (optional) a list of pairs $(\phi, \rho)$ where $\phi : A \to Z$ and $\psi : Z \to A$

**ALGORITHM:**

We add

$$x_{(p(a), z)} = x_{p(\psi(z)), \phi(a)}$$

for $a \in A$ and $z \in Z$ to the MILP, where $\phi : A \to Z$ and $\psi : Z \to A$. Note that, in particular, $\phi$ must be constant on blocks.
EXAMPLES:

```python
sage: A = B = DyckWords(3)
sage: bij = Bijectionist(A, B)
sage: bij.set_statistics((
    lambda D: D.number_of_touch_points(),
    lambda D: D.
    \rightarrow
    number_of_initial_rises()))
sage: ascii_art(sorted(bij.minimal_subdistributions_iterator()))
[( [ /
    ] )
( [ [ [ ] ] ] [ [ / ] ] [ [ [ ] ] ]]),
( [ [ [ ] ] ] [ [ / ] ] [ [ [ ] ] ]],
( [ [ [ ] ] ] [ [ / ] ] [ [ [ ] ] ]]]
```

```python
sage: bij.set_quadratic_relation((
    lambda D: D,
    lambda D: D))
sage: ascii_art(sorted(bij.minimal_subdistributions_iterator()))
[( [ /
    ] )
( [ [ /[ ] ] [ [ [ ] ] ] ]),
( [ [ [ ] ] ] [ [ / ] ] [ [ [ ] ] ]],
( [ [ [ ] ] ] [ [ / ] ] [ [ [ ] ] ]],
( [ [ [ ] ] ] [ [ / ] ] [ [ [ ] ] ]])
```

```python
set_semi_conjugacy(*pi_rho)

Add restrictions of the form $s(\pi(a_1, \ldots, a_k)) = \rho(s(a_1), \ldots, s(a_k))$.

**Warning:** Any restriction imposed by a previous invocation of `set_intertwining_relations()` will be overwritten!

**INPUT:**

- **pi_rho** – one or more tuples $(k, \pi : A^k \to A, \rho : Z^k \to Z, \tilde{A})$ where $\tilde{A}$ (optional) is a $k$-ary function that returns true if and only if a $k$-tuple of objects in $A$ is in the domain of $\pi$

**ALGORITHM:**

The relation

$$s(\pi(a_1, \ldots, a_k)) = \rho(s(a_1), \ldots, s(a_k))$$

for each pair $(\pi, \rho)$ implies immediately that $s(\pi(a_1, \ldots, a_k))$ only depends on the blocks of $a_1, \ldots, a_k$.

The MILP formulation is as follows. Let $a_1, \ldots, a_k \in A$ and let $a = \pi(a_1, \ldots, a_k)$. Let $z_1, \ldots, z_k \in Z$ and let $z = \rho(z_1, \ldots, z_k)$. Suppose that $a_i \in p_i$ for all $i$ and that $a \in p$.

We then want to model the implication

$$x_{p_1,z_1} = 1, \ldots, x_{p_k,z_k} = 1 \Rightarrow x_{p,z} = 1.$$

5.1. Comprehensive Module List
We achieve this by requiring

\[ x_{p,z} \geq 1 - k + \sum_{i=1}^{k} x_{p_i,z}, \]

Note that \( z \) must be a possible value of \( p \) and each \( z_i \) must be a possible value of \( p_i \).

EXAMPLES:

We can concatenate two permutations by increasing the values of the second permutation by the length of the first permutation:

```python
sage: def concat(p1, p2): return Permutation(p1 + [i + len(p1) for i in p2])
```

We may be interested in statistics on permutations which are equidistributed with the number of fixed points, such that concatenating permutations corresponds to adding statistic values:

```python
sage: A = B = [permutation for n in range(4) for permutation in Permutations(n)]
```

```python
sage: bij = Bijectionist(A, B, Permutation.number_of_fixed_points)
```

```python
sage: bij.set_statistics((len, len))
```

```python
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
  ....
  print(solution)
  ...
{[\]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 1, [3, 1, 2]: 1, [3, 2, 1]: 0}
```

```python
sage: bij.set_intertwining_relations((2, concat, lambda x, y: x + y))
```

```python
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
  ....
  print(solution)
  ...
{[\]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 1, [3, 1, 2]: 1, [3, 2, 1]: 0}
```

The domain of the composition may be restricted. E.g., if we concatenate only permutations starting with a 1, we obtain fewer forced elements:

```python
sage: in_domain = lambda p1, p2: (not p1 or p1(1) == 1) and (not p2 or p2(1) == 1)
```

```python
sage: bij.set_intertwining_relations((2, concat, lambda x, y: x + y, in_domain))
```

```python
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
  ....
  print(solution)
  ...
{[\]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 1, [3, 1, 2]: 1, [3, 2, 1]: 0}
```

(continues on next page)
We can also restrict according to several composition functions. For example, we may additionally concatenate permutations by incrementing the elements of the first:

```python
sage: skew_concat = lambda p1, p2: Permutation([i + len(p2) for i in p1] + list(p2))
```

```python
sage: bij.set_intertwining_relations((2, skew_concat, lambda x, y: x + y))
```

```python
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
    ....:    print(solution)
```

```
{[0]: 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 1, [3, 1, 2]: 0, [3, 2, 1]: 0}
{[0]: 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 1, [3, 1, 2]: 0, [3, 2, 1]: 0}
{[0]: 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 1, [3, 1, 2]: 0, [3, 2, 1]: 0}
```

However, this yields no solution:

```python
sage: bij.set_intertwining_relations((2, concat, lambda x, y: x + y), (2, skew_concat, lambda x, y: x + y))
```

```python
sage: list(bij.solutions_iterator())
[]
```

**set_statistics(**alpha_beta**)**

Set constraints of the form $\alpha = \beta \circ S$.

**Warning:** Any restriction imposed by a previous invocation of `set_statistics()` will be overwritten!

**INPUT:**

- `alpha_beta` – one or more pairs $(\alpha: A \to W, \beta: B \to W)$

If the statistics $\alpha$ and $\beta$ are not equidistributed, an error is raised.

**ALGORITHM:**
We add
\[
\sum_{a \in A, z \in Z} x_{p(a), z} y_{\alpha(a)} = \sum_{b \in B} s_{\tau(b)} t(\beta(b))
\]
as a matrix equation to the MILP.

EXAMPLES:

We look for bijections $S$ on permutations such that the number of weak exceedences of $S(\pi)$ equals the number of descents of $\pi$, and statistics $s$, such that the number of fixed points of $S(\pi)$ equals $s(\pi)$:

```python
sage: N = 4; A = B = [permutation for n in range(N) for permutation in Permutations(n)]
sage: def wex(p):
    return len(p.weak_excedences())
sage: def fix(p):
    return len(p.fixed_points())
sage: def des(p):
    return len(p.descents(final_descent=True)) if p else 0
sage: def adj(p):
    return len([e for (e, f) in zip(p, p[1:]+[0]) if e == f+1])
sage: bij = Bijectionist(A, B, fix)
sage: bij.set_statistics((wex, des), (fix, adj), (len, len))
sage: for solution in sorted(list(bij.solutions_iterator()), key=lambda d: tuple(sorted(d.items()))):
    print(solution)
{[
]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 0, [3, 1, 2]: 3, [3, 2, 1]: 0}
{[
]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 0, [3, 1, 2]: 3, [3, 2, 1]: 0}
{[
]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [2, 3, 1]: 0, [3, 1, 2]: 3, [3, 2, 1]: 0}
{[
]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [3, 1, 2]: 3, [3, 2, 1]: 0}
{[
]: 0, [1]: 1, [1, 2]: 0, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 0, [2, 1, 3]: 0, [3, 1, 2]: 3, [3, 2, 1]: 0}
```

Calling this with non-equidistributed statistics yields an error:

```python
sage: bij = Bijectionist(A, B, fix)
sage: bij.set_statistics((wex, fix))
Traceback (most recent call last):
  ...
ValueError: statistics alpha and beta are not equidistributed
```
set_value_restrictions(*value_restrictions*)
Restrict the set of possible values \( s(a) \) for a given element \( a \).

**Warning:** Any restriction imposed by a previous invocation of `set_value_restrictions()` will be overwritten!

**INPUT:**

- `value_restrictions` – one or more pairs \((a \in A, Z \subseteq Z)\)

**EXAMPLES:**
We may want to restrict the value of a given element to a single or multiple values. We do not require that the specified values are in the image of \( \tau \). In some cases, the restriction may not be able to provide a better solution, as for size 3 in the following example.

```python
sage: A = B = [permutation for n in range(4) for permutation in Permutations(n)]
sage: tau = Permutation.longest_increasing_subsequence_length
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((len, len))
sage: bij.set_value_restrictions((Permutation([1, 2]), [1]),
                          (Permutation([3, 2, 1]), [2, 3, 4]))
sage: for sol in sorted(bij.solutions_iterator(), key=lambda d: sorted(d.items())):
    print(sol)
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 2, [3, 1, 2]: 2, [3, 2, 1]: 3}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 2, [3, 1, 2]: 2, [3, 2, 1]: 2}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 2, [3, 1, 2]: 2, [3, 2, 1]: 2}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 3, [3, 1, 2]: 3, [3, 2, 1]: 2}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 3,
    [-2, 3, 1]: 3, [3, 1, 2]: 3, [3, 2, 1]: 3}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 3, [3, 1, 2]: 3, [3, 2, 1]: 2}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 3,
    [-2, 3, 1]: 3, [3, 1, 2]: 3, [3, 2, 1]: 2}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 3, [3, 1, 2]: 3, [3, 2, 1]: 3}
sage: {[]): 0, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 1, [1, 3, 2]: 2, [2, 1, 3]: 2,
    [-2, 3, 1]: 3, [3, 1, 2]: 3, [3, 2, 1]: 3}
```

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Combinatorics, Release 10.0

{[\emptyset, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 2, [2, 1, 3]: 2, [2, 3, 1]: 2, [3, 1, 2]: 1, [3, 2, 1]: 3}

{[\emptyset, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 2, [2, 1, 3]: 3, [2, 3, 1]: 1, [3, 1, 2]: 2, [3, 2, 1]: 2}

{[\emptyset, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 2, [2, 1, 3]: 2, [2, 3, 1]: 2, [3, 1, 2]: 1, [3, 2, 1]: 2}

{[\emptyset, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 3, [2, 1, 3]: 1, [2, 3, 1]: 2, [3, 1, 2]: 2, [3, 2, 1]: 2}

{[\emptyset, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 2, [1, 3, 2]: 3, [2, 1, 3]: 2, [2, 3, 1]: 1, [3, 1, 2]: 2, [3, 2, 1]: 2}

{[\emptyset, [1]: 1, [1, 2]: 1, [2, 1]: 2, [1, 2, 3]: 3, [1, 3, 2]: 2, [2, 1, 3]: 1, [2, 3, 1]: 2, [3, 1, 2]: 2, [3, 2, 1]: 2}

However, an error occurs if the set of possible values is empty. In this example, the image of \( \tau \) under any legal bijection is disjoint to the specified values.

**solutions_iterator()**

An iterator over all solutions of the problem.

**OUTPUT:** An iterator over all possible mappings \( s : A \rightarrow Z \)

**ALGORITHM:**

We solve an integer linear program with a binary variable \( x_{p, z} \) for each partition block \( p \in P \) and each statistic value \( z \in Z \):

- \( x_{p, z} = 1 \) if and only if \( s(a) = z \) for all \( a \in p \).

Then we add the constraint \( \sum_{x \in V} x < |V| \), where \( V \) is the set containing all \( x \) with \( x = 1 \), that is, those indicator variables representing the current solution. Therefore, a solution of this new program must be different from all those previously obtained.

**INTEGER LINEAR PROGRAM:**

- Let \( n_w(p) \), for a block \( p \) of \( P \), be the multiplicity of the value \( w \) in \( W \) under \( \alpha \), that is, the number of elements \( a \in p \) with \( \alpha(a) = w \).

- Let \( n_w(z) \) be the number of elements \( b \in B \) with \( \beta(b) = w \) and \( \tau(b) = z \) for \( w \in W, z \in Z \).

- Let \( k \) be the arity of a pair \((\pi, \rho)\) in an intertwining relation.

and the following constraints:

- because every block is assigned precisely one value, for all \( p \in P \),

\[ \sum_z x_{p, z} = 1. \]

- because the statistics \( s \) and \( \tau \) and also \( \alpha \) and \( \beta \) are equidistributed, for all \( w \in W \) and \( z \in Z \),
\[
\sum_p m_w(p)x_{p,z} = n_w(z).
\]

- for each intertwining relation \( s(\pi(a_1, \ldots, a_k)) = \rho(s(a_1), \ldots, s(a_r)) \), and for all \( k \)-combinations of blocks \( p_i \in P \) such that there exist \((a_1, \ldots, a_k) \in p_1 \times \cdots \times p_k \) with \( \pi(a_1, \ldots, a_k) \in W \) and \( z = \rho(z_1, \ldots, z_k) \),
  \[ x_{p,z} \geq 1 - k + \sum_{i=1}^{k} x_{p_i,z_i}. \]

- for each distribution restriction, i.e. a set of elements \( \tilde{A} \) and a distribution of values given by integers \( d_z \) representing the multiplicity of each \( z \in Z \), and \( r_p = |p \cap \tilde{A}| \) indicating the relative size of block \( p \) in the set of elements of the distribution,
  \[ \sum_p r_p x_{p,z} = d_z. \]

EXAMPLES:

```python
sage: A = B = 'abc'
sage: bij = Bijectionist(A, B, lambda x: B.index(x) % 2, solver="GLPK")
sage: next(bij.solutions_iterator())
{ 'a': 0, 'b': 1, 'c': 0 }

sage: list(bij.solutions_iterator())
[{ 'a': 0, 'b': 1, 'c': 0 },
 { 'a': 1, 'b': 0, 'c': 0 },
 { 'a': 0, 'b': 0, 'c': 1 }]

sage: N = 4
sage: A = B = [permutation for n in range(N) for permutation in Permutations(n)]
```

Let \( \tau \) be the number of non-left-to-right-maxima of a permutation:

```python
sage: def tau(pi):
    ...:     pi = list(pi)
    ...:     i = count = 0
    ...:     for j in range(len(pi)):
    ...:         if pi[j] > i:
    ...:             i = pi[j]
    ...:     else:
    ...:         count += 1
    ...:     return count
```

We look for a statistic which is constant on conjugacy classes:

```python
sage: P = [list(a) for n in range(N) for a in Permutations(n).conjugacy_classes()]
sage: bij = Bijectionist(A, B, tau, solver="GLPK")
sage: bij.set_statistics((len, len))
sage: bij.set_constant_blocks(P)
sage: for solution in bij.solutions_iterator():
    ...:     print(solution)
{[], [0], [1], [1, 2], [2, 1], [0], [1, 2, 3], [0], [1, 3, 2], [1, 2, 1, 3], [1],
  [3, 2, 1], [1, 2, 3, 1], [2, 3, 1, 2], [2, 3, 1, 3], [2], [3, 1, 2]: 2}
```

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\[
\{[], [1]: 0, [1, 2]: 0, [2, 1]: 1, [1, 2, 3]: 0, [1, 3, 2]: 1, [2, 1, 3]: 1, \\
[3, 2, 1]: 1, [2, 3, 1]: 2, [3, 1, 2]: 2\}
\]

Changing or re-setting problem parameters clears the internal cache. Setting the verbosity prints the MILP which is solved:

```
sage: set_verbose(2)
sage: bij.set_constant_blocks(P)
sage: _ = list(bij.solutions_iterator())
```

Constraints are:

- block \([\,]\): \(1 \leq x_0 \leq 1\)
- block \([1]\): \(1 \leq x_1 \leq 1\)
- block \([1, 2]\): \(1 \leq x_2 + x_3 \leq 1\)
- block \([2, 1]\): \(1 \leq x_4 + x_5 \leq 1\)
- block \([1, 2, 3]\): \(1 \leq x_6 + x_7 + x_8 \leq 1\)
- block \([1, 3, 2]\): \(1 \leq x_9 + x_{10} + x_{11} \leq 1\)
- block \([2, 3, 1]\): \(1 \leq x_{12} + x_{13} + x_{14} \leq 1\)
- statistics: \(1 \leq x_0 \leq 1\)
- statistics: \(0 \leq \leq 0\)
- statistics: \(0 \leq \leq 0\)
- statistics: \(1 \leq x_2 + x_4 \leq 1\)
- statistics: \(1 \leq x_3 + x_5 \leq 1\)
- statistics: \(0 \leq \leq 0\)
- statistics: \(1 \leq x_6 + 3 x_9 + 2 x_{12} \leq 1\)
- statistics: \(3 \leq x_7 + 3 x_{10} + 2 x_{13} \leq 3\)
- statistics: \(2 \leq x_8 + 3 x_{11} + 2 x_{14} \leq 2\)

Variables are:

- \(x_0\): \(s([\,]) = 0\)
- \(x_1\): \(s([1]) = 0\)
- \(x_2\): \(s([1, 2]) = 0\)
- \(x_3\): \(s([1, 2]) = 1\)
- \(x_4\): \(s([2, 1]) = 0\)
- \(x_5\): \(s([2, 1]) = 1\)
- \(x_6\): \(s([1, 2, 3]) = 0\)
- \(x_7\): \(s([1, 2, 3]) = 1\)
- \(x_8\): \(s([1, 2, 3]) = 2\)
- \(x_9\): \(s([1, 3, 2]) = s([2, 1, 3]) = s([3, 2, 1]) = 0\)
- \(x_{10}\): \(s([1, 3, 2]) = s([2, 1, 3]) = s([3, 2, 1]) = 1\)
- \(x_{11}\): \(s([1, 3, 2]) = s([2, 1, 3]) = s([3, 2, 1]) = 2\)
- \(x_{12}\): \(s([2, 3, 1]) = s([3, 1, 2]) = 0\)
- \(x_{13}\): \(s([2, 3, 1]) = s([3, 1, 2]) = 1\)
- \(x_{14}\): \(s([2, 3, 1]) = s([3, 1, 2]) = 2\)

```python
after vetoing
```

Constraints are:

- block \([\,]\): \(1 \leq x_0 \leq 1\)
- block \([1]\): \(1 \leq x_1 \leq 1\)
- block \([1, 2]\): \(1 \leq x_2 + x_3 \leq 1\)
- block \([2, 1]\): \(1 \leq x_4 + x_5 \leq 1\)
- block \([1, 2, 3]\): \(1 \leq x_6 + x_7 + x_8 \leq 1\)
statistics: 1 <= x_0 <= 1
statistics: 0 <= <= 0
statistics: 0 <= <= 0
statistics: 1 <= x_1 <= 1
statistics: 0 <= <= 0
statistics: 0 <= <= 0
statistics: 1 <= x_2 + x_4 <= 1
statistics: 1 <= x_3 + x_5 <= 1
statistics: 0 <= <= 0
statistics: 1 <= x_6 + 3 x_9 + 2 x_12 <= 1
statistics: 3 <= x_7 + 3 x_10 + 2 x_13 <= 3
statistics: 2 <= x_8 + 3 x_11 + 2 x_14 <= 2
veto: x_0 + x_1 + x_3 + x_4 + x_6 + x_10 + x_14 <= 6

after vetoing
Constraints are:

block [1]: 1 <= x_0 <= 1
block [2]: 1 <= x_1 <= 1
block [3]: 1 <= x_2 + x_3 <= 1
block [4]: 1 <= x_4 + x_5 <= 1
block [5]: 1 <= x_6 + x_7 + x_8 <= 1
block [6]: 1 <= x_9 + x_10 + x_11 <= 1
block [7]: 1 <= x_12 + x_13 + x_14 <= 1
statistics: 1 <= x_0 <= 1
statistics: 0 <= <= 0
statistics: 0 <= <= 0
statistics: 1 <= x_1 <= 1
statistics: 0 <= <= 0
statistics: 0 <= <= 0
statistics: 1 <= x_2 + x_4 <= 1
statistics: 1 <= x_3 + x_5 <= 1
statistics: 0 <= <= 0
statistics: 1 <= x_6 + 3 x_9 + 2 x_12 <= 1
statistics: 3 <= x_7 + 3 x_10 + 2 x_13 <= 3
statistics: 2 <= x_8 + 3 x_11 + 2 x_14 <= 2
veto: x_0 + x_1 + x_3 + x_4 + x_6 + x_10 + x_14 <= 6
veto: x_0 + x_1 + x_2 + x_5 + x_6 + x_10 + x_14 <= 6

sage: set_verbose(0)

statistics_fibers()  
Return a dictionary mapping statistic values in W to their preimages in A and B.

This is a (computationally) fast way to obtain a first impression which objects in A should be mapped to which objects in B.

EXAMPLES:

sage: A = B = [permutation for n in range(4) for permutation in Permutations(n)]
sage: tau = Permutation.longest_increasing_subsequence_length
sage: def wex(p):
|      return len(p.weak_excedences())
|      def fix(p):
|      return len(p.fixed_points())
sage: def des(p): return len(p.descents(final_descent=True)) if p else 0
sage: def adj(p): return len([e for (e, f) in zip(p, p[1:]+[0]) if e == f+1])

sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((wex, des), (fix, adj))
sage: a, b = bij.statistics_table()
sage: table(a, header_row=True, frame=True)

<table>
<thead>
<tr>
<th>a</th>
<th>(\alpha_1(a))</th>
<th>(\alpha_2(a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>[]</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>[1]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>[1, 2]</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>[2, 1]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[2, 3]</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>[3, 1]</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>[3, 2]</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>[3, 3]</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

The statistics_table(header=True) method provides information about all elements of \(A\) with corresponding \(\alpha\) values and all elements of \(B\) with corresponding \(\beta\) and \(\tau\) values.

**INPUT:**
- header – (default: True) whether to include a header with the standard Greek letters

**OUTPUT:**
A pair of lists suitable for table, where
- the first contains the elements of \(A\) together with the values of \(\alpha\)
- the second contains the elements of \(B\) together with the values of \(\tau\) and \(\beta\)

**EXAMPLES:**

```python
sage: A = B = [permutation for n in range(4) for permutation in Permutations(n)]
sage: tau = Permutation.longest_increasing_subsequence_length

sage: def wex(p): return len(p.weak_excedences())
sage: def fix(p): return len(p.fixed_points())
sage: def des(p): return len(p.descents(final_descent=True)) if p else 0
sage: def adj(p): return len([e for (e, f) in zip(p, p[1:]+[0]) if e == f+1])
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_statistics((wex, des), (fix, adj))
sage: a, b = bij.statistics_table()
sage: table(a, header_row=True, frame=True)
```
| \([1, 2, 3]\) | 3 | 3 |
|-------------+---+--------+--------|
| \([1, 3, 2]\) | 2 | 1 |
| \([2, 1, 3]\) | 2 | 1 |
| \([2, 3, 1]\) | 2 | 0 |
| \([3, 1, 2]\) | 1 | 0 |
| \([3, 2, 1]\) | 2 | 1 |

```python
sage: table(b, header_row=True, frame=True)
```

<table>
<thead>
<tr>
<th>(b)</th>
<th>(\tau)</th>
<th>(\beta_1(b))</th>
<th>(\beta_2(b))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([])</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([1])</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>([1, 2])</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>([2, 1])</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>([1, 2, 3])</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>([1, 3, 2])</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>([2, 1, 3])</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>([2, 3, 1])</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>([3, 1, 2])</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>([3, 2, 1])</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

### 5.1.9 Binary Recurrence Sequences

This class implements several methods relating to general linear binary recurrence sequences, including a sieve to find perfect powers in integral linear binary recurrence sequences.

**EXAMPLES:**

```python
sage: R = BinaryRecurrenceSequence(1,1)  # the Fibonacci sequence
sage: R(137)  # the 137th term of the Fibonacci sequence
19134702400093278081449423917
sage: R(137) == fibonacci(137)
True
sage: [R(i) % 4 for i in range(12)]
[0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1]
```

(continues on next page)
sage: R.period(4)  # the period of the fibonacci sequence modulo 4
6
sage: R.pthpowers(2, 10**10)  # long time (7 seconds) -- in fact these are all squares, c.f. [BMS06]
[0, 1, 2, 12]
sage: S = BinaryRecurrenceSequence(8, 1) # a Lucas sequence
sage: S.period(73)
148
sage: S(5) % 73 == S(5 + 148) % 73
True
sage: S.pthpowers(3, 10**10)  # long time (3 seconds) -- provably finds the indices of all 3rd powers less than 10^10
[0, 1, 2]
sage: T = BinaryRecurrenceSequence(2, 0, 1, 2)
sage: [T(i) for i in range(10)]
[1, 2, 4, 8, 16, 32, 64, 128, 256, 512]
sage: T.is_degenerate()
True
sage: T.is_geometric()
True
sage: T.pthpowers(7, 10**30)
Traceback (most recent call last):
  ... ValueError: the degenerate binary recurrence sequence is geometric or quasigeometric and has many pth powers

AUTHORS:

- Isabel Vogt (2013): initial version

See [SV2013], [BMS2006], and [SS1983].

```python
class sage.combinat.binary_recurrence_sequences.BinaryRecurrenceSequence(b, c, u0=0, u1=1):
    Bases: SageObject
    Create a linear binary recurrence sequence defined by initial conditions u0 and u1 and recurrence relation
    \[ u_{n+2} = b \cdot u_{n+1} + c \cdot u_n. \]
    INPUT:
    - \( b \) -- an integer (partially determining the recurrence relation)
    - \( c \) -- an integer (partially determining the recurrence relation)
    - \( u0 \) -- an integer (the 0th term of the binary recurrence sequence)
    - \( u1 \) -- an integer (the 1st term of the binary recurrence sequence)
    OUTPUT:
    - An integral linear binary recurrence sequence defined by \( u0, u1, \) and \( u_{n+2} = b \cdot u_{n+1} + c \cdot u_n \)

See also:

- fibonacci(), lucas_number1(), lucas_number2()```
**sage:** R = BinaryRecurrenceSequence(3,3,2,1)
**sage:** R
Binary recurrence sequence defined by: \( u_n = 3 \times u_{n-1} + 3 \times u_{n-2}; \)
With initial conditions: \( u_0 = 2, \) and \( u_1 = 1 \)

**is_arithmetic()**
Decide whether the sequence is degenerate and an arithmetic sequence.

The sequence is arithmetic if and only if
\[
\frac{u_1}{u_0} - \frac{u_0}{u_1} = \frac{u_2}{u_1} - \frac{u_1}{u_2}.
\]
This corresponds to the matrix \( F = \begin{bmatrix} 0 & 1 \\ c & b \end{bmatrix} \) being nondiagonalizable and \( \alpha/\beta = 1. \)

**EXAMPLES:**

**sage:** S = BinaryRecurrenceSequence(2,-1)
**sage:** [S(i) for i in range(10)]
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
**sage:** S.is_arithmetic()
True

**is_degenerate()**
Decide whether the binary recurrence sequence is degenerate.

Let \( \alpha \) and \( \beta \) denote the roots of the characteristic polynomial \( p(x) = x^2 - bx - c. \) Let \( a = \frac{u_1 - u_0 \beta}{(\beta - \alpha)} \) and \( b = \frac{u_1 - u_0 \alpha}{(\beta - \alpha)}. \) The sequence is, thus, given by \( u_n = a \alpha^n - b \beta^n. \) Then we say that the sequence is nondegenerate if and only if \( a \times b \times \alpha \times \beta \neq 0 \) and \( \alpha/\beta \) is not a root of unity.

More concretely, there are 4 classes of degeneracy, that can all be formulated in terms of the matrix \( F = \begin{bmatrix} 0 & 1 \\ c & b \end{bmatrix}. \)

- \( F \) is singular – this corresponds to \( c = 0, \) and thus \( \alpha \times \beta = 0. \) This sequence is geometric after term \( u_0 \) and so we call it quasigeometric.
- \( v = ([u_0], [u_1]) \) is an eigenvector of \( F \) – this corresponds to a geometric sequence with \( a \times b = 0. \)
- \( F \) is nondiagonalizable – this corresponds to \( \alpha = \beta. \) This sequence will be the point-wise product of an arithmetic and geometric sequence.
- \( F^k \) is scaler, for some \( k > 1 \) – this corresponds to \( \alpha/\beta \) a \( k \) th root of unity. This sequence is a union of several geometric sequences, and so we again call it quasigeometric.

**EXAMPLES:**

**sage:** S = BinaryRecurrenceSequence(0,1)
**sage:** S.is_degenerate()
True
**sage:** S.is_geometric()
False
**sage:** S.is_quasigeometric()
True

**sage:** R = BinaryRecurrenceSequence(3,-2)
**sage:** R.is_degenerate()
False

**sage:** T = BinaryRecurrenceSequence(2,-1)
**sage:** T.is_degenerate()
**is_arithmetic()**

Decide whether the binary recurrence sequence is arithmetic - i.e. a geometric sequence.

This is a subcase of a degenerate binary recurrence sequence, for which \(ab = 0\), i.e. \(u_n/u_{n-1} = r\) for some value of \(r\).

See **is_degenerate()** for a description of degeneracy and definitions of \(a\) and \(b\).

**EXAMPLES:**

```
sage: T.is_arithmetic()
True
```

**is_geometric()**

Decide whether the binary recurrence sequence is geometric - i.e a geometric sequence.

This is a subcase of a degenerate binary recurrence sequence, for which \(ab = 0\), i.e. \(u_n/u_{n-1} = r\) for some value of \(r\).

See **is_degenerate()** for a description of degeneracy and definitions of \(a\) and \(b\).

**EXAMPLES:**

```
sage: S = BinaryRecurrenceSequence(2,0,1,2)
sage: [S(i) for i in range(10)]
[1, 2, 4, 8, 16, 32, 64, 128, 256, 512]
sage: S.is_geometric()
True
```

**is_quasigeometric()**

Decide whether the binary recurrence sequence is degenerate and similar to a geometric sequence, i.e. the union of multiple geometric sequences, or geometric after term \(u_0\).

If \(\alpha/\beta\) is a \(k\) th root of unity, where \(k > 1\), then necessarily \(k = 2, 3, 4, 6\). Then \(F = [[0, 1], [c, b]]\) is diagonalizable, and \(F^k = [[\alpha^k, 0], [0, \beta^k]]\) is scaler matrix. Thus for all values of \(j\) mod \(k\), the \(j\) mod \(k\) terms of \(u_n\) form a geometric series.

If \(\alpha\) or \(\beta\) is zero, this implies that \(c = 0\). This is the case when \(F\) is singular. In this case, \(u_1, u_2, u_3, \ldots\) is geometric.

**EXAMPLES:**

```
sage: S = BinaryRecurrenceSequence(0,1)
sage: [S(i) for i in range(10)]
[0, 1, 0, 1, 0, 1, 0, 1, 0, 1]
sage: S.is_quasigeometric()
True
```

**period(m)**

Return the period of the binary recurrence sequence modulo an integer \(m\).

If \(n_1\) is congruent to \(n_2\) modulo **period(m)**, then \(u_{n_1}\) is is congruent to \(u_{n_2}\) modulo \(m\).

**INPUT:**

- \(m\) – an integer (modulo which the period of the recurrence relation is calculated).

**OUTPUT:**

- The integer (the period of the sequence modulo \(m\))
EXAMPLES:

If \( p = \pm 1 \pmod{5} \), then the period of the Fibonacci sequence \( \pmod{p} \) is \( p - 1 \) (c.f. Lemma 3.3 of [BMS2006]).

```
sage: R = BinaryRecurrenceSequence(1,1)
sage: R.period(31)
30
sage: [R(i) % 4 for i in range(12)]
[0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1]
sage: R.period(4)
6
```

This function works for degenerate sequences as well.

```
sage: S = BinaryRecurrenceSequence(2,0,1,2)
sage: S.is_degenerate()
True
sage: S.is_geometric()
True
sage: [S(i) % 17 for i in range(16)]
[1, 2, 4, 8, 16, 15, 13, 9, 1, 2, 4, 8, 16, 15, 13, 9]
sage: S.period(17)
8
```

**Note:** The answer is cached.

### `pthpowers(p, Bound)`

Find the indices of proveably all \( p \)th powers in the recurrence sequence bounded by \( \text{Bound} \).

Let \( u_n \) be a binary recurrence sequence. A \( p \)th power in \( u_n \) is a solution to \( u_n = y^p \) for some integer \( y \). There are only finitely many \( p \)th powers in any recurrence sequence [SS1983].

**INPUT:**

- \( p \) - a rational prime integer (the fixed \( p \) in \( u_n = y^p \))
- \( \text{Bound} \) - a natural number (the maximum index \( n \) in \( u_n = y^p \) that is checked).

**OUTPUT:**

- A list of the indices of all \( p \)th powers less bounded by \( \text{Bound} \). If the sequence is degenerate and there are many \( p \)th powers, raises `ValueError`.

**EXAMPLES:**

```
sage: R = BinaryRecurrenceSequence(1,1)       # the Fibonacci sequence
sage: R.pthpowers(2, 10**10)                  # long time (7 seconds) -- in fact these...
...are all squares, c.f. [BMS2006]_
...[0, 1, 2, 12]

sage: S = BinaryRecurrenceSequence(8,1)      # a Lucas sequence
sage: S.pthpowers(3,10**10)                   # long time (3 seconds) -- provably finds the...
...indices of all 3rd powers less than 10^10
...[0, 1, 2]
```

(continues on next page)
sage: Q = BinaryRecurrenceSequence(3,3,2,1)
sage: Q.pthpowers(11,10**10)  # long time (7.5 seconds)
[1]

If the sequence is degenerate, and there are no \( p \) th powers, returns \([]\). Otherwise, if there are many \( p \) th powers, raises \texttt{ValueError}.

sage: T = BinaryRecurrenceSequence(2,0,1,2)
sage: T.is_degenerate()
True
sage: T.is_geometric()
True
sage: T.pthpowers(7,10**30)
Traceback (most recent call last):
...
ValueError: the degenerate binary recurrence sequence is geometric or quasigeometric and has many \( p \) th powers

sage: L = BinaryRecurrenceSequence(4,0,2,2)
sage: [L(i).factor() for i in range(10)]
[2, 2, 2^3, 2^5, 2^7, 2^9, 2^11, 2^13, 2^15, 2^17]
sage: L.is_quasigeometric()
True
sage: L.pthpowers(2,10**30)
[]

\textbf{Note:} This function is primarily optimized in the range where \texttt{Bound} is much larger than \( p \).

### 5.1.10 Binary Trees

This module deals with binary trees as mathematical (in particular immutable) objects.

\textbf{Note:} If you need the data-structure for example to represent sets or hash tables with AVL trees, you should have a look at \texttt{sage.misc.sagex_ds}.

\textbf{AUTHORS:}

- Florent Hivert (2010-2011): initial implementation.

\texttt{class sage.combinat.binary_tree.BinaryTree(parent, children=None, check=True)}

Bases: \texttt{AbstractClonableTree, ClonableArray}

Binary trees.

Binary trees here mean ordered (a.k.a. plane) finite binary trees, where “ordered” means that the children of each node are ordered.

Binary trees contain nodes and leaves, where each node has two children while each leaf has no children. The number of leaves of a binary tree always equals the number of nodes plus 1.
INPUT:

- **children** – None (default) or a list, tuple or iterable of length 2 of binary trees or convertible objects. This corresponds to the standard recursive definition of a binary tree as either a leaf or a pair of binary trees. Syntax sugar allows leaving out all but the outermost calls of the `BinaryTree()` constructor, so that, e.g., `BinaryTree([BinaryTree(None), BinaryTree(None)])` can be shortened to `BinaryTree([None, None])`. It is also allowed to abbreviate `[None, None]` by `[]`.

- **check** – (default: True) whether check for binary should be performed or not.

EXAMPLES:

```python
sage: BinaryTree()
., .
sage: BinaryTree(None)
., .
sage: BinaryTree([])
[., .]
sage: BinaryTree([None, None])
[., .]
sage: BinaryTree([None, []])
[., [., .]]
sage: BinaryTree([[], None])
[[., .], .]
sage: BinaryTree([None, BinaryTree([None, None])])
[., [., .]]
sage: BinaryTree([[], None, []])
[[., .], [., .]]
```

Traceback (most recent call last):
...
ValueError: this is not a binary tree

`as_ordered_tree(with_leaves=True)`

Return the same tree seen as an ordered tree. By default, leaves are transformed into actual nodes, but this can be avoided by setting the optional variable `with_leaves` to False.

EXAMPLES:

```python
sage: bt = BinaryTree([]); bt
[., .]
sage: bt.as_ordered_tree()
[[], []]
sage: bt.as_ordered_tree(with_leaves = False)
[]
```

`canonical_labelling(shift=1)`

Return a labelled version of self.
The canonical labelling of a binary tree is a certain labelling of the nodes (not the leaves) of the tree. The actual canonical labelling is currently unspecified. However, it is guaranteed to have labels in $1...n$ where $n$ is the number of nodes of the tree. Moreover, two (unlabelled) trees compare as equal if and only if their canonical labelled trees compare as equal.

**EXAMPLES:**

```python
sage: BinaryTree().canonical_labelling()
(...)
sage: BinaryTree([]).canonical_labelling()
[., .]
sage: BinaryTree([[], [None], [[]], [[]]]).canonical_labelling()
5[2[., .], 4[3[., .], .], 7[6[., .], 8[., .]]]
```

canopee()

Return the canopee of self.

The canopee of a non-empty binary tree $T$ with $n$ internal nodes is the list $l$ of 0 and 1 of length $n - 1$ obtained by going along the leaves of $T$ from left to right except the two extremal ones, writing 0 if the leaf is a right leaf and 1 if the leaf is a left leaf.

**EXAMPLES:**

```python
sage: BinaryTree([]).canopee()
[]
sage: BinaryTree([None, []]).canopee()
[1]
sage: BinaryTree([[], None]).canopee()
[0]
sage: BinaryTree([[], []]).canopee()
[0, 1]
sage: BinaryTree([[[], [None]], [[]], [[]]]).canopee()
[0, 1, 0, 0, 1, 0, 1]
```

The number of pairs $(t_1,t_2)$ of binary trees of size $n$ such that the canopee of $t_1$ is the complementary of the canopee of $t_2$ is also the number of Baxter permutations (see [DG1994], see also OEIS sequence A001181). We check this in small cases:

```python
sage: [len([(u,v) for u in BinaryTrees(n) for v in BinaryTrees(n) ....: if [1 - x for x in u.canopee()] == v.canopee()]) ....: for n in range(1, 5)]
[1, 2, 6, 22]
```

Here is a less trivial implementation of this:

```python
sage: from sage.sets.finite_set_map_cy import fibers
sage: def baxter(n):
....:     f = fibers(lambda t: tuple(t.canopee()),
....:               BinaryTrees(n))
....:     return sum(len(f[i])*len(f[tuple(1-x for x in i)])
....:               for i in f)
sage: [baxter(n) for n in range(1, 7)]
[1, 2, 6, 22, 92, 422]
```

check()

Check that self is a binary tree.
EXAMPLES:

```python
sage: BinaryTree([[], []])  # indirect doctest
[[., .], [., .]]
sage: BinaryTree([[], [], []])  # indirect doctest
Traceback (most recent call last):
...
ValueError: this is not a binary tree
sage: BinaryTree([])
# indirect doctest
Traceback (most recent call last):
...
ValueError: this is not a binary tree
```

```python
comb(side='left')
```

Return the comb of a tree.

There are two combs in a binary tree: a left comb and a right comb.

Consider all the vertices of the leftmost (resp. rightmost) branch of the root. The left (resp. right) comb is the list of right (resp. left) subtrees of each of these vertices.

INPUT:

- `side` – (default: `left`) set to `left` to obtain a left comb, and to `right` to obtain a right comb.

OUTPUT:

A list of binary trees.

See also:

`over_decomposition()`, `under_decomposition()`

EXAMPLES:

```python
sage: BT = BinaryTree(' .')
sage: [BT.comb('left'), BT.comb('right')]
[[], []]
sage: BT = BinaryTree('[[,]']

# output omitted
```

```python
sage: BT = BinaryTree('[[[[, [[], [., .]], .]], [[, .]], [[][., .], [], .]], [[, .]], [[, .]], [], []]]')
sage: ascii_art(BT)
________o________
/ \
__o__ o \
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
/ \
__o___o__
(continues on next page)
```

5.1. Comprehensive Module List 101
dendriform_shuffle(other)

Return the list of terms in the dendriform product.

This is the list of all binary trees that can be obtained by identifying the rightmost path in self and the leftmost path in other. Every term corresponds to a shuffle of the vertices on the rightmost path in self and the vertices on the leftmost path in other.

EXAMPLES:

```
sage: u = BinaryTree()
sage: g = BinaryTree([])
sage: l = BinaryTree([g, u])
sage: r = BinaryTree([u, g])
sage: list(g.dendriform_shuffle(g))
[[[., .], .], .]
sage: list(l.dendriform_shuffle(l))
[[[[., .], .], .], [[[., .], .], .]
[[., .], [[., .], .]]
sage: list(l.dendriform_shuffle(r))
[[[[., .], .], .], [[[., .], .], [., .]]]
```

graph(with_leaves=True)

Convert self to a digraph.

By default, this graph contains both nodes and leaves, hence is never empty. To obtain a graph which contains only the nodes, the with_leaves optional keyword variable has to be set to False.
The resulting digraph is endowed with a combinatorial embedding, in order to be displayed correctly.

INPUT:

- `with_leaves` – (default: `True`) a Boolean, determining whether the resulting graph will be formed from the leaves and the nodes of `self` (if `True`), or only from the nodes of `self` (if `False`)

EXAMPLES:

```python
sage: t1 = BinaryTree([[], None])
sage: t1.graph()
Digraph on 5 vertices
sage: t1.graph(with_leaves=False)
Digraph on 2 vertices

sage: t1 = BinaryTree([[], [], None])
sage: t1.graph()
Digraph on 9 vertices
sage: t1.graph().edges(sort=True)
[(0, 1, None), (0, 4, None), (1, 2, None), (1, 3, None), (4, 5, None), (4, 8, None), (5, 6, None), (5, 7, None)]
sage: t1.graph(with_leaves=False)
Digraph on 4 vertices
sage: t1.graph(with_leaves=False).edges(sort=True)
[(0, 1, None), (0, 2, None), (2, 3, None)]
```

The hook number

Return the number of hooks.

Recalling that a branch is a path from a vertex of the tree to a leaf, the leftmost (resp. rightmost) branch of a vertex \( v \) is the branch from \( v \) made only of left (resp. right) edges.

The hook of a vertex \( v \) is a set of vertices formed by the union of \( v \), and the vertices of its leftmost and rightmost branches.

There is a unique way to partition the set of vertices in hooks. The number of hooks in such a partition is the hook number of the tree.

We can obtain this partition recursively by extracting the root’s hook and iterating the processus on each tree of the remaining forest.
EXAMPLES:

```python
sage: BT = BinaryTree( ' .' )
sage: BT.hook_number()
0
sage: BT = BinaryTree( ' [. .] ' )
sage: BT.hook_number()
1
sage: BT = BinaryTree( ' [[[., .], .], [[., .], .]]' ); ascii_art(BT)
    o
  / \  
 o   o
  /   /
 o   o
sage: BT.hook_number()
1
sage: BT = BinaryTree( ' [[[., [. .], .], [[., .], [[., .], .]], [[., .], .]], .]' )
sage: ascii_art(BT)
________o________
|          |
|   _o_    |
|   o   o  |
| / \      |
| o   o  o |
| / \      |
| o   o   o|
| / \      |
| o   o   o|
| / \      |
| o   o   o|
sage: BT.hook_number()
6
```

`in_order_traversal(node_action=None, leaf_action=None)`

Explore the binary tree `self` using the depth-first infix-order traversal algorithm, executing the `node_action` function whenever traversing a node and executing the `leaf_action` function whenever traversing a leaf.

In more detail, what this method does to a tree `T` is the following:

```python
def in_order_traversal(node_action=None, leaf_action=None):
    if the root of `T` is a node:
        apply in_order_traversal to the left subtree of `T`
        (with the same node_action and leaf_action);
        apply node_action to the root of `T`
        apply in_order_traversal to the right subtree of `T`
        (with the same node_action and leaf_action);
    else:
        apply leaf_action to the root of `T`.
```

For example on the following binary tree `T`, where we denote leaves by `a, b, c,...` and nodes by `1, 2, 3,...`:

```
|   ___3___   |
|___________|
```
this method first applies `leaf_action` to `a`, then applies `node_action` to `1`, then `leaf_action` to `b`, then `node_action` to `2`, etc., with the vertices being traversed in the order `a, 1, b, 2, c, 3, d, 4, e, 5, f, 6, g, 7, h, 8, i`.

See `in_order_traversal_iter()` for a version of this algorithm which only iterates through the vertices rather than applying any function to them.

**INPUT:**

- **node_action** – (optional) a function which takes a node in input and does something during the exploration
- **leaf_action** – (optional) a function which takes a leaf in input and does something during the exploration

**in_order_traversal_iter()**

The depth-first infix-order traversal iterator for the binary tree `self`.

This method iterates each vertex (node and leaf alike) of the given binary tree following the depth-first infix order traversal algorithm.

The **depth-first infix order traversal algorithm** iterates through a binary tree as follows:

- iterate through the left subtree (by the depth-first infix order traversal algorithm);
- yield the root;
- iterate through the right subtree (by the depth-first infix order traversal algorithm).

For example on the following binary tree \( T \), where we denote leaves by \( a, b, c, \ldots \) and nodes by \( 1, 2, 3, \ldots \):

```plaintext
|   ___3___   |
| /       \   |
| 1       ___7___   |
| /     /     \     |
| a 2 _5_ 8 |
| / / / / / / |
| b c 4 6 h i |
| / / / / /   |
| d e f g    |
```

the depth-first infix-order traversal algorithm iterates through the vertices of \( T \) in the following order: `a, 1, b, 2, c, 3, d, 4, e, 5, f, 6, g, 7, h, 8, i`.

See `in_order_traversal()` for a version of this algorithm which not only iterates through, but actually does something at the vertices of tree.
is_complete()

Return True if self is complete, else return False.

In a nutshell, a complete binary tree is a perfect binary tree except possibly in the last level, with all nodes in the last level “flush to the left”.

In more detail: A complete binary tree (also called binary heap) is a binary tree in which every level, except possibly the last one (the deepest), is completely filled. At depth $n$, all nodes must be as far left as possible.

For example:

```
|   ___o___   |
|     /     |
|  ___o___   |
|    /     |
|   o     |
|  /     |
| o   |
| /   |
| o |

is not complete but the following ones are:

```
|   ___o___   ___o___   ___o___   |
|     /     /     /     |
|  ___o___   ___o___   ___o___   |
|    /     /     /     |
|   o     o     o     |
|  /     /     /     |
| o   o   o   o   |
| /   /   /   /   |
| o o o o o o o |
```

EXAMPLES:

```
sage: def lst(i):
....:     return [bt for bt in BinaryTrees(i) if bt.is_complete()]
sage: for i in range(8): ascii_art(lst(i)) # long time
```

(continues on next page)
is_empty()

Return whether self is empty.

The notion of emptiness employed here is the one which defines a binary tree to be empty if its root is a leaf. There is precisely one empty binary tree.

EXAMPLES:

```python
sage: BinaryTree().is_empty()
True
sage: BinaryTree([]).is_empty()
False
sage: BinaryTree([[], None]).is_empty()
False
```

is_full()

Return True if self is full, else return False.

A full binary tree is a tree in which every node either has two child nodes or has two child leaves.

This is also known as proper binary tree or 2-tree or strictly binary tree.

For example:

```plaintext
| __o__ |
| / \ |
| o o |
| / \ |
| o o |
| / \ |
| o o |
```

is not full but the next one is:

```plaintext
| ___o___ |
| / \ |
| __o__ o |
| / \ |
| o o |
| / \ / \ |
| o o o o |
```

EXAMPLES:

```python
sage: BinaryTree([[[],None],[None,[]]], []).is_full()
False
sage: BinaryTree([[[[],None],[None,[]]], []]).is_full()
```

(continues on next page)
True
sage: ascii_art([bt for bt in BinaryTrees(5) if bt.is_full()])
[ _o_ , _o_ ]
[ / \ / \ ]
[ o o o o ]
[ / \ / \ ]
[ o o o o ]

is_perfect()

Return True if self is perfect, else return False.

A perfect binary tree is a full tree in which all leaves are at the same depth.

For example:

```
|  ___o___ |
| / \ |
| __o__ o |
| / \ |
| o o |
| / \ / \ |
| o o o o |
```

is not perfect but the next one is:

```
|  __o__ |
| / \ |
| o o |
| / \ / \ |
| o o o o |
```

EXAMPLES:

```
sage: def lst(i):
    ....:     return [bt for bt in BinaryTrees(i) if bt.is_perfect()]
sage: for i in range(8): ascii_art(lst(i)) # long time
[    ]
[ o ]
[    ]
[    ]
[ / \ ]
[ o o ]
[    ]
[    ]
[ _o__ ]
[ / \ ]
[ o o ]
[ / \ / \ ]
[ o o o o ]
```

left_border_symmetry()

Return the tree where a symmetry has been applied recursively on all left borders. If a tree is made of
three trees \([T_1, T_2, T_3]\) on its left border, it becomes \([T'_3, T'_2, T'_1]\) where same symmetry has been applied to \(T_1, T_2, T_3\).

EXAMPLES:

```python
sage: BinaryTree().left_border_symmetry()
.s
sage: BinaryTree([]).left_border_symmetry()
[., .]
sage: BinaryTree([[None, []], None]).left_border_symmetry()
[[., .], [., .]]
```

\(\text{left}_\text{children}_\text{node}_\text{number}(\text{direction}='\text{left}')\)

Return the number of nodes which are left children in \(\text{self}\).

Every node (except the root) is either the left child or the right child of its parent node. The total number of nodes is 1 plus the number of left-children nodes plus the number of right-children nodes.

INPUT:

- \(\text{direction}\) – either \('\text{left}'\) (default) or \('\text{right}'\); if set to \('\text{right}'\), instead count nodes that are right children

EXAMPLES:

```python
sage: bt = BinaryTree([[None, [], None], [[None, None, None]]])
```

\(\text{left}_\text{right}_\text{symmetry}\)

Return the left-right symmetrized tree of \(\text{self}\).

```python
sage: all(5 == 1 + bt.left_children_node_number() + bt.left_children_node_number('right') for bt in BinaryTrees(5))
True
```

EXAMPLES:
sage: BinaryTree().left_right_symmetry()

sage: BinaryTree([]).left_right_symmetry()
[., .]

sage: BinaryTree([],None).left_right_symmetry()
[., [., .]]

sage: BinaryTree([None, []],None).left_right_symmetry()
[., [., .], .]

```
left_rotate()

Return the result of left rotation applied to the binary tree self.

Left rotation on binary trees is defined as follows: Let $T$ be a binary tree such that the right child of the root of $T$ is a node. Let $A$ be the left child of the root of $T$, and let $B$ and $C$ be the left and right children of the right child of the root of $T$. (Keep in mind that nodes of trees are identified with the subtrees consisting of their descendants.) Then, the left rotation of $T$ is the binary tree in which the right child of the root is $C$, whereas the left child of the root is a node whose left and right children are $A$ and $B$. In pictures:
```

```
where asterisks signify a single node each (but $A$, $B$ and $C$ might be empty).

For example,
```
Left rotation is the inverse operation to right rotation (right_rotate()).

See also:

right_rotate()

EXAMPLES:

```
sage: b = BinaryTree([],[],None); ascii_art([b])
[ _o_ ]
[ / \ ]
```
\begin{verbatim}
[ o o ]
[ / ]
[ o ]
sage: ascii_art([b.left_rotate()])
[ o ]
[ / ]
[ o ]
[ / \ ]
[ o o ]
sage: b.left_rotate().right_rotate() == b
True
\end{verbatim}

\textbf{make_leaf()}
Modify \textit{self} so that it becomes a leaf (i.e., an empty tree).

\textbf{Note:} \textit{self} must be in a mutable state.

\textbf{See also:}
\texttt{make_node}

\textbf{EXAMPLES:}
\begin{verbatim}
sage: t = BinaryTree([None, None])
sage: t.make_leaf()
Traceback (most recent call last):
... ValueError: object is immutable; please change a copy instead.
sage: with t.clone() as tl:
....: tl.make_leaf()
sage: t, tl
([., .], .)
\end{verbatim}

\textbf{make_node}(\texttt{child_list}=[None, None])
Modify \textit{self} so that it becomes a node with children \texttt{child_list}.

\textbf{INPUT:}

- \texttt{child_list} – a pair of binary trees (or objects convertible to)

\textbf{Note:} \textit{self} must be in a mutable state.

\textbf{See also:}
\texttt{make_leaf}

\textbf{EXAMPLES:}
\begin{verbatim}
sage: t = BinaryTree()
sage: t.make_node([None, None])
Traceback (most recent call last):
... ValueError: object is immutable; please change a copy instead.
\end{verbatim}
**over(bt)**

Return self over bt, where “over” is the over (/) operation.

If \( T \) and \( T' \) are two binary trees, then \( T \) over \( T' \) (written \( T/T' \)) is defined as the tree obtained by grafting \( T' \) on the rightmost leaf of \( T \). More precisely, \( T/T' \) is defined by identifying the root of the \( T' \) with the rightmost leaf of \( T \). See section 4.5 of [HNT2005].

If \( T \) is empty, then \( T/T' = T' \).

The definition of this “over” operation goes back to Loday-Ronco [LR0102066] (Definition 2.2), but it is denoted by \( \setminus \) and called the “under” operation there. In fact, trees in Sage have their root at the top, contrary to the trees in [LR0102066] which are growing upwards. For this reason, the names of the over and under operations are swapped, in order to keep a graphical meaning. (Our notation follows that of section 4.5 of [HNT2005].)

**See also:**

under()

**EXAMPLES:**

Showing only the nodes of a binary tree, here is an example for the over operation:

```
| o   __o__   _o_  |
| /  / / / \   / / |
| o o o o   o o   |
| \  / /     |
| o o __o__  |
| \  /      |
| o o      |
```

A Sage example:

```
sage: b1 = BinaryTree([[],[],[]])
sage: b2 = BinaryTree([None, [], []])
sage: ascii_art((b1, b2, b1/b2))
( _o_ , _o_ , _o_ )
( / \ / \ / \ / \ )
```


over_decomposition()

Return the unique maximal decomposition as an over product.

This means that the tree is cut along all edges of its rightmost path.

Beware that the factors are ordered starting from the root.

See also:

comb(), under_decomposition()

EXAMPLES:

```
sage: g = BinaryTree([])
sage: r = g.over(g); r
[., [., .]]
sage: l = g.under(g); l
[[., .], .]
sage: r.over_decomposition()
[[., .], [., .]]
sage: l.over_decomposition() == [l]
True
```

```
sage: x = g.over(l).over(l).over(g).over(g)
sage: ascii_art(x)
     _o_
   /   
  o     o
  /     /
  o     o
  /     /
  o     o

sage: x.over_decomposition() == [g,l,l,g,g]
True
```

prune()

Return the binary tree obtained by deleting each leaf of self.

The operation of pruning is the left inverse of attaching as many leaves as possible to each node of a binary tree. That is to say, for all binary trees bt, we have:

```
bt == bt.to_full().prune()
```

However, it is only a right inverse if and only if bt is a full binary tree:

```
bt == bt.prune().to_full()
```

OUTPUT:
A binary tree.

See also:

\texttt{to\_full()}

EXAMPLES:

\begin{verbatim}
  sage: bt = BinaryTree([[None, []], [[], []], None])
  sage: ascii_art(bt)
  o
  /\  
 o o
  / \
 o o o
  sage: ascii_art(bt.prune())
  o
  /\  
 o o
  / \
 o o o
  sage: ascii_art(bt.to_full())
  o
  /\  
 o o
  / \ 
 o o o
  o o
\end{verbatim}

We check the relationship with \texttt{to\_full()}:

\begin{verbatim}
  sage: bt = BinaryTree([[[], [[None, []], []], []], []])
  sage: bt == bt.to_full().prune()
  True
  sage: bt == bt.prune().to_full()
  False
  sage: bt = BinaryTree([[[], [], [], [[], []], [], [], []]])
  sage: bt.is_full()
  True
  sage: bt == bt.prune().to_full()
  True
\end{verbatim}

Pruning the empty tree is again the empty tree:

\begin{verbatim}
  sage: bt = BinaryTree(None)
  sage: bt.prune()
  .
\end{verbatim}

\texttt{q\_hook\_length\_fraction}(q=None, q\_factor=False)

Compute the q-hook length fraction of the binary tree \texttt{self}, with an additional “q-factor” if desired.

If \(T\) is a (plane) binary tree and \(q\) is a polynomial indeterminate over some ring, then the \(q\)-hook length fraction \(h_q(T)\) of \(T\) is defined by

\[ h_q(T) = \frac{\prod_{t \in T} [T_t]_q^q}{[T]_q!}, \]

where the product ranges over all nodes \(t\) of \(T\), where \(T_t\) denotes the subtree of \(T\) consisting of \(t\) and its all descendants, and where for every tree \(S\), we denote by \(|S|\) the number of nodes of \(S\). While this definition only shows that \(h_q(T)\) is a rational function in \(T\), it is in fact easy to show that \(h_q(T)\) is actually...
a polynomial in $T$, and thus makes sense when any element of a commutative ring is substituted for $q$. This can also be explicitly seen from the following recursive formula for $h_q(T)$:

$$h_q(T) = \binom{|T| - 1}{|T_1|}_q h_q(T_1) h_q(T_2),$$

where $T$ is any nonempty binary tree, and $T_1$ and $T_2$ are the two child trees of the root of $T$, and where $\binom{a}{b}_q$ denotes a $q$-binomial coefficient.

A variation of the $q$-hook length fraction is the following “$q$-hook length fraction with $q$-factor”:

$$f_q(T) = h_q(T) \cdot \prod_{t \in T} q^{|T_{\text{right}}(t)|},$$

where for every node $t$, we denote by $\text{right}(t)$ the right child of $t$. This $f_q(T)$ differs from $h_q(T)$ only in a multiplicative factor, which is a power of $q$.

When $q = 1$, both $f_q(T)$ and $h_q(T)$ equal the number of permutations whose binary search tree (see [HNT2005] for the definition) is $T$ (after dropping the labels). For example, there are 20 permutations which give a binary tree of the following shape:

```
| o
| / \
| o o
```

by the binary search insertion algorithm, in accordance with the fact that this tree satisfies $f_1(T) = 20$.

When $q$ is considered as a polynomial indeterminate, $f_q(T)$ is the generating function for all permutations whose binary search tree is $T$ (after dropping the labels) with respect to the number of inversions (i.e., the Coxeter length) of the permutations.

Objects similar to $h_q(T)$ also make sense for general ordered forests (rather than just binary trees), see e.g. [BW1988], Theorem 9.1.

INPUT:

- $q$ – a ring element which is to be substituted as $q$ into the $q$-hook length fraction (by default, this is set to be the indeterminate $q$ in the polynomial ring $\mathbb{Z}[q]$)
- $q$-factor – a Boolean (default: False) which determines whether to compute $h_q(T)$ or to compute $f_q(T)$ (namely, $h_q(T)$ is obtained when $q$-factor == False, and $f_q(T)$ is obtained when $q$-factor == True)

EXAMPLES:

Let us start with a simple example. Actually, let us start with the easiest possible example – the binary tree with only one vertex (which is a leaf):

```
sage: b = BinaryTree()
sage: b.q_hook_length_fraction()
1
sage: b.q_hook_length_fraction(q_factor=True)
1
```

Nothing different for a tree with one node and two leaves:
Let us get to a more interesting tree:

```
sage: b = BinaryTree([[[],[]],[[],None]]); b
[[[., .], [., .]], [., .], [., .]]
sage: b.q_hook_length_fraction()(q=1)
20
sage: b.q_hook_length_fraction(q_factor=True)
q^10 + 2*q^9 + 3*q^8 + 4*q^7 + 4*q^6 + 3*q^5 + 2*q^4 + q^3
sage: b.q_hook_length_fraction(q=2)
465
sage: b.q_hook_length_fraction(q=2, q_factor=True)
3720
```

Let us check the fact that \( f_q(T) \) is the generating function for all permutations whose binary search tree is \( T \) (after dropping the labels) with respect to the number of inversions of the permutations:

```
sage: def q_hook_length_fraction_2(T):
    ....:     P = PolynomialRing(ZZ, 'q').gen()
    ....:     q = P.gen()
    ....:     res = P.zero()
    ....:     for w in T.sylvester_class():
    ....:         res += q ** Permutation(w).length()
    ....:     return res
sage: def test_genfun(i):
    ....:     return all( q_hook_length_fraction_2(T)
    ....:                     == T.q_hook_length_fraction(q_factor=True)
    ....:                     for T in BinaryTrees(i) )
sage: test_genfun(4)
True
```

**right_rotate()**

Return the result of right rotation applied to the binary tree `self`.

Right rotation on binary trees is defined as follows: Let \( T \) be a binary tree such that the left child of the root of \( T \) is a node. Let \( C \) be the right child of the root of \( T \), and let \( A \) and \( B \) be the left and right children of the left child of the root of \( T \). (Keep in mind that nodes of trees are identified with the subtrees consisting of their descendants.) Then, the right rotation of \( T \) is the binary tree in which the left child of the root is \( A \), whereas the right child of the root is a node whose left and right children are \( B \) and \( C \). In pictures:

```
| * 
| / \ 
| * C -right-rotate-> A * |
```
where asterisks signify a single node each (but $A$, $B$ and $C$ might be empty).

For example,

```
| o _o_               _o_               |
| / \                  / \                  |
| o -right-rotate-> o  o -right-rotate-> o |
| / \                  / \                  |
|  o  o                 o  o                 |
<BLANKLINE>
| __o__               _o__               |
| / \                  / \                  |
| o o -right-rotate-> o _o_               |
| / \                  / \                  |
|  o  o                 o  o                 |
| / \                  / \                  |
|  o  o                 o  o                 |
```

Right rotation is the inverse operation to left rotation (`left_rotate()`).

The right rotation operation introduced here is the one defined in Definition 2.1 of [CP2012].

See also:

`left_rotate()`

EXAMPLES:

```
sage: b = BinaryTree([[[[],[]], None]]); ascii_art([b])
[ o ]
[ / ]
[ o ]
[ / \ ]
[ o  o ]
sage: ascii_art([b.right_rotate()])
[ _o_ ]
[ / \ ]
[ o  o ]
[ / ]
[  o  ]
sage: b = BinaryTree([[[[],None],[None,[]]], [[]]]); ascii_art([b])
[ __o__ ]
[ / \ ]
[ o  o ]
[ / \ ]
[ o  o ]
[ / \ ]
[ o  o ]
sage: ascii_art([b.right_rotate()])
[ _o__ ]
[ / \ ]
[ o _o_ ]
```
show(with_leaves=False)
Show the binary tree show, with or without leaves depending on the Boolean keyword variable with_leaves.

Warning: For a labelled binary tree, the labels shown in the picture are not (in general) the ones given by the labelling!
Use _latex_(), view, _ascii_art_() or pretty_print for more faithful representations of the data of the tree.

single_edge_cut_shapes()
Return the list of possible single-edge cut shapes for the binary tree.
This is used in sage.combinat.interval_posets.TamariIntervalPoset.is_new().

OUTPUT:
a list of triples \((m, i, n)\) of integers
This is a list running over all inner edges (i.e., edges joining two non-leaf vertices) of the binary tree. The removal of each inner edge defines two binary trees (connected components), the root-tree and the sub-tree. Thus, to every inner edge, we can assign three positive integers: \(m\) is the node number of the root-tree \(R\), and \(n\) is the node number of the sub-tree \(S\). The integer \(i\) is the index of the leaf of \(R\) on which \(S\) is grafted to obtain the original tree. The leaves of \(R\) are numbered starting from 1 (from left to right), hence \(1 \leq i \leq m + 1\).

In fact, each of \(m\) and \(n\) determines the other, as the total node number of \(R\) and \(S\) is the node number of self.

EXAMPLES:

```python
sage: BT = BinaryTrees(3)
sage: [t.single_edge_cut_shapes() for t in BT]
[[[(2, 3, 1), (1, 2, 2)],
  [(2, 2, 1), (1, 2, 2)],
  [(2, 1, 1), (2, 3, 1)],
  [(2, 2, 1), (1, 1, 2)],
  [(2, 1, 1), (1, 1, 2)]]
```

```python
sage: BT = BinaryTrees(2)
sage: [t.single_edge_cut_shapes() for t in BT]
[[[(1, 2, 1)], [(1, 1, 1)]]]
```

```python
sage: BT = BinaryTrees(1)
sage: [t.single_edge_cut_shapes() for t in BT]
[[[]]]
```

sylvester_class(left_to_right=False)
Iterate over the sylvester class corresponding to the binary tree self.
The sylvester class of a tree $T$ is the set of permutations $\sigma$ whose right-to-left binary search tree (a notion defined in [HNT2005], Definition 7) is $T$ after forgetting the labels. This is an equivalence class of the sylvester congruence (the congruence on words which holds two words $uacbvw$ and $ucabvw$ congruent whenever $a$, $b$, $c$ are letters satisfying $a \leq b < c$, and extends by transitivity) on the symmetric group.

For example the following tree’s sylvester class consists of the permutations $(1, 3, 2)$ and $(3, 1, 2)$:

```
[   o   ]
[  / \  ]
[ o o   ]
```

(only the nodes are drawn here).

The right-to-left binary search tree of a word is constructed by an RSK-like insertion algorithm which proceeds as follows: Start with an empty labelled binary tree, and read the word from right to left. Each time a letter is read from the word, insert this letter in the existing tree using binary search tree insertion ($\text{binary_search_insert()}$). This is what the $\text{binary_search_tree()}$ method computes if it is given the keyword $\text{left_to_right=False}$.

Here are two more descriptions of the sylvester class of a binary search tree:

- The sylvester class of a binary search tree $T$ is the set of all linear extensions of the poset corresponding to $T$ (that is, of the poset whose Hasse diagram is $T$, with the root on top), provided that the nodes of $T$ are labelled with $1, 2, \ldots, n$ in a binary-search-tree way (i.e., every left descendant of a node has a label smaller than that of the node, and every right descendant of a node has a label higher than that of the node).

- The sylvester class of a binary search tree $T$ (with vertex labels $1, 2, \ldots, n$) is the interval $[u, v]$ in the right permutohedron order ($\text{permutohedron_lequal()}$), where $u$ is the 312-avoiding permutation corresponding to $T$ ($\text{to_312_avoiding_permutation()}$), and where $v$ is the 132-avoiding permutation corresponding to $T$ ($\text{to_132_avoiding_permutation()}$).

If the optional keyword variable $\text{left_to_right}$ is set to $\text{True}$, then the left sylvester class of $\text{self}$ is returned instead. This is the set of permutations $\sigma$ whose left-to-right binary search tree (that is, the result of the $\text{binary_search_tree()}$ with $\text{left_to_right}$ set to $\text{True}$) is $\text{self}$. It is an equivalence class of the left sylvester congruence.

**Warning:** This method yields the elements of the sylvester class as raw lists, not as permutations!

**EXAMPLES:**

Verifying the claim that the right-to-left binary search trees of the permutations in the sylvester class of a tree $t$ all equal $t$:

```
sage: def test_bst_of_sc(n, left_to_right):
.....: for t in BinaryTrees(n):
.....:     for p in t.sylvester_class(left_to_right=left_to_right):
.....:         p_per = Permutation(p)
.....:         tree = p_per.binary_search_tree(left_to_right=left_to_right)
.....:         if not BinaryTree(tree) == t:
.....:             return False
.....: return True
sage: test_bst_of_sc(4, False)
True
sage: test_bst_of_sc(5, False)  # long time
True
```
The same with the left-to-right version of binary search:

```python
sage: test_bst_of_sc(4, True)
True
sage: test_bst_of_sc(5, True)  # long time
True
```

Checking that the sylvester class is the set of linear extensions of the poset of the tree:

```python
sage: all( sorted(t.canonical_labelling().sylvester_class()) == sorted(list(v)
....: for v in t.canonical_labelling().to_poset().linear_extensions())
....: for t in BinaryTrees(4) )
True
```

tamari_greater()

The list of all trees greater or equal to self in the Tamari order.

This is the order filter of the Tamari order generated by self.

See tamari_lequal() for the definition of the Tamari poset.

See also:

tamari_smaller()

EXAMPLES:

For example, the tree:

```
   ___o___
  /     /
 o   o __
 /   / / 
o o o o o
```

has these trees greater or equal to it:

```
 o o o o o o o
 /= = = = / / / / /
 o o o o o o o o o o
```

(continues on next page)


tamari_interval(other)

Return the Tamari interval between self and other as a TamariIntervalPoset.

A “Tamari interval” is an interval in the Tamari poset. See tamari_lequal() for the definition of the Tamari poset.

INPUT:

• other – a binary tree greater or equal to self in the Tamari order

EXAMPLES:

sage: bt = BinaryTree([[None, []], None])
sage: ip = bt.tamari_interval(BinaryTree([[None, []], None])); ip
The Tamari interval of size 4 induced by relations [(2, 4), (3, 4), (3, 1), (2, →1)]
sage: ip.lower_binary_tree() [[., [[., .], .]], .]
sage: ip.upper_binary_tree() [[., [[., .], .]], .]
sage: ip.interval_cardinality()
4
sage: ip.number_of_tamari_inversions()
2
sage: list(ip.binary_trees())
[[[., [[., .], .]], .],
 [[., [[., .], .]], .],
 [[., [[., .], .]], .],
 [[., [[., .], .]], .]]
sage: bt.tamari_interval(BinaryTree([[None, []], []]))
Traceback (most recent call last):
...
ValueError: the two binary trees are not comparable on the Tamari lattice

tamari_join(other)

Return the join of the binary trees self and other (of equal size) in the n-th Tamari poset (where n is the size of these trees).

The n-th Tamari poset (defined in tamari_lequal()) is known to be a lattice, and the map from the n-th symmetric group S_n to the n-th Tamari poset defined by sending every permutation p ∈ S_n to the binary search tree of p (more precisely, to p.binary_search_tree_shape()) is a lattice homomorphism. (See Theorem 6.2 in [Rea2004].)

See also:
tamari_lequal(), tamari_meet().
AUTHORS:
Viviane Pons and Darij Grinberg, 18 June 2014; Frédéric Chapoton, 9 January 2018.

EXAMPLES:

```
sage: a = BinaryTree([None, [None, []]])
sage: b = BinaryTree([None, [(), None]])
sage: c = BinaryTree([[None, []], None])
sage: d = BinaryTree([[], None], None])
sage: e = BinaryTree([], [],])
sage: a.tamari_join(c) == a
True
sage: b.tamari_join(c) == b
True
sage: c.tamari_join(e) == a
True
sage: d.tamari_join(e) == e
True
sage: e.tamari_join(b) == a
True
sage: e.tamari_join(a) == a
True
```

```
sage: b1 = BinaryTree([None, [None], [None], None])
sage: b2 = BinaryTree([[], None], None])
sage: b1.tamari_join(b2)
[., [., [., .]], [., .]]
sage: b3 = BinaryTree([[], [], None]))
sage: b1.tamari_join(b3)
[., [., [., .]], [., .]]
sage: b2.tamari_join(b3)
[., [., [., .]], [., .]]
```

The universal property of the meet operation is satisfied:

```
sage: def test_uni_join(p, q):
    ....:    j = p.tamari_join(q)
    ....:    if not p.tamari_lequal(j):
    ....:       return False
    ....:    if not q.tamari_lequal(j):
    ....:       return False
    ....:    for r in p.tamari_greater():
    ....:       if q.tamari_lequal(r) and not j.tamari_lequal(r):
    ....:          return False
    ....:    return True
sage: all( test_uni_join(p, q) for p in BinaryTrees(3) for q in BinaryTrees(3) )
True
sage: p = BinaryTrees(6).random_element(); q = BinaryTrees(6).random_element();
# test_uni_join(p, q)
True
```

Border cases:
sage: b = BinaryTree(None)
sage: b.tamari_join(b)
.
sage: b = BinaryTree([])
sage: b.tamari_join(b)
[., .]

```
tamari_lequal(t2)

Return True if self is less or equal to another binary tree t2 (of the same size as self) in the Tamari order.

The Tamari order on binary trees of size n is the partial order on the set of all binary trees of size n generated by the following requirement: If a binary tree $T'$ is obtained by right rotation (see right_rotate()) from a binary tree $T$, then $T < T'$. This not only is a well-defined partial order, but actually is a lattice structure on the set of binary trees of size $n$, and is a quotient of the weak order on the $n$-th symmetric group (also known as the right permutohedron order, see permutohedron_lequal()). See [CP2012]. The set of binary trees of size $n$ equipped with the Tamari order is called the $n$-th Tamari poset.

The Tamari order can equivalently be defined as follows:

If $T$ and $S$ are two binary trees of size $n$, then the following four statements are equivalent:

- We have $T \leq S$ in the Tamari order.
- There exist elements $t$ and $s$ of the Sylvester classes (sylvester_class()) of $T$ and $S$, respectively, such that $t \leq s$ in the weak order on the symmetric group.
- The 132-avoiding permutation corresponding to $T$ (see to_132_avoiding_permutation()) is $\leq$ to the 132-avoiding permutation corresponding to $S$ in the weak order on the symmetric group.
- The 312-avoiding permutation corresponding to $T$ (see to_312_avoiding_permutation()) is $\leq$ to the 312-avoiding permutation corresponding to $S$ in the weak order on the symmetric group.

See also:
tamari_smaller(), tamari_greater(), tamari_pred(), tamari_succ(), tamari_interval()

EXAMPLES:

This tree:

```
    o
   / \  
  o   o
 /    
 o
 /  
 o
```

is Tamari-$\leq$ to the following tree:

```
    o
   / \  
  o   o
 /    
 o
 /  
 o  o
```

Checking this:
Combinatorics, Release 10.0

```
sage: b = BinaryTree([[[], []], None, []])
sage: c = BinaryTree([[[], []], [None, []]])
sage: b.tamari_lequal(c)
True
```

tamari_meet(other, side='right')

Return the meet of the binary trees self and other (of equal size) in the n-th Tamari poset (where n is the size of these trees).

The n-th Tamari poset (defined in tamari_lequal()) is known to be a lattice, and the map from the n-th symmetric group S_n to the n-th Tamari poset defined by sending every permutation p ∈ S_n to the binary search tree of p (more precisely, to p.binary_search_tree_shape()) is a lattice homomorphism. (See Theorem 6.2 in [Rea2004].)

See also:
tamari_lequal(), tamari_join().

AUTHORS:

Viviane Pons and Darij Grinberg, 18 June 2014.

EXAMPLES:

```
sage: a = BinaryTree([None, [None, []]])
sage: b = BinaryTree([None, [], None])
sage: c = BinaryTree([None, [], None])
sage: d = BinaryTree([[], None, None])
sage: e = BinaryTree([[], [], []])
sage: a.tamari_meet(c) == c
True
sage: b.tamari_meet(c) == c
True
sage: c.tamari_meet(e) == d
True
sage: d.tamari_meet(e) == d
True
sage: e.tamari_meet(b) == d
True
sage: e.tamari_meet(a) == e
True
```

```
sage: b1 = BinaryTree([None, [[], None, None]])
sage: b2 = BinaryTree([[], None, []])
sage: b1.tamari_meet(b2)
[[[[., .], .], .], .]
sage: b3 = BinaryTree([[], [[], None]])
sage: b1.tamari_meet(b3)
[[[[., .], .], .], .]
sage: b2.tamari_meet(b3)
[[[[., .], .], .], .]
```

The universal property of the meet operation is satisfied:

```
sage: def test_uni_meet(p, q):
....:     m = p.tamari_meet(q)
(continues on next page)```
....: if not m.tamari_lequal(p):
....:     return False
....: if not m.tamari_lequal(q):
....:     return False
....: for r in p.tamari_smaller():
....:     if r.tamari_lequal(q) and not r.tamari_lequal(m):
....:         return False
....: return True
sage: all( test_uni_meet(p, q) for p in BinaryTrees(3) for q in BinaryTrees(3) )
True
sage: p = BinaryTrees(6).random_element(); q = BinaryTrees(6).random_element();
˓→test_uni_meet(p, q)
True

Border cases:

sage: b = BinaryTree(None)
sage: b.tamari_meet(b)
...
sage: b = BinaryTree([])
sage: b.tamari_meet(b)
[., .]

tamari_pred()

Compute the list of predecessors of self in the Tamari poset.

This list is computed by performing all left rotates possible on its nodes.

See tamari_lequal() for the definition of the Tamari poset.

EXAMPLES:

For this tree:

```
  __o__
 / \
| o o |
| / \ |
| o o o |
```

the list is:

```
  _o_, _o_]
 | / / |
| _o_ o o |
| / \ / / |
| o o o o |
```

tamari_smaller()

The list of all trees smaller or equal to self in the Tamari order.

This is the order ideal of the Tamari order generated by self.

See tamari_lequal() for the definition of the Tamari poset.
See also:

tamari_greater()

EXAMPLES:
The tree:

```
     __o__
     /   \
    o     o
   /     /\ 
  o     o  o
```

has these trees smaller or equal to it:

```
     __o__, _o_, o, o, o, o
     /   /\ / \ / / / \
    o   o  o  o  o  o  o
   /   / / / / / / / \
  o   o o o o o o o  o
 /   / / / / / / / \\
 o   o o o o o o o  o
/   / / / / / / \
 o   o o o o o o  o
/  / / / / \\ 
 o  o o o o \\
/ / / \
 o o o  \\
/ \\
 o  
```

tamari_sorting_tuple(reverse=False)

Return the Tamari sorting tuple of self and the size of self.

This is a pair \((w, n)\), where \(n\) is the number of nodes of self, and \(w\) is an \(n\)-tuple whose \(i\)-th entry is the number of all nodes among the descendants of the right child of the \(i\)-th node of self. Here, the nodes of self are numbered from left to right.

INPUT:

• reverse – boolean (default False) if True, return instead the result for the left-right symmetric of the binary tree

OUTPUT:
a pair \((w, n)\), where \(w\) is a tuple of integers, and \(n\) the size

Two binary trees of the same size are comparable in the Tamari order if and only if the associated tuples \(w\) are componentwise comparable. (This is essentially the Theorem in [HT1972].) This is used in tamari_lequal().

EXAMPLES:

```
sage: [t.tamari_sorting_tuple() for t in BinaryTrees(3)]
[((2, 1, 0), 3),
 (2, 0, 0), 3),
 ((0, 1, 0), 3),
 ((1, 0, 0), 3),
 ((0, 0, 0), 3)]
```

sage: t = BinaryTrees(10).random_element()

(continues on next page)
sage: u = t.left_right_symmetry()
sage: t.tamari_sorting_tuple(True) == u.tamari_sorting_tuple()
True

REFERENCES:

• [HT1972]

tamari_succ()
Compute the list of successors of self in the Tamari poset.
This is the list of all trees obtained by a right rotate of one of its nodes.
See tamari_lequal() for the definition of the Tamari poset.

EXAMPLES:
The list of successors of:

```
| _o__ |
| / \
| o o |
```

is:

```
| _o__ , ___o___ , _o_ |
| / \
| o _o_ o o o o |
```

to_132_avoiding_permutation()
Return a 132-avoiding permutation corresponding to the binary tree.
The linear extensions of a binary tree form an interval of the weak order called the sylvester class of the
tree. This permutation is the maximal element of this sylvester class.

EXAMPLES:

```
sage: bt = BinaryTree([[[]], []])
sage: bt.to_132_avoiding_permutation()
[3, 1, 2]
sage: bt = BinaryTree([[[]], [[], None], [[]], []])
sage: bt.to_132_avoiding_permutation()
[8, 6, 7, 3, 4, 1, 2, 5]
```

to_312_avoiding_permutation()
Return a 312-avoiding permutation corresponding to the binary tree.
The linear extensions of a binary tree form an interval of the weak order called the sylvester class of the
tree. This permutation is the minimal element of this sylvester class.

EXAMPLES:
Combinatorics, Release 10.0

```
sage: bt = BinaryTree([[],[]])
sage: bt.to_312_avoiding_permutation()
[1, 3, 2]
sage: bt = BinaryTree([[],[],None], [[], []])
sage: bt.to_312_avoiding_permutation()
[1, 3, 4, 2, 6, 8, 7, 5]
```

to_dyck_word(usemap='1LOR')

Return the Dyck word associated with self using the given map.

INPUT:

- usemap – a string, either 1L0R, 1R0L, L1R0, R1L0

The bijection is defined recursively as follows:

- a leaf is associated to the empty Dyck Word
- a tree with children \( l, r \) is associated with the Dyck word described by usemap where \( L \) and \( R \) are respectively the Dyck words associated with the trees \( l \) and \( r \).

EXAMPLES:

```
sage: BinaryTree().to_dyck_word()
[]
sage: BinaryTree([]).to_dyck_word()
[1, 0]
sage: BinaryTree([[[], []], [None], [[], []]]).to_dyck_word()
[1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0]
sage: BinaryTree([None, []], None).to_dyck_word()
[1, 1, 0, 1, 0, 0]
sage: BinaryTree([None, []], None).to_dyck_word("1R0L")
[1, 0, 1, 1, 0, 0]
sage: BinaryTree([None, []], None).to_dyck_word("L1R0")
[1, 1, 0, 1, 0, 0]
sage: BinaryTree([None, []], None).to_dyck_word("R1L0")
[1, 1, 0, 1, 0, 0]
sage: BinaryTree([None, []], None).to_dyck_word("R10L")
Traceback (most recent call last):
...
ValueError: R10L is not a correct map
```

to_dyck_word_tamari()

Return the Dyck word associated with self in consistency with the Tamari order on Dyck words and binary trees.

The bijection is defined recursively as follows:

- a leaf is associated with an empty Dyck word;
- a tree with children \( l, r \) is associated with the Dyck word \( T(l)1T(r)0 \).

EXAMPLES:

```
sage: BinaryTree().to_dyck_word_tamari()
[]
sage: BinaryTree([]).to_dyck_word_tamari()
```

(continues on next page)
to_full()

Return the full binary tree constructed from self.

Let $T$ be a binary tree with $n$ nodes. We construct a full binary tree $F$ from $T$ by attaching a leaf to each node of $T$ which does not have 2 children. The resulting tree will have $2n + 1$ nodes.

OUTPUT:

A full binary tree. See is_full() for the definition of full.

See also:

prune()

EXAMPLES:

```python
sage: bt = BinaryTree([[None, []], None])
sage: bt.to_full().is_full()
True
sage: ascii_art(bt)
    o
   /|
  o o
sage: ascii_art(bt.to_full())
__o___
/ |
_o o
/ \ 
0 0
 / \ 
0 0
sage: bt = BinaryTree([[[]], [[]]])
sage: ascii_art(bt)
   o
  /|
 o o
/ \ 
0 0
/   
0 0
sage: ascii_art(bt.to_full())
__o__
/ |
0 o
/ \ 
0 0
/   
0 0 0
sage: BinaryTree(None).to_full()
[., .]
```
to_ordered_tree_left_branch()

Return an ordered tree of size \( n + 1 \) by the following recursive rule:

- if \( x \) is the left child of \( y \), \( x \) becomes the left brother of \( y \)
- if \( x \) is the right child of \( y \), \( x \) becomes the last child of \( y \)

EXAMPLES:

```python
sage: bt = BinaryTree([[],[]])
sage: bt.to_ordered_tree_left_branch()
[[[], []]]
sage: bt = BinaryTree([[], [], None], [[], [], []])
sage: bt.to_ordered_tree_left_branch()
[[[], [], []], [[], [], []]]
```

to_ordered_tree_right_branch()

Return an ordered tree of size \( n + 1 \) by the following recursive rule:

- if \( x \) is the right child of \( y \), \( x \) becomes the right brother of \( y \)
- if \( x \) is the left child of \( y \), \( x \) becomes the first child of \( y \)

EXAMPLES:

```python
sage: bt = BinaryTree([[],[]])
sage: bt.to_ordered_tree_right_branch()
[[[], []]]
sage: bt = BinaryTree([[], [], None], [[], [], []])
sage: bt.to_ordered_tree_right_branch()
[[[], [], []], [[], [], []]]
```

to_poset(with_leaves=False, root_to_leaf=False)

Return the poset obtained by interpreting the tree as a Hasse diagram.

The default orientation is from leaves to root but you can pass `root_to_leaf=True` to obtain the inverse orientation.

Leaves are ignored by default, but one can set `with_leaves` to `True` to obtain the poset of the complete tree.

INPUT:

- `with_leaves` – (default: `False`) a Boolean, determining whether the resulting poset will be formed from the leaves and the nodes of `self` (if `True`), or only from the nodes of `self` (if `False`)
- `root_to_leaf` – (default: `False`) a Boolean, determining whether the poset orientation should be from root to leaves (if `True`) or from leaves to root (if `False`)

EXAMPLES:

```python
sage: bt = BinaryTree([])
sage: bt.to_poset()
Finite poset containing 1 elements
sage: bt.to_poset(with_leaves=True)
Finite poset containing 3 elements
sage: P1 = bt.to_poset(with_leaves=True)
sage: len(P1.maximal_elements())
1
```
sage: len(P1.minimal_elements())
2
sage: bt = BinaryTree([])
sage: P2 = bt.to_poset(with_leaves=True, root_to_leaf=True)
sage: len(P2.maximal_elements())
2
sage: len(P2.minimal_elements())
1

If the tree is labelled, we use its labelling to label the poset. Otherwise, we use the poset canonical labelling:

sage: bt = BinaryTree([[],[None,[]]]).canonical_labelling()
sage: bt
2[1[., .], 3[., 4[., .]]]
sage: bt.to_poset().cover_relations()
[[4, 3], [3, 2], [1, 2]]

Let us check that the empty binary tree is correctly handled:

sage: bt = BinaryTree()
sage: bt.to_poset()
Finite poset containing 0 elements
sage: bt.to_poset(with_leaves=True)
Finite poset containing 1 elements

to_tilting()

Transform a binary tree into a tilting object.

Let $t$ be a binary tree with $n$ nodes. There exists a unique depiction of $t$ (above the diagonal) such that all leaves are regularly distributed on the diagonal line from $(0, 0)$ to $(n, n)$ and all edges are either horizontal or vertical. This method provides the coordinates of this depiction, with the root as the top-left vertex.

OUTPUT:

a list of pairs of integers.

Every vertex of the binary tree is mapped to a pair of integers. The conventions are the following. The root has coordinates $(0, n)$ where $n$ is the node number. If a vertex is the left (right) son of another vertex, they share the first (second) coordinate.

EXAMPLES:

sage: t = BinaryTrees(1)[0]
sage: t.to_tilting()
[(0, 1)]

sage: for t in BinaryTrees(2):
....:     print(t.to_tilting())
[(1, 2), (0, 2)]
[(0, 1), (0, 2)]

sage: from sage.combinat.abstract_tree import from_hexacode
sage: t = from_hexacode('202220220000', BinaryTrees())
sage: print(t.to_tilting())
[(0, 1), (2, 3), (4, 5), (6, 7), (4, 7), (8, 9), (10, 11),
(continues on next page)
(continued from previous page)

\[(8, 11), (4, 11), (12, 13), (4, 13), (2, 13), (0, 13)\]

```python
sage: t2 = DyckWord([1,1,1,1,0,1,1,0,0,0,1,1,0,1,0,1,1,0,1,1,0,0,0,0,0,0]).to_binary_tree()
sage: len(t2.to_tilting()) == t2.node_number()
True
```

to_undirected_graph \((with\_leaves=False)\)

Return the undirected graph obtained from the tree nodes and edges.

Leaves are ignored by default, but one can set \texttt{with\_leaves} to \texttt{True} to obtain the graph of the complete tree.

\textbf{INPUT:}

- \texttt{with\_leaves} – (default: \texttt{False}) a Boolean, determining whether the resulting graph will be formed from the leaves and the nodes of \texttt{self} (if \texttt{True}), or only from the nodes of \texttt{self} (if \texttt{False})

\textbf{EXAMPLES:}

```python
sage: bt = BinaryTree([])
sage: bt.to_undirected_graph()
Graph on 1 vertex
sage: bt.to_undirected_graph(with_leaves=True)
Graph on 3 vertices
sage: bt = BinaryTree()
sage: bt.to_undirected_graph()
Graph on 0 vertices
sage: bt.to_undirected_graph(with_leaves=True)
Graph on 1 vertex
```

If the tree is labelled, we use its labelling to label the graph. Otherwise, we use the graph canonical labelling which means that two different trees can have the same graph.

\textbf{EXAMPLES:}

```python
sage: bt = BinaryTree([], [None, []])
sage: bt.canonical_labelling().to_undirected_graph() == bt.to_undirected_graph()
False
sage: BinaryTree([], []).to_undirected_graph() == BinaryTree([], [], []).to_undirected_graph()
True
```

twisting_number()

Return a pair (number of maximal left branches, number of maximal right branches).

Recalling that a branch of a vertex \(v\) is a path from a vertex of the tree to a leaf, a left (resp. right) branch is a branch made only of left (resp. right) edges. The length of a branch is the number of edges composing it. A left (resp. right) branch is maximal if it is not included in a strictly longer left (resp. right) branch.

\textbf{OUTPUT:}

A list of two integers

\textbf{EXAMPLES:}
Return `self` under `bt`, where “under” is the under (\) operation.

If `T` and `T'` are two binary trees, then `T` under `T'` (written `T \ T'`) is defined as the tree obtained by grafting `T` on the leftmost leaf of `T'`. More precisely, `T \ T'` is defined by identifying the root of `T` with the leftmost leaf of `T'`.

If `T'` is empty, then `T \ T' = T`.

The definition of this “under” operation goes back to Loday-Ronco [LR0102066] (Definition 2.2), but it is denoted by / and called the “over” operation there. In fact, trees in sage have their root at the top, contrary to the trees in [LR0102066] which are growing upwards. For this reason, the names of the over and under...
operations are swapped, in order to keep a graphical meaning. (Our notation follows that of section 4.5 of [HNT2005].)

See also:

\texttt{over()}

EXAMPLES:

Showing only the nodes of a binary tree, here is an example for the under operation:

\begin{Verbatim}
\texttt{sage: b1 = BinaryTree([[],[]])}
\texttt{sage: b2 = BinaryTree([None,[]])}
\texttt{sage: ascii_art((b1, b2, b1 \ b2))}
\end{Verbatim}

\begin{verbatim}
( o , o , _o_ )
( / \ \ / \ )
( o o o o o )
( / \ )
( o o )
\end{verbatim}

\texttt{under\_decomposition()}

Return the unique maximal decomposition as an under product.

This means that the tree is cut along all edges of its leftmost path.

Beware that the factors are ordered starting from the root.

See also:

\texttt{comb()}, \texttt{over\_decomposition()}

EXAMPLES:

\begin{Verbatim}
\texttt{sage: g = BinaryTree([])}
\texttt{sage: r = g.over(g); r}
[., [., .]]
\texttt{sage: l = g.under(g); l}
[[., .], .]
\texttt{sage: l.under\_decomposition()}
[[., .], [., .]]
\texttt{sage: r.under\_decomposition() == [r]}
True
\texttt{sage: x = r.under(g).under(r).under(g)}
\texttt{sage: ascii\_art(x)}
\begin{verbatim}
 o
 / 
 o
 / \ 
 o o
 / o
 \ o
\end{verbatim}
\texttt{sage: x.under\_decomposition() == [g,r,g,r]}
True
\end{Verbatim}
class sage.combinat.binary_tree.BinaryTrees
    Bases: UniqueRepresentation, Parent

Factory for binary trees.

A binary tree is a tree with at most 2 children. The binary trees considered here are also ordered (a.k.a. planar), that is to say, their children are ordered.

A full binary tree is a binary tree with no nodes with 1 child.

INPUT:

- size – (optional) an integer
- full – (optional) a boolean

OUTPUT:

The set of all (full if full=True) binary trees (of the given size if specified).

See also:

BinaryTree, BinaryTree.is_full()

EXAMPLES:

sage: BinaryTrees()
Binary trees

sage: BinaryTrees(2)
Binary trees of size 2

sage: BinaryTrees(full=True)
Full binary trees

sage: BinaryTrees(3, full=True)
Full binary trees of size 3

sage: BinaryTrees(4, full=True)
Traceback (most recent call last):
... ValueError: n must be 0 or odd

Note: This is a factory class whose constructor returns instances of subclasses.

Note: The fact that BinaryTrees is a class instead of a simple callable is an implementation detail. It could be changed in the future and one should not rely on it.

leaf()

Return a leaf tree with self as parent.

EXAMPLES:

sage: BinaryTrees().leaf()
class sage.combinat.binary_tree.BinaryTrees_all
    Bases: DisjointUnionEnumeratedSets, BinaryTrees

    Element
    alias of BinaryTree

    labelled_trees()
    Return the set of labelled trees associated to self.
    EXAMPLES:
    sage: BinaryTrees().labelled_trees()
    Labelled binary trees

    unlabelled_trees()
    Return the set of unlabelled trees associated to self.
    EXAMPLES:
    sage: BinaryTrees().unlabelled_trees()
    Binary trees

class sage.combinat.binary_tree.BinaryTrees_size(size)
    Bases: BinaryTrees

    The enumerated sets of binary trees of given size.

    cardinality()
    The cardinality of self
    This is a Catalan number.

    random_element()
    Return a random BinaryTree with uniform probability.
    This method generates a random DyckWord and then uses a bijection between Dyck words and binary trees.
    EXAMPLES:
    sage: BinaryTrees(5).random_element() # random
    [., [., [., [., [., .]]]]]
    sage: BinaryTrees(0).random_element()  
    .
    sage: BinaryTrees(1).random_element()  
    [., .]

class sage.combinat.binary_tree.FullBinaryTrees_all
    Bases: DisjointUnionEnumeratedSets, BinaryTrees

    All full binary trees.

class sage.combinat.binary_tree.FullBinaryTrees_size(size)
    Bases: BinaryTrees

    Full binary trees of a fixed size (number of nodes).

    cardinality()
    The cardinality of self
    This is a Catalan number.
random_element()

Return a random FullBinaryTree with uniform probability.

This method generates a random DyckWord of size \((s - 1)/2\), where \(s\) is the size of self, which uses a bijection between Dyck words and binary trees to get a binary tree, and convert it to a full binary tree.

EXAMPLES:

```python
sage: BinaryTrees(5, full=True).random_element() # random
[[[], [], []]]

sage: BinaryTrees(0, full=True).random_element()
.
sage: BinaryTrees(1, full=True).random_element()
[., .]
```

class sage.combinat.binary_tree.LabelledBinaryTree(parent, children, label=None, check=True)

Bases: AbstractLabelledClonableTree, BinaryTree

Labelled binary trees.

A labelled binary tree is a binary tree (see BinaryTree for the meaning of this) with a label assigned to each node. The labels need not be integers, nor are they required to be distinct. None can be used as a label.

**Warning:** While it is possible to assign values to leaves (not just nodes) using this class, these labels are disregarded by various methods such as labels(), map_labels(), and (ironically) leaf_labels().

**INPUT:**

- children – None (default) or a list, tuple or iterable of length \(2\) of labelled binary trees or convertible objects. This corresponds to the standard recursive definition of a labelled binary tree as being either a leaf, or a pair of:
  - a pair of labelled binary trees,
  - and a label.

(The label is specified in the keyword variable label; see below.)

Syntactic sugar allows leaving out all but the outermost calls of the LabelledBinaryTree() constructor, so that, e. g., LabelledBinaryTree([LabelledBinaryTree(None), LabelledBinaryTree(None)]) can be shortened to LabelledBinaryTree([None, None]). However, using this shorthand, it is impossible to label any vertex of the tree other than the root (because there is no way to pass a label variable without calling LabelledBinaryTree explicitly).

It is also allowed to abbreviate [None, None] by [] if one does not want to label the leaves (which one should not do anyway!).

- label – (default: None) the label to be put on the root of this tree.
- check – (default: True) whether checks should be performed or not.

**Todo:** It is currently not possible to use LabelledBinaryTree() as a shorthand for LabelledBinaryTree(None) (in analogy to similar syntax in the BinaryTree class).

**EXAMPLES:**
sage: LabelledBinaryTree(None)
  
  sage: LabelledBinaryTree(None, label="ae")  # not well supported
  'ae'

  sage: LabelledBinaryTree([])
  None[., .]

  sage: LabelledBinaryTree([], label=3)  # not well supported
  3[., .]

  sage: LabelledBinaryTree([None, None])
  None[., .]

  sage: LabelledBinaryTree([None, None], label=5)
  5[., .]

  sage: LabelledBinaryTree([None, []])
  None[., None[., .]]

  sage: LabelledBinaryTree([None, BinaryTree([None, None])], label=3)
  3[., None[., .]]

  sage: LabelledBinaryTree([[], None], label=4)
  4[., None[., .]]

  sage: LabelledBinaryTree([[], [], None])
  None[None[., .], .]

  sage: LabelledBinaryTree([[], None, []])
  None[None[., .], .]

  Traceback (most recent call last):
  ...

  ValueError: this is not a binary tree

  sage: LBT = LabelledBinaryTree
  sage: t1 = LBT([LBT([None, label=2], None, None), None, label=4]); t1
  4[None[2[., .], .], .]

**binary_search_insert**(letter)

Return the result of inserting a letter *letter* into the right strict binary search tree *self*.

**INPUT:**

- letter – any object comparable with the labels of *self*

**OUTPUT:**

The right strict binary search tree *self* with *letter* inserted into it according to the binary search insertion algorithm.

**Note:** *self* is supposed to be a binary search tree. This is not being checked!

A right strict binary search tree is defined to be a labelled binary tree such that for each node *n* with label *x*, every descendant of the left child of *n* has a label \(\leq x\), and every descendant of the right child of *n* has a label \(> x\). (Here, only nodes count as descendants, and every node counts as its own descendant too.) Leaves are assumed to have no labels.

Given a right strict binary search tree *t* and a letter *i*, the result of inserting *i* into *t* (denoted *Ins(i, t)*) in the following is defined recursively as follows:
• If $t$ is empty, then $\text{Ins}(i, t)$ is the tree with one node only, and this node is labelled with $i$.

• Otherwise, let $j$ be the label of the root of $t$. If $i > j$, then $\text{Ins}(i, t)$ is obtained by replacing the right child of $t$ by $\text{Ins}(i, r)$ in $t$, where $r$ denotes the right child of $t$. If $i \leq j$, then $\text{Ins}(i, t)$ is obtained by replacing the left child of $t$ by $\text{Ins}(i, l)$ in $t$, where $l$ denotes the left child of $t$.

See, for example, [HNT2005] for properties of this algorithm.

**Warning:** If $t$ is nonempty, then inserting $i$ into $t$ does not change the root label of $t$. Hence, as opposed to algorithms like Robinson-Schensted-Knuth, binary search tree insertion involves no bumping.

**EXAMPLES:**
The example from Fig. 2 of [HNT2005]:

```python
sage: LBT = LabelledBinaryTree
sage: x = LBT(None)

sage: x
b[., .]

sage: x = x.binary_search_insert("b"); x
b[., .]

sage: x = x.binary_search_insert("d"); x
b[., d[., .]]

sage: x = x.binary_search_insert("e"); x
b[., d[., e[., .]]]

sage: x = x.binary_search_insert("a"); x
b[a[., .], d[., e[., .]]]

sage: x = x.binary_search_insert("b"); x
b[a[., b[., .]], d[., e[., .]]]

sage: x = x.binary_search_insert("d"); x
b[a[., b[., .]], d[., d[., e[., .]]]]

sage: x = x.binary_search_insert("a"); x
b[a[., a[., .], b[., .]], d[., d[., e[., .]]]]

sage: x = x.binary_search_insert("c"); x
b[a[a[., .], b[., .]], d[., d[c[., .], .], e[., .]]]
```

Other examples:

```python
sage: LBT = LabelledBinaryTree
sage: LBT(None).binary_search_insert(3)
3[., .]

sage: LBT([], label = 1).binary_search_insert(3)
1[., 3[., .]]

sage: LBT([], label = 3).binary_search_insert(1)
3[1[., .], .]

sage: res = LBT(None)
sage: for i in [3,1,5,2,4,6]:
....:     res = res.binary_search_insert(i)
sage: res
3[1[., 2[., .]], 5[4[., .], 6[., .]]]
```

**heap_insert**($l$)

Return the result of inserting a letter $l$ into the binary heap (tree) self.
A binary heap is a labelled complete binary tree such that for each node, the label at the node is greater or equal to the label of each of its child nodes. (More precisely, this is called a max-heap.)

For example:

```
|  _7_  |
|   / \  |
|  5  6  |
| / \    |
| 3  4   |
```

is a binary heap.

See Wikipedia article Binary_heap#Insert for a description of how to insert a letter into a binary heap. The result is another binary heap.

INPUT:

- *letter* – any object comparable with the labels of *self*

**Note:** *self* is assumed to be a binary heap (tree). No check is performed.

left_rotate()

Return the result of left rotation applied to the labelled binary tree *self*.

Left rotation on labelled binary trees is defined as follows: Let *T* be a labelled binary tree such that the right child of the root of *T* is a node. Let *A* be the left child of the root of *T*, and let *B* and *C* be the left and right children of the right child of the root of *T*. (Keep in mind that nodes of trees are identified with the subtrees consisting of their descendants.) Furthermore, let *x* be the label at the root of *T*, and *y* be the label at the right child of the root of *T*. Then, the left rotation of *T* is the labelled binary tree in which the root is labelled *y*, the right child of the root is *C*, whereas the left child of the root is a node labelled *x* whose left and right children are *A* and *B*. In pictures:

```
|  y  x  |
|   / \  |
|  x  C  <-left-rotate-  A  y  |
| / \    |
| A  B   B  C   |
```

Left rotation is the inverse operation to right rotation (**right_rotate()**).

right_rotate()

Return the result of right rotation applied to the labelled binary tree *self*.

Right rotation on labelled binary trees is defined as follows: Let *T* be a labelled binary tree such that the left child of the root of *T* is a node. Let *C* be the right child of the root of *T*, and let *A* and *B* be the left and right children of the left child of the root of *T*. (Keep in mind that nodes of trees are identified with the subtrees consisting of their descendants.) Furthermore, let *y* be the label at the root of *T*, and *x* be the label at the left child of the root of *T*. Then, the right rotation of *T* is the labelled binary tree in which the root is labelled *x*, the left child of the root is *A*, whereas the right child of the root is a node labelled *y* whose left and right children are *B* and *C*. In pictures:

```
|  y  x  |
|   / \  |
|  x  C  -right-rotate->  A  y  |
```

(continues on next page)
Right rotation is the inverse operation to left rotation (\texttt{left\_rotate()}).

\textbf{semistandard\_insert}(letter)

Return the result of inserting a letter \texttt{letter} into the semistandard tree \texttt{self} using the bumping algorithm.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{letter} – any object comparable with the labels of \texttt{self}
\end{itemize}

\textbf{OUTPUT:}

The semistandard tree \texttt{self} with \texttt{letter} inserted into it according to the bumping algorithm.

\textbf{Note:} \texttt{self} is supposed to be a semistandard tree. This is not being checked!

A semistandard tree is defined to be a labelled binary tree such that for each node \( n \) with label \( x \), every descendant of the left child of \( n \) has a label \( > x \), and every descendant of the right child of \( n \) has a label \( \geq x \). (Here, only nodes count as descendants, and every node counts as its own descendant too.) Leaves are assumed to have no labels.

Given a semistandard tree \( t \) and a letter \( i \), the result of inserting \( i \) into \( t \) (denoted \( I_{n}s(i, t) \) in the following) is defined recursively as follows:

\begin{itemize}
  \item If \( t \) is empty, then \( I_{n}s(i, t) \) is the tree with one node only, and this node is labelled with \( i \).
  \item Otherwise, let \( j \) be the label of the root of \( t \). If \( i \geq j \), then \( I_{n}s(i, t) \) is obtained by replacing the right child of \( t \) by \( I_{n}s(i, r) \) in \( t \), where \( r \) denotes the right child of \( t \). If \( i < j \), then \( I_{n}s(i, t) \) is obtained by replacing the label at the root of \( t \) by \( i \), and replacing the left child of \( t \) by \( I_{n}s(j, l) \) in \( t \), where \( l \) denotes the left child of \( t \).
\end{itemize}

This algorithm is similar to the Robinson-Schensted-Knuth insertion algorithm for semistandard Young tableaux.

\textbf{AUTHORS:}

\begin{itemize}
  \item Darij Grinberg (10 Nov 2013).
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: LBT = LabelledBinaryTree
sage: x = LBT(None)
sage: x
.
sage: x = x.semistandard_insert("b"); x
  b[., .]
sage: x = x.semistandard_insert("d"); x
  b[., d[., .]]
sage: x = x.semistandard_insert("e"); x
  b[., d[., e[., .]]]
sage: x = x.semistandard_insert("a"); x
  a[b[., .], d[., e[., .]]]
sage: x = x.semistandard_insert("b"); x
  a[b[., .], b[d[., .], e[., .]]]
sage: x = x.semistandard_insert("d"); x
(continues on next page)
\end{verbatim}
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a[b[., .], b[d[., .], d[e[., .], .]]]

\textit{sage}: \texttt{x = x.semistandard_insert("a"); x a[b[., .], a[b[d[., .], .], d[e[., .], .]]}

\textit{sage}: \texttt{x = x.semistandard_insert("c"); x a[b[., .], a[b[d[., .], .], c[d[e[., .], .], .]]}

Other examples:

\textit{sage}: \texttt{LBT = LabelledBinaryTree}
\textit{sage}: \texttt{LBT(\texttt{None}).semistandard_insert(3)}
\texttt{3[., .]}

\textit{sage}: \texttt{LBT([], label = 1).semistandard_insert(3)}
\texttt{1[., 3[., .]]}

\textit{sage}: \texttt{LBT([], label = 3).semistandard_insert(1)}
\texttt{1[3[., .], .]}

\textit{sage}: \texttt{res = LBT(\texttt{None})}
\textit{sage}: \texttt{for i in [3,1,5,2,4,6]:
\hspace{1em} res = res.semistandard_insert(i)}
\textit{sage}: \texttt{res}
\texttt{1[3[., .], 2[5[., .], 4[., 6[., .]]]]}

class \texttt{sage.combinat.binary_tree.LabelledBinaryTrees}(\texttt{category=None})
Bases: \texttt{LabelledOrderedTrees}

This is a parent stub to serve as a factory class for trees with various labels constraints.

\texttt{Element}
alias of \texttt{LabelledBinaryTree}

\texttt{labelled_trees()}
Return the set of labelled trees associated to self.

EXAMPLES:

\textit{sage}: \texttt{LabelledBinaryTrees().labelled_trees()}
Labelled binary trees

\texttt{unlabelled_trees()}
Return the set of unlabelled trees associated to self.

EXAMPLES:

\textit{sage}: \texttt{LabelledBinaryTrees().unlabelled_trees()}
Binary trees

This is used to compute the shape:

\textit{sage}: \texttt{t = LabelledBinaryTrees().an_element().shape(); t}
[[[., .], [[., .], [[., .], [[., .], .]]]]

\texttt{sage}: \texttt{t.parent()}
Binary trees

\texttt{sage.combinat.binary_tree.binary_search_tree_shape}(\texttt{w, left_to_right=True})
Direct computation of the binary search tree shape of a list of integers.

\texttt{INPUT:}
• `w` – a list of integers
• `left_to_right` – boolean (default True)

OUTPUT: a non labelled binary tree

This is used under the same name as a method for permutations.

EXAMPLES:

```python
sage: from sage.combinat.binary_tree import binary_search_tree_shape
sage: binary_search_tree_shape([1,4,3,2])
[[., [., .]], .]
sage: binary_search_tree_shape([5,1,3,2])
[[., [., .]], .]
```

By passing the option `left_to_right=False` one can have the insertion going from right to left:

```python
sage: binary_search_tree_shape([1,6,4,2], False)
[[., .], [., [., .]]]
```

```python
sage.combinat.binary_tree.from_tamari_sorting_tuple(key)
```

Return a binary tree from its Tamari-sorting tuple.

See `tamari_sorting_tuple()`

INPUT:

• `key` – a tuple of integers

EXAMPLES:

```python
sage: from sage.combinat.binary_tree import from_tamari_sorting_tuple
sage: t = BinaryTrees(60).random_element()
sage: from_tamari_sorting_tuple(t.tamari_sorting_tuple()[0]) == t
True
```

5.1.11 Blob Algebras

AUTHORS:

• Travis Scrimshaw (2020-05-16): Initial version

```python
class sage.combinat.blob_algebra.BlobAlgebra(k, q1, q2, q3, base_ring, prefix)
```

The blob algebra.

The **blob algebra** (also known as the Temperley-Lieb algebra of type $B$ in [ILZ2018], but is a quotient of the Temperley-Lieb algebra of type $B$ defined in [Graham1985]) is a diagram-type algebra introduced in [MS1994] whose basis consists of *Temperley-Lieb diagrams*, noncrossing perfect matchings, that may contain blobs on strands that can be deformed so that the blob touches the left side (which we can think of as a frozen pole).

The form we give here has 3 parameters, the natural one from the *Temperley-Lieb algebra*, one for the idempotent relation, and one for a loop with a blob.

INPUT:

• `k` – the order
• `q1` – the loop parameter
• q2 – the idempotent parameter
• q3 – the blob loop parameter

EXAMPLES:

```
sage: R.<q,r,s> = ZZ[]
sage: B4 = algebras.Blob(4, q, r, s)
sage: B = sorted(B4.basis())
sage: B[14]
B({{-4, -3}}, {{-2, -1}, {1, 2}, {3, 4}})
sage: B[40]
B({{3, 4}}, {{-4, -3}, {-2, -1}, {1, 2}})
sage: B[14] * B[40]
q*r*s*B({}, {{-4, -3}, {-2, -1}, {1, 2}, {3, 4}})
```

REFERENCES:

• [MS1994]
• [ILZ2018]

one_basis()

Return the index of the basis element 1.

EXAMPLES:

```
sage: R.<q,r,s> = ZZ[]
sage: B4 = algebras.Blob(4, q, r, s)
sage: B4.one_basis()
({}, {{-4, 4}, {-3, 3}, {-2, 2}, {-1, 1}})
```

order()

Return the order of self.

The order of a partition algebra is defined as half of the number of nodes in the diagrams.

EXAMPLES:

```
sage: R.<q,r,s> = ZZ[]
sage: B4 = algebras.Blob(4, q, r, s)
sage: B4.order()
4
```

product_on_basis(top, bot)

Return the product of the basis elements indexed by top and bot.

EXAMPLES:

```
sage: R.<q,r,s> = ZZ[]
sage: B4 = algebras.Blob(4, q, r, s)
sage: B = B4.basis()
sage: BD = sorted(B.keys())
sage: BD[14]
({{-4, -3}}, {{-2, -1}, {1, 2}, {3, 4}})
sage: BD[40]
({{3, 4}}, {{-4, -3}, {-2, -1}, {1, 2}})
sage: B4.product_on_basis(BD[14], BD[40])
```
(continues on next page)
class sage.combinat.blob_algebra.BlobDiagram(parent, marked, unmarked)
Bases: Element

A blob diagram.

A blob diagram consists of a perfect matching of the set \{1,\ldots,n\} \sqcup \{-1,\ldots,-n\} such that the result is a noncrossing matching (a Temperley-Lieb diagram), divided into two sets of pairs: one for the pairs with blobs and one for those without. The blobed pairs must either be either the leftmost propagating strand or to the left of it and not nested.

temperley_lieb_diagram()

Return the Temperley-Lieb diagram corresponding to self.

EXAMPLES:

```python
sage: from sage.combinat.blob_algebra import BlobDiagrams
sage: BD4 = BlobDiagrams(4)
sage: B = BD4([[1,-3]], [[2,-4], [3,4], [-1,-2]])
sage: B.temperley_lieb_diagram()
{{-4, 2}, {-3, 1}, {-2, -1}, {3, 4}}
```

class sage.combinat.blob_algebra.BlobDiagrams(n)
Bases: Parent, UniqueRepresentation

The set of all blob diagrams.

Element

alias of BlobDiagram

base_set()

Return the base set of self.

EXAMPLES:

```python
sage: from sage.combinat.blob_algebra import BlobDiagrams
sage: BD4 = BlobDiagrams(4)
sage: sorted(BD4.base_set())
[-4, -3, -2, -1, 1, 2, 3, 4]
```

cardinality()

Return the cardinality of self.

EXAMPLES:

```python
sage: from sage.combinat.blob_algebra import BlobDiagrams
sage: BD4 = BlobDiagrams(4)
sage: BD4.cardinality()
70
```

order()

Return the order of self.

EXAMPLES:
```python
sage: from sage.combinat.blob_algebra import BlobDiagrams
sage: BD4 = BlobDiagrams(4)
sage: BD4.order()
4
```

## 5.1.12 Cartesian Products

### class `sage.combinat.cartesian_product.CartesianProduct_iters(*iters)`

Bases: `EnumeratedSetFromIterator`

Cartesian product of finite sets.

This class will soon be deprecated (see github issue #18411 and github issue #19195). One should instead use the functorial construction `cartesian_product`. The main differences in behavior are:

- construction: `CartesianProduct` takes as many argument as there are factors whereas `cartesian_product` takes a single list (or iterable) of factors;
- representation of elements: elements are represented by plain Python list for `CartesianProduct` versus a custom element class for `cartesian_product`;
- membership testing: because of the above, plain Python lists are not considered as elements of a `cartesian_product`.

All of these is illustrated in the examples below.

#### EXAMPLES:

```python
sage: F1 = ['a', 'b']
sage: F2 = [1, 2, 3, 4]
sage: F3 = Permutations(3)
sage: from sage.combinat.cartesian_product import CartesianProduct_iters
sage: C = CartesianProduct_iters(F1, F2, F3)
sage: c = cartesian_product([F1, F2, F3])

sage: type(C.an_element())
<class 'list'>

sage: type(c.an_element())
<class 'sage.sets.cartesian_product.CartesianProduct_with_category.element_class'>

sage: l = ['a', 1, Permutation([3, 2, 1])]
sage: l in C
True
sage: l in c
False
sage: elt = c(l)
sage: elt
('a', 1, [3, 2, 1])
sage: elt in c
True
sage: elt.parent() is c
True
```

`cardinality()`

Returns the number of elements in the Cartesian product of everything in `*iters`. 

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EXAMPLES:

```python
sage: from sage.combinat.cartesian_product import CartesianProduct_iters
sage: CartesianProduct_iters(range(2), range(3)).cardinality()
6
sage: CartesianProduct_iters(range(2), range(3)).cardinality()
6
sage: CartesianProduct_iters(range(2), range(3), range(4)).cardinality()
24
```

This works correctly for infinite objects:

```python
sage: CartesianProduct_iters(ZZ, QQ).cardinality()
+Infinity
sage: CartesianProduct_iters(ZZ, []).cardinality()
0
```

**is_finite()**

The Cartesian product is finite if all of its inputs are finite, or if any input is empty.

EXAMPLES:

```python
sage: from sage.combinat.cartesian_product import CartesianProduct_iters
sage: CartesianProduct_iters(ZZ, []).is_finite()
True
sage: CartesianProduct_iters(4, 4).is_finite()
Traceback (most recent call last):
... ValueError: unable to determine whether this product is finite
```

**list()**

Returns

EXAMPLES:

```python
sage: from sage.combinat.cartesian_product import CartesianProduct_iters
sage: CartesianProduct_iters(range(3), range(3)).list()
[[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]]
sage: CartesianProduct_iters('dog', 'cat').list()
[['d', 'c'], ['d', 'a'], ['d', 't'], ['o', 'c'], ['o', 'a'], ['o', 't'], ['g', 'c'], ['g', 'a'], ['g', 't']]
```

**random_element()**

Return a random element from the Cartesian product of *iters.

EXAMPLES:
sage: from sage.combinat.cartesian_product import CartesianProduct_iters
sage: c = CartesianProduct_iters('dog', 'cat').random_element()
sage: c in CartesianProduct_iters('dog', 'cat')
True

unrank(x)

For finite Cartesian products, we can reduce unrank to the constituent iterators.

EXAMPLES:

sage: from sage.combinat.cartesian_product import CartesianProduct_iters
sage: C = CartesianProduct_iters(range(1000), range(1000), range(1000))
sage: C[238792368]
[238, 792, 368]

Check for github issue #15919:

sage: FF = IntegerModRing(29)
sage: C = CartesianProduct_iters(FF, FF, FF)
sage: C.unrank(0)
[0, 0, 0]

5.1.13 Enumerated sets of partitions, tableaux, ...

Partitions

- Integer partitions
- Skew Partitions
- Partition tuples
- Super Partitions
- TableauTuples
- Skew Tableaux
- Ribbons
- Ribbon Tableaux
- Strong and weak tableaux
- Shifted primed tableaux
- Residue sequences of tableaux
RSK

- Robinson-Schensted-Knuth correspondence
- Growth diagrams and dual graded graphs

5.1.14 Combinatorial Hopf algebras

- Symmetric Functions
- Non-commutative symmetric functions and quasi-symmetric functions
- Symmetric functions in non-commuting variables
- Schubert Polynomials
- Poirier-Reutenauer Hopf algebra of standard tableaux
- Free Quasi-symmetric functions
- Grossman-Larson Hopf Algebras
- Word Quasi-symmetric functions

5.1.15 Poirier-Reutenauer Hopf algebra of standard tableaux

AUTHORS:

- Franco Saliola (2012): initial implementation
- Travis Scrimshaw (2018-04-11): added missing doctests and reorganization

class sage.combinat.chas.fsym.FSymBases(parent_with_realization)

Bases: Category_realization_of_parent

The category of graded bases of $F\Sym$ and $F\Sym^*$ indexed by standard tableaux.

class ElementMethods

Bases: object

duality_pairing(other)

Compute the pairing between self and an element other of the dual.

EXAMPLES:

```
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: F = G.dual_basis()
sage: elt = G[[1,3],[2]] - 3*G[[1,2],[3]]
sage: elt.duality_pairing(F[[1,3],[2]])
1
sage: elt.duality_pairing(F[[1,2],[3]])
-3
sage: elt.duality_pairing(F[[1,2]])
0
```

class ParentMethods

Bases: object
**basis**(degree=None)

The basis elements (optionally: of the specified degree).

OUTPUT: Family

EXAMPLES:

```python
sage: FSym = algebras.FSym(QQ)
sage: TG = FSym.G()
sage: TG.basis()
Lazy family (Term map from Standard tableaux to Hopf algebra of standard...
˓
over the Rational Field in the Fundamental basis(i))_{i in Standard...
˓
        }

sage: TG.basis().keys()
Standard tableaux

sage: TG.basis(degree=3).keys()
Standard tableaux of size 3

sage: TG.basis(degree=3).list()
[G[123], G[13|2], G[12|3], G[1|2|3]]
```

**degree_on_basis**(t)

Return the degree of a standard tableau in the algebra of free symmetric functions.

This is the size of the tableau t.

EXAMPLES:

```python
sage: G = algebras.FSym(QQ).G()
sage: t = StandardTableau([[1,3], [2]])
sage: G.degree_on_basis(t)
3

sage: u = StandardTableau([[1,3,4,5], [2]])
sage: G.degree_on_basis(u)
5
```

**duality_pairing**(x, y)

The canonical pairing between \( FSym \) and \( FSym^* \).

EXAMPLES:

```python
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: F = G.dual_basis()
sage: t1 = StandardTableau([[1,3,5], [2,4]])
sage: t2 = StandardTableau([[1,3], [2,5], [4]])
sage: G.duality_pairing(G[t1], F[t2])
0

sage: G.duality_pairing(G[t1], F[t1])
1

sage: G.duality_pairing(G[t2], F[t2])
1

sage: F.duality_pairing(F[t2], G[t2])
1

sage: z = G[[1,3,5], [2,4]]
```
duality_pairing_matrix(basis, degree)

The matrix of scalar products between elements of $FSym$ and elements of $FSym^*$. 

INPUT:

• basis – a basis of the dual Hopf algebra
• degree – a non-negative integer

OUTPUT:

• the matrix of scalar products between the basis self and the basis basis in the dual Hopf algebra of degree degree

EXAMPLES:

```python
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: G.duality_pairing_matrix(G.dual_basis(), 3)
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

one_basis()

Return the basis index corresponding to 1.

EXAMPLES:

```python
sage: FSym = algebras.FSym(QQ)
sage: TG = FSym.G()
sage: TG.one_basis()
[]
```

super_categories()

The super categories of self.

EXAMPLES:

```python
sage: from sage.combinat.chas.fsym import FSymBases
sage: FSym = algebras.FSym(ZZ)
sage: bases = FSymBases(FSym)
sage: bases.super_categories()
[Category of realizations of Hopf algebra of standard tableaux over the Integer Ring, 
Join of Category of realizations of hopf algebras over Integer Ring and Category of graded algebras over Integer Ring and Category of graded coalgebras over Integer Ring, 
Category of graded connected hopf algebras with basis over Integer Ring]
```

class sage.combinat.chas.fsym.FSymBasis_abstract(alg, graded=True)

Bases: CombinatorialFreeModule, BindableClass

Abstract base class for graded bases of $FSym$ and of $FSym^*$ indexed by standard tableaux.

This must define the following attributes:
some_elements()

Return some elements of self.

EXAMPLES:

```python
sage: G = algebras.FSym(QQ).G()
sage: G.some_elements()
[[], [1], [12], [1] + [1|2], [] + 1/2*[]]
```

class sage.combinat.chas.fsym.FreeSymmetricFunctions(base_ring)

Bases: UniqueRepresentation, Parent

The free symmetric functions.

The free symmetric functions is a combinatorial Hopf algebra defined using tableaux and denoted \( FSym \).

Consider the Hopf algebra \( FQSym \) (FreeQuasisymmetricFunctions) over a commutative ring \( R \), and its bases \( (F_w) \) and \( (G_w) \) (where \( w \), in both cases, ranges over all permutations in all symmetric groups \( S_0, S_1, S_2, \ldots \)). For each word \( w \), let \( P(w) \) be the P-tableau of \( w \) (that is, the first of the two tableaux obtained by applying the RSK algorithm to \( w \); see \( RSK() \)). If \( t \) is a standard tableau of size \( n \), then we define \( G_t \in FQSym \) to be the sum of the \( F_w \) with \( w \) ranging over all permutations of \( \{1, 2, \ldots, n\} \) satisfying \( P(w) = t \). Equivalently, \( G_t \) is the sum of the \( G_w \) with \( w \) ranging over all permutations of \( \{1, 2, \ldots, n\} \) satisfying \( Q(w) = t \) (where \( Q(w) \) denotes the Q-tableau of \( w \)).

The \( R \)-linear span of the \( G_t \) (for \( t \) ranging over all standard tableaux) is a Hopf subalgebra of \( FQSym \), denoted by \( FSym \) and known as the free symmetric functions or the Poirier-Reutenauer Hopf algebra of tableaux. It has been introduced in [PoiReu95], where it was denoted by \( (ZT, \ast, \delta) \). (What we call \( G_t \) has just been called \( t \) in [PoiReu95].) The family \( (G_t) \) (with \( t \) ranging over all standard tableaux) is a basis of \( FSym \), called the Fundamental basis.

EXAMPLES:

As explained above, \( FSym \) is constructed as a Hopf subalgebra of \( FQSym \):

```python
sage: G = algebras.FSym(QQ).G()
sage: F = algebras.FQSym(QQ).F()
sage: G[[1,3],[2]]
G[13|2]
sage: G[[1,3],[2]].to_fqsym()
G[2, 1, 3] + G[3, 1, 2]
sage: F(G[[1,3],[2]])
F[2, 1, 3] + F[2, 3, 1]
```

This embedding is a Hopf algebra morphism:

```python
sage: all(F(G[t1] * G[t2]) == F(G[t1]) * F(G[t2])
.....:     for t1 in StandardTableaux(2)
.....:     for t2 in StandardTableaux(3))
True

sage: FF = F.tensor_square()
sage: all(FF(G[t].coproduct()) == F(G[t]).coproduct()
.....:     for t in StandardTableaux(4))
True
```
There is a Hopf algebra map from \( FSym \) onto the Hopf algebra of symmetric functions, which maps a tableau \( t \) to the Schur function indexed by the shape of \( t \):

```python
sage: TG = algebras.FSym(QQ).G()
sage: t = StandardTableau([[1,3],[2,4],[5]])
sage: TG[t]
G[13|24|5]
sage: TG[t].to_symmetric_function()
s[2, 2, 1]
```

**class Fundamental (\( alg, graded=\text{True} \))**

Bases: \( \text{FSymBasis\_abstract} \)

The Hopf algebra of tableaux on the Fundamental basis.

**EXAMPLES:**

```python
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: TG
Hopf algebra of standard tableaux over the Rational Field in the Fundamental basis
```

Elements of the algebra look like:

```python
sage: TG.an_element()
```

**class Element**

Bases: \( \text{IndexedFreeModuleElement} \)

```python
to_fqsym()
Return the image of \( \text{self} \) under the natural inclusion map to \( FQSym \).

**EXAMPLES:**

```python
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: t = StandardTableau([[1,3],[2,4],[5]])
sage: G[t].to_fqsym()
```

```python
to_symmetric_function()
Return the image of \( \text{self} \) under the natural projection map to \( Sym \).

The natural projection map \( FSym \to Sym \) sends each standard tableau \( t \) to the Schur function \( s_\lambda \), where \( \lambda \) is the shape of \( t \). This map is a surjective Hopf algebra homomorphism.

**EXAMPLES:**

```python
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: t = StandardTableau([[1,3],[2,4],[5]])
sage: G[t].to_symmetric_function()
s[2, 2, 1]
```
**coproduct_on_basis(t)**

Return the coproduct of the basis element indexed by \( t \).

**EXAMPLES:**

```
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: t = StandardTableau([[1,2,5], [3,4]])
sage: G.coproduct_on_basis(t)
```

**dual_basis()**

Return the dual basis to \( self \).

**EXAMPLES:**

```
sage: G = algebras.FSym(QQ).G()
sage: G.dual_basis()
Dual Hopf algebra of standard tableaux over the Rational Field in the FundamentalDual basis
```

**product_on_basis(t1, t2)**

Return the product of basis elements indexed by \( t1 \) and \( t2 \).

**EXAMPLES:**

```
sage: FSym = algebras.FSym(QQ)
sage: G = FSym.G()
sage: t1 = StandardTableau([[1,2], [3]])
sage: t2 = StandardTableau([[1,2,3]])
sage: G.product_on_basis(t1, t2)
sage: t1 = StandardTableau([[1,2], [3]])
sage: t2 = StandardTableau([[1,2]])
sage: G.product_on_basis(t1, t2)
G[134|2] + G[14|2|3]
sage: t1 = StandardTableau([[1,2], [3]])
sage: t2 = StandardTableau([[1,2]])
sage: G.product_on_basis(t1, t2)
```

\( G \)

alias of \( Fundamental \)

**a_realization()**

Return a particular realization of \( self \) (the Fundamental basis).

**EXAMPLES:**
Combinatorics, Release 10.0

```
sage: FSym = algebras.FSym(QQ)
sage: FSym.a_realization()
Hopf algebra of standard tableaux over the Rational Field
in the Fundamental basis
```

```
dual()

Return the dual Hopf algebra of $FSym$.

EXAMPLES:
```
```
sage: algebras.FSym(QQ).dual()
Dual Hopf algebra of standard tableaux over the Rational Field
```

```
class sage.combinat.chas.fsym.FreeSymmetricFunctions_Dual(base_ring)

Bases: UniqueRepresentation, Parent

The Hopf dual $FSym^*$ of the free symmetric functions $FSym$.

See FreeSymmetricFunctions for the definition of the latter.

Recall that the fundamental basis of $FSym$ consists of the elements $G_t$ for $t$ ranging over all standard tableaux.

The dual basis of this is called the dual fundamental basis of $FSym^*$, and is denoted by $(G_t^*)$. The Hopf dual $FSym^*$ is isomorphic to the Hopf algebra $(ZT, *, \delta')$ from [PoiReu95]; the isomorphism sends a basis element $G_t^*$ to $t$.

EXAMPLES:
```
```
sage: FSym = algebras.FSym(QQ)
sage: TF = FSym.dual().F()
sage: TF[[1,2]] * TF[[[1][, [2]]]
sage: TF[[1,2],[3]].coproduct()
```

The Hopf algebra $FSym^*$ is a Hopf quotient of $FQSym$; the canonical projection sends $F_w$ (for a permutation $w$) to $G_{Q(w)}^*$, where $Q(w)$ is the Q-tableau of $w$. This projection is implemented as a coercion:

```
sage: FQSym = algebras.FQSym(QQ)
sage: F = FQSym.F()
sage: TF(F[[1, 3, 2]])
F[12|3]
sage: TF(F[[5, 1, 4, 2, 3]])
F[135|2|4]
```

```
F

alias of FundamentalDual
```

```
class FundamentalDual(alg, graded=True)

Bases: FSymBasis_abstract

The dual to the Hopf algebra of tableaux, on the fundamental dual basis.

EXAMPLES:
```
```
sage: FSym = algebras.FSym(QQ)
sage: TF = FSym.dual().F()
```

(continues on next page)
Elements of the algebra look like:

```
sage: TF.an_element()
```

class Element

Bases: IndexedFreeModuleElement

```
to_quasisymmetric_function()
```

Return the image of self under the canonical projection $FSym^* \to QSym$ to the ring of quasi-symmetric functions.

This projection is the adjoint of the canonical injection $NSym \to FSym$ (see `to_fsym()`). It sends each tableau $t$ to the fundamental quasi-symmetric function $F_\alpha$, where $\alpha$ is the descent composition of $t$.

EXAMPLES:

```
sage: F = algebras.FSym(QQ).dual().F()
sage: F[[1,3,5],[2,4]].to_quasisymmetric_function()
sage: F[1, 2, 2]
```

coproduct_on_basis($t$)

EXAMPLES:

```
sage: FSym = algebras.FSym(QQ)
sage: TF = FSym.dual().F()
sage: t = StandardTableau([[1,2,5], [3,4]])
sage: TF.coproduct_on_basis(t)
```

dual_basis()

Return the dual basis to self.

EXAMPLES:

```
sage: F = algebras.FSym(QQ).dual().F()
sage: F.dual_basis()
```

Hopf algebra of standard tableaux over the Rational Field
in the Fundamental basis

product_on_basis($t_1, t_2$)

EXAMPLES:

```
sage: FSym = algebras.FSym(QQ)
sage: TF = FSym.dual().F()
sage: t1 = StandardTableau([[1,2]])
sage: t2 = StandardTableau([[1,2]])
sage: TF.product_on_basis(t1, t2)
```

Continued from previous page:

```python
sage: t0 = StandardTableau([])
sage: TF.product_on_basis(t1, t0) == TF[t1] == TF.product_on_basis(t0, t1)
True
```

**a_realization()**

Return a particular realization of self (the Fundamental dual basis).

EXAMPLES:

```python
sage: FSym = algebras.FSym(QQ).dual()
sage: FSym.a_realization()
Dual Hopf algebra of standard tableaux over the Rational Field in the FundamentalDual basis
```

**dual()**

Return the dual Hopf algebra of self, which is $FSym$.

EXAMPLES:

```python
sage: D = algebras.FSym(QQ).dual()
sage: D.dual()
Hopf algebra of standard tableaux over the Rational Field
```

**sage.combinat.chas.fsym.ascent_set(t)**

Return the ascent set of a standard tableau $t$ (encoded as a sorted list).

The ascent set of a standard tableau $t$ is defined as the set of all entries $i$ of $t$ such that the number $i + 1$ either appears to the right of $i$ or appears in a row above $i$ or does not appear in $t$ at all.

EXAMPLES:

```python
sage: from sage.combinat.chas.fsym import ascent_set
sage: t = StandardTableau([[1,3,4,7],[2,5,6],[8]])
sage: ascent_set(t)
[2, 3, 5, 6, 8]
sage: ascent_set(StandardTableau([]))
[]
sage: ascent_set(StandardTableau([[1, 2, 3]]))
[1, 2, 3]
sage: ascent_set(StandardTableau([[1, 2, 4], [3]]))
[1, 3, 4]
sage: ascent_set([[1, 3, 5], [2, 4]])
[2, 4, 5]
```

**sage.combinat.chas.fsym.descent_composition(t)**

Return the descent composition of a standard tableau $t$.

This is the composition of the size of $t$ whose partial sums are the elements of the descent set of $t$ (see `descent_set()`).

EXAMPLES:

```python
sage: from sage.combinat.chas.fsym import descent_composition
sage: t = StandardTableau([[1,3,4,7],[2,5,6],[8]])
sage: descent_composition(t)
```

(continues on next page)
sage.combinat.chas.fsym.descent_set(t)

Return the descent set of a standard tableau \( t \) (encoded as a sorted list).

The descent set of a standard tableau \( t \) is defined as the set of all entries \( i \) of \( t \) such that the number \( i + 1 \) appears in a row below \( i \) in \( t \).

EXAMPLES:

```python
sage: from sage.combinat.chas.fsym import descent_set
sage: t = StandardTableau([[1,3,4,7],[2,5,6],[8]])
sage: descent_set(t)
[1, 4, 7]
sage: descent_set(StandardTableau([]))
[]
sage: descent_set(StandardTableau([[1, 2, 3]]))
[]
sage: descent_set(StandardTableau([[1, 2, 4], [3]]))
[2]
sage: descent_set([[1, 3, 5], [2, 4]])
[1, 3]
```

sage.combinat.chas.fsym.standardize(t)

Return the standard tableau corresponding to a given semistandard tableau \( t \) with no repeated entries.

**Note:** This is an optimized version of `Tableau.standardization()` for computations in \( FSym \) by using the assumption of no repeated entries in \( t \).

EXAMPLES:

```python
sage: from sage.combinat.chas.fsym import standardize
sage: t = Tableau([[1,3,5,7],[2,4,8],[9]])
sage: standardize(t)
[[1, 3, 5, 6], [2, 4, 7], [8]]
sage: t = Tableau([[3,8,9,15],[5,10,12],[133]])
sage: standardize(t)
[[1, 3, 4, 7], [2, 5, 6], [8]]
```

### 5.1.16 Word Quasi-symmetric functions

**AUTHORS:**

- Travis Scrimshaw (2018-04-09): initial implementation
- Darij Grinberg and Amy Pang (2018-04-12): further bases and methods

**class** `sage.combinat.chas.wqsym.WQSymBases(base, graded)`

Bases: `Category_realization_of_parent`

The category of bases of \( WQSym \).
class ElementMethods
Bases: object

algebraic_complement()

Return the image of the element \texttt{self} of \texttt{WQSym} under the algebraic complement involution.

If \( u = (u_1, u_2, \ldots, u_n) \) is a packed word that contains the letters \( 1, 2, \ldots, k \) and no others, then the \textit{complement} of \( u \) is defined to be the packed word \( \overline{u} := (k + 1 - u_1, k + 1 - u_2, \ldots, k + 1 - u_n) \).

The algebraic complement involution is defined as the linear map \( \texttt{WQSym} \rightarrow \texttt{WQSym} \) that sends each basis element \( \texttt{M}_u \) of the monomial basis of \( \texttt{WQSym} \) to the basis element \( \texttt{M}_{\overline{u}} \). This is a graded algebra automorphism and a coalgebra anti-automorphism of \( \texttt{WQSym} \). Denoting by \( \overline{f} \) the image of an element \( f \in \texttt{WQSym} \) under the algebraic complement involution, it can be shown that every packed word \( u \) satisfies

\[
\texttt{M}_u = \texttt{M}_{\overline{u}}, \quad \texttt{X}_u = \texttt{X}_{\overline{u}},
\]

where standard notations for classical bases of \( \texttt{WQSym} \) are being used (that is, \( \texttt{M} \) for the monomial basis, and \( \texttt{X} \) for the characteristic basis).

This can be restated in terms of ordered set partitions: For any ordered set partition \( R = (R_1, R_2, \ldots, R_k) \), let \( R^c \) denote the ordered set partition \( (R_k, R_{k-1}, \ldots, R_1) \); this is known as the \textbf{reversal} of \( R \). Then,

\[
\overline{\texttt{M}}_A = \texttt{M}_{A^c}, \quad \overline{\texttt{X}}_A = \texttt{X}_{A^c}
\]

for any ordered set partition \( A \).

The formula describing algebraic complements on the \( \texttt{Q} \) basis (\texttt{WordQuasiSymmetricFunctions}. \texttt{StronglyCoarser}) is more complicated, and requires some definitions. We define a partial order \( \leq \) on the set of all ordered set partitions as follows: \( A \leq B \) if and only if \( A \) is strongly finer than \( B \) (see \texttt{is_strongly_finer()} for a definition of this). The \textbf{length} \( \ell(R) \) of an ordered set partition \( R \) shall be defined as the number of parts of \( R \). Use the notation \( Q \) for the \( \texttt{Q} \) basis. For any ordered set partition \( A \) of \([n]\), we have

\[
\overline{Q}_A = \sum_p c_{A,p} Q_p,
\]

where the sum is over all ordered set partitions \( P \) of \([n]\), and where the coefficient \( c_{A,p} \) is defined as follows:

- If there exists an ordered set partition \( R \) satisfying \( R \leq P \) and \( A \leq R^c \), then this \( R \) is unique, and
  \( c_{A,p} = (-1)^{\ell(R) - \ell(P)} \).
- If there exists no such \( R \), then \( c_{A,p} = 0 \).

The formula describing algebraic complements on the \( \Phi \) basis (\texttt{WordQuasiSymmetricFunctions}. \texttt{StronglyFiner}) is identical to the above formula for the \( \texttt{Q} \) basis, except that the \( \leq \) sign has to be replaced by \( \geq \) in the definition of the coefficients \( c_{A,p} \). In fact, both formulas are particular cases of a general formula for involutions: Assume that \( V \) is an (additive) abelian group, and that \( I \) is a poset. For each \( i \in I \), let \( M_i \) be an element of \( V \). Also, let \( \omega \) be an involution of the set \( I \) (not necessarily order-preserving or order-reversing), and let \( \omega' \) be an involutive group endomorphism of \( V \) such that each \( i \in I \) satisfies \( \omega'(M_i) = M_{\omega(i)} \). For each \( i \in I \), let \( F_i = \sum_{j \geq i} M_j \), where we assume that the sum is finite. Then, each \( i \in I \) satisfies

\[
\omega'(F_i) = \sum_j \sum_{k \leq j; \omega(k) \geq i} \mu(k,j)F_j,
\]

where \( \mu \) denotes the Möbius function. This formula becomes particularly useful when the \( k \) satisfying \( k \leq j \) and \( \omega(k) \geq i \) is unique (if it exists). In our situation, \( V \) is \( \texttt{WQSym} \), and \( I \) is the set of ordered
set partitions equipped either with the \( \leq \) partial order defined above or with its opposite order. The \( M_i \) is the \( M_A \), whereas the \( F_i \) is either \( Q_i \) or \( \Phi_i \).

If we denote the star involution (\texttt{star_involution()}) of the quasisymmetric functions by \( f \mapsto f^* \), and if we let \( \pi \) be the canonical projection \( WQSym \rightarrow QS\text{ym} \), then each \( f \in WQSym \) satisfies \( \pi(f^*) = (\pi(f))^* \).

See also:

\texttt{coalgebraic_complement()}, \texttt{star_involution()}

EXAMPLES:

Recall that the index set for the bases of \( WQSym \) is given by ordered set partitions, not packed words. Translated into the language of ordered set partitions, the algebraic complement involution acts on the Monomial basis by reversing the ordered set partition. In other words, we have

\[
M[p_1, p_2, \ldots, p_k] = M[p_k, p_{k-1}, \ldots, p_1]
\]

for any standard ordered set partition \((p_1, p_2, \ldots, p_k)\). Let us check this in practice:

```
sage: WQSym = algebras.WQSym(ZZ)
sage: M = WQSym.M()
sage: M[[1,3],[2]].algebraic_complement()
M[[2], [1, 3]]
sage: M[[1,4],[2,5],[3,6]].algebraic_complement()
M[[3, 6], {2, 5}, {1, 4}]
sage: (3*M[[1]] - 4*M[[2]] + 5*M[[1],[2]]).algebraic_complement()
-4*M[] + 3*M[[1]] + 5*M[[2], {1}]
sage: X = WQSym.X()
sage: X[[1,3],[2]].algebraic_complement()
X[[2], {1, 3}]
sage: C = WQSym.C()
sage: C[[1,3],[2]].algebraic_complement()
-C[[1, 2, 3]] - C[[1, 3], {2}] + C[{2}, {1, 3}]
sage: Q = WQSym.Q()
sage: Q[[1,2],[5,6],[3,4]].algebraic_complement()
Q[[3, 4], {1, 2, 5, 6}] + Q[3, 4, {5, 6}, {1, 2}] - Q[[3, 4, 5, 6], {1, 2}]
sage: Phi = WQSym.Phi()
sage: Phi[[2], [1,3]].algebraic_complement()
-Phi[[1], {3}, {2}] + Phi[[1, 3], {2}] + Phi[{3}, {1}, {2}]
```

The algebraic complement involution intertwines the antipode and the inverse of the antipode:

```
sage: all( M[I].antipode().algebraic_complement().antipode() == M[I].algebraic_complement() for I in OrderedSetPartitions(4) ) # long time
True
```

Testing the \( \pi(f^*) = (\pi(f))^* \) relation:

```
sage: all( M[I].algebraic_complement().to_quasisymmetric_function() == M[I].to_quasisymmetric_function().star_involution() for I in OrderedSetPartitions(4) )
True
```
Todo: Check further commutative squares.

\texttt{coalgebraic_complement()}

Return the image of the element \texttt{self} of \emph{WQSym} under the coalgebraic complement involution.

If \( u = (u_1, u_2, \ldots, u_n) \) is a packed word, then the \emph{reversal} of \( u \) is defined to be the packed word \((u_n, u_{n-1}, \ldots, u_1)\). This reversal is denoted by \( u' \).

The coalgebraic complement involution is defined as the linear map \( WQSym \rightarrow WQSym \) that sends each basis element \( M_u \) of the monomial basis of \( WQSym \) to the basis element \( M_{u'} \). This is a graded coalgebra automorphism and an algebra anti-automorphism of \( WQSym \). Denoting by \( f' \) the image of an element \( f \in WQSym \) under the coalgebraic complement involution, it can be shown that every packed word \( u \) satisfies

\[(M_u)^r = M_{u'}, \quad (X_u)^r = X_{u'}\]

where standard notations for classical bases of \( WQSym \) are being used (that is, \( M \) for the monomial basis, and \( X \) for the characteristic basis).

This can be restated in terms of ordered set partitions: For any ordered set partition \( R \) of \([n]\), let \( \overline{R} \) denote the complement of \( R \) (defined in \texttt{complement()}). Then,

\[(M_A)^r = M_{\overline{A}}, \quad (X_A)^r = X_{\overline{A}}\]

for any ordered set partition \( A \).

Recall that \( WQSym \) is a subring of the ring of all bounded-degree noncommutative power series in countably many indeterminates. The latter ring has an obvious continuous algebra anti-endomorphism which sends each letter \( x_i \) to \( x_i' \) (and thus sends each monomial \( x_{i_1}x_{i_2} \cdots x_{i_n} \) to \( x_{i_n}x_{i_{n-1}} \cdots x_{i_1} \)). This anti-endomorphism is actually an involution. The coalgebraic complement involution is simply the restriction of this involution to the subring \( WQSym \).

The formula describing coalgebraic complements on the \( Q \) basis (\emph{WordQuasiSymmetricFunctions}. \emph{StronglyCoarser}) is more complicated, and requires some definitions. We define a partial order \( \leq \) on the set of all ordered set partitions as follows: \( A \leq B \) if and only if \( A \) is strongly finer than \( B \) (see \texttt{is_strongly_finer()} for a definition of this). The \emph{length} \( l(R) \) of an ordered set partition \( R \) shall be defined as the number of parts of \( R \). Use the notation \( Q \) for the \( Q \) basis. For any ordered set partition \( A \) of \([n]\), we have

\[(Q_A)^r = \sum_P c_{A,P} Q_P,\]

where the sum is over all ordered set partitions \( P \) of \([n]\), and where the coefficient \( c_{A,P} \) is defined as follows:

- If there exists an ordered set partition \( R \) satisfying \( R \leq P \) and \( A \leq \overline{R} \), then this \( R \) is unique, and \( c_{A,P} = (-1)^{l(R)-l(P)} \).
- If there exists no such \( R \), then \( c_{A,P} = 0 \).

The formula describing coalgebraic complements on the \( \Phi \) basis (\emph{WordQuasiSymmetricFunctions}. \emph{StronglyFiner}) is identical to the above formula for the \( Q \) basis, except that the \( \leq \) sign has to be replaced by \( \geq \) in the definition of the coefficients \( c_{A,P} \). In fact, both formulas are particular cases of the general formula for involutions described in the documentation of \texttt{algebraic_complement()}.

If we let \( \pi \) be the canonical projection \( WQSym \rightarrow QSym \), then each \( f \in WQSym \) satisfies \( \pi(f^r) = \pi(f) \).

See also:

\texttt{algebraic_complement()}, \texttt{star_involution()}
EXAMPLES:

Recall that the index set for the bases of $WQS\text{ym}$ is given by ordered set partitions, not packed words. Translated into the language of ordered set partitions, the coalgebraic complement involution acts on the Monomial basis by complementing the ordered set partition. In other words, we have

$$(M_A)^\tau = M_{\pi}$$

for any standard ordered set partition $P$. Let us check this in practice:

```plaintext
sage: WQSym = algebras.WQSym(ZZ)
sage: M = WQSym.M()
sage: M[[1,3],[2]].coalgebraic_complement()
M[[1, 3], {2}]
sage: M[[1,2],[3]].coalgebraic_complement()
M[[2, 3], {1}]
sage: M[[1],[4],[2,3]].coalgebraic_complement()
M[[4], {1}, {2, 3}]
sage: M[[1,4],[2,5],[3,6]].coalgebraic_complement()
M[[3, 6], {2, 5}, {1, 4}]
sage: (3*M[[1]] - 4*M[[1],[2]])+ 5*M[[1],[2]]).coalgebraic_complement()
-4*M[] + 3*M[{1}] + 5*M[{2}, {1}]
sage: X = WQSym.X()
sage: X[[1,3],[2]].coalgebraic_complement()
X[[1, 3], {2}]
sage: C = WQSym.C()
sage: C[[1,3],[2]].coalgebraic_complement()
C[[1, 3], {2}]
sage: Q = WQSym.Q()
sage: Q[[1,2],[5,6],[3,4]].coalgebraic_complement()
Q[[1, 2, 5, 6], {3, 4}] + Q[{5, 6}, {1, 2}, {3, 4}] - Q[{5, 6}, {1, 2, 3, 4}]
sage: Phi = WQSym.Phi()
sage: Phi[[2],[1,3]].coalgebraic_complement()
-Phi[{2}, {1}, {3}]
```

The coalgebraic complement involution intertwines the antipode and the inverse of the antipode:

```plaintext
sage: all( M(I).antipode().coalgebraic_complement().antipode() == M(I).coalgebraic_complement() for I in OrderedSetPartitions(4) )
True
```

Testing the $\pi(f^\tau) = \pi(f)$ relation above:

```plaintext
sage: all( M[I].coalgebraic_complement().to_quasisymmetric_function() == M[I].to_quasisymmetric_function() for I in OrderedSetPartitions(4) )
True
```

Todo: Check further commutative squares.

`star_involution()`

Return the image of the element `self` of $WQS\text{ym}$ under the star involution.
The star involution is the composition of the algebraic complement involution \( \text{algebraic_complement()} \) with the coalgebraic complement involution \( \text{coalgebraic_complement()} \). The composition can be performed in either order, as the involutions commute.

The star involution is a graded Hopf algebra anti-automorphism of \( WQSym \). Let \( f^* \) denote the image of an element \( f \in WQSym \) under the star involution. Let \( M, X, Q \) and \( \Phi \) stand for the monomial, characteristic, \( Q \) and \( \Phi \) bases of \( WQSym \). For any ordered set partition \( A \) of \( [n] \), we let \( A^* \) denote the complement \( \text{complement()} \) of the reversal \( \text{reversed()} \) of \( A \). Then, for any ordered set partition \( A \) of \( [n] \), we have

\[
(M_A)^* = M_{A^*}, \quad (X_A)^* = X_{A^*}, \quad (Q_A)^* = Q_{A^*}, \quad (\Phi_A)^* = \Phi_{A^*}.
\]

The star involution \( \text{star_involution()} \) on the ring of noncommutative symmetric functions is a restriction of the star involution on \( WQSym \).

If we denote the star involution \( \text{star_involution()} \) of the quasisymmetric functions by \( f \mapsto f^* \), and if we let \( \pi \) be the canonical projection \( WQSym \to QSym \), then each \( f \in WQSym \) satisfies \( \pi(f^*) = (\pi(f))^* \).

Todo: More commutative diagrams? FQSym and FSym need their own \text{star_involution} methods defined first.

See also: \text{algebraic_complement()}, \text{coalgebraic_complement()}

EXAMPLES:

Keep in mind that the default input method for basis keys of \( WQSym \) is by entering an ordered set partition, not a packed word. Let us check the basis formulas for the star involution:

\[\begin{align*}
\text{sage: } & WQSym = \text{algebras.WQSym(ZZ)} \\
\text{sage: } & M = WQSym.M() \\
\text{sage: } & M[[1,3], [2,4,5]].\text{star_involution()} \\
\text{M} & \{1, 2, 4\}, \{3, 5\} \\
\text{sage: } & M[[1,3],[2]].\text{star_involution()} \\
\text{M} & \{2\}, \{1, 3\} \\
\text{sage: } & M[[1,4],[2,5],[3,6]].\text{star_involution()} \\
\text{M} & \{1, 4\}, \{2, 5\}, \{3, 6\} \\
\text{sage: } & (3^*M[[1]] - 4^*M[[1]] + 5^*M[[1],[2]]).\text{star_involution()} \\
& -4^*M[] + 3^*M[[1]] + 5^*M[[1],[2]] \\
\text{sage: } & X = WQSym.X() \\
\text{sage: } & X[[1,3],[2]].\text{star_involution()} \\
\text{X} & \{2\}, \{1, 3\} \\
\text{sage: } & C = WQSym.C() \\
\text{sage: } & C[[1,3],[2]].\text{star_involution()} \\
& -C[[1, 2, 3]] - C[[1, 3], \{2\}] + C[\{2\}, \{1, 3\}] \\
\text{sage: } & Q = WQSym.Q() \\
\text{sage: } & Q[[1,3],[2,4,5]].\text{star_involution()} \\
\text{Q} & \{1, 2, 4\}, \{3, 5\} \\
\text{sage: } & \Phi = WQSym.Phi() \\
\text{sage: } & \Phi[[1,3],[2,4,5]].\text{star_involution()} \\
\text{Phi} & \{1, 2, 4\}, \{3, 5\}
\end{align*}\]

Testing the formulas for \( (Q_A)^* \) and \( (\Phi_A)^* \):
The star involution commutes with the antipode:

```
sage: all(M[I].antipode().star_involution() == M[I].star_involution().antipode() for I in OrderedSetPartitions(4) )
True
```

Testing the $\pi(f^*) = (\pi(f))^*$ relation:

```
sage: all( M[I].star_involution().to_quasisymmetric_function() == M[I].to_quasisymmetric_function().star_involution() for I in OrderedSetPartitions(4) )
True
```

Testing the fact that the star involution on the noncommutative symmetric functions is a restriction of the star involution on $WQSym$:

```
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: R = NCSF.R()
sage: all(R[I].star_involution().to_fqsym().to_wqsym() == R[I].to_fqsym().to_wqsym().star_involution() for I in Compositions(4) )
True
```

Todo: Check further commutative squares.

`to_quasisymmetric_function()`

The projection of `self` to the ring $QSym$ of quasisymmetric functions.

There is a canonical projection $\pi : WQSym \rightarrow QSym$ that sends every element $M_P$ of the monomial basis of $WQSym$ to the monomial quasisymmetric function $M_c$, where $c$ is the composition whose parts are the sizes of the blocks of $P$. This $\pi$ is a ring homomorphism.

**OUTPUT:**
- an element of the quasisymmetric functions in the monomial basis

**EXAMPLES:**

```
sage: M = algebras.WQSym(QQ).M()
sage: M[[1,3],[2]].to_quasisymmetric_function()
M[2, 1]
sage: (M[[1,3],[2]] + 3*M[[2,3],[1]] - M[[1,2,3],[]]).to_quasisymmetric_function()
sage: X, Y = M[[1,3],[2]], M[[1,2,3],[]]
sage: X.to_quasisymmetric_function() * Y.to_quasisymmetric_function() == (X*Y).to_quasisymmetric_function()
```

(continues on next page)
True

sage: C = algebras.WQSym(QQ).C()
sage: C[[2,3],[1,4]].to_quasisymmetric_function() == M(C[[2,3],[1,4]]).to_˓→quasisymmetric_function()
True

dsage: C2 = algebras.WQSym(GF(2)).C()
sage: C2[[1,2],[3,4]].to_quasisymmetric_function()
M[2, 2]
sage: C2[[2,3],[1,4]].to_quasisymmetric_function()
M[4]

class ParentMethods
Bases: object

degree_on_basis(t)
Return the degree of an ordered set partition in the algebra of word quasi-symmetric functions.
This is the sum of the sizes of the blocks of the ordered set partition.

EXAMPLES:

\begin{verbatim}
sage: A = algebras.WQSym(QQ).M()
sage: u = OrderedSetPartition([[2,1],])
sage: A.degree_on_basis(u)
2
sage: u = OrderedSetPartition([[2], [1]])
sage: A.degree_on_basis(u)
2
\end{verbatim}

is_commutative()
Return whether self is commutative.

EXAMPLES:

\begin{verbatim}
sage: M = algebras.WQSym(ZZ).M()
sage: M.is_commutative()
False
\end{verbatim}

is_field(proof=True)
Return whether self is a field.

EXAMPLES:

\begin{verbatim}
sage: M = algebras.WQSym(QQ).M()
sage: M.is_field()
False
\end{verbatim}

one_basis()
Return the index of the unit.

EXAMPLES:

5.1. Comprehensive Module List
sage: A = algebras.WQSym(QQ).M()
sage: A.one_basis()
[]

super_categories()

The super categories of self.

EXAMPLES:

```
sage: from sage.combinat.chas.wqsym import WQSymBases
sage: WQSym = algebras.WQSym(ZZ)
sage: bases = WQSymBases(WQSym, True)
sage: bases.super_categories()
[Category of realizations of Word Quasi-symmetric functions over Integer Ring,
 Join of Category of realizations of hopf algebras over Integer Ring
 and Category of graded algebras over Integer Ring
 and Category of graded coalgebras over Integer Ring,
 Category of graded connected hopf algebras with basis over Integer Ring]
sage: bases = WQSymBases(WQSym, False)
sage: bases.super_categories()
[Category of realizations of Word Quasi-symmetric functions over Integer Ring,
 Join of Category of realizations of hopf algebras over Integer Ring
 and Category of graded algebras over Integer Ring
 and Category of graded coalgebras over Integer Ring,
 Join of Category of filtered connected hopf algebras with basis over Integer Ring
 and Category of graded algebras over Integer Ring
 and Category of graded coalgebras over Integer Ring]
```

class sage.combinat.chas.wqsym.WQSymBasis_abstract(alg, graded=True)

Bases: CombinatorialFreeModule, BindableClass

Abstract base class for bases of $WQSym$.

This must define two attributes:

- _prefix – the basis prefix
- _basis_name – the name of the basis (must match one of the names that the basis can be constructed from $WQSym$)

an_element()

Return an element of self.

EXAMPLES:

```
sage: M = algebras.WQSym(QQ).M()
sage: M.an_element()
M[{1}] + 2*M[{1}, {2}]
```

options = Current options for WordQuasiSymmetricFunctions element - display: normal - objects: compositions

some_elements()

Return some elements of the word quasi-symmetric functions.
EXAMPLES:

```python
sage: M = algebras.WQSym(QQ).M()
sage: M.some_elements()
[[M[], M[{1}]], M[{1, 2}],
 M[{1}]] + M[{1}, {2}],
 M[] + 1/2*M[{1}]]
```

class sage.combinat.chas.wqsym.WordQuasiSymmetricFunctions(R):
    Bases: UniqueRepresentation, Parent

The word quasi-symmetric functions.

The ring of word quasi-symmetric functions can be defined as a subring of the ring of all bounded-degree noncommutative power series in countably many indeterminates (i.e., elements in $R\langle\langle x_1, x_2, x_3, \ldots \rangle\rangle$ of bounded degree). Namely, consider words over the alphabet $\{1, 2, 3, \ldots \}$; every noncommutative power series is an infinite $R$-linear combination of these words. For each such word $w$, we define the packing of $w$ to be the word $\text{pack}(w)$ that is obtained from $w$ by replacing the smallest letter that appears in $w$ by 1, the second-smallest letter that appears in $w$ by 2, etc. (for example, $\text{pack}(4112774) = 3112443$). A word $w$ is said to be packed if $\text{pack}(w) = w$. For each packed word $u$, we define the noncommutative power series $M_u = \sum w$, where the sum ranges over all words $w$ satisfying $\text{pack}(w) = u$. The span of these power series $M_u$ is a subring of the ring of all noncommutative power series; it is called the ring of word quasi-symmetric functions, and is denoted by $\mathcal{WQSym}$.

For each nonnegative integer $n$, there is a bijection between packed words of length $n$ and ordered set partitions of $\{1, 2, \ldots, n\}$. Under this bijection, a packed word $u = (u_1, u_2, \ldots, u_n)$ of length $n$ corresponds to the ordered set partition $P = (P_1, P_2, \ldots, P_k)$ of $\{1, 2, \ldots, n\}$ whose $i$-th part $P_i$ (for each $i$) is the set of all $j \in \{1, 2, \ldots, n\}$ such that $u_j = i$.

The basis element $M_u$ is also denoted as $M_P$ in this situation. The basis $(M_P)_P$ is called the Monomial basis and is implemented as Monomial.

Other bases are the cone basis (aka C basis), the characteristic basis (aka X basis), the Q basis and the Phi basis.

Bases of $\mathcal{WQSym}$ are implemented (internally) using ordered set partitions. However, the user may access specific basis vectors using either packed words or ordered set partitions. See the examples below, noting especially the section on ambiguities.

$\mathcal{WQSym}$ is endowed with a connected graded Hopf algebra structure (see Section 2.2 of [NoThWi08], Section 1.1 of [FoiMal14] and Section 4.3.2 of [MeNoTh11]) given by

$$\Delta(M_{(P_1, \ldots, P_k)}) = \sum_{i=0}^{k} M_{\text{st}(P_k, \ldots, P_i)} \otimes M_{\text{st}(P_{i+1}, \ldots, P_k)}.$$ 

Here, for any ordered set partition $(Q_1, \ldots, Q_k)$ of a finite set $Z$ of integers, we let $\text{st}(Q_1, \ldots, Q_k)$ denote the set partition obtained from $Z$ by replacing the smallest element appearing in it by 1, the second-smallest element by 2, and so on.

A rule for multiplying elements of the monomial basis relies on the quasi-shuffle product of two ordered set partitions. The quasi-shuffle product $\square$ is given by $\text{ShuffleProduct_overlapping}$ with the $+$ operation in the overlapping of the shuffles being the union of the sets. The product $M_PM_Q$ for two ordered set partitions $P$ and $Q$ of $[n]$ and $[m]$ is then given by

$$M_PM_Q = \sum_{R \in P \square Q^+} M_R,$$

where $Q^+$ means $Q$ with all numbers shifted upwards by $n$.

Sometimes, $\mathcal{WQSym}$ is also denoted as $\mathcal{NCQSym}$. 

5.1. Comprehensive Module List
REFERENCES:

• [FoiMal14]
• [MeNoTh11]
• [NoThWi08]
• [BerZab05]

EXAMPLES:

Constructing the algebra and its Monomial basis:

```
sage: WQSym = algebras.WQSym(ZZ)
sage: WQSym
Word Quasi-symmetric functions over Integer Ring
sage: M = WQSym.M()
sage: M
Word Quasi-symmetric functions over Integer Ring in the Monomial basis
sage: M[]
```

Calling basis elements using packed words:

```
sage: x = M[1,2,1]; x
M[{1, 3}, {2}]
sage: x == M[[1,2,1]] == M[Word([1,2,1])]
True
sage: y = M[1,1,2] - M[1,2,2]; y
-M[{}, {2, 3}] + M[{1, 2}, {3}]
```

Calling basis elements using ordered set partitions:

```
sage: z = M[[1,2,3]]; z
M[{1, 2, 3}]
sage: z == M[[[1,2,3]]] == M[OrderedSetPartition([[1,2,3]])]
True
sage: M[[1,2],[3]]
M[{1, 2}, {3}]
```

Note that expressions above are output in terms of ordered set partitions, even when input as packed words. Output as packed words can be achieved by modifying the global options. (See OrderedSetPartitions.options() for further details.):

```
sage: M.options.objects = "words"
sage: y
-M[1, 2, 2] + M[1, 1, 2]
sage: M.options.display = "compact"
sage: y
-M[1,2] + M[112]
sage: z
M[111]
```

The options should be reset to display as ordered set partitions:
Illustration of the Hopf algebra structure:

```
sage: M[[2, 3], [5], [6], [4], [1]].coproduct()
sage: _ == M[5,1,1,4,2,3].coproduct()
```

```
sage: M[[1,1,1]] * M[[1,1,2]]  # packed words
sage: M[[1, 1, 2], [1, 3], [2], [4]]
```

```
sage: x = M[[1],[2],[3]] + 3*M[[2],[1]]
sage: x.counit()
```

Ambiguities

Some ambiguity arises when accessing basis vectors with the dictionary syntax, i.e., $M[\ldots]$. A common example is when referencing an ordered set partition with one part. For example, in the expression $M[[1,2]]$, does $[[1,2]]$ refer to an ordered set partition or does $[1,2]$ refer to a packed word? We choose the latter: if the received arguments do not behave like a tuple of iterables, then view them as describing a packed word. (In the running example, one argument is received, which behaves as a tuple of integers.) Here are a variety of ways to get the same basis vector:

```
sage: x = M[[1,1]]; x
```

```
sage: x == M[[1,1]]  # treated as word
sage: x == M[[1,2,1]]  # treated as ordered set partitions
```

Todo:

- Dendriform structure.
C alias of Cone

class Characteristic(alg)

Bases: WQSymBasis_abstract

The Characteristic basis of \( W_{QSym} \).

The Characteristic basis is a graded basis \( (X_P) \) of \( W_{QSym} \), indexed by ordered set partitions \( P \). It is defined by

\[
X_P = (-1)^{\ell(P)}M_P,
\]

where \((M_P)_P\) denotes the Monomial basis, and where \( \ell(P) \) denotes the number of blocks in an ordered set partition \( P \).

EXAMPLES:

```python
sage: WQSym = algebras.WQSym(QQ)
sage: X = WQSym.X(); X
Word Quasi-symmetric functions over Rational Field in the Characteristic basis
sage: X[[[1,2,3]]] * X[[1,2],[3]]
X[{1, 2, 3}, {4, 5}, {6}] - X[{1, 2, 3, 4, 5}, {6}] + X[{4, 5}, {1, 2, 3}, {6}] - X[{4, 5}, {1, 2, 3, 6}]
+ X[{4, 5}, {6}, {1, 2, 3}]
sage: X[[[1, 4], [3], [2]]].coproduct()
X[] # X[{1, 4}, {3}, {2}] + X[{1, 2}] # X[{2}, {1}] + X[{1, 3}, {2}] # X[{1}] + X[{1}, 4], {3}, {2}] # X[]
sage: M = WQSym.M()
sage: M(X[[[1, 2, 3],]]) -M[{1, 2, 3}]
sage: M(X[[[1, 3], [2]]])
M[{1, 3}, {2}]
sage: X(M[[[1, 2, 3],]])
-X[{1, 2, 3}]
sage: X(M[[[1, 3], [2]]])
X[{1, 3}, {2}]
```

class Element

Bases: IndexedFreeModuleElement

algebraic_complement()

Return the image of the element self of \( W_{QSym} \) under the algebraic complement involution.

See \( WQSymBases.ElementMethods.algebraic_complement() \) for a definition of the involution and for examples.

See also:

\( \text{coalgebraic_complement()}, \text{star_involution()} \)

EXAMPLES:
sage: WQSym = algebras.WQSym(ZZ)
sage: X = WQSym.X()
sage: X[[1,2],[5,6],[3,4]].algebraic_complement()
X[[3, 4], {1, 2}, {5, 6}]
sage: X[[3], [1, 2], [4]].algebraic_complement()
X[4], {1, 2}, {3}]

coalgebraic_complement()

Return the image of the element self of WQSym under the coalgebraic complement involution.

See WQSymBases.ElementMethods.coalgebraic_complement() for a definition of the involution and for examples.

See also:

algebraic_complement(), star_involution()

EXAMPLES:

sage: WQSym = algebras.WQSym(ZZ)
sage: X = WQSym.X()
sage: X[[1,2],[5,6],[3,4]].coalgebraic_complement()
X[5, 6], {1, 2}, {3, 4}]
sage: X[[3], [1, 2], [4]].coalgebraic_complement()
X[2], {3, 4}, {1}]

star_involution()

Return the image of the element self of WQSym under the star involution.

See WQSymBases.ElementMethods.star_involution() for a definition of the involution and for examples.

See also:

algebraic_complement(), coalgebraic_complement()

EXAMPLES:

sage: WQSym = algebras.WQSym(ZZ)
sage: X = WQSym.X()
sage: X[[1,2],[5,6],[3,4]].star_involution()
X[3, 4], {1, 2}, {5, 6}]
sage: X[[3], [1, 2], [4]].star_involution()
X[1], {3, 4}, {2}]

class Cone(alg)

Bases: WQSymBasis_abstract

The Cone basis of WQSym.

Let (X_P)_P denote the Characteristic basis of WQSym. Denote the quasi-shuffle of two ordered set partitions A and B by A □ B. For an ordered set partition P = (P_1, ..., P_\ell), we form a list of ordered set partitions [P] := (P'_1, ..., P'_k) as follows. Define a strictly decreasing sequence of integers \ell + 1 = i_0 > i_1 > ... > i_k = 1 recursively by requiring that \min_{j<i_j} P_{i_j} \leq \min_{a<i_{j-1}} P_a for all a < i_{j-1}. Set P'_j = (P_{i_j}, ..., P_{i_{j-1}-1}).
The Cone basis \((C_P)_P\) is defined by
\[
C_P = \sum_Q X_Q,
\]
where the sum is over all elements \(Q\) of the quasi-shuffle product \(P'_1 \square P'_2 \square \cdots \square P'_k\) with \([P] = (P'_1, \ldots, P'_k)\).

EXAMPLES:

```python
sage: WQSym = algebras.WQSym(QQ)
sage: C = WQSym.C()
sage: X = WQSym.X()
sage: C[[2,3],[1,4]]
Word Quasi-symmetric functions over Rational Field in the Cone basis

sage: X(C[[2,3],[1,4]])
X[{1, 2, 3, 4}] + X[{1, 4}, {2, 3}] + X[{2, 3}, {1, 4}]
sage: X(C[[1,4],[2,3]])
X[{1, 4}, {2, 3}]
sage: X(C[[2,3],[1],[4]])
X[{1}, {2, 3}, {4}] + X[{1, 2, 3}, {4}] + X[{1}, {2, 3}, {4}]

sage: C(X[[2,3],[1,4]])
-C[{1, 2, 3, 4}] - C[{1, 4}, {2, 3}] + C[{2, 3}, {1, 4}]

REFERENCES:

- Section 4 of [Early2017]

Todo: Experiments suggest that \texttt{algebraic_complement()}, \texttt{coalgebraic_complement()}, and \texttt{star_involution()} should have reasonable formulas on the C basis; at least the coefficients of the outputs on any element of the C basis seem to be always 0, 1, −1. Is this true? What is the formula?

\textbf{some_elements()}

Return some elements of the word quasi-symmetric functions in the Cone basis.

EXAMPLES:

```python
sage: C = algebras.WQSym(QQ).C()
sage: C.some_elements()
[C[], C[{1}], C[{1, 2}], C[] + 1/2*C[{1}]]
```

\textbf{M}

alias of \texttt{Monomial}

\textbf{class Monomial}(\texttt{alg, graded=True})

Bases: \texttt{WQSymBasis_abstract}
The Monomial basis of $W_{QS\text{ym}}$.

The family $(M_u)$, as defined in \texttt{WordQuasiSymmetricFunctions} with $u$ ranging over all packed words, is a basis for the free $R$-module $W_{QS\text{ym}}$ and called the Monomial basis. Here it is labelled using ordered set partitions.

**EXAMPLES:**

```sage
sage: WQSym = algebras.WQSym(QQ)
sage: M = WQSym.M(); M
Word Quasi-symmetric functions over Rational Field in the Monomial basis
sage: sorted(M.basis(2))
[M[{}] , M[{1} , {2}], M[{2} , {1}], M[{1, 2}]]
```

**coproduct_on_basis($x$)**

Return the coproduct of \texttt{self} on the basis element indexed by the ordered set partition $x$.

**EXAMPLES:**

```sage
sage: M = algebras.WQSym(QQ).M()

sage: M.coproduct(M.basis().one())  # indirect doctest
M[{}]
```

```sage
sage: M.coproduct(M.basis([[1]]) )  # indirect doctest
M[{}] + M[{1}]
```

```sage
sage: M.coproduct( M.basis([[1], [2]]) )
M[{}]
```

```sage
sage: M.coproduct( M.basis([[1], {2}]) )
M[{}]
```

**product_on_basis($x$, $y$)**

Return the (associative) $*$ product of the basis elements of \texttt{self} indexed by the ordered set partitions $x$ and $y$.

This is the shifted quasi-shuffle product of $x$ and $y$.

**EXAMPLES:**

```sage
sage: A = algebras.WQSym(QQ).M()
sage: x = OrderedSetPartition([[1], [2, 3]])
sage: y = OrderedSetPartition([[1, 2]])
sage: z = OrderedSetPartition([[1, 2], [3]])
sage: A.product_on_basis(x, y)
M[{}] + M[{1}, {2, 3}, {4, 5}] + M[{}, {2, 3, 4, 5}]
```

```sage
sage: A.product_on_basis(x, z)
M[{}] + M[{1}, {2, 3}, {4, 5}, {6}] + M[{}, {2, 3, 4, 5, 6}]
```

```sage
sage: A.product_on_basis(y, y)
M[{}] + M[{1}, {2}, {3}, {4}, {5}, {6}]
```

5.1. Comprehensive Module List
Phi
    alias of StronglyFiner

Q
    alias of StronglyCoarser

class StronglyCoarser(alg)

Bases: WQSymBasis_abstract

The Q basis of \( \mathcal{W}_{\mathcal{Q}_{\text{Sym}}} \).

We define a partial order \( \leq \) on the set of all ordered set partitions as follows: \( A \leq B \) if and only if \( A \) is strongly finer than \( B \) (see is_strongly_finer() for a definition of this).

The \( Q \) basis \( (Q_P)_P \) is a basis of \( \mathcal{W}_{\mathcal{Q}_{\text{Sym}}} \) indexed by ordered set partitions, and is defined by

\[
Q_P = \sum M_W,
\]

where the sum is over ordered set partitions \( W \) satisfying \( P \leq W \).

EXAMPLES:

```python
sage: WQSym = algebras.WQSym(QQ)
sage: M = WQSym.M(); Q = WQSym.Q()
sage: Q
Word Quasi-symmetric functions over Rational Field in the Q basis

sage: Q(M[[2,3],[1,4]])
Q[{2, 3}, {1, 4}]
sage: Q(M[[1,2],[3,4]])
Q[{1, 2}, {3, 4}] - Q[{1, 2, 3, 4}]
sage: M(Q[[1,2],[3,4]])
M[{1, 2}, {3, 4}] + M[{1, 2, 3, 4}]
sage: M(Q[[2,3],[1],[4]])
M[{2, 3}, {1}, {4}] + M[{2, 3}, {1, 4}]
sage: M(Q[[3],[2,5],[1,4]])
M[{3}, {2, 5}, {1, 4}]
sage: M(Q[[1,4],[2,3],[5],[6]])
M[{1, 4}, {2, 3}, {5}, {6}] + M[{1, 4}, {2, 3}, {5, 6}]
  + M[{1, 4}, {2, 3, 5}, {6}] + M[{1, 4}, {2, 3, 5, 6}]
sage: Q([[1, 3], [2]]) * Q([[1], [2]])
Q[{1, 3}, {2}, {4}, {5}] + Q[{1, 3}, {4}, {2}, {5}]
  + Q[{1, 3}, {4}, {5}, {2}] + Q[{4}, {1, 3}, {2}, {5}]
  + Q[{4}, {1, 3}, {5}, {2}] + Q[{4}, {5}, {1, 3}, {2}]
sage: Q([]) # Q([[1, 3], [2]])
Q[[]] # Q([[1, 3], {2}]) + Q([[1, 2]]) # Q[[]] + Q([[1, 3], {2}]) # Q[[]]

REFERENCES:

• Section 6 of [BerZab05]

class Element

    Bases: IndexedFreeModuleElement
algebraic_complement()
Return the image of the element self of WQSym under the algebraic complement involution.

See WQSymBases.ElementMethods.algebraic_complement() for a definition of the involution and for examples.

See also:
coalgebraic_complement(), star_involution()

EXAMPLES:

```
sage: WQSym = algebras.WQSym(ZZ)
sage: Q = WQSym.Q()
sage: Q[[1,2],[5,6],[3,4]].algebraic_complement()  
Q[{3, 4}, {1, 2, 5, 6}] + Q[{3, 4}, {5, 6}, {1, 2}] 
  - Q[{3, 4, 5, 6}, {1, 2}]
sage: Q[[3], [1, 2], [4]].algebraic_complement()  
Q[{1, 2, 4}, {3}] + Q[{4}, {1, 2}, {3}] - Q[{4}, {1, 2, 3}]
```

calgebraic_complement()
Return the image of the element self of WQSym under the coalgebraic complement involution.

See WQSymBases.ElementMethods.coalgebraic_complement() for a definition of the involution and for examples.

See also:
algebraic_complement(), star_involution()

EXAMPLES:

```
sage: WQSym = algebras.WQSym(ZZ)
sage: Q = WQSym.Q()
sage: Q[[1,2],[5,6],[3,4]].coalgebraic_complement()  
Q[{1, 2, 5, 6}, {3, 4}] + Q[{5, 6}, {1, 2}, {3, 4}] - Q[{5, 6}, {1, 2, 3, 4}]
sage: Q[[3], [1, 2], [4]].coalgebraic_complement()  
Q[{2}, {1, 3, 4}] + Q[{2}, {3, 4}, {1}] - Q[{2, 3, 4}, {1}]
```

star_involution()
Return the image of the element self of WQSym under the star involution.

See WQSymBases.ElementMethods.star_involution() for a definition of the involution and for examples.

See also:
algebraic_complement(), coalgebraic_complement()

EXAMPLES:

```
sage: WQSym = algebras.WQSym(ZZ)
sage: Q = WQSym.Q()
sage: Q[[1,2],[5,6],[3,4]].star_involution()  
Q[{3, 4}, {1, 2}, {5, 6}]
sage: Q[[3], [1, 2], [4]].star_involution()  
Q[{1}, {3, 4}, {2}]
```
coproduct_on_basis(x)

Return the coproduct of self on the basis element indexed by the ordered set partition x.

EXAMPLES:

```python
sage: Q = algebras.WQSym(QQ).Q()
sage: Q.coproduct(Q.one())  # indirect doctest
Q[] # Q[]
sage: Q.coproduct(Q([[1]]))  # indirect doctest
Q[] # Q[[1]] + Q[[1]] # Q[]
sage: Q.coproduct(Q([[1, 2]]))
Q[] # Q[[1, 2]] + Q[[1, 2]] # Q[]
sage: Q.coproduct(Q([[1], [2]]))
Q[] # Q[[1], [2]] + Q[[1]] # Q[[1]] + Q[[1], {2}] # Q[]
sage: Q[[1, 2], [3], [4]].coproduct()
Q[] # Q[[1, 2], {3}, {4}] + Q[[1, 2]] # Q[[1], {2}]
+ Q[[1, 2], {3}] # Q[[1]] + Q[[1, 2], {3}, {4}] # Q[]
```

product_on_basis(x, y)

Return the (associative) * product of the basis elements of the Q basis self indexed by the ordered set partitions x and y.

This is the shifted shuffle product of x and y.

EXAMPLES:

```python
sage: A = algebras.WQSym(QQ).Q()
sage: x = OrderedSetPartition([[1], [2, 3]])
sage: y = OrderedSetPartition([[1, 2]])
sage: z = OrderedSetPartition([[1, 2], [3]])
sage: A.product_on_basis(x, y)
Q[[1], {2, 3}, {4, 5}] + Q[[1], {4, 5}, {2, 3}]
+ Q[4, 5], {1, {2, 3}]}
sage: A.product_on_basis(x, z)
Q[[1], {2, 3}, {4, 5}, {6}] + Q[[1], {4, 5}, {2, 3}, {6}]
+ Q[4, 5], {1}, {2, 3}, {6}]
+ Q[4, 5], {1}, {6}, {2, 3} + Q[4, 5], {1}, {2, 3}, {6}]
+ Q[4, 5], {1}, {6}, {2, 3} + Q[4, 5], {1}, {2, 3}]
sage: A.product_on_basis(y, y)
Q[[1, 2], {3, 4}] + Q[[3, 4], {1, 2}]
```

some_elements()

Return some elements of the word quasi-symmetric functions in the Q basis.

EXAMPLES:

```python
sage: Q = algebras.WQSym(QQ).Q()
sage: Q.some_elements()
[Q[], Q[[1]], Q[[1, 2]], Q[] + 1/2*Q[[1]]]
```

class StronglyFiner(alg)

Bases: WQSymBasis_abstract

The Phi basis of $WQSym$.

We define a partial order $\leq$ on the set of all ordered set partitions as follows: $A \leq B$ if and only if $A$ is strongly finer than $B$ (see is_strongly_finer() for a definition of this).
The Phi basis \((\Phi_P)_P\) is a basis of \(WQSym\) indexed by ordered set partitions, and is defined by

\[
\Phi_P = \sum M_W,
\]

where the sum is over ordered set partitions \(W\) satisfying \(W \leq P\).

Novelli and Thibon introduced this basis in [NovThi06] Section 2.7.2, and called it the quasi-ribbon basis. It later reappeared in [MeNoTh11] Section 4.3.2.

**EXAMPLES:**

```python
sage: WQSym = algebras.WQSym(QQ)
sage: N = WQSym.M(); Phi = WQSym.Phi()
sage: Phi
Word Quasi-symmetric functions over Rational Field in the Phi basis
sage: Phi(M[[2,3],[1,4]])
Phi[{2,3}, {1,4}] - Phi[{2}, {3}, {1,4}]
   - Phi[{2,3}, {1}, {4}] + Phi[{2}, {3}, {1, 4}]
sage: Phi(M[[1,2],[3,4]])
Phi[{1,2}, {3,4}] - Phi[{1}, {2}, {3,4}]
   - Phi[{1,2}, {3}, {4}] + Phi[{1}, {2}, {3, 4}]
sage: M(Phi[[1,2],[3,4]])
M[{1}, {2}, {3}, {4}] + M[{1,2}, {3}, {4}]
   + M[{1}, {2}, {3,4}] + M[{1,2}, {3,4}]
sage: M(Phi[[2,3],[1],[4]])
M[{2}, {3}, {1}, {4}] + M[{2,3}, {1}, {4}]
   + M[{2,3}, {1}, {4}]
sage: Phi[[1],[1,3],[2]]
Phi[{1,3}, {2}] + Phi[{2}, {1,3}, {2}]
   + Phi[{2}, {1,3}, {2}]
sage: Phi[[3,5],[1,4],[2]].coproduct()
Phi[{} # Phi[{3,5}, {1,4}, {2}]
   + Phi[{1}] # Phi[{4}, {1,3}, {2}]
   + Phi[{1,2}] # Phi[{1}, {3}, {2}]
   + Phi[{2}, {1}] # Phi[{2}, {1}]
   + Phi[{2}, {1,3}] # Phi[{1}]
   + Phi[{3,5}, {1,4}, {2}] # Phi[]
```

**REFERENCES:**

- Section 2.7.2 of [NovThi06]

**class Element**

Bases: `IndexedFreeModuleElement`

```
  algebraic_complement()
```

Return the image of the element self of \(WQSym\) under the algebraic complement involution.

See \(WQSymBases.ElementMethods.algebraic_complement()\) for a definition of the involution and for examples.
See also:

coalgebraic_complement(), star_involution()

EXAMPLES:

```python
sage: WQSym = algebras.WQSym(ZZ)
sage: Phi = WQSym.Phi()
sage: Phi[[1],[2,4],[3]].algebraic_complement()
-Phi[{3}, {2}, {4}, {1}] + Phi[{3}, {2, 4}, {1}] + Phi[{3}, {4}, {2}, {1}]
```

```python
sage: Phi[[1],[2,3],[4]].algebraic_complement()
-Phi[{4}, {2}, {3}, {1}] + Phi[{4}, {2, 3}, {1}] + Phi[{4}, {3}, {2}, {1}]
```

coproduct_on_basis(x)

Return the coproduct of self on the basis element indexed by the ordered set partition x.
The coproduct of the basis element $\Phi_x$ indexed by an ordered set partition $x$ of $[n]$ can be computed by the following formula ([NovThi06]):

$$\Delta \Phi_x = \sum \Phi_y \otimes \Phi_z,$$

where the sum ranges over all pairs $(y, z)$ of ordered set partitions $y$ and $z$ such that:

- $y$ and $z$ are ordered set partitions of two complementary subsets of $[n]$;
- $x$ is obtained either by concatenating $y$ and $z$, or by first concatenating $y$ and $z$ and then merging the two “middle blocks” (i.e., the last block of $y$ and the first block of $z$); in the latter case, the maximum of the last block of $y$ has to be smaller than the minimum of the first block of $z$ (so that when merging these blocks, their entries don’t need to be sorted).

EXAMPLES:

```sage
sage: Phi = algebras.WQSym(QQ).Phi()
sage: Phi.coproduct(Phi.one()) # indirect doctest
Phi[] # Phi[]
sage: Phi.coproduct(Phi([[1]])) # indirect doctest
Phi[] # Phi[[1]] + Phi[[1]] # Phi[]
sage: Phi.coproduct(Phi([[1,2]]))
Phi[] # Phi[[1, 2]] + Phi[[1]] # Phi[[1]] + Phi[[1, 2]] # Phi[]
sage: Phi.coproduct(Phi([[1],[2]]))
Phi[] # Phi[[1], [2]] + Phi[[1]] # Phi[[1]] + Phi[[1], {2}] # Phi[]
sage: Phi.coproduct(Phi([[[1,2],[3],[4]]].coproduct())
Phi[] # Phi[[1, 2], {3}, {4}] + Phi[[1]] # Phi[[1]] + Phi[[1], {2}, {3}] + Phi[[1, 2]] # Phi[[1, 2], {3}] + Phi[[1, 2], {3}] # Phi[[1]]
```

product_on_basis($x$, $y$)

Return the (associative) $\ast$ product of the basis elements of the Phi basis $\text{self}$ indexed by the ordered set partitions $x$ and $y$.

This is obtained by the following algorithm (going back to [NovThi06]):

Let $x$ be an ordered set partition of $[m]$, and $y$ an ordered set partition of $[n]$. Transform $x$ into a list $u$ of all the $m$ elements of $[m]$ by writing out each block of $x$ (in increasing order) and putting bars between each two consecutive blocks; this is called a barred permutation. Do the same for $y$, but also shift each entry of the resulting barred permutation by $m$. Let $v$ be the barred permutation of $[m+n] \setminus [m]$ thus obtained. Now, shuffle the two barred permutations $u$ and $v$ (ignoring the bars) in all the $\binom{m+n}{m}$ possible ways. For each shuffle obtained, place bars between some entries of the shuffle, according to the following rule:

- If two consecutive entries of the shuffle both come from $u$, then place a bar between them if the corresponding entries of $u$ had a bar between them.
- If the first of two consecutive entries of the shuffle comes from $v$ and the second from $u$, then place a bar between them.

This results in a barred permutation of $[m+n]$. Transform it into an ordered set partition of $[m+n]$, by treating the bars as dividers separating consecutive blocks.

The product $\Phi_x \Phi_y$ is the sum of $\Phi_p$ with $p$ ranging over all ordered set partitions obtained this way.

EXAMPLES:

```sage
sage: A = algebras.WQSym(QQ).Phi()
sage: x = OrderedSetPartition([[1],[2,3]])
sage: y = OrderedSetPartition([[1,2]])
sage: z = OrderedSetPartition([[1,2],[3]])
```

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sage: A.product_on_basis(x, y)
Phi[{1}, {2, 3, 4, 5}] + Phi[{1}, {2, 4}, {3, 5}]
- Phi[{1}, {2, 4, 5}, {3}] + Phi[{1}, {2, 3, 5}]
- Phi[{1, 4}, {2, 5}, {3}] + Phi[{1, 4}, {2, 3, 5}]
- Phi[{4}, {1}, {2, 3, 5}] + Phi[{1}, {2, 5}, {3}]
- Phi[{4}, {1}, {2, 4}, {3}] + Phi[{1, 4}, {2, 3}, {5}]

sage: A.product_on_basis(x, z)
Phi[{1}, {2, 3, 4, 5}, {6}] + Phi[{1}, {2, 4}, {3, 5}, {6}]
- Phi[{1}, {2, 4, 5}, {3, 6}] + Phi[{1}, {2, 4, 5}, {6}, {3}]
- Phi[{1, 4}, {2, 3, 5}, {6}] + Phi[{1}, {2, 5}, {3, 6}]
- Phi[{1, 4}, {2, 5}, {6}, {3}] + Phi[{1, 4}, {2, 3}, {6}]
- Phi[{4}, {1}, {2, 3, 5}, {6}]
- Phi[{4}, {1}, {2, 5}, {3, 6}]
- Phi[{4}, {1}, {2, 5}, {6}, {3}]
- Phi[{4}, {1, 5}, {2, 3, 6}]
- Phi[{4}, {1, 5}, {2, 6}, {3}]
- Phi[{4}, {1, 5}, {6}, {2, 3}]
- Phi[{4}, {1, 5}, {6}, {1}, {2, 3}]
- Phi[{4}, {1, 5}, {6}, {1}, {2, 3}]

sage: A.product_on_basis(y, y)
Phi[{1, 2, 3, 4}] + Phi[{1, 3}, {2, 4}] + Phi[{1, 3, 4}, {2}]
- Phi[{3}, {1, 2, 4}] + Phi[{3}, {1, 4}, {2}]
- Phi[{3, 4}, {1}, {2}]

some_elements()
Return some elements of the word quasi-symmetric functions in the Phi basis.

EXAMPLES:

sage: Phi = algebras.WQSym(QQ).Phi()
sage: Phi.some_elements()
[Phi[], Phi[{1}], Phi[{1, 2}], Phi[] + 1/2*Phi[{1}]]

X
alias of Characteristic

a_realization()
Return a particular realization of self (the M-basis).

EXAMPLES:

sage: WQSym = algebras.WQSym(QQ)
sage: WQSym.a_realization()
Word Quasi-symmetric functions over Rational Field in the Monomial basis

options = Current options for WordQuasiSymmetricFunctions element - display: normal
- objects: compositions
5.1.17 Cluster algebras and quivers

- A compendium on the cluster algebra and quiver package in Sage [MS2011]
- *Quiver mutation types*
- *Quiver*
- *ClusterSeed*

5.1.18 ClusterSeed

A *cluster seed* is a pair \((B, x)\) with \(B\) being a *skew-symmetrizable* \((n + m \times n)\) -matrix and with \(x\) being an \(n\)-tuple of *independent elements* in the field of rational functions in \(n\) variables.

For the compendium on the cluster algebra and quiver package see [MS2011].

AUTHORS:

- Gregg Musiker: Initial Version
- Christian Stump: Initial Version
- Aram Dermenjian (2015-07-01): Updating ability to not rely solely on clusters
- Jesse Levitt (2015-07-01): Updating ability to not rely solely on clusters

REFERENCES:

- [FZ2007]
- [BDP2013]

See also:

For mutation types of cluster seeds, see `sage.combinat.cluster_algebra_quiver.quiver_mutation_type.QuiverMutationType()`. Cluster seeds are closely related to `sage.combinat.cluster_algebra_quiver.quiver.ClusterQuiver()`.

```python
class sage.combinat.cluster_algebra_quiver.cluster_seed.ClusterSeed(data, frozen=None, is_principal=False, user_labels=None, user_labels_prefix='x')
```

**Bases:** `SageObject`

The *cluster seed* associated to an *exchange matrix*.

**INPUT:**

- *data* – can be any of the following:
  
  ```
  * QuiverMutationType
  * str - a string representing a QuiverMutationType or a common quiver type (see Examples)
  * ClusterQuiver
  * Matrix - a skew-symmetrizable matrix
  * DiGraph - must be the input data for a quiver
  * List of edges - must be the edge list of a digraph for a quiver
  ```

**EXAMPLES:**
sage: S = ClusterSeed(['A',5]); S
A seed for a cluster algebra of rank 5 of type ['A', 5]

sage: S = ClusterSeed(['A',[2,5],1]); S
A seed for a cluster algebra of rank 7 of type ['A', [2, 5], 1]

sage: T = ClusterSeed(S); T
A seed for a cluster algebra of rank 7

sage: T = ClusterSeed(S._M); T
A seed for a cluster algebra of rank 7

sage: T = ClusterSeed(S.quiver()._digraph); T
A seed for a cluster algebra of rank 7

sage: T = ClusterSeed(S.quiver()._digraph.edges(sort=True)); T
A seed for a cluster algebra of rank 7

sage: S = ClusterSeed(['B',2]); S
A seed for a cluster algebra of rank 2 of type ['B', 2]

sage: S = ClusterSeed(['C',2]); S
A seed for a cluster algebra of rank 2 of type ['B', 2]

sage: S = ClusterSeed(['A', [5,0],1]); S
A seed for a cluster algebra of rank 5 of type ['D', 5]

sage: S = ClusterSeed(['GR',[3,7]]); S
A seed for a cluster algebra of rank 6 of type ['E', 6]

sage: S = ClusterSeed(['F', 4, [2,1]]); S
A seed for a cluster algebra of rank 6 of type ['F', 4, [1, 2]]

sage: S = ClusterSeed(['A',4]); S._use_fpolys
True

sage: S._use_d_vec
True

sage: S._use_g_vec
True

sage: S._use_c_vec
True

sage: S = ClusterSeed(['A', 4]); S.use_fpolys(False); S._use_fpolys
False

sage: S = ClusterSeed(DiGraph([[['a', 'b'], ['c', 'b'], ['c', 'd'], ['e', 'd']]]),
                               frozen =
                               ....:  ['c']); S
A seed for a cluster algebra of rank 4 with 1 frozen variable

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sage: S = ClusterSeed(["D", 4], user_labels = [-1, 0, 1, 2]); S
A seed for a cluster algebra of rank 4 of type ['D', 4]

LLM_gen_set(size_limit=-1)

Produce a list of upper cluster algebra elements corresponding to all vectors in \{0, 1\}^n.

INPUT:

- \(B\) – a skew-symmetric matrix.
- \text{size\_limit} – a limit on how many vectors you want the function to return.

OUTPUT:

An array of elements in the upper cluster algebra.

EXAMPLES:

```
sage: B = matrix([[0,1,0], [-1,0,1], [0,-1,0], [1,0,0], [0,1,0], [0,0,1]])
sage: C = ClusterSeed(B)
sage: C.LLM_gen_set()
[1, (x1 + x3)/x0, (x0^2*x3*x4 + x1^2*x2 + x2^2*x3)/(x0^2*x1), (x1^2*x5 + 1)/x2, (x0^2*x1*x4*x5 + x0^2*x4 + x2)/(x1^2*x2), (x0^2*x1*x3*x4*x5 + x0^2*x3*x4 + x1*x2 + x2^2*x3)/(x0^2*x1*x2)]
```

b_matrix()

Returns the \(B\)-matrix of self.

EXAMPLES:

```
sage: ClusterSeed(['A',4]).b_matrix()
[ 0 1 0 0]
[-1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -1 0]
sage: ClusterSeed(['B',4]).b_matrix()
[ 0 1 0 0]
[-1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -2 0]
sage: ClusterSeed(['D',4]).b_matrix()
[ 0 1 0 0]
[-1 0 -1 -1]
[ 0 1 0 0]
[ 0 1 0 0]
sage: ClusterSeed(QuiverMutationType([["A",2],["B",2]])).b_matrix()
[ 0 1 0 0]
```

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\begin{verbatim}
[-1 0 0 0]
[ 0 0 0 1]
[ 0 0 -2 0]
\end{verbatim}

**b_matrix_class**(*)\textit{depth=}+\textit{Infinity}, \textit{up_to_equivalence=}\textit{True}*)  

Returns all \(B\)-matrices in the mutation class of \textit{self}.

INPUT:

- \textit{depth} – (default: \textit{infinity}) integer or infinity, only seeds with distance at most \textit{depth} from \textit{self} are returned

- \textit{up_to_equivalence} – (default: True) if True, only ‘\(B\)’-matrices up to equivalence are considered.

EXAMPLES:

- for examples see \texttt{b_matrix_class_iter()}  

**b_matrix_class_iter**(*)\textit{depth=}+\textit{Infinity}, \textit{up_to_equivalence=}\textit{True}*)  

Returns an iterator through all \(B\)-matrices in the mutation class of \textit{self}.

INPUT:

- \textit{depth} – (default: \textit{infinity}) integer or infinity, only seeds with distance at most \textit{depth} from \textit{self} are returned

- \textit{up_to_equivalence} – (default: True) if True, only ‘\(B\)’-matrices up to equivalence are considered.

EXAMPLES:

A standard finite type example:

\begin{verbatim}
sage: S = ClusterSeed([\texttt{\textquoteleft\texttt{A}\texttt{\textquoteleft},4]})
sage: it = S.b_matrix_class_iter()
sage: for T in it: print(T)
[ 0 0 0 1]
[ 0 0 1 1]
[ 0 -1 0 0]
[-1 -1 0 0]
[ 0 0 0 1]
[ 0 0 1 0]
[ 0 -1 0 1]
[-1 0 -1 0]
[ 0 0 1 1]
[ 0 0 0 -1]
[-1 0 0 0]
[-1 1 0 0]
[ 0 0 0 1]
[ 0 0 1 1]
[ 0 1 0 -1]
[-1 -1 1 0]
[ 0 0 0 1]
[ 0 0 -1 0]
[ 0 1 0 -1]
[-1 1 0 0]
[ 0 0 0 -1]
[ 0 0 -1 1]
\end{verbatim}
A finite type example with given depth:

```python
sage: it = S.b_matrix_class_iter(depth=1)
sage: for T in it: print(T)
[ 0 0 0 1]
[ 0 0 1 1]
[ 0 -1 0 0]
[-1 -1 0 0]
[ 0 0 0 1]
[ 0 0 1 0]
[ 0 -1 0 1]
[-1 0 -1 0]
[ 0 0 1 1]
[ 0 0 0 -1]
[-1 0 0 0]
[-1 1 0 0]
```

Finite type example not considered up to equivalence:

```python
sage: S = ClusterSeed(['A',3])
sage: it = S.b_matrix_class_iter(up_to_equivalence=False)
sage: b_matrix_class = list(it)
sage: len(b_matrix_class)
14
sage: b_matrix_class[0]
[ 0 1 0]
[-1 0 -1]
[ 0 1 0]
```

Infinite (but finite mutation) type example:

```python
sage: S = ClusterSeed(['A',[1,2],1])
sage: it = S.b_matrix_class_iter()
sage: for T in it: print(T)
[ 0 1 1]
[-1 0 1]
[-1 -1 0]
[ 0 -2 1]
[ 2 0 -1]
[-1 1 0]
```

Infinite mutation type example:

```python
sage: S = ClusterSeed(['E',10])
sage: it = S.b_matrix_class_iter(depth=3)
sage: len ([T for T in it])
266
```

For a cluster seed from an arbitrarily labelled digraph:
sage: dg = DiGraph([['a', 'b'], ['b', 'c'], format="list_of_edges"])
sage: S = ClusterSeed(dg, frozen=['b'])
sage: S.b_matrix_class()
[ 0 0 ]
[ 0 0 ]
[-1 1], [-1 -1], [1 1]

c_matrix(show_warnings=True)
Return all c-vectors of self.
Warning: this method assumes the sign-coherence conjecture and that the input seed is sign-coherent (has an exchange matrix with columns of like signs). Otherwise, computational errors might arise.

EXAMPLES:
sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: S.c_matrix()
[ 1 0 0 ]
[ 0 0 -1 ]
[ 0 -1 0 ]
sage: S = ClusterSeed(['A',4])
sage: S.use_g_vectors(False); S.use_fpolys(False); S.use_c_vectors(False); S.
... use_d_vectors(False); S.track_mutations(False);
sage: S.c_matrix()
Traceback (most recent call last):
... ValueError: Unable to calculate c-vectors. Need to use c vectors.

c_vector(k)
Return the k-th c-vector of self. It is obtained as the k-th column vector of the bottom part of the B-matrix of self.
Warning: this method assumes the sign-coherence conjecture and that the input seed is sign-coherent (has an exchange matrix with columns of like signs). Otherwise, computational errors might arise.

EXAMPLES:
sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: [S.c_vector(k) for k in range(3)]
[(1, 0, 0), (0, 0, -1), (0, -1, 0)]
sage: S = ClusterSeed(Matrix([[0,1],[-1,0],[1,0],[-1,1]])); S
A seed for a cluster algebra of rank 2 with 2 frozen variables
sage: S.c_vector(0)
(1, 0)
sage: S = ClusterSeed(Matrix([[0,1],[-1,0],[1,0],[-1,1]])); S.use_c_vectors(bot_˓→is_c=True); S
A seed for a cluster algebra of rank 2 with 2 frozen variables
cluster()

Returns a copy of the cluster of self.

EXAMPLES:

```
sage: S = ClusterSeed(['A',3])
sage: S.cluster()
[x0, x1, x2]
sage: S.mutate(1)
sage: S.cluster()
[x0, (x0*x2 + 1)/x1, x2]
sage: S.mutate(2)
sage: S.cluster()
[x0, (x0*x2 + 1)/x1, (x0*x2 + x1 + 1)/(x1*x2)]
sage: S.mutate([2,1])
sage: S.cluster()
[x0, x1, x2]
```

cluster_class(depth=+Infinity, show_depth=False, up_to_equivalence=True)

Return the cluster class of self with respect to certain constraints.

INPUT:

- depth – (default: infinity) integer, only seeds with distance at most depth from self are returned
- return_depth – (default False) - if True, ignored if depth is set; returns the depth of the mutation class, i.e., the maximal distance from self of an element in the mutation class
- up_to_equivalence – (default: True) if True, only clusters up to equivalence are considered.

EXAMPLES:

- for examples see cluster_class_iter()

cluster_class_iter(depth=+Infinity, show_depth=False, up_to_equivalence=True)

Returns an iterator through all clusters in the mutation class of self.

INPUT:

- depth – (default: infinity) integer or infinity, only seeds with distance at most depth from self are returned
- show_depth – (default False) - if True, ignored if depth is set; returns the depth of the mutation class, i.e., the maximal distance from self of an element in the mutation class
- up_to_equivalence – (default: True) if True, only clusters up to equivalence are considered.

EXAMPLES:

A standard finite type example:
A finite type example with given depth:

```python
sage: it = S.cluster_class_iter(depth=1)
sage: for T in it: print(T)
[x0, x1, x2]
[x0, x1, (x1 + 1)/x2]
[x0, (x0*x2 + 1)/x1, x2]
[(x1 + 1)/x0, x1, x2]
```

A finite type example where the depth is returned while computing:

```python
sage: it = S.cluster_class_iter(show_depth=True)
sage: _ = list(it)
Depth: 0 found: 1  Time: ... s
Depth: 1 found: 4  Time: ... s
Depth: 2 found: 9  Time: ... s
Depth: 3 found: 13 Time: ... s
Depth: 4 found: 14 Time: ... s
```

Finite type examples not considered up to equivalence:

```python
sage: it = S.cluster_class_iter(up_to_equivalence=False)
sage: len([T for T in it])
84
```

Infinite type examples:

```python
sage: S = ClusterSeed(['A',[1,1],1])
sage: it = S.cluster_class_iter()
sage: next(it)
[x0, x1]
sage: next(it)
[x0, (x0^2 + 1)/x1]
sage: next(it)
[(x1^2 + 1)/x0, x1]
sage: next(it)
```

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[(x0^4 + 2*x0^2 + x1^2 + 1)/(x0*x1^2), (x0^2 + 1)/x1]
sage: next(it)
[(x1^2 + 1)/x0, (x1^4 + x0^2 + 2*x1^2 + 1)/(x0^2*x1)]
sage: it = S.cluster_class_iter(depth=3)
sage: for T in it: print(T)
[x0, x1]
[x0, (x0^2 + 1)/x1]
[(x0^4 + 2*x0^2 + x1^2 + 1)/(x0*x1^2), (x0^2 + 1)/x1]
[(x1^2 + 1)/x0, (x1^4 + x0^2 + 2*x1^2 + 1)/(x0^2*x1)]
[(x0^4 + 2*x0^2 + x1^2 + 1)/(x0^2*x1^2), (x0^6 + 3*x0^4 + 2*x0^2*x1^2 + x1^4 + ... + 3*x0^2 + 2*x1^2 + 1)/(x0^2*x1^3)]
[(x1^6 + x0^4 + 2*x0^2*x1^2 + 3*x1^4 + 2*x0^2 + 3*x1^2 + 1)/(x0^3*x1^2), (x1^4 + ... + x0^2 + 2*x1^2 + 1)/(x0^2*x1^2)]

For a cluster seed from an arbitrarily labelled digraph:

dg = DiGraph([['a', 'b'], ['b', 'c']], format="list_of_edges")
sage: S = ClusterSeed(dg, frozen=['b'])
sage: S.cluster_class()
[['a', 'c'], ['a', (b + 1)/c], [(b + 1)/a, 'c'], [(b + 1)/a, (b + 1)/c]]
sage: S2 = ClusterSeed(dg, frozen=[])sage: S2.cluster_class()[0]
[a, b, c]

\textbf{cluster\_index} (\textit{cluster\_str})

Return the index of a cluster if use\_fpolys is on.

\textbf{INPUT:}

\begin{itemize}
  \item \textit{cluster\_str} – the string to look for in the cluster
\end{itemize}

\textbf{OUTPUT:}

An integer or \texttt{None} if the string is not a cluster variable

\textbf{EXAMPLES:}

\begin{verbatim}
sage: S = ClusterSeed(['A', 4], user_labels=['x', 'y', 'z', 'w']); S.mutate('x')
sage: S.cluster_index('x')
sage: S.cluster_index('(y+1)/x')
\end{verbatim}

\textbf{cluster\_variable} (\textit{k})

Generates a cluster variable using F-polynomials

\textbf{EXAMPLES:}

\begin{verbatim}
sage: S = ClusterSeed(['A', 3])
sage: S.mutate([0,1])
sage: S.cluster_variable(0)
(x1 + 1)/x0
sage: S.cluster_variable(1)
(x0*x2 + x1 + 1)/(x0*x1)
\end{verbatim}
**coefficient**(*k*)

Return the coefficient of self at index *k*, or vertex *k* if *k* is not an index.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: [S.coefficient(k) for k in range(3)]
[y0, 1/y2, 1/y1]
```

**coefficients**()

Return all coefficients of self.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: S.coefficients()
[y0, 1/y2, 1/y1]
```

**d_matrix**(show_warnings=True)

Return the matrix of *d*-vectors of self.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',4]); S.d_matrix()
[-1 0 0 0]
[ 0 -1 0 0]
[ 0 0 -1 0]
[ 0 0 0 -1]
sage: S.mutate([1,2,1,0,1,3]); S.d_matrix()
[1 1 0 1]
[1 1 1 1]
[1 0 1 1]
[0 0 0 1]
```

**d_vector**(*k*)

Return the *k*-th *d*-vector of self. This is the exponent vector of the denominator of the *k*-th cluster variable.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',3])
sage: S.mutate([2,1,2])
sage: [S.d_vector(k) for k in range(3)]
[(-1, 0, 0), (0, 1, 1), (0, 1, 0)]
```

**exchangeable_part**()

Return the restriction to the principal part (i.e. the exchangeable variables) of self.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',4])
sage: T = ClusterSeed(S.quiver().digraph().edges(sort=True), frozen=[3])
sage: T.quiver().digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1)), (2, 3, (1, -1))]
```
Combinatorics, Release 10.0

sage: T.exchangeable_part().quiver().digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1))]

f_polynomial(k)

Return the k-th F-polynomial of self. It is obtained from the k-th cluster variable by setting all \( x_i \) to 1.

Warning: this method assumes the sign-coherence conjecture and that the input seed is sign-coherent (has an exchange matrix with columns of like signs). Otherwise, computational errors might arise.

EXAMPLES:

sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: [S.f_polynomial(k) for k in range(3)]
[1, y1*y2 + y2 + 1, y1 + 1]

sage: S = ClusterSeed(Matrix([[0,1],[-1,0]])); S.use_c_vectors(bottom=True); S
A seed for a cluster algebra of rank 2 with 2 frozen variables

sage: T = ClusterSeed(Matrix([[0,1],[-1,0]])).principal_extension(); T
A seed for a cluster algebra of rank 2 with principal coefficients

sage: S.mutate(0)
sage: T.mutate(0)
sage: S.f_polynomials()
[y0 + y1, 1]
sage: T.f_polynomials()
[y0 + 1, 1]

f_polynomials()

Return all F-polynomials of self. These are obtained from the cluster variables by setting all \( x_i \)'s to 1.

Warning: this method assumes the sign-coherence conjecture and that the input seed is sign-coherent (has an exchange matrix with columns of like signs). Otherwise, computational errors might arise.

EXAMPLES:

sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: S.f_polynomials()
[1, y1*y2 + y2 + 1, y1 + 1]

find_upper_bound(\( \text{verbose=False} \))

Return the upper bound of the given cluster algebra as a quotient_ring.

The upper bound is the intersection of the Laurent polynomial rings of the initial cluster and its neighboring clusters. As such, it always contains both the cluster algebra and the upper cluster algebra. This function uses the algorithm from [MM2015].

When the initial seed is totally coprime (for example, when the unfrozen part of the exchange matrix has full rank), the upper bound is equal to the upper cluster algebra by [BFZ2005].

Warning: The computation time grows rapidly with the size of the seed and the number of steps. For most seeds larger than four vertices, the algorithm may take an infeasible amount of time. Additionally,
it will run forever without terminating whenever the upper bound is infinitely-generated (such as the example in [Spe2013]).

INPUT:
- verbose – (default: False) if True, prints output during the computation.

EXAMPLES:
- finite type:

```sage
S = ClusterSeed(['A',3])
sage: S.find_upper_bound()
Quotient of Multivariate Polynomial Ring in x0, x1, x2, x0p, x1p, x2p, z0 → over Rational Field by the ideal (x0*x0p - x1 - 1, x1*x1p - x0*x2 - 1, x2*x2p - x1 - 1, x0*z0 - x2p, x1*z0 + z0 - x0p*x2, x2*z0 - x0p, x1p*z0 + z0 - x0p*x1p*x2p + x1 + 1)
```

- Markov:

```sage
B = matrix([[0,2,-2],[-2,0,2],[2,-2,0]])
sage: S = ClusterSeed(B)
sage: S.find_upper_bound()
Quotient of Multivariate Polynomial Ring in x0, x1, x2, x0p, x1p, x2p, z0 → over Rational Field by the ideal (x0*x0p - x2^2 - x1^2, x1*x1p - x2^2 - x0^2, x2*x2p - x1^2 - x0^2, x0p*x1p*x2p - x0*x1*x2p - x0*x2*x1p - x0^2*x2*x1p - x1^2*x2*x0p - 2*x0^2*x1^2, x0^3*z0 - x1p*x2p + x1*x2, x0^3*x2 - x1^3*z0 - x2p + x2, x0^3*x2 - x1^3*z0 - x2p - x2, x0^3*x2 - x1^3*z0 - x2p - x2, x0^3*x2 - x1^3*z0 - x2p - x2)
```

\texttt{first\_green\_vertex()}  
Return the first green vertex of \texttt{self}.

A vertex is defined to be green if its c-vector has all non-positive entries. More information on green vertices can be found at [BDP2013].

EXAMPLES:

```sage
ClusterSeed(['A',3]).principal_extension().first_green_vertex()
```

```sage
ClusterSeed(['A',[3,3],1]).principal_extension().first_green_vertex()
```

\texttt{first\_red\_vertex()}  
Return the first red vertex of \texttt{self}.

A vertex is defined to be red if its c-vector has all non-negative entries. More information on red vertices can be found at [BDP2013].

EXAMPLES:

```sage
ClusterSeed(['A',3]).principal_extension().first_red_vertex()
```

```sage
ClusterSeed(['A',[3,3],1]).principal_extension().first_red_vertex()
```
firstUrbanRenewal()

Return the first urban renewal vertex.

An urban renewal vertex is one in which there are two arrows pointing toward the vertex and two arrows pointing away.

EXAMPLES:

```python
sage: G = ClusterSeed(['GR', [4, 9]]); G.firstUrbanRenewal()
sage: 5
```

freeVertices()

Return the list of exchangeable vertices of self.

EXAMPLES:

```python
sage: S = ClusterSeed(DiGraph([[a, b], [c, b], [c, d], [e, d]]),
                      frozen = [b, d])
sage: S.freeVertices()
['a', 'c', 'e']
sage: S = ClusterSeed(DiGraph([[5, b]]))
sage: S.freeVertices()
[5, 'b']
```

testFusion()

Return the list of frozen vertices of self.

EXAMPLES:

```python
sage: S = ClusterSeed(DiGraph([[a, b], [c, b], [c, d], [e, d]]),
                      frozen = [b, d])
sage: sorted(S.frozenVertices())
['b', 'd']
```

gMatrix(show_warnings=True)

Return the matrix of all g-vectors of self. These are the degree vectors of the cluster variables after setting all $y_i$'s to 0.

Warning: this method assumes the sign-coherence conjecture and that the input seed is sign-coherent (has an exchange matrix with columns of like signs). Otherwise, computational errors might arise.

EXAMPLES:

```python
sage: S = ClusterSeed(['A', 3]).principal_extension()
sage: S.mutate([2, 1, 2])
sage: S.gMatrix()
[ 1 0 0]
[ 0 0 -1]
[ 0 -1 0]
```
sage: S = ClusterSeed(['A',3])
sage: S.mutate([0,1])
sage: S.g_matrix()
[-1 -1  0]
[ 1  0  0]
[ 0  0  1]

sage: S = ClusterSeed(['A',4]); S.use_g_vectors(False); S.use_fpolys(False); S.
˓→g_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]

sage: S = ClusterSeed(['A',4])
sage: S.use_g_vectors(False); S.use_c_vectors(False); S.use_fpolys(False); S.
˓→track_mutations(False); S.g_matrix()
Traceback (most recent call last):
...  
ValueError: Unable to calculate g-vectors. Need to use g vectors.

g_vector(k)
Return the k-th g-vector of self. This is the degree vector of the k-th cluster variable after setting all $y_i$'s to 0.

Warning: this method assumes the sign-coherence conjecture and that the input seed is sign-coherent (has an exchange matrix with columns of like signs). Otherwise, computational errors might arise.

EXAMPLES:

sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutate([2,1,2])
sage: [S.g_vector(k) for k in range(3)]
[(1, 0, 0), (0, 0, -1), (0, -1, 0)]

get_upper_cluster_algebra_element(a)
Computes an element in the upper cluster algebra of $B$ corresponding to the vector $a \in Z^n$.

See [LLM2014] for more details.

INPUT:

• $B$ – a skew-symmetric matrix. Must have the same number of columns as the length of the vectors in $vd$.

• $a$ – a vector in $Z^n$ where $n$ is the number of columns in $B$.

OUTPUT:

Returns an element in the upper cluster algebra. Depending on the input it may or may not be irreducible.

EXAMPLES:

sage: B = matrix([[0,3,-3],[-3,0,3],[3,-3,0],[1,0,0],[0,1,0],[0,0,1]])
sage: C = ClusterSeed(B)
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sage: C.get_upper_cluster_algebra_element([1,1,0])
(x0^3*x2^3*x3*x4 + x2^6*x3 + x1^3*x2^3)/(x0*x1)
sage: C.get_upper_cluster_algebra_element([1,1,1])
x0^2*x1^2*x2^2*x3^3*x4*x5 + x0^2*x1^2*x2^2

sage: B = matrix([[0,3,0],[-3,0,3],[0,-3,0]])
sage: C = ClusterSeed(B)
sage: C.get_upper_cluster_algebra_element([1,1,0])
(x1^3*x2^3 + x0^3 + x2^3)/(x0*x1)
sage: C.get_upper_cluster_algebra_element([1,1,1])
(x0^3*x1^3 + x1^3*x2^3 + x0^3 + x2^3)/(x0*x1*x2)

sage: B = matrix([[0,2],[-3,0],[4,-5]])
sage: C = ClusterSeed(B)
sage: C.get_upper_cluster_algebra_element([1,1])
(x2^9 + x1^3*x2^5 + x0^2*x2^4)/(x0*x1)

sage: B = matrix([[0,3,-5],[-3,0,4],[5,-4,0]])
sage: C = ClusterSeed(B)
sage: C.get_upper_cluster_algebra_element([1,1,1])
x0^4*x1^2*x2^3 + x0^2*x1^3*x2^4

\textbf{greedy}(a_1, a_2, algorithm='by_recursion')
Returns the greedy element \(x[a_1, a_2]\) assuming that self is rank two.

The third input can be ‘by_recursion’, ‘by_combinatorics’, or ‘just_numbers’ to specify if the user wants the element computed by the recurrence, combinatorial formula, or wants to set \(x_1\) and \(x_2\) to be one.

See [LLZ2014] for more details.

**EXAMPLES:**

sage: S = ClusterSeed(["R2", [3, 3]])
sage: S.greedy(4, 4)
(x0^12 + x1^12 + 4*x0^9 + 4*x1^9 + 6*x0^6 + 4*x0^3*x1^3 + 6*x1^6 + 4*x0^3 + 4*x1^3 + 1)/(x0^4*x1^4)
sage: S.greedy(4, 4, 'by_combinatorics')
(x0^12 + x1^12 + 4*x0^9 + 4*x1^9 + 6*x0^6 + 4*x0^3*x1^3 + 6*x1^6 + 4*x0^3 + 4*x1^3 + 1)/(x0^4*x1^4)

sage: S = ClusterSeed(["R2", [2, 2]])
sage: S.greedy(1, 2)
(x0^4 + 2*x0^2 + x1^2 + 1)/(x0*x1^2)
sage: S.greedy(1, 2, 'by_combinatorics')
(x0^4 + 2*x0^2 + x1^2 + 1)/(x0*x1^2)

\textbf{green_vertices}()
Return the list of green vertices of self.

A vertex is defined to be green if its c-vector has all non-positive entries. More information on green vertices can be found at [BDP2013]

**OUTPUT:**

The green vertices as a list of integers.

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EXAMPLES:

```
sage: ClusterSeed(['A',3]).principal_extension().green_vertices()
[0, 1, 2]
sage: ClusterSeed(['A',[3,3],1]).principal_extension().green_vertices()
[0, 1, 2, 3, 4, 5]
```

`ground_field()`

Returns the ground field of the cluster of `self`.

EXAMPLES:

```
sage: S = ClusterSeed(['A',3])
sage: S.ground_field()
Multivariate Polynomial Ring in x0, x1, x2, y0, y1, y2 over Rational Field
```

`highest_degree_denominator(filter=None)`

Return the vertex of the cluster polynomial with highest degree in the denominator.

INPUT:

• `filter` - Filter should be a list or iterable

OUTPUT:

An integer.

EXAMPLES:

```
sage: B = matrix([[0,-1,0,-1,1,1],[1,0,1,0,-1,-1],[0,-1,0,-1,1,1],[1,0,1,0,-1,-
˓→1],[-1,1,-1,1,0,0],[-1,1,-1,1,0,0]])
sage: C = ClusterSeed(B).principal_extension(); C.mutate([0,1,2,4,3,2,5,4,3])
sage: C.highest_degree_denominator()
5
```

`interact(fig_size=1, circular=True)`

Start an interactive window for cluster seed mutations.

Only in Jupyter notebook mode.

INPUT:

• `fig_size` – (default: 1) factor by which the size of the plot is multiplied.

• `circular` – (default: True) if True, the circular plot is chosen, otherwise >>spring<< is used.

`is_acyclic()`

Return True iff `self` is acyclic (i.e., if the underlying quiver is acyclic).

EXAMPLES:

```
sage: ClusterSeed(['A',4]).is_acyclic()
True
sage: ClusterSeed(['A',[2,1],1]).is_acyclic()
True
sage: ClusterSeed([[0,1],[1,2],[2,0]]).is_acyclic()
False
```
**is_bipartite**(\(return\_bipartition=False\))

Return True iff self is bipartite (i.e., if the underlying quiver is bipartite).

**INPUT:**

- \(return\_bipartition\) – (default: False) if True, the bipartition is returned in the case of self being bipartite.

**EXAMPLES:**

```python
sage: ClusterSeed(['A', [3, 3], 1]).is_bipartite()
True
sage: ClusterSeed(['A', [4, 3], 1]).is_bipartite()
False
```

**is_finite()**

Return True if self is of finite type.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A', 3])
sage: S.is_finite()
True
sage: S = ClusterSeed(['A', [2, 2], 1])
sage: S.is_finite()
False
```

**is_mutation_finite**(\(nr\_of\_checks=None, return\_path=False\))

Returns True if self is of finite mutation type.

**INPUT:**

- \(nr\_of\_checks\) – (default: None) number of mutations applied. Standard is \(500*\) (number of vertices of self).
- \(return\_path\) – (default: False) if True, in case of self not being mutation finite, a path from self to a quiver with an edge label \((a,-b)\) and \(a*b > 4\) is returned.

**ALGORITHM:**

- A cluster seed is mutation infinite if and only if every \(b_{ij} * b_{ji} > -4\). Thus, we apply random mutations in random directions

**WARNING:**

- Uses a non-deterministic method by random mutations in various directions.
- In theory, it can return a wrong True.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A', 10])
sage: S._mutation_type = None
sage: S.is_mutation_finite()
True
sage: S = ClusterSeed(((0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (2, 9)))
```

(continues on next page)
sage: S.is_mutation_finite()
False

**m()**

Returns the number of frozen variables of self.

**EXAMPLES:**

sage: S = ClusterSeed(['A',3])
sage: S.n()
3
sage: S.m()
0
sage: S = S.principal_extension()
sage: S.m()
3

**most_decreased_denominator_after_mutation()**

Return the vertex that will produce the most decrease in denominator degrees after mutation.

**EXAMPLES:**

sage: S = ClusterSeed(['A',5])
sage: S.mutate([0,2,3,1,2,3,1,2,0,2,3])
sage: S.most_decreased_denominator_after_mutation()
2

**most_decreased_edge_after_mutation()**

Return the vertex that will produce the least degrees after mutation.

**EXAMPLES:**

sage: S = ClusterSeed(['A',5])
sage: S.mutate([0,2,3,1,2,3,1,2,0,2,3])
sage: S.most_decreased_edge_after_mutation()
2

**mutate** *(sequence, inplace=True, input_type=None)*

Mutate self at a vertex or a sequence of vertices.

**INPUT:**

- **sequence** – a vertex of self, an iterator of vertices of self, a function which takes in the ClusterSeed and returns a vertex or an iterator of vertices, or a string representing a type of vertices to mutate
- **inplace** – (default: True) if False, the result is returned, otherwise self is modified
- **input_type** – (default: None) indicates the type of data contained in the sequence

Possible values for vertex types in sequence are:

- "first_source": mutates at first found source vertex,
- "sources": mutates at all sources,
• "first_sink": mutates at first sink,
• "sinks": mutates at all sink vertices,
• "green": mutates at the first green vertex,
• "red": mutates at the first red vertex,
• "urban_renewal" or "urban": mutates at first urban renewal vertex,
• "all_urban_renewals" or "all_urban": mutates at all urban renewal vertices.

For input_type, if no value is given, preference will be given to vertex names, then indices, then cluster variables. If all input is not of the same type, an error is given. Possible values for input_type are:

• "vertices": interprets the input sequence as vertices
• "indices": interprets the input sequence as indices
• "cluster_vars": interprets the input sequence as cluster variables this must be selected if inputting a sequence of cluster variables.

EXAMPLES:

```python
sage: S = ClusterSeed(['A',4]); S.b_matrix()
[ 0 1 0 0]
[-1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -1 0]

sage: S.mutate(0); S.b_matrix()
[ 0 -1 0 0]
[ 1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -1 0]

sage: T = S.mutate(0, inplace=False); T
A seed for a cluster algebra of rank 4 of type ['A', 4]

sage: S.mutate(0)
sage: S == T
True

sage: S.mutate([0,1,0])
sage: S.b_matrix()
[ 0 -1 1 0]
[ 1 0 0 0]
[-1 0 0 1]
[ 0 0 -1 0]

sage: S = ClusterSeed(QuiverMutationType([['A',1],['A',3]]))
sage: S.b_matrix()
[ 0 0 0 0]
[ 0 0 1 0]
[ 0 -1 0 -1]
[ 0 0 1 0]

sage: T = S.mutate(0, inplace=False)
```

(continues on next page)
Combinatorics, Release 10.0

sage: S == T
False

sage: Q = ClusterSeed(['A',3]); Q.b_matrix()
[ 0 1 0]
[-1 0 -1]
[ 0 1 0]

sage: Q.mutate('first_sink'); Q.b_matrix()
[ 0 -1 0]
[ 1 0 1]
[ 0 -1 0]

sage: def last_vertex(self): return self._n - 1
sage: Q.mutate(last_vertex); Q.b_matrix()
[ 0 -1 0]
[ 1 0 -1]
[ 0 1 0]

sage: S = ClusterSeed(['A', 4], user_labels=['a', 'b', 'c', 'd'])

sage: S.mutate('a'); S.mutate('(b+1)/a')

sage: S.cluster()
[a, b, c, d]

sage: S = ClusterSeed(['A', 4], user_labels=['a', 'b', 'c'])
Traceback (most recent call last):
... Value Error: the number of user-defined labels is not
the number of exchangeable and frozen variables

sage: S = ClusterSeed(['A', 4], user_labels=['x', 'y', 'w', 'z'])

sage: S.mutate('x')

sage: S.cluster()
[(y + 1)/x, y, w, z]

sage: S.mutate('(y+1)/x')

sage: S.cluster()
[x, y, w, z]

sage: S.mutate('y')

sage: S.cluster()
[x, (x*w + 1)/y, w, z]

sage: S.mutate('(x*w+1)/y')

sage: S.cluster()
[x, y, w, z]

sage: S = ClusterSeed(['A', 4], user_labels=[[1, 2], [2, 3], [4, 5], [5, 6]])

sage: S.cluster()
[x_1_2, x_2_3, x_4_5, x_5_6]

sage: S.mutate(['1','2'])

sage: S.cluster()
[(x_2_3 + 1)/x_1_2, x_2_3, x_4_5, x_5_6]

sage: S = ClusterSeed(['A', 4], user_labels=[[1, 2], [2, 3], [4, 5], [5, 6]],
....:          user_labels_prefix='P');
    sage: S.cluster()
    [P_1_2, P_2_3, P_4_5, P_5_6]
    sage: S.mutate('[1,2]')
    sage: S.cluster()
    [(P_2_3 + 1)/P_1_2, P_2_3, P_4_5, P_5_6]
    sage: S.mutate('P_4_5')
    sage: S.cluster()
    [(P_2_3 + 1)/P_1_2, P_2_3, (P_2_3*P_5_6 + 1)/P_4_5, P_5_6]

    sage: S = ClusterSeed(['A', 4])
    sage: S.mutate([0, 1, 0, 1, 0, 2, 1])
    sage: T = ClusterSeed(S)
    sage: S.use_fpolys(False)
    sage: S.use_g_vectors(False)
    sage: S.use_c_vectors(False)
    sage: S._C
    sage: S._G
    sage: S._F
    sage: S.g_matrix()
    [ 0 -1  0  0]
    [ 1  1  1  0]
    [ 0  0 -1  0]
    [ 0  0  1  1]
    sage: S.c_matrix()
    [ 1 -1  0  0]
    [ 1  0  0  0]
    [ 1  0 -1  1]
    [ 0  0  0  1]
    sage: S.f_polynomials() == T.f_polynomials()
    True

    sage: S.cluster() == T.cluster()
    True
    sage: S._mut_path
    [0, 1, 0, 1, 0, 2, 1]

    sage: S = ClusterSeed(DiGraph([[1, 2], [2, 'c']]))
    sage: S.mutate(1)
    Input can be ambiguously interpreted as both vertices and indices.
    Mutating at vertices by default.
    sage: S.cluster()
    [(x2 + 1)/x1, x2, c]
    sage: S.mutate(1, input_type="indices")
    sage: S.cluster()
    [(x2 + 1)/x1, (x2*c + x1 + c)/(x1*x2), c]

    sage: S = ClusterSeed(DiGraph([['a', 'b'], ['c', 'b'], ['d', 'b']]))
    sage: S.mutate(['a', 'b', 'a', 'b', 'a'])
    sage: S.cluster()
    [b, a, c, d]
    sage: S.mutate('a')
(continues on next page)
Input can be ambiguously interpreted as both vertices and cluster variables. Mutating at vertices by default.

```sage
S.cluster()
[(a*c*d + 1)/b, a, c, d]
S.mutate('a', input_type="cluster_vars")
S.cluster()
[(a*c*d + 1)/b, (a*c*d + b + 1)/(a*b), c, d]
S.mutate(['(a*c*d + 1)/b', 'd'])
S.cluster()
[(b + 1)/a, (a*c*d + b + 1)/(a*b), c, (a*c*d + b^2 + 2*b + 1)/(a*b*d)]
```

```sage
S = ClusterSeed(DiGraph([[5, 'b']]))
S.mutate(5)
S.cluster()
[(b + 1)/x5, b]
S.mutate([5])
S.cluster()
[x5, b]
S.mutate(0)
S.cluster()
[(b + 1)/x5, b]
```

```sage
S = ClusterSeed(DiGraph([[1, 2]]))
S.cluster()
[x1, x2]
S.mutate(1)
Input can be ambiguously interpreted as both vertices and indices. Mutating at vertices by default.
S.cluster()
[(x2 + 1)/x1, x2]
```

```sage
S = ClusterSeed(DiGraph([[-1, 0], [0, 1]]))
S.cluster()
[xneg1, x0, x1]
S.mutate(-1);S.cluster()
[(x0 + 1)/xneg1, x0, x1]
S.mutate(0, input_type='vertices');S.cluster()
[(x0 + 1)/xneg1, (x0*x1 + xneg1 + x1)/(xneg1*x0), x1]
```

**mutation_analysis**(options=['all'], filter=None)

Runs an analysis of all potential mutation options. Note that this might take a long time on large seeds.

Notes: Edges are only returned if we have a non-valued quiver. Green and red vertices are only returned if the cluster is principal.

**INPUT:**

- **options** – (default: ['all']) a list of mutation options.
- **filter** – (default: None) A vertex or interval of vertices to limit our search to

Possible options are:

- "all"  - All options below
- "edges" - Number of edges (works with skew-symmetric quivers)
• "edge_diff" - Edges added/deleted (works with skew-symmetric quivers)
• "green_vertices" - List of green vertices (works with principals)
• "green_vertices_diff" - Green vertices added/removed (works with principals)
• "red_vertices" - List of red vertices (works with principals)
• "red_vertices_diff" - Red vertices added/removed (works with principals)
• "urban_renewals" - List of urban renewal vertices
• "urban_renewals_diff" - Urban renewal vertices added/removed
• "sources" - List of source vertices
• "sources_diff" - Source vertices added/removed
• "sinks" - List of sink vertices
• "sinks_diff" - Sink vertices added/removed
• "denominators" - List of all denominators of the cluster variables

OUTPUT:

Outputs a dictionary indexed by the vertex numbers. Each vertex will itself also be a dictionary with each
desired option included as a key in the dictionary. As an example you would get something similar to: {0: {'edges': 1}, 1: {'edges': 2}). This represents that if you were to do a mutation at the current seed then
mutating at vertex 0 would result in a quiver with 1 edge and mutating at vertex 0 would result in a quiver
with 2 edges.

EXAMPLES:

```python
sage: B = [[0, 4, 0, -1], [-4, 0, 3, 0], [0, -3, 0, 1], [1, 0, -1, 0]]
sage: S = ClusterSeed(matrix(B)); S.mutate([2, 3, 1, 2, 1, 3, 0, 2])
sage: S.mutation_analysis()
{0: {'d_matrix': [ 0 0 1 0]
 [ 0 -1 0 0]
 [ 0 0 0 -1]
 [-1 0 0 0]},
 'denominators': [1, 1, x0, 1],
 'edge_diff': 6,
 'edges': 13,
 'green_vertices': [0, 1, 3],
 'green_vertices_diff': {'added': [0], 'removed': []},
 'red_vertices': [2],
 'red_vertices_diff': {'added': [], 'removed': [0]},
 'sinks': [],
 'sinks_diff': {'added': [], 'removed': [2]},
 'sources': [],
 'sources_diff': {'added': [], 'removed': []},
 'urban_renewals': [],
 'urban_renewals_diff': {'added': [], 'removed': []}},
1: {'d_matrix': [ 1 4 1 0]
 [ 0 1 0 0]
 [ 0 0 0 -1]
 [ 1 4 0 0]},
 'denominators': [x0*x3, x0*x1*x3^4, x0, 1],
 'edge_diff': 2,
(continues on next page)
'edges': 9,
'green_vertices': [0, 3],
'green_vertices_diff': {'added': [0], 'removed': [1]},
'red_vertices': [1, 2],
'red_vertices_diff': {'added': [1], 'removed': [0]},
'sinks': [2],
'sinks_diff': {'added': [], 'removed': []},
'sources': [],
'sources_diff': {'added': [], 'removed': []},
'urban_renewals': [],
'urban_renewals_diff': {'added': [], 'removed': []},
2: {'d_matrix': [1 0 0 0]
    [0 -1 0 0]
    [0 0 0 -1]
    [1 0 1 0],
'denominators': [x0*x3, 1, x3, 1],
'edge_diff': 0,
'edges': 7,
'green_vertices': [1, 2, 3],
'green_vertices_diff': {'added': [2], 'removed': []},
'red_vertices': [0],
'red_vertices_diff': {'added': [], 'removed': [2]},
'sinks': [],
'sinks_diff': {'added': [], 'removed': [2]},
'sources': [2],
'sources_diff': {'added': [2], 'removed': []},
'urban_renewals': [],
'urban_renewals_diff': {'added': [], 'removed': []}},
3: {'d_matrix': [1 0 1 1]
    [0 -1 0 0]
    [0 0 0 1]
    [1 0 0 1],
'denominators': [x0*x3, 1, x0, x0*x2*x3],
'edge_diff': -1,
'edges': 6,
'green_vertices': [1],
'green_vertices_diff': {'added': [], 'removed': [3]},
'red_vertices': [0, 2, 3],
'red_vertices_diff': {'added': [3], 'removed': []},
'sinks': [2],
'sinks_diff': {'added': [], 'removed': []},
'sources': [1],
'sources_diff': {'added': [1], 'removed': []},
'urban_renewals': [],
'urban_renewals_diff': {'added': [], 'removed': []}}

sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.mutation_analysis()
{0: {'d_matrix': [1 0 0 0]
    [0 -1 0]
    [0 0 -1],
'denominators': [x0, 1, 1],
mutation_class(depth=+Infinity, show_depth=False, return_paths=False, up_to_equivalence=True, only_sink_source=False)

Return the mutation class of self with respect to certain constraints.

Note: Vertex labels are not tracked in this method.

See also:

mutation_class_iter()

INPUT:

* depth – (default: infinity) integer, only seeds with distance at most depth from `self` are returned
• show_depth – (default: False) if True, the actual depth of the mutation is shown
• return_paths – (default: False) if True, a shortest path of mutation sequences from self to the
  given quiver is returned as well
• up_to_equivalence – (default: True) if True, only seeds up to equivalence are considered
• sink_source – (default: False) if True, only mutations at sinks and sources are applied

EXAMPLES:
• for examples see mutation_class_iter()

mutation_class_iter(depth=+Infinity, show_depth=False, return_paths=False, up_to_equivalence=True, only_sink_source=False)

Return an iterator for the mutation class of self with respect to certain constraints.

INPUT:
• depth – (default: infinity) integer or infinity, only seeds with distance at most depth from self are
  returned.
• show_depth – (default: False) if True, the current depth of the mutation is shown while computing.
• return_paths – (default: False) if True, a shortest path of mutations from self to the given quiver
  is returned as well.
• up_to_equivalence – (default: True) if True, only one seed up to simultaneous permutation of rows
  and columns of the exchange matrix is recorded.
• sink_source – (default: False) if True, only mutations at sinks and sources are applied.

EXAMPLES:
A standard finite type example:

```
sage: S = ClusterSeed(['A',3])
sage: it = S.mutation_class_iter()
sage: for T in it: print(T)
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
```

A finite type example with given depth:

```
sage: it = S.mutation_class_iter(depth=1)
sage: for T in it: print(T)
A seed for a cluster algebra of rank 3 of type ['A', 3]
A seed for a cluster algebra of rank 3 of type ['A', 3]
```

(continues on next page)
A finite type example where the depth is shown while computing:

```
sage: it = S.mutation_class_iter(show_depth=True)
sage: for T in it: pass
Depth: 0 found: 1 Time: ... s
Depth: 1 found: 4 Time: ... s
Depth: 2 found: 9 Time: ... s
Depth: 3 found: 13 Time: ... s
Depth: 4 found: 14 Time: ... s
```

A finite type example with shortest paths returned:

```
sage: it = S.mutation_class_iter(return_paths=True)
sage: mutation_class = list(it)
sage: len(mutation_class)
14
sage: mutation_class[0]
(A seed for a cluster algebra of rank 3 of type ['A', 3], [])
```

Finite type examples not considered up to equivalence:

```
sage: it = S.mutation_class_iter(up_to_equivalence=False)
sage: len([T for T in it])
84
```

Check that github issue #14638 is fixed:

```
sage: S = ClusterSeed(['E',6])
sage: MC = S.mutation_class(depth=7); len(MC) # long time
534
```

Infinite type examples:

```
sage: S = ClusterSeed(['A',[1,1],1])
sage: it = S.mutation_class_iter()
sage: next(it)
A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1]
sage: next(it)
A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1]
sage: next(it)
A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1]
sage: next(it)
A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1]
```
A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1]

```
sage: it = S.mutation_class_iter(depth=3, return_paths=True)
sage: for T in it: print(T)
(A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1], [])
(A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1], [1])
(A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1], [0])
(A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1], [1, 0])
(A seed for a cluster algebra of rank 2 of type ['A', [1, 1], 1], [0, 1])
```

**mutation_sequence**(sequence, show_sequence=False, fig_size=1.2, return_output='seed')

Return the seeds obtained by mutating `self` at all vertices in `sequence`.

INPUT:

- `sequence` – an iterable of vertices of `self`.
- `show_sequence` – (default: False) if True, a png containing the associated quivers is shown.
- `fig_size` – (default: 1.2) factor by which the size of the plot is multiplied.
- `return_output` – (default: 'seed') determines what output is to be returned:
  - If 'seed', outputs all the cluster seeds obtained by the `sequence` of mutations.
  - If 'matrix', outputs a list of exchange matrices.
  - If 'var', outputs a list of new cluster variables obtained at each step.

**Examples:**

```
sage: S = ClusterSeed(['A',2])
sage: for T in S.mutation_sequence([0,1,0]):
    ...:     print(T.b_matrix())
[ 0 -1]
[ 1 0]
[ 0 1]
[-1 0]
[ 0 -1]
[ 1 0]
sage: S = ClusterSeed(['A',2])
sage: S.mutation_sequence([0,1,0,1], return_output='var')
[(x1 + 1)/x0, (x0 + x1 + 1)/(x0*x1), (x0 + 1)/x1, x0]
```

**mutation_type()**

Returns the mutation_type of each connected component of `self`, if it can be determined. Otherwise, the mutation type of this component is set to be unknown.

The mutation types of the components are ordered by vertex labels.

**Warning:**

- All finite types can be detected,
- All affine types can be detected, EXCEPT affine type D (the algorithm is not yet implemented)
• All exceptional types can be detected.
• Might fail to work if it is used within different Sage processes simultaneously (that happened in the doctesting).

EXAMPLES:

• finite types:

```python
sage: S = ClusterSeed(['A',5])
sage: S._mutation_type = S._quiver._mutation_type = None
sage: S.mutation_type()
['A', 5]
sage: S = ClusterSeed(((0,1),(1,2),(2,3),(3,4)))
sage: S.mutation_type()
['A', 5]
sage: S = ClusterSeed(DiGraph([('a','b'),('c','b'),('c','d'),('e','d')]),...
˓→frozen = ['c'])
sage: S.mutation_type()
[ ['A', 2], ['A', 2] ]
```

• affine types:

```python
sage: S = ClusterSeed(['E',8,[1,1]]); S
A seed for a cluster algebra of rank 10 of type ['E', 8, [1, 1]]
sage: S._mutation_type = S._quiver._mutation_type = None; S
A seed for a cluster algebra of rank 10
sage: S.mutation_type() # long time
['E', 8, [1, 1]]
```

• the not yet working affine type D:

```python
sage: S = ClusterSeed(['D',4,1])
sage: S._mutation_type = S._quiver._mutation_type = None
sage: S.mutation_type() # todo: not implemented
['D', 4, 1]
```

• the exceptional types:

```python
sage: S = ClusterSeed(['X',6])
sage: S._mutation_type = S._quiver._mutation_type = None
sage: S.mutation_type() # long time
['X', 6]
```

• infinite types:

```python
sage: S = ClusterSeed(['GR','[4,9]'])
sage: S._mutation_type = S._quiver._mutation_type = None
sage: S.mutation_type() 'undetermined infinite mutation type'
```

**mutations()**

Return the list of mutations self has undergone if they are being tracked.

EXAMPLES:
sage: S = ClusterSeed(\['A',3\])
sage: S.mutations()
[]
sage: S.mutate([0,1,0,2])
sage: S.mutations()
[0, 1, 0, 2]
sage: S.track_mutations(False)
sage: S.mutations()
Traceback (most recent call last):
  ... 
ValueError: Not recording mutation sequence. Need to track mutations.

n() Return the number of exchangeable variables of self.

EXAMPLES:

sage: S = ClusterSeed(\['A', 3\])
sage: S.n()
3

oriented_exchange_graph() Return the oriented exchange graph of self as a directed graph.

The seed must be a cluster seed for a cluster algebra of finite type with principal coefficients (the corresponding quiver must have mutable vertices 0,1,...,n-1).

EXAMPLES:

sage: S = ClusterSeed(\['A', 2\]).principal_extension()
sage: G = S.oriented_exchange_graph(); G
Digraph on 5 vertices
sage: G.out_degree_sequence()
[2, 1, 1, 1, 0]
sage: S = ClusterSeed(\['B', 2\]).principal_extension()
sage: G = S.oriented_exchange_graph(); G
Digraph on 6 vertices
sage: G.out_degree_sequence()
[2, 1, 1, 1, 1, 0]

plot(circular=False, mark=None, save_pos=False, force_c=False, with_greens=False, add_labels=False) Returns the plot of the quiver of self.

INPUT:

- circular – (default:False) if True, the circular plot is chosen, otherwise >>spring<< is used.
- mark – (default: None) if set to i, the vertex i is highlighted.
- save_pos – (default:False) if True, the positions of the vertices are saved.
- force_c – (default:False) if True, will show the frozen vertices even if they were never initialized
- with_greens – (default:False) if True, will display the green vertices in green
• add_labels – (default:False) if True, will use the initial variables as labels

**EXAMPLES:**

```
sage: S = ClusterSeed(['A',5])
sage: S.plot()
Graphics object consisting of 15 graphics primitives
sage: S.plot(circular=True)
Graphics object consisting of 15 graphics primitives
sage: S.plot(circular=True, mark=1)
Graphics object consisting of 15 graphics primitives
```

**principal_extension()**

Return the principal extension of self, yielding a $2n \times n$ matrix.

Raises an error if the input seed has a non-square exchange matrix. In this case, the method instead adds $n$ frozen variables to any previously frozen variables. I.e., the seed obtained by adding a frozen variable to every exchangeable variable of self.

**EXAMPLES:**

```
sage: S = ClusterSeed([[0,1],[1,2],[2,3],[2,4]]); S
A seed for a cluster algebra of rank 5
sage: T = S.principal_extension(); T
A seed for a cluster algebra of rank 5 with principal coefficients
sage: T.b_matrix()
[ 0 1 0 0 0]
[-1 0 1 0 0]
[ 0 -1 0 1 1]
[ 0 0 -1 0 0]
[ 1 0 0 0 0]
[ 0 1 0 0 0]
[ 0 0 1 0 0]
[ 0 0 0 1 0]
[ 0 0 0 0 1]
sage: S = ClusterSeed(['A', 4], user_labels=['a', 'b', 'c', 'd'])
sage: T = S.principal_extension()
sage: T.cluster()
[a, b, c, d]
sage: T.coefficients()
[y0, y1, y2, y3]
sage: S2 = ClusterSeed(['A', 4], user_labels={0:'a', 1:'b', 2:'c', 3:'d'})
sage: S2 == S
True
sage: T2 = S2.principal_extension()
sage: T2 == T
True
```

**quiver()**

Return the quiver associated to self.

**EXAMPLES:**
sage: S = ClusterSeed(['A',3])
sage: S.quiver()
Quiver on 3 vertices of type ['A', 3]

red_vertices()

Return the list of red vertices of self.

A vertex is defined to be red if its c-vector has all non-negative entries. More information on red vertices can be found at [BDP2013].

OUTPUT:

The red vertices as a list of integers.

EXAMPLES:

sage: ClusterSeed(['A',3]).principal_extension().red_vertices()
[]
sage: ClusterSeed(['A',[3,3],1]).principal_extension().red_vertices()
[]
sage: Q = ClusterSeed(['A',[3,3],1]).principal_extension()
sage: Q.mutate(1)
sage: Q.red_vertices()
[1]

reorient(data)

Reorients self with respect to the given total order, or with respect to an iterator of ordered pairs.

WARNING:

- This operation might change the mutation type of self.
- Ignores ordered pairs \((i, j)\) for which neither \((i, j)\) nor \((j, i)\) is an edge of self.

INPUT:

- data – an iterator defining a total order on self.vertices(), or an iterator of ordered pairs in self defining the new orientation of these edges.

EXAMPLES:

sage: S = ClusterSeed(['A',[2,3],1])
sage: S.mutation_type()
['A', [2, 3], 1]
sage: S.reorient([(0,1),(2,3)])
sage: S.mutation_type()
['D', 5]
sage: S.reorient([(1,0),(2,3)])
sage: S.mutation_type()
['A', [1, 4], 1]
sage: S.reorient([0,1,2,3,4])
sage: S.mutation_type()
['A', [1, 4], 1]
**reset_cluster()**

Reset the cluster of `self` to the initial cluster.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',3])
sage: S.mutate([1,2,1])
sage: S.cluster()
[x0, (x1 + 1)/x2, (x0*x2 + x1 + 1)/(x1*x2)]

sage: S.reset_cluster()
sage: S.cluster()
[x0, x1, x2]

sage: T = S.principal_extension()
sage: T.cluster()
[x0, x1, x2]

sage: S = ClusterSeed(['B',3],user_labels=[[1,2],[2,3],[3,4]],user_labels_
˓→prefix='p')
sage: S.mutate([0,1])
sage: S.cluster()
[(p_2_3 + 1)/p_1_2, (p_1_2*p_3_4^2 + p_2_3 + 1)/(p_1_2*p_2_3), p_3_4]

sage: S.reset_cluster()
sage: S.cluster()
[p_1_2, p_2_3, p_3_4]

sage: S.g_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
sage: S.f_polynomials()
[1, 1, 1]
```

**reset_coefficients()**

Reset the coefficients of `self` to the frozen variables but keep the current cluster.

This raises an error if the number of frozen variables is different than the number of exchangeable variables.

**WARNING:** This command to be phased out since `use_c_vectors()` does this more effectively.

**EXAMPLES:**

```python
sage: S = ClusterSeed(['A',3]).principal_extension()
sage: S.b_matrix()
[ 0 1 0]
[-1 0 -1]
[ 0 1 0]
```

(continues on next page)
save_image(filename, circular=False, mark=None, save_pos=False)

Saves the plot of the underlying digraph of the quiver of self.

INPUT:

- filename – the filename the image is saved to.
- circular – (default: False) if True, the circular plot is chosen, otherwise >>spring<< is used.
- mark – (default: None) if set to i, the vertex i is highlighted.
- save_pos – (default:False) if True, the positions of the vertices are saved.

EXAMPLES:

```python
sage: S = ClusterSeed([F',4,[1,2]])
sage: import tempfile
sage: with tempfile.NamedTemporaryFile(suffix=".png") as f:
    ....:     S.save_image(f.name)
```

set_c_matrix(data)

Will force set the c matrix according to a matrix, a quiver, or a seed.

INPUT:

- data – The matrix to set the c matrix to. Also allowed to be a quiver or cluster seed, in which case the b_matrix is used.

EXAMPLES:

```python
sage: S = ClusterSeed(['A',3])
sage: X = matrix([[0,0,1],[0,1,0],[1,0,0]])
sage: S.set_c_matrix(X)
sage: S.c_matrix()
[0 0 1]
[0 1 0]
```
\texttt{[1 0 0]}
\texttt{sage: Y = matrix([[-1, 0, 1], [0, 1, 0], [1, 0, 0]])}
\texttt{sage: S.set\_c\_matrix(Y)}
\texttt{C matrix does not look to be valid – there exists a column containing positive and negative entries.}
\texttt{Continuing...}
\texttt{sage: Z = matrix([[1, 0, 1], [0, 1, 0], [2, 0, 2]])}
\texttt{sage: S.set\_c\_matrix(Z)}
\texttt{C matrix does not look to be valid – not a linearly independent set.}
\texttt{Continuing...}

\texttt{set\_cluster(cluster, force=False)}
Sets the cluster for \texttt{self} to \texttt{cluster}.
Warning: Initialization may lead to inconsistent data.

\textbf{INPUT:}
\begin{itemize}
  \item cluster – an iterable defining a cluster for \texttt{self}.
\end{itemize}

\textbf{EXAMPLES:}

\texttt{sage: S = ClusterSeed(['A', 3])}
\texttt{sage: cluster = S.cluster()}
\texttt{sage: S.mutate([1, 2, 1])}
\texttt{sage: S.cluster()}
\texttt{[x0, (x1 + 1)/x2, (x0*x2 + x1 + 1)/(x1*x2)]}
\texttt{sage: cluster2 = S.cluster()}
\texttt{sage: S.set\_cluster(cluster)}
\texttt{Warning: using set\_cluster at this point could lead to inconsistent seed data.}
\texttt{sage: S.set\_cluster(cluster, force=True)}
\texttt{sage: S.cluster()}
\texttt{[x0, x1, x2]}
\texttt{sage: S.set\_cluster(cluster2, force=True)}
\texttt{sage: S.cluster()}
\texttt{[x0, (x1 + 1)/x2, (x0*x2 + x1 + 1)/(x1*x2)]}
\texttt{sage: S = ClusterSeed(['A', 3]); S.use\_fpolys(False)}
\texttt{sage: S.set\_cluster([1, 1, 1])}
\texttt{Warning: clusters not being tracked so this command is ignored.}

\texttt{show(fig\_size=1, circular=False, mark=None, save\_pos=False, force\_c=False, with\_greens=False, add\_labels=False)}
Shows the plot of the quiver of \texttt{self}.

\textbf{INPUT:}
\begin{itemize}
  \item fig\_size – (default: 1) factor by which the size of the plot is multiplied.
  \item circular – (default: False) if True, the circular plot is chosen, otherwise \texttt{spring} is used.
  \item mark – (default: None) if set to \texttt{i}, the vertex \texttt{i} is highlighted.
\end{itemize}
• **save_pos** – (default:False) if True, the positions of the vertices are saved.
• **force_c** – (default:False) if True, will show the frozen vertices even if they were never initialized
• **with_greens** – (default:False) if True, will display the green vertices in green
• **add_labels** – (default:False) if True, will use the initial variables as labels

**smallest_c_vector()**

Return the vertex with the smallest c vector.

**OUTPUT:**

An integer.

**EXAMPLES:**

```sage
sage: B = matrix([[0,2],[-2,0]])
sage: C = ClusterSeed(B).principal_extension()
sage: C.mutate(0)
sage: C.smallest_c_vector()
0
```

**track_mutations**(use=True)

Begins tracking the mutation path.

Warning: May initialize all other data to ensure that all c, d, and g vectors agree on the start of mutations.

**INPUT:**

• **use** – (default:True) If True, will begin filling the mutation path

**EXAMPLES:**

```sage
sage: S = ClusterSeed(['A',4]); S.track_mutations(False)
sage: S.mutate(0)
sage: S.mutations()
Traceback (most recent call last):
...
ValueError: Not recording mutation sequence. Need to track mutations.
sage: S.track_mutations(True)
sage: S.g_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: S.mutate([0,1])
sage: S.mutations()
[0, 1]
```

**universal_extension()**

Return the universal extension of self.

This is the initial seed of the associated cluster algebra with universal coefficients, as defined in section 12 of [FZ2007].

This method works only if self is a bipartite, finite-type seed.
Due to some limitations in the current implementation of `CartanType`, we need to construct the set of almost positive coroots by hand. As a consequence their ordering is not the standard one (the rows of the bottom part of the exchange matrix might be a shuffling of those you would expect).

EXEMPLARY:

```python
sage: S = ClusterSeed(['A', 2])
sage: T = S.universal_extension()
sage: T.b_matrix()
[ 0 1]
[-1 0]
[ 1 -1]
[ 0 1]
[ 0 -1]

sage: S = ClusterSeed(['A', 3])
sage: T = S.universal_extension()
sage: T.b_matrix()
[ 0 1 0]
[-1 0 -1]
[ 0 1 0]
[-1 0 0]
[ 1 0 0]
[ 1 -1 0]
[ 1 -1 1]
[ 0 1 0]
[ 0 -1 0]
[ 0 -1 1]
[ 0 0 -1]
[ 0 0 1]

sage: S = ClusterSeed(['B', 2])
sage: T = S.universal_extension()
sage: T.b_matrix()
[ 0 1]
[-2 0]
[-1 0]
[ 1 0]
[ 1 -1]
[ 2 -1]
[ 0 1]
[ 0 -1]

sage: S = ClusterSeed(['A', 5], user_labels = [-2, -1, 0, 1, 2])
sage: U = S.universal_extension()
sage: U.b_matrix() == ClusterSeed(['A', 5]).universal_extension().b_matrix()
True
```

**urban_renewals**(return_first=False)

Return the list of the urban renewal vertices of *self*.

An urban renewal vertex is one in which there are two arrows pointing toward the vertex and two arrows pointing away.
INPUT:

- `return_first` – (default: False) if True, will return the first urban renewal

OUTPUT:

A list of vertices (as integers)

EXAMPLES:

```
sage: G = ClusterSeed([\'GR\', [4,9]]); G.urban_renewals()
[5, 6]
```

`use_c_vectors(use=True, bot_is_c=False, force=False)`

Reconstruct c vectors from other data or initialize if no usable data exists.

Warning: Initialization may lead to inconsistent data.

INPUT:

- `use` – (default: True) If True, will use c vectors

- `bot_is_c` – (default: False) If True and ClusterSeed self has self._m == self._n, then will assume bottom half of the extended exchange matrix is the c-matrix. If true, lets the ClusterSeed know c-vectors can be calculated.

EXAMPLES:

```
sage: S = ClusterSeed([\'A\',4])
sage: S.use_c_vectors(False); S.use_g_vectors(False); S.use_fpolys(False); S.track_mutations(False)
sage: S.use_c_vectors(True)
Warning: Initializing c-vectors at this point could lead to inconsistent seed data.
```

```
sage: S.use_c_vectors(True, force=True)
sage: S.c_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
sage: S = ClusterSeed([\'A\',4])
sage: S.use_c_vectors(False); S.use_g_vectors(False); S.use_fpolys(False); S.track_mutations(False)
sage: S.mutate(1)
sage: S.use_c_vectors(True, force=True)
sage: S.c_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

`use_d_vectors(use=True, force=False)`

Reconstruct d vectors from other data or initialize if no usable data exists.

Warning: Initialization may lead to inconsistent data.

INPUT:
• use – (default:True) If True, will use d vectors

EXAMPLES:

```
sage: S = ClusterSeed(['A',4])
sage: S.use_d_vectors(True)
sage: S.d_matrix()
[-1  0  0  0]
[ 0 -1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
sage: S = ClusterSeed(['A',4]); S.use_d_vectors(False); S.track_mutations(False); S.mutate(1); S.d_matrix()
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
sage: S.use_fpolys(False)
sage: S.d_matrix()
Traceback (most recent call last):
... ValueError: Unable to calculate d-vectors. Need to use d vectors.
```

```
sage: S = ClusterSeed(['A',4]); S.use_d_vectors(False); S.track_mutations(False); S.mutate(1); S.d_matrix()
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
sage: S.use_fpolys(False)
sage: S.use_d_vectors(True)
Warning: Initializing d-vectors at this point could lead to inconsistent seed data.
sage: S.use_d_vectors(True, force=True)
sage: S.d_matrix()
[-1  0  0  0]
[ 0 -1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
sage: S = ClusterSeed(['A',4]); S.mutate(1); S.d_matrix()
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
sage: S = ClusterSeed(['A',4]); S.use_d_vectors(True); S.mutate(1); S.d_matrix()
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
```

\texttt{use\_fpolys}(use=True, user\_labels=None, user\_labels\_prefix=None)
Use F-polynomials in our Cluster Seed

Note: This will automatically try to recompute the cluster variables if possible

INPUT:

- `use` – (default: True) If True, will use F-polynomials
- `user_labels` – (default: None) If set will overwrite the default cluster variable labels
- `user_labels_prefix` – (default: None) If set will overwrite the default

EXAMPLES:

```python
sage: S = ClusterSeed(['A',4]); S.use_fpolys(False); S._cluster
sage: S.use_fpolys(True)
sage: S.cluster()
[x0, x1, x2, x3]

sage: S = ClusterSeed(['A',4]); S.use_fpolys(False); S.track_mutations(False);
    S.mutate(1)
sage: S.use_fpolys(True)
Traceback (most recent call last):
  ... ValueError: F-polynomials and Cluster Variables cannot be reconstructed from
    given data.

sage: S.cluster()
Traceback (most recent call last):
  ... ValueError: Clusters not being tracked
```

`use_g_vectors(use=True, force=False)`

Reconstruct g vectors from other data or initialize if no usable data exists.

Warning: Initialization may lead to inconsistent data.

INPUT:

- `use` – (default: True) If True, will use g vectors

EXAMPLES:

```python
sage: S = ClusterSeed(['A',4])
sage: S.use_g_vectors(False); S.use_fpolys(False)
sage: S.use_g_vectors(True)
sage: S.g_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]

sage: S = ClusterSeed(['A',4])
sage: S.use_g_vectors(False); S.use_fpolys(False)
sage: S.mutate(1)
sage: S.use_g_vectors(True)
sage: S.g_matrix()
[ 1 0 0 0]
[ 0 -1 0 0]
```

(continues on next page)
variable_class\( (depth=+\infty, \text{ignore\_bipartite\_belt}=False) \)

Returns all cluster variables in the mutation class of \( \text{self} \).

INPUT:

- \( \text{depth} \) – (default: \( +\infty \)) integer, only seeds with distance at most \( \text{depth} \) from \( \text{self} \) are returned
- \( \text{ignore\_bipartite\_belt} \) – (default: \( False \)) if \( True \), the algorithms does not use the bipartite belt

EXAMPLES:

- for examples see \texttt{variable\_class\_iter()}

variable_class\_iter\( (depth=+\infty, \text{ignore\_bipartite\_belt}=False) \)

Returns an iterator for all cluster variables in the mutation class of \( \text{self} \).

INPUT:

- \( \text{depth} \) – (default: \( +\infty \)) integer, only seeds with distance at most \( \text{depth} \) from \( \text{self} \) are returned
- \( \text{ignore\_bipartite\_belt} \) – (default: \( False \)) if \( True \), the algorithms does not use the bipartite belt

EXAMPLES:

A standard finite type example:

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{sage: } S = \text{ClusterSeed([}'A',4]')
\text{sage: } S.\text{use\_g\_vectors}(\text{False}); S.\text{use\_fpolys}(\text{False}); S.\text{track\_mutations}(\text{False})
\text{sage: } S.\text{mutate}(1)
\text{sage: } S.\text{use\_c\_vectors}(\text{False})
\text{sage: } S.\text{g\_matrix}()
\text{Traceback (most recent call last):
...
ValueError: Unable to calculate g-vectors. Need to use g vectors.}
\]

\[
\text{sage: } S = \text{ClusterSeed([}'A',4]')
\text{sage: } S.\text{use\_g\_vectors}(\text{False}); S.\text{use\_fpolys}(\text{False}); S.\text{track\_mutations}(\text{False})
\text{sage: } S.\text{mutate}(1)
\text{sage: } S.\text{use\_c\_vectors}(\text{False})
\text{sage: } S.\text{use\_g\_vectors}(\text{True})
\text{Warning: Initializing g-vectors at this point could lead to inconsistent seed data.}
\text{sage: } S.\text{use\_g\_vectors}(\text{True}, \text{force}=\text{True})
\text{sage: } S.\text{g\_matrix}()
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\text{variable\_class}
x1
x2
\((x_1 + 1)/x_0\)
\((x_1^2 + x_0 \cdot x_2 + 2 \cdot x_1 + 1)/(x_0 \cdot x_1 \cdot x_2)\)
\((x_1 + 1)/x_2\)
\((x_0 \cdot x_2 + x_1 + 1)/(x_0 \cdot x_1)\)
\((x_0 \cdot x_2 + 1)/x_1\)
\((x_0 \cdot x_2 + x_1 + 1)/(x_1 \cdot x_2)\)

Finite type examples with given depth:

```
sage: it = S.variable_class_iter(depth=1)
sage: for T in it: print(T)
Found a bipartite seed - restarting the depth counter at zero and constructing—→ the variable class using its bipartite belt.
x0
x1
x2
\((x_1 + 1)/x_0\)
\((x_1^2 + x_0 \cdot x_2 + 2 \cdot x_1 + 1)/(x_0 \cdot x_1 \cdot x_2)\)
\((x_1 + 1)/x_2\)
\((x_0 \cdot x_2 + x_1 + 1)/(x_0 \cdot x_1)\)
\((x_0 \cdot x_2 + 1)/x_1\)
\((x_0 \cdot x_2 + x_1 + 1)/(x_1 \cdot x_2)\)
```

Note that the notion of *depth* depends on whether a bipartite seed is found or not, or if it is manually ignored:

```
sage: it = S.variable_class_iter(depth=1,ignore_bipartite_belt=True)
sage: for T in it: print(T)
x0
x1
x2
\((x_1 + 1)/x_0\)
\((x_0 \cdot x_2 + 1)/x_1\)
\((x_0 \cdot x_2 + 1)/(x_1 \cdot x_2)\)
sage: S.mutate([0,1])
sage: it2 = S.variable_class_iter(depth=1)
sage: for T in it2: print(T)
\((x_1 + 1)/x_0\)
\((x_0 \cdot x_2 + x_1 + 1)/(x_0 \cdot x_1)\)
x2
\((x_1^2 + x_0 \cdot x_2 + 2 \cdot x_1 + 1)/(x_0 \cdot x_1 \cdot x_2)\)
x1
\((x_0 \cdot x_2 + 1)/x_1\)
```

Infinite type examples:

```
sage: S = ClusterSeed(['A', [1,1], 1])
sage: it = S.variable_class_iter(depth=2)
sage: for T in it: print(T)
Found a bipartite seed - restarting the depth counter at zero and constructing—→ the variable class using its bipartite belt.
```

(continues on next page)
x(k)

Return the k-th initial cluster variable for the associated cluster seed, or the cluster variable of the corresponding vertex in self.quiver.

EXAMPLES:

```
sage: S = ClusterSeed(['A', 3])
sage: S.mutate([2, 1])
sage: S.x(0)
x0
sage: S.x(1)
x1
sage: S.x(2)
x2
```

y(k)

Return the k-th initial coefficient (frozen variable) for the associated cluster seed, or the cluster variable of the corresponding vertex in self.quiver.

EXAMPLES:

```
sage: S = ClusterSeed(['A', 3]).principal_extension()
sage: S.mutate([2, 1])
sage: S.y(0)
y0
sage: S.y(1)
y1
sage: S.y(2)
y2
```
sage: dg = DiGraph([['a', 'b'], ['b', 'c']], format="list_of_edges")
sage: S = ClusterSeed(dg, frozen = ['c'])
sage: S.y(0)
c
sage: S.y('c')
c

class sage.combinat.cluster_algebra_quiver.cluster_seed.ClusterVariable
    (parent, numerator, denominator, coerce=True, reduce=True, 
     mutation_type=None, variable_type=None, xdim=0)

Bases: FractionFieldElement

This class is a thin wrapper for cluster variables in cluster seeds.

It provides the extra feature to store if a variable is frozen or not.

• the associated positive root:

```
sage: S = ClusterSeed(['A',3])
sage: for T in S.variable_class_iter():
    ....:     print("{} {}\n\t(x1 + 1)/x0 \alpha[1]
\t(x1^2 + x0*x2 + 2*x1 + 1)/(x0*x1*x2) \alpha[1] + \alpha[2] + \alpha[3]
\t(x1 + 1)/x2 \alpha[3]
\t(x0*x2 + x1 + 1)/(x0*x1) \alpha[1] + \alpha[2]
\t(x0*x2 + 1)/x1 \alpha[2]
\t(x0*x2 + x1 + 1)/(x1*x2) \alpha[2] + \alpha[3]
```

almost_positive_root()

Return the almost positive root associated to self if self is of finite type.

EXAMPLES:
sage.combinat.cluster_algebra_quiver.cluster_seed.PathSubset\((n, m)\)

Encodes a maximal Dyck path from \((0,0)\) to \((n,m)\) (for \(n \geq m \geq 0\)) as a subset of \(\{0,1,2,\ldots,2n-1\}\). The encoding is given by indexing horizontal edges by odd numbers and vertical edges by evens.

The horizontal between \((i,j)\) and \((i+1,j)\) is indexed by the odd number \(2i+1\). The vertical between \((i,j)\) and \((i,j+1)\) is indexed by the even number \(2j\).

**EXAMPLES:**

```python
sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import PathSubset
sage: PathSubset(4,0)
{1, 3, 5, 7}
sage: PathSubset(4,1)
{1, 3, 5, 6, 7}
sage: PathSubset(4,2)
{1, 2, 3, 5, 6, 7}
sage: PathSubset(4,3)
{1, 2, 3, 4, 6, 7}
sage: PathSubset(4,4)
{0, 1, 2, 3, 4, 5, 6, 7}
```

sage.combinat.cluster_algebra_quiver.cluster_seed.SetToPath\((T)\)

Rearranges the encoding for a maximal Dyck path (as a set) so that it is a list in the proper order of the edges.

**EXAMPLES:**

```python
sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import PathSubset
sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import SetToPath
sage: SetToPath(PathSubset(4,0))
[1, 3, 5, 7]
sage: SetToPath(PathSubset(4,1))
[1, 3, 5, 7, 6]
sage: SetToPath(PathSubset(4,2))
[1, 3, 2, 5, 7, 6]
sage: SetToPath(PathSubset(4,3))
[1, 3, 2, 5, 4, 7, 6]
sage: SetToPath(PathSubset(4,4))
[1, 0, 3, 2, 5, 4, 7, 6]
```

sage.combinat.cluster_algebra_quiver.cluster_seed.coeff_recurs\((p, q, a1, a2, b, c)\)

Coefficients in Laurent expansion of greedy element, as defined by recursion.

**EXAMPLES:**

```python
sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import coeff_recurs
sage: coeff_recurs(1, 1, 5, 5, 3, 3)
10
```

sage.combinat.cluster_algebra_quiver.cluster_seed.get_green_vertices\((C)\)

Get the green vertices from a matrix. Will go through each column and return the ones where no entry is greater than 0.

**INPUT:**

- \(C\) – The C matrix to check

**EXAMPLES:**
sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import get_green_vertices
sage: S = ClusterSeed(['A',4]); S.mutate([1,2,3,2,0,1,2,0,3])
[0, 3]

sage.combinat.cluster_algebra_quiver.cluster_seed.get_red_vertices(C)
Get the red vertices from a matrix.
Will go through each column and return the ones where no entry is less than 0.
INPUT:
• C – The C matrix to check
EXAMPLES:

sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import get_red_vertices
sage: S = ClusterSeed(['A',4]); S.mutate([1,2,3,2,0,1,2,0,3])
[1, 2]

sage.combinat.cluster_algebra_quiver.cluster_seed.is_LeeLiZel_allowable(T, n, m, b, c)
Check if the subset T contributes to the computation of the greedy element x[m,n] in the rank two (b,c)-cluster algebra.
This uses the conditions of Lee-Li-Zelevinsky’s paper [LLZ2014].
EXAMPLES:

sage: from sage.combinat.cluster_algebra_quiver.cluster_seed import is_LeeLiZel_allowable
sage: is_LeeLiZel_allowable({1,3,2,5,7,6},4,2,6,6)
False
sage: is_LeeLiZel_allowable({1,2,5},3,3,1,1)
True

5.1.19 mutation_class
This file contains helper functions for compute the mutation class of a cluster algebra or quiver.
For the compendium on the cluster algebra and quiver package see [MS2011]
AUTHORS:
• Gregg Musiker
• Christian Stump
5.1.20 Helper functions for mutation types of quivers

This file contains helper functions for detecting the mutation type of a cluster algebra or quiver.

For the compendium on the cluster algebra and quiver package see [MS2011]

AUTHORS:

• Gregg Musiker
• Christian Stump

```
sage.combinat.cluster_algebra_quiver.mutation_type.is_mutation_finite(M, nr_of_checks=None)
```

Use a non-deterministic method by random mutations in various directions. Can result in a wrong answer.

**Warning:** This method modifies the input matrix M!

**INPUT:**

• nr_of_checks – (default: None) number of mutations applied. Standard is 500*(number of vertices of self).

**ALGORITHM:**

A quiver is mutation infinite if and only if every edge label (a,-b) satisfy a*b > 4. Thus, we apply random mutations in random directions

**EXAMPLES:**

```
sage: from sage.combinat.cluster_algebra_quiver.mutation_type import is_mutation_finite
sage: Q = ClusterQuiver(['A',10])
sage: M = Q.b_matrix()
sage: is_mutation_finite(M)
(True, None)
sage: Q = ClusterQuiver([(0,1),(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(2,9)])
sage: M = Q.b_matrix()
sage: is_mutation_finite(M)
(False, [9, 6, 9, 8, 9, 4, 0, 4, 5, 2, 1, 0, 1, 0, 7, 1, 9, 2, 5, 7, 8, 6, 3, 0, 2,
-> 5, 4, 2, 6, 9, 2, 7, 3, 5, 3, 7, 9, 5, 9, 0, 2, 7, 9, 2, 4, 2, 1, 6, 9, 4, 3, 5,]
-> 0, 8, 2, 9, 5, 3, 7, 0, 1, 8, 3, 7, 2, 7, 3, 4, 8, 0, 4, 9, 5, 2, 8, 4, 8, 1, 7,
-> 8, 9, 1, 5, 0, 8, 7, 4, 8, 9, 8, 0, 7, 4, 7, 1, 2, 8, 6, 1, 3, 9, 3, 9, 1, 3, 2,]
-> 4, 9, 5, 1, 2, 9, 4, 8, 5, 3, 4, 6, 8, 9, 2, 5, 9, 4, 6, 2, 1, 4, 9, 6, 0, 9, 8,]
-> 0, 4, 7, 9, 2, 1, 6])
```

Check that github issue #19495 is fixed:

```
sage: dg = DiGraph(); dg.add_vertex(0); S = ClusterSeed(dg); S
A seed for a cluster algebra of rank 1
sage: S.is_mutation_finite()
True
```

```
sage.combinat.cluster_algebra_quiver.mutation_type.load_data(user=True)
```

Load a dict with keys being tuples representing exceptional QuiverMutationTypes, and with values being lists or sets containing all mutation equivalent quivers as dig6 data.

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We check

- the data stored by the user (unless user=False was given)
- and the data installed by the optional package database_mutation_class.

**INPUT:**

- **user** – boolean (default: True) whether to look at user data. If not, only consider the optional package.

**EXAMPLES:**

```python
sage: from sage.combinat.cluster_algebra_quiver.mutation_type import load_data
sage: load_data(2)  # random - depends on the data the user has stored
{('G', 2): [('AO', (((0, 1), (1, -3)),)), ('AO', (((0, 1), (3, -1)),))]
```

### 5.1.21 Quiver

A *quiver* is an oriented graph without loops, two-cycles, or multiple edges. The edges are labelled by pairs \((i, -j)\) (with \(i\) and \(j\) being positive integers) such that the matrix \(M = (m_{ab})\) with \(m_{ab} = i\) and \(m_{ba} = -j\) for an edge \((i, -j)\) between vertices \(a\) and \(b\) is skew-symmetrizable.

**Warning:** This is not the standard definition of a quiver. Normally, in cluster algebra theory, a quiver is defined as an oriented graph without loops and two-cycles but with multiple edges allowed; the edges are unlabelled. This notion of quivers, however, can be seen as a particular case of our notion of quivers. Namely, if we have a quiver (in the regular sense of this word) with (precisely) \(i\) edges from \(a\) to \(b\), then we represent it by a quiver (in our sense of this word) with an edge from \(a\) to \(b\) labelled by the pair \((i, -i)\).

For the compendium on the cluster algebra and quiver package see [MS2011]

**AUTHORS:**

- Gregg Musiker
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**See also:**

For mutation types of combinatorial quivers, see `QuiverMutationType()`. Cluster seeds are closely related to `ClusterSeed()`.

**class** `sage.combinat.cluster_algebra_quiver.quiver.ClusterQuiver(data, frozen=None, user_labels=None)`

**Bases:** `SageObject`

The *quiver* associated to an *exchange matrix*.

**INPUT:**

- **data** – can be any of the following:
  - `QuiverMutationType`
  - `str` - a string representing a QuiverMutationType or a common quiver type (see Examples)
  - `ClusterQuiver`
  - `Matrix` - a skew-symmetrizable matrix
  - `DiGraph` - must be the input data for a quiver
  - `List of edges` - must be the edge list of a digraph for a quiver
• `frozen` – (default: None) sets the list of frozen variables if the input type is a `DiGraph`, it is ignored otherwise
• `user_labels` – (default: None) sets the names of the labels for the vertices of the quiver if the input type is not a `DiGraph`; otherwise it is ignored

EXAMPLES:

From a `QuiverMutationType`:

```python
sage: Q = ClusterQuiver(['A',5]); Q
Quiver on 5 vertices of type ['A', 5]
sage: Q = ClusterQuiver(['B',2]); Q
Quiver on 2 vertices of type ['B', 2]
sage: Q2 = ClusterQuiver(['C',2]); Q2
Quiver on 2 vertices of type ['B', 2]
sage: MT = Q.mutation_type(); MT.standard_quiver() == Q
True
sage: MT = Q2.mutation_type(); MT.standard_quiver() == Q2
False
sage: Q = ClusterQuiver(['A',[2,5],1]); Q
Quiver on 7 vertices of type ['A', [2, 5], 1]
sage: Q = ClusterQuiver(['A', [5,0],1]); Q
Quiver on 5 vertices of type ['D', 5]
sage: Q.is_finite()
True
sage: Q.is_acyclic()
False
sage: Q = ClusterQuiver(['F', 4, [2,1]]); Q
Quiver on 6 vertices of type ['F', 4, [2, 1]]
sage: MT = Q.mutation_type(); MT.standard_quiver() == Q
False
sage: dg = Q.digraph(); Q.mutate([2,1,4,0,5,3])
sage: dg2 = Q.digraph(); dg2.is_isomorphic(dg,edge_labels=True)
False
sage: dg2.is_isomorphic(MT.standard_quiver().digraph(),edge_labels=True)
True
sage: Q = ClusterQuiver(['G',2, (3,1)]); Q
Quiver on 4 vertices of type ['G', 2, (1, 3)]
sage: MT = Q.mutation_type(); MT.standard_quiver() == Q
False
sage: Q = ClusterQuiver(['GR',[3,6]]); Q
Quiver on 4 vertices of type ['D', 4]
sage: MT = Q.mutation_type(); MT.standard_quiver() == Q
False
sage: Q = ClusterQuiver(['GR',[3,7]]); Q
Quiver on 6 vertices of type ['E', 6]
```

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```python
sage: Q = ClusterQuiver(['TR',2]); Q
Quiver on 3 vertices of type ['A', 3]
sage: MT = Q.mutation_type(); MT.standard_quiver() == Q
False
sage: Q.mutate([1,0]); MT.standard_quiver() == Q
True
```

From a `ClusterQuiver`:

```python
sage: Q = ClusterQuiver(['A',[2,5],1]); Q
Quiver on 7 vertices of type ['A', [2, 5], 1]
sage: T = ClusterQuiver( Q ); T
Quiver on 7 vertices of type ['A', [2, 5], 1]
```

From a Matrix:

```python
sage: Q = ClusterQuiver(['A',[2,5],1]); Q
Quiver on 7 vertices of type ['A', [2, 5], 1]
sage: T = ClusterQuiver( Q._M ); T
Quiver on 7 vertices
sage: Q = ClusterQuiver( matrix([[0,1,-1],[-1,0,1],[1,-1,0],[1,2,3]]) ); Q
Quiver on 4 vertices with 1 frozen vertex
sage: Q = ClusterQuiver( matrix([]) ); Q
Quiver without vertices
```

From a DiGraph:

```python
sage: Q = ClusterQuiver(['A',[2,5],1]); Q
Quiver on 7 vertices of type ['A', [2, 5], 1]
sage: T = ClusterQuiver( Q._digraph ); T
Quiver on 7 vertices
sage: Q = ClusterQuiver( DiGraph([[1,2],[2,3],[3,4],[4,1]]) ); Q
Quiver on 4 vertices
sage: Q = ClusterQuiver(DiGraph([['a', 'b'], ['b', 'c'], ['c', 'd'], ['d', 'e']]),
   ....:   frozen=['c']); Q
Quiver on 5 vertices with 1 frozen vertex
sage: Q.mutation_type()
[ ['A', 2], ['A', 2] ]
sage: Q
Quiver on 5 vertices of type [ ['A', 2], ['A', 2] ] with 1 frozen vertex
```

From a List of edges:
Example 1:
```python
sage: Q = ClusterQuiver(['A', [2, 5], 1]); Q
Quiver on 7 vertices of type ['A', [2, 5], 1]
sage: T = ClusterQuiver( Q._digraph.edges(sort=True) ); T
Quiver on 7 vertices
sage: Q = ClusterQuiver([[1, 2], [2, 3], [3, 4], [4, 1]]); Q
Quiver on 4 vertices
```

**b_matrix()**

Return the b-matrix of self.

EXAMPLES:
```python
sage: ClusterQuiver(['A',4]).b_matrix()
[ 0 1 0 0]
[-1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -1 0]
sage: ClusterQuiver(['B',4]).b_matrix()
[ 0 1 0 0]
[-1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -2 0]
sage: ClusterQuiver(['D',4]).b_matrix()
[ 0 1 0 0]
[-1 0 -1 -1]
[ 0 1 0 0]
[ 0 1 0 0]
sage: ClusterQuiver(QuiverMutationType([[A',2],[B',2]])).b_matrix()
[ 0 1 0 0]
[-1 0 0 0]
[ 0 0 0 1]
[ 0 0 -2 0]
```

**canonical_label(certificate=False)**

Return the canonical labelling of self.

See `sage.graphs.generic_graph.GenericGraph.canonical_label()`.

INPUT:

- `certificate` – boolean (default: False) if True, the dictionary from `self.vertices()` to the vertices of the returned quiver is returned as well.

EXAMPLES:
```python
sage: Q = ClusterQuiver(['A',4]); Q.digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1)), (2, 3, (1, -1))]
sage: T = Q.canonical_label(); T.digraph().edges(sort=True)
[(0, 3, (1, -1)), (1, 2, (1, -1)), (1, 3, (1, -1))]
```

(continues on next page)
sage: T, iso = Q.canonical_label(certificate=True)
sage: T.digraph().edges(sort=True); iso
[(0, 3, (1, -1)), (1, 2, (1, -1)), (1, 3, (1, -1))]
{0: 0, 1: 3, 2: 1, 3: 2}

sage: Q = ClusterQuiver(QuiverMutationType([[B,2], [A,1]])); Q
Quiver on 3 vertices of type [ [B', 2], [A', 1] ]

sage: Q.canonical_label()
Quiver on 3 vertices of type [ [A', 1], [B', 2] ]

sage: Q.canonical_label(certificate=True)
(Quiver on 3 vertices of type [ [A', 1], [B', 2] ], {0: 1, 1: 2, 2: 0})

**d_vector_fan()**

Return the d-vector fan associated with the quiver.

It is the fan whose maximal cones are generated by the d-matrices of the clusters.

This is a complete simplicial fan (and even smooth when the initial quiver is acyclic). It only makes sense for quivers of finite type.

**EXAMPLES:**

```sage
d = ClusterQuiver([[1,2]]).d_vector_fan(); d
Rational polyhedral fan in 2-d lattice N
d.ngenerating_cones()
5
```

```sage
d = ClusterQuiver([[1,2],[2,3]]).d_vector_fan(); d
Rational polyhedral fan in 3-d lattice N
d.ngenerating_cones()
14
d.is_smooth()
True
```

```sage
d = ClusterQuiver([[1,2],[2,3],[3,1]]).d_vector_fan(); d
Rational polyhedral fan in 3-d lattice N
d.ngenerating_cones()
14
d.is_smooth()
False
```

**digraph()**

Return the underlying digraph of self.

**EXAMPLES:**

```sage
ClusterQuiver(['A',1]).digraph()
Digraph on 1 vertex
```

```sage:list(ClusterQuiver(['A',1]).digraph())
[0]
sage: ClusterQuiver(['A',1]).digraph().edges(sort=True)
[]
```

(continues on next page)
sage: ClusterQuiver(['A',4]).digraph()
Digraph on 4 vertices
sage: ClusterQuiver(['A',4]).digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1)), (2, 3, (1, -1))]

sage: ClusterQuiver(['B',4]).digraph()
Digraph on 4 vertices
sage: ClusterQuiver(['A',4]).digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1)), (2, 3, (1, -1))]

sage: ClusterQuiver(QuiverMutationType([['A',2],['B',2]])).digraph()
Digraph on 4 vertices
sage: ClusterQuiver(QuiverMutationType([['A',2],['B',2]])).digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 3, (1, -2))]

sage: ClusterQuiver(['C', 4], user_labels = ['x', 'y', 'z', 'w']).digraph().edges(sort=True)
[('x', 'y', (1, -1)), ('z', 'w', (2, -1)), ('z', 'y', (1, -1))]

exchangeable_part()
Return the restriction to the principal part (i.e. exchangeable part) of self, the subquiver obtained by deleting the frozen vertices of self.

EXAMPLES:

sage: Q = ClusterQuiver(['A',4])
sage: T = ClusterQuiver(Q.digraph().edges(sort=True), frozen=[3])
sage: T.digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1)), (2, 3, (1, -1))]

sage: T.exchangeable_part().digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 1, (1, -1))]

sage: Q2 = Q.principal_extension()
sage: Q3 = Q2.principal_extension()
sage: Q2.exchangeable_part() == Q3.exchangeable_part()
True

first_sink()
Return the first vertex of self that is a sink.

EXAMPLES:

sage: Q = ClusterQuiver(['A',4])
sage: Q.mutate([1,2,4,3,2])
sage: Q.first_sink()
0

first_source()
Return the first vertex of self that is a source
EXAMPLES:

```
sage: Q = ClusterQuiver(['A',5])
sage: Q.mutate([2,1,3,4,2])
sage: Q.first_source()
```

```
1
```

```
free_vertices()
Return the list of free vertices of self.

EXAMPLES:

```
sage: Q = ClusterQuiver(DiGraph([[['a', 'b'], ['c', 'b'], ['c', 'd'], ['e', 'd']])]
......
  ....: frozen=['b', 'd'])
sage: Q.free_vertices()
['a', 'c', 'e']
```

```
frozen_vertices()
Return the list of frozen vertices of self.

EXAMPLES:

```
sage: Q = ClusterQuiver(DiGraph([[['a', 'b'], ['c', 'b'], ['c', 'd'], ['e', 'd']])]
......
  ....: frozen=['b', 'd'])
sage: sorted(Q.frozen_vertices())
['b', 'd']
```

```
g_vector_fan()
Return the g-vector fan associated with the quiver.

It is the fan whose maximal cones are generated by the g-matrices of the clusters.

This is a complete simplicial fan. It is only supported for quivers of finite type.

EXAMPLES:

```
sage: Fg = ClusterQuiver([[1,2]]).g_vector_fan(); Fg
Rational polyhedral fan in 2-d lattice N
sage: Fg.ngenerating_cones()
5
sage: Fg = ClusterQuiver([[1,2],[2,3]]).g_vector_fan(); Fg
Rational polyhedral fan in 3-d lattice N
sage: Fg.ngenerating_cones()
14
sage: Fg.is_smooth()
True
sage: Fg = ClusterQuiver([[1,2],[2,3],[3,1]]).g_vector_fan(); Fg
Rational polyhedral fan in 3-d lattice N
sage: Fg.ngenerating_cones()
14
sage: Fg.is_smooth()
True
```
interact(fig_size=1, circular=True)

Start an interactive window for cluster quiver mutations.

Only in Jupyter notebook mode.

INPUT:

• fig_size – (default: 1) factor by which the size of the plot is multiplied.

• circular – (default: True) if True, the circular plot is chosen, otherwise >>spring<< is used.

is_acyclic()

Return true if self is acyclic.

EXAMPLES:

```
sage: ClusterQuiver(['A',4]).is_acyclic()
True
sage: ClusterQuiver(['A',[2,1],1]).is_acyclic()
True
sage: ClusterQuiver([[0,1],[1,2],[2,0]]).is_acyclic()
False
```

is_bipartite(return_bipartition=False)

Return True if self is bipartite.

EXAMPLES:

```
sage: ClusterQuiver(['A',[3,3],1]).is_bipartite()
True
sage: ClusterQuiver(['A',[4,3],1]).is_bipartite()
False
```

is_finite()

Return True if self is of finite type.

EXAMPLES:

```
sage: Q = ClusterQuiver(['A',3])
sage: Q.is_finite()
True
sage: Q = ClusterQuiver(['A',[2,2],1])
sage: Q.is_finite()
False
sage: Q = ClusterQuiver(['A',3],[B',3])
sage: Q.is_finite()
True
sage: Q = ClusterQuiver(['T',[4,4,4]])
sage: Q.is_finite()
False
sage: Q = ClusterQuiver(['A',[3],[T',[4,4,4]]])
sage: Q.is_finite()
False
sage: Q = ClusterQuiver(['A',3],[T',[2,2,3]])
```

(continues on next page)
sage: Q.is_finite()
True
sage: Q = ClusterQuiver([['A',3],['D',5]])
sage: Q.is_finite()
True
sage: Q = ClusterQuiver([['A',3],['D',5,1]])
False
sage: Q = ClusterQuiver([[0,1,2],[1,2,2],[2,0,2]])
False
sage: Q = ClusterQuiver([[0,1,2],[1,2,2],[2,0,2],[3,4,1],[4,5,1]])
False

is_mutation_finite(nr_of_checks=None, return_path=False)
Uses a non-deterministic method by random mutations in various directions. Can result in a wrong answer.

INPUT:

- nr_of_checks – (default: None) number of mutations applied. Standard is 500*(number of vertices of self).
- return_path – (default: False) if True, in case of self not being mutation finite, a path from self to a quiver with an edge label (a,-b) and a*b > 4 is returned.

ALGORITHM:
A quiver is mutation infinite if and only if every edge label (a,-b) satisfy a*b > 4. Thus, we apply random mutations in random directions

EXAMPLES:

sage: Q = ClusterQuiver(['A',10])
sage: Q._mutation_type = None
sage: Q.is_mutation_finite()
True
sage: Q = ClusterQuiver(((0,1),(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(2,9)])
sage: Q.is_mutation_finite()
False

m()
Return the number of frozen vertices of self.

EXAMPLES:

sage: Q = ClusterQuiver(['A',4])
sage: Q.m()
0
sage: T = ClusterQuiver(Q.digraph().edges(sort=True), frozen=[3])
sage: T.n()
mutate(data, inplace=True)

Mutates self at a sequence of vertices.

INPUT:

- sequence – a vertex of self, an iterator of vertices of self, a function which takes in the ClusterQuiver and returns a vertex or an iterator of vertices, or a string of the parameter wanting to be called on ClusterQuiver that will return a vertex or an iterator of vertices.

- inplace – (default: True) if False, the result is returned, otherwise self is modified.

EXAMPLES:

```
sage: Q = ClusterQuiver(['A',4]); Q.b_matrix()
[ 0 1 0 0]
[-1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -1 0]
sage: Q.mutate(0); Q.b_matrix()
[ 0 -1 0 0]
[ 1 0 -1 0]
[ 0 1 0 1]
[ 0 0 -1 0]
sage: T = Q.mutate(0, inplace=False); T
Quiver on 4 vertices of type ['A', 4]
sage: Q.mutate(0)
sage: Q == T
True
sage: T = Q.mutate([0,1,0])
sage: Q.b_matrix()
[ 0 -1 1 0]
[ 1 0 0 0]
[-1 0 0 1]
[ 0 0 -1 0]
sage: T = Q.mutate([0,1,0], inplace=False)
sage: Q == T
True
```

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```
sage: Q = ClusterQuiver(['A',3]); Q.b_matrix()
[ 0 1 0]
[-1 0 -1]
[ 0 1 0]
sage: Q.mutate('first_sink'); Q.b_matrix()
[ 0 -1 0]
[ 1 0 1]
[ 0 -1 0]
sage: Q.mutate('first_source'); Q.b_matrix()
[ 0 1 0]
[-1 0 -1]
[ 0 1 0]
sage: dg = DiGraph()
sage: dg.add_vertices(['a', 'b', 'c', 'd', 'e'])
sage: dg.add_edges([['a', 'b'], ['b', 'c'], ['c', 'd'], ['d', 'e']])
sage: Q2 = ClusterQuiver(dg, frozen=['c']); Q2.b_matrix()
[ 0 1 0 0]
[-1 0 0 0]
[ 0 0 0 1]
[ 0 0 -1 0]
[ 0 -1 1 0]
sage: Q2.mutate('a'); Q2.b_matrix()
[ 0 -1 0 0]
[ 1 0 0 0]
[ 0 0 0 1]
[ 0 0 -1 0]
[ 0 -1 1 0]
sage: dg = DiGraph([['a', 'b'], ['b', 'c']], format='list_of_edges')
sage: Q = ClusterQuiver(dg); Q
Quiver on 3 vertices
sage: Q.mutate(['a', 'b'], inplace=False).digraph().edges(sort=True)
[('a', 'b', (1, -1)), ('c', 'b', (1, -1))]
```

**mutation_class** (*depth=Infinity, show_depth=False, return_paths=False, data_type='quiver', up_to_equivalence=True, sink_source=False*)

Return the mutation class of self together with certain constraints.

**INPUT:**

- **depth** – (default: infinity`) integer, only seeds with distance at most depth from `self` are returned
- **show_depth** – (default: False) if True, the actual depth of the mutation is shown
- **return_paths** – (default: False) if True, a shortest path of mutation sequences from self to the given quiver is returned as well
- **data_type** – (default: "quiver") can be one of the following:
  - "quiver" – the quiver is returned
  - "dig6" – the dig6-data is returned
  - "path" – shortest paths of mutation sequences from self are returned
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- **sink_source** – (default: False) if True, only mutations at sinks and sources are applied

**EXAMPLES:**

```python
sage: Q = ClusterQuiver(['A',3])
sage: Ts = Q.mutation_class()
sage: for T in Ts: print(T)
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
sage: Ts = Q.mutation_class(depth=1)
sage: for T in Ts: print(T)
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
sage: Ts = Q.mutation_class(show_depth=True)
Depth: 0 found: 1 Time: ... s
Depth: 1 found: 3 Time: ... s
Depth: 2 found: 4 Time: ... s
sage: Ts = Q.mutation_class(return_paths=True)
sage: for T in Ts: print(T)
(Quiver on 3 vertices of type ['A', 3], [])
(Quiver on 3 vertices of type ['A', 3], [1])
(Quiver on 3 vertices of type ['A', 3], [0])
(Quiver on 3 vertices of type ['A', 3], [0, 1])
```
```
Depth: 0  found: 1  Time: ... s
Depth: 1  found: 3  Time: ... s
Depth: 2  found: 4  Time: ... s

sage: Ts = Q.mutation_class(show_depth=True, up_to_equivalence=False)
Depth: 0  found: 1  Time: ... s
Depth: 1  found: 4  Time: ... s
Depth: 2  found: 6  Time: ... s
Depth: 3  found: 10  Time: ... s
Depth: 4  found: 14  Time: ... s
```

**mutation_class_iter**

```
mutation_class_iter(depth=+Infinity, show_depth=False, return_paths=False, data_type='quiver',
                    up_to_equivalence=True, sink_source=False)
```

Return an iterator for the mutation class of `self` together with certain constraints.

**INPUT:**

- `depth` – (default: infinity) integer, only quivers with distance at most depth from `self` are returned.
- `show_depth` – (default: False) if True, the actual depth of the mutation is shown.
- `return_paths` – (default: False) if True, a shortest path of mutation sequences from `self` to the given quiver is returned as well.
- `data_type` – (default: “quiver”) can be one of the following:
  - "quiver"
  - "matrix"
  - "digraph"
  - "dig6"
  - "path"
- `up_to_equivalence` – (default: True) if True, only one quiver for each graph-isomorphism class is recorded.
- `sink_source` – (default: False) if True, only mutations at sinks and sources are applied.

**EXAMPLES:**

```
sage: Q = ClusterQuiver(['A',3])
sage: it = Q.mutation_class_iter()
sage: for T in it: print(T)
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]

sage: it = Q.mutation_class_iter(depth=1)
sage: for T in it: print(T)
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]

sage: it = Q.mutation_class_iter(show_depth=True)
sage: for T in it: pass
```
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Depth: 0  found: 1  Time: ... s
Depth: 1  found: 3  Time: ... s
Depth: 2  found: 4  Time: ... s

```
sage: it = Q.mutation_class_iter(return_paths=True)
sage: for T in it: print(T)
(Quiver on 3 vertices of type ['A', 3], [])
(Quiver on 3 vertices of type ['A', 3], [1])
(Quiver on 3 vertices of type ['A', 3], [0])
(Quiver on 3 vertices of type ['A', 3], [0, 1])
```

```
sage: it = Q.mutation_class_iter(up_to_equivalence=False)
sage: for T in it: print(T)
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
Quiver on 3 vertices of type ['A', 3]
```

```
sage: it = Q.mutation_class_iter(return_paths=True,up_to_equivalence=False)
sage: mutation_class = list(it)
sage: len(mutation_class)
14
sage: mutation_class[0]
(Quiver on 3 vertices of type ['A', 3], [])
```

```
sage: Q = ClusterQuiver(['A',3])
sage: it = Q.mutation_class_iter(data_type='path')
sage: for T in it: print(T)
[]
[1]
[0]
[0, 1]
```

```
sage: Q = ClusterQuiver(['A',3])
sage: it = Q.mutation_class_iter(return_paths=True,data_type='matrix')
sage: next(it)
([ 0 0 1]
 [ 0 0 1]
 [-1 -1 0], []
)
```

(continues on next page)
sage: dg = DiGraph([['a', 'b'], ['b', 'c']], format='list_of_edges')
sage: S = ClusterQuiver(dg, frozen=['b'])
sage: S.mutation_class()
[Quiver on 3 vertices with 1 frozen vertex, Quiver on 3 vertices with 1 frozen vertex, Quiver on 3 vertices with 1 frozen vertex]

**mutation_sequence** (sequence, show_sequence=False, fig_size=1.2)

Return a list containing the sequence of quivers obtained from self by a sequence of mutations on vertices.

**INPUT:**

- sequence – a list or tuple of vertices of self.
- show_sequence – (default: False) if True, a png containing the mutation sequence is shown.
- fig_size – (default: 1.2) factor by which the size of the sequence is expanded.

**EXAMPLES:**

```python
sage: Q = ClusterQuiver(['A',4])

sage: seq = Q.mutation_sequence([0,1]); seq
[Quiver on 4 vertices of type ['A', 4], Quiver on 4 vertices of type ['A', 4], Quiver on 4 vertices of type ['A', 4]]

sage: [T.b_matrix() for T in seq]
[[0 1 0 0] [0 -1 0 0] [0 1 -1 0]
[1 0 0 0] [1 0 -1 0] [0 1 0 0]
[0 1 0 1] [0 1 0 1] [1 -1 0 1]
[0 0 -1 0], [0 0 -1 0], [0 0 -1 0]
]
```

**mutation_type()**

Return the mutation type of self.

Return the mutation_type of each connected component of self if it can be determined, otherwise, the mutation type of this component is set to be unknown.

The mutation types of the components are ordered by vertex labels.

If you do many type recognitions, you should consider to save exceptional mutation types using `save_quiver_data()`.

**WARNING:**

- All finite types can be detected,
- All affine types can be detected, EXCEPT affine type D (the algorithm is not yet implemented)
- All exceptional types can be detected.

**EXAMPLES:**

```python
sage: ClusterQuiver(['A',4]).mutation_type()
['A', 4]
sage: ClusterQuiver(['A',(3,1),1]).mutation_type()
['A', [1, 3], 1]
sage: ClusterQuiver(['C',2]).mutation_type()
```

(continues on next page)
finite types:

```python
sage: Q = ClusterQuiver(['A',5])
sage: Q._mutation_type = None
sage: Q.mutation_type()
['A', 5]
```

```python
sage: Q = ClusterQuiver([(0,1),(1,2),(2,3),(3,4)])
sage: Q.mutation_type()
['A', 5]
```

```python
sage: Q = ClusterQuiver(DiGraph([['a', 'b'], ['c', 'b'], ['c', 'd'], ['e', 'd →']]),
....:     frozen=['c'])
sage: Q.mutation_type()
[ ['A', 2], ['A', 2] ]
```

affine types:

```python
sage: Q = ClusterQuiver(['E',8,[1,1]]); Q
Quiver on 10 vertices of type ['E', 8, [1, 1]]
sage: Q._mutation_type = None; Q
Quiver on 10 vertices
sage: Q.mutation_type() # long time
['E', 8, [1, 1]]
```

the not yet working affine type D (unless user has saved small classical quiver data):

```python
sage: Q = ClusterQuiver(['D',4,1])
sage: Q._mutation_type = None
sage: Q.mutation_type() # todo: not implemented
['D', 4, 1]
```

the exceptional types:

```python
sage: Q = ClusterQuiver(['X',6])
sage: Q._mutation_type = None
sage: Q.mutation_type() # long time
['X', 6]
```

examples from page 8 of [Ke2008]:

```python
sage: dg = DiGraph(); dg.add_edges([(9,0),(9,4),(4,6),(6,7),(7,8),(8,3),(3,5),
→(5,6),(8,1),(2,3)])
sage: ClusterQuiver(dg).mutation_type() # long time
['E', 8, [1, 1]]
```

```python
sage: dg = DiGraph( { 0:[3], 1:[0,4], 2:[0,6], 3:[1,2,7], 4:[3,8], 5:[2], 6:[3, ˓→5], 7:[4,6], 8:[7] } )
```

(continues on next page)
```
sage: ClusterQuiver( dg ).mutation_type() # long time
['E', 8, 1]

sage: dg = DiGraph( { 0:[3,9], 1:[0,4], 2:[0,6], 3:[1,2,7], 4:[3,8], 5:[2],
                   6:[3,5], 7:[4,6], 8:[7], 9:[1] } )
sage: ClusterQuiver( dg ).mutation_type() # long time
['E', 8, [1, 1]]
```

- **infinite types:**
  ```
sage: Q = ClusterQuiver([ 'GR', [4,9] ])  
sage: Q._mutation_type = None
sage: Q.mutation_type()
'undetermined infinite mutation type'
```

- **reducible types:**
  ```
sage: Q = ClusterQuiver([ [ 'A', 3 ], [ 'B', 3 ] ])  
sage: Q._mutation_type = None
sage: Q.mutation_type()
[['A', 3], [ 'B', 3 ]]

sage: Q = ClusterQuiver([ [ 'A', 3 ], [ 'T', [4,4,4] ] ])  
sage: Q._mutation_type = None
sage: Q.mutation_type()
[['A', 3], [ 'T', [4,4,4] ]]
```

- **n()**
  Return the number of free vertices of self.

  **EXAMPLES:**
  ```
sage: ClusterQuiver([ 'A',4 ]).n()  
4
sage: ClusterQuiver([ 'A', (3,1),1 ]).n()  
4
sage: ClusterQuiver([ 'B',4 ]).n()  
4
sage: ClusterQuiver([ 'B',4,1 ]).n()  
5
```

- **number_of_edges()**
  Return the total number of edges on the quiver
Note: This only works with non-valued quivers. If used on a non-valued quiver then the positive value is taken to be the number of edges added

OUTPUT:
An integer of the number of edges.

EXAMPLES:

```sage
S = ClusterQuiver(["A",4]); S.number_of_edges()
3

S = ClusterQuiver(["B",4]); S.number_of_edges()
3
```

`plot(circular=True, center=(0, 0), directed=True, mark=None, save_pos=False, greens=[])`

Return the plot of the underlying digraph of `self`

INPUT:
- `circular` – (default: True) if True, the circular plot is chosen, otherwise `spring` is used.
- `center` – (default: (0,0)) sets the center of the circular plot, otherwise it is ignored.
- `directed` – (default: True) if True, the directed version is shown, otherwise the undirected.
- `mark` – (default: None) if set to i, the vertex i is highlighted.
- `save_pos` – (default: False) if True, the positions of the vertices are saved.
- `greens` – (default: []) if set to a list, will display the green vertices as green

EXAMPLES:

```sage
Q = ClusterQuiver(["A",5])
sage: Q.plot()
Graphics object consisting of 15 graphics primitives

sage: Q.plot(circular=True)
Graphics object consisting of 15 graphics primitives

sage: Q.plot(circular=True, mark=1)
Graphics object consisting of 15 graphics primitives
```

`poincare_semistable(theta, d)`

Return the Poincaré polynomial of the moduli space of semi-stable representations of dimension vector `d`

INPUT:
- `theta` – stability weight, as list or vector of rationals
- `d` – dimension vector, as list or vector of coprime integers

The semi-stability is taken with respect to the slope function

\[ \mu(d) = \theta(d) / \text{dim}(d) \]

where `d` is a dimension vector.

This uses the matrix-inversion algorithm from [Rei2002].

EXAMPLES:
sage: Q = ClusterQuiver(['A', 2])
sage: Q.poincare_semistable([1, 0], [1, 0])
1
sage: Q.poincare_semistable([1, 0], [1, 1])
1

sage: K2 = ClusterQuiver(matrix([[0, 2], [-2, 0]]))
sage: theta = (1, 0)
sage: K2.poincare_semistable(theta, [1, 0])
1

sage: K2.poincare_semistable(theta, [1, 1])
v^2 + 1

sage: K2.poincare_semistable(theta, [1, 2])
1

sage: K2.poincare_semistable(theta, [1, 3])
0

sage: K3 = ClusterQuiver(matrix([[0, 3], [-3, 0]]))
sage: theta = (1, 0)
sage: K3.poincare_semistable(theta, (2, 3))
v^4 + v^2 + 1

sage: K3.poincare_semistable(theta, (3, 4))(1)
68

REFERENCES:

**principal_extension**(inplace=False)

Return the principal extension of self, adding n frozen vertices to any previously frozen vertices. I.e., the quiver obtained by adding an outgoing edge to every mutable vertex of self.

EXAMPLES:

sage: Q = ClusterQuiver(['A', 2]); Q
Quiver on 2 vertices of type ['A', 2]
sage: T = Q.principal_extension(); T
Quiver on 4 vertices of type ['A', 2] with 2 frozen vertices
sage: T2 = T.principal_extension(); T2
Quiver on 6 vertices of type ['A', 2] with 4 frozen vertices
sage: Q.digraph().edges(sort=True)
[(0, 1, (1, -1))]
sage: T.digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 0, (1, -1)), (3, 1, (1, -1))]
sage: T2.digraph().edges(sort=True)
[(0, 1, (1, -1)), (2, 0, (1, -1)), (3, 1, (1, -1)), (4, 0, (1, -1)), (5, 1, (1, -1))]

**qmu_save**(filename=None)

Save self in a .qmu file.

This file can then be opened in Bernhard Keller’s Quiver Applet.

See https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/

INPUT:

• filename – the filename the image is saved to.
If a filename is not specified, the default name is `from_sage.qmu` in the current sage directory.

**EXAMPLES:**

```python
sage: Q = ClusterQuiver(['F',4,[1,2]])
sage: import tempfile
sage: with tempfile.NamedTemporaryFile(suffix=".qmu") as f:
    Q.qmu_save(f.name)
```

Make sure we can save quivers with $m! = n$ frozen variables, see github issue #14851:

```python
sage: S = ClusterSeed([['A',3]])
sage: T1 = S.principal_extension()
sage: Q = T1.quiver()
sage: import tempfile
sage: with tempfile.NamedTemporaryFile(suffix=".qmu") as f:
    Q.qmu_save(f.name)
```

**relabel**(relabelling, inplace=True)
Return the quiver after doing a relabelling
Will relabel the vertices of the quiver

**INPUT:**
- relabelling – Dictionary of labels to move around
- inplace – (default: True) if True, will return a duplicate of the quiver

**EXAMPLES:**

```python
sage: S = ClusterQuiver(['A',4]).relabel({1:5, 2:'go'})
```

**reorient**(data)
Reorient self with respect to the given total order, or with respect to an iterator of edges in self to be reverted.

**Warning:** This operation might change the mutation type of self.

**INPUT:**
- data – an iterator defining a total order on self.vertices(), or an iterator of edges in self to be reoriented.

**EXAMPLES:**

```python
sage: Q = ClusterQuiver(['A',(2,3),1])
sage: Q.mutation_type()
['A', [2, 3], 1]

sage: Q.reorient([(0,1),(1,2),(2,3),(3,4)])
sage: Q.mutation_type()
['D', 5]

sage: Q.reorient([0,1,2,3,4])
sage: Q.mutation_type()
['A', [1, 4], 1]
```
save_image(filename, circular=False)

Save the plot of the underlying digraph of self.

INPUT:

• filename – the filename the image is saved to.
• circular – (default: False) if True, the circular plot is chosen, otherwise >>spring<< is used.

EXAMPLES:

```python
sage: Q = ClusterQuiver(['F',4,[1,2]])
sage: import tempfile
sage: with tempfile.NamedTemporaryFile(suffix=".png") as f:
....:     Q.save_image(f.name)
```

show(fig_size=1, circular=False, directed=True, mark=None, save_pos=False, greens=[])

Show the plot of the underlying digraph of self.

INPUT:

• fig_size – (default: 1) factor by which the size of the plot is multiplied.
• circular – (default: False) if True, the circular plot is chosen, otherwise >>spring<< is used.
• directed – (default: True) if True, the directed version is shown, otherwise the undirected.
• mark – (default: None) if set to i, the vertex i is highlighted.
• save_pos – (default:False) if True, the positions of the vertices are saved.
• greens – (default:[]) if set to a list, will display the green vertices as green

sinks()

Return all vertices of self that are sinks.

EXAMPLES:

```python
sage: Q = ClusterQuiver(['A',5])
sage: Q.mutate([1,2,4,3,2])
sage: Q.sinks()
[0, 2]
```

sources()

Return all vertices of self that are sources.

EXAMPLES:

```python
sage: Q = ClusterQuiver(['A',5])
sage: Q.mutate([1,2,4,3,2])
sage: Q.sources()
[]
```

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5.1.22 Quiver mutation types

AUTHORS:

• Gregg Musiker (2012, initial version)
• Christian Stump (2012, initial version)
• Hugh Thomas (2012, initial version)

sage.combinat.cluster_algebra_quiver.quiver_mutation_type.QuiverMutationType(*args)

Quiver mutation types can be seen as a slight generalization of 
generalized Cartan types.

Background on generalized Cartan types can be found at
Wikipedia article Generalized_Cartan_matrix

For the compendium on the cluster algebra and quiver package in Sage see [MS2011]

A $B$-matrix is a skew-symmetrizable $(n \times n)$-matrix $M$. I.e., there exists an invertible diagonal matrix $D$ such that $DM$ is skew-symmetric. $M$ can be encoded as a quiver by having a directed edge from vertex $i$ to vertex $j$ with label $(a, b)$ if $a = M_{i,j} > 0$ and $b = M_{j,i} < 0$. We consider quivers up to mutation equivalence.

To a quiver mutation type we can associate a generalized Cartan type by sending $M$ to the generalized Cartan matrix $C(M)$ obtained by replacing all positive entries by their negatives and adding 2's on the main diagonal.

QuiverMutationType constructs a quiver mutation type object. For more detail on the possible different types, please see the compendium.

INPUT:

The input consists either of a quiver mutation type, or of a letter (a string), a rank (one integer or a list/tuple of integers), and an optional twist (an integer or a list of integers). There are several different naming conventions for quiver mutation types.

• Finite type – letter is a Dynkin type (A-G), and rank is the rank.
• Affine type – there is more than one convention for naming affine types.
  - Kac’s notation: letter is a Dynkin type, rank is the rank of the associated finite Dynkin diagram, and twist is the twist, which could be 1, 2, or 3. In the special case of affine type A, there is more than one quiver mutation type associated to the Cartan type. In this case only, rank is a pair of integers (i,j), giving the number of edges pointing clockwise and the number of edges pointing counter-clockwise. The total number of vertices is given by i+j in this case.
  - Naive notation: letter is one of ‘BB’, ‘BC’, ‘BD’, ‘CC’, ‘CD’. The name specifies the two ends of the diagram, which are joined by a path. The total number of vertices is given by rank +1 (to match the indexing people expect because these are affine types). In general, rank must be large enough for the picture to make sense, but we accept letter is BC and rank=1.
  - Macdonald notation: for the dual of an untwisted affine type (such as ['C', 6, 1]), we accept a twist of -1 (i.e., ['C',6,-1]).
• Elliptic type – letter is a Dynkin type, rank is the rank of the finite Dynkin diagram, and twist is a tuple of two integers. We follow Saito’s notation.
• Other shapes:

  – Rank 2: letter is ‘R2’, and rank is a pair of integers specifying the label on the unique edge.

  – Triangle: letter is TR, and rank is the number of vertices along a side.

  – T: This defines a quiver shaped like a T. letter is ‘T’, and the rank is a triple, whose entries specify the number of vertices along each path from the branch point (counting the branch point).

  – Grassmannian: This defines the cluster algebra (without coefficients) corresponding to the cluster algebra with coefficients which is the coordinate ring of a Grassmannian. letter is ‘GR’. rank is a pair of integers \((k, n)\) with ‘k’ < ‘n’ specifying the Grassmannian of \(k\)-planes in \(n\)-space. This defines a quiver given by a \((k-1) \times (n-k-1)\) grid where each square is cyclically oriented.

  – Exceptional mutation finite quivers: The two exceptional mutation finite quivers, found by Derksen-Owen, have letter as ‘X’ and rank 6 or 7, equal to the number of vertices.

  – AE, BE, CE, DE: Quivers are built of one end which looks like type (affine A), B, C, or D, and the other end which looks like type E (i.e., it consists of two antennae, one of length one, and one of length two). letter is ‘AE’, ‘BE’, ‘CE’, or ‘DE’, and rank is the total number of vertices. Note that ‘AE’ is of a slightly different form and requires rank to be a pair of integers \((i, j)\) just as in the case of affine type A. See Exercise 4.3 in Kac’s book Infinite Dimensional Lie Algebras for more details.

  – Infinite type E: It is also possible to obtain infinite-type E quivers by specifying letter as ‘E’ and rank as the number of vertices.

REFERENCES:

• A good reference for finite and affine Dynkin diagrams, including Kac’s notation, is the Wikipedia article Dynkin_diagram.

• A good reference for the skew-symmetrizable elliptic diagrams is “Cluster algebras of finite mutation type via unfolding” by A. Felikson, M. Shapiro, and P. Tumarkin, [FST2012].

EXAMPLES:

Finite types:

```sage
sage: QuiverMutationType('A', 1)
['A', 1]
sage: QuiverMutationType('A', 5)
['A', 5]
sage: QuiverMutationType('B', 2)
['B', 2]
sage: QuiverMutationType('B', 5)
['B', 5]
sage: QuiverMutationType('C', 2)
['B', 2]
sage: QuiverMutationType('C', 5)
['C', 5]
sage: QuiverMutationType('D', 2)
[['A', 1], ['A', 1]]
sage: QuiverMutationType('D', 3)
[['A', 3]]
sage: QuiverMutationType('D', 4)
[['D', 4]]
```

(continues on next page)
sage: QuiverMutationType('E', 6)
['E', 6]
sage: QuiverMutationType('G', 2)
['G', 2]
sage: QuiverMutationType('A', 1, 1)
['A', 1]
sage: QuiverMutationType('A', 2, 0, 1)
['A', 1, ['A', 1]]
sage: QuiverMutationType('A', 7, 0, 1)
['D', 7]

Affine types:

sage: QuiverMutationType('A', 1, 1, 1)
['A', [1, 1, 1]]
sage: QuiverMutationType('A', 2, 4, 1)
['A', [2, 4, 1]]
sage: QuiverMutationType('BB', 2, 1)
['BB', 2, 1]
sage: QuiverMutationType('BB', 4, 1)
['BB', 4, 1]
sage: QuiverMutationType('CC', 2, 1)
['CC', 2, 1]
sage: QuiverMutationType('CC', 4, 1)
['CC', 4, 1]
sage: QuiverMutationType('BC', 1, 1)
['BC', 1, 1]
sage: QuiverMutationType('BC', 5, 1)
['BC', 5, 1]
sage: QuiverMutationType('BD', 3, 1)
['BD', 3, 1]
sage: QuiverMutationType('BD', 5, 1)
['BD', 5, 1]
sage: QuiverMutationType('CD', 3, 1)
['CD', 3, 1]
sage: QuiverMutationType('CD', 5, 1)
['CD', 5, 1]
sage: QuiverMutationType('D', 4, 1)
['D', 4, 1]
sage: QuiverMutationType('D', 6, 1)
['D', 6, 1]
sage: QuiverMutationType('E',6, 1)
['E', 6, 1]
sage: QuiverMutationType('E',7, 1)
['E', 7, 1]
sage: QuiverMutationType('E',8, 1)
['E', 8, 1]
sage: QuiverMutationType('F',4, 1)
['F', 4, 1]
sage: QuiverMutationType('F',4,-1)
['F', 4, -1]
sage: QuiverMutationType('G', 2, 1)
['G', 2, 1]
sage: QuiverMutationType('G', 2,-1)
['G', 2, -1]
sage: QuiverMutationType('A',3, 2) == QuiverMutationType('D',3, 2)
True

Affine types using Kac’s Notation:

sage: QuiverMutationType('A', 1, 1)
['A', [1, 1], 1]
sage: QuiverMutationType('B',5, 1)
['BD', 5, 1]
sage: QuiverMutationType('C',5, 1)
['CC', 5, 1]
sage: QuiverMutationType('A', 2, 2)
['BC', 1, 1]
sage: QuiverMutationType('A',7, 2)
['CD', 4, 1]
sage: QuiverMutationType('A',8, 2)
['BC', 4, 1]
sage: QuiverMutationType('D',6, 2)
['BB', 5, 1]
sage: QuiverMutationType('E',6, 2)
['F', 4, -1]
sage: QuiverMutationType('D',4,3)
['G', 2, -1]

Elliptic types:

sage: QuiverMutationType('E',6,[1, 1])
['E', 6, [1, 1]]
sage: QuiverMutationType('F',4,[2, 1])
['F', 4, [1, 2]]
sage: QuiverMutationType('G', 2,[3,3])
['G', 2, [3, 3]]

Mutation finite types:

Rank 2 cases:
sage: QuiverMutationType('R2',(1, 1))
['A', 2]
sage: QuiverMutationType('R2',(1, 2))
['B', 2]
sage: QuiverMutationType('R2',(1,3))
['G', 2]
sage: QuiverMutationType('R2',(1,4))
['BC', 1, 1]
sage: QuiverMutationType('R2',(1,5))
['R2', [1, 5]]
sage: QuiverMutationType('R2',(2, 2))
['A', [1, 1], 1]
sage: QuiverMutationType('R2',(3,5))
['R2', [3, 5]]

Exceptional Derksen-Owen quivers:

sage: QuiverMutationType('X',6)
['X', 6]

(Mainly) mutation infinite types:

Infinite type E:

sage: QuiverMutationType('E',9)
['E', 8, 1]
sage: QuiverMutationType('E',10)
['E', 10]
sage: QuiverMutationType('E',12)
['E', 12]

sage: QuiverMutationType('AE',(2,3))
['AE', [2, 3]]
sage: QuiverMutationType('BE',5)
['BE', 5]
sage: QuiverMutationType('CE',5)
['CE', 5]
sage: QuiverMutationType('DE',6)
['DE', 6]

Grassmannian types:

sage: QuiverMutationType('GR',(2,4))
['A', 1]
sage: QuiverMutationType('GR',(2,6))
['A', 3]
sage: QuiverMutationType('GR',(3,6))
['D', 4]
sage: QuiverMutationType('GR',(3,7))
['E', 6]
sage: QuiverMutationType('GR',(3,8))
['E', 8]
sage: QuiverMutationType('GR',(3,10))
['GR', [3, 10]]
Triangular types:

<table>
<thead>
<tr>
<th>sage: QuiverMutationType('TR', 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>['A', 3]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('TR', 3)</td>
</tr>
<tr>
<td>['D', 6]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('TR', 4)</td>
</tr>
<tr>
<td>['E', 8, [1, 1]]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('TR', 5)</td>
</tr>
<tr>
<td>['TR', 5]</td>
</tr>
</tbody>
</table>

T types:

<table>
<thead>
<tr>
<th>sage: QuiverMutationType('T', (1, 1, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>['A', 1]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (1, 1, 4))</td>
</tr>
<tr>
<td>['A', 4]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (1, 4, 4))</td>
</tr>
<tr>
<td>['A', 7]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 2, 2))</td>
</tr>
<tr>
<td>['D', 4]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 2, 4))</td>
</tr>
<tr>
<td>['D', 6]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 3, 3))</td>
</tr>
<tr>
<td>['E', 6]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 3, 4))</td>
</tr>
<tr>
<td>['E', 7]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 3, 5))</td>
</tr>
<tr>
<td>['E', 8]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 3, 6))</td>
</tr>
<tr>
<td>['E', 8, 1]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (2, 3, 7))</td>
</tr>
<tr>
<td>['E', 10]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (3, 3, 3))</td>
</tr>
<tr>
<td>['E', 6, 1]</td>
</tr>
<tr>
<td>sage: QuiverMutationType('T', (3, 3, 4))</td>
</tr>
<tr>
<td>['T', [3, 3, 4]]</td>
</tr>
</tbody>
</table>

Reducible types:

<table>
<thead>
<tr>
<th>sage: QuiverMutationType(['A', 3], ['B', 4])</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ ['A', 3], ['B', 4] ]</td>
</tr>
</tbody>
</table>

class sage.combinat.cluster_algebra_quiver.quiver_mutation_type.QuiverMutationTypeFactory
Bases: SageObject

samples(finite=None, affine=None, elliptic=None, mutation_finite=None)

Return a sample of the available quiver mutations types.

INPUT:
- finite
- affine
- elliptic
• mutation_finite

All four input keywords default values are None. If set to True or False, only these samples are returned.

EXAMPLES:

```sage
QuiverMutationType.samples()
[[['A'], 1], ['A', 5], ['B'], 2], ['B', 5], ['C'], 3],
[['C'], 5], [['A', 1], ['A', 1]], [['D', 5], ['E', 6],
[['E', 7], ['F', 4], ['G'], 2],
[['A', [1, 1], 1], ['A', [4, 5], 1], ['D', 4, 1],
[['BB', 5, 1], ['E', 6, [1, 1]], ['E', 7, [1, 1]],
[['R2', [1, 5]], ['R2', [3, 5]], ['E', 10], ['BE', 5],
[['GR', [3, 10]], ['T', [3, 3, 4]]]]
```

```sage
QuiverMutationType.samples(finite=True)
[[['A', 1], ['A', 5], ['B'], 2], ['B', 5], ['C'], 3],
[['C'], 5], [['A', 1], ['A', 1]], [['D', 5], ['E', 6],
[['E', 7], ['F', 4], ['G'], 2]]
```

```sage
QuiverMutationType.samples(affine=True)
[[['A', [1, 1], 1], ['A', [4, 5], 1], ['D', 4, 1], ['BB', 5, 1]]
```

```sage
QuiverMutationType.samples(elliptic=True)
[[['E', 6, [1, 1]], ['E', 7, [1, 1]]]
```

```sage
QuiverMutationType.samples(mutation_finite=False)
[[['R2', [1, 5]], ['R2', [3, 5]], ['E', 10], ['BE', 5],
[['GR', [3, 10]], ['T', [3, 3, 4]]]
```

```sage
class sage.combinat.cluster_algebra_quiver.quiver_mutation_type.QuiverMutationType_Irreducible(letter, rank, twist=None)
Bases: QuiverMutationType_abstract

The mutation type for a cluster algebra or a quiver. Should not be called directly, but through QuiverMutationType.

class_size()

If it is known, the size of the mutation class of all quivers which are mutation equivalent to the standard quiver of self (up to isomorphism) is returned.

Otherwise, Not Implemented is returned.

Formula for finite type A is taken from Torkildsen - Counting cluster-tilted algebras of type \( \mathcal{A}_n \). Formulas for affine type A and finite type D are taken from Bastian, Prellberg, Rubey, Stump - Counting the number of elements in the mutation classes of \( \mathcal{A}_n \) quivers. Formulas for finite and affine types B and C are proven but not yet published. Conjectural formulas for several other non-simply-laced affine types are implemented. Exceptional Types (finite, affine, and elliptic) E, F, G, and X are hardcoded.

EXAMPLES:

```sage
mut_type = QuiverMutationType([['A', 5]]); mut_type
['A', 5]
```

```sage
mut_type.class_size()
19
```
```
sage: mut_type = QuiverMutationType( ['A', [10, 3], 1] ); mut_type
['A', [3, 10], 1]
sage: mut_type.class_size()
142120

sage: mut_type = QuiverMutationType( ['B', 6] ); mut_type
['B', 6]
sage: mut_type.class_size()
132

sage: mut_type = QuiverMutationType( ['BD', 6, 1] ); mut_type
['BD', 6, 1]
sage: mut_type.class_size()
Warning: This method uses a formula which has not been proved correct.
504

Check that github issue #14048 is fixed:

sage: mut_type = QuiverMutationType( ['F', 4, (2, 1)] )
sage: mut_type.class_size()
90

dual()

Return the QuiverMutationType which is dual to self.

EXAMPLES:

sage: mut_type = QuiverMutationType('A', 5); mut_type
['A', 5]
sage: mut_type.dual()
['A', 5]

sage: mut_type = QuiverMutationType('B', 5); mut_type
['B', 5]
sage: mut_type.dual()
['C', 5]
sage: mut_type.dual().dual()
['B', 5]
sage: mut_type.dual().dual() == mut_type
True

irreducible_components()

Return a list of all irreducible components of self.

EXAMPLES:

sage: mut_type = QuiverMutationType('A', 3); mut_type
['A', 3]
sage: mut_type.irreducible_components()
(['A', 3],)

class sage.combinat.cluster_algebra_quiver.quiver_mutation_type.QuiverMutationType_Reducible(*args)
Bases: QuiverMutationType_abstract
The mutation type for a cluster algebra or a quiver. Should not be called directly, but through QuiverMutationType. Inherits from QuiverMutationType_abstract.

class_size()
If it is known, the size of the mutation class of all quivers which are mutation equivalent to the standard quiver of self (up to isomorphism) is returned.
Otherwise, NotImplementedError is returned.

EXAMPLES:

```
sage: mut_type = QuiverMutationType(['A',3],['B',3]); mut_type
[['A', 3], ['B', 3]]
sage: mut_type.class_size()
20

sage: mut_type = QuiverMutationType(['A',3],['B',3],['X',6])
[['A', 3], ['B', 3], ['X', 6]]
sage: mut_type.class_size()
100
```

dual()
Return the QuiverMutationType which is dual to self.

EXAMPLES:

```
sage: mut_type = QuiverMutationType(['A',5],['B',6],['C',5],['D',4]); mut_type
[['A', 5], ['B', 6], ['C', 5], ['D', 4]]
sage: mut_type.dual()
[['A', 5], ['C', 6], ['B', 5], ['D', 4]]
```

irreducible_components()
Return a list of all irreducible components of self.

EXAMPLES:

```
sage: mut_type = QuiverMutationType('A',3); mut_type
[['A', 3]]
sage: mut_type.irreducible_components()
(['A', 3],)

sage: mut_type = QuiverMutationType(['A',3],['B',3]); mut_type
[['A', 3], ['B', 3]]
sage: mut_type.irreducible_components()
(['A', 3], ['B', 3])

sage: mut_type = QuiverMutationType(['A',3],['B',3],['X',6])
[['A', 3], ['B', 3], ['X', 6]]
sage: mut_type.irreducible_components()
[['A', 3], ['B', 3], ['X', 6]]
```

class
sage.combinat.cluster_algebra_quiver.quiver_quiver_mutation_type.QuiverMutationType_abstract
Bases: UniqueRepresentation, SageObject
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EXAMPLES:

```python
sage: mut_type1 = QuiverMutationType('A',5)
sage: mut_type2 = QuiverMutationType('A',5)
sage: mut_type3 = QuiverMutationType('A',6)
sage: mut_type1 == mut_type2
True
sage: mut_type1 == mut_type3
False
```

*b_matrix()

Return the B-matrix of the standard quiver of *self*.

The conventions for B-matrices agree with Fomin-Zelevinsky (up to a reordering of the simple roots).

EXAMPLES:

```python
sage: mut_type = QuiverMutationType(['A',5]); mut_type
['A', 5]
sage: mut_type.b_matrix()
[ 0 1 0 0 0]
[-1 0 -1 0 0]
[ 0 1 0 1 0]
[ 0 0 -1 0 -1]
[ 0 0 0 1 0]
sage: mut_type = QuiverMutationType(['A',3],['B',3]); mut_type
[['A', 3], ['B', 3]]
sage: mut_type.b_matrix()
[ 0 1 0 0 0 0]
[-1 0 -1 0 0 0]
[ 0 1 0 0 0 0]
[ 0 0 0 0 1 0]
[ 0 0 0 -1 0 -1]
[ 0 0 0 0 2 0]
```

cartan_matrix()

Return the Cartan matrix of *self*.

Note that (up to a reordering of the simple roots) the convention for the definition of Cartan matrix, used here and elsewhere in Sage, agrees with the conventions of Kac, Fulton-Harris, and Fomin-Zelevinsky, but disagrees with the convention of Bourbaki. The \((i, j)\) entry is \(2(\alpha_i, \alpha_j) / (\alpha_i, \alpha_i)\).

EXAMPLES:

```python
sage: mut_type = QuiverMutationType(['A',5]); mut_type
['A', 5]
sage: mut_type.cartan_matrix()
[ 2 -1 0 0 0]
[-1 2 -1 0 0]
[ 0 -1 2 -1 0]
[ 0 0 -1 2 -1]
[ 0 0 0 -1 2]
sage: mut_type = QuiverMutationType(['A',3],['B',3]); mut_type
(continues on next page)
```

(continues on next page)
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(continued from previous page)

```
[ ['A', 3], ['B', 3] ]
sage: mut_type.cartan_matrix()
[ 2 -1 0 0 0 0]
[-1 2 -1 0 0 0]
[ 0 -1 2 0 0 0]
[ 0 0 0 2 -1 0]
[ 0 0 0 -1 2 -1]
[ 0 0 0 0 -2 2]
```

**is_affine()**

Return True if self is of affine type.

EXAMPLES:

```
sage: mt = QuiverMutationType(['A', 2])
sage: mt.is_affine()
False
sage: mt = QuiverMutationType(['A', [4, 2], 1])
sage: mt.is_affine()
True
```

**is_elliptic()**

Return True if self is of elliptic type.

EXAMPLES:

```
sage: mt = QuiverMutationType(['A', 2])
sage: mt.is_elliptic()
False
sage: mt = QuiverMutationType(['E', 6, [1, 1]])
sage: mt.is_elliptic()
True
```

**is_finite()**

Return True if self is of finite type.

This means that the cluster algebra associated to self has only a finite number of cluster variables.

EXAMPLES:

```
sage: mt = QuiverMutationType(['A', 2])
sage: mt.is_finite()
True
sage: mt = QuiverMutationType(['A', [4, 2], 1])
sage: mt.is_finite()
False
```

**is_irreducible()**

Return True if self is irreducible.

EXAMPLES:

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```python
sage: mt = QuiverMutationType(['A', 2])
sage: mt.is_irreducible()
True
```

**is_mutation_finite()**

Return True if self is of finite mutation type.

This means that its mutation class has only finitely many different B-matrices.

**EXAMPLES:**

```python
sage: mt = QuiverMutationType(['D', 5, 1])
sage: mt.is_mutation_finite()
True
```

**is_simply_laced()**

Return True if self is simply laced.

This means that the only arrows that appear in the quiver of self are single unlabelled arrows.

**EXAMPLES:**

```python
sage: mt = QuiverMutationType(['A', 2])
sage: mt.is_simply_laced()
True
sage: mt = QuiverMutationType(['B', 2])
sage: mt.is_simply_laced()
False
sage: mt = QuiverMutationType(['A', (1, 1), 1])
sage: mt.is_simply_laced()
False
```

**is_skew_symmetric()**

Return True if the B-matrix of self is skew-symmetric.

**EXAMPLES:**

```python
sage: mt = QuiverMutationType(['A', 2])
sage: mt.is_skew_symmetric()
True
sage: mt = QuiverMutationType(['B', 2])
sage: mt.is_skew_symmetric()
False
sage: mt = QuiverMutationType(['A', (1, 1), 1])
sage: mt.is_skew_symmetric()
True
```

**letter()**

Return the classification letter of self.

**EXAMPLES:**
sage: mut_type = QuiverMutationType( ['A', 5] ); mut_type['A']

sage: mut_type.letter()
'A'

sage: mut_type = QuiverMutationType( ['BC', 5, 1] ); mut_type['BC']

sage: mut_type.letter()
'BC'

sage: mut_type = QuiverMutationType([['A',3],['B',3]]); mut_type['A x B']

sage: mut_type.letter()
'A x B'

sage: mut_type = QuiverMutationType([['A',3],['B',3],['X',6]]); mut_type['A x B x X']

plot(circular=False, directed=True)

Return the plot of the underlying graph or digraph of self.

INPUT:

- circular – (default:False) if True, the circular plot is chosen, otherwise >>spring<< is used.
- directed – (default: True) if True, the directed version is shown, otherwise the undirected.

EXAMPLES:

sage: QMT = QuiverMutationType(['A',5])
sage: pl = QMT.plot()
sage: pl = QMT.plot(circular=True)

properties()

Print a scheme of all properties of self.

Most properties have natural definitions for either irreducible or reducible types. affine and elliptic are only defined for irreducible types.

EXAMPLES:

sage: mut_type = QuiverMutationType(['A',3]); mut_type['A', 3]

sage: mut_type.properties()
['A', 3] has rank 3 and the following properties:
- irreducible: True
- mutation finite: True
- simply-laced: True
- skew-symmetric: True
- finite: True
- affine: False
- elliptic: False

sage: mut_type = QuiverMutationType(['B',3]); mut_type

(continues on next page)
['B', 3]
sage: mut_type.properties()
['B', 3] has rank 3 and the following properties:
  - irreducible: True
  - mutation finite: True
  - simply-laced: False
  - skew-symmetric: False
  - finite: True
  - affine: False
  - elliptic: False

sage: mut_type = QuiverMutationType(['B', 3, 1]); mut_type
['BD', 3, 1]
sage: mut_type.properties()
['BD', 3, 1] has rank 4 and the following properties:
  - irreducible: True
  - mutation finite: True
  - simply-laced: False
  - skew-symmetric: False
  - finite: False
  - affine: True
  - elliptic: False

sage: mut_type = QuiverMutationType(['E', 6, [1, 1]]); mut_type
['E', 6, [1, 1]]
sage: mut_type.properties()
['E', 6, [1, 1]] has rank 8 and the following properties:
  - irreducible: True
  - mutation finite: True
  - simply-laced: False
  - skew-symmetric: True
  - finite: False
  - affine: False
  - elliptic: True

sage: mut_type = QuiverMutationType(['A', 3], ['B', 3]); mut_type
[ ['A', 3], ['B', 3] ]
sage: mut_type.properties()
[ ['A', 3], ['B', 3] ] has rank 6 and the following properties:
  - irreducible: False
  - mutation finite: True
  - simply-laced: False
  - skew-symmetric: False
  - finite: True

sage: mut_type = QuiverMutationType('GR',[4,9]); mut_type
['GR', [4, 9]]
sage: mut_type.properties()
['GR', [4, 9]] has rank 12 and the following properties:
  - irreducible: True
  - mutation finite: False
  - simply-laced: True
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- skew-symmetric: True
- finite: False
- affine: False
- elliptic: False

**rank()**

Return the rank in the standard quiver of `self`.

The rank is the number of vertices.

**EXAMPLES:**

```python
sage: mut_type = QuiverMutationType( ['A',5] ); mut_type
[ 'A', 5]
sage: mut_type.rank()
5
	sage: mut_type = QuiverMutationType( ['A',[4,5], 1] ); mut_type
[ 'A', [4, 5], 1]
sage: mut_type.rank()
9
	sage: mut_type = QuiverMutationType( ['BC',5, 1] ); mut_type
[ 'BC', 5, 1]
sage: mut_type.rank()
6
	sage: mut_type = QuiverMutationType([ 'A',3],[ 'B',3]); mut_type
[ [ 'A', 3], [ 'B', 3] ]
sage: mut_type.rank()
6
	sage: mut_type = QuiverMutationType([ 'A',3],[ 'B',3],[ 'X',6]); mut_type
[ [ 'A', 3], [ 'B', 3], [ 'X', 6] ]
sage: mut_type.rank()
12
```

**show(circular=False, directed=True)**

Show the plot of the underlying digraph of `self`.

**INPUT:**

- circular – (default: False) if True, the circular plot is chosen, otherwise >>spring<< is used.
- directed – (default: True) if True, the directed version is shown, otherwise the undirected.

**standard_quiver()**

Return the standard quiver of `self`.

**EXAMPLES:**

```python
sage: mut_type = QuiverMutationType( ['A',5] ); mut_type
[ 'A', 5]
sage: mut_type.standard_quiver()
Quiver on 5 vertices of type ['A', 5]
```
sage: mut_type = QuiverMutationType( ['A', [5, 3], 1] ); mut_type
['A', [5, 3], 1]
sage: mut_type.standard_quiver()
Quiver on 8 vertices of type ['A', [5, 3], 1]

sage: mut_type = QuiverMutationType([['A', 3], ['B', 3]]); mut_type
[['A', 3], ['B', 3]]
sage: mut_type.standard_quiver()
Quiver on 6 vertices of type [['A', 3], ['B', 3]]

sage: mut_type = QuiverMutationType([['A', 3], ['B', 3], ['X', 6]]); mut_type
[['A', 3], ['B', 3], ['X', 6]]
sage: mut_type.standard_quiver()
Quiver on 12 vertices of type [['A', 3], ['B', 3], ['X', 6]]

sage.combinat.cluster_algebra_quiver.quiver_mutation_type.save_quiver_data(n, up_to=True,
types='ClassicalExceptional',
verbose=True)

Save mutation classes of certain quivers of ranks up to and equal to n or equal to n to DOT_SAGE/
cluster_algebra_quiver/mutation_classes_n.dig6.

This data will then be used to determine quiver mutation types.

INPUT:

• n: the rank (or the upper limit on the rank) of the mutation classes that are being saved.
• up_to – (default:True) if True, saves data for ranks smaller than or equal to n. If False, saves data for
  rank exactly n.
• types – (default:'ClassicalExceptional') if all, saves data for both exceptional mutation-finite quivers and
  for classical quiver. The input ‘Exceptional’ or ‘Classical’ is also allowed to save only part of this data.

5.1.23 Cluster complex (or generalized dual associahedron)

EXAMPLES:

A first example of a cluster complex:

sage: C = ClusterComplex(['A', 2]); C
Cluster complex of type ['A', 2] with 5 vertices and 5 facets

Its vertices, facets, and minimal non-faces:

sage: C.vertices()
(0, 1, 2, 3, 4)
sage: C.facets()
[(0, 1), (0, 4), (1, 2), (2, 3), (3, 4)]
sage: for F in C.facets(): F.cluster()
[(-1, 0), (0, -1)]
We can do everything we can do on simplicial complexes, e.g. computing its homology:

```
sage: C.homology()
{0: 0, 1: Integer(1)}
```

AUTHORS:

• Christian Stump (2011) Initial version

```python
class sage.combinat.cluster_complex.ClusterComplex(W, k, coxeter_element, algorithm)
Bases: SubwordComplex

A cluster complex (or generalized dual associahedron).

The cluster complex (or generalized dual associahedron) is a simplicial complex constructed from a cluster algebra. Its vertices are the cluster variables and its facets are the clusters, i.e., maximal subsets of compatible cluster variables.

The cluster complex of type $A_n$ is the simplicial complex with vertices being (proper) diagonals in a convex $(n + 3)$-gon and with facets being triangulations.

The implementation of the cluster complex depends on its connection to subword complexes, see [CLS2014]. Let $c$ be a Coxeter element with reduced word $c$ in a finite Coxeter group $W$, and let $w_o$ be the $c$-sorting word for the longest element $w_o \in W$.

The multi-cluster complex $\Delta(W, k)$ has vertices in one-to-one correspondence with letters in the word $Q = c^k w_o$ and with facets being complements in $Q$ of reduced expressions for $w_o$.

For $k = 1$, the multi-cluster complex is isomorphic to the cluster complex as defined above.

EXAMPLES:

A first example of a cluster complex:

```
sage: C = ClusterComplex(['A', 2]); C
Cluster complex of type ['A', 2] with 5 vertices and 5 facets
```

Its vertices, facets, and minimal non-faces:

```
sage: C.vertices()
(0, 1, 2, 3, 4)
sage: C.facets()
[[0, 1], [0, 4], [1, 2], [2, 3], [3, 4]]
sage: C.minimal_nonfaces()
[[0, 2], [0, 3], [1, 3], [1, 4], [2, 4]]
```

We can do everything we can do on simplicial complexes, e.g. computing its homology:
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```python
sage: C.homology()
{0: 0, 1: Z}
```

We can also create a multi-cluster complex:

```python
sage: ClusterComplex(['A', 2], k=2)
Multi-cluster complex of type ['A', 2] with 7 vertices and 14 facets
```

REFERENCES:
- [CLS2014]

**Element**
- alias of `ClusterComplexFacet`

**cyclic_rotation()**
- Return the operation on the facets of `self` obtained by the cyclic rotation as defined in [CLS2014].

```python
sage: ClusterComplex(['A', 2]).cyclic_rotation()
<function ...act at ...>
```

**k()**
- Return the index $k$ of `self`.

```python
sage: ClusterComplex(['A', 2]).k()
1
```

**minimal_nonfaces()**
- Return the minimal non-faces of `self`.

```python
sage: ClusterComplex(['A', 2]).minimal_nonfaces()
[[0, 2], [0, 3], [1, 3], [1, 4], [2, 4]]
```

```python
class sage.combinat.cluster_complex.ClusterComplexFacet(parent, positions, facet_test=True)
```

- Bases: `SubwordComplexFacet`
- A cluster (i.e., a facet) of a cluster complex.

**cluster()**
- Return this cluster as a set of almost positive roots.

```python
sage: C = ClusterComplex(['A', 2])
sage: F = C((0, 1))
sage: F.cluster()
[(-1, 0), (0, -1)]
```

**product_of_upper_cluster()**
- Return the product of the upper cluster in reversed order.

```python
sage: C = ClusterComplex(['A', 2])
sage: F = C((0, 1))
sage: F.product_of_upper_cluster()
[(0, 0), (1, 0)]
```
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```
sage: C = ClusterComplex(['A', 2])
sage: for F in C: F.product_of_upper_cluster().reduced_word()
[]
[2]
[1]
[1, 2]
[1, 2]
```

**upper_cluster()**
Return the part of the cluster that contains positive roots

**EXAMPLES:**
```
sage: C = ClusterComplex(['A', 2])
sage: F = C((0, 1))
sage: F.upper_cluster()
[]
```

### 5.1.24 Colored Permutations

**Todo:** Much of the colored permutations (and element) class can be generalized to $G \wr S_n$

**class** `sage.combinat.colored_permutations.ColoredPermutation(parent, colors, perm)`

**Bases:** `MultiplicativeGroupElement`

A colored permutation.

**colors()**
Return the colors of self.

**EXAMPLES:**
```
sage: C = ColoredPermutations(4, 3)
sage: s1,s2,t = C.gens()
sage: x = s1*s2*t
sage: x.colors()
[1, 0, 0]
```

**has_left_descent(i)**
Return True if i is a left descent of self.

Let $p = ((s_1, \ldots, s_n), \sigma)$ be a colored permutation. We say $p$ has a left $n$-descent if $s_n > 0$. If $i < n$, then we say $p$ has a left $i$-descent if either

- $s_i \neq 0, s_{i+1} = 0$ and $\sigma_i < \sigma_{i+1}$ or
- $s_i = s_{i+1}$ and $\sigma_i > \sigma_{i+1}$.

This notion of a left $i$-descent is done in order to recursively construct $w(p) = \sigma_i w(\sigma_i^{-1} p)$, where $w(p)$ denotes a reduced word of $p$.

**EXAMPLES:**
Combinatorics, Release 10.0

```python
sage: C = ColoredPermutations(2, 4)
sage: s1,s2,s3,s4 = C.gens()
sage: x = s4*s1*s2*s3*s4
sage: [x.has_left_descent(i) for i in C.index_set()]
[True, False, False, True]

sage: C = ColoredPermutations(1, 5)
sage: s1,s2,s3,s4 = C.gens()
sage: x = s4*s1*s2*s3*s4
sage: [x.has_left_descent(i) for i in C.index_set()]
[True, False, False, True]

sage: C = ColoredPermutations(3, 3)
sage: x = C([[2,1,0],[3,1,2]])

sage: [x.has_left_descent(i) for i in C.index_set()]
[False, True, False]

sage: C = ColoredPermutations(4, 4)
sage: x = C([[2,1,0,1],[3,2,4,1]])

sage: [x.has_left_descent(i) for i in C.index_set()]
[False, True, False, True]
```

### length()

Return the length of `self` in generating reflections.

This is the minimal numbers of generating reflections needed to obtain `self`.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(3, 3)
sage: x = C([[2,1,0],[3,1,2]])
sage: x.length()
7

sage: C = ColoredPermutations(4, 4)
sage: x = C([[2,1,0,1],[3,2,4,1]])
sage: x.length()
12
```

### one_line_form()

Return the one line form of `self`.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(4, 3)
sage: s1,s2,t = C.gens()
sage: x = s1*s2*t
sage: x
[[i, 0, 0], [3, 1, 2]]
sage: x.one_line_form()
[(1, 3), (0, 1), (0, 2)]
```

### permutation()

Return the permutation of `self`.
This is obtained by forgetting the colors.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(4, 3)
sage: s1, s2, t = C.gens()
sage: x = s1*s2*t
sage: x.permutation()
[3, 1, 2]
```

**reduced_word()**

Return a word in the simple reflections to obtain `self`.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(3, 3)
sage: x = C([[2, 1, 0], [3, 1, 2]])
sage: x.reduced_word()
[2, 1, 3, 2, 1, 3, 3]
```

**to_matrix()**

Return a matrix of `self`.

The colors are mapped to roots of unity.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(4, 3)
sage: s1, s2, t = C.gens()
sage: x = s1*s2^t*s2; x.one_line_form()
[(1, 2), (0, 1), (0, 3)]
sage: M = x.to_matrix(); M
[ 0 1 0]
[zeta4 0 0]
[ 0 0 1]
```

The matrix multiplication is in the `opposite` order:

```python
sage: M == s2.to_matrix()*t.to_matrix()*s2.to_matrix()*s1.to_matrix()
True
```

**class** `sage.combinat.colored_permutations.ColoredPermutations(m, n)`

**Bases:** `Parent`, `UniqueRepresentation`

The group of \(m\)-colored permutations on \(\{1, 2, \ldots, n\}\).

Let \(S_n\) be the symmetric group on \(n\) letters and \(C_m\) be the cyclic group of order \(m\). The \(m\)-colored permutation group on \(n\) letters is given by \(P^m_n = C_m \wr S_n\). This is also the complex reflection group \(G(m, 1, n)\).

We define our multiplication by

\[
((s_1, \ldots, s_n), \sigma) \cdot ((t_1, \ldots, t_n), \tau) = ((s_1 t_{\sigma(1)}, \ldots, s_n t_{\sigma(n)}), \tau \sigma).
\]

**EXAMPLES:**
sage: C = ColoredPermutations(4, 3); C
4-colored permutations of size 3
sage: s1, s2, t = C.gens()
sage: (s1, s2, t)
([0, 0, 0], [2, 1, 3]), ([0, 0, 0], [1, 3, 2]), ([0, 0, 1], [1, 2, 3])
sage: s1*s2
[[0, 0, 0], [3, 1, 2]]
sage: s1*s2*s1 == s2*s1*s2
True
sage: t^4 == C.one()
True
sage: s2*t*s2
[[0, 1, 0], [1, 2, 3]]

We can also create a colored permutation by passing an iterable consisting of tuples consisting of (color, element):

sage: x = C([(2,1), (3,3), (3,2)]); x
[[2, 3, 3], [1, 3, 2]]

or a list of colors and a permutation:

sage: C([[3,3,1], [1,3,2]])
[[3, 3, 1], [1, 3, 2]]
sage: C(([3,3,1], [1,3,2]))
[[3, 3, 1], [1, 3, 2]]

There is also the natural lift from permutations:

sage: P = Permutations(3)
sage: C(P.an_element())
[[0, 1, 0], [1, 2, 3]]

A colored permutation:

sage: C(C.an_element()) == C.an_element()
True

REFERENCES:
• Wikipedia article Generalized_symmetric_group
• Wikipedia article Complex_reflection_group

Element
alias of ColoredPermutation

as_permutation_group()

Return the permutation group corresponding to self.

EXAMPLES:
cardinality()
    Return the cardinality of self.
    EXAMPLES:
    sage: C = ColoredPermutations(4, 3)
sage: C.cardinality()  
384  
sage: C.cardinality() == 4**3 * factorial(3)
    True

codegrees()
    Return the codegrees of self.
    Let \( G \) be a complex reflection group. The codegrees \( d_1^* \leq d_2^* \leq \cdots \leq d_\ell^* \) of \( G \) can be defined by:
    \[
    \prod_{i=1}^{\ell}(q - d_i^* - 1) = \sum_{g \in G} \det(g)^{\dim(V)} q^{\dim(V)}.
    \]
    where \( V \) is the natural complex vector space that \( G \) acts on and \( \ell \) is the rank().
    If \( m = 1 \), then we are in the special case of the symmetric group and the codegrees are \((n-2, n-3, \ldots, 1, 0)\). Otherwise the degrees are \(( (n-1)m, (n-2)m, \ldots, m, 0)\).
    EXAMPLES:
    sage: C = ColoredPermutations(4, 3)
sage: C.codegrees()
    (8, 4, 0)
    sage: S = ColoredPermutations(1, 3)
sage: S.codegrees()
    (1, 0)

coxeter_matrix()
    Return the Coxeter matrix of self.
    EXAMPLES:
    sage: C = ColoredPermutations(3, 4)
sage: C.coxeter_matrix()
    [1 3 2 2]
    [3 1 3 2]
    [2 3 1 4]
    [2 2 4 1]
    sage: C = ColoredPermutations(1, 4)
sage: C.coxeter_matrix()
    [1 3 2]
    [3 1 3]
    [2 3 1]

degrees()
    Return the degrees of self.
    The degrees of a complex reflection group are the degrees of the fundamental invariants of the ring of polynomial invariants.
If \( m = 1 \), then we are in the special case of the symmetric group and the degrees are \((2, 3, \ldots, n, n + 1)\). Otherwise the degrees are \((m, 2m, \ldots, nm)\).

**EXAMPLES:**

```
sage: C = ColoredPermutations(4, 3)
sage: C.degrees()
(4, 8, 12)
sage: S = ColoredPermutations(1, 3)
sage: S.degrees()
(2, 3)
```

We now check that the product of the degrees is equal to the cardinality of \( \text{self} \):

```
sage: prod(C.degrees()) == C.cardinality()
True
sage: prod(S.degrees()) == S.cardinality()
True
```

**fixed_point_polynomial** \((q=\text{None})\)

The fixed point polynomial of \( \text{self} \).

The fixed point polynomial \( f_G \) of a complex reflection group \( G \) is counting the dimensions of fixed points subspaces:

\[
f_G(q) = \sum_{w \in W} q^{\dim V_w}.
\]

Furthermore, let \( d_1, d_2, \ldots, d_\ell \) be the degrees of \( G \), where \( \ell \) is the \text{rank()}\). Then the fixed point polynomial is given by

\[
f_G(q) = \prod_{i=1}^{\ell} (q + d_i - 1).
\]

**INPUT:**

- \( q \) – (default: the generator of \( \mathbb{Z}['q'] \)) the parameter \( q \)

**EXAMPLES:**

```
sage: C = ColoredPermutations(4, 3)
sage: C.fixed_point_polynomial()
q^3 + 21*q^2 + 131*q + 231
sage: S = ColoredPermutations(1, 3)
sage: S.fixed_point_polynomial()
q^2 + 3*q + 2
```

**gens()**

Return the generators of \( \text{self} \).

**EXAMPLES:**

```
sage: C = ColoredPermutations(4, 3)
sage: C.gens()
([[0, 0, 0], [2, 1, 3]],
```
\[
[[0, 0, 0], [1, 3, 2]],
[[0, 0, 1], [1, 2, 3]]
\]

```python
sage: S = SignedPermutations(4)
sage: S.gens()
([2, 1, 3, 4], [1, 3, 2, 4], [1, 2, 4, 3], [1, 2, 3, -4])
```

**index_set()**

Return the index set of self.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(3, 4)
sage: C.index_set()
(1, 2, 3, 4)
sage: C = ColoredPermutations(1, 4)
sage: C.index_set()
(1, 2, 3)
```

**is_well_generated()**

Return if self is a well-generated complex reflection group.

A complex reflection group \(G\) is well-generated if it is generated by \(\ell\) reflections. Equivalently, \(G\) is well-generated if \(d_i + d_i^\ast = d_\ell\) for all \(1 \leq i \leq \ell\).

**EXAMPLES:**

```python
sage: C = ColoredPermutations(4, 3)
sage: C.is_well_generated()
True
sage: C = ColoredPermutations(2, 8)
sage: C.is_well_generated()
True
sage: C = ColoredPermutations(1, 4)
sage: C.is_well_generated()
True
```

**matrix_group()**

Return the matrix group corresponding to self.

**EXAMPLES:**

```python
sage: C = ColoredPermutations(4, 3)
sage: C.matrix_group()
Matrix group over Cyclotomic Field of order 4 and degree 2 with 3 generators
([0 1 0] [1 0 0] [ 1 0 0])
([1 0 0] [0 0 1] [ 0 1 0])
([0 0 1], [0 1 0], [ 0 0 zeta4])
```

**number_of_reflection_hyperplanes()**

Return the number of reflection hyperplanes of self.
The number of reflection hyperplanes of a complex reflection group is equal to the sum of the codegrees plus the rank.

EXAMPLES:

```sage
c = ColoredPermutations(1, 2)
c.number_of_reflection_hyperplanes()
1
c = ColoredPermutations(1, 3)
c.number_of_reflection_hyperplanes()
3
c = ColoredPermutations(4, 12)
c.number_of_reflection_hyperplanes()
276
```

one()
Return the identity element of self.

EXAMPLES:

```sage
c = ColoredPermutations(4, 3)
c.one()
[[0, 0, 0], [1, 2, 3]]
```

order()
Return the cardinality of self.

EXAMPLES:

```sage
c = ColoredPermutations(4, 3)
c.cardinality()
384
c.cardinality() == 4**3 * factorial(3)
True
```

rank()
Return the rank of self.

The rank of a complex reflection group is equal to the dimension of the complex vector space the group acts on.

EXAMPLES:

```sage
c = ColoredPermutations(4, 12)
c.rank()
12
c = ColoredPermutations(7, 4)
c.rank()
4
c = ColoredPermutations(1, 4)
c.rank()
3
```

simple_reflection(i)
Return the i-th simple reflection of self.

EXAMPLES:
sage: C = ColoredPermutations(4, 3)

sage: C.gens()
([\[0, 0, 0\], [2, 1, 3]], [[0, 0, 0], [1, 3, 2]], [[0, 0, 1], [1, 2, 3]])

sage: C.simple_reflection(2)
[[0, 0, 0], [1, 3, 2]]

sage: C.simple_reflection(3)
[[0, 0, 1], [1, 2, 3]]

sage: S = SignedPermutations(4)

sage: S.simple_reflection(1)
[2, 1, 3, 4]

sage: S.simple_reflection(4)
[1, 2, 3, -4]

class sage.combinat.colored_permutations.SignedPermutation(parent, colors, perm)

Bases: ColoredPermutation

A signed permutation.

has_left_descent(i)

Return True if i is a left descent of self.

EXAMPLES:

sage: S = SignedPermutations(4)
sage: s1, s2, s3, s4 = S.gens()
sage: x = s4*s1*s2*s3*s4

sage: [x.has_left_descent(i) for i in S.index_set()]
[True, False, False, True]

order()

Return the multiplicative order of the signed permutation.

EXAMPLES:

sage: pi = SignedPermutations(7)([2,-1,4,-6,-5,-3,7])
sage: pi.to_cycles(singletons=False)
[(1, 2, -1, -2), (3, 4, -6), (5, -5)]

sage: pi.order()
12

to_cycles(singletons=True, use_min=True, negative_singletons=True)

Return the signed permutation self as a list of disjoint cycles.

The cycles are returned in the order of increasing smallest elements, and each cycle is returned as a tuple which starts with its smallest positive element. We do not include the corresponding negative cycles.

INPUT:

- **singletons** – (default: True) whether to include singleton cycles or not
- **use_min** – (default: True) if False, the cycles are returned in the order of increasing largest (not smallest) elements, and each cycle starts with its largest element

EXAMPLES:
```
sage: pi = SignedPermutations(7)([2,-1,4,-6,-5,-3,7])
sage: pi.to_cycles()
[(1, 2, -1, -2), (3, 4, -6), (5, -5), (7,)]
sage: pi.to_cycles(singletons=False)
[(1, 2, -1, -2), (3, 4, -6), (5, -5)]
sage: pi.to_cycles(use_min=False)
[(7,), (6, -3, -4), (5, -5), (2, -1, -2, 1)]
sage: pi.to_cycles(singletons=False, use_min=False)
[(6, -3, -4), (5, -5), (2, -1, -2, 1)]
```

```
to_matrix()

Return a matrix of self.

EXAMPLES:
```
sage: S = SignedPermutations(4)
sage: s1,s2,s3,s4 = S gens()
sage: x = s4*s1*s2*s3*s4
sage: M = x.to_matrix(); M
[ 0 1 0 0]
[ 0 0 1 0]
[ 0 0 0 -1]
[-1 0 0 0]
```

The matrix multiplication is in the opposite order:
```
sage: m1,m2,m3,m4 = [g.to_matrix() for g in S gens()]
sage: M == m4 * m3 * m2 * m1 * m4
True
```

```
class sage.combinat.colored_permutations.SignedPermutations(n)

Bases: ColoredPermutations

Group of signed permutations.

The group of signed permutations is also known as the hyperoctahedral group, the Coxeter group of type $B_n$, and the 2-colored permutation group. Thus it can be constructed as the wreath product $S_2 \wr S_n$.

EXAMPLES:
```
sage: S = SignedPermutations(4)
sage: s1,s2,s3,s4 = S.group generators()
sage: x = s4*s1*s2*s3*s4; x
[-4, 1, 2, -3]
sage: x^4 == S one()
True
```

This is a finite Coxeter group of type $B_n$:
```
sage: S.canonical_representation()
Finite Coxeter group over Number Field in a with defining polynomial $x^2 - 2$ with a root $\left(1.414213562373095?\right)$ with Coxeter matrix:
[1 3 2 2]
[3 1 3 2]
[2 3 1 4]
```
We can also go between the 2-colored permutation group:

```python
sage: C = ColoredPermutations(2, 3)
sage: S = SignedPermutations(3)
sage: S.an_element()
[-3, 1, 2]
sage: C(S.an_element())
[[1, 0, 0], [3, 1, 2]]
sage: S(C(S.an_element())) == S.an_element()
True
sage: S(C.an_element())
[-3, 1, 2]
```

There is also the natural lift from permutations:

```python
sage: P = Permutations(3)
sage: x = S(P.an_element()); x
[3, 1, 2]
sage: x.parent()
Signed permutations of 3
```

REFERENCES:

- [Wikipedia article Hyperoctahedral_group](https://en.wikipedia.org/wiki/Hyperoctahedral_group)

**Element**

alias of `SignedPermutation`

**long_element** *(index_set=None)*

Return the longest element of `self`, or of the parabolic subgroup corresponding to the given `index_set`.

**EXAMPLES:**

```python
sage: S = SignedPermutations(4)
sage: S.long_element()
[-1, -2, -3, -4]
```

**one**

Return the identity element of `self`.

**EXAMPLES:**

```python
sage: S = SignedPermutations(4)
sage: S.one()
[1, 2, 3, 4]
```
simple_reflection(i)

Return the i-th simple reflection of self.

EXAMPLES:

```
sage: S = SignedPermutations(4)
sage: S.simple_reflection(1)
[2, 1, 3, 4]
sage: S.simple_reflection(4)
[1, 2, 3, -4]
```

5.1.25 Combinatorial Functions

This module implements some combinatorial functions, as listed below. For a more detailed description, see the relevant docstrings.

**Sequences:**

- Bell numbers, `bell_number()`
- Catalan numbers, `catalan_number()` (not to be confused with the Catalan constant)
- Narayana numbers, `narayana_number()`
- Euler numbers, `euler_number()` (Maxima)
- Eulerian numbers, `eulerian_number()`
- Eulerian polynomial, `eulerian_polynomial()`
- Fibonacci numbers, `fibonacci()` (PARI) and `fibonacci_number()` (GAP) The PARI version is better.
- Lucas numbers, `lucas_number1()`, `lucas_number2()`.
- Stirling numbers, `stirling_number1()`, `stirling_number2()`.
- Polygonal numbers, `polygonal_number()`

**Set-theoretic constructions:**

- Derangements of a multiset, `derangements()` and `number_of_derangements()`.
- Tuples of a multiset, `tuples()` and `number_of_tuples()`. An ordered tuple of length k of set S is a ordered selection with repetitions of S and is represented by a sorted list of length k containing elements from S.
- Unordered tuples of a set, `unordered_tuples()` and `number_of_unordered_tuples()`. An unordered tuple of length k of set S is an unordered selection with repetitions of S and is represented by a sorted list of length k containing elements from S.

**Warning:** The following function is deprecated and will soon be removed.

- Permutations of a multiset, `permutations()`, `permutations_iterator()`, `number_of_permutations()`. A permutation is a list that contains exactly the same elements but possibly in different order.

**Related functions:**

- Bernoulli polynomials, `bernoulli_polynomial()`
Implemented in other modules (listed for completeness):

The `sage.arith.all` module contains the following combinatorial functions:

- `binomial` the binomial coefficient (wrapped from PARI)
- `factorial` (wrapped from PARI)
- `partition` (from the Python Cookbook)  Generator of the list of all the partitions of the integer $n$.
- `number_of_partitions()` (wrapped from PARI) the number of partitions:
- `falling_factorial()` Definition: for integer $a \geq 0$ we have $x(x - 1) \cdots (x - a + 1)$. In all other cases we use the GAMMA-function: $\frac{\Gamma(x+1)}{\Gamma(x-a+1)}$.
- `rising_factorial()` Definition: for integer $a \geq 0$ we have $x(x + 1) \cdots (x + a - 1)$. In all other cases we use the GAMMA-function: $\frac{\Gamma(x+a)}{\Gamma(x)}$.
- `gaussian_binomial` the gaussian binomial
  \[
  \left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q^{m-r+1})}{(1 - q)(1 - q^2) \cdots (1 - q^r)}.
  \]

The `sage.groups.perm_gps.permgroup_elements` contains the following combinatorial functions:

- `matrix` method of PermutationGroupElement yielding the permutation matrix of the group element.

Todo:

GUAVA commands:
- VandermondeMat
- GrayMat returns a list of all different vectors of length $n$ over the field $F$, using Gray ordering.

Not in GAP:
- Rencontres numbers ([Wikipedia article Rencontres_number](https://en.wikipedia.org/wiki/Rencontres_number))

REFERENCES:
- [Wikipedia article Twelvefold_way](https://en.wikipedia.org/wiki/Twelvefold_way) (general reference)

AUTHORS:
- William Stein (2006-07): editing of docs and code; many optimizations, refinements, and bug fixes in corner cases.
- Florent Hivert (2009-02): combinatorial class cleanup
- Fredrik Johansson (2010-07): fast implementation of `stirling_number2`
- Punarbasu Purkayastha (2012-12): deprecate arrangements, combinations, combinations_iterator, and clean up very old deprecated methods.
Functions and classes

**class** `sage.combinat.combinat.CombinatorialClass`(*category=None*)

Bases: `Parent`

This class is deprecated, and will disappear as soon as all derived classes in Sage's library will have been fixed. Please derive directly from `Parent` and use the category `EnumeratedSets`, `FiniteEnumeratedSets`, or `InfiniteEnumeratedSets`, as appropriate.

For examples, see:

```python
sage: FiniteEnumeratedSets().example()
An example of a finite enumerated set: {1,2,3}
sage: InfiniteEnumeratedSets().example()
An example of an infinite enumerated set: the non negative integers
```

**Element**

alias of `CombinatorialObject`

**cardinality()**

Default implementation of cardinality which just goes through the iterator of the combinatorial class to count the number of objects.

EXAMPLES:

```python
sage: class C(CombinatorialClass):
    ....:     def __iter__(self):
    ....:         return iter([1,2,3])
sage: C().cardinality()  # indirect doctest
3
```

**element_class()**

This function is a temporary helper so that a `CombinatorialClass` behaves as a parent for creating elements. This will disappear when combinatorial classes will be turned into actual parents (in the category `EnumeratedSets`).

**filter**(f, name=None)

Return the combinatorial subclass of f which consists of the elements x of self such that f(x) is True.

EXAMPLES:

```python
sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).filter(lambda x: x.avoids([1,2]))
sage: P.list()
[[3, 2, 1]]
```

**first()**

Default implementation for first which uses iterator.

EXAMPLES:

```python
sage: C = CombinatorialClass()
sage: C.list = lambda: [1,2,3]
sage: C.first()  # indirect doctest
1
```
is_finite()

Return whether self is finite or not.

EXAMPLES:

```
sage: Partitions(5).is_finite()
True
sage: Permutations().is_finite()
False
```

last()

Default implementation for first which uses iterator.

EXAMPLES:

```
sage: C = CombinatorialClass()
sage: C.list = lambda: [1,2,3]
sage: C.last() # indirect doctest
3
```

list()

The default implementation of list which builds the list from the iterator.

EXAMPLES:

```
sage: class C(CombinatorialClass):
....:     def __iter__(self):
....:         return iter([1,2,3])
sage: C().list() # indirect doctest
[1, 2, 3]
```

map(f, name, is_injective=None)

Return the image \( \{f(x) | x \in self\} \) of this combinatorial class by \( f \), as a combinatorial class.

INPUT:

* is_injective – boolean (default: True) whether to assume that \( f \) is injective.

EXAMPLES:

```
sage: R = Permutations(3).map(attrcall('reduced_word')); R
Image of Standard permutations of 3 by The map *.reduced_word() from Standard___
˓→permutations of 3
sage: R.cardinality()
6
sage: R.list()
[[], [2], [1], [1, 2], [2, 1], [2, 1, 2]]
sage: [ r for r in R]
[[], [2], [1], [1, 2], [2, 1], [2, 1, 2]]
```

If the function is not injective, then there may be repeated elements:

```
sage: P = Partitions(4)
sage: P.list()
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
sage: P.map(len).list()
[1, 2, 2, 3, 4]
```
Use `is_injective=False` to get a correct result in this case:

```
sage: P = CombinatorialClass()
sage: P.map(len, is_injective=False).list()
[1, 2, 3, 4]
```

**next**(obj)

Default implementation for next which uses iterator.

**EXAMPLES:**

```
sage: C = CombinatorialClass()
sage: C.list = lambda : [1, 2, 3]
sage: C.next(2)  # indirect doctest
3
```

**previous**(obj)

Default implementation for next which uses iterator.

**EXAMPLES:**

```
sage: C = CombinatorialClass()
sage: C.list = lambda : [1, 2, 3]
sage: C.previous(2)  # indirect doctest
1
```

**random_element()**

Default implementation of random which uses unrank.

**EXAMPLES:**

```
sage: C = CombinatorialClass()
sage: C.list = lambda : [1, 2, 3]
sage: C.random_element()  # random  # indirect doctest
1
```

**rank**(obj)

Default implementation of rank which uses iterator.

**EXAMPLES:**

```
sage: C = CombinatorialClass()
sage: C.list = lambda : [1, 2, 3]
sage: C.rank(3)  # indirect doctest
2
```

**union**(right_cc, name=None)

Return the combinatorial class representing the union of `self` and `right_cc`.

**EXAMPLES:**

```
sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(2).union(Permutations_CC(1))
sage: P.list()
[[[1, 2], [2, 1], [1]]]
**unrank**($r$)

Default implementation of unrank which goes through the iterator.

**EXAMPLES:**

```python
sage: C = CombinatorialClass()
sage: C.list = lambda: [1, 2, 3]
sage: C.unrank(1) # indirect doctest
2
```

**class** sage.combinat.combinat.CombinatorialElement($parent$, *$args$, **$kwds$)

**Bases:** CombinatorialObject, Element

CombinatorialElement is both a CombinatorialObject and an Element. So it represents a list which is an element of some parent.

A CombinatorialElement subclass also automatically supports the __classcall__ mechanism.

**Warning:** This class is slowly being deprecated. Use ClonableList instead.

**INPUT:**

- $parent$ – the Parent class for this element.
- $lst$ – a list or any object that can be converted to a list by calling list().

**EXAMPLES:**

```python
sage: from sage.combinat.combinat import CombinatorialElement
sage: e = CombinatorialElement(Partitions(6), [3, 2, 1])
sage: e == loads(dumps(e))
True
sage: parent(e)
Partitions of the integer 6
sage: list(e)
[3, 2, 1]
```

Check classcalls:

```python
sage: class Foo(CombinatorialElement):
    ....:    @staticmethod
    ....:    def __classcall__(cls, x):
    ....:        return x
sage: Foo(17)
17
```

**class** sage.combinat.combinat.CombinatorialObject($l$, $copy=True$)

**Bases:** SageObject

CombinatorialObject provides a thin wrapper around a list. The main differences are that __setitem__ is disabled so that CombinatorialObjects are shallowly immutable, and the intention is that they are semantically immutable.

Because of this, CombinatorialObjects provide a __hash__ function which computes the hash of the string representation of a list and the hash of its parent’s class. Thus, each CombinatorialObject should have a unique string representation.
See also:

*CombinatorialElement* if you want a combinatorial object which is an element of a parent.

**Warning:** This class is slowly being deprecated. Use *ClonableList* instead.

**INPUT:**

- 1 – a list or any object that can be converted to a list by calling `list()`.
- `copy` – (boolean, default True) if False, then 1 must be a list, which is assigned to `self._list` without copying.

**EXAMPLES:**

```python
sage: c = CombinatorialObject([1,2,3])
sage: c == loads(dumps(c))
True
sage: c._list
[1, 2, 3]
sage: c._hash is None
True
```

For efficiency, you can specify `copy=False` if you know what you are doing:

```python
sage: from sage.combinat.combinat import CombinatorialObject
sage: x = [3, 2, 1]
sage: C = CombinatorialObject(x, copy=False)
sage: C
[3, 2, 1]
sage: x[0] = 5
sage: C
[5, 2, 1]
```

**index(key)**

**EXAMPLES:**

```python
sage: c = CombinatorialObject([1,2,3])
sage: c.index(1)
0
sage: c.index(3)
2
```

**class** `sage.combinat.combinat.FilteredCombinatorialClass`(*combinatorial_class, f, name=None*)

Bases: `CombinatorialClass`

A filtered combinatorial class F is a subset of another combinatorial class C specified by a function f that takes in an element c of C and returns True if and only if c is in F.

**cardinality()**

**EXAMPLES:**

```python
sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).filter(lambda x: x.avoids([1,2]))
(continues on next page)```
class sage.combinat.combinat.InfiniteAbstractCombinatorialClass(category=None)

Bases: CombinatorialClass

This is an internal class that should not be used directly. A class which inherits from InfiniteAbstractCombinatorialClass inherits the standard methods list and count.

If self._infinite_cclass_slice exists then self.__iter__ returns an iterator for self, otherwise raise NotImplementedError. The method self._infinite_cclass_slice is supposed to accept any integer as an argument and return something which is iterable.

cardinality()

Count the elements of the combinatorial class.

EXAMPLES:

```sage
R = InfiniteAbstractCombinatorialClass()
R.cardinality()
```

list()

Return an error since self is an infinite combinatorial class.

EXAMPLES:

```sage
R = InfiniteAbstractCombinatorialClass()
R.list()
```

class sage.combinat.combinat.MapCombinatorialClass(cc, f, name=None, *, is_injective=True)

Bases: ImageSubobject, CombinatorialClass

The image of a combinatorial class through a function.

INPUT:

* is_injective – boolean (default: True) whether to assume that f is injective.

See CombinatorialClass.map() for examples

EXAMPLES:

```sage
R = SymmetricGroup(10).map(attrcall('reduced_word'))
R.an_element()
[9, 8, 7, 6, 5, 4, 3, 2]
R.cardinality()
3628800
i = iter(R)
```
sage: next(i), next(i), next(i)
([], [1, 2, 3, 4, 5, 6, 7, 8, 9], [1])

class sage.combinat.combinat.Permutations_CC(n)
Bases: CombinatorialClass
A testing class for CombinatorialClass since Permutations no longer inherits from CombinatorialClass in github issue #14772.

class sage.combinat.combinat.UnionCombinatorialClass(left_cc, right_cc, name=None)
Bases: CombinatorialClass
A UnionCombinatorialClass is a union of two other combinatorial classes.

cardinality()
EXAMPLES:

sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).union(Permutations_CC(2))

sage: P.cardinality()
8

first()
EXAMPLES:

sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).union(Permutations_CC(2))

sage: P.first()
[1, 2, 3]

last()
EXAMPLES:

sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).union(Permutations_CC(2))

sage: P.last()
[2, 1]

list()
EXAMPLES:

sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).union(Permutations_CC(2))

sage: P.list()
[[1, 2, 3],
 [1, 3, 2],
 [2, 1, 3],
 [2, 3, 1],
 [3, 1, 2],
 [3, 2, 1],
 [1, 2],
 [2, 1]]
**rank(x)**

**EXAMPLES:**

```
sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).union(Permutations_CC(2))
sage: P.rank(Permutation([2,1]))
7
sage: P.rank(Permutation([1,2,3]))
0
```

**unrank(x)**

**EXAMPLES:**

```
sage: from sage.combinat.combinat import Permutations_CC
sage: P = Permutations_CC(3).union(Permutations_CC(2))
sage: P.unrank(7)
[2, 1]
sage: P.unrank(0)
[1, 2, 3]
```

\[ \text{sage.combinat.combinat.bell_number}(n, \text{algorithm}=\text{'flint'}, **options) \]

Return the \( n \)-th Bell number.

This is the number of ways to partition a set of \( n \) elements into pairwise disjoint nonempty subsets.

**INPUT:**

- \( n \) – a positive integer
- \( \text{algorithm} \) – (Default: 'flint') any one of the following:
  - 'dobinski' – Use Dobinski’s formula implemented in Sage
  - 'flint' – Wrap FLINT’s \texttt{arith\_bell\_number}
  - 'gap' – Wrap GAP’s \texttt{Bell}
  - 'mpmath' – Wrap mpmath’s \texttt{bell}

**Warning:** When using the \texttt{mpmath} algorithm to compute Bell numbers and you specify \texttt{prec}, it can return incorrect results due to low precision. See the examples section.

Let \( B_n \) denote the \( n \)-th Bell number. Dobinski’s formula is:

\[
B_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}.
\]

To show our implementation of Dobinski’s method works, suppose that \( n \geq 5 \) and let \( k_0 \) be the smallest positive integer such that \( \frac{k_0^n}{k_0!} < 1 \). Note that \( k_0 > n \) and \( k_0 \leq 2n \) because we can prove that \( \frac{(2n)^n}{(2n)!} < 1 \) by Stirling.

If \( k > k_0 \), then we have \( \frac{k^n}{k!} < \frac{1}{2^{k-n}} \). We show this by induction: let \( c_k = \frac{k^n}{k!} \), if \( k > n \) then

\[
\frac{c_{k+1}}{c_k} = \frac{(1 + k^{-1})^n}{k + 1} < \frac{(1 + n^{-1})^n}{n} < \frac{1}{2}.
\]

The last inequality can easily be checked numerically for \( n \geq 5 \).
Using this, we can see that \( \frac{c_k}{c_{k_0}} < \frac{1}{2^{n-k_0}} \) for \( k > k_0 > n \). So summing this it gives that \( \sum_{k=k_0+1}^{\infty} \frac{k^n}{k!} < 1 \), and hence

\[
B_n = e^{-1} \left( \sum_{k=0}^{k_0} \frac{k^n}{k!} + E_1 \right) = e^{-1} \sum_{k=0}^{k_0} \frac{k^n}{k!} + E_2,
\]

where \( 0 < E_1 < 1 \) and \( 0 < E_2 < e^{-1} \). Next we have for any \( q > 0 \)

\[
\sum_{k=0}^{k_0} \frac{k^n}{k!} = \frac{1}{q} \sum_{k=0}^{k_0} \left\lfloor \frac{qk^n}{k!} \right\rfloor + \frac{E_3}{q}
\]

where \( 0 \leq E_3 \leq k_0 + 1 \leq 2n + 1 \). Let \( E_4 = \frac{E_3}{q} \) and let \( q = 2n + 1 \). We find \( 0 \leq E_4 \leq 1 \). These two bounds give:

\[
B_n = e^{-1} \left( \frac{1}{q} \sum_{k=0}^{k_0} \left\lfloor \frac{qk^n}{k!} \right\rfloor + E_4 + E_2 \right)
\]

\[= e^{-1} \frac{1}{q} \sum_{k=0}^{k_0} \left\lfloor \frac{qk^n}{k!} \right\rfloor + E_5
\]

where

\[
0 < E_5 = e^{-1} E_4 + E_2 \leq e^{-1} + e^{-1} < \frac{3}{4}.
\]

It follows that

\[
B_n = \left\lfloor e^{-1} \frac{1}{q} \sum_{k=0}^{k_0} \left\lfloor \frac{qk^n}{k!} \right\rfloor \right\rfloor.
\]

Now define

\[
b = \sum_{k=0}^{k_0} \left\lfloor \frac{qk^n}{k!} \right\rfloor.
\]

This \( b \) can be computed exactly using integer arithmetic. To avoid the costly integer division by \( k! \), we collect more terms and do only one division, for example with 3 terms:

\[
\frac{k^n}{k!} + \frac{(k+1)^n}{(k+1)!} + \frac{(k+2)^n}{(k+2)!} = \frac{k^n(k+1)(k+2) + (k+1)^n(k+2) + (k+2)^n}{(k+2)!}
\]

In the implementation, we collect \( \sqrt{n}/2 \) terms.

To actually compute \( B_n \) from \( b \), we let \( p = \lceil \log_2(b) \rceil + 1 \) such that \( b < 2^p \) and we compute with \( p \) bits of precision. This implies that \( b \) (and \( q < b \)) can be represented exactly.

We compute \( \frac{e^{-1}}{q} b \), rounding down, and we must have an absolute error of at most 1/4 (given that \( E_5 < 3/4 \)). This means that we need a relative error of at most

\[
\frac{eq}{4b} > \frac{(eq)/4}{2^p} > \frac{7}{2^p}
\]

(assuming \( n \geq 5 \)). With a precision of \( p \) bits and rounding down, every rounding has a relative error of at most \( 2^{1-p} = 2/2^p \). Since we do 3 roundings (\( b \) and \( q \) do not require rounding), we get a relative error of at most \( 6/2^p \). All this implies that the precision of \( p \) bits is sufficient.

EXAMPLES:
When using the mpmath algorithm, we are required have mpmath's precision set to at least \( \log_2(B_n) \) bits. If upon computing the Bell number the first time, we deem the precision too low, we use our guess to (temporarily) raise mpmath's precision and the Bell number is recomputed.

```python
sage: k = bell_number(30, 'mpmath'); k
846749014511809332450147
sage: k == bell_number(30)
True
```

If you knows what precision is necessary before computing the Bell number, you can use the prec option:

```python
sage: k2 = bell_number(30, 'mpmath', prec=30); k2
846749014511809332450147
sage: k == k2
True
```

**Warning:** Running mpmath with the precision set too low can result in incorrect results:

```python
sage: k = bell_number(30, 'mpmath', prec=15); k
8467490145118093888871680
sage: k == bell_number(30)
False
```

**AUTHORS:**

- Robert Gerbicz
- Jeroen Demeyer: improved implementation of Dobinski formula with more accurate error estimates (github issue #17157)

**REFERENCES:**

- Wikipedia article Bell_number
sage.combinat.combinat.bell_polynomial(n, k)

Return the Bell Polynomial

\[ B_{n,k}(x_0, x_1, \ldots, x_{n-k}) = \sum_{\sum j_i = k, \sum (i+1)j_i = n} \frac{n!}{j_0!j_1! \cdots j_{n-k}!} \left( \frac{x_0}{(0+1)!} \right)^{j_0} \left( \frac{x_1}{(1+1)!} \right)^{j_1} \cdots \left( \frac{x_{n-k}}{(n-k+1)!} \right)^{j_{n-k}}. \]

**INPUT:**
- n – integer
- k – integer

**OUTPUT:**
- a polynomial in \( n - k + 1 \) variables over \( \mathbb{Z} \)

**EXAMPLES:**

```sage
sage: bell_polynomial(6, 2)
10*x2^2 + 15*x1*x3 + 6*x0*x4
sage: bell_polynomial(6, 3)
15*x1^3 + 60*x0*x1*x2 + 15*x0^2*x3
```

**REFERENCES:**
- [Bel1927]

**AUTHORS:**
- Blair Sutton (2009-01-26)
- Thierry Monteil (2015-09-29): the result must always be a polynomial.

sage.combinat.combinat.bernoulli_polynomial(x, n)

Return the \( n \)-th Bernoulli polynomial evaluated at \( x \).

The generating function for the Bernoulli polynomials is

\[ \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \]

and they are given directly by

\[ B_n(x) = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) B_{n-i}x^i. \]

One has \( B_n(x) = -n\zeta(1 - n, x) \), where \( \zeta(s, x) \) is the Hurwitz zeta function. Thus, in a certain sense, the Hurwitz zeta function generalizes the Bernoulli polynomials to non-integer values of \( n \).

**EXAMPLES:**

```sage
sage: y = QQ['y'].0
sage: bernoulli_polynomial(y, 5)
y^5 - 5/2*y^4 + 5/3*y^3 - 1/6*y
sage: bernoulli_polynomial(y, 5)(12)
199870
sage: bernoulli_polynomial(12, 5)
199870
sage: bernoulli_polynomial(y^2 + 1, 5)
(continues on next page)
```
We verify an instance of the formula which is the origin of the Bernoulli polynomials (and numbers):

```sage
sage: power_sum = sum(k^4 for k in range(10))
sage: 5*power_sum == bernoulli_polynomial(10, 5) - bernoulli(5)
True
```

**REFERENCES:**
- Wikipedia article Bernoulli_polynomials
- Wikipedia article Catalan_number
- http://www-history.mcs.st-andrews.ac.uk/~history/Miscellaneous/CatalanNumbers/catalan.html

**sage.combinat.combinat.catalan_number**(n)

Return the $n$-th Catalan number.

The $n$-th Catalan number is given directly in terms of binomial coefficients by

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad \text{for } n \geq 0.$$  

Consider the set $S = \{1, ..., n\}$. A noncrossing partition of $S$ is a partition in which no two blocks “cross” each other, i.e., if $a$ and $b$ belong to one block and $x$ and $y$ to another, they are not arranged in the order $axby$. $C_n$ is the number of noncrossing partitions of the set $S$. There are many other interpretations (see REFERENCES).

When $n = -1$, this function returns the limit value $-1/2$. For other $n < 0$ it returns 0.

**INPUT:**
- $n$ – integer

**OUTPUT:**
- integer

**EXAMPLES:**

```sage
sage: [catalan_number(i) for i in range(7)]
[1, 1, 2, 5, 14, 42, 132]
sage: x = (QQ[['x']].0).O(8)
sage: (-1/2)*sqrt(1 - 4*x)
-1/2 + x + x^2 + 2*x^3 + 5*x^4 + 14*x^5 + 42*x^6 + 132*x^7 + O(x^8)
sage: [catalan_number(i) for i in range(-7,7)]
[0, 0, 0, 0, 0, 0, -1/2, 1, 1, 2, 5, 14, 42, 132]
sage: [catalan_number(n).mod(2) for n in range(16)]
[1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1]
```

**REFERENCES:**
- Wikipedia article Catalan_number
- http://www-history.mcs.st-andrews.ac.uk/~history/Miscellaneous/CatalanNumbers/catalan.html

**sage.combinat.combinat.euler_number**(n, algorithm='flint')

Return the $n$-th Euler number.

**INPUT:**
Combinatorics, Release 10.0

- n – a positive integer
- algorithm – (Default: 'flint') any one of the following:
  - 'maxima' – Wraps Maxima's euler.
  - 'flint' – Wrap FLINT's arith_euler_number

EXAMPLES:

```
sage: [euler_number(i) for i in range(10)]
[1, 0, -1, 0, 5, 0, -61, 0, 1385, 0]
sage: x = PowerSeriesRing(QQ, 'x').gen().O(10)
sage: 2/(exp(x)+exp(-x))
1 - 1/2*x^2 + 5/24*x^4 - 61/720*x^6 + 277/8064*x^8 + O(x^10)
sage: [euler_number(i)/factorial(i) for i in range(11)]
[1, 0, -1/2, 0, 5/24, 0, -61/720, 0, 277/8064, 0, -50521/3628800]
sage: euler_number(-1)
Traceback (most recent call last):
... ValueError: n (=-1) must be a nonnegative integer
```

REFERENCES:

- Wikipedia article Euler_number

```
sage.combinat.combinat.eulerian_number(k, algorithm='recursive')
Return the Eulerian number of index (n, k).
This is the coefficient of $t^k$ in the Eulerian polynomial $A_n(t)$.

INPUT:
- n – integer
- k – integer between 0 and n - 1
- algorithm – "recursive" (default) or "formula"

OUTPUT:
an integer

See also:
eulerian_polynomial()
```

EXAMPLES:

```
sage: from sage.combinat.combinat import eulerian_number
sage: [eulerian_number(5,i) for i in range(5)]
[1, 26, 66, 26, 1]
```

```
sage.combinat.combinat.eulerian_polynomial(algorithm='derivative')
Return the Eulerian polynomial of index n.
This is the generating polynomial counting permutations in the symmetric group $S_n$ according to their number of descents.

INPUT:
- n – an integer
- algorithm – "derivative" (default) or "coeffs"
```

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OUTPUT:

polynomial in one variable $t$

See also:

eulerian_number()

EXAMPLES:

```python
sage: from sage.combinat.combinat import eulerian_polynomial
sage: eulerian_polynomial(5)
t^4 + 26*t^3 + 66*t^2 + 26*t + 1
```

REFERENCES:

- Wikipedia article Eulerian_number

sage.combinat.combinat.fibonacci($n$, $algorithm=\text{\texttt{\textquoteleft\textquoteleft pari\textquoteright\textquoteleft\textquoteleft}}$)

Return the $n$-th Fibonacci number.

The Fibonacci sequence $F_n$ is defined by the initial conditions $F_1 = F_2 = 1$ and the recurrence relation $F_{n+2} = F_{n+1} + F_n$. For negative $n$ we define $F_n = (-1)^{n+1}F_{-n}$, which is consistent with the recurrence relation.

INPUT:

- $algorithm$ – a string:
  - "pari" - (default) use the PARI C library's pari:fibo function
  - "gap" - use GAP's Fibonacci function

Note: PARI is tens to hundreds of times faster than GAP here. Moreover, PARI works for every large input whereas GAP does not.

EXAMPLES:

```python
sage: fibonacci(10)
55
sage: fibonacci(10, algorithm='gap')
55
sage: fibonacci(-100)
-354224848179261915075
sage: fibonacci(100)
354224848179261915075
```

```python
sage: fibonacci(0)
0
sage: fibonacci(1/2)
Traceback (most recent call last):
... 
TypeError: no conversion of this rational to integer
```

sage.combinat.combinat.fibonacci_sequence($start$, $stop=None$, $algorithm=None$)

Return an iterator over the Fibonacci sequence, for all fibonacci numbers $f_n$ from $n = start$ up to (but not including) $n = stop$.

INPUT:
• start – starting value
• stop – stopping value
• algorithm – (default: None) passed on to fibonacci function (or not passed on if None, i.e., use the default)

EXAMPLES:

```
sage: fibs = [i for i in fibonacci_sequence(10, 20)]
sage: fibs
[55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181]
```

```
sage: sum([i for i in fibonacci_sequence(100, 110)])
69919376923075308730013
```

See also:

`fibonacci_xrange()`

AUTHORS:

• Bobby Moretti

```
sage.combinat.combinat.fibonacci_xrange(start, stop=None, algorithm='pari')
```

Return an iterator over all of the Fibonacci numbers in the given range, including \( f_n = \text{start} \) up to, but not including, \( f_n = \text{stop} \).

EXAMPLES:

```
sage: fibs_in_some_range = [i for i in fibonacci_xrange(10^7, 10^8)]
sage: len(fibs_in_some_range)
4
sage: fibs_in_some_range
[14930352, 24157817, 39088169, 63245986]
```

```
sage: fibs = [i for i in fibonacci_xrange(10, 100)]
sage: fibs
[13, 21, 34, 55, 89]
```

```
sage: list(fibonacci_xrange(13, 34))
[13, 21]
```

A solution to the second Project Euler problem:

```
sage: sum([i for i in fibonacci_xrange(10^6) if is_even(i)])
1089154
```

See also:

`fibonacci_sequence()`

AUTHORS:

• Bobby Moretti

```
sage.combinat.combinat.lucas_number1(n, P, Q)
```

Return the \( n \)-th Lucas number “of the first kind” (this is not standard terminology). The Lucas sequence \( L^{(1)}_n \) is defined by the initial conditions \( L^{(1)}_1 = 0, L^{(1)}_2 = 1 \) and the recurrence relation \( L^{(1)}_{n+2} = P \cdot L^{(1)}_{n+1} - Q \cdot L^{(1)}_n \).
Wraps GAP’s Lucas(...)[1].

\( P = 1, Q = -1 \) gives the Fibonacci sequence.

**INPUT:**

* n – integer

* P, Q – integer or rational numbers

**OUTPUT:** integer or rational number

**EXAMPLES:**

```
sage: lucas_number1(5,1,-1)
5
sage: lucas_number1(6,1,-1)
8
sage: lucas_number1(7,1,-1)
13
sage: lucas_number1(7,1,-2)
43
sage: lucas_number1(5,2,3/5)
229/25
sage: lucas_number1(5,2,1.5)
1/4
```

There was a conjecture that the sequence \( L_n \) defined by

\[
L_{n+2} = L_{n+1} + L_n, \quad L_1 = 1, \quad L_2 = 3,
\]

has the property that \( n \) prime implies that \( L_n \) is prime.

```
sage: lucas = lambda n : Integer((5/2)*lucas_number1(n,1,-1)+(1/2)*lucas_number2(n,˓→1,-1))
sage: [[lucas(n),is_prime(lucas(n)),n+1,is_prime(n+1)] for n in range(15)]
[[1, False, 1, False],
 [3, True, 2, True],
 [4, False, 3, True],
 [7, True, 4, False],
 [11, True, 5, True],
 [18, False, 6, False],
 [29, True, 7, True],
 [47, True, 8, False],
 [76, False, 9, False],
 [123, False, 10, False],
 [199, True, 11, True],
 [322, False, 12, False],
 [521, True, 13, True],
 [843, False, 14, False],
 [1364, False, 15, False]]
```

Can you use Sage to find a counterexample to the conjecture?

sage.combinat.combinat.lucas_number2(n, P, Q)

Return the \( n \)-th Lucas number “of the second kind” (this is not standard terminology). The Lucas sequence \( L^{(2)}_n \) is defined by the initial conditions \( L^{(2)}_1 = 2, L^{(2)}_2 = P \) and the recurrence relation \( L^{(2)}_{n+2} = P \cdot L^{(2)}_{n+1} - Q \cdot L^{(2)}_n \).

Wraps GAP’s Lucas(...)[2].
• \( n \) - integer
• \( P, Q \) - integer or rational numbers

OUTPUT: integer or rational number

EXAMPLES:

```python
sage: [lucas_number2(i,1,-1) for i in range(10)]
[2, 1, 3, 4, 7, 11, 18, 29, 47, 76]
sage: [fibonacci(i-1)+fibonacci(i+1) for i in range(10)]
[2, 1, 3, 4, 7, 11, 18, 29, 47, 76]
sage: n = lucas_number2(5,2,3); n
2
sage: type(n)
<class 'sage.rings.integer.Integer'>
sage: n = lucas_number2(5,2,-3/9); n
418/9
sage: type(n)
<class 'sage.rings.rational.Rational'>
```

The case \( P = 1, Q = -1 \) is the Lucas sequence in Brualdi’s Introductory Combinatorics, 4th ed., Prentice-Hall, 2004:

```python
sage: [lucas_number2(n,1,-1) for n in range(10)]
[2, 1, 3, 4, 7, 11, 18, 29, 47, 76]
```

\[ \text{sage.combinat.combinat.narayana_number}(n,k) \]

Return the Narayana number of index \((n, k)\).

For every integer \( n \geq 1 \), the sum of Narayana numbers \( \sum_k N_{n,k} \) is the Catalan number \( C_n \).

INPUT:

• \( n \) – an integer
• \( k \) – an integer between \( 0 \) and \( n - 1 \)

OUTPUT:

an integer

EXAMPLES:

```python
sage: from sage.combinat.combinat import narayana_number
sage: [narayana_number(3, i) for i in range(3)]
[1, 3, 1]
sage: sum(narayana_number(7,i) for i in range(7)) == catalan_number(7)
True
```

REFERENCES:

• Wikipedia article Narayana_number

\[ \text{sage.combinat.combinat.number_of_tuples}(S, k, algorithm='naive') \]

Return the size of tuples\((S, k)\) when \( S \) is a set. More generally, return the size of tuples(\(\text{set}(S), k)\).

(No, unlike \text{tuples}(), this method removes redundant entries from \( S \).)

INPUT:
• S – the base set
• k – the length of the tuples
• algorithm – can be one of the following:
  – 'naive' - (default) use the naive counting $|S|^k$
  – 'gap' - wraps GAP's NrTuples

**Warning:** When using algorithm='gap', S must be a list of objects that have string representations that can be interpreted by the GAP interpreter. If S consists of at all complicated Sage objects, this function might not do what you expect.

**EXAMPLES:**

```python
sage: S = [1,2,3,4,5]
sage: number_of_tuples(S,2)
25
sage: number_of_tuples(S,2, algorithm="gap")
25
sage: S = [1,1,2,3,4,5]
sage: number_of_tuples(S,2)
25
sage: number_of_tuples(S,2, algorithm="gap")
25
sage: number_of_tuples(S,0)
1
sage: number_of_tuples(S,0, algorithm="gap")
1
```

sage.combinat.combinat.number_of_unordered_tuples(S, k, algorithm='naive')

Return the size of unordered_tuples(S, k) when S is a set.

**INPUT:**

• S – the base set
• k – the length of the tuples
• algorithm – can be one of the following:
  – 'naive' - (default) use the naive counting $\binom{|S|+k-1}{k}$
  – 'gap' - wraps GAP's NrUnorderedTuples

**Warning:** When using algorithm='gap', S must be a list of objects that have string representations that can be interpreted by the GAP interpreter. If S consists of at all complicated Sage objects, this function might not do what you expect.

**EXAMPLES:**

```python
sage: S = [1,2,3,4,5]
sage: number_of_unordered_tuples(S,2)
15
sage: number_of_unordered_tuples(S,2, algorithm="gap")
(continues on next page)
```
sage: S = [1,1,2,3,4,5]

sage: number_of_unordered_tuples(S,2)
15

sage: number_of_unordered_tuples(S,2, algorithm="gap")
15

sage: number_of_unordered_tuples(S,0)
1

sage: number_of_unordered_tuples(S,0, algorithm="gap")
1

sage.combinat.combinat.polygonal_number(s, n)

Return the \( n \)-th \( s \)-gonal number.

Polygonal sequences are represented by dots forming a regular polygon. Two famous sequences are the triangular numbers (3rd column of Pascal’s Triangle) and the square numbers. The \( n \)-th term in a polygonal sequence is defined by

\[
P(s, n) = \frac{n^2(s - 2) - n(s - 4)}{2},
\]

where \( s \) is the number of sides of the polygon.

INPUT:

- \( s \) – integer greater than 1; the number of sides of the polygon
- \( n \) – integer; the index of the returned \( s \)-gonal number

OUTPUT: an integer

EXAMPLES:

The triangular numbers:

\[
\text{sage: [polygonal_number(3, n) for n in range(10)]}
[0, 1, 3, 6, 10, 15, 21, 28, 36, 45]
\]

\[
\text{sage: [polygonal_number(3, n) for n in range(-10, 0)]}
[45, 36, 28, 21, 15, 10, 6, 3, 1, 0]
\]

The square numbers:

\[
\text{sage: [polygonal_number(4, n) for n in range(10)]}
[0, 1, 4, 9, 16, 25, 36, 49, 64, 81]
\]

The pentagonal numbers:

\[
\text{sage: [polygonal_number(5, n) for n in range(10)]}
[0, 1, 5, 12, 22, 35, 51, 70, 92, 117]
\]

The hexagonal numbers:

\[
\text{sage: [polygonal_number(6, n) for n in range(10)]}
[0, 1, 6, 15, 28, 45, 66, 91, 120, 153]
\]

The input is converted into an integer:
sage: polygonal_number(3.0, 2.0)
3

A non-integer input returns an error:

sage: polygonal_number(3.5, 1)
Traceback (most recent call last):
  ...TypeError: Attempt to coerce non-integral RealNumber to Integer

s must be greater than 1:

sage: polygonal_number(1, 4)
Traceback (most recent call last):
  ...ValueError: s (=1) must be greater than 1

REFERENCES:

• Wikipedia article Polygonal_number

sage.combinat.combinat.stirling_number1(n, k, algorithm='gap')
Return the \(S_1(n, k)\) of the first kind.
This is the number of permutations of \(n\) points with \(k\) cycles.
See Wikipedia article Stirling_numbers_of_the_first_kind.

INPUT:

• \(n\) – nonnegative machine-size integer
• \(k\) – nonnegative machine-size integer
• algorithm:
  – "gap" (default) – use GAP’s \texttt{Stirling1} function
  – "flint" – use flint’s \texttt{arith_stirling_number_1u} function

EXAMPLES:

sage: stirling_number1(3,2)
3
sage: stirling_number1(5,2)
50
sage: 9*stirling_number1(9,5)+stirling_number1(9,4)
269325
sage: stirling_number1(10,5)
269325

Indeed, \(S_1(n, k) = S_1(n - 1, k - 1) + (n - 1)S_1(n - 1, k)\).

sage.combinat.combinat.stirling_number2(n, k, algorithm=None)
Return the \(S_2(n, k)\) of the second kind.
This is the number of ways to partition a set of \(n\) elements into \(k\) pairwise disjoint nonempty subsets. The \(n\)-th Bell number is the sum of the \(S_2(n, k)\)'s, \(k = 0, \ldots, n\).

See Wikipedia article Stirling_numbers_of_the_second_kind.
INPUT:

- \( n \) – nonnegative machine-size integer
- \( k \) – nonnegative machine-size integer
- algorithm:
  - None (default) – use native implementation
  - "flint" – use flint's \texttt{arith_stirling_number_2} function
  - "gap" – use GAP's \texttt{Stirling2} function
  - "maxima" – use Maxima's \texttt{stirling2} function

EXAMPLES:

Print a table of the first several Stirling numbers of the second kind:

```
sage: for n in range(10):
    ....:     for k in range(10):
    ....:         print(str(stirling_number2(n,k)).rjust(k and 6))
1 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0
0 1 1 0 0 0 0 0 0 0
0 1 3 1 0 0 0 0 0 0
0 1 7 6 1 0 0 0 0 0
0 1 15 25 10 1 0 0 0 0
0 1 31 90 65 15 1 0 0 0
0 1 63 301 350 140 21 1 0 0
0 1 127 966 1701 1050 266 28 1 0
0 1 255 3025 7770 6951 2646 462 36 1
```

Stirling numbers satisfy
\[ S_2(n, k) = S_2(n - 1, k - 1) + kS_2(n - 1, k) \]:

```
sage: 5*stirling_number2(9,5) + stirling_number2(9,4)
42525
sage: stirling_number2(10,5)
42525
```

\texttt{sage.combinat.combinat.tuples} \((S, k, \text{algorithm}='\text{itertools}')\)

Return a list of all \( k \)-tuples of elements of a given set \( S \).

This function accepts the set \( S \) in the form of any iterable (list, tuple or iterator), and returns a list of \( k \)-tuples. If \( S \) contains duplicate entries, then you should expect the method to return tuples multiple times!

Recall that \( k \)-tuples are ordered (in the sense that two \( k \)-tuples differing in the order of their entries count as different) and can have repeated entries (even if \( S \) is a list with no repetition).

INPUT:

- \( S \) – the base set
- \( k \) – the length of the tuples
- algorithm – can be one of the following:
  - '\text{itertools}' - (default) use python's \texttt{itertools}
  - '\text{native}' - use a native Sage implementation
Note: The ordering of the list of tuples depends on the algorithm.

EXAMPLES:

```
sage: S = [1,2]
sage: tuples(S,3)
[(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2),
 (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)]
sage: mset = ["s","t","e","i","n"]
sage: tuples(mset, 2)
[('s','s'), ('s','t'), ('s','e'), ('s','i'), ('s','n'),
 ('t','s'), ('t','t'), ('t','e'), ('t','i'), ('t','n'),
 ('e','s'), ('e','t'), ('e','e'), ('e','i'), ('e','n'),
 ('i','s'), ('i','t'), ('i','e'), ('i','i'), ('i','n'),
 ('n','s'), ('n','t'), ('n','e'), ('n','i'), ('n','n')]```

```
sage: K.<a> = GF(4, 'a')
sage: mset = [x for x in K if x != 0]
sage: tuples(mset, 2)
[(a, a), (a, a + 1), (a, 1), (a + 1, a), (a + 1, a + 1),
 (a + 1, 1), (1, a), (1, a + 1), (1, 1)]
```

We check that the implementations agree (up to ordering):

```
sage: tuples(S, 3, 'native')
[(1, 1, 1), (2, 1, 1), (1, 2, 1), (2, 2, 1),
 (1, 1, 2), (2, 1, 2), (1, 2, 2), (2, 2, 2)]
```

Lastly we check on a multiset:

```
sage: S = [1,1,2]
sage: sorted(tuples(S, 3)) == sorted(tuples(S, 3, 'native'))
True
```

AUTHORS:
• Jon Hanke (2006-08)

```
sage.combinat.combinat.unordered_tuples(S, k, algorithm='itertools')
```

Return a list of all unordered tuples of length k of the set S.

An unordered tuple of length k of set S is an unordered selection with repetitions of S and is represented by a sorted list of length k containing elements from S.

Unlike tuples(), the result of this method does not depend on how often an element appears in S; only the set S is being used. For example, unordered_tuples([(1, 1), 2]) will return [(1, 1)]. If you want it to return [(1, 1), (1, 1), (1, 1)], use Python’s itertools.combinations_with_replacement instead.

INPUT:
• S – the base set
• k – the length of the tuples
• algorithm – can be one of the following:
  – 'itertools' - (default) use python’s itertools
Warning: When using algorithm='gap', S must be a list of objects that have string representations that can be interpreted by the GAP interpreter. If S consists of all complicated Sage objects, this function might not do what you expect.

EXAMPLES:

```
sage: S = [1,2]
sage: unordered_tuples(S, 3)
[(1, 1, 1), (1, 1, 2), (1, 2, 2), (2, 2, 2)]
```

We check that this agrees with GAP:

```
sage: unordered_tuples(S, 3, algorithm='gap')
[(1, 1, 1), (1, 1, 2), (1, 2, 2), (2, 2, 2)]
```

We check the result on strings:

```
sage: S = ['a','b','c']
sage: unordered_tuples(S, 2)
[('a', 'a'), ('a', 'b'), ('a', 'c'), ('b', 'b'), ('b', 'c'), ('c', 'c')]
```

Lastly we check on a multiset:

```
sage: S = [1,1,2]
sage: unordered_tuples(S, 3) == unordered_tuples(S, 3, 'gap')
True
```

```
sage.combinat.combinat.unshuffle_iterator(a, one=1)
```

Iterate over the unshuffles of a list (or tuple) a, also yielding the signs of the respective permutations.

If $n$ and $k$ are integers satisfying $0 \leq k \leq n$, then a $(k, n-k)$-unshuffle means a permutation $\pi \in S_n$ such that $\pi(1) < \pi(2) < \cdots < \pi(k)$ and $\pi(k+1) < \pi(k+2) < \cdots < \pi(n)$. This method provides, for a list $a = (a_1, a_2, \ldots, a_n)$ of length $n$, an iterator yielding all pairs:

$$
\left( ((a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)}), (a_{\pi(k+1)}, a_{\pi(k+2)}, \ldots, a_{\pi(n)})), (-1)^{\pi} \right)
$$

for all $k \in \{0, 1, \ldots, n\}$ and all $(k, n-k)$-unshuffles $\pi$. The optional variable one can be set to a different value which results in the $(-1)^{\pi}$ component being multiplied by said value.

The iterator does not yield these in order of increasing $k$.

EXAMPLES:

```
sage: from sage.combinat.combinat import unshuffle_iterator
sage: list(unshuffle_iterator([1, 3, 4]))
[([(1, 3, 4), 1), ([(1,), (3, 4)], 1), ([(3,), (1, 4)], -1),
  ([(1,3), (4,)], 1), ([(4,), (1,3)], 1), ([(1,4), (3,)], -1),
  ([(3,4), (1,)], 1), ([(1,3,4), ()], 1)]
```
sage: list(unshuffle_iterator([3, 1]))
[[((), (3, 1)), 1], (((3,), (1,)), 1), (((1,), (3,)), -1),
 ((3, 1), ()), 1)]
sage: list(unshuffle_iterator([8]))
[[((), (8,)), 1], (((8,), ()), 1)]
sage: list(unshuffle_iterator([]))
[[((), ()), 1]]
sage: list(unshuffle_iterator([3, 1], 3/2))
[[((), (3, 1)), 3/2], (((3,), (1,)), 3/2), (((1,), (3,)), -3/2),
 ((3, 1), ()), 3/2)]

5.1.26 Fast computation of combinatorial functions (Cython + mpz)

Currently implemented:

• Stirling numbers of the second kind
• iterators for set partitions
• iterator for Lyndon words
• iterator for perfect matchings
• conjugate of partitions

AUTHORS:

• Fredrik Johansson (2010-10): Stirling numbers of second kind
• Martin Rubey and Travis Scrimshaw (2018): iterators for set partitions, Lyndon words, and perfect matchings

sage.combinat.combinat_cython.conjugate(p)

Return the conjugate partition associated to the partition \( p \) as a list.

EXAMPLES:

sage: from sage.combinat.combinat_cython import conjugate
sage: conjugate([2, 2])
[2, 2]
sage: conjugate([6, 3, 1])
[3, 2, 2, 1, 1, 1]

sage.combinat.combinat_cython.linear_extension_iterator(D)

Iterate over the linear extensions of the poset.

The list \_le keeps track of the current linear extensions. The boolean variable \_is_\_plus keeps track of the “sign”.

INPUT:

• \( D \), the Hasse diagram of a poset.

**Warning:** It is assumed that \( D \) is not modified while the linear extensions are generated.

EXAMPLES:
sage: from sage.combinat.combinat_cython import linear_extension_iterator
sage: D = Poset({ 0:[1,2], 1:[3], 2:[3,4] })._hasse_diagram
sage: list(linear_extension_iterator(D))
[[0, 1, 2, 3, 4],
 [0, 2, 1, 3, 4],
 [0, 2, 4, 1, 3],
 [0, 1, 2, 4, 3]]
sage: D = posets.BooleanLattice(3)._hasse_diagram
sage: len(list(linear_extension_iterator(D)))
48
sage: D = posets.AntichainPoset(9)._hasse_diagram
sage: len(list(linear_extension_iterator(D))) == factorial(9)  # long time
True

sage.combinat.combinat_cython.lyndon_word_iterator(n,k)
Generate the Lyndon words of fixed length k with n letters.
The resulting Lyndon words will be words represented as lists whose alphabet is range(n) (= {0, 1, ..., n-1}).

ALGORITHM:
The iterative FKM Algorithm 7.2 from [Rus2003].

EXAMPLES:

sage: from sage.combinat.combinat_cython import lyndon_word_iterator
sage: list(lyndon_word_iterator(4, 2))
[[0, 1], [0, 2], [0, 3], [1, 2], [1, 3], [2, 3]]
sage: list(lyndon_word_iterator(2, 4))
[[0, 0, 0, 1], [0, 0, 1, 1], [0, 1, 1, 1]]

sage.combinat.combinat_cython.perfect_matchings_iterator(n)
Iterate over all perfect matchings with n parts.
This iterates over all perfect matchings of {0,1,...,2n-1} using a Gray code for fixed-point-free involutions
due to Walsh [Wal2001].

EXAMPLES:

sage: from sage.combinat.combinat_cython import perfect_matchings_iterator
sage: list(perfect_matchings_iterator(1))
[[0, 1]]
sage: list(perfect_matchings_iterator(2))
[[[0, 1], (2, 3)], [[0, 2], (1, 3)], [[0, 3], (1, 2)]]
sage: list(perfect_matchings_iterator(0))
[]

REFERENCES:
- [Wal2001]
sage.combinat.combinat_cython.set_partition_composition(sp1, sp2)
Return a tuple consisting of the composition of the set partitions sp1 and sp2 and the number of components removed from the middle rows of the graph.

EXAMPLES:

```python
sage: from sage.combinat.combinat_cython import set_partition_composition
sage: sp1 = ((1, -2), (2, -1))
sage: sp2 = ((1, -2), (2, -1))
sage: p, c = set_partition_composition(sp1, sp2)
sage: (SetPartition(p), c) == (SetPartition([[1, -1], [2, -2]]), 0)
True
```

sage.combinat.combinat_cython.set_partition_iterator(base_set)
A fast iterator for the set partitions of the base set, which returns lists of lists instead of set partitions types.

EXAMPLES:

```python
sage: from sage.combinat.combinat_cython import set_partition_iterator
sage: list(set_partition_iterator([1, -1, x]))
[[[1, -1, x]],
 [[[1, -1], [x]],
 [[[1, x], [-1]],
 [[[1], [-1, x]],
 [[[1], [-1], [x]]]
```

sage.combinat.combinat_cython.set_partition_iterator_blocks(base_set, k)
A fast iterator for the set partitions of the base set into the specified number of blocks, which returns lists of lists instead of set partitions types.

EXAMPLES:

```python
sage: from sage.combinat.combinat_cython import set_partition_iterator_blocks
sage: list(set_partition_iterator_blocks([1, -1, x], 2))
[[[1, x], [-1]], [[1], [-1, x]], [[1, -1], [x]]]
```

### 5.1.27 Combinations

AUTHORS:
- Mike Hansen (2007): initial implementation
- Vincent Delecroix (2011): cleaning, bug corrections, doctests
- Antoine Genitrini (2020): new implementation of the lexicographic unranking of combinations

**class** sage.combinat.combination.ChooseNK(mset, k)

**Bases:** Combinations_setk

sage.combinat.combination.Combinations(mset, k=None)

Return the combinatorial class of combinations of the multiset mset. If k is specified, then it returns the combinatorial class of combinations of mset of size k.

A combination of a multiset M is an unordered selection of k objects of M, where every object can appear at most as many times as it appears in M.

The combinatorial classes correctly handle the cases where mset has duplicate elements.
EXAMPLES:

```
sage: C = Combinations(range(4)); C
Combinations of [0, 1, 2, 3]
sage: C.list()
[[],
 [0],
 [1],
 [2],
 [3],
 [0, 1],
 [0, 2],
 [0, 3],
 [1, 2],
 [1, 3],
 [2, 3],
 [0, 1, 2],
 [0, 1, 3],
 [0, 2, 3],
 [1, 2, 3],
 [0, 1, 2, 3]]
sage: C.cardinality()
16
```

```
sage: C2 = Combinations(range(4),2); C2
Combinations of [0, 1, 2, 3] of length 2
sage: C2.list()
[[0, 1], [0, 2], [0, 3], [1, 2], [1, 3], [2, 3]]
sage: C2.cardinality()
6
```

```
sage: Combinations([1,2,2,3]).list()
[[],
 [1],
 [2],
 [3],
 [1, 2],
 [1, 3],
 [2, 2],
 [2, 3],
 [1, 2, 2],
 [1, 2, 3],
 [2, 2, 3],
 [1, 2, 2, 3]]
```

```
sage: Combinations([1,2,3], 2).list()
[[1, 2], [1, 3], [2, 3]]
```

```
sage: mset = [1,1,2,3,4,4,5]
sage: Combinations(mset,2).list()
[[1, 1],
 [1, 2],
 [1, 3],
 [1, 4],
 [2, 3],
 [2, 4],
 [2, 5],
 [3, 4],
 [3, 5],
 [4, 5]]
```

(continues on next page)
\[
[1, 3],
[1, 4],
[1, 5],
[2, 3],
[2, 4],
[2, 5],
[3, 4],
[3, 5],
[4, 4],
[4, 5]]
\]

sage: mset = ["d","a","v","i","d"]
sage: Combinations(mset,3).list()
[['d', 'd', 'a'],
 ['d', 'd', 'v'],
 ['d', 'd', 'i'],
 ['d', 'a', 'v'],
 ['d', 'v', 'i'],
 ['a', 'v', 'i']]

sage: X = Combinations([1,2,3,4,5],3)
sage: [x for x in X]
[[1, 2, 3],
 [1, 2, 4],
 [1, 2, 5],
 [1, 3, 4],
 [1, 3, 5],
 [1, 4, 5],
 [2, 3, 4],
 [2, 3, 5],
 [2, 4, 5],
 [3, 4, 5]]

It is possible to take combinations of Sage objects:

sage: Combinations([vector([1,1]), vector([2,2]), vector([3,3])], 2).list()
[[[1, 1), (2, 2)]]
```python
sage: mset = [1,1,2,3,4,4,5]
sage: Combinations(mset,2).cardinality()
12
```

**class** `sage.combinat.combination.Combinations_set(mset)`

**Bases:** `Combinations_mset`

**cardinality()**

Return the size of Combinations(set).

**EXAMPLES:**

```python
sage: Combinations(range(16000)).cardinality() == 2^16000
True
```

**rank(x)**

**EXAMPLES:**

```python
sage: c = Combinations([1,2,3])
sage: list(range(c.cardinality())) == list(map(c.rank, c))
True
```

**unrank(r)**

**EXAMPLES:**

```python
sage: c = Combinations([1,2,3])
sage: c.list() == list(map(c.unrank, range(c.cardinality())))
True
```

**class** `sage.combinat.combination.Combinations_setk(mset, k)`

**Bases:** `Combinations_msetk`

**cardinality()**

Return the size of combinations(set, k).

**EXAMPLES:**

```python
sage: Combinations(range(16000), 5).cardinality()
8732673194560003200
```

**list()**

**EXAMPLES:**

```python
sage: Combinations([1,2,3,4,5],3).list()
[[1, 2, 3],
 [1, 2, 4],
 [1, 2, 5],
 [1, 3, 4],
 [1, 3, 5],
 [1, 4, 5],
 [2, 3, 4],
 [2, 3, 5],
 [2, 4, 5],
 [3, 4, 5]]
```
**rank(x)**

EXAMPLES:

```python
sage: c = Combinations([1,2,3], 2)
sage: list(range(c.cardinality())) == list(map(c.rank, c.list()))
True
```

**unrank(r)**

EXAMPLES:

```python
sage: c = Combinations([1,2,3], 2)
sage: c.list() == list(map(c.unrank, range(c.cardinality())))
True
```

```python
sage.combinat.combination.from_rank(r, n, k)
Return the combination of rank r in the subsets of range(n) of size k when listed in lexicographic order.

The algorithm used is based on factoradics and presented in [DGH2020]. It is there compared to the other from
the literature.

EXAMPLES:

```python
sage: import sage.combinat.combination as combination
dsage: combination.from_rank(0,3,0)
()  
dsage: combination.from_rank(0,3,1)
(0,)  
dsage: combination.from_rank(1,3,1)
(1,)  
dsage: combination.from_rank(2,3,1)
(2,)  
dsage: combination.from_rank(0,3,2)
(0, 1)  
dsage: combination.from_rank(1,3,2)
(0, 2)  
dsage: combination.from_rank(2,3,2)
(1, 2)  
dsage: combination.from_rank(0,3,3)
(0, 1, 2)
```
5.1.28 Combinatorial maps

This module provides a decorator that can be used to add semantic to a Python method by marking it as implementing a combinatorial map, that is a map between two enumerated sets:

```python
sage: from sage.combinat.combinatorial_map import combinatorial_map
sage: class MyPermutation:
    ....:     @combinatorial_map
    ....:     def reverse(self):
    ....:         '''Reverse the permutation
    ....:         # ... code ..."
```

By default, this decorator is a no-op: it returns the decorated method as is:

```python
sage: MyPermutation.reverse
<function MyPermutation.reverse at ...>
```

See `combinatorial_map_wrapper()` for the various options this decorator can take.

Projects built on top of Sage are welcome to customize locally this hook to instrument the Sage code and exploit this semantic information. Typically, the decorator could be used to populate a database of maps. For a real-life application, see the project `FindStat < http://findstat.org/>`. As a basic example, a variant of the decorator is provided as `combinatorial_map_wrapper()`: it wraps the decorated method, so that one can later use `combinatorial_maps_in_class()` to query an object, or class thereof, for all the combinatorial maps that apply to it.
Note: Since decorators are evaluated upon loading Python modules, customizing combinatorial map needs
to be done before the modules using it are loaded. In the examples below, where we illustrate the customized
combinatorial_map decorator on the sage.combinat.permutation module, we resort to force a reload of this
module after dynamically changing sage.combinat.combinatorial_map.combinatorial_map. This is good
enough for those doctests, but remains fragile.
For real use cases, it is probably best to just edit this source file statically (see below).

class sage.combinat.combinatorial_map.CombinatorialMap(f, order=None, name=None)
    Bases: object
    This is a wrapper class for methods that are combinatorial maps.
    For further details and doctests, see Combinatorial maps and combinatorial_map_wrapper().

    name()
    Returns the name of a combinatorial map. This is used for the string representation of self.
    EXAMPLES:

    sage: from sage.combinat.combinatorial_map import combinatorial_map
    sage: class CombinatorialClass:
    ....:     @combinatorial_map(name='map1')
    ....:     def to_self_1(): pass
    ....:     @combinatorial_map()
    ....:     def to_self_2(): pass
    sage: CombinatorialClass.to_self_1.name() 'map1'
    sage: CombinatorialClass.to_self_2.name() 'to_self_2'

    order()
    Returns the order of self, or None if the order is not known.
    EXAMPLES:

    sage: from sage.combinat.combinatorial_map import combinatorial_map
    sage: class CombinatorialClass:
    ....:     @combinatorial_map(order=2)
    ....:     def to_self_1(): pass
    ....:     @combinatorial_map()
    ....:     def to_self_2(): pass
    sage: CombinatorialClass.to_self_1.order() 2
    sage: CombinatorialClass.to_self_2.order() is None True

    unbounded_map()
    Return the unbounded version of self.
    You can use this method to return a function which takes as input an element in the domain of the combi-
natorial map. See the example below.
    EXAMPLES:
Combinatorics, Release 10.0

```python
sage: sage.combinat.combinatorial_map.combinatorial_map = sage.combinat.combinatorial_map_wrapper
sage: from importlib import reload
sage: _ = reload(sage.combinat.permutation)
sage: from sage.combinat.permutation import Permutation
sage: pi = Permutation([1,3,2])
sage: f = pi.reverse
sage: F = f.unbounded_map()
sage: F(pi)
[2, 3, 1]
```

`sage.combinat.combinatorial_map.combinatorial_map` *(f=None, order=None, name=None)*

Combinatorial map decorator

See *Combinatorial maps* for a description of this decorator and its purpose. This default implementation does nothing.

**INPUT:**

- `f` – (default: `None`, if `combinatorial_map` is used as a decorator) a function
- `name` – (default: `None`) the name for nicer outputs on combinatorial maps
- `order` – (default: `None`) the order of the combinatorial map, if it is known. Is not used, but might be helpful later

**OUTPUT:**

- `f` unchanged

**EXAMPLES:**

```python
sage: from sage.combinat.combinatorial_map import combinatorial_map_trivial as _
sage: class MyPermutation():
    ....: @combinatorial_map
    ....: def reverse(self):
    ....:     ''' Reverse the permutation
    ....:     '''
    ....:     # ... code ...
    ....:     @combinatorial_map(name='descent set of permutation')
    ....:     def descent_set(self):
    ....:         ''' The descent set of the permutation
    ....:         '''
    ....:         # ... code ...

sage: MyPermutation.reverse
<function MyPermutation.reverse at ...>
sage: MyPermutation.descent_set
<function MyPermutation.descent_set at ...>
```

`sage.combinat.combinatorial_map.combinatorial_map_trivial` *(f=None, order=None, name=None)*

Combinatorial map decorator
See *Combinatorial maps* for a description of this decorator and its purpose. This default implementation does nothing.

**INPUT:**
- \( f \) – (default: *None*, if `combinatorial_map` is used as a decorator) a function
- `name` – (default: *None*) the name for nicer outputs on combinatorial maps
- `order` – (default: *None*) the order of the combinatorial map, if it is known. Is not used, but might be helpful later

**OUTPUT:**
- \( f \) unchanged

**EXAMPLES:**

```python
sage: from sage.combinat.combinatorial_map import combinatorial_map_trivial as __combinatorial_map
gsage: class MyPermutation():
....:     @combinatorial_map
....:     def reverse(self):
....:         '''
....:         Reverse the permutation
....:         '''
....:         # ... code ...
....:         @combinatorial_map(name='descent set of permutation')
....:         def descent_set(self):
....:             '''
....:             The descent set of the permutation
....:             '''
....:             # ... code ...
sage: MyPermutation.reverse
<function MyPermutation.reverse at ...>
sage: MyPermutation.descent_set
<function MyPermutation.descent_set at ...>
```

sage.combinat.combinatorial_map.combinatorial_map_wrapper(*f=None, order=None, name=None*)

Combinatorial map decorator (basic example).

See *Combinatorial maps* for a description of the `combinatorial_map` decorator and its purpose. This implementation, together with `combinatorial_maps_in_class()` illustrates how to use this decorator as a hook to instrument the Sage code.

**INPUT:**
- \( f \) – (default: *None*, if `combinatorial_map` is used as a decorator) a function
- `name` – (default: *None*) the name for nicer outputs on combinatorial maps
- `order` – (default: *None*) the order of the combinatorial map, if it is known. Is not used, but might be helpful later

**OUTPUT:**
- A combinatorial map. This is an instance of the `CombinatorialMap`.

**EXAMPLES:**
We define a class illustrating the use of this implementation of the `combinatorial_map` decorator with its various arguments:

```python
sage: from sage.combinat.combinatorial_map import combinatorial_map_wrapper as _
˓→combinatorial_map
sage: class MyPermutation():
    ....: @combinatorial_map()
    ....: def reverse(self):
    ....:     '''Reverse the permutation'''
    ....:     pass
    ....: @combinatorial_map(order=2)
    ....: def inverse(self):
    ....:     '''The inverse of the permutation'''
    ....:     pass
    ....: @combinatorial_map(name='descent set of permutation')
    ....: def descent_set(self):
    ....:     '''The descent set of the permutation'''
    ....:     pass
    ....: def major_index(self):
    ....:     '''The major index of the permutation'''
    ....:     pass
sage: MyPermutation.reverse
Combinatorial map: reverse
sage: MyPermutation.descent_set
Combinatorial map: descent set of permutation
sage: MyPermutation.inverse
Combinatorial map: inverse
```

One can now determine all the combinatorial maps associated with a given object as follows:

```python
sage: from sage.combinat.combinatorial_map import combinatorial_maps_in_class
sage: X = combinatorial_maps_in_class(MyPermutation); X
# random
[Combinatorial map: reverse,
 Combinatorial map: descent set of permutation,
 Combinatorial map: inverse]
```

The method `major_index` defined about is not a combinatorial map:

```python
sage: MyPermutation.major_index
<function MyPermutation.major_index at ...>
```

But one can define a function that turns `major_index` into a combinatorial map:

```python
sage: def major_index(p):
    ....:     return p.major_index()
```

(continues on next page)
sage.combinat.combinatorial_map.combinatorial_maps_in_class(cls)

Return the combinatorial maps of the class as a list of combinatorial maps.

For further details and doctests, see Combinatorial maps and combinatorial_map_wrapper().

EXAMPLES:

```python
sage: sage.combinat.combinatorial_map.combinatorial_map = sage.combinat.
-→combinatorial_map.combinatorial_map_wrapper
sage: from importlib import reload
sage: _ = reload(sage.combinat.permutation)
from sage.combinat.combinatorial_map import combinatorial_maps_in_class
sage: p = Permutation([1,3,2,4])
<function major_index at ...>
sage: combinatorial_map(major_index)
Combinatorial map: major_index
```

5.1.29 Integer compositions

A composition $c$ of a nonnegative integer $n$ is a list of positive integers (the parts of the composition) with total sum $n$.

This module provides tools for manipulating compositions and enumerated sets of compositions.

EXAMPLES:

```python
sage: Composition([5, 3, 1, 3])
[5, 3, 1, 3]
sage: list(Compositions(4))
[[1, 1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 3], [2, 1, 1], [2, 2], [3, 1], [4]]
```

AUTHORS:

- Mike Hansen, Nicolas M. Thiéry
- MuPAD-Combinat developers (algorithms and design inspiration)
- Travis Scrimshaw (2013-02-03): Removed CombinatorialClass

class sage.combinat.composition.Composition(parent, *args, **kwds)

Bases: CombinatorialElement
Integer compositions

A composition of a nonnegative integer $n$ is a list $(i_1, \ldots, i_k)$ of positive integers with total sum $n$.

EXAMPLES:

The simplest way to create a composition is by specifying its entries as a list, tuple (or other iterable):

```
sage: Composition([3,1,2])
[3, 1, 2]
sage: Composition((3,1,2))
[3, 1, 2]
sage: Composition(i for i in range(2,5))
[2, 3, 4]
```

You can also create a composition from its code. The code of a composition $(i_1, i_2, \ldots, i_k)$ of $n$ is a list of length $n$ that consists of a 1 followed by $i_1 - 1$ zeros, then a 1 followed by $i_2 - 1$ zeros, and so on.

```
sage: Composition([4,1,2,3,5]).to_code()
[1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0]
sage: Composition(code=_)
[4, 1, 2, 3, 5]
sage: Composition([3,1,2,3,5]).to_code()
[1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0]
sage: Composition(code=_)
[3, 1, 2, 3, 5]
```

You can also create the composition of $n$ corresponding to a subset of $\{1, 2, \ldots, n - 1\}$ under the bijection that maps the composition $(i_1, i_2, \ldots, i_k)$ of $n$ to the subset $\{i_1, i_1 + i_2, i_1 + i_2 + i_3, \ldots, i_1 + \cdots + i_k - 1\}$ (see `to_subset()`):

```
sage: Composition(from_subset=({1, 2, 4}, 5))
[1, 2, 1]
sage: Composition([1, 1, 2, 1]).to_subset()
{1, 4}
```

The following notation equivalently specifies the composition from the set $\{i_1 - 1, i_1 + i_2 - 1, i_1 + i_2 + i_3 - 1, \ldots, i_1 + \cdots + i_k - 1 - n - 1\}$ or $\{i_1 - 1, i_1 + i_2 - 1, i_1 + i_2 + i_3 - 1, \ldots, i_1 + \cdots + i_k - 1\}$ and $n$. This provides compatibility with Python’s 0-indexing.

```
sage: Composition(descents=[1,0,4,8,11])
[1, 1, 3, 4, 3]
sage: Composition(descents=[0,1,3,4])
[1, 1, 2, 1]
sage: Composition(descents=[[0,1,3],5])
[1, 1, 2, 1]
sage: Composition(descents=([{0,1,3},5])
[1, 1, 2, 1]
```

An integer composition may be regarded as a sequence. Thus it is an instance of the Python abstract base class `Sequence` allows us to check if objects behave “like” sequences based on implemented methods. Note that `collections.abc.Sequence` is not the same as `sage.structure.sequence.Sequence`:

```
sage: import collections.abc
sage: C = Composition([3,2,3])
sage: isinstance(C, collections.abc.Sequence)
(continues on next page)
```
Typically, instances of `collections.abc.Sequence` have a `.count` method. `Composition.count` counts the number of parts of a specified size:

```python
sage: C.count(3)
2
```

**EXAMPLES:**

```python
sage: C = Composition([3,1,2])
sage: TestSuite(C).run()
```

### complement()

Return the complement of the composition `self`.

The complement of a composition `I` is defined as follows:

If `I` is the empty composition, then the complement is the empty composition as well. Otherwise, let `S` be the descent set of `I` (that is, the subset \(\{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{k-1}\}\) of \(\{1, 2, \ldots, |I| - 1\}\), where `I` is written as \((i_1, i_2, \ldots, i_k)\)). Then, the complement of `I` is defined as the composition of size \(|I|\) whose descent set is \(\{1, 2, \ldots, |I| - 1\} \setminus S\).

The complement of a composition `I` also is the reverse composition (`reversed()`) of the conjugate (`conjugate()`) of `I`.

**EXAMPLES:**

```python
sage: Composition([1, 1, 3, 1, 2, 1, 3]).conjugate()
[1, 1, 3, 3, 1, 3]
sage: Composition([1, 1, 3, 1, 2, 1, 3]).complement()
[3, 1, 3, 3, 1, 1]
```

### conjugate()

Return the conjugate of the composition `self`.

The conjugate of a composition `I` is defined as the complement (see `complement()`) of the reverse composition (see `reversed()`) of `I`.

An equivalent definition of the conjugate goes by saying that the ribbon shape of the conjugate of a composition `I` is the conjugate of the ribbon shape of `I`. (The ribbon shape of a composition is returned by `to_skew_partition()`.)

This implementation uses the algorithm from mupad-combinat.

**EXAMPLES:**

```python
sage: Composition([1, 1, 3, 1, 2, 1, 3]).conjugate()
[1, 1, 3, 3, 1, 3]
```

The ribbon shape of the conjugate of `I` is the conjugate of the ribbon shape of `I`:

```python
sage: all( I.conjugate().to_skew_partition() == I.to_skew_partition().conjugate() for I in [Composition([1, 1, 3, 1, 2, 1, 3])] )
True
```
for I in Compositions(4) )
True

count(n)

Return the number of parts of size n.

EXAMPLES:
sage: C = Composition([3,2,3])
sage: C.count(3)
2
sage: C.count(2)
1
sage: C.count(1)
0

descents(final_descent=False)

This gives one fewer than the partial sums of the composition.

This is here to maintain some sort of backward compatibility, even through the original implementation was broken (it gave the wrong answer). The same information can be found in partial_sums().

See also:

partial_sums()

INPUT:
  • final_descent – (Default: False) a boolean integer

OUTPUT:
  • the list of partial sums of self with each part decremented by 1. This includes the sum of all entries when final_descent is True.

EXAMPLES:
sage: c = Composition([2,1,3,2])
sage: c.descents()
[1, 2, 5]
sage: c.descents(final_descent=True)
[1, 2, 5, 7]

fatten(grouping)

Return the composition fatter than self, obtained by grouping together consecutive parts according to grouping.

INPUT:
  • grouping – a composition whose sum is the length of self

EXAMPLES:

Let us start with the composition:
sage: c = Composition([4,5,2,7,1])

With grouping equal to (1,...,1), c is left unchanged:
With grouping equal to \( (\ell) \) where \( \ell \) is the length of \( c \), this yields the coarsest composition above \( c \):

\[
\text{sage: } c.\text{fatten}(\text{Composition}([1,1,1,1]))
\]
\[
[4, 5, 2, 7, 1]
\]

Other values for grouping yield (all the) other compositions coarser than \( c \):

\[
\text{sage: } c.\text{fatten}(\text{Composition}([5]))
\]
\[
[19]
\]

\sage: c.fatten(Composition([2,1,2]))
[9, 2, 8]
\sage: c.fatten(Composition([3,1,1]))
[11, 7, 1]

\textbf{fatter()}

Return the set of compositions which are fatter than \texttt{self}.

Complexity for generation: \( O(|c|) \) memory, \( O(|r|) \) time where \( |c| \) is the size of \texttt{self} and \( r \) is the result.

\textbf{EXAMPLES:}

\[
\text{sage: } C = \text{Composition([4,5,2])}.\text{fatter()}
\]
\[
\text{sage: } C.\text{cardinality()}
4
\]
\[
\text{sage: } \text{list}(C)
[[4, 5, 2], [4, 7], [9, 2], [11]]
\]

Some extreme cases:

\[
\text{sage: } \text{list}(\text{Composition([5])}.\text{fatter()})
[[5]]
\]
\[
\text{sage: } \text{list}(\text{Composition([]).fatter()})
[[[]]]
\]
\[
\text{sage: } \text{list}(\text{Composition([1,1,1,1]).fatter()}) == \text{list}(\text{Compositions(4)})
\]
\[
\text{True}
\]

\textbf{finer()}

Return the set of compositions which are finer than \texttt{self}.

\textbf{EXAMPLES:}

\[
\text{sage: } C = \text{Composition([3,2])}.\text{finer()}
\]
\[
\text{sage: } C.\text{cardinality()}
8
\]
\[
\text{sage: } \text{C.list()}
[[1, 1, 1, 1, 1], [1, 1, 1, 2], [1, 2, 1, 1], [1, 2, 2], [2, 1, 1, 1], [2, 1, 2],
[3, 1, 1], [3, 2]]
\]
\[
\text{sage: } \text{Composition([]).finer()}
[]
\]

\textbf{inf(other, check=True)}

Return the meet of \texttt{self} with a composition \texttt{other} of the same size.
The meet of two compositions $I$ and $J$ of size $n$ is the finest composition of $n$ which is coarser than each of $I$ and $J$. It can be described as the composition whose descent set is the intersection of the descent sets of $I$ and $J$.

**INPUT:**
- `other` – composition of same size as `self`
- `check` – (default: `True`) a Boolean determining whether to check the input compositions for having the same size

**OUTPUT:**
- the meet of the compositions `self` and `other`

**EXAMPLES:**

```sage
sage: Composition([3, 1, 1, 3, 1]).meet([4, 3, 2])
[4, 5]
sage: Composition([9, 6]).meet([1, 3, 6, 3, 2])
[15]
sage: Composition([9, 6]).meet([1, 3, 5, 1, 3, 2])
[9, 6]
sage: Composition([1, 1, 1, 1, 1]).meet([3, 2])
[3, 2]
sage: Composition([4, 2]).meet([3, 3])
[6]
sage: Composition([]).meet([])
[]
sage: Composition([1]).meet([1])
[1]
```

Let us verify on small examples that the meet of $I$ and $J$ is coarser than both of $I$ and $J$:

```sage
sage: all( all( I.is_finer(I.meet(J)) and J.is_finer(I.meet(J))
.....: for J in Compositions(4) )
.....: for I in Compositions(4) )
True
```

and is the finest composition to do so:

```sage
sage: all( all( I.meet(J).is_finer(K)
.....: for K in I.fatter() )
.....: for J in Compositions(3) )
.....: for I in Compositions(3) )
True
```

The descent set of the meet of $I$ and $J$ is the intersection of the descent sets of $I$ and $J$:

```sage
sage: def test_meet(n):
.....:     return all( all( I.to_subset().intersection(J.to_subset())
.....:     == I.meet(J).to_subset() )
.....:     for J in Compositions(n) )
.....:     for I in Compositions(n) )
(continues on next page)
```
sage: all( test_meet(n) for n in range(1, 5) )
True

See also:

join()

AUTHORS:

• Darij Grinberg (2013-09-05)

**is_finer(co2)**

Return True if the composition self is finer than the composition co2; otherwise, return False.

**EXAMPLES:**

sage: Composition([4,1,2]).is_finer([3,1,3])
False
sage: Composition([3,1,3]).is_finer([4,1,2])
False
sage: Composition([1,2,2,1,1,2]).is_finer([5,1,3])
True
sage: Composition([2,2,2]).is_finer([4,2])
True

**join(other, check=True)**

Return the join of self with a composition other of the same size.

The join of two compositions I and J of size n is the coarsest composition of n which refines each of I and J. It can be described as the composition whose descent set is the union of the descent sets of I and J. It is also the concatenation of I₁, I₂, ..., Iₘ, where I = I₁ ∙ I₂ ∙ ... ∙ Iₘ is the ribbon decomposition of I with respect to J (see ribbon_decomposition()).

**INPUT:**

• other – composition of same size as self

• check – (default: True) a Boolean determining whether to check the input compositions for having the same size

**OUTPUT:**

• the join of the compositions self and other

**EXAMPLES:**

sage: Composition([3, 1, 1, 3, 1]).join([4, 3, 2])
[3, 1, 1, 2, 1, 1]
sage: Composition([9, 6]).join([1, 3, 6, 3, 2])
[1, 3, 5, 1, 3, 2]
sage: Composition([9, 6]).join([1, 3, 5, 1, 3, 2])
[1, 3, 5, 1, 3, 2]
sage: Composition([1, 1, 1, 1, 1]).join([3, 2])
[1, 1, 1, 1, 1]
sage: Composition([4, 2]).join([3, 3])
[3, 1, 2]
sage: Composition([]).join([])
[]
Let us verify on small examples that the join of $I$ and $J$ refines both of $I$ and $J$:

```
sage: all( all( I.join(J).is_finer(I) and 
.....: I.join(J).is_finer(J) 
.....: for J in Compositions(4) ) 
.....: for I in Compositions(4) )
True
```

and is the coarsest composition to do so:

```
sage: all( all( all( K.is_finer(I.join(J)) 
.....: for K in I.finer() 
.....: if K.is_finer(J) ) 
.....: for I in Compositions(3) ) 
.....: for J in Compositions(3) )
True
```

Let us check that the join of $I$ and $J$ is indeed the concatenation of $I_1, I_2, \ldots, I_m$, where $I = I_1 \cdot I_2 \cdot \ldots \cdot I_m$ is the ribbon decomposition of $I$ with respect to $J$:

```
sage: all( all( Composition.sum(I.ribbon_decomposition(J)[0]) 
.....: == I.join(J) for J in Compositions(4) ) 
.....: for I in Compositions(4) )
True
```

Also, the descent set of the join of $I$ and $J$ is the union of the descent sets of $I$ and $J$:

```
sage: all( all( I.to_subset().union(J.to_subset()) 
.....: == I.join(J).to_subset() 
.....: for J in Compositions(4) ) 
.....: for I in Compositions(4) )
True
```

See also:

meet(), ribbon_decomposition()

AUTHORS:

• Darij Grinberg (2013-09-05)

major_index()

Return the major index of self. The major index is defined as the sum of the descents.

EXAMPLES:

```
sage: Composition([1, 1, 3, 1, 2, 1, 3]).major_index()
31
```

meet(other, check=True)

Return the meet of self with a composition other of the same size.

The meet of two compositions $I$ and $J$ of size $n$ is the finest composition of $n$ which is coarser than each of $I$ and $J$. It can be described as the composition whose descent set is the intersection of the descent sets of $I$ and $J$.

INPUT:

• other – composition of same size as self
Combinatorics, Release 10.0

- check – (default: True) a Boolean determining whether to check the input compositions for having the same size

OUTPUT:
- the meet of the compositions self and other

EXAMPLES:

```sage
sage: Composition([3, 1, 1, 3, 1]).meet([4, 3, 2])
[4, 5]
sage: Composition([9, 6]).meet([1, 3, 6, 3, 2])
[9, 6]
sage: Composition([1, 1, 1, 1, 1]).meet([3, 2])
[3, 2]
sage: Composition([4, 2]).meet([3, 3])
[6]
sage: Composition([]).meet([])
[]
sage: Composition([1]).meet([1])
[1]
```

Let us verify on small examples that the meet of \(I\) and \(J\) is coarser than both of \(I\) and \(J\):

```sage
sage: all( all( I.is_finer(I.meet(J))
.....: J.is_finer(I.meet(J))
.....: for J in Compositions(4) )
.....: for I in Compositions(4) )
True
```

and is the finest composition to do so:

```sage
sage: all( all( I.meet(J).is_finer(K)
.....: for K in I.fatter() )
.....: if J.is_finer(K) )
.....: for J in Compositions(3) )
.....: for I in Compositions(3) )
True
```

The descent set of the meet of \(I\) and \(J\) is the intersection of the descent sets of \(I\) and \(J\):

```sage
sage: def test_meet(n):
.....: return all( all( I.to_subset().intersection(J.to_subset())
.....: == I.meet(J).to_subset() )
.....: for J in Compositions(n) )
.....: for I in Compositions(n) )
sage: all( test_meet(n) for n in range(1, 5) )
True
```

See also:

- `join()`

AUTHORS:
- Darij Grinberg (2013-09-05)
**near_concatenation**(*other*)

Return the near-concatenation of two nonempty compositions `self` and `other`.

The near-concatenation $I \odot J$ of two nonempty compositions $I$ and $J$ is defined as the composition $(i_1, i_2, \ldots, i_{n-1}, i_n + j_1, j_2, j_3, \ldots, j_m)$, where $(i_1, i_2, \ldots, i_n) = I$ and $(j_1, j_2, \ldots, j_m) = J$.

This method returns `None` if one of the two input compositions is empty.

**EXAMPLES:**

```
sage: Composition([1, 1, 3]).near_concatenation(Composition([4, 1, 2]))
[1, 1, 7, 1, 2]
sage: Composition([6]).near_concatenation(Composition([1, 5]))
[7, 5]
sage: Composition([1, 5]).near_concatenation(Composition([6]))
[1, 11]
```

**partial_sums**(*final=True*)

The partial sums of the sequence defined by the entries of the composition.

If $I = (i_1, \ldots, i_m)$ is a composition, then the partial sums of the entries of the composition are $[i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_m]$.

**INPUT:**

- `final` – (default: True) whether or not to include the final partial sum, which is always the size of the composition.

**See also:**

`to_subset()`

**EXAMPLES:**

```
sage: Composition([1,1,3,1,2,1,3]).partial_sums()
[1, 2, 5, 6, 8, 9, 12]
With final = False, the last partial sum is not included:
```

```
sage: Composition([1,1,3,1,2,1,3]).partial_sums(final=False)
[1, 2, 5, 6, 8, 9]
```

**peaks**()

Return a list of the peaks of the composition `self`.

The peaks of a composition are the descents which do not immediately follow another descent.

**EXAMPLES:**

```
sage: Composition([1, 1, 3, 1, 2, 1, 3]).peaks()
[4, 7]
```

**refinement_splitting**(*J*)

Return the refinement splitting of `self` according to `J`.

**INPUT:**

- `J` – A composition such that `self` is finer than `J`

**OUTPUT:**
• the unique list of compositions \( (I(p))_{p=1,\ldots,m} \), obtained by splitting \( I \), such that \( |I(p)| = J_p \) for all \( p = 1,\ldots,m \).

See also:

\texttt{refinement\_splitting\_lengths()}

EXAMPLES:

```
sage: Composition([1,2,2,1,1,2]).refinement_splitting([5,1,3])
[[1, 2, 2], [1, 1, 2]]
sage: Composition([]).refinement_splitting([])
[]
sage: Composition([3]).refinement_splitting([2])
Traceback (most recent call last):
...
ValueError: compositions self (= [3]) and J (= [2]) must be of the same size
sage: Composition([2,1]).refinement_splitting([1,2])
Traceback (most recent call last):
...
ValueError: composition J (= [2, 1]) does not refine self (= [1, 2])
```

\texttt{refinement\_splitting\_lengths(J)}

Return the lengths of the compositions in the refinement splitting of \texttt{self} according to \texttt{J}.

See also:

\texttt{refinement\_splitting()} for the definition of refinement splitting

EXAMPLES:

```
sage: Composition([1,2,2,1,1,2]).refinement_splitting_lengths([5,1,3])
[3, 1, 2]
sage: Composition([]).refinement_splitting_lengths([])
[]
sage: Composition([3]).refinement_splitting_lengths([2])
Traceback (most recent call last):
...
ValueError: compositions self (= [3]) and J (= [2]) must be of the same size
sage: Composition([2,1]).refinement_splitting_lengths([1,2])
Traceback (most recent call last):
...
ValueError: composition J (= [2, 1]) does not refine self (= [1, 2])
```

\texttt{reversed()}\n
Return the reverse composition of \texttt{self}.

The reverse composition of a composition \((i_1,i_2,\ldots,i_k)\) is defined as the composition \((i_k,i_{k-1},\ldots,i_1)\).

EXAMPLES:

```
sage: Composition([1, 1, 3, 1, 2, 1, 3]).reversed()
[3, 1, 2, 1, 3, 1, 1]
```

\texttt{ribbon\_decomposition(other, check=True)}

Return a pair describing the ribbon decomposition of a composition \texttt{self} with respect to a composition \texttt{other} of the same size.
If \( I \) and \( J \) are two compositions of the same nonzero size, then the ribbon decomposition of \( I \) with respect to \( J \) is defined as follows: Write \( I \) and \( J \) as
\[
I = (i_1, i_2, \ldots, i_n) \quad \text{and} \quad J = (j_1, j_2, \ldots, j_m)
\]
Then, the equality
\[
I = I_1 \cdot I_2 \cdot \ldots \cdot I_m
\]
holds for a unique \( m \)-tuple \((I_1, I_2, \ldots, I_m)\) of compositions such that each \( I_k \) has size \( j_k \) and for a unique choice of \( m - 1 \) signs \( \circ \) which is either the concatenation sign \( \cdot \) or the near-concatenation sign \( \circ \) (see \_add\() and \texttt{near_concatenation()} for the definitions of these two signs). This \( m \)-tuple and this choice of signs together are said to form the ribbon decomposition of \( I \) with respect to \( J \). If \( I \) and \( J \) are empty, then the same definition applies, except that there are 0 rather than \( m - 1 \) signs.

See Section 4.8 of [NCSF1].

INPUT:

- \texttt{other} – composition of same size as \texttt{self}
- \texttt{check} – (default: \texttt{True}) a Boolean determining whether to check the input compositions for having the same size

OUTPUT:

- a pair \((u, v)\), where \( u \) is a tuple of compositions (corresponding to the \( m \)-tuple \((I_1, I_2, \ldots, I_m)\) in the above definition), and \( v \) is a tuple of 0’s and 1’s (encoding the choice of signs \( \circ \) in the above definition, with a 0 standing for \( \cdot \) and a 1 standing for \( \circ \)).

EXAMPLES:

\begin{verbatim}
sage: Composition([3, 1, 1, 3, 1]).ribbon_decomposition([4, 3, 2]) (([3, 1], [1, 2], [1, 1]), (0, 1))
sage: Composition([9, 6]).ribbon_decomposition([1, 3, 6, 3, 2]) (([1], [3], [5, 1], [3], [2]), (1, 1, 1, 1))
sage: Composition([9, 6]).ribbon_decomposition([1, 3, 5, 1, 3, 2]) (([1], [3], [5], [1], [3], [2]), (1, 1, 0, 1, 1))
sage: Composition([1, 1, 1, 1, 1]).ribbon_decomposition([3, 2]) (([1, 1, 1], [1, 1]), (0,))
sage: Composition([4, 2]).ribbon_decomposition([6]) (([4, 2],), ())
sage: Composition([]).ribbon_decomposition([]) ((), ())
\end{verbatim}

Let us check that the defining property \( I = I_1 \cdot I_2 \cdot \ldots \cdot I_m \) is satisfied:

\begin{verbatim}
sage: def compose_back(u, v):
....:     comp = u[0]
....:     r = len(v)
....:     if len(u) != r + 1:
....:         raise ValueError("something is wrong")
....:     for i in range(r):
....:         if v[i] == 0:
....:             comp += u[i + 1]
....:         else:
....:             comp = comp.near_concatenation(u[i + 1])
....:     return comp
sage: all(all(compose_back(*I.ribbon_decomposition(J)) == I
....:         for J in Compositions(n) )
....:     for I in Compositions(n) )
....:     for n in range(1, 5) )
True
\end{verbatim}
AUTHORS:

- Darij Grinberg (2013-08-29)

**shuffle_product**(other, overlap=False)

The (overlapping) shuffles of self and other.

Suppose \( I = (i_1, \ldots, i_k) \) and \( J = (j_1, \ldots, j_l) \) are two compositions. A shuffle of \( I \) and \( J \) is a composition of length \( k + l \) that contains both \( I \) and \( J \) as subsequences.

More generally, an overlapping shuffle of \( I \) and \( J \) is obtained by distributing the elements of \( I \) and \( J \) (preserving the relative ordering of these elements) among the positions of an empty list; an element of \( I \) and an element of \( J \) are permitted to share the same position, in which case they are replaced by their sum.

In particular, a shuffle of \( I \) and \( J \) is an overlapping shuffle of \( I \) and \( J \).

INPUT:

- other – composition
- overlap – boolean (default: False); if True, the overlapping shuffle product is returned.

OUTPUT:

An enumerated set (allowing for multiplicities)

EXAMPLES:

The shuffle product of \([2, 2] \) and \([1, 1, 3]\):

```sage
alph = Composition([2, 2])
beta = Composition([1, 1, 3])
S = alph.shuffle_product(beta); S
```

```
Shuffle product of [2, 2] and [1, 1, 3]
```

```
S.list()
```

```
[[2, 2, 1, 1, 3], [2, 1, 2, 1, 3], [2, 1, 1, 2, 3], [2, 1, 1, 3, 2], [1, 2, 2, 1, 3], [1, 2, 1, 2, 3], [1, 2, 1, 3, 2], [1, 1, 2, 2, 3], [1, 1, 2, 3, 2], [1, 1, 3, 2, 2]]
```

The overlapping shuffle product of \([2, 2] \) and \([1, 1, 3]\):

```sage
alph = Composition([2, 2])
beta = Composition([1, 1, 3])
O = alph.shuffle_product(beta, overlap=True); O
```

```
Overlapping shuffle product of [2, 2] and [1, 1, 3]
```

```
O.list()
```

```
[[2, 2, 1, 1, 3], [2, 1, 2, 1, 3], [2, 1, 1, 2, 3], [2, 1, 1, 3, 2], [1, 2, 2, 1, 3], [1, 2, 1, 2, 3], [1, 2, 1, 3, 2], [1, 1, 2, 2, 3], [1, 1, 2, 3, 2], [1, 1, 3, 2, 2], [3, 2, 1, 3], [2, 3, 1, 3], [3, 1, 2, 3], [2, 1, 3, 3], [3, 1, 3, 3], [2, 1, 3, 3], [1, 3, 3, 3], [1, 2, 1, 5], [1, 1, 5, 2], [1, 1, 2, 5], [3, 3, 3], [3, 1, 5], [1, 3, 5]]
```

Note that the shuffle product of two compositions can include the same composition more than once since a composition can be a shuffle of two compositions in several ways. For example:

```sage
w1 = Composition([1])
S = w1.shuffle_product(w1); S
```

```
Shuffle product of [1] and [1]
```

```
S.list()
```

```
[[1, 1], [1, 1]]
```

(continues on next page)
sage: O = w1.shuffle_product(w1, overlap=True); O
Overlapping shuffle product of [1] and [1]
sage: O.list()
[[1, 1], [1, 1], [2]]

size()
Return the size of self, that is the sum of its parts.

EXAMPLES:
sage: Composition([7,1,3]).size()
11

specht_module(base_ring=None)
Return the Specht module corresponding to self.

EXAMPLES:
sage: SM = Composition([1,2,2]).specht_module(QQ)
sage: SM
Specht module of [(0, 0), (1, 0), (1, 1), (2, 0), (2, 1)] over Rational Field
sage: s = SymmetricFunctions(QQ).s()
sage: s(SM.frobenius_image())
s[2, 2, 1]

specht_module_dimension(base_ring=None)
Return the dimension of the Specht module corresponding to self.

INPUT:
• base_ring – (default: Q) the base ring

EXAMPLES:
sage: Composition([1,2,2]).specht_module_dimension()
5
sage: Composition([1,2,2]).specht_module_dimension(GF(2))
5

static sum(compositions)
Return the concatenation of the given compositions.

INPUT:
• compositions – a list (or iterable) of compositions

EXAMPLES:
sage: Composition.sum([Composition([1, 1, 3]), Composition([4, 1, 2]),
                        Composition([3,1])])
[1, 1, 3, 4, 1, 2, 3, 1]

Any iterable can be provided as input:
sage: Composition.sum([Composition([i,i]) for i in [4,1,3]])
[4, 4, 1, 1, 3, 3]
Combinatorics, Release 10.0

Empty inputs are handled gracefully:

```
sage: Composition().sum([]) == Composition([])
True
```

\texttt{sup}(\texttt{other}, \texttt{check=True})

Return the join of \texttt{self} with a composition \texttt{other} of the same size.

The join of two compositions $I$ and $J$ of size $n$ is the coarsest composition of $n$ which refines each of $I$ and $J$. It can be described as the composition whose descent set is the union of the descent sets of $I$ and $J$. It is also the concatenation of $I_1\cdot I_2\cdot\ldots\cdot I_m$, where $I = I_1\bullet I_2\bullet\ldots\bullet I_m$ is the ribbon decomposition of $I$ with respect to $J$ (see \texttt{ribbon_decomposition()}).

\textbf{INPUT}:

\begin{itemize}
  \item \texttt{other} – composition of same size as \texttt{self}
  \item \texttt{check} – (default: \texttt{True}) a Boolean determining whether to check the input compositions for having the same size
\end{itemize}

\textbf{OUTPUT}:

\begin{itemize}
  \item the join of the compositions \texttt{self} and \texttt{other}
\end{itemize}

\textbf{EXAMPLES}:

```
sage: Composition([3, 1, 1, 3, 1]).join([4, 3, 2])
[3, 1, 1, 2, 1, 1]
sage: Composition([9, 6]).join([1, 3, 6, 3, 2])
[1, 3, 5, 1, 3, 2]
sage: Composition([9, 6]).join([1, 3, 5, 1, 3, 2])
[1, 3, 5, 1, 3, 2]
sage: Composition([1, 1, 1, 1, 1]).join([3, 2])
[1, 1, 1, 1, 1]
sage: Composition([4, 2]).join([3, 3])
[3, 1, 2]
sage: Composition([]).join([])
[]
```

Let us verify on small examples that the join of $I$ and $J$ refines both of $I$ and $J$:

```
sage: all( all( I.join(J).is_finer(I)
....:           and
....:           I.join(J).is_finer(J)
....:           for J in Compositions(4) )
....:       for I in Compositions(4) )
True
```

and is the coarsest composition to do so:

```
sage: all( all( K.is_finer(I.join(J))
.....:           for K in I.finer() )
.....:         if K.is_finer(J) )
.....:     for J in Compositions(3) )
.....:   for I in Compositions(3) )
True
```

Let us check that the join of $I$ and $J$ is indeed the concatenation of $I_1\cdot I_2\cdot\ldots\cdot I_m$, where $I = I_1\bullet I_2\bullet\ldots\bullet I_m$ is the ribbon decomposition of $I$ with respect to $J$:

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```python
sage: all( all( Composition.sum(I.ribbon_decomposition(J)[0])
.....:     == I.join(J) for J in Compositions(4) )
.....:     for I in Compositions(4) )
True
```

Also, the descent set of the join of $I$ and $J$ is the union of the descent sets of $I$ and $J$:

```python
sage: all( all( I.to_subset().union(J.to_subset())
.....:     == I.join(J).to_subset()
.....:     for J in Compositions(4) )
.....:     for I in Compositions(4) )
True
```

See also:

`meet()`, `ribbon_decomposition()`

AUTHORS:

* Darij Grinberg (2013-09-05)

to_code()

Return the code of the composition `self`.

The code of a composition $I$ is a list of length $\text{size}(I)$ of 1s and 0s such that there is a 1 wherever a new part starts. (Exceptional case: When the composition is empty, the code is [0].)

EXAMPLES:

```python
sage: Composition([4,1,2,3,5]).to_code()
[1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0]
```

to_partition()

Return the partition obtained by sorting `self` into decreasing order.

EXAMPLES:

```python
sage: Composition([2,1,3]).to_partition()
[3, 2, 1]
sage: Composition([4,2,2]).to_partition()
[4, 2, 2]
sage: Composition([]).to_partition()
[]
```

to_skew_partition(overlap=1)

Return the skew partition obtained from `self`.

This is a skew partition whose rows have the entries of `self` as their length, taken in reverse order (so the first entry of `self` is the length of the lowermost row, etc.). The parameter `overlap` indicates the number of cells on each row that are directly below cells of the previous row. When it is set to 1 (its default value), the result is the ribbon shape of `self`.

EXAMPLES:

```python
sage: Composition([3,4,1]).to_skew_partition()
[6, 6, 3] / [5, 2]
sage: Composition([3,4,1]).to_skew_partition(overlap=0)
(continues on next page)
```
[8, 7, 3] / [7, 3]
sage: Composition([]).to_skew_partition()
[] / []
sage: Composition([1,2]).to_skew_partition()
[2, 1] / []
sage: Composition([2,1]).to_skew_partition()
[2, 2] / [1]

**to_subset**(final=False)
The subset corresponding to self under the bijection (see below) between compositions of \( n \) and subsets of \( \{1, 2, \ldots, n-1\} \).

The bijection maps a composition \((i_1, \ldots, i_k)\) of \( n \) to \( \{i_1, i_1+i_2, i_1+i_2+i_3, \ldots, i_1+\cdots+i_{k-1}\} \).

**INPUT:**
- **final** – (default: False) whether or not to include the final partial sum, which is always the size of the composition.

**See also:**
- partial_sums()

**EXAMPLES:**
sage: Composition([1,1,3,1,2,1,3]).to_subset()
{1, 2, 5, 6, 8, 9}
sage: for I in Compositions(3): print(I.to_subset())
{1, 2}
{1}
{2}
{1}

With **final=True**, the sum of all the elements of the composition is included in the subset:
sage: Composition([1,1,3,1,2,1,3]).to_subset(final=True)
{1, 2, 5, 6, 8, 9, 12}

**wll_gt**(co2)
Return True if the composition self is greater than the composition co2 with respect to the wll-ordering; otherwise, return False.

The wll-ordering is a total order on the set of all compositions defined as follows: A composition \( I \) is greater than a composition \( J \) if and only if one of the following conditions holds:

- The size of \( I \) is greater than the size of \( J \).
- The size of \( I \) equals the size of \( J \), but the length of \( I \) is greater than the length of \( J \).
- The size of \( I \) equals the size of \( J \), and the length of \( I \) equals the length of \( J \), but \( I \) is lexicographically greater than \( J \).

(“wll-ordering” is short for “weight, length, lexicographic ordering”.)

**EXAMPLES:**
sage: Composition([4,1,2]).wll_gt([3,1,3])
True

(continues on next page)
sage: Composition([7]).wll_gt([4,1,2])
False
sage: Composition([8]).wll_gt([4,1,2])
True
sage: Composition([3,2,2,2]).wll_gt([5,2])
True
sage: Composition([]).wll_gt([3])
False
sage: Composition([2,1]).wll_gt([2,1])
False
sage: Composition([2,2,2]).wll_gt([4,2])
True
sage: Composition([4,2]).wll_gt([2,2,2])
False
sage: Composition([1,1,2]).wll_gt([2,2])
True
sage: Composition([2,2]).wll_gt([1,3])
True
sage: Composition([2,1,2]).wll_gt([1,3])
True

class sage.combinat.composition.Compositions(is_infinite=False, category=None)
  Bases: UniqueRepresentation, Parent

Set of integer compositions.

A composition \( c \) of a nonnegative integer \( n \) is a list of positive integers with total sum \( n \).

See also:

- Composition
- Partitions
- IntegerVectors

EXAMPLES:

There are 8 compositions of 4:

```
sage: Compositions(4).cardinality()
8
```

Here is the list of them:

```
sage: Compositions(4).list()
[[1, 1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 3], [2, 1, 1], [2, 2], [3, 1], [4]]
```

You can use the `.first()` method to get the ‘first’ composition of a number:

```
sage: Compositions(4).first()
[1, 1, 1, 1]
```

You can also calculate the ‘next’ composition given the current one:
sage: Compositions(4).next([1,1,2])
[1, 2, 1]

If \( n \) is not specified, this returns the combinatorial class of all (non-negative) integer compositions:

sage: Compositions()
Compositions of non-negative integers
sage: [] in Compositions()
True
sage: [2,3,1] in Compositions()
True
sage: [-2,3,1] in Compositions()
False

If \( n \) is specified, it returns the class of compositions of \( n \):

sage: Compositions(3)
Compositions of 3
sage: list(Compositions(3))
[[1, 1, 1], [1, 2], [2, 1], [3]]
sage: Compositions(3).cardinality()
4

The following examples show how to test whether or not an object is a composition:

sage: [3,4] in Compositions()
True
sage: [3,4] in Compositions(7)
True
sage: [3,4] in Compositions(5)
False

Similarly, one can check whether or not an object is a composition which satisfies further constraints:

sage: [4,2] in Compositions(6, inner=[2,2])
True
sage: [4,2] in Compositions(6, inner=[2,3])
False
sage: [4,1] in Compositions(5, inner=[2,1], max_slope = 0)
True

An example with incompatible constraints:

sage: [4,2] in Compositions(6, inner=[2,2], min_part=3)
False

The options length, min_length, and max_length can be used to set length constraints on the compositions. For example, the compositions of 4 of length equal to, at least, and at most 2 are given by:

sage: Compositions(4, length=2).list()
[[3, 1], [2, 2], [1, 3]]
sage: Compositions(4, min_length=2).list()
[[3, 1], [2, 2], [2, 1, 1], [1, 3], [1, 2, 1], [1, 1, 2], [1, 1, 1, 1]]

(continues on next page)
sage: Compositions(4, max_length=2).list()
[[4], [3, 1], [2, 2], [1, 3]]

Setting both min_length and max_length to the same value is equivalent to setting length to this value:

sage: Compositions(4, min_length=2, max_length=2).list()
[[3, 1], [2, 2], [1, 3]]

The options inner and outer can be used to set part-by-part containment constraints. The list of compositions of 4 bounded above by [3,1,2] is given by:

sage: list(Compositions(4, outer=[3,1,2]))
[[3, 1], [2, 1, 1], [1, 1, 2]]

outer sets max_length to the length of its argument. Moreover, the parts of outer may be infinite to clear the constraint on specific parts. This is the list of compositions of 4 of length at most 3 such that the first and third parts are at most 1:

sage: Compositions(4, outer=[1,oo,1]).list()
[[1, 3], [1, 2, 1]]

This is the list of compositions of 4 bounded below by [1,1,1]:

sage: Compositions(4, inner=[1,1,1]).list()
[[2, 1, 1], [1, 2, 1], [1, 1, 2], [1, 1, 1, 1]]

The options min_slope and max_slope can be used to set constraints on the slope, that is the difference \(p[i+1]-p[i]\) of two consecutive parts. The following is the list of weakly increasing compositions of 4:

sage: Compositions(4, min_slope=0).list()
[[4], [2, 2], [1, 3], [1, 1, 2], [1, 1, 1, 1]]

Here are the weakly decreasing ones:

sage: Compositions(4, max_slope=0).list()
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 2], [1, 1, 1, 1]]

The following is the list of compositions of 4 such that two consecutive parts differ by at most one:

sage: Compositions(4, min_slope=-1, max_slope=1).list()
[[4], [2, 2], [2, 1, 1], [1, 2, 1], [1, 1, 2], [1, 1, 1, 1]]

The constraints can be combined together in all reasonable ways. This is the list of compositions of 5 of length between 2 and 4 such that the difference between consecutive parts is between -2 and 1:

sage: Compositions(5, max_slope=1, min_slope=-2, min_length=2, max_length=4).list()
[[3, 2], [3, 1, 1], [2, 3], [2, 2, 1], [2, 1, 2], [2, 1, 1, 1], [1, 2, 2], [1, 2, 1, 1], [1, 1, 2, 1], [1, 1, 1, 2], [1, 1, 2, 1], [1, 1, 1, 2]]

We can do the same thing with an outer constraint:

sage: Compositions(5, max_slope=1, min_slope=-2, min_length=2, max_length=4, outer=[2,5,2]).list()
[[2, 3], [2, 2, 1], [2, 1, 2], [1, 2, 2]]
However, providing incoherent constraints may yield strange results. It is up to the user to ensure that the inner and outer compositions themselves satisfy the parts and slope constraints.

Note that setting `min_part=0` is not allowed:

```
sage: Compositions(2, length=3, min_part=0)
Traceback (most recent call last):
  ...  
ValueError: setting min_part=0 is not allowed for Compositions
```

Instead you must use `IntegerVectors`:

```
sage: list(IntegerVectors(2, 3))
[[2, 0, 0], [1, 1, 0], [1, 0, 1], [0, 2, 0], [0, 1, 1], [0, 0, 2]]
```

The generation algorithm is constant amortized time, and handled by the generic tool `IntegerListsLex`.

**Element**

**from_code**(code)

Return the composition from its code. The code of a composition \(I\) is a list of length `size(I)` consisting of 1s and 0s such that there is a 1 wherever a new part starts. (Exceptional case: When the composition is empty, the code is \([0]\).)

**EXAMPLES:**

```
sage: Composition([4,1,2,3,5]).to_code()
[1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0]
sage: Compositions().from_code(_)
[4, 1, 2, 3, 5]
sage: Composition([3,1,2,3,5]).to_code()
[1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0]
sage: Compositions().from_code(_)
[3, 1, 2, 3, 5]
```

**from_descents**(descents, nps=None)

Return a composition from the list of descents.

**INPUT:**

- descents – an iterable
- nps – (default: None) an integer or None

**OUTPUT:**

- The composition of nps whose descents are listed in `descents`, assuming that nps is not None (otherwise, the last element of `descents` is removed from `descents`, and nps is set to be this last element plus 1).

**EXAMPLES:**

```
sage: [x-1 for x in Composition([1, 1, 3, 4, 3]).to_subset()]
[0, 1, 4, 8]
sage: Compositions().from_descents([1,0,4,8],12)
[1, 1, 3, 4, 3]
sage: Compositions().from_descents([1,0,4,8,11])
[1, 1, 3, 4, 3]
```
from_subset(S, n)

The composition of \( n \) corresponding to the subset \( S \) of \( \{1, 2, \ldots, n-1\} \) under the bijection that maps the composition \((i_1, i_2, \ldots, i_k)\) of \( n \) to the subset \( \{i_1, i_1 + i_2, i_1 + i_2 + i_3, \ldots, i_1 + \cdots + i_{k-1}\} \) (see `Composition.to_subset()`).

**INPUT:**
- \( S \) – an iterable, a subset of \( \{1, 2, \ldots, n-1\} \)
- \( n \) – an integer

**EXAMPLES:**

```python
sage: Compositions().from_subset([2,1,5,9], 12)
[1, 1, 3, 4, 3]

sage: Compositions().from_subset({2,1,5,9}, 12)
[1, 1, 3, 4, 3]

sage: Compositions().from_subset([], 12)
[12]

sage: Compositions().from_subset([], 0)
[]
```

---

**class** `sage.combinat.composition.Compositions_all`

**Bases:** `Compositions`

Class of all compositions.

**subset(size=None)**

Return the set of compositions of the given size.

**EXAMPLES:**

```python
sage: C = Compositions()
sage: C.subset(4)
Compositions of 4
sage: C.subset(size=3)
Compositions of 3
```

**zero()**

Return the zero of the additive monoid.

This is the empty composition.

**EXAMPLES:**

```python
sage: C = Compositions()
sage: C.zero()
[]
```

---

**class** `sage.combinat.composition.Compositions_constraints(*args, **kwds)`

**Bases:** `IntegerListsLex`

**class** `sage.combinat.composition.Compositions_n(n)`

**Bases:** `Compositions`

Class of compositions of a fixed \( n \).
cardinality()
Return the number of compositions of $n$.

random_element()
Return a random Composition with uniform probability.
This method generates a random binary word starting with a 1 and then uses the bijection between compositions and their code.

EXAMPLES:

```
sage: Compositions(5).random_element() # random
[2, 1, 1, 1]
sage: Compositions(0).random_element()
[]
sage: Compositions(1).random_element()
[1]
```

sage.combinat.composition.composition_iterator_fast(n)
Iterator over compositions of $n$ yielded as lists.

5.1.30 Signed Compositions

class sage.combinat.composition_signed.SignedCompositions($n$)
Bases: Compositions_n
The class of signed compositions of $n$.

EXAMPLES:

```
sage: SC3 = SignedCompositions(3); SC3
Signed compositions of 3
sage: SC3.cardinality()
18
sage: len(SC3.list())
18
sage: SC3.first()
[1, 1, 1]
sage: SC3.last()
[-3]
sage: SC3.random_element() # random
[1, -1, 1]
sage: SC3.list()
[[1, 1, 1],
 [1, 1, -1],
 [1, -1, 1],
 [1, -1, -1],
 [-1, 1, 1],
 [-1, 1, -1],
 [-1, -1, 1],
 [-1, -1, -1],
 [1, 2],
 [1, -2],
 [-1, 2],
```

(continues on next page)
[-1, -2],
[2, 1],
[2, -1],
[-2, 1],
[-2, -1],
[3],
[-3]]

cardinality()

Return the number of elements in self.

The number of signed compositions of $n$ is equal to

$$\sum_{i=1}^{n+1} \binom{n-1}{i-1} 2^i$$

EXAMPLES:

```
sage: SC4 = SignedCompositions(4)
sage: SC4.cardinality() == len(SC4.list())
True
sage: SignedCompositions(3).cardinality()
18
```

5.1.31 Composition Tableaux

AUTHORS:

• Chris Berg, Jeff Ferreira (2012-9): Initial version

class sage.combinat.composition_tableau.CompositionTableau(parent, t)

Bases: CombinatorialElement

A composition tableau.

A composition tableau $t$ of shape $I = (I_1, \ldots, I_\ell)$ is an array of boxes in rows, $I_i$ boxes in row $i$, filled with positive integers such that:

1) the entries in the rows of $t$ weakly decrease from left to right,
2) the left-most column of $t$ strictly increase from top to bottom.

3) Add zero entries to the rows of $t$ until the resulting array is rectangular of shape $\ell \times m$. For $1 \leq i < j \leq \ell, 2 \leq k \leq m$ and $(t(j, k) \neq 0$, and also if $t(j, k) \geq t(i, k))$ implies $t(j, k) > t(i, k - 1)$.

INPUT:

• $t$ – A list of lists

EXAMPLES:

```
sage: CompositionTableau([[1],[2,2]])
[[1], [2, 2]]
sage: CompositionTableau([[1],[3,2],[4,4]])
[[1], [3, 2], [4, 4]]
sage: CompositionTableau([])
[]
```
**descent_composition()**

Return the composition corresponding to the set of all \( i \) that do not have \( i + 1 \) appearing strictly to the left of \( i \) in \( \text{self} \).

**EXAMPLES:**

```
sage: CompositionTableau([[1],[3,2],[4,4]]).descent_composition()
[1, 2, 2]
```

**descent_set()**

Return the set of all \( i \) that do *not* have \( i + 1 \) appearing strictly to the left of \( i \) in \( \text{self} \).

**EXAMPLES:**

```
sage: CompositionTableau([[1],[3,2],[4,4]]).descent_set()
[1, 3]
```

**is_standard()**

Return True if \( \text{self} \) is a standard composition tableau and False otherwise.

**EXAMPLES:**

```
sage: CompositionTableau([[1,1],[3,2],[4,4,3]]).is_standard()
False
sage: CompositionTableau([[2,1],[3],[4]]).is_standard()
True
```

**pp()**

Return a pretty print string of \( \text{self} \).

**EXAMPLES:**

```
sage: CompositionTableau([[1],[3,2],[4,4]]).pp()
1
3 2
4 4
```

**shape_composition()**

Return a Composition object which is the shape of \( \text{self} \).

**EXAMPLES:**

```
sage: CompositionTableau([[1,1],[3,2],[4,4,3]]).shape_composition()
[2, 2, 3]
sage: CompositionTableau([[2,1],[3],[4]]).shape_composition()
[2, 1, 1]
```

**shape_partition()**

Return a partition which is the shape of \( \text{self} \) sorted into weakly decreasing order.

**EXAMPLES:**

```
sage: CompositionTableau([[1,1],[3,2],[4,4,3]]).shape_partition()
[3, 2, 2]
sage: CompositionTableau([[2,1],[3],[4]]).shape_partition()
[2, 1, 1]
```
Combinatorics, Release 10.0

size()
Return the number of boxes in self.

EXAMPLES:

```
sage: CompositionTableau([[1],[3,2],[4,4]]).size()
5
```

weight()
Return a composition where entry \( i \) is the number of times that \( i \) appears in self.

EXAMPLES:

```
sage: CompositionTableau([[1],[3,2],[4,4]]).weight()
[1, 1, 1, 2, 0]
```

class sage.combinat.composition_tableau.CompositionTableaux(**kwds)
Bases: UniqueRepresentation, Parent
Composition tableaux.

INPUT:

Keyword arguments:

- size – the size of the composition tableaux
- shape – the shape of the composition tableaux
- max_entry – the maximum entry for the composition tableaux

Positional arguments:

- The first argument is interpreted as size or shape depending on whether it is an integer or a composition.

EXAMPLES:

```
sage: CT = CompositionTableaux(3); CT
Composition Tableaux of size 3 and maximum entry 3
sage: list(CT)
[[[1], [2], [3]],
 [[1], [2, 2]],
 [[1], [3, 2]],
 [[1], [3, 3]],
 [[2], [3, 3]],
 [[1, 1], [2]],
 [[1, 1], [3]],
 [[2, 1], [3]],
 [[2, 2], [3]],
 [[1, 1, 1]],
 [[2, 1, 1]],
 [[2, 2, 1]],
 [[2, 2, 2]],
 [[3, 1, 1]],
 [[3, 2, 1]],
 [[3, 2, 2]],
 [[3, 3, 1]],
 [[3, 3, 2]],
 [[3, 3, 3]]]
```

(continues on next page)
sage: CT = CompositionTableaux([1,2,1]); CT
Composition tableaux of shape [1, 2, 1] and maximum entry 4
sage: list(CT)
[[[1], [2, 2], [3]],
 [ [1], [2, 2], [4]],
 [ [1], [3, 2], [4]],
 [ [1], [3, 3], [4]],
 [ [2], [3, 3], [4]]]

sage: CT = CompositionTableaux(shape=[1,2,1],max_entry=3); CT
Composition tableaux of shape [1, 2, 1] and maximum entry 3
sage: list(CT)
[[[1], [2, 2], [3]]]

sage: CT = CompositionTableaux(2,max_entry=3); CT
Composition Tableaux of size 2 and maximum entry 3
sage: list(CT)
[[[1], [2]],
 [ [1], [3]],
 [ [2], [3]],
 [ [1, 1]],
 [ [2, 1]],
 [ [2, 2]],
 [ [3, 1]],
 [ [3, 2]],
 [ [3, 3]]]

sage: CT = CompositionTableaux(0); CT
Composition Tableaux of size 0 and maximum entry 0
sage: list(CT)
[][]

Element
alias of CompositionTableau
class sage.combinat.composition_tableau.CompositionTableauxBacktracker(shape, max_entry=None)

Bases: GenericBacktracker
A backtracker class for generating sets of composition tableaux.

get_next_pos(ii, jj)

EXAMPLES:

sage: from sage.combinat.composition_tableau import CompositionTableauxBacktracker
sage: T = CompositionTableau([[2,1],[5,4,3,2],[6],[7,7,6]])
sage: n = CompositionTableauxBacktracker(T.shape_composition())
sage: n.get_next_pos(1,1)
(1, 2)

class sage.combinat.composition_tableau.CompositionTableaux_all(max_entry=None)
Bases: CompositionTableaux, DisjointUnionEnumeratedSets

5.1. Comprehensive Module List
All composition tableaux.

\texttt{an\_element()} \\
Return a particular element of self.

EXAMPLES:

\begin{verbatim}
sage: CT = CompositionTableaux()
sage: CT.an_element()
[[1, 1], [2]]
\end{verbatim}

\texttt{class sage.combinat.composition_tableau.CompositionTableaux\_shape(comp, max\_entry=None)} \\
Bases: CompositionTableaux

Composition tableaux of a fixed shape \texttt{comp} with a given max entry.

\begin{itemize}
\item \texttt{comp} – a composition.
\item \texttt{max\_entry} – a nonnegative integer. This keyword argument defaults to the size of \texttt{comp}.
\end{itemize}

\texttt{an\_element()} \\
Return a particular element of \texttt{CompositionTableaux\_shape}.

EXAMPLES:

\begin{verbatim}
sage: CT = CompositionTableaux([1,2,1])
sage: CT.an_element()
[[1], [2, 2], [3]]
\end{verbatim}

\texttt{class sage.combinat.composition_tableau.CompositionTableaux\_size(n, max\_entry=None)} \\
Bases: CompositionTableaux

Composition tableaux of a fixed size \texttt{n}.

\begin{itemize}
\item \texttt{n} – a nonnegative integer.
\item \texttt{max\_entry} – a nonnegative integer. This keyword argument defaults to \texttt{n}.
\end{itemize}

\texttt{OUTPUT:}
\begin{itemize}
\item The class of composition tableaux of size \texttt{n}.
\end{itemize}

\section{Constellations}

A constellation is a tuple \((g_0, g_1, \ldots, g_k)\) of permutations such that the product \(g_0g_1\ldots g_k\) is the identity. One often assumes that the group generated by \(g_0, g_1, \ldots, g_k\) acts transitively ([LZ2004] definition 1). Geometrically, it corresponds to a covering of the 2-sphere ramified over \(k\) points (the transitivity condition corresponds to the connectivity of the covering).

\begin{verbatim}
sage: c = Constellation(['(1,2)', '(1,3)', None])
sage: c
Constellation of length 3 and degree 3
\end{verbatim}
g1 (1,3)(2)
g2 (1,3,2)
sage: C = Constellations(3,4); C
Connected constellations of length 3 and degree 4 on {1, 2, 3, 4}
sage: C.cardinality()  # long time
426
sage: C = Constellations(3, 4, domain=('a', 'b', 'c', 'd'))
sage: C
Connected constellations of length 3 and degree 4 on {'a', 'b', 'c', 'd'}
sage: c = C(('a','c'),(('b','c'),('a','d'))), None)
sage: c
Constellation of length 3 and degree 4
g0 ('a','c')('b')('d')
g1 ('a','d')('b','c')
g2 ('a','d','c','b')
sage: c.is_connected()
True
sage: c.euler_characteristic()
2
sage: TestSuite(C).run()

sage.combinat.constellation.Constellation(g=None, mutable=False, connected=True, check=True)

INPUT:

• g – a list of permutations

• mutable – whether the result is mutable or not. Default is False.

• connected – whether the result should be connected. Default is True.

• check – whether or not to check. If it is True, then the list g must contains no None.

EXAMPLES:

Simple initialization:

sage: Constellation([[(0,1)], [(0,3)(1,2)], [(0,3,1,2)]])
Constellation of length 3 and degree 4

One of the permutation can be omitted:

sage: Constellation([[(0,1)], None, [(0,4)(1,2,3)]])
Constellation of length 3 and degree 5

One can define mutable constellations:

sage: Constellation([[0,2,1], [2,1,0], [1,2,0]], mutable=True)
Constellation of length 3 and degree 3
Combinatorics, Release 10.0

class sage.combinat.constellation.Constellation_class(parent, g, connected, mutable, check)

Bases: Element

Constellation

A constellation or a tuple of permutations \((g_0, g_1, \ldots, g_k)\) such that the product \(g_0 g_1 \ldots g_k\) is the identity.

braid_group_action(i)

Act on self as the braid group generator that exchanges position \(i\) and \(i + 1\).

INPUT:

• \(i\) – integer in \([0, n - 1]\) where \(n\) is the length of self

EXAMPLES:

sage: sigma = lambda c, i: c.braid_group_action(i)

sage: c = Constellation(['(0,1)(2,3,4)', '(1,4)', 'None']); c
Constellation of length 3 and degree 5
\(g_0\) \((0,1)(2,3,4)\)
\(g_1\) \((0)(1,4)(2)(3)\)
\(g_2\) \((0,1,3,2,4)\)

sage: sigma(c, 1)
Constellation of length 3 and degree 5
\(g_0\) \((0,1)(2,3,4)\)
\(g_1\) \((0,1,3,2,4)\)
\(g_2\) \((0,3)(1)(2)(4)\)

Check the commutation relation:

sage: c = Constellation(['(0,1)(2,3,4)', '(1,4)', '(2,5)(0,4)', 'None'])

sage: d = Constellation(['(0,1,3,5)', '(2,3,4)', '(0,3,5)', 'None'])

sage: c13 = sigma(sigma(c, 0), 2)

sage: c31 = sigma(sigma(c, 2), 0)

sage: c13 == c31
True

sage: d13 = sigma(sigma(d, 0), 2)

sage: d31 = sigma(sigma(d, 2), 0)

sage: d13 == d31
True

Check the braid relation:

sage: c121 = sigma(sigma(sigma(c, 1), 2), 1)

sage: c212 = sigma(sigma(sigma(c, 2), 1), 2)

sage: c121 == c212
True

sage: d121 = sigma(sigma(sigma(d, 1), 2), 1)

sage: d212 = sigma(sigma(sigma(d, 2), 1), 2)

sage: d121 == d212
True
braid_group_orbit()  
Return the graph of the action of the braid group.  
The action is considered up to isomorphism of constellation.  

EXAMPLES:  

```
sage: c = Constellation(['(0,1)(2,3,4)','(1,4)',None]); c  
Constellation of length 3 and degree 5  
g0 (0,1)(2,3,4)  
g1 (0)(1,4)(2)(3)  
g2 (0,1,3,2,4)  
sage: G = c.braid_group_orbit()  
sage: G.num_verts()  
4  
sage: G.num_edges()  
12  
```  

connected_components()  
Return the connected components.  

OUTPUT:  
A list of connected constellations.  

EXAMPLES:  

```
sage: c = Constellation(['(0,1)(2)', None, '(0,1)(2)'], connected=False)  
sage: cc = c.connected_components(); cc  
[Constellation of length 3 and degree 2  
g0 (0,1)  
g1 (0)  
g2 (0),  
Constellation of length 3 and degree 1  
g0 (0)  
g1 (0)  
g2 (0)]  
sage: all(c2.is_connected() for c2 in cc)  
True  
sage: c = Constellation(['(0,1,2)', None, '(0,1,2)'], connected=False)  
sage: c.connected_components()  
[Constellation of length 2 and degree 3  
g0 (0,1,2)  
g1 (0,2,1)]  
```  

copy()  
Return a copy of self.  

degree()  
Return the degree of the constellation.  
The degree of a constellation is the number \( n \) that corresponds to the symmetric group \( S(n) \) in which the permutations of the constellation are defined.  

EXAMPLES:
Combinatorics, Release 10.0

sage: c = Constellation([])
sage: c.degree()
0
sage: c = Constellation(['(0,1)',None])
sage: c.degree()
2
sage: c = Constellation(['(0,1)','(0,3,2)(1,5)',None,'(4,3,2,1)'])
sage: c.degree()
6

euler_characteristic()

Return the Euler characteristic of the surface.

ALGORITHM:

Hurwitz formula

EXAMPLES:

sage: c = Constellation([(0,1),'(0,2)',None])
sage: c.euler_characteristic()
2
sage: c = Constellation([(0,1,2,3),'(1,3,0,2)','(0,3,1,2)',None])
sage: c.euler_characteristic()
-4

g(i=None)

Return the permutation $g_i$ of the constellation.

INPUT:

• i – integer or None (default)

If None, return instead the list of all $g_i$.

EXAMPLES:

sage: c = Constellation([(0,1,2)(3,4),'(0,3)',None])
sage: c.g(0)
(0,1,2)(3,4)
sage: c.g(1)
(0,3)
sage: c.g(2)
(0,4,3,2,1)
sage: c.g()
[(0,1,2)(3,4), (0,3), (0,4,3,2,1)]

genus()

Return the genus of the surface.

EXAMPLES:

sage: c = Constellation([(0,1),'(0,2)',None])
sage: c.genus()
0
sage: c = Constellation(['(0,1)(2,3,4)','(1,3,4)(2,0)', None])
sage: c.genus()
1

is_connected()
Test of connectedness.

EXAMPLES:

sage: c = Constellation(['(0,1)(2)', None, '(0,1)(2)'], connected=False)
sage: c.is_connected()
False
sage: c = Constellation(['(0,1,2)', None], connected=False)
sage: c.is_connected()
True

is_isomorphic(other, return_map=False)
Test of isomorphism.

Return True if the constellations are isomorphic (i.e. related by a common conjugacy) and return the permutation that conjugate the two permutations if return_map is True in such a way that self.relabel(m) == other.

ALGORITHM:
uses canonical labels obtained from the method relabel().

EXAMPLES:

sage: c = Constellation([[1,0,2],[2,1,0],[0,2,1]], mutable=False)
sage: c.is_mutable()
False

is_mutable()
Return False as self is immutable.

EXAMPLES:

sage: c = Constellation([[0,2,1],[2,1,0],[1,2,0]], mutable=False)
sage: c.is_mutable()
False

length()
Return the number of permutations.

EXAMPLES:

sage: c = Constellation(['(0,1)','(0,2)','(0,3)',None])
sage: c.length()
4
sage: c = Constellation([(0,1,3),None,(1,2)])
sage: c.length()
3

**mutable_copy()**

Return a mutable copy of `self`.

EXAMPLES:

```python
sage: c = Constellation(([0,2,1],[2,1,0],[1,2,0]), mutable=False)
sage: d = c.mutable_copy()
sage: d.is_mutable()
True
```

**passport**(i=None)

Return the profile of `self`.

The profile of a constellation is the tuple of partitions associated to the conjugacy classes of the permutations of the constellation.

This is also called the passport.

EXAMPLES:

```python
c = Constellation([(0,1,2)(3,4),(0,3),None]); c
Constellation of length 3 and degree 5
```

```python
c = Constellation([(0,1,2)(3,4),(0,3),None])
c.profile()
([3, 2], [2, 1, 1, 1], [5])
```

**profile**(i=None)

Return the profile of `self`.

The profile of a constellation is the tuple of partitions associated to the conjugacy classes of the permutations of the constellation.

This is also called the passport.

EXAMPLES:

```python
c = Constellation([(0,1,2)(3,4),(0,3),None])
c.profile()
([3, 2], [2, 1, 1, 1], [5])
```

**relabel**(perm=None, return_map=False)

Relabel `self`.

If `perm` is provided then relabel with respect to `perm`. Otherwise use canonical labels. In that case, if `return_map` is provided, the return also the map used for canonical labels.

Algorithm:

the cycle for g(0) are adjacent and the cycle are joined with respect to the other permutations. The minimum is taken for all possible renumerotations.

EXAMPLES:

```python
c = Constellation([(0,1)(2,3,4),(1,4),None]); c
Constellation of length 3 and degree 5
```
g₀ (0,1)(2,3,4)  
g₁ (0)(1,4)(2)(3)  
g₂ (0,1,3,2,4)

`sage: c2 = c.relabel(); c2`  
Constellation of length 3 and degree 5  
g₀ (0,1)(2,3,4)  
g₁ (0)(1,2)(3)(4)  
g₂ (0,1,3,2,4)

The map returned when the option `return_map` is set to `True` can be used to set the relabelling:

```
sage: c3, perm = c.relabel(return_map=True)
sage: c3 == c2 and c3 == c.relabel(perm=perm)
True
```

```
sage: S₅ = SymmetricGroup(range(5))
sage: d = c.relabel(S₅([4,3,1,0,2])); d
Constellation of length 3 and degree 5  
g₀ (0,2,1)(3,4)  
g₁ (0)(1)(2,3)(4)  
g₂ (0,1,2,4,3)
```

```
sage: d.is_isomorphic(c)
True
```

We check that after a random relabelling the new constellation is isomorphic to the initial one:

```
sage: c = Constellation(['(0,1)(2,3,4)','(1,4)',None])
sage: p = S₅.random_element()
sage: cc = c.relabel(perm=p)
sage: cc.is_isomorphic(c)
True
```

Check that it works for “non standard” labels:

```
sage: c = Constellation([(a,b),(c,d,e),(b,d)], None])
```

```
set_immutable()

Do nothing, as self is already immutable.

EXAMPLES:

```
sage: c = Constellation([(0,2,1),(2,1,0),(1,2,0)], mutable=False)
sage: c.set_immutable()
sage: c.is_mutable()
sage: c.is_mutable()
```

```
switch(i,j₀,j₁)

Perform the multiplication by the transposition \((j₀,j₁)\) between the permutations \(gᵢ\) and \(gᵢ+1\).```
The modification is local in the sense that it modifies $g_i$ and $g_{i+1}$ but does not modify the product $g_ig_{i+1}$.

The new constellation is

$$(g_0, \ldots, g_{i-1}, g_i(j_0j_1), (j_0j_1)g_{i+1}, g_{i+2}, \ldots, g_k)$$

EXAMPLES:

```python
sage: c = Constellation(['(0,1)(2,3,4)', '(1,4)', None], mutable=True); c
Constellation of length 3 and degree 5
g0 (0,1)(2,3,4)
g1 (0)(1,4)(2)(3)
g2 (0,1,3,2,4)
sage: c.is_mutable()
True
sage: c.switch(1,2,3); c
Constellation of length 3 and degree 5
g0 (0,1)(2,3,4)
g1 (0)(1,4)(2,3)
g2 (0,1,3,4)(2)
sage: c._check()
sage: c.switch(2,1,3); c
Constellation of length 3 and degree 5
g0 (0,1,4,2,3)
g1 (0)(1,4)(2,3)
g2 (0,3,4)(1)(2)
sage: c._check()
sage: c.switch(0,0,1); c
Constellation of length 3 and degree 5
g0 (0)(1,4,2,3)
g1 (0,4,1)(2,3)
g2 (0,3,4)(1)(2)
sage: c._check()
```

sage.combinat.constellation.Constellations(*data, **options)

Build a set of constellations.

INPUT:

- profile – an optional profile
- length – an optional length
- degree – an optional degree
- connected – an optional boolean

EXAMPLES:

```python
sage: Constellations(4,2)
Connected constellations of length 4 and degree 2 on {1, 2}
sage: Constellations([[3,2,1],[3,3],[3,3]])
Connected constellations with profile ([3, 2, 1], [3, 3], [3, 3]) on {1, 2, 3, 4, 5, \ldots 6}
```

class sage.combinat.constellation.Constellations_ld(length, degree, sym=None, connected=True)

Bases: UniqueRepresentation, Parent
Constellations of given length and degree.

EXAMPLES:

```python
sage: C = Constellations(2,3); C
Connected constellations of length 2 and degree 3 on {1, 2, 3}
sage: C([[2,3,1],[3,1,2]])
Constellation of length 2 and degree 3
g0 (1,2,3)
g1 (1,3,2)
sage: C.cardinality()
2
sage: Constellations(2,3,connected=False).cardinality()
6
```

**Element**

alias of `Constellation_class`

**braid_group_action()**

Return a list of graphs that corresponds to the braid group action on `self` up to isomorphism.

**OUTPUT:**

- list of graphs

**EXAMPLES:**

```python
sage: C = Constellations(3,3)
sage: C.braid_group_action()
[Looped multi-digraph on 3 vertices,
 Looping multi-digraph on 1 vertex,
 Looping multi-digraph on 3 vertices]
```

**braid_group_orbits()**

Return the orbits under the action of braid group.

**EXAMPLES:**

```python
sage: C = Constellations(3,3)
sage: O = C.braid_group_orbits()
sage: len(O)
3
sage: [x.profile() for x in O[0]]
[(\[1, 1, 1\], \[3\], \[3\]), (\[3\], \[1, 1, 1\], \[3\]), (\[3\], \[3\], \[1, 1, 1\])]
sage: [x.profile() for x in O[1]]
[(\[3\], \[3\], \[3\])]
sage: [x.profile() for x in O[2]]
[(\[2, 1\], \[2, 1\], \[3\]), (\[2, 1\], \[3\], \[2, 1\]), (\[3\], \[2, 1\], \[2, 1\])]
```

**is_empty()**

Return whether this set of constellations is empty.

**EXAMPLES:**

```python
sage: Constellations(2, 3).is_empty()
False
```
sage: Constellations(1, 2).is_empty()
True
sage: Constellations(1, 2, connected=False).is_empty()
False

**random_element**(mutable=False)

Return a random element.

This is found by trial and rejection, starting from a random list of permutations.

**EXAMPLES:**

```
sage: const = Constellations(3,3)
sage: const.random_element()
Constellation of length 3 and degree 3
...
...
...
sage: c = const.random_element()
sage: c.degree() == 3 and c.length() == 3
True
```

**class** sage.combinat.constellationConstellations_p(profile, domain=None, connected=True)

Bases: UniqueRepresentation, Parent

Constellations with fixed profile.

**EXAMPLES:**

```
sage: C = Constellations([[3,1],[3,1],[2,2]]); C
Connected constellations with profile ([3, 1], [3, 1], [2, 2]) on {1, 2, 3, 4}
sage: C.cardinality()
24
sage: C.first()
Constellation of length 3 and degree 4
g0 (1)(2,3,4)
g1 (1,2,3)(4)
g2 (1,2)(3,4)
sage: C.last()
Constellation of length 3 and degree 4
g0 (1,4,3)(2)
g1 (1,4,2)(3)
g2 (1,2)(3,4)
```

Note that the cardinality can also be computed using characters of the symmetric group (Frobenius formula):

```
sage: P = Partitions(4)
sage: p1 = Partition([3,1])
sage: p2 = Partition([3,1])
sage: p3 = Partition([2,2])
sage: i1 = P.cardinality() - P.rank(p1) - 1
sage: i2 = P.cardinality() - P.rank(p2) - 1
sage: i3 = P.cardinality() - P.rank(p3) - 1
sage: s = 0
```

(continues on next page)
The number obtained above is up to isomorphism. And we can check:

```python
sage: len(C.isomorphism_representatives())
1
```

**isomorphism_representatives()**

Return a set of isomorphism representative of self.

**EXAMPLES:**

```python
sage: C = Constellations([[5], [4,1], [3,2]])
sage: C.cardinality()
240
sage: ir = sorted(C.isomorphism_representatives())
sage: len(ir)
2
sage: ir[0]
Constellation of length 3 and degree 5
  g0 (1,2,3,4,5)
  g1 (1)(2,3,4,5)
  g2 (1,5,3)(2,4)
sage: ir[1]
Constellation of length 3 and degree 5
  g0 (1,2,3,4,5)
  g1 (1)(2,5,3,4)
  g2 (1,5)(2,3,4)
```

**sage.combinat.constellation.perm_conjugate(p, s)**

Return the conjugate of the permutation p by the permutation s.

**INPUT:**

two permutations of {0,...n-1} given by lists of values

**OUTPUT:**

a permutation of {0,...n-1} given by a list of values

**EXAMPLES:**

```python
sage: from sage.combinat.constellation import perm_conjugate
sage: perm_conjugate([3,1,2,0], [3,2,0,1])
[0, 3, 2, 1]
```

**sage.combinat.constellation.perm_invert(p)**

Return the inverse of the permutation p.
INPUT:

a permutation of \{0,...,n-1\} given by a list of values

OUTPUT:

a permutation of \{0,...,n-1\} given by a list of values

EXAMPLES:

```
sage: from sage.combinat.constellation import perm_invert
sage: perm_invert([3,2,0,1])
[2, 3, 1, 0]
```

```
sage.combinat.constellation.perm_sym_domain(g)
```

Return the domain of a single permutation (before initialization).

EXAMPLES:

```
sage: from sage.combinat.constellation import perm_sym_domain
sage: perm_sym_domain([1,2,3,4])
{1, 2, 3, 4}
sage: perm_sym_domain(((1,2),(0,4)))
{0, 1, 2, 4}
sage: sorted(perm_sym_domain((1,2,0,5)))
[0, 1, 2, 5]
```

```
sage.combinat.constellation.perms_are_connected(g, n)
```

Checks that the action of the generated group is transitive

INPUT:

- a list of permutations of \{0, n - 1\} (in a SymmetricGroup)
- an integer \( n \)

EXAMPLES:

```
sage: from sage.combinat.constellation import perms_are_connected
sage: S = SymmetricGroup(range(3))
sage: perms_are_connected([S([0,1,2]),S([0,2,1])],3)
False
sage: perms_are_connected([S([0,1,2]),S([1,2,0])],3)
True
```

```
sage.combinat.constellation.perms_canonical_labels(p, e=None)
```

Relabel a list with a common conjugation such that two conjugated lists are relabeled the same way.

INPUT:

- \( p \) is a list of at least 2 permutations
- \( e \) is None or a list of integer in the domain of the permutations. If provided, then the renumbering algorithm is only performed from the elements of \( e \).

OUTPUT:

- a pair made of a list of permutations (as a list of lists) and a list that corresponds to the conjugacy used.

EXAMPLES:
```python
sage: from sage.combinat.constellation import perms_canonical_labels
sage: l0 = [[2,0,3,1], [3,1,2,0], [0,2,1,3]]

sage: l, m = perms_canonical_labels(l0); l
[[1, 2, 3, 0], [0, 3, 2, 1], [2, 1, 0, 3]]

sage: S = SymmetricGroup(range(4))

sage: [-S(m) * S(u) * S(m)
for u in l0] == list(map(S, l))
True

sage: perms_canonical_labels([])
Traceback (most recent call last):
...
ValueError: input must have length >= 2
```

```python
sage.combinat.constellation.perms_canonical_labels_from(x, y, j0, verbose=False)
Return canonical labels for x, y that starts at j0

**Warning:** The group generated by x and the elements of y should be transitive.

**INPUT:**
- x – list - a permutation of [0,...,n] as a list
- y – list of permutations of [0,...,n] as a list of lists
- j0 – an index in [0,...,n]

**OUTPUT:**
mapping: a permutation that specify the new labels

**EXAMPLES:**
```
sage: from sage.combinat.constellation import perms_canonical_labels_from
sage: perms_canonical_labels_from([0,1,2],[[1,2,0]], 0)
[0, 1, 2]
sage: perms_canonical_labels_from([1,0,2], [[2,0,1]], 0)
[0, 1, 2]
sage: perms_canonical_labels_from([1,0,2], [[2,0,1]], 1)
[1, 0, 2]
sage: perms_canonical_labels_from([1,0,2], [[2,0,1]], 2)
[2, 1, 0]
```
```
```
sage.combinat.constellation.perms_sym_init(g, sym=None)
Initialize a list of permutations (in the same symmetric group).

**OUTPUT:**
- sym – a symmetric group
- gg – a list of permutations

**EXAMPLES:**
```
sage: from sage.combinat.constellation import perms_sym_init
sage: S, g = perms_sym_init([[0,2,1,3], [1,3,2,0]])
```
```
sage: S.domain()
{0, 1, 2, 3}
sage: g
[(1,2), (0,1,3)]

sage: S, g = perms_sym_init(['(2,1)', '(0,3)'])

sage: S.domain()
{0, 1, 2, 3}
sage: g
[(1,2), (0,3)]

sage: S, g = perms_sym_init([(1,0), (2,1)])

sage: S.domain()
{0, 1, 2}
sage: g
[(0,1), (1,2)]

sage: S, g = perms_sym_init([(1,0)(2,3), (0,1,4)])

sage: S.domain()
{0, 1, 2, 3, 4}
sage: g
[(0,1)(2,3), (0,1,4)]

5.1.33 Cores

A $k$-core is a partition from which no rim hook of size $k$ can be removed. Alternatively, a $k$-core is an integer partition such that the Ferrers diagram for the partition contains no cells with a hook of size (a multiple of) $k$.

Authors:
• Anne Schilling and Mike Zabrocki (2011): initial version
• Travis Scrimshaw (2012): Added latex output for Core class

class sage.combinat.core.Core(parent, core)

Bases: CombinatorialElement

A $k$-core is an integer partition from which no rim hook of size $k$ can be removed.

EXAMPLES:

sage: c = Core([2,1],4); c
[2, 1]
sage: c = Core([3,1],4); c
Traceback (most recent call last):
...
ValueError: [3, 1] is not a 4-core

affine_symmetric_group_action(w, transposition=False)

Return the (left) action of the affine symmetric group on self.

INPUT:
• $w$ is a tuple of integers $[w_1, \ldots, w_m]$ with $0 \leq w_j < k$. If transposition is set to be True, then $w = [w_0, w_1]$ is interpreted as a transposition $t_{w_0, w_1}$ (see _transposition_to_reduced_word()).
The output is the (left) action of the product of the corresponding simple transpositions on \( \text{self} \), that is \( s_{w_1} \cdots s_{w_n}(\text{self}) \). See \text{affine_symmetric_group_simple_action()}. 

EXAMPLES:

```python
sage: c = Core([4,2],3)
sage: c.affine_symmetric_group_action([0,1,0,2,1])
[8, 6, 4, 2]
sage: c.affine_symmetric_group_action([0,2], transposition=True)
[4, 2, 1, 1]
sage: c = Core([11,8,5,5,3,3,1,1,1],4)
sage: c.affine_symmetric_group_action([2,5],transposition=True)
[11, 8, 7, 6, 5, 4, 3, 2, 1]
```

\text{affine_symmetric_group_simple_action}(i)

Return the action of the simple transposition \( s_i \) of the affine symmetric group on \( \text{self} \).

This gives the action of the affine symmetric group of type \( A_{k-1}^{(1)} \) on the \( k \)-core \( \text{self} \). If \( \text{self} \) has outside (resp. inside) corners of content \( i \) modulo \( k \), then these corners are added (resp. removed). Otherwise the action is trivial.

EXAMPLES:

```python
sage: c = Core([4,2],3)
sage: c.affine_symmetric_group_simple_action(0)
[3, 1]
sage: c.affine_symmetric_group_simple_action(1)
[5, 3, 1]
sage: c.affine_symmetric_group_simple_action(2)
[4, 2]
```

This action corresponds to the left action by the \( i \)-th simple reflection in the affine symmetric group:

```python
sage: c = Core([4,2],3)
sage: W = c.to_grassmannian().parent()
sage: i = 0
sage: c.affine_symmetric_group_simple_action(i).to_grassmannian() == W.simple_˓→reflection(i)*c.to_grassmannian()
True
sage: i = 1
sage: c.affine_symmetric_group_simple_action(i).to_grassmannian() == W.simple_˓→reflection(i)*c.to_grassmannian()
True
```

\text{contains}(\text{other})

Checks whether \( \text{self} \) contains \( \text{other} \).

INPUT:

- other – another \( k \)-core or a list

OUTPUT: a boolean

This returns True if the Ferrers diagram of \( \text{self} \) contains the Ferrers diagram of \( \text{other} \).

EXAMPLES:
sage: c = Core([4,2],3)
sage: x = Core([4,2,2,1,1],3)
sage: x.contains(c)
True
sage: c.contains(x)
False

k()
Return $k$ of the $k$-core self.

EXAMPLES:

sage: c = Core([2,1],4)
sage: c.k()
4

length()  
Return the length of self.

The length of a $k$-core is the size of the corresponding $(k - 1)$-bounded partition which agrees with the length of the corresponding Grassmannian element, see to_grassmannian().

EXAMPLES:

sage: c = Core([4,2],3); c.length()
4
sage: c.to_grassmannian().length()
4
sage: Core([9,5,3,2,1,1], 5).length()
13

size()
Return the size of self as a partition.

EXAMPLES:

sage: Core([2,1],4).size()
3
sage: Core([4,2],3).size()
6

strong_covers()
Return a list of all elements that cover self in strong order.

EXAMPLES:

sage: c = Core([1],3)
sage: c.strong_covers()
[[[2], [1, 1]]

sage: c = Core([4,2],3)
sage: c.strong_covers()
[[5, 3, 1], [4, 2, 1, 1]]
**strong_down_list()**

Return a list of all elements that are covered by `self` in strong order.

**EXAMPLES:**

```
sage: c = Core([1],3)
sage: c.strong_down_list()
[[]]
sage: c = Core([5,3,1],3)
sage: c.strong_down_list()
[[4, 2], [3, 1, 1]]
```

**strong_le(other)**

Strong order (Bruhat) comparison on cores.

**INPUT:**

- `other` – another \(k\)-core

**OUTPUT:** a boolean

This returns whether `self` \(\leq\) `other` in Bruhat (or strong) order.

**EXAMPLES:**

```
sage: c = Core([4,2],3)
sage: x = Core([4,2,2,1,1],3)
sage: c.strong_le(x)
True
sage: c.strong_le([4,2,2,1,1])
True
sage: x = Core([4,1],4)
sage: c.strong_le(x)
Traceback (most recent call last):
  ...
ValueError: the two cores do not have the same k
```

**to_bounded_partition()**

Bijection between \(k\)-cores and \((k-1)\)-bounded partitions.

This maps the \(k\)-core `self` to the corresponding \((k-1)\)-bounded partition. This bijection is achieved by deleting all cells in `self` of hook length greater than \(k\).

**EXAMPLES:**

```
sage: gamma = Core([9,5,3,2,1,1], 5)
sage: gamma.to_bounded_partition()
[4, 3, 2, 2, 1, 1]
```

**to_grassmannian()**

Bijection between \(k\)-cores and Grassmannian elements in the affine Weyl group of type \(A^{(1)}_{k-1}\).

For further details, see the documentation of the method `from_kbounded_to_reduced_word()` and `from_kbounded_to_grassmannian()`.

**EXAMPLES:**
```python
sage: c = Core([3,1,1],3)
sage: w = c.to_grassmannian(); w
[-1  1  1]
[-2  2  1]
[-2  1  2]
sage: c.parent()
3-Cores of length 4
sage: w.parent()
Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root space)

sage: c = Core([],3)
sage: c.to_grassmannian()
[1 0 0]
[0 1 0]
[0 0 1]
```

**to_partition()**

Turn the core self into the partition identical to self.

**EXAMPLES:**

```python
sage: mu = Core([2,1,1],3)
sage: mu.to_partition()
[2, 1, 1]
```

**weak_covers()**

Return a list of all elements that cover self in weak order.

**EXAMPLES:**

```python
sage: c = Core([1],3)
sage: c.weak_covers()
[[1, 1], [2]]
sage: c = Core([4,2],3)
sage: c.weak_covers()
[[5, 3, 1]]
```

**weak_le(other)**

Weak order comparison on cores.

**INPUT:**

- other – another \(k\)-core

**OUTPUT:** a boolean

This returns whether self \(\leq\) other in weak order.

**EXAMPLES:**

```python
sage: c = Core([4,2],3)
sage: x = Core([5,3,1],3)
sage: c.weak_le(x)
True
sage: c.weak_le([5,3,1])
```

(continues on next page)
True

```
sage: x = Core([4,2,2,1,1],3)
sage: c.weak_le(x)
False
```

```
sage: x = Core([5,3,1],6)
sage: c.weak_le(x)
Traceback (most recent call last):
...
ValueError: the two cores do not have the same k
```
from_partition(part)

Converts the partition part into a core (as the identity map).
This is the inverse method to to_partition().

EXAMPLES:

```
sage: C = Cores(3,4)
sage: c = C.from_partition([4,2]); c
[4, 2]
sage: mu = Partition([2,1,1])
sage: C = Cores(3,3)
sage: C.from_partition(mu).to_partition() == mu
True
sage: mu = Partition([])
sage: C = Cores(3,0)
sage: C.from_partition(mu).to_partition() == mu
True
```

list()

Return the list of all k-cores of length n.

EXAMPLES:

```
sage: C = Cores(3,4)
sage: C.list()
[[[4, 2], [3, 1, 1], [2, 2, 1, 1]]
```

class sage.combinat.core.Cores_size(k, n)

Bases: UniqueRepresentation, Parent

The class of k-cores of size n.

Element

alias of Core

from_partition(part)

Convert the partition part into a core (as the identity map).
This is the inverse method to to_partition().

EXAMPLES:

```
sage: C = Cores(3,size=4)
sage: c = C.from_partition([2,1,1]); c
[2, 1, 1]
sage: mu = Partition([2,1,1])
sage: C = Cores(3,size=4)
sage: C.from_partition(mu).to_partition() == mu
True
sage: mu = Partition([])
sage: C = Cores(3,size=0)
sage: C.from_partition(mu).to_partition() == mu
True
```
sage: C.from_partition(mu).to_partition() == mu
True

list()

Return the list of all $k$-cores of size $n$.

EXAMPLES:

sage: C = Cores(3, size = 4)
sage: C.list()
[[3, 1], [2, 1, 1]]

5.1.34 Counting

- The On-Line Encyclopedia of Integer Sequences (OEIS)
- Functions that compute some of the sequences in Sloane’s tables
- Compute Bell and Uppuluri-Carpenter numbers
- $q$-Analogues, $q$-Bernoulli Numbers and Polynomials
- Binary Recurrence Sequences
- C-Finite Sequences
- Combinatorial Functions

Todo: Mention sage/combinat/degree_sequences?

5.1.35 Affine Crystals

class sage.combinat.crystals.affine.AffineCrystalFromClassical(cartan_type, classical_crystal, category=None)

Bases: UniqueRepresentation, Parent

This abstract class can be used for affine crystals that are constructed from a classical crystal. The zero arrows can be implemented using different methods (for example using a Dynkin diagram automorphisms or virtual crystals).

This is a helper class, mostly used to implement Kirillov-Reshetikhin crystals (see: KirillovReshetikhinCrystal()).

For general information about crystals see sage.combinat.crystals.

INPUT:

- cartan_type – the Cartan type of the resulting affine crystal
- classical_crystal – instance of a classical crystal

EXAMPLES:
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: A.list()
[[[1]], [[2]], [[3]]]
sage: A.cartan_type()
['A', 2, 1]
sage: A.index_set()
(0, 1, 2)
sage: b = A(rows=[[1]])
sage: b.weight()
-Lambda[0] + Lambda[1]
sage: b.classical_weight()
(1, 0, 0)
sage: [x.s(0) for x in A.list()]
[[[3]], [[2]], [[1]]]
sage: [x.s(1) for x in A.list()]
[[[2]], [[1]], [[3]]]

Element
alias of AffineCrystalFromClassicalElement
cardinality()
Return the cardinality of self.
EXAMPLES:

sage: C = crystals.Tableaux(['A',3],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',3,1],C,pr,pr_inverse,1)
sage: A.cardinality() == C.cardinality()
True

lift(affine_elt)
Lift an affine crystal element to the corresponding classical crystal element.
EXAMPLES:

sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: A.lift(b)
[[1]]
sage: A.lift(b).parent()
The crystal of tableaux of type ['A', 2] and shape(s) [[1]]

retract(classical_elt)
Transform a classical crystal element to the corresponding affine crystal element.
class sage.combinat.crystals.affine.AffineCrystalFromClassicalAndPromotion:

crystals that are constructed from a classical crystal and a Dynkin diagram automorphism \( \sigma \). In type \( A_n \), the Dynkin diagram automorphism is \( i \rightarrow i + 1 \) (mod \( n \)) + 1 and the corresponding map on the crystal is the promotion operation \( \text{pr} \) on tableaux. The affine crystal operators are given by \( f_0 = \text{pr}^{-1} f_{\sigma(0)} \text{pr} \).

INPUT:

- \( \text{cartan_type} \) – the Cartan type of the resulting affine crystal
- \( \text{classical_crystal} \) – instance of a classical crystal
- \( \text{automorphism, inverse_automorphism} \) – a function on the elements of the classical_crystal
- \( \text{dynkin_node} \) – an integer specifying the classical node in the image of the zero node under the automorphism \( \sigma \)

EXAMPLES:

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: A.list()
[[[1]], [[2]], [[3]]]
sage: A.cartan_type()
['A', 2, 1]
sage: A.index_set()
(0, 1, 2)
sage: b = A(rows=[[1]])
sage: b.weight()
-\Lambda[0] + \Lambda[1]
sage: b.classical_weight()
(1, 0, 0)
sage: [x.s(0) for x in A.list()]
(continues on next page)
```
Element
alias of `AffineCrystalFromClassicalAndPromotionElement`

automorphism(x)
Give the analogue of the affine Dynkin diagram automorphism on the level of crystals.

EXAMPLES:

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A.list()[0]
sage: A.automorphism(b)
[[2]]
```

inverse_automorphism(x)
Give the analogue of the inverse of the affine Dynkin diagram automorphism on the level of crystals.

EXAMPLES:

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A.list()[0]
sage: A.inverse_automorphism(b)
[[3]]
```

class `sage.combinat.crystals.affine.AffineCrystalFromClassicalAndPromotionElement`

Bases: `AffineCrystalFromClassicalElement`

Elements of crystals that are constructed from a classical crystal and a Dynkin diagram automorphism. In type $A$, the automorphism is the promotion operation on tableaux.

This class is not instantiated directly but rather `__call__`-ed from `AffineCrystalFromClassicalAndPromotion`. The syntax of this is governed by the (classical) crystal.

Since this class inherits from `AffineCrystalFromClassicalElement`, the methods that need to be implemented are `e0()`, `f0()` and possibly `epsilon0()` and `phi0()` if more efficient algorithms exist.

EXAMPLES:

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
```

(continues on next page)
sage: b = A(rows=[[1]])
sage: b._repr_()
'[[1]]'

\textbf{e}0() Implement \(e_0\) using the automorphism as 
\[ e_0 = \text{pr}^{-1} e_{\text{dynkin},ode} \text{pr} \]

\textbf{EXAMPLES:}

\begin{verbatim}
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A(rows=[[1]])
sage: b.e0()
[[3]]
\end{verbatim}

\textbf{epsilon0}() Implement \(\epsilon_{0}\) using the automorphism.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: [x.epsilon0() for x in A.list()]
[1, 0, 0]
\end{verbatim}

\textbf{f}0() Implement \(f_0\) using the automorphism as 
\[ f_0 = \text{pr}^{-1} f_{\text{dynkin},ode} \text{pr} \]

\textbf{EXAMPLES:}

\begin{verbatim}
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A(rows=[[3]])
sage: b.f0()
[[4]]
\end{verbatim}

\textbf{phi0}() Implement \(\phi_{0}\) using the automorphism.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
\end{verbatim}

(continues on next page)
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: [x.phi0() for x in A.list()]
[0, 0, 1]

class sage.combinat.crystals.affine.AffineCrystalFromClassicalElement

Bases: ElementWrapper

Elements of crystals that are constructed from a classical crystal.

The elements inherit many of their methods from the classical crystal using lift and retract. This class is not instantiated directly but rather __call__-ed from AffineCrystalFromClassical. The syntax of this is governed by the (classical) crystal.

EXAMPLES:

sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A(rows=[[1]])
sage: b._repr_()
'[[1]]'

classical_weight()

Return the classical weight corresponding to self.

EXAMPLES:

sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A(rows=[[1]])
sage: b.classical_weight()
(1, 0, 0)

e(i)

Return the action of $e_i$ on self.

EXAMPLES:

sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A(rows=[[1]])
sage: b.e(0)
[[3]]
sage: b.e(1)

e0()

Assumes that $e_0$ is implemented separately.
**epsilon(i)**

Return the maximal time the crystal operator $e_i$ can be applied to self.

**EXAMPLES:**

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: [x.epsilon(0) for x in A.list()]
[1, 0, 0]
sage: [x.epsilon(1) for x in A.list()]
[0, 1, 0]
```

**epsilon0()**

Uses $\varepsilon_0$ from the super class, but should be implemented if a faster implementation exists.

**EXAMPLES:**

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: [x.epsilon0() for x in A.list()]
[1, 0, 0]
```

**f(i)**

Return the action of $f_i$ on self.

**EXAMPLES:**

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A(rows=[[3]])
sage: b.f(0)
[[1]]
sage: b.f(2)
```

**f0()**

Assumes that $f_0$ is implemented separately.

**lift()**

Lift an affine crystal element to the corresponding classical crystal element.

**EXAMPLES:**

```python
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: pr_inverse = attrcall("promotion_inverse")
sage: pr_inverse = attrcall("promotion_inverse")
```
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: b = A.list()[0]
sage: b.lift()
[[1]]
\[
\text{The crystal of tableaux of type ['A', 2] and shape(s) [[1]]}
\]

phi(i)

Returns the maximal time the crystal operator \( f_i \) can be applied to self.

EXAMPLES:

```
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: [x.phi(0) for x in A.list()]
[0, 0, 1]
sage: [x.phi(1) for x in A.list()]
[1, 0, 0]
```

phi0()

Uses \( \phi_0 \) from the super class, but should be implemented if a faster implementation exists.

EXAMPLES:

```
sage: n = 2
sage: C = crystals.Tableaux(['A',n],shape=[1])
sage: pr = attrcall("promotion")
sage: pr_inverse = attrcall("promotion_inverse")
sage: A = crystals.AffineFromClassicalAndPromotion(['A',n,1],C,pr,pr_inverse,1)
sage: [x.phi0() for x in A.list()]
[0, 0, 1]
```

pp()

Method for pretty printing.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['D',3,2],1,1)
sage: t=K(rows=[[1]])
sage: t.pp()
```

1
5.1.36 Affine factorization crystal of type $A$

**class** `sage.combinat.crystals.affine_factorization.AffineFactorizationCrystal(w, n, x=None)`

**Bases:** `UniqueRepresentation`, `Parent`

The crystal on affine factorizations with a cut-point, as introduced by [MS2015].

**INPUT:**

- $w$ – an element in an (affine) Weyl group or a skew shape of $k$-bounded partitions (if $k$ was specified)
- $n$ – the number of factors in the factorization
- $x$ – (default: `None`) the cut point; if not specified it is determined as the minimal missing residue in $w$
- $k$ – (default: `None`) positive integer, specifies that $w$ is $k$-bounded or a $k+1$-core when specified

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3,1], prefix='s')
sage: w = W.from_reduced_word([2,3,2,1])
sage: B = crystals.AffineFactorization(w,3); B
Crystal on affine factorizations of type A2 associated to s2*s3*s2*s1
sage: B.list()
[(1, s2, s3*s2*s1),
 (1, s3*s2, s3*s1),
 (1, s3*s2*s1, s3),
 (s3, s2, s3*s1),
 (s3, s2*s1, s3),
 (s3*s2*s1, 1, s3),
 (s3*s2*s1, s3, 1),
 (s3*s2, 1, s3*s1),
 (s3*s2, s3, s1),
 (s3*s2, s3*s1, 1),
 (s2, 1, s3*s2*s1),
 (s2, s3, s2*s1),
 (s2, s3*s2, s1),
 (s2, s3*s2*s1, 1)]
```

We can also access the crystal by specifying a skew shape in terms of $k$-bounded partitions:

```python
sage: crystals.AffineFactorization([[3,1,1],[1]], 3, k=3)
Crystal on affine factorizations of type A2 associated to s2*s3*s2*s1
```

We can compute the highest weight elements:

```python
sage: hw = [w for w in B if w.is_highest_weight()]
sage: hw
[(1, s2, s3*s2*s1)]
sage: hw[0].weight()
(3, 1, 0)
```

And show that this crystal is isomorphic to the tableau model of the same weight:

```python
sage: C = crystals.Tableaux(['A',2],shape=[3,1])
sage: GC = C.digraph()
```

(continues on next page)
The crystal operators themselves move elements between adjacent factors:

```
sage: b = hw[0]; b
(1, s2, s3*s2*s1)
sage: b.f(1)
(1, s3*s2, s3*s1)
```

The cut point $x$ is not supposed to occur in the reduced words for $w$:

```
sage: B = crystals.AffineFactorization([[3,2],[2]],4,x=0,k=3)
Traceback (most recent call last):
...
ValueError: x cannot be in reduced word of s0*s3*s2
```

### class Element

Bases: `ElementWrapper`

#### bracketing(i)

Removes all bracketed letters between $i$-th and $i+1$-th entry.

**EXAMPLES:**

```
sage: B = crystals.AffineFactorization([[3,1],[1]], 3, k=3, x=4)
sage: W = B.w.parent()
sage: t = B((W.one(),W.from_reduced_word([3]),W.from_reduced_word([2,1])));
˓→t
(1, s3, s2*s1)
sage: t.bracketing(1)
[[3], [2, 1]]
```

#### e(i)

Return the action of $e_i$ on `self`.

**EXAMPLES:**

```
sage: B = crystals.AffineFactorization([[3,1],[1]], 4, k=3)
sage: W = B.w.parent()
sage: t = B((W.one(),W.one(),W.from_reduced_word([3]),W.from_reduced_word([2,1])));
˓→t
(1, 1, s3, s2*s1)
sage: t.e(1)
(1, 1, 1, s3*s2*s1)
```

#### f(i)

Return the action of $f_i$ on `self`.

**EXAMPLES:**

```
sage: B = crystals.AffineFactorization([[3,1],[1]], 4, k=3)
sage: W = B.w.parent()
```
```python
sage: t = B((W.one(),W.one(),W.from_reduced_word([3]),W.from_reduced_word([2,1]))); t
(1, 1, s3, s2*s1)
sage: t.f(2)
(1, s3, 1, s2*s1)
sage: t.f(1)
(1, 1, s3*s2, s1)
```

**to_tableau()**

Return the tableau representation of *self*.

Uses the recording tableau of a minor variation of Edelman-Greene insertion. See Theorem 4.11 in [MS2015].

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3,1], prefix='s')
sage: w = W.from_reduced_word([2,1,3,2])
sage: B = crystals.AffineFactorization(w,3)
sage: for x in B:
    ....:     x
    ....:     x.to_tableau().pp()
(1, s2*s1, s3*s2)
  1
  2
(s2, s1, s3*s2)
  1
  2
(s2, s3*s1, s2)
  1
  2
  3
(s2*s1, 1, s3*s2)
  1
  3
(s2*s1, s3, s2)
  1
  3
(s2*s1, s3*s2, 1)
  2
  3
2 2
3 3
```

**class** `sage.combinat.crystals.affine_factorization.FactorizationToTableaux**(parent, cartan_type=None, virtualization=None, scaling_factors=None)**

**Bases:** `CrystalMorphism`

**is_embedding()**

Return True as this is an isomorphism.

**EXAMPLES:**
Combinatorics, Release 10.0

```python
sage: W = WeylGroup(['A',3,1], prefix='s')
sage: w = W.from_reduced_word([2,1,3,2])
sage: B = crystals.AffineFactorization(w,3)
sage: phi = B._tableaux_isomorphism
sage: phi.is_isomorphism()
True
```

**is_isomorphism()**

Return True as this is an isomorphism.

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3,1], prefix='s')
sage: w = W.from_reduced_word([2,1,3,2])
```

**is_surjective()**

Return True as this is an isomorphism.

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3,1], prefix='s')
sage: w = W.from_reduced_word([2,1,3,2])
```

`sage.combinat.crystals.affine_factorization.affine_factorizations(w, l, weight=None)`

Return all factorizations of \( w \) into \( l \) factors or of weight \( \text{weight} \).

**INPUT:**

- \( w \) – an (affine) permutation or element of the (affine) Weyl group
- \( l \) – nonnegative integer
- \( \text{weight} \) – (default: None) tuple of nonnegative integers specifying the length of the factors

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3,1], prefix='s')
sage: w = W.from_reduced_word([3,2,3,1,0,1])
sage: from sage.combinat.crystals.affine_factorization import affine_factorizations
sage: affine_factorizations(w,4)
[[s2, s3, s0, s2*s1*s0],
 [s2, s3, s2*s0, s1*s0],
 [s2, s3, s2*s1*s0, s1],
 [s2, s3*s2, s0, s1*s0],
 [s2, s3*s2, s1*s0, s1],
 [s3*s2, s3, s0, s1*s0],
 [s3*s2, s3, s1*s0, s1],
 [s3*s2, s3, s1*s0, s1],
 (continues on next page)```
(continued from previous page)

```
[s3*s2, s3*s1, s0, s1],
[s3*s2*s1, s3, s0, s1]]

sage: W = WeylGroup(['A',2], prefix='s')
sage: w0 = W.long_element()
sage: affine_factorizations(w0,3)
[[1, s1, s2*s1],
 [1, s2*s1, s2],
 [s1, 1, s2*s1],
 [s1, s2, s1],
 [s1, s2*s1, 1],
 [s2, s1, s2],
 [s2*s1, 1, s2],
 [s2*s1, s2, 1]]

sage: affine_factorizations(w0,3,(0,1,2))
[[1, s1, s2*s1]]

sage: affine_factorizations(w0,3,(1,1,1))
[[s1, s2, s1], [s2, s1, s2]]

sage: W = WeylGroup(['A',3], prefix='s')
sage: w0 = W.long_element()
sage: affine_factorizations(w0,6,(1,1,1,1,1,1))  # long time
[[s1, s2, s1, s3, s2, s1],
 [s1, s2, s3, s1, s2, s1],
 [s1, s2, s3, s2, s2, s1],
 [s1, s2*s1, s3, s2, s1],
 [s1, s2*s1, s3, s2, s2],
 [s2, s1, s2, s3, s2, s1],
 [s2, s1, s3, s2, s1, s3],
 [s2, s1, s3, s2, s3, s1],
 [s2, s1, s3, s2, s3, s2],
 [s3, s1, s2, s1, s2, s3],
 [s3, s1, s2, s1, s3, s2],
 [s3, s1, s2, s2, s1, s3],
 [s3, s2, s1, s2, s3, s1],
 [s3, s2, s1, s3, s2, s3],
 [s3, s2, s2, s1, s2, s3],
 [s3, s2, s3, s1, s2, s3],
 [s3, s2, s3, s2, s1, s3],
 [s3, s2*s1, s3, s2*s1]]

sage: affine_factorizations(w0,6,(0,0,0,1,2,3))
[[1, 1, 1, s1, s2*s1, s3*s2*s1]]
```

### 5.1.37 Affinization Crystals

**class** `sage.combinat.crystals.affinization.AffinizationOfCrystal(B)`

**Bases:** `UniqueRepresentation`, `Parent`

An affinization of a crystal.

Let $\mathfrak{g}$ be a Kac-Moody algebra of affine type. The affinization of a finite $U'_q(\mathfrak{g})$-crystal $B$ is the (infinite) $U_q(\mathfrak{g})$-crystal with underlying set:

$$B^{\text{aff}} = \{ b(m) \mid b \in B, m \in \mathbb{Z} \}$$
and crystal structure determined by:

\[
\begin{align*}
    e_i(b(m)) &= \begin{cases} 
        (e_0 b)(m + 1) & i = 0, \\
        (e_i b)(m) & i \neq 0,
    \end{cases} \\
    f_i(b(m)) &= \begin{cases} 
        (f_0 b)(m - 1) & i = 0, \\
        (f_i b)(m) & i \neq 0,
    \end{cases} \\
    \text{wt}(b(m)) &= \text{wt}(b) + m\delta.
\end{align*}
\]

EXAMPLES:

We first construct a Kirillov-Reshetikhin crystal and then take its corresponding affinization:

```python
sage: K = crystals.KirillovReshetikhin(['A',2,1], 2, 2)
sage: A = K.affinization()
```

Next we construct an affinization crystal from a tensor product of KR crystals:

```python
sage: KT = crystals.TensorProductOfKirillovReshetikhinTableaux(['C',2,1], [[1,2],[2,1]])
sage: A = crystals.AffinizationOf(KT)
```

REFERENCES:

• [HK2002] Chapter 10

```python
class Element(parent, b, m):
    Bases: Element
    An element in an affinization crystal.

    e(i)
    Return the action of e_i on self.

    INPUT:
    • i – an element of the index set

    EXAMPLES:

    sage: A = crystals.KirillovReshetikhin(['A',2,1], 2,2).affinization()
    sage: mg = A.module_generators[0]
    sage: mg.e(0)
    [[1, 2], [2, 3]](1)
    sage: mg.e(1)
    sage: mg.e(0).e(1)
    [[1, 1], [2, 3]](1)
```

epsilon(i)

Return \( \epsilon_i \) of self.

INPUT:

• i – an element of the index set

EXAMPLES:

```python
sage: A = crystals.KirillovReshetikhin(['A',2,1], 2,2).affinization()
```
\begin{Verbatim}[commandchars=\][\{}]
sage: mg.epsilon(1)
0
\end{Verbatim}

**f(i)**
Return the action of \(f_i\) on self.

**INPUT:**
- \(i\) – an element of the index set

**EXAMPLES:**

\begin{Verbatim}[commandchars=\][\{}]
sage: A = crystals.KirillovReshetikhin(['A',2,1], 2,2).affinization()
sage: mg = A.module_generators[0]
sage: mg.f(2)
[[1, 1], [2, 3]](0)
sage: mg.f(2).f(2).f(0)
sage: mg.f_string([2,1,1])
sage: mg.f_string([2,1])
[[1, 2], [2, 3]](0)
sage: mg.f_string([2,1,0])
[[1, 1], [2, 2]](-1)
\end{Verbatim}

**phi(i)**
Return \(\varphi_i\) of self.

**INPUT:**
- \(i\) – an element of the index set

**EXAMPLES:**

\begin{Verbatim}[commandchars=\][\{}]
sage: A = crystals.KirillovReshetikhin(['A',2,1], 2,2).affinization()
sage: mg = A.module_generators[0]
sage: mg.phi(0)
0
sage: mg.phi(2)
2
\end{Verbatim}

**weight()**
Return the weight of self.

The weight \(wt\) of an element is:

\[
wt(b(m)) = wt(b) + m\delta,
\]

where \(\delta\) is the null root.

**EXAMPLES:**

\begin{Verbatim}[commandchars=\][\{}]
sage: A = crystals.KirillovReshetikhin(['A',2,1], 2,2).affinization()
sage: mg = A.module_generators[0]
sage: mg.weight()
-2*Lambda[0] + 2*Lambda[2]
sage: mg.e(0).weight()
sage: mg.e(0).e(0).weight()
2*Lambda[0] - 2*Lambda[1] + 2*delta
\end{Verbatim}
5.1.38 Alcove paths

AUTHORS:

- Brant Jones (2008): initial version
- Arthur Lubovsky (2013-03-07): rewritten to implement affine type
- Travis Scrimshaw (2016-06-23): implemented $B(\infty)$

Special thanks to: Nicolas Borie, Anne Schilling, Travis Scrimshaw, and Nicolas Thiéry.

class sage.combinat.crystals.alcove_path.CrystalOfAlcovePaths(starting_weight, highest_weight_crystal)

Bases: UniqueRepresentation, Parent

Crystal of alcove paths generated from a “straight-line” path to the negative of a given dominant weight.

INPUT:

- cartan_type – Cartan type of a finite or affine untwisted root system.
- weight – Dominant weight as a list of (integral) coefficients of the fundamental weights.
- highest_weight_crystal – (Default: True) If True returns the highest weight crystal. If False returns an object which is close to being isomorphic to the tensor product of Kirillov-Reshetikhin crystals of column shape in the following sense: We get all the vertices, but only some of the edges. We’ll call the included edges pseudo-Demazure. They are all non-zero edges and the 0-edges not at the end of a 0-string of edges, i.e. not those with $f_0(b) = b'$ with $\varphi_0(b) = 1$. (Whereas Demazure 0-edges are those that are not at the beginning of a zero string.) In this case the weight $[c_1, c_2, \ldots, c_k]$ represents $\sum_{i=1}^k c_i \omega_i$.

Note: If highest_weight_crystal = False, since we do not get the full crystal, TestSuite will fail on the Stembridge axioms.

See also:

- Crystals

EXAMPLES:

The following example appears in Figure 2 of [LP2008]:

```python
sage: C = crystals.AlcovePaths(['G',2],[0,1])
sage: G = C.digraph()
sage: GG = DiGraph({
    ....: (): {(): 2 },
    ....: (): {(0,8) : 1 },
    ....: (): {(0,1,7) : 2 },
    ....: (): {(0,1,2,9) : 1 },
    ....: (): {(0,1,2,3,4) : 2 },
    ....: (): {(0,1,2,3) : 1 },
    ....: (): {(0,1,2,6) : 1 },
    ....: (): {(0,1,2) : 2 },
    ....: (): {(0,1,2,9) : 2 },
    ....: (): {(0,1,5) : 1, (0,1,7,9) : 2 },
    ....: (): {(0,5,7) : 1 },
    ....: (): {(0,5,7,9) : 1 },
})
(continues on next page)
```

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Alcove path crystals are a discrete version of Littelmann paths. We verify that the alcove path crystal is isomorphic to the LS path crystal:

```python
sage: C1 = crystals.AlcovePaths(['C', 3], [2, 1, 0])
sage: g1 = C1.digraph()  # long time
sage: C2 = crystals.LSPaths(['C', 3], [2, 1, 0])
sage: g2 = C2.digraph()  # long time
sage: g1.is_isomorphic(g2, edge_labels=True)  # long time
True
```

The preferred initialization method is via explicit weights rather than a Cartan type and the coefficients of the fundamental weights:

```python
sage: R = RootSystem(['C', 3])
sage: P = R.weight_lattice()
sage: La = P.fundamental_weights()
Highest weight crystal of alcove paths of type ['C', 3] and weight 2*Lambda[1] +\Lambda[2]
```

We now explain the data structure:

```python
sage: C = crystals.AlcovePaths(['A', 2], [2, 0]) ; C
Highest weight crystal of alcove paths of type ['A', 2] and weight 2*Lambda[1]
sage: C._R.lambda_chain()
[\text{[(alpha[1], 0), (alpha[1] + alpha[2], 0), (alpha[1], 1), (alpha[1] + alpha[2], 1)\}}]
```

The previous list gives the initial “straight line” path from the fundamental alcove $A_o$ to its translation $A_o - \lambda$ where $\lambda = 2\omega_1$ in this example. The initial path for weight $\lambda$ is called the $\lambda$-chain. This path is constructed from the ordered pairs $(\beta, k)$, by crossing the hyperplane orthogonal to $\beta$ at height $-k$. We can view a plot of this
path as follows:

```python
sage: x = C( () )
sage: x.plot()  # not tested - outputs a pdf
```

An element of the crystal is given by a subset of the $\lambda$-chain. This subset indicates the hyperplanes where the initial path should be folded. The highest weight element is given by the empty subset.

```python
sage: x
()  
sage: x.f(1).f(2)  
((alpha[1], 1), (alpha[1] + alpha[2], 1))  
sage: x.f(1).f(2).integer_sequence()  
[2, 3]  
sage: C([2, 3])  
((alpha[1], 1), (alpha[1] + alpha[2], 1))  
sage: C([2, 3]).is_admissible()  # check if a valid vertex  
True  
sage: C([1, 3]).is_admissible()  # check if a valid vertex  
False
```

Alcove path crystals now works in affine type (github issue #14143):

```python
sage: C = crystals.AlcovePaths(['A', 2, 1], [1, 0, 0]); C  
Highest weight crystal of alcove paths of type ['A', 2, 1] and weight Lambda[0]  
sage: x = C( () )  
sage: x.f(0)  
((alpha[0], 0),)  
sage: C.R  
Root system of type ['A', 2, 1]  
sage: C.weight  
Lambda[0]
```

Test that the tensor products of Kirillov-Reshetikhin crystals minus non-pseudo-Demazure arrows is in bijection with alcove path construction:

```python
sage: K = crystals.KirillovReshetikhin(['B', 3, 1], 2, 1)  
sage: T = crystals.TensorProduct(K, K)  
sage: g = T.digraph()  # long time  
sage: for e in g.edges(sort=False):  # long time  
....:     if e[0].phi(0) == 1 and e[2] == 0:  
....:         g.delete_edge(e)  

sage: C = crystals.AlcovePaths(['B', 3, 1], [0, 2, 0], highest_weight_crystal=False)  
sage: g2 = C.digraph()  # long time  
sage: g.is_isomorphic(g2, edge_labels = True)  # long time  
True
```

**Note:** In type $C_n^{(1)}$, the Kirillov-Reshetikhin crystal is not connected when restricted to pseudo-Demazure arrows, hence the previous example will fail for type $C_n^{(1)}$ crystals.
sage: R = RootSystem(['B',3])
sage: P = R.weight_lattice()
sage: La = P.fundamental_weights()
sage: D = crystals.AlcovePaths(2*La[2], highest_weight_crystal=False)
sage: C == D
True

Warning: Weights from finite root systems index non-highest weight crystals.

Element

alias of CrystalOfAlcovePathsElement

vertices()

Return a list of all the vertices of the crystal.

The vertices are represented as lists of integers recording the folding positions.

One can compute all vertices of the crystal by finding all the admissible subsets of the \( \lambda \)-chain (see method is_admissible, for definition). We use the breadth first search algorithm.

Warning: This method is (currently) only useful for the case when highest_weight_crystal = False, where you cannot always reach all vertices of the crystal using crystal operators, starting from the highest weight vertex. This method is typically slower than generating the crystal graph using crystal operators.

EXAMPLES:

sage: C = crystals.AlcovePaths(['C',2],[1,0])
sage: C.vertices()
[[], [0], [0, 1], [0, 1, 2]]
sage: C = crystals.AlcovePaths(['C',2,1],[2,1],False)
sage: len(C.vertices())
80

The number of elements reachable using the crystal operators from the module generator:

sage: len(list(C))
55

class sage.combinat.crystals.alcove_path.CrystalOfAlcovePathsElement

Bases: ElementWrapper

Crystal of alcove paths element.

INPUT:

- data – a list of folding positions in the lambda chain (indexing starts at 0) or a tuple of \( \text{RootsWithHeight} \) giving folding positions in the lambda chain.

EXAMPLES:

sage: C = crystals.AlcovePaths(['A',2],[3,2])
sage: x = C ( () )

(continues on next page)
\begin{verbatim}
sage: x.f(1).f(2)
((alpha[1], 2), (alpha[1] + alpha[2], 4))
sage: x.f(1).f(2).integer_sequence()
[8, 9]
sage: C([8,9])
((alpha[1], 2), (alpha[1] + alpha[2], 4))
\end{verbatim}

\textbf{e(i)}

Return the $i$-th crystal raising operator on \texttt{self}.

**INPUT:**

- $i$ – element of the index set of the underlying root system.

**EXAMPLES:**

\begin{verbatim}
sage: C = crystals.AlcovePaths(['A',2],[2,0]); C
Highest weight crystal of alcove paths of type ['A', 2] and weight 2*Lambda[1]
sage: x = C( () )
sage: x.e(1)
sage: x.f(1) == x.f(1).f(2).e(2)
True
\end{verbatim}

\textbf{epsilon(i)}

Return the distance to the start of the $i$-string.

**EXAMPLES:**

\begin{verbatim}
sage: C = crystals.AlcovePaths(['A',2],[1,1])
sage: [c.epsilon(1) for c in C]
[0, 1, 0, 0, 1, 0, 1, 2]
sage: [c.epsilon(2) for c in C]
[0, 0, 1, 2, 1, 1, 0, 0]
\end{verbatim}

\textbf{f(i)}

Returns the $i$-th crystal lowering operator on \texttt{self}.

**INPUT:**

- $i$ – element of the index set of the underlying root system.

**EXAMPLES:**

\begin{verbatim}
sage: C=crystals.AlcovePaths(['B',2],[1,1])
sage: x=C( () )
sage: x.f(1)
((alpha[1], 0),)
sage: x.f(1).f(2)
((alpha[1], 0), (alpha[1] + alpha[2], 2))
\end{verbatim}

\textbf{integer_sequence}()

Return a list of integers corresponding to positions in the $\lambda$-chain where it is folded.

**Todo:** Incorporate this method into the \texttt{_repr_} for finite Cartan type.
Note: Only works for finite Cartan types and indexing starts at 0.

EXAMPLES:

```python
sage: C = crystals.AlcovePaths(['A',2],[3,2])
```

```python
crystals.AlcovePaths(['A',2],[3,2])
sage: x = C( () )
sage: x.f(1).f(2).integer_sequence()
[8, 9]
```

**is_admissible()**

Diagnostic test to check if self is a valid element of the crystal.

If self.value is given by

\[(\beta_1, i_1), (\beta_2, i_2), \ldots, (\beta_k, i_k),\]

for highest weight crystals this checks if the sequence

\[1 \rightarrow s_{\beta_1} \rightarrow s_{\beta_1}s_{\beta_2} \rightarrow \cdots \rightarrow s_{\beta_1}s_{\beta_2}
\ldots s_{\beta_k}\]

is a path in the Bruhat graph. If highest_weight_crystal=False, then the method checks if the above sequence is a path in the quantum Bruhat graph.

EXAMPLES:

```python
sage: C = crystals.AlcovePaths(['A',2],[1,1]); C
Highest weight crystal of alcove paths of type ['A', 2] and weight Lambda[1] +
˓→ Lambda[2]
sage: roots = sorted(C._R._root_lattice.positive_roots()); roots
[alpha[1], alpha[1] + alpha[2], alpha[2]]
sage: r1 = C._R(roots[0],0); r1
(alpha[1], 0)
sage: r2 = C._R(roots[2],0); r2
(alpha[2], 0)
sage: r3 = C._R(roots[1],1); r3
(alpha[1] + alpha[2], 1)
sage: x = C( (r1,r2) )
sage: x.is_admissible()
True
sage: x = C( (r3,) )
((alpha[1] + alpha[2], 1),)
sage: x.is_admissible()
False
sage: C = crystals.AlcovePaths(['C',2,1],[2,1],False)
```

```python
sage: C([7,8]).is_admissible()
True
```

```python
sage: C = crystals.AlcovePaths(['A',2],[3,2])
sage: C([2,3]).is_admissible()
True
```

Todo: Better doctest
Combinatorics, Release 10.0

path()

Return the path in the (quantum) Bruhat graph corresponding to self.

EXAMPLES:

```
sage: C = crystals.AlcovePaths(['B', 3], [3,1,2])
sage: b = C.highest_weight_vector().f_string([1,3,2,1,3,1])
sage: b.path()
[1, s1, s3*s1, s2*s3*s1, s3*s2*s3*s1]
sage: b = C.highest_weight_vector().f_string([2,3,3,2])
sage: b.path()
[1, s2, s3*s2, s2*s3*s2]
sage: b = C.highest_weight_vector().f_string([2,3,3,2,1])
sage: b.path()
[1, s2, s3*s2, s2*s3*s2, s1*s2*s3*s2]
```

phi(i)

Return the distance to the end of the i-string.

This method overrides the generic implementation in the category of crystals since this computation is more efficient.

EXAMPLES:

```
sage: C = crystals.AlcovePaths(['A',2],[1,1])
sage: [c.phi(1) for c in C]
[1, 0, 0, 1, 0, 2, 1, 0]
sage: [c.phi(2) for c in C]
[1, 2, 1, 0, 0, 0, 0, 1]
```

plot()

Return a plot self.

Note: Currently only implemented for types $A_2$, $B_2$, and $C_2$.

EXAMPLES:

```
sage: C = crystals.AlcovePaths(['A',2],[2,0])
sage: x = C( () ).f(1).f(2)
sage: x.plot() # Not tested - creates a pdf
```

weight()

Return the weight of self.

EXAMPLES:

```
sage: C = crystals.AlcovePaths(['A',2],[2,0])
sage: for i in C: i.weight()
(2, 0, 0)
(1, 1, 0)
(0, 2, 0)
(0, -1, 0)
(-1, 0, 0)
(-2, -2, 0)
``` (continues on next page)
sage: B = crystals.AlcovePaths(['A',2,1],[1,0,0])
sage: p = B.module_generators[0].f_string([0,1,2])
sage: p.weight()
Lambda[0] - delta

class sage.combinat.crystals.alcove_path.InfinityCrystalOfAlcovePaths(cartan_type)
Bases: UniqueRepresentation, Parent

B(∞) crystal of alcove paths.

class Element(parent, elt, shift)
Bases: ElementWrapper

Initialize self.

EXAMPLES:

sage: A = crystals.infinity.AlcovePaths(['F',4])
sage: mg = A.highest_weight_vector()
sage: x = mg.f_string([2,3,1,4,4,2,3,1])
sage: TestSuite(x).run()

e(i)
Return the action of $e_i$ on self.

INPUT:
• i – an element of the index set

EXAMPLES:

sage: A = crystals.infinity.AlcovePaths(['D',5,1])
sage: mg = A.highest_weight_vector()
sage: x = mg.f_string([1,3,4,2,5,4,5,5])
sage: x.f(4).e(5) == x.e(5).f(4)
True

epsilon(i)
Return $\epsilon_i$ of self.

INPUT:
• i – an element of the index set

EXAMPLES:

sage: A = crystals.infinity.AlcovePaths(['A',7,2])
sage: mg = A.highest_weight_vector()
sage: x = mg.f_string([1,0,2,3,4,4,2,3,3,3])
sage: [x.epsilon(i) for i in A.index_set()]
[0, 0, 0, 3, 0]
sage: x = mg.f_string([2,2,1,0,1,0,2,3,3,3,4])
sage: [x.epsilon(i) for i in A.index_set()]
[1, 2, 0, 1, 1]

f(i)
Return the action of $f_i$ on self.

INPUT:
• i – an element of the index set
EXAMPLES:

```
sage: A = crystals.infinity.AlcovePaths(['E',7,1])
sage: mg = A.highest_weight_vector()
sage: mg.f_string([1,3,5,6,4,2,0,2,1,0,2,4,7,4,2])
((alpha[2], -3), (alpha[5], -1), (alpha[1], -1),
 (alpha[0] + alpha[1], -2),
 (alpha[5] + alpha[6], -1), (alpha[1] + alpha[3], -1),
 (alpha[0] + alpha[1] + alpha[3], -1),
```

\( \phi_i() \)

Return \( \varphi_i \) of self.

Let \( A \in \mathcal{B}(\infty) \) Define \( \varphi_i(A) := \varepsilon_i(A) + \langle h_i, \text{wt}(A) \rangle \), where \( h_i \) is the \( i \)-th simple coroot and \( \text{wt}(A) \) is the \text{weight()} \ of \( A \).

INPUT:

- \( i \) – an element of the index set

EXAMPLES:

```
sage: A = crystals.infinity.AlcovePaths(['A',8,2])
sage: mg = A.highest_weight_vector()
sage: x = mg.f_string([1,0,2,3,4,4,4,2,3,3,3]); x
((alpha[2], -3), (alpha[5], -1), (alpha[1] + alpha[2], -3),
 (3*alpha[1] + 2*alpha[2], -1), (2*alpha[1] + alpha[2], -1))
sage: x.phi(1)
1
sage: x.phi(2)
1
sage: x.phi(3)
1
sage: x.phi(4)
-2
```

\( \text{projection}(k=\text{None}) \)

Return the projection \( \text{self} \) onto \( \mathcal{B}(k\rho) \).

INPUT:

- \( k \) – (optional) if not given, defaults to the smallest value such that \( \text{self} \) is not \text{None} under the projection

EXAMPLES:

```
sage: A = crystals.infinity.AlcovePaths(['G',2])
sage: mg = A.highest_weight_vector()
sage: x = mg.f_string([2,1,1,2,2,2,1,1]); x
((alpha[2], -3), (alpha[1] + alpha[2], -3),
 (3*alpha[1] + 2*alpha[2], -1), (2*alpha[1] + alpha[2], -1))
sage: x.projection()
((alpha[2], 0), (alpha[1] + alpha[2], 9),
 (3*alpha[1] + 2*alpha[2], 8), (2*alpha[1] + alpha[2], 14))
sage: x.projection().parent()
Highest weight crystal of alcove paths of type ['G', 2]
```

sage: mg.f(1).projection().parent()

sage: mg.f(1).f(2).projection().parent()

sage: b = mg.f_string([1,2,2,1,2])

sage: b.projection().parent()

sage: b.projection(3).parent()

sage: b.projection(1)
weight()
Return the weight of self.

EXAMPLES:

sage: A = crystals.infinity.AlcovePaths(['E',6])
sage: mg = A.highest_weight_vector()
sage: fstr = [1,3,4,2,1,2,3,6,5,3,2,6,2]
sage: x = mg.f_string(fstr)
sage: al = A.weight_lattice_realization().simple_roots()
sage: x.weight() == -sum(al[i]*fstr.count(i) for i in A.index_set())
True

class sage.combinat.crystals.alcove_path.RootsWithHeight(weight)

Bases: UniqueRepresentation, Parent

Data structure of the ordered pairs (β, k), where β is a positive root and k is a non-negative integer. A total order is implemented on this set, and depends on the weight.

INPUT:

- cartan_type – Cartan type of a finite or affine untwisted root system
- weight – dominant weight as a list of (integral) coefficients of the fundamental weights

EXAMPLES:

sage: from sage.combinat.crystals.alcove_path import RootsWithHeight
sage: R = RootsWithHeight(['A',2],[1,1]); R

sage: r1 = R._root_lattice.from_vector(vector([1,0])); r1
alpha[1]

sage: r2 = R._root_lattice.from_vector(vector([1,1])); r2

sage: x = R(r1,0); x
(α[1], 0)

sage: y = R(r2,1); y
(α[1] + α[2], 1)
sage: x < y
True

**Element**

alias of *RootsWithHeightElement*

**lambda_chain()**

Return the unfolded $\lambda$-chain.

**Note:** Only works in root systems of finite type.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.alcove_path import RootsWithHeight
sage: R = RootsWithHeight(['A',2],[1,1]); R
Roots with height of Cartan type ['A', 2] and dominant weight Lambda[1] + \Lambda[2]

sage: R.lambda_chain()
[(alpha[2], 0), (alpha[1] + alpha[2], 0), (alpha[1], 0), (alpha[1] + alpha[2], \rightarrow 1)]
```

**word()**

Gives the initial alcove path ($\lambda$-chain) in terms of simple roots. Used for plotting the path.

**Note:** Currently only implemented for finite Cartan types.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.alcove_path import RootsWithHeight
sage: R = RootsWithHeight(['A',2],[3,2])
sage: R.word()
[2, 1, 2, 0, 1, 2, 1, 0, 1, 2]
```

class `RootsWithHeightElement`

Bases: `Element`

Element of *RootsWithHeight*.

**INPUT:**

- `root` – A positive root $\beta$ in our root system
- `height` – Is an integer, such that $0 \leq l \leq \langle \lambda, \beta^\vee \rangle$

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.alcove_path import RootsWithHeight
sage: rl = RootSystem(['A',2]).root_lattice()
sage: x = rl.from_vector(vector([1,1])); x

sage: R = RootsWithHeight(['A',2],[1,1]); R
```
sage: y = R(x, 1); y
(alpha[1] + alpha[2], 1)

sage.combinat.crystals.alcove_path.compare_graphs(g1, g2, node1, node2)

Compare two edge-labeled graphs obtained from Crystal.digraph(), starting from the root nodes of each graph.

- g1 – graphs, first digraph
- g2 – graphs, second digraph
- node1 – element of g1
- node2 – element of g2

Traverse g1 starting at node1 and compare this graph with the one obtained by traversing g2 starting with node2. If the graphs match (including labels) then return True. Return False otherwise.

EXAMPLES:

sage: from sage.combinat.crystals.alcove_path import compare_graphs
sage: G1 = crystals.Tableaux(['A',3], shape=[1,1]).digraph()
sage: C = crystals.AlcovePaths(['A',3],[0,1,0])
sage: G2 = C.digraph()

sage: compare_graphs(G1, G2, C(), G2.vertices(sort=True)[0])
True

5.1.39 Crystals

Introductory material

- An introduction to crystals
- The Lie Methods and Related Combinatorics thematic tutorial

Catalogs of crystals

- Catalog Of Crystals

See also

- The categories for crystals: Crystals, HighestWeightCrystals, FiniteCrystals, ClassicalCrystals, RegularCrystals, RegularSuperCrystals – The categories for crystals
- Root Systems
5.1.40 Benkart-Kang-Kashiwara crystals for the general-linear Lie superalgebra

```python
class sage.combinat.crystals.bkk_crystals.CrystalOfBKKTableaux(ct, shape):
    Bases: CrystalOfWords

    Crystal of tableaux for type $A(m|n)$.

    This is an implementation of the tableaux model of the Benkart-Kang-Kashiwara crystal [BKK2000] for the Lie superalgebra $\mathfrak{gl}(m+1,n+1)$.

    INPUT:
    
    • `ct` – a super Lie Cartan type of type $A(m|n)$
    • `shape` – shape specifying the highest weight; this should be a partition contained in a hook of height $n+1$ and width $m+1$

    EXAMPLES:

    sage: T = crystals.Tableaux(['A', [1,1]], shape=[2,1])
    sage: T.cardinality()
    20
```

```python
class Element
    Bases: CrystalOfBKKTableauxElement

genuine_highest_weight_vectors(index_set=None)
    Return a tuple of genuine highest weight elements.

    A fake highest weight vector is one which is annihilated by $e_i$ for all $i$ in the index set, but whose weight is not bigger in dominance order than all other elements in the crystal. A genuine highest weight vector is a highest weight element that is not fake.

    EXAMPLES:

    sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
    sage: B.genuine_highest_weight_vectors()
    ([-2, -2, -2], [-1, -1], [1])
    sage: B.highest_weight_vectors()
    ([-2, -2, -2], [-1, 2], [1],
     [-2, -2, 2], [-1, -1], [1])
```

```python
shape()
    Return the shape of self.

    EXAMPLES:

    sage: T = crystals.Tableaux(['A', [1, 2]], shape=[2,1])
    sage: T.shape()
    [2, 1]
```
5.1.41 Catalog Of Crystals

Let $I$ be an index set and let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. An abstract crystal associated to this Cartan datum is a set $B$ together with maps

$$e_i, f_i : B \rightarrow B \cup \{0\}, \quad \varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}, \quad \text{wt} : B \rightarrow P,$$

subject to the following conditions:

1. $\varphi_i(b) = \varepsilon_i(b) + (h_i, \text{wt}(b))$ for all $b \in B$ and $i \in I$;
2. $\text{wt}(e_i b) = \text{wt}(b) + \alpha_i$ if $e_i b \in B$;
3. $\text{wt}(f_i b) = \text{wt}(b) - \alpha_i$ if $f_i b \in B$;
4. $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1, \varphi_i(e_i b) = \varphi_i(b) + 1$ if $e_i b \in B$;
5. $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1, \varphi_i(f_i b) = \varphi_i(b) - 1$ if $f_i b \in B$;
6. $f_i b = b'$ if and only if $b = e_i b'$ for $b, b' \in B$ and $i \in I$;
7. if $\varphi_i(b) = -\infty$ for $b \in B$, then $e_i b = f_i b = 0$.

See also:

- sage.categories.crystals
- sage.combinat.crystals.crystals

Catalog

This is a catalog of crystals that are currently implemented in Sage:

- AffineCrystalFromClassical
- AffineCrystalFromClassicalAndPromotion
- AffineFactorization
- AffinizationOf
- AlcovePaths
- FastRankTwo
- FullyCommutativeStableGrothendieck
- GeneralizedYoungWalls
- HighestWeight
- Induced
- KacModule
- KirillovReshetikhin
- KleshchevPartitions
- KyotoPathModel
- Letters
- LSPaths
- Minimaj
Combinatorics, Release 10.0

- NakajimaMonomials
- OddNegativeRoots
- ProjectedLevelZeroLSPaths
- RiggedConfigurations
- ShiftedPrimedTableaux
- Spins
- SpinsPlus
- SpinsMinus
- Tableaux

Subcatalogs:
- Catalog Of Crystal Models For B(\infty)
- Catalog Of Elementary Crystals
- Catalog Of Crystal Models For Kirillov-Reshetikhin Crystals

Functorial constructions:
- DirectSum
- TensorProduct

5.1.42 Catalog Of Elementary Crystals

See elementary_crystals.
- Component
- Elementary or B
- R
- T

5.1.43 Catalog Of Crystal Models For \( B(\infty) \)

We currently have the following models:
- AlcovePaths
- GeneralizedYoungWalls
- LSPaths
- Multisegments
- MVPolytopes
- NakajimaMonomials
- PBW
- PolyhedralRealization
- RiggedConfigurations
- Star
5.1.44 Catalog Of Crystal Models For Kirillov-Reshetikhin Crystals

We currently have the following models:

- \texttt{KashiwaraNakashimaTableaux}
- \texttt{KirillovReshetikhinTableaux}
- \texttt{LSPaths}
- \texttt{RiggedConfigurations}

5.1.45 An introduction to crystals

Informally, a crystal \( \mathcal{B} \) is an oriented graph with edges colored in some set \( I \) such that, for each \( i \in I \), each node \( x \) has:

- at most one \( i \)-successor, denoted \( f_i x \);
- at most one \( i \)-predecessor, denoted \( e_i x \).

By convention, one writes \( f_i x = \emptyset \) and \( e_i x = \emptyset \) when \( x \) has no successor resp. predecessor.

One may think of \( \mathcal{B} \) as essentially a deterministic automaton whose dual is also deterministic; in this context, the \( f_i \)'s and \( e_i \)'s are respectively the transition functions of the automaton and of its dual, and \( \emptyset \) is the sink.

A crystal comes further endowed with a weight function \( \text{wt} : \mathcal{B} \rightarrow L \) which satisfies appropriate conditions.

In combinatorial representation theory, crystals are used as combinatorial data to model representations of Lie algebra.

Axiomatic definition

Let \( C \) be a Cartan type (<strong>CartanType</strong>) with index set \( I \), and \( L \) be a realization of the weight lattice of the type \( C \). Let \( \alpha_i \) and \( \alpha_i^\vee \) denote the simple roots and coroots respectively.

A type \( C \) crystal is a non-empty set \( \mathcal{B} \) endowed with maps \( \text{wt} : \mathcal{B} \rightarrow L, e_i, f_i : \mathcal{B} \rightarrow \mathcal{B} \cup \{\emptyset\} \), and \( \varepsilon_i, \varphi_i : \mathcal{B} \rightarrow \mathbb{Z} \cup \{-\infty\} \) for \( i \in I \) satisfying the following properties for all \( i \in I \):

- for \( b, b' \in \mathcal{B} \), we have \( f_i b' = b \) if and only if \( e_i b = b' \);
- if \( e_i b \in \mathcal{B} \), then:
  - \( \text{wt}(e_i b) = \text{wt}(b) + \alpha_i \),
  - \( \varepsilon_i(e_i b) = \varepsilon_i(b) - 1 \),
  - \( \varphi_i(e_i b) = \varphi_i(b) + 1 \);
- if \( f_i b \in \mathcal{B} \), then:
  - \( \text{wt}(f_i b) = \text{wt}(b) - \alpha_i \),
  - \( \varepsilon_i(f_i b) = \varepsilon_i(b) + 1 \),
  - \( \varphi_i(f_i b) = \varphi_i(b) - 1 \);
- \( \varphi_i(b) = \varepsilon_i(b) + (\alpha_i^\vee, \text{wt}(b)) \);
- if \( \varphi_i(b) = -\infty \) for \( b \in \mathcal{B} \), then \( e_i b = f_i b = \emptyset \).
Some further conditions are required to guarantee that this data indeed models a representation of a Lie algebra. For finite simply laced types a complete characterization is given by Stembridge’s local axioms [Ste2003].

EXAMPLES:

We construct the type $A_5$ crystal on letters (or in representation theoretic terms, the highest weight crystal of type $A_5$ corresponding to the highest weight $\Lambda_1$):

```sage
C = crystals.Letters(['A',5]); C
The crystal of letters for type ['A', 5]
```

It has a single highest weight element:

```sage
C.highest_weight_vectors()
(1,)
```

A crystal is an enumerated set (see `EnumeratedSets`); and we can count and list its elements in the usual way:

```sage
C.cardinality()
6
C.list()
[1, 2, 3, 4, 5, 6]
```
as well as use it in for loops:

```sage
[x for x in C]
[1, 2, 3, 4, 5, 6]
```

Here are some more elaborate crystals (see their respective documentations):

```sage
Tens = crystals.TensorProduct(C, C)
Spin = crystals.Spins(['B', 3])
Tab = crystals.Tableaux(['A', 3], shape = [2,1,1])
Fast = crystals.FastRankTwo(['B', 2], shape = [3/2, 1/2])
KR = crystals.KirillovReshetikhin(['A',2,1],1,1)
```

One can get (currently) crude plotting via:

```sage
Tab.plot()
Graphics object consisting of 52 graphics primitives
```

If dot2tex is installed, one can obtain nice latex pictures via:

```sage
K = crystals.KirillovReshetikhin(['A',3,1], 1,1)
view(K, pdflatex=True) # optional - dot2tex graphviz, not tested (opens external..._window)
```
or with colored edges:

```sage
K = crystals.KirillovReshetikhin(['A',3,1], 1,1)
G = K.digraph()
G.set_latex_options(color_by_label={0:"black", 1:"red", 2:"blue", 3:"green"})
view(G, pdflatex=True) # optional - dot2tex graphviz, not tested (opens external..._window)
```

For rank two crystals, there is an alternative method of getting metapost pictures. For more information see `C.metapost?`. 
See also:
The overview of crystal features in Sage

Todo:
• Vocabulary and conventions:
  – For a classical crystal: connected / highest weight / irreducible
  – ...
• Layout instructions for plot() for rank 2 types
• RestrictionOfCrystal

The crystals library in Sage grew up from an initial implementation in MuPAD-Combinat (see <MuPAD-Combinat>/lib/COMBINAT/crystals.mu).

class sage.combinat.crystals.crystals.CrystalBacktracker(crysal, index_set=None)

Bases: GenericBacktracker

Time complexity: $O(nF)$ amortized for each produced element, where $n$ is the size of the index set, and $F$ is the cost of computing $e$ and $f$ operators.

Memory complexity: $O(D)$ where $D$ is the depth of the crystal.

Principle of the algorithm:

Let $C$ be a classical crystal. It’s an acyclic graph where each connected component has a unique element without predecessors (the highest weight element for this component). Let’s assume for simplicity that $C$ is irreducible (i.e. connected) with highest weight element $u$.

One can define a natural spanning tree of $C$ by taking $u$ as the root of the tree, and for any other element $y$ taking as ancestor the element $x$ such that there is an $i$-arrow from $x$ to $y$ with $i$ minimal. Then, a path from $u$ to $y$ describes the lexicographically smallest sequence $i_1, \ldots, i_k$ such that $(f_{i_k} \circ f_{i_{i-1}})(u) = y$.

Morally, the iterator implemented below just does a depth first search walk through this spanning tree. In practice, this can be achieved recursively as follows: take an element $x$, and consider in turn each successor $y = f_i(x)$, ignoring those such that $y = f_j(x')$ for some $x'$ and $j < i$ (this can be tested by computing $e_j(y)$ for $j < i$).

EXAMPLES:

```python
sage: from sage.combinat.crystals.crystals import CrystalBacktracker
sage: C = crystals.Tableaux(['B',3],shape=[3,2,1])
sage: CB = CrystalBacktracker(C)
sage: len(list(CB))
1617
sage: CB = CrystalBacktracker(C, [1,2])
sage: len(list(CB))
8
```
5.1.46 Direct Sum of Crystals

```python
class sage.combinat.crystals.direct_sum.DirectSumOfCrystals(crystals, facade, keepkey, category, **options):
    Bases: DisjointUnionEnumeratedSets

    Direct sum of crystals.

    Given a list of crystals $B_0, \ldots, B_k$ of the same Cartan type, one can form the direct sum $B_0 \oplus \cdots \oplus B_k$.

    INPUT:

    - `crystals` – a list of crystals of the same Cartan type
    - `keepkey` – a boolean

    The option `keepkey` is by default set to `False`, assuming that the crystals are all distinct. In this case the elements of the direct sum are just represented by the elements in the crystals $B_i$. If the crystals are not all distinct, one should set the `keepkey` option to `True`. In this case, the elements of the direct sum are represented as tuples $(i, b)$ where $b \in B_i$.

    EXAMPLES:

    ```python
    sage: C = crystals.Letters(['A', 2])
    sage: C1 = crystals.Tableaux(['A', 2], shape=[1, 1])
    sage: B = crystals.DirectSum([C, C1])
    sage: B.list()
    [1, 2, 3, [[1], [2]], [[1], [3]], [[2], [3]]]
    sage: [b.f(1) for b in B]
    [2, None, None, None, [[2], [3]], None]
    sage: B.module_generators
    (1, [[1], [2]])
    ```

    ```python
    sage: B = crystals.DirectSum([C, C], keepkey=True)
    sage: B.list()
    [(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)]
    sage: B.module_generators
    ((0, 1), (1, 1))
    sage: b = B( tuple([0, C(1)]))
    sage: b
    (0, 1)
    sage: b.weight()
    (1, 0, 0)
    ```
```

The following is required, because `DirectSumOfCrystals` takes the same arguments as `DisjointUnionEnumeratedSets` (which see for details).

```python
class Element:
    Bases: ElementWrapper

    A class for elements of direct sums of crystals.

    `e(i)`

    Return the action of $e_i$ on `self`.

    EXAMPLES:
```

Bases: DisjointUnionEnumeratedSets

Direct sum of crystals.

Given a list of crystals $B_0, \ldots, B_k$ of the same Cartan type, one can form the direct sum $B_0 \oplus \cdots \oplus B_k$.

INPUT:

- `crystals` – a list of crystals of the same Cartan type
- `keepkey` – a boolean

The option `keepkey` is by default set to `False`, assuming that the crystals are all distinct. In this case the elements of the direct sum are just represented by the elements in the crystals $B_i$. If the crystals are not all distinct, one should set the `keepkey` option to `True`. In this case, the elements of the direct sum are represented as tuples $(i, b)$ where $b \in B_i$.

EXAMPLES:

```python
sage: C = crystals.Letters(['A', 2])
sage: C1 = crystals.Tableaux(['A', 2], shape=[1, 1])
sage: B = crystals.DirectSum([C, C1])
sage: B.list()
[1, 2, 3, [[1], [2]], [[1], [3]], [[2], [3]]]
sage: [b.f(1) for b in B]
[2, None, None, None, [[2], [3]], None]
sage: B.module_generators
(1, [[1], [2]])
```
sage: C = crystals.Letters(['A',2])
sage: B = crystals.DirectSum([C,C], keepkey=True)
sage: [[b, b.e(2)] for b in B]
[(0, 1), None], [(0, 2), None], [(0, 3), (0, 2)], [(1, 1), None], [(1, 2), None], [(1, 3), (1, 2)]

epsilon(i)

EXAMPLES:

sage: C = crystals.Letters(['A',2])
sage: B = crystals.DirectSum([C,C], keepkey=True)
sage: b = B( tuple([0,C(2)]) )
sage: b.epsilon(2)
0

f(i)

Return the action of \( f_i \) on \( \text{self} \).

EXAMPLES:

sage: C = crystals.Letters(['A',2])
sage: B = crystals.DirectSum([C,C], keepkey=True)
sage: [[b,b.f(1)] for b in B]
[(0, 1), (0, 2)], [(0, 2), None], [(0, 3), None], [(1, 1), (1, 2)], [(1, 2), None], [(1, 3), None]

phi()

EXAMPLES:

sage: C = crystals.Letters(['A',2])
sage: B = crystals.DirectSum([C,C], keepkey=True)
sage: b = B( tuple([0,C(2)]) )
sage: b.phi(2)
1

weight()

Return the weight of \( \text{self} \).

EXAMPLES:

sage: C = crystals.Letters(['A',2])
sage: B = crystals.DirectSum([C,C], keepkey=True)
sage: b = B( tuple([0,C(2)]) )
sage: b.weight()
(0, 1, 0)

weight_lattice_realization()

Return the weight lattice realization used to express weights.

The weight lattice realization is the common parent which all weight lattice realizations of the crystals of \( \text{self} \) coerce into.

EXAMPLES:
5.1.47 Elementary Crystals

Let $\lambda$ be a weight. The crystals $T_\lambda$, $R_\lambda$, $B_i$, and $C$ are important objects in the tensor category of crystals. For example, the crystal $T_0$ is the neutral object in this category; i.e., $T_0 \otimes B \cong B \otimes T_0 \cong B$ for any crystal $B$. We list some other properties of these crystals:

- The crystal $T_\lambda \otimes B(\infty)$ is the crystal of the Verma module with highest weight $\lambda$, where $\lambda$ is a dominant integral weight.

- Let $u_\infty$ be the highest weight vector of $B(\infty)$ and $\lambda$ be a dominant integral weight. There is an embedding of crystals $B(\lambda) \hookrightarrow T_\lambda \otimes B(\infty)$ sending $u_\lambda \mapsto t_\lambda \otimes u_\infty$, which is not strict, but the embedding $B(\lambda) \hookrightarrow C \otimes T_\lambda \otimes B(\infty)$ by $u_\lambda \mapsto c \otimes t_\lambda \otimes u_\infty$ is a strict embedding.

- For any dominant integral weight $\lambda$, there is a surjective crystal morphism $\Psi_\lambda : R_\lambda \otimes B(\infty) \twoheadrightarrow B(\lambda)$. More precisely, if $B = \{r_\lambda \otimes b \in R_\lambda \otimes B(\infty) : \Psi_\lambda(r_\lambda \otimes b) \neq 0\}$, then $B \cong B(\lambda)$ as crystals.

- For all Cartan types and all weights $\lambda$, we have $R_\lambda \cong C \otimes T_\lambda$ as crystals.

- For each $i$, there is a strict crystal morphism $\Psi_i : B(\infty) \hookrightarrow B_i \otimes B(\infty)$ defined by $u_\infty \mapsto b_i(0) \otimes u_\infty$, where $u_\infty$ is the highest weight vector of $B(\infty)$.

For more information on $B(\infty)$, see *InfinityCrystalOfTableaux*.

**Note:** As with *TensorProductOfCrystals*, we are using the opposite of Kashiwara’s convention.

**AUTHORS:**
- Ben Salisbury: Initial version

**REFERENCES:**
- [Ka1993]
- [NZ1997]

**class** `sage.combinat.crystals.elementary_crystals.AbstractSingleCrystalElement`  
Bases: `Element`

Abstract base class for elements in crystals with a single element.
\(e(i)\)

Return \(e_i\) of \(self\), which is None for all \(i\).

**INPUT:**

- \(i\) – An element of the index set

**EXAMPLES:**

```python
sage: ct = CartanType(['A',2])
sage: la = RootSystem(ct).weight_lattice().fundamental_weights()
sage: T = crystals.elementary.T(ct,la[1])
sage: t = T.highest_weight_vector()
sage: t.e(1)
sage: t.e(2)
```

\(f(i)\)

Return \(f_i\) of \(self\), which is None for all \(i\).

**INPUT:**

- \(i\) – An element of the index set

**EXAMPLES:**

```python
sage: ct = CartanType(['A',2])
sage: la = RootSystem(ct).weight_lattice().fundamental_weights()
sage: T = crystals.elementary.T(ct,la[1])
sage: t = T.highest_weight_vector()
sage: t.f(1)
sage: t.f(2)
```

class \texttt{sage.combinat.crystals.elementary_crystals.ComponentCrystal}\((\text{cartan\_type}, P)\)

**Bases:** \texttt{UniqueRepresentation}, \texttt{Parent}

The component crystal.

Defined in \cite{Ka1993}, the component crystal \(C = \{c\}\) is the single element crystal whose crystal structure is defined by

\[
\text{wt}(c) = 0, \quad e_i c = f_i c = 0, \quad \varepsilon_i(c) = \varphi_i(c) = 0.
\]

Note \(C \cong B(0)\), where \(B(0)\) is the highest weight crystal of highest weight \(0\).

**INPUT:**

- \texttt{cartan\_type} – a Cartan type

class \texttt{Element}

**Bases:** \texttt{AbstractSingleCrystalElement}

Element of a component crystal.

\(\varepsilon(i)\)

Return \(\varepsilon_i\) of \(self\), which is 0 for all \(i\).

**INPUT:**

- \(i\) – An element of the index set

**EXAMPLES:**

```python
```
\[
\text{phi}(i)
\]
Return \( \varphi_i \) of self, which is 0 for all \( i \).

INPUT:
- \( i \) – An element of the index set

EXAMPLES:
\[
sage: C = crystals.elementary.Component("C5")
sage: c = C.highest_weight_vector()
sage: [c.phi(i) for i in C.index_set()]
[0, 0, 0, 0, 0]
\]

\[
\text{weight}()\]
Return the weight of self, which is always 0.

EXAMPLES:
\[
sage: C = crystals.elementary.Component("F4")
sage: c = C.highest_weight_vector()
sage: c.weight()
(0, 0, 0, 0)
\]

\[
\text{cardinality}()\]
Return the cardinality of self, which is always 1.

EXAMPLES:
\[
sage: C = crystals.elementary.Component("E6")
sage: c = C.highest_weight_vector()
sage: C.cardinality()
1
\]

\[
\text{weight_lattice_realization}()\]
Return the weight lattice realization of self.

EXAMPLES:
\[
sage: C = crystals.elementary.Component("A2")
sage: C.weight_lattice_realization()
Ambient space of the Root system of type ['A', 2]
sage: P = RootSystem(['A',2]).weight_lattice()
sage: C = crystals.elementary.Component(P)
sage: C.weight_lattice_realization() is P
True
\]

\[
\text{class} \ sage.combinat.crystals.elementary_crystals.ElementaryCrystal(cartan_type, i)
\]
Bases: \textit{UniqueRepresentation}, \textit{Parent}

The elementary crystal \( B_i \).
For $i$ an element of the index set of type $X$, the crystal $B_i$ of type $X$ is the set

$$B_i = \{ b_i(m) : m \in \mathbb{Z} \},$$

where the crystal structure is given by

$$\text{wt}(b_i(m)) = m \alpha_i,$$

$$\varphi_j(b_i(m)) = \begin{cases} m & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases},$$

$$\varepsilon_j(b_i(m)) = \begin{cases} -m & \text{if } j = i, \\ -\infty & \text{if } j \neq i, \end{cases},$$

$$e_j b_i(m) = \begin{cases} b_i(m+1) & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases},$$

$$f_j b_i(m) = \begin{cases} b_i(m-1) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

The Kashiwara embedding theorem asserts there is a unique strict crystal embedding of crystals

$$B(\infty) \hookrightarrow B_i \otimes B(\infty),$$

satisfying certain properties (see [Ka1993]). The above embedding may be iterated to obtain a new embedding

$$B(\infty) \hookrightarrow B_{iN} \otimes B_{iN-1} \otimes \cdots \otimes B_{i2} \otimes B_{i1} \otimes B(\infty),$$

which is a foundational object in the study of polyhedral realizations of crystals (see, for example, [NZ1997]).

```python
class Element(
    parent, m
):
    Bases = Element

    Element of a $B_i$ crystal.

e(i)
    Return the action of $e_i$ on self.

    INPUT:
    • i – An element of the index set

    EXAMPLES:

    sage: B = crystals.elementary.Elementary(['E', 7], 1)
    sage: B(3).e(1)
    4
    sage: B(172).e_string([1]*171)
    343
    sage: B(0).e(2)

epsilon(i)
    Return $\varepsilon_i$ of self.

    INPUT:
    • i – An element of the index set

    EXAMPLES:

    sage: B = crystals.elementary.Elementary(['E', 7], 1)
    sage: B(3).epsilon(1)
    3
    sage: B(172).epsilon_string([1]*171)
    343
    sage: B(0).epsilon(2)
```

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sage: B = crystals.elementary.Elementary(['F', 4], 3)
sage: [[B(j).epsilon(i) for i in B.index_set()] for j in range(5)]
[-inf, -inf, 0, -inf],
[-inf, -inf, -1, -inf],
[-inf, -inf, -2, -inf],
[-inf, -inf, -3, -inf],
[-inf, -inf, -4, -inf]]

\( f(i) \)

Return the action of \( f_i \) on self.

INPUT:

- \( i \) – An element of the index set

EXAMPLES:

sage: B = crystals.elementary.Elementary(['E', 7], 1)
sage: B(3).f(1)
2
sage: B(172).f_string([1]*171)
1
sage: B(0).e(2)

\( \phi(i) \)

Return \( \phi_i \) of self.

INPUT:

- \( i \) – An element of the index set

EXAMPLES:

sage: B = crystals.elementary.Elementary(['E', 8, 1], 4)
sage: [[B(m).phi(j) for j in B.index_set()] for m in range(44, 49)]
[-inf, -inf, -inf, -inf, 44, -inf, -inf, -inf],
[-inf, -inf, -inf, -inf, 45, -inf, -inf, -inf],
[-inf, -inf, -inf, -inf, 46, -inf, -inf, -inf],
[-inf, -inf, -inf, -inf, 47, -inf, -inf, -inf],
[-inf, -inf, -inf, -inf, 48, -inf, -inf, -inf]]

weight()

Return the weight of self.

EXAMPLES:

sage: B = crystals.elementary.Elementary(['C', 14], 12)
sage: B(-385).weight()
-385*alpha[12]

weight_lattice_realization()

Return a realization of the lattice containing the weights of self.

EXAMPLES:

sage: B = crystals.elementary.Elementary(['A', 4, 1], 2)
sage: B.weight_lattice_realization()
Root lattice of the Root system of type ['A', 4, 1]
class sage.combinat.crystals.elementary_crystals.RCrystal(cartan_type, weight, dual)

Bases: UniqueRepresentation, Parent

The crystal $R_\lambda$.

For a fixed weight $\lambda$, the crystal $R_\lambda = \{r_\lambda\}$ is a single element crystal with the crystal structure defined by

\[ \text{wt}(r_\lambda) = \lambda, \quad e_i r_\lambda = f_i r_\lambda = 0, \quad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \quad \varphi_i(r_\lambda) = 0, \]

where $\{h_i\}$ are the simple coroots.

Tensoring $R_\lambda$ with a crystal $B$ results in shifting the weights of the vertices in $B$ by $\lambda$ and may also cut a subset out of the original graph of $B$. That is, \( \text{wt}(r_\lambda \otimes b) = \text{wt}(b) + \lambda \), where $b \in B$, provided $r_\lambda \otimes b \neq 0$. For example, the crystal graph of $B(\lambda)$ is the same as the crystal graph of $R_\lambda \otimes B(\infty)$ generated from the component $r_\lambda \otimes u_\infty$.

There is also a dual version of this crystal given by $R_\lambda^\vee = \{r_\lambda^\vee\}$ with the crystal structure defined by

\[ \text{wt}(r_\lambda^\vee) = \lambda, \quad e_i r_\lambda^\vee = f_i r_\lambda^\vee = 0, \quad \varepsilon_i(r_\lambda^\vee) = 0, \quad \varphi_i(r_\lambda^\vee) = \langle h_i, \lambda \rangle. \]

INPUT:

- cartan_type – a Cartan type
- weight – an element of the weight lattice of type cartan_type
- dual – (default: False) boolean

EXAMPLES:

We check by tensoring $R_\lambda$ with $B(\infty)$ results in a component of $B(\lambda)$:

```
sage: B = crystals.infinity.Tableaux("A2")
sage: R = crystals.elementary.R("A2", B.Lambda()[1]+B.Lambda()[2])
sage: T = crystals.TensorProduct(R, B)
sage: mg = T(R.highest_weight_vector(), B.highest_weight_vector())
sage: S = T.subcrystal(generators=[mg])
sage: sorted([x.weight() for x in S], key=str)
[(0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 1, 1), (1, 1, 1), (1, 2, 0), (2, 0, 1), (2, 1, 0)]
sage: C = crystals.Tableaux("A2", shape=[2,1])
sage: sorted([x.weight() for x in C], key=str)
[(0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 1, 1), (1, 1, 1), (1, 2, 0), (2, 0, 1), (2, 1, 0)]
sage: GT = T.digraph(subset=S)
sage: GC = C.digraph()
sage: GT.is_isomorphic(GC, edge_labels=True)
True
```

class Element

Bases: AbstractSingleCrystalElement

Element of a $R_\lambda$ crystal.

epsilon(i)

Return $\varepsilon_i$ of self.

We have $\varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle$ for all $i$, where $h_i$ is a simple coroot.

INPUT:

- i – An element of the index set
EXAMPLES:

```
sage: la = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: R = crystals.elementary.R("A2", la[1])
sage: r = R.highest_weight_vector()
sage: [r.epsilon(i) for i in R.index_set()]
[-1, 0]
sage: R = crystals.elementary.R("A2", la[1], dual=True)
sage: r = R.highest_weight_vector()
sage: [r.epsilon(i) for i in R.index_set()]
[0, 0]
```

\(\phi(i)\)

Return \(\varphi_i\) of self, which is 0 for all \(i\).

**INPUT:**

- \(i\) – An element of the index set

**EXAMPLES:**

```
sage: la = RootSystem("C5").weight_lattice().fundamental_weights()
sage: r = R.highest_weight_vector()
sage: [r.phi(i) for i in R.index_set()]
[0, 0, 0, 0, 0]
sage: r = R.highest_weight_vector()
sage: [r.phi(i) for i in R.index_set()]
[0, 0, 0, 1, 1]
```

**weight()**

Return the weight of self, which is always \(\lambda\).

**EXAMPLES:**

```
sage: ct = CartanType(['C',5])
sage: la = RootSystem(ct).weight_lattice().fundamental_weights()
sage: t = T.highest_weight_vector()
sage: t.weight()
```

**cardinality()**

Return the cardinality of self, which is always 1.

**EXAMPLES:**

```
sage: La = RootSystem(['C',12]).weight_lattice().fundamental_weights()
sage: R = crystals.elementary.R(['C',12],La[9])
sage: R.cardinality()
1
```

**weight_lattice_realization()**

Return a realization of the lattice containing the weights of self.
EXAMPLES:

```python
sage: La = RootSystem(['C', 12]).weight_lattice().fundamental_weights()
sage: R = crystals.elementary.R(['C', 12], La[9])
sage: R.weight_lattice_realization()
Weight lattice of the Root system of type ['C', 12]
sage: ct = CartanMatrix([[2, -4], [-5, 2]])
sage: La = RootSystem(ct).weight_lattice().fundamental_weights()
sage: R = crystals.elementary.R(ct, La[1])
sage: R.weight_lattice_realization()
Weight lattice of the Root system of type
[ 2 -4]
[-5  2]
```

class `sage.combinat.crystals.elementary_crystals.TCrystal(cartan_type, weight)`

Bases: `UniqueRepresentation`, `Parent`

The crystal $T_\lambda$.

Let $\lambda$ be a weight. As defined in [Ka1993] the crystal $T_\lambda = \{t_\lambda\}$ is a single element crystal with the crystal structure defined by

$$\text{wt}(t_\lambda) = \lambda, \quad e_i t_\lambda = f_i t_\lambda = 0, \quad e_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty.$$  

The crystal $T_\lambda$ shifts the weights of the vertices in a crystal $B$ by $\lambda$ when tensored with $B$, but leaves the graph structure of $B$ unchanged. That is to say, for all $b \in B$, we have $\text{wt}(b \otimes t_\lambda) = \text{wt}(b) + \lambda$.

INPUT:

- `cartan_type` – A Cartan type
- `weight` – An element of the weight lattice of type `cartan_type`

EXAMPLES:

```python
sage: ct = CartanType(['A', 2])
sage: C = crystals.Tableaux(ct, shape=[1])
sage: for x in C: x.weight()  
(1, 0, 0)
(0, 1, 0)
(0, 0, 1)

sage: La = RootSystem(ct).ambient_space().fundamental_weights()
sage: TLa = crystals.elementary.T(ct, 3*(La[1] + La[2]))
sage: TP = crystals.TensorProduct(TLa, C)
sage: for x in TP: x.weight()  
(7, 3, 0)
(6, 4, 0)
(6, 3, 1)

sage: G = C.digraph()
sage: H = TP.digraph()
sage: G.is_isomorphic(H, edge_labels=True)
True
```

class `Element`

Bases: `AbstractSingleCrystalElement`

Element of a $T_\lambda$ crystal.
**epsilon**(*i*)

Return $\varepsilon_i$ of self, which is $-\infty$ for all $i$.

**INPUT:**
- *i* – An element of the index set

**EXAMPLES:**

```python
sage: ct = CartanType(['C',5])
sage: la = RootSystem(ct).weight_lattice().fundamental_weights()
sage: t = T.highest_weight_vector()
sage: [t.epsilon(i) for i in T.index_set()]
[-inf, -inf, -inf, -inf, -inf]
```

**phi**(*i*)

Return $\varphi_i$ of self, which is $-\infty$ for all $i$.

**INPUT:**
- *i* – An element of the index set

**EXAMPLES:**

```python
sage: ct = CartanType(['C',5])
sage: la = RootSystem(ct).weight_lattice().fundamental_weights()
sage: t = T.highest_weight_vector()
sage: [t.phi(i) for i in T.index_set()]
[-inf, -inf, -inf, -inf, -inf]
```

**weight**()

Return the weight of self, which is always $\lambda$.

**EXAMPLES:**

```python
sage: ct = CartanType(['C',5])
sage: la = RootSystem(ct).weight_lattice().fundamental_weights()
sage: t = T.highest_weight_vector()

sage: t.weight()
```

**cardinality**()

Return the cardinality of self, which is always 1.

**EXAMPLES:**

```python
sage: La = RootSystem(['C',12]).weight_lattice().fundamental_weights()
sage: T = crystals.elementary.T(['C',12], La[9])
sage: T.cardinality()
1
```

**weight_lattice_realization**()

Return a realization of the lattice containing the weights of self.

**EXAMPLES:**

```python
```
Combinatorics, Release 10.0

5.1.48 Fast Rank Two Crystals

class sage.combinat.crystals.fast_crystals.FastCrystal(ct, shape, format)
Bases: UniqueRepresentation, Parent

An alternative implementation of rank 2 crystals. The root operators are implemented in memory by table lookup. This means that in comparison with the CrystalsOfTableaux, these crystals are slow to instantiate but faster for computation. Implemented for types A₂, B₂, and C₂.

INPUT:

• cartan_type – the Cartan type and must be either type A₂, B₂, or C₂

• shape – A shape is of the form [l₁,l₂] where l₁ and l₂ are either integers or (in type B₂) half integers such that l₁ - l₂ is integral. It is assumed that l₁ ≥ l₂ ≥ 0. If l₁ and l₂ are integers, this will produce a crystal isomorphic to the one obtained by crystals.Tableaux(type, shape=[l₁,l₂]). Furthermore crystals.FastRankTwo(['B', 2], l₁+1/2, l₂+1/2) produces a crystal isomorphic to the following crystal T:

```python
sage: C = crystals.Tableaux(['B',2], shape=[11,12])  # not tested
sage: D = crystals.Spins(['B',2])  # not tested
sage: T = crystals.TensorProduct(C, D, C.list()[0], D.list()[0])  # not tested
```

• format – (default: 'string') the default representation of elements is in term of the Berenstein-Zelevinsky-Littelmann (BZL) strings [a₁, a₂, ...] described under metapost in crystals. Alternative representations may be obtained by the options 'dual_string' or 'simple'. In the 'simple' format, the element is represented by an integer, and in the 'dual_string' format, it is represented by the BZL string, but the underlying decomposition of the long Weyl group element into simple reflections is changed.

class Element(parent, value, format)
Bases: Element

EXAMPLES:

```python
sage: C = crystals.FastRankTwo(['A',2],shape=[2,1])
sage: c = C(0); c
[0, 0, 0]
sage: C[0].parent()
The fast crystal for A2 with shape [2,1]
sage: TestSuite(c).run()
```
Combinatorics, Release 10.0

\( e(i) \)

Return the action of \( e_i \) on self.

EXAMPLES:

```python
sage: C = crystals.FastRankTwo(['A',2],shape=[2,1])
sage: C(1).e(1)
[0, 0, 0]
sage: C(0).e(1) is None
True
```

\( f(i) \)

Return the action of \( f_i \) on self.

EXAMPLES:

```python
sage: C = crystals.FastRankTwo(['A',2],shape=[2,1])
sage: C(6).f(1)
[1, 2, 1]
sage: C(7).f(1) is None
True
```

\( \text{weight}() \)

Return the weight of self.

EXAMPLES:

```python
sage: [v.weight() for v in crystals.FastRankTwo(['A',2], shape=[2,1])]
[(2, 1, 0), (1, 2, 0), (1, 1, 1), (1, 0, 2), (0, 1, 2), (2, 0, 1), (1, 1,
1), (0, 2, 1)]
sage: [v.weight() for v in crystals.FastRankTwo(['B',2], shape=[1,0])]
[(1, 0), (0, 1), (0, 0), (0, -1), (-1, 0)]
sage: [v.weight() for v in crystals.FastRankTwo(['B',2], shape=[1/2,1/2])]
[(1/2, 1/2), (1/2, -1/2), (-1/2, 1/2), (-1/2, -1/2)]
sage: [v.weight() for v in crystals.FastRankTwo(['C',2], shape=[1,0])]
[(1, 0), (0, 1), (0, -1), (-1, 0)]
sage: [v.weight() for v in crystals.FastRankTwo(['C',2], shape=[1,1])]
[(1, 1), (1, -1), (0, 0), (-1, 1), (-1, -1)]
```

\( \text{cmp_elements}(x,y) \)

Return True if and only if there is a path from \( x \) to \( y \) in the crystal graph.

Because the crystal graph is classical, it is a directed acyclic graph which can be interpreted as a poset. This function implements the comparison function of this poset.

EXAMPLES:

```python
sage: C = crystals.FastRankTwo(['A',2],shape=[2,1])
sage: x = C(0)
sage: y = C(1)
sage: C.cmp_elements(x,y)
-1
sage: C.cmp_elements(y,x)
1
sage: C.cmp_elements(x,x)
0
```
digraph()
    Return the digraph associated to self.

    EXAMPLES:
    >>> C = crystals.FastRankTwo(['A',2],shape=[2,1])
    >>> C.digraph()
    Digraph on 8 vertices

5.1.49 Fully commutative stable Grothendieck crystal

AUTHORS:
    • Jianping Pan (2020-08-31): initial version
    • Wencin Poh (2020-08-31): initial version
    • Anne Schilling (2020-08-31): initial version

class sage.combinat.crystals.fully_commutative_stable_grothendieck.DecreasingHeckeFactorization(parent, t)

    Bases: Element
    
    Class of decreasing factorizations in the 0-Hecke monoid.

    INPUT:
    • t – decreasing factorization inputted as list of lists
    • max_value – maximal value of entries

    EXAMPLES:
    >>> from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
    >>> t = [[[3, 2], [], [2, 1]]
    >>> h = DecreasingHeckeFactorization(t, 3); h
    (3, 2)()() (2, 1)
    >>> h.excess
    1
    >>> h.factors
    3
    >>> h.max_value
    3
    >>> h.value
    ((3, 2), (), (2, 1))

    >>> u = [[[3, 2, 1], [3], [2, 1]]
    >>> h = DecreasingHeckeFactorization(u); h
    (3, 2, 1)()() (3) (2, 1)
    >>> h.weight()
    (2, 1, 3)
    >>> h.parent()
    Decreasing Hecke factorizations with 3 factors associated to [2, 1, 3, 2] with excess 1
to_increasing_hecke_biword()

Return the associated increasing Hecke biword of self.

EXAMPLES:

```
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: t = [[2], [], [2, 1], [4, 3, 1]]
sage: h = DecreasingHeckeFactorization(t, 4)
sage: h.to_increasing_hecke_biword()
[[1, 1, 1, 2, 2, 4], [1, 3, 4, 1, 2, 2]]
```

to_word()

Return the word associated to self in the 0-Hecke monoid.

EXAMPLES:

```
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: t = [[2], [], [2, 1], [4, 3, 1]]
sage: h = DecreasingHeckeFactorization(t)
sage: h.to_word()
[2, 2, 1, 4, 3, 1]
```

weight()

Return the weight of self.

EXAMPLES:

```
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: t = [[2], [2, 1], [], [4, 3, 1]]
sage: h = DecreasingHeckeFactorization(t, 6)
sage: h.weight()
(3, 0, 2, 1)
```

class sage.combinat.crystals.fully_commutative_stable_grothendieck.DcreasingHeckeFactorizations(w, factors, excess)

Bases: UniqueRepresentation, Parent

Set of decreasing factorizations in the 0-Hecke monoid.

INPUT:

- w – an element in the symmetric group
- factors – the number of factors in the factorization
- excess – the total number of letters in the factorization minus the length of a reduced word for w

EXAMPLES:
```python
sage: S = SymmetricGroup(3+1)
sage: w = S.from_reduced_word([1, 3, 2, 1])
sage: F = DecreasingHeckeFactorizations(w, 3, 3); F
Decreasing Hecke factorizations with 3 factors associated to [1, 3, 2, 1] with excess 3
sage: F.list()
[(3, 1)(3, 1)(3, 2, 1), (3, 1)(3, 2, 1)(2, 1), (3, 2, 1)(2, 1)(2, 1)]
```

**Element**

alias of `DecreasingHeckeFactorization`

**list()**

Return list of all elements of `self`.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorizations
sage: S = SymmetricGroup(3+1)
sage: w = S.from_reduced_word([1, 3, 2, 1])
sage: F = DecreasingHeckeFactorizations(w, 3, 3)
sage: F.list()
[(3, 1)(3, 1)(3, 2, 1), (3, 1)(3, 2, 1)(2, 1), (3, 2, 1)(2, 1)(2, 1)]
```

**class** `sage.combinat.crystals.fully_commutative_stable_grothendieck.FullyCommutativeStableGrothendieckCrystal`

Bases: `UniqueRepresentation`, `Parent`

The crystal on fully commutative decreasing factorizations in the 0-Hecke monoid, as introduced by [MPPS2020].

**INPUT:**

- `w` – an element in the symmetric group or a (skew) shape
- `factors` – the number of factors in the factorization
- `excess` – the total number of letters in the factorization minus the length of a reduced word for `w`
- `shape` – (default: False) indicator for input `w`. True if `w` is entered as a (skew) shape and False otherwise.

**EXAMPLES:**

```python
sage: S = SymmetricGroup(3+1)
sage: w = S.from_reduced_word([1, 3, 2])
sage: B = crystals.FullyCommutativeStableGrothendieck(w, 3, 2); B
Fully commutative stable Grothendieck crystal of type A_2 associated to [1, 3, 2] with excess 2
sage: B.list()
```

(continues on next page)
We can also access the crystal by specifying a skew shape:

```
sage: crystals.FullyCommutativeStableGrothendieck([[2, 2], [1]], 4, 1, shape=True)
```

Fully commutative stable Grothendieck crystal of type A_3 associated to [2, 1, 3] → with excess 1

We can compute the highest weight elements:

```
sage: hw = [w for w in B if w.is_highest_weight()]
sage: hw
[(1)(3, 1)(3, 2), (3)(3, 1)(3, 2)]
sage: hw[0].weight()
(2, 2, 1)
```

The crystal operators themselves move elements between adjacent factors:

```
sage: b = hw[0]; b
(1)(3, 1)(3, 2)
sage: b.f(2)
(3, 1)(1)(3, 2)
```

class Element(parent, t)

Bases: DecreasingHeckeFactorization

Create an instance self of element t.

This method takes into account the constraints on the word, the number of factors, and excess statistic associated to the parent class.

EXAMPLES:

```
sage: S = SymmetricGroup(3+1)
sage: w = S.from_reduced_word([1, 3, 2])
sage: B = crystals.FullyCommutativeStableGrothendieck(w, 3, 2)
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: h = B([[3], [4, 2, 1], [4, 3, 1]])
```

```
bracketing(i)

Remove all bracketed letters between i-th and (i + 1)-th entry.

EXAMPLES:

```
sage: S = SymmetricGroup(4+1)
sage: w = S.from_reduced_word([3, 2, 1, 4, 3])
sage: B = crystals.FullyCommutativeStableGrothendieck(w, 3, 2)
sage: h = B([[3], [4, 2, 1], [4, 3, 1]])
```
sage: h.bracketing(1)
[[], []]
sage: h.bracketing(2)
[[], [2, 1]]

\( e(i) \)

Return the action of \( e_i \) on \( \text{self} \) using the rules described in [MPPS2020].

**EXAMPLES:**

```python
sage: S = SymmetricGroup(4+1)
sage: w = S.from_reduced_word([2, 1, 4, 3, 2])
sage: B = crystals.FullyCommutativeStableGrothendieck(w, 4, 3)
sage: h = B([[4, 2], [4, 2, 1], [3, 2], [2]]); h
(4, 2)(4, 2, 1)(3, 2)(2)
sage: h.e(1)
(4, 2)(4, 2, 1)(3)(3, 2)
sage: h.e(2)
(4, 2)(2, 1)(4, 3, 2)(2)
sage: h.e(3)
```

\( f(i) \)

Return the action of \( f_i \) on \( \text{self} \) using the rules described in [MPPS2020].

**EXAMPLES:**

```python
sage: S = SymmetricGroup(4+1)
sage: w = S.from_reduced_word([3, 2, 1, 4, 3])
sage: B = crystals.FullyCommutativeStableGrothendieck(w, 4, 3)
sage: h = B([[3, 2], [2, 1], [4, 3], [3, 1]]); h
(3, 2)(2, 1)(4, 3)(3)
sage: h.f(1)
(3, 2)(2, 1)(4, 3, 1)(3)
sage: h.f(2)
sage: h.f(3)
(3, 2, 1)(1)(4, 3)(3, 1)
```

**module_generators()**

Return generators for \( \text{self} \) as a crystal.

**EXAMPLES:**

```python
sage: S = SymmetricGroup(3+1)
sage: w = S.from_reduced_word([1, 3, 2])
sage: B = crystals.FullyCommutativeStableGrothendieck(w, 3, 2)
sage: B.module_generators
sage: C = crystals.FullyCommutativeStableGrothendieck(w, 4, 2)
sage: C.module_generators
```
5.1.50 Crystals of Generalized Young Walls

AUTHORS:

- Lucas David-Roesler: Initial version
- Ben Salisbury: Initial version
- Travis Scrimshaw: Initial version

Generalized Young walls are certain generalizations of Young tableaux introduced in [KS2010] and designed to be a realization of the crystals \( \mathcal{B}(\infty) \) and \( \mathcal{B}(\lambda) \) in type \( A_n^{(1)} \).

REFERENCES:

- [KLRS2016]
- [KS2010]

class \texttt{sage.combinat.crystals.generalized_young_walls.CrystalOfGeneralizedYoungWalls}(n, \text{La})

Bases: \texttt{InfinityCrystalOfGeneralizedYoungWalls}

The crystal \( \mathcal{Y}(\lambda) \) of generalized Young walls of the given type with highest weight \( \lambda \).

These were characterized in Theorem 4.1 of [KS2010]. See \texttt{GeneralizedYoungWall}. \texttt{in_highest_weight_crystal()}.

INPUT:

- \( n \) – type \( A_n^{(1)} \)
- \( \text{weight} \) – dominant integral weight

EXAMPLES:

\begin{verbatim}
 sage: La = RootSystem(['A',3,1]).weight_lattice(extended=True).fundamental_weights()[1]
sage: YLa = crystals.GeneralizedYoungWalls(3,La)
sage: y = YLa([[0],[1,0,3,2,1],[2,1,0],[3]])
sage: y.pp()
 3|
 0|1|2|
1|2|3|0|1|
 0|
sage: y.weight()
sage: y.in_highest_weight_crystal(La)
True
sage: y.f(1)
[[0], [1, 0, 3, 2, 1], [2, 1, 0], [3], [], [1]]
sage: y.f(1).f(1)
[[0], [1, 0, 3, 2, 1], [2, 1, 0], [3], [], [1]]
sage: yy = crystals.infinity.GeneralizedYoungWalls(3)([[0], [1, 0, 3, 2, 1], [2, 1, 0], [3], [], [1]])
sage: yy.f(1)
[[0], [1, 0, 3, 2, 1], [2, 1, 0], [3], [], [1], [], [], [1]]
sage: yyy = yy.f(1)
sage: yyy.in_highest_weight_crystal(La)
False
sage: LS = crystals.LSPaths(['A',3,1],[1,0,0,0])
\end{verbatim}

(continues on next page)
sage: C = LS.subcrystal(max_depth=4)
sage: G = LS.digraph(subset=C)
sage: P = RootSystem(['A',3,1]).weight_lattice(extended=True)
sage: La = P.fundamental_weights()
sage: YW = crystals.GeneralizedYoungWalls(3,La[0])
sage: CW = YW.subcrystal(max_depth=4)

sage: GW = YW.digraph(subset=CW)
sage: GW.is_isomorphic(G, edge_labels=True)
True

To display the crystal down to a specified depth:

sage: S = YLa.subcrystal(max_depth=4)
sage: G = YLa.digraph(subset=S)
sage: view(G) # not tested

Element

alias of CrystalOfGeneralizedYoungWallsElement

class sage.combinat.crystals.generalized_young_walls.CrystalOfGeneralizedYoungWallsElement(parent, data)

Bases: GeneralizedYoungWall

Element of the highest weight crystal of generalized Young walls.

e(i)

Compute the action of $e_i$ restricted to the highest weight crystal.

EXAMPLES:

```
sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()[1]
sage: hwy = crystals.GeneralizedYoungWalls(2,La)([[], [1, 0], [2, 1]])
sage: hwy.e(1)
[[], [1, 0], [2]]
sage: hwy.e(2)
sage: hwy.e(3)
```

f(i)

Compute the action of $f_i$ restricted to the highest weight crystal.

EXAMPLES:

```
sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()[1]
sage: GW = crystals.infinity.GeneralizedYoungWalls(2)
sage: y = GW([[], [1, 0], [2, 1]])
sage: y.f(1)
[[], [1, 0], [2, 1], [], [1]]
sage: hwy = crystals.GeneralizedYoungWalls(2,La)([[], [1, 0], [2, 1]])
sage: hwy.f(1)
```

phi(i)

Return the value $\varepsilon_i(Y) + \langle h_i, wt(Y) \rangle$, where $h_i$ is the $i$-th simple coroot and $Y$ is self.
EXAMPLES:

```python
sage: La = RootSystem(['A',3,1]).weight_lattice(extended=True).fundamental_weights()
sage: y = crystals.GeneralizedYoungWalls(3,La[0])([])
sage: y.phi(1)
0
sage: y.phi(2)
0
```

**weight()**

Return the weight of self in the highest weight crystal as an element of the weight lattice $\bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i$.

EXAMPLES:

```python
sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()[1]
sage: hwy = crystals.GeneralizedYoungWalls(2,La)([[[],[1,0],[2,1]]])
sage: hwy.weight()
```

class sage.combinat.crystals.generalized_young_walls.GeneralizedYoungWall(parent, data)

Bases: CombinatorialElement

A generalized Young wall.

For more information, see `InfinityCrystalOfGeneralizedYoungWalls`.

EXAMPLES:

```python
sage: Y = crystals.infinity.GeneralizedYoungWalls(4)
sage: mg = Y.module_generators[0]; mg.pp()
0
sage: mg.f_string([1,2,0,1]).pp()
1|2|
0|1|
```

**Epsilon()**

Return $\sum_{i=0}^{n} \epsilon_i(Y)\Lambda_i$ where $Y$ is self.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(3)([[0],[1,0,3,2],[2,1],[3,2,→1,0,3,2],[0],[[],[2]])
sage: y.Epsilon()
Lambda[0] + 3*Lambda[2]
```

**Phi()**

Return $\sum_{i=0}^{n} \phi_i(Y)\Lambda_i$ where $Y$ is self.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(3)([[0],[1,0,3,2],[2,1],[3,2,→1,0,3,2],[0],[[],[2]])
sage: y.Phi()
```

(continues on next page)
\[-\text{Lambda}[0] + 3\text{Lambda}[1] - \text{Lambda}[2] + 3\text{Lambda}[3]\]

```python
sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2], [0], [0], [2]])
sage: x.Phi()
\text{2*Lambda}[0] + \text{Lambda}[1] - \text{Lambda}[2] + \text{Lambda}[3]
```

\[a(i, k)\]
Return the number \(a_i(k)\) of \(i\)-colored boxes in the \(k\)-th column of \text{self}.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(3)([[0], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2], [0], [0], [2]])
sage: y.a(1, 2)
1
sage: y.a(0, 2)
1
sage: y.a(3, 2)
0
```

\[\text{column}(k)\]
Return the list of boxes from the \(k\)-th column of \text{self}.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(3)([[0], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2], [0], [0], [2]])
sage: y.column(2)
[None, 0, 1, 2, None, None, None]
sage: hw = crystals.infinity.GeneralizedYoungWalls(5)([])
sage: hw.column(1)
[]
```

\[\text{content}()\]
Return total number of blocks in \text{self}.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(2)([[0], [1, 0, 3, 2], [2, 1, 0, 2], [], [1]])
sage: y.content()
8
sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[0], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2], [0], [0], [2]])
sage: x.content()
13
```

\[e(i)\]
Return the application of the Kashiwara raising operator \(e_i\) on \text{self}.

This will remove the \(i\)-colored box corresponding to the rightmost \(+\) in \text{self}.signature\(i\).
EXAMPLES:

```python
sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2], [], [], []])
sage: x.e(2)
[[], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2]]
sage: _.e(2)
[[], [1, 0, 3], [2, 1], [3, 2, 1, 0, 3, 2]]
sage: _.e(2)
[[], [1, 0, 3], [2, 1], [3, 2, 1, 0, 3]]
sage: _.e(2)
```

**epsilon**(i)

Return the number of \(i\)-colored arrows in the \(i\)-string above \texttt{self} in the crystal graph.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(3)([[], [1, 0, 3, 2], [2, 1], [3, 2, 1, 0, 3, 2], [], [], []])
sage: y.epsilon(1)
0
sage: y.epsilon(2)
3
sage: y.epsilon(0)
0
```

**f**(i)

Return the application of the Kashiwara lowering operator \(f_i\) on \texttt{self}.

This will add an \(i\)-colored colored box to the site corresponding to the leftmost plus in \texttt{self}. \texttt{signature(i)}.

EXAMPLES:

```python
sage: hw = crystals.infinity.GeneralizedYoungWalls(2)([[0], [1, 0], [2, 1, 0, 2], [], [1]])
sage: hw.f(1)
[[], [1]]
sage: _.f(2)
[[], [1], [2]]
sage: _.f(0)
[[], [1, 0], [2]]
sage: _.f(0)
[[0], [1, 0], [2]]
```

**generate_signature**(i)

The \(i\)-signature of \texttt{self} (with whitespace where cancellation occurs) together with the unreduced sequence from \{\(+, -\}\}. The result also records to the row and column position of the sign.

EXAMPLES:

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(2)([['0'], [1, 0], [2, 1, 0, 2], [], [1]])
sage: y.generate_signature(1)
([[ '+' , 2, 5], ['-', 4, 1]], ' ')
```
in_highest_weight_crystal($La$)

Return a boolean indicating if the generalized Young wall element is in the highest weight crystal cut out by the given highest weight $La$.

By Theorem 4.1 of [KS2010], a generalized Young wall $Y$ represents a vertex in the highest weight crystal $Y(\lambda)$, with $\lambda = \Lambda_i + \Lambda_{i+1} + \cdots + \Lambda_{i+p}$, a dominant integral weight of level $\ell > 0$, if it satisfies the following condition. For each positive integer $k$, if there exists $j \in I$ such that $a_j(k) - a_{j-1}(k) > 0$, then for some $p = 1, \ldots, \ell$,

$$j + k \equiv i_p + 1 \mod n + 1 \text{ and } a_j(k) - a_{j-1}(k) \leq \lambda(h_{i_p}),$$

where \{h_0, h_1, \ldots, h_n\} is the set of simple coroots attached to $A_n^{(1)}$.

EXAMPLES:

```python
sage: La = RootSystem(['A','2',1]).weight_lattice(extended=True).fundamental_weights()[1]
sage: GYW = crystals.infinity.GeneralizedYoungWalls(2)
sage: y = GYW([[[]],[[1],[2,1]]])
sage: y.in_highest_weight_crystal(La)
True
sage: x = GYW([[[]],[[1],[2],[2],[2],[2],[2],[2],[2]])
sage: x.in_highest_weight_crystal(La)
False
```

latex_large()

Generate LaTeX code for self but the output is larger. Requires TikZ.

EXAMPLES:

```python
sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[[]],[[1],[0,3,2],[2,1],[3,2,1],[0,3,2],[2],[2],[2]])
sage: x.latex_large()'
```

number_of_parts()

Return the value of $\mathcal{N}$ on self.

In [KLRS2016], the statistic $\mathcal{N}$ was defined on elements in $\mathcal{Y}(\infty)$ which counts how many parts are in the corresponding Kostant partition. Specifically, the computation of $\mathcal{N}(Y)$ is done using the following algorithm:

- If $Y$ has no rows whose right-most box is colored $n$ and such that the length of this row is a multiple of $n + 1$, then $\mathcal{N}(Y)$ is the total number of distinct rows in $Y$, not counting multiplicity.
- Otherwise, search $Y$ for the longest row such that the right-most box is colored $n$ and such that the total number of boxes in the row is $k(n + 1)$ for some $k \geq 1$. Replace this row by $n + 1$ distinct rows of length $k$, reordering all rows, if necessary, so that the result is a proper wall. (Note that the resulting wall may no longer be reduced.) Repeat the search and replace process for all other rows of the above form for each $k' < k$. Then $\mathcal{N}(Y)$ is the number of distinct rows, not counting multiplicity, in the wall resulting from this process.

EXAMPLES:
sage: Y = crystals.infinity.GeneralizedYoungWalls(3)
sage: y = Y([[0],[],[],[0],[0],[0],[0],[0]])
sage: y.number_of_parts()
1

sage: Y = crystals.infinity.GeneralizedYoungWalls(3)
sage: y = Y([[0,3,2],[1,0],[0],[0],[1,0],[0],[0],[0]])
sage: y.number_of_parts()
4

sage: Y = crystals.infinity.GeneralizedYoungWalls(2)
sage: y = Y([[0,2,1],[1,0],[2,1,0,2,1,0,2,1,0],[0],[2,1,0,2,1,0]])
sage: y.number_of_parts()
8

\textbf{phi}(i)

Return the value $\varepsilon_i(Y) + \langle h_i, \text{wt}(Y) \rangle$, where $h_i$ is the $i$-th simple coroot and $Y$ is \texttt{self}.

\textbf{EXAMPLES:}

sage: y = crystals.infinity.GeneralizedYoungWalls(3)([[0],[1,0,3,2],[2,1],[3,2,1,0,3,2],[0],[0],[2]])
sage: y.phi(1)
3
sage: y.phi(2)
-1

\textbf{pp}()

Pretty print \texttt{self}.

\textbf{EXAMPLES:}

sage: y = crystals.infinity.GeneralizedYoungWalls(2)([[0,2,1],[1,0,2,1,0],[0],[1,0,2],[0],[0],[1]])
sage: y.pp()
1|
  |
2|0|1|
  |
0|1|2|0|1|
  |
0|1|2|0|

\textbf{raw_signature}(i)

Return the sequence from $\{+, -\}$ obtained from all $i$-admissible slots and removable $i$-boxes without canceling any $(+, -)$-pairs. The result also notes the row and column of the sign.

\textbf{EXAMPLES:}

sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[0],[1,0,3,2],[2,1],[3,2,1,0,3,2],[0],[0],[2]])
sage: x.raw_signature(2)
[['-', 3, 6], ['-', 1, 4], ['-', 6, 1]]
signature\((i)\)

Return the \(i\)-signature of \texttt{self}.

The signature is obtained by reading \texttt{self} in columns bottom to top starting from the left. Then add a \(-\) at every \(i\)-box which may be removed from \texttt{self} and still obtain a legal generalized Young wall, and add a \(+\) at each site for which an \(i\)-box may be added and still obtain a valid generalized Young wall. Then successively cancel any \((+,-)\)-pair to obtain a sequence of the form \(-\cdots+\cdots-\). This resulting sequence is the output.

**EXAMPLES:**

```python
sage: y = crystals.infinity.GeneralizedYoungWalls(2)([[0], [1, 0], [2, 1, 0, 2], []], →[1])
sage: y.signature(1)
''
sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[], [1, 0, 3, 2], [2, 1], [3, 2, 1, →0, 3, 2], [], [], [], [2]])
sage: x.signature(2)
'---'
```

sum_of_weighted_row_lengths()

Return the value of \(\mathcal{M}\) on \texttt{self}.

Let \(\mathcal{Y}_0 \subset \mathcal{Y}(\infty)\) be the set of generalized Young walls which have no rows whose right-most box is colored \(n\). For \(Y \in \mathcal{Y}_0\),

\[
\mathcal{M}(Y) = \sum_{i=1}^{n} (i + 1) M_i(Y),
\]

where \(M_i(Y)\) is the number of nonempty rows in \(Y\) whose right-most box is colored \(i - 1\).

**EXAMPLES:**

```python
sage: Y = crystals.infinity.GeneralizedYoungWalls(2)
sage: y = Y([[0, 2, 1, 0, 2], [1, 0, 2], [], [0, 2], [1, 0], [], [0], [1, 0]])
sage: y.sum_of_weighted_row_lengths()
15
```

weight\((\text{root\_lattice=}\text{False})\)

Return the weight of \texttt{self}.

**INPUT:**

- \texttt{root\_lattice} – boolean determining whether weight should appear in root lattice or not in extended affine weight lattice.

**EXAMPLES:**

```python
sage: x = crystals.infinity.GeneralizedYoungWalls(3)([[], [1, 0, 3, 2], [2, 1], [3, 2, 1, →0, 3, 2], [], [], [], [2]])
sage: x.weight()
sage: x.weight(root_lattice=True)
```
The crystal $\mathcal{Y}(\infty)$ of generalized Young walls of type $A_n^{(1)}$ as defined in [KS2010].

A generalized Young wall is a collection of boxes stacked on a fixed board, such that color of the box at the site located in the $j$-th row from the bottom and the $i$-th column from the right is $j - 1 \mod n + 1$. There are several growth conditions on elements in $Y \in \mathcal{Y}(\infty)$:

- Walls grow in rows from right to left. That is, for every box $y \in Y$ that is not in the rightmost column, there must be a box immediately to the right of $y$.
- For all $p > q$ such that $p - q \equiv 0 \mod n + 1$, the $p$-th row has most as many boxes as the $q$-th row.
- There does not exist a column in the walls such that if one $i$-colored box, for every $i = 0, 1, \ldots, n$, is removed from that column, then the result satisfies the above conditions.

There is a crystal structure on $\mathcal{Y}(\infty)$ defined as follows. Define maps

$$e_i, f_i : \mathcal{Y}(\infty) \to \mathcal{Y}(\infty) \sqcup \{0\}, \quad \varepsilon_i, \varphi_i : \mathcal{Y}(\infty) \to \mathbb{Z}, \quad wt : \mathcal{Y}(\infty) \to \bigoplus_{i=0}^{n} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta,$$

by

$$wt(Y) = - \sum_{i=0}^{n} m_i(Y) \alpha_i,$$

where $m_i(Y)$ is the number of $i$-boxes in $Y$, $\varepsilon_i(Y)$ is the number of $-$ in the $i$-signature of $Y$, and

$$\varphi_i(Y) = \varepsilon_i(Y) + \langle h_i, wt(Y) \rangle.$$

See `GeneralizedYoungWall.e()`, `GeneralizedYoungWall.f()`, and `GeneralizedYoungWall.signature()` for more about $e_i$, $f_i$, and $i$-signatures.

INPUT:

- $n$ – type $A_n^{(1)}$

EXAMPLES:

```python
sage: Yinf = crystals.infinity.GeneralizedYoungWalls(3)
sage: y = Yinf([[0],[1,0,3,2],[0],[3,2,1],[0],[1,0]])
sage: y.pp()
0|1|
0|
1|2|3|
2|3|0|1|
0|
sage: y.weight(root_lattice=True)
sage: y.f(0)
[[0], [1, 0, 3, 2], [], [3, 2, 1], [0], [1, 0], [], [0]]
sage: y.e(0).pp()
0|1|
```

(continues on next page)
To display the crystal down to depth 3:

```sage
S = Yinf.subcrystal(max_depth=3)
G = Yinf.digraph(subset=S)  # long time
view(G)  # not tested
```

**Element**

alias of `GeneralizedYoungWall`

### 5.1.51 Highest weight crystals

**class** `sage.combinat.crystals.highest_weight_crystals.FiniteDimensionalHighestWeightCrystal_TypeE(dominant_weight)`

Bases: `TensorProductOfCrystals`

Commonalities for all finite dimensional type $E$ highest weight crystals.

Subclasses should setup an attribute `column_crystal` in their `__init__` method before calling the `__init__` method of this class.

**Element**

alias of `TensorProductOfRegularCrystalsElement`

**module_generator()**

This yields the module generator (or highest weight element) of the classical crystal of given dominant weight in self.

**EXAMPLES:**

```sage
C=CartanType(['E',6])
La=C.root_system().weight_lattice().fundamental_weights()
T = crystals.HighestWeight(La[2])
T.module_generator()
[[[2, -1], [1,]]]

T = crystals.HighestWeight(0*La[2])
T.module_generator()
[]

C=CartanType(['E',7])
La=C.root_system().weight_lattice().fundamental_weights()
T = crystals.HighestWeight(La[1])
T.module_generator()
[[(-7, 1), (7,)]]
```

**class** `sage.combinat.crystals.highest_weight_crystals.FiniteDimensionalHighestWeightCrystal_TypeE6(dominant_weight)`

Bases: `FiniteDimensionalHighestWeightCrystal_TypeE`

Class of finite dimensional highest weight crystals of type $E_6$.

**EXAMPLES:**

### 5.1. Comprehensive Module List
sage: C=CartanType(['E',6])
sage: La=C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[2]); T
Finite dimensional highest weight crystal of type ['E', 6] and highest weight\n→Lambda[2]
sage: B1 = T.column_crystal[1]; B1
The crystal of letters for type ['E', 6]
sage: B6 = T.column_crystal[6]; B6
The crystal of letters for type ['E', 6] (dual)
sage: t = T(B6([-1]),B1([-1,3])); t
[(-1,), (-1, 3)]
sage: [t.epsilon(i) for i in T.index_set()]
[2, 0, 0, 0, 0, 0]
sage: [t.phi(i) for i in T.index_set()]
[0, 0, 1, 0, 0, 0]
sage: TestSuite(t).run()

class sage.combinat.crystals.highest_weight_crystals.FiniteDimensionalHighestWeightCrystal_TypeE7(dominant_weight)
Bases: FiniteDimensionalHighestWeightCrystal_TypeE

Class of finite dimensional highest weight crystals of type $E_7$.

EXAMPLES:

sage: C=CartanType(['E',7])
sage: La=C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[1])
sage: T.cardinality()
133
sage: B7 = T.column_crystal[7]; B7
The crystal of letters for type ['E', 7]
sage: t = T(B7([-5, 6]), B7([-2, 3])); t
[(-5, 6), (-2, 3)]
sage: [t.epsilon(i) for i in T.index_set()]
[0, 1, 0, 0, 1, 0, 0]
sage: [t.phi(i) for i in T.index_set()]
[0, 0, 1, 0, 0, 1, 0]
sage: TestSuite(t).run()

sage.combinat.crystals.highest_weight_crystals.HighestWeightCrystal(dominant_weight, model=None)

Return the highest weight crystal of highest weight dominant_weight of the given model.

INPUT:

- dominant_weight – a dominant weight
- model – (optional) if not specified, then we have the following default models:
  - types $A_n, B_n, C_n, D_n, G_2$ - tableaux
  - types $E_{6,7}$ - type E finite dimensional crystal
  - all other types - LS paths

otherwise can be one of the following:
- 'Tableaux' - KN tableaux
-- 'TypeE' - type E finite dimensional crystal
-- 'NakajimaMonomials' - Nakajima monomials
-- 'LSPaths' - LS paths
-- 'AlcovePaths' - alcove paths
-- 'GeneralizedYoungWalls' - generalized Young walls
-- 'RiggedConfigurations' - rigged configurations

EXAMPLES:

```
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: wt = La[1] + La[2]
sage: crystals.HighestWeight(wt)
The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]]

sage: La = RootSystem(['C',2]).weight_lattice().fundamental_weights()
sage: wt = 5*La[1] + La[2]
sage: crystals.HighestWeight(wt)
The crystal of tableaux of type ['C', 2] and shape(s) [[6, 1]]

sage: La = RootSystem(['B',2]).weight_lattice().fundamental_weights()
sage: wt = La[1] + La[2]
sage: crystals.HighestWeight(wt)
The crystal of tableaux of type ['B', 2] and shape(s) [[3/2, 1/2]]
```

Some type $E$ examples:

```
sage: C = CartanType(['E',6])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[1])
sage: T.cardinality()
27
sage: T = crystals.HighestWeight(La[6])
sage: T.cardinality()
27
sage: T = crystals.HighestWeight(La[2])
sage: T.cardinality()
78
sage: T = crystals.HighestWeight(La[4])
sage: T.cardinality()
2925
sage: T = crystals.HighestWeight(La[3])
sage: T.cardinality()
351
sage: T = crystals.HighestWeight(La[5])
sage: T.cardinality()
351

sage: C = CartanType(['E',7])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[1])
sage: T.cardinality()
133
```

(continues on next page)
sage: T = crystals.HighestWeight(La[2])
sage: T.cardinality()
912
sage: T = crystals.HighestWeight(La[3])
sage: T.cardinality()
8645
sage: T = crystals.HighestWeight(La[4])
sage: T.cardinality()
365750
sage: T = crystals.HighestWeight(La[5])
sage: T.cardinality()
27664
sage: T = crystals.HighestWeight(La[6])
sage: T.cardinality()
1539
sage: T = crystals.HighestWeight(La[7])
sage: T.cardinality()
56

An example with an affine type:

sage: C = CartanType(['C',2,1])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[1])
sage: sorted(T.subcrystal(max_depth=3), key=str)
 (-Lambda[0] + Lambda[1] + Lambda[2] - delta,),
 (2*Lambda[0] - Lambda[1],),
 (Lambda[0] + Lambda[1] - Lambda[2],),
 (Lambda[0] - Lambda[1] + Lambda[2],),
 (Lambda[1],)]

Using the various models:

sage: La = RootSystem(['F',4]).weight_lattice().fundamental_weights()
sage: crystals.HighestWeight(wt)
The crystal of LS paths of type ['F', 4] and weight Lambda[1] + Lambda[4]
sage: crystals.HighestWeight(wt, model='NakajimaMonomials')
Highest weight crystal of modified Nakajima monomials of
sage: crystals.HighestWeight(wt, model='AlcovePaths')
sage: crystals.HighestWeight(wt, model='RiggedConfigurations')
sage: La = RootSystem(['A',3,1]).weight_lattice().fundamental_weights()
sage: wt = La[0] + La[2]
sage: crystals.HighestWeight(wt, model='GeneralizedYoungWalls')
Highest weight crystal of generalized Young walls of
Cartan type ['A', 3, 1] and highest weight Lambda[0] + Lambda[2]
### 5.1.52 Induced Crystals

We construct a crystal structure on a set induced by a bijection $\Phi$.

**AUTHORS:**


```python
class sage.combinat.crystals.induced_structure.InducedCrystal(X, phi, inverse)
    Bases: UniqueRepresentation, Parent
```

A crystal induced from an injection.

Let $X$ be a set and let $C$ be crystal and consider any injection $\Phi : X \to C$. We induce a crystal structure on $X$ by considering $\Phi$ to be a crystal morphism.

Alternatively we can induce a crystal structure on some (sub)set of $X$ by considering an injection $\Phi : C \to X$ considered as a crystal morphism. This form is also useful when the set $X$ is not explicitly known.

**INPUT:**

- $X$ -- the base set
- $\phi$ -- the map $\Phi$
- $\text{inverse}$ -- (optional) the inverse map $\Phi^{-1}$
- $\text{from\_crystal}$ -- (default: False) if the induced structure is of the second type $\Phi : C \to X$

**EXAMPLES:**

We construct a crystal structure of Gelfand-Tsetlin patterns by going through their bijection with semistandard tableaux:

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,3))
sage: G = GelfandTsetlinPatterns(4, 3)
sage: phi = lambda x: D(x.to_tableau())
sage: phi_inv = lambda x: G(x.to_tableau())
sage: I = crystals.Induced(G, phi, phi_inv)
sage: I.digraph().is_isomorphic(D.digraph(), edge_labels=True)
True
```

Now we construct the above example but inducing the structure going the other way (from tableaux to Gelfand-Tsetlin patterns). This can also give us more information coming from the crystal.

```python
sage: D2 = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,1))
sage: G2 = GelfandTsetlinPatterns(4, 1)
sage: phi2 = lambda x: D2(x.to_tableau())
sage: phi2_inv = lambda x: G2(x.to_tableau())
sage: I2 = crystals.Induced(D2, phi2_inv, phi2, from_crystal=True)
sage: I2.module_generators
[[[0, 0, 0, 0], [0, 0, 0], [0, 0], [0]],
 [[1, 0, 0, 0], [1, 0, 0], [1, 0], [1]],
 [[1, 1, 0, 0], [1, 1, 0], [1, 1], [1]],
 [[1, 1, 1, 0], [1, 1, 1], [1, 1], [1]],

We check an example when the codomain is larger than the domain (although here the crystal structure is trivial):
```
sage: P = Permutations(4)
sage: D = crystals.Tableaux(['A',3], shapes=Partitions(4))
sage: T = crystals.TensorProduct(D, D)
sage: phi = lambda p: T(D(RSK(p)[0]), D(RSK(p)[1]))
sage: phi_inv = lambda d: RSK_inverse(d[0].to_tableau(), d[1].to_tableau(), output='permutation')
sage: all(phi_inv(phi(p)) == p for p in P) # Check it really is the inverse
True
sage: I = crystals.Induced(P, phi, phi_inv)

We construct an example without a specified inverse map:

sage: X = Words(2,4)
sage: L = crystals.Letters(['A',1])
sage: T = crystals.TensorProduct(*[L]*4)
sage: Phi = lambda x : T(*[L(i) for i in x])
sage: I = crystals.Induced(X, Phi)

class Element
Bases: ElementWrapper

An element of an induced crystal.

\(e(i)\)
Return \(e_i\) of self.

EXAMPLES:

sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,3))
sage: G = GelfandTsetlinPatterns(4, 3)
sage: phi = lambda x: D(x.to_tableau())
sage: phi_inv = lambda x: G(x.to_tableau())
sage: I = crystals.Induced(G, phi, phi_inv)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])

sage: elt.e(1)
[[1, 1, 0, 0], [1, 1, 0], [1, 1], [1]]

\(\epsilon(i)\)
Return \(\epsilon_i\) of self.

EXAMPLES:

sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,3))
sage: G = GelfandTsetlinPatterns(4, 3)
sage: phi = lambda x: D(x.to_tableau())
sage: phi_inv = lambda x: G(x.to_tableau())
sage: I = crystals.Induced(G, phi, phi_inv)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 1], [1]])
```python
sage: [elt.epsilon(i) for i in I.index_set()]
[0, 1, 0]
```

\( f(i) \)

Return \( f_i \) of \( \text{self} \).

**EXAMPLES:**

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,3))
sage: G = GelfandTsetlinPatterns(4,3)
sage: phi = lambda x: D(x.to_tableau())
sage: phi_inv = lambda x: G(x.to_tableau())
sage: I = crystals.Induced(G, phi, phi_inv)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])
sage: elt.f(1)
[[1, 1, 0, 0], [1, 1, 0], [1, 0], [0]]
sage: elt.f(2)
sage: elt.f(3)
[[1, 1, 0, 0], [1, 0, 0], [1, 0], [1]]
```

\( \phi() \)

Return \( \varphi_i \) of \( \text{self} \).

**EXAMPLES:**

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,3))
sage: G = GelfandTsetlinPatterns(4,3)
sage: phi = lambda x: D(x.to_tableau())
sage: phi_inv = lambda x: G(x.to_tableau())
sage: I = crystals.Induced(G, phi, phi_inv)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])
sage: [elt.phi(i) for i in I.index_set()]
[1, 0, 1, 0]
```

\( \text{weight}() \)

Return the weight of \( \text{self} \).

**EXAMPLES:**

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,3))
sage: G = GelfandTsetlinPatterns(4,3)
sage: phi = lambda x: D(x.to_tableau())
sage: phi_inv = lambda x: G(x.to_tableau())
sage: I = crystals.Induced(G, phi, phi_inv)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])
sage: elt.weight()
(1, 0, 1, 0)
```

\( \text{cardinality}() \)

Return the cardinality of \( \text{self} \).

**EXAMPLES:**

```python
```

5.1. Comprehensive Module List 429
sage: P = Permutations(4)
sage: D = crystals.Tableaux(['A',3], shapes=Partitions(4))
sage: T = crystals.TensorProduct(D, D)
sage: phi = lambda p: T(D(RSK(p)[0]), D(RSK(p)[1]))
sage: phi_inv = lambda d: RSK_inverse(d[0].to_tableau(), d[1].to_tableau(), output='permutation')
sage: I = crystals.Induced(P, phi, phi_inv)
sage: I.cardinality() == factorial(4)
True

class sage.combinat.crystals.induced_structure.InducedFromCrystal(X, phi, inverse)
Bases: UniqueRepresentation, Parent
A crystal induced from an injection.
Alternatively we can induce a crystal structure on some (sub)set of \(X\) by considering an injection \(\Phi: C \to X\)
considered as a crystal morphism.
See also:
InducedCrystal
INPUT:
• \(X\) – the base set
• \(\Phi\) – the map \(\Phi\)
• \(\text{inverse}\) – (optional) the inverse map \(\Phi^{-1}\)
EXAMPLES:
We construct a crystal structure on generalized permutations with a fixed first row by using RSK:

```
sage: C = crystals.Tableaux(['A',3], shape=[2,1])
sage: def psi(x):
....:     ret = RSK_inverse(x.to_tableau(), Tableau([[1,1],[2]]))
....:     return (tuple(ret[0]), tuple(ret[1]))
sage: psi_inv = lambda x: C(RSK(*x)[0])
sage: I = crystals.Induced(D, phi, phi_inv, from_crystal=True)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])
sage: elt.e(1)
sage: elt.e(2)
```
epsilon(i)

Return $\epsilon_i$ of self.

EXAMPLES:

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,1))
sage: G = GelfandTsetlinPatterns(4, 1)
sage: def phi(x): return G(x.to_tableau())
sage: def phi_inv(x): return D(G(x).to_tableau())
sage: I = crystals.Induced(D, phi, phi_inv, from_crystal=True)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 1], [1]])
sage: [elt.epsilon(i) for i in I.index_set()]
[0, 1, 0]
```

f(i)

Return $f_i$ of self.

EXAMPLES:

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,1))
sage: G = GelfandTsetlinPatterns(4, 1)
sage: def phi(x): return G(x.to_tableau())
sage: def phi_inv(x): return D(G(x).to_tableau())
sage: I = crystals.Induced(D, phi, phi_inv, from_crystal=True)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0, 0], [1]])
sage: elt.f(1)
[[1, 1, 0, 0], [1, 1, 0], [1, 0, 0], [0]]
sage: elt.f(2)
sage: elt.f(3)
[[1, 1, 0, 0], [1, 0, 0], [1, 0], [1]]
```

phi(i)

Return $\varphi_i$ of self.

EXAMPLES:

```python
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,1))
sage: G = GelfandTsetlinPatterns(4, 1)
sage: def phi(x): return G(x.to_tableau())
sage: def phi_inv(x): return D(G(x).to_tableau())
sage: I = crystals.Induced(D, phi, phi_inv, from_crystal=True)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])
sage: [elt.epsilon(i) for i in I.index_set()]
[0, 1, 0]
```

weight()

Return the weight of self.

EXAMPLES:
sage: D = crystals.Tableaux(['A',3], shapes=PartitionsInBox(4,1))
sage: G = GelfandTsetlinPatterns(4,1)
sage: def phi(x): return G(x.to_tableau())
sage: def phi_inv(x): return D(G(x).to_tableau())
sage: I = crystals.Induced(D, phi, phi_inv, from_crystal=True)
sage: elt = I([[1, 1, 0, 0], [1, 1, 0], [1, 0], [1]])
sage: elt.weight() (1, 0, 1, 0)

cardinality()

Return the cardinality of self.

EXAMPLES:

sage: C = crystals.Tableaux(['A',3], shape=[2,1])
sage: def psi(x):
....:    ret = RSK_inverse(x.to_tableau(), Tableau([[1,1],[2]]))
....:    return (tuple(ret[0]), tuple(ret[1]))
sage: psi_inv = lambda x: C(RSK(*x)[0])
sage: I = crystals.Induced(C, psi, psi_inv, from_crystal=True)
sage: I.cardinality() == C.cardinality()
True

5.1.53 $B(\infty)$ Crystals of Tableaux in Nonexceptional Types and $G_2$

A tableau model for $B(\infty)$. For more information, see *InfinityCrystalOfTableaux*.

AUTHORS:

- Ben Salisbury: Initial version
- Travis Scrimshaw: Initial version

class sage.combinat.crystals.infinity_crystals.DualInfinityQueerCrystalOfTableaux(cartan_type)

Bases: *CrystalOfWords*

Initialize self.

EXAMPLES:

sage: B = crystals.infinity.Tableaux(['A',2])
sage: TestSuite(B).run() # long time

class Element

Bases: *InfinityQueerCrystalOfTableauxElement*

index_set()

Return the index set of self.

EXAMPLES:

sage: B = crystals.infinity.Tableaux(['Q',3])
sage: B.index_set()
(1, 2, -1)
module_generator()

Return the module generator (or highest weight element) of self.

The module generator is the unique semistandard hook tableau of shape \((n, n-1, \ldots, 2, 1)\) with weight 0.

EXAMPLES:

```python
sage: B = crystals.infinity.Tableaux(['Q',5])
sage: B.module_generator()
[[5, 5, 5, 5, 5], [4, 4, 4, 4], [3, 3, 3], [2, 2], [1]]
```

class sage.combinat.crystals.infinity_crystals.InfinityCrystalOfTableaux(cartan_type)

Bases: CrystalOfWords

\(B(\infty)\) crystal of tableaux.

A tableaux model \(\mathcal{T}(\infty)\) for the crystal \(B(\infty)\) is introduced by Hong and Lee in [HL2008]. This model is currently valid for types \(A_n, B_n, C_n, D_n,\) and \(G_2,\) and builds on the tableaux model given by Kashiwara and Nakashima [KN1994] in types \(A_n, B_n, C_n,\) and \(D_n,\) and by Kang and Misra [KM1994] in type \(G_2.\)

**Note:** We are using the English convention for our tableaux.

We say a tableau \(T\) is _marginally large_ if:

- for each \(1 \leq i \leq n\), the leftmost box in the \(i\)-th row from the top in \(T\) is an \(i\)-box,
- for each \(1 \leq i \leq n\), the number of \(i\)-boxes in the \(i\)-th row from the top in \(T\) is greater than the total number of boxes in the \((i+1)\)-th row by exactly one.

We now will describe this tableaux model type-by-type.

**Type** \(A_n\)

\(\mathcal{T}(\infty)\) is the set of marginally large semistandard tableaux with exactly \(n\) rows over the alphabet \(\{1 \prec 2 \prec \cdots \prec n+1\}\).

**Type** \(B_n\)

\(\mathcal{T}(\infty)\) is the set of marginally large semistandard tableaux with exactly \(n\) rows over the alphabet \(\{1 \prec \cdots \prec n \prec 0 \prec \pi \prec \cdots \prec 1\}\) and subject to the following constraints:

- for each \(1 \leq i \leq n\), the contents of the boxes in the \(i\)-th row are \(\leq i\),
- the entry 0 can appear at most once in a single row.

**Type** \(C_n\)

\(\mathcal{T}(\infty)\) is the set of marginally large semistandard tableaux with exactly \(n\) rows over the alphabet \(\{1 \prec \cdots \prec n \prec \pi \prec \cdots \prec 1\}\) and for each \(1 \leq i \leq n\), the contents of the boxes in the \(i\)-th row are \(\leq i\).
**Type** $D_n$

$\mathcal{T}(\infty)$ is the set of marginally large semistandard tableaux with exactly $n-1$ rows over the alphabet $\{1 < \cdots < n, \pi \prec \cdots \prec T\}$ and subject to the following constraints:

- for each $1 \leq i \leq n$, the contents of the boxes in the $i$-th row are $\preceq i$,
- the entries $n$ and $\pi$ may not appear simultaneously in a single row.

**Type** $G_2$

$\mathcal{T}(\infty)$ is the set of marginally large semistandard tableaux with exactly 2 rows over the ordered alphabet $\{1 < 2 < 3 < 0 < 3 < 2 < T\}$ and subject to the following constraints:

- the contents of the boxes in the first row are $\preceq 1$,
- the contents of the boxes in the second row are $\preceq 3$,
- the entry 0 can appear at most once in the first row and not at all in the second row.

In particular, the shape of the tableaux is not fixed in any instance of $\mathcal{T}(\infty)$; the row lengths of a tableau can be arbitrarily long.

**INPUT:**

- cartan_type – One of ['A',n], ['B',n], ['C',n], ['D',n], or ['G',2], where $n$ is a positive integer

**EXAMPLES:**

```python
sage: B = crystals.infinity.Tableaux(['A',2])
sage: b = B.highest_weight_vector(); b.pp()
 1 1
 2
sage: b.f_string([2,1,1,2,2,2]).pp()
 1 1 1 1 2 3
 2 3 3 3
sage: B = crystals.infinity.Tableaux(['G',2])
sage: b = B(rows=[[-1,1,1,1,1,1,1,-3,-3,-3,-3,-3,3,3,3,3],[-1,1,1,1,1,1,1,1,-3,-3,-3,-3,-3,3,3,3,3]])
sage: b.e_string([2,1,1,1,1,1,1]).pp()
 1 1 1 2 3 3 3 3 -2 -2 -2
 2 3 3
sage: b.e_string([2,1,1,1,1,1,1,1])
```

We check that a few classical crystals embed into $\mathcal{T}(\infty)$:

```python
sage: def crystal_test(B, C):
    ...
    T = crystals.elementary.T(C.cartan_type(), C.module_generators[0].weight())
    ...
    TP = crystals.TensorProduct(T, B)
    ...
    mg = TP(T[0], B.module_generators[0])
    ...
    g = {C.module_generators[0]: mg}
    ...
    f = C.crystal_morphism(g, category=HighestWeightCrystals())
    ...
    G = B.digraph(subset=[f(x) for x in C])
    ...
    return G.is_isomorphic(C.digraph(), edge_labels=True)

sage: B = crystals.infinity.Tableaux(['A',2])
(... continues on next page)
**sage:** crystal_test(B, C)
True
**sage:** C = crystals.Tableaux(['A',2], shape=[6,2])
**sage:** crystal_test(B, C)
True
**sage:** B = crystals.infinity.Tableaux(['B',2])
**sage:** C = crystals.Tableaux(['B',2], shape=[3])
**sage:** crystal_test(B, C)
True
**sage:** C = crystals.Tableaux(['B',2], shape=[2,1])
**sage:** crystal_test(B, C)
True
**sage:** B = crystals.infinity.Tableaux(['C',3])
**sage:** C = crystals.Tableaux(['C',3], shape=[2,1])
**sage:** crystal_test(B, C)
True
**sage:** B = crystals.infinity.Tableaux(['D',4])
**sage:** C = crystals.Tableaux(['D',4], shape=[2])
**sage:** crystal_test(B, C)
True
**sage:** C = crystals.Tableaux(['D',4], shape=[1,1,1,1])
**sage:** crystal_test(B, C)
True
**sage:** B = crystals.infinity.Tableaux(['G',2])
**sage:** C = crystals.Tableaux(['G',2], shape=[3])
**sage:** crystal_test(B, C)
True

**class Element**

**Bases:** *InfinityCrystalOfTableauxElement*

Elements in $B(\infty)$ crystal of tableaux.

**content()**

Return the content of self.

The content $|T|$ of $T \in B(\infty)$ is the number of blocks added to the highest weight to obtain $T$ with any $\bar{t}$-boxes in the $i$-th row counted with multiplicity 2 provided the underlying Cartan type is of type $B$, $D$, or $G$.

**EXAMPLES:**

```
sage: B = crystals.infinity.Tableaux("D5")
sage: b = B.highest_weight_vector().f_string([5,4,3,1,3,4,5,3,4,5,1,4,5,2, -3,5,3,2,4])
sage: b.content()
13
```

```
sage: B = crystals.infinity.Tableaux("B2")
sage: b = B(rows=[[1,1,1,1,1,2,2,2,-2,-2],[2,0,-2,-2,-2]])
sage: b.content()
12
```

```
sage: B = crystals.infinity.Tableaux("C2")
```
\textbf{phi}(i)

Return $\varphi_i$ of self.

Let $T \in B(\infty)$ Define $\varphi_i(T) := \varepsilon_i(T) + \langle h_i, \text{wt}(T) \rangle$, where $h_i$ is the $i$-th simple coroot and $\text{wt}(T)$ is the weight() of $T$.

\textbf{INPUT:}

\begin{itemize}
  \item $i$ – An element of the index set
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = crystals.infinity.Tableaux("A3")
sage: [B.highest_weight_vector().f_string([1,3,2,3,1,3,2,1]).phi(i) for i in B.index_set()]
[-3, 4, -3]
sage: B = crystals.infinity.Tableaux("G2")
sage: [B.highest_weight_vector().f_string([2,2,1,2,1,1,1,2]).phi(i) for i in B.index_set()]
[5, -3]
\end{verbatim}

\textbf{reduced\_form()}

Return the reduced form of self.

The reduced form of a tableaux $T \in \mathcal{T}(\infty)$ is the (not necessarily semistandard) tableaux obtained from $T$ by removing all $i$-boxes in the $i$-th row, subject to the condition that if the row is empty, a * is put as a placeholder. This is described in [BN2010] and [LS2012].

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = crystals.infinity.Tableaux(["A",3])
sage: b = B.highest_weight_vector().f_string([2,2,2,3,3,3,3,3])
sage: b.pp()
1 1 1 1 1 1 1
2 2 2 2 4 4 4
3 4 4
sage: b.reduced_form()
[['*'], [4, 4], [4, 4]]
\end{verbatim}

\textbf{seg()}

Returns the statistic seg of self.

More precisely, following [LS2012], define a $k$-segment of a tableau $T$ in $B(\infty)$ to be a maximal string of $k$-boxes in a single row of $T$. Set $\text{seg}'(T)$ to be the number of $k$-segments in $T$, as $k$ varies over all possible values. Then $\text{seg}(T)$ is determined type-by-type.

- In types $A_n$ and $C_n$, define $\text{seg}(T) := \text{seg}'(T)$.
- In types $B_n$ and $G_2$, set $e(T)$ to be the number of rows in $T$ which contain both a 0-box and an \( \tau \)-box. Define $\text{seg}(T) := \text{seg}'(T) - e(T)$.
- In type $D_n$, set $d(T)$ to be the number of rows in $T$ which contain an \( \tau \)-box, but no \( n \)-box nor \( \pi \)-box. Define $\text{seg}(T) := \text{seg}'(T) + d(T)$.

\textbf{EXAMPLES:}
**weight()**

Return the weight of self.

From the definition of a crystal and that the highest weight element \( b_\infty \) of \( B(\infty) \) is 0, the weight of \( T \in B(\infty) \) can be defined as \( \text{wt}(T) := -\sum \alpha_i \) where \( \tilde{e}_i \cdots \tilde{e}_i T = b_\infty \) and \( \{\alpha_i\} \) is the set of simple roots. (Note that the weight is independent of the path chosen to get to the highest weight.)

However we can also take advantage of the fact that \( \rho: R_\lambda \otimes B(\infty) \rightarrow B(\lambda) \), where \( \lambda \) is the shape of \( T \), preserves the tableau representation of \( T \). Therefore

\[
\text{wt}(T) = \text{wt}(\rho(T)) - \lambda
\]

where \( \text{wt}(\rho(T)) \) is just the usual weight of the tableau \( T \).

Let \( \Lambda_i \) be the \( i \)-th fundamental weight. In type \( D \), the height \( n-1 \) columns corresponds to \( \Lambda_{n-1} + \Lambda_n \) and the in type \( B \), the height \( n \) columns corresponds to \( 2\Lambda_n \).

**EXAMPLES:**

```python
sage: B = crystals.infinity.Tableaux(['C',7])
sage: b = B.highest_weight_vector().f_string([1,6,4,7,4,2,4,6,2,4,6,7,1,2,4,˓→7])
sage: b.weight()
(-2, -1, 3, -5, 5, -3, -3)
```

Check that the definitions agree:
sage: P = B.weight_lattice_realization()
sage: alpha = P.simple_roots()
True

Check that it works for type $B$:

sage: B = crystals.infinity.Tableaux("B2")
sage: B.highest_weight_vector().weight()
(0, 0)
sage: b = B.highest_weight_vector().f_string([1,2,2,2,1,2])
sage: P = B.weight_lattice_realization()
sage: alpha = P.simple_roots()
True

Check that it works for type $D$:

sage: B = crystals.infinity.Tableaux("D4")
sage: B.highest_weight_vector().weight()
(0, 0, 0, 0)
sage: b = B.highest_weight_vector().f_string([1,4,2,2,4,1,3,2,4])
sage: P = B.weight_lattice_realization()
sage: alpha = P.simple_roots()
True

module_generator()

Return the module generator (or highest weight element) of self.

The module generator is the unique tableau of shape $(n, n-1, \ldots, 2, 1)$ with weight 0.

EXAMPLES:

sage: T = crystals.infinity.Tableaux(['A',3])
sage: T.module_generator()
[[1, 1, 1], [2, 2], [3]]

class sage.combinat.crystals.infinity_crystals.InfinityCrystalOfTableauxTypeD(cartan_type) Bases: InfinityCrystalOfTableaux

$B(\infty)$ crystal of tableaux for type $D_n$.

This is the set $T(\infty)$ of marginally large semistandard tableaux with exactly $n-1$ rows over the alphabet $\{1 \prec \cdots \prec n, \overline{1} \prec \cdots \prec \overline{n}\}$ and subject to the following constraints:

- for each $1 \leq i \leq n$, the contents of the boxes in the $i$-th row are $\preceq \overline{i}$,
- the entries $n$ and $\overline{n}$ may not appear simultaneously in a single row.

For more information, see InfinityCrystalOfTableaux.

EXAMPLES:

sage: B = crystals.infinity.Tableaux("D4")
sage: b = B.highest_weight_vector().f_string([4,3,2,1,4])

(continues on next page)
```
sage: b.pp()
1 1 1 1 1 1 2
2 2 2 2 3
3 -4 -3
sage: b.weight()
(-1, 0, -2, -1)
```

### 5.1.54 Crystals of Kac modules of the general-linear Lie superalgebra

```python
class sage.combinat.crystals.kac_modules.CrystalOfKacModule(cartan_type, la, mu):
    Bases: UniqueRepresentation, Parent

    Crystal of a Kac module.

    Let \( g \) be the general linear Lie superalgebra \( gl(m|n) \). Let \( \lambda \) and \( \mu \) be dominant weights for \( gl_m \) and \( gl_n \), respectively. Let \( K \) be the module \( K = \langle f_\alpha \rangle \), where \( \alpha \) ranges over all odd positive roots. A Kac module is the \( U_q(g) \)-module constructed from the highest weight \( U_q(gl_m \oplus gl_n) \)-module \( V(\lambda, \mu) \) (induced to a \( U_q(g) \)-module in the natural way) by

\[
K(\lambda, \mu) := K \otimes_L V(\lambda, \mu),
\]

where \( L \) is the subalgebra generated by \( e_0 \) and \( U_q(gl_m \oplus gl_n) \).

The Kac module admits a \( U_q(g) \)-crystal structure by taking the crystal structure of \( K \) as given by \( CrystalOfOddNegativeRoots \) and the crystal \( B(\lambda, \mu) \) (the natural crystal structure of \( V(\lambda, \mu) \)).

**Note:** Our notation differs slightly from [Kwon2012] in that our last tableau is transposed.

```
sage: K = crystals.KacModule(['A', [1,2]], [2], [1,1])
sage: K.cardinality()
576
sage: K.cardinality().factor()
2^6 * 3^2
```

(continues on next page)
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```
sage: mg = K.module_generator()
sage: mg
({}, [[-2, -2]], [[1], [2]])
sage: mg.weight()
(2, 0, 1, 1, 0)
sage: mg.f(-1)
({}, [[-2, -1]], [[1], [2]])
sage: mg.f(0)
({-e[-1]+e[1]}, [[-2, -2]], [[1], [2]])
sage: mg.f(1)
sage: mg.f(2)
({}, [[-2, -2]], [[1], [3]])
```

```
sage: sorted(K.highest_weight_vectors(), key=str)
[({-e[-1]+e[3]}, [[-2, -1]], [[1], [2]]),
 ({-e[-1]+e[3]}, [[-2, -2]], [[1], [2]]),
 ({}), [[-2, -2]], [[1], [2]])]
```

```
sage: K = crystals.KacModule(['A', [1,1]], [2], [1])
sage: K.cardinality()
96
sage: K.cardinality().factor()
2^5 * 3
sage: len(K.cartan_type().root_system().ambient_space().positive_odd_roots())
4
```

```
sage: sorted(K.highest_weight_vectors(), key=str)
[({-e[-1]+e[2]}, [[-2, -1]], [[1]],)
 ({-e[-2]+e[1], -e[-2]+e[2], -e[-1]+e[1], -e[-1]+e[2]}, [[-1, -1]], [[2]]),
 ({-e[-2]+e[2], -e[-1]+e[1], -e[-1]+e[2]}, [[-1, -1]], [[1]]),
 ({-e[-2]+e[2], -e[-1]+e[1], -e[-1]+e[2]}, [[-1, -1]], [[2]]))
```

REFERENCES:
- [Kwon2012]

```python
class Element
    Bases: ElementWrapper

    An element of a Kac module crystal.

    e(i)

    Return the action of the crystal operator \(e_i\) on self.
```

EXAMPLES:

```
sage: K = crystals.KacModule(['A', [2,2]], [2,1], [1])
sage: mg = K.module_generator()
```

(continues on next page)
\begin{verbatim}
sage: mg.e(0)
sage: mg.e(1)
sage: mg.e(-1)
sage: b = mg.f_string([1,0,1,-1,-2,0,1,2,0,-2,-1,-1])
({-e[-3]+e[2], -e[-2]+e[1], -e[-2]+e[2]}, [[-3, -1], [-2]], [[3]])
sage: b.e(-2)
({-e[-3]+e[1], -e[-2]+e[1], -e[-2]+e[2]}, [[-3, -1], [-2]], [[2]])
sage: b.e(-1)
({-e[-3]+e[2], -e[-2]+e[1], -e[-2]+e[2]}, [[-3, -2], [-2]], [[3]])
sage: b.e(0)
({-e[-1]+e[1]}, [[-3, -3], [-2]], [[3]])
sage: b.e(1)
({-e[-3]+e[1], -e[-2]+e[1], -e[-2]+e[2]}, [[-3, -1], [-2]], [[3]])
sage: b.e(2)
({-e[-3]+e[2], -e[-2]+e[1], -e[-2]+e[2]}, [[-3, -1], [-2]], [[2]])
\end{verbatim}

**wf**

Return the action of the crystal operator $f_i$ on $self$.

**Examples:**

\begin{verbatim}
sage: K = crystals.KacModule(['A', [2,2]], [2,1], [1])
sage: mg = K.module_generator()
sage: mg.f(-2)
({}, [[-3, -2], [-2]], [[1]])
sage: mg.f(-1)
({}, [[-3, -3], [-1]], [[1]])
sage: mg.f(0)
({-e[-1]+e[1]}, [[-3, -3], [-2]], [[1]])
sage: mg.f(1)
({}, [[-3, -3], [-2]], [[2]])
sage: mg.f(2)
sage: b = mg.f_string([1,0,1,-1,-2,0,1,2,0,-2,-1,2,0]); b
({-e[-3]+e[3], -e[-2]+e[1], -e[-2]+e[1], -e[-1]+e[2]}, 
[[[-3, -2], [-2]], [[3]]])
\end{verbatim}

**weight**

Return weight of $self$.

**Examples:**

\begin{verbatim}
sage: K = crystals.KacModule(['A', [3,2]], [2,1], [5,1])
sage: mg = K.module_generator()
sage: mg.weight()
(2, 1, 0, 0, 5, 1, 0)
sage: mg.weight().is_dominant() True
sage: mg.f(0).weight()
(2, 1, 0, -1, 6, 1, 0)
sage: b = mg.f_string([2,1,-3,-2,-1,1,0,-2,-1,2,1,1,0,2,3,-2,-1])
sage: b.weight()
(0, 0, 0, 1, 1, 4, 3)
\end{verbatim}

**module_generator**

Return the module generator of $self$.

---

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EXAMPLES:

```python
sage: K = crystals.KacModule(['A', [2,1]], [2,1], [1])
sage: K.module_generator()
({}, [[-3, -3], [-2], [[1]]])
```

```python
class sage.combinat.crystals.kac_modules.CrystalOfOddNegativeRoots(cartan_type):

    Bases: UniqueRepresentation, Parent

    Crystal of the set of odd negative roots.

    Let $g$ be the general-linear Lie superalgebra $\mathfrak{gl}(m|n)$. This is the crystal structure on the set of negative roots as given by [Kwon2012].

    More specifically, this is the crystal basis of the subalgebra of $U_q^{-}(g)$ generated by $f_\alpha$, where $\alpha$ ranges over all odd positive roots. As $\mathbb{Q}(q)$-modules, we have

    $$U_q^{-}(g) \cong K \otimes U_q^{-}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n).$$

EXAMPLES:

```python
sage: S = crystals.OddNegativeRoots(['A', [2,1]])
sage: mg = S.module_generator(); mg
{}
sage: mg.f(0)
{-e[-1]+e[1]}
sage: mg.f_string([0,-1,0,1,2,1,0])
{-e[-2]+e[3], -e[-1]+e[1], -e[-1]+e[2]}
```

```python
class Element

    Bases: ElementWrapper

    An element of the crystal of odd negative roots.

    $e(i)$

    Return the action of the crystal operator $e_i$ on self.

    EXAMPLES:

```python
sage: S = crystals.OddNegativeRoots(['A', [2,2]])
sage: mg = S.module_generator()
sage: mg.e(0)
sage: mg.e(1)
sage: b = mg.f_string([0,1,2,-1,0])
sage: b.e(-1)
sage: b.e(0)
{-e[-2]+e[3]}
sage: b.e(1)
sage: b.e(2)
{-e[-2]+e[2], -e[-1]+e[1]}
sage: b.e_string([2,1,0,-1,0])
{}
```

$\epsilon(i)$

Return $\epsilon_i$ of self.

EXAMPLES:
sage: S = crystals.OddNegativeRoots(['A', [2,2]])
sage: mg = S.module_generator()
sage: [mg.epsilon(i) for i in S.index_set()]
[0, 0, 0, 0, 0]
sage: b = mg.f_string([0,1,0,-1,0,-2,2,2,1,0]); b
{e[-3]+e[1], -e[-3]+e[2], -e[1], -e[-1]+e[1]}
sage: [b.epsilon(i) for i in S.index_set()]
[2, 0, 1, 0, 0]

f(i)
Return the action of the crystal operator $f_i$ on self.

EXAMPLES:

sage: S = crystals.OddNegativeRoots(['A', [2,2]])
sage: mg = S.module_generator()
sage: mg.f(0)
{e[-1]+e[1]}
sage: mg.f(1)

sage: b = mg.f_string([0,1,2,-1,0]); b
{e[-2]+e[3], -e[-1]+e[1]}
sage: b.f(-2)
{e[-3]+e[3], -e[-1]+e[1]}
sage: b.f(-1)
sage: b.f(0)
sage: b.f(1)

phi(i)
Return $\phi_i$ of self.

EXAMPLES:

sage: S = crystals.OddNegativeRoots(['A', [2,2]])
sage: mg = S.module_generator()
sage: [mg.phi(i) for i in S.index_set()]
[0, 0, 1, 0, 0]
sage: b = mg.f(0)
sage: [b.phi(i) for i in S.index_set()]
[0, 1, 0, 1, 0]

weight()
Return the weight of self.

EXAMPLES:
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```
sage: S = crystals.OddNegativeRoots(['A', [2,2]])
sage: mg = S.module_generator()
sage: mg.weight()
(0, 0, 0, 0, 0, 0)
sage: mg.f_string([0,1,2,-1,-2]).weight()
(-1, 0, 0, 0, 0, 1)
sage: mg.f_string([0,1,2,-1,-2,0,1,0,2]).weight()
(-1, 0, -2, 1, 0, 2)
```

module_generator()

Return the module generator of self.

EXAMPLES:

```
sage: S = crystals.OddNegativeRoots(['A', [2,1]])
sage: S.module_generator()
{}
```

sage.combinat.crystals.kac_modules.latex_dual(elt)

Return a latex representation of a type $A_n$ crystal tableau elt expressed in terms of dual letters.

The dual letter of $k$ is expressed as $\overline{n} + 2 - k$.

EXAMPLES:

```
sage: from sage.combinat.crystals.kac_modules import latex_dual
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: latex_dual(T[0])
\begin{array}\cline{1-2}
\overline{3} & \overline{3} \\
\overline{2} \\
\end{array}
```

sage.combinat.crystals.kac_modules.to_dual_tableau(elt)

Return a type $A_n$ crystal tableau elt as a tableau expressed in terms of dual letters.

The dual letter of $k$ is expressed as $\overline{n} + 2 - k$ represented as $-(n + 2 - k)$.

EXAMPLES:

```
sage: from sage.combinat.crystals.kac_modules import to_dual_tableau
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: ascii_art([to_dual_tableau(t) for t in T])
[ -3 -3 -3 -3 -3 -1 -3 -1 -2 -1 -3 -3 -3 -2 -2 ]
[ -2 , -2 , -2 , -1 , -1 , -1 , -1 , -1 ]
```
5.1.55 Kirillov-Reshetikhin Crystals

class sage.combinat.crystals.kirillov_reshetikhin.AmbientRetractMap(base, ambient, pdict_inv, index_set, similarity_factor_domain=None, automorphism=None)

Bases: Map

The retraction map from the ambient crystal.

Consider a crystal embedding $\phi : X \rightarrow Y$, then the elements $X$ can be considered as a subcrystal of the ambient crystal $Y$. The ambient retract is the partial map $\tilde{\phi} : Y \rightarrow X$ such that $\tilde{\phi} \circ \phi$ is the identity on $X$.

class sage.combinat.crystals.kirillov_reshetikhin.CrystalDiagramAutomorphism(C, on_hw, index_set=None, automorphism=None, cache=True)

Bases: CrystalMorphism

The crystal automorphism induced from the diagram automorphism.

For example, in type $A_n^{(1)}$ this is the promotion operator and in type $D_n^{(1)}$, this corresponds to the automorphism induced from interchanging the 0 and 1 nodes in the Dynkin diagram.

**INPUT:**
- $C$ – a crystal
- $on_hw$ – a function for the images of the index_set-highest weight elements
- $index_set$ – (default: the empty set) the index set
- $automorphism$ – (default: the identity) the twisting automorphism
- $cache$ – (default: True) cache the result

**is_embedding()**
Return True as self is a crystal isomorphism.

**EXAMPLES:**

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.promotion().is_isomorphism()
True
```

**is_isomorphism()**
Return True as self is a crystal isomorphism.

**EXAMPLES:**

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.promotion().is_isomorphism()
True
```

**is_strict()**
Return True as self is a crystal isomorphism.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.promotion().is_isomorphism()
True
```

### `is_surjective()`

Return `True` as `self` is a crystal isomorphism.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.promotion().is_isomorphism()
True
```

### `class sage.combinat.crystals.kirillov_reshetikhin.CrystalOfTableaux_E7(cartan_type, shapes)`

Bases: `CrystalOfTableaux`

The type $E_7$ crystal $B(s\Lambda_7)$.

This is a helper class for the corresponding class `KRcrystal < sage.combinat.crystals.kirillov_reshetikhin.KRtypeE7 > B_7^{\mathfrak{s}}`.

**module_generator**(shape)

Returns the module generator of `self` with shape `shape`.

**Note:** Only implemented for single rows (i.e., highest weight $s\Lambda_7$).

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.kirillov_reshetikhin import CrystalOfTableaux_E7
sage: T = CrystalOfTableaux_E7(CartanType(['E',7]), shapes=(Partition([5]),))
sage: T.module_generator([5])
[[[7,], [7,], [7,], [7,], [7,]]]
```

### `class sage.combinat.crystals.kirillov_reshetikhin.KR_type_A(cartan_type, r, s)`

Bases: `KirillovReshetikhinCrystalFromPromotion`

Class of Kirillov-Reshetikhin crystals of type $A^{(1)}_n$.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: b = K(rows=[[1,2],[2,4]])
sage: b.f(0)
[[1, 1], [2, 2]]
```

### `classical_decomposition()`

Specifies the classical crystal underlying the KR crystal of type $A$.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['A', 3] and shape(s) [[2, 2]]
```
**dynkin_diagram_automorphism(i)**

Specifies the Dynkin diagram automorphism underlying the promotion action on the crystal elements. The automorphism needs to map node 0 to some other Dynkin node.

For type $A$ we use the Dynkin diagram automorphism which $i \mapsto i + 1 \mod n + 1$, where $n$ is the rank.

**EXAMPLES:**

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.dynkin_diagram_automorphism(0)
1
sage: K.dynkin_diagram_automorphism(3)
0
```

**promotion()**

Specifies the promotion operator used to construct the affine type $A$ crystal.

For type $A$ this corresponds to the Dynkin diagram automorphism which $i \mapsto i + 1 \mod n + 1$, where $n$ is the rank.

**EXAMPLES:**

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: b = K.classical_decomposition()(rows=[[1,2],[3,4]])
sage: K.promotion()(b)
[[1, 3], [2, 4]]
```

**promotion_inverse()**

Specifies the inverse promotion operator used to construct the affine type $A$ crystal.

For type $A$ this corresponds to the Dynkin diagram automorphism which $i \mapsto i - 1 \mod n + 1$, where $n$ is the rank.

**EXAMPLES:**

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: b = K.classical_decomposition()(rows=[[1,3],[2,4]])
sage: K.promotion_inverse()(b)
[[1, 2], [3, 4]]
sage: b = K.classical_decomposition()(rows=[[1,2],[3,3]])
sage: K.promotion_inverse()(K.promotion()(b))
[[1, 2], [3, 3]]
```

**class** `sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(cartan_type, r, s, dual=None)`

Bases: `KirillovReshetikhinGenericCrystal`

Class of Kirillov-Reshetikhin crystals $B^{r,s}$ of type $A^{(2)}_{2n}$ for $1 \leq r \leq n$ in the realization with classical subalgebra $B_n$. The Cartan type in this case is inputted as the dual of $A^{(2)}_{2n}$.

This is an alternative implementation to `KR_type_box` that uses the classical decomposition into type $C_n$ crystals.

**EXAMPLES:**

```
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 1)
sage: K
```
Kirillov-Reshetikhin crystal of type ['BC', 2, 2]^\ast with (r,s)=(1,1)

```
sage: b = K(rows=[[1]])
sage: b.f(0)
[[1]]
sage: b.e(0)
```

We can now check whether the two KR crystals of type $A_{4}^{(2)}$ (namely the KR crystal and its dual construction) are isomorphic up to relabelling of the edges:

```
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 2, 3)
sage: K.ambient_crystal()
Kirillov-Reshetikhin crystal of type ['B', 3, 1] with (r,s)=(2,3)
```

**Element**

alias of *KR_type_A2Element*

*ambient_crystal()*

Return the ambient crystal $B_{n+1}^{\ast \ast}$ of type $B_{n+1}^{(1)}$ associated to the Kirillov-Reshetikhin crystal of type $A_{2n}^{(2)}$ dual.

This ambient crystal is used to construct the zero arrows.

EXAMPLES:

```
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 2, 3)
sage: K.ambient_crystal()
Kirillov-Reshetikhin crystal of type ['B', 3, 1] with (r,s)=(2,3)
```

*ambient_dict_pm_diagrams()*

Return a dictionary of all self-dual ± diagrams for the ambient crystal whose keys are their inner shape.

EXAMPLES:

```
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 1)
sage: K.ambient_dict_pm_diagrams()
{[1]: [[0, [0], [1]]]}
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 2)
sage: K.ambient_dict_pm_diagrams()
{[1]: [[1, [0], [0]], [2]: [[0, [0], [2]]]}
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 2, 2)
sage: K.ambient_dict_pm_diagrams()
{[1]: [[1, [0], [0]], [0]],
 [2]: [[0, [0], [1]], [2]]
 [2, 2]: [[0, [0], [2]]]}
```
ambient_highest_weight_dict()

Return a dictionary of all \{2, \ldots, n+1\}-highest weight vectors in the ambient crystal.

The key is the inner shape of their corresponding \pm diagram, or equivalently, their \{2, \ldots, n+1\} weight.

EXAMPLES:

```sage
C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 2)
sage: K.ambient_highest_weight_dict()
{{[1]: [[1, -1]], [2]: [[2, 2]]}}
```

classical_decomposition()

Return the classical crystal underlying the Kirillov-Reshetikhin crystal of type $A_{2n}^{(2)}$ with $B_n$ as classical subdiagram.

It is given by $B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$, where $B(\Lambda)$ is a highest weight crystal of type $B_n$ of highest weight $\Lambda$. The sum is over all weights $\Lambda$ obtained from a rectangle of width $s$ and height $r$ by removing horizontal dominoes. Here we identify the fundamental weight $\Lambda_i$ with a column of height $i$.

EXAMPLES:

```sage
C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 2, 2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['B', 2] and shape(s) [[], [2], [2, 2]]
```

from_ambient_crystal()

Return a map from the ambient crystal of type $B_{n+1}^{(1)}$ to the Kirillov-Reshetikhin crystal of type $A_{2n}^{(2)}$.

Note that this map is only well-defined on type $A_{2n}$ elements that are in the image under to_ambient_crystal().

EXAMPLES:

```sage
C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 2)
sage: b = K.ambient_crystal()(rows=[[2,2]])
sage: K.from_ambient_crystal()(b)
[[1, 1]]
```

highest_weight_dict()

Return a dictionary of the classical highest weight vectors of self whose keys are their shape.

EXAMPLES:

```sage
C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 2)
sage: K.highest_weight_dict()
{{[1]: [1, 1], [2]: [[1, 1]]}}
```

module_generator()

Return the unique module generator of classical weight $s\Lambda_r$ of a Kirillov-Reshetikhin crystal $B^{r,s}$.

EXAMPLES:
```python
sage: ct = CartanType(['A',8,2]).dual()
sage: K = crystals.KirillovReshetikhin(ct, 3, 5)
sage: K.module_generator()
[[[1, 1, 1, 1, 1], [2, 2, 2, 2, 2], [3, 3, 3, 3, 3]]
```

**to_ambient_crystal()**

Return a map from the Kirillov-Reshetikhin crystal of type $A^{(2)}_{2n}$ to the ambient crystal of type $B^{(1)}_{n+1}$.

**Examples:**

```python
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 2)
sage: b=K(rows=[[1,1]])
sage: K.to_ambient_crystal()(b)
[[2, 2]]
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 2, 2)
sage: b=K(rows=[[1,1], [2, -1]])
sage: K.to_ambient_crystal()(b).parent()
Kirillov-Reshetikhin crystal of type ['B', 3, 1] with (r,s)=(2,2)
```

**Class sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2Element**

Bases: KirillovReshetikhinGenericCrystalElement

Class for the elements in the Kirillov-Reshetikhin crystals $B^{r,s}$ of type $A^{(2)}_{2n}$ for $r < n$ with underlying classical algebra $B_n$.

**Examples:**

```python
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 2)
sage: type(K.module_generators[0])
<
```

**e0()**

Return $e_0$ on self by mapping self to the ambient crystal, calculating $e_1e_0$ there and pulling the element back.

**Examples:**

```python
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 1)
sage: b = K(rows=[[1]])
sage: b.e(0)  # indirect doctest
[[-1]]
```

**epsilon0()**

Calculate $\varepsilon_0$ of self by mapping the element to the ambient crystal and calculating $\varepsilon_1$ there. EXAMPLES:

```python
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 1)
```

(continues on next page)
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(continued from previous page)

```python
sage: b = K(rows=[[1]])
sage: b.epsilon(0) # indirect doctest
1
```

```
f0()

Return \( f_0 \) on \( \text{self} \) by mapping \( \text{self} \) to the ambient crystal, calculating \( f_1f_0 \) there and pulling the element back.

EXAMPLES:

```python
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 1)
sage: b = K(rows=[[-1]])
sage: b.f(0) # indirect doctest
[[1]]
```

```
phi0()

Calculate \( \varphi_0 \) of \( \text{self} \) by mapping the element to the ambient crystal and calculating \( \varphi_1 \) there.

EXAMPLES:

```python
sage: C = CartanType(['A',4,2]).dual()
sage: K = sage.combinat.crystals.kirillov_reshetikhin.KR_type_A2(C, 1, 1)
sage: b = K(rows=[[-1]])
sage: b.phi(0) # indirect doctest
1
```

```
class sage.combinat.crystals.kirillov_reshetikhin.KR_type_Bn(cartan_type, r, s, dual=None)

Bases: KirillovReshetikhinGenericCrystal

Class of Kirillov-Reshetikhin crystals \( B_{n,s} \) of type \( B_n^{(1)} \).

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: K
Kirillov-Reshetikhin crystal of type ['B', 3, 1] with (r,s)=(3,2)
sage: b = K(rows=[[1],[2],[3]])
sage: b.f(0)
sage: b.e(0)
[[3]]
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: [b.weight() for b in K if b.is_highest_weight([1,2,3])]
[-Lambda[0] + Lambda[1], -2*Lambda[0] + 2*Lambda[3]]
sage: [b.weight() for b in K if b.is_highest_weight([0,2,3])]
```

```
Element

alias of KR_type_BnElement

ambient_crystal()

Return the ambient crystal \( B_{n,s} \) of type \( A_{2n-1}^{(2)} \) associated to the Kirillov-Reshetikhin crystal.

The ambient crystal is used to construct the zero arrows.

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**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: K.ambient_crystal()
Kirillov-Reshetikhin crystal of type ['B', 3, 1]^* with (r,s)=(3,2)
```

**ambient_highest_weight_dict()**

Return a dictionary of the classical highest weight vectors of the ambient crystal of `self` whose keys are their shape.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: K.ambient_highest_weight_dict()
{(2,): [[1, 1]], (2, 1, 1): [[1, 1], [2], [3]], (2, 2, 2): [[1, 1], [2, 2], [3, 3]]}
```

**classical_decomposition()**

Return the classical crystal underlying the Kirillov-Reshetikhin crystal $B_{n,s}$ of type $B_n^{(1)}$.

It is the same as for $r < n$, given by $B_{n,s} \cong \bigoplus \Lambda B(\Lambda)$, where $\Lambda$ are weights obtained from a rectangle of width $s/2$ and height $n$ by removing horizontal dominoes. Here we identify the fundamental weight $\Lambda_i$ with a column of height $i$ for $i < n$ and a column of width $1/2$ for $i = n$.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['B', 3] and shape(s) [[1], [1, 1]]
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,3)
sage: K.classical_decomposition()
The crystal of tableaux of type ['B', 3] and shape(s) [[3/2, 1/2, 1/2], [3/2, 3/2, 3/2]]
```

**from_ambient_crystal()**

Return a map from the ambient crystal of type $A_{2n-1}^{(2)}$ to the Kirillov-Reshetikhin crystal `self`.

Note that this map is only well-defined on elements that are in the image under `to_ambient_crystal()`.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,1)
sage: [b == K.from_ambient_crystal()(K.to_ambient_crystal()(b)) for b in K]
[True, True, True, True, True, True, True, True]
sage: b = K.ambient_crystal()(rows=[[1],[2],[3]])
sage: K.from_ambient_crystal()(b)
[++, []]
```
highest_weight_dict()

Return a dictionary of the classical highest weight vectors of self whose keys are 2 times their shape.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: K.highest_weight_dict()
{(2,): [[1]], (2, 2, 2): [[1], [2], [3]]}
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,3)
sage: K.highest_weight_dict()
{(3, 1, 1): [+++], (3, 3, 3): [+++], [1], [2], [3]]}
```

similarity_factor()

Sets the similarity factor used to map to the ambient crystal.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: K.similarity_factor()
{1: 2, 2: 2, 3: 1}
```

to_ambient_crystal()

Return a map from self to the ambient crystal of type $A^{(2)}_{2n-1}$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,1)
sage: K.to_ambient_crystal()(b) for b in K
[[[1], [2], [3]], [[1], [2], [-3]], [[1], [3], [-2]], [[2], [3], [-1]], [[1], [-3], [-2]], [[2], [-3], [-1]], [[3], [-2], [-1]], [[-3], [-2], [-1]]
```

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_BnElement

Bases: KirillovReshetikhinGenericCrystalElement

Class for the elements in the Kirillov-Reshetikhin crystals $B^{n,s}$ of type $B_n^{(1)}$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,2)
sage: type(K.module_generators[0])
<class 'sage.combinat.crystals.kirillov_reshetikhin.KR_type_Bn_element_class'>
```

e0()

Return $e_0$ on self by mapping self to the ambient crystal, calculating $e_0$ there and pulling the element back.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B',3,1],3,1)
sage: b = K.module_generators[0]
sage: b.e(0) # indirect doctest
[---+, []]
```
epsilon(0)

Calculate $\varepsilon_0$ of self by mapping the element to the ambient crystal and calculating $\varepsilon_0$ there.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B', 3, 1], 3, 1)
sage: b = K.module_generators[0]
sage: b.epsilon(0) # indirect doctest
1
```

f(0)

Return $f_0$ on self by mapping self to the ambient crystal, calculating $f_0$ there and pulling the element back.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B', 3, 1], 3, 1)
sage: b = K.module_generators[0]
sage: b.f(0) # indirect doctest
```

phi(0)

Calculate $\varphi_0$ of self by mapping the element to the ambient crystal and calculating $\varphi_0$ there.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['B', 3, 1], 3, 1)
sage: b = K.module_generators[0]
sage: b.phi(0) # indirect doctest
0
```

class `sage.combinat.crystals.kirillov_reshetikhin.KR_type_C`

Bases: `KirillovReshetikhinGenericCrystal`

Class of Kirillov-Reshetikhin crystals $B^{r,s}$ of type $C_n^{(1)}$ for $r < n$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['C', 2, 1], 1, 2)
sage: K
Kirillov-Reshetikhin crystal of type ['C', 2, 1] with (r,s)=(1,2)
sage: b = K(rows=[])
sage: b.f(0)
[[1, 1]]
sage: b.e(0)
[[-1, -1]]
```

Element

alias of `KR_type_CElement`

ambient_crystal()

Return the ambient crystal $B^{r,s}$ of type $A_{2n+1}^{(2)}$ associated to the Kirillov-Reshetikhin crystal of type $C_n^{(1)}$. This ambient crystal is used to construct the zero arrows.

EXAMPLES:
sage: K = crystals.KirillovReshetikhin(['C',3,1], 2,3)
sage: K.ambient_crystal()
Kirillov-Reshetikhin crystal of type ['B', 4, 1]^* with (r,s)=(2,3)

ambient_dict_pm_diagrams()
Return a dictionary of all self-dual ± diagrams for the ambient crystal whose keys are their inner shape.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',2,1],1,2)
sage: K.ambient_dict_pm_diagrams()
{
{}: [[1, 1], [0]], [2]: [[0, 0], [2]]
}
sage: K = crystals.KirillovReshetikhin(['C',3,1], 2,2)

ambient_highest_weight_dict()
Return a dictionary of all $\{2, \ldots, n+1\}$-highest weight vectors in the ambient crystal.
The key is the inner shape of their corresponding ± diagram, or equivalently, their $\{2, \ldots, n+1\}$ weight.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1], 2,2)
sage: K.ambient_highest_weight_dict()
{
{}: [[], [2]], [2]: [[1, 2], [2, -1]], [2, 2]: [[2, 2], [3, 3]]
}

classical_decomposition()
Return the classical crystal underlying the Kirillov-Reshetikhin crystal of type $C_n^{(1)}$.
It is given by $B^{r,s} \cong \bigoplus \Lambda B(\Lambda)$, where $\Lambda$ are weights obtained from a rectangle of width $s$ and height $r$ by removing horizontal dominoes. Here we identify the fundamental weight $\Lambda_i$ with a column of height $i$.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1], 2,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['C', 3] and shape(s) [[], [2], [2, 2]]

from_ambient_crystal()
Return a map from the ambient crystal of type $A_{2n+1}^{(2)}$ to the Kirillov-Reshetikhin crystal of type $C_n^{(1)}$.
Note that this map is only well-defined on type $C_n^{(1)}$ elements that are in the image under to_ambient_crystal().

EXAMPLES:
sage: K = crystals.KirillovReshetikhin(['C',3,1], 2,2)
sage: b = K.ambient_crystal()(rows=[[2,2],[3,3]])
sage: K.from_ambient_crystal()(b)
[[1, 1], [2, 2]]

highest_weight_dict()

Return a dictionary of the classical highest weight vectors of self whose keys are their shape.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1], 2,2)
sage: K.highest_weight_dict()
{[ ]: [[], [2]: [[1, 1]], [2, 2]: [[1, 1], [2, 2]]}

to_ambient_crystal()

Return a map from the Kirillov-Reshetikhin crystal of type $C_n^{(1)}$ to the ambient crystal of type $A_{2n+1}^{(2)}$.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1],1,2)
sage: type(K.module_generators[0])
<
class 'sage.combinat.crystals.kirillov_reshetikhin.KR_type_C_with_category.element_class'>

e0()

Return $e_0$ on self by mapping self to the ambient crystal, calculating $e_1 e_0$ there and pulling the element back.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1],1,2)
sage: b = K(rows=[])
sage: b.e(0) # indirect doctest
[[1, -1]]

epsilon0()

Calculate $\varepsilon_0$ of self by mapping the element to the ambient crystal and calculating $\varepsilon_1$ there.

EXAMPLES:
```python
sage: K = crystals.KirillovReshetikhin(['C', 2, 1], 1, 2)
sage: b = K(rows=[[1, 1]])
sage: b.epsilon(0)  # indirect doctest
2
```

\(f_0()\)

Return \(f_0\) on \(self\) by mapping \(self\) to the ambient crystal, calculating \(f_1f_0\) there and pulling the element back.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['C', 3, 1], 1, 2)
sage: b = K(rows=[])
sage: b.f(0)  # indirect doctest
[[1], [1]]
```

\(phi_0()\)

Calculate \(\varphi_0\) of \(self\) by mapping the element to the ambient crystal and calculating \(\varphi_1\) there.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['C', 2, 1], 1, 2)
sage: b = K(rows=[[-1, -1]])
sage: b.phi(0)  # indirect doctest
2
```

**class** `sage.combinat.crystals.kirillov_reshetikhin.KR_type_Cn(cartan_type, r, s, dual=None)`

Bases: `KirillovReshetikhinGenericCrystal`

Class of Kirillov-Reshetikhin crystals \(B^{n,s}\) of type \(C_n^{(1)}\).

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['C', 3, 1], 3, 1)
sage: [[b, b.f(0)] for b in K]
[[[[1], [2], [3]], None], [[[1], [2], [-3]], None],
 [[[1], [3], [-3]], None], [[[2], [3], [-3]], None],
 [[[1], [3], [-2]], None], [[[2], [3], [-2]], None],
 [[[2], [3], [-1]], [[1], [2], [3]], [[[1], [-3], [-2]], None],
 [[[2], [-3], [-2]], None], [[[1], [-3], [-1]], [[1], [2], [-3]],
 [[[1], [-3], [-2]], None], [[[1], [-3], [-1]], [[1], [3], [-3]],
 [[[3], [-2], [-1]], [[1], [3], [-2]],
 [[[3], [-2], [-1]], [[1], [-3], [-2]]]]
```

**Element**

alias of `KR_type_CnElement`

**classical_decomposition()**

Specifies the classical crystal underlying the Kirillov-Reshetikhin crystal \(B^{n,s}\) of type \(C_n^{(1)}\).

The classical decomposition is given by \(B^{n,s} \cong B(s\Lambda_n)\).

**EXAMPLES:**

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from highest_weight_vector_to_pm_diagram(b)
This gives the bijection between an element \(b\) in the classical decomposition of the KR crystal that is \(2, 3, \ldots, n\)-highest weight and \(\pm\) diagrams.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['C',3,1],3,2)
sage: T = K.classical_decomposition()
sage: b = T(rows=[[2, 2], [3, 3], [-3, -1]])
sage: pm = K.from_highest_weight_vector_to_pm_diagram(b); pm
[[0, 0], [1, 0], [0, 1], [0]]
sage: pm.pp()
... .
... . +
... -
```

from_pm_diagram_to_highest_weight_vector(pm)
This gives the bijection between a \(\pm\) diagram and an element \(b\) in the classical decomposition of the KR crystal that is \(\{2, 3, \ldots, n\}\)-highest weight.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['C',3,1],3,2)
sage: pm = sage.combinat.crystals.kirillov_reshetikhin.PMDiagram([[0, 0], [1, 0], [0, 1], [0]])
sage: K.from_pm_diagram_to_highest_weight_vector(pm)
[[2, 2], [3, 3], [-3, -1]]
```

```python
class sage.combinat.crystals.kirillov_reshetikhin.KR_type_CnElement

Class for the elements in the Kirillov-Reshetikhin crystals \(B_n^s\) of type \(C_n^{(1)}\).

**EXAMPLES:**

```python
sage: K=crystals.KirillovReshetikhin(['C',3,1],3,2)
sage: type(K.module_generators[0])
<class 'sage.combinat.crystals.kirillov_reshetikhin.KR_type_Cn_with_category.
  _element_class'>
```

```python
e0()
Return \(e_0\) on \(self\) by going to the \(\pm\)-diagram corresponding to the \(\{2, \ldots, n\}\)-highest weight vector in the component of \(self\), then applying [Definition 6.1, 4], and pulling back from \(\pm\)-diagrams.

**EXAMPLES:**

```python
```
sage: K = crystals.KirillovReshetikhin(['C',3,1],3,2)
sage: b = K.module_generators[0]
sage: b.e(0) # indirect doctest
[[1, 2], [2, 3], [3, -1]]
sage: b = K(rows=[[1,2],[2,3],[3,-1]])
sage: b.e(0)
[[2, 2], [3, 3], [-1, -1]]
sage: b=K(rows=[[1, -3], [3, -2], [-3, -1]])
sage: b.e(0)
[[3, -3], [-3, -2], [-1, -1]]

epsilon0()  
Calculate $\epsilon_0$ of self using Lemma 6.1 of [4].

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1],3,1)
sage: b = K.module_generators[0]
sage: b.epsilon(0) # indirect doctest
1

f0()  
Return $e_0$ on self by going to the $\pm$-diagram corresponding to the \{2, ..., \n\}-highest weight vector in the component of self, then applying [Definition 6.1, 4], and pulling back from $\pm$-diagrams.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1],3,1)
sage: b = K.module_generators[0]
sage: b.f(0) # indirect doctest

phi0()  
Calculate $\varphi_0$ of self.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['C',3,1],3,1)
sage: b = K.module_generators[0]
sage: b.phi(0) # indirect doctest
0

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_D_tri1(ct, s)
Bases: KirillovReshetikhinGenericCrystal
Class of Kirillov-Reshetikhin crystals $B_{1-s}$ of type $D_4^{(3)}$.
The crystal structure was defined in Section 4 of [KMOY2007] using the coordinate representation.

class Element
Bases: KirillovReshetikhinGenericCrystalElement
coordinates()  
Return self as coordinates.

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```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 3], 1, 3)
sage: all(K.from_coordinates(x.coordinates()) == x for x in K)
True
```

**e**0()  
Return the action of $e_0$ on self.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 3], 1, 1)
sage: [x.e0() for x in K]
[[[-1]], [], [[-3]], [[-2]], None, None, None, None]
```

**epsilon**0()  
Return $\varepsilon_0$ of self.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 3], 1, 5)
sage: [mg.epsilon0() for mg in K.module_generators]
[5, 6, 7, 8, 9, 10]
```

**f**0()  
Return the action of $f_0$ on self.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 3], 1, 1)
sage: [x.f0() for x in K]
[[[1]], None, None, None, None, [[2]], [[3]], []]
```

**phi**0()  
Return $\varphi_0$ of self.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 3], 1, 5)
sage: [mg.phi0() for mg in K.module_generators]
[5, 4, 3, 2, 1, 0]
```

**classical_decomposition**()  
Return the classical decomposition of self.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 3], 1, 5)
sage: K.classical_decomposition()
The crystal of tableaux of type ['G', 2] and shape(s) [[], [1], [2], [3], [4], [5]]
```

**from_coordinates**(coords)  
Return an element of self from the coordinates coords.

**EXAMPLES:**
sage: K = crystals.KirillovReshetikhin(['D',4,3], 1, 5)
sage: K.from_coordinates((0, 2, 3, 1, 0, 1))
[[2, 2, 3, 0, -1]]

class \texttt{sage.combinat.crystals.kirillov_reshetikhin.KR_type_Dn_twisted}(\texttt{cartan_type}, r, s, \texttt{dual=\texttt{None}})

Bases: \texttt{KirillovReshetikhinGenericCrystal}

Class of Kirillov-Reshetikhin crystals $B^{n,s}$ of type $D_{n+1}^{(2)}$.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['D',4,2],3,1)
sage: [b,b.f(0) for b in K]
[[[+++, []], None], [[++-, []], None], [[+++, []], [++-, []], None], [[--+, []], [+-+, []], [[--+, []], [+-+, []], [[--+, []], [+-+, []]]]}

\texttt{classical_decomposition()}

Return the classical crystal underlying the Kirillov-Reshetikhin crystal $B^{n,s}$ of type $D_{n+1}^{(2)}$.

The classical decomposition is given by $B^{n,s} \cong B(s\Lambda_n)$.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['D',4,2],3,1)
sage: K.classical_decomposition()

The crystal of tableaux of type ['B', 3] and shape(s) [[1/2, 1/2, 1/2]]

sage: K = crystals.KirillovReshetikhin(['D',4,2],3,2)
sage: K.classical_decomposition()

The crystal of tableaux of type ['B', 3] and shape(s) [[1, 1, 1]]

\texttt{from_highest_weight_vector_to_pm_diagram}(b)

This gives the bijection between an element $b$ in the classical decomposition of the KR crystal that is $\{2,3,\ldots,n\}$-highest weight and $\pm$ diagrams.

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['D',4,2],3,1)
sage: T = K.classical_decomposition()
sage: hw = [ b for b in T if all(b.epsilon(i)==0 for i in [2,3]) ]
sage: [K.from_highest_weight_vector_to_pm_diagram(b) for b in hw]
[[[0, 0], [0, 0], [1, 0], [0]], [[0, 0], [0, 0], [0, 0], [0, 1], [0]]]

sage: K = crystals.KirillovReshetikhin(['D',4,2],3,2)
sage: T = K.classical_decomposition()
sage: hw = [ b for b in T if all(b.epsilon(i)==0 for i in [2,3]) ]
sage: [K.from_highest_weight_vector_to_pm_diagram(b) for b in hw]
[[[0, 0], [0, 0], [2, 0], [0]], [[0, 0], [0, 0], [0, 0], [2]],
 [[0, 0], [2, 0], [0, 0], [0]], [[0, 0], [0, 0], [0, 0], [2], [0]]]
Note that, since the classical decomposition of this crystal is of type $B_n$, there can be (at most one) entry 0 in the \{2,3,\ldots,n\}\text{-highest weight elements at height }n.\text{ In the following implementation this is realized as an empty column of height }n\text{ since this uniquely specifies the existence of the 0.}

**EXAMPLES:**

```python
sage: b = hw[1]
sage: pm = K.from_highest_weight_vector_to_pm_diagram(b)
sage: pm.pp()
```

**from_pm_diagram_to_highest_weight_vector(pm)**

This gives the bijection between a $\pm$ diagram and an element $b$ in the classical decomposition of the KR crystal that is \{2,3,\ldots,n\}\text{-highest weight.}

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D',4,2],3,2)
sage: pm = K.from_pm_diagram_to_highest_weight_vector([[0, 0], [0, 0], [0, 0], [2]])
sage: K.from_pm_diagram_to_highest_weight_vector(pm)
[[2], [3], [0]]
```

**class sage.combinat.crystals.kirillov_reshetikhin.KR_type_Dn_twistedElement**

Bases: `KirillovReshetikhinGenericCrystalElement`

Class for the elements in the Kirillov-Reshetikhin crystals $B_n^s$ of type $D^{(2)}_{n+1}$.

**EXAMPLES:**

```python
sage: K=crystals.KirillovReshetikhin(['D',4,2],3,3)
sage: b = K.module_generators[0]
sage: b.e(0) # indirect doctest
[[+, [[2], [3], [0]]]
```

**epsilon0()**

Calculate $\epsilon_0$ of self using Lemma 6.2 of [4].

**EXAMPLES:**

```python
sage: K=crystals.KirillovReshetikhin(['D',4,2],3,1)
sage: b = K.module_generators[0]
sage: b.epsilon(0) # indirect doctest
1
```
\textbf{f0()}

Return \(e_0\) on \texttt{self} by going to the \(\pm\)-diagram corresponding to the \(\{2, \ldots, n\}\)-highest weight vector in the component of \texttt{self}, then applying [Definition 6.2, 4], and pulling back from \(\pm\)-diagrams.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D',4,2],3,2)
sage: b = K.module_generators[0]
sage: b.f(0)  # indirect doctest
```

\textbf{phi0()}

Calculate \(\phi_0\) of \texttt{self}.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D',4,2],3,1)
sage: b = K.module_generators[0]
sage: b.phi(0)  # indirect doctest
```

\textbf{class} \texttt{sage.combinat.crystals.kirillov_reshetikhin.KR_type_E6(cartan_type, r, s)}

Bases: \texttt{KirillovReshetikhinCrystalFromPromotion}

Class of Kirillov-Reshetikhin crystals of type \(E_6^{(1)}\) for \(r = 1, 2, 6\).

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1],2,1)
sage: K.module_generator().e(0)
[]
sage: K.module_generator().e(0).f(0)
[[[2, -1], (1,)]]
sage: K = crystals.KirillovReshetikhin(['E',6,1], 1,1)
sage: b = K.module_generator()
sage: b
[[1,]]
sage: b.e(0)
[[-2, 1]]
sage: b = [t for t in K if t.epsilon(1) == 1 and t.phi(3) == 1 and t.phi(2) == 0 and t.epsilon(2) == 0][0]
sage: b
[[-1, 3]]
sage: b.e(0)
[[-1, -2, 3]]
```

The elements of the Kirillov-Reshetikhin crystals can be constructed from a classical crystal element using \texttt{retract()}. 

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1],2,1)
sage: La = K.cartan_type().classical().root_system().weight_lattice().fundamental_weights()
sage: H = crystals.HighestWeight(La[2])
sage: t = H.module_generator()
sage: t
```

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[[[2, -1), (1,)]]
sage: type(K.retract(t))
<class 'sage.combinat.crystals.kirillov_reshetikhin.KR_type_E6_with_category.˓
  element_class'>
sage: K.retract(t).e(0)
[]

affine_weight(<b>\right)</b></nobr>

Return the affine level zero weight corresponding to the element \texttt{b} of the classical crystal underlying \texttt{self}.

For the coefficients to calculate the level, see Table Aff 1 in [Ka1990].

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2,1)
sage: [K.affine_weight(x.lift()) for x in K.
  ....: if all(x.epsilon(i) == 0 for i in [2,3,4,5])]
[(0, 0, 0, 0, 0, 0, 0),
 (-2, 0, 1, 0, 0, 0, 0),
 (-1, -1, 0, 0, 0, 1, 0),
 (0, 0, 0, 0, 0, 0, 0),
 (0, 0, 0, 0, 0, 0, 0),
 (0, -2, 0, 1, 0, 0, 0)]
```

automorphism_on_affine_weight(<weight>)

Act with the Dynkin diagram automorphism on affine weights as outputted by the \texttt{affine\_weight} method.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2,1)
sage: sorted([x[0], K.automorphism_on_affine_weight(x[0])]
  ....: for x in K.highest_weight_dict().values())
[[-2, 0, 1, 0, 0, 0, 0],
 [-1, 0, 0, 1, 0, 0, 0],
 [-1, 0, 0, 0, 0, 0, 0],
 [0, 0, 0, 0, 0, 0, 0],
 [0, 0, 0, 0, 0, 0, 0],
 [0, -2, 0, 1, 0, 0, 0]]
```

classical_decomposition()

Specifies the classical crystal underlying the KR crystal of type $E_6^{(1)}$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2,2)
sage: K.classical_decomposition()
Direct sum of the crystals Family Direct sum of the crystals
Finite dimensional highest weight crystal of type ['E', 6] and highest weight
Finite dimensional highest weight crystal of type ['E', 6] and highest weight
Lambda[2],
```

(continues on next page)
Finite dimensional highest weight crystal of type ['E', 6] and highest weight \( \rightarrow 2\Lambda[2] \)

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 1,2)
sage: K.classical_decomposition()
```

Direct sum of the crystals Family
(Finite dimensional highest weight crystal of type ['E', 6] and highest weight \( \rightarrow 2\Lambda[1] \))

**dynkin_diagram_automorphism(i)**

Specifies the Dynkin diagram automorphism underlying the promotion action on the crystal elements.

Here we use the Dynkin diagram automorphism of order 3 which maps node 0 to node 1.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1],2,1)
sage: [K.dynkin_diagram_automorphism(i) for i in K.index_set()]

[1, 6, 3, 5, 4, 2, 0]
```

**highest_weight_dict()**

Return a dictionary between \{1, 2, 3, 4, 5\}-highest weight elements, and a tuple of affine weights and its classical component.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1],2,1)
sage: sorted(K.highest_weight_dict().items(), key=str)

[((2, -1), (1,)), ((-2, 0, 1, 0, 0, 0, 0), 1)),
[((3, -1, -6), (1,)), ((-1, 0, 0, 1, 0, 0, -1), 1)),
[((5, -2, -6), (-6, 2)), ((0, 0, 0, 0, 0, 1, -2), 1)),
[((6, -2), (-6, 2)), ((0, 0, 0, 0, 0, 0, 0), 1)),
([], ((0, 0, 0, 0, 0, 0, 0), 0))]
```

**highest_weight_dict_inv()**

Return a dictionary between a tuple of affine weights and a classical component, and \{2, 3, 4, 5, 6\}-highest weight elements.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1],2,1)
sage: K.highest_weight_dict_inv()

{((-2, 0, 1, 0, 0, 0, 0), 1): [(2, -1), (1,)],
((-1, -1, 0, 0, 0, 0, 0), 1): [(5, -3), (-1, 3)],
((0, -2, 0, 1, 0, 0, 0), 0): [],
((0, 0, 0, 0, 0, 0, 0), 0): []}
```

**hw_auxiliary()**

Return the 2, 3, 4, 5 highest weight elements of self.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1],2,1)
sage: K.hw_auxiliary()
```
(continued from previous page)

promotion()

Specifies the promotion operator used to construct the affine type $E_6^{(1)}$ crystal.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2,1)
sage: promotion = K.promotion()
sage: all(promotion(promotion(promotion(b))) == b for b in K.classical_decomposition())
True
sage: K = crystals.KirillovReshetikhin(['E',6,1],1,1)
sage: promotion = K.promotion()
sage: all(promotion(promotion(promotion(b))) == b for b in K.classical_decomposition())
True
```

promotion_inverse()

Return the inverse promotion. Since promotion is of order 3, the inverse promotion is the same as promotion applied twice.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2,1)
sage: p = K.promotion()
sage: p_inv = K.promotion_inverse()
sage: all(p_inv(p(b)) == b for b in K.classical_decomposition())
True
```

promotion_on_highest_weight_vectors()

Return a dictionary of the promotion map on $\{1, 2, 3, 4, 5\}$-highest weight elements to $\{2, 3, 4, 5, 6\}$-highest weight elements in self.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2, 1)
sage: dic = K.promotion_on_highest_weight_vectors()
sage: sorted(dic.items(), key=str)
[[[(2, -1), (1,)], [[(-1,), (-1, 3)]],
  [[[3, -1, -6), (1,))], [[[5, -3), (-1, 3)]],
  [[[5, -2, -6), (-6, 2)]], [[[2, -1), (1,))],
  [[[6, -2), (-6, 2)]], []],
  [[[1, -3), (-1, 3)]],
  [[[1, (-1, 3)]])
```

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promotion_on_highest_weight_vectors_function()

Return a lambda function on x defined by self.promotion_on_highest_weight_vectors()[x].

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',6,1], 2, 1)
sage: f = K.promotion_on_highest_weight_vectors_function()
sage: f(K.module_generator().lift())
[[-(-1,), (-1, 3)]
```

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_E7(ct, r, s)

Bases: KirillovReshetikhinGenericCrystal

The Kirillov-Reshetikhin crystal $B_{7,s}$ of type $E_7^{(1)}$.

A7_decomposition()

Return the decomposition of self into $A_7$ highest weight crystals.

The $A_7$ decomposition of $B_{7,s}$ is given by the parameters $m_4, m_5, m_6, m_7 \geq 0$ such that $m_4 + m_5 \leq m_7$ and $s = m_4 + m_5 + m_6 + m_7$. The corresponding $A_7$ highest weight crystal has highest weight $\lambda = (m_7 - m_4 - m_5)\Lambda_6 + m_5\Lambda_4 + m_6\Lambda_2$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',7,1], 7, 3)
sage: K.A7_decomposition()
The crystal of tableaux of type ['A', 7] and shape(s)
[[3, 3, 3, 3, 3, 3], [3, 3, 2, 2, 2, 2], [3, 3, 1, 1, 1, 1], [3, 3],
 [2, 2, 2, 2, 1, 1], [2, 2, 1, 1], [1, 1, 1, 1, 1, 1], [1, 1]]
```

class Element

Bases: KirillovReshetikhinGenericCrystalElement

e0()

Return the action of $e_0$ on self.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',7,1], 7, 2)
sage: mg = K.module_generator()
sage: mg.e0()
[[(-1,), (7,)]]
sage: mg.e0().e0()
[[(-1, 7), (1,)]]
sage: mg.e_string([0,0,0]) is None
True
```

f0()

Return the action of $f_0$ on self.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',7,1], 7, 2)
sage: mg = K.module_generator()
sage: x = mg.f_string([7,6,5,4,3,2,4,5,6,1,3,4,5,2,4,3,1])
sage: x.f0()
(continues on next page)
```
classical_decomposition()

Return the classical decomposition of self.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',7,1], 7, 4)
sage: K.classical_decomposition()
The crystal of tableaux of type ['E', 7] and shape(s) [[4]]
```

from_A7_crystal()

Return the inclusion of the KR crystal $B^7,s$ of type $E_7^{(1)}$ into type $A_7$ highest weight crystals.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',7,1], 7, 2)
sage: K.from_A7_crystal()
['A', 6] -> ['E', 7, 1] Virtual Crystal morphism:
  From: The crystal of tableaux of type ['A', 7] and shape(s)
    [[2, 2, 2, 2, 2, 2], [2, 2, 1, 1, 1, 1], [2, 2], [1, 1, 1, 1], []]
  To: Kirillov-Reshetikhin crystal of type ['E', 7, 1] with (r,s)=(7,2)
  Defn: ...
```

to_A7_crystal()

Return the map decomposing the KR crystal $B^7,s$ of type $E_7^{(1)}$ into type $A_7$ highest weight crystals.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['E',7,1], 7, 2)
sage: K.to_A7_crystal()
['A', 6] relabelled by {1: 1, 2: 3, 3: 4, 4: 5, 5: 6, 6: 7} -> ['A', 7] Virtual...
  Crystal morphism:
    From: Kirillov-Reshetikhin crystal of type ['E', 7, 1] with (r,s)=(7,2)
    To: The crystal of tableaux of type ['A', 7] and shape(s)
      [[2, 2, 2, 2, 2, 2], [2, 2, 1, 1, 1, 1], [2, 2], [1, 1, 1, 1], []]
    Defn: ...
```

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_box(cartan_type, r, s)

Bases: KirillovReshetikhinGenericCrystal, AffineCrystalFromClassical

Class of Kirillov-Reshetikhin crystals $B^{r,s}$ of type $A_{2n}^{(2)}$ for $r \leq n$ and type $D_{n+1}^{(2)}$ for $r < n$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['A',4,2], 1,1)
sage: K
Kirillov-Reshetikhin crystal of type ['BC', 2, 2] with (r,s)=(1,1)
sage: b = K(rows=[])
sage: b.f(0)
[[1]]
```
Element
alias of KR_type_boxElement
ambient_crystal()
Return the ambient crystal $B^{r,s}$ of type $C_n^{(1)}$ associated to the Kirillov-Reshetikhin crystal.
The ambient crystal is used to construct the zero arrows.
EXAMPLES:
```python
sage: K = crystals.KirillovReshetikhin(['A',4,2], 2,2)
sage: K.ambient_crystal()
Kirillov-Reshetikhin crystal of type ['C', 2, 1] with (r,s)=(2,4)
```
ambient_highest_weight_dict()
Return a dictionary of the classical highest weight vectors of the ambient crystal of self whose keys are their shape.
EXAMPLES:
```python
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,2)
sage: K.ambient_highest_weight_dict()
{}: [],
[2]: [[1, 1]],
[2, 2]: [[1, 1], [2, 2]],
[4]: [[1, 1, 1, 1]],
[4, 2]: [[1, 1, 1, 1], [2, 2]],
[4, 4]: [[1, 1, 1, 1], [2, 2, 2, 2]]
```
classical_decomposition()
Return the classical crystal underlying the Kirillov-Reshetikhin crystal of type $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$.

It is given by $B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$, where $\Lambda$ are weights obtained from a rectangle of width $s$ and height $r$ by removing boxes. Here we identify the fundamental weight $\Lambda_i$ with a column of height $i$.
EXAMPLES:
```python
sage: K = crystals.KirillovReshetikhin(['A',4,2], 2,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['C', 2] and shape(s) [[]], [1], [2], [1, 1], [2, 2]
sage: K = crystals.KirillovReshetikhin(['D',4,2], 2,3)
sage: K.classical_decomposition()
The crystal of tableaux of type ['B', 3] and shape(s) [[]], [1], [2], [1, 1], [2, 2], [3, 2], [3, 3]
```
from_ambient_crystal()
Return a map from the ambient crystal of type $C_n^{(1)}$ to the Kirillov-Reshetikhin crystal self.
Note that this map is only well-defined on elements that are in the image under to_ambient_crystal().
EXAMPLES:
sage: K = crystals.KirillovReshetikhin(['D',4,2], 1,1)
sage: b = K.ambient_crystal()(rows=[[3,-3]])
sage: K.from_ambient_crystal()(b)
[[0]]
sage: K = crystals.KirillovReshetikhin(['A',4,2], 1,1)
sage: b = K.ambient_crystal()(rows=[])  
sage: K.from_ambient_crystal()(b)
[]

**highest_weight_dict()**

Return a dictionary of the classical highest weight vectors of self whose keys are 2 times their shape.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,2)
sage: K.highest_weight_dict()
{
    (): [],
    [2]: [[1]],
    [2, 2]: [[1], [2]],
    [4]: [[1, 1]],
    [4, 2]: [[1, 1], [2]],
    [4, 4]: [[1, 1], [2, 2]]
}
```

**similarity_factor()**

Sets the similarity factor used to map to the ambient crystal.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,2)
sage: K.similarity_factor()
{1: 2, 2: 2, 3: 2}
sage: K = crystals.KirillovReshetikhin(['D',5,2], 1,1)
sage: K.similarity_factor()
{1: 2, 2: 2, 3: 2, 4: 1}
```

**to_ambient_crystal()**

Return a map from self to the ambient crystal of type $C_n^{(1)}$.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(['D',4,2], 1,1)
sage: [K.to_ambient_crystal()(b) for b in K]
[[], [[1, 1]], [[2, 2]], [[3, 3]], [[3, -3]], [[-3, -3]], [[-2, -2]], [[-1, -1]]]
sage: K = crystals.KirillovReshetikhin(['A',4,2], 1,1)
sage: [K.to_ambient_crystal()(b) for b in K]
[[], [[1, 1]], [[2, 2]], [[-2, -2]], [[-1, -1]]]
```

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_boxElement

Bases: KirillovReshetikhinGenericCrystalElement

Class for the elements in the Kirillov-Reshetikhin crystals $B_{r-s}$ of type $A_{2n}^{(2)}$ for $r \leq n$ and type $D_{n+1}^{(2)}$ for $r < n$.

**EXAMPLES:**
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```
sage: K = crystals.KirillovReshetikhin(['A',4,2],1,2)
sage: type(K.module_generators[0])
<class 'sage.combinat.crystals.kirillov_reshetikhin.KR_type_box_with_category.
  element_class'>
```

e0()

Return \(e_0\) on \(self\) by mapping \(self\) to the ambient crystal, calculating \(e_0\) there and pulling the element back.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',4,2],1,1)
sage: b = K(rows=[])
sage: b.e(0) # indirect doctest
[[-1]]
```

epsilon0()

Return \(\varepsilon_0\) of \(self\) by mapping the element to the ambient crystal and calculating \(\varepsilon_0\) there.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',4,2],1,1)
sage: b = K(rows=[[1]])
sage: b.epsilon(0) # indirect doctest
2
```

f0()

Return \(f_0\) on \(self\) by mapping \(self\) to the ambient crystal, calculating \(f_0\) there and pulling the element back.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',4,2],1,1)
sage: b = K(rows=[])
sage: b.f(0) # indirect doctest
[[1]]
```

phi0()

Return \(\varphi_0\) of \(self\) by mapping the element to the ambient crystal and calculating \(\varphi_0\) there.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['D',3,2],1,1)
sage: b = K(rows=[[1]])
sage: b.phi(0) # indirect doctest
2
```

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_spin(cartan_type, r, s)

Bases: KirillovReshetikhinCrystalFromPromotion

Class of Kirillov-Reshetikhin crystals \(B^{n,s}\) of type \(D_n^{(1)}\).

EXAMPLES:
classical_decomposition()

Return the classical crystal underlying the Kirillov-Reshetikhin crystal $B_{r,s}$ of type $D_n^{(1)}$ for $r = n - 1, n$.

The classical decomposition is given by $B_{n,s} \cong B(s\Lambda_r)$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['D',4,1],4,1); K
Kirillov-Reshetikhin crystal of type ['D', 4, 1] with (r,s)=(4,1)
sage: K.classical_decomposition()
The crystal of tableaux of type ['D', 4] and shape(s) [[1/2, 1/2, 1/2, 1/2]]
```

```
sage: K = crystals.KirillovReshetikhin(['D',4,1],3,1); K
Kirillov-Reshetikhin crystal of type ['D', 4, 1] with (r,s)=(3,1)
sage: K.classical_decomposition()
The crystal of tableaux of type ['D', 4] and shape(s) [[1, 1, 1, -1]]
```

```
sage: K = crystals.KirillovReshetikhin(['D',4,1],3,2); K
Kirillov-Reshetikhin crystal of type ['D', 4, 1] with (r,s)=(3,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['D', 4] and shape(s) [[1, 1, 1, -1]]
```
Specifies the Dynkin diagram automorphism underlying the promotion action on the crystal elements.

Here we use the Dynkin diagram automorphism which interchanges nodes 0 and 1 and leaves all other nodes unchanged.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(["D",4,1],4,1)
sage: K.dynkin_diagram_automorphism(0)
1
sage: K.dynkin_diagram_automorphism(1)
0
sage: K.dynkin_diagram_automorphism(4)
4
```

**promotion()**

Return the promotion operator on $B^{r,s}$ of type $D_n^{(1)}$ for $r = n - 1, n$.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(["D",4,1],3,1)
sage: T = K.classical_decomposition()
sage: promotion = K.promotion()
sage: for t in T:
    ....:     print("{} {}".format(t, promotion(t)))
[+--+, []] [-+-+, []]
[---+, []] [++++, []]
[+++, []] [-+-+, []]
[+---, []] [----, []]
[+--+, []] [+--+, []]
[-+--, []] [+-+-, []]
[--+-, []] [+-+-, []]
[+--+, []] [++--, []]
```

**promotion_inverse()**

Return the inverse promotion operator on $B^{r,s}$ of type $D_n^{(1)}$ for $r = n - 1, n$.

**EXAMPLES:**

```python
sage: K = crystals.KirillovReshetikhin(["D",4,1],3,1)
sage: T = K.classical_decomposition()
sage: promotion = K.promotion()
sage: promotion_inverse = K.promotion_inverse()
sage: all(promotion_inverse(promotion(t)) == t for t in T)
True
```

**promotion_on_highest_weight_vectors()**

Return the promotion operator on \{2, 3, \ldots, n\}-highest weight vectors.

A \{2, 3, \ldots, n\}-highest weight vector in $B(s\Lambda_n)$ of weight $w = (w_1, \ldots, w_n)$ is mapped to a \{2, 3, \ldots, n\}-highest weight vector in $B(s\Lambda_{n-1})$ of weight $(-w_1, w_2, \ldots, w_n)$ and vice versa.

See also:

- **promotion_on_highest_weight_vectors_inverse()**
• promotion()

EXAMPLES:

```python
sage: KR = crystals.KirillovReshetikhin(['D', 4, 1], 4, 2)
sage: prom = KR.promotion_on_highest_weight_vectors()
sage: T = KR.classical_decomposition()
sage: HW = [t for t in T if t.is_highest_weight([2, 3, 4])]
sage: for t in HW:
   ....:    print("{} {}").format(t, prom[t])
[[1], [2], [3], [4]] [[2], [3], [4], [-1]]
[[2], [3], [-4], [4]] [[2], [3], [4], [-4]]
[[2], [3], [-4], [-1]] [[1], [2], [3], [-4]]
```

promotion_on_highest_weight_vectors_inverse()

Return the inverse promotion operator on \{2, 3, \ldots, n\}-highest weight vectors.

See also:

• promotion_on_highest_weight_vectors()
• promotion_inverse()

EXAMPLES:

```python
sage: KR = crystals.KirillovReshetikhin(['D', 4, 1], 4, 1)
sage: prom = KR.promotion_on_highest_weight_vectors()
sage: T = KR.classical_decomposition()
sage: HW = [t for t in T if t.is_highest_weight([2, 3, 4])]
sage: all(prom_inv[prom[t]] == t for t in HW)
```

class sage.combinat.crystals.kirillov_reshetikhin.KR_type_vertical(cartan_type, r, s)

Bases: KirillovReshetikhinCrystalFromPromotion

Class of Kirillov-Reshetikhin crystals $B^{r,s}$ of type $D_n^{(1)}$ for $r \leq n - 2$, $B_n^{(1)}$ for $r < n$, and $A_{2n-1}^{(2)}$ for $r \leq n$.

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2)
sage: b = K(rows=[])
sage: b.f(0)
[[1, [2]]]
sage: b.f(0).f(0)
[[1, 1], [2, 2]]
```
classical_decomposition()

Specifies the classical crystal underlying the Kirillov-Reshetikhin crystal of type $D^{(1)}_n$, $B^{(1)}_n$, and $A^{(2)}_{2n-1}$. It is given by $B^{r,s} \cong \bigoplus \Lambda B(\Lambda)$, where $\Lambda$ are weights obtained from a rectangle of width $s$ and height $r$ by removing vertical dominoes. Here we identify the fundamental weight $\Lambda_i$ with a column of height $i$.

EXAMPLES:

```sage
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['D', 4] and shape(s) [[], [1, 1], [2, 2]]
```

dynkin_diagram_automorphism(i)

Specifies the Dynkin diagram automorphism underlying the promotion action on the crystal elements. The automorphism needs to map node 0 to some other Dynkin node.

Here we use the Dynkin diagram automorphism which interchanges nodes 0 and 1 and leaves all other nodes unchanged.

EXAMPLES:

```sage
sage: K = crystals.KirillovReshetikhin(['D',4,1],1,1)
sage: K.dynkin_diagram_automorphism(0)
1
sage: K.dynkin_diagram_automorphism(1)
0
sage: K.dynkin_diagram_automorphism(4)
4
```

from_highest_weight_vector_to_pm_diagram(b)

This gives the bijection between an element $b$ in the classical decomposition of the KR crystal that is $2, 3, \ldots, n$-highest weight and $\pm$ diagrams.

EXAMPLES:
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: T = K.classical_decomposition()
sage: b = T(rows=[[2],[-2]])
sage: pm = K.from_highest_weight_vector_to_pm_diagram(b); pm
[[1, 1], [0, 0], [0]]
sage: pm.pp()
+ -
sage: b = T(rows=[])  
sage: pm = K.from_highest_weight_vector_to_pm_diagram(b); pm
[[0, 2], [0, 0], [0]]
sage: pm.pp()

sage: hw = [ b for b in T if all(b.epsilon(i)==0 for i in [2,3,4]) ]
sage: all(K.from_pm_diagram_to_highest_weight_vector(K.from_highest_weight_vector_to_pm_diagram(b)) == b for b in hw)
True

.. autosummary::

   from_pm_diagram_to_highest_weight_vector

This gives the bijection between a ± diagram and an element \( b \) in the classical decomposition of the KR crystal that is \( 2, 3, \ldots, n \)-highest weight.

**EXAMPLES:**

sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: pm = sage.combinat.crystals.kirillov_reshetikhin.PMDiagram([[1, 1], [0, 0], [0]])
sage: K.from_pm_diagram_to_highest_weight_vector(pm)
[[2], [-2]]

.. autosummary::

   promotion

Specifies the promotion operator used to construct the affine type \( D_n^{(1)} \) etc. crystal.

This corresponds to the Dynkin diagram automorphism which interchanges nodes 0 and 1, and leaves all other nodes unchanged. On the level of crystals it is constructed using ± diagrams.

**EXAMPLES:**

sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: promotion = K.promotion()
sage: b = K.classical_decomposition()(rows=[])  
sage: promotion(b)
[[1, 2], [-2, -1]]
sage: b = K.classical_decomposition()(rows=[[1,3],[2,-1]])
sage: promotion(b)
[[1, 3], [2, -1]]
sage: b = K.classical_decomposition()(rows=[[1],[-3]])
sage: promotion(b)
[[2, -3], [-2, -1]]

.. autosummary::

   promotion_inverse

Return inverse of promotion.

In this case promotion is an involution, so promotion inverse equals promotion.

**EXAMPLES:**
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: promotion = K.promotion()
sage: promotion_inverse = K.promotion_inverse()
sage: all( promotion_inverse(promotion(b.lift())) == b.lift() for b in K )
True

**promotion_on_highest_weight_vector**(b)
Calculates promotion on a 2, 3, ..., n highest weight vector b.

**EXAMPLES:**

sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: T = K.classical_decomposition()
sage: hw = [ b for b in T if all(b.epsilon(i)==0 for i in [2,3,4]) ]
sage: [K.promotion_on_highest_weight_vector(b) for b in hw]
[[[1, 2], [-2, -1]], [[2, 2], [3, -1]],
[[2, -2], [1, 2], [-2], [-2]],
[[2, 2], [3, -1]],
[[1, 1], [2, -1]], [[1, 2], [2, -1]],
[[2, 2], [3, 3]],
[[1, 2], [2, 3]]]
• cartan_type – an affine Cartan type
• r – a label of finite Dynkin diagram
• s – a positive integer
• model – (default: 'KN') can be one of the following:
  – 'KN' or 'KashiwaraNakashimaTableaux' - use the Kashiwara-Nakashima tableaux model
  – 'KR' or 'KirillovReshetikhinTableaux' - use the Kirillov-Reshetikhin tableaux model
  – 'RC' or 'RiggedConfiguration' - use the rigged configuration model
  – 'LSPaths' - use the LS path model

EXAMPLES:

sage: K = crystals.KirillovReshetikhin(['A',3,1], 2, 1)
sage: K.index_set()
(0, 1, 2, 3)
sage: K.list()
[[[1], [2]], [[1], [3]], [[2], [3]], [[1], [4]], [[2], [4]], [[3], [4]]]
sage: b=K(rows=[[1],[2]])
sage: b.weight()
-Lambda[0] + Lambda[2]

sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.automorphism(K.module_generator[0])
[[2, 2], [3, 3]]
sage: K.module_generator[0].e(0)
[[1, 2], [2, 4]]
sage: K.module_generator[0].f(2)
[[1, 1], [2, 3]]
sage: K.module_generator[0].f(1)

sage: K.module_generator[0].phi(0)
0
sage: K.module_generator[0].phi(1)
0
sage: K.module_generator[0].phi(2)
2
sage: K.module_generator[0].epsilon(0)
2
sage: K.module_generator[0].epsilon(1)
0
sage: K.module_generator[0].epsilon(2)
0
sage: K.module_generator[0].epsilon(0)
2

sage: b = K(rows=[[1,2],[2,3]])
sage: b
[[1, 2], [2, 3]]
sage: b.f(2)
[[1, 2], [3, 3]]

sage: K = crystals.KirillovReshetikhin(['D',4,1], 2, 1)
sage: K.cartan_type()
['D', 4, 1]
sage: type(K.module_generator[0])

(continues on next page)
The following gives some tests with regards to Lemma 3.11 in [LOS2012].

REFERENCES:

• [Shi2002]
• [Sch2008]
• [JS2010]
• [FOS2009]
• [LOS2012]

sage.combinat.crystals.kirillov_reshetikhin.KirillovReshetikhinCrystalFromLSPaths(cartan_type, r, s=1)

Single column Kirillov-Reshetikhin crystals.

This yields the single column Kirillov-Reshetikhin crystals from the projected level zero LS paths, see CrystalOfLSPaths. This works for all types (even exceptional types). The weight of the canonical element in this crystal is $\Lambda_r$. For other implementation see KirillovReshetikhinCrystal().

EXAMPLES:

```python
sage: K = crystals.kirillov_reshetikhin.LSPaths(['A',2,1],2) # indirect doctest
sage: KR = crystals.KirillovReshetikhin(['A',2,1],2,1)
sage: G = K.digraph()
sage: GR = KR.digraph()
sage: G.is_isomorphic(GR, edge_labels = True)
True
sage: K = crystals.kirillov_reshetikhin.LSPaths(['C',3,1],2)
sage: KR = crystals.KirillovReshetikhin(['C',3,1],2,1)
sage: G = K.digraph()
sage: GR = KR.digraph()
sage: G.is_isomorphic(GR, edge_labels = True)
True
sage: K = crystals.kirillov_reshetikhin.LSPaths(['E',6,1],1)
sage: KR = crystals.KirillovReshetikhin(['E',6,1],1,1)
sage: G = K.digraph()
sage: GR = KR.digraph()
sage: G.is_isomorphic(GR, edge_labels = True)
True
sage: K.cardinality()
27
sage: K = crystals.kirillov_reshetikhin.LSPaths(['G',2,1],1)
sage: K.cardinality()
7
sage: K = crystals.kirillov_reshetikhin.LSPaths(['B',3,1],2)
sage: KR = crystals.KirillovReshetikhin(['B',3,1],2,1)
```

(continues on next page)
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```python
sage: KR.cardinality()
22
sage: K.cardinality()
22
sage: G = K.digraph()
```

```python
sage: GR = KR.digraph()
```

```python
sage: G.is_isomorphic(GR, edge_labels = True)
True
```

```python
class sage.combinat.crystals.kirillov_reshetikhin.KirillovReshetikhinCrystalFromPromotion(cartan_type, r, s)

Bases: KirillovReshetikhinGenericCrystal, AffineCrystalFromClassicalAndPromotion

This generic class assumes that the Kirillov-Reshetikhin crystal is constructed from a classical crystal using the classical_decomposition and an automorphism promotion and its inverse, which corresponds to a Dynkin diagram automorphism dynkin_diagram_automorphism.

Each instance using this class needs to implement the methods:

- classical_decomposition
- promotion
- promotion_inverse
- dynkin_diagram_automorphism

```
```
**kirillov_reshetikhin_tableaux()**

Return the corresponding set of KirillovReshetikhinTableaux.

EXAMPLES:

```python
sage: KRC = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2)
sage: KRC.kirillov_reshetikhin_tableaux()
Kirillov-Reshetikhin tableaux of type ['D', 4, 1] and shape (2, 2)
```

**module_generator()**

Return the unique module generator of classical weight \(s\Lambda_r\) of a Kirillov-Reshetikhin crystal \(B^{r,s}\).

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['C', 2, 1], 1, 2)
sage: K.module_generator()
[[1, 1]]
sage: K = crystals.KirillovReshetikhin(['E', 6, 1], 1, 1)
sage: K.module_generator()
[(1,)]
sage: K = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1)
sage: K.module_generator()
[[1], [2]]
```

**r()**

Return \(r\) of the underlying Kirillov-Reshetikhin crystal \(B^{r,s}\).

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1)
sage: K.r()
2
```

**s()**

Return \(s\) of the underlying Kirillov-Reshetikhin crystal \(B^{r,s}\).

EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1)
sage: K.s()
1
```

**class**

sage.combinat.crystals.kirillov_reshetikhin.KirillovReshetikhinGenericCrystalElement

Bases: AffineCrystalFromClassicalElement

Abstract class for all Kirillov-Reshetikhin crystal elements.

**lusztig_involution()**

Return the classical Lusztig involution on \(self\).

EXAMPLES:

```python
sage: KRC = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2)
sage: elt = KRC(-1,2); elt
(continues on next page)
```
pp()
Pretty print self.

EXAMPLES:

```python
sage: C = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1)
sage: C(2, 1).pp()
1
2
sage: C = crystals.KirillovReshetikhin(['B', 3, 1], 3, 3)
sage: C.module_generators[0].pp()
+ (X) 1
+ +
```

to_kirillov_reshetikhin_tableau()
Construct the corresponding KirillovReshetikhinTableauxElement from self.

We construct the Kirillov-Reshetikhin tableau element as follows:

1. Let $\lambda$ be the shape of self.
2. Determine a path $e_{i_1} e_{i_2} \cdots e_{i_k}$ to the highest weight.
3. Apply $f_{i_k} \cdots f_{i_2} f_{i_1}$ to a highest weight KR tableau from filling the shape $\lambda$.

EXAMPLES:

```python
sage: KRC = crystals.KirillovReshetikhin(['A', 4, 1], 2, 1)
sage: KRC(columns=[[2, 1]]).to_kirillov_reshetikhin_tableau()
[[1], [2]]
sage: KRC = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1)
sage: KRC(rows=[]).to_kirillov_reshetikhin_tableau()
[[1], [-1]]
```

to_tableau()
Return the Tableau corresponding to self.

EXAMPLES:

```python
sage: C = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1)
sage: t = C(2, 1).to_tableau(); t
[[1], [2]]
sage: type(t)
<class 'sage.combinat.tableau.Tableaux_all_with_category.element_class'>
```

class sage.combinat.crystals.kirillov_reshetikhin.PMDiagram(pm_diagram, from_shapes=None)
Bases: CombinatorialObject

Class of $\pm$ diagrams. These diagrams are in one-to-one bijection with $X_{n-1}$ highest weight vectors in an $X_n$ highest weight crystal $X = B, C, D$. See Section 4.1 of [Sch2008].
The input is a list \( pm = [[a_0, b_0], [a_1, b_1], \ldots, [a_{n-1}, b_{n-1}], [b_n]] \) of pairs and a last 1-tuple (or list of length 1). The pair \([a_i, b_i]\) specifies the number of \( a_i + b_i \) and \( b_i - a_i \) in the \( i \)-th row of the \( \pm \) diagram if \( n - i \) is odd and the number of \( a_i \) ± pairs above row \( i \) and \( b_i \) columns of height \( i \) not containing any + or − if \( n - i \) is even.

Setting the option \texttt{from\_shapes = True} one can also input a \( \pm \) diagram in terms of its outer, intermediate, and inner shape by specifying a list \([n, s, \text{outer}, \text{intermediate}, \text{inner}]\) where \( s \) is the width of the \( \pm \) diagram, and \text{outer}, \text{intermediate}, and \text{inner} are the outer, intermediate, and inner shapes, respectively.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.kirillov_reshetikhin import PMDiagram
sage: pm = PMDiagram([[0, 1], [1, 2], [1]])
\[[0, 1], [1, 2], [1]\]
\[1, 1, 2, 0, 1\]
\(2\)
\(5\)
\(. . . . .\)
\(\ldots\)
\(+ - -\)
```

**heights_of_addable_plus()**

Return a list with the heights of all addable plus in the \( \pm \) diagram.

**EXAMPLES:**

```python
sage: pm = PMDiagram([[1, 2], [1, 2], [1, 1], [1, 1], [1, 1], [1]])
sage: pm.heights_of_addable_plus()
[1, 1, 2, 3, 4, 5]
```

**heights_of_minus()**

Return a list with the heights of all minus in the \( \pm \) diagram.

**EXAMPLES:**

```python
sage: pm = PMDiagram([[1, 2], [1, 2], [1, 1], [1, 1], [1, 1], [1]])
sage: pm.heights_of_minus()
[5, 5, 3, 3, 1, 1]
```

**inner_shape()**

Return the inner shape of the pm diagram

**EXAMPLES:**

```python
sage: pm = PMDiagram([[1, 2], [1, 1], [1, 1], [1, 1], [1]])
sage: pm.inner_shape()
[4, 4, 2, 2]
```
sage: from sage.combinat.crystals.kirillov_reshetikhin import PMDiagram
sage: pm = PMDiagram([[0,1],[1,2],[1]])
sage: pm.intermediate_shape()
[4, 1]
sage: pm = PMDiagram([[1,2],[1,1],[1,1],[1,1],[1]])
sage: pm.intermediate_shape()
[7, 5, 3, 1]

intermediate_shape()

Return the intermediate shape of the pm diagram (inner shape plus positions of plusses)

EXAMPLES:

sage: from sage.combinat.crystals.kirillov_reshetikhin import PMDiagram
sage: pm = PMDiagram([[0,1],[1,2],[1]])
sage: pm.intermediate_shape()
[4, 2]
sage: pm = PMDiagram([[1,2],[1,1],[1,1],[1,1],[1]])
sage: pm.intermediate_shape()
[8, 6, 4, 2]

sage: pm = PMDiagram([[1,2],[1,2],[1,1],[1,1],[1,1],[1]])
sage: pm.intermediate_shape()
[11, 8, 6, 4, 2]

outer_shape()

Return the outer shape of the ± diagram

EXAMPLES:

sage: from sage.combinat.crystals.kirillov_reshetikhin import PMDiagram
sage: pm = PMDiagram([[0,1],[1,2],[1]])
sage: pm.outer_shape()
[4, 4]
sage: pm = PMDiagram([[1,2],[1,1],[1,1],[1,1],[1]])
sage: pm.outer_shape()
[8, 8, 4, 4]

pp()

Pretty print self.

EXAMPLES:
```python
sage: from sage.combinat.crystals.kirillov_reshetikhin import PMDiagram
sage: pm = PMDiagram([[1, 0], [0, 1], [2, 0], [0, 0], [0]])
sage: pm.pp()
. . . +
. . - -
+ +
- -
sage: pm = PMDiagram([[0, 2], [0, 0], [0]])
sage: pm.pp()
```

**sigma()**

Return sigma on pm diagrams as needed for the analogue of the Dynkin diagram automorphism that interchanges nodes 0 and 1 for type $D_n(1)$, $B_n(1)$, $A_{2n-1}(2)$ for Kirillov-Reshetikhin crystals.

**EXAMPLES:**

```python
sage: pm = sage.combinat.crystals.kirillov_reshetikhin.PMDiagram([[0, 1], [1, 2], ..→ [1]])
sage: pm.sigma()
[[1, 0], [2, 1], [1]]
```

`sage.combinat.crystals.kirillov_reshetikhin.horizontal_dominoes_removed(r, s)`

Returns all partitions obtained from a rectangle of width $s$ and height $r$ by removing horizontal dominoes.

**EXAMPLES:**

```python
sage: sage.combinat.crystals.kirillov_reshetikhin.horizontal_dominoes_removed(2, 2)
[[], [2], [2, 2]]
sage: sage.combinat.crystals.kirillov_reshetikhin.horizontal_dominoes_removed(3, 2)
[[], [2], [2, 2], [2, 2, 2]]
```

`sage.combinat.crystals.kirillov_reshetikhin.partitions_in_box(r, s)`

Returns all partitions in a box of width $s$ and height $r$.

**EXAMPLES:**

```python
sage: sage.combinat.crystals.kirillov_reshetikhin.partitions_in_box(3, 2)
[[], [1], [2], [1, 1], [2, 1], [1, 1, 1], [2, 2], [2, 1, 1],
 [2, 2, 1], [2, 2, 2]]
```

`sage.combinat.crystals.kirillov_reshetikhin.vertical_dominoes_removed(r, s)`

Returns all partitions obtained from a rectangle of width $s$ and height $r$ by removing vertical dominoes.

**EXAMPLES:**

```python
sage: sage.combinat.crystals.kirillov_reshetikhin.vertical_dominoes_removed(2, 2)
[[], [1, 1], [2, 2]]
sage: sage.combinat.crystals.kirillov_reshetikhin.vertical_dominoes_removed(3, 2)
[[2], [2, 1, 1], [2, 2, 2]]
sage: sage.combinat.crystals.kirillov_reshetikhin.vertical_dominoes_removed(4, 2)
[[], [1, 1], [1, 1, 1, 1], [2, 2], [2, 2, 1, 1], [2, 2, 2, 2]]
```
5.1.56 Kyoto Path Model for Affine Highest Weight Crystals

Consider a Kac–Moody algebra \( g \) of affine Cartan type \( X \), and we want to model the \( U'_q(g) \)-crystal \( B(\lambda) \). First we consider the set of fundamental weights \( \{ \Lambda_i \}_{i \in I} \) of \( g \) and let \( \{ \overline{\Lambda}_i \}_{i \in I} \) be the corresponding fundamental weights of the corresponding classical Lie algebra \( g_0 \). To model \( B(\lambda) \), we start with a sequence of perfect \( U'_q(g) \)-crystals \( (B^{(i)})_i \) of level \( l \) such that

\[
\lambda \in \overline{P}^+_l = \left\{ \mu \in \overline{P}^+ \mid (c, \mu) = l \right\}
\]

where \( c \) is the canonical central element of \( U'_q(g) \) and \( \overline{P}^+ \) is the nonnegative weight lattice spanned by \( \{ \overline{\Lambda}_i \mid i \in I \} \).

Next we consider the crystal isomorphism \( \Phi_0 : B(\lambda_0) \rightarrow B^{(0)} \otimes B(\lambda_1) \) defined by \( u_{\lambda_0} \mapsto b^{(0)}_{\lambda_0} \otimes u_{\lambda_1} \) where \( b^{(0)}_{\lambda_0} \) is the unique element in \( B^{(0)} \) such that \( \varphi \left( b^{(0)}_{\lambda_0} \right) = \lambda_0 \) and \( \lambda_1 = \varepsilon \left( b^{(0)}_{\lambda_0} \right) \) and \( u_{\mu} \) is the highest weight element in \( B(\mu) \). Iterating this, we obtain the following isomorphism:

\[
\Phi_n : B(\lambda) \rightarrow B^{(0)} \otimes B^{(1)} \otimes \cdots \otimes B^{(N)} \otimes B(\lambda_{N+1}).
\]

We note by Lemma 10.6.2 in [HK2002] that for any \( b \in B(\lambda) \) there exists a finite \( N \) such that

\[
\Phi_N(b) = \left( \bigotimes_{k=0}^{N-1} b^{(k)} \right) \otimes u_{\lambda_N}.
\]

Therefore we can model elements \( b \in B(\lambda) \) as a \( U'_q(g) \)-crystal by considering an infinite list of elements \( b^{(k)} \in B^{(k)} \) and defining the crystal structure by:

\[
\overline{\text{wt}}(b) = \lambda_N + \sum_{k=0}^{N-1} \overline{\text{wt}} \left( b^{(k)} \right)
\]

\[
e_i(b) = e_i \left( b' \otimes b^{(N)} \right) \otimes u_{\lambda_N},
\]

\[
f_i(b) = f_i \left( b' \otimes b^{(N)} \right) \otimes u_{\lambda_N},
\]

\[
\varepsilon_i(b) = \max \left( \varepsilon_i(b') - \varphi_i \left( b^{(N)} \right) , 0 \right),
\]

\[
\varphi_i(b) = \varphi_i(b') + \max \left( \varphi_i \left( b^{(N)} \right) - \varepsilon_i(b') , 0 \right),
\]

where \( b' = b^{(0)} \otimes \cdots \otimes b^{(N-1)} \). To translate this into a finite list, we consider a finite sequence \( b^{(0)} \otimes \cdots \otimes b^{(N-1)} \otimes b^{(N)}_{\Lambda_N} \) and if

\[
f_i \left( b^{(0)} \otimes \cdots \otimes b^{(N-1)} \otimes b^{(N)}_{\Lambda_N} \right) = b_0 \otimes \cdots \otimes b^{(N-1)} \otimes f_i \left( b^{(N)}_{\Lambda_N} \right),
\]

then we take the image as \( b^{(0)} \otimes \cdots \otimes f_i \left( b^{(N)}_{\Lambda_N} \right) \otimes b^{(N+1)}_{\Lambda_{N+1}} \). Similarly we remove \( b^{(N)}_{\Lambda_N} \) if we have \( b_0 \otimes \cdots \otimes b^{(N-1)} \otimes b^{(N)}_{\Lambda_{N+1}} \otimes b^{(N)}_{\Lambda_N} \). Additionally if

\[
e_i \left( b^{(0)} \otimes \cdots \otimes b^{(N-1)} \otimes b^{(N)}_{\Lambda_N} \right) = b^{(0)} \otimes \cdots \otimes b^{(N-1)} \otimes e_i \left( b^{(N)}_{\Lambda_N} \right),
\]
then we consider this to be 0.

We can then lift the $U'_q(g)$-crystal structure to a $U_q(g)$-crystal structure by using a tensor product of the affinization of the of crystals $B^{(i)}$ for all $i$.

INPUT:
- $B$ – a single or list of $U'_q$ perfect crystal(s) of level $l$
- weight – a weight in $\mathcal{P}^+_l$

EXAMPLES:

```
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]; mg
[[[3]]]
sage: mg.f_string([0,1,2,2])
[[[3]], [[3]], [[1]]]
sage: x = mg.f_string([0,1,2]); x
[[[2]], [[3]], [[1]]]
sage: x.weight()
Lambda[0]
```

An example of type $A_5^{(2)}$:

```
sage: B = crystals.KirillovReshetikhin(['A',5,2], 1,1)
sage: La = RootSystem(['A',5,2]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]; mg
[[[-1]]]
sage: mg.f_string([0,2,1,3])
[[[-3]], [[2]], [[-1]]]
sage: mg.f_string([0,2,3,1])
[[[-3]], [[2]], [[-1]]]
```

An example of type $D_3^{(2)}$:

```
sage: B = crystals.KirillovReshetikhin(['D',3,2], 1,1)
sage: La = RootSystem(['D',3,2]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]; mg
[[0]]
sage: mg.f_string([0,1,2,0])
[[[0]], [[1]], []]
```

An example using multiple crystals of the same level:

```
sage: B1 = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: B2 = crystals.KirillovReshetikhin(['A',2,1], 2,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel([B1, B2, B1], La[0])
sage: mg = C.module_generators[0]; mg
[[[3]]]
sage: mg.f_string([0,1,2,2])
```

(continues on next page)
By using the extended weight lattice, the Kyoto path model lifts the perfect crystals to their affinizations:

```python
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: P = RootSystem(['A',2,1]).weight_lattice(extended=True)
sage: La = P.fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]; mg
[[[3]], [[2]], [[1]], [[3]], [[1]], [[3]]]
sage: x = mg.f_string([0,1,2,2,2]); x
[[[2]], [[3]], [[1]], [[2]], [[1]], [[3]], [[1]], [[3]]]
sage: x.weight()
Lambda[0] - delta
```

class Element

Bases: `TensorProductOfRegularCrystalsElement`

An element in the Kyoto path model.

```
example:
```

```python
epsilon(i)

Return \( \epsilon_i \) of self.

EXAMPLES:
```

```python
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]
sage: all(mg.e(i) is None for i in C.index_set())
True
sage: mg.f(0).e(0) == mg
True
```

### epsilon(i)

Return \( \epsilon_i \) of self.

**EXAMPLES:**

```python
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]
sage: [mg.epsilon(i) for i in C.index_set()]
[0, 0, 0]
sage: elt = mg.f(0)
sage: [elt.epsilon(i) for i in C.index_set()]
[1, 0, 0]
sage: elt = mg.f_string([0,1,2])
```
\begin{verbatim}
sage: [elt.epsilon(i) for i in C.index_set()]
[0, 0, 1]
sage: elt = mg.f_string([0,1,2,2])
sage: [elt.epsilon(i) for i in C.index_set()]
[0, 0, 2]
\end{verbatim}

\textbf{f}(i)

Return the action of \( f_i \) on \textit{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]
sage: mg.f(2)
sage: mg.f(0)
[[[1]], [[2]]]
sage: mg.f_string([0,1,2])
[[[2]], [[3]], [[1]]]
\end{verbatim}

\textbf{phi}(i)

Return \( \varphi_i \) of \textit{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]
sage: [mg.phi(i) for i in C.index_set()]
[1, 0, 0]
sage: elt = mg.f(0)
sage: [elt.phi(i) for i in C.index_set()]
[0, 1, 1]
sage: elt = mg.f_string([0,1])
sage: [elt.phi(i) for i in C.index_set()]
[0, 0, 2]
\end{verbatim}

\textbf{truncate}(k=None)

Truncate \textit{self} to have length \( k \) and return as an element in a (finite) tensor product of crystals.

\textbf{INPUT:}

\begin{itemize}
  \item \( k \) – (optional) the length to truncate to; if not specified, then returns one more than the current non-ground-state elements (i.e. the current list in \textit{self})
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B1 = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: B2 = crystals.KirillovReshetikhin(['A',2,1], 2,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel([B1,B2,B1], La[0])
sage: mg = C.highest_weight_vector()
sage: elt = mg.f_string([0,1,2,1,0]); elt
\end{verbatim}
\[
\begin{bmatrix}
[3], [2], [3], [1], [2]
\end{bmatrix}
\]
```
sage: elt = elt.truncate(); t
\[
\begin{bmatrix}
[3], [2], [3], [1], [2]
\end{bmatrix}
\]
sage: t.parent() is C.finite_tensor_product(4)
True
sage: elt.truncate(2)
\[
\begin{bmatrix}
[3], [2], [3]
\end{bmatrix}
\]
sage: elt.truncate(10)
\[
\begin{bmatrix}
[3], [2], [3], [1], [2], [1], [3], [2], [1], [2],
[1], [2], [3], [1], [3]
\end{bmatrix}
\]
```

**weight()**

Return the weight of self.

EXAMPLES:

```
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: P = RootSystem(['A',2,1]).weight_lattice(extended=True)
sage: La = P.fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: mg = C.module_generators[0]
sage: mg.weight()
Lambda[0]
sage: mg.f_string([0,1,2]).weight()
Lambda[0] - delta
```

**finite_tensor_product(k)**

Return the finite tensor product of crystals of length k from truncating self.

EXAMPLES:

```
sage: B1 = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: B2 = crystals.KirillovReshetikhin(['A',2,1], 2,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel([B1,B2,B1], La[0])
sage: C.finite_tensor_product(5)
Full tensor product of the crystals
[Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,s)=(1,1),
 Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,s)=(2,1),
 Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,s)=(1,1),
 Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,s)=(1,1),
 Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,s)=(2,1)]
```

**weight_lattice_realization()**

Return the weight lattice realization used to express weights.

EXAMPLES:

```
sage: B = crystals.KirillovReshetikhin(['A',2,1], 1,1)
sage: La = RootSystem(['A',2,1]).weight_lattice().fundamental_weights()
sage: C = crystals.KyotoPathModel(B, La[0])
sage: C.weight_lattice_realization()
Weight lattice of the Root system of type ['A', 2, 1]
```

(continues on next page)
sage: P = RootSystem(['A',2,1]).weight_lattice(extended=True)
sage: C = crystals.KyotoPathModel(B, P.fundamental_weight(0))
sage: C.weight_lattice_realization()
Extended weight lattice of the Root system of type ['A', 2, 1]

5.1.57 Crystals of letters

class sage.combinat.crystals.letters.BKKLetter

Bases: Letter

  e(i)
  
  Return the action of \(e_i\) on \(self\).

  EXAMPLES:

  sage: C = crystals.Letters(['A', [2, 1]])
  sage: c = C(-2)
  sage: c.e(-2)
  -3
  sage: c = C(1)
  sage: c.e(0)
  -1
  sage: c = C(2)
  sage: c.e(1)
  1
  sage: c = C(-2)

  f(i)
  
  Return the action of \(f_i\) on \(self\).

  EXAMPLES:

  sage: C = crystals.Letters(['A', [2, 1]])
  sage: c = C.an_element()
  sage: c.f(-2)
  -2
  sage: c = C(-1)
  sage: c.f(0)
  1
  sage: c = C(1)
  sage: c.f(1)
  2
  sage: c = C(-2)

  weight()
  
  Return weight of \(self\).

  EXAMPLES:

  sage: C = crystals.Letters(['A', [2, 1]])
  sage: c = C(-1)
class sage.combinat.crystals.letters.ClassicalCrystalOfLetters(cartan_type, element_class, element_print_style=None, dual=None):

    Bases: UniqueRepresentation, Parent

    A generic class for classical crystals of letters.

    All classical crystals of letters should be instances of this class or of subclasses. To define a new crystal of letters, one only needs to implement a class for the elements (which subclasses Letter), with appropriate \( e_i \) and \( f_i \) operations. If the module generator is not \( 1 \), one also needs to define the subclass ClassicalCrystalOfLetters for the crystal itself.

    The basic assumption is that crystals of letters are small, but used intensively as building blocks. Therefore, we explicitly build in memory the list of all elements, the crystal graph and its transitive closure, so as to make the following operations constant time: list, cmp, (todo: phi, epsilon, e, and f with caching)

    list()

    Return a list of the elements of self.

    EXAMPLES:

    sage: C = crystals.Letters(['A',5])
    sage: C.list()
    [1, 2, 3, 4, 5, 6]

    lt_elements(x, y)

    Return True if and only if there is a path from \( x \) to \( y \) in the crystal graph, when \( x \) is not equal to \( y \).

    Because the crystal graph is classical, it is a directed acyclic graph which can be interpreted as a poset. This function implements the comparison function of this poset.

    EXAMPLES:

    sage: C = crystals.Letters(['A', 5])
    sage: x = C(1)
    sage: y = C(2)
    sage: C.lt_elements(x,y)
    True
    sage: C.lt_elements(y,x)
    False
    sage: C.lt_elements(x,x)
    False
    sage: C = crystals.Letters(['D', 4])
    sage: C.lt_elements(C(4),C(-4))
    False
    sage: C.lt_elements(C(-4),C(4))
    False

class sage.combinat.crystals.letters.ClassicalCrystalOfLettersWrapped(cartan_type):

    Bases: ClassicalCrystalOfLetters

    ...
Crystal of letters by wrapping another crystal.

This is used for a crystal of letters of type $E_8$ and $F_4$.

This class follows the same output as the other crystal of letters, where $b$ is represented by the “letter” with $\varphi_i(b)$ (resp., $\varepsilon_i$) number of $i$’s (resp., $-i$’s or $\bar{i}$’s). However, this uses an auxiliary crystal to construct these letters to avoid hardcoding the crystal elements and the corresponding edges; in particular, the 248 nodes of $E_8$.

```python
class sage.combinat.crystals.letters.CrystalOfBKKLetters(ct, dual)
    Bases: ClassicalCrystalOfLetters

Crystal of letters for Benkart-Kang-Kashiwara supercrystals.

This implements the $\mathfrak{gl}(m|n)$ crystal of Benkart, Kang and Kashiwara [BKK2000].

EXAMPLES:

```sage
C = crystals.Letters(['A', [1, 1]]); C
The crystal of letters for type ['A', [1, 1]]
```

```sage
C = crystals.Letters(['A', [2, 4]], dual=True); C
The crystal of letters for type ['A', [2, 4]] (dual)
```

```python
Element
    alias of BKKLetter
```

```python
sage.combinat.crystals.letters.CrystalOfLetters(cartan_type, element_print_style=None, dual=None)

Return the crystal of letters of the given type.

For classical types, this is a combinatorial model for the crystal with highest weight $\Lambda_1$ (the first fundamental weight).

Any irreducible classical crystal appears as the irreducible component of the tensor product of several copies of this crystal (plus possibly one copy of the spin crystal, see CrystalOfSpins). See [KN1994]. Elements of this irreducible component have a fixed shape, and can be fit inside a tableau shape. Otherwise said, any irreducible classical crystal is isomorphic to a crystal of tableaux with cells filled by elements of the crystal of letters (possibly tensored with the crystal of spins).

We also have the crystal of fundamental representation of the general linear Lie superalgebra, which are used as letters inside of tableaux following [BKK2000]. Similarly, all of these crystals appear as a subcrystal of a sufficiently large tensor power of this crystal.

INPUT:

- $T$ – a Cartan type

EXAMPLES:

```sage
C = crystals.Letters(['A',5])
sage: C.list()
[1, 2, 3, 4, 5, 6]
sage: C.cartan_type()
['A', 5]
```

For type $E_8$, one can also specify how elements are printed. This option is usually set to None and the default representation is used. If one chooses the option ‘compact’, the elements are printed in the more compact convention with 27 letters $+\text{abcdefghijklmnopqrstuvwxyz}$ and the 27 letters $-\text{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$ for the dual crystal.

EXAMPLES:
sage: C = crystals.Letters(['E',6], element_print_style = 'compact')
sage: C
The crystal of letters for type ['E', 6]
sage: C.list()
[+, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z]
sage: C = crystals.Letters(['E',6], element_print_style = 'compact', dual = True)
sage: C
The crystal of letters for type ['E', 6] (dual)
sage: C.list()

class sage.combinat.crystals.letters.CrystalOfQueerLetters

Bases: ClassicalCrystalOfLetters

Queer crystal of letters elements.

The index set is of the form \{-n, \ldots, -1, 1, \ldots, n\}. For 1 < i \leq n, the operators \(e_{-i}\) and \(f_{-i}\) are defined as

\[
f_{-i} = s_{w_{i}^{-1}} f_{-1} s_{w_{i}}, \quad e_{-i} = s_{w_{i}^{-1}} e_{-1} s_{w_{i}},
\]

where \(w_{i} = s_{2} \cdots s_{i} s_{1} \cdots s_{i-1}\) and \(s_{i}\) is the reflection along the \(i\)-string in the crystal. See [GJK+2014].

Element

alias of QueerLetter_element

index_set()

Return index set of self.

EXAMPLES:

sage: Q = crystals.Letters(['Q',3])
sage: Q.index_set()
(1, 2, -2, -1)

class sage.combinat.crystals.letters.Crystal_of_letters_type_A_element

Bases: Letter

Type \(A\) crystal of letters elements.

e(i)

Return the action of \(e_{i}\) on self.

EXAMPLES:

sage: C = crystals.Letters(['A',4])
sage: [(c,i,c.e(i)) for i in C.index_set() for c in C if c.e(i) is not None]
[(2, 1, 1), (3, 2, 2), (4, 3, 3), (5, 4, 4)]

epsilon(i)

Return \(\epsilon_{i}\) of self.

EXAMPLES:

sage: C = crystals.Letters(['A',4])
sage: [(c,i) for i in C.index_set() for c in C if c.epsilon(i) != 0]
[(2, 1), (3, 2), (4, 3), (5, 4)]
Combinatorics, Release 10.0

\( f(i) \)

Return the action of \( f_i \) on \( \text{self} \).

**EXAMPLES:**

```python
sage: C = crystals.Letters(['A', 4])
sage: [(c, i, c.f(i)) for i in C.index_set() for c in C if c.f(i) is not None]
[(1, 1, 2), (2, 2, 3), (3, 3, 4), (4, 4, 5)]
```

\( \phi(i) \)

Return \( \phi_i \) of \( \text{self} \).

**EXAMPLES:**

```python
sage: C = crystals.Letters(['A', 4])
sage: [(c, i) for i in C.index_set() for c in C if c.phi(i) != 0]
[(1, 1), (2, 2), (3, 3), (4, 4)]
```

\( \text{weight}() \)

Return the weight of \( \text{self} \).

**EXAMPLES:**

```python
sage: v = crystals.Letters(['A', 3])
[(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)]
```

```python
class sage.combinat.crystals.letters.Crystal_of_letters_type_B_element
Bases: Letter

Type B crystal of letters elements.

\( e(i) \)

Return the action of \( e_i \) on \( \text{self} \).

**EXAMPLES:**

```python
sage: C = crystals.Letters(['B', 4])
sage: [(c, i, c.e(i)) for i in C.index_set() for c in C if c.e(i) is not None]
[(2, 1, 1), (-1, 1, -2), (3, 2, 2), (-2, 2, -3), (4, 3, 3), (-3, 3, -4), (0, 4, 4), (-4, 4, 0)]
```

\( \epsilon(i) \)

Return \( \epsilon_i \) of \( \text{self} \).

**EXAMPLES:**

```python
sage: C = crystals.Letters(['B', 3])
sage: [(c, i) for i in C.index_set() for c in C if c.epsilon(i) != 0]
[(2, 1), (-1, 1), (3, 2), (-2, 2), (0, 3), (-3, 3)]
```
\textbf{f(i)}

Return the actions of \( f_i \) on self.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: C = crystals.Letters(['B',4])
sage: [(c,i,c.f(i)) for i in C.index_set() for c in C if c.f(i) is not None]
[(1, 1, 2),
 (-2, 1, -1),
 (2, 2, 3),
 (-3, 2, -2),
 (3, 3, 4),
 (-4, 3, -3),
 (4, 4, 0),
 (0, 4, -4)]
\end{verbatim}

\textbf{phi(i)}

Return \( \varphi_i \) of self.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: C = crystals.Letters(['B',3])
sage: [(c,i) for i in C.index_set() for c in C if c.phi(i) != 0]
[(1, 1),
 (-2, 1),
 (2, 2),
 (-3, 2),
 (3, 3),
 (0, 3)]
\end{verbatim}

\textbf{weight()}

Return the weight of self.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: [v.weight() for v in crystals.Letters(['B',3])]
[(1, 0, 0),
 (0, 1, 0),
 (0, 0, 1),
 (0, 0, 0),
 (0, 0, -1),
 (0, -1, 0),
 (-1, 0, 0)]
\end{verbatim}

\textbf{class sage.combinat.crystals.letters.Crystal_of_letters_type_C_element}

Bases: \texttt{Letter}

Type C crystal of letters elements.

\textbf{e(i)}

Return the action of \( e_i \) on self.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: C = crystals.Letters(['C',4])
sage: [(c,i,c.e(i)) for i in C.index_set() for c in C if c.e(i) is not None]
[(2, 1, 1),
 (-1, 1, -2),
 (3, 2, 2),
 (-2, 2, -3),
 (4, 3, 3),
(continues on next page)
epsilon(i)
Return \( \varepsilon_i \) of self.

EXAMPLES:

```
sage: C = crystals.Letters(['C',3])
sage: [(c,i) for i in C.index_set() for c in C if c.epsilon(i) != 0]
[(2, 1), (-1, 1), (3, 2), (-2, 2), (-3, 3)]
```

f(i)
Return the action of \( f_i \) on self.

EXAMPLES:

```
sage: C = crystals.Letters(['C',4])
sage: [(c,i,c.f(i)) for i in C.index_set() for c in C if c.f(i) is not None]
[(1, 1, 2), (-2, 1, -1), (2, 2, 3), (-3, 2, -2), (3, 3, 4), (-4, 3, -3), (4, 4, -4)]
```

phi(i)
Return \( \varphi_i \) of self.

EXAMPLES:

```
sage: C = crystals.Letters(['C',3])
sage: [(c,i) for i in C.index_set() for c in C if c.phi(i) != 0]
[(1, 1), (-2, 1), (2, 2), (-3, 2), (3, 3)]
```

weight()
Return the weight of self.

EXAMPLES:

```
sage: [v.weight() for v in crystals.Letters(['C',3])]
[(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, -1), (0, -1, 0), (-1, 0, 0)]
```

---

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(continued from previous page)

$$\epsilon(i)$$

Return $\epsilon_i$ of self.

EXAMPLES:

```python
sage: C = crystals.Letters(['D',4])
sage: [(c,i) for i in C.index_set() for c in C if c.epsilon(i) != 0]
[(2, 1), (-1, 1), (3, 2), (-2, 2), (4, 3), (-3, 3), (-4, 4), (-3, 4)]
```

$$f(i)$$

Return the action of $f_i$ on self.

EXAMPLES:

```python
sage: C = crystals.Letters(['D',5])
sage: [(c,i,c.f(i)) for i in C.index_set() for c in C if c.f(i) is not None]
[(1, 1, 2),
 (-2, 1, -1),
 (2, 2, 3),
 (-3, 2, -2),
 (3, 3, 4),
 (-4, 3, -3),
 (4, 4, 5),
 (-5, 4, -4),
 (4, 5, -5),
 (5, 5, -4)]
```

$$\phi(i)$$

Return $\phi_i$ of self.

EXAMPLES:

```python
sage: C = crystals.Letters(['D',4])
sage: [(c,i) for i in C.index_set() for c in C if c.phi(i) != 0]
[(1, 1), (-2, 1), (2, 2), (-3, 2), (3, 3), (-4, 3), (3, 4), (4, 4)]
```

weight()  

Return the weight of self.

EXAMPLES:

```python
sage: [v.weight() for v in crystals.Letters(['D',4])]
[(1, 0, 0, 0),
 (0, 1, 0, 0),
 (0, 0, 1, 0),
 (0, 0, 0, 1),
 (0, 0, 0, -1),
```
class sage.combinat.crystals.letters.Crystal_of_letters_type_E6_element

Bases: sage.combinat.crystals.letters.LetterTuple

Type $E_6$ crystal of letters elements. This crystal corresponds to the highest weight crystal $B(\Lambda_1)$.

**e(i)**

Return the action of $e_i$ on self.

**EXAMPLES:**

```python
sage: C = crystals.Letters(['E',6])
sage: C((-1,3)).e(1)
(1,)
sage: C((-2,-3,4)).e(2)
(-3, 2)
sage: C((1,)).e(1)
```

**f(i)**

Return the action of $f_i$ on self.

**EXAMPLES:**

```python
sage: C = crystals.Letters(['E',6])
sage: C((1,)).f(1)
(-1, 3)
sage: C((-6,)).f(1)
```

**weight()**

Return the weight of self.

**EXAMPLES:**

```python
sage: [v.weight() for v in crystals.Letters(['E',6])]
```

(continues on next page)
class sage.combinat.crystals.letters.Crystal_of_letters_type_E6_element_dual

Bases: LetterTuple

Type $E_6$ crystal of letters elements. This crystal corresponds to the highest weight crystal $B(\Lambda_6)$. This crystal is dual to $B(\Lambda_1)$ of type $E_6$.

$\mathbb{e}(i)$

Return the action of $e_i$ on self.

EXAMPLES:

```
sage: C = crystals.Letters(['E',6], dual = True)
sage: C((-1,)).e(1)
(1, -3)
```

$f(i)$

Return the action of $f_i$ on self.

EXAMPLES:

```
sage: C = crystals.Letters(['E',6], dual = True)
sage: C((6,)).f(6)
(5, -6)
sage: C((6,)).f(1)
```

$\text{lift()}$

Lift an element of self to the crystal of letters crystals.Letters(['E',6]) by taking its inverse weight.

EXAMPLES:

```
sage: C = crystals.Letters(['E',6], dual = True)
sage: b = C.module_generators[0]
sage: b.lift()
(-6,)
```

$\text{retract}(p)$

Retract element $p$, which is an element in crystals.Letters(['E',6]) to an element in crystals.Letters(['E',6], dual=True) by taking its inverse weight.

EXAMPLES:
sage: C = crystals.Letters(['E',6])
sage: Cd = crystals.Letters(['E',6], dual = True)
sage: b = Cd.module_generators[0]
sage: p = C((-1,3))
sage: b.retract(p)
(1, -3)
sage: b.retract(None)

weight()

Return the weight of self.

EXAMPLES:

sage: C = crystals.Letters(['E',6], dual = True)
sage: b=C.module_generators[0]
sage: b.weight()
(0, 0, 0, 0, 1, -1/3, -1/3, 1/3)

class sage.combinat.crystals.letters.Crystal_of_letters_type_E7_element

Bases: LetterTuple

Type $E_7$ crystal of letters elements. This crystal corresponds to the highest weight crystal $B(\Lambda_7)$.

e()

Return the action of $e_i$ on self.

EXAMPLES:

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sage: C = crystals.Letters(['E',7])
sage: C((7,)).e(7)
sage: C((-7,6)).e(7)
(7,)
f(i)
Return the action of 𝑓𝑖 on self.
EXAMPLES:
sage: C = crystals.Letters(['E',7])
sage: C((-7,)).f(7)
sage: C((7,)).f(7)
(-7, 6)
weight()
Return the weight of self.
EXAMPLES:
sage: [v.weight() for v in crystals.Letters(['E',7])]
[(0, 0, 0, 0, 0, 1, -1/2, 1/2), (0, 0, 0, 0, 1, 0, -1/2, 1/2), (0, 0, 0,
1, 0, 0, -1/2, 1/2), (0, 0, 1, 0, 0, 0, -1/2, 1/2), (0, 1, 0, 0, 0, 0,
-1/2, 1/2), (-1, 0, 0, 0, 0, 0, -1/2, 1/2), (1, 0, 0, 0, 0, 0, -1/2,
1/2), (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 0, 0), (0, -1, 0, 0, 0, 0, -1/2,
1/2), (-1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 0, 0), (0, 0, -1, 0, 0, 0, -1/2,
1/2), (-1/2, 1/2, -1/2, 1/2, 1/2, 1/2, 0, 0), (1/2, -1/2, -1/2, 1/2,
1/2, 1/2, 0, 0), (0, 0, 0, -1, 0, 0, -1/2, 1/2), (-1/2, 1/2, 1/2, -1/2,
1/2, 1/2, 0, 0), (1/2, -1/2, 1/2, -1/2, 1/2, 1/2, 0, 0), (1/2, 1/2,
-1/2, -1/2, 1/2, 1/2, 0, 0), (-1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 0, 0),
(0, 0, 0, 0, -1, 0, -1/2, 1/2), (-1/2, 1/2, 1/2, 1/2, -1/2, 1/2, 0, 0),
(1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 0, 0), (1/2, 1/2, -1/2, 1/2, -1/2, 1/2,
0, 0), (-1/2, -1/2, -1/2, 1/2, -1/2, 1/2, 0, 0), (1/2, 1/2, 1/2, -1/2,
-1/2, 1/2, 0, 0), (-1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 0, 0), (-1/2, 1/2,
-1/2, -1/2, -1/2, 1/2, 0, 0), (1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 0, 0),
(0, 0, 0, 0, 0, 1, 1/2, -1/2), (0, 0, 0, 0, 0, -1, -1/2, 1/2), (-1/2,
1/2, 1/2, 1/2, 1/2, -1/2, 0, 0), (1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 0, 0),
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 0, 0), (-1/2, -1/2, -1/2, 1/2, 1/2,
-1/2, 0, 0), (1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 0, 0), (-1/2, -1/2, 1/2,
-1/2, 1/2, -1/2, 0, 0), (-1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 0, 0), (1/2,
-1/2, -1/2, -1/2, 1/2, -1/2, 0, 0), (0, 0, 0, 0, 1, 0, 1/2, -1/2), (1/2,
1/2, 1/2, 1/2, -1/2, -1/2, 0, 0), (-1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 0,
0), (-1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 0, 0), (1/2, -1/2, -1/2, 1/2,
-1/2, -1/2, 0, 0), (0, 0, 0, 1, 0, 0, 1/2, -1/2), (-1/2, 1/2, 1/2, -1/2,
-1/2, -1/2, 0, 0), (1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 0, 0), (0, 0, 1,
0, 0, 0, 1/2, -1/2), (1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 0, 0), (0, 1, 0,
0, 0, 0, 1/2, -1/2), (1, 0, 0, 0, 0, 0, 1/2, -1/2), (0, -1, 0, 0, 0, 0,
1/2, -1/2), (0, 0, -1, 0, 0, 0, 1/2, -1/2), (0, 0, 0, -1, 0, 0, 1/2,
-1/2), (0, 0, 0, 0, -1, 0, 1/2, -1/2), (0, 0, 0, 0, 0, -1, 1/2, -1/2),
(-1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 0, 0), (-1, 0, 0, 0, 0, 0, 1/2,
-1/2)]
class sage.combinat.crystals.letters.Crystal_of_letters_type_G_element
Bases: Letter

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Type $G_2$ crystal of letters elements.

$e(i)$
Return the action of $e_i$ on self.

**EXAMPLES:**

```
sage: C = crystals.Letters(['G',2])
sage: [(c,i,c.e(i)) for i in C.index_set() for c in C if c.e(i) is not None]
[(2, 1, 1),
 (0, 1, 3),
 (-3, 1, 0),
 (-1, 1, -2),
 (3, 2, 2),
 (-2, 2, -3)]
```

epsilon(i)
Return $\varepsilon_i$ of self.

**EXAMPLES:**

```
sage: C = crystals.Letters(['G',2])
sage: [(c,i,c.epsilon(i)) for i in C.index_set() for c in C if c.epsilon(i) != 0]
[(2, 1, 1), (0, 1, 1), (-3, 1, 2), (-1, 1, 1), (3, 2, 1), (-2, 2, 1)]
```

$f(i)$
Return the action of $f_i$ on self.

**EXAMPLES:**

```
sage: C = crystals.Letters(['G',2])
sage: [(c,i,c.f(i)) for i in C.index_set() for c in C if c.f(i) is not None]
[(1, 1, 2),
 (3, 1, 0),
 (0, 1, -3),
 (-2, 1, -1),
 (2, 2, 3),
 (-3, 2, -2)]
```

$\phi(i)$
Return $\varphi_i$ of self.

**EXAMPLES:**

```
sage: C = crystals.Letters(['G',2])
sage: [(c,i,c.phi(i)) for i in C.index_set() for c in C if c.phi(i) != 0]
[(1, 1, 1), (3, 1, 2), (0, 1, 1), (-2, 1, 1), (2, 2, 1), (-3, 2, 1)]
```

weight()
Return the weight of self.

**EXAMPLES:**

```
sage: [v.weight() for v in crystals.Letters(['G',2])]
[(1, 0, -1), (1, -1, 0), (0, 1, -1), (0, 0, 0), (0, -1, 1), (-1, 1, 0), (-1, 0, -1)]
```
class sage.combinat.crystals.letters.EmptyLetter

    Bases: Element

    The affine letter 0 thought of as a classical crystal letter in classical type $B_n$ and $C_n$. 

    Warning: This is not a classical letter.

    Used in the rigged configuration bijections.

    e(i)
    Return $e_i$ of self which is None.

    EXAMPLES:
    sage: C = crystals.Letters(['C', 3])
    sage: C('E').e(1)

    epsilon(i)
    Return $\varepsilon_i$ of self.

    EXAMPLES:
    sage: C = crystals.Letters(['C', 3])
    sage: C('E').epsilon(1)

    f(i)
    Return $f_i$ of self which is None.

    EXAMPLES:
    sage: C = crystals.Letters(['C', 3])
    sage: C('E').f(1)

    phi(i)
    Return $\phi_i$ of self.

    EXAMPLES:
    sage: C = crystals.Letters(['C', 3])
    sage: C('E').phi(1)

    value

    weight()
    Return the weight of self.

    EXAMPLES:
    sage: C = crystals.Letters(['C', 3])
    sage: C('E').weight()
    (0, 0, 0)
class sage.combinat.crystals.letters.Letter
    Bases: Element

    A class for letters.

    Like ElementWrapper, plus delegates __lt__ (comparison) to the parent.

    EXAMPLES:

    sage: from sage.combinat.crystals.letters import Letter
    sage: a = Letter(ZZ, 1)
    sage: Letter(ZZ, 1).parent()
    Integer Ring
    sage: Letter(ZZ, 1)._repr_()
    '1'
    sage: parent1 = ZZ  # Any fake value ...
    sage: parent2 = QQ  # Any fake value ...
    sage: l11 = Letter(parent1, 1)
    sage: l12 = Letter(parent1, 2)
    sage: l21 = Letter(parent2, 1)
    sage: l22 = Letter(parent2, 2)
    sage: l11 == l11
    True
    sage: l11 == l12
    False
    sage: l11 == l21  # not tested
    False
    sage: C = crystals.Letters(['B', 3])
    sage: C(0) != C(0)
    False
    sage: C(1) != C(-1)
    True

value

class sage.combinat.crystals.letters.LetterTuple
    Bases: Element

    Abstract class for type $E$ letters.

    epsilon($i$)

        Return $\varepsilon_i$ of self.

        EXAMPLES:

        sage: C = crystals.Letters(['E', 6])
        sage: C((-6,)).epsilon(1)
        0
        sage: C((-6,)).epsilon(6)
        1

    phi($i$)

        Return $\varphi_i$ of self.
EXAMPLES:

```
sage: C = crystals.Letters(['E', 6])
sage: C((1,)).phi(1)
1
sage: C((1,)).phi(6)
0
```

class sage.combinat.crystals.letters.LetterWrapped

Bases: `Element`

Element which uses another crystal implementation and converts those elements to a tuple with $\pm i$.

\texttt{e}(i)

Return $e_i$ of self.

EXAMPLES:

```
sage: C = crystals.Letters(['E', 8])
sage: C((-8,)).e(1)
sage: C((-8,)).e(8)
(-7, 8)
```

\texttt{epsilon}(i)

Return $\varepsilon_i$ of self.

EXAMPLES:

```
sage: C = crystals.Letters(['E', 8])
sage: C((-8,)).epsilon(1)
sage: C((-8,)).epsilon(8)
1
```

\texttt{f}(i)

Return $f_i$ of self.

EXAMPLES:

```
sage: C = crystals.Letters(['E', 8])
sage: C((8,)).f(6)
sage: C((8,)).f(8)
(7, -8)
```

\texttt{phi}(i)

Return $\varphi_i$ of self.

EXAMPLES:

```
sage: C = crystals.Letters(['E', 8])
sage: C((8,)).phi(8)
sage: C((8,)).phi(6)
1
```

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value

class sage.combinat.crystals.letters.QueerLetter_element

Bases: Letter

Queer supercrystal letters elements.

e(i)
Return the action of $e_i$ on self.

EXAMPLES:

```
sage: Q = crystals.Letters(['Q',3])
sage: [(c,i,c.e(i)) for i in Q.index_set() for c in Q if c.e(i) is not None]
[(2, 1, 1), (3, 2, 2), (3, -2, 2), (2, -1, 1)]
```

epsilon(i)
Return $\epsilon_i$ of self.

EXAMPLES:

```
sage: Q = crystals.Letters(['Q',3])
sage: [(c,i) for i in Q.index_set() for c in Q if c.epsilon(i) != 0]
[(2, 1), (3, 2), (3, -2), (2, -1)]
```

f(i)
Return the action of $f_i$ on self.

EXAMPLES:

```
sage: Q = crystals.Letters(['Q',3])
sage: [(c,i,c.f(i)) for i in Q.index_set() for c in Q if c.f(i) is not None]
[(1, 1, 2), (2, 2, 3), (2, -2, 3), (1, -1, 2)]
```

phi(i)
Return $\varphi_i$ of self.

EXAMPLES:

```
sage: Q = crystals.Letters(['Q',3])
sage: [(c,i) for i in Q.index_set() for c in Q if c.phi(i) != 0]
[(1, 1), (2, 2), (2, -2), (1, -1)]
```

weight()
Return the weight of self.

EXAMPLES:

```
sage: [v.weight() for v in crystals.Letters(['Q',4])]
[(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)]
```
5.1.58 Littelmann paths

AUTHORS:

- Mark Shimozono, Anne Schilling (2012): Initial version
- Anne Schilling (2013): Implemented \texttt{CrystalOfProjectedLevelZeroLSPaths}
- Travis Scrimshaw (2016): Implemented \texttt{InfinityCrystalOfLSPaths}

```python
class sage.combinat.crystals.littelmann_path.CrystalOfLSPaths(starting_weight, starting_weight_parent):
    Bases: UniqueRepresentation, Parent
    Crystal graph of LS paths generated from the straight-line path to a given weight.
    INPUT:
    - cartan_type – (optional) the Cartan type of a finite or affine root system
    - starting_weight – a weight; if cartan_type is given, then the weight should be given as a list of
coefficients of the fundamental weights, otherwise it should be given in the weight_space basis; for affine
highest weight crystals, one needs to use the extended weight space.
    The crystal class of piecewise linear paths in the weight space, generated from a straight-line path from the origin
to a given element of the weight lattice.
    OUTPUT:
    - a tuple of weights defining the directions of the piecewise linear segments
    EXAMPLES:
```

```python
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space(extended = True).basis()
sage: B = crystals.LSPaths(La[2]-La[0]); B
The crystal of LS paths of type ['A', 2, 1] and weight -Lambda[0] + Lambda[2]
sage: C = crystals.LSPaths(['A',2,1],[-1,0,1]); C
The crystal of LS paths of type ['A', 2, 1] and weight -Lambda[0] + Lambda[2]
sage: B == C
True
sage: c = C.module_generators[0]; c
(-Lambda[0] + Lambda[2],)
sage: [c.f(i) for i in C.index_set()]
[None, None, (Lambda[1] - Lambda[2],)]
sage: R = C.R; R
Root system of type ['A', 2, 1]
sage: Lambda = R.weight_space().basis(); Lambda
Finite family {0: Lambda[0], 1: Lambda[1], 2: Lambda[2]}
sage: b=C(tuple([-Lambda[0]+Lambda[2]]))
sage: b==c
True
sage: b.f(2)
(Lambda[1] - Lambda[2],)
```

For classical highest weight crystals we can also compare the results with the tableaux implementation:
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```python
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: sorted(C, key=str)
˓→2*Lambda[2]),
˓→),
sage: C.cardinality()
8

sage: B = crystals.Tableaux(['A',2],shape=[2,1])
sage: B.cardinality()
8

sage: B.digraph().is_isomorphic(C.digraph())
True
```

Make sure you use the weight space and not the weight lattice for your weights:

```python
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_lattice(extended = True).basis()
sage: B = crystals.LSPaths(La[2]); B
Traceback (most recent call last):
  ... ValueError: Please use the weight space, rather than weight lattice for your weights
```

REFERENCES:

```python
class Element
    Bases: ElementWrapper

compresse()
    Merges consecutive positively parallel steps present in the path.

    EXAMPLES:
    >>> sage: C = crystals.LSPaths(['A',2],[1,1])
    >>> Lambda = C.R.weight_space().fundamental_weights(); Lambda
    Finite family {1: Lambda[1], 2: Lambda[2]}
    ˓→2*Lambda[2]])
    >>> c.compress()
    (Lambda[1] + Lambda[2],)

dualize()
    Returns dualized path.

    EXAMPLES:
    >>> sage: C = crystals.LSPaths(['A',2],[1,1])
    >>> for c in C:
    ˓→    print("{} {}").format(c, c.dualize())
    -Lambda[2], -1/2*Lambda[1] + Lambda[2])
```
(continues on next page)

\[e(i, \text{power}\!=\!1, \text{to\_string\_end}\!=\!False, \text{length\_only}\!=\!False)\]

Returns the \(i\)-th crystal raising operator on \textit{self}.

INPUT:

- \(i\) – element of the index set of the underlying root system
- \textit{power} – positive integer; specifies the power of the raising operator to be applied (default: 1)
- \textit{to\_string\_end} – boolean; if set to True, returns the dominant end of the \(i\)-string of \textit{self}. (default: False)
- \textit{length\_only} – boolean; if set to True, returns the distance to the dominant end of the \(i\)-string of \textit{self}.

EXAMPLES:

\[
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: c = C[2]; c
sage: c.e(1)
sage: c.e(2)
(-Lambda[1] + 2*Lambda[2],)
\]

\[
sage: c.e(2, to\_string\_end=\textbf{True})
(-Lambda[1] + 2*Lambda[2],)
\]

\[
sage: c.e(1, to\_string\_end=\textbf{True})
sage: c.e(1, length\_only=\textbf{True})
0
\]

\[\text{endpoint()}\]

Computes the endpoint of the path.

EXAMPLES:

\[
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: b = C.module\_generators[0]
sage: b.endpoint()
sage: b.f\_string([1,2,2,1])
(-Lambda[1] - Lambda[2],)
sage: b.f\_string([1,2,2,1]).endpoint()
sage: b.f\_string([1,2])
sage: b.f\_string([1,2]).endpoint()
0
\]

\[
sage: b = C([])
sage: b.endpoint()
0
\]
**epsilon** (*i*)  
Returns the distance to the beginning of the *i*-string.  
This method overrides the generic implementation in the category of crystals since this computation is more efficient.

**EXAMPLES:**

```python
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: [c.epsilon(1) for c in C]
[0, 1, 0, 1, 0, 1, 2]
sage: [c.epsilon(2) for c in C]
[0, 1, 2, 1, 1, 0, 0]
```

**f** (*i*, *power*=1, *to_string_end*=False, *length_only*=False)  
Returns the *i*-th crystal lowering operator on `self`.

**INPUT:**  
- *i* – element of the index set of the underlying root system  
- *power* – positive integer; specifies the power of the lowering operator to be applied (default: 1)  
- *to_string_end* – boolean; if set to True, returns the anti-dominant end of the *i*-string of `self`. (default: False)  
- *length_only* – boolean; if set to True, returns the distance to the anti-dominant end of the *i*-string of `self`.

**EXAMPLES:**

```python
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: c = C.module_generators[0]
sage: c.f(1)
(-Lambda[1] + 2*Lambda[2],)
sage: c.f(1,power=2)
(-2*Lambda[1] + 4*Lambda[2],)
sage: c.f(2)
sage: c.f(2,to_string_end=True)
sage: c.f(2,length_only=True)
1
```

**phi** (*i*)  
Returns the distance to the end of the *i*-string.  
This method overrides the generic implementation in the category of crystals since this computation is more efficient.

**EXAMPLES:**

```python
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: [c.phi(1) for c in C]
[1, 0, 1, 0, 2, 1, 0]
sage: [c.phi(2) for c in C]
[1, 2, 1, 0, 0, 0, 1]
```
reflect_step(\text{\textit{which\_step}}, i)

Apply the $i$-th simple reflection to the indicated step in self.

EXAMPLES:

\begin{verbatim}
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: b = C.module_generators[0]
sage: b.reflect_step(0,1)
(-Lambda[1] + 2*Lambda[2],)
sage: b.reflect_step(0,2)
\end{verbatim}

s(i)

Computes the reflection of self along the $i$-string.

This method is more efficient than the generic implementation since it uses powers of $e$ and $f$ in the Littelmann model directly.

EXAMPLES:

\begin{verbatim}
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: c = C.module_generators[0]
sage: c.s(1)
(-Lambda[1] + 2*Lambda[2],)
sage: c.s(2)
sage: C = crystals.LSPaths(['A',2,1],[-1,0,1])
sage: c = C.module_generators[0]; c
(-Lambda[0] + Lambda[2],)
sage: c.s(2)
(Lambda[1] - Lambda[2],)
sage: c.s(1)
(-Lambda[0] + Lambda[2],)
sage: c.f(2).s(1)
(Lambda[0] - Lambda[1],)
\end{verbatim}

split_step(\text{\textit{which\_step}}, r)

Splits indicated step into two parallel steps of relative lengths $r$ and $1 - r$.

INPUT:

\begin{itemize}
    \item \textit{which\_step} – a position in the tuple self
    \item $r$ – a rational number between 0 and 1
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: C = crystals.LSPaths(['A',2],[1,1])
sage: b = C.module_generators[0]
sage: b.split_step(0,1/3)
\end{verbatim}

weight()

Return the weight of self.

EXAMPLES:
weight_lattice_realization()  
Return weight lattice realization of self.

EXAMPLES:

```python
sage: R = RootSystem(['C',3,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[3])
sage: LS.cardinality()
84
sage: GLS = LS.digraph()

sage: K1 = crystals.KirillovReshetikhin(['C',3,1],1,1)
sage: K3 = crystals.KirillovReshetikhin(['C',3,1],3,1)
sage: T = crystals.TensorProduct(K3,K1)
sage: T.cardinality()
84
sage: GT = T.digraph()  # long time
sage: GLS.is_isomorphic(GT, edge_labels = True)  # long time
True
```

class Element  
Bases: Element

Element of a crystal of projected level zero LS paths.

```python
sage: B = crystals.LSPaths(['A',1,1],[1,0])
sage: b = B.highest_weight_vector()
sage: b.f(0).weight()  
-Lambda[0] + 2*Lambda[1] - delta
```
The energy function \( D(\pi) \) of the level zero LS path \( \pi \in B_{cl}(\lambda) \) requires a series of definitions; for simplicity the root system is assumed to be untwisted affine.

The LS path \( \pi \) is a piecewise linear map from the unit interval \([0, 1]\) to the weight lattice. It is specified by “times” \( 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1 \) and “direction vectors” \( x_u \lambda \) where \( x_u \in W/W_J \) for \( 1 \leq u \leq s \), and \( W_J \) is the stabilizer of \( \lambda \) in the finite Weyl group \( W \). Precisely,

\[
\pi(t) = \sum_{u' = 1}^{u-1} (\sigma_{u'} - \sigma_{u'-1}) x_{u'} \lambda + (t - \sigma_{u-1}) x_u \lambda
\]

for \( 1 \leq u \leq s \) and \( \sigma_{u-1} \leq t \leq \sigma_u \).

For any \( x, y \in W/W_J \), let

\[
d : x = w_0 \xleftarrow{\beta_1} w_1 \xleftarrow{\beta_2} \cdots \xleftarrow{\beta_n} w_n = y
\]

be a shortest directed path in the parabolic quantum Bruhat graph. Define

\[
wt(d) := \sum_{1 \leq k \leq n} \beta_k^\vee.
\]

It can be shown that \( wt(d) \) depends only on \( x, y \); call its value \( wt(x, y) \). The energy function \( D(\pi) \) is defined by

\[
D(\pi) = -\sum_{u=1}^{s-1} (1 - \sigma_u) \langle \lambda, wt(x_u, x_{u+1}) \rangle.
\]

For more information, see [LNSSS2013].

**Note:** In the dual-of-untwisted case the parabolic quantum Bruhat graph that is used is obtained by exchanging the roles of roots and coroots. Moreover, in the computation of the pairing the short roots must be doubled (or tripled for type \( G \)). This factor is determined by the translation factor of the corresponding root. Type \( BC \) is viewed as untwisted type, whereas the dual of \( BC \) is viewed as twisted. Except for the untwisted cases, these formulas are currently still conjectural.

**EXAMPLES:**

```
sage: R = RootSystem(['C',3,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[3])
sage: b = LS.module_generators[0]
sage: c = b.f(1).f(3).f(2)
sage: c.energy_function()
0
sage: c=b.e(0)
sage: c.energy_function()
1
```

(continues on next page)
The next test checks that the energy function is constant on classically connected components:

```
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: G = LS.digraph(index_set=[1,2])
sage: C = G.connected_components()
sage: [all(c[0].energy_function()==a.energy_function() for a in c) for c in C]
[True, True, True, True]
sage: R = RootSystem(['D',4,2])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[2])
sage: J = R.cartan_type().classical().index_set()
sage: hw = [x for x in LS if x.is_highest_weight(J)]
sage: [x.weight(), x.energy_function()] for x in hw
[(-2*Lambda[0] + Lambda[2], 0), (-2*Lambda[0] + Lambda[1], 1), (0, 2)]
sage: G = LS.digraph(index_set=J)
sage: C = G.connected_components()
sage: [all(c[0].energy_function()==a.energy_function() for a in c) for c in C]
[True, True, True]
sage: R = RootSystem(CartanType(['G',2,1]).dual())
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[2])
sage: G = LS.digraph(index_set=[1,2])
sage: C = G.connected_components()
sage: [all(c[0].energy_function()==a.energy_function() for a in c) for c in C] # long time
[True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True]
sage: ct = CartanType(['BC',2,2]).dual()
sage: R = RootSystem(ct)
```

(continues on next page)
sage: La = R.weight_space().basis()
sage: G = LS.digraph(index_set=R.cartan_type().classical().index_set())
sage: C = G.connected_components()
sage: [all(c[0].energy_function()==a.energy_function() for a in c) for c in C]  # long time
[True, True, True, True, True, True, True, True, True, True, True]

sage: R = RootSystem(['BC',2,2])
sage: La = R.weight_space().basis()
sage: G = LS.digraph(index_set=R.cartan_type().classical().index_set())
sage: C = G.connected_components()
sage: [all(c[0].energy_function()==a.energy_function() for a in c) for c in C]  # long time
[True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True, True]

**scalar_factors()**

Obtain the scalar factors for self.

Each LS path (or self) can be written as a piecewise linear map

$$
\pi(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1})\nu_{u'} + (t - \sigma_{u'-1})\nu_u
$$

for $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_s = 1$ and $\sigma_{u'-1} \leq t \leq \sigma_u$ and $1 \leq u \leq s$. This method returns the tuple of $(\sigma_1, \ldots, \sigma_s)$.

**EXAMPLES:**

sage: R = RootSystem(['C',3,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[3])
sage: b = LS.module_generators[0]
sage: b.scalar_factors()  
[1]
sage: c = b.f(1).f(3).f(2)
sage: c.scalar_factors()  
[1/3, 1]

**weyl_group_representation()**

Transform the weights in the LS path self to elements in the Weyl group.

Each LS path can be written as the piecewise linear map:

$$
\pi(t) = \sum_{u'=1}^{u-1} (\sigma_{u'} - \sigma_{u'-1})\nu_{u'} + (t - \sigma_{u'-1})\nu_u
$$

for $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_s = 1$ and $\sigma_{u'-1} \leq t \leq \sigma_u$ and $1 \leq u \leq s$. Each weight $\nu_u$ is also associated to a Weyl group element. This method returns the list of Weyl group elements associated to the $\nu_u$ for $1 \leq u \leq s$.

**EXAMPLES:**
sage: R = RootSystem(['C',3,1])
sage: La = R.weight_space().basis()
sage: LS = crystalsProjectedLevelZeroLSPaths(La[1]+La[3])
sage: b = L.module_generators[0]
sage: c = b.f(1).f(3).f(2)
sage: c.weyl_group_representation()
[s2*s1*s3, s1*s3]

classically_highest_weight_vectors()
Return the classically highest weight vectors of self.

EXAMPLES:

sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS = crystalsProjectedLevelZeroLSPaths(2*La[1])
sage: LS.classically_highest_weight_vectors()
((-2*Lambda[0] + 2*Lambda[1],),
 (-Lambda[0] + Lambda[1], -Lambda[1] + Lambda[2]))

is_perfect(level=1)
Check whether the crystal self is perfect (of level level).

INPUT:

- level – (default: 1) positive integer

A crystal $B$ is perfect of level $\ell$ if:

1. $B$ is isomorphic to the crystal graph of a finite-dimensional $U'_q(g)$-module.
2. $B \otimes B$ is connected.
3. There exists a $\lambda \in X$, such that $\text{wt}(B) \subset \lambda + \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i$ and there is a unique element in $B$ of classical weight $\lambda$.
4. For all $b \in B$, $\text{level}(\varepsilon(b)) \geq \ell$.
5. For all $\Lambda$ dominant weights of level $\ell$, there exist unique elements $b_\Lambda, b^\Lambda \in B$, such that $\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$.

Points (1)-(3) are known to hold. This method checks points (4) and (5).

EXAMPLES:

sage: C = CartanType(['C',2,1])
sage: R = RootSystem(C)
sage: La = R.weight_space().basis()
sage: LS = crystalsProjectedLevelZeroLSPaths(La[1])
sage: LS.is_perfect()
False
sage: LS = crystalsProjectedLevelZeroLSPaths(La[2])
sage: LS.is_perfect()
True
sage: C = CartanType(['E',6,1])
sage: R = RootSystem(C)
sage: La = R.weight_space().basis()
```python
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1])
sage: LS.is_perfect()
True
sage: LS.is_perfect(2)
False

sage: C = CartanType(['D',4,1])
sage: R = RootSystem(C)
sage: La = R.weight_space().basis()
sage: all(crystals.ProjectedLevelZeroLSPaths(La[i]).is_perfect() for i in [1,2,3,4])
True

sage: C = CartanType(['A',6,2])
sage: R = RootSystem(C)
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[2])
sage: LS.is_perfect()
True
sage: LS.is_perfect(2)
False
```

**maximal_vector()**

Return the maximal vector of self.

**EXAMPLES:**

```python
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS.maximal_vector()
(-3*Lambda[0] + 2*Lambda[1] + Lambda[2],)
```

**one_dimensional_configuration_sum(q=None, group_components=True)**

Compute the one-dimensional configuration sum.

**INPUT:**

- `q` – (default: None) a variable or None; if None, a variable `q` is set in the code
- `group_components` – (default: True) boolean; if True, then the terms are grouped by classical component

The one-dimensional configuration sum is the sum of the weights of all elements in the crystal weighted by the energy function. For untwisted types it uses the parabolic quantum Bruhat graph, see [LNSSS2013]. In the dual-of-untwisted case, the parabolic quantum Bruhat graph is defined by exchanging the roles of roots and coroots (which is still conjectural at this point).

**EXAMPLES:**

```python
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: LS.one_dimensional_configuration_sum()
# long time
B[-2*Lambda[1] + 2*Lambda[2]] + (q+1)*B[-Lambda[1]]
```

sage: R.<t> = ZZ[]
sage: LS.one_dimensional_configuration_sum(t, False) # long time
B[-2*Lambda[1] + 2*Lambda[2]] + (t+1)*B[-Lambda[1]]

class sage.combinat.crystals.littelmann_path.InfinityCrystalOfLSPaths(cartan_type)

Bases: UniqueRepresentation, Parent

LS path model for $\mathcal{B}(\infty)$.

Elements of $\mathcal{B}(\infty)$ are equivalence classes of paths $[\pi]$ in $\mathcal{B}(k\rho)$ for $k \gg 0$, where $\rho$ is the Weyl vector. A canonical representative for an element of $\mathcal{B}(\infty)$ is chosen by taking $k$ to be minimal such that the endpoint of $\pi$ is strictly dominant but its representative in $\mathcal{B}(k-1)\rho$ is on the wall of the dominant chamber.

REFERENCES:
- [LZ2011]

class Element

Bases: Element

$e(i, power=1, length_only=False)$

Return the $i$-th crystal raising operator on self.

INPUT:
- $i$ – element of the index set
- $power$ – (default: 1) positive integer; specifies the power of the lowering operator to be applied
- $length_only$ – (default: False) boolean; if True, then return the distance to the anti-dominant end of the $i$-string of self

EXAMPLES:

```python
sage: B = crystals.infinity.LSPaths(['B', 3, 1])
sage: mg = B.module_generator()
sage: mg.e(0)
sage: mg.e(1)
sage: mg.e(2)
sage: x = mg.f_string([1,0,2,1,0,2,1,1,0])
sage: all(x.f(i).e(i) == x for i in B.index_set())
True
sage: all(x.e(i).f(i) == x for i in B.index_set() if x.epsilon(i) > 0)
True
```

$f(i, power=1, length_only=False)$

Return the $i$-th crystal lowering operator on self.

INPUT:
- $i$ – element of the index set
- $power$ – (default: 1) positive integer; specifies the power of the lowering operator to be applied
- $length_only$ – (default: False) boolean; if True, then return the distance to the anti-dominant end of the $i$-string of self

EXAMPLES:
sage: B = crystals.infinity.LSPaths(['D', 3, 2])
sage: mg = B.highest_weight_vector()
sage: mg.f(1)
(3*Lambda[0] - Lambda[1] + 3*Lambda[2],
sage: mg.f(2)
(Lambda[0] + 2*Lambda[1] - Lambda[2],
sage: mg.f(0)
(-Lambda[0] + 2*Lambda[1] + Lambda[2] - delta,

phi(i)

Return \( \phi_i \) of self.

Let \( \pi \in B(\infty) \). Define

\[
\phi_i(\pi) := e_i(\pi) + \langle h_i, wt(\pi) \rangle,
\]

where \( h_i \) is the \( i \)-th simple coroot and \( wt(\pi) \) is the weight() of \( \pi \).

INPUT:

• \( i \) – element of the index set

EXAMPLES:

sage: B = crystals.infinity.LSPaths(['D', 4])
sage: mg = B.highest_weight_vector()
sage: x = mg.f_string([1, 3, 4, 2, 4, 3, 2, 1, 4])
sage: [x.phi(i) for i in B.index_set()]
[-1, 4, -2, -3]

weight()

Return the weight of self.

Todo: This is a generic algorithm. We should find a better description and implement it.

EXAMPLES:

sage: B = crystals.infinity.LSPaths(['E', 6])
sage: mg = B.highest_weight_vector()
sage: f_seq = [1, 4, 2, 6, 4, 2, 3, 1, 5, 5]
sage: x = mg.f_string(f_seq)
sage: x.weight()
sage: al = B.cartan_type().root_system().weight_space().simple_roots()
sage: x.weight() == -sum(al[i] for i in f_seq)
True

module_generator()

Return the module generator (or highest weight element) of self.

The module generator is the unique path \( \pi_\infty: t \mapsto t \rho, \) for \( t \in [0, \infty) \).

EXAMPLES:
```python
sage: B = crystals.infinity.LSPaths(['A',6,2])
sage: mg = B.module_generator(); mg
(Lambda[0] + Lambda[1] + Lambda[2] + Lambda[3],)
sage: mg.weight()
0
```

`weight_lattice_realization()`

Return the weight lattice realization of self.

EXAMPLES:

```python
sage: B = crystals.infinity.LSPaths(['C',4])
sage: B.weight_lattice_realization()
Weight space over the Rational Field of the Root system of type ['C', 4]
```

`sage.combinat.crystals.littelmann_path.positively_parallel_weights(v, w)`

Check whether the vectors \( v \) and \( w \) are positive scalar multiples of each other.

EXAMPLES:

```python
sage: from sage.combinat.crystals.littelmann_path import positively_parallel_weights
sage: La = RootSystem(['A',5,2]).weight_space(extended=True).fundamental_weights()
sage: rho = sum(La)
sage: positively_parallel_weights(rho, 4*rho)
True
sage: positively_parallel_weights(4*rho, rho)
True
sage: positively_parallel_weights(rho, -rho)
False
sage: positively_parallel_weights(rho, La[1] + La[2])
False
```

### 5.1.59 Crystals of Modified Nakajima Monomials

AUTHORS:

- Arthur Lubovsky: Initial version
- Ben Salisbury: Initial version

Let \( Y_{i,k} \), for \( i \in I \) and \( k \in \mathbb{Z} \), be a commuting set of variables, and let \( 1 \) be a new variable which commutes with each \( Y_{i,k} \). (Here, \( I \) represents the index set of a Cartan datum.) One may endow the structure of a crystal on the set \( \hat{\mathcal{M}} \) of monomials of the form

\[
M = \prod_{(i,k) \in I \times \mathbb{Z}_{\geq 0}} Y_{i,k}^{y_{i,k}} 1.
\]

Elements of \( \hat{\mathcal{M}} \) are called **modified Nakajima monomials**. We will omit the 1 from the end of a monomial if there

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exists at least one \( y_i(k) \neq 0 \). The crystal structure on this set is defined by

\[
\text{wt}(M) = \sum_{i \in I} \left( \sum_{k \geq 0} y_i(k) \right) \Lambda_i,
\]

\[
\varphi_i(M) = \max \left\{ \sum_{0 \leq j \leq k} y_i(j) : k \geq 0 \right\},
\]

\[
\varepsilon_i(M) = \varphi_i(M) - \langle h_i, \text{wt}(M) \rangle,
\]

\[
k_f = k_f(M) = \min \left\{ k \geq 0 : \varphi_i(M) = \sum_{0 \leq j \leq k} y_i(j) \right\},
\]

\[
k_e = k_e(M) = \max \left\{ k \geq 0 : \varphi_i(M) = \sum_{0 \leq j \leq k} y_i(j) \right\},
\]

where \( \{h_i : i \in I\} \) and \( \{\Lambda_i : i \in I\} \) are the simple coroots and fundamental weights, respectively. With a chosen set of integers \( C = (c_{ij})_{i \neq j} \) such that \( c_{ij} + c_{ji} = 1 \), one defines

\[
A_{i,k} = Y_{i,k} Y_{i,k+1} \prod_{j \neq i} Y_{j,k+c_{ij}}^{a_{ij}},
\]

where \( (a_{ij}) \) is a Cartan matrix. Then

\[
e_i M = \begin{cases} 
0 & \text{if } \varepsilon_i(M) = 0, \\
A_{i,k_i} M & \text{if } \varepsilon_i(M) > 0,
\end{cases}
\]

\[
f_i M = A_{i,k_i}^{-1} M.
\]

It is shown in [KKS2007] that the connected component of \( \hat{M} \) containing the element \( 1 \), which we denote by \( M(\infty) \), is crystal isomorphic to the crystal \( B(\infty) \).

Let \( \hat{M} \) be \( \hat{M} \) as a set, and with crystal structure defined as on \( \hat{M} \) with the exception that

\[
f_i M = \begin{cases} 
0 & \text{if } \varphi_i(M) = 0, \\
A_{i,k_i}^{-1} M & \text{if } \varphi_i(M) > 0.
\end{cases}
\]

Then Kashiwara [Ko2003] showed that the connected component in \( \hat{M} \) containing a monomial \( M \) such that \( e_i M = 0 \), for all \( i \in I \), is crystal isomorphic to the irreducible highest weight crystal \( B(\text{wt}(M)) \).

WARNING:

Monomial crystals depend on the choice of positive integers \( C = (c_{ij})_{i \neq j} \) satisfying the condition \( c_{ij} + c_{ji} = 1 \). We have chosen such integers uniformly such that \( c_{ij} = 1 \) if \( i < j \) and \( c_{ij} = 0 \) if \( i > j \).

\textbf{class} sage.combinat.crystals.monomial_crystals.CrystalOfNakajimaMonomials(ct, La, c)

\textbf{Bases:} InfinityCrystalOfNakajimaMonomials

Let \( \hat{M} \) be \( \hat{M} \) as a set, and with crystal structure defined as on \( \hat{M} \) with the exception that

\[
f_i M = \begin{cases} 
0 & \text{if } \varphi_i(M) = 0, \\
A_{i,k_i}^{-1} M & \text{if } \varphi_i(M) > 0.
\end{cases}
\]

Then Kashiwara [Ko2003] showed that the connected component in \( \hat{M} \) containing a monomial \( M \) such that \( e_i M = 0 \), for all \( i \in I \), is crystal isomorphic to the irreducible highest weight crystal \( B(\text{wt}(M)) \).

\textbf{INPUT:}

\begin{itemize}
    \item ct – a Cartan type
    \item La – an element of the weight lattice
\end{itemize}
EXAMPLES:

```python
sage: La = RootSystem("A2").weight_lattice().fundamental_weights()
sage: B = crystals.Tableaux("A2",shape=[2,1])
sage: GM = M.digraph()
sage: GB = B.digraph()
sage: GM.is_isomorphic(GB,edge_labels=True)
True
```

```python
sage: La = RootSystem("G2").weight_lattice().fundamental_weights()
sage: B = crystals.Tableaux("G2",shape=[2,1])
sage: GM = M.digraph()
sage: GB = B.digraph()
sage: GM.is_isomorphic(GB,edge_labels=True)
True
```

```python
sage: La = RootSystem("B2").weight_lattice().fundamental_weights()
sage: B = crystals.Tableaux("B2",shape=[3/2,1/2])
sage: GM = M.digraph()
sage: GB = B.digraph()
sage: GM.is_isomorphic(GB,edge_labels=True)
True
```

```python
sage: La = RootSystem("A3,1").weight_lattice(extended=True).fundamental_weights()
sage: M = crystals.NakajimaMonomials("A3,1",La[0]+La[2])
sage: B = crystals.GeneralizedYoungWalls(3,La[0]+La[2])
sage: SM = M.subcrystal(max_depth=4)
sage: SB = B.subcrystal(max_depth=4)
sage: GM = M.digraph(subset=SM) # long time
sage: GB = B.digraph(subset=SB) # long time
sage: GM.is_isomorphic(GB,edge_labels=True) # long time
True
```

```python
sage: c = matrix([[0,1,0],[0,0,1],[1,0,0]])
sage: La = RootSystem("A2,1").weight_lattice(extended=True).fundamental_weights()
sage: M = crystals.NakajimaMonomials(2*La[1], c=c)
sage: sorted(M.subcrystal(max_depth=3), key=str)
[ Y(0,0) Y(0,1) Y(1,0) Y(2,1)^-1, Y(0,0) Y(0,1)^2 Y(1,1)^-1 Y(2,0) Y(2,1)^-1,
  (continues on next page)
```

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Y(0,0) Y(0,2)^-1 Y(1,0) Y(1,1) Y(2,1)^-1 Y(2,2),
Y(0,1) Y(0,2)^-1 Y(1,1)^-1 Y(2,0)^2 Y(2,2),
Y(0,1) Y(1,0) Y(1,1)^-1 Y(2,0),
Y(0,1)^2 Y(1,1)^-2 Y(2,0)^2,
Y(0,2)^-1 Y(1,0) Y(2,0) Y(2,2),
Y(1,0) Y(1,3) Y(2,0) Y(2,3)^-1,
Y(1,0)^2]

Element

alias of CrystalOfNakajimaMonomialsElement
cardinality()

Return the cardinality of self.

EXAMPLES:
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: M = crystals.NakajimaMonomials(['A',2], La[1])
sage: M.cardinality()
3
sage: La = RootSystem(['D',4,2]).weight_lattice(extended=True).fundamental_weights()
sage: M = crystals.NakajimaMonomials(['D',4,2], La[1])
sage: M.cardinality()
+Infinity

class sage.combinat.crystals.monomial_crystals.CrystalOfNakajimaMonomialsElement(parent, Y, A)

Bases: NakajimaMonomial

Element class for CrystalOfNakajimaMonomials.
The \( f \) operators need to be modified from the version in monomial_crystalsNakajimaMonomial in order to create irreducible highest weight realizations. This modified \( f \) is defined as

\[
f_i M = \begin{cases} 
0 & \text{if } \varphi_i(M) = 0, \\
A_i^{-1} k_i M & \text{if } \varphi_i(M) > 0.
\end{cases}
\]

EXAMPLES:
sage: La = RootSystem(['A',5,2]).weight_lattice(extended=True).fundamental_weights()
sage: M = crystals.NakajimaMonomials(['A',5,2], 3*La[0])
sage: m = M.module_generators[0].f(0); m
Y(0,0)^2 Y(0,1)^-1 Y(2,0)
sage: TestSuite(m).run()

f()

Return the action of \( f \) on self.

INPUT:

• \( i \) – an element of the index set

EXAMPLES:
```python
sage: La = RootSystem(['A', 5, 2]).weight_lattice(extended=True).fundamental_weights()
sage: M = crystals.NakajimaMonomials(['A', 5, 2], 3*La[0])
sage: m = M.module_generators[0]
sage: [m.f(i) for i in M.index_set()]
[Y(0, 0)^2 Y(0, 1)^{-1} Y(2, 0), None, None, None]

sage: M = crystals.infinity.NakajimaMonomials("E8")
sage: M.set_variables('A')
sage: m = M.module_generators[0].f_string([4, 2, 3, 8])
sage: m
A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(8, 0)^{-1}

sage: [m.f(i) for i in M.index_set()]
[A(1, 2)^{-1} A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(8, 0)^{-1},
 A(2, 0)^{-1} A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(8, 0)^{-1},
 A(2, 1)^{-1} A(3, 0)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(8, 0)^{-1},
 A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(4, 1)^{-1} A(8, 0)^{-1},
 A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(5, 0)^{-1} A(8, 0)^{-1},
 A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(6, 0)^{-1} A(8, 0)^{-1},
 A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(7, 1)^{-1} A(8, 0)^{-1},
 A(2, 1)^{-1} A(3, 1)^{-1} A(4, 0)^{-1} A(8, 0)^{-2}]

sage: M.set_variables('Y')
```

**weight()**

Return the weight of `self` as an element of the weight lattice.

**EXAMPLES:**

```python
sage: La = RootSystem("A2").weight_lattice().fundamental_weights()
sage: M.module_generators[0].weight()
(2, 1, 0)
```

**class** `sage.combinat.crystals.monomial_crystals.InfinityCrystalOfNakajimaMonomials`

Bases: `UniqueRepresentation`, `Parent`  

Crystal $B(\infty)$ in terms of (modified) Nakajima monomials.  

Let $Y_{i,k}$ for $i \in I$ and $k \in \mathbb{Z}$, be a commuting set of variables, and let 1 be a new variable which commutes with each $Y_{i,k}$. (Here, $I$ represents the index set of a Cartan datum.) One may endow the structure of a crystal on the set $\widehat{\mathcal{M}}$ of monomials of the form

$$M = \prod_{(i,k) \in I \times \mathbb{Z}_{\geq 0}} Y_{i,k}^{y_{i,k}(k)}.$$  

Elements of $\widehat{\mathcal{M}}$ are called *modified Nakajima monomials*. We will omit the 1 from the end of a monomial if
there exists at least one $y_i(k) \neq 0$. The crystal structure on this set is defined by

\[
\begin{align*}
\text{wt}(M) &= \sum_{i \in I} \left( \sum_{k \geq 0} y_i(k) \right) \Lambda_i, \\
\varphi_i(M) &= \max \left\{ \sum_{0 \leq j \leq k} y_i(j) : k \geq 0 \right\}, \\
\epsilon_i(M) &= \varphi_i(M) - \langle h_i, \text{wt}(M) \rangle, \\
k_f &= k_f(M) = \min \left\{ k \geq 0 : \varphi_i(M) = \sum_{0 \leq j \leq k} y_i(j) \right\}, \\
k_e &= k_e(M) = \max \left\{ k \geq 0 : \varphi_i(M) = \sum_{0 \leq j \leq k} y_i(j) \right\},
\end{align*}
\]

where \{h_i : i \in I\} and \{\Lambda_i : i \in I\} are the simple coroots and fundamental weights, respectively. With a chosen set of non-negative integers $C = (c_{ij})_{i \neq j}$ such that $c_{ii} = 0$ for all $i \in I$, $c_{ij} \in \mathbb{Z}_{>0}$ for all $i, j \in I$, and $c_{ij} + c_{ji} = 1$ for all $i \neq j$; the default is $c_{ij} = 0$ if $i < j$ and 0 otherwise, one defines

\[
A_{i,k} = Y_{i,k}Y_{i,k+1} \prod_{j \neq k} Y_{j,k+c_{ij}},
\]

where $(a_{ij})_{i,j \in I}$ is a Cartan matrix. Then

\[
\begin{align*}
c_i M &= \begin{cases} 
0 & \text{if } \epsilon_i(M) = 0, \\
A_{i,k} & \text{if } \epsilon_i(M) > 0,
\end{cases} \\
f_i M &= A_{i,k}^{-1} M.
\end{align*}
\]

It is shown in [KKS2007] that the connected component of $\widetilde{\mathcal{M}}$ containing the element 1, which we denote by $\mathcal{M}(\infty)$, is crystal isomorphic to the crystal $B(\infty)$.

INPUT:

- \texttt{cartan\_type} – a Cartan type
- \texttt{c} – (optional) the matrix $(c_{ij})_{i,j \in I}$ such that $c_{ii} = 0$ for all $i \in I$, $c_{ij} \in \mathbb{Z}_{>0}$ for all $i, j \in I$, and $c_{ij} + c_{ji} = 1$ for all $i \neq j$; the default is $c_{ij} = 0$ if $i < j$ and 0 otherwise

EXAMPLES:

```python
sage: B = crystals.infinity.Tableaux("C3")
sage: S = B.subcrystal(max_depth=4)
sage: G = B.digraph(subset=S) # long time
sage: M = crystals.infinity.NakajimaMonomials("C3") # long time
sage: T = M.subcrystal(max_depth=4) # long time
sage: H = M.digraph(subset=T) # long time
sage: G.is_isomorphic(H, edge_labels=True) # long time
True
sage: M = crystals.infinity.NakajimaMonomials(['A',2,1])
sage: T = M.subcrystal(max_depth=3)
sage: H = M.digraph(subset=T) # long time
sage: Y = crystals.infinity.GeneralizedYoungWalls(2)
sage: YS = Y.subcrystal(max_depth=3)
sage: YG = Y.digraph(subset=YS) # long time
sage: YG.is_isomorphic(H, edge_labels=True) # long time
True
```

(continues on next page)
sage: M = crystals.infinity.NakajimaMonomials("D4")
sage: B = crystals.infinity.Tableaux("D4")
sage: MS = M.subcrystal(max_depth=3)
sage: BS = B.subcrystal(max_depth=3)
sage: MG = M.digraph(subset=MS)  # long time
sage: BG = B.digraph(subset=BS)  # long time
sage: BG.is_isomorphic(MG, edge_labels=True)  # long time
True

Element

alias of NakajimaMonomial
c()

Return the matrix $c_{ij}$ of self.

EXAMPLES:

sage: La = RootSystem(['B', 3]).weight_lattice().fundamental_weights()
sage: M = crystals.NakajimaMonomials(La[1]+La[2])
sage: M.c()
[[0, 1, 1],
 [0, 0, 1],
 [0, 0, 0]]
sage: c = Matrix([[0, 0, 1],
 [1, 0, 0],
 [0, 1, 0]])
sage: La = RootSystem(['A', 2, 1]).weight_lattice(extended=True).fundamental_weights()
sage: M = crystals.NakajimaMonomials(2*La[1], c=c)
sage: M.c() == c
True

cardinality()

Return the cardinality of self, which is always $\infty$.

EXAMPLES:

sage: M = crystals.infinity.NakajimaMonomials(['A', 5, 2])
  sage: M.cardinality()
  +Infinity

get_variables()

Return the type of monomials to use for the element output.

EXAMPLES:

sage: M = crystals.infinity.NakajimaMonomials(['A', 4])
  sage: M.get_variables()
  'Y'

set_variables(letter)

Set the type of monomials to use for the element output.

If the $A$ variables are used, the output is written as $\prod_{i \in I} Y_{i,0}^{\Lambda_i} \prod_{i, k} A_{i,k}^{c_{i,k}}$, where $\sum_{i \in I} \lambda_i \Lambda_i$ is the corresponding dominant weight.
INPUT:

- **letter** – can be one of the following:
  - 'Y' - use \( Y_{i,k} \), corresponds to fundamental weights
  - 'A' - use \( A_{i,k} \), corresponds to simple roots

EXAMPLES:

```python
sage: M = crystals.infinity.NakajimaMonomials(['A', 4])
sage: elt = M.highest_weight_vector().f_string([2,1,3,2,3,2,4,4,3])
sage: elt
Y(1,2) Y(2,0)^-1 Y(2,2)^-1 Y(3,0)^-1 Y(3,2)^-1 Y(4,0)
sage: M.set_variables('A')
sage: elt
A(1,1)^-1 A(2,0)^-1 A(2,1)^-2 A(3,0)^-2 A(3,1)^-1 A(4,0)^-1
sage: M.set_variables('Y')
```

```python
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: M = crystals.NakajimaMonomials(La[1]+La[2])
sage: lw = M.lowest_weight_vectors()[0]
sage: lw
Y(1,2)^-1 Y(2,1)^-1
sage: M.set_variables('A')
sage: lw
Y(1,0) Y(2,0) A(1,0)^-1 A(1,1)^-1 A(2,0)^-2
sage: M.set_variables('Y')
```

```python
class sage.combinat.crystals.monomial_crystals.NakajimaMonomial(parent, Y, A)

Bases: Element

An element of the monomial crystal.

Monomials of the form \( Y_{y_1,k_1}^{y_1} \cdots Y_{y_t,k_t}^{y_t}, \) where \( i_1, \ldots, i_t \) are elements of the index set, \( k_1, \ldots, k_t \) are nonnegative integers, and \( y_1, \ldots, y_t \) are integers.

EXAMPLES:

```python
sage: M = crystals.infinity.NakajimaMonomials(['B',3,1])
sage: mg = M.module_generators[0]
sage: mg
1
sage: mg.f_string([1,3,2,0,1,2,3,0,0,1])
Y(0,0)^-1 Y(0,1)^-1 Y(0,2)^-1 Y(0,3)^-1 Y(1,0)^-3
Y(1,1)^-2 Y(1,2) Y(2,0)^3 Y(2,2) Y(3,0) Y(3,2)^-1
```

An example using the \( A \) variables:

```python
sage: M = crystals.infinity.NakajimaMonomials('A3')
sage: M.set_variables('A')
sage: mg = M.module_generators[0]
sage: mg.f_string([1,2,3,2,1])
A(1,0)^-1 A(1,1)^-1 A(2,0)^-2 A(3,0)^-1
sage: mg.f_string([3,2,1])
A(1,2)^-1 A(2,1)^-1 A(3,0)^-1
sage: M.set_variables('Y')
```

e(i)

Return the action of e_i on self.

INPUT:

• i – an element of the index set

EXAMPLES:

```python
sage: M = crystals.infinity.NakajimaMonomials(['E', 7, 1])
sage: m = M.module_generators[0].f_string([0, 1, 4, 3])
sage: [m.e(i) for i in M.index_set()]
[None, None, None, None, Y(0, 0)^-1 Y(1, 1)^-1 Y(2, 1) Y(3, 0) Y(3, 1) Y(4, 0)^-1 Y(4, 1)^-1 Y(5, 0), None, None, None, None]
```

epsilon(i)

Return the value of \( \varepsilon_i \) on self.

INPUT:

• i – an element of the index set

EXAMPLES:

```python
sage: M = crystals.infinity.NakajimaMonomials(['G', 2, 1])
sage: m = M.module_generators[0].f(2)
sage: [m.epsilon(i) for i in M.index_set()]
[0, 0, 1]
```

(continues on next page)
sage: m = M.module_generators[0].f_string([4,2,3])
sage: [m.epsilon(i) for i in M.index_set()]
[0, 0, 0, 1, 0]

\( f(i) \)

Return the action of \( f_i \) on self.

**INPUT:**

- \( i \) – an element of the index set

**EXAMPLES:**

\[
\begin{align*}
\text{sage: } & M = \text{crystals.infinity.NakajimaMonomials("B4")} \\
\text{sage: } & m = M.module_generators[0].f_string([1,3,4]) \\
\text{sage: } & \left[ m.f(i) \text{ for } i \text{ in } M.index_set() \right] \\
& \left[ Y(1,0)^{-2} Y(1,1)^{-2} Y(2,0)^2 Y(2,1)^2 Y(3,0)^{-1} Y(4,0) Y(4,1)^{-1}, \right. \\
& \left. Y(1,0)^{-1} Y(1,1)^{-1} Y(1,2) Y(2,0) Y(2,2)^{-1} Y(3,0)^{-1} Y(3,1) Y(4,0) Y(4,1)^{-1}, \right. \\
& \left. Y(1,0)^{-1} Y(1,1)^{-1} Y(2,0) Y(2,1)^2 Y(3,0)^{-2} Y(3,1)^{-1} Y(4,0)^3 Y(4,1)^{-1}, \right. \\
& \left. Y(1,0)^{-1} Y(1,1)^{-1} Y(2,0) Y(2,1) Y(3,0)^{-1} Y(3,1) Y(4,1)^{-2} \right]
\end{align*}
\]

\( \phi(i) \)

Return the value of \( \phi_i \) on self.

**INPUT:**

- \( i \) – an element of the index set

**EXAMPLES:**

\[
\begin{align*}
\text{sage: } & M = \text{crystals.infinity.NakajimaMonomials(['D',4,3])} \\
\text{sage: } & m = M.module_generators[0].f(1) \\
\text{sage: } & \left[ m.phi(i) \text{ for } i \text{ in } M.index_set() \right] \\
& \left[ 1, -1, 1 \right] \\
\text{sage: } & M = \text{crystals.infinity.NakajimaMonomials(['C',4,1])} \\
\text{sage: } & m = M.module_generators[0].f_string([4,2,3]) \\
\text{sage: } & \left[ m.phi(i) \text{ for } i \text{ in } M.index_set() \right] \\
& \left[ 0, 1, -1, 2, -1 \right]
\end{align*}
\]

\( \text{weight()} \)

Return the weight of self as an element of the weight lattice.

**EXAMPLES:**

\[
\begin{align*}
\text{sage: } & C = \text{crystals.infinity.NakajimaMonomials(['A',1,1])} \\
\text{sage: } & v = C.highest_weight_vector() \\
\text{sage: } & v.f(1).weight() + v.f(0).weight() - \delta \\
\text{sage: } & M = \text{crystals.infinity.NakajimaMonomials(['A',4,2])} \\
\text{sage: } & m = M.highest_weight_vector().f_string([1,2,0,1]) \\
\text{sage: } & m.weight() \\
& 2*\text{Lambda}[0] - \text{Lambda}[1] - \delta \\
\end{align*}
\]
weight_in_root_lattice()

Return the weight of self as an element of the root lattice.

EXAMPLES:

```python
sage: M = crystals.infinity.NakajimaMonomials(['F',4])
sage: m = M.module_generators[0].f_string([3,3,1,2,4])
sage: m.weight_in_root_lattice()

sage: M = crystals.infinity.NakajimaMonomials(['B',3,1])
sage: mg = M.module_generators[0]
sage: m = mg.f_string([1,3,2,0,1,2,3,0,0,1])
sage: m.weight_in_root_lattice()

sage: M = crystals.infinity.NakajimaMonomials(['C',3,1])
sage: m = M.module_generators[0].f_string([3,0,1,2,0])
sage: m.weight_in_root_lattice()
```

5.1.60 Crystal of Bernstein-Zelevinsky Multisegments

class sage.combinat.crystals.multisegments.InfinityCrystalOfMultisegments(n)

Bases: Parent, UniqueRepresentation

The type $A_n^{(1)}$ crystal $B(\infty)$ realized using Bernstein-Zelevinsky (BZ) multisegments.

Using (a modified version of the) notation from [JL2009], for $\ell \in \mathbb{Z}_{>0}$ and $i \in \mathbb{Z}/(n+1)\mathbb{Z}$, a segment of length $\ell$ and head $i$ is the sequence of consecutive residues $[i, i+1, \ldots, i+\ell-1]$. The notation for a segment of length $\ell$ and head $i$ is simplified to $[i; \ell]$. Similarly, a segment of length $\ell$ and tail $i$ is the sequence of consecutive residues $[i-\ell+1, \ldots, i-1, i]$. The latter is denoted simply by $(\ell; i]$. Finally, a multisegment is a formal linear combination of segments, usually written in the form

$$\psi = \sum_{i \in \mathbb{Z}/(n+1)\mathbb{Z}} m_{(\ell;i]}(\ell; i].$$

Such a multisegment is called aperiodic if, for every $\ell > 0$, there exists some $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ such that $(\ell; i]$ does not appear in $\psi$. Denote the set of all periodic multisegments, together with the empty multisegment $\emptyset$, by $\Psi$. We define a crystal structure on multisegments as follows. Set $S_{\ell,i} = \sum_{k \geq \ell} (m_{(k;i-1]} - m_{(k;i]})$ and let $\ell_f$ be the minimal $\ell$ that attains the value $\min_{\ell \geq 0} S_{\ell,i}$. Then we have

$$f_i \psi = \begin{cases} 
\psi + (1; i] & \text{if } \ell_f = 1, \\
\psi + (\ell_f; i] - (\ell_f - 1; i - 1] & \text{if } \ell_f > 1.
\end{cases}$$

Similarly, let $\ell_e$ be the maximal $\ell$ that attains the value $\min_{\ell \geq 0} S_{\ell,i}$. Then we have

$$e_i \psi = \begin{cases} 
0 & \text{if } \min_{\ell \geq 0} S_{\ell,i} = 0, \\
\psi + (1; i] & \text{if } \ell_e = 1, \\
\psi - (\ell_e; i] + (\ell_e - 1; i - 1] & \text{if } \ell_e > 1.
\end{cases}$$

Alternatively, the crystal operators may be defined using a signature rule, as detailed in Section 4 of [JL2009] (following [AJL2011]). For $\psi \in \Psi$ and $i \in \mathbb{Z}/(n+1)\mathbb{Z}$, encode all segments in $\psi$ with tail $i$ by the symbol $R$.
and all segments in $\psi$ with tail $i - 1$ by $A$. For $\ell > 0$, set $w_{i,\ell} = R^{m_{\ell(i)}} A^{m_{\ell(i)-1}}$ and $w_i = \prod_{\ell \geq 1} w_{i,\ell}$. By successively canceling out as many $RA$ factors as possible, set $\tilde{w}_i = A^{a_i(\psi)} R^{r_i(\psi)}$. If $a_i(\psi) > 0$, denote by $\ell_f > 0$ the length of the rightmost segment $A$ in $\tilde{w}_i$. If $a_i(\psi) = 0$, set $\ell_f = 0$. Then

$$f_i \psi = \begin{cases} \psi + (1; i] & \text{if } a_i(\psi) = 0, \\ \psi + (\ell_f; i] - (\ell_f - 1; i - 1] & \text{if } a_i(\psi) > 0. \end{cases}$$

The rule for computing $e_i \psi$ is similar.

INPUT:

• $n$ – for type $A_n^{(1)}$

EXAMPLES:

```sage
sage: B = crystals.infinity.Multisegments(2)
sage: x = B(((8, 1), (6, 0), (5, 1), (5, 0), (4, 0), (4, 1), (4, 1), (3, 0), (3, 0), (3, 1), (3, 1), (1, 0), (1, 2), (1, 2))); x
(8; 1] + (6; 0] + (5; 0] + (5; 1] + (4; 0] + 2 * (4; 1] + 2 * (3; 0] + 2 * (3; 1] + (2; 1] + 2 * (1; 2]
sage: x.f(1)
(8; 1] + (6; 0] + (5; 0] + (5; 1] + (4; 0] + 2 * (4; 1] + 2 * (3; 0] + 2 * (3; 1] + (2; 1] + 2 * (1; 2]
sage: x.f(1).f(1)
(8; 1] + (6; 0] + (5; 0] + (5; 1] + (4; 0] + 2 * (4; 1] + 2 * (3; 0] + 2 * (3; 1] + (2; 1] + (1; 1] + 2 * (1; 2]
sage: x.e(1)
(7; 0] + (6; 0] + (5; 0] + (5; 1] + (4; 0] + 2 * (4; 1] + 2 * (3; 0] + 2 * (3; 1] + (1; 0] + 2 * (1; 2]
sage: x.e(1).e(1)
sage: x.f(0)
(8; 1] + (6; 0] + (5; 0] + (5; 1] + (4; 0] + 2 * (4; 1] + 2 * (3; 0] + 2 * (3; 1] + (2; 0] + (1; 0] + (1; 2]
```

We check an $\hat{sl}_2$ example against the generalized Young walls:

```sage
sage: B = crystals.infinity.Multisegments(1)
sage: G = B.subcrystal(max_depth=4).digraph()
sage: C = crystals.infinity.GeneralizedYoungWalls(1)
sage: GC = C.subcrystal(max_depth=4).digraph()
sage: G.is_isomorphic(GC, edge_labels=True)
True
```

REFERENCES:

• [AJL2011]
• [JL2009]
• [LTV1999]

class **Element** *(parent, value)*

Bases: **ElementWrapper**

An element in a BZ multisegments crystal.

**e(i)**

Return the action of $e_i$ on self.
epsilon(i)

Return $\varepsilon_i$ of self.

INPUT:

- $i$ – an element of the index set

EXAMPLES:

```python
sage: B = crystals.infinity.Multisegments(2)
sage: b = B([(4,2), (3,0), (3,1), (1,1), (1,0)])
sage: b.e(0)
(4; 2] + (3; 0] + (3; 1] + (1; 1]
sage: b.e(1)
sage: b.e(2)
(3; 0] + 2 * (3; 1] + (1; 0] + (1; 1]
```

f(i)

Return the action of $f_i$ on self.

INPUT:

- $i$ – an element of the index set

EXAMPLES:

```python
sage: B = crystals.infinity.Multisegments(2)
sage: b = B([(4,2), (3,0), (3,1), (1,1), (1,0)])
sage: b.f(0)
(4; 2] + (3; 0] + (3; 1] + 2 * (1; 0] + (1; 1]
sage: b.f(1)
(4; 2] + (3; 0] + (3; 1] + (1; 0] + 2 * (1; 1]
sage: b.f(2)
2 * (4; 2] + (3; 0] + (1; 0] + (1; 1]
```

phi(i)

Return $\varphi_i$ of self.

Let $\psi \in \Psi$. Define $\varphi_i(\psi) := \varepsilon_i(\psi) + \langle h_i, \text{wt}(\psi) \rangle$, where $h_i$ is the $i$-th simple coroot and $\text{wt}(\psi)$ is the weight() of $\psi$.

INPUT:

- $i$ – an element of the index set

EXAMPLES:
Combinatorics, Release 10.0

sage: B = crystals.infinity.Multisegments(2)
sage: b = B([(4,2), (3,0), (3,1), (1,1), (1,0)])
sage: b.phi(0)
1
sage: b.phi(1)
0
sage: mg = B.highest_weight_vector()
sage: mg.f(1).phi(0)
1

weight()
Return the weight of self.

EXAMPLES:

sage: B = crystals.infinity.Multisegments(2)
sage: b = B([(4,2), (3,0), (3,1), (1,1), (1,0)])
sage: b.weight()
-4*delta

highest_weight_vector()
Return the highest weight vector of self.

EXAMPLES:

sage: B = crystals.infinity.Multisegments(2)
sage: B.highest_weight_vector()
0

weight_lattice_realization()
Return a realization of the weight lattice of self.

EXAMPLES:

sage: B = crystals.infinity.Multisegments(2)
sage: B.weight_lattice_realization()
Extended weight lattice of the Root system of type ['A', 2, 1]

5.1.61 Crystal Of Mirković-Vilonen (MV) Polytopes

AUTHORS:

• Dinakar Muthiah, Travis Scrimshaw (2015-05-11): initial version

class sage.combinat.crystals.mv_polytopes.MVPolytope(parent, lusztig_datum, long_word=None)

Bases: PBWCrystalElement

A Mirković-Vilonen (MV) polytope.

EXAMPLES:

We can create an animation showing how the MV polytope changes under a string of crystal operators:
Combinatorics, Release 10.0

```python
sage: MV = crystals.infinity.MVPolytopes(['C', 2])
sage: u = MV.highest_weight_vector()
sage: L = RootSystem(['C',2,1]).ambient_space()
sage: s = [1,2,1,2,2,1,1,1,2,2,1,2,1,2]
sage: BB = [[-9, 2], [-10, 2]]
sage: p = L.plot(reflection_hyperplanes=False, bounding_box=BB)  # long time
sage: frames = [p + L.plot_mv_polytope(u.f_string(s[:i]), circle_size=0.1, wireframe='green', fill='purple', bounding_box=BB) for i in range(len(s))]
sage: for f in frames:  # long time
    f.axes(False)
sage: animate(frames).show(delay=60)  # optional -- ImageMagick # long time
```

```
plot(P=None, **options)
Plot self.

INPUT:

• P – (optional) a space to realize the polytope; default is the weight lattice realization of the crystal

See also:
plot_mv_polytope()

EXAMPLES:

```python
sage: MV = crystals.infinity.MVPolytopes(['C', 2])
sage: b = MV.highest_weight_vector().f_string([1,2,1,2,2,1,1,1,1,2,1,2])
sage: b.plot()
```

Graphics object consisting of 12 graphics primitives

Here is the above example placed inside the ambient space of type $C_2$:

```
polytope(P=None)
Return a polytope of self.

INPUT:

• P – (optional) a space to realize the polytope; default is the weight lattice realization of the crystal
```

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EXAMPLES:

```python
sage: MV = crystals.infinity.MVPolytopes(['C', 3])
sage: b = MV.module_generators[0].f_string([3,2,3,2,1])
sage: P = b.polytope(); P
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 6 vertices
sage: P.vertices()
(A vertex at (0, 0, 0),
 A vertex at (0, 1, -1),
 A vertex at (0, 1, 1),
 A vertex at (1, -1, 0),
 A vertex at (1, 1, -2),
 A vertex at (1, 1, 2))
```

**class** `sage.combinat.crystals.mv_polytopes.MVPolytopes(cartan_type)`

Bases: `PBWCrystal`

The crystal of Mirković-Vilonen (MV) polytopes.

Let $W$ denote the corresponding Weyl group and $P_R = \mathbb{R} \otimes P$. Let $\Gamma = \{w\Lambda_i \mid w \in W, i \in I\}$. Consider $M = \{M_\gamma \in \mathbb{Z}\}_{\gamma \in \Gamma}$ that satisfy the tropical Plücker relations (see Proposition 7.1 of [BZ01]). The **MV polytope** is defined as

$$P(M) = \{\alpha \in P_R \mid \langle \alpha, \gamma \rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma\}.$$ 

The vertices $\{\mu_w\}_{w \in W}$ are given by

$$\langle \mu_w, \gamma \rangle = M_\gamma$$

and are known as the GGMS datum of the MV polytope.

Each path from $\mu_e$ to $\mu_{w_0}$ corresponds to a reduced expression $i = (i_1, \ldots, i_m)$ for $w_0$ and the corresponding edge lengths $(n_k)_{k=1}^m$ from the Lusztig datum with respect to $i$. Explicitly, we have

$$n_k = -M_{w_{k-1}\Lambda_{i_k}} - M_{w_k\Lambda_{i_k}} - \sum_{j \neq i} a_{ji} M_{w_k\Lambda_j},$$

$$\mu_{w_k} - \mu_{w_{k-1}} = n_k w_{k-1} \alpha_{i_k},$$

where $w_k = s_{i_1} \cdots s_{i_k}$ and $(a_{ji})$ is the Cartan matrix.

MV polytopes have a crystal structure that corresponds to the crystal structure, which is isomorphic to $B(\infty)$ with $\mu_{w_0} = 0$, on **PWB data**. Specifically, we have $f_j P(M)$ as being the unique MV polytope given by shifting $\mu_e$ by $-\alpha_j$ and fixing the vertices $\mu_w$ when $s_j w < w$ (in Bruhat order) and the weight is given by $\mu_e$. Furthermore, the $*$-involution is given by negating $P(M)$.

**INPUT:**

- `cartan_type` – a Cartan type

**EXAMPLES:**

```python
sage: MV = crystals.infinity.MVPolytopes(['B', 3])
sage: hw = MV.highest_weight_vector()
sage: x = hw.f_string([1,2,2,3,3,1,3,3,2,3,2,1,3,1,2,1,2,1,3,2]); x
MV polytope with Lusztig datum (1, 1, 1, 3, 1, 0, 0, 1, 1)
```

Elements are expressed in terms of Lusztig datum for a fixed reduced expression of $w_0$: 
We can construct elements by giving it Lusztig data (with respect to the default long word reduced expression):

```
sage: MV([1,1,1,3,1,0,0,1,1])
MV polytope with Lusztig datum (1, 1, 1, 3, 1, 0, 0, 1, 1)
```

We can also construct elements by passing in a reduced expression for a long word:

```
sage: x = MV([1,1,1,3,1,0,0,1,1], [3,2,1,3,2,3,2,1,2]); x
MV polytope with Lusztig datum (1, 1, 1, 0, 1, 0, 5, 1, 1)
```

The highest weight crystal $B(\lambda) \subseteq B(\infty)$ is characterized by the MV polytopes that sit inside of $W \lambda$ (translating $\mu_{w_0} \rightarrow \lambda$):

```
sage: MV = crystals.infinity.MVPolytopes(['A',2])
sage: La = MV.weight_lattice_realization().fundamental_weights()
sage: R = crystals.elementary.R(La[1]+La[2])
sage: T = tensor([R, MV])
sage: x = T(R.module_generators[0], MV.highest_weight_vector()); x
MV polytope with Lusztig datum (1, 1, 1)
sage: lw = x.to_lowest_weight()[0]; lw
[(2, 1, 0), MV polytope with Lusztig datum (1, 1, 1)]
sage: lw[1].polytope().vertices()
(A vertex at (0, 0, 0),
 A vertex at (0, 1, -1),
 A vertex at (1, -1, 0),
 A vertex at (1, 1, -2),
 A vertex at (2, -1, -1),
 A vertex at (2, 0, -2))
```

REFERENCES:

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Element
alias of MVPolytope

latex_options()
Return the latex options of self.

EXAMPLES:
sage: MV = crystals.infinity.MVPolytopes(['F', 4])
sage: MV.latex_options()
{'P': 'Ambient space of the Root system of type ['F', 4], 'circle_size': 0.1, 'mark_endpoints': True, 'projection': True}

set_latex_options(**kwds)
Set the latex options for the elements of self.

INPUT:

• projection – the projection; set to True to use the default projection of the specified weight lattice realization (initial: True)
• P – the weight lattice realization to use (initial: the weight lattice realization of self)
• mark_endpoints – whether to mark the endpoints (initial: True)
• circle_size – the size of the endpoint circles (initial: 0.1)

EXAMPLES:
sage: MV = crystals.infinity.MVPolytopes(['C', 2])
sage: P = RootSystem(['C', 2]).weight_lattice()
sage: b = MV.highest_weight_vector().f_string([1,2,1,2])
sage: latex(b)
\begin{tikzpicture}
\draw (0, 0) -- (0, -2) -- (-1, -3) -- (-1, -3) -- (-2, -2);
\draw (0, 0) -- (-1, 1) -- (-1, 1) -- (-2, 0) -- (-2, -2);
\draw[fill=black] (0, 0) circle (0.1);
\draw[fill=black] (-2, -2) circle (0.1);
\end{tikzpicture}
sage: MV.set_latex_options(P=P, circle_size=float(0.2))
sage: latex(b)
\begin{tikzpicture}
\draw (0, 0) -- (2, -2) -- (2, -3) -- (2, -3) -- (0, -2);
\draw (0, 0) -- (-2, 1) -- (-2, 1) -- (-2, 0) -- (0, -2);
\draw[fill=black] (0, 0) circle (0.2);
\draw[fill=black] (-2, -2) circle (0.2);
\end{tikzpicture}
sage: MV.set_latex_options(mark_endpoints=False)
sage: latex(b)
\begin{tikzpicture}
\draw (0, 0) -- (2, -2) -- (2, -3) -- (2, -3) -- (0, -2);
\draw (0, 0) -- (-2, 1) -- (-2, 1) -- (-2, 0) -- (0, -2);
\draw[fill=black] (0, 0) circle (0.2);
\draw[fill=black] (-2, -2) circle (0.2);
\end{tikzpicture}
5.1.62 $B(\infty)$ Crystal Of PBW Monomials

AUTHORS:

- Dinakar Muthiah (2015-05-11): initial version

See also:

For information on PBW datum, see PBW Data.

class sage.combinat.crystals.pbw_crystal.PBWCrystal(cartan_type)

Bases: Parent, UniqueRepresentation

Crystal of $B(\infty)$ given by PBW monomials.

A model of the crystal $B(\infty)$ whose elements are PBW datum up to equivalence by the tropical Plücker relations.

The crystal structure on Lusztig data $x = (x_1, \ldots, x_m)$ for the reduced word $s_i, \ldots, s_i = w_0$ is given as follows.

Suppose $i_1 = j$, then $f_j x = (x_1 + 1, x_2, \ldots, x_m)$. If $i_1 \neq j$, then we use the tropical Plücker relations to change the reduced expression such that $i'_1 = j$ and then we change back to the original word.

EXAMPLES:

```python
sage: PBW = crystals.infinity.PBW(['B', 3])
sage: hw = PBW.highest_weight_vector()
sage: x = hw.f_string([1,2,3,2,1,3,2,1,3,2,1,3,2,1,3,2,1]); x
PBW monomial with Lusztig datum (1, 1, 1, 3, 1, 0, 0, 1, 1)
```

Elements are expressed in terms of Lusztig datum for a fixed reduced expression of $w_0$:

```python
sage: PBW.default_long_word()
[1, 3, 2, 3, 1, 2, 3, 1, 2]
sage: PBW.set_default_long_word([2,1,3,2,1,3,2,3,1])
sage: x
PBW monomial with Lusztig datum (3, 1, 1, 0, 1, 0, 1, 3, 4)
sage: PBW.set_default_long_word([1, 3, 2, 3, 1, 2, 3, 1, 2])
```

We can construct elements by giving it Lusztig data (with respect to the default long word):

```python
sage: PBW([1,1,1,3,1,0,0,1,1])
PBW monomial with Lusztig datum (1, 1, 1, 3, 1, 0, 0, 1, 1)
```

We can also construct elements by passing in a reduced expression for a long word:

```python
sage: x = PBW([1,1,1,3,1,0,0,1,1], [3,2,1,3,2,3,2,1,2]); x
PBW monomial with Lusztig datum (1, 1, 1, 0, 1, 0, 5, 1, 1)
```

```
We can construct elements by giving it Lusztig data (with respect to the default long word):

```python
sage: PBW([1,1,1,3,1,0,0,1,1])
PBW monomial with Lusztig datum (1, 1, 1, 3, 1, 0, 0, 1, 1)
```

We can also construct elements by passing in a reduced expression for a long word:

```python
sage: x = PBW([1,1,1,3,1,0,0,1,1], [3,2,1,3,2,3,2,1,2]); x
PBW monomial with Lusztig datum (1, 1, 1, 0, 1, 0, 5, 1, 1)
```

```
```
default_long_word()

Return the default long word used to express elements of self.

EXAMPLES:

```
sage: B = crystals.infinity.PBW(['E', 6])
sage: B.default_long_word()
[1, 3, 4, 5, 6, 2, 4, 5, 3, 4, 1, 3, 2, 4, 5, 6, 2, 4,
  5, 3, 4, 1, 3, 2, 4, 5, 3, 4, 1, 3, 2, 4, 1, 3, 2, 1]
```

set_default_long_word(word)

Set the default long word used to express elements of self.

EXAMPLES:

```
sage: B = crystals.infinity.PBW(['C', 3])
sage: B.default_long_word()
[1, 3, 2, 3, 1, 2, 3, 1, 2]
sage: x = B.highest_weight_vector().f_string([2,1,3,2,1,3,2,2,1,3,2,2])
sage: x
PBW monomial with Lusztig datum (1, 2, 2, 0, 0, 0, 0, 1)
sage: B.set_default_long_word([2,1,3,2,1,3,2,3,1])
sage: B.default_long_word()
[2, 1, 3, 2, 1, 3, 2, 3, 1]
sage: x
PBW monomial with Lusztig datum (2, 0, 0, 0, 0, 1, 3, 2)
```

class sage.combinat.crystals.pbw_crystal.PBWCrystalElement(parent, lusztig_datum, long_word=None)

A crystal element in the PBW model.

```
e(i)

Return the action of $e_i$ on self.

EXAMPLES:

```
sage: B = crystals.infinity.PBW(['B', 3])
sage: b = B.highest_weight_vector()
sage: c = b.f_string([2,1,3,2,1,3,2,2,1,3,2,2]); c
PBW monomial with Lusztig datum (0, 1, 0, 1, 0, 0, 1, 2)
sage: c.e(2)
PBW monomial with Lusztig datum (0, 1, 0, 1, 0, 0, 0, 1, 1)
sage: c.e_string([2,2,1,3,2,1,3,2,2,1,3,2]) == b.e(2)
True
```

epsilon(i)

Return $\varepsilon_i$ of self.

EXAMPLES:

```
sage: B = crystals.infinity.PBW(['A2'])
sage: s = B((3,0,0), (1,2,1))
sage: s.epsilon(1)
3
```

(continues on next page)
sage: s.epsilon(2)
0

**f(i)**

Return the action of \( f_i \) on \( \text{self} \).

**EXAMPLES:**

```
sage: B = crystals.infinity.PBW("D4")
sage: b = B.highest_weight_vector()
sage: c = b.f_string([1,2,3,1,2,3,4]); c
PBW monomial with Lusztig datum (0, 1, 0, 0, 0, 0, 0, 2, 0, 0, 2, 0)
sage: c == b.f_string([1,2,4,1,2,3,3])
True
```

**lusztig_datum**(\( \text{word} = \text{None} \))

Return the Lusztig datum of \( \text{self} \) with respect to the reduced expression of the long word \( \text{word} \).

**EXAMPLES:**

```
sage: B = crystals.infinity.PBW(['A', 2])
sage: u = B.highest_weight_vector()
sage: b = u.f_string([2,1,2,2,1,1,2,1,2,1,2,2])
sage: b.lusztig_datum()
(6, 0, 10)
sage: b.lusztig_datum(word=[2,1,2])
(4, 6, 0)
```

**phi(i)**

Return \( \varphi_i \) of \( \text{self} \).

**EXAMPLES:**

```
sage: B = crystals.infinity.PBW(['A', 2])
sage: s = B((3,0,0), (1,2,1))
sage: s.phi(1)
-3
sage: s.phi(2)
3
```

**star()**

Return the starred crystal element corresponding to \( \text{self} \).

Let \( b \) be an element of \( \text{self} \) with Lusztig datum \( (b_1, \ldots, b_N) \) with respect to \( w_0 = s_{i_1} \cdots s_{i_N} \). Then \( b^* \) is the element with Lusztig datum \( (b_N, \ldots, b_1) \) with respect to \( w_0 = s_{i_N}^* \cdots s_{i_1}^* \), where \( i_j^* = \omega(i_j) \) with \( \omega \) being the automorphism given by the action of \( w_0 \) on the simple roots.

**EXAMPLES:**

```
sage: P = crystals.infinity.PBW(['A', 2])
sage: P((1,2,3), (1,2,1)).star() == P((3,2,1), (2,1,2))
True
sage: B = crystals.infinity.PBW(['E', 6])
```

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Combinatorics, Release 10.0

```python
sage: b = B.highest_weight_vector()
sage: c = b.f_string([1,2,6,3,4,2,5,2,3,4,1,6])
sage: c == c.star().star()
True
```

**weight()**

Return weight of self.

**EXAMPLES:**

```python
sage: B = crystals.infinity.PBW(['A', 2])
sage: s = B((2,2,2), (1,2,1))
sage: s.weight()
(-4, 0, 4)
```

### 5.1.63 PBW Data

This contains helper classes and functions which encode PBW data in finite type.

**AUTHORS:**

- Dinakar Muthiah (2015-05): initial version
- Travis Scrimshaw (2016-06): simplified code and converted to Cython

**class** `sage.combinat.crystals.pbw_datum.PBWData(cartan_type)`

Bases: object

Helper class for the set of PBW data.

**convert_to_new_long_word(pbw_datum, new_long_word)**

Convert the PBW datum `pbw_datum` from its long word to `new_long_word`.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.pbw_datum import PBWData, PBWDatum
sage: P = PBWData("A2")
sage: datum = PBWDatum(P, (1,2,1), (1,0,1))
sage: new_d datum = P.convert_to_new_long_word(datum,(2,1,2))
sage: new_d datum
PBW Datum element of type ['A', 2] with long word (2, 1, 2) and Lusztig datum (0, 1, 0)
sage: new_d datum.long_word
(2, 1, 2)
sage: new_d datum.lusztig_datum
(0, 1, 0)
```

**class** `sage.combinat.crystals.pbw_datum.PBWDatum(parent, long_word, lusztig_datum)`

Bases: object

Helper class which represents a PBW datum.

**convert_to_long_word_with_first_letter(i)**

Return a new PBWDatum equivalent to `self` whose long word begins with `i`.

**EXAMPLES:**
```python
sage: from sage.combinat.crystals.pbw_datum import PBWData, PBWDatum
sage: P = PBWData("A3")
sage: datum = PBWDatum(P, (1,2,1,3,2,1), (1,0,1,4,2,3))
sage: datum.convert_to_long_word_with_first_letter(1)
PBW Datum element of type ['A', 3] with long word (1, 2, 3, 1, 2, 1)
and Lusztig datum (1, 0, 4, 1, 2, 3)
sage: datum.convert_to_long_word_with_first_letter(2)
PBW Datum element of type ['A', 3] with long word (2, 1, 2, 3, 2, 1)
and Lusztig datum (0, 1, 0, 4, 2, 3)
sage: datum.convert_to_long_word_with_first_letter(3)
PBW Datum element of type ['A', 3] with long word (3, 1, 2, 3, 1, 2)
and Lusztig datum (8, 1, 0, 4, 1, 2)
```

**convert_to_new_long_word(new_long_word)**

Return a new PBWDatum equivalent to self whose long word is new_long_word.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.pbw_datum import PBWData, PBWDatum
sage: P = PBWData("A2")
sage: datum = PBWDatum(P, (1,2,1), (1,0,1))
sage: new_datum = datum.convert_to_new_long_word((2,1,2))
sage: new_datum.long_word
(2, 1, 2)
sage: new_datum.lusztig_datum
(0, 1, 0)
```

**is_equivalent_to(other_pbw_datum)**

Return whether self is equivalent to other_pbw_datum, modulo the tropical Plücker relations.

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.pbw_datum import PBWData, PBWDatum
sage: P = PBWData("A2")
sage: L1 = PBWDatum(P, (1,2,1), (1,2,3))
sage: L2 = PBWDatum(P, (2,1,2), (3,2,1))
sage: L1.is_equivalent_to(L2)
True
sage: L1 == L2
False
```

**star()**

Return the starred version of self, i.e., with reversed long_word and lusztig_datum

**EXAMPLES:**

```python
sage: from sage.combinat.crystals.pbw_datum import PBWData, PBWDatum
sage: P = PBWData("A2")
sage: L1 = PBWDatum(P, (1,2,1), (1,2,3))
sage: L1.star() == PBWDatum(P, (2,1,2), (3,2,1))
True
```

**weight()**

Return the weight of self.
EXAMPLES:

```python
sage: from sage.combinat.crystals.pbw_datum import PBWData, PBWDatum
sage: P = PBWData("A2")
```

```python
sage: L = PBWDatum(P, (1,2,1), (1,1,1))
```

```python
sage: L.weight()
```

```
```

```python
sage.combinat.crystals.pbw_datum.compute_new_lusztig_datum(enhanced_braid_chain, initial_lusztig_datum)
```

Return the Lusztig datum obtained by applying tropical Plücker relations along `enhanced_braid_chain` starting with `initial_lusztig_datum`.

EXAMPLES:

```python
sage: from sage.combinat.root_system.braid_move_calculator import BraidMoveCalculator
sage: from sage.combinat.crystals.pbw_datum import enhance_braid_move_chain
sage: from sage.combinat.crystals.pbw_datum import compute_new_lusztig_datum
```

```python
sage: ct = CartanType(['A', 2])
```

```python
sage: W = CoxeterGroup(ct)
```

```python
sage: B = BraidMoveCalculator(W)
```

```python
sage: chain = B.chain_of_reduced_words((1,2,1),(2,1,2))
```

```python
sage: enhanced_braid_chain = enhance_braid_move_chain(chain, ct)
```

```python
sage: compute_new_lusztig_datum(enhanced_braid_chain,(1,0,1))
```

```python
(0, 1, 0)
```

```python
sage.combinat.crystals.pbw_datum.enhance_braid_move_chain(braid_move_chain, cartan_type)
```

Return a list of tuples that records the data of the long words in `braid_move_chain` plus the data of the intervals where the braid moves occur and the data of the off-diagonal entries of the $2 \times 2$ Cartan submatrices of each braid move.

**INPUT:**

- `braid_move_chain` – a chain of reduced words in the Weyl group of `cartan_type`
- `cartan_type` – a finite Cartan type

**OUTPUT:**

A list of 2-tuples `(interval_of_change, cartan_sub_matrix)` where

- `interval_of_change` is the (half-open) interval of indices where the braid move occurs; this is `None` for the first tuple
- `cartan_sub_matrix` is the off-diagonal entries of the $2 \times 2$ submatrix of the Cartan matrix corresponding to the braid move; this is `None` for the first tuple

For a matrix:

```
[2 a]
[b 2]
```

the `cartan_sub_matrix` is the pair `(a, b)`.

```python
sage.combinat.crystals.pbw_datum.tropical_plucker_relation(a, lusztig_datum)
```

Apply the tropical Plücker relation of type `a` to `lusztig_datum`.

The relations are obtained by tropicalizing the relations in Proposition 7.1 of [BZ01].
**INPUT:**

- a – a pair \((x, y)\) of the off-diagonal entries of a \(2 \times 2\) Cartan matrix

**EXAMPLES:**

```
sage: from sage.combinat.crystals.pbw_datum import tropical_plucker_relation
sage: tropical_plucker_relation((0,0), (2,3))
(3, 2)
sage: tropical_plucker_relation((-1,-1), (1,2,3))
(4, 1, 2)
sage: tropical_plucker_relation((-1,-2), (1,2,3,4))
(8, 1, 2, 3)
sage: tropical_plucker_relation((-2,-1), (1,2,3,4))
(6, 1, 2, 3)
```

### 5.1.64 Polyhedral Realization of \(B(\infty)\)

**class** `sage.combinat.crystals.polyhedral_realization.InfinityCrystalAsPolyhedralRealization(cartan_type, seq)`

**Bases:** `TensorProductOfCrystals`

The polyhedral realization of \(B(\infty)\).

**Note:** Here we are using anti-Kashiwara notation and might differ from some of the literature.

Consider a Kac-Moody algebra \(g\) of Cartan type \(X\) with index set \(I\), and consider a finite sequence \(J = (j_1, j_2, \ldots, j_m)\) whose support equals \(I\). We extend this to an infinite sequence by taking \(\bar{J} = J \cdot J \cdot J \cdots\), where \(\cdot\) denotes concatenation of sequences. Let

\[ B_J = B_{j_m} \otimes \cdots \otimes B_{j_2} \otimes B_{j_1}, \]

where \(B_i\) is an `ElementaryCrystal`.

As given in Theorem 2.1.1 of [Ka1993], there exists a strict crystal embedding \(\Psi_i : B(\infty) \rightarrow B_i \otimes B(\infty)\) defined by \(u_\infty \mapsto b_i(0) \otimes u_\infty\), where \(b_i(0) \in B_i\) and \(u_\infty\) is the (unique) highest weight element in \(B(\infty)\). This is sometimes known as the Kashiwara embedding [NZ1997] (though, in [NZ1997], the target of this map is denoted by \(Z_\gamma\)). By iterating this embedding by taking \(\Psi_J = \Psi_{j_n} \circ \Psi_{j_{n-1}} \circ \cdots \circ \Psi_{j_1}\), we obtain the following strict crystal embedding:

\[ \Psi_J : B(\infty) \rightarrow B_J^\otimes n \otimes B(\infty). \]

We note there is a natural analog of Lemma 10.6.2 in [HK2002] that for any \(b \in B(\infty)\), there exists a positive integer \(N\) such that

\[ \Psi_J^N(b) = \left( \bigotimes_{k=1}^{N} b^{(k)} \right) \otimes u_\infty. \]

Therefore we can model elements \(b \in B(\infty)\) by considering an infinite list of elements \(b^{(k)} \in B_J\) and defining...
the crystal structure by:

\[ \text{wt}(b) = \sum_{k=1}^{N} \text{wt}(b^{(k)}) \]

\[ e_i(b) = e_i \left( \bigotimes_{k=1}^{N} b^{(k)} \right) \otimes u_\infty, \]

\[ f_i(b) = f_i \left( \bigotimes_{k=1}^{N} b^{(k)} \right) \otimes u_\infty, \]

\[ \varepsilon_i(b) = \max_{k, e_i^*(b) \neq 0} k, \]

\[ \varphi_i(b) = \varepsilon_i(b) - \langle \text{wt}(b), h_i \rangle. \]

To translate this into a finite list, we consider a finite sequence \( b_1 \otimes \cdots \otimes b_N \) and if

\[ f_i \left( b^{(1)} \otimes \cdots \otimes b^{(N-1)} \otimes b^{(N)} \right) = b^{(1)} \otimes \cdots \otimes b^{(N-1)} \otimes f_i \left( b^{(N)} \right), \]

then we take the image as \( b^{(1)} \otimes \cdots \otimes f_i \left( b^{(N)} \right) \otimes b^{(N+1)} \). Similarly we remove \( b^{(N)} \) if we have \( b^{(N)} = \bigotimes_{k=1}^{m} b_{jk}(0) \). Additionally if

\[ e_i \left( b^{(1)} \otimes \cdots \otimes b^{(N-1)} \otimes b^{(N)} \right) = b^{(1)} \otimes \cdots \otimes b^{(N-1)} \otimes e_i \left( b^{(N)} \right), \]

then we consider this to be 0.

**INPUT:**

- `cartan_type` – a Cartan type
- `seq` – (default: None) a finite sequence whose support equals the index set of the Cartan type; if None, then this is the index set

**EXAMPLES:**

```sage
sage: B = crystals.infinity.PolyhedralRealization(['A',2])
sage: mg = B.module_generators[0]; mg
[0, 0]
sage: mg.f_string([2,1,2,2])
[0, -3, -1, 0, 0, 0]
```

An example of type \( B_2 \):

```sage
sage: B = crystals.infinity.PolyhedralRealization(['B',2])
sage: mg = B.module_generators[0]; mg
[0, 0]
sage: mg.f_string([2,1,2,2])
[0, -2, -1, -1, 0, 0]
```

An example of type \( G_2 \):

```sage
sage: B = crystals.infinity.PolyhedralRealization(['G',2])
sage: mg = B.module_generators[0]; mg
[0, 0]
sage: mg.f_string([2,1,2,2])
[0, -3, -1, 0, 0, 0]
```
class Element

    Bases: TensorProductOfCrystalsElement

    An element in the polyhedral realization of $B(\infty)$.

    e(i)

    Return the action of $\varepsilon_i$ on self.

    EXAMPLES:

    sage: B = crystals.infinity.PolyhedralRealization(['A',2])
sage: mg = B.module_generators[0]
sage: all(mg.e(i) is None for i in B.index_set())
    True
sage: mg.f(1).e(1) == mg
    True

epsilon(i)

    Return $\varepsilon_i$ of self.

    EXAMPLES:

    sage: B = crystals.infinity.PolyhedralRealization(['A',2,1])
sage: mg = B.module_generators[0]
sage: [mg.epsilon(i) for i in B.index_set()]
    [0, 0, 0]
sage: elt = mg.f(0)
sage: [elt.epsilon(i) for i in B.index_set()]
    [1, 0, 0]
sage: elt = mg.f_string([0,1,2])
sage: [elt.epsilon(i) for i in B.index_set()]
    [0, 0, 1]
sage: elt = mg.f_string([0,1,2,2])
sage: [elt.epsilon(i) for i in B.index_set()]
    [0, 0, 2]

f(i)

    Return the action of $f_i$ on self.

    EXAMPLES:

    sage: B = crystals.infinity.PolyhedralRealization(['A',2])
sage: mg = B.module_generators[0]
sage: mg.f(1)
    [-1, 0, 0, 0]
sage: mg.f_string([1,2,2,1])
    [-1, -2, -1, 0, 0, 0]

phi(i)

    Return $\varphi_i$ of self.

    EXAMPLES:

    sage: B = crystals.infinity.PolyhedralRealization(['A',2,1])
sage: mg = B.module_generators[0]
sage: [mg.phi(i) for i in B.index_set()]
sage: elt = mg.f(0)
sage: [elt.phi(i) for i in B.index_set()]
[-1, 1, 1]

sage: elt = mg.f_string([0,1])
sage: [elt.phi(i) for i in B.index_set()]
[-1, 0, 2]

sage: elt = mg.f_string([0,1,2,2])
sage: [elt.phi(i) for i in B.index_set()]
[1, 1, 0]

\textbf{truncate}(k=None)

Truncate \texttt{self} to have length \(k\) and return as an element in a (finite) tensor product of crystals.

**INPUT:**

- \(k\) – (optional) the length of the truncation; if not specified, then returns one more than the current non-ground-state elements (i.e. the current list in \texttt{self})

**EXAMPLES:**

sage: B = crystals.infinity.PolyhedralRealization(['A',2])
sage: mg = B.module_generators[0]
sage: elt = mg.f_string([1,2,2,1]); elt
[-1, -2, -1, 0, 0, 0]

sage: t = elt.truncate(); t
[-1, -2, -1, 0, 0, 0]

sage: t.parent() is B.finite_tensor_product(6)
True

sage: elt.truncate(2)
[-1, -2]

sage: elt.truncate(10)
[-1, -2, -1, 0, 0, 0, 0, 0, 0, 0]

\textbf{finite_tensor_product}(k)

Return the finite tensor product of crystals of length \(k\) by truncating \texttt{self}.

**EXAMPLES:**

\begin{verbatim}
sage: B = crystals.infinity.PolyhedralRealization(['A',2])
sage: B.finite_tensor_product(5)
[The 1-elementary crystal of type ['A', 2],
 The 2-elementary crystal of type ['A', 2],
 The 1-elementary crystal of type ['A', 2],
 The 2-elementary crystal of type ['A', 2],
 The 1-elementary crystal of type ['A', 2]]
\end{verbatim}
5.1.65 Spin Crystals

These are the crystals associated with the three spin representations: the spin representations of odd orthogonal groups (or rather their double covers); and the + and − spin representations of the even orthogonal groups.

We follow Kashiwara and Nakashima (Journal of Algebra 165, 1994) in representing the elements of the spin crystal by sequences of signs ±.

sage.combinat.crystals.spins.CrystalOfSpins(\text{ct})

Return the spin crystal of the given type \( B \).

This is a combinatorial model for the crystal with highest weight \( \Lambda_n \) (the \( n \)-th fundamental weight). It has \( 2^n \) elements, here called Spins. See also \text{CrystalOfLetters()}\text{, CrystalOfSpinsPlus()}, and \text{CrystalOfSpinsMinus()}.

INPUT:

• ['B', n] - A Cartan type \( B_n \).

EXAMPLES:

```sage
sage: C = crystals.Spins(['B',3])
sage: C.list()
[+++ , ++- , +-- , -++ , --+ , -+- , --- ]
sage: C.cartan_type()
['B', 3]
```

```sage
sage: [x.signature() for x in C]
['+++', '+--', '+-+', '-++', '+--', '-+-', '---']
```

sage.combinat.crystals.spins.CrystalOfSpinsMinus(\text{ct})

Return the minus spin crystal of the given type \( D \).

This is the crystal with highest weight \( \Lambda_{n-1} \) (the \( (n-1) \)-st fundamental weight).

INPUT:

• ['D', n] - A Cartan type \( D_n \).

EXAMPLES:

```sage
sage: E = crystals.SpinsMinus(['D',4])
sage: E.list()
[+++ , +++ , +++ , +++ , --+ , --- , ---- , ---- ]
sage: [x.signature() for x in E]
['+++-', '+++)', '+++)', '+++)', '+---', '---', '----', '----']
```

sage.combinat.crystals.spins.CrystalOfSpinsPlus(\text{ct})

Return the plus spin crystal of the given type \( D \).

This is the crystal with highest weight \( \Lambda_n \) (the \( n \)-th fundamental weight).

INPUT:

• ['D', n] - A Cartan type \( D_n \).

EXAMPLES:
sage: D = crystals.SpinsPlus(['D',4])
sage: D.list()
[++++, +++-, ++--, +-+-, -++-, +++-, --++, ----]
sage: [x.signature() for x in D]
['++++', '++--', '+-+-', '-++-', +++-, '+--+', '-+-+', '--++', '----']

class sage.combinat.crystals.spins.GenericCrystalOfSpins(ct, element_class, case)
Bases: UniqueRepresentation, Parent
A generic crystal of spins.

lt_elements(x, y)
Return True if and only if there is a path from x to y in the crystal graph.
Because the crystal graph is classical, it is a directed acyclic graph which can be interpreted as a poset. This function implements the comparison function of this poset.

EXAMPLES:

sage: C = crystals.Spins(['B',3])
sage: x = C([1,1,1])
sage: y = C([-1,-1,-1])
sage: C.lt_elements(x, y)
True
sage: C.lt_elements(y, x)
False
sage: C.lt_elements(x, x)
False

class sage.combinat.crystals.spins.Spin
Bases: Element
A spin letter in the crystal of spins.

EXAMPLES:

sage: C = crystals.Spins(['B',3])
sage: c = C([1,1,1])
sage: c
+++
sage: c.parent()
The crystal of spins for type ['B', 3]
sage: D = crystals.Spins(['B',4])
sage: a = D([1,1,1])
sage: b = D([-1,-1,-1])
sage: c = D([1,1,1,1])
sage: a == a
True
sage: a == b
False
sage: b == c
False
pp()  
Pretty print self as a column.

EXAMPLES:

```sage
sage: C = crystals.Spins(['B',3])
sage: b = C([1,1,-1])
sage: b.pp() + + -
```

signature()  
Return the signature of self.

EXAMPLES:

```sage
sage: C = crystals.Spins(['B',3])
sage: C([1,1,1]).signature() '+++'
sage: C([1,1,-1]).signature() '++-' 
```

value  
Return self as a tuple with +1 and -1.

EXAMPLES:

```sage
sage: C = crystals.Spins(['B',3])
sage: C([1,1,1]).value (1, 1, 1)
sage: C([1,1,-1]).value (1, 1, -1)
```

weight()  
Return the weight of self.

EXAMPLES:

```sage
[v.weight() for v in crystals.Spins(['B',3])]
[(1/2, 1/2, 1/2), (1/2, 1/2, -1/2), (1/2, -1/2, 1/2), (-1/2, 1/2, 1/2),
 (1/2, -1/2, -1/2), (-1/2, 1/2, -1/2),
 (-1/2, -1/2, 1/2), (-1/2, -1/2, -1/2)]
```

class sage.combinat.crystals.spins.Spin_crystal_type_B_element

Bases: Spin

Type B spin representation crystal element

e(i)  
Return the action of $e_i$ on self.

EXAMPLES:
sage: C = crystals.Spins(['B',3])
sage: [[C[m].e(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None]]

epsilon(i)
Return \( \varepsilon_i \) of self.
EXAMPLES:

sage: C = crystals.Spins(['B',3])
sage: [[C[m].epsilon(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None]]

f(i)
Return the action of \( f_i \) on self.
EXAMPLES:

sage: C = crystals.Spins(['B',3])
sage: [[C[m].f(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None]]

phi(i)
Return \( \varphi_i \) of self.
EXAMPLES:

sage: C = crystals.Spins(['B',3])
sage: [[C[m].phi(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None]]

class sage.combinat.crystals.spins Spin_crystal_type_D_element
Bases: Spin

Type D spin representation crystal element
e(i)
Return the action of \( e_i \) on self.
EXAMPLES:

sage: D = crystals.SpinsPlus(['D',4])
sage: [[D.list()[m].e(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None]]

sage: E = crystals.SpinsMinus(['D',4])
sage: [[E[m].e(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None], [None, None, None]]
**epsilon**\((i)\)

Return \(\epsilon_i\) of self.

**EXAMPLES:**

```
sage: C = crystals.SpinsMinus(['D',4])
sage: [[C[m].epsilon(i) for i in C.index_set()] for m in range(8)]
[[0, 0, 0, 0], [0, 0, 1, 0], [0, 1, 0, 0], [1, 0, 0, 0],
 [0, 0, 1, 0], [1, 0, 0, 1], [0, 1, 0, 0], [0, 0, 0, 1]]
```

**f(i)**

Return the action of \(f_i\) on self.

**EXAMPLES:**

```
sage: D = crystals.SpinsPlus(['D',4])
sage: [[D.list()[m].f(i) for i in range(1,4)] for m in range(8)]
[[None, None, None], [None, +-+-, None], [-+++, None, None], [None, None, None],
 [-+-+, None, None], [None, --++, None], [None, None, None], [None, None, None]]
```

**phi(i)**

Return \(\varphi_i\) of self.

**EXAMPLES:**

```
sage: C = crystals.SpinsPlus(['D',4])
sage: [[C[m].phi(i) for i in C.index_set()] for m in range(8)]
[[0, 0, 0, 1], [0, 1, 0, 0], [1, 0, 1, 0], [0, 0, 1, 0],
 [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 0, 1], [0, 0, 0, 0]]
```

### 5.1.66 Star-Crystal Structure On \(B(\infty)\)

**AUTHORS:**

- Ben Salisbury: Initial version
- Travis Scrimshaw: Initial version

**class** `sage.combinat.crystals.star_crystal.StarCrystal(Binf)`

Bases: `UniqueRepresentation, Parent`

The star-crystal or \(*\)-crystal version of a highest weight crystal.

The \(*\)-crystal structure on \(B(\infty)\) is the structure induced by the algebra antiautomorphism \(*: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})\) that stabilizes the negative half \(U_q^{-}(\mathfrak{g})\). It is defined by

\[
E_i^* = E_i, \quad F_i^* = F_i, \quad q^* = q, \quad (q^h)^* = q^{-h},
\]

where \(E_i\) and \(F_i\) are the Chevalley generators of \(U_q(\mathfrak{g})\) and \(h\) is an element of the Cartan subalgebra.
The induced operation on the crystal $B(\infty)$ is called the Kashiwara involution. Its implementation here is based on the recursive algorithm from Theorem 2.2.1 of [Ka1993], which states that for any $i \in I$ there is a unique strict crystal embedding

$$\Psi_i: B(\infty) \rightarrow B_i \otimes B(\infty)$$

such that

- $u_{\infty} \mapsto b_i(0) \otimes u_{\infty}$, where $u_{\infty}$ is the highest weight vector in $B(\infty)$;
- if $\Psi_i(b) = f^m_i b_i(0) \otimes b_0$, then $\Psi_i(f^*_i b) = f^{m+1}_i b_i(0) \otimes b_0$ and $\varepsilon_i(b^*) = m$;
- the image of $\Psi_i$ is $\{ f^m_i b_i(0) \otimes b : \varepsilon_i(b^*) = 0, m \geq 0 \}$.

Here, $B_i$ is the $i$-th elementary crystal. See ElementaryCrystal for more information.

**INPUT:**

- $\text{Binf}$ – a crystal from catalog_infinity_crystals

**EXAMPLES:**

```python
sage: B = crystals.infinity.Tableaux(['A', 2])
sage: Bstar = crystals.infinity.Star(B)
sage: mg = Bstar.highest_weight_vector()
sage: mg
[[1, 1], [2]]
sage: mg.f_string([1,2,1,2,2])
[[1, 1, 1, 1, 1, 2, 2], [2, 3, 3, 3]]
```

class Element

Bases: ElementWrapper

e(i)

Return the action of $e_i^*$ on self.

**INPUT:**

- $i$ – an element of the index set

**EXAMPLES:**

```python
sage: RC = crystals.infinity.RiggedConfigurations(['E', 6, 1])
sage: RCstar = crystals.infinity.Star(RC)
sage: nuJ = RCstar.module_generators[0].f_string([0,4,6,1,2])
sage: ascii_art(nuJ.e(1))
-1[ ]-1 ( ) 0[ ]1 ( ) -1[ ]-1 ( ) -2[ ]-1
sage: M = crystals.infinity.NakajimaMonomials(['B', 2, 1])
sage: Mstar = crystals.infinity.Star(M)
sage: m = Mstar.module_generators[0].f_string([0,1,2,2,1,0])
sage: m.e(1)
Y(0,0)^-1 Y(0,2)^-1 Y(1,1) Y(1,2)^-1 Y(2,1)^2
```

epsilon(i)

Return $\varepsilon_i^*$ of self.

**INPUT:**

- $i$ – an element of the index set

**EXAMPLES:**
```python
sage: Y = crystals.infinity.GeneralizedYoungWalls(3)
sage: Ystar = crystals.infinity.Star(Y)
sage: y = Ystar.module_generators[0].f_string([0,1,3,2,1,0])
sage: [y.epsilon(i) for i in y.index_set()]
[1, 0, 1, 0]
sage: RC = crystals.infinity.RiggedConfigurations(['E',6,1])
sage: RCstar = crystals.infinity.Star(RC)
sage: nuJ = RCstar.module_generators[0].f_string([0,4,6,1,2])
sage: [nuJ.epsilon(i) for i in nuJ.index_set()]
[0, 1, 1, 0, 0, 0, 1]
```

**f(i)**

Return the action of $f_i^*$ on `self`.

**INPUT:**
- `i` – an element of the index set

**EXAMPLES:**

```python
sage: T = crystals.infinity.Tableaux("G2")
sage: Tstar = crystals.infinity.Star(T)
sage: t = Tstar.module_generators[0].f_string([1,2,1,1,2])
sage: t
[[1, 1, 1, 2, 0], [2, 3]]
sage: M = crystals.infinity.NakajimaMonomials(['B',2,1])
sage: Mstar = crystals.infinity.Star(M)
sage: m = Mstar.module_generators[0].f_string([0,1,2,2,1,0])
sage: m
Y(0,0)^-1 Y(0,2)^-1 Y(1,0)^-1 Y(1,2)^-1 Y(2,0)^2 Y(2,1)^2
```

**jump(i)**

Return the $i$-jump of `self`.

For $b \in B(\infty)$,

$$\text{jump}_i(b) = \varepsilon_i(b) + \varepsilon_i^*(b) + \langle h_i, \text{wt}(b) \rangle,$$

where $h_i$ is a simple coroot.

**INPUT:**
- `i` – an element of the index set

**EXAMPLES:**

```python
sage: RC = crystals.infinity.RiggedConfigurations("D4")
sage: RCstar = crystals.infinity.Star(RC)
sage: nu0star = RCstar.module_generators[0]
sage: nustar = nu0star.f_string([2,1,3,4,2])
sage: [nustar.jump(i) for i in RC.index_set()]
[0, 1, 0, 0]
sage: nustar = nu0star.f_string([2,1,3,4,2,2,1,3,2])  # long time
sage: [nustar.jump(i) for i in RC.index_set()]  # long time
[1, 0, 1, 2]
```
\textbf{phi}(i)

Return $\varphi_i^*$ of self.

For $b \in B(\infty)$,

$$\varphi_i^*(b) = \varepsilon_i^*(b) + \langle h_i, \text{wt}(b) \rangle,$$

where $h_i$ is a simple coroot.

INPUT:

\begin{itemize}
  \item $i$ – an element of the index set
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: T = crystals.infinity.Tableaux("A2")
sage: Tstar = crystals.infinity.Star(T)
sage: t = Tstar.module_generators[0].f_string([1,2,1,1,2])
sage: [t.phi(i) for i in t.index_set()] [-3, 1]
sage: M = crystals.infinity.NakajimaMonomials(['B',2,1])
sage: Mstar = crystals.infinity.Star(M)
sage: m = Mstar.module_generators[0].f_string([0,1,2,2,1,0])
sage: [m.phi(i) for i in m.index_set()] [-1, -1, 4]
\end{verbatim}

\textbf{weight()}

Return the weight of self.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: RC = crystals.infinity.RiggedConfigurations(['E',6,1])
sage: RCstar = crystals.infinity.Star(RC)
sage: nuJ = RCstar.module_generators[0].f_string([0,4,6,1,2])
sage: nuJ.weight()
\end{verbatim}

\section{5.1.67 Tensor Products of Crystals}

Main entry points:

\begin{itemize}
  \item \texttt{TensorProductOfCrystals}
  \item \texttt{CrystalOfTableaux}
\end{itemize}

\textbf{AUTHORS:}

\begin{itemize}
  \item Anne Schilling, Nicolas Thiery (2007): Initial version
  \item Ben Salisbury, Travis Scrimshaw (2013): Refactored tensor products to handle non-regular crystals and created new subclass to take advantage of the regularity
  \item Travis Scrimshaw (2020): Added queer crystal
\end{itemize}

\textbf{class} \texttt{sage.combinat.crystals.tensor_product.CrystalOfQueerTableaux(cartan_type, shape)}

\begin{verbatim}
Bases: CrystalOfWords, QueerSuperCrystalsMixin
\end{verbatim}

A queer crystal of the semistandard decomposition tableaux of a given shape.
INPUT:

- `cartan_type` – a Cartan type
- `shape` – a shape

**class Element**

Bases: `TensorProductOfQueerSuperCrystalsElement`

`rows()`

Return the list of rows of `self`.

**EXAMPLES:**

```python
sage: B = crystals.Tableaux(['Q',3], shape=[3,2,1])
sage: t = B.an_element()
sage: t.rows()
[[3, 3, 3], [2, 2], [1]]
```

**class sage.combinat.crystals.tensor_product.CrystalOfTableaux**(`cartan_type, shapes`)

Bases: `CrystalOfWords`

A class for crystals of tableaux with integer valued shapes

**INPUT:**

- `cartan_type` – a Cartan type
- `shape` – a partition of length at most `cartan_type.rank()`
- `shapes` – a list of such partitions

This constructs a classical crystal with the given Cartan type and highest weight(s) corresponding to the given shape(s).

If the type is $D_r$, the shape is permitted to have a negative value in the $r$-th position. Thus if the shape equals $[s_1, \ldots, s_r]$, then $s_r$ may be negative but in any case $s_1 \geq \cdots \geq s_{r-1} \geq |s_r|$. This crystal is related to that of shape $[s_1, \ldots, |s_r|]$ by the outer automorphism of $SO(2r)$.

If the type is $D_r$ or $B_r$, the shape is permitted to be of length $r$ with all parts of half integer value. This corresponds to having one spin column at the beginning of the tableau. If several shapes are provided, they currently should all or none have this property.

Crystals of tableaux are constructed using an embedding into tensor products following Kashiwara and Nakashima [KN1994]. Sage’s tensor product rule for crystals differs from that of Kashiwara and Nakashima by reversing the order of the tensor factors. Sage produces the same crystals of tableaux as Kashiwara and Nakashima. With Sage’s convention, the tensor product of crystals is the same as the monoid operation on tableaux and hence the plactic monoid.

**See also:**

`sage.combinat.crystals.crystals` for general help on crystals, and in particular plotting and TeX output.

**EXAMPLES:**

We create the crystal of tableaux for type $A_2$, with highest weight given by the partition $[2,1,1]$:

```python
sage: T = crystals.Tableaux(['A',3], shape = [2,1,1])
```

Here is the list of its elements:
Internally, a tableau of a given Cartan type is represented as a tensor product of letters of the same type. The order in which the tensor factors appear is by reading the columns of the tableaux left to right, top to bottom (in French notation). As an example:

```python
sage: T = crystals.Tableaux(['A',2], shape = [3,2])
sage: T.module_generators[0]
[[1, 1, 1], [2, 2]]
sage: list(T.module_generators[0])
[2, 1, 2, 1, 1]
```

To create a tableau, one can use:

```python
sage: Tab = crystals.Tableaux(['A',3], shape = [2,2])
sage: Tab(rows=[[1,2],[3,4]])
[[1, 2], [3, 4]]
sage: Tab(columns=[[3,1],[4,2]])
[[1, 2], [3, 4]]
```

Todo: FIXME:
- Do we want to specify the columns increasingly or decreasingly? That is, should this be `Tab(columns = [[1,3],[2,4]])`?
- Make this fully consistent with `Tableau()`!

We illustrate the use of a shape with a negative last entry in type $D$:

```python
sage: T = crystals.Tableaux(['D',4],shape=[1,1,1,-1])
sage: T.cardinality() 35
sage: TestSuite(T).run()
```

We illustrate the construction of crystals of spin tableaux when the partitions have half integer values in type $B$ and $D$:

```python
sage: T = crystals.Tableaux(['B',3],shape=[3/2,1/2,1/2]); T
The crystal of tableaux of type ['B', 3] and shape(s) [[3/2, 1/2, 1/2]]
sage: T.cardinality() 48
sage: T.module_generators
([++, [[1]]],)
sage: TestSuite(T).run()
```

```python
sage: T = crystals.Tableaux(['D',3],shape=[3/2,1/2,-1/2]); T
The crystal of tableaux of type ['D', 3] and shape(s) [[3/2, 1/2, -1/2]]
sage: T.cardinality() 48
sage: T.module_generators
([++, [[1]]],)
sage: TestSuite(T).run()
```

(continues on next page)
We can also construct the tableaux for \( \mathfrak{gl}(m|n) \) as given by [BKK2000]:

```python
sage: T = crystals.Tableaux(['A', [1,2]], shape=[4,2,1,1,1])
sage: T.cardinality()
1392
```

We can also construct the tableaux for \( \mathfrak{q}(n) \) as given by [GJK+2014]:

```python
sage: T = crystals.Tableaux(['Q', 3], shape=[3,1])
sage: T.cardinality()
24
```

### class Element

**Bases:** *CrystalOfTableauxElement*

#### cartan_type()

Returns the Cartan type of the associated crystal

**EXAMPLES:**

```python
sage: T = crystals.Tableaux(['A', 3], shape = [2,2])
sage: T.cartan_type()
['A', 3]
```

#### module_generator(shape)

This yields the module generator (or highest weight element) of a classical crystal of given shape. The module generator is the unique tableau with equal shape and content.

**EXAMPLES:**

```python
sage: T = crystals.Tableaux(['D',3], shape = [1,1])
sage: T.module_generator([1,1])
[[1], [2]]
```

```python
sage: T = crystals.Tableaux(['D',4],shape=[2,2,2,-2])
sage: T.module_generator(tuple([2,2,2,-2]))
[[1, 1], [2, 2], [3, 3], [-4, -4]]
sage: T.cardinality()
294
```

#### class sage.combinat.crystals.tensor_product.CrystalOfWords

**Bases:** *UniqueRepresentation, Parent*

Auxiliary class to provide a call method to create tensor product elements. This class is shared with several tensor product classes and is also used in *CrystalOfTableaux* to allow tableaux of different tensor product structures
in column-reading (and hence different shapes) to be considered elements in the same crystal.

**class Element**

Bases: `TensorProductOfCrystalsElement`

**class** `sage.combinat.crystals.tensor_product.FullTensorProductOfCrystals`(*crystals, **options*)

Bases: `TensorProductOfCrystals`

Full tensor product of crystals.

**Todo:** Merge this into `TensorProductOfCrystals`.

**cardinality()**

Return the cardinality of `self`.

**EXAMPLES:**

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C)
sage: T.cardinality()
sage: 9
```

**weight_lattice_realization()**

Return the weight lattice realization used to express weights.

The weight lattice realization is the common parent which all weight lattice realizations of the crystals of `self` coerce into.

**EXAMPLES:**

```
sage: B = crystals.elementary.B(['A',4], 2)
sage: B.weight_lattice_realization()
Root lattice of the Root system of type ['A', 4]
sage: T = crystals.infinity.Tableaux(['A',4])
sage: T.weight_lattice_realization()
Ambient space of the Root system of type ['A', 4]
sage: TP = crystals.TensorProduct(B, T)
sage: TP.weight_lattice_realization()
Ambient space of the Root system of type ['A', 4]
```

**class** `sage.combinat.crystals.tensor_product.FullTensorProductOfQueerSuperCrystals`(*crystals, **options*)

Bases: `FullTensorProductOfCrystals, QueerSuperCrystalsMixin`

Tensor product of queer super crystals.

**class Element**

Bases: `TensorProductOfQueerSuperCrystalsElement`

**class** `sage.combinat.crystals.tensor_product.FullTensorProductOfRegularCrystals`(*crystals, **options*)

Bases: `FullTensorProductOfCrystals`

Full tensor product of regular crystals.
class Element

Bases: TensorProductOfRegularCrystalsElement

class sage.combinat.crystals.tensor_product.FullTensorProductOfSuperCrystals(crystals, **options)

Bases: FullTensorProductOfCrystals

Tensor product of super crystals.

EXAMPLES:

```
sage: L = crystals.Letters(['A', [1,1]])
sage: T = tensor([L,L,L])
sage: T.cardinality()
64
```

class Element

Bases: TensorProductOfSuperCrystalsElement

class sage.combinat.crystals.tensor_product.QueerSuperCrystalsMixin

Bases: object

Mixin class with methods for a finite queer supercrystal.

index_set()

Return the enlarged index set.

EXAMPLES:

```
sage: Q = crystals.Letters(['Q', 3])
sage: T = tensor([Q,Q])
sage: T.index_set()
(-4, -3, -2, -1, 1, 2)
```

class sage.combinat.crystals.tensor_product.TensorProductOfCrystals

Bases: CrystalOfWords

Tensor product of crystals.

Given two crystals $B$ and $B'$ of the same Cartan type, one can form the tensor product $B \otimes B'$. As a set $B \otimes B'$ is the Cartesian product $B \times B'$. The crystal operators $f_i$ and $e_i$ act on $b \otimes b' \in B \otimes B'$ as follows:

$$
    f_i(b \otimes b') = \begin{cases}
    f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\
    b \otimes f_i(b') & \text{otherwise}
    \end{cases}
$$

and

$$
    e_i(b \otimes b') = \begin{cases}
    e_i(b) \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b') \\
    b \otimes e_i(b') & \text{otherwise}
    \end{cases}
$$

We also define:

$$
    \varphi_i(b \otimes b') = \max (\varphi_i(b), \varphi_i(b')) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle, \\
    \varepsilon_i(b \otimes b') = \max (\varepsilon_i(b'), \varepsilon_i(b) - \langle \alpha_i^\vee, \mathrm{wt}(b') \rangle).
$$

Note: This is the opposite of Kashiwara’s convention for tensor products of crystals.
Since tensor products are associative \((B \otimes C) \otimes D \cong B \otimes (C \otimes D)\) via the natural isomorphism \((b \otimes c) \otimes d \mapsto b \otimes (c \otimes d)\), we can generalizing this to arbitrary tensor products. Thus consider \(B_N \otimes \cdots \otimes B_1\), where each \(B_k\) is an abstract crystal. The underlying set of the tensor product is \(B_N \times \cdots \times B_1\), while the crystal structure is given as follows. Let \(I\) be the index set, and fix some \(i \in I\) and \(b_N \otimes \cdots \otimes b_1 \in B_N \otimes \cdots \otimes B_1\). Define

\[a_i(k) := \varepsilon_i(b_k) - \sum_{j=1}^{k-1} (\alpha_i^\vee, \text{wt}(b_j))\]

Then

\[
\text{wt}(b_N \otimes \cdots \otimes b_1) = \text{wt}(b_N) + \cdots + \text{wt}(b_1),
\]

\[
\varepsilon_i(b_N \otimes \cdots \otimes b_1) = \max_{1 \leq k \leq n} \left( \sum_{j=1}^{k} \varepsilon_i(b_j) - \sum_{j=1}^{k-1} \varphi_i(b_j) \right) = \max_{1 \leq k \leq N} (a_i(k)),
\]

\[
\varphi_i(b_N \otimes \cdots \otimes b_1) = \max_{1 \leq k \leq N} \left( \varphi_i(b_N) + \sum_{j=k}^{N-1} (\varphi_i(b_j) - \varepsilon_i(b_{j+1})) \right) = \max_{1 \leq k \leq N} (\lambda_i + a_i(k))
\]

where \(\lambda_i = (\alpha_i^\vee, \text{wt}(b_N \otimes \cdots \otimes b_1))\). Then for \(k = 1, \ldots, N\) the action of the Kashiwara operators is determined as follows.

- If \(a_i(k) > a_i(j)\) for \(1 \leq j < k\) and \(a_i(k) \geq a_i(j)\) for \(k < j \leq N\):
  \[e_i(b_N \otimes \cdots \otimes b_1) = b_N \otimes \cdots \otimes e_ib_k \otimes \cdots \otimes b_1.\]

- If \(a_i(k) \geq a_i(j)\) for \(1 \leq j < k\) and \(a_i(k) > a_i(j)\) for \(k < j \leq N\):
  \[f_i(b_N \otimes \cdots \otimes b_1) = b_N \otimes \cdots \otimes f_ib_k \otimes \cdots \otimes b_1.\]

Note that this is just recursively applying the definition of the tensor product on two crystals. Recall that \((\alpha_i^\vee, \text{wt}(b_j)) = \varphi_i(b_j) - \varepsilon_i(b_j)\) by the definition of a crystal.

**Regular crystals**

Now if all crystals \(B_k\) are regular crystals, all \(\varepsilon_i\) and \(\varphi_i\) are non-negative and we can define tensor product by the *signature rule*. We start by writing a word in \(+\) and \(\pm\) as follows:

\[
\varphi_i(b_N) \text{ times } \varepsilon_i(b_N) \text{ times } \cdots \varphi_i(b_1) \text{ times } \varepsilon_i(b_1) \text{ times },
\]

and then canceling ordered pairs of \(+\) until the word is in the reduced form:

\[
\varphi_i \text{ times } \varepsilon_i \text{ times }.
\]

Here \(e_i\) acts on the factor corresponding to the leftmost \(+\) and \(f_i\) on the factor corresponding to the rightmost \(\pm\). If there is no \(+\) or \(\pm\) respectively, then the result is 0 (None).

**EXAMPLES:**

We construct the type \(A_2\)-crystal generated by \(2 \otimes 1 \otimes 1\):
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C,generators=[[C(2),C(1),C(1)]])

It has 8 elements:

```
sage: T.list()
[[2, 1, 1], [2, 1, 2], [2, 1, 3], [3, 1, 3], [3, 2, 3], [3, 1, 1], [3, 1, 2], [3, 2, 2]]
```

One can also check the Cartan type of the crystal:

```
sage: T.cartan_type()
['A', 2]
```

Other examples include crystals of tableaux (which internally are represented as tensor products obtained by reading the tableaux columnwise):

```
sage: C = crystals.Tableaux(['A',3], shape=[1,1,0])
sage: D = crystals.Tableaux(['A',3], shape=[1,0,0])
sage: T = crystals.TensorProduct(C,D, generators=[[C(rows=[1], [2]), D(rows=[1])], [C(rows=[2], [3]), D(rows=[1])]])
sage: T.cardinality()
24
sage: TestSuite(T).run()
sage: T.module_generators
([1, 1], [1, 2], [1, 3], [2, 1], [2, 2], [2, 3], [3, 1], [3, 2], [3, 3])
```

If no module generators are specified, we obtain the full tensor product:

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C)
sage: T.list()
[[1, 1], [1, 2], [1, 3], [2, 1], [2, 2], [2, 3], [3, 1], [3, 2], [3, 3]]
sage: T.cardinality()
9
```

For a tensor product of crystals without module generators, the default implementation of module_generators contains all elements in the tensor product of the crystals. If there is a subset of elements in the tensor product that still generates the crystal, this needs to be implemented for the specific crystal separately:

```
sage: T.module_generators.list()
[[1, 1], [1, 2], [1, 3], [2, 1], [2, 2], [2, 3], [3, 1], [3, 2], [3, 3]]
```

For classical highest weight crystals, it is also possible to list all highest weight elements:

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C,generators=[[C(2),C(1),C(1)], [C(1),C(2), C(1)]])
sage: T.highest_weight_vectors()
([2, 1, 1], [1, 2, 1])
```

Examples with non-regular and infinite crystals (these did not work before github issue #14402):
sage: B = crystals.infinity.Tableaux(['D',10])
sage: T = crystals.TensorProduct(B,B)
sage: T
Full tensor product of the crystals
[The infinity crystal of tableaux of type ['D', 10],
The infinity crystal of tableaux of type ['D', 10]]

sage: B = crystals.infinity.GeneralizedYoungWalls(15)
sage: T = crystals.TensorProduct(B,B,B)
sage: T
Full tensor product of the crystals
[Crystal of generalized Young walls of type ['A', 15, 1],
Crystal of generalized Young walls of type ['A', 15, 1],
Crystal of generalized Young walls of type ['A', 15, 1]]

sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()
sage: B = crystals.GeneralizedYoungWalls(2,La[0]+La[1])
sage: C = crystals.GeneralizedYoungWalls(2,2*La[2])
sage: D = crystals.GeneralizedYoungWalls(2,3*La[0]+La[2])
sage: T = crystals.TensorProduct(B,C,D)
sage: T
Full tensor product of the crystals
[Highest weight crystal of generalized Young walls of Cartan type ['A', 2, 1] and
→highest weight Lambda[0] + Lambda[1],
Highest weight crystal of generalized Young walls of Cartan type ['A', 2, 1] and
→highest weight 2*Lambda[2],
Highest weight crystal of generalized Young walls of Cartan type ['A', 2, 1] and
→highest weight 3*Lambda[0] + Lambda[2]]

There is also a global option for setting the convention (by default Sage uses anti-Kashiwara):

sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C)
sage: elt = T(C(1), C(2)); elt
[1, 2]
sage: crystals.TensorProduct.options.convention = "Kashiwara"
sage: elt
[2, 1]
sage: crystals.TensorProduct.options._reset()

options = Current options for TensorProductOfCrystals - convention: antiKashiwara

class sage.combinat.crystals.tensor_product.TensorProductOfCrystalsWithGenerators(crystals, generators, cartan_type)

Bases: TensorProductOfCrystals

Tensor product of crystals with a generating set.

Todo: Deprecate this class in favor of using subcrystal().
class sage.combinat.crystals.tensor_product.TensorProductOfRegularCrystalsWithGenerators(crystals, generators, cartan_type)

Bases: TensorProductOfCrystalsWithGenerators
Tensor product of regular crystals with a generating set.

class Element
    Bases: TensorProductOfRegularCrystalsElement

5.1.68 Tensor Products of Crystal Elements

AUTHORS:

• Anne Schilling, Nicolas Thiery (2007): Initial version
• Ben Salisbury, Travis Scrimshaw (2013): Refactored tensor products to handle non-regular crystals and created
  new subclass to take advantage of the regularity
• Travis Scrimshaw (2017): Cythonized element classes
• Franco Saliola (2017): Tensor products for crystal of super algebras
• Anne Schilling (2018): Tensor products for crystals of queer super algebras

class sage.combinat.crystals.tensor_product_element.CrystalOfBKKTableauxElement
    Bases: TensorProductOfSuperCrystalsElement
    Element class for the crystal of tableaux for Lie superalgebras of [BKK2000].

pp()
    Pretty print self.

    EXAMPLES:

    sage: C = crystals.Tableaux(['A',[1,2]], shape=[1,1])
    sage: c = C.an_element()
    sage: c.pp()
    [-2]
    [-1]

    to_tableau()
    Return the Tableau object corresponding to self.

    EXAMPLES:

    sage: C = crystals.Tableaux(['A',[1,2]], shape=[1,1])
    sage: c = C.an_element()
    sage: c.to_tableau()
    [[-2], [-1]]
    sage: type(c.to_tableau())
    <class 'sage.combinat.tableau.Tableaux_all_with_category.element_class'>
    sage: type(c)
class sage.combinat.crystals.tensor_product_element.CrystalOfTableauxElement

Bases: TensorProductOfRegularCrystalsElement

Element in a crystal of tableaux.

pp()

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape = [2,2])
sage: t = T(rows=[[1,2],[3,4]])
sage: t.pp()
1 2
3 4
```

promotion()

Return the result of applying promotion on self.

Promotion for type A crystals of tableaux of rectangular shape. This method only makes sense in type A with rectangular shapes.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',3], shape = [3,3,3])
sage: t = C(Tableau([[1,1,1],[2,2,3],[3,4,4]]))
sage: t
[[1, 1, 1], [2, 2, 3], [3, 4, 4]]
sage: t.promotion()
[[1, 1, 2], [2, 3, 3], [4, 4, 4]]
sage: t.promotion().parent()
The crystal of tableaux of type ['A', 3] and shape(s) [[3, 3, 3]]
```

promotion_inverse()

Return the result of applying inverse promotion on self.

Inverse promotion for type A crystals of tableaux of rectangular shape. This method only makes sense in type A with rectangular shapes.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',3], shape = [3,3,3])
sage: t = C(Tableau([[1,1,1],[2,2,3],[3,4,4]]))
sage: t
[[1, 1, 1], [2, 2, 3], [3, 4, 4]]
sage: t.promotion_inverse()
[[1, 1, 2], [2, 3, 3], [4, 4, 4]]
sage: t.promotion_inverse().parent()
The crystal of tableaux of type ['A', 3] and shape(s) [[3, 3, 3]]
```

shape()

Return the shape of the tableau corresponding to self.

OUTPUT: an instance of Partition
See also:

```
to_tableau()
```

EXAMPLES:

```
sage: C = crystals.Tableaux("A", 2, shape=[2,1])
sage: x = C.an_element()
sage: x.to_tableau().shape()
[2, 1]
sage: x.shape()
[2, 1]
```

```
to_tableau()
```

Return the Tableau object corresponding to self.

EXAMPLES:

```
sage: T = crystals.Tableaux("A", 3, shape = [2,2])
sage: t = T(rows=[[1,2],[3,4]]).to_tableau(); t
[[1, 2], [3, 4]]
sage: type(t)
<class 'sage.combinat.tableau.Tableaux_all_with_category.element_class'>
sage: type(t[0][0])
<int>
sage: T = crystals.Tableaux("D", 3, shape = [1,1])
sage: t = T(rows=[[3],[3],[-3]]).to_tableau(); t
[[3], [-3], [3]]
sage: T = crystals.Tableaux("B", 2, shape = [1,1])
sage: t = T(rows=[[0],[0]]).to_tableau(); t
[[0], [0]]
```

```
class sage.combinat.crystals.tensor_product_element.ImmutableListWithParent
Bases: ClonableArray

A class for lists having a parent

Specification: any subclass C should implement __init__ which accepts the following form C(parent, list=list)
```

```
class sage.combinat.crystals.tensor_product_element.InfinityCrystalOfTableauxElement
Bases: CrystalOfTableauxElement

```

```
e(i)
```

Return the action of \( \tilde{e}_i \) on self.

INPUT:

- \( i \) – an element of the index set

EXAMPLES:

```
sage: B = crystals.infinity.Tableaux(['B',3])
sage: b = B(rows=[[1,1,1,1,1,1,1,1,1,1,1,2,0,-3,-1,-1,-1,-1,-1,-1,-1,-1,-1]])
sage: b.e(3).pp()
1 1 1 1 1 1 1 1 2 0 -3 -1 -1 -1 -1 -1 -1 -1 -1 -1
```

(continues on next page)
\texttt{sage}: \ \texttt{b.e(1).pp()}
\begin{verbatim}
   1 1 1 1 1 1 0 -3 -1 -1 -1 -1
   2 2 2 2 -2 -2
   3 -3 -3
\end{verbatim}

\texttt{f(i)}

Return the action of $\tilde{f}_i$ on \texttt{self}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{i} – an element of the index set
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = crystals.infinity.Tableaux(['C',4])
sage: b = B.highest_weight_vector()
sage: b.f(1).pp()
1 1 1 1 2
2 2 2
3 3
4
\end{verbatim}

\begin{verbatim}
sage: b.f(3).pp()
1 1 1 1 1
2 2 2 2
3 3 4
4
\end{verbatim}

\begin{verbatim}
sage: b.f(3).f(4).pp()
1 1 1 1 1
2 2 2 2
3 3 -4
4
\end{verbatim}

\texttt{class sage.combinat.crystals.tensor_product_element.InfinityCrystalOfTableauxElementTypeD}

\texttt{Bases: InfinityCrystalOfTableauxElement}

\texttt{e(i)}

Return the action of $\tilde{e}_i$ on \texttt{self}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{i} – an element of the index set
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = crystals.infinity.Tableaux(['D',4])
sage: b = B.highest_weight_vector().f_string([1,4,3,1,2]); b.pp()
1 1 1 1 2 3
2 2 2
3 -3
\end{verbatim}

\begin{verbatim}
sage: b.e(2).pp()
1 1 1 1 2 2
2 2 2
3 -3
\end{verbatim}
\( f(i) \)

Return the action of \( \tilde{f}_i \) on self.

**INPUT:**

- \( i \) – an element of the index set

**EXAMPLES:**

```sage
sage: B = crystals.infinity.Tableaux(['D', 5])
sage: b = B.highest_weight_vector().f_string([1,4,3,1,5]); b.pp()
1 1 1 1 1 1 2 2
2 2 2 2 2
3 3 3 -5
4 5
sage: b.f(1).pp()
1 1 1 1 1 1 2 2 2
2 2 2 2 2
3 3 3 -5
4 5
sage: b.f(5).pp()
1 1 1 1 1 1 2 2
2 2 2 2 2
3 3 3 -5
4 -4
```

**class** `sage.combinat.crystals.tensor_product_element.InfinityQueerCrystalOfTableauxElement`

Bases: `TensorProductOfQueerSuperCrystalsElement`

Initialize self.

**EXAMPLES:**

```sage
sage: B = crystals.infinity.Tableaux(['Q', 4])
sage: t = B([[4,4,4,4,4,2,1], [3,3,3], [2,2,1], [1]])
sage: t
[[4, 4, 4, 4, 4, 2, 1], [3, 3, 3], [2, 2, 1], [1]]
sage: TestSuite(t).run()
```

\( e(i) \)

Return the action of \( e_i \) on self.

**INPUT:**

- \( i \) – an element of the index set

**EXAMPLES:**

```sage
sage: B = crystals.infinity.Tableaux(['Q', 4])
sage: t = B([[4,4,4,4,4,2,1], [3,3,3], [2,2,1], [1]])
sage: t.e(1)
[[4, 4, 4, 4, 4, 4, 2, 1], [3, 3, 3, 3], [2, 2, 1, 1], [1]]
sage: t.e(3)
[[4, 4, 4, 4, 4, 3, 2, 1], [3, 3, 3], [2, 2, 1], [1]]
sage: t.e(-1)
```
epsilon(i)
Return $\epsilon_i$ of self.

INPUT:
• $i$ – an element of the index set

EXAMPLES:
```python
sage: B = crystals.infinity.Tableaux(['Q',4])
sage: t = B([[4,4,4,4,4,2,1],[3,3,3,3],[2,2,1],[1]])
sage: [t.epsilon(i) for i in B.index_set()]
[-1, 1, -2, 0]
```

f(i)
Return the action of $f_i$ on self.

INPUT:
• $i$ – an element of the index set

EXAMPLES:
```python
sage: B = crystals.infinity.Tableaux(['Q',4])
sage: t = B([[4,4,4,4,4,2,1],[3,3,3,3],[2,2,1],[1]])
sage: t.f(1)
[[4, 4, 4, 4, 4, 2, 2], [3, 3, 3, 3], [2, 2, 1], [1]]
sage: t.f(3)
sage: t.f(-1)
[[4, 4, 4, 4, 4, 2, 2], [3, 3, 3, 3], [2, 2, 1], [1]]
```

rows()
Return the list of rows of self.

EXAMPLES:
```python
sage: B = crystals.infinity.Tableaux(['Q',4])
sage: t = B([[4,4,4,4,4,2,1],[3,3,3,3],[2,2,1],[1]])
sage: t.rows()
[[1, 2, 4, 4, 4, 4, 4], [3, 3, 3, 3], [1, 2, 2], [1]]
```

weight()
Return the weight of self.

EXAMPLES:
```python
sage: B = crystals.infinity.Tableaux(['Q',4])
sage: t = B([[4,4,4,4,4,2,1],[3,3,3,3],[2,2,1],[1]])
sage: t.weight()
(4, 2, 2, 0)
```

class sage.combinat.crystals.tensor_product_element.TensorProductOfCrystalsElement

A class for elements of tensor products of crystals.
e(i)
Return the action of $e_i$ on self.

INPUT:

• $i$ – an element of the index set

EXAMPLES:

```
sage: B = crystals.infinity.Tableaux("D4")
sage: T = crystals.TensorProduct(B,B)
sage: b1 = B.highest_weight_vector().f_string([1,4,3])
sage: b2 = B.highest_weight_vector().f_string([2,2,3,1,4])
sage: t = T(b2, b1)
sage: t.e(1)
[[[1, 1, 1, 1, 1], [2, 2, 3, -3], [3]], [[1, 1, 1, 1, 2], [2, 2, 2], [3, -3]]]
sage: t.e(2)

epsilon(i)
Return $\epsilon_i$ of self.

INPUT:

• $i$ – an element of the index set

EXAMPLES:

```
sage: B = crystals.infinity.Tableaux("G2")
sage: T = crystals.TensorProduct(B,B)
sage: b1 = B.highest_weight_vector().f(2)
sage: b2 = B.highest_weight_vector().f_string([2,2,1])
sage: t = T(b2, b1)
sage: [t.epsilon(i) for i in B.index_set()]
[0, 3]
```

f(i)
Return the action of $f_i$ on self.

INPUT:

• $i$ – an element of the index set

EXAMPLES:

```
sage: La = RootSystem(['A',3,1]).weight_lattice(extended=True).fundamental_weights()
sage: B = crystals.GeneralizedYoungWalls(3,La[0])
sage: T = crystals.TensorProduct(B,B,B)
sage: b1 = B.highest_weight_vector().f_string([0,3])
sage: b2 = B.highest_weight_vector().f_string([0])
sage: b3 = B.highest_weight_vector()
sage: t = T(b3, b2, b1)
sage: t.f(0)
```

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(continued from previous page)

sage: t.f(1)
[[], [[0]], [[0, 3]]]
sage: t.f(2)
[[], [[0]], [[0, 3, 2]]]
sage: t.f(3)
[[], [[0, 3]], [[0, 3]]]

\(\phi(i)\)

Return \(\phi_i\) of self.

INPUT:

- \(i\) – an element of the index set

EXAMPLES:

```
sage: La = RootSystem(['A', 2, 1]).weight_lattice(extended=True).fundamental_weights()
sage: B = crystals.GeneralizedYoungWalls(2,La[0]+La[1])
sage: T = crystals.TensorProduct(B,B)
sage: b1 = B.highest_weight_vector().f_string([1,0])
sage: b2 = B.highest_weight_vector().f_string([0,1])
sage: t = T(b2, b1)
sage: [t.phi(i) for i in B.index_set()]
[1, 1, 4]
```

\texttt{pp()}

Pretty print self.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',3], shape=[3,1])
sage: D = crystals.Tableaux(['A',3], shape=[1])
sage: E = crystals.Tableaux(['A',3], shape=[2,2,2])
sage: T = crystals.TensorProduct(C,D,E)
sage: T.module_generators[0].pp()
 1 1 1 (X) 1 (X) 1 1
 2 2 2
 3 3
```

\texttt{weight()}

Return the weight of self.

EXAMPLES:

```
sage: B = crystals.infinity.Tableaux("A3")
sage: T = crystals.TensorProduct(B,B)
sage: b1 = B.highest_weight_vector().f_string([2,1,3])
sage: b2 = B.highest_weight_vector().f(1)
sage: t = T(b2, b1)
sage: t
[[[1, 1, 1, 2], [2, 2], [3]], [[1, 1, 1, 1, 2], [2, 2, 4], [3]]]
sage: t.weight()
(-2, 1, 0, 1)
```
sage: C = crystals.Letters(['A', 3])
sage: T = crystals.TensorProduct(C, C)
sage: T(C(1), C(2)).weight()
(1, 1, 0, 0)
sage: T = crystals.Tableaux(['D', 4], shape=[])
sage: T.list()[0].weight()
(0, 0, 0, 0)

class

Bases: TensorProductOfRegularCrystalsElement

Element class for a tensor product of crystals for queer Lie superalgebras.

This implements the tensor product rule for crystals of Grantcharov et al. [GJK+2014]. Given crystals $B_1$ and $B_2$ of type $q_{n+1}$, we define the tensor product $b_1 \otimes b_2 \in B_1 \otimes B_2$, where $b_1 \in B_1$ and $b_2 \in B_2$, as the following:

In addition to the tensor product rule for type $A_n$, the tensor product rule for $e^{-1}$ and $f^{-1}$ on $b_1 \otimes b_2$ are given by

$$
e^{-1}(b_1 \otimes b_2) = \begin{cases} b_1 \otimes e^{-1} b_2 & \text{if } \text{wt}(b_1)_1 = \text{wt}(b_1)_2 = 0, \\
e^{-1} b_1 \otimes b_2 & \text{otherwise}, \end{cases}$$
$$f^{-1}(b_1 \otimes b_2) = \begin{cases} b_1 \otimes f^{-1} b_2 & \text{if } \text{wt}(b_1)_1 = \text{wt}(b_1)_2 = 0, \\
f^{-1} b_1 \otimes b_2 & \text{otherwise}. \end{cases}$$

For $1 < i \leq n$, the operators $e_{-i}$ and $f_{-i}$ are defined as

$$e_{-i} = s_{w_i}^{-1} e_{-1} s_{w_i}, \quad f_{-i} = s_{w_i}^{-1} f_{-1} s_{w_i},$$

Here, $w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}$ and $s_i$ is the reflection along the $i$-string in the crystal. Moreover, for $1 < i \leq n$, we define the operators $e_{-i'}$ and $f_{-i'}$ as

$$e_{-i'} = s_{w_0} f_{-(n+1-i)} s_{w_0}, \quad f_{-i'} = s_{w_0} e_{-(n+1-i)} s_{w_0},$$

where $w_0$ is the longest element in the symmetric group $\mathcal{S}_{n+1}$ generated by $s_1, \ldots, s_n$. In this implementation, we use the integers $-2n, \ldots, -(n+1)$ to respectively denote the indices $-n', \ldots, -1'$.

Return $e_i$ on self.

EXAMPLES:

```python
sage: Q = crystals.Letters(['Q', 3])
sage: T = tensor([Q, Q])
sage: t = T(Q(1), Q(1))
sage: t.e(-1)
[1, 1]
sage: T = tensor([Q, Q, Q, Q])
sage: t = T(Q(1), Q(3), Q(2), Q(1))
sage: t.e(-2)
[2, 2, 1, 1]
```
epsilon(i)
Return \( \varepsilon_i \) on self.

EXAMPLES:
```
sage: Q = crystals.Letters(['Q', 3])
sage: T = tensor([Q, Q, Q, Q])
sage: t = T(Q(1), Q(3), Q(2), Q(1))
sage: t.epsilon(-2)
1
```

f(i)
Return \( f_i \) on self.

EXAMPLES:
```
sage: Q = crystals.Letters(['Q', 3])
sage: T = tensor([Q, Q])
sage: t = T(Q(1), Q(1))
sage: t.f(-1)
[2, 1]
```

phi(i)
Return \( \varphi_i \) on self.

EXAMPLES:
```
sage: Q = crystals.Letters(['Q', 3])
sage: T = tensor([Q, Q, Q, Q])
sage: t = T(Q(1), Q(3), Q(2), Q(1))
sage: t.phi(-2)
0
sage: t.phi(-1)
1
```

class sage.combinat.crystals.tensor_product_element.TensorProductOfRegularCrystalsElement

Element class for a tensor product of regular crystals.

e(i)
Return the action of \( e_i \) on self.

EXAMPLES:
```
sage: C = crystals.Letters(['A', 5])
sage: T = crystals.TensorProduct(C, C)
sage: T(C(1),C(2)).e(1) == T(C(1),C(1))
True
sage: T(C(2),C(1)).e(1) is None
True
sage: T(C(2),C(2)).e(1) == T(C(1),C(2))
True
```

epsilon(i)
Return \( \varepsilon_i \) of self.
EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(1),C(1)).epsilon(1)
0
sage: T(C(1),C(2)).epsilon(1)
1
sage: T(C(2),C(1)).epsilon(1)
0
```

\(f(i)\)

Return the action of \(f_i\) on self.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(1),C(1)).f(1)
[1, 2]
sage: T(C(1),C(2)).f(1)
[2, 2]
sage: T(C(2),C(1)).f(1) is None
True
```

\(\phi(i)\)

Return \(\phi_i\) of self.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(1),C(1)).phi(1)
2
sage: T(C(1),C(2)).phi(1)
1
sage: T(C(2),C(1)).phi(1)
0
```

\(\text{position\_of\_first\_unmatched\_plus}(i)\)

Return the position of the first unmatched + or None if there is no unmatched +.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(2),C(1)).position_of_first_unmatched_plus(1)
sage: T(C(1),C(2)).position_of_first_unmatched_plus(1)
1
```

\(\text{position\_of\_last\_unmatched\_minus}(i)\)

Return the position of the last unmatched − or None if there is no unmatched −.

EXAMPLES:
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```python
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(2),C(1)).position_of_last_unmatched_minus(1)
0

positions_of_unmatched_minus(i, dual=False, reverse=False)

EXAMPLES:

```python
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(2),C(1)).positions_of_unmatched_minus(1)
[]
sage: T(C(1),C(2)).positions_of_unmatched_minus(1)
[0]
```

```python
positions_of_unmatched_plus(i)

EXAMPLES:
```python
sage: C = crystals.Letters(['A',5])
sage: T = crystals.TensorProduct(C,C)
sage: T(C(2),C(1)).positions_of_unmatched_plus(1)
[]
sage: T(C(1),C(2)).positions_of_unmatched_plus(1)
[1]
```

class sage.combinat.crystals.tensor_product_element.TensorProductOfSuperCrystalsElement

Bases: TensorProductOfRegularCrystalsElement

Element class for a tensor product of crystals for Lie superalgebras.

This implements the tensor product rule for crystals of Lie superalgebras of [BKK2000].

```python
e(i)

Return e_i on self.

EXAMPLES:
```python
sage: C = crystals.Letters(['A', [2, 1]])
sage: T = tensor([C,C])
sage: t = T(C(1),C(1))
sage: t.e(0)
[-1, 1]
```

```python
epsilon(i)

Return \epsilon_i on self.

EXAMPLES:
```python
sage: C = crystals.Letters(['A', [2, 1]])
sage: T = tensor([C,C])
sage: t = T(C(1),C(1))
sage: t.epsilon(0)
1
```
\( f(i) \)

Return \( f_i \) on self.

EXAMPLES:

```python
sage: C = crystals.Letters(['A', [2, 1]])
sage: T = tensor([C,C])
sage: t = T(C(1),C(1))
sage: t.f(0)
[]
sage: t.f(1)
[1, 2]
```

\( \phi(i) \)

Return \( \phi_i \) on self.

EXAMPLES:

```python
sage: C = crystals.Letters(['A', [2, 1]])
sage: T = tensor([C,C])
sage: t = T(C(1),C(1))
sage: t.phi(0)
0
```

### 5.1.69 Cyclic sieving phenomenon

Implementation of the Cyclic Sieving Phenomenon as described by Reiner, Stanton, and White in [RSW2004].

We define the \texttt{CyclicSievingPolynomial()} of a finite set \( S \) together with cyclic action \( \text{cyc\_act} \) (of order \( n \)) to be the unique polynomial \( P(q) \) of order \( < n \) such that the triple \((S, \text{cyc\_act}, P(q))\) exhibits the cyclic sieving phenomenon.

AUTHORS:

- Christian Stump

REFERENCES:

\texttt{sage.combinat.cyclic_sieving_phenomenon.CyclicSievingCheck}(L, cyc\_act, f, order=None)

Return whether the triple \((L, \text{cyc\_act}, f)\) exhibits the cyclic sieving phenomenon.

If \( \text{cyc\_act} \) is None, \( L \) is expected to contain the orbit lengths.

INPUT:

- \( \text{L} \) – if \( \text{cyc\_act} \) is None: list of orbit sizes, otherwise list of objects
- \( \text{cyc\_act} \) – (default:None) bijective function from \( L \) to \( L \)
- \( \text{order} \) – (default:None) if set to an integer, this cyclic order of \( \text{cyc\_act} \) is used (must be an integer multiple of the order of \( \text{cyc\_act} \)) otherwise, the order of \( \text{cyc\_act} \) is used

EXAMPLES:

```python
sage: from sage.combinat.cyclic_sieving_phenomenon import *
sage: from sage.combinat.q_analogues import q_binomial
sage: S42 = Subsets([1,2,3,4], 2)
sage: def cyc_act(S): return Set(i.mod(4) + 1 for i in S)
```

(continues on next page)
sage: cyc_act([1,3])
{2, 4}
sage: cyc_act([1,4])
{1, 2}
sage: p = q_binomial(4,2); p
q^4 + q^3 + 2*q^2 + q + 1
sage: CyclicSievingPolynomial(S42, cyc_act)
q^3 + 2*q^2 + q + 2
sage: CyclicSievingCheck(S42, cyc_act, p)
True

sage.combinat.cyclic_sieving_phenomenon.CyclicSievingPolynomial(L, cyc_act=None, order=None, get_order=False)

Return the unique polynomial $p$ of degree smaller than $order$ such that the triple $(L, cyc_act, p)$ exhibits
the Cyclic Sieving Phenomenon.

If $cyc_act$ is None, $L$ is expected to contain the orbit lengths.

INPUT:

- $L$ – if $cyc_act$ is None: list of orbit sizes, otherwise list of objects
- $cyc_act$ – (default:None) bijective function from $L$ to $L$
- $order$ – (default:None) if set to an integer, this
cyclic order of $cyc_act$ is used (must be an integer multiple of the order of $cyc_act$) otherwise, the
order of $cyc_action$ is used
- $get_order$ – (default:False) if True, a tuple $[p,n]$ is returned where $p$ is as above, and $n$ is the order

EXAMPLES:

sage: from sage.combinat.cyclic_sieving_phenomenon import CyclicSievingPolynomial
sage: S42 = Subsets([1,2,3,4], 2)
sage: def cyc_act(S):
    return Set(i.mod(4) + 1 for i in S)
sage: cyc_act([1,3])
{2, 4}
sage: cyc_act([1,4])
{1, 2}
sage: CyclicSievingPolynomial(S42, cyc_act)
q^3 + 2*q^2 + q + 2
sage: CyclicSievingPolynomial(S42, cyc_act, get_order=True)
[q^3 + 2*q^2 + q + 2, 4]
sage: CyclicSievingPolynomial(S42, cyc_act, order=8)
q^6 + 2*q^4 + q^2 + 2
sage: CyclicSievingPolynomial([4,2])
q^3 + 2*q^2 + q + 2

sage.combinat.cyclic_sieving_phenomenon.orbit_decomposition(L, cyc_act)

Return the orbit decomposition of $L$ by the action of $cyc_act$.

INPUT:

- $L$ – list
- $cyc_act$ – bijective function from $L$ to $L$

OUTPUT:
• a list of lists, the orbits under the cyc_act acting on L

EXAMPLES:

```python
sage: from sage.combinat.cyclic_sieving_phenomenon import *
sage: S42 = Subsets([1,2,3,4], 2); S42
Subsets of {1, 2, 3, 4} of size 2
sage: def cyc_act(S):
    return Set(i.mod(4) + 1 for i in S)
sage: cyc_act([1,3])
{2, 4}
sage: cyc_act([1,4])
{1, 2}
sage: orbits = orbit_decomposition(S42, cyc_act)
sage: sorted([sorted(orb, key=sorted) for orb in orbits], key=len)
[[{1, 3}, {2, 4}], [{1, 2}, {1, 4}, {2, 3}, {3, 4}]]
```

5.1.70 De Bruijn sequences

A De Bruijn sequence is defined as the shortest cyclic sequence that incorporates all substrings of a certain length of an alphabet.

For instance, the $2^3 = 8$ binary strings of length 3 are all included in the following string:

```python
sage: DeBruijnSequences(2,3).an_element()
[0, 0, 0, 1, 0, 1, 1, 1]
```

They can be obtained as a subsequence of the cyclic De Bruijn sequence of parameters $k = 2$ and $n = 3$:

```python
sage: seq = DeBruijnSequences(2,3).an_element()
sage: print(Word(seq).string_rep())
00010111
```

This sequence is of length $k^n$, which is best possible as it is the number of $k$-ary strings of length $n$. One can equivalently define a De Bruijn sequence of parameters $k$ and $n$ as a cyclic sequence of length $k^n$ in which all substring of length $n$ are different.

See also Wikipedia article De_Bruijn_sequence.

AUTHOR:

• Eviatar Bach (2011): initial version
• Nathann Cohen (2011): Some work on the documentation and defined the __contain__ method
class sage.combinat.debruijn_sequence.DeBruijnSequences(k, n)

Bases: UniqueRepresentation, Parent

Represents the De Bruijn sequences of given parameters $k$ and $n$.

A De Bruijn sequence of parameters $k$ and $n$ is defined as the shortest cyclic sequence that incorporates all sub-strings of length $n$ a $k$-ary alphabet.

This class can be used to generate the lexicographically smallest De Bruijn sequence, to count the number of existing De Bruijn sequences or to test whether a given sequence is De Bruijn.

INPUT:

- $k$ – A natural number to define arity. The letters used are the integers $0..k - 1$.
- $n$ – A natural number that defines the length of the substring.

EXAMPLES:

Obtaining a De Bruijn sequence:

```
sage: seq = DeBruijnSequences(2, 3).an_element()
sage: seq
[0, 0, 0, 1, 0, 1, 1, 1]
```

Testing whether it is indeed one:

```
sage: seq in DeBruijnSequences(2, 3)
True
```

The total number for these parameters:

```
sage: DeBruijnSequences(2, 3).cardinality()
2
```

Note: This function only generates one De Bruijn sequence (the smallest lexicographically). Support for generating all possible ones may be added in the future.

an_element()

Returns the lexicographically smallest De Bruijn sequence with the given parameters.

ALGORITHM:

The algorithm is described in the book “Combinatorial Generation” by Frank Ruskey. This program is based on a Ruby implementation by Jonas Elfström, which is based on the C program by Joe Sadawa.

EXAMPLES:

```
sage: DeBruijnSequences(2, 3).an_element()
[0, 0, 0, 1, 0, 1, 1, 1]
```

cardinality()

Returns the number of distinct De Bruijn sequences for the object's parameters.

EXAMPLES:

```
sage: DeBruijnSequences(2, 5).cardinality()
2048
```
ALGORITHM:

The formula for cardinality is $k!^{k^{n-1}}/k^n$ [Ros2002].

sage.combinat.debruijn_sequence.debruijn_sequence$(k, n)$
The generating function for De Bruijn sequences. This avoids the object creation, so is significantly faster than accessing from DeBruijnSequence. For more information, see the documentation there. The algorithm used is from Frank Ruskey’s “Combinatorial Generation”.

INPUT:

• $k$ – Arity. Must be an integer.
• $n$ – Substring length. Must be an integer.

EXAMPLES:

sage: from sage.combinat.debruijn_sequence import debruijn_sequence
sage: debruijn_sequence(3, 1)
[0, 1, 2]

sage.combinat.debruijn_sequence.is_debruijn_sequence$(seq, k, n)$
Given a sequence of integer elements in $0 .. k - 1$, tests whether it corresponds to a De Bruijn sequence of parameters $k$ and $n$.

INPUT:

• $seq$ – Sequence of elements in $0 .. k - 1$.
• $n, k$ – Integers.

EXAMPLES:

sage: from sage.combinat.debruijn_sequence import is_debruijn_sequence
sage: s = DeBruijnSequences(2, 3).an_element()
sage: is_debruijn_sequence(s, 2, 3)
True
sage: is_debruijn_sequence(s + [0], 2, 3)
False
sage: is_debruijn_sequence([1] + s[1:], 2, 3)
False

5.1.71 Degree sequences

The present module implements the DegreeSequences class, whose instances represent the integer sequences of length $n$:

sage: DegreeSequences(6)
Degree sequences on 6 elements

With the object DegreeSequences$(n)$, one can:

• Check whether a sequence is indeed a degree sequence:

sage: DS = DegreeSequences(5)
sage: [4, 3, 3, 3, 3] in DS
True

(continues on next page)
• List all the possible degree sequences of length \( n \):

```python
sage: for seq in DegreeSequences(4):
    print(seq)
[0, 0, 0, 0]
[1, 1, 0, 0]
[2, 1, 1, 0]
[3, 1, 1, 1]
[1, 1, 1, 1]
[2, 2, 1, 1]
[2, 2, 2, 0]
[3, 2, 2, 1]
[2, 2, 2, 2]
[3, 3, 2, 2]
[3, 3, 3, 3]
```

**Note:** Given a degree sequence, one can obtain a graph realizing it by using `DegreeSequence()`. For instance:

```python
sage: ds = [3, 3, 2, 2, 2, 2, 2, 1, 1, 0]
sage: g = graphs.DegreeSequence(ds)
sage: g.degree_sequence()
[3, 3, 2, 2, 2, 2, 2, 1, 1, 0]
```

### Definitions

A sequence of integers \( d_1, \ldots, d_n \) is said to be a *degree sequence* (or *graphic sequence*) if there exists a graph in which vertex \( i \) is of degree \( d_i \). It is often required to be *non-increasing*, i.e. that \( d_1 \geq \ldots \geq d_n \). Finding a graph with given degree sequence is known as the *graph realization problem*.

An integer sequence need not necessarily be a degree sequence. Indeed, in a degree sequence of length \( n \) no integer can be larger than \( n - 1 \) – the degree of a vertex is at most \( n - 1 \) – and the sum of them is at most \( n(n - 1) \).

Degree sequences are completely characterized by a result from Erdos and Gallai:

**Erdos and Gallai:** The sequence of integers \( d_1, \ldots, d_n \) is a degree sequence if and only if \( \sum_i d_i \) is even and \( \forall i \)

\[
\sum_{j \leq i} d_j \leq j(j - 1) + \sum_{j > i} \min(d_j, i).
\]

Alternatively, a degree sequence can be defined recursively:

**Havel and Hakimi:** The sequence of integers \( d_1, \ldots, d_n \) is a degree sequence if and only if \( d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n \) is also a degree sequence.

Or equivalently:

**Havel and Hakimi (bis):** If there is a realization of an integer sequence as a graph (i.e. if the sequence is a degree sequence), then it can be realized in such a way that the vertex of maximum degree \( \Delta \) is adjacent to the \( \Delta \) vertices of highest degree (except itself, of course).
Algorithms

Checking whether a given sequence is a degree sequence

This is tested using Erdos and Gallai’s criterion. It is also checked that the given sequence is non-increasing and has length $n$.

Iterating through the sequences of length $n$

From Havel and Hakimi’s recursive definition of a degree sequence, one can build an enumeration algorithm as done in [RCES1994]. It consists in trying to extend a current degree sequence on $n$ elements into a degree sequence on $n + 1$ elements by adding a vertex of degree larger than those already present in the sequence. This can be seen as reversing the reduction operation described in Havel and Hakimi’s characterization. This operation can appear in several different ways:

- Extensions of a degree sequence that do not change the value of the maximum element
  - If the maximum element of a given degree sequence is 0, then one can remove it to reduce the sequence, following Havel and Hakimi’s rule. Conversely, if the maximum element of the (current) sequence is 0, then one can always extend it by adding a new element of degree 0 to the sequence.

\[
\begin{array}{c}
0, 0, 0 \xrightarrow{Extension} 0, 0, 0 \\
\vdots \\
0, 0, 0, \ldots, 0, 0, 0 \xrightarrow{Extension} 0, 0, 0 \\
\end{array}
\]

- If there are at least $\Delta + 1$ elements of (maximum) degree $\Delta$ in a given degree sequence, then one can reduce it by removing a vertex of degree $\Delta$ and decreasing the values of $\Delta$ elements of value $\Delta$ to $\Delta - 1$. Conversely, if the maximum element of the (current) sequence is $d > 0$, then one can add a new element of degree $d$ to the sequence if it can be linked to $d$ elements of (current) degree $d - 1$. Those $d$ vertices of degree $d - 1$ hence become vertices of degree $d$, and so $d$ elements of degree $d - 1$ are removed from the sequence while $d + 1$ elements of degree $d$ are added to it.

\[
\begin{array}{c}
3, 2, 2, 1 \xrightarrow{Extension} 3, 3, (2 + 1), (2 + 1), (2 + 1), 1 = 3, 3, 3, 3, 1 \xrightarrow{Reduction} 3, 2, 2, 1 \\
\end{array}
\]

- Extension of a degree sequence that changes the value of the maximum element:
  - In the general case, i.e. when the number of elements of value $\Delta, \Delta - 1$ is small compared to $\Delta$ (i.e. the maximum element of a given degree sequence), reducing a sequence strictly decreases the value of the maximum element. According to Havel and Hakimi’s characterization there is only one way to reduce a sequence, but reversing this operation is more complicated than in the previous cases. Indeed, the following extensions are perfectly valid according to the reduction rule.

\[
\begin{array}{c}
2, 1, 1, 0, 0 \xrightarrow{Extension} 3, (2 + 1), (1 + 1), (1 + 1), 0, 0 = 3, 3, 2, 2, 0, 0 \xrightarrow{Reduction} 2, 1, 1, 0, 0 \\
2, 1, 1, 0, 0 \xrightarrow{Extension} 3, (2 + 1), (1 + 1), 1, (0 + 1), 0 = 3, 3, 2, 1, 0 \xrightarrow{Reduction} 2, 1, 1, 0, 0 \\
2, 1, 1, 0, 0 \xrightarrow{Extension} 3, (2 + 1), 1, (0 + 1), (0 + 1) = 3, 3, 1, 1, 1, 1 \xrightarrow{Reduction} 2, 1, 1, 0, 0 \\
\vdots \\
\end{array}
\]

In order to extend a current degree sequence while strictly increasing its maximum degree, it is equivalent to pick a set $I$ of elements of the degree sequence with $|I| > \Delta$ in such a way that the $(d_i + 1)_{i \in I}$ are the $|I|$ maximum elements of the sequence $(d_i + 1)_{1 \leq i \leq n}$, and to add to this new sequence an element of value $|I|$. The non-increasing sequence containing the elements $|I|$ and $(d_i + 1)_{1 \leq i \leq n}$ can be reduced to $(d_i)_{1 \leq i \leq n}$ by Havel and Hakimi’s rule.

\[
...1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 5, 7, ... \xrightarrow{Extension} ...1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 5, 7, ...
\]

The number of possible sets $I$ having this property (i.e. the number of possible extensions of a sequence) is smaller than it seems. Indeed, by definition, if $j \notin I$ then for all $i \in I$ the inequality $d_j \leq d_i + 1$ holds.
Hence, each set $I$ is entirely determined by the largest element $d_k$ of the sequence that it does not contain (hence $I$ contains $\{1, ..., k - 1\}$), and by the cardinalities of $\{i \in I : d_i = d_k\}$ and $\{i \in I : d_i = d_k - 1\}$.

$$I = \{i \in I : d_i = d_k\} \cup \{i \in I : d_i = d_k - 1\} \cup \{i : d_i > d_k\}.$$ The number of possible extensions is hence at most cubic, and is easily enumerated.

**About the implementation**

In the actual implementation of the enumeration algorithm, the degree sequence is stored differently for reasons of efficiency.

Indeed, when enumerating all the degree sequences of length $n$, Sage first allocates an array `seq` of $n + 1$ integers where `seq[i]` is the number of elements of value $i$ in the current sequence. Obviously, `seq[n]=0` holds in permanence : it is useful to allocate a larger array than necessary to simplify the code. The `seq` array is a global variable.

The recursive function `enum(depth, maximum)` is the one building the list of sequences. It builds the list of degree sequences of length $n$ which extend the sequence currently stored in `seq[0]...seq[depth-1]`. When it is called, `maximum` must be set to the maximum value of an element in the partial sequence `seq[0]...seq[depth-1]`.

If during its run the function `enum` heavily works on the content of the `seq` array, the value of `seq` is the same before and after the run of `enum`.

**Extending the current partial sequence**

The two cases for which the maximum degree of the partial sequence does not change are easy to detect. It is (slightly) harder to enumerate all the sets $I$ corresponding to possible extensions of the partial sequence. As said previously, to each set $I$ one can associate an integer `current_box` such that $I$ contains all the $i$ satisfying $d_i > current_box$.

The variable `taken` represents the number of all such elements $i$, so that when enumerating all possible sets $I$ in the algorithm we have the equality

$$I = \text{taken} + \text{number of elements of value } current\_box + \text{number of elements of value } current\_box - 1.$$ 

**REFERENCES:**

- [RCES1994]

**AUTHORS:**

- Nathann Cohen

**Warning:** For the moment, iterating over all degree sequences involves building the list of them first, then iterate on this list. This is obviously bad, as it requires uselessly a lot of memory for large values of $n$.

This should be changed. Updating the code does not require more than a couple of minutes.

class sage.combinat.degree_sequences.DegreeSequences(n)

Bases: object

Degree Sequences

An instance of this class represents the degree sequences of graphs on a given number $n$ of vertices. It can be used to list and count them, as well as to test whether a sequence is a degree sequence. For more information, please refer to the documentation of the `DegreeSequence` module.

**EXAMPLES:**
5.1.72 Derangements

AUTHORS:

- Alasdair McAndrew (2010-05): Initial version
- Travis Scrimshaw (2013-03-30): Put derangements into category framework

```python
sage: D = Derangements(4)
sage: elt = D([4, 3, 2, 1])
sage: TestSuite(elt).run()
```

```python
def to_permutation()
    Return the permutation corresponding to self.
```

```python
sage: p = D([4, 3, 2, 1]).to_permutation(); p
[4, 3, 2, 1]
sage: type(p)
<class 'sage.combinat.permutation.StandardPermutations_all_with_category.
    _element_class'>
```

```python
sage: D = Derangements([1, 3, 3, 4])
sage: D[0].to_permutation()
Traceback (most recent call last):
...
ValueError: can only convert to a permutation for derangements of [1, 2, ..., n]
```

```python
sage: D = Derangements(4)
sage: p = D([4, 3, 2, 1]).to_permutation(); p
[4, 3, 2, 1]
sage: type(p)
<class 'sage.combinat.permutation.StandardPermutations_all_with_category.
    _element_class'>
```

```python
class sage.combinat.derangements.Derangements(x)
    Bases: UniqueRepresentation, Parent
    
The class of all derangements of a set or multiset.
```

A derangement on a set \( S \) is a permutation \( \sigma \) such that \( \sigma(x) \neq x \) for all \( x \in S \), i.e. \( \sigma \) is a permutation of \( S \) with no fixed points.

INPUT:
• $x$ – Can be an integer which corresponds to derangements of $\{1, 2, 3, \ldots, x\}$, a list, or a string

REFERENCES:
• [BV2004]
• Wikipedia article Derangement

EXAMPLES:

```sage
def derangements(x):
    return [list(range(1, n+1)) for n in range(1, x+1) if is_derangement(list(range(1, n+1)))]

def is_derangement(lst):
    return all(lst[x] != x for x in range(len(lst)))
```

```sage
d = derangements(4)
d
```

```
[[1, 2, 3, 4],
 [1, 3, 2, 4],
 [1, 2, 4, 3],
 [1, 4, 2, 3],
 [2, 1, 3, 4],
 [2, 3, 1, 4],
 [2, 1, 4, 3],
 [2, 4, 1, 3],
 [3, 1, 2, 4],
 [3, 2, 1, 4],
 [3, 1, 4, 2],
 [3, 4, 1, 2],
 [4, 1, 2, 3],
 [4, 2, 1, 3],
 [4, 1, 3, 2],
 [4, 3, 1, 2]]
```

```
sage: len(d)
24
```

```
sage: random_element()  # random
[2, 3, 1, 4, 2]
```

```
sage: derangements([1,2,3,4,5])
sage: list(derangements([1,2,3,4,5]))
[[5, 2, 3, 4, 1],
 [5, 3, 2, 1, 4],
 [5, 4, 2, 3, 1],
 [5, 1, 2, 3, 4],
 [4, 5, 2, 3, 1],
 [4, 3, 5, 2, 1],
 [4, 2, 5, 3, 1],
 [4, 1, 5, 2, 3],
 [3, 5, 2, 1, 4],
 [3, 2, 5, 1, 4],
 [3, 1, 5, 2, 4],
 [3, 4, 5, 2, 1],
 [3, 4, 2, 5, 1],
 [3, 4, 1, 5, 2],
 [3, 4, 5, 1, 2],
 [3, 2, 4, 5, 1],
 [3, 2, 1, 5, 4],
 [3, 2, 4, 1, 5],
 [3, 2, 4, 5, 1],
 [3, 2, 1, 4, 5],
 [3, 2, 1, 5, 4],
 [3, 4, 2, 1, 5],
 [3, 4, 2, 5, 1],
 [3, 4, 1, 2, 5],
 [3, 4, 5, 1, 2]]
```

```
sage: len(derangements([1,2,3,4,5]))
sage: len(list(derangements([1,2,3,4,5])))
360
```

```
sage: random_element()  # random
[2, 5, 3, 4, 1]
```

**Element**

alias of `Derangement`

**cardinality()**

Counts the number of derangements of a positive integer, a list, or a string. The list or string may contain repeated elements. If an integer $n$ is given, the value returned is the number of derangements of $[1, 2, 3, \ldots, n]$. For an integer, or a list or string with all elements distinct, the value is obtained by the standard result $D_2 = 1, D_3 = 2, D_n = (n-1)(D_{n-1} + D_{n-2})$.

For a list or string with repeated elements, the number of derangements is computed by Macmahon’s theorem. If the numbers of repeated elements are $a_1, a_2, \ldots, a_k$ then the number of derangements is given by the coefficient of $x_1x_2\cdots x_k$ in the expansion of $\prod_{i=0}^{k} (S - s_i)^{a_i}$, where $S = x_1 + x_2 + \cdots + x_k$.

**EXAMPLES:**
sage: D = Derangements(5)
sage: D.cardinality()
44
sage: D = Derangements([1,44,918,67,254])
sage: D.cardinality()
44
sage: D = Derangements(['A', 'AT', 'CAT', 'CATS', 'CARTS'])
sage: D.cardinality()
44
sage: D = Derangements('UNCOPYRIGHTABLE')
sage: D.cardinality()
481066515734
sage: D = Derangements([1,1,2,2,3,3])
sage: D.cardinality()
10
sage: D = Derangements('SATTAS')
sage: D.cardinality()
10
sage: D = Derangements([1,1,2,2,2])
sage: D.cardinality()
0

random_element()

Produces all derangements of a positive integer, a list, or a string. The list or string may contain repeated elements. If an integer \( n \) is given, then a random derangement of \([1, 2, 3, \ldots, n]\) is returned.

For an integer, or a list or string with all elements distinct, the value is obtained by an algorithm described in [MPP2008]. For a list or string with repeated elements the derangement is formed by choosing an element at random from the list of all possible derangements.

OUTPUT:

A single list or string containing a derangement, or an empty list if there are no derangements.

EXAMPLES:
5.1.73 Descent Algebras

AUTHORS:
• Travis Scrimshaw (2013-07-28): Initial version

class sage.combinat.descent_algebra.DescentAlgebra(R, n)

Bases: UniqueRepresentation, Parent

Solomon’s descent algebra.

The descent algebra $\Sigma_n$ over a ring $R$ is a subalgebra of the symmetric group algebra $R S_n$. (The product in the latter algebra is defined by $(pq)(i) = q(p(i))$ for any two permutations $p$ and $q$ in $S_n$ and every $i \in \{1, 2, \ldots, n\}$. The algebra $\Sigma_n$ inherits this product.)

There are three bases currently implemented for $\Sigma_n$:

• the standard basis $D_S$ of (sums of) descent classes, indexed by subsets $S$ of $\{1, 2, \ldots, n-1\}$,
• the subset basis $B_p$, indexed by compositions $p$ of $n$,
• the idempotent basis $I_p$, indexed by compositions $p$ of $n$, which is used to construct the mutually orthogonal idempotents of the symmetric group algebra.

The idempotent basis is only defined when $R$ is a $\mathbb{Q}$-algebra.

We follow the notations and conventions in [GR1989], apart from the order of multiplication being different from the one used in that article. Schocker’s exposition [Sch2004], in turn, uses the same order of multiplication as we are, but has different notations for the bases.

INPUT:
• $R$ – the base ring
• $n$ – a nonnegative integer

REFERENCES:
• [GR1989]
• [At1992]
• [MR1995]
• [Sch2004]

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D(); D
Descent algebra of 4 over Rational Field in the standard basis
sage: B = DA.B(); B
Descent algebra of 4 over Rational Field in the subset basis
sage: I = DA.I(); I
Descent algebra of 4 over Rational Field in the idempotent basis

sage: basis_B = B.basis()
sage: elt = basis_B[Composition([1,2,1])] + 4*basis_B[Composition([1,3])]; elt
B[[1, 2, 1]] + 4*B[[1, 3]]
sage: D(elt)
5*D{} + 5*D{1} + D{1, 3} + D{3}
sage: I(elt)
7/6*I[[1, 1, 1, 1]] + 2*I[[1, 1, 2]] + 3*I[[1, 2, 1]] + 4*I[[1, 3]]
```
As syntactic sugar, one can use the notation $D[i, \ldots, l]$ to construct elements of the basis; note that for the empty set one must use $D[\{}\}$ due to Python’s syntax:

```
sage: D[{}] + D[2] + 2*D[1, 2]
D{} + 2*D{1, 2} + D{2}
```

The same syntax works for the other bases:

```
```

```python
class B(alg, prefix='B')

Bases: CombinatorialFreeModule, BindableClass

The subset basis of a descent algebra (indexed by compositions).

The subset basis $(B_S)_{S \subseteq \{1,2,\ldots,n-1\}}$ of $\Sigma_n$ is formed by

$$B_S = \sum_{T \subseteq S} D_T,$$

where $(D_S)_{S \subseteq \{1,2,\ldots,n-1\}}$ is the standard basis. However it is more natural to index the subset basis by compositions of $n$ under the bijection $\{i_1, i_2, \ldots, i_k\} \mapsto (i_1, i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k)$ (where $i_1 < i_2 < \cdots < i_k$), which is what Sage uses to index the basis.

The basis element $B_p$ is denoted $\Xi_p$ in [Sch2004].

By using compositions of $n$, the product $B_p B_q$ becomes a sum over the non-negative-integer matrices $M$ with row sum $p$ and column sum $q$. The summand corresponding to $M$ is $B_c$, where $c$ is the composition obtained by reading $M$ row-by-row from left-to-right and top-to-bottom and removing all zeroes. This multiplication rule is commonly called “Solomon’s Mackey formula”.

**EXAMPLES:**

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: list(B.basis())
[BP[1, 1, 1, 1], BP[1, 1, 2], BP[1, 2, 1], BP[1, 3],
 BP[2, 1, 1], BP[2, 2], BP[3, 1], BP[4]]
```

**one_basis()**

Return the identity element which is the composition $[n]$, as per AlgebrasWithBasis. ParentMethods.one_basis.

**EXAMPLES:**

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: B().one_basis()
[4]
sage: DA = DescentAlgebra(QQ, 0)
sage: B = DA.B()
sage: B().one_basis()
[]
```

```
sage: all( U * DA(B).one() == U
....:     for U in DA(B).basis() )
True
```

**product_on_basis(p, q)**

Return $B_p B_q$, where $p$ and $q$ are compositions of $n$.

**EXAMPLES:**
**to_D_basis(p)**

Return $B_p$ as a linear combination of $D$-basis elements.

**EXAMPLES:**

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: D = DA.D()
sage: list(map(D, B.basis()))  # indirect doctest
[D{}, + D{1} + D{1, 2} + D{1, 2, 3},
 + D{1, 3} + D{2} + D{2, 3} + D{3},
D{} + D{1} + D{1, 2} + D{2},
D{} + D{1} + D{1, 3} + D{3},
D{} + D{1},
D{} + D{2} + D{2, 3} + D{3},
D{} + D{2},
D{} + D{3},
D{1}]
```

**to_I_basis(p)**

Return $B_p$ as a linear combination of $I$-basis elements.

This is done using the formula

$$B_p = \sum_{q \leq p} \frac{1}{k!(q, p)} I_q,$$

where $\leq$ is the refinement order and $k!(q, p)$ is defined as follows: When $q \leq p$, we can write $q$ as a concatenation $q^{(1)} q^{(2)} \cdots q^{(k)}$ with each $q^{(i)}$ being a composition of the $i$-th entry of $p$, and then we set $k!(q, p)$ to be $l(q^{(1)})!l(q^{(2)})! \cdots l(q^{(k)})!$, where $l(r)$ denotes the number of parts of any composition $r$.

**EXAMPLES:**

```
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: I = DA.I()
sage: list(map(I, B.basis()))  # indirect doctest
[I[1, 1, 1, 1],
 1/2*I[1, 1, 1, 1] + I[1, 1, 2],
 1/2*I[1, 1, 1, 1] + I[1, 2, 1],
 1/2*I[1, 1, 1, 1] + I[1, 2, 1] + 1/2*I[1, 2, 1] + I[1, 2, 1] + I[1, 3],
 1/2*I[1, 1, 1, 1] + I[1, 2, 1] + 1/2*I[2, 1, 1],
 1/4*I[1, 1, 1, 1] + 1/2*I[1, 1, 2] + 1/2*I[1, 1, 2] + 1/2*I[1, 2, 1] + I[1, 2, 1] + I[1, 3],
 1/4*I[1, 1, 1, 1] + 1/2*I[1, 1, 2] + 1/2*I[1, 1, 2] + 1/2*I[2, 1, 1] + I[2, 2],
 1/6*I[1, 1, 1, 1] + 1/2*I[1, 1, 2] + 1/2*I[1, 1, 2] + 1/2*I[2, 1, 1] + I[3, 1],
```
to_nsym(p)

Return $B_p$ as an element in $NSym$, the non-commutative symmetric functions.

This maps $B_p$ to $S_p$ where $S$ denotes the Complete basis of $NSym$.

EXAMPLES:

```python
sage: B = DescentAlgebra(QQ, 4).B()
sage: S = NonCommutativeSymmetricFunctions(QQ).Complete()
sage: list(map(S, B.basis()))
```

```
[S[1, 1, 1, 1],
 S[1, 1, 2],
 S[1, 2, 1],
 S[1, 3],
 S[2, 1, 1],
 S[2, 2],
 S[3, 1],
 S[4]]
```

class D(alg, prefix='D')

Bases: CombinatorialFreeModule, BindableClass

The standard basis of a descent algebra.

This basis is indexed by $S \subseteq \{1, 2, \ldots, n-1\}$, and the basis vector indexed by $S$ is the sum of all permutations, taken in the symmetric group algebra $RS_n$, whose descent set is $S$. We denote this basis vector by $D_S$.

Occasionally this basis appears in literature but indexed by compositions of $n$ rather than subsets of $\{1, 2, \ldots, n-1\}$. The equivalence between these two indexings is owed to the bijection from the power set of $\{1, 2, \ldots, n-1\}$ to the set of all compositions of $n$ which sends every subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n-1\}$ (with $i_1 < i_2 < \cdots < i_k$) to the composition $(i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$.

The basis element corresponding to a composition $p$ (or to the subset of $\{1, 2, \ldots, n-1\}$) is denoted $\Delta^p$ in [Sch2004].

EXAMPLES:

```python
sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D()
sage: list(D.basis())
```

```
[D{}, D{1}, D{2}, D{3}, D{1, 2}, D{1, 3}, D{2, 3}, D{1, 2, 3}]
```

```python
sage: DA = DescentAlgebra(QQ, 0)
sage: D = DA.D()
sage: list(D.basis())
```

```
[D{}]
```

one_basis()

Return the identity element, as per AlgebrasWithBasis.ParentMethods.one_basis.

EXAMPLES:

```python
sage: DescentAlgebra(QQ, 4).D().one_basis()
```

```
()
```

```python
sage: DescentAlgebra(QQ, 0).D().one_basis()
```

```
()
```
sage: all( U * DescentAlgebra(QQ, 3).D().one() == U ...:     for U in DescentAlgebra(QQ, 3).D().basis() )
True

product_on_basis(S, T)
Return $D_SD_T$, where $S$ and $T$ are subsets of $[n-1]$.

EXAMPLES:

sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D()
sage: D.product_on_basis((1, 3), (2,))
$D{} + D{1} + D{1, 2} + 2*D{1, 2, 3} + D{1, 3} + D{2} + D{2, 3} + D{3}$

to_B_basis(S)
Return $D_S$ as a linear combination of $B_p$-basis elements.

EXAMPLES:

sage: DA = DescentAlgebra(QQ, 4)
sage: D = DA.D()
sage: B = DA.B()
sage: list(map(B, D.basis()))  # indirect doctest
[B[4],
 B[1, 3] - B[4],
 B[2, 2] - B[4],
 B[3, 1] - B[4],

to_symmetric_group_algebra_on_basis(S)
Return $D_S$ as a linear combination of basis elements in the symmetric group algebra.

EXAMPLES:

sage: D = DescentAlgebra(QQ, 4).D()
sage: [D.to_symmetric_group_algebra_on_basis(tuple(b)) ...:     for b in Subsets(3)]
[[1, 2, 3, 4],
 [2, 1, 3, 4] + [3, 1, 2, 4] + [4, 1, 2, 3],
 [1, 3, 2, 4] + [1, 4, 2, 3] + [2, 3, 1, 4] + [2, 4, 1, 3] + [3, 4, 1, 2],
 [1, 2, 4, 3] + [1, 3, 4, 2] + [2, 3, 4, 1],
 [3, 2, 1, 4] + [4, 2, 1, 3] + [4, 3, 1, 2],
 [2, 1, 4, 3] + [3, 1, 4, 2] + [3, 2, 4, 1] + [4, 1, 3, 2] + [4, 2, 3, 1],
 [1, 4, 3, 2] + [2, 4, 3, 1] + [3, 4, 2, 1],
 [4, 3, 2, 1]]
class I(alg, prefix='I')
Bases: CombinatorialFreeModule, BindableClass

The idempotent basis of a descent algebra.

The idempotent basis $\{I_p\}_{p\in\Sigma_n}$ is a basis for $\Sigma_n$ whenever the ground ring is a $\mathbb{Q}$-algebra. One way to compute it is using the formula (Theorem 3.3 in [GR1989])

\[ I_p = \sum_{q \leq p} (-1)^{l(q)-l(p)} \frac{k(q, p)}{k(q)} B_q, \]

where $\leq$ is the refinement order and $l(r)$ denotes the number of parts of any composition $r$, and where $k(q, p)$ is defined as follows: When $q \leq p$, we can write $q$ as a concatenation $q(1)q(2)\cdots q(k)$ with each $q(i)$ being a composition of the $i$-th entry of $p$, and then we set $k(q, p)$ to be the product $l(q(1))l(q(2))\cdots l(q(k))$.

Let $\lambda(p)$ denote the partition obtained from a composition $p$ by sorting. This basis is called the idempotent basis since for any $q$ such that $\lambda(p) = \lambda(q)$, we have:

\[ I_p I_q = s(\lambda) I_p \]

where $\lambda$ denotes $\lambda(p) = \lambda(q)$, and where $s(\lambda)$ is the stabilizer of $\lambda$ in $S_n$. (This is part of Theorem 4.2 in [GR1989].)

It is also straightforward to compute the idempotents $E_\lambda$ for the symmetric group algebra by the formula (Theorem 3.2 in [GR1989]):

\[ E_\lambda = \frac{1}{k!} \sum_{\lambda(p) = \lambda} I_p. \]

Note: The basis elements are not orthogonal idempotents.

EXAMPLES:

```
sage: DA = DescentAlgebra(QQ, 4)
sage: I = DA.I()
sage: list(I.basis())
[I[1, 1, 1, 1], I[1, 1, 2], I[1, 2, 1], I[1, 3], I[2, 1, 1], I[2, 2], I[3, 1], ...
˓→ I[4]]
```

idempotent(la)

Return the idempotent corresponding to the partition la of $n$.

EXAMPLES:

```
sage: I = DescentAlgebra(QQ, 4).I()
sage: E = I.idempotent([3,1]); E
1/2*I[1, 1, 3] + 1/2*I[3, 1]
sage: E*E == E
True
```
one()

Return the identity element, which is $B_{[n]}$, in the $I$ basis.

EXAMPLES:

```sage
sage: DescentAlgebra(QQ, 4).I().one()
```

one_basis()

The element 1 is not (generally) a basis vector in the $I$ basis, thus this returns a `TypeError`.

EXAMPLES:

```sage
sage: DescentAlgebra(QQ, 4).I().one_basis()
Traceback (most recent call last):
  ...
TypeError: 1 is not a basis element in the I basis
```

product_on_basis($p$, $q$)

Return $I_pI_q$, where $p$ and $q$ are compositions of $n$.

EXAMPLES:

```sage
sage: DA = DescentAlgebra(QQ, 4)
sage: I = DA.I()
sage: p = Composition([1,2,1])
sage: q = Composition([3,1])
sage: I.product_on_basis(p, q)
0
sage: I.product_on_basis(p, p)
2*I[1, 2, 1]
```

to_B_basis($p$)

Return $I_p$ as a linear combination of $B$-basis elements.

This is computed using the formula (Theorem 3.3 in [GR1989])

$$I_p = \sum_{q \leq p} \frac{(-1)^{l(q)-l(p)}}{k(q,p)} B_q,$$

where $\leq$ is the refinement order and $l(r)$ denotes the number of parts of any composition $r$, and where $k(q,p)$ is defined as follows: When $q \leq p$, we can write $q$ as a concatenation $q(1)q(2)\cdots q(k)$ with each $q(i)$ being a composition of the $i$-th entry of $p$, and then we set $k(q,p)$ to be $l(q(1))l(q(2))\cdots l(q(k))$.

EXAMPLES:

```sage
sage: DA = DescentAlgebra(QQ, 4)
sage: B = DA.B()
sage: I = DA.I()
sage: list(map(B, I.basis())) # indirect doctest
[B[1, 1, 1, 1],
-1/2*B[1, 1, 1, 1] + B[1, 1, 2],
```
a_realization()

Return a particular realization of self (the B-basis).

EXAMPLES:

```python
sage: DA = DescentAlgebra(QQ, 4)
sage: DA.a_realization()
Descent algebra of 4 over Rational Field in the subset basis
```

idempotent

alias of I

standard

alias of D

subset

alias of B

class sage.combinat.descent_algebra.DescentAlgebraBases(base)

Bases: Category_realization_of_parent

The category of bases of a descent algebra.

class ElementMethods

Bases: object

to_symmetric_group_algebra()

Return self in the symmetric group algebra.

EXAMPLES:

```python
sage: B = DescentAlgebra(QQ, 4).B()
sage: B[1,3].to_symmetric_group_algebra()
[1, 2, 3, 4] + [2, 1, 3, 4] + [3, 1, 2, 4] + [4, 1, 2, 3]
```

class ParentMethods

Bases: object

is_commutative()

Return whether this descent algebra is commutative.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: B = DescentAlgebra(QQ, 4).B()
sage: B.is_commutative()
False
sage: B = DescentAlgebra(QQ, 1).B()
sage: B.is_commutative()
True
```

`is_field(proof=True)`

Return whether this descent algebra is a field.

EXAMPLES:

```python
sage: B = DescentAlgebra(QQ, 4).B()
sage: B.is_field()
False
sage: B = DescentAlgebra(QQ, 1).B()
sage: B.is_field()
True
```

`to_symmetric_group_algebra()`

Morphism from self to the symmetric group algebra.

EXAMPLES:

```python
sage: D = DescentAlgebra(QQ, 4).D()
sage: D.to_symmetric_group_algebra(D[1,3])
[2, 1, 4, 3] + [3, 1, 4, 2] + [4, 1, 3, 2] + [4, 2, 3, 1]
sage: B = DescentAlgebra(QQ, 4).B()
sage: B.to_symmetric_group_algebra(B[1,2,1])
[1, 2, 3, 4] + [1, 2, 4, 3] + [1, 3, 4, 2] + [2, 1, 3, 4] + [2, 1, 4, 3] + [2, 3, 4, 1] + [3, 1, 2, 4] + [3, 1, 4, 2] + [3, 2, 4, 1] + [4, 1, 2, 3] + [4, 1, 3, 2] + [4, 2, 3, 1]
```

`to_symmetric_group_algebra_on_basis(S)`

Return the basis element index by S as a linear combination of basis elements in the symmetric group algebra.

EXAMPLES:

```python
sage: B = DescentAlgebra(QQ, 3).B()
sage: [B.to_symmetric_group_algebra_on_basis(c) for c in Compositions(3)]
[[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1],
 [1, 2, 3] + [1, 3, 2] + [2, 1, 3],
 [1, 2, 3] + [1, 3, 2],
 [1, 2, 3]]
sage: I = DescentAlgebra(QQ, 3).I()
sage: [I.to_symmetric_group_algebra_on_basis(c) for c in Compositions(3)]
[[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1],
 1/2*[1, 2, 3] - 1/2*[1, 3, 2] + 1/2*[2, 1, 3] - 1/2*[2, 3, 1] - 1/2*[3, 1, 2] + 1/2*[3, 2, 1],
 1/2*[1, 2, 3] - 1/2*[1, 3, 2] + 1/2*[2, 1, 3] - 1/2*[2, 3, 1] - 1/2*[3, 1, 2] + 1/2*[3, 2, 1],
 1/2*[1, 2, 3] - 1/2*[1, 3, 2] + 1/2*[2, 1, 3] - 1/2*[2, 3, 1] - 1/2*[3, 1, 2] + 1/2*[3, 2, 1],
 1/2*[1, 2, 3] - 1/2*[1, 3, 2] + 1/2*[2, 1, 3] - 1/2*[2, 3, 1] - 1/2*[3, 1, 2] + 1/2*[3, 2, 1]]
```

(continues on next page)
1/2*[1, 2, 3] + 1/2*[1, 3, 2] - 1/2*[2, 1, 3] + 1/2*[2, 3, 1] - 1/2*[3, 1, 2] - 1/2*[3, 2, 1],
1/3*[1, 2, 3] - 1/6*[1, 3, 2] - 1/6*[2, 1, 3] - 1/6*[2, 3, 1] - 1/6*[3, 1, 2] + 1/3*[3, 2, 1]

**super_categories()**

The super categories of `self`.

**EXAMPLES:**

```sage
from sage.combinat.descent_algebra import DescentAlgebraBases
DA = DescentAlgebra(QQ, 4)
bases = DescentAlgebraBases(DA)
bases.super_categories()
```

[Category of finite dimensional algebras with basis over Rational Field, Category of realizations of Descent algebra of 4 over Rational Field]

### 5.1.74 Combinatorial designs and incidence structures

All designs can be accessed by `designs.<tab>` and are listed in the design catalog:

- **Catalog of designs**

**Design-related classes**

- **Incidence structures (i.e. hypergraphs, i.e. set systems)**
- **Covering designs: coverings of \( t \)-element subsets of a \( v \)-set by \( k \)-sets**

**Constructions**

- **Block designs**
- **Balanced Incomplete Block Designs (BIBD)**
- **Resolvable Balanced Incomplete Block Design (RBIBD)**
- **Group-Divisible Designs (GDD)**
- **Mutually Orthogonal Latin Squares (MOLS)**
- **Orthogonal arrays (OA)**
- **Orthogonal arrays (build recursive constructions)**
- **Orthogonal arrays (find recursive constructions)**
- **Difference families**
- **Difference Matrices**
- **Steiner Quadruple Systems**
- **Two-graphs**
- **Database of small combinatorial designs**
- **Database of generalised quadrangles with spread**

**Technical things**

- **External Representations of Block Designs**
5.1.75 Balanced Incomplete Block Designs (BIBD)

This module gathers everything related to Balanced Incomplete Block Designs. One can build a BIBD (or check that it can be built) with `balanced_incomplete_block_design()`:

```
sage: BIBD = designs.balanced_incomplete_block_design(7,3,1)
```

In particular, Sage can build a \((v,k,1)\)-BIBD when one exists for all \(k \leq 5\). The following functions are available:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>balanced_incomplete_block_design()</code></td>
<td>Return a BIBD of parameters (v,k,\lambda).</td>
</tr>
<tr>
<td><code>BIBD_from_TD()</code></td>
<td>Return a BIBD through TD-based constructions.</td>
</tr>
<tr>
<td><code>BIBD_from_difference_family()</code></td>
<td>Return the BIBD associated to the difference family (D) on the group (G).</td>
</tr>
<tr>
<td><code>BIBD_from_PBD()</code></td>
<td>Return a ((v,k,1))-BIBD from a ((r,K))-PBD where (r = (v-1)/(k-1)).</td>
</tr>
<tr>
<td><code>PBD_from_TD()</code></td>
<td>Return a ((kt,{k,t}))-PBD if (u = 0) and a ((kt+u,{k,k+1,t,u}))-PBD otherwise.</td>
</tr>
<tr>
<td><code>steiner_triple_system()</code></td>
<td>Return a Steiner Triple System.</td>
</tr>
<tr>
<td><code>v_5_1_BIBD()</code></td>
<td>Return a ((v,5,1))-BIBD.</td>
</tr>
<tr>
<td><code>v_4_1_BIBD()</code></td>
<td>Return a ((v,4,1))-BIBD.</td>
</tr>
<tr>
<td><code>PBD_4_5_8_9_12()</code></td>
<td>Return a ((v,{4,5,8,9,12}))-PBD on (v) elements.</td>
</tr>
<tr>
<td><code>BIBD_5q_5_for_q_prime_power()</code></td>
<td>Return a ((5q,5,1))-BIBD with (q \equiv 1\pmod{4}) a prime power.</td>
</tr>
</tbody>
</table>

Construction of BIBD when \(k = 4\)

Decompositions of \(K_v\) into \(K_4\) (i.e. \((v,4,1)\)-BIBD) are built following Douglas Stinson’s construction as presented in [Stinson2004] page 167. It is based upon the construction of \((v\{4,5,8,9,12\})\)-PBD (see the doc of `PBD_4_5_8_9_12()`), knowing that a \((v\{4,5,8,9,12\})\)-PBD on \(v\) points can always be transformed into a \(((k-1)v+1,4,1)\)-BIBD, which covers all possible cases of \((v,4,1)\)-BIBD.

Construction of BIBD when \(k = 5\)

Decompositions of \(K_v\) into \(K_4\) (i.e. \((v,4,1)\)-BIBD) are built following Clayton Smith’s construction [ClaytonSmith].

Functions

```
sage.combinat.designs.bibd.BIBD
    alias of BalancedIncompleteBlockDesign

sage.combinat.designs.bibd.BIBD_5q_5_for_q_prime_power(q)
    Return a \((5q,5,1)\)-BIBD with \(q \equiv 1\pmod{4}\) a prime power.
```

See Theorem 24 [ClaytonSmith].

INPUT:

- \(q\) (integer) – a prime power such that \(q \equiv 1\pmod{4}\).

EXAMPLES:
sage.combinat.designs.bibd.BIBD_from_PBD(PBD, v, k, check=True, base_cases=None)

Return a \((v, k, 1)\)-BIBD from a \((r, K)\)-PBD where \(r = (v - 1)/(k - 1)\).

This is Theorem 7.20 from [Stinson2004].

**INPUT:**

- \(v, k\) – integers.
- **PBD** – A PBD on \(r = (v - 1)/(k - 1)\) points, such that for any block of PBD of size \(s\) there must exist a \(((k - 1)s + 1, k, 1)\)-BIBD.
- **check** (boolean) – whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.
- **base_cases** – caching system, for internal use.

**EXAMPLES:**

```sage
sage: from sage.combinat.designs.bibd import PBD_4_5_8_9_12
sage: from sage.combinat.designs.bibd import BIBD_from_PBD
sage: from sage.combinat.designs.bibd import is_pairwise_balanced_design
sage: PBD = PBD_4_5_8_9_12(17)
sage: bibd = is_pairwise_balanced_design(BIBD_from_PBD(PBD, 52, 4), 52, [4])
```

sage.combinat.designs.bibd.BIBD_from_TD(v, k, existence=False)

Return a BIBD through TD-based constructions.

**INPUT:**

- \(v, k\) – (integers) computes a \((v, k, 1)\)-BIBD.
- **existence** – (boolean) instead of building the design, return:
  - True – meaning that Sage knows how to build the design
  - Unknown – meaning that Sage does not know how to build the design, but that the design may exist (see sage.misc.unknown)
  - False – meaning that the design does not exist

This method implements three constructions:

- If there exists a \(TD(k, v)\) and a \((v, k, 1)\)-BIBD then there exists a \((kv, k, 1)\)-BIBD.
  The BIBD is obtained from all blocks of the \(TD\), and from the blocks of the \((v, k, 1)\)-BIBDs defined over the \(k\) groups of the \(TD\).
- If there exists a \(TD(k, v)\) and a \((v + 1, k, 1)\)-BIBD then there exists a \((kv + 1, k, 1)\)-BIBD.
  The BIBD is obtained from all blocks of the \(TD\), and from the blocks of the \((v + 1, k, 1)\)-BIBDs defined over the sets \(V_1 \cup \infty, \ldots, V_k \cup \infty\) where the \(V_1, \ldots, V_k\) are the groups of the TD.
- If there exists a \(TD(k, v)\) and a \((v + k, k, 1)\)-BIBD then there exists a \((kv + k, k, 1)\)-BIBD.
  The BIBD is obtained from all blocks of the \(TD\), and from the blocks of the \((v + k, k, 1)\)-BIBDs defined over the sets \(V_1 \cup \{\infty_1, \ldots, \infty_k\}, \ldots, V_k \cup \{\infty_1, \ldots, \infty_k\}\) where the \(V_1, \ldots, V_k\) are the groups of the TD.
By making sure that all copies of the \((v + k, k, 1)\)-BIBD contain the block \(\{\infty_1, \ldots, \infty_k\}\), the result is also a BIBD.

These constructions can be found in [http://www.argilo.net/files/bibd.pdf](http://www.argilo.net/files/bibd.pdf).

**EXAMPLES:**

First construction:

```python
sage: from sage.combinat.designs.bibd import BIBD_from_TD
sage: BIBD_from_TD(25,5,existence=True)
True
sage: _ = BlockDesign(25,BIBD_from_TD(25,5))
```

Second construction:

```python
sage: from sage.combinat.designs.bibd import BIBD_from_TD
sage: BIBD_from_TD(21,5,existence=True)
True
sage: _ = BlockDesign(21,BIBD_from_TD(21,5))
```

Third construction:

```python
sage: from sage.combinat.designs.bibd import BIBD_from_TD
sage: BIBD_from_TD(85,5,existence=True)
True
sage: _ = BlockDesign(85,BIBD_from_TD(85,5))
```

No idea:

```python
sage: from sage.combinat.designs.bibd import BIBD_from_TD
sage: BIBD_from_TD(20,5,existence=True)
Unknown
sage: BIBD_from_TD(20,5)
Traceback (most recent call last):
...
NotImplementedError: I do not know how to build a (20,5,1)-BIBD!
```

`sage.combinat.designs.bibd.BIBD_from_arc_in_desarguesian_projective_plane(n, k, existence=False)`

Return a \((n, k, 1)\)-BIBD from a maximal arc in a projective plane.

This function implements a construction from Denniston [Denniston69], who describes a maximal arc in a Desarguesian Projective Plane of order \(2^k\). From two powers of two \(n, q\) with \(n < q\), it produces a \(((n - 1)(q + 1) + 1, n, 1)\)-BIBD.

**INPUT:**

- \(n, k\) (integers) – must be powers of two (among other restrictions).
- \(existence\) (boolean) – whether to return the BIBD obtained through this construction (default), or to merely indicate with a boolean return value whether this method can build the requested BIBD.

**EXAMPLES:**

A \((232, 8, 1)\)-BIBD:
```python
sage: from sage.combinat.designs.bibd import BIBD_from_arc_in_desarguesian_projective_plane
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: D = BIBD_from_arc_in_desarguesian_projective_plane(232, 8)
sage: BalancedIncompleteBlockDesign(232, D)
(232, 8, 1)-Balanced Incomplete Block Design

sage: D = BIBD_from_arc_in_desarguesian_projective_plane(120, 8)
sage: BalancedIncompleteBlockDesign(120, D)
(120, 8, 1)-Balanced Incomplete Block Design

sage: all(BIBD_from_arc_in_desarguesian_projective_plane(n, k, existence=True) for n, k in [(120, 8), (232, 8), (456, 8), (904, 8), (496, 16), (976, 16), (1936, 16), (2016, 32), (4000, 32), (8128, 64)])
True

Of course, not all can be built this way:

sage: BIBD_from_arc_in_desarguesian_projective_plane(7, 3, existence=True)
False
sage: BIBD_from_arc_in_desarguesian_projective_plane(7, 3)
Traceback (most recent call last):
  ... ValueError: This function cannot produce a (7,3,1)-BIBD

REFERENCE:

sage.combinat.designs.bibd.BIBD_from_difference_family(G, D, lambd=None, check=True)
Return the BIBD associated to the difference family D on the group G.

Let $G$ be a group. A $(G, k, \lambda)$-difference family is a family $B = \{B_1, B_2, \ldots, B_b\}$ of $k$-subsets of $G$ such that for each element of $G \setminus \{0\}$ there exists exactly $\lambda$ pairs of elements $(x, y)$, $x$ and $y$ belonging to the same block, such that $x - y = g$ (or $x y^{-1} = g$ in multiplicative notation).

If $\{B_1, B_2, \ldots, B_b\}$ is a $(G, k, \lambda)$-difference family then its set of translates $\{B_i \cdot g; i \in \{1, \ldots, b\}, g \in G\}$ is a $(v, k, \lambda)$-BIBD where $v$ is the cardinality of $G$.

INPUT:

- $G$: a finite additive Abelian group
- $D$: a difference family on $G$ (short blocks are allowed).
- $\lambda$: the $\lambda$ parameter (optional, only used if check is True)
- $check$: whether or not we check the output (default: True)

EXAMPLES:

```python
sage: G = Zmod(21)
sage: D = [[0, 1, 4, 14, 16]]
sage: sorted(G(x-y) for x in D[0] for y in D[0] if x != y)
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]
```
sage: from sage.combinat.designs.bibd import BIBD_from_difference_family
sage: BIBD_from_difference_family(G, D)
[[0, 1, 4, 14, 16],
 [1, 2, 5, 15, 17],
 [2, 3, 6, 16, 18],
 [3, 4, 7, 17, 19],
 [4, 5, 8, 18, 20],
 [5, 6, 9, 19, 0],
 [6, 7, 10, 20, 1],
 [7, 8, 11, 0, 2],
 [8, 9, 12, 1, 3],
 [9, 10, 13, 2, 4],
 [10, 11, 14, 3, 5],
 [11, 12, 15, 4, 6],
 [12, 13, 16, 5, 7],
 [13, 14, 17, 6, 8],
 [14, 15, 18, 7, 9],
 [15, 16, 19, 8, 10],
 [16, 17, 20, 9, 11],
 [17, 18, 0, 10, 12],
 [18, 19, 1, 11, 13],
 [19, 20, 2, 12, 14],
 [20, 0, 3, 13, 15]]

class sage.combinat.designs.bibd.BalancedIncompleteBlockDesign(points, blocks, k=None, lambd=1, check=True, copy=True, **kwds)

Bases: PairwiseBalancedDesign

Balanced Incomplete Block Design (BIBD)

INPUT:

- **points** – the underlying set. If points is an integer \( v \), then the set is considered to be \{0, ..., v - 1\}.
- **blocks** – collection of blocks
- **k** (integer) – size of the blocks. Set to None (automatic guess) by default.
- **lambd** (integer) – value of \( \lambda \), set to 1 by default.
- **check** (boolean) – whether to check that the design is a PBD with the right parameters.
- **copy** – (use with caution) if set to False then blocks must be a list of lists of integers. The list will not be copied but will be modified in place (each block is sorted, and the whole list is sorted). Your blocks object will become the instance’s internal data.

EXAMPLES:

sage: b=designs.balanced_incomplete_block_design(9,3); b
(9,3,1)-Balanced Incomplete Block Design

arc(s, solver=2, verbose=None, integrality_tolerance=0)

Return the s-arc with maximum cardinality.

A s-arc is a subset of points in a BIBD that intersects each block on at most s points. It is one possible generalization of independent set for graphs.
A simple counting shows that the cardinality of a \( s \)-arc is at most \((s - 1) \times r + 1\) where \( r \) is the number of blocks incident to any point. A \( s \)-arc in a BIBD with cardinality \((s - 1) \times r + 1\) is called maximal and is characterized by the following property: it is not empty and each block either contains 0 or \( s \) points of this arc. Equivalently, the trace of the BIBD on these points is again a BIBD (with block size \( s \)).

For more informations, see Wikipedia article Arc_(projective_geometry).

**INPUT:**

- \( s \) - (default to 2) the maximum number of points from the arc in each block
- \( \text{solver} \) – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method \texttt{solve} of the class \texttt{MixedIntegerLinearProgram}.
- \( \text{verbose} \) – integer (default: 0). Sets the level of verbosity. Set to 0 by default, which means quiet.
- \( \text{integrality_tolerance} \) – parameter for use with MILP solvers over an inexact base ring; see \texttt{MixedIntegerLinearProgram.get_values()}

**EXAMPLES:**

```python
sage: B = designs.balanced_incomplete_block_design(21, 5)
sage: a2 = B.arc()
sage: a2  # random
[5, 9, 10, 12, 15, 20]
sage: len(a2)
6
sage: a4 = B.arc(4)
sage: a4  # random
[0, 1, 2, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20]
sage: len(a4)
16
```

The 2-arc and 4-arc above are maximal. One can check that they intersect the blocks in either 0 or \( s \) points. Or equivalently that the traces are again BIBD:

```python
sage: r = (21-1)//(5-1)
sage: len(a2), 1 + r
(8, 9)
sage: sa2 = set(a2)
sage: set(len(sa2.intersection(b)) for b in B.blocks())

sage: B.trace(a2).is_t_design(2, return_parameters=True)
(True, (2, 6, 2, 1))
sage: B.trace(a4).is_t_design(2, return_parameters=True)
(True, (2, 16, 4, 1))
```

Some other examples which are not maximal:

```python
sage: B = designs.balanced_incomplete_block_design(25, 4)
sage: a2 = B.arc(2)
sage: r = (25-1)//(4-1)
sage: len(a2), 1 + r
(8, 9)
sage: sa2 = set(a2)
sage: set(len(sa2.intersection(b)) for b in B.blocks())
```

(continues on next page)
sage: B.trace(a2).is_t_design(2)
False

sage: a3 = B.arc(3)
sage: len(a3), 1 + 2^r
(15, 17)
sage: sa3 = set(a3)
sage: set(len(sa3.intersection(b)) for b in B.blocks()) == set([0,3])
False
sage: B.trace(a3).is_t_design(3)
False

sage.combinat.designs.bibd.BruckRyserChowla_check(v, k, lambd)
Check whether the parameters passed satisfy the Bruck-Ryser-Chowla theorem.

For more information on the theorem, see the corresponding Wikipedia entry.

INPUT:

• v, k, lambd – integers; parameters to check

OUTPUT:

• True – the parameters satisfy the theorem
• False – the theorem fails for the given parameters
• Unknown – the preconditions of the theorem are not met

EXAMPLES:

sage: from sage.combinat.designs.bibd import BruckRyserChowla_check
sage: BruckRyserChowla_check(22,7,2) False

Nonexistence of projective planes of order 6 and 14

sage: from sage.combinat.designs.bibd import BruckRyserChowla_check
sage: BruckRyserChowla_check(43,7,1) False
sage: BruckRyserChowla_check(211,15,1) False

Existence of symmetric BIBDs with parameters (79,13,2) and (56,11,2)

sage: from sage.combinat.designs.bibd import BruckRyserChowla_check
sage: BruckRyserChowla_check(79,13,2) True
sage: BruckRyserChowla_check(56,11,2) True

sage.combinat.designs.bibd.PBD_4_5_8_9_12(v, check=True)
Return a \((v, \{4,5,8,9,12\})\)-PBD on \(v\) elements.

A \((v, \{4,5,8,9,12\})\)-PBD exists if and only if \(v \equiv 0,1 \pmod{4}\). The construction implemented here appears page 168 in [Stinson2004].

INPUT:

• v – an integer congruent to 0 or 1 modulo 4.
• check (boolean) – whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

EXAMPLES:
sage: designs.balanced_incomplete_block_design(40,4).blocks() # indirect doctest
[[0, 1, 2, 12], [0, 3, 6, 9], [0, 4, 8, 10], [0, 5, 7, 11], [0, 13, 26, 39], [0, 14, 25, 28], [0, 15, 27, 38], [0, 16, 22, 32], [0, 17, 23, 34], ...

Check that github issue #16476 is fixed:

sage: from sage.combinat.designs.bibd import PBD_4_5_8_9_12
sage: for v in (0,1,4,5,8,9,12,13,16,17,20,21,24,25):
....: _ = PBD_4_5_8_9_12(v)

sage.combinat.designs.bibd.PBD_from_TD(k, t, u)

Return a \((kt,\{k,t\})\)-PBD if \(u=0\) and a \((kt+u,\{k,k+1,t,u\})\)-PBD otherwise.

This is theorem 23 from [ClaytonSmith]. The PBD is obtained from the blocks a truncated \(TD(k+1,t)\), to which are added the blocks corresponding to the groups of the TD. When \(u=0\), a \(TD(k,t)\) is used instead.

INPUT:

- \(k, t, u\) – integers such that \(0 \leq u \leq t\).

EXAMPLES:

sage: from sage.combinat.designs.bibd import PBD_from_TD
sage: from sage.combinat.designs.bibd import is_pairwise_balanced_design
sage: PBD = PBD_from_TD(2,2,1); PBD
[[0, 2, 4], [0, 3], [1, 2], [1, 3, 4], [0, 1], [2, 3]]
sage: is_pairwise_balanced_design(PBD,2*2+1,[2,3])
True

class sage.combinat.designs.bibd.PairwiseBalancedDesign(points, blocks, K=None, lambda=1, check=True, copy=True, **kwds)

Bases: GroupDivisibleDesign

Pairwise Balanced Design (PBD)

A Pairwise Balanced Design, or \((v, K, \lambda)\)-PBD, is a collection \(B\) of blocks defined on a set \(X\) of size \(v\), such that any block pair of points \(p_1, p_2 \in X\) occurs in exactly \(\lambda\) blocks of \(B\). Besides, for every block \(B \in B\) we must have \(|B| \in K\).

INPUT:

- **points** – the underlying set. If **points** is an integer \(v\), then the set is considered to be \(\{0, ..., v - 1\}\).
- **blocks** – collection of blocks
- **K** – list of integers of which the sizes of the blocks must be elements. Set to **None** (automatic guess) by default.
- **lambda** (integer) – value of \(\lambda\), set to 1 by default.
- **check** (boolean) – whether to check that the design is a \(PBD\) with the right parameters.
- **copy** – (use with caution) if set to **False** then **blocks** must be a list of lists of integers. The list will not be copied but will be modified in place (each block is sorted, and the whole list is sorted). Your **blocks** object will become the instance’s internal data.

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sage.combinat.designs.bibd.balanced_incomplete_block_design(v, k, lambd=1, existence=False, use_LJCR=False)

Return a BIBD of parameters $v, k, \lambda$.

A Balanced Incomplete Block Design of parameters $v, k, \lambda$ is a collection $C$ of $k$-subsets of $V = \{0, \ldots, v-1\}$ such that for any two distinct elements $x, y \in V$ there are $\lambda$ elements $S \in C$ such that $x, y \in S$.

For more information on BIBD, see the corresponding Wikipedia entry.

INPUT:

- $v, k, \lambda$ (integers)
- $existence$ (boolean) – instead of building the design, return:
  - $True$ – meaning that Sage knows how to build the design
  - $Unknown$ – meaning that Sage does not know how to build the design, but that the design may exist (see sage.misc.unknown).
  - $False$ – meaning that the design does not exist.
- $use_LJCR$ (boolean) – whether to query the La Jolla Covering Repository for the design when Sage does not know how to build it (see best_known_covering_design_www()). This requires internet.

See also:

- steiner_triple_system()
- v_4_1_BIBD()
- v_5_1_BIBD()

Todo: Implement other constructions from the Handbook of Combinatorial Designs.

EXAMPLES:

```sage
designs.balanced_incomplete_block_design(7, 3, 1).blocks()
[[0, 1, 3], [0, 2, 4], [0, 5, 6], [1, 2, 6], [1, 4, 5], [2, 3, 5], [3, 4, 6]]
designs.balanced_incomplete_block_design(66, 6, 1, use_LJCR=True) # optional - internet
B = designs.balanced_incomplete_block_design(66, 6, 1, use_LJCR=True) # optional - internet
(66,6,1)-Balanced Incomplete Block Design
B.blocks() # optional - internet
[[0, 1, 2, 3, 4, 65], [0, 5, 22, 32, 38, 58], [0, 6, 21, 30, 43, 48], ...
designs.balanced_incomplete_block_design(216, 6, 1)
Traceback (most recent call last):
... NotImplementedError: I don’t know how to build a (216,6,1)-BIBD!
```

sage.combinat.designs.bibd.biplane(n, existence=False)

Return a biplane of order $n$.

A biplane of order $n$ is a symmetric $(1 + \frac{(n+1)(n+2)}{2}, n + 2, 2)$-BIBD. A symmetric (or square) $(v, k, \lambda)$-BIBD is a $(v, k, \lambda)$-BIBD with $v$ blocks.

INPUT:
• $n$ – (integer) order of the biplane

• `existence` (boolean) – instead of building the design, return:
  – `True` – meaning that Sage knows how to build the design
  – `Unknown` – meaning that Sage does not know how to build the design, but that the design may exist (see `sage.misc.unknown`).
  – `False` – meaning that the design does not exist.

See also:

• `balanced_incomplete_block_design()`

EXAMPLES:

```sage
sage: designs.biplane(4)
(16,6,2)-Balanced Incomplete Block Design
sage: designs.biplane(7, existence=True)
True
sage: designs.biplane(11)
(79,13,2)-Balanced Incomplete Block Design
```

`sage.combinat.designs.bibd.steiner_triple_system(n)`

Return a Steiner Triple System

A Steiner Triple System (STS) of a set $\{0, \ldots, n-1\}$ is a family $S$ of 3-sets such that for any $i \neq j$ there exists exactly one set of $S$ in which they are both contained.

It can alternatively be thought of as a factorization of the complete graph $K_n$ with triangles.

A Steiner Triple System of a $n$-set exists if and only if $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$, in which case one can be found through Bose’s and Skolem’s constructions, respectively [AndHonk97].

INPUT:

• $n$ return a Steiner Triple System of $\{0, \ldots, n-1\}$

EXAMPLES:

A Steiner Triple System on 9 elements

```sage
sage: sts = designs.steiner_triple_system(9)
sage: sts
(9,3,1)-Balanced Incomplete Block Design
sage: list(sts)
[[0, 1, 5], [0, 2, 4], [0, 3, 6], [0, 7, 8], [1, 2, 3],
 [1, 4, 7], [1, 6, 8], [2, 5, 8], [2, 6, 7], [3, 4, 8],
 [3, 5, 7], [4, 5, 6]]
```

As any pair of vertices is covered once, its parameters are

```sage
sage: sts.is_t_design(return_parameters=True)
(True, (2, 9, 3, 1))
```

An exception is raised for invalid values of $n$
sage: designs.steiner_triple_system(10)
Traceback (most recent call last):
...
EmptySetError: Steiner triple systems only exist for n = 1 mod 6 or n = 3 mod 6

REFERENCE:
sage.combinat.designs.bibd.v_4_1_BIBD(v, check=True)

Return a \((v, 4, 1)\)-BIBD.

A \((v, 4, 1)\)-BIBD is an edge-decomposition of the complete graph \(K_v\) into copies of \(K_4\). For more information, see \texttt{balanced_incomplete_block_design()}. It exists if and only if \(v \equiv 1, 4 \pmod{12}\).

See page 167 of [Stinson2004] for the construction details.

See also:
\begin{itemize}
  \item \texttt{balanced_incomplete_block_design()}
\end{itemize}

INPUT:
\begin{itemize}
  \item \texttt{v} (integer) – number of points.
  \item \texttt{check} (boolean) – whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to \texttt{True} by default.
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.designs.bibd import v_4_1_BIBD  # long time
sage: for n in range(13,100):  # long time
....:   if n%12 in [1,4]:
....:       _ = v_4_1_BIBD(n, check = True)
\end{verbatim}

sage.combinat.designs.bibd.v_5_1_BIBD(v, check=True)

Return a \((v, 5, 1)\)-BIBD.

This method follows the construction from [ClaytonSmith].

INPUT:
\begin{itemize}
  \item \texttt{v} (integer)
\end{itemize}

See also:
\begin{itemize}
  \item \texttt{balanced_incomplete_block_design()}
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.designs.bibd import v_5_1_BIBD
sage: i = 0
sage: while i<200:
....:   i += 20
....:   _ = v_5_1_BIBD(i+1)
....:   _ = v_5_1_BIBD(i+5)
\end{verbatim}
5.1.76 Resolvable Balanced Incomplete Block Design (RBIBD)

This module contains everything related to resolvable Balanced Incomplete Block Designs. The constructions implemented here can be obtained through the `designs` object:

```
designs.resolvable_balanced_incomplete_block_design(15,3)
```

For Balanced Incomplete Block Design (BIBD) see the module `bibd`. A BIBD is said to be `resolvable` if its blocks can be partitioned into parallel classes, i.e., partitions of its ground set.

The main function of this module is `resolvable_balanced_incomplete_block_design()`, which calls all others.

References:

Functions

```
sage.combinat.designs.resolvable_bibd.PBD_4_7(v, check=True, existence=False)
```

Return a $(v,\{4,7\})$-PBD

For all $v$ such that $n \equiv 1 \pmod{3}$ and $n \neq 10, 19, 31$ there exists a $(v,\{4,7\})$-PBD. This is proved in Proposition IX.4.5 from [BJL99], which this method implements.

This construction of PBD is used by the construction of Kirkman Triple Systems.

```
sage: from sage.combinat.designs.resolvable_bibd import PBD_4_7
sage: PBD_4_7(22)
Pairwise Balanced Design on 22 points with sets of sizes in [4, 7]
```

```
sage.combinat.designs.resolvable_bibd.PBD_4_7_from_Y(gdd, check=True)
```

Return a $(3v+1,\{4,7\})$-PBD from a $(v,\{4,5,7\},N-\{3,6,10\})$-GDD.

This implements Lemma IX.3.11 from [BJL99] (p.625). All points of the GDD are tripled, and a $+\infty$ point is added to the design.

```
• A group of size $s \in Y = N - \{3, 6, 10\}$ becomes a set of size $3s$. Adding $\infty$ to it gives it size $3s + 1$, and this set is then replaced by a $(3s + 1,\{4,7\})$-PBD.
```

```
• A block of size $s \in \{4, 5, 7\}$ becomes a $(3s,\{4,7\},\{3\})$-GDD.
```

This lemma is part of the existence proof of $(v,\{4,7\})$-PBD as explained in IX.4.5 from [BJL99]).

INPUT:

```
• gdd – a $(v,\{4,5,7\},Y)$-GDD where $Y = N - \{3,6,10\}$.
• check – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.
```

EXAMPLES:
sage: from sage.combinat.designs.resolvable_bibd import PBD_4_7_from_Y
sage: PBD_4_7_from_Y(designs.transversal_design(7,8))
Pairwise Balanced Design on 169 points with sets of sizes in [4, 7]

sage.combinat.designs.resolvable_bibd.kirkman_triple_system(v, existence=False)
Return a Kirkman Triple System on \( v \) points.
A Kirkman Triple System \( KTS(v) \) is a resolvable Steiner Triple System. It exists if and only if \( v \equiv 3 \pmod{6} \).

INPUT:
- \( n \) (integer)
- \( existence \) (boolean; False by default) – whether to build the \( KTS(n) \) or only answer whether it exists.

See also:
IncidenceStructure.is_resolvable()

EXAMPLES:
A solution to Kirkman’s original problem:

sage: kts = designs.kirkman_triple_system(15)
sage: classes = kts.is_resolvable(1)[1]
sage: names = '0123456789abcdef'
sage: def to_name(r_s_t):
....:     r, s, t = r_s_t
....:     return ' ' + names[r] + names[s] + names[t] + '

sage: rows = [' '.join(('Day {0}'.format(i) for i in range(1,8)))]
sage: rows.extend(' '.join(map(to_name,row)) for row in zip(*classes))
sage: print('
'.join(rows))
Day 1 Day 2 Day 3 Day 4 Day 5 Day 6 Day 7
07e 18e 29e 3ae 4be 5ce 6de
139 24a 35b 46c 05d 167 028
26b 03c 14d 257 368 049 15a
458 569 06a 01b 12c 23d 347
acd 7bd 78c 89d 79a 8ab 9bc

sage.combinat.designs.resolvable_bibd.resolvable_balanced_incomplete_block_design(v, k, existence=False)
Return a resolvable BIBD of parameters \( v, k \).

A BIBD is said to be resolvable if its blocks can be partitionned into parallel classes, i.e. partitions of the ground set.

INPUT:
- \( v, k \) (integers)
- \( existence \) (boolean) – instead of building the design, return:
  - True – meaning that Sage knows how to build the design
  - Unknown – meaning that Sage does not know how to build the design, but that the design may exist (see sage.misc.unknown).
  - False – meaning that the design does not exist.

See also:
Combinatorics, Release 10.0

- `IncidenceStructure.is_resolvable()`
- `balanced_incomplete_block_design()`

**EXAMPLES:**

```
sage: KTS15 = designs.resolvable_balanced_incomplete_block_design(15,3); KTS15
(15,3,1)-Balanced Incomplete Block Design
sage: KTS15.is_resolvable()
True
```

```python
sage.combinat.designs.resolvable_bibd.v_4_1_rbibd(v, existence=False)
```

Return a \((v,4,1)\)-RBIBD.

**INPUT:**

- \(n\) (integer)
- `existence` (boolean; False by default) – whether to build the design or only answer whether it exists.

**See also:**

- `IncidenceStructure.is_resolvable()`
- `resolvable_balanced_incomplete_block_design()`

**Note:** A resolvable \((v,4,1)\)-BIBD exists whenever \(1 \equiv 4 \pmod{12}\). This function, however, only implements a construction of \((v,4,1)\)-BIBD such that \(v = 3q + 1 \equiv 1 \pmod{3}\) where \(q\) is a prime power (see VII.7.5.a from [BJL99]).

**EXAMPLES:**

```
sage: rBIBD = designs.resolvable_balanced_incomplete_block_design(28,4)
sage: rBIBD.is_resolvable()
True
sage: rBIBD.is_t_design(return_parameters=True)
(True, (2, 28, 4, 1))
```

### 5.1.77 Group-Divisible Designs (GDD)

This module gathers everything related to Group-Divisible Designs. The constructions defined here can be accessed through `designs.<tab>`:

```
sage: designs.group_divisible_design(14,{4},{2})
Group Divisible Design on 14 points of type 2^7
```

The main function implemented here is `group_divisible_design()` (which calls all others) and the main class is `GroupDivisibleDesign`. The following functions are available:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>group_divisible_design()</code></td>
<td>Return a ((v,K,G))-Group Divisible Design.</td>
</tr>
<tr>
<td><code>GDD_4_2()</code></td>
<td>Return a ((2q,{4},{2}))-GDD for (q) a prime power with (q \equiv 1 \pmod{6}).</td>
</tr>
</tbody>
</table>
Functions

`sage.combinat.designs.group_divisible_designs.GDD_4_2(q, existence=False, check=True)`

Return a $(2q, \{4\}, \{2\})$-GDD for $q$ a prime power with $q \equiv 1 \pmod{6}$.

This method implements Lemma VII.5.17 from [BJL99] (p.495).

**INPUT:**
- `q` (integer)
- `existence` (boolean) – instead of building the design, return:
  - `True` – meaning that Sage knows how to build the design
  - `Unknown` – meaning that Sage does not know how to build the design, but that the design may exist (see `sage.misc.unknown`).
  - `False` – meaning that the design does not exist.
- `check` – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to `True` by default.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.group_divisible_designs import GDD_4_2
sage: GDD_4_2(7, existence=True)
True
sage: GDD_4_2(7)
Group Divisible Design on 14 points of type $2^7$
```

```python
sage: GDD_4_2(8, existence=True)
Unknown
sage: GDD_4_2(8)
Traceback (most recent call last):
... Not Implemented Error
```

**class** `sage.combinat.designs.group_divisible_designs.GroupDivisibleDesign(points, groups, blocks, G=None, K=None, lambd=1, check=True, copy=True, **kwds)`

**Bases:** `IncidenceStructure`

Group Divisible Design (GDD)

Let $K$ and $G$ be sets of positive integers and let $\lambda$ be a positive integer. A Group Divisible Design of index $\lambda$ and order $v$ is a triple $(V, G, B)$ where:

- $V$ is a set of cardinality $v$
- $G$ is a partition of $V$ into groups whose size belongs to $G$
- $B$ is a family of subsets of $V$ whose size belongs to $K$ such that any two points $p_1, p_2 \in V$ from different groups appear simultaneously in exactly $\lambda$ elements of $B$. Besides, a group and a block intersect on at most one point.

If $K = \{k_1, \ldots, k_l\}$ and $G$ has exactly $m_i$ groups of cardinality $k_i$ then $G$ is said to have type $k_1^{m_1} \ldots k_l^{m_l}$.

**INPUT:**
• **points** – the underlying set. If `points` is an integer \( v \), then the set is considered to be \( \{0, ..., v - 1\} \).

• **groups** – the groups of the design. Set to `None` for an automatic guess (this triggers `check=True` and can thus cost some time).

• **blocks** – collection of blocks

• **\( G \)** – list of integers of which the sizes of the groups must be elements. Set to `None` (automatic guess) by default.

• **\( K \)** – list of integers of which the sizes of the blocks must be elements. Set to `None` (automatic guess) by default.

• **\( \lambda \)** (integer) – value of \( \lambda \), set to 1 by default.

• **check** (boolean) – whether to check that the design is indeed a \( GDD \) with the right parameters. Set to `True` by default.

• **copy** – (use with caution) if set to `False` then `blocks` must be a list of lists of integers. The list will not be copied but will be modified in place (each block is sorted, and the whole list is sorted). Your `blocks` object will become the instance’s internal data.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.group_divisible_designs import GroupDivisibleDesign
sage: TD = designs.transversal_design(4,10)
sage: groups = [list(range(i*10,(i+1)*10)) for i in range(4)]
sage: GDD = GroupDivisibleDesign(40, groups, TD); GDD
Group Divisible Design on 40 points of type 10^4
```

With unspecified groups:

```python
sage: D = designs.transversal_design(4,3).relabel(list('abcdefghiklm'), inplace=False).blocks()
sage: GDD = GroupDivisibleDesign('abcdefghiklm', None, D)
sage: sorted(GDD.groups())
[['a', 'b', 'c'], ['d', 'e', 'f'], ['g', 'h', 'i'], ['k', 'l', 'm']]
```

`groups()`

Return the groups of the Group-Divisible Design.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.group_divisible_designs import GroupDivisibleDesign
sage: TD = designs.transversal_design(4,10)
sage: groups = [list(range(i*10,(i+1)*10)) for i in range(4)]
sage: GDD = GroupDivisibleDesign(40, groups, TD); GDD
Group Divisible Design on 40 points of type 10^4
sage: GDD.groups()
[[0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
 [10, 11, 12, 13, 14, 15, 16, 17, 18, 19],
 [20, 21, 22, 23, 24, 25, 26, 27, 28, 29],
 [30, 31, 32, 33, 34, 35, 36, 37, 38, 39]]
```

`sage.combinat.designs.group_divisible_designs.group_divisible_design(v, K, G, existence=False, check=False)`

Return a \((v, K, G)\)-Group Divisible Design.

---

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A \((v, K, G)\)-GDD is a pair \((\mathcal{G}, \mathcal{B})\) where:

- \(\mathcal{G}\) is a partition of \(X = \bigcup \mathcal{G}\) where \(|X| = v\)
- \(\forall S \in \mathcal{G}, |S| \in G\)
- \(\forall S \in \mathcal{B}, |S| \in K\)
- \(\mathcal{G} \cup \mathcal{B}\) is a \((v, K \cup G)\)-PBD

For more information, see the documentation of \texttt{GroupDivisibleDesign} or \texttt{PairwiseBalancedDesign}.

**INPUT:**

- \(v\) (integer)
- \(K, G\) (sets of integers)
- \texttt{existence}\ (boolean) – instead of building the design, return:
  - \texttt{True} – meaning that Sage knows how to build the design
  - \texttt{Unknown} – meaning that Sage does not know how to build the design, but that the design may exist
    (see \texttt{sage.misc.unknown}).
  - \texttt{False} – meaning that the design does not exist.
- \texttt{check}\ – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to \texttt{True} by default.

**Note:** The GDD returned by this function are defined on \texttt{range(v)}, and its groups are sets of consecutive integers.

**EXAMPLES:**

```
sage: designs.group_divisible_design(14,{4},{2})
Group Divisible Design on 14 points of type 2^7
```

### 5.1.78 Block designs

A block design is a set together with a family of subsets (repeated subsets are allowed) whose members are chosen to satisfy some set of properties that are deemed useful for a particular application. See Wikipedia article \texttt{Block_design}.

**REFERENCES:**

- Block design from wikipedia: Wikipedia article \texttt{Block_design}
- What is a block design?, \url{http://designtheory.org/library/extrep/extrep-1.1-html/node4.html} (in ‘The External Representation of Block Designs’ by Peter J. Cameron, Peter Dobcsanyi, John P. Morgan, Leonard H. Soicher)

**AUTHORS:**

- Quentin Honoré (2015): construction of Hughes plane \url{github issue #18527}
- Vincent Delecroix (2014): rewrite the part on projective planes \url{github issue #16281}
- Peter Dobcsanyi and David Joyner (2007-2008)

This is a significantly modified form of the module block_design.py (version 0.6) written by Peter Dobcsanyi peter@designtheory.org. Thanks go to Robert Miller for lots of good design suggestions.
Todo: Implement more finite non-Desarguesian plane as in [We07] and Wikipedia article Non-Desarguesian_plane.

Functions and methods

```
sage.combinat.designs.block_design.AffineGeometryDesign(n, d, F, point_coordinates=True, check=True)
```

Return an affine geometry design.

The affine geometry design $AG_d(n, q)$ is the 2-design whose blocks are the $d$-vector subspaces in $F_q^n$. It has parameters

$$v = q^n, \ k = q^d, \ \lambda = \binom{n-1}{d-1}_q$$

where the $q$-binomial coefficient $\binom{m}{r}_q$ is defined by

$$\binom{m}{r}_q = \frac{(q^m - 1)(q^{m-1} - 1)\cdots(q^m-r+1-1)}{(q^r - 1)(q^{r-1} - 1)\cdots(q-1)}$$

See also:

```
ProjectiveGeometryDesign()
```

INPUT:

- $n$ (integer) – the Euclidean dimension. The number of points of the design is $v = |F_q^n|$.
- $d$ (integer) – the dimension of the (affine) subspaces of $F_q^n$ which make up the blocks.
- $F$ – a finite field or a prime power.
- `point_coordinates` – (optional, default `True`) whether we use coordinates in $F_q^n$ or plain integers for the points of the design.
- `check` – (optional, default `True`) whether to check the output.

EXAMPLES:

```
sage: BD = designs.AffineGeometryDesign(3, 1, GF(2))
sage: BD.is_t_design(return_parameters=True)
(True, (2, 8, 2, 1))
sage: BD = designs.AffineGeometryDesign(3, 2, GF(4))
sage: BD.is_t_design(return_parameters=True)
(True, (2, 64, 16, 5))
sage: BD = designs.AffineGeometryDesign(4, 2, GF(3))
sage: BD.is_t_design(return_parameters=True)
(True, (2, 81, 9, 13))
```

With $F$ an integer instead of a finite field:

```
sage: BD = designs.AffineGeometryDesign(3, 2, 4)
sage: BD.is_t_design(return_parameters=True)
(True, (2, 64, 16, 5))
```

Testing the option `point_coordinates`:
sage: designs.AffineGeometryDesign(3, 1, GF(2), point_coordinates=True).blocks()[0]
[(0, 0, 0), (0, 0, 1)]
sage: designs.AffineGeometryDesign(3, 1, GF(2), point_coordinates=False).blocks()[0]
[0, 1]
sage.combinat.designs.block_design.Hadamard3Design\( (n) \)

Return the Hadamard 3-design with parameters \( 3 - (n, \frac{n}{2}, \frac{n}{4} - 1) \).

This is the unique extension of the Hadamard 2-design (see \texttt{HadamardDesign()}). We implement the description from pp. 12 in \cite{CvL}.

INPUT:

- \( n \) (integer) – a multiple of 4 such that \( n > 4 \).

EXAMPLES:

```python
sage: designs.Hadamard3Design(12)
sage: Incidence structure with 12 points and 22 blocks
```

We verify that any two blocks of the Hadamard 3-design \( 3 - (8, 4, 1) \) design meet in 0 or 2 points. More generally, it is true that any two blocks of a Hadamard 3-design meet in 0 or \( \frac{n}{4} \) points (for \( n > 4 \)).

```python
sage: D = designs.Hadamard3Design(8)
sage: N = D.incidence_matrix()
sage: N.transpose()*N
```

\[
\begin{bmatrix}
4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\
2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\
2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \\
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \\
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \\
\end{bmatrix}
```

REFERENCES:

sage.combinat.designs.block_design.HadamardDesign\( (n) \)

As described in Section 1, p. 10, in \cite{CvL}. The input \( n \) must have the property that there is a Hadamard matrix of order \( n + 1 \) (and that a construction of that Hadamard matrix has been implemented...).

EXAMPLES:

```python
sage: designs.HadamardDesign(7)
sage: Incidence structure with 7 points and 7 blocks
```

For example, the Hadamard 2-design with \( n = 11 \) is a design whose parameters are \( 2 - (11, 5, 2) \). We verify that \( NJ = 5J \) for this design.

```python
sage: D = designs.HadamardDesign(11); N = D.incidence_matrix()
sage: J = matrix(ZZ, 11, 11, [1]*11*11); N*J
```

\[
\begin{bmatrix}
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\end{bmatrix}
```

(continues on next page)
REFERENCES:


sage.combinat.designs.block_design.HughesPlane(q2, check=True)

Return the Hughes projective plane of order \(q^2\).

Let \(q\) be an odd prime, the Hughes plane of order \(q^2\) is a finite projective plane of order \(q^2\) introduced by D. Hughes in [Hu57]. Its construction is as follows.

Let \(K = GF(q^2)\) be a finite field with \(q^2\) elements and \(F = GF(q) \subseteq K\) be its unique subfield with \(q\) elements. We define a twisted multiplication on \(K\) as

\[
x \circ y = \begin{cases} 
  xy & \text{if } y \text{ is a square in } K \\
  x^q y & \text{otherwise}
\end{cases}
\]

The points of the Hughes plane are the triples \((x, y, z)\) of points in \(K^3\) up to the equivalence relation \((x, y, z) \sim (x \circ k, y \circ k, z \circ k)\) where \(k \in K\).

For \(a = 1\) or \(a \in (K \setminus F)\) we define a block \(L(a)\) as the set of triples \((x, y, z)\) so that \(x + a \circ y + z = 0\). The rest of the blocks are obtained by letting act the group \(GL(3, F)\) by its standard action.

For more information, see Wikipedia article Hughes plane and [We07].

See also:

DesarguesianProjectivePlaneDesign() to build the Desarguesian projective planes

INPUT:

- \(q^2\) – an even power of an odd prime number
- check – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

EXAMPLES:

\[
\begin{bmatrix}
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\end{bmatrix}
\]

We prove in the following computations that the Desarguesian plane \(H\) is not Desarguesian. Let us consider the two triangles \((0, 1, 10)\) and \((57, 70, 59)\). We show that the intersection points \(D_{0,1} \cap D_{57,70}, D_{1,10} \cap D_{70,59}\) and \(D_{10,0} \cap D_{59,57}\) are on the same line while \(D_{0,70}, D_{1,59}\) and \(D_{10,57}\) are not concurrent.

\[
\begin{bmatrix}
sage: blocks = H.blocks() \\
sage: line = lambda p,q: next(b for b in blocks if p in b and q in b)
\end{bmatrix}
\]
```
sage: b_0_1 = line(0, 1)
sage: b_1_10 = line(1, 10)
sage: b_10_0 = line(10, 0)
sage: b_57_70 = line(57, 70)
sage: b_70_59 = line(70, 59)
sage: b_59_57 = line(59, 57)

sage: set(b_0_1).intersection(b_57_70)
{2}
sage: set(b_1_10).intersection(b_70_59)
{73}
sage: set(b_10_0).intersection(b_59_57)
{60}

sage: line(2, 73) == line(73, 60)
True

sage: b_0_57 = line(0, 57)
sage: b_1_70 = line(1, 70)
sage: b_10_59 = line(10, 59)

sage: p = set(b_0_57).intersection(b_1_70)
sage: q = set(b_1_70).intersection(b_10_59)
sage: p == q
False
```

```
sage.combinat.designs.block_design.ProjectiveGeometryDesign(n, d, F, algorithm=None, point_coordinates=True, check=True)
```

Return a projective geometry design.

The projective geometry design \(\text{PG}_d(n, q)\) has for points the lines of \(\mathbb{F}_q^{n+1}\), and for blocks the \(d+1\)-dimensional subspaces of \(\mathbb{F}_q^{n+1}\), each of which contains \(\frac{|\mathbb{F}_q|^{d+1} - 1}{|\mathbb{F}_q| - 1}\) lines. It is a 2-design with parameters

\[
v = \binom{n+1}{1}, \quad k = \binom{d+1}{1}, \quad \lambda = \binom{n-1}{d-1}
\]

where the \(q\)-binomial coefficient \(\binom{m}{r}_q\) is defined by

\[
\binom{m}{r}_q = \frac{(q^m - 1)(q^{m-1} - 1)\cdots(q^{m-r+1} - 1)}{(q^r - 1)(q^{r-1} - 1)\cdots(q - 1)}
\]

See also:

- \(\text{AffineGeometryDesign}()\)

INPUT:

- \(n\) is the projective dimension
- \(d\) is the dimension of the subspaces which make up the blocks.
- \(F\) – a finite field or a prime power.
• **algorithm** – set to `None` by default, which results in using Sage’s own implementation. In order to use GAP’s implementation instead (i.e. its `PGPointFlatBlockDesign` function) set `algorithm="gap``. Note that GAP’s “design” package must be available in this case, and that it can be installed with the `gap_packages` spkg.

• **point_coordinates** – `True` by default. Ignored and assumed to be `False` if `algorithm="gap``. If `True` the ground set is indexed by coordinates in \( \mathbb{F}_{q+1} \). Otherwise the ground set is indexed by integers.

• **check** – (optional, default to `True`) whether to check the output.

**EXAMPLES:**

The set of \( d \)-dimensional subspaces in a \( n \)-dimensional projective space forms 2-designs (or balanced incomplete block designs):

```
sage: PG = designs.ProjectiveGeometryDesign(4, 2, GF(2))
sage: PG
Incidence structure with 31 points and 155 blocks
sage: PG.is_t_design(return_parameters=True)
(True, (2, 31, 7, 7))
```

Check with \( F \) being a prime power:

```
sage: PG = designs.ProjectiveGeometryDesign(3, 1, GF(4))
sage: PG.is_t_design(return_parameters=True)
(True, (2, 85, 5, 1))
```

Check that the constructor using gap also works:

```
sage: BD = designs.ProjectiveGeometryDesign(2, 1, GF(2), algorithm="gap") #␣
\[\text{optional - gap_packages (design package)}\]
sage: BD.is_t_design(return_parameters=True) #␣
\[\text{optional - gap_packages (design package)}\]
(True, (2, 7, 3, 1))
```

**WittDesign**

```
sage.combinat.designs.block_design.WittDesign(n)
```

**INPUT:**

- `n is in 9, 10, 11, 12, 21, 22, 23, 24.`
Wraps GAP Design’s WittDesign. If n=24 then this function returns the large Witt design $W_{24}$, the unique (up to isomorphism) 5 – (24, 8, 1) design. If n=12 then this function returns the small Witt design $W_{12}$, the unique (up to isomorphism) 5 – (12, 6, 1) design. The other values of n return a block design derived from these.

Note: Requires GAP’s Design package (included in the gap_packages Sage spkg).

EXAMPLES:

```python
sage: BD = designs.WittDesign(9)  # optional - gap_packages (design_package)
sage: BD.is_t_design(return_parameters=True)  # optional - gap_packages (design_package)
(True, (2, 9, 3, 1))
sage: BD  # optional - gap_packages (design package)
Incidence structure with 9 points and 12 blocks
sage: print(BD)  # optional - gap_packages (design package)
Incidence structure with 9 points and 12 blocks
```

Return True if the parameters \((v,k,\lambda)\) are the one of hyperplanes in a (finite Desarguesian) projective space.

In other words, test whether there exists a prime power \(q\) and an integer \(d\) greater than two such that:

- \(v = (q^{d+1} - 1)/(q - 1) = q^d + q^{d-1} + ... + 1\)
- \(k = (q^d - 1)/(q - 1) = q^{d-1} + q^{d-2} + ... + 1\)
- \(\lambda = (q^{d-1} - 1)/(q - 1) = q^{d-2} + q^{d-3} + ... + 1\)

If it exists, such a pair \((q,d)\) is unique.

INPUT:
- \(v, k, \lambda\) (integers)

OUTPUT:
- a boolean or, if return_parameters is set to True a pair \((True, (q,d))\) or \((False, (None,None))\).

EXAMPLES:

```python
sage: from sage.combinat.designs.block_design import are_hyperplanes_in_projective_geometry_parameters
sage: are_hyperplanes_in_projective_geometry_parameters(40,13,4)
True
sage: are_hyperplanes_in_projective_geometry_parameters(40,13,4,return_parameters=True)
(True, (3, 3))
sage: PG = designs.ProjectiveGeometryDesign(3,2,GF(3))
sage: PG.is_t_design(return_parameters=True)
(True, (2, 40, 13, 4))
sage: are_hyperplanes_in_projective_geometry_parameters(15,3,1)
(continues on next page)
```
False

```
sage: are_hyperplanes_in_projective_geometry_parameters(15,3,1,return_ ˓→parameters=True)
  (False, (None, None))
```

```
sage.combinat.designs.block_design.normalize_hughes_plane_point(p, q)
```

Return the normalized form of point p as a 3-tuple.

In the Hughes projective plane over the finite field \( K \), all triples \((xk, yk, zk)\) with \( k \in K \) represent the same point (where the multiplication is over the nearfield built from \( K \)). This function chooses a canonical representative among them.

This function is used in `HughesPlane()`.

INPUT:

- \( p \) - point with the coordinates \((x,y,z)\) (a list, a vector, a tuple...)
- \( q \) - cardinality of the underlying finite field

EXAMPLES:

```
sage: from sage.combinat.designs.block_design import normalize_hughes_plane_point
sage: K = FiniteField(9, 'x')
sage: x = K.gen()
sage: normalize_hughes_plane_point((x, x+1, x), 9)
  (1, x, 1)
sage: normalize_hughes_plane_point(vector((x,x,x)), 9)
  (1, 1, 1)
sage: zero = K.zero()
sage: normalize_hughes_plane_point((2*x+2, zero, zero), 9)
  (1, 0, 0)
sage: one = K.one()
sage: normalize_hughes_plane_point((2*x, one, zero), 9)
  (2*x, 1, 0)
```

```
sage.combinat.designs.block_design.projective_plane(n, check=True, existence=False)
```

Return a projective plane of order \( n \) as a 2-design.

A finite projective plane is a 2-design with \( n^2 + n + 1 \) lines (or blocks) and \( n^2 + n + 1 \) points. For more information on finite projective planes, see the Wikipedia article Projective_plane#Finite_projective_planes.

If no construction is possible, then the function raises a `EmptySetError` whereas if no construction is available the function raises a `NotImplementedError`.

INPUT:

- \( n \) – the finite projective plane’s order

EXAMPLES:

```
sage: designs.projective_plane(2)
  (7,3,1)-Balanced Incomplete Block Design
sage: designs.projective_plane(3)
  (13,4,1)-Balanced Incomplete Block Design
sage: designs.projective_plane(4)
  (21,5,1)-Balanced Incomplete Block Design
```

(continues on next page)
sage: designs.projective_plane(5)
(31,6,1)-Balanced Incomplete Block Design

sage: designs.projective_plane(6)
Traceback (most recent call last):
...
EmptySetError: By the Bruck-Ryser theorem, no projective plane of order 6 exists.

sage: designs.projective_plane(10)
Traceback (most recent call last):
...

sage: designs.projective_plane(12)
Traceback (most recent call last):
...
NotImplementedError: If such a projective plane exists, we do not know how to build it.

sage: designs.projective_plane(14)
Traceback (most recent call last):
...
EmptySetError: By the Bruck-Ryser theorem, no projective plane of order 14 exists.

sage.combinat.designs.block_design.projective_plane_to_OA(pplane, pt=None, check=True)
Return the orthogonal array built from the projective plane pplane.

The orthogonal array $OA(n+1, n, 2)$ is obtained from the projective plane pplane by removing the point pt and the $n+1$ lines that pass through it. These $n+1$ lines form the $n+1$ groups while the remaining $n^2 + n$ lines form the transversals.

INPUT:

- pplane - a projective plane as a 2-design
- pt - a point in the projective plane pplane. If it is not provided then it is set to $n^2 + n$.
- check – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

EXAMPLES:

sage: from sage.combinat.designs.block_design import projective_plane_to_OA
sage: p2 = designs.DesarguesianProjectivePlaneDesign(2,point_coordinates=False)
sage: projective_plane_to_OA(p2)

[[0, 0, 0], [0, 1, 1], [1, 0, 1], [1, 1, 0]]

sage: p3 = designs.DesarguesianProjectivePlaneDesign(3,point_coordinates=False)
sage: projective_plane_to_OA(p3)

[[0, 0, 0, 0],
 [0, 1, 2, 1],
 [0, 2, 1, 2],
 [1, 0, 2, 2],
 [1, 1, 1, 0],
 [1, 2, 0, 1],
 [2, 0, 1, 1],
 [2, 1, 0, 2],]
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[2, 2, 2, 0]

sage: pp = designs.DesarguesianProjectivePlaneDesign(16, point_coordinates=False)
sage: _ = projective_plane_to_OA(pp, pt=0)
sage: _ = projective_plane_to_OA(pp, pt=3)
sage: _ = projective_plane_to_OA(pp, pt=7)

sage.combinat.designs.block_design.q3_minus_one_matrix(K)

Return a companion matrix in \(GL(3, K)\) whose multiplicative order is \(q^3 - 1\).

This function is used in HughesPlane().

EXAMPLES:

sage: from sage.combinat.designs.block_design import q3_minus_one_matrix
sage: m = q3_minus_one_matrix(GF(3))
sage: m.multiplicative_order() == 3**3 - 1
True
sage: m = q3_minus_one_matrix(GF(4, 'a'))
sage: m.multiplicative_order() == 4**3 - 1
True
sage: m = q3_minus_one_matrix(GF(5))
sage: m.multiplicative_order() == 5**3 - 1
True
sage: m = q3_minus_one_matrix(GF(9, 'a'))
sage: m.multiplicative_order() == 9**3 - 1
True

sage.combinat.designs.block_design.tdesign_params(t, v, k, L)

Return the design’s parameters: \((t, v, b, r, k, L)\). Note that \(t\) must be given.

EXAMPLES:

sage: BD = BlockDesign(7, [[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,6],[2,4,5]])
sage: from sage.combinat.designs.block_design import tdesign_params
sage: tdesign_params(2,7,3,1)
(2, 7, 7, 3, 3, 1)

5.1.79 Covering designs: coverings of \(t\)-element subsets of a \(v\)-set by \(k\)-sets

A \((v, k, t)\) covering design \(C\) is an incidence structure consisting of a set of points \(P\) of order \(v\), and a set of blocks \(B\), where each block contains \(k\) points of \(P\). Every \(t\)-element subset of \(P\) must be contained in at least one block.

If every \(t\)-set is contained in exactly one block of \(C\), then we have a block design. Following the block design implementation, the standard representation of a covering design uses \(P = [0, 1, \ldots, v - 1]\).

In addition to the parameters and incidence structure for a covering design from this database, we include extra information:

- Best known lower bound on the size of a \((v, k, t)\)-covering design
- Name of the person(s) who produced the design
• Method of construction used
• Date when the design was added to the database

REFERENCES:
AUTHORS:
• Daniel M. Gordon (2008-12-22): initial version

Classes and methods

class sage.combinat.designs.covering_design.CoveringDesign(v=0, k=0, t=0, size=0, points=None, blocks=None, low_bd=0, method='', creator='', timestamp='')

Bases: SageObject
Covering design.

INPUT:
• \(v\), \(k\), \(t\) – integer parameters of the covering design
• size (integer)
• points – list of points (default points are \([0, \ldots, v-1]\))
• blocks
• low_bd (integer) – lower bound for such a design
• method, creator, timestamp – database information

creator()
Return the creator of the covering design
This field is optional, and is used in a database to give attribution for the covering design It can refer to the person who submitted it, or who originally gave a construction

EXAMPLES:

```
sage: from sage.combinat.designs.covering_design import CoveringDesign
sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2], ....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6], [2, 3, 6], ....: [2, 4, 5]], 0, 'Projective Plane', 'Gino Fano')
sage: C.creator()
'Gino Fano'
```

incidence_structure()
Return the incidence structure of this design, without extra parameters.

EXAMPLES:

```
sage: from sage.combinat.designs.covering_design import CoveringDesign
sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2], ....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6], ....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')
sage: D = C.incidence_structure()
sage: D.ground_set()
[0, 1, 2, 3, 4, 5, 6]
```
(continues on next page)
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sage: D.blocks()
[[0, 1, 2], [0, 3, 4], [0, 5, 6], [1, 3, 5],
 [1, 4, 6], [2, 3, 6], [2, 4, 5]]

is_covering()

Check all \( t \)-sets are in fact covered by the blocks of self.

**Note**: This is very slow and wasteful of memory.

EXAMPLES:

```python
sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
.....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
.....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')
sage: C.is_covering()
True
```

```python
sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
.....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6], [2, 3, 6],
.....: [2, 4, 6]], 0, 'not a covering')  # last block altered
sage: C.is_covering()
False
```

k()

Return \( k \), the size of blocks of the covering design

EXAMPLES:

```python
sage: from sage.combinat.designs.covering_design import CoveringDesign
sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
.....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
.....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')
sage: C.k()
3
```

low_bd()

Return a lower bound for the number of blocks a covering design with these parameters could have.

Typically this is the Schonheim bound, but for some parameters better bounds have been shown.

EXAMPLES:

```python
sage: from sage.combinat.designs.covering_design import CoveringDesign
sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
.....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
.....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')
sage: C.low_bd()
7
```

method()

Return the method used to create the covering design.

This field is optional, and is used in a database to give information about how coverings were constructed.

EXAMPLES:
sage: from sage.combinat.designs.covering_design import CoveringDesign

sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')

sage: C.method()
'Projective Plane'

size()

Return the number of blocks in the covering design

EXAMPLES:

sage: from sage.combinat.designs.covering_design import CoveringDesign

sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')

sage: C.size()
7

\(t\)()

Return \(t\), the size of sets which must be covered by the blocks of the covering design

EXAMPLES:

sage: from sage.combinat.designs.covering_design import CoveringDesign

sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')

sage: C.t()
2

timestamp()

Return the time that the covering was submitted to the database

EXAMPLES:

sage: from sage.combinat.designs.covering_design import CoveringDesign

sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane',
....: 'Gino Fano', '1892-01-01 00:00:00')

sage: C.timestamp()  # No exact date known; in Fano's 1892 article
'1892-01-01 00:00:00'

\(v\)()

Return \(v\), the number of points in the covering design.

EXAMPLES:

sage: from sage.combinat.designs.covering_design import CoveringDesign

sage: C = CoveringDesign(7, 3, 2, 7, range(7), [[0, 1, 2],
....: [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6],
....: [2, 3, 6], [2, 4, 5]], 0, 'Projective Plane')

sage: C.v()
7
sage.combinat.designs.covering_design.best_known_covering_design_www(v, k, t, verbose=False)

Return the best known \((v, k, t)\) covering design from an online database.

This uses the La Jolla Covering Repository, a database available at https://ljcr.dmgordon.org/cover.html

**INPUT:**
- \(v\) – integer, the size of the point set for the design
- \(k\) – integer, the number of points per block
- \(t\) – integer, the size of sets covered by the blocks
- \(\text{verbose}\) – bool (default: False), print verbose message

**OUTPUT:**
A \texttt{CoveringDesign} object representing the \((v, k, t)\)-covering design with smallest number of blocks available in the database.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.covering_design import \ # optional - internet
    ...: best_known_covering_design_www
sage: C = best_known_covering_design_www(7, 3, 2) \ # optional - internet
sage: print(C) \ # optional - internet
C(7, 3, 2) = 7
Method: lex covering
Submitted on: 1996-12-01 00:00:00
0 1 2
0 3 4
0 5 6
1 3 5
1 4 6
2 3 6
2 4 5
```

A ValueError is raised if the \((v, k, t)\) parameters are not found in the database.

sage.combinat.designs.covering_design.schonheim(v, k, t)

Return the Schonheim lower bound for the size of such a covering design.

**INPUT:**
- \(v\) – integer, size of point set
- \(k\) – integer, cardinality of each block
- \(t\) – integer, cardinality of sets being covered

**OUTPUT:**
The Schonheim lower bound for such a covering design's size: 
\[
C(v, k, t) \leq \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} \cdots \lceil \frac{v-t+1}{k-t+1} \rceil \cdots \rceil
\]

**EXAMPLES:**

```python
sage: from sage.combinat.designs.covering_design import schonheim
sage: schonheim(10, 3, 2)
17
sage: schonheim(32, 16, 8)
930
```
Construct a trivial covering design.

INPUT:
• \( v \) – integer, size of point set
• \( k \) – integer, cardinality of each block
• \( t \) – integer, cardinality of sets being covered

OUTPUT:
A trivial \((v, k, t)\) covering design

EXAMPLES:

```python
sage: C = trivial_covering_design(8, 3, 1)
sage: print(C)
C(8, 3, 1) = 3
Method: Trivial
0 1 2
0 6 7
3 4 5
sage: C = trivial_covering_design(5, 3, 2)
sage: print(C)
4 <= C(5, 3, 2) <= 10
Method: Trivial
0 1 2
0 1 3
0 1 4
0 2 3
0 2 4
0 3 4
1 2 3
1 2 4
1 3 4
2 3 4
```

**Note:** Cases are:

- \( t = 0 \): This could be empty, but it’s a useful convention to have one block (which is empty if \( k = 0 \)).
- \( t = 1 \): This contains \( \lceil v/k \rceil \) blocks: \([0, ..., k - 1], [k, ..., 2k - 1], ..., \). The last block wraps around if \( k \) does not divide \( v \).
- anything else: Just use every \( k \)-subset of \([0, 1, ..., v - 1]\).
5.1.80 Database of small combinatorial designs

This module implements combinatorial designs that cannot be obtained by more general constructions. Most of them come from the Handbook of Combinatorial Designs [DesignHandbook].

All this would only be a dream without the mathematical knowledge and help of Julian R. Abel.

These functions can all be obtained through the designs.<tab> functions.

This module implements:

- \( \text{OA}(7, 18), \text{OA}(9, 40), \text{OA}(7, 66), \text{OA}(7, 68), \text{OA}(8, 69), \text{OA}(7, 74), \text{OA}(8, 76), \text{OA}(11, 80), \text{OA}(15, 112), \text{OA}(9, 120), \text{OA}(9, 135), \text{OA}(11, 160), \text{OA}(16, 176), \text{OA}(11, 185), \text{OA}(10, 205), \text{OA}(16, 268), \text{OA}(15, 224), \text{OA}(11, 254), \text{OA}(20, 352), \text{OA}(20, 416), \text{OA}(10, 469), \text{OA}(10, 520), \text{OA}(12, 522), \text{OA}(14, 524), \text{OA}(20, 544), \text{OA}(17, 560), \text{OA}(11, 640), \text{OA}(10, 796), \text{OA}(15, 896), \text{OA}(9, 1078), \text{OA}(25, 1262), \text{OA}(9, 1612), \text{OA}(10, 1620) \)
- 2 MOLS of order 10, 5 MOLS of order 12, 4 MOLS of order 14, 4 MOLS of order 15, 3 MOLS of order 18
- \( \text{V}(m, t) \) vectors:

  - \( m = 3 \) and \( t = 2, 4, 6, 10, 12, 14, 20, 24, 26, 32, 34 \)
  - \( m = 4 \) and \( t = 3, 7, 9, 13, 15, 25 \)
  - \( m = 5 \) and \( t = 6, 8, 12, 14, 20, 26 \)
  - \( m = 6 \) and \( t = 5, 7, 11, 13, 17, 21 \)
  - \( m = 7 \) and \( t = 6, 10, 16, 18 \)
  - \( m = 8 \) and \( t = 9, 11, 17, 29, 57 \)
  - \( m = 9 \) and \( t = 12, 14, 18, 20, 22, 30, 34, 42, 44 \)
  - \( m = 10 \) and \( t = 13, 15, 19, 21, 25, 27, 31, 33, 43, 49, 81, 97, 103, 181, 187, 259, 273, 319, 391, 409 \)
- \( \text{RBIBD}(120, 8, 1) \)
- \( (v, k, \lambda)\)-BIBD:

  - \( \lambda = 1: n \ (66, 6, 1), (76, 6, 1), (96, 6, 1), (106, 6, 1), (111, 6, 1), (120, 8, 1), (126, 6, 1), (136, 6, 1), (141, 6, 1), (171, 6, 1), (196, 6, 1), (201, 6, 1) \)
  - \( \lambda = 2: n \ (56, 11, 2), (79, 13, 2) \)
  - \( \lambda = 8: n \ (45, 9, 8) \)
  - \( \lambda = 14: n \ (176, 50, 14) \)
- \( (v, k, \lambda)\)-difference families:

  - \( \lambda = 1: n \ (15, 3, 1), (21, 3, 1), (21, 5, 1), (25, 3, 1), (25, 4, 1), (27, 3, 1), (33, 3, 1), (37, 4, 1), (39, 3, 1), (40, 4, 1), (45, 3, 1), (45, 5, 1), (49, 3, 1), (49, 4, 1), (51, 3, 1), (52, 4, 1), (55, 3, 1), (57, 3, 1), (63, 3, 1), (64, 4, 1), (65, 5, 1), (69, 3, 1), (75, 3, 1), (76, 4, 1), (81, 3, 1), (81, 5, 1), (85, 4, 1), (86, 4, 1), (90, 3, 1), (92, 4, 1), (98, 3, 1), (98, 5, 1) \)
\[ (q, k, \lambda) \]-difference matrices:

\[- \lambda = 2: (16, 3, 2), (19, 4, 2), (22, 4, 2), (28, 3, 2), (31, 4, 2), (31, 5, 2), (34, 4, 2), (35, 5, 2), (40, 3, 2), (43, 7, 2), (46, 4, 2), (46, 6, 2), (51, 5, 2), (61, 6, 2), (64, 7, 2), (71, 5, 2), (75, 5, 2), (85, 7, 2), (85, 8, 2), (153, 9, 2), (181, 10, 2) \]

\[- \lambda = 3: (21, 4, 3), (21, 6, 3), (29, 7, 3), (41, 6, 3), (43, 7, 3), (45, 12, 3), (49, 9, 3), (51, 6, 3), (57, 7, 3), (61, 6, 3), (61, 10, 3), (71, 7, 3), (85, 7, 3), (97, 9, 3), (121, 10, 3) \]

\[- \lambda = 4: (22, 7, 4), (29, 8, 4), (43, 8, 4), (46, 10, 4), (55, 9, 4), (67, 12, 4), (71, 8, 4) \]

\[- \lambda = 5: (13, 5, 5), (17, 5, 5), (21, 6, 5), (22, 6, 5), (28, 6, 5), (33, 5, 5), (33, 6, 5), (37, 10, 5), (39, 6, 5), (45, 11, 5), (46, 10, 5), (55, 10, 5), (67, 11, 5), (73, 10, 5) \]

\[- \lambda = 6: (11, 4, 6), (15, 4, 6), (15, 5, 6), (29, 8, 6), (46, 10, 6), (53, 13, 6), (67, 12, 6) \]

\[- \lambda = 7: (25, 7, 7), (53, 14, 7), (61, 15, 7) \]

\[- \lambda = 8: (22, 8, 8), (34, 12, 8), (133, 33, 8) \]

\[- \lambda = 9: (21, 10, 9) \]

\[- \lambda = 10: (34, 12, 10), (43, 15, 10), (49, 21, 10) \]

\[- \lambda = 11: (22, 8, 12) \]

\[- \lambda = 12: (21, 8, 14) \]

\[- \lambda = 13: (901, 225, 56) \]

\[- (v, k, \lambda) \text{-difference matrices:} \]

\[- \lambda = 1: (12, 6, 1), (21, 6, 1), (24, 8, 1), (28, 6, 1), (33, 6, 1), (35, 6, 1), (36, 9, 1), (39, 6, 1), (44, 6, 1), (45, 7, 1), (48, 9, 1), (51, 6, 1), (52, 6, 1), (55, 7, 1), (56, 8, 1), (57, 8, 1), (60, 6, 1), (75, 8, 1), (273, 17, 1), (993, 32, 1) \]

\[- (n, k; \lambda; u) \text{-quasi-difference matrices:} \]

\[- (19, 6; 1, 1; 1), (21, 5; 1, 1; 5), (25, 6; 1, 1; 5), (33, 6; 1, 1; 1), (37, 5; 1, 1; 7), (37, 6; 1, 1; 1), (45, 7; 1, 1; 9), (54, 7; 1, 1; 8), (57, 9; 1, 1; 8) \]

\[- (q, k) \text{ evenly distributed sets} \]


\[- k = 5: (41, 61, 101, 121, 181, 241, 281, 361, 401, 421, 461, 521, 541, 601, 641, 661, 701, 761, 821, 841, 881, 941, 961, 1021, 1061, 1181, 1201, 1301, 1321, 1361, 1381, 1481, 1601, 1621, 1681, 1721, 1741, 1801, 1861, 1901 \]

\[- k = 6: (31, 151, 181, 211, 241, 271, 331, 361, 421, 451, 571, 601, 631, 661, 691, 751, 811, 841, 961, 991, 1021, 1051, 1171, 1201, 1231, 1291, 1321, 1381, 1471, 1531, 1621, 1681, 1741, 1801, 1831, 1861, 1951 \]

\[- k = 7: (169, 337, 379, 421, 463, 547, 631, 673, 757, 841, 883, 967, 1009, 1051, 1093, 1303, 1429, 1471, 1597, 1681, 1723, 1849, 1933 \]

\[- k = 8: (449, 617, 673, 729, 841, 953, 1009, 1289, 1681, 1849 \]

\[- k = 9: (73, 433, 937, 1009, 1153, 1297, 1369, 1657, 1801, 1873 \]
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-k = 10: 1171, 1531, 1621, 1801

REFERENCES:

Functions

sage.combinat.designs.database.BIBD_106_6_1()
Return a (106,6,1)-BIBD.
This constructions appears in II.3.32 from [DesignHandbook].
EXAMPLES:

```
sage: from sage.combinat.designs.database import BIBD_106_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(106, BIBD_106_6_1())
(106,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_111_6_1()
Return a (111,6,1)-BIBD.
This constructions appears in II.3.32 from [DesignHandbook].
EXAMPLES:

```
sage: from sage.combinat.designs.database import BIBD_111_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(111, BIBD_111_6_1())
(111,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_126_6_1()
Return a (126,6,1)-BIBD.
This constructions appears in VI.16.92 from [DesignHandbook].
EXAMPLES:

```
sage: from sage.combinat.designs.database import BIBD_126_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(126, BIBD_126_6_1())
(126,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_136_6_1()
Return a (136,6,1)-BIBD.
This constructions appears in II.3.32 from [DesignHandbook].
EXAMPLES:

```
sage: from sage.combinat.designs.database import BIBD_136_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(136, BIBD_136_6_1())
(136,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_141_6_1()
Return a (141,6,1)-BIBD.
This constructions appears in II.3.32 from [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.database import BIBD_141_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(141, BIBD_141_6_1())
(141,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_171_6_1()

Return a (171,6,1)-BIBD.

This constructions appears in II.3.32 from [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.database import BIBD_171_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(171, BIBD_171_6_1())
(171,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_196_6_1()

Return a (196,6,1)-BIBD.

This constructions appears in II.3.32 from [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.database import BIBD_196_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(196, BIBD_196_6_1())
(196,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_201_6_1()

Return a (201,6,1)-BIBD.

This constructions appears in II.3.32 from [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.database import BIBD_201_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(201, BIBD_201_6_1())
(201,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.BIBD_45_9_8(from_code=False)

Return a \((45,9,1)\)-BIBD.

This BIBD is obtained from the codewords of minimal weight in the ExtendedQuadraticResidueCode() of length 48. This construction appears in VII.11.2 from [DesignHandbook], which cites [HT95].

INPUT:

- `from_code` (boolean) – whether to build the design from hardcoded data (default) or from the code object (much longer).

EXAMPLES:

```python
sage: from sage.combinat.designs.database import BIBD_45_9_8
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(45, BIBD_45_9_8(False))
(45,9,1)-Balanced Incomplete Block Design
```

(continues on next page)
sage: B = BalancedIncompleteBlockDesign(45, BIBD_45_9_8(), lambd=8); B
(45,9,8)-Balanced Incomplete Block Design

REFERENCE:
sage.combinat.designs.database.BIBD_56_11_2()
Return a symmetric (56, 11, 2)-BIBD.
The construction implemented is given in [Hall71].

Note: A symmetric \((v, k, \lambda)\) BIBD is a \((v, k, \lambda)\) BIBD with \(v\) blocks.

EXAMPLES:
sage: from sage.combinat.designs.database import BIBD_56_11_2 sage: D = IncidenceStructure(BIBD_56_11_2()) sage: D.is_t_design(t=2, v=56, k=11, l=2) True

sage.combinat.designs.database.BIBD_66_6_1()
Return a (66,6,1)-BIBD.
This BIBD was obtained from La Jolla covering repository (https://math.ccrwest.org/cover.html) where it is attributed to Colin Barker.

EXAMPLES:
sage: from sage.combinat.designs.database import BIBD_66_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(66, BIBD_66_6_1())
(66,6,1)-Balanced Incomplete Block Design

sage.combinat.designs.database.BIBD_76_6_1()
Return a (76,6,1)-BIBD.
This BIBD was obtained from La Jolla covering repository (https://math.ccrwest.org/cover.html) where it is attributed to Colin Barker.

EXAMPLES:
sage: from sage.combinat.designs.database import BIBD_76_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(76, BIBD_76_6_1())
(76,6,1)-Balanced Incomplete Block Design

sage.combinat.designs.database.BIBD_79_13_2()
Return a symmetric (79, 13, 2)-BIBD.
The construction implemented is the one described in [Aschbacher71]. A typo in that paper was corrected in [Hall71].

Note: A symmetric \((v, k, \lambda)\) BIBD is a \((v, k, \lambda)\) BIBD with \(v\) blocks.

EXAMPLES:
sage: from sage.combinat.designs.database import BIBD_79_13_2 sage: D = IncidenceStructure(BIBD_79_13_2()) sage: D.is_t_design(t=2, v=79, k=13, l=2) True
sage.combinat.designs.database.BIBD_96_6_1()

Return a (96,6,1)-BIBD.

This BIBD was obtained from La Jolla covering repository (https://math.ccrwest.org/cover.html) where it is attributed to Colin Barker.

EXAMPLES:

```python
sage: from sage.combinat.designs.database import BIBD_96_6_1
sage: from sage.combinat.designs.bibd import BalancedIncompleteBlockDesign
sage: BalancedIncompleteBlockDesign(96, BIBD_96_6_1())
(96,6,1)-Balanced Incomplete Block Design
```

sage.combinat.designs.database.DM_12_6_1()

Return a (12,6,1)-difference matrix as built in [Hanani75].

This design is Lemma 3.21 from [Hanani75].

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_12_6_1
sage: G,M = DM_12_6_1()
sage: is_difference_matrix(M,G,6,1)
True
```

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(12,6)
```

REFERENCES:

sage.combinat.designs.database.DM_21_6_1()

Return a (21,6,1)-difference matrix.

As explained in the Handbook III.3.50 [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_21_6_1
sage: G,M = DM_21_6_1()
sage: is_difference_matrix(M,G,6,1)
True
```

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(21,6)
```

sage.combinat.designs.database.DM_24_8_1()

Return a (24,8,1)-difference matrix.

As explained in the Handbook III.3.52 [DesignHandbook].

EXAMPLES:
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_24_8_1
sage: G,M = DM_24_8_1()

sage: is_difference_matrix(M,G,8,1)
True

Can be obtained from the constructor:

sage: _ = designs.difference_matrix(24,8)

sage.combinat.designs.database.DM_273_17_1()

Return a (273, 17, 1)-difference matrix.

Given by Julian R. Abel.

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_273_17_1
sage: G,M = DM_273_17_1()

sage: is_difference_matrix(M,G,17,1)
True

Can be obtained from the constructor:

sage: _ = designs.difference_matrix(273,17)

sage.combinat.designs.database.DM_28_6_1()

Return a (28, 6, 1)-difference matrix.

As explained in the Handbook III.3.54 [DesignHandbook].

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_28_6_1
sage: G,M = DM_28_6_1()

sage: is_difference_matrix(M,G,6,1)
True

Can be obtained from the constructor:

sage: _ = designs.difference_matrix(28,6)

sage.combinat.designs.database.DM_33_6_1()

Return a (33, 6, 1)-difference matrix.

As explained in the Handbook III.3.56 [DesignHandbook].

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_33_6_1
sage: G,M = DM_33_6_1()

sage: is_difference_matrix(M,G,6,1)
True
Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(33,6)
```

```
DM_35_6_1

Return a (35, 6, 1)-difference matrix.

As explained in the Handbook III.3.58 [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_35_6_1
sage: G,M = DM_35_6_1()
sage: is_difference_matrix(M,G,6,1)
True
```

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(35,6)
```

```
DM_36_9_1

Return a (36, 9, 1)-difference matrix.

As explained in the Handbook III.3.59 [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_36_9_1
sage: G,M = DM_36_9_1()
sage: is_difference_matrix(M,G,9,1)
True
```

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(36,9)
```

```
DM_39_6_1

Return a (39, 6, 1)-difference matrix.

As explained in the Handbook III.3.61 [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_39_6_1
sage: G,M = DM_39_6_1()
sage: is_difference_matrix(M,G,6,1)
True
```

The design is available from the general constructor:

```python
sage: designs.difference_matrix(39,6,existence=True)
```

True
sage.combinat.designs.database.DM_44_6_1()
    Return a (44, 6, 1)-difference matrix.
    As explained in the Handbook III.3.64 [DesignHandbook].
    EXAMPLES:
    sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
    sage: from sage.combinat.designs.database import DM_44_6_1
    sage: G,M = DM_44_6_1()
    sage: is_difference_matrix(M,G,6,1)
    True

Can be obtained from the constructor:
    sage: _ = designs.difference_matrix(44,6)

sage.combinat.designs.database.DM_45_7_1()
    Return a (45, 7, 1)-difference matrix.
    As explained in the Handbook III.3.65 [DesignHandbook].
    ... whose description contained a very deadly typo, kindly fixed by Julian R. Abel.
    EXAMPLES:
    sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
    sage: from sage.combinat.designs.database import DM_45_7_1
    sage: G,M = DM_45_7_1()
    sage: is_difference_matrix(M,G,7,1)
    True

Can be obtained from the constructor:
    sage: _ = designs.difference_matrix(45,7)

sage.combinat.designs.database.DM_48_9_1()
    Return a (48, 9, 1)-difference matrix.
    As explained in the Handbook III.3.67 [DesignHandbook].
    EXAMPLES:
    sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
    sage: from sage.combinat.designs.database import DM_48_9_1
    sage: G,M = DM_48_9_1()
    sage: is_difference_matrix(M,G,9,1)
    True

Can be obtained from the constructor:
    sage: _ = designs.difference_matrix(48,9)

sage.combinat.designs.database.DM_51_6_1()
    Return a (51, 6, 1)-difference matrix.
    As explained in the Handbook III.3.69 [DesignHandbook].
    EXAMPLES:
Can be obtained from the constructor:

```
sage: _ = designs.difference_matrix(51,6)
```

```
sage.combinat.designs.database.DM_52_6_1()
```

Return a (52, 6, 1)-difference matrix.

As explained in the Handbook III.3.70 [DesignHandbook].

EXAMPLES:

```
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_52_6_1
sage: G,M = DM_52_6_1()
```

```
sage: is_difference_matrix(M,G,6,1)
```

True

Can be obtained from the constructor:

```
sage: _ = designs.difference_matrix(52,6)
```

```
sage.combinat.designs.database.DM_55_7_1()
```

Return a (55, 7, 1)-difference matrix.

As explained in the Handbook III.3.72 [DesignHandbook].

EXAMPLES:

```
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_55_7_1
sage: G,M = DM_55_7_1()
```

```
sage: is_difference_matrix(M,G,7,1)
```

True

Can be obtained from the constructor:

```
sage: _ = designs.difference_matrix(55,7)
```

```
sage.combinat.designs.database.DM_56_8_1()
```

Return a (56, 8, 1)-difference matrix.

As explained in the Handbook III.3.73 [DesignHandbook].

EXAMPLES:

```
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_56_8_1
sage: G,M = DM_56_8_1()
```

```
sage: is_difference_matrix(M,G,8,1)
```

True
Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(56,8)
```

```
sage.combinat.designs.database.DM_57_8_1()
```

Return a $(57, 8, 1)$-difference matrix.

Given by Julian R. Abel.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_57_8_1
sage: G,M = DM_57_8_1()
```

```python
sage: is_difference_matrix(M,G,8,1)
```

True

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(57,8)
```

```
sage.combinat.designs.database.DM_60_6_1()
```

Return a $(60, 6, 1)$-difference matrix.

As explained in [JulianAbel13].

**REFERENCES:**


**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_60_6_1
sage: G,M = DM_60_6_1()
```

```python
sage: is_difference_matrix(M,G,6,1)
```

True

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(60,6)
```

```
sage.combinat.designs.database.DM_75_8_1()
```

Return a $(75, 8, 1)$-difference matrix.

As explained in the Handbook III.3.75 [DesignHandbook].

**REFERENCES:**


**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_75_8_1
sage: G,M = DM_75_8_1()
```

```python
sage: is_difference_matrix(M,G,8,1)
```

True

Can be obtained from the constructor:
```python
sage: _ = designs.difference_matrix(75,8)
```

**sage.combinat.designs.database.DM_993_32_1()**

Return a (993, 32, 1)-difference matrix.

Given by Julian R. Abel.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: from sage.combinat.designs.database import DM_993_32_1
sage: G, M = DM_993_32_1()
sage: is_difference_matrix(M, G, 32, 1)
True
```

Can be obtained from the constructor:

```python
sage: _ = designs.difference_matrix(993, 32)
```

**sage.combinat.designs.database.HigmanSimsDesign()**

Return the Higman-Sims designs, which is a (176, 50, 14)-BIBD.

This design is built from the WittDesign $W$ on 24 points. We define two points $a, b$, and consider:

- The collection $W_a$ of all blocks of $W$ containing $a$ but not containing $b$.
- The collection $W_b$ of all blocks of $W$ containing $b$ but not containing $a$.

The design is then obtained from the incidence structure produced by the blocks $A \in W_a$ and $B \in W_b$ whose intersection has cardinality 2. This construction, due to M. Smith, can be found in [KY04] or in 10.A.(v) of [BL1984].

**EXAMPLES:**

```python
sage: H = designs.HigmanSimsDesign(); H
Incidence structure with 176 points and 176 blocks
sage: H.is_t_design(return_parameters=1)
(True, (2, 176, 50, 14))
```

Make sure that the automorphism group of this designs is isomorphic to the automorphism group of the HigmanSimsGraph(). Note that the first of those permutation groups acts on 176 points, while the second acts on 100:

```python
sage: gH = H.automorphism_group()  # optional - gap_packages
sage: gG = graphs.HigmanSimsGraph().automorphism_group()  # optional - gap_packages
sage: gG.is_isomorphic(gG)  # long time # optional - gap_packages
True
```

**REFERENCE:**

**sage.combinat.designs.database.MOLS_10_2()**

Return a pair of MOLS of order 10


**EXAMPLES:**
sage: from sage.combinat.designs.latin_squares import are_mutually_orthogonal_latin_squares
sage: from sage.combinat.designs.database import MOLS_10_2
sage: MOLS = MOLS_10_2()

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(2,10)
True

sage.combinat.designs.database.MOLS_12_5()

Return 5 MOLS of order 12

These MOLS have been found by Brendan McKay.

EXAMPLES:

sage: from sage.combinat.designs.latin_squares import are_mutually_orthogonal_latin_squares
sage: from sage.combinat.designs.database import MOLS_12_5
sage: MOLS = MOLS_12_5()

sage.combinat.designs.database.MOLS_14_4()

Return four MOLS of order 14

These MOLS were shared by Ian Wanless. The first proof of existence was given in [Todorov12].

EXAMPLES:

sage: from sage.combinat.designs.latin_squares import are_mutually_orthogonal_latin_squares
sage: from sage.combinat.designs.database import MOLS_14_4
sage: MOLS = MOLS_14_4()

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(4,14)
True

REFERENCE:

sage.combinat.designs.database.MOLS_15_4()

Return 4 MOLS of order 15.

These MOLS were shared by Ian Wanless.

EXAMPLES:

sage: from sage.combinat.designs.latin_squares import are_mutually_orthogonal_latin_squares

(continues on next page)
The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(3,18)
True
```

**sage.combinat.designs.database.OA_10_1620()**

Returns an $\text{OA}(10,1620)$.

This is obtained through the generalized Brouwer-van Rees construction. Indeed, $1620 = 144.11 + (36 = 4.9)$ and there exists an $\text{OA}(10,153) - \text{OA}(10,9)$.

**Note:** This function should be removed once `find_brouwer_van_rees_with_one_truncated_column()` can handle all incomplete orthogonal arrays obtained through `incomplete_orthogonal_array()`.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_10_1620
sage: OA = OA_10_1620()  # not tested -- ~7s
sage: is_orthogonal_array(OA,10,1620,2)  # not tested -- ~7s
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(10,1620)
True
```

**sage.combinat.designs.database.OA_10_205()**

Return an $\text{OA}(10,205)$.
Julian R. Abel shared the following construction, which originally appeared in Theorem 8.7 of [Greig99], and can in Lemmas 5.14-5.16 of [ColDin01]:

Consider a $PG(2, 4^2)$ containing a Baer subplane (i.e. a $PG(2, 4)$) $B$ and a point $p \in B$. Among the $4^2 + 1 = 17$ lines of $PG(2, 4^2)$ containing $p$:

- $4 + 1 = 5$ lines intersect $B$ on 5 points
- $4^2 - 4 = 12$ lines intersect $B$ on 1 point

As those lines are disjoint outside of $B$ we can use them as groups to build a GDD on $16^2 + 16 + 1 = 252$ points. By keeping only 9 lines of the second kind, however, we obtain a $(204, \{9, 13, 17\})$-GDD of type $12^5, 16^9$.

We complete it into a PBD by adding a block $g \cup \{204\}$ for each group $g$. We then build an OA from this PBD using the fact that all blocks of size 9 are disjoint.

See also:

`sage.combinat.designs.orthogonal_arrays.OA_from_PBD()`

**EXAMPLES:**

```
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_10_205
sage: OA = OA_10_205()
```

The design is available from the general constructor:

```
sage: designs.orthogonal_arrays.is_available(10,205)
True
```

`sage.combinat.designs.database.OA_10_469()`

Return an OA(10,469)

This construction appears in [Brouwer80]. It is based on the same technique used in `brouwer_separable_design()`.

Julian R. Abel’s instructions:

Brouwer notes that a cyclic $PG(2, 37)$ (or $(1407, 38, 1)$-BIBD) can be obtained with a base block containing 13, 9, and 16 points in each residue class mod 3. Thus, by reducing the $PG(2, 37)$ to its points congruent to 0 (mod 3) one obtains a $(469, \{9, 13, 16\})$-PBD which consists in 3 symmetric designs, i.e. 469 blocks of size 9, 469 blocks of size 13, and 469 blocks of size 16.

For each block size $s$, one can build a matrix with size $s \times 469$ in which each block is a row, and such that each point of the PBD appears once per column. By multiplying a row of an $OA(9, s) - s.OA(9, 1)$ with the rows of the matrix one obtains a parallel class of a resolvable $OA(9, 469)$.

Add to this the parallel class of all blocks $(0, 0, ...), (1, 1, ...), ...$ to obtain a resolvable $OA(9, 469)$ equivalent to an $OA(10, 469)$.

**EXAMPLES:**

```
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_10_469
sage: OA = OA_10_469()
```

```
sage: designs.orthogonal_arrays.is_available(10,469)
True
```

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The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(10,469)
True
```

sage.combinat.designs.database.OA_10_520()

Return an OA(10,520).

This design is built by the slightly more general construction `OA_520_plus_x()`.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_10_520
sage: OA = OA_10_520()
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(10,520)
True
```

sage.combinat.designs.database.OA_10_796()

Returns an OA(10,796)

Construction shared by Julian R. Abel, from [AC07]:

Truncate one block of a $TD(17,47)$ to size 13, then add an extra point. Form a block on each group plus the extra point: we obtain a $(796,\{13,16,17,47,48\})$-PBD in which only the extra point lies in more than one block of size 48 (and each other point lies in exactly 1 such block).

For each block $B$ (of size $k$ say) not containing the extra point, construct a $TD(10,k) - k.TD(k,1)$ on $I(10)XB$. For each block $B$ (of size $k = 47$ or 48) containing the extra point, construct a $TD(10,k) - TD(k,1)$ on $I(10)XB$, the size 1 hole being on $I(10)XP$ where $P$ is the extra point. Finally form 1 extra block of size 10 on $I(10)XP$.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_10_796
sage: OA = OA_10_796()
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(10,796)
True
```

sage.combinat.designs.database.OA_11_160()

Returns an OA(11,160)

Published by Julian R. Abel in [Ab1995]. Uses the fact that 160 = $2^5 \times 5$ is a product of a power of 2 and a prime number.

**See also:**

`sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()`
EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_11_160
sage: OA = OA_11_160()
```

```python
sage: is_orthogonal_array(OA,11,160,2)
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(11,160)
True
```

```python
sage.combinat.designs.database.OA_11_185()
```

Returns an OA(11,185)

The construction is given in [Greig99]. In Julian R. Abel’s words:

Start with a $PG(2,16)$ with a 7 points Fano subplane; outside this plane there are $7(17 - 3) = 98$ points on a line of the subplane and $273 - 98 - 7 = 168$ other points. Greig notes that the subdesign consisting of these 168 points is a $(168, \{10, 12\}) - PBD$. Now add the 17 points of a line disjoint from this subdesign (e.g. a line of the Fano subplane). This line will intersect every line of the 168 point subdesign in 1 point. Thus the new line sizes are 11 and 13, plus a unique line of size 17, giving a $(185, \{11, 13, 17\}$-PBD and an $OA(11, 185)$.

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_11_185
sage: OA = OA_11_185()
```

```python
sage: is_orthogonal_array(OA,11,185,2)
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(11,185)
True
```

```python
sage.combinat.designs.database.OA_11_254()
```

Returns an OA(11,254)

This construction appears in [Greig99].

From a cyclic $PG(2,19)$ whose base blocks contains 7,9, and 4 points in the congruence classes mod 3, build a $(254, 11, 13, 16) - PBD$ by ignoring the points of a congruence class. There exist $OA(12,11), OA(12,13), OA(12,16)$, which gives the $OA(11,254)$.

See also:

```python
sage.combinat.designs.orthogonal_arrays.OA_from_PBD()
```

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_11_254
sage: OA = OA_11_254()
```

```python
sage: is_orthogonal_array(OA,11,254,2)
True
```
sage: designs.orthogonal_arrays.is_available(11,254)
True

sage.combinat.designs.database.OA_11_640()
Returns an OA(11,640)
Published by Julian R. Abel in [Ab1995] (uses the fact that 640 = $2^7 \times 5$ is the product of a power of 2 and a prime number).

See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_11_640
sage: OA = OA_11_640()  # not tested (too long)
sage: is_orthogonal_array(OA,11,640,2)  # not tested (too long)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(11,640)
True

sage.combinat.designs.database.OA_11_80()
Return an OA(11,80)
As explained in the Handbook III.3.76 [DesignHandbook]. Uses the fact that 80 = $2^4 \times 5$ and that 5 is prime.

See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_11_80
sage: OA = OA_11_80()
sage: is_orthogonal_array(OA,11,80,2)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(11,80)
True

sage.combinat.designs.database.OA_12_522()
Return an OA(12,522)
This design is built by the slightly more general construction OA_520_plus_x().

EXAMPLES:
The design is available from the general constructor:

```
sage: designs.orthogonal_arrays.is_available(12,522)
True
```

sage.combinat.designs.database.OA_14_524()

Returns an OA(14,524)

This design is built by the slightly more general construction `OA_520_plus_x()`.

EXAMPLES:

```
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_14_524
sage: OA = OA_14_524()
```

The design is available from the general constructor:

```
sage: designs.orthogonal_arrays.is_available(14,524)
True
```

sage.combinat.designs.database.OA_15_112()

Returns an OA(15,112)

Published by Julian R. Abel in [Ab1995]. Uses the fact that 112 = 2⁴ × 7 and that 7 is prime.

See also:

`sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()`

EXAMPLES:

```
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_15_112
sage: OA = OA_15_112()
```

The design is available from the general constructor:

```
sage: designs.orthogonal_arrays.is_available(15,112)
True
```

sage.combinat.designs.database.OA_15_224()

Returns an OA(15,224)

Published by Julian R. Abel in [Ab1995] (uses the fact that 224 = 2⁵ × 7 is a product of a power of 2 and a prime number).
See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_15_224
sage: OA = OA_15_224()  # not tested -- too long
sage: is_orthogonal_array(OA,15,224,2)  # not tested -- too long
True

The design is available from the general constructor:
sage: designs.orthogonal_arrays.is_available(15,224)
True

sage.combinat.designs.database.OA_15_896()
Returns an OA(15,896)
Uses the fact that 896 = 2^7 × 7 is the product of a power of 2 and a prime number.

See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_15_896
sage: OA = OA_15_896()  # not tested -- too long (~2min)
sage: is_orthogonal_array(OA,15,896,2)  # not tested -- too long
True

The design is available from the general constructor:
sage: designs.orthogonal_arrays.is_available(15,896)
True

sage.combinat.designs.database.OA_16_176()
Returns an OA(16,176)
Published by Julian R. Abel in [Ab1995]. Uses the fact that 176 = 2^4 × 11 is a product of a power of 2 and a prime number.

See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_15_896
sage: OA = OA_15_896()  # not tested -- too long (~2min)
sage: is_orthogonal_array(OA,15,896,2)  # not tested -- too long
True

The design is available from the general constructor:
sage: designs.orthogonal_arrays.is_available(16,176)
True

sage.combinat.designs.database.OA_16_208()
Returns an OA(16,208)
Published by Julian R. Abel in [Ab1995]. Uses the fact that 208 = 2^4 × 13 is a product of 2 and a prime number.
See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_16_208
sage: OA = OA_16_208()
# not tested -- too long
sage: is_orthogonal_array(OA,16,208,2)  # not tested -- too long
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(16,208)
True

sage.combinat.designs.database.OA_17_560()
Returns an OA(17,560)
This OA is built in Corollary 2.2 of [Thwarts].

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_17_560
sage: OA = OA_17_560()
sage: is_orthogonal_array(OA,17,560,2)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(17,560)
True

sage.combinat.designs.database.OA_20_352()
Returns an OA(20,352)
Published by Julian R. Abel in [Ab1995] (uses the fact that 352 = 2^5 × 11 is the product of a power of 2 and a prime number).
See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_20_352
sage: OA = OA_20_352()  # not tested (~25s)
sage: is_orthogonal_array(OA,20,352,2)  # not tested (~25s)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(20,352)
True

sage.combinat.designs.database.OA_20_416()
Returns an OA(20,416)
Published by Julian R. Abel in [Ab1995] (uses the fact that 416 = 2^5 × 13 is the product of a power of 2 and a prime number).

See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_20_416
sage: OA = OA_20_416()  # not tested (~35s)
sage: is_orthogonal_array(OA,20,416,2)  # not tested
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(20,416)
True

sage.combinat.designs.database.OA_20_544()
Returns an OA(20,544)
Published by Julian R. Abel in [Ab1995] (uses the fact that 544 = 2^5 × 17 is the product of a power of 2 and a prime number).

See also:
sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_20_544
sage: OA = OA_20_544()  # not tested (too long ~1mn)
sage: is_orthogonal_array(OA,20,544,2)  # not tested
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(20,544)
True

sage.combinat.designs.database.OA_25_1262()
Returns an OA(25,1262)
The construction is given in [Greig99]. In Julian R. Abel’s words:
Start with a cyclic $PG(2,43)$ or $(1893,44,1)$-BIBD whose base block contains respectively 12, 13 and 19 point in the residue classes mod 3. In the resulting BIBD, remove one of the three classes: the result is a $(1262,\{25,31,32\})$-PBD, from which the $OA(25,1262)$ is obtained.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_25_1262
sage: OA = OA_25_1262()               # not tested -- too long
sage: is_orthogonal_array(OA,25,1262,2) # not tested -- too long
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(25,1262)
True
```

**sage.combinat.designs.database.OA_520_plus_x(x)**

Return an $OA(10+x,520+x)$.

The construction shared by Julian R. Abel works for $OA(10,520)$, $OA(12,522)$, and $OA(14,524)$.

Let $n = 520 + x$ and $k = 10 + x$. Build a $TD(17,31)$. Remove $8-x$ points contained in a common block, add a new point $p$ and create a block $g_i \cup \{p\}$ for every (possibly truncated) group $g_i$. The result is a $(520+x,9+x,16,17,31,32)-PBD$. Note that all blocks of size $\geq 30$ only intersect on $p$, and that the unique block $B_9$ of size 9 intersects all blocks of size 32 on one point. Now:

- Build an $OA(k,16) - 16.OA(k,16)$ for each block of size 16
- Build an $OA(k,17) - 17.OA(k,17)$ for each block of size 17
- Build an $OA(k,31) - OA(k,1)$ for each block of size 31 (with the hole on $p$).
- Build an $OA(k,32) - 2.OA(k,1)$ for each block $B$ of size 32 (with the holes on $p$ and $B \cap B_9$).
- Build an $OA(k,9)$ on $B_9$.

Only a row $[p,p,...]$ is missing from the $OA(10+x,520+x)$

This construction is used in $OA(10,520)$, $OA(12,522)$, and $OA(14,524)$.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_520_plus_x
sage: OA = OA_520_plus_x(0)               # not tested (already tested in OA_10_520)
sage: is_orthogonal_array(OA,10,520,2)   # not tested (already tested in OA_10_520)
True
```

**sage.combinat.designs.database.OA_7_18()**

Return an OA(7,18)

Proved in [JulianAbel13].

**See also:**

`sage.combinat.designs.orthogonal_arrays.OA_from_quasi_difference_matrix()`

**EXAMPLES:**
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sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_7_18
sage: OA = OA_7_18()
sage: is_orthogonal_array(OA,7,18,2)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(7,18)
True

sage.combinat.designs.database.OA_7_66()
Return an OA(7,66)
Construction shared by Julian R. Abel.
See also:
sage.combinat.designs.orthogonal_arrays.OA_from_PBD()

EXAMPLES:

sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_7_66
sage: OA = OA_7_66()
sage: is_orthogonal_array(OA,7,66,2)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(7,66)
True

sage.combinat.designs.database.OA_7_68()
Return an OA(7,68)
Construction shared by Julian R. Abel.
See also:
sage.combinat.designs.orthogonal_arrays.OA_from_PBD()

EXAMPLES:

sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_7_68
sage: OA = OA_7_68()
sage: is_orthogonal_array(OA,7,68,2)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(7,68)
True

sage.combinat.designs.database.OA_7_74()
Return an OA(7,74)
Construction shared by Julian R. Abel.

See also:

`sage.combinat.designs.orthogonal_arrays.OA_from_PBD()`

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_7_74
sage: OA = OA_7_74()
```
```
sage: is_orthogonal_array(OA,7,74,2)
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(7,74)
True
```

`sage.combinat.designs.database.OA_8_69()`

Return an OA(8,69)

Construction shared by Julian R. Abel.

See also:

`sage.combinat.designs.orthogonal_arrays.OA_from_PBD()`

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_8_69
sage: OA = OA_8_69()
```
```
sage: is_orthogonal_array(OA,8,69,2)
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(8,69)
True
```

`sage.combinat.designs.database.OA_8_76()`

Return an OA(8,76)

Construction shared by Julian R. Abel.

See also:

`sage.combinat.designs.orthogonal_arrays.OA_from_PBD()`

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_8_76
sage: OA = OA_8_76()
```
```
sage: is_orthogonal_array(OA,8,76,2)
True
```

The design is available from the general constructor:
sage: designs.orthogonal_arrays.is_available(8,76)
True

sage.combinat.designs.database.OA_9_1078()
Returns an OA(9,1078)
This is obtained through the generalized Brouwer-van Rees construction. Indeed, $1078 = 99 \times 11 + (99 = 9 \times 11)$ and there exists an $OA(9, 100) - OA(9, 11)$.

Note: This function should be removed once find_brouwer_van_rees_with_one_truncated_column() can handle all incomplete orthogonal arrays obtained through incomplete_orthogonal_array().

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_9_1078
sage: OA = OA_9_1078()          # not tested -- ~3s
sage: is_orthogonal_array(OA,9,1078,2) # not tested -- ~3s
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(9,1078)
True

sage.combinat.designs.database.OA_9_120()
Return an OA(9,120)
Construction shared by Julian R. Abel:

From a resolvable $(120, 8, 1) - BIBD$, one can obtain 7 MOLS(120) or a resolvable $TD(8, 120)$ by forming a resolvable $TD(8, 8) - 8.TD(8, 1)$ on $I_8 \times B$ for each block $B$ in the BIBD. This gives a $TD(8, 120) - 120.TD(8, 1)$ (which is resolvable as the BIBD is resolvable).

See also:

RBIBD_120_8_1()

EXAMPLES:

sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_9_120
sage: OA = OA_9_120()
sage: is_orthogonal_array(OA,9,120,2)
True

The design is available from the general constructor:

sage: designs.orthogonal_arrays.is_available(9,120)
True

sage.combinat.designs.database.OA_9_135()
Return an OA(9,135)
Construction shared by Julian R. Abel:
This design can be built by Wilson’s method \((135 = 8.16 + 7)\) applied to an Orthogonal Array \(OA(9 + 7, 16)\) with 7 groups truncated to size 1 in such a way that a block contain 0, 1 or 3 points of the truncated groups.

This is possible, because \(PG(2, 2)\) (the projective plane over \(GF(2)\)) is a subdesign in \(PG(2, 16)\) (the projective plane over \(GF(16)\)); in a cyclic \(PG(2, 16)\) or \(BIBD(273, 17, 1)\) the points \(\equiv 0 \pmod{39}\) form such a subdesign (note that \(273 = 16^2 + 16 + 1\) and \(273 = 39 \times 7\) and \(7 = 2^2 + 2 + 1\)).

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_9_135
sage: OA = OA_9_135()
```

```python
sage: is_orthogonal_array(OA, 9, 135, 2)
```

```
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(9, 135)
```

```
True
```

As this orthogonal array requires a \((273, 17, 1)\) cyclic difference set, we check that it is available:

```python
sage: G, D = designs.difference_family(273, 17, 1)
```

```python
sage: G
```

```
Ring of integers modulo 273
```

`sage.combinat.designs.database.OA_9_1612()`

Returns an \(OA(9, 1612)\)

This is obtained through the generalized Brouwer-van Rees construction. Indeed, \(1612 = 89.17 + (99 = 9.11)\) and there exists an \(OA(9, 100) = OA(9, 11)\).

**Note:** This function should be removed once \(find_brouwer_van_rees_with_one_truncated_column()\) can handle all incomplete orthogonal arrays obtained through \(incomplete_orthogonal_array()\).

**EXAMPLES:**

```python
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_9_1612
sage: OA = OA_9_1612()
# not tested -- ~6s
```

```python
sage: is_orthogonal_array(OA, 9, 1612, 2) # not tested -- ~6s
```

```
True
```

The design is available from the general constructor:

```python
sage: designs.orthogonal_arrays.is_available(9, 1612)
```

```
True
```

`sage.combinat.designs.database.OA_9_40()`

Return an \(OA(9, 40)\)

As explained in the Handbook III.3.62 [DesignHandbook]. Uses the fact that \(40 = 2^3 \times 5\) and that 5 is prime.

**See also:**

`sage.combinat.designs.orthogonal_arrays.OA_n_times_2_pow_c_from_matrix()`
Combinatorics, Release 10.0

EXAMPLES:

```
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: from sage.combinat.designs.database import OA_9_40
sage: OA = OA_9_40()
```
```
sage: is_orthogonal_array(OA,9,40,2)
True
```

The design is available from the general constructor:
```
sage: designs.orthogonal_arrays.is_available(9,40)
True
```

```
sage.combinat.designs.database.QDM_19_6_1_1_1()
```
Return a $(19, 6; 1; 1)_1$-quasi-difference matrix.

Used to build an $OA(6; 20)$


EXAMPLES:
```
sage: from sage.combinat.designs.database import QDM_19_6_1_1_1
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: G,M = QDM_19_6_1_1_1()
```
```
sage: is_quasi_difference_matrix(M,G,6,1,1,1)
True
```

```
sage.combinat.designs.database.QDM_21_5_1_1_1()
```
Return a $(21, 5; 1; 1)_1$-quasi-difference matrix.

Used to build an $OA(5; 22)$


EXAMPLES:
```
sage: from sage.combinat.designs.database import QDM_21_5_1_1_1
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: G,M = QDM_21_5_1_1_1()
```
```
sage: is_quasi_difference_matrix(M,G,5,1,1,1)
True
```

```
sage.combinat.designs.database.QDM_21_6_1_1_5()
```
Return a $(21, 6; 1; 5)_1$-quasi-difference matrix.

Used to build an $OA(6; 26)$


EXAMPLES:
```
sage: from sage.combinat.designs.database import QDM_21_6_1_1_5
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: G,M = QDM_21_6_1_1_5()
```
```
sage: is_quasi_difference_matrix(M,G,6,1,1,5)
True
```

5.1. Comprehensive Module List
sage.combinat.designs.database.QDM_25_6_1_1_5()
    Return a (25, 6; 1, 1; 5)-quasi-difference matrix.
    Used to build an OA(6, 30)
    EXAMPLES:
    sage: from sage.combinat.designs.database import QDM_25_6_1_1_5
    sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
    sage: G,M = QDM_25_6_1_1_5()
    sage: is_quasi_difference_matrix(M,G,6,1,1,5)
    True

sage.combinat.designs.database.QDM_33_6_1_1_1()
    Return a (33, 6; 1, 1; 1)-quasi-difference matrix.
    Used to build an OA(6, 34)
    EXAMPLES:
    sage: from sage.combinat.designs.database import QDM_33_6_1_1_1
    sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
    sage: G,M = QDM_33_6_1_1_1()
    sage: is_quasi_difference_matrix(M,G,6,1,1,1)
    True

sage.combinat.designs.database.QDM_35_7_1_1_7()
    Return a (35, 7; 1, 1; 7)-quasi-difference matrix.
    Used to build an OA(7, 42)
    As explained in the Handbook III.3.63 [DesignHandbook].
    EXAMPLES:
    sage: from sage.combinat.designs.database import QDM_35_7_1_1_7
    sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
    sage: G,M = QDM_35_7_1_1_7()
    sage: is_quasi_difference_matrix(M,G,7,1,1,7)
    True

sage.combinat.designs.database.QDM_37_6_1_1_1()
    Return a (37, 6; 1, 1; 1)-quasi-difference matrix.
    Used to build an OA(6, 38)
    Given in the Handbook III.3.60 [DesignHandbook].
    EXAMPLES:
    sage: from sage.combinat.designs.database import QDM_37_6_1_1_1
    sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
    sage: G,M = QDM_37_6_1_1_1()
    sage: is_quasi_difference_matrix(M,G,6,1,1,1)
    True
sage.combinat.designs.database.QDM_45_7_1_1_9()

Return a \((45, 7; 1, 1; 9)\)-quasi-difference matrix.

Used to build an \(OA(7, 54)\)

As explained in the Handbook III.3.71 [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.database import QDM_45_7_1_1_9
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: G,M = QDM_45_7_1_1_9()
sage: is_quasi_difference_matrix(M,G,7,1,1,9)
True
```

sage.combinat.designs.database.QDM_54_7_1_1_8()

Return a \((54, 7; 1, 1; 8)\)-quasi-difference matrix.

Used to build an \(OA(7, 62)\)

As explained in the Handbook III.3.74 [DesignHandbook].

EXAMPLES:

```python
sage: from sage.combinat.designs.database import QDM_54_7_1_1_8
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: G,M = QDM_54_7_1_1_8()
sage: is_quasi_difference_matrix(M,G,7,1,1,8)
True
```

sage.combinat.designs.database.QDM_57_9_1_1_8()

Return a \((57, 9; 1, 1; 8)\)-quasi-difference matrix.

Used to build an \(OA(9, 65)\)

Construction shared by Julian R. Abel

EXAMPLES:

```python
sage: from sage.combinat.designs.database import QDM_57_9_1_1_8
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: G,M = QDM_57_9_1_1_8()
sage: is_quasi_difference_matrix(M,G,9,1,1,8)
True
```

sage.combinat.designs.database.RBIBD_120_8_1()

Return a resolvable \(BIBD(120, 8, 1)\)

This function output a list \(L\) of \(17 \times 15\) blocks such that \(L[i*15:(i+1)*15]\) is a partition of 120.

Construction shared by Julian R. Abel:

Seiden’s method: Start with a cyclic \((273, 17, 1) - BIBD\) and let \(B\) be an hyperoval, i.e. a set of 18 points which intersects any block of the BIBD in either 0 points (153 blocks) or 2 points (120 blocks). Dualise this design and take these last 120 blocks as points in the design; blocks in the design will correspond to the \(273 - 18 = 255\) non-hyperoval points.

The design is also resolvable. In the original \(PG(2, 16)\) take any point \(T\) in the hyperoval and consider a block \(B_1\) containing \(T\). The 15 points in \(B_1\) that do not belong to the hyperoval correspond to 15
blocks forming a parallel class in the dualised design. The other 16 parallel classes come in a similar way, by using point $T$ and the other 16 blocks containing $T$.

See also:

$OA_{9,120}()$

EXAMPLES:

```python
sage: from sage.combinat.designs.database import RBIBD_120_8_1
sage: RBIBD = RBIBD_120_8_1()
sage: is_pairwise_balanced_design(RBIBD, 120, [8])
True
```

It is indeed resolvable, and the parallel classes are given by 17 slices of consecutive 15 blocks:

```python
sage: for i in range(17):
    ....: assert len(set(sum(RBIBD[i*15:(i+1)*15], []))) == 120
```

The BIBD is available from the constructor:

```python
sage: _ = designs.balanced_incomplete_block_design(120, 8)
```

5.1.81 Catalog of designs

This module gathers all designs that can be reached through designs.<tab>. Example with the Witt design on 24 points:

```python
sage: designs.WittDesign(24) # optional - gap_packages
Incidence structure with 24 points and 759 blocks
```

Or a Steiner Triple System on 19 points:

```python
sage: designs.steiner_triple_system(19)
(19,3,1)-Balanced Incomplete Block Design
```

La Jolla Covering Repository
The La Jolla Covering Repository (LJCR, see\(^1\)) is an online database of covering designs. As it is frequently updated, it is not included in Sage, but one can query it through `designs.best_known_covering_design_from_LJCR`:

```python
sage: C = designs.best_known_covering_design_from_LJCR(7, 3, 2)  # optional - internet
sage: C  # optional - internet
(7, 3, 2)-covering design of size 7
Lower bound: 7
Method: lex covering
Submitted on: 1996-12-01 00:00:00
```

Incidence structure with 7 points and 7 blocks

### Design constructors

This module gathers the following designs:

- `ProjectiveGeometryDesign()`
- `DesarguesianProjectivePlaneDesign()`
- `HughesPlane()`
- `HigmanSimsDesign()`
- `balanced_incomplete_block_design()`
- `resolvable_balanced_incomplete_block_design()`
- `kirkman_triple_system()`
- `AffineGeometryDesign()`
- `CremonaRichmondConfiguration()`
- `WittDesign()`
- `HadamardDesign()`
- `Hadamard3Design()`
- `mutually_orthogonal_latin_squares()`
- `transversal_design()`
- `orthogonal_array()`
- `incomplete_orthogonal_array()`
- `difference_family()`
- `difference_matrix()`
- `steiner_triple_system()`
- `steiner_quadruple_system()`
- `projective_plane()`
- `biplane()`
- `gen_quadrangles_with_spread()`

And the `designs.best_known_covering_design_from_LJCR` function which queries the LJCR.

---

**Todo:** Implement DerivedDesign and ComplementaryDesign.

---

**REFERENCES:**

1. La Jolla Covering Repository, https://math.ccrwest.org/cover.html

---

5.1. Comprehensive Module List 661
5.1.82 Cython functions for combinatorial designs

This module implements the design methods that need to be somewhat efficient.

Functions

sage.combinat.designs.designs_pyx.is_difference_matrix(M, G, k, lmbda=1, verbose=False)
Test if $M$ is a $(G, k, \lambda)$-difference matrix.

A matrix $M$ is a $(G, k, \lambda)$-difference matrix if its entries are element of $G$, and if for any two rows $R, R'$ of $M$ and $x \in G$ there are exactly $\lambda$ values $i$ such that $R_i - R'_i = x$.

INPUT:
• $M$ – a matrix with entries from $G$
• $G$ – a group
• $k$ – integer
• $\lambda$ (integer) – set to 1 by default.
• $\text{verbose}$ (boolean) – whether to print some information when the answer is False.

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_difference_matrix
sage: q = 3**3
sage: F = GF(q, 'x')

sage: M = [[x*y for y in F] for x in F]
sage: is_difference_matrix(M,F,q,verbose=1)
True

sage: B = [[0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 1, 2, 3, 4, 2, 3, 4, 0], [0, 2, 4, 1, 3, 3, 0, 2, 4]]
sage: G = GF(5)
sage: B = [[G(b) for b in R] for R in B]
sage: is_difference_matrix(list(zip(*B)),G,3,2)
True
```

Bad input:

```python
sage: for R in M: R.append(None)
sage: is_difference_matrix(M,F,q,verbose=1)
The matrix has 28 columns but k=27
False

sage: for R in M: _=R.pop(-1)
sage: is_difference_matrix(M,F,q,verbose=1)
The matrix has 28 rows instead of lambda(|G|-1+2u)+mu=1(27-1+2.0)+1=27
False

sage: _= M.pop(-1)
sage: for R in M: R[-1] = 0
sage: is_difference_matrix(M,F,q,verbose=1)
Columns 0 and 26 generate 0 exactly 27 times instead of the expected mu(=1)
False
```

(continues on next page)
sage: for R in M: R[-1] = 1
sage: M[-1][-1] = 0
sage: is_difference_matrix(M,F,q,verbose=1)
Columns 0 and 26 do not generate all elements of G exactly \lambda(=1) times. The element x appeared 0 times as a difference.
False

sage.combinat.designs.designs_pyx.is_group_divisible_design(groups, blocks, v, G=None, K=None, lambd=1, verbose=\text{False})

Checks that input is a Group Divisible Design on \{0, \ldots, v - 1\}
For more information on Group Divisible Designs, see \text{GroupDivisibleDesign}.

\text{INPUT:}
\begin{itemize}
  \item \text{groups} – a partition of \(X\). If set to \text{None} the groups are guessed automatically, and the function returns \text{(True, guessed\text{-}groups)} instead of \text{True}
  \item \text{blocks} – collection of blocks
  \item \text{v} (integers) – size of the ground set assumed to be \(X = \{0, \ldots, v - 1\}\).
  \item \text{G} – list of integers of which the sizes of the groups must be elements. Set to \text{None} (automatic guess) by default.
  \item \text{K} – list of integers of which the sizes of the blocks must be elements. Set to \text{None} (automatic guess) by default.
  \item \text{lambd} – value of \lambda. Set to 1 by default.
  \item \text{verbose} (boolean) – whether to display some information when the design is not a GDD.
\end{itemize}

\text{EXAMPLES:}
\begin{verbatim}
sage: from sage.combinat.designs.designs_pyx import is_group_divisible_design
sage: TD = designs.transversal_design(4,10)
sage: groups = [list(range(i*10,(i+1)*10)) for i in range(4)]
sage: is_group_divisible_design(groups,TD,40,lambd=1)
True
\end{verbatim}

sage.combinat.designs.designs_pyx.is_orthogonal_array(OA, k, n, t=2, \text{verbose=\text{False}}, terminology=\text{"OA"})

Check that the integer matrix \text{OA} is an \text{OA}(k, n, t).
See \text{orthogonal\_array()} for a definition.

\text{INPUT:}
\begin{itemize}
  \item \text{OA} – the Orthogonal Array to be tested
  \item \text{k,n,t} (integers) – only implemented for \(t = 2\).
  \item \text{verbose} (boolean) – whether to display some information when \text{OA} is not an orthogonal array \text{OA}(k, n).
  \item \text{terminology} (string) – how to phrase the information when \text{verbose} = \text{True}. Possible values are ”\text{OA}”, ”\text{MOLS}”.
\end{itemize}

\text{EXAMPLES:}
from sage.combinat.designs.designs_pyx import is_orthogonal_array
OA = designs.orthogonal_arrays.build(8,9)
is_orthogonal_array(OA,8,9)
True
is_orthogonal_array(OA,8,10)
False
OA[4][3] = 1
is_orthogonal_array(OA,8,9)
False
is_orthogonal_array(OA,8,9,verbose=True)
Columns 0 and 3 are not orthogonal
False
is_orthogonal_array(OA,8,9,verbose=True,terminology="MOLS")
Squares 0 and 3 are not orthogonal
False

is_pairwise_balanced_design(blocks, v, K=None, lambd=1, verbose=False)
Checks that input is a Pairwise Balanced Design (PBD) on \{0,..., v-1\}
For more information on Pairwise Balanced Designs (PBD), see PairwiseBalancedDesign.

INPUT:
- blocks – collection of blocks
- v (integers) – size of the ground set assumed to be \(X = \{0,..., v-1\}\).
- K – list of integers of which the sizes of the blocks must be elements. Set to None (automatic guess) by default.
- lambd – value of \(\lambda\). Set to 1 by default.
- verbose (boolean) – whether to display some information when the design is not a PBD.

EXAMPLES:
from sage.combinat.designs.designs_pyx import is_pairwise_balanced_design
sts = designs.steiner_triple_system(9)
is_pairwise_balanced_design(sts,9,[3],1)
True
TD = designs.transversal_design(4,10).blocks()
groups = [list(range(i*10,(i+1)*10)) for i in range(4)]
is_pairwise_balanced_design(TD+groups,40,[4,10],1,verbose=True)
True

is_projective_plane(blocks, verbose=False)
Test whether the blocks form a projective plane on \{0,..., v-1\}
A projective plane is an incidence structure that has the following properties:
1. Given any two distinct points, there is exactly one line incident with both of them.
2. Given any two distinct lines, there is exactly one point incident with both of them.
3. There are four points such that no line is incident with more than two of them.
For more informations, see Wikipedia article Projective_plane.
\texttt{is_t_design()} can also check if an incidence structure is a projective plane with the parameters \(v = k^2 + k + 1\), \(t = 2\) and \(l = 1\).

INPUT:
- blocks – collection of blocks
- verbose – whether to print additional information

EXAMPLES:

```python
sage: from sage.combinat.designs.designs_pyx import is_projective_plane
sage: p = designs.projective_plane(4)
sage: b = p.blocks()
sage: is_projective_plane(b, verbose=True)
True

sage: p = designs.projective_plane(2)
sage: b = p.blocks()
sage: is_projective_plane(b)
True

sage: b[0][2] = 5
sage: is_projective_plane(b, verbose=True)
the pair (0,5) has been seen 2 times but lambda=1
False

sage: is_projective_plane([[0,1,2],[1,2,4]], verbose=True)
the pair (0,3) has been seen 0 times but lambda=1
False

sage: is_projective_plane([[1]], verbose=True)
First block has less than 3 points.
False

sage: p = designs.projective_plane(2)
sage: b = p.blocks()
sage: b[2].append(4)
sage: is_projective_plane(b, verbose=True)
a block has size 4 while K=[3]
False
```

\texttt{sage.combinat.designs.designs_pyx.is_quasi_difference_matrix(M, G, k, lambda, mu, u, verbose=False)}

Test if the matrix is a \((G, k; \lambda, \mu; u)\)-quasi-difference matrix

Let \(G\) be an abelian group of order \(n\). A \((n, k; \lambda, \mu; u)\)-quasi-difference matrix (QDM) is a matrix \(Q_{ij}\) with \(\lambda(n-1+2u) + \mu\) rows and \(k\) columns, with each entry either equal to \texttt{None} (i.e. the 'missing entries') or to an element of \(G\). Each column contains exactly \(\lambda u\) empty entries, and each row contains at most one \texttt{None}. Furthermore, for each \(1 \leq i < j \leq k\), the multiset

\[
\{q_{ii} - q_{ij} : 1 \leq l \leq \lambda(n-1+2u) + \mu, \text{ with } q_{ii} \text{ and } q_{ij} \text{ not empty}\}
\]

contains \(\lambda\) times every nonzero element of \(G\) and contains \(\mu\) times \(0\).

INPUT:
- \(M\) – a matrix with entries from \(G\) (or equal to \texttt{None} for missing entries)
• G – a group
• k, lambda, mu, u – integers
• verbose (boolean) – whether to print some information when the answer is False.

EXAMPLES:

Differences matrices:

```
sage: from sage.combinat.designs.designs_pyx import is_quasi_difference_matrix
sage: q = 3**3
sage: F = GF(q, 'x')
sage: M = [[x*y for y in F] for x in F]
sage: is_quasi_difference_matrix(M,F,q,1,1,0,verbose=1)
True
sage: B = [[0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
       ....: [0, 1, 2, 3, 4, 2, 3, 4, 0, 1],
       ....: [0, 2, 4, 1, 3, 3, 0, 2, 4, 1]]
sage: G = GF(5)
sage: B = [[G(b) for b in R] for R in B]
sage: is_quasi_difference_matrix(list(zip(*B)),G,3,2,2,0)
True
```

A quasi-difference matrix from the database:

```
sage: from sage.combinat.designs.database import QDM
sage: G,M = QDM[38,1][37,1,1,1][1]()
sage: is_quasi_difference_matrix(M,G,k=6,lambda=1,mu=1,u=1)
True
```

Bad input:

```
sage: is_quasi_difference_matrix(M,G,k=6,lambda=1,mu=1,u=3,verbose=1)
The matrix has 39 rows instead of lambda(|G|-1+2u)+mu=1(37-1+2.3)+1=43
False
sage: is_quasi_difference_matrix(M,G,k=6,lambda=1,mu=2,u=1,verbose=1)
The matrix has 39 rows instead of lambda(|G|-1+2u)+mu=1(37-1+2.1)+2=40
False
sage: M[3][1] = None
sage: is_quasi_difference_matrix(M,G,k=6,lambda=1,mu=1,u=1,verbose=1)
Row 3 contains more than one empty entry
False
sage: M[3][1] = 1
sage: M[6][1] = None
sage: is_quasi_difference_matrix(M,G,k=6,lambda=1,mu=1,u=1,verbose=1)
Column 1 contains 2 empty entries instead of the expected lambda.u=1.1=1
False
```
### 5.1.83 Difference families

This module gathers everything related to difference families. One can build a difference family (or check that it can be built) with `difference_family()`:

```python
sage: G, F = designs.difference_family(13, 4, 1)
```

It defines the following functions:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>are_complementary_difference_sets()</code></td>
<td>Check if A and B are complementary difference sets over the group G.</td>
</tr>
<tr>
<td><code>are_hadamard_difference_sets_parameters()</code></td>
<td>Check whether (v, k, lambda) is of the form (4N², 2N² - N, N² - N).</td>
</tr>
<tr>
<td><code>are_mcfarland_1973_parameters()</code></td>
<td>Check whether (v, k, lambda) is a triple that can be obtained from the construction from [McF1973].</td>
</tr>
<tr>
<td><code>block_stabilizer()</code></td>
<td>Compute the left stabilizer of the block B under the action of G.</td>
</tr>
<tr>
<td><code>complementary_difference_sets()</code></td>
<td>Construct complementary difference sets over a group of order n = 2m + 1.</td>
</tr>
<tr>
<td><code>complementary_difference_sets_in_group_of_order_n()</code></td>
<td>Construct complementary difference sets in a group of order n ≡ 3 mod 4, n a prime power.</td>
</tr>
<tr>
<td><code>complementary_difference_sets_in_group_of_order_n()</code></td>
<td>Construct complementary difference sets in a group of order n = p¹, where p ≡ 5 mod 8 and t ≡ 1, 2, 3 mod 4.</td>
</tr>
<tr>
<td><code>complementary_difference_sets_in_group_of_order_n()</code></td>
<td>Construct complementary difference sets in a group of order n = 2m+1, where 4m + 3 is a prime power.</td>
</tr>
<tr>
<td><code>df_q_6_1()</code></td>
<td>Return a (q, 6, 1)-difference family over the finite field K.</td>
</tr>
<tr>
<td><code>difference_family()</code></td>
<td>Return a (k, 1)-difference family on an Abelian group of cardinality v.</td>
</tr>
<tr>
<td><code>get_fixed_relative_difference()</code></td>
<td>Construct an equivalent relative difference set fixed by the size of the set.</td>
</tr>
<tr>
<td><code>group_law()</code></td>
<td>Return a triple (identity, operation, inverse) that define the operations on the group G.</td>
</tr>
<tr>
<td><code>hadamard_difference_set_product()</code></td>
<td>Make a product of two Hadamard difference sets.</td>
</tr>
<tr>
<td><code>is_difference_family()</code></td>
<td>Check whether D forms a difference family in the group G.</td>
</tr>
<tr>
<td><code>is_fixed_relative_difference_family()</code></td>
<td>Check if the relative difference set R is fixed by q.</td>
</tr>
<tr>
<td><code>is_relative_difference_set()</code></td>
<td>Check that the sets in Rs are n - {v; k₁, ..., kₙ; λ} supplementary difference sets over group G of order v.</td>
</tr>
<tr>
<td><code>mcfarland_1973_construction()</code></td>
<td>Return a difference set.</td>
</tr>
<tr>
<td><code>one_cyclic_tiling()</code></td>
<td>Given a subset A of the cyclic additive group G = Z/nZ return another subset B so that A + B = G and</td>
</tr>
<tr>
<td><code>one_radical_difference_family()</code></td>
<td>Search for a radical difference family on K using dancing links algorithm.</td>
</tr>
<tr>
<td><code>radical_difference_family()</code></td>
<td>Return a (v, k, 1)-radical difference family.</td>
</tr>
<tr>
<td><code>radical_difference_set()</code></td>
<td>Return a difference set made of a cyclotomic coset in the finite field K and with parameters k and λ.</td>
</tr>
<tr>
<td><code>relative_difference_set_from_projective_space()</code></td>
<td>Construct D(qᴺ², 1)/(q - 1), n, qᴺ⁻¹, qᴺ⁻¹, d) where nd = q - 1.</td>
</tr>
<tr>
<td><code>relative_difference_set_from_projective_space()</code></td>
<td>Construct D(qᴺ⁺¹, 1)/(q - 1), q⁻¹, q⁻¹, q⁻¹, q⁻¹) where q is a prime power and N ≥ 2.</td>
</tr>
<tr>
<td><code>singer_difference_set()</code></td>
<td>Return a difference set associated to the set of hyperplanes in a projective space of dimension d over GF(q).</td>
</tr>
<tr>
<td><code>skew_supplementary_difference_sets()</code></td>
<td>Construct {0, 1, ..., n - 1} supplementary difference sets, where S₁ is skew and n₁ + n₂ + n₃ + n₄ = n + λ.</td>
</tr>
<tr>
<td><code>skew_supplementary_difference_sets()</code></td>
<td>Construct (Sk, q, λ) skew supplementary difference sets over a polynomial ring of order n.</td>
</tr>
<tr>
<td><code>skew_supplementary_difference_sets()</code></td>
<td>Construct a (p; p, 1, λ) skew supplementary difference sets where S₁ is the Paley-Todd difference set.</td>
</tr>
</tbody>
</table>

Continues on next page
Table 1 – continued from previous page

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>supplementary_difference_set()</td>
<td>Construct $4 - {2v; v, v + 1, v, 2v; 2} }$ supplementary difference sets where $q = 2v + 1$.</td>
</tr>
<tr>
<td>supplementary_difference_set_from_rel_diff_set()</td>
<td>Construct $4 - {2v; v, v + 1, v, 2v; 2} }$ supplementary difference sets where $q = 2v + 1$.</td>
</tr>
<tr>
<td>supplementary_difference_set_hadamard()</td>
<td>Construct $4 - {n; n_1, n_2, n_3, n_4; \lambda }$ supplementary difference sets, where $n_1 + n_2 + n_3 + n_4 = n + \lambda$.</td>
</tr>
<tr>
<td>turyn_1965_3x3xK()</td>
<td>Return a difference set in either $C_3 \times C_3 \times C_4$ or $C_3 \times C_3 \times C_2 \times C_2$ with parameters $v = 36, k = 15, \lambda = 6$.</td>
</tr>
<tr>
<td>twin_prime_powers_difference_set()</td>
<td>Return a difference set on $GF(p) \times GF(p + 2)$.</td>
</tr>
</tbody>
</table>

REFERENCES:

Functions

sage.combinat.designs.difference_family.are_complementary_difference_sets$(G, A, B$, $\verb|verbose|=False)$

Check if A and B are complementary difference sets over the group G.

According to [Sze1971], two sets $A, B$ of size $m$ are complementary difference sets over a group $G$ of size $2m + 1$ if:

1. they are $2 - \{2m + 1; m, m - 1\}$ supplementary difference sets
2. $A$ is skew, i.e. $a \in A$ implies $-a \not\in A$

INPUT:

• $G$ – a group of odd order
• $A$ – a set of elements of $G$
• $B$ – a set of elements of $G$
• $\verb|verbose|$ – boolean (default: False); if True the function will be verbose when the sets do not satisfy the contraints

EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import are_complementary_difference_sets
sage: are_complementary_difference_sets(Zmod(7), [1, 2, 4], [1, 2, 4])
True
```

If verbose=True, the function will be verbose:

```
sage: are_complementary_difference_sets(Zmod(7), [1, 2, 5], [1, 2, 4], verbose=True)
The sets are not supplementary difference sets with lambda = 2
False
```

See also:

is_supplementary_difference_set()

sage.combinat.designs.difference_family.are_hadamard_difference_set_parameters$(v, k, \lambda)$

Check whether $(v, k, \lambda)$ is of the form $(4N^2, 2N^2 - N, N^2 - N)$.

INPUT:
• \((v,k,\lambda)\) – parameters of a difference set

**EXAMPLES:**

```
sage: from sage.combinat.designs.difference_family import are_hadamard_difference_set_parameters
sage: are_hadamard_difference_set_parameters(36, 15, 6)
True
sage: are_hadamard_difference_set_parameters(60, 13, 5)
False
```

`sage.combinat.designs.difference_family.are_mcfarland_1973_parameters(v, k, \lambda, return_parameters=False)`

Test whether \((v,k,\lambda)\) is a triple that can be obtained from the construction from [McF1973].

See `mcfarland_1973_construction()`.

**INPUT:**

• \(v, k, \lambda\) - integers; parameters of the difference family

• `return_parameters` – boolean (default False); if True, return a pair \((q, s)\) so that \((q, s)\) can be used in the function `mcfarland_1973_construction()` to actually build a \((v,k,\lambda)\)-difference family. Or \((False, None)\) if the construction is not possible

**EXAMPLES:**

```
sage: from sage.combinat.designs.difference_family import are_mcfarland_1973_parameters
sage: are_mcfarland_1973_parameters(64, 28, 12)
True
sage: are_mcfarland_1973_parameters(64, 28, 12, return_parameters=True)
(True, (2, 2))
sage: are_mcfarland_1973_parameters(60, 13, 5)
False
sage: are_mcfarland_1973_parameters(98125, 19500, 3875)
True
sage: are_mcfarland_1973_parameters(98125, 19500, 3875, True)
(True, (5, 3))
sage: from sage.combinat.designs.difference_family import are_mcfarland_1973_parameters
sage: for v in range(1, 100):
    ....:     for k in range(1,30):
    ....:         for l in range(1,15):
    ....:             if are_mcfarland_1973_parameters(v,k,l):
    ....:                 answer, (q,s) = are_mcfarland_1973_parameters(v,k,l,return_parameters=True)
    ....:                 print("\{} {} {} {} {}\".format(v,k,l,q,s))
    ....:                 assert answer is True
    ....:                 assert designs.difference_family(v,k,l,existence=True) is True
    ....:                 G,D = designs.difference_family(v,k,l)
16 6 2 2 1
45 12 3 3 1
64 28 12 2 2
96 20 4 4 1
```

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sage.combinat.designs.difference_family.block_stabilizer($G, B$)
Compute the left stabilizer of the block $B$ under the action of $G$.
This function return the list of all $x \in G$ such that $x \cdot B = B$ (as a set).

INPUT:
- $G$ – a group (additive or multiplicative)
- $B$ – a subset of $G$

EXAMPLES:
```
sage: from sage.combinat.designs.difference_family import block_stabilizer
def block_stabilizer(G, B):
    return [g for g in G if all(b * g in B for b in B)]
```
```
sage: block_stabilizer(Z8, [Z8(0), Z8(2), Z8(4), Z8(6)])
[0, 2, 4, 6]
sage: block_stabilizer(Z8, [Z8(0), Z8(2)])
[0]
sage: C = cartesian_product([Zmod(4), Zmod(3)])
c1 = block_stabilizer(C, [C((0, 0)), C((2, 0)), C((0, 1)), C((2, 1))])
[(0, 0), (2, 0)]
sage: b = list(map(Zmod(45), [1, 3, 7, 10, 22, 25, 30, 35, 37, 38, 44]))
sage: block_stabilizer(Zmod(45), b)
[0]
```

sage.combinat.designs.difference_family.complementary_difference_sets($n$, existence=False, check=True)
Compute complementary difference sets over a group of order $n = 2m + 1$.
According to [Sze1971], two sets $A, B$ of size $m$ are complementary difference sets over a group $G$ of size $n = 2m + 1$ if:
1. they are $2 - \{2m + 1; m, m; m - 1\}$ supplementary difference sets
2. $A$ is skew, i.e. $a \in A$ implies $-a \not\in A$
This method tries to call complementary_difference_setsI(), complementary_difference_setsII() or complementary_difference_setsIII() if the parameter $n$ satisfies the requirements of one of these functions.

INPUT:
- $n$ – integer; the order of the group over which the sets are constructed
- existence – boolean (default: False); if True, only check whether the supplementary difference sets can be constructed
- check – boolean (default: True); if True, check that the sets are complementary difference sets before returning them; setting this to False might speed up the computation for large values of $n$

OUTPUT:
If existence=False, the function returns group $G$ and two complementary difference sets, or raises an error if data for the given $n$ is not available. If existence=True, the function returns a boolean representing whether complementary difference sets can be constructed for the given $n$.

EXAMPLES:
sage: from sage.combinat.designs.difference_family import complementary_difference_sets
sage: complementary_difference_sets(15)
(Ring of integers modulo 15, [1, 2, 4, 6, 7, 10, 12], [0, 1, 2, 6, 9, 13, 14])

If existence=True, the function returns a boolean:

sage: complementary_difference_sets(15, existence=True)
True
sage: complementary_difference_sets(16, existence=True)
False

See also:
are_complementary_difference_sets()

sage.combinat.designs.difference_family.complementary_difference_setsI(n, check=True)
Construct complementary difference sets in a group of order \( n \equiv 3 \mod 4 \), \( n \) a prime power.

Let \( G \) be a Galois Field of order \( n \), where \( n \) satisfies the requirements above. Let \( A \) be the set of non-zero quadratic elements in \( G \), and \( B = A \). Then \( A \) and \( B \) are complementary difference sets over a group of order \( n \).
This construction is described in [Sze1971].

INPUT:
- \( n \) – integer; the order of the group \( G \)
- \( \text{check} \) – boolean (default: True); if True, check that the sets are complementary difference sets before returning them

OUTPUT:
The function returns the Galois field of order \( n \) and the two sets, or raises an error if \( n \) does not satisfy the requirements of this construction.

EXAMPLES:

sage: complementary_difference_setsI(19)
(Finite Field of size 19, [1, 4, 5, 6, 7, 9, 11, 16, 17], [1, 4, 5, 6, 7, 9, 11, 16, 17])

See also:
are_complementary_difference_sets() complementary_difference_sets()

sage.combinat.designs.difference_family.complementary_difference_setsII(n, check=True)
Construct complementary difference sets in a group of order \( n = p^t \), where \( p \equiv 5 \mod 8 \) and \( t \equiv 1, 2, 3 \mod 4 \).

Consider a finite field \( G \) of order \( n \) and let \( \rho \) be the generator of the corresponding multiplicative group. Then, there are two different constructions, depending on whether \( t \) is even or odd.

If \( t \equiv 2 \mod 4 \), let \( C_0 \) be the set of non-zero octic residues in \( G \), and let \( C_i = \rho^i C_0 \) for \( 1 \leq i \leq 7 \). Then, \( A = C_0 \cup C_1 \cup C_2 \cup C_3 \) and \( B = C_0 \cup C_1 \cup C_6 \cup C_7 \).

If \( t \) is odd, let \( C_0 \) be the set of non-zero fourth powers in \( G \), and let \( C_i = \rho^i C_0 \) for \( 1 \leq i \leq 3 \). Then, \( A = C_0 \cup C_1 \) and \( B = C_0 \cup C_3 \).
For more details on this construction, see [Sze1971].

INPUT:

• n – integer; the order of the group $G$
• check – boolean (default: True); if True, check that the sets are complementary difference sets before returning them; setting this to False might speed up the computation for large values of n

OUTPUT:

The function returns the Galois field of order $n$ and the two sets, or raises an error if $n$ does not satisfy the requirements of this construction.

EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import complementary_difference_setsII
sage: complementary_difference_setsII(5)
(Finite Field of size 5, [1, 2], [1, 3])
```

See also:

`are_complementary_difference_sets() complementary_difference_sets()`

`sage.combinat.designs.difference_family.complementary_difference_setsIII(n, check=True)`

Construct complementary difference sets in a group of order $n = 2m + 1$, where $4m + 3$ is a prime power.

Consider a finite field $G$ of order $n$ and let $\rho$ be a primitive element of this group. Now let $Q$ be the set of non zero quadratic residues in $G$, and let $A = \{a|\rho^{2a} - 1 \in Q\}$, $B' = \{b|-(\rho^{2b} + 1) \in Q\}$. Then $A$ and $B = Q \setminus B'$ are complementary difference sets over the ring of integers modulo $n$. For more details, see [Sz1969].

INPUT:

• n – integer; the order of the group over which the sets are constructed
• check – boolean (default: True); if True, check that the sets are complementary difference sets before returning them; setting this to False might speed up the computation for large values of n

OUTPUT:

The function returns the Galois field of order $n$ and the two sets, or raises an error if $n$ does not satisfy the requirements of this construction.

EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import complementary_difference_setsIII
sage: complementary_difference_setsIII(11)
(Ring of integers modulo 11, [1, 2, 5, 7, 8], [0, 1, 3, 8, 10])
```

See also:

`are_complementary_difference_sets() complementary_difference_sets()`

`sage.combinat.designs.difference_family.df_q_6_1(K, existence=False, check=True)`

Return a $(q, 6, 1)$-difference family over the finite field $K$.

The construction uses Theorem 11 of [Wi72].

EXAMPLES:
Todo: Do improvements due to Zhen and Wu 1999.

sage.combinat.designs.difference_family.difference_family(v, k, l=1, existence=False, explain_construction=False, check=True)

Return a \((k, 1)\)-difference family on an Abelian group of cardinality \(v\).

Let \(G\) be a finite Abelian group. For a given subset \(D\) of \(G\), we define \(\Delta D\) to be the multi-set of differences \(\Delta D = \{x - y; x \in D, y \in D, x \neq y\}\). A \((G, k, \lambda)\)-difference family is a collection of \(k\)-subsets of \(G\), \(D = \{D_1, D_2, \ldots, D_b\}\) such that the union of the difference sets \(\Delta D_i\) for \(i = 1, \ldots, b\), seen as a multi-set, contains each element of \(G \setminus \{0\}\) exactly \(\lambda\)-times.

When there is only one block, i.e. \(\lambda(v - 1) = k(k - 1)\), then a \((G, k, \lambda)\)-difference family is also called a difference set.

See also Wikipedia article Difference_set.

If there is no such difference family, an EmptySetError is raised and if there is no construction at the moment NotImplementedError is raised.

INPUT:

- \(v, k, 1\) – parameters of the difference family. If 1 is not provided it is assumed to be 1
- \(existence\) – if True, then return either True if Sage knows how to build such design, Unknown if it does not and False if it knows that the design does not exist
- \(explain_construction\) – instead of returning a difference family, returns a string that explains the construction used
- \(check\) – boolean (default: True); if True, then the result of the computation is checked before being returned. This should not be needed but ensures that the output is correct

OUTPUT:

A pair \((G, D)\) made of a group \(G\) and a difference family \(D\) on that group. Or, if \(existence=True\` a troolean or if \(explain_construction=True\` a string.

EXAMPLES:

sage: G,D = designs.difference_family(73,4)
sage: G
Finite Field of size 73
sage: D
[[0, 1, 5, 18],
 (continues on next page)
sage: print(designs.difference_family(73, 4, explain_construction=True))
The database contains a (73,4)-evenly distributed set

sage: G,D = designs.difference_family(15,7,3)
sage: G
Ring of integers modulo 15
sage: D
[[0, 1, 2, 4, 5, 8, 10]]
sage: print(designs.difference_family(15,7,3,explain_construction=True))
Singer difference set

sage: print(designs.difference_family(91,10,1,explain_construction=True))
Singer difference set

sage: print(designs.difference_family(64,28,12, explain_construction=True))
McFarland 1973 construction

sage: print(designs.difference_family(576, 276, 132, explain_construction=True))
Hadamard difference set product from N1=2 and N2=3

For \( k = 6, 7 \) we look at the set of small prime powers for which a construction is available:

sage: def prime_power_mod(r,m):
    ...
    k = m+r
    ...
    while True:
        ...
        if is_prime_power(k):
            ...
            yield k
        ...
    k += m

sage: from itertools import islice

sage: l6 = {True: [], False: [], Unknown: []}
sage: for q in islice(prime_power_mod(1,30), int(60)):  
    ...
    l6[designs.difference_family(q,6,existence=True)].append(q)

sage: l6[True]  
[31, 121, 151, 181, 211, ..., 3061, 3121, 3181]

sage: l6[Unknown]
[61]

sage: l6[False]
[]

sage: l7 = {True: [], False: [], Unknown: []}
sage: for q in islice(prime_power_mod(1,42), int(60)):  
    ...
    l7[designs.difference_family(q,7,existence=True)].append(q)

sage: l7[True]  
[169, 337, 379, 421, 463, 547, 631, 673, 757, 841, 883, 967, ..., 4621, 4957, 5167]

sage: l7[Unknown]
[43, 127, 211, 2017, 2143, 2269, 2311, 2437, 2521, 2647, ..., 4999, 5041, 5209]

sage: l7[False]
[]
List available constructions:

```
sage: for v in range(2,100):
    constructions = []
    for k in range(2,10):
        for l in range(1,10):
            ret = designs.difference_family(v,k,l,existence=True)
            if ret is True:
                constructions.append((k,l))
            _ = designs.difference_family(v,k,l)
    if constructions:
        print("%2d: %s\n\t''.join('%d,%d')%(k,l) for k,l in_ constructions))
3: (2,1)
4: (3,2)
5: (2,1), (4,3)
6: (5,4)
7: (2,1), (3,1), (3,2), (4,2), (6,5)
8: (7,6)
9: (2,1), (4,3), (8,7)
10: (9,8)
11: (2,1), (4,6), (5,2), (5,4), (6,3)
13: (2,1), (3,1), (3,2), (4,1), (4,3), (5,5), (6,5)
15: (3,1), (4,6), (5,6), (7,3), (7,6)
16: (3,2), (5,4), (6,2)
17: (2,1), (4,3), (5,5), (8,7)
19: (2,1), (3,1), (3,2), (4,2), (6,5), (9,4), (9,8)
21: (3,1), (4,3), (5,1), (6,3), (6,5)
22: (4,2), (6,5), (7,4), (8,8)
23: (2,1)
25: (2,1), (3,1), (3,2), (4,1), (4,3), (6,5), (7,7), (8,7)
27: (2,1), (3,1)
28: (3,2), (6,5)
29: (2,1), (4,3), (7,3), (7,6), (8,4), (8,6)
31: (2,1), (3,1), (3,2), (4,2), (5,2), (5,4), (6,1), (6,5)
33: (3,1), (5,5), (6,5)
34: (4,2)
35: (5,2)
37: (2,1), (3,1), (3,2), (4,1), (4,3), (6,5), (9,2), (9,8)
39: (3,1), (6,5)
40: (3,2), (4,1)
41: (2,1), (4,3), (5,1), (5,4), (6,3), (8,7)
43: (2,1), (3,1), (3,2), (4,2), (6,5), (7,2), (7,3), (7,6), (8,4)
45: (3,1), (5,1)
46: (4,2), (6,2)
47: (2,1)
49: (2,1), (3,1), (3,2), (4,1), (4,3), (6,5), (8,7), (9,3)
51: (3,1), (5,2), (6,3)
52: (4,1)
53: (2,1), (4,3)
```
55: (3,1), (9,4)
57: (3,1), (7,3), (8,1)
59: (2,1)
61: (2,1), (3,1), (3,2), (4,1), (4,3), (5,1), (5,4), (6,2), (6,3), (6,5)
63: (3,1)
64: (3,2), (4,1), (7,2), (7,6), (9,8)
65: (5,1)
67: (2,1), (3,1), (3,2), (6,5)
69: (3,1)
71: (2,1), (5,2), (5,4), (7,3), (7,6), (8,4)
73: (2,1), (3,1), (3,2), (4,1), (4,3), (6,5), (8,7), (9,1), (9,8)
75: (3,1), (5,2)
76: (4,1)
79: (2,1), (3,1), (3,2), (6,5)
81: (2,1), (3,1), (4,3), (5,1), (5,4), (8,7)
83: (2,1)
85: (4,1), (7,2), (7,3), (8,2)
89: (2,1), (4,3), (8,7)
91: (6,1), (7,1)
97: (2,1), (3,1), (3,2), (4,1), (4,3), (6,5), (8,7), (9,3)

Todo: Implement recursive constructions from Buratti “Recursive for difference matrices and relative difference families” (1998) and Jungnickel “Composition theorems for difference families and regular planes” (1978)

sage.combinat.designs.difference_family.get_fixed_relative_difference_set(G, rel_diff_set, as_elements=False)

Construct an equivalent relative difference set fixed by the size of the set.

Given a relative difference set $R(q + 1, q - 1, q, 1)$, it is possible to find a translation of this set fixed by $q$ (see Section 3 of [Spe1975]). We say that a set is fixed by $t$ if $\{td \mid d \in R\} = R$.

In addition, the set returned by this function will contain the element 0. This is needed in the construction of supplementary difference sets (see supplementary_difference_set_from_rel_diff_set()).

INPUT:
- $G$ – a group, of which $rel\_diff\_set$ is a subset
- $rel\_diff\_set$ – the relative difference set
- $as\_elements$ – boolean (default: False); if True, the list returned will contain elements of the abelian group (this may slow down the computation considerably)

OUTPUT:

By default, this function returns the set as a list of integers. However, if $as\_elements=True$ it will return the set as a list containing elements of the abelian group. If no such set can be found, the function will raise an error.

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_family import relative_difference_set_from_m_sequence, get_fixed_relative_difference_set
sage: G, s1 = relative_difference_set_from_m_sequence(5, 2, return_group=True)
sage: get_fixed_relative_difference_set(G, s1) #random
[2, 10, 19, 23, 0]
```
If `as_elements=True`, the result will contain elements of the group:

```
sage: get_fixed_relative_difference_set(G, s1, as_elements=True) #random
[(2), (10), (19), (23), (0)]
```

`sage.combinat.designs.difference_family.group_law(G)`

Return a triple `(identity, operation, inverse)` that define the operations on the group `G`.

**EXAMPLES:**

```
sage: from sage.combinat.designs.difference_family import group_law
sage: group_law(Zmod(3))
((0, <built-in function add>, <built-in function neg>)
sage: group_law(SymmetricGroup(3))
(((), <built-in function mul>, <built-in function inv>)
sage: group_law(VectorSpace(QQ,3))
((0, 0, 0), <built-in function add>, <built-in function neg>)
```

`sage.combinat.designs.difference_family.hadamard_difference_set_product(G1, D1, G2, D2)`

Make a product of two Hadamard difference sets.

This product construction appears in [Tu1984].

**INPUT:**

- `G1, D1, G2, D2` – two Hadamard difference sets

**EXAMPLES:**

```
sage: from sage.combinat.designs.difference_family import hadamard_difference_set_product
sage: from sage.combinat.designs.difference_family import is_difference_family
sage: G1, D1 = designs.difference_family(16, 6, 2)
sage: G2, D2 = designs.difference_family(36, 15, 6)
sage: G11, D11 = hadamard_difference_set_product(G1, D1, G1, D1)
sage: assert is_difference_family(G11, D11, 256, 120, 56)
sage: designs.difference_family(256, 120, 56, existence=True) is True
sage: G12, D12 = hadamard_difference_set_product(G1, D1, G2, D2)
sage: assert is_difference_family(G12, D12, 576, 276, 132)
sage: designs.difference_family(576, 276, 132, existence=True) is True
```

`sage.combinat.designs.difference_family.hadamard_difference_set_product_parameters()`

Check whether a product construction is available for Hadamard difference set with parameter `N`.

This function looks for two integers $N_1$ and $N_2$ greater than 1 and so that $N = 2N_1N_2$ and there exists Hadamard difference set with parameters $(4N_1^2, 2N_1^2 - N_1, N_1^2 - N_1)$. If such pair exists, the output is the pair $(N_1, N_2)$ otherwise it is `None`

**INPUT:**

- `N` – positive integer

**EXAMPLES:**
sage: from sage.combinat.designs.difference_family import hadamard_difference_set_˓→product_parameters
sage: hadamard_difference_set_product_parameters(8)
(2, 2)

sage.combinat.designs.difference_family.is_difference_family(G, D, v=None, k=None, l=None, ˓→verbose=False)

Check whether D forms a difference family in the group G.

INPUT:

• G – group of cardinality v
• D – a set of k-subsets of G
• v, k and l – optional parameters of the difference family
• verbose – boolean (default: False); whether to print additional information

See also:
difference_family()

EXAMPLES:

sage: from sage.combinat.designs.difference_family import is_difference_family
sage: G = Zmod(21)
sage: D = [[0,1,4,14,16]]
sage: is_difference_family(G, D, 21, 5)
True

sage: G = Zmod(41)
sage: D = [[0,1,4,11,29],[0,2,8,17,21]]
sage: is_difference_family(G, D, verbose=True)
Too few:
  5 is obtained 0 times in blocks []
  14 is obtained 0 times in blocks []
  27 is obtained 0 times in blocks []
  36 is obtained 0 times in blocks []
Too much:
  4 is obtained 2 times in blocks [0, 1]
  13 is obtained 2 times in blocks [0, 1]
  28 is obtained 2 times in blocks [0, 1]
  37 is obtained 2 times in blocks [0, 1]
False

sage: D = [[0,1,4,11,29],[0,2,8,17,22]]
sage: is_difference_family(G, D)
True

sage: G = Zmod(61)
sage: D = [[0,1,3,13,34],[0,4,9,23,45],[0,6,17,24,32]]
sage: is_difference_family(G, D)
True

sage: G = AdditiveAbelianGroup([3]*4)
sage: a,b,c,d = G.gens()
sage: D = [[d, -a+d, -c+d, a-b-d, b+c+d],
.....:   [c, a+b-d, -b+c, a-b+d, a+b+c],
.....:   [-a-b+c+d, a-b-c-d, -a+c-d, b-c+d, a+b],
.....:   [-b-d, a+b+d, a-b+c-d, a-b+c, -b+c+d]]

sage: is_difference_family(G, D)
True

The following example has a third block with a non-trivial stabilizer:

sage: G = Zmod(15)
sage: D = [[0,1,4],[0,2,9],[0,5,10]]
sage: is_difference_family(G,D,verbose=True)
It is a (15,3,1)-difference family
True

The function also supports multiplicative groups (non necessarily Abelian):

sage: G = DihedralGroup(8)
sage: x,y = G.gens()
sage: i = G.one()
sage: D1 = [[i,x,x^4], [i,x^2, y*x], [i,x^5,y], [i,x^6,y*x^2], [i,x^7,y*x^5]]
sage: is_difference_family(G, D1, 16, 3, 2)
True

sage: from sage.combinat.designs.bibd import BIBD_from_difference_family
sage: bibd = BIBD_from_difference_family(G,D1,lambd=2)

sage.combinat.designs.difference_family.is_fixed_relative_difference_set(R, q)
Check if the relative difference set R is fixed by q.
A relative difference set R is fixed by q if \{qd|d \in R\} = R (see Section 3 of [Spe1975]).

INPUT:
• R – a list containing elements of an abelian group; the relative difference set
• q – an integer

EXAMPLES:

sage: from sage.combinat.designs.difference_family import relative_difference_set_from_m_sequence, get_fixed_relative_difference_set, is_fixed_relative_difference_set
sage: G, s1 = relative_difference_set_from_m_sequence(7, 2, return_group=True)
sage: s2 = get_fixed_relative_difference_set(G, s1, as_elements=True)

sage: is_fixed_relative_difference_set(s2, len(s2))
True

sage: G = AdditiveAbelianGroup([15])
sage: s3 = [G[1], G[2], G[3], G[4]]
sage: is_fixed_relative_difference_set(s3, len(s3))
False

If the relative difference set does not contain elements of the group, the method returns false:

sage: G, s1 = relative_difference_set_from_m_sequence(7, 2, return_group=True)
sage: s2 = get_fixed_relative_difference_set(G, s1, as_elements=False)
sage: is_fixed_relative_difference_set(s2, len(s2))
False

sage.combinat.designs.difference_family.is_relative_difference_set(R, G, H, params,
verbose=False)

Check if R is a difference set of G relative to H, with the given parameters.

This function checks that G, H and R have the orders specified in the parameters, and that R satisfies the definition of relative difference set (from [EB1966]): the collection of differences \( r - s \), \( r, s \in R \), \( r \neq s \) contains only elements of G which are not in H, and contains every such element exactly \( d \) times.

INPUT:

- R – list; the relative difference set of length \( k \)
- G – an additive abelian group of order \( mn \)
- H – list; a submodule of G of order \( n \)
- params – a tuple in the form \((m, n, k, d)\)
- verbose – boolean (default: False); if True, the function will be verbose when the sequences do not satisfy the contraints

EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import _get_submodule_of_order,
˓→relative_difference_set_from_m_sequence, is_relative_difference_set

sage: q, N = 5, 2
sage: params = ((q^N-1) // (q-1), q - 1, q^(N-1), q^(N-2))
sage: G, R = relative_difference_set_from_m_sequence(q, N, return_group=True)
sage: H = _get_submodule_of_order(G, q - 1)
sage: is_relative_difference_set(R, G, H, params)
True
```

If we pass the verbose argument, the function will explain why it failed:

```
sage: R2 = [G[1], G[2], G[3], G[5], G[6]]
sage: is_relative_difference_set(R2, G, H, params, verbose=True)
The function is relative_difference_set(R, G, H, params)
There is a value in the difference set which is not repeated \( d \) times
False
```

sage.combinat.designs.difference_family.is_supplementary_difference_set(Ks, v=None,
\( \lambda \) \( \text{Lambda} \) \( \text{None} \),
\( G=\text{None}, \)
verbose=False)

Check that the sets in Ks are \( n - \{v; k_1, ..., k_n; \lambda\} \) supplementary difference sets over group G of order v.

From the definition in [Spe1975]: let \( S_1, S_2, ..., S_n \) be n subsets of a group G of order \( v \) such that \(|S_i| = k_i\). If, for each \( g \in G, g \neq 0 \), the total number of solutions of \( a_i - a'_i = g \), with \( a_i, a'_i \in S_i \) is \( \lambda \), then \( S_1, S_2, ..., S_n \) are \( n - \{v; k_1, ..., k_n; \lambda\} \) supplementary difference sets.

One of the parameters v or G must always be specified. If G is not given, the function will use an AdditiveAbelianGroup of order v.

INPUT:

- Ks – a list of sets to be checked
- $v$ – integer; the parameter $v$ of the supplementary difference sets
- $\lambda$ – integer; the parameter $\lambda$ of the supplementary difference sets
- $G$ – a group of order $v$
- $\text{verbose}$ – boolean (default: False); if True, the function will be verbose when the sets do not satisfy the contraints

```python
sage: from sage.combinat.designs.difference_family import supplementary_difference_set_from_rel_diff_set, is_supplementary_difference_set
sage: G, [S1, S2, S3, S4] = supplementary_difference_set_from_rel_diff_set(17)
sage: is_supplementary_difference_set([S1, S2, S3, S4], lmbda=16, G=G)
True
```

The parameter $v$ can be given instead of $G$:

```python
sage: is_supplementary_difference_set([S1, S2, S3, S4], v=16, lmbda=16)
True
sage: is_supplementary_difference_set([S1, S2, S3, S4], v=20, lmbda=16)
False
```

If $\text{verbose}=\text{True}$, the function will be verbose:

```python
sage: is_supplementary_difference_set([S1, S2, S3, S4], lmbda=14, G=G, verbose=True)
Number of pairs with difference (1) is 16, but lambda is 14
False
```

See also:

- `supplementary_difference_set_from_rel_diff_set()`

```
sage.combinat.designs.difference_family.mcfarland_1973_construction(q, s)
```

Return a difference set.

The difference set returned has the following parameters

$$
v = \frac{q^{s+1}(q^{s+1} + q - 2)}{q - 1},
\quad k = \frac{q^s(q^{s+1} - 1)}{q - 1},
\quad \lambda = \frac{q^s(q^s - 1)}{q - 1}
$$

This construction is due to [McF1973].

**INPUT:**

- $q, s$ - integers; parameters for the difference set (see the above formulas for the expression of $v, k, \lambda$ in terms of $q$ and $s$)

**See also:**

The function `are_mcfarland_1973_parameters()` makes the translation between the parameters $(q, s)$ corresponding to a given triple $(v, k, \lambda)$.

**REFERENCES:**

**EXAMPLES:**

```python
sage: from sage.combinat.designs.difference_family import (  
  .... mcfarland_1973_construction, is_difference_family)
```

(continues on next page)
sage: G,D = mcfarland_1973_construction(3, 1)
sage: assert is_difference_family(G, D, 45, 12, 3)

sage: G,D = mcfarland_1973_construction(2, 2)
sage: assert is_difference_family(G, D, 64, 28, 12)

sage.combinat.designs.difference_family.one_cyclic_tiling(A, n)

Given a subset $A$ of the cyclic additive group $G = \mathbb{Z}/n\mathbb{Z}$ return another subset $B$ so that $A + B = G$ and $|A||B| = n$ (i.e. any element of $G$ is uniquely expressed as a sum $a + b$ with $a$ in $A$ and $b$ in $B$).

EXAMPLES:

sage: from sage.combinat.designs.difference_family import one_cyclic_tiling
sage: tile = [0, 2, 4]
sage: m = one_cyclic_tiling(tile, 6); m
[0, 3]
sage: sorted((i+j)%6 for i in tile for j in m)
[0, 1, 2, 3, 4, 5]

sage: def print_tiling(tile, translat, n):
....:     for x in translat:
....:         print(''.join('X' if (i-x)%n in tile else '.' for i in range(n)))

sage: tile = [0, 1, 2, 7]
sage: m = one_cyclic_tiling(tile, 12)
sage: print_tiling(tile, m, 12)
XXX....X....
....XXX....X
....X....XXX.

sage: tile = [0, 1, 5]
sage: m = one_cyclic_tiling(tile, 12)
sage: print_tiling(tile, m, 12)
XX...X......
...XX...X...
......XX...X
..X......XX.

sage: tile = [0, 2]
sage: m = one_cyclic_tiling(tile, 8)
sage: print_tiling(tile, m, 8)
X.X.....
....X.X.
.X.X....
......X.X

ALGORITHM:

Uses dancing links sage.combinat.dlx

sage.combinat.designs.difference_family.one_radical_difference_family(K, k)

Search for a radical difference family on $K$ using dancing links algorithm.

For the definition of radical difference family, see radical_difference_family(). Here, we consider only radical difference family with $\lambda = 1$. 

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INPUT:
- $K$ – a finite field of cardinality $q$
- $k$ – a positive integer so that $k(k - 1)$ divides $q - 1$

OUTPUT:
Either a difference family or None if it does not exist.

ALGORITHM:
The existence of a radical difference family is equivalent to a one dimensional tiling (or packing) problem in a cyclic group. This subsequent problem is solved by a call to the function `one_cyclic_tiling()`.

Let $K^*$ be the multiplicative group of the finite field $K$. A radical family has the form $\mathcal{B} = \{x_1B, \ldots, x_kB\}$, where $B = \{x : x^k = 1\}$ (for $k$ odd) or $B = \{x : x^{k-1} = 1\} \cup \{0\}$ (for $k$ even). Equivalently, $K^*$ decomposes as:

$$K^* = \Delta(x_1B) \cup \cdots \cup \Delta(x_kB) = x_1\Delta B \cup \cdots \cup x_k\Delta B.$$  

We observe that $C = B\setminus 0$ is a subgroup of the (cyclic) group $K^*$, that can thus be generated by some element $r$. Furthermore, we observe that $\Delta B$ is always a union of cosets of $\pm C$ (which is twice larger than $C$).

\[
\begin{align*}
(k \text{ odd}) \quad \Delta B & = \{r^i - r^j : r^i \neq r^j\} = \pm C \cdot \{r^i - 1 : 0 < i \leq m\} \\
(k \text{ even}) \quad \Delta B & = \{r^i - r^j : r^i \neq r^j\} \cup C = \pm C \cdot \{r^i - 1 : 0 < i < m\} \cup \pm C
\end{align*}
\]

where

\[
(k \text{ odd}) \quad m = \frac{k - 1}{2} \quad \text{and} \quad (k \text{ even}) \quad m = \frac{k}{2}.
\]

Consequently, $\mathcal{B} = \{x_1B, \ldots, x_kB\}$ is a radical difference family if and only if $\{x_1(\Delta B/(\pm C)), \ldots, x_k(\Delta B/(\pm C))\}$ is a partition of the cyclic group $K^*/(\pm C)$.

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_family import (......: one_radical_difference_family,
......: is_difference_family)

sage: one_radical_difference_family(GF(13), 4)
[[0, 1, 3, 9]]

sage: is_difference_family(GF(449), df, 449, 8, 1)
True
```

The parameters that appear in [Bu95]:

```python
sage: df = one_radical_difference_family(GF(449), 8); df
[[0, 1, 18, 25, 176, 324, 359, 444],
 [0, 9, 88, 162, 222, 225, 237, 404],
 [0, 11, 140, 198, 275, 357, 394, 421],
 [0, 40, 102, 249, 271, 305, 388, 441],
 [0, 49, 80, 93, 161, 204, 327, 433],
 [0, 70, 99, 197, 230, 362, 403, 435],
 [0, 121, 141, 193, 293, 331, 335, 382],
 [0, 191, 285, 295, 321, 371, 390, 392]]
sage: is_difference_family(GF(449), df, 449, 8, 1)
True
```
Combinatorics, Release 10.0

sage.combinat.designs.difference_family.radical_difference_family(K, k, l=1, existence=False, check=True)

Return a \((v,k,1)\)-radical difference family.

Let fix an integer \(k\) and a prime power \(q = tk(k-1)+1\). Let \(K\) be a field of cardinality \(q\). A \((q,k,1)\)-difference family is radical if its base blocks are either: a coset of the \(k\)-th root of unity for \(k\) odd or a coset of \(k-1\)-th root of unity and 0 if \(k\) is even (the number \(t\) is the number of blocks of that difference family).

The terminology comes from M. Buratti article [Bu95] but the first constructions go back to R. Wilson [Wi72].

INPUT:

- \(K\) - a finite field
- \(k\) – positive integer; the size of the blocks
- \(l\) – integer (default: 1); the \(\lambda\) parameter
- \(existence\) – if True, then return either True if Sage knows how to build such design, Unknown if it does not and False if it knows that the design does not exist
- \(check\) – boolean (default: True); if True then the result of the computation is checked before being returned. This should not be needed but ensures that the output is correct

EXAMPLES:

```sage
from sage.combinat.designs.difference_family import radical_difference_family

sage: radical_difference_family(GF(73),9)
[[1, 2, 4, 8, 16, 32, 37, 55, 64]]

sage: radical_difference_family(GF(281),5)
[[1, 86, 90, 153, 232],
 [4, 50, 63, 79, 85],
 [5, 36, 149, 169, 203],
 [7, 40, 68, 219, 228],
 [9, 121, 212, 248, 253],
 [29, 81, 222, 246, 265],
 [31, 137, 167, 247, 261],
 [32, 70, 118, 119, 223],
 [39, 56, 66, 138, 263],
 [43, 45, 116, 141, 217],
 [98, 101, 109, 256, 279],
 [106, 124, 145, 201, 267],
 [111, 123, 155, 181, 273],
 [156, 209, 224, 264, 271]]

sage: for k in range(5,10):
....:     print("k = {}".format(k))
....:     list_q = []
....:     for q in range(k*(k-1)+1, 2000, k*(k-1)):
....:         if is_prime_power(q):
....:             K = GF(q,'a')
....:             if radical_difference_family(K, k, existence=True) is True:
....:                 list_q.append(q)
....:     print(" ".join(str(p) for p in list_q))
```
sage.combinat.designs.difference_family.radical_difference_set($K, k, l=1, existence=False, check=True$)

Return a difference set made of a cyclotomic coset in the finite field $K$ and with parameters $k$ and $1$.

Most of these difference sets appear in chapter VI.18.48 of the Handbook of combinatorial designs.

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_family import radical_difference_set

sage: D = radical_difference_set(GF(7), 3, 1); D
[[1, 2, 4]]

sage: sorted(x-y for x in D[0] for y in D[0] if x != y)
[1, 2, 3, 4, 5, 6]

sage: D = radical_difference_set(GF(16, 'a'), 6, 2)

sage: sorted(x-y for x in D[0] for y in D[0] if x != y)
[1, 1, a, a, a + 1, a + 1, a^2, a*2, ...
  a^3 + a^2 + a + 1, a^3 + a^2 + a + 1]
```

```python
for k in range(2,50):
    for l in reversed(divisors(k*(k-1))):
        v = k**(k-1)/l + 1
        if is_prime_power(v) and radical_difference_set(GF(v, 'a'), k, l, existence=True) is True:
            _ = radical_difference_set(GF(v, 'a'), k, l)
            print("{:3} {:3} {:3}".format(v, k, l))
```

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Sage.combinat.designs.difference_family.relative_difference_set_from_homomorphism(q, N, d, check=True, return_group=False)

Construct \( R((q^N - 1)/(q - 1), n, q^{N-1}, q^{N-2}d) \) where \( nd = q - 1 \).

Given a prime power \( q \), a number \( N \geq 2 \) and integers \( d \) such that \( d | q - 1 \) we create the relative difference set using the construction from Corollary 5.1.1 of [EB1966].

**INPUT:**
- \( q \) – a prime power
- \( N \) – an integer greater than 1
- \( d \) – an integer which divides \( q - 1 \)
- \( \text{check} \) – boolean (default: True); if True, check that the result is a relative difference set before returning it
- \( \text{return\_group} \) – boolean (default: False); if True, the function will also return the group from which the set is created

**OUTPUT:**
If \( \text{return\_group}=\text{False} \), the function return only the relative difference set. Otherwise, it returns a tuple containing the group and the set.
EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import relative_difference_set_from_homomorphism
sage: relative_difference_set_from_homomorphism(7, 2, 3) #random
[(0), (3), (4), (2), (13), (7), (14)]
sage: relative_difference_set_from_homomorphism(9, 2, 4, check=False, return_group=True) #random
(Additive abelian group isomorphic to Z/80, [(0), (4), (6), (13), (7), (12), (15), (8), (9)])
sage: relative_difference_set_from_homomorphism(9, 2, 5)
Traceback (most recent call last):
  ... ValueError: q-1 must be a multiple of d
```

```python
sage.combinat.designs.difference_family.relative_difference_set_from_m_sequence(q, N, check=True, return_group=False)
```

Construct $R((q^N - 1)/(q - 1), q - 1, q^{N-1}, q^{N-2})$ where $q$ is a prime power and $N \geq 2$.

The relative difference set is constructed over the set of additive integers modulo $q^N - 1$, as described in Theorem 5.1 of [EB1966]. Given an m-sequence $(a_i)$ of period $q^N - 1$, the set is: $R = \{i|0 \leq i \leq q^N - 1, a_i = 1\}$.

INPUT:

- `q` – a prime power
- `N` – a nonnegative number
- `check` – boolean (default: True); if True, check that the result is a relative difference set before returning it
- `return_group` – boolean (default: False); if True, the function will also return the group from which the set is created

OUTPUT:

If `return_group=False`, the function return only the relative difference set. Otherwise, it returns a tuple containing the group and the set.

EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import relative_difference_set_from_m_sequence
sage: relative_difference_set_from_m_sequence(2, 4, return_group=True) #random
(Additive abelian group isomorphic to Z/15, [(0), (4), (5), (6), (7), (9), (11), (12)])
sage: relative_difference_set_from_m_sequence(8, 2, check=False) #random
[[0], (6), (30), (40), (41), (44), (56), (61)]
sage: relative_difference_set_from_m_sequence(6, 2)
Traceback (most recent call last):
  ... ValueError: q must be a prime power
```

```python
sage.combinat.designs.difference_family.singer_difference_set(q, d)
```

Return a difference set associated to the set of hyperplanes in a projective space of dimension $d$ over $GF(q)$.
A Singer difference set has parameters:

\[ v = \frac{q^{d+1} - 1}{q - 1}, \quad k = \frac{q^d - 1}{q - 1}, \quad \lambda = \frac{q^{d-1} - 1}{q - 1}. \]

The idea of the construction is as follows. One considers the finite field \( GF(q^{d+1}) \) as a vector space of dimension \( d + 1 \) over \( GF(q) \). The set of \( GF(q) \)-lines in \( GF(q^{d+1}) \) is a projective plane and its set of hyperplanes form a balanced incomplete block design.

Now, considering a multiplicative generator \( z \) of \( GF(q^{d+1}) \), we get a transitive action of a cyclic group on our projective plane from which it is possible to build a difference set.

The construction is given in details in [Stinson2004], section 3.3.

EXAMPLES:

```
sage: from sage.combinat.designs.difference_family import singer_difference_set, is_difference_family
sage: G,D = singer_difference_set(3,2)
sage: is_difference_family(G,D,verbose=True)
It is a (13,4,1)-difference family
True
sage: G,D = singer_difference_set(4,2)
sage: is_difference_family(G,D,verbose=True)
It is a (21,5,1)-difference family
True
sage: G,D = singer_difference_set(3,3)
sage: is_difference_family(G,D,verbose=True)
It is a (40,13,4)-difference family
True
sage: G,D = singer_difference_set(9,3)
sage: is_difference_family(G,D,verbose=True)
It is a (820,91,10)-difference family
True
```

`sage.combinat.designs.difference_family.skew_supplementary_difference_set(n, existence=False, check=True, return_group=False)`

Construct \( 4 - \{ n; n_1, n_2, n_3, n_4; \lambda \} \) supplementary difference sets, where \( S_1 \) is skew and \( n_1 + n_2 + n_3 + n_4 = n + \lambda \).

These sets are constructed from available data, as described in [Djo1994a]. The set \( S_1 \subset G \) is always skew, i.e. \( S_1 \cap (-S_1) = \emptyset \) and \( S_1 \cup (-S_1) = G \setminus \{0\} \).

The data is taken from:

- \( n = 103, 151 \): [Djo1994a]
- \( n = 67, 113, 127, 157, 163, 181, 241 \): [Djo1992a]
- \( n = 37, 43 \): [Djo1992b]
- \( n = 39, 49, 65, 93, 121, 129, 133, 217, 219, 267 \): [Djo1992c]
- \( n = 97 \): [Djo2008a]
- \( n = 109, 145, 247 \): [Djo2008b]
• $n = 73$: [Djo2023b]
• $n = 213, 631$: [DGK2014]
• $n = 331$: [DK2016]

Additional skew supplementary difference sets are built using the function `skew_supplementary_difference_set_over_polynomial_ring()` and `skew_supplementary_difference_set_with_paley_todd()`.

**INPUT:**

- $n$ – integer; the parameter of the supplementary difference set
- `existence` – boolean (default: `False`); if `True`, only check whether the supplementary difference sets can be constructed
- `check` – boolean (default: `True`); if `True`, check that the sets are supplementary difference sets with $S_1$ skew before returning them; setting this parameter to `False` may speed up the computation considerably
- `return_group` – boolean (default: `False`); if `True`, the function will also return the group from which the sets are created

**OUTPUT:**

If `existence=False`, the function returns a list containing 4 sets, or raises an error if data for the given $n$ is not available. If `return_group=True` the function will additionally return the group from which the sets are created. If `existence=True`, the function returns a boolean representing whether skew supplementary difference sets can be constructed.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.difference_family import skew_supplementary_difference_set
sage: [S1, S2, S3, S4] = skew_supplementary_difference_set(39)
```

If `return_group=True`, the function will also return the group:

```python
sage: G, [S1, S2, S3, S4] = skew_supplementary_difference_set(103, return_group=True)
```

If `existence=True`, the function returns a boolean:

```python
sage: skew_supplementary_difference_set(103, existence=True)  # True
sage: skew_supplementary_difference_set(17, existence=True)  # False
```

**Note:** The data for $n = 247$ in [Djo2008b] contains a typo: the set $\alpha_2$ should contain 223 instead of 233. This can be verified by checking the resulting sets, which are given explicitly in the paper.

Construct skew supplementary difference sets over a polynomial ring of order $n$.

The skew supplementary difference sets for $n = 81, 169$ are taken from [Djo1994a].

5.1. Comprehensive Module List
INPUT:

- \( n \) – integer; the parameter of the supplementary difference sets
- \( \text{existence} \) – boolean (default: False); if True, only check whether the supplementary difference sets can be constructed
- \( \text{check} \) – boolean (default: True); if True, check that the sets are supplementary difference sets with \( S_1 \) skew before returning them; setting this parameter to False may speed up the computation considerably

OUTPUT:

If \( \text{existence}=\text{False} \), the function returns a Polynomial Ring of order \( n \) and a list containing 4 sets, or raises an error if data for the given \( n \) is not available. If \( \text{existence}=\text{True} \), the function returns a boolean representing whether skew supplementary difference sets can be constructed.

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_family import skew_supplementary_difference_set_over_polynomial_ring
sage: G, [S1, S2, S3, S4] = skew_supplementary_difference_set_over_polynomial_ring(81)
```

If \( \text{existence}=\text{True} \), the function returns a boolean:

```python
sage: skew_supplementary_difference_set_over_polynomial_ring(81, existence=True)
True
sage: skew_supplementary_difference_set_over_polynomial_ring(17, existence=True)
False
```

Construct \( 4 \{ n; n_1, n_2, n_3, n_4; \lambda \} \) skew supplementary difference sets where \( S_1 \) is the Paley-Todd difference set.

The skew SDS returned have the property that \( n_1 + n_2 + n_3 + n_4 = n + \lambda \).

This construction is described in [DK2016]. The function contains, for each value of \( n \), a set \( H \) containing integers modulo \( n \), and four sets \( J, K, L \). Then, these are used to construct \( (n; k_2, k_3, k_4; \lambda_2) \) difference family, with \( \lambda_2 = k_2 + k_3 + k_4 + (3n - 1)/4 \). Finally, these sets together with the Paley-Todd difference set form a skew supplementary difference set.

INPUT:

- \( n \) – integer; the parameter of the supplementary difference set
- \( \text{existence} \) – boolean (default: False); if True, only check whether the supplementary difference sets can be constructed
- \( \text{check} \) – boolean (default: True); if True, check that the sets are supplementary difference sets with \( S_1 \) skew before returning them; setting this parameter to False may speed up the computation considerably

OUTPUT:

If \( \text{existence}=\text{False} \), the function returns the group \( G \) of integers modulo \( n \) and a list containing 4 sets, or raises an error if data for the given \( n \) is not available. If \( \text{existence}=\text{True} \), the function returns a boolean representing whether skew supplementary difference sets can be constructed.

EXAMPLES:
sage: from sage.combinat.designs.difference_family import skew_supplementary_difference_set_with_paley_todd
sage: G, [S1, S2, S3, S4] = skew_supplementary_difference_set_with_paley_todd(239)

If existence is True, the function returns a boolean:

sage: skew_supplementary_difference_set_with_paley_todd(239, existence=True)
True
sage: skew_supplementary_difference_set_with_paley_todd(17, existence=True)
False

sage.combinat.designs.difference_family.supplementary_difference_set(q, existence=False, check=True)

Construct $4 - \{2v; v, v+1, v, v; 2v\}$ supplementary difference sets where $q = 2v + 1$.

This is a deprecated version of `supplementary_difference_set_from_rel_diff_set()`, please use that instead.

sage.combinat.designs.difference_family.supplementary_difference_set_from_rel_diff_set(q, existence=False, check=True)

Construct $4 - \{2v; v, v+1, v, v; 2v\}$ supplementary difference sets where $q = 2v + 1$.

The sets are created from relative difference sets as detailed in Theorem 3.3 of [Spe1975]. This construction requires that $q$ is an odd prime power and that there exists $s \geq 0$ such that $(q - (2^s + 1))/2^s + 1$ is an odd prime power.

Note that the construction from [Spe1975] states that the resulting sets are $4 - \{2v; v+1, v, v, v; 2v\}$ supplementary difference sets. However, the implementation of that construction returns $4 - \{2v; v, v+1, v, v; 2v\}$ supplementary difference sets. This is not important, since the supplementary difference sets are not ordered.

INPUT:

• $q$ – an odd prime power
• `existence` – boolean (default: False); If True, only check whether the supplementary difference sets can be constructed
• `check` – boolean (default: True); If True, check that the sets are supplementary difference sets before returning them

OUTPUT:

If `existence=False`, the function returns the 4 sets (containing integers), or raises an error if $q$ does not satisfy the constraints. If `existence=True`, the function returns a boolean representing whether supplementary difference sets can be constructed.

EXAMPLES:
If `existence=True`, the function returns a boolean:

```
sage: supplementary_difference_set_from_rel_diff_set(7, existence=True)
False
sage: supplementary_difference_set_from_rel_diff_set(17, existence=True)
True
```

See also:

- `is_supplementary_difference_set()`

```
sage.combinat.designs.difference_family.supplementary_difference_set_hadamard(n, existence=False, check=True)
```

Construct $4 - \{n; n_1, n_2, n_3, n_4; \lambda\}$ supplementary difference sets, where $n_1 + n_2 + n_3 + n_4 = n + \lambda$.

These sets are constructed from available data, as described in [Djo1994a]. The data is taken from:

- $n = 191$: [Djo2008c]
- $n = 239$: [Djo1994b]
- $n = 251$: [DGK2014]

Additional SDS are constructed using `skew_supplementary_difference_set()`.

**INPUT:**

- `n` – integer; the parameter of the supplementary difference set
- `existence` – boolean (default: `False`); if True, only check whether the supplementary difference sets can be constructed
- `check` – boolean (default: `True`); if True, check that the sets are supplementary difference sets before returning them; Setting this parameter to `False` may speed up the computation considerably

**OUTPUT:**

If `existence=False`, the function returns the ring of integers modulo $n$ and a list containing the 4 sets, or raises an error if data for the given $n$ is not available. If `existence=True`, the function returns a boolean representing whether skew supplementary difference sets can be constructed.

**EXAMPLES:**

```
sage: from sage.combinat.designs.difference_family import supplementary_difference_set_hadamard
sage: G, [S1, S2, S3, S4] = supplementary_difference_set_hadamard(191)
```

If `existence=True`, the function returns a boolean:

```
sage: supplementary_difference_set_hadamard(191, existence=True)
True
sage: supplementary_difference_set_hadamard(17, existence=True)
False
```
sage.combinat.designs.difference_family.turyn_1965_3x3xK(k=4)
Return a difference set in either \(C_3 \times C_3 \times C_4\) or \(C_3 \times C_3 \times C_2 \times C_2\) with parameters \(v = 36, k = 15, \lambda = 6\).
This example appears in [Tu1965].
INPUT:
• \(k\) – either 2 (to get a difference set in \(C_3 \times C_3 \times C_2 \times C_2\)) or 4 (to get a difference set in \(C_3 \times C_3 \times C_3 \times C_4\))

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_family import turyn_1965_3x3xK
sage: from sage.combinat.designs.difference_family import is_difference_family
sage: G,D = turyn_1965_3x3xK(4)
sage: assert is_difference_family(G, D, 36, 15, 6)
sage: G,D = turyn_1965_3x3xK(2)
sage: assert is_difference_family(G, D, 36, 15, 6)
```

sage.combinat.designs.difference_family.twin_prime_powers_difference_set(p, check=True)
Return a difference set on \(GF(p) \times GF(p+2)\).
The difference set is built from the following element of the Cartesian product of finite fields \(GF(p) \times GF(p+2)\):
• \((x, 0)\) with any \(x\)
• \((x, y)\) with \(x\) and \(y\) squares
• \((x, y)\) with \(x\) and \(y\) non-squares
For more information see Wikipedia article Difference_set.
INPUT:
• check – boolean (default: True); if True, then the result of the computation is checked before being returned. This should not be needed but ensures that the output is correct

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_family import twin_prime_powers_difference_set
sage: G,D = twin_prime_powers_difference_set(3)
sage: G
The Cartesian product of (Finite Field of size 3, Finite Field of size 5)
sage: D
\[
\begin{bmatrix}
(1, 1), & (1, 4), & (2, 2), & (2, 3), & (0, 0), & (1, 0), & (2, 0)
\end{bmatrix}
\]
```

5.1.84 Difference Matrices

This module gathers code related to difference matrices. One can build those objects (or know if they can be built) with `difference_matrix()`:

```python
sage: G,DM = designs.difference_matrix(9,5,1)
```
sage.combinat.designs.difference_matrices.difference_matrix(g, k, lmbda=1, existence=False, check=True)

Return a \((g, k, \lambda)\)-difference matrix

A matrix \(M\) is a \((g, k, \lambda)\)-difference matrix if it has size \(\lambda g \times k\), its entries belong to the group \(G\) of cardinality \(g\), and for any two rows \(R, R'\) of \(M\) and \(x \in G\) there are exactly \(\lambda\) values \(i\) such that \(R_i - R'_i = x\).

INPUT:

- \(k\) – (integer) number of columns. If \(k=\text{None}\) it is set to the largest value available.
- \(g\) – (integer) cardinality of the group \(G\)
- \(\lambda\) – (integer; default: 1) – number of times each element of \(G\) appears as a difference.
- \(\text{check}\) – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to \text{True} by default.
- \(\text{existence}\) (boolean) – instead of building the design, return:
  - \text{True}\) – meaning that Sage knows how to build the design
  - \text{Unknown}\) – meaning that Sage does not know how to build the design, but that the design may exist (see \text{sage.misc.unknown}).
  - \text{False}\) – meaning that the design does not exist.

**Note:** When \(k=\text{None}\) and \(\text{existence=\text{True}}\) the function returns an integer, i.e. the largest \(k\) such that we can build a \((g, k, \lambda)\)-DM.

**EXAMPLES:**

```
sage: G,M = designs.difference_matrix(25,10); G
Finite Field in x of size 5^2
sage: designs.difference_matrix(993,None,existence=1)
32
```

Here we print for each \(g\) the maximum possible \(k\) for which Sage knows how to build a \((g, k, 1)\)-difference matrix:

```
sage: for g in range(2,30):
    ....:     k_max = designs.difference_matrix(g,g=None,existence=True)
    ....:     print("{:2} {}").format(g, k_max)
    ....:     _ = designs.difference_matrix(g,k_max)
2 2
3 3
4 4
5 5
6 2
7 7
8 8
9 9
10 2
11 11
12 6
```
sage.combinat.designs.difference_matrices.difference_matrix_product\(k, M1, G1, \lambda_1, M2, G2, \lambda_2, \text{check=True}\)

Return the product of the \((G1,k,\lambda_1)\) and \((G2,k,\lambda_2)\) difference matrices \(M1\) and \(M2\).

The result is a \((G1 \times G2, k, \lambda_1 \lambda_2)\)-difference matrix.

**INPUT:**
- \(k, \lambda_1, \lambda_2\) – positive integer
- \(G1, G2\) – groups
- \(M1, M2\) – \((G1,k,\lambda_1)\) and \((G,k,\lambda_2)\) difference matrices
- \text{check} (boolean) – if True (default), the output is checked before being returned.

**EXAMPLES:**

```
sage: from sage.combinat.designs.difference_matrices import (  
    ...: difference_matrix_product,  
    ...: is_difference_matrix)  
  sage: G1,M1 = designs.difference_matrix(11,6)  
  sage: G2,M2 = designs.difference_matrix(7,6)  
  sage: G,M = difference_matrix_product(6,M1,G1,1,M2,G2,1)  
  sage: G1  
  Finite Field of size 11  
  sage: G2  
  Finite Field of size 7  
  sage: G  
  The Cartesian product of (Finite Field of size 11, Finite Field of size 7)  
  sage: is_difference_matrix(M,G,6,1)  
  True
```

sage.combinat.designs.difference_matrices.find_product_decomposition\(k, \lambda_1=1\)

Try to find a product decomposition construction for difference matrices.

**INPUT:**
- \(g,k,\lambda_1\) – integers, parameters of the difference matrix
OUTPUT:
A pair of pairs \((g_1, \lambda_1), (g_2, \lambda_2)\) if Sage knows how to build \((g_1, k, \lambda_1)\) and \((g_2, k, \lambda_2)\) difference matrices and False otherwise.

EXAMPLES:

```python
sage: from sage.combinat.designs.difference_matrices import find_product_decomposition
sage: find_product_decomposition(77,6)
((7, 1), (11, 1))
sage: find_product_decomposition(616,7)
((7, 1), (88, 1))
sage: find_product_decomposition(24,10)
False
```

### 5.1.85 Evenly distributed sets in finite fields

This module consists of a simple class `EvenlyDistributedSetsBacktracker`. Its main purpose is to iterate through the evenly distributed sets of a given finite field.

The naive backtracker implemented here is not directly used to generate difference family as even for small parameters it already takes time to run. Instead, its output has been stored into a database `sage.combinat.designs.database`. If the backtracker is improved, then one might want to update this database with more values.

#### Classes and methods

```python
class sage.combinat.designs.evenly_distributed_sets.EvenlyDistributedSetsBacktracker

Set of evenly distributed subsets in finite fields.

**Definition:** Let \( K \) be a finite field of cardinality \( q \) and \( k \) an integer so that \( k(k-1) \) divides \( q-1 \). Let \( H = K^* \) be the multiplicative group of invertible elements in \( K \). A \( k \)-evenly distributed set in \( K \) is a set \( B = \{ b_1, b_2, \ldots, b_k \} \) of \( k \) elements of \( K \) so that the \( k(k-1) \) differences \( \Delta B = \{ b_i - b_j; i \neq j \} \) hit each coset modulo \( H^{2(q-1)/(k(k-1))} \) exactly twice.

Evenly distributed sets were introduced by Wilson [Wi72] (see also [BJL99-1], Chapter VII.6). He proved that for any \( k \), and for any prime power \( q \) large enough such that \( k(k-1) \) divides \( q \) there exists a \( k \)-evenly distributed set in the field of cardinality \( q \). This existence result based on a counting argument (using Dirichlet theorem) does not provide a simple method to generate them.

From a \( k \)-evenly distributed set, it is straightforward to build a \((q,k,1)\)-difference family (see `to_difference_family()`). Another approach to generate a difference family, somehow dual to this one, is via radical difference family (see in particular `radical_difference_family()` from the module `difference_family`). By default, this backtracker only considers evenly distributed sets up to affine automorphisms, i.e. \( B \) is considered equivalent to \( sB + t \) for any invertible element \( s \) and any element \( t \) in the field \( K \). Note that the set of differences is just multiplicatively translated by \( s \) as \( \Delta(sB + t) = s(\Delta B) \), and so that \( B \) is an evenly distributed set if and only if \( sB \) is one too. This behaviour can be modified via the argument `up_to_isomorphism` (see the input section and the examples below).

**INPUT:**

- \( K \) – a finite field of cardinality \( q \)
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- $k$ – a positive integer such that $k(k-1)$ divides $q-1$

- `up_to_isomorphism` (boolean, default `True`) whether only consider evenly distributed sets up to automorphisms of the field of the form $x \mapsto ax + b$. If set to `False` then the iteration is over all evenly distributed sets that contain 0 and 1.

- `check` – boolean (default is `False`). Whether you want to check intermediate steps of the iterator. This is mainly intended for debugging purpose. Setting it to `True` will considerably slow the iteration.

**EXAMPLES:**
The main part of the code is contained in the iterator. To get one evenly distributed set just do:

```python
def get_evenly_distributed_set(k):
    return set(range(k))
```

The class has a method to convert it to a difference family:

```python
def to_difference_family(self):
    return [set(range(self.q))]
```

It is also possible to run over all evenly distributed sets:

```python
for k in range(13, 200, 12):
    if is_prime_power(k):
        K = GF(k, 'a')
        E1 = EvenlyDistributedSetsBacktracker(K, 4, False)
        E2 = EvenlyDistributedSetsBacktracker(K, 4, True)
        print(f"{{:3}} {{:3}} {{:3}}".format(k, E1.cardinality(), E2.cardinality()))
```

(continues on next page)
Note that by definition, the number of evenly distributed sets up to isomorphisms is at most \( k(k - 1) \) times smaller than without isomorphisms. But it might not be exactly \( k(k - 1) \) as some of them might have symmetries:

```python
sage: B = EvenlyDistributedSetsBacktracker(Zmod(13), 4).an_element()
sage: B
[0, 1, 11, 5]
sage: [9*x + 5 for x in B]
[5, 1, 0, 11]
sage: [3*x + 11 for x in B]
[11, 1, 5, 0]
```

**an_element()**

Return an evenly distributed set.

If there is no such subset raise an `EmptySetError`.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.evenly_distributed_sets import EvenlyDistributedSetsBacktracker
sage: E = EvenlyDistributedSetsBacktracker(Zmod(41),5)
sage: E.an_element()
[0, 1, 13, 38, 31]
sage: E = EvenlyDistributedSetsBacktracker(Zmod(61),6)
sage: E.an_element()
Traceback (most recent call last):
...
EmptySetError: no 6-evenly distributed set in Ring of integers modulo 61
```

**cardinality()**

Return the number of evenly distributed sets.

Use with precaution as there can be a lot of such sets and this method might be very long to answer!

**EXAMPLES:**

```python
sage: from sage.combinat.designs.evenly_distributed_sets import EvenlyDistributedSetsBacktracker
sage: E = EvenlyDistributedSetsBacktracker(GF(25, 'a'),4)
sage: E 4-evenly distributed sets (up to isomorphism) in Finite Field in a of size 5^2
sage: E.cardinality()
4
```
to_difference_family(B, check=True)

Given an evenly distributed set B convert it to a difference family.

As for any \( x \in K^* = H \) we have \( |\Delta B \cap xH^{2(q-1)/(k(k-1))}| = 2 \) (see EvenlyDistributedSetsBacktracker), the difference family is produced as \( \{xB : x \in H^{2(q-1)/(k(k-1))} \} \)

This method is useful if you want to obtain the difference family from the output of the iterator.

**INPUT:**

- **B** – an evenly distributed set
- **check** – (boolean, default True) whether to check the result

**EXAMPLES:**

```python
sage: from sage.combinat.designs.evenly_distributed_sets import EvenlyDistributedSetsBacktracker
sage: E = EvenlyDistributedSetsBacktracker(Zmod(41),5)

sage: B = E.an_element(); B
[0, 1, 13, 38, 31]

sage: D = E.to_difference_family(B); D
[[[0, 1, 13, 38, 31], [0, 32, 6, 27, 8]]

sage: from sage.combinat.designs.difference_family import is_difference_family
sage: is_difference_family(Zmod(41),D,41,5,1)
True
```

Setting check to False is much faster:

```python
sage: timeit("df = E.to_difference_family(B, check=True)") # random
625 loops, best of 3: 117 µs per loop

sage: timeit("df = E.to_difference_family(B, check=False)") # random
625 loops, best of 3: 1.83 µs per loop
```

5.1.86 External Representations of Block Designs

The “ext_rep” module is an API to the abstract tree represented by an XML document containing the External Representation of a list of block designs. The module also provides the related I/O operations for reading/writing ext-rep files or data. The parsing is based on expat.

This is a modified form of the module ext_rep.py (version 0.8) written by Peter Dobcsanyi [Do2009] peter@designtheory.org.

**Todo:** The XML data from the designtheory.org database contains a wealth of information about things like automorphism groups, transitivity, cycle type representatives, etc, but none of this data is made available through the current
implementation.

**Functions**

```python
class sage.combinat.designs.ext_rep.XTree(node)
    Bases: object

    A lazy class to wrap a rooted tree representing an XML document. The tree's nodes are tuples of the structure:
    
    (name, {dictionary of attributes}, [list of children])

    Methods and services of an XTree object t:
    
    • t.attribute – attribute named
    • t.child – first child named
    • t[i] – i-th child
    • for child in t: – iterate over t's children
    • len(t) – number of t's children

    If child is not an empty subtree, return the subtree as an XTree object. If child is an empty subtree, return _name
    of the subtree. Otherwise return the child itself.

    The lazy tree idea originated from a utility class of the pyRXP 0.9 package by Robin Becker at ReportLab.
```

```python
class sage.combinat.designs.ext_rep.XTreeProcessor
    Bases: object

    An incremental event-driven parser for ext-rep documents. The processing stages:
    
    • <list_of_designs ...> opening element. call-back: list_of_designs_proc
    • <list_definition> subtree. call-back: list_definition_proc
    • <info> subtree. call-back: info_proc
    • iterating over <designs> processing each <block_design> separately. call-back: block_design_proc
    • finishing with closing </designs> and </list_of_designs>.

    parse(xml_source)
        The main parsing function. Given an XML source (either a file handle or a string), parse the entire XML
        source.

        EXAMPLES:

        sage: from sage.combinat.designs import ext_rep
        sage: file_loc = ext_rep.dump_to_tmpfile(ext_rep.v2_b2_k2_icgsa)
        sage: proc = ext_rep.XTreeProcessor()
        sage: proc.save_designs = True
        sage: f = ext_rep.open_extrep_file(file_loc)
        sage: proc.parse(f)
        sage: f.close()
        sage: os.remove(file_loc)
        sage: proc.list_of_designs[0]
        ([2, [[0, 1], [0, 1]]])
```

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sage.combinat.designs.ext_rep.check_dtrs_protocols(input_name, input_pv)

Check that the XML data is in a valid format. We can currently handle version 2.0. For more information see http://designtheory.org/library/extrep/

EXAMPLES:

```
sage: from sage.combinat.designs import ext_rep
sage: ext_rep.check_dtrs_protocols('source', '2.0')
sage: ext_rep.check_dtrs_protocols('source', '3.0')
```

```
Traceback (most recent call last):
...!
RuntimeError: Incompatible dtrs_protocols: program: 2.0 source: 3.0
```

sage.combinat.designs.ext_rep.designs_from_XML(fname)

Return a list of designs contained in an XML file fname. The list contains tuples of the form (v, bs) where v is the number of points of the design and bs is the list of blocks.

EXAMPLES:

```
sage: from sage.combinat.designs import ext_rep
sage: file_loc = ext_rep.dump_to_tmpfile(ext_rep.v2_b2_k2_icgsa)
(2, [[0, 1], [0, 1]])
sage: os.remove(file_loc)
```

```
sage: from sage.combinat.designs import ext_rep
sage: from sage.combinat.designs.block_design import BlockDesign
sage: file_loc = ext_rep.dump_to_tmpfile(ext_rep.v2_b2_k2_icgsa)
sage: v, blocks = ext_rep.designs_from_XML(file_loc)[0]
sage: d = BlockDesign(v, blocks)
sage: d.blocks()
[[0, 1], [0, 1]]
sage: d.is_t_design(t=2)
True
sage: d.is_t_design(return_parameters=True)
(True, (2, 2, 2, 2))
```

sage.combinat.designs.ext_rep.designs_from_XML_url(url)

Return a list of designs contained in an XML file named by a URL. The list contains tuples of the form (v, bs) where v is the number of points of the design and bs is the list of blocks.

EXAMPLES:

```
sage: from sage.combinat.designs import ext_rep
sage: file_loc = ext_rep.dump_to_tmpfile(ext_rep.v2_b2_k2_icgsa)
sage: ext_rep.designs_from_XML_url("file://" + file_loc)[0]
(2, [[0, 1], [0, 1]])
sage: os.remove(file_loc)
```

```
sage: from sage.combinat.designs import ext_rep
sage: ext_rep.designs_from_XML_url("http://designtheory.org/database/v-b-k/v3-b6-k2.icgsa.txt.bz2")
# optional - internet
[(3, [[0, 1], [0, 1], [0, 1], [0, 1], [0, 1], [0, 2]]),
 (3, [[0, 1], [0, 1], [0, 1], [0, 1], [0, 2], [0, 2]]),
 (3, [[0, 1], [0, 1], [0, 1], [0, 1], [0, 2], [1, 2]])]`
```
sage.combinat.designs.ext_rep.dump_to_tmpfile(s)
Utility function to dump a string to a temporary file.
EXAMPLES:

```
sage: from sage.combinat.designs import ext_rep
sage: file_loc = ext_rep.dump_to_tmpfile("boo")
sage: os.remove(file_loc)
```

sage.combinat.designs.ext_rep.open_extrep_file(fname)
Try to guess the compression type from extension and open the extrep file.
EXAMPLES:

```
sage: from sage.combinat.designs import ext_rep
sage: file_loc = ext_rep.dump_to_tmpfile(ext_rep.v2_b2_k2_icgsa)
sage: proc = ext_rep.XTreeProcessor()
sage: f = ext_rep.open_extrep_file(file_loc)
sage: proc.parse(f)
sage: f.close()
sage: os.remove(file_loc)
```

sage.combinat.designs.ext_rep.open_extrep_url(url)
Try to guess the compression type from extension and open the extrep file pointed to by the url. This function (unlike open_extrep_file) returns the uncompressed text contained in the file.
EXAMPLES:

```
sage: from sage.combinat.designs import ext_rep
sage: file_loc = ext_rep.dump_to_tmpfile(ext_rep.v2_b2_k2_icgsa)
sage: proc = ext_rep.XTreeProcessor()
sage: s = ext_rep.open_extrep_url("file://" + file_loc)
sage: proc.parse(s)
sage: os.remove(file_loc)
```

5.1.87 Database of generalised quadrangles with spread

This module implements some construction of generalised quadrangles with spread.
EXAMPLES:

```
sage: GQ, S = designs.generalised_quadrangle_with_spread(4, 16, check=False)
sage: GQ
Incidence structure with 325 points and 1105 blocks
sage: GQ2, O = designs.generalised_quadrangle_hermitian_with_ovoid(4)
```

(continues on previous page)
sage: GQ2
Incidence structure with 1105 points and 325 blocks
sage: GQ3 = GQ.dual()
sage: set(GQ3._points) == set(GQ2._points)
True
sage: GQ2.is_isomorphic(GQ3) # long time
True

REFERENCES:
- [PT2009]
- [TP1994]
- Wikipedia article Generalized_quadrangle

AUTHORS:
- Ivo Maffei (2020-07-26): initial version

sage.combinat.designs.gen_quadrangles_with_spread.dual_GQ_ovoid(GQ, O)
Compute the dual incidence structure of GQ and return the image of O under the dual map

INPUT:
- GQ – IncidenceStructure; the generalised quadrangle we want the dual of
- O – iterable; the iterable of blocks we want to compute the dual

OUTPUT:
A pair (D, S) where D is the dual of GQ and S is the dual of O

EXAMPLES:

sage: from sage.combinat.designs.gen_quadrangles_with_spread import dual_GQ_ovoid
....: dual_GQ_ovoid
sage: t = designs.generalised_quadrangle_hermitian_with_ovoid(3)
sage: t[0].is_generalized_quadrangle(parameters=True)
(9, 3)
sage: t = dual_GQ_ovoid(*t)
sage: t[0].is_generalized_quadrangle(parameters=True)
(3, 9)
sage: all([x in t[0] for x in t[1]])
True

sage.combinat.designs.gen_quadrangles_with_spread.generalised_quadrangle_hermitian_with_ovoid(q)
Construct the generalised quadrangle $H(3, q^2)$ with an ovoid.
The GQ has order $(q^2, q)$.

INPUT:
- q – integer; a prime power

OUTPUT:
A pair (D, O) where D is an IncidenceStructure representing the generalised quadrangle and O is a list of points of D which constitute an ovoid of D

EXAMPLES:
sage: t = designs.generalised_quadrangle_hermitian_with_void(4)
sage: t[0]
Incidence structure with 1105 points and 325 blocks
sage: len(t[1])
65
sage: G = t[0].intersection_graph([1])  # line graph
sage: G.is_strongly_regular(True)
(325, 68, 3, 17)
set(t[0].block_sizes())
{17}

REFERENCES:
For more on $H(3, q^2)$ and the construction implemented here see [PT2009] or [TP1994].

sage.combinat.designs.gen_quadrangles_with_spread.generalised_quadrangle_with_spread($s, t,$
existence=False, check=True)

Construct a generalised quadrangle $GQ$ of order $(s, t)$ with a spread $S$.

INPUT:
- $s, t$ – integers; order of the generalised quadrangle
- existence – boolean;
- check – boolean; if True, then Sage checks that the object built is correct. (default: True)

OUTPUT:
A pair $(GQ, S)$ where $GQ$ is a $IncidenceStructure$ representing the generalised quadrangle and $S$ is a list
of blocks of $GQ$ representing the spread of $GQ$.

EXAMPLES:

sage: t = designs.generalised_quadrangle_with_spread(3, 9)
sage: t[0]
Incidence structure with 112 points and 280 blocks
sage: designs.generalised_quadrangle_with_spread(5, 25, existence=True)
True
sage: (designs.generalised_quadrangle_with_spread(4, 16, check=False))[0]
Incidence structure with 325 points and 1105 blocks
sage: designs.generalised_quadrangle_with_spread(0, 2, existence=True)
False

REFERENCES:
For more on generalised quadrangles and their spread see [PT2009] or [TP1994].

sage.combinat.designs.gen_quadrangles_with_spread.is_GQ_with_spread($GQ, S, s=None, t=None$)

Check if $GQ$ is a generalised quadrangle of order $(s, t)$ and check that $S$ is a spread of $GQ$.

INPUT:
- $GQ$ – $IncidenceStructure$; the incidence structure that is supposed to be a generalised quadrangle
- $S$ – iterable; the spread of $GQ$ as an iterable of the blocks of $GQ$
- $s, t$ – integers (optional); if $(s, t)$ are given, then we check that $GQ$ has order $(s, t)$
EXAMPLES:

```python
sage: from sage.combinat.designs.gen_quadrangles_with_spread import *

 sage: t = generalised_quadrangle_hermitian_with_ovoid(3)
 sage: is_GQ_with_spread(*t)
 Traceback (most recent call last):
   ...
   TypeError: 'int' object is not iterable

 sage: t = dual_GQ_ovoid(*t)
 sage: is_GQ_with_spread(*t)
 True
 sage: is_GQ_with_spread(*t, s=3)
 True
```

5.1.88 Incidence structures (i.e. hypergraphs, i.e. set systems)

An incidence structure is specified by a list of points, blocks, or an incidence matrix \((1,2)\). *IncidenceStructure* instances have the following methods:

- **automorphism_group()**: Return the subgroup of the automorphism group of the incidence graph which respects the P B partition. It is (isomorphic to) the automorphism group of the block design, although the degrees differ.
- **block_sizes()**: Return the set of block sizes.
- **blocks()**: Return the list of blocks.
- **canonical_label()**: Return a canonical label for the incidence structure.
- **coloring()**: Compute a (weak) \(k\)-coloring of the hypergraph.
- **complement()**: Return the complement of the incidence structure.
- **copy()**: Return a copy of the incidence structure.
- **degree()**: Return the degree of a point \(p\) (or a set of points).
- **degrees()**: Return the degree of all sets of given size, or the degree of all points.
- **dual()**: Return the dual of the incidence structure.
- **edge_coloring()**: Compute a proper edge-coloring.
- **ground_set()**: Return the ground set (i.e. the list of points).
- **incidence_graph()**: Return the incidence graph of the incidence structure.
- **incidence_matrix()**: Return the incidence matrix \(A\) of the design. \(A\) is a \((v \times b)\) matrix defined by: \(A[i,j] = 1\) if \(i\) is in block \(B_j\) and \(0\) otherwise.
- **induced_substructure()**: Return the substructure induced by a set of points.
- **intersection_graph()**: Return the intersection graph of the incidence structure.
- **is_berge_cyclic()**: Check whether *self* is a Berge-Cyclic uniform hypergraph.
- **is_connected()**: Test whether the design is connected.
- **is_generalized_quadrangle()**: Test if the incidence structure is a generalized quadrangle.
- **is_isomorphic()**: Return whether the two incidence structures are isomorphic.
- **is_regular()**: Test whether the incidence structure is \(r\)-regular.
- **is_resolvable()**: Test whether the hypergraph is resolvable.
- **is_simple()**: Test whether this design is simple (i.e. no repeated block).
- **is_spread()**: Check whether the input is a spread for *self*.
- **is_t_design()**: Test whether *self* is a \(t-(v,k,l)\) design.
- **is_uniform()**: Test whether the incidence structure is \(k\)-uniform.

continues on next page

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1. Block designs and incidence structures from wikipedia. Wikipedia article Block_design Wikipedia article Incidence_structure
### Table 2 – continued from previous page

<table>
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<tr>
<th>Method</th>
<th>Description</th>
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<tbody>
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<td>isomorphic_substructures()</td>
<td>Iterate over all copies of $H^2$ contained in self.</td>
</tr>
<tr>
<td>num_blocks()</td>
<td>Return the number of blocks.</td>
</tr>
<tr>
<td>num_points()</td>
<td>Return the size of the ground set.</td>
</tr>
<tr>
<td>packing()</td>
<td>Return a maximum packing</td>
</tr>
<tr>
<td>rank()</td>
<td>Return the rank of the hypergraph (the maximum size of a block).</td>
</tr>
<tr>
<td>relabel()</td>
<td>Relabel the ground set</td>
</tr>
<tr>
<td>trace()</td>
<td>Return the trace of a set of points.</td>
</tr>
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</table>

### REFERENCES:

**AUTHORS:**

- Peter Dobcsanyi and David Joyner (2007-2008)
  
  This is a significantly modified form of part of the module block_design.py (version 0.6) written by Peter Dobcsanyi peter@designtheory.org.

- Vincent Delecroix (2014): major rewrite

### Methods

```python
class sage.combinat.designs.incidence_structures.IncidenceStructure(points=None, blocks=None, incidence_matrix=None, name=None, check=True, copy=True):
```

**Bases:** object

A base class for incidence structures (i.e., hypergraphs, i.e., set systems)

An incidence structure (i.e., hypergraph, i.e., set system) can be defined from a collection of blocks (i.e., sets, i.e., edges), optionally with an explicit ground set (i.e., point set, i.e., vertex set). Alternatively they can be defined from a binary incidence matrix.

**INPUT:**

- **points** – (i.e., ground set, i.e., vertex set) the underlying set. If points is an integer $v$, then the set is considered to be $\{0, ..., v - 1\}$.

**Note:** The following syntax, where points is omitted, automatically defines the ground set as the union of the blocks:

```python
sage: H = IncidenceStructure([[‘a’,’b’,’c’],[‘c’,’d’,’e’]])
sage: sorted(H.ground_set())
[‘a’, ‘b’, ‘c’, ‘d’, ‘e’]
```

- **blocks** – (i.e., edges, i.e., sets) the blocks defining the incidence structure. Can be any iterable.
- **incidence_matrix** – a binary incidence matrix. Each column represents a set.
- **name** (a string, such as “Fano plane”).
- **check** – whether to check the input
- **copy** – (use with caution) if set to False then blocks must be a list of lists of integers. The list will not be copied but will be modified in place (each block is sorted, and the whole list is sorted). Your blocks object will become the IncidenceStructure instance’s internal data.
EXAMPLES:

An incidence structure can be constructed by giving the number of points and the list of blocks:

```
sage: IncidenceStructure(7, [[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,6],[2,4,5]])
Incidence structure with 7 points and 7 blocks
```

Only providing the set of blocks is sufficient. In this case, the ground set is defined as the union of the blocks:

```
sage: IncidenceStructure([[1,2,3],[2,3,4]])
Incidence structure with 4 points and 2 blocks
```

Or by its adjacency matrix (a \{0,1\}-matrix in which rows are indexed by points and columns by blocks):

```
sage: m = matrix([[0,1,0],[0,0,1],[1,0,1],[1,1,1]])
sage: IncidenceStructure(m)
Incidence structure with 4 points and 3 blocks
```

The points can be any (hashable) object:

```
sage: V = [(0,'a'),(0,'b'),(1,'a'),(1,'b')]
sage: B = [(V[0],V[1],V[2]), (V[1],V[2]), (V[0],V[2])]
sage: I = IncidenceStructure(V, B)
sage: I.ground_set()
[(0,'a'), (0,'b'), (1,'a'), (1,'b')]
sage: I.blocks()
[((0,'a'), (0,'b'), (1,'a'), (1,'b'))]
```

The order of the points and blocks does not matter as they are sorted on input (see github issue #11333):

```
sage: A = IncidenceStructure([[0,1,2], [[0],[0,2]])
sage: B = IncidenceStructure([[1,0,2], [[0],[2,0]])
sage: B == A
True
```

If you care for speed, you can set `copy` to `False`, but in that case, your input must be a list of lists and the ground set must be 0, ..., \(v-1\):

```
sage: blocks = [[0,1],[2,0],[1,2]]  # a list of lists of integers
sage: I = IncidenceStructure(3, blocks, copy=False)
sage: I._blocks is blocks
True
```

`automorphism_group()`

Return the subgroup of the automorphism group of the incidence graph which respects the \(P \; B\) partition.

It is (isomorphic to) the automorphism group of the block design, although the degrees differ.

EXAMPLES:
Combinatorics, Release 10.0

sage: P = designs.DesarguesianProjectivePlaneDesign(2); P
(7,3,1)-Balanced Incomplete Block Design
sage: G = P.automorphism_group()
sage: G.is_isomorphic(PGL(3,2))
True
sage: G
Permutation Group with generators [...]
sage: G.cardinality()
168

A non self-dual example:

sage: IS = IncidenceStructure(list(range(4)), [[0,1,2,3],[1,2,3]])
sage: IS.automorphism_group().cardinality()
6
sage: IS.dual().automorphism_group().cardinality()
1

Examples with non-integer points:

sage: I = IncidenceStructure(['abc', ('ab','ac','bc')])
sage: I.automorphism_group()
Permutation Group with generators [('b','c'), ('a','b')]
sage: IncidenceStructure([[1,2],[3,4]]).automorphism_group()
Permutation Group with generators [[(1,2),(3,4)]]

block_sizes()

Return the set of block sizes.

EXAMPLES:

sage: BD = IncidenceStructure(8, [[0,1,3],[1,4,5,6],[1,2],[5,6,7]])
sage: BD.block_sizes()
[3, 2, 4, 3]
sage: BD = IncidenceStructure(7,[[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,6],[2,4,5]])
sage: BD.block_sizes()
[3, 3, 3, 3, 3, 3, 3]

blocks()

Return the list of blocks.

EXAMPLES:

sage: BD = IncidenceStructure(7,[[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,6],[2,4,5]])
sage: BD.blocks()
[[0, 1, 2], [0, 3, 4], [0, 5, 6], [1, 3, 5], [1, 4, 6], [2, 3, 6], [2, 4, 5]]

canonical_label()

Return a canonical label for the incidence structure.

A canonical label is relabeling of the points into integers \{0, ..., n - 1\} such that isomorphic incidence structures are relabelled to equal objects.

EXAMPLES:
sage: fano1 = designs.balanced_incomplete_block_design(7,3)
sage: fano2 = designs.projective_plane(2)
sage: fano1 == fano2
False
sage: fano1.relabel(fano1.canonical_label())
sage: fano2.relabel(fano2.canonical_label())
sage: fano1 == fano2
True

coloring($k$, solver=None, verbose=None, integrality_tolerance=0)

Compute a (weak) $k$-coloring of the hypergraph

A weak coloring of a hypergraph $\mathcal{H}$ is an assignment of colors to its vertices such that no set is monochromatic.

INPUT:

- $k$ (integer) – compute a coloring with $k$ colors if an integer is provided, otherwise returns an optimal coloring (i.e. with the minimum possible number of colors).
- solver – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method `solve` of the class `MixedIntegerLinearProgram`.
- verbose – non-negative integer (default: 0). Set the level of verbosity you want from the linear program solver. Since the problem is $NP$-complete, its solving may take some time depending on the graph. A value of 0 means that there will be no message printed by the solver.
- integrality_tolerance – parameter for use with MILP solvers over an inexact base ring; see `MixedIntegerLinearProgram.get_values()`.

EXAMPLES:

The Fano plane has chromatic number 3:

sage: len(designs.steiner_triple_system(7).coloring())
3

One admissible 3-coloring:

```
sage: designs.steiner_triple_system(7).coloring()  # not tested - architecture-dependent
[[0, 2, 5, 1], [4, 3], [6]]
```

The chromatic number of a graph is equal to the chromatic number of its 2-uniform corresponding hypergraph:

```
sage: g = graphs.PetersenGraph()
sage: H = IncidenceStructure(g.edges(sort=True, labels=False))
sage: len(g.coloring())
3
sage: len(H.coloring())
3
```

complement($uniform=False$)

Return the complement of the incidence structure.

Two different definitions of “complement” are made available, according to the value of $uniform$. 

5.1. Comprehensive Module List
INPUT:

- uniform (boolean) –
  - if set to False (default), returns the incidence structure whose blocks are the complements of all blocks of the incidence structure.
  - If set to True and the incidence structure is $k$-uniform, returns the incidence structure whose blocks are all $k$-sets of the ground set that do not appear in self.

EXAMPLES:

The complement of a `BalancedIncompleteBlockDesign` is also a 2-design:

```python
sage: bibd = designs.balanced_incomplete_block_design(13,4)
sage: bibd.is_t_design(return_parameters=True)
(True, (2, 13, 4, 1))
sage: bibd.complement().is_t_design(return_parameters=True)
(True, (2, 13, 9, 6))
```

The “uniform” complement of a graph is a graph:

```python
sage: g = graphs.PetersenGraph()
sage: G = IncidenceStructure(g.edges(sort=True, labels=False))
sage: H = G.complement(uniform=True)
sage: h = Graph(H.blocks())
sage: g == h
False
sage: g == h.complement()
True
```

`copy()`

Return a copy of the incidence structure.

EXAMPLES:

```python
sage: IS = IncidenceStructure([[1,2,3,"e"]],name="Test")
sage: IS
Incidence structure with 4 points and 1 blocks
sage: copy(IS)
Incidence structure with 4 points and 1 blocks
sage: [1, 2, 3, 'e'] in copy(IS)
True
sage: copy(IS)._name
'Test'
```

degree($p=None, \text{subset}=False$)

Return the degree of a point $p$ (or a set of points).

The degree of a point (or set of points) is the number of blocks that contain it.

INPUT:

- $p$ – a point (or a set of points) of the incidence structure.
- subset (boolean) – whether to interpret the argument as a set of point (subset=True) or as a point (subset=False, default).

EXAMPLES:
sage: designs.steiner_triple_system(9).degree(3)
4
sage: designs.steiner_triple_system(9).degree({1,2},subset=True)
1

degrees(size=None)
Return the degree of all sets of given size, or the degree of all points.

The degree of a point (or set of point) is the number of blocks that contain it.

INPUT:
• size (integer) – return the degree of all subsets of points of cardinality size. When size=None, the function outputs the degree of all points.

Note: When size=None the output is indexed by the points. When size=1 it is indexed by tuples of size 1. This is the same information, stored slightly differently.

OUTPUT:
A dictionary whose values are degrees and keys are either:
• the points of the incidence structure if size=None (default)
• the subsets of size size of the points stored as tuples

EXAMPLES:
sage: IncidenceStructure([[1,2,3],[1,4]]).degrees(2)
{(1, 2): 1, (1, 3): 1, (1, 4): 1, (2, 3): 1, (2, 4): 0, (3, 4): 0}

In a Steiner triple system, all pairs have degree 1:
sage: S13 = designs.steiner_triple_system(13)
sage: all(v == 1 for v in S13.degrees(2).values())
True

dual(algorithm=None)
Return the dual of the incidence structure.

INPUT:
• algorithm – whether to use Sage’s implementation (algorithm=None, default) or use GAP’s (algorithm="gap").

Note: The algorithm="gap" option requires GAP’s Design package (included in the gap_packages Sage spkg).

EXAMPLES:
The dual of a projective plane is a projective plane:
sage: PP = designs.DesarguesianProjectivePlaneDesign(4)
sage: PP.dual().is_t_design(return_parameters=True)
(True, (2, 21, 5, 1))

REFERENCE:

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edge_coloring()

Compute a proper edge-coloring.

A proper edge-coloring is an assignment of colors to the sets of the incidence structure such that two sets with non-empty intersection receive different colors. The coloring returned minimizes the number of colors.

OUTPUT:

A partition of the sets into color classes.

EXAMPLES:

```sage
H = Hypergraph([{1,2,3},{2,3,4},{3,4,5},{4,5,6}]); H
Incidence structure with 6 points and 4 blocks
sage: C = H.edge_coloring()
sage: C # random
[[[3, 4, 5]], [[2, 3, 4]], [[4, 5, 6], [1, 2, 3]]]
sage: Set(map(Set,sum(C,[]))) == Set(map(Set,H.blocks()))
True
```

ground_set()

Return the ground set (i.e the list of points).

EXAMPLES:

```sage
IncidenceStructure(3, [[0,1],[0,2]]).ground_set()
[0, 1, 2]
```

incidence_graph(labels=False)

Return the incidence graph of the incidence structure

A point and a block are adjacent in this graph whenever they are incident.

INPUT:

- labels (boolean) – whether to return a graph whose vertices are integers, or labelled elements.
  - labels is False (default) – in this case the first vertices of the graphs are the elements of `ground_set()`, and appear in the same order. Similarly, the following vertices represent the elements of `blocks()`, and appear in the same order.
  - labels is True, the points keep their original labels, and the blocks are `Set` objects.

Note that the labelled incidence graph can be incorrect when blocks are repeated, and on some (rare) occasions when the elements of `ground_set()` mix `Set()` and non-`Set` objects.

EXAMPLES:

```sage
BD = IncidenceStructure(7, [[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,→6],[2,4,5]])
sage: BD.incidence_graph()
Bipartite graph on 14 vertices
sage: A = BD.incidence_matrix()
sage: Graph(block_matrix([[A*0,A],[A.transpose(),A*0]])) == BD.incidence_graph()
True
```
\textbf{incidence\_matrix()}

Return the incidence matrix $A$ of the design. $A$ is a $(v \times b)$ matrix defined by: $A[i,j] = 1$ if $i$ is in block $B_j$ and 0 otherwise.

**EXAMPLES:**

\begin{verbatim}
sage: BD = IncidenceStructure(7, [[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,6],[2,4,5]]) sage: BD.block_sizes() [3, 3, 3, 3, 3, 3, 3] sage: BD.incidence_matrix() \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
sage: I = IncidenceStructure(('abc', ('ab','abc','ac','c'))) sage: I.incidence_matrix() \begin{bmatrix} 1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1 \end{bmatrix}
\end{verbatim}

\textbf{induced\_substructure(points)}

Return the substructure induced by a set of points.

The substructure induced in $\mathcal{H}$ by a set $X \subseteq V(\mathcal{H})$ of points is the incidence structure $\mathcal{H}_X$ defined on $X$ whose sets are all $S \in \mathcal{H}$ such that $S \subseteq X$.

**INPUT:**

- `points` – a set of points.

**Note:** This method goes over all sets of `self` before building a new \texttt{IncidenceStructure} (which involves some relabelling and sorting). It probably should not be called in a performance-critical code.

**EXAMPLES:**

A Fano plane with one point removed:

\begin{verbatim}
sage: F = designs.steiner_triple_system(7) sage: F.induced_substructure([0..5]) Incidence structure with 6 points and 4 blocks
\end{verbatim}

\textbf{intersection\_graph(sizes=None)}

Return the intersection graph of the incidence structure.

The vertices of this graph are the \texttt{blocks()} of the incidence structure. Two of them are adjacent if the size of their intersection belongs to the set \texttt{sizes}.

**INPUT:**

- `sizes` – a list/set of integers. For convenience, setting \texttt{sizes} to 5 has the same effect as \texttt{sizes=[5]}. When set to `None` (default), behaves as \texttt{sizes=PositiveIntegers()}.  

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EXAMPLES:

The intersection graph of a `balanced_incomplete_block_design()` is a strongly regular graph (when it is not trivial):

```python
sage: BIBD = designs.balanced_incomplete_block_design(19,3)
sage: G = BIBD.intersection_graph(1)
sage: G.is_strongly_regular(parameters=True)
(57, 24, 11, 9)
```

`is_berge_cyclic()`

Check whether self is a Berge-Cyclic uniform hypergraph.

A $k$-uniform Berge cycle (named after Claude Berge) of length $\ell$ is a cyclic list of distinct $k$-sets $F_1, \ldots, F_\ell$, $\ell > 1$, and distinct vertices $C = \{v_1, \ldots, v_\ell\}$ such that for each $1 \leq i \leq \ell$, $F_i$ contains $v_i$ and $v_{i+1}$ (where $v_{\ell+1} = v_1$).

A uniform hypergraph is Berge-cyclic if its incidence graph is cyclic. It is called “Berge-acyclic” otherwise.

For more information, see [Fag1983] and Wikipedia article Hypergraph.

EXAMPLES:

```python
sage: Hypergraph(5, [[1, 2, 3], [2, 3 ,4]]).is_berge_cyclic()
True
sage: Hypergraph(6, [[1, 2, 3], [3 ,4, 5]]).is_berge_cyclic()
False
```

`is_connected()`

Test whether the design is connected.

EXAMPLES:

```python
sage: IncidenceStructure(3, [[0,1],[0,2]]).is_connected()
True
sage: IncidenceStructure(4, [[0,1],[2,3]]).is_connected()
False
```

`is_generalized_quadrangle(verbose=False, parameters=False)`

Test if the incidence structure is a generalized quadrangle.

An incidence structure is a generalized quadrangle iff (see [BH2012], section 9.6):

- two blocks intersect on at most one point.

- For every point $p$ not in a block $B$, there is a unique block $B'$ intersecting both $\{p\}$ and $B$

It is a regular generalized quadrangle if furthermore:

- it is $s + 1$-uniform for some positive integer $s$.

- it is $t + 1$-regular for some positive integer $t$.

For more information, see the Wikipedia article Generalized_quadrangle.

Note: Some references (e.g. [PT2009] or Wikipedia article Generalized_quadrangle) only allow regular generalized quadrangles. To use such a definition, see the parameters optional argument described below, or the methods `is_regular()` and `is_uniform()`.

INPUT:
• **verbose** (boolean) – whether to print an explanation when the instance is not a generalized quadrangle.

• **parameters** (boolean; \(\text{False}\)) – if set to \(\text{True}\), the function returns a pair \((s, t)\) instead of \(\text{True}\) answers. In this case, \(s\) and \(t\) are the integers defined above if they exist (each can be set to \(\text{False}\) otherwise).

**EXAMPLES:**

```python
sage: h = designs.CremonaRichmondConfiguration()
sage: h.is_generalized_quadrangle()
True
```

This is actually a *regular* generalized quadrangle:

```python
sage: h.is_generalized_quadrangle(parameters=\text{True})
(2, 2)
```

**is_isomorphic**(*other*, certificate=\(\text{False}\))

Return whether the two incidence structures are isomorphic.

**INPUT:**

• **other** – an incidence structure.

• **certificate** (boolean) – whether to return an isomorphism from \(\text{self}\) to \(\text{other}\) instead of a boolean answer.

**EXAMPLES:**

```python
sage: fano1 = designs.balanced_incomplete_block_design(7,3)
sage: fano2 = designs.projective_plane(2)
sage: fano1.is_isomorphic(fano2)
True
sage: fano1.is_isomorphic(fano2, certificate=\text{True})
{0: 0, 1: 1, 2: 2, 3: 6, 4: 4, 5: 3, 6: 5}
```

**is_regular**(*\(r=\text{None}\)*)

Test whether the incidence structure is \(r\)-regular.

An incidence structure is said to be \(r\)-regular if all its points are incident with exactly \(r\) blocks.

**INPUT:**

• **\(r\)** (integer)

**OUTPUT:**

If \(r\) is defined, a boolean is returned. If \(r\) is set to \(\text{None}\) (default), the method returns either \(\text{False}\) or the integer \(r\) such that the incidence structure is \(r\)-regular.

**Warning:** In case of 0-regular incidence structure, beware that *if not H.is_regular()* is a satisfied condition.

**EXAMPLES:**

```python
sage: designs.balanced_incomplete_block_design(7,3).is_regular()
3
sage: designs.balanced_incomplete_block_design(7,3).is_regular(\text{r=3})
```

(continues on next page)
is_resolvable(certificate, solver=False, verbose=None, check=0, integrality_tolerance=True)

Test whether the hypergraph is resolvable

A hypergraph is said to be resolvable if its sets can be partitioned into classes, each of which is a partition of the ground set.

Note: This problem is solved using an Integer Linear Program, and GLPK (the default LP solver) has been reported to be very slow on some instances. If you hit this wall, consider installing a more powerful MILP solver (CPLEX, Gurobi, ...).

INPUT:

- certificate (boolean) – whether to return the classes along with the binary answer (see examples below).
- solver – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method solve of the class MixedIntegerLinearProgram.
- verbose – integer (default: 0). Sets the level of verbosity. Set to 0 by default, which means quiet.
- check (boolean) – whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.
- integrality_tolerance – parameter for use with MILP solvers over an inexact base ring; see MixedIntegerLinearProgram.get_values().

EXAMPLES:

Some resolvable designs:

\begin{verbatim}
sage: TD = designs.transversal_design(2,2,resolvable=True)
sage: TD.is_resolvable()
True

sage: AG = designs.AffineGeometryDesign(3,1,GF(2))
sage: AG.is_resolvable()
True
\end{verbatim}
A non-resolvable design:

```
sage: Fano = designs.balanced_incomplete_block_design(7,3)
sage: Fano.is_resolvable()
False
sage: Fano.is_resolvable(True)
(False, [])
```

**is_simple()**

Test whether this design is simple (i.e. no repeated block).

**EXAMPLES:**

```
sage: IncidenceStructure(3, [[0,1],[1,2],[0,2]]).is_simple()
True
sage: IncidenceStructure(3, [[0],[0]]).is_simple()
False
```

**is_spread()**

Check whether the input is a spread for self.

A spread of an incidence structure \((P, B)\) is a subset of \(B\) which forms a partition of \(P\).

**INPUT:**

- spread – iterable; defines the spread

**EXAMPLES:**

```
sage: E = IncidenceStructure([[1, 2, 3], [4, 5, 6], [1, 5, 6]])
sage: E.is_spread([[1, 2, 3], [4, 5, 6]])
True
sage: E.is_spread([1, 2, 3, 4, 5, 6])
Traceback (most recent call last):
...
  TypeError: 'sage.rings.integer.Integer' object is not iterable
```

(continues on next page)
Combinatorics, Release 10.0

sage: E.is_spread([[1, 2, 3, 4], [5, 6]])
False

Order of blocks or of points within each block doesn’t matter:

sage: E = IncidenceStructure([[1, 2, 3], [4, 5, 6], [1, 5, 6]])
sage: E.is_spread([[5, 6, 4], [3, 1, 2]])
True

is_t_design(t=None, v=None, k=None, l=None, return_parameters=False)
Test whether self is a $t-(v,k,l)$ design.

A $t- (v,k,\lambda)$ (sometimes called $t$-design for short) is a block design in which:

• the underlying set has cardinality $v$
• the blocks have size $k$
• each $t$-subset of points is covered by $\lambda$ blocks

INPUT:

• $t, v, k, l$ (integers) – their value is set to None by default. The function tests whether the design is a $t- (v,k,\lambda)$ design using the provided values and guesses the others. Note that $l$ cannot be specified if $t$ is not.
• return_parameters (boolean) – whether to return the parameters of the $t$-design. If set to True, the function returns a pair (boolean_answer, (t,v,k,l)).

EXAMPLES:

sage: fano_blocks = [[0,1,2],[0,3,4],[0,5,6],[1,3,5],[1,4,6],[2,3,6],[2,4,5]]
sage: BD = IncidenceStructure(7, fano_blocks)
sage: BD.is_t_design()
True
sage: BD.is_t_design(return_parameters=True)
(True, (2, 7, 3, 1))
sage: BD.is_t_design(2, 7, 3, 1)
True
sage: BD.is_t_design(1, 7, 3, 3)
True
sage: BD.is_t_design(0, 7, 3, 7)
True
sage: BD.is_t_design(0, 6, 3, 7) or BD.is_t_design(0, 7, 4, 7) or BD.is_t_design(0, 7, 3, 8)
False
sage: BD = designs.AffineGeometryDesign(3, 1, GF(2))
sage: BD.is_t_design(1)
True
sage: BD.is_t_design(2)
True

Steiner triple and quadruple systems are other names for $2- (v,3,1)$ and $3- (v,4,1)$ designs:
Some examples of Witt designs that need the gap database:

```python
sage: BD = designs.WittDesign(9)  # optional - gap_packages
sage: BD.is_t_design(2,9,3,1)    # optional - gap_packages
    True
sage: W12 = designs.WittDesign(12)  # optional - gap_packages
sage: W12.is_t_design(5,12,6,1)  # optional - gap_packages
    True
sage: W12.is_t_design(4)  # optional - gap_packages
    True
```

Further examples:

```python
sage: D = IncidenceStructure(4, [[], []])
sage: D.is_t_design(return_parameters=True)
    (True, (0, 4, 0, 2))

sage: D = IncidenceStructure(4, [[0, 1], [0, 2], [0, 3]])
sage: D.is_t_design(return_parameters=True)
    (True, (0, 4, 2, 3))

sage: D = IncidenceStructure(4, [[0], [1], [2], [3]])
sage: D.is_t_design(return_parameters=True)
    (True, (1, 4, 1, 1))

sage: D = IncidenceStructure(4, [[0, 1], [2, 3]])
sage: D.is_t_design(return_parameters=True)
    (True, (1, 4, 2, 1))

sage: D = IncidenceStructure(4, [list(range(4))])
sage: D.is_t_design(return_parameters=True)
    (True, (4, 4, 4, 1))
```

**is_uniform**(*k=None*)

Test whether the incidence structure is *k*-uniform

An incidence structure is said to be *k*-uniform if all its blocks have size *k*.

**INPUT:**

- *k* (integer)
OUT:  
If \( k \) is defined, a boolean is returned. If \( k \) is set to \( \text{None} \) (default), the method returns either \( \text{False} \) or the integer \( k \) such that the incidence structure is \( k \)-uniform.

**Warning:** In case of 0-uniform incidence structure, beware that if not \( \text{H.is_uniform()} \) is a satisfied condition.

**EXAMPLES:**

```python
sage: designs.balanced_incomplete_block_design(7,3).is_uniform()
3
sage: designs.balanced_incomplete_block_design(7,3).is_uniform(k=3)
True
sage: designs.balanced_incomplete_block_design(7,3).is_uniform(k=4)
False
```

**isomorphic_substructures_iterator** \((H2, induced=False)\)

Iterates over all copies of \( H2 \) contained in \( self \).

A hypergraph \( H_1 \) contains an isomorphic copy of a hypergraph \( H_2 \) if there exists an injection \( f: V(H_2) \rightarrow V(H_1) \) such that for any set \( S_2 \in E(H_2) \) the set \( S_1 = f(S_2) \) belongs to \( E(H_1) \).

It is an *induced* copy if no other set of \( E(H_1) \) is contained in \( f(V(H_2)) \), i.e. \(|E(H_2)| = \{ S : S \in E(H_1) \text{ and } f(V(H_2)) \}\).

This function lists all such injections. In particular, the number of copies of \( H \) in itself is equal to the size of its automorphism group.

See **subhypergraph_search** for more information.

**INPUT:**

- \( H2 \) an *IncidenceStructure* object.
- \( induced \) (boolean) – whether to require the copies to be induced. Set to \( \text{False} \) by default.

**EXAMPLES:**

How many distinct \( C_5 \) in Petersen’s graph ?

```python
sage: P = graphs.PetersenGraph()
sage: C = graphs.CycleGraph(5)
sage: IP = IncidenceStructure(P.edges(sort=True, labels=False))
sage: IC = IncidenceStructure(C.edges(sort=True, labels=False))
sage: IP.isomorphic_substructures_iterator(IC))
120
```

As the automorphism group of \( C_5 \) has size 10, the number of distinct unlabelled copies is 12. Let us check that all functions returned correspond to an actual \( C_5 \) subgraph:

```python
sage: for f in IP.isomorphic_substructures_iterator(IC):
   ....:   assert all(P.has_edge(f[x],f[y]) for x,y in C.edges(sort=True,
   ....:                            labels=False))
```

The number of induced copies, in this case, is the same:
They begin to differ if we make one vertex universal:

```python
sage: P.add_edges([(0,x) for x in P], loops=False)
sage: IP = IncidenceStructure(P.edges(sort=True, labels=False))
sage: IC = IncidenceStructure(C.edges(sort=True, labels=False))
sage: sum(1 for _ in IP.isomorphic_substructures_iterator(IC))
420
sage: sum(1 for _ in IP.isomorphic_substructures_iterator(IC, induced=True))
60
```

The number of copies of $H$ in itself is the size of its automorphism group:

```python
sage: H = designs.projective_plane(3)
sage: sum(1 for _ in H.isomorphic_substructures_iterator(H))
5616
sage: H.automorphism_group().cardinality()
5616
```

**num_blocks()**

Return the number of blocks.

**num_points()**

Return the size of the ground set.

**packing**(solver, verbose=None, integrality_tolerance=0)

Return a maximum packing

A maximum packing in a hypergraph is collection of disjoint sets/blocks of maximal cardinality. This problem is NP-complete in general, and in particular on 3-uniform hypergraphs. It is solved here with an Integer Linear Program.

For more information, see the Wikipedia article Packing_in_a_hypergraph.

**INPUT:**

- **solver** – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on LP solvers and which default solver is used, see the method `solve` of the class `MixedIntegerLinearProgram`.

---

**EXAMPLES:**

```python
sage: designs.DesarguesianProjectivePlaneDesign(2).num_blocks()
7
sage: B = IncidenceStructure(4, [[0,1],[0,2],[0,3],[1,2],[1,2,3]])
sage: B.num_blocks()
5
```

```python
sage: designs.DesarguesianProjectivePlaneDesign(2).num_points()
7
sage: B = IncidenceStructure(4, [[0,1],[0,2],[0,3],[1,2],[1,2,3]])
sage: B.num_points()
4
```
• **verbose** – integer (default: 0). Sets the level of verbosity. Set to 0 by default, which means quiet.

• **integrality_tolerance** – parameter for use with MILP solvers over an inexact base ring; see `MixedIntegerLinearProgram.get_values()`.

EXAMPLES:

```python
sage: P = IncidenceStructure([[1,2],[3,4],[2,3]]).packing()
sage: sorted(sorted(b) for b in P)
[[1, 2], [3, 4]]
sage: len(designs.steiner_triple_system(9).packing())
3
```

**rank()**

Return the rank of the hypergraph (the maximum size of a block).

EXAMPLES:

```python
sage: h = Hypergraph(8, [[0,1,3],[1,4,5,6],[1,2]])
sage: h.rank()
4
```

**relabel**(perm=None, inplace=True)

Relabel the ground set

INPUT:

• *perm* – can be one of

  – a dictionary – then each point `p` (which should be a key of `d`) is relabeled to `d[p]`

  – a list or a tuple of length `n` – the first point returned by `ground_set()` is relabeled to `1[0]`, the second to `1[1]`, ...

  – `None` – the incidence structure is relabeled to be on `{0, 1, ..., n − 1}` in the ordering given by `ground_set()`.

• *inplace* – If `True` then return a relabeled graph and does not touch `self` (default is `False`).

EXAMPLES:

```python
sage: TD=designs.transversal_design(5,5)
sage: TD.relabel({i:chr(97+i) for i in range(25)})
sage: TD.blocks()[:3]
[['a', 'f', 'k', 'p', 'u'], ['a', 'g', 'm', 's', 'y'], ['a', 'h', 'o', 'q', 'x']]
```

Relabel to integer points:

```python
sage: TD.relabel()
sage: TD.blocks()[:3]
[[0, 5, 10, 15, 20], [0, 6, 12, 18, 24], [0, 7, 14, 16, 23]]
```

**trace**(points, min_size=1, multiset=True)

Return the trace of a set of points.
Given an hypergraph $\mathcal{H}$, the *trace* of a set $X$ of points in $\mathcal{H}$ is the hypergraph whose blocks are all non-empty $S \cap X$ where $S \in \mathcal{H}$.

**INPUT:**

- `points` – a set of points.
- `min_size` (integer; default 1) – minimum size of the sets to keep. By default all empty sets are discarded, i.e. `min_size=1`.
- `multiset` (boolean; default `True`) – whether to keep multiple copies of the same set.

**Note:** This method goes over all sets of `self` before building a new `IncidenceStructure` (which involves some relabelling and sorting). It probably should not be called in a performance-critical code.

**EXAMPLES:**

A Baer subplane of order 2 (i.e. a Fano plane) in a projective plane of order 4:

```python
sage: P4 = designs.projective_plane(4)
sage: F = designs.projective_plane(2)
sage: for x in Subsets(P4.ground_set(),7):
    if P4.trace(x,min_size=2).is_isomorphic(F):
        break
sage: subplane = P4.trace(x,min_size=2); subplane
Incidence structure with 7 points and 7 blocks
sage: subplane.is_isomorphic(F)
True
```

### 5.1.89 Mutually Orthogonal Latin Squares (MOLS)

The main function of this module is `mutually_orthogonal_latin_squares()` and can be be used to generate MOLS (or check that they exist):

```python
sage: MOLS = designs.mutually_orthogonal_latin_squares(4,8)
```

For more information on MOLS, see the Wikipedia entry on MOLS. If you are only interested by latin squares, see `latin`.

The functions defined here are

- `mutually_orthogonal_latin_squares()` Return $k$ Mutually Orthogonal $n \times n$ Latin Squares.
- `are_mutually_orthogonal_latin_squares()` Check that the list `l` of matrices in are MOLS.
- `latin_square_product()` Return the product of two (or more) latin squares.
- `MOLS_table()` Prints the MOLS table.

**Table of MOLS**

Sage can produce a table of MOLS similar to the one from the Handbook of Combinatorial Designs [DesignHandbook] (available here).

```python
sage: from sage.combinat.designs.latin_squares import MOLS_table
sage: MOLS_table(600) # long time
```

(continues on next page)
Comparison with the results from the Handbook of Combinatorial Designs (2ed) [DesignHandbook]:

```sage
sage: MOLS_table(600, compare=True) # long time
```

<table>
<thead>
<tr>
<th>0</th>
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<th>+</th>
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</thead>
<tbody>
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<td>600</td>
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</tbody>
</table>

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Todo: Look at [ColDin01].

REFERENCES:

Functions

sage.combinat.designs.latin_squares.MOLS_table(start, stop=None, compare=False, width=None)

Prints the MOLS table that Sage can produce.

INPUT:

- start, stop (integers) – print the table of MOLS for value of n such that start<=n<stop. If only one integer is given as input, it is interpreted as the value of stop with start=0 (same behaviour as range).
- compare (boolean) – if sets to True the MOLS displays with + and − entries its difference with the table from the Handbook of Combinatorial Designs (2ed).
- width (integer) – the width of each column of the table. By default, it is computed from range of values determined by the parameters start and stop.

EXAMPLES:

```
sage: from sage.combinat.designs.latin_squares import MOLS_table
sage: MOLS_table(100)

| 0 | oo 1 oo 2 3 4 5 6 7 8 9 10 5 12 4 4 15 16 5 18 |
| 20| 4 5 3 22 7 24 4 26 5 28 4 30 31 5 4 5 8 36 4 5 |
| 40| 7 40 5 52 5 52 5 6 67 7 7 7 7 5 5 5 5 58 |
| 60| 5 60 5 6 6 66 7 5 66 5 6 67 7 7 7 5 6 6 6 6 6 6 78 |
| 80| 9 80 8 82 6 6 6 6 6 8 88 6 7 6 6 6 7 96 6 8 |
```

```
sage: MOLS_table(100, width=4)

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
</table>
```

(continues on next page)
sage.combinat.designs.latin_squares.are_mutually_orthogonal_latin_squares(l, verbose=False)

Check whether the list of matrices in \( l \) form mutually orthogonal latin squares.

**INPUT:**

- **verbose** - if True then print why the list of matrices provided are not mutually orthogonal latin squares

**EXAMPLES:**

```sage
sage: from sage.combinat.designs.latin_squares import are_mutually_orthogonal_latin_squares

sage: m1 = matrix([[0,1,2],[2,0,1],[1,2,0]])
sage: m2 = matrix([[0,1,2],[1,2,0],[2,0,1]])
sage: m3 = matrix([[0,1,2],[2,0,1],[1,2,0]])
sage: are_mutually_orthogonal_latin_squares([m1,m2])
True
sage: are_mutually_orthogonal_latin_squares([m1,m3])
False
sage: are_mutually_orthogonal_latin_squares([m2,m3])
True
sage: are_mutually_orthogonal_latin_squares([m1,m2,m3], verbose=True)
Squares 0 and 2 are not orthogonal
False
```

sage.combinat.designs.latin_squares.latin_square_product(M, N, *others)
Return the product of two (or more) latin squares.

Given two Latin Squares $M, N$ of respective sizes $m, n$, the direct product $M \times N$ of size $mn$ is defined by

$$(M \times N)((i_1, i_2), (j_1, j_2)) = (M(i_1, j_1), N(i_2, j_2))$$

where $i_1, j_1 \in [m], i_2, j_2 \in [n]$

Each pair of values $(i, j) \in [m] \times [n]$ is then relabeled to $in + j$.

This is Lemma 6.25 of [Stinson2004].

INPUT:

An arbitrary number of latin squares (greater than 2).

EXAMPLES:

```python
sage: from sage.combinat.designs.latin_squares import latin_square_product
sage: m=designs.mutually_orthogonal_latin_squares(3,4)[0]
sage: latin_square_product(m,m,m)
64 x 64 sparse matrix over Integer Ring (use the '.str()' method to see the entries)
```

sage.combinat.designs.latin_squares.mutually_orthogonal_latin_squares($k, n, partitions=False, check=True$)

Return $k$ Mutually Orthogonal $n \times n$ Latin Squares (MOLS).

For more information on Mutually Orthogonal Latin Squares, see `latin_squares`.

INPUT:

- $k$ (integer) – number of MOLS. If $k=None$ it is set to the largest value available.
- $n$ (integer) – size of the latin square.
- $partitions$ (boolean) – a Latin Square can be seen as 3 partitions of the $n^2$ cells of the array into $n$ sets of size $n$, respectively:
  - The partition of rows
  - The partition of columns
  - The partition of number (cells numbered with 0, cells numbered with 1, ...)

These partitions have the additional property that any two sets from different partitions intersect on exactly one element.

When $partitions$ is set to $True$, this function returns a list of $k + 2$ partitions satisfying this intersection property instead of the $k + 2$ MOLS (though the data is exactly the same in both cases).

- $check$ – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to $True$ by default.

EXAMPLES:

```python
sage: designs.mutually_orthogonal_latin_squares(4,5)
[0 2 4 1 3] [0 3 1 4 2] [0 4 3 2 1] [0 1 2 3 4]
[4 1 3 0 2] [3 1 4 2 0] [2 1 0 4 3] [4 0 1 2 3]
[3 0 2 4 1] [1 4 2 0 3] [4 3 2 1 0] [3 4 0 1 2]
[2 4 1 3 0] [4 2 0 3 1] [1 0 4 3 2] [2 3 4 0 1]
[1 3 0 2 4], [2 0 3 1 4], [3 2 1 0 4], [1 2 3 4 0]
```

(continues on next page)
sage: designs.mutually_orthogonal_latin_squares(3,7)
[[0 2 4 6 1 3 5]
 [0 3 6 2 5 1 4]
 [0 4 1 5 2 6 3]
 [6 1 3 5 0 2 4]
 [5 1 4 0 3 6 2]
 [4 1 5 2 6 3 0]
 [5 0 2 4 6 1 3]
 [3 6 2 5 1 4 0]
 [1 5 2 6 3 0 4]
 [4 6 1 3 5 0 2]
 [1 4 0 3 6 2 5]
 [5 2 6 3 0 4 1]
 [3 5 0 2 4 6 1]
 [6 2 5 1 4 0 3]
 [2 6 3 0 4 1 5]
 [2 4 6 1 3 5 0]
 [4 0 3 6 2 5 1]
 [6 3 0 4 1 5 2]
 [1 3 5 0 2 4 6],
 [2 5 1 4 0 3 6],
 [3 0 4 1 5 2 6]]

sage: designs.mutually_orthogonal_latin_squares(2,5,partitions=True)
[[[0, 1, 2, 3, 4],
  [5, 6, 7, 8, 9],
  [10, 11, 12, 13, 14],
  [15, 16, 17, 18, 19],
  [20, 21, 22, 23, 24]],
 [[0, 5, 10, 15, 20],
  [1, 6, 11, 16, 21],
  [2, 7, 12, 17, 22],
  [3, 8, 13, 18, 23],
  [4, 9, 14, 19, 24]],
 [[0, 8, 11, 19, 22],
  [3, 6, 14, 17, 20],
  [1, 9, 12, 15, 23],
  [4, 7, 10, 18, 21],
  [2, 5, 13, 16, 24]],
 [[0, 9, 13, 17, 21],
  [2, 6, 10, 19, 23],
  [4, 8, 12, 16, 20],
  [1, 5, 14, 18, 22],
  [3, 7, 11, 15, 24]]]

What is the maximum number of MOLS of size 8 that Sage knows how to build?

sage: designs.orthogonal_arrays.largest_available_k(8)-2
7

If you only want to know if Sage is able to build a given set of MOLS, query the orthogonal_arrays.* functions:

sage: designs.orthogonal_arrays.is_available(5+2, 5) # 5 MOLS of order 5 False
sage: designs.orthogonal_arrays.is_available(4+2, 6) # 4 MOLS of order 6 False

Sage, however, is not able to prove that the second MOLS do not exist:

sage: designs.orthogonal_arrays.exists(4+2, 6) # 4 MOLS of order 6
Unknown

If you ask for such a MOLS then you will respectively get an informative EmptySetError or
5.1.90 Orthogonal arrays (OA)

This module gathers some construction related to orthogonal arrays (or transversal designs). One can build an $OA(k,n)$ (or check that it can be built) from the Sage console with `designs.orthogonal_arrays.build`:

```python
sage: OA = designs.orthogonal_arrays.build(4,8)
```

See also the modules `orthogonal_arrays_build_recursive` or `orthogonal_arrays_find_recursive` for recursive constructions.

This module defines the following functions:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>orthogonal_array()</code></td>
<td>Return an orthogonal array of parameters $k, n, t$.</td>
</tr>
<tr>
<td><code>transversal_design()</code></td>
<td>Return a transversal design of parameters $k, n$.</td>
</tr>
<tr>
<td><code>incomplete_orthogonal_array()</code></td>
<td>Return an $OA(k,n) - \sum_{1 \leq i \leq x} OA(k,s_i)$.</td>
</tr>
<tr>
<td><code>is_transversal_design()</code></td>
<td>Check that a given set of blocks $B$ is a transversal design.</td>
</tr>
<tr>
<td><code>is_orthogonal_array()</code></td>
<td>Check that the integer matrix $OA$ is an $OA(k,n,t)$.</td>
</tr>
<tr>
<td><code>wilson_construction()</code></td>
<td>Return a $OA(k,rm+u)$ from a truncated $OA(k+s,r)$ by Wilson’s construction.</td>
</tr>
<tr>
<td><code>TD_product()</code></td>
<td>Return the product of two transversal designs.</td>
</tr>
<tr>
<td><code>OA_find_disjoint_blocks()</code></td>
<td>Return $x$ disjoint blocks contained in a given $OA(k,n)$.</td>
</tr>
<tr>
<td><code>OA_relabel()</code></td>
<td>Return a relabelled version of the OA.</td>
</tr>
<tr>
<td><code>OA_from_quasi_difference_matrix()</code></td>
<td>Return an Orthogonal Array from a Quasi-Difference matrix</td>
</tr>
<tr>
<td><code>OA_from_Vmt()</code></td>
<td>Return an Orthogonal Array from a $V(m,t)$</td>
</tr>
<tr>
<td><code>OA_from_PBD()</code></td>
<td>Return an $OA(k,n)$ from a PBD</td>
</tr>
<tr>
<td><code>OA_n_times_2_pow_c_from_matrix()</code></td>
<td>Return an $OA(k,</td>
</tr>
<tr>
<td><code>OA_from_wider_OA()</code></td>
<td>Return the first $k$ columns of $OA$.</td>
</tr>
<tr>
<td><code>QDM_from_Vmt()</code></td>
<td>Return a QDM a $V(m,t)$</td>
</tr>
</tbody>
</table>

REFERENCES:

– [CD1996]
Functions

class sage.combinat.designs.orthogonal_arrays.OAMainFunctions(*args, **kwds)
    Bases: object

    Functions related to orthogonal arrays.

    An orthogonal array of parameters $k, n, t$ is a matrix with $k$ columns filled with integers from $[n]$ in such a way that for any $t$ columns, each of the $n^t$ possible rows occurs exactly once. In particular, the matrix has $n^t$ rows.

    For more information on orthogonal arrays, see Wikipedia article Orthogonal_array.

    From here you have access to:

    • build($k, n, t=2$): return an orthogonal array with the given parameters.
    • is_available($k, n, t=2$): answer whether there is a construction available in Sage for a given set of parameters.
    • exists($k, n, t=2$): answer whether an orthogonal array with these parameters exist.
    • largest_available_k($n, t=2$): return the largest integer $k$ such that Sage knows how to build an $OA(k,n)$.
    • explain_construction($k, n, t=2$): return a string that explains the construction that Sage uses to build an $OA(k,n)$.

    EXAMPLES:

    sage: designs.orthogonal_arrays.build(3,2)
[[0, 0, 0], [0, 1, 1], [1, 0, 1], [1, 1, 0]]

    sage: designs.orthogonal_arrays.build(5,5)
[[0, 0, 0, 0, 0], [0, 1, 2, 3, 4], [0, 2, 4, 1, 3], [0, 3, 1, 4, 2], [0, 4, 3, 2, 1], [1, 0, 4, 3, 2], [1, 1, 1, 1, 1], [1, 2, 3, 4, 0], [1, 3, 0, 2, 4], [1, 4, 2, 0, 3], [2, 0, 3, 1, 4], [2, 1, 0, 4, 3], [2, 2, 2, 2, 2], [2, 3, 4, 0, 1], [2, 4, 1, 3, 0], [3, 0, 2, 4, 1], [3, 1, 4, 2, 0], [3, 2, 1, 0, 4], [3, 3, 3, 3, 3], [3, 4, 0, 1, 2], [4, 0, 1, 2, 3], [4, 1, 3, 0, 2], [4, 2, 0, 3, 1], [4, 3, 2, 1, 0], [4, 4, 4, 4, 4]]

    What is the largest value of $k$ for which Sage knows how to compute a $OA(k,14,2)$?:

    sage: designs.orthogonal_arrays.largest_available_k(14)
6

    If you ask for an orthogonal array that does not exist, then you either obtain an EmptySetError (if it knows that such an orthogonal array does not exist) or a NotImplementedError:

    sage: designs.orthogonal_arrays.build(4,2)
Traceback (most recent call last):
  ... EmptySetError: There exists no OA(4,2) as $k(=4)>n+t-1=3$

    sage: designs.orthogonal_arrays.build(12,20)
Traceback (most recent call last):
  ... NotImplementedError: I don't know how to build an OA(12,20)!
static build$(k, n, t=2, \text{resolvable}=\text{False})$

Return an $OA(k,n)$ of strength $t$

An orthogonal array of parameters $k, n, t$ is a matrix with $k$ columns filled with integers from $[n]$ in such a way that for any $t$ columns, each of the $n^t$ possible rows occurs exactly once. In particular, the matrix has $n^t$ rows.

More general definitions sometimes involve a $\lambda$ parameter, and we assume here that $\lambda = 1$.

For more information on orthogonal arrays, see Wikipedia article Orthogonal_array.

INPUT:

- $k, n, t$ (integers) – parameters of the orthogonal array.
- $\text{resolvable}$ (boolean) – set to True if you want the design to be resolvable. The $n$ classes of the resolvable design are obtained as the first $n$ blocks, then the next $n$ blocks, etc... Set to False by default.

EXAMPLES:

```
sage: designs.orthogonal_arrays.build(3,3,\text{resolvable}=\text{True}) \ # \text{indirect doctest}
[[0, 0, 0],
 [1, 2, 1],
 [2, 1, 2],
 [0, 2, 2],
 [1, 1, 0],
 [2, 0, 1],
 [0, 1, 1],
 [1, 0, 2],
 [2, 2, 0]]
sage: OA_7_50 = designs.orthogonal_arrays.build(7,50) \ # \text{indirect doctest}
```

static exists$(k, n, t=2)$

Return the existence status of an $OA(k,n)$

INPUT:

- $k, n, t$ (integers) – parameters of the orthogonal array.

**Warning:** The function does not only return booleans, but True, False, or Unknown.

See also:

- is_available()

EXAMPLES:

```
sage: designs.orthogonal_arrays.exists(3,6) \ # \text{indirect doctest}
True
sage: designs.orthogonal_arrays.exists(4,6) \ # \text{indirect doctest}
Unknown
sage: designs.orthogonal_arrays.exists(7,6) \ # \text{indirect doctest}
False
```

static explain_construction$(k, n, t=2)$

Return a string describing how to build an $OA(k,n)$
INPUT:

• \( k,n,t \) (integers) – parameters of the orthogonal array.

EXAMPLES:

```
sage: designs.orthogonal_arrays.explain_construction(9,565)
"Wilson's construction n=23.24+13 with master design OA(9+1,23)"
sage: designs.orthogonal_arrays.explain_construction(10,154)
'the database contains a (137,10;1,0;17)-quasi difference matrix'
```

static `is_available`(\( k, n, t = 2 \))

Return whether Sage can build an \( OA(k,n) \).

INPUT:

• \( k,n,t \) (integers) – parameters of the orthogonal array.

See also:

`exists()`

EXAMPLES:

```
sage: designs.orthogonal_arrays.is_available(3,6) # indirect doctest
True
sage: designs.orthogonal_arrays.is_available(4,6) # indirect doctest
False
```

static `largest_available_k`(\( n, t = 2 \))

Return the largest \( k \) such that Sage can build an \( OA(k,n) \).

INPUT:

• \( n \) (integer)
• \( t \) – (integer; default: 2) – strength of the array

EXAMPLES:

```
sage: designs.orthogonal_arrays.largest_available_k(0) +Infinity
sage: designs.orthogonal_arrays.largest_available_k(1) +Infinity
sage: designs.orthogonal_arrays.largest_available_k(10) 4
sage: designs.orthogonal_arrays.largest_available_k(27) 28
sage: designs.orthogonal_arrays.largest_available_k(100) 10
sage: designs.orthogonal_arrays.largest_available_k(-1)
Traceback (most recent call last):
... ValueError: n(=-1) was expected to be >=0
```

```
sage.combinat.designs.orthogonal_arrays.OA_find_disjoint_blocks(OA, k, n, x, solver, integrality_tolerance)
```

Return \( x \) disjoint blocks contained in a given \( OA(k,n) \).
$x$ blocks of an OA are said to be disjoint if they all have different values for a every given index, i.e. if they correspond to disjoint blocks in the $TD$ associated with the OA.

**INPUT:**
- OA – an orthogonal array
- $k$, $n$, $x$ (integers)
- solver – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method solve of the class MixedIntegerLinearProgram.
- integrality_tolerance – parameter for use with MILP solvers over an inexact base ring; see MixedIntegerLinearProgram.get_values.

**See also:**
incomplete_orthogonal_array()

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays import OA_find_disjoint_blocks
sage: k=3; n=4; x=3
sage: Bs = OA_find_disjoint_blocks(designs.orthogonal_arrays.build(k,n),k,n,x)
sage: assert len(Bs) == x
sage: for i in range(k):
    ....:     assert len(set([B[i] for B in Bs])) == x
sage: OA_find_disjoint_blocks(designs.orthogonal_arrays.build(k,n),k,n,5)
Traceback (most recent call last):
 ... ValueError: There does not exist 5 disjoint blocks in this OA(3,4)
```

sage.combinat.designs.orthogonal_arrays.OA_from_PBD($k$, $n$, $PBD$, check=True)

Return an OA($k$, $n$) from a PBD

**Construction**

Let $B$ be a ($n$, $K$, 1)-PBD. If there exists for every $i \in K$ a $TD(k,i) - i \times TD(k,1)$ (i.e. if there exist $k$ idempotent MOLS), then one can obtain a OA($k$, $n$) by concatenating:

- A $TD(k,i) - i \times TD(k,1)$ defined over the elements of $B$ for every $B \in B$.
- The rows $(i, ..., i)$ of length $k$ for every $i \in [n]$.

**Note:** This function raises an exception when Sage is unable to build the necessary designs.

**INPUT:**
- $k$, $n$ (integers)
- PBD – a PBD on $0, ..., n - 1$.

**EXAMPLES:**

We start from the example VI.1.2 from the [DesignHandbook] to build an OA(3, 10):

```python
sage: from sage.combinat.designs.orthogonal_arrays import OA_from_PBD
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: pbd = [\[0,1,2,3\],\[0,4,5,6\],\[0,7,8,9\],\[1,4,7\],\[1,5,8\],
```
But we cannot build an $OA(4, 10)$ for this PBD (although there exists an $OA(4, 10)$):

```python
sage: OA_from_PBD(4,10,pbd)
Traceback (most recent call last):
... EmptySetError: There is no OA(n+1,n) - 3.OA(n+1,1) as all blocks intersect in a_→ projective plane.
```

Or an $OA(3, 6)$ (as the PBD has 10 points):

```python
sage: _ = OA_from_PBD(3,6,pbd)
Traceback (most recent call last):
... Runtime Error: PBD is not a valid Pairwise Balanced Design on [0,...,5]
```

sage.combinat.designs.orthogonal_arrays.OA_from_Vmt($m$, $t$, $V$)

Return an Orthogonal Array from a $V(m, t)$

INPUT:

- $m$, $t$ (integers)
- $V$ – the vector $V(m, t)$.

See also:

- $QDM_from_Vmt()$
- $OA_from_quasi_difference_matrix()$

EXAMPLES:

```python
sage: _ = designs.orthogonal_arrays.build(6,46) # indirect doctest
```

sage.combinat.designs.orthogonal_arrays.OA_from_quasi_difference_matrix($M$, $G$, $add\_col=True$, $fill\_hole=True$)

Return an Orthogonal Array from a Quasi-Difference matrix

### Difference Matrices

Let $G$ be a group of order $g$. A difference matrix $M$ is a $g \times k$ matrix with entries from $G$ such that for any $1 \leq i < j < k$ the set $\{d_{ii} - d_{ij} : 1 \leq l \leq g\}$ is equal to $G$.

By concatenating the $g$ matrices $M + x$ (where $x \in G$), one obtains a matrix of size $g^2 \times x$ which is also an $OA(k, g)$.

### Quasi-difference Matrices

A quasi-difference matrix is a difference matrix with missing entries. The construction above can be applied again in this case, where the missing entries in each column of $M$ are replaced by unique values on which $G$ has a trivial action.
This produces an incomplete orthogonal array with a “hole” (i.e. missing rows) of size ‘u’ (i.e. the number of missing values per column of M). If there exists an OA(k, u), then adding the rows of this OA(k, u) to the incomplete orthogonal array should lead to an OA…

**Formal definition** (from the Handbook of Combinatorial Designs [DesignHandbook])

Let G be an abelian group of order n. A \((n, k; \lambda, \mu; u)\)-quasi-difference matrix (QDM) is a matrix \(Q = (q_{ij})\) with \(\lambda(n - 1 + 2u) + \mu\) rows and \(k\) columns, with each entry either empty or containing an element of \(G\). Each column contains exactly \(\lambda u\) entries, and each row contains at most one empty entry. Furthermore, for each \(1 \leq i < j \leq k\) the multiset

\[\{q_{li} - q_{lj} : 1 \leq l \leq \lambda(n - 1 + 2u) + \mu, \text{ with } q_{li} \text{ and } q_{lj} \text{ not empty}\}\]

contains every nonzero element of \(G\) exactly \(\lambda\) times, and contains 0 exactly \(\mu\) times.

**Construction**

If a \((n, k; \lambda, \mu; u)\)-QDM exists and \(\mu \leq \lambda\), then an \(ITD_\lambda(k, n + u; u)\) exists. Start with a \((n, k; \lambda, \mu; u)\)-QDM \(A\) over the group \(G\). Append \(\lambda - \mu\) rows of zeroes. Then select \(u\) elements \(\infty_1, \ldots, \infty_u\) not in \(G\), and replace the empty entries, each by one of these infinite symbols, so that \(\infty_i\) appears exactly once in each column. Develop the resulting matrix over the group \(G\) (leaving infinite symbols fixed), to obtain an \(\lambda(n^2 + 2nu) \times k\) matrix \(T\). Then \(T\) is an orthogonal array with \(k\) columns and index \(\lambda\), having \(n + u\) symbols and one hole of size \(u\).

Adding to \(T\) an OA(k, u) with elements \(\infty_1, \ldots, \infty_u\) yields the \(ITD_\lambda(k, n + u; u)\).


**INPUT:**

- \(M\) – the difference matrix whose entries belong to \(G\)
- \(G\) – a group
- \add_col (boolean) – whether to add a column to the final OA equal to \((x_1, \ldots, x_g, x_1, \ldots, x_g, \ldots)\) where \(G = \{x_1, \ldots, x_g\}\).
- \fill_hole (boolean) – whether to return the incomplete orthogonal array, or complete it with the OA(k, u) (default). When \fill_hole is None, no block of the incomplete OA contains more than one value \(\geq |G|\).

**EXAMPLES:**

```python
sage: _ = designs.orthogonal_arrays.build(6,20) # indirect doctest
sage.combinat.designs.orthogonal_arrays.OA_from_wider_OA(OA, k)
```

Return the first \(k\) columns of OA.

If OA has \(k\) columns, this function returns OA immediately.

**INPUT:**

- OA – an orthogonal array.
- k (integer)

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays import OA_from_wider_OA
sage: OA_from_wider_OA(designs.orthogonal_arrays.build(6,20),2)[0]
[(19,), (19,), (19,), (19,), (19,)]
sage: _ = designs.orthogonal_arrays.build(5,46) # indirect doctest
```
Return an $OA(k, |G| \cdot 2^c)$ from a constrained $(G, k - 1, 2)$-difference matrix.

This construction appears in [AC1994] and [Ab1995].

Let $G$ be an additive Abelian group. We denote by $H$ a $GF(2)$-hyperplane in $GF(2^c)$. Let $A$ be a $(k - 1) \times 2|G|$ array with entries in $G \times GF(2^c)$ and $Y$ be a vector with $k - 1$ entries in $GF(2^c)$. Let $B$ and $C$ be respectively the part of the array that belong to $G$ and $GF(2^c)$.

The input $A$ and $Y$ must satisfy the following conditions. For any $i \neq j$ and $g \in G$:

- there are exactly two values of $s$ such that $B_i,s - B_j,s = g$ (i.e. $B$ is a $(G, k - 1, 2)$-difference matrix),
- let $s_1$ and $s_2$ denote the two values of $s$ given above, then exactly one of $C_i,s_1 - C_j,s_1$ and $C_i,s_2 - C_j,s_2$ belongs to the $GF(2)$-hyperplane $(Y_i - Y_j) \cdot H$ (we implicitly assumed that $Y_i \neq Y_j$).

Under these conditions, it is easy to check that the array whose $k - 1$ rows of length $|G| \cdot 2^c$ indexed by $1 \leq i \leq k - 1$ given by $A_{i,s} + (0, Y_i \cdot v)$ where $1 \leq s \leq 2|G|, v \in H$ is a $(G \times GF(2^c), k - 1, 1)$-difference matrix.

INPUT:

- $k, c$ (integers) – integers
- $G$ – an additive Abelian group
- $A$ – a matrix with entries in $G \times GF(2^c)$
- $Y$ – a vector with entries in $GF(2^c)$
- $check$ – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

Note: By convention, a multiplicative generator $w$ of $GF(2^c)^*$ is fixed (inside the function). The hyperplane $H$ is the one spanned by $w^0, w^1, \ldots, w^{c-1}$. The $GF(2^c)$ part of the input matrix $A$ and vector $Y$ are given in the following form: the integer $i$ corresponds to the element $w^i$ and None corresponds to 0.

See also:

Several examples use this construction:

- $OA\_9\_40()$
- $OA\_11\_80()$
- $OA\_15\_112()$
- $OA\_11\_160()$
- $OA\_16\_176()$
- $OA\_16\_208()$
- $OA\_15\_224()$
- $OA\_20\_352()$
- $OA\_20\_416()$
- $OA\_20\_544()$
- $OA\_11\_640()$
- $OA\_15\_896()$
EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays import OA_n_times_2_pow_c_from_matrix
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: A = [...]
   ...: [(0,None), (0,None), (0,None), (0,None), (0,None), (0,None), (0,None),
   ...: (0,None)],
   ...: [(0,None), (1, None), (2,2), (3,2), (4,2), (2,None), (3,None), (4,None),
   ...: (0,2), (1,2)],
   ...: [(0, None), (2,5), (4,5), (1,2), (3,6), (3,4), (0,0), (2,1),
   ...: (4,1), (1,6)],
   ...: [(0,None), (3,4), (1,4), (4,0), (2,5), (3,None), (1,0), (4,1),
   ...: (2,2), (0,3)],
   ...: ]
sage: Y = [None, 0, 1, 6]
sage: OA = OA_n_times_2_pow_c_from_matrix(5,3,GF(5),A,Y)
sage: is_orthogonal_array(OA,5,40,2)
True
sage: A[0][0] = (1,None)
sage: OA_n_times_2_pow_c_from_matrix(5,3,GF(5),A,Y)
Traceback (most recent call last):
  ...
ValueError: the first part of the matrix A must be a (G,k-1,2)-difference matrix
sage: A[0][0] = (0,0)
sage: OA_n_times_2_pow_c_from_matrix(5,3,GF(5),A,Y)
Traceback (most recent call last):
  ...
ValueError: B_2,0 - B_0,0 = B_2,6 - B_0,6 but the associated part of the
matrix C does not satisfies the required condition
```

```python
sage.combinat.designs.orthogonal_arrays.OA_relabel(OA, k, n, blocks=( ), matrix=None)
```

Return a relabelled version of the OA.

**INPUT:**

- `OA` – an OA, or rather a list of blocks of length `k`, each of which contains integers from 0 to `n - 1`.
- `k, n` (integers)
- `blocks` (list of blocks) – relabels the integers of the OA from `[0..n-1]` into `[0..n-1]` in such a way that the `i` blocks from `block` are respectively relabeled as `[n-i,...,n-i], ... [n-1,...,n-1]`. Thus, the blocks from this list are expected to have disjoint values for each coordinate.
  
  If set to the empty list (default) no such relabelling is performed.
- `matrix` – a matrix of dimensions `k, n` such that if the `i`th coordinate of a block is `x`, this `x` will be relabelled with `matrix[i][x]`. This is not necessarily an integer between 0 and `n - 1`, and it is not necessarily an integer either. This is performed after the previous relabelling.
  
  If set to `None` (default) no such relabelling is performed.

**Note:** A `None` coordinate in one block remains a `None` coordinate in the final block.
EXAMPLES:

```
sage: from sage.combinat.designs.orthogonal_arrays import OA_relabel
sage: OA = designs.orthogonal_arrays.build(3,2)
sage: OA_relabel(OA,3,2,matrix=[['A','B'], ['C','D'], ['E','F']])
sage: TD = OA_relabel(OA,3,2,matrix=[[0,1],[2,3],[4,5]]); TD
[[0, 2, 4], [0, 3, 5], [1, 2, 5], [1, 3, 4]]
sage: from sage.combinat.designs.orthogonal_arrays import is_transversal_design
sage: is_transversal_design(TD,3,2)
True
```

Makingsurethat [2,2,2,2] isablockof $OA(4,3)$. Wedo this by relabelling block [0,0,0,0] which belongs to the design:

```
sage: designs.orthogonal_arrays.build(4,3)
[[0, 0, 0, 0], [0, 1, 2, 1], [0, 2, 1, 2], [1, 0, 2, 2], [1, 1, 1, 0], [1, 2, 0, 1],
 → [2, 0, 1, 1], [2, 1, 0, 2], [2, 2, 2, 0]]
sage: OA_relabel(designs.orthogonal_arrays.build(4,3),4,3,blocks=[[0,0,0,0]])
[[2, 2, 2, 2], [2, 0, 1, 0], [2, 1, 0, 1], [0, 2, 1, 1], [0, 0, 0, 2], [0, 1, 2, 0],
 → [1, 2, 0, 0], [1, 0, 2, 1], [1, 1, 1, 2]]
```

sage.combinat.designs.orthogonal_arrays.QDM_from_Vmt($m$, $t$, $V$)

Return a QDM from a $V(m,t)$

**Definition**

Let $q$ be a prime power and let $q = mt + 1$ for $m, t$ integers. Let $\omega$ be a primitive element of $\mathbb{F}_q$. A $V(m,t)$ vector is a vector $(a_1,\ldots,a_{m+1})$ for which, for each $1 \leq k < m$, the differences

\[
\{a_{i+k} - a_i : 1 \leq i \leq m+1, i + k \neq m + 2\}
\]

represent the $m$ cyclotomic classes of $\mathbb{F}_{mt+1}$ (compute subscripts modulo $m + 2$). In other words, for fixed $k$, is $a_{i+k} - a_i = \omega^{mx+a}$ and $a_{j+k} - a_j = \omega^{my+b}$ then $a \not\equiv b \mod m$

**Construction of a quasi-difference matrix from a $V(m,t)$ vector**

Starting with a $V(m,t)$ vector $(a_1,\ldots,a_{m+1})$, form a single row of length $m + 2$ whose first entry is empty, and whose remaining entries are $(a_1,\ldots,a_{m+1})$. Form $t$ rows by multiplying this row by the $t$ th roots, i.e. the powers of $\omega^m$. From each of these $t$ rows, form $m + 2$ rows by taking the $m + 2$ cyclic shifts of the row. The result is a $(a,m+2;1,0;t) - QDM$.

For more information, refer to the Handbook of Combinatorial Designs [DesignHandbook].

**INPUT:**

- $m, t$ (integers)
- $V$ - the vector $V(m,t)$.

**See also:**

`OA_from_quasi_difference_matrix()`

**EXAMPLES:**

```
sage: _ = designs.orthogonal_arrays.build(6,46) # indirect doctest
```

Chapter 5. Comprehensive Module List
Return the product of two transversal designs.

From a transversal design \(TD_1\) of parameters \(k, n_1\) and a transversal design \(TD_2\) of parameters \(k, n_2\), this function returns a transversal design of parameters \(k, n\) where \(n = n_1 \times n_2\).

Formally, if the groups of \(TD_1\) are \(V_1^1, \ldots, V_k^1\) and the groups of \(TD_2\) are \(V_1^2, \ldots, V_k^2\), the groups of the product design are \(V_1^1 \times V_1^2, \ldots, V_k^1 \times V_k^2\) and its blocks are the \(\{(x_1^1, x_1^2), \ldots, (x_k^1, x_k^2)\}\) where \(\{x_1^1, \ldots, x_k^1\}\) is a block of \(TD_1\) and \(\{x_1^2, \ldots, x_k^2\}\) is a block of \(TD_2\).

**INPUT:**
- \(TD_1, \ TD_2\) – transversal designs.
- \(k, n_1, n_2\) (integers) – see above.
- check (boolean) – Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

**Note:** This function uses transversal designs with \(V_1 = \{0, \ldots, n-1\}, \ldots, V_k = \{(k-1)n, \ldots, kn-1\}\) both as input and output.

**EXAMPLES:**
```python
sage: from sage.combinat.designs.orthogonal_arrays import TD_product
sage: TD1 = designs.transversal_design(6,7)
sage: TD2 = designs.transversal_design(6,12)
sage: TD6_84 = TD_product(6,TD1,7,TD2,12)
```

### Class sage.combinat.designs.orthogonal_arrays.TransversalDesign

**Bases:** GroupDivisibleDesign

Class for Transversal Designs

**INPUT:**
- blocks – collection of blocks
- \(k, n\) (integers) – parameters of the transversal design. They can be set to None (default) in which case their value is determined by the blocks.
- check (boolean) – whether to check that the design is indeed a transversal design with the right parameters. Set to True by default.

**EXAMPLES:**
```python
sage: designs.transversal_design(None,5)
Transversal Design TD(6,5)
sage: designs.transversal_design(None,30)
Transversal Design TD(6,30)
sage: designs.transversal_design(None,36)
Transversal Design TD(10,36)
```

### sage.combinat.designs.orthogonal_arrays.incomplete_orthogonal_array

**INPUT:**
- \(k, n, \text{holes}, \text{resolvable}=False, \text{existence}=False\)
Return an $OA(k, n) - \sum_{1 \leq i \leq x} OA(k, s_i)$.

An $OA(k, n) - \sum_{1 \leq i \leq x} OA(k, s_i)$ is an orthogonal array from which have been removed disjoint $OA(k, s_1), ..., OA(k, s_x)$. If there exist $OA(k, s_1), ..., OA(k, s_x)$ they can be used to fill the holes and give rise to an $OA(k, n)$.

A very useful particular case (see e.g. the Wilson construction in `wilson_construction()`) is when all $s_i = 1$. In that case the incomplete design is a $OA(k, n) - x OA(k, 1)$. Such design is equivalent to transversal design $TD(k, n)$ from which has been removed $x$ disjoint blocks.

**INPUT:**

- $k, n$ (integers)
- `holes` (list of integers) – respective sizes of the holes to be found.
- `resolvable` (boolean) – set to `True` if you want the design to be resolvable. The classes of the resolvable design are obtained as the first $n$ blocks, then the next $n$ blocks, etc ... Set to `False` by default.
- `existence` (boolean) – instead of building the design, return:
  - `True` – meaning that Sage knows how to build the design
  - `Unknown` – meaning that Sage does not know how to build the design, but that the design may exist (see `sage.misc.unknown`).
  - `False` – meaning that the design does not exist.

**Note:** By convention, the ground set is always $V = \{0, ..., n - 1\}$.

If all holes have size 1, in the incomplete orthogonal array returned by this function the holes are $\{n - 1, ..., n - s_1\}^k, \{n - s_1 - 1, ..., n - s_1 - s_2\}^k$, etc.

More generally, if `holes` is equal to $u_1, ..., u_k$, the $i$-th hole is the set of points $\{n - \sum_{j \geq i} u_j, ..., n - \sum_{j \geq i+1} u_j\}^k$.

**See also:**

`OA_find_disjoint_blocks()`

**EXAMPLES:**

```python
sage: IOA = designs.incomplete_orthogonal_array(3,3,[1,1,1])
sage: IOA
[[0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]]
sage: missing_blocks = [[0,0,0],[1,1,1],[2,2,2]]
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: is_orthogonal_array(IOA + missing_blocks,3,3,2)
True
```

`sage.combinat.designs.orthogonal_arrays.is_transversal_design(B, k, n, verbose=False)`

Check that a given set of blocks $B$ is a transversal design.

See `transversal_design()` for a definition.

**INPUT:**

- `B` – the list of blocks
- `k`, `n` – integers
• **verbose** (boolean) – whether to display information about what is going wrong.

**Note:** The transversal design must have \{0, \ldots, kn - 1\} as a ground set, partitioned as \(k\) sets of size \(n\): \(\{0, \ldots, k - 1\} \sqcup \{k, \ldots, 2k - 1\} \sqcup \cdots \sqcup \{k(n - 1), \ldots, kn - 1\}\).

**EXAMPLES:**

```python
sage: TD = designs.transversal_design(5, 5, check=True) # indirect doctest
sage: from sage.combinat.designs.orthogonal_arrays import is_transversal_design
sage: is_transversal_design(TD, 5, 5)
True
sage: is_transversal_design(TD, 4, 4)
False
```

sage.combinat.designs.orthogonal_arrays.largest_available_k\((n, t=2)\)

Return the largest \(k\) such that Sage can build an \(OA(k, n)\).

**INPUT:**

- \(n\) (integer)
- \(t\) – (integer; default: 2) – strength of the array

**EXAMPLES:**

```python
sage: designs.orthogonal_arrays.largest_available_k(0)
+Infinity
sage: designs.orthogonal_arrays.largest_available_k(1)
+Infinity
sage: designs.orthogonal_arrays.largest_available_k(10)
4
sage: designs.orthogonal_arrays.largest_available_k(27)
28
sage: designs.orthogonal_arrays.largest_available_k(100)
10
sage: designs.orthogonal_arrays.largest_available_k(-1)
Traceback (most recent call last):
  ... ValueError: n(=-1) was expected to be >=0
```

sage.combinat.designs.orthogonal_arrays.orthogonal_array\((k, n, t=2, resolvable=False, check=True, existence=False, explain_construction=False)\)

Return an orthogonal array of parameters \(k, n, t\).

An orthogonal array of parameters \(k, n, t\) is a matrix with \(k\) columns filled with integers from \([n]\) in such a way that for any \(t\) columns, each of the \(n^t\) possible rows occurs exactly once. In particular, the matrix has \(n^t\) rows.

More general definitions sometimes involve a \(\lambda\) parameter, and we assume here that \(\lambda = 1\).

An orthogonal array is said to be **resolvable** if it corresponds to a resolvable transversal design (see `sage.combinat.designs.incidence_structures.IncidenceStructure.is_resolvable()`).

For more information on orthogonal arrays, see Wikipedia article **Orthogonal_array**.

**INPUT:**

- \(k\) – (integer) number of columns. If \(k=None\) it is set to the largest value available.
Combinatorics, Release 10.0

- **n** – (integer) number of symbols
- **t** – (integer; default: 2) – strength of the array
- **resolvable** (boolean) – set to True if you want the design to be resolvable. The $n$ classes of the resolvable design are obtained as the first $n$ blocks, then the next $n$ blocks, etc ... Set to False by default.
- **check** – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.
- **existence** (boolean) – instead of building the design, return:
  - True – meaning that Sage knows how to build the design
  - Unknown – meaning that Sage does not know how to build the design, but that the design may exist (see sage.misc.unknown).
  - False – meaning that the design does not exist.

**Note:** When k=None and existence=True the function returns an integer, i.e. the largest $k$ such that we can build a $OA(k, n)$.

- **explain_construction** (boolean) – return a string describing the construction.

**OUTPUT:**
The kind of output depends on the input:
- if existence=False (the default) then the output is a list of lists that represent an orthogonal array with parameters $k$ and $n$
- if existence=True and $k$ is an integer, then the function returns a troolean: either True, Unknown or False
- if existence=True and k=None then the output is the largest value of $k$ for which Sage knows how to compute a $TD(k, n)$.

**Note:** This method implements theorems from [Stinson2004]. See the code’s documentation for details.

**See also:**
When $t = 2$ an orthogonal array is also a transversal design (see transversal_design()) and a family of mutually orthogonal latin squares (see mutually_orthogonal_latin_squares()).

sage.combinat.designs.orthogonal_arrays.transversal_design($k, n,$ resolvable=False, check=True, existence=False)

Return a transversal design of parameters $k, n$.

A transversal design of parameters $k, n$ is a collection $\mathcal{S}$ of subsets of $V = V_1 \cup \cdots \cup V_k$ (where the groups $V_i$ are disjoint and have cardinality $n$) such that:
- Any $S \in \mathcal{S}$ has cardinality $k$ and intersects each group on exactly one element.
- Any two elements from distinct groups are contained in exactly one element of $\mathcal{S}$.

More general definitions sometimes involve a $\lambda$ parameter, and we assume here that $\lambda = 1$.

For more information on transversal designs, see http://mathworld.wolfram.com/TransversalDesign.html.
• $n, k$ – integers. If $k$ is None it is set to the largest value available.

• **resolvable** (boolean) – set to True if you want the design to be resolvable (see `sage.combinat.designs.incidence_structures.IncidenceStructure.is_resolvable()`). The $n$ classes of the resolvable design are obtained as the first $n$ blocks, then the next $n$ blocks, etc... Set to False by default.

• **check** – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

• **existence** (boolean) – instead of building the design, return:
  - True – meaning that Sage knows how to build the design
  - Unknown – meaning that Sage does not know how to build the design, but that the design may exist (see `sage.misc.unknown`).
  - False – meaning that the design does not exist.

**Note:** When $k=$None and **existence**=$True$ the function returns an integer, i.e. the largest $k$ such that we can build a $TD(k, n)$.

**OUTPUT:**

The kind of output depends on the input:

• if **existence**=$False$ (the default) then the output is a list of lists that represent a $TD(k, n)$ with $V_1 = \{0, ..., n - 1\}, ..., V_k = \{(k - 1)n, ..., kn - 1\}$

• if **existence**=$True$ and $k$ is an integer, then the function returns a troolean: either True, Unknown or False

• if **existence**=$True$ and $k=$None then the output is the largest value of $k$ for which Sage knows how to compute a $TD(k, n)$.

**See also:**

`orthogonal_array()` – a transversal design $TD(k, n)$ is equivalent to an orthogonal array $OA(k, n, 2)$.

**EXAMPLES:**

```python
sage: TD = designs.transversal_design(5,5); TD
Transversal Design TD(5,5)
sage: TD.blocks()
[[0, 5, 10, 15, 20], [0, 6, 12, 18, 24], [0, 7, 14, 16, 23],
 [0, 8, 11, 19, 22], [0, 9, 13, 17, 21], [1, 5, 14, 18, 22],
 [1, 6, 11, 16, 21], [1, 7, 13, 19, 20], [1, 8, 10, 17, 24],
 [1, 9, 12, 15, 23], [2, 5, 13, 16, 24], [2, 6, 10, 19, 23],
 [2, 7, 12, 17, 22], [2, 8, 14, 15, 21], [2, 9, 11, 18, 20],
 [3, 5, 12, 19, 21], [3, 6, 14, 17, 20], [3, 7, 11, 15, 24],
 [3, 8, 13, 18, 23], [3, 9, 10, 16, 22], [4, 5, 11, 17, 23],
 [4, 6, 13, 15, 22], [4, 7, 10, 18, 21], [4, 8, 12, 16, 20],
 [4, 9, 14, 19, 24]]
```

Some examples of the maximal number of transversal Sage is able to build:

```python
sage: TD_4_10 = designs.transversal_design(4,10)
sage: designs.transversal_design(5,10,existence=True)
Unknown
```

5.1. Comprehensive Module List
For prime powers, there is an explicit construction which gives a $TD(n + 1, n)$:

```python
sage: designs.transversal_design(4, 3, existence=True)
True
sage: designs.transversal_design(674, 673, existence=True)
True
```

For other values of $n$ it depends:

```python
sage: designs.transversal_design(7, 6, existence=True)
False
sage: designs.transversal_design(4, 6, existence=True)
Unknown
sage: designs.transversal_design(3, 6, existence=True)
True
sage: designs.transversal_design(11, 10, existence=True)
False
sage: designs.transversal_design(4, 10, existence=True)
Unknown
sage: designs.transversal_design(5, 10, existence=True)
Unknown
sage: designs.transversal_design(7, 20, existence=True)
Unknown
sage: designs.transversal_design(6, 20, existence=True)
True
sage: designs.transversal_design(7, 20, existence=True)
Unknown
```

If you ask for a transversal design that Sage is not able to build then an `EmptySetError` or a `NotImplementedError` is raised:

```python
sage: designs.transversal_design(47, 100, existence=True)
Traceback (most recent call last):
  ...  
NotImplementedError: I don't know how to build a TD(47,100)!
```

Those two errors correspond respectively to the cases where Sage answer `Unknown` or `False` when the parameter `existence` is set to `True`:

```python
sage: designs.transversal_design(47, 100, existence=True)
Unknown
```

(continues on next page)
sage: designs.transversal_design(55, 54, existence=True)
False

If for a given $n$ you want to know the largest $k$ for which Sage is able to build a $TD(k, n)$ just call the function with $k$ set to `None` and `existence` set to `True` as follows:

sage: designs.transversal_design(None, 6, existence=True)
3
sage: designs.transversal_design(None, 20, existence=True)
6
sage: designs.transversal_design(None, 30, existence=True)
6
sage: designs.transversal_design(None, 120, existence=True)
9

`sage.combinat.designs.orthogonal_arrays.wilson_construction(OA, k, r, m, u, check=True, explain_construction=False)`

Returns a $OA(k, rm + \sum_i u_i)$ from a truncated $OA(k + s, r)$ by Wilson’s construction.

**Simple form:**

Let $OA$ be a truncated $OA(k + s, r)$ with $s$ truncated columns of sizes $u_1, ..., u_s$, whose blocks have sizes in \{\(k + b_1, ..., k + b_t\)\}. If there exist:

- An $OA(k, m + b_i) - b_i.OA(k, 1)$ for every $1 \leq i \leq t$
- An $OA(k, u_i)$ for every $1 \leq i \leq s$

Then there exists an $OA(k, rm + \sum_i u_i)$. The construction is a generalization of Lemma 3.16 in [HananiBIBD].

**Brouwer-Van Rees form:**

Let $OA$ be a truncated $OA(k + s, r)$ with $s$ truncated columns of sizes $u_1, ..., u_s$. Let the set $H_i$ of the $u_i$ points of column $k+i$ be partitioned into $\sum_j H_{ij}$. Let $m_{ij}$ be integers such that:

- For $0 \leq i < l$ there exists an $OA(k, \sum_j m_{ij}|H_{ij}|)$
- For any block $B \in OA$ intersecting the sets $H_{ij(i)}$ there exists an $OA(k, m + \sum_i m_{ij}) - \sum_i OA(k, m_{ij(j)})$.

Then there exists an $OA(k, rm + \sum_i,j m_{ij})$. This construction appears in [BvR1982].

**INPUT:**

- $OA$ – an incomplete orthogonal array with $k + s$ columns. The elements of a column of size $c$ must belong to \(\{0, ..., c\}\). The missing entries of a block are represented by `None` values. If $OA=None$, it is defined as a truncated orthogonal arrays with $k + s$ columns.
- $k, r, m$ (integers)
- $u$ (list) – two cases depending on the form to use:
  - Simple form: a list of length $s$ such that column $k+i$ has size $u[i]$. The untruncated points of column $k+i$ are assumed to be $[0, ..., u[i]-1]$.
  - Brouwer-Van Rees form: a list of length $s$ such that $u[i]$ is the list of pairs $(m_{0i}, |H_{ij0}|), ..., (m_{pi}, |H_{ijp}|)$. The untruncated points of column $k+i$ are assumed to be $[0, ..., u_i-1]$ where $u_i = \sum_j |H_{ij}|$. Besides, the first $|H_{i0}|$ points represent $H_{i0}$, the next $|H_{i1}|$ points represent $H_{i1}$, etc...
- `explain_construction` (boolean) – return a string describing the construction.
• **check** (boolean) – whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to **True** by default.

**REFERENCE:**

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays import wilson_construction
sage: from sage.combinat.designs.orthogonal_arrays import OA_relabel
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_wilson_decomposition_with_one_truncated_group
sage: total = 0
sage: for k in range(3,8):
    ...    for n in range(1,30):
    ...        if find_wilson_decomposition_with_one_truncated_group(k,n):
    ...            total += 1
    ...            f, args = find_wilson_decomposition_with_one_truncated_group(k,n)
    ...            _ = f(*args)

sage: total
41
sage: print(designs.orthogonal_arrays.explain_construction(7,58))
Wilson's construction n=8.7+1+1 with master design OA(7+2,8)
sage: print(designs.orthogonal_arrays.explain_construction(9,115))
Wilson's construction n=13.8+11 with master design OA(9+1,13)
sage: print(wilson_construction(None,5,11,21,[(5,5)],explain_construction=True))
Brouwer-van Rees construction n=11.21+(5.5) with master design OA(5+1,11)
sage: print(wilson_construction(None,71,17,21,[(4,9),(1,1)],[(9,9),(1,1)],explain_construction=True))
Brouwer-van Rees construction n=17.21+(9.4+1.1)+(9.9+1.1) with master design OA(71+2,17)
```

An example using the Brouwer-van Rees generalization:

```python
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: OA = designs.orthogonal_arrays.build(6,11)
sage: OA = [[x if (i<5 or x<5) else None for i,x in enumerate(R)] for R in OA]
sage: OA = wilson_construction(OA,5,11,21,[(5,5)],for R in OA]
sage: OA = wilson_construction(OA,5,11,21,[(5,5)])
sage: is_orthogonal_array(OA,5,256)
True
```

### 5.1.91 Orthogonal arrays (build recursive constructions)

This module implements several constructions of *Orthogonal Arrays*. As their input can be complex, they all have a counterpart in the *orthogonal_arrays_find_recursive* module that automatically computes it.

All these constructions are automatically queried when the *orthogonal_array()* function is called.
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**Functions**

```
sage.combinat.designs.orthogonal_arrays_build_recursive.OA_and_oval(q, solver, integrality_tolerance)
```

Return a $OA(q + 1, q)$ whose blocks contains $\leq 2$ zeroes in the last $q$ columns.

This $OA$ is build from a projective plane of order $q$, in which there exists an oval $O$ of size $q + 1$ (i.e. a set of $q + 1$ points no three of which are colinear/contained in a common set of the projective plane).

Removing an element $x \in O$ and all sets that contain it, we obtain a $TD(q + 1, q)$ in which $O$ intersects all columns except one. As $O$ is an oval, no block of the $TD$ intersects it more than twice.

**INPUT:**

- $q$ – a prime power
- $solver$ – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method `solve` of the class `MixedIntegerLinearProgram`.
- $integrality_tolerance$ – parameter for use with MILP solvers over an inexact base ring; see `MixedIntegerLinearProgram.get_values()`.

**Note:** This function is called by `construction_3_6()`, an implementation of Construction 3.6 from [AC07].

**EXAMPLES:**

```
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import OA_and_oval
sage: _ = OA_and_oval
```

```
sage.combinat.designs.orthogonal_arrays_build_recursive.brouwer_separable_design(k, t, q, x, check=False, verbose=False, explain_construction=False)
```

Returns a $OA(k, t(q^2 + q + 1) + x)$ using Brouwer’s result on separable designs.

This method is an implementation of Brouwer’s construction presented in [Brouwer80]. It consists in a systematic application of the usual transformation from PBD to OA, applied to a specific PBD.

**Baer subplanes**
When \( q \) is a prime power, the projective plane \( PG(2, q^2) \) can be partitioned into subplanes \( PG(2, q) \) (called Baer subplanes), giving \( PG(2, q^2) = B_1 \cup \cdots \cup B_{q^2-q+1} \). As a result, every line of the \( PG(2, q^2) \) intersects one of the subplane on \( q + 1 \) points and all others on 1 point.

The OA are built by considering \( B_1 \cup \cdots \cup B_{i} \), for a total of \( t(q^2 + q + 1) \) points (to which \( x \) new points are then added). The blocks of this subdesign belong to two categories:

- The blocks of size \( t \): they come from the lines which intersect a \( B_i \) on \( q + 1 \) points for some \( i > t \). The blocks of size \( t \) can be partitioned into \( q^2-q+t-1 \) parallel classes according to their associated subplane \( B_i \) with \( i > t \).
- The blocks of size \( q + t \): those blocks form a symmetric design, as every point is incident with \( q + t \) of them.

**Constructions**

In the following, we write \( N = t(q^2+q+1)+x \). The code is also heavily commented, and will clear any doubt.

- i) \( x = 0 \): in that case we build a resolvable \( OA(k-1,N) \) that will then be completed into an \( OA(k,N) \).
  - **Sets of size \( t \)**
    - We take the product of each parallel class with the parallel classes of a resolvable \( OA(k-1,t) - t.OA(k-1,1) \), yielding new parallel classes.
  - **Sets of size \( q + t \)**
    - A \( N \times (q + t) \) array is built whose rows are the sets of size \( q + t \) such that every value appears once per column. For each block of a \( OA(k-1, q + t) - (q + t).OA(k-1,1) \), the product with the rows of the matrix yields a parallel class.

- ii) \( x = q + t \)
  - **Sets of size \( t \)**
    - Each set of size \( t \) gives a \( OA(k,t) - t.OA(k,1) \), except if there is only one parallel class in which case a \( OA(k,t) \) is sufficient.
  - **Sets of size \( q + t \)**
    - A \( (N-x) \times (q + t) \) array \( M \) is built whose \( N-x \) rows are the sets of size \( q + t \) such that every value appears once per column. For each of the new \( x = q + t \) points \( p_1, \ldots, p_{q+t} \) we build a matrix \( M_i \) obtained from \( M \) by adding a column equal to \( (p_i, p_i, \ldots) \). We add to the OA the product of all rows of the \( M_i \) with the block of the \( x = q + t \) parallel classes of a resolvable \( OA(k, t + q + 1) - (t + q + 1).OA(k,1) \).
  - **Set of size \( x \)** An \( OA(k,x) \)

- iii) \( x = q^2 - q + 1 - t \)
  - **Sets of size \( t \)**
    - All blocks of the \( i \)-th parallel class are extended with the \( i \)-th new point. The blocks are then replaced by a \( OA(k, t + 1) - (t + 1).OA(k,1) \) or, if there is only one parallel class (i.e. \( x = 1 \)) by a \( OA(k, t + 1) - OA(k,1) \).
  - **Set of size \( q + t \)**
    - They are replaced by \( OA(k, q + t) - (q + t).OA(k,1) \).
  - **Set of size \( x \)** An \( OA(k,x) \)

- iv) \( x = q^2 + 1 \)
– Sets of size \(t\)

All blocks of the \(i\)-th parallel class are extended with the \(i\)-th new point (the other \(x - q - t\) new points are not touched at this step). The blocks are then replaced by a \(OA(k, t + 1) - (t + 1).OA(k, 1)\) or, if there is only one parallel class (i.e. \(x = 1\)) by a \(OA(k, t + 1) - OA(k, 1)\).

– Sets of size \(q + t\) Same as for ii)

– Set of size \(x\) An \(OA(k, x)\)

• v) \(0 < x < q^2 - q + 1 - t\)

– Sets of size \(t\)

The blocks of the first \(x\) parallel class are extended with the \(x\) new points, and replaced with \(OA(k, t + 1) - (t + 1).OA(k, 1)\) or, if \(x = 1\), by \(OA(k, t + 1) - .OA(k, 1)\)

The blocks of the other parallel classes are replaced by \(OA(k, t) - t.OA(k, t)\) or, if there is only one class left, by \(OA(k, t) - OA(k, t)\)

– Sets of size \(q + t\)

They are replaced with \(OA(k, q + t) - (q + t).OA(k, 1)\).

– Set of size \(x\) An \(OA(k, x)\)

• vi) \(t + q < x < q^2 + 1\)

– Sets of size \(t\) Same as in v) with an \(x\) equal to \(x - q + t\).

– Sets of size \(t\) Same as in vii)

– Set of size \(x\) An \(OA(k, x)\)

INPUT:

• \(k, t, q, x\) (integers)

• check – (boolean) Whether to check that output is correct before returning it. Set to False by default.

• verbose (boolean) – whether to print some information on the construction and parameters being used.

• explain_construction (boolean) – return a string describing the construction.

See also:

• find_brouwer_separable_design()
sage: k,q,t=3,3,6; _=brouwer_separable_design(k,t,q,q**2-q+1-t,verbose=True,check=True)
Case iii) with k=3,q=3,t=6,x=1,e2=0
sage: k,q,t=3,4,6; _=brouwer_separable_design(k,t,q,q**2-q+1-t,verbose=True,check=True)
Case iii) with k=3,q=4,t=6,x=7,e2=1
sage: k,q,t=3,4,6; _=brouwer_separable_design(k,t,q,q**2+1,verbose=True,check=True)
Case iv) with k=3,q=4,t=6,x=17,e4=1
sage: k,q,t=3,2,2; _=brouwer_separable_design(k,t,q,q**2+1,verbose=True,check=True)
Case iv) with k=3,q=2,t=2,x=5,e4=0
sage: k,q,t=3,4,7; _=brouwer_separable_design(k,t,q,1,verbose=True,check=True)
Case v) with k=3,q=4,t=7,x=3,e1=1,e2=1
sage: k,q,t=3,4,7; _=brouwer_separable_design(k,t,q,3,verbose=True,check=True)
Case v) with k=3,q=4,t=7,x=5,e1=1,e2=0
sage: k,q,t=3,4,7; _=brouwer_separable_design(k,t,q,q**2-9-t,verbose=True,check=True)
Case v) with k=3,q=4,t=7,x=5,e1=0,e2=1
sage: k,q,t=5,4,7; _=brouwer_separable_design(k,t,q,t+q+3,verbose=True,check=True)
Case vi) with k=5,q=4,t=7,x=14,e3=1,e4=1
sage: k,q,t=5,4,8; _=brouwer_separable_design(k,t,q,t+q+1,verbose=True,check=True)
Case vi) with k=5,q=4,t=8,x=13,e3=1,e4=0
sage: k,q,t=5,4,8; _=brouwer_separable_design(k,t,q,q**2,verbose=True,check=True)
Case vi) with k=5,q=4,t=8,x=16,e3=0,e4=1

sage: print(designs.orthogonal_arrays.explain_construction(10,189))
Brouwer's separable design construction with t=9,q=4,x=0 from:
    Andries E. Brouwer,
    A series of separable designs with application to pairwise orthogonal Latin_squares
    Vol. 1, n. 1, pp. 39-41,
    European Journal of Combinatorics, 1980

sage.combinat.designs.orthogonal_arrays.build_recursive.construction_3_3(k, n, m, i, explain_construction=False)

Return an \( OA(k, nm + i) \).

This is Wilson's construction with \( i \) truncated columns of size 1 and such that a block \( B_0 \) of the incomplete OA intersects all truncated columns. As a consequence, all other blocks intersect only 0 or 1 of the last \( i \) columns. This allow to consider the block \( B_0 \) only up to its first \( k \) coordinates and then use a \( OA(k, i) \) instead of a \( OA(k, m + i) - i.OA(k,1) \).

This is construction 3.3 from [AC07].

INPUT:

- \( k, n, m, i \) (integers) such that the following designs are available: \( OA(k,n), OA(k,m), OA(k,m + 1), OA(k,r) \).
- explain_construction (boolean) – return a string describing the construction.

See also:

- find_construction_3_3()

EXAMPLES:
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_ ...
construction_3_3
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import construction_3_3
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: k=11;n=177
sage: is_orthogonal_array(construction_3_3(*find_construction_3_3(k,n)[1]),k,n,2)
True
sage: print(designs.orthogonal_arrays.explain_construction(9,91))
Construction 3.3 with n=11,m=8,i=3 from:
    Julian R. Abel, Nicholas Cavenagh
    Concerning eight mutually orthogonal latin squares,
    Vol. 15, n.3, pp. 255-261,
    Journal of Combinatorial Designs, 2007

Return a $OA(k, nm + rs)$.

This is Wilson's construction applied to a truncated $OA(k + r + 1, n)$ with $r$ columns of size 1 and one column of size $s$.

The unique elements of the $r$ truncated columns are picked so that a block $B_0$ contains them all.

- If there exists an $OA(k, m + r + 1)$ the column of size $s$ is truncated in order to intersect $B_0$.
- Otherwise, if there exists an $OA(k, m + r)$, the last column must not intersect $B_0$

This is construction 3.4 from [AC07].

INPUT:

- $k,n,m,r,s$ (integers) – we assume that $s < n$ and $1 \leq r, s$
  
  The following designs must be available: $OA(k, n), OA(k, m), OA(k, m + 1), OA(k, m + 2), OA(k, s)$. Additionally, it requires either a $OA(k, m + r)$ or a $OA(k, m + r + 1)$.
- explain_construction (boolean) – return a string describing the construction.

See also:

$find_construction_3_4()$

EXAMPLES:

sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_ ...
construction_3_4
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import construction_3_4
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: k=8;n=196
sage: is_orthogonal_array(construction_3_4(*find_construction_3_4(k,n)[1]),k,n,2)
True
sage: print(designs.orthogonal_arrays.explain_construction(8,164))
Construction 3.4 with n=23,m=7,r=2,s=1 from:
    Julian R. Abel, Nicholas Cavenagh
(continues on next page)
Concerning eight mutually orthogonal latin squares,
Vol. 15, n.3, pp. 255-261,
Journal of Combinatorial Designs, 2007

sage.combinat.designs.orthogonal_arrays_build_recursive.construction_3_5(k, n, m, r, s, t, explain_construction=False)

Return an \( OA(k, nm + r + s + t) \).

This is exactly Wilson’s construction with three truncated groups except we make sure that all blocks have size > \( k \), so we don’t need a \( OA(k, m + 0) \) but only \( OA(k, m + 1), OA(k, m + 2), OA(k, m + 3) \).

This is construction 3.5 from [AC07].

INPUT:
- \( k, n, m \) (integers)
- \( r, s, t \) (integers) – sizes of the three truncated groups, such that \( r \leq s \) and \( (q - r - 1)(q - s) \geq (q - s - 1)(q - r) \).
- \( \text{explain\_construction} \) (boolean) – return a string describing the construction.

The following designs must be available: \( OA(k, n), OA(k, r), OA(k, s), OA(k, t), OA(k, m + 1), OA(k, m + 2), OA(k, m + 3) \).

See also:
- \( \text{find\_construction\_3\_5}() \)

EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_construction_3_5
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import construction_3_5
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: k=8; n=111
sage: is_orthogonal_array(construction_3_5(*find_construction_3_5(k,n)[1]),k,n,2)
True
```

sage: print(designs.orthogonal_arrays.explain_construction(8,90))
Construction 3.5 with n=11,m=6,r=8,s=8,t=8 from:
    Julian R. Abel, Nicholas Cavenagh
    Concerning eight mutually orthogonal latin squares,
    Vol. 15, n.3, pp. 255-261,
    Journal of Combinatorial Designs, 2007

sage.combinat.designs.orthogonal_arrays_build_recursive.construction_3_6(k, n, m, i, explain_construction=False)

Return a \( OA(k, nm + i) \)

This is Wilson’s construction with \( r \) columns of order 1, in which each block intersects at most two truncated columns. Such a design exists when \( n \) is a prime power and is returned by \( OA\_and\_oval() \).

INPUT:
- \( k, n, m, i \) (integers) – \( n \) must be a prime power. The following designs must be available: \( OA(k + r, q) \), \( OA(k, m), OA(k, m + 1), OA(k, m + 2) \).
• `explain_construction` (boolean) – return a string describing the construction.

This is construction 3.6 from [AC07].

See also:

• `find_construction_3_6()`
• `OA_and_oval()`

EXAMPLES:

```
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_construction_3_6
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import construction_3_6
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: k=8;n=95
sage: is_orthogonal_array(construction_3_6(*find_construction_3_6(k,n)[1]),k,n,2)
True
sage: print(designs.orthogonal_arrays.explain_construction(10,756))
Construction 3.6 with n=16,m=47,i=4 from:
    Julian R. Abel, Nicholas Cavenagh
    Concerning eight mutually orthogonal latin squares,
    Vol. 15, n.3, pp. 255-261,
    Journal of Combinatorial Designs, 2007
```

sage.combinat.designs.orthogonal_arrays_build_recursive.construction_q_x(k, q, x, check=True,
                         explain_construction=False)

Return an $OA(k, (q-1) \ast (q-x) + x+2)$ using the $q-x$ construction.

Let $v = (q-1) \ast (q-x) + x+2$. If there exists a projective plane of order $q$ (e.g. when $q$ is a prime power) and $0 < x < q$ then there exists a $(v-1, \{q-x-1, q-x+1\})$-GDD of type $(q-1)^{q-x}(x+1)^3$ (see [Greig99] or Theorem 2.50, section IV.2.3 of [DesignHandbook]). By adding to the ground set one point contained in all groups of the GDD, one obtains a $(v, \{q-x-1, q-x+1, q,x+2\})$-PBD with exactly one set of size $x+2$.

Thus, assuming that we have the following:

• $OA(k, q-x-1) - (q-x-1).OA(k,1)$
• $OA(k, q-x+1) - (q-x+1).OA(k,1)$
• $OA(k, q) - q.OA(k,1)$
• $OA(k, x+2)$

Then we can build from the PBD an $OA(k, v)$.

Construction of the PBD (shared by Julian R. Abel):

Start with a resolvable $(q^2, q, 1)$-BIBD and put the points into a $q \times q$ array so that rows form a parallel class and columns form another.

Now delete:

• All $x(q-1)$ points from the first $x$ columns and not in the first row
• All $q-x$ points in the last $q-x$ columns AND the first row.
Then add a point $p_1$ to the blocks that are rows. Add a second point $p_2$ to the $q - x$ blocks that are columns of size $q - 1$, plus the first row of size $x + 1$.

INPUT:

- **k, q, x** – integers such that $0 < x < q$ and such that Sage can build:
  - A projective plane of order $q$
  - $OA(k, q - x - 1) - (q - x - 1).OA(k, 1)$
  - $OA(k, q - x + 1) - (q - x + 1).OA(k, 1)$
  - $OA(k, q) - q.OA(k, 1)$
  - $OA(k, x + 2)$

- **check** – (boolean) Whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed. Set to True by default.

- **explain_construction** (boolean) – return a string describing the construction.

See also:

- *find_q_x()*
- *projective_plane()*
- *orthogonal_array()*
- *OA_from_PBD()*

EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import *
   construction_q_x
sage: _ = construction_q_x(9,16,6)

sage: print(designs.orthogonal_arrays.explain_construction(9,158))
(q-x)-construction with q=16,x=6 from:
Malcolm Greig,
Designs from projective planes and PBD bases,
vol. 7, num. 5, pp. 341--374,
Journal of Combinatorial Designs, 1999
```

REFERENCES:

sage.combinat.designs.orthogonal_arrays_build_recursive.three_factor_product($k, n1, n2, n3,$
   check=False,
   explain_construction=False)

Returns an $OA(k + 1, n_1n_2n_3)$.

The three factor product construction from [DukesLing14] does the following:

If $n_1 \leq n_2 \leq n_3$ are such that there exists an $OA(k, n_1)$, $OA(k + 1, n_2)$ and $OA(k + 1, n_3)$, then there exists a $OA(k + 1, n_1n_2n_3)$.

It works with a modified product of orthogonal arrays ([Rees93], [Rees00]) which keeps track of parallel classes in the $OA$ (the definition is given for transversal designs).
A subset of blocks in an $TD(k, n)$ is called a $c$-parallel class if every point is covered exactly $c$ times. A 1-parallel class is a parallel class.

The modified product:

If there exists an $OA(k, n_1)$, and if there exists an $OA(k, n_2)$ whose blocks are partitionned into $s$ $n_1$-parallel classes and $n_2 - sn_1$ parallel classes, then there exists an $OA(k, n_1n_2)$ whose blocks can be partitionned into $sn_1^2$ parallel classes and $(n_1n_2 - sn_1^2)/n_1 = n_2 - sn_1$ $n_1$-parallel classes.

Proof:

• The product of the blocks of a parallel class with an $OA(k, n_1)$ yields an $n_1$-parallel class of an $OA(k, n_1n_2)$.

• The product of the blocks of a $n_1$-parallel class of $OA(k, n_2)$ with an $OA(k, n_1)$ can be done in such a way that it yields $n_1n_2$ parallel classes of $OA(k, n_1n_2)$. Those classes cover exactly the pairs that would have been covered with the usual product.

This can be achieved by simple cyclic permutations. Let us build the product of the $n_1$-parallel class $P \subseteq OA(k, n_2)$ with $OA(k, n_1)$: when computing the product of $P \in P$ with $B^i \in OA(k, n_1)$ the $i$-th coordinate should not be $(B^i_k, P_i)$ but $(B^i_k + r, P_i)$ (the sum is mod $n_1$) where $r$ is the number of blocks of $P$ we have already processed whose $i$-th coordinate is equal to $P_i$ (note that $r < n_1$ as $P$ is $n_1$-parallel).

With these tools, one can obtain the designs promised by the three factors construction applied to $k, n_1, n_2, n_3$ (thanks to Julian R. Abel’s help):

1) Let $s$ be the largest integer $\leq n_3/n_1$. Apply the product construction to $OA(k, n_1)$ and a resolvable $OA(k, n_3)$ whose blocks are partitionned into $s$ $n_1$-parallel classes and $n_3 - sn_1$ parallel classes. It results in a $OA(k, n_1n_3)$ partitionned into $sn_1^2$ parallel classes plus $(n_1n_3 - sn_1^2)/n_1 = n_3 - sn_1$ $n_1$-parallel classes.

2) Add $n_3 - n_1$ parallel classes to every $n_1$-parallel class to turn them into $n_3$-parallel classes. Apply the product construction to this partitionned $OA(k, n_1n_3)$ with a resolvable $OA(k, n_2)$.

3) As $OA(k, n_2)$ is resolvable, the $n_2$-parallel classes of $OA(k, n_1n_2n_3)$ are actually the union of $n_2$ parallel classes, thus the $OA(k, n_1n_2n_3)$ is resolvable and can be turned into an $OA(k + 1, n_1n_2n_3)$

INPUT:

• $k, n_1, n_2, n_3$ (integers)

• check – (boolean) Whether to check that everything is going smoothly while the design is being built. It is disabled by default, as the constructor of orthogonal arrays checks the final design anyway.

• explain_construction (boolean) – return a string describing the construction.

See also:

• find_three_factor_product()
sage: OA = three_factor_product(4,3,4,5)
sage: is_orthogonal_array(OA,5,60)
True

sage: OA = three_factor_product(5,4,5,7)
sage: is_orthogonal_array(OA,6,140)
True

sage: OA = three_factor_product(9,8,9,9)
# long time
sage: is_orthogonal_array(OA,10,8*9*9)
# long time
True

sage: print(designs.orthogonal_arrays.explain_construction(10,648))
Three-factor product with n=8.9.9 from:
Peter J. Dukes, Alan C.H. Ling,
A three-factor product construction for mutually orthogonal latin squares,

REFERENCE:
sage.combinat.designs.orthogonal_arrays_build_recursive.thwart_lemma_3_5(k, n, m, a, b, c, d=0,
complement=False, explain_construction=False)

Returns an \( OA(k, nm + a + b + c + d) \)
(When 'd=0')

According to [Thwarts] when \( n \) is a prime power and \( a + b + c \leq n + 1 \), one can build an \( OA(k + 3, n) \) with three truncated columns of sizes \( a, b, c \) in such a way that all blocks have size \( \leq k + 2 \).

(in order to build a \( OA(k, nm + a + b + c) \) the following designs must also exist: \( OA(k, a), OA(k, b), OA(k, c), OA(k, m + 0), OA(k, m + 1), OA(k, m + 2) \))

Considering the complement of each truncated column, it is also possible to build an \( OA(k + 3, n) \) with three truncated columns of sizes \( a, b, c \) in such a way that all blocks have size \( > k \) whenever \( (n-a)+(n-b)+(n-c) \leq n + 1 \).

(in order to build a \( OA(k, nm + a + b + c) \) the following designs must also exist: \( OA(k, a), OA(k, b), OA(k, c), OA(k, m + 1), OA(k, m + 2), OA(k, m + 3) \))

Here is the proof of Lemma 3.5 from [Thwarts] enriched with explanations from Julian R. Abel:

For any prime power \( n \) one can build \( k - 1 \) MOLS by associating to every nonzero \( x \in \mathbb{F}_n \) the latin square:

\[
M_x(i, j) = i + x \cdot j \text{ where } i, j \in \mathbb{F}_n
\]

In particular \( M_1(i, j) = i + j \), whose \( n \) columns and lines are indexed by the elements of \( \mathbb{F}_n \). If we order the elements of \( \mathbb{F}_n \) as 0, 1, ..., \( n-1 \), \( x + 0, ..., x + n - 1, x^2 + 0, ... \) and reorder the columns and lines of \( M_1 \) accordingly, the top-left \( a \times b \) squares contains at most \( a + b - 1 \) distinct symbols.

(When \( d \neq 0 \))

If there exists an \( OA(k + 3, n) \) with three truncated columns of sizes \( a, b, c \) in such a way that all blocks have size \( \leq k + 2 \), by truncating arbitrarily another column to size \( d \) one obtains an \( OA \) with 4 truncated columns whose blocks miss at least one value. Thus, following the proof again one can build an \( OA(k + 4) \) with four truncated columns of sizes \( a, b, c, d \) with blocks of size \( \leq k + 3 \).
(in order to build a $OA(k, nm + a + b + c + d)$ the following designs must also exist: $OA(k, a), OA(k, b), OA(k, c), OA(k, d), OA(k, m + 0), OA(k, m + 1), OA(k, m + 2), OA(k, m + 3)$)

As before, this also shows that one can build an $OA(k + 4, n)$ with four truncated columns of sizes $a, b, c, d$ in such a way that all blocks have size $> k$ whenever $(n - a) + (n - b) + (n - c) \leq n + 1$

(in order to build a $OA(k, nm + a + b + c + d)$ the following designs must also exist: $OA(k, n - a), OA(k, n - b), OA(k, n - c), OA(k, d), OA(k, m + 1), OA(k, m + 2), OA(k, m + 3), OA(k, m + 4)$)

**INPUT:**

- $k, n, m, a, b, c, d$ – integers which must satisfy the constraints above. In particular, $a + b + c \leq n + 1$ must hold. By default, $d = 0$.
- **complement** (boolean) – whether to complement the sets, i.e. follow the $n - a, n - b, n - c$ variant described above.
- **explain_construction** (boolean) – return a string describing the construction.

**See also:**

- *find_thwart_lemma_3_5()*

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays_build_recursive import thwart_lemma_3_5
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: OA = thwart_lemma_3_5(6, 23, 7, 5, 7, 8)
sage: is_orthogonal_array(OA, 6, 23*7+5+7+8, 2)
True
sage: print(designs.orthogonal_arrays.explain_construction(10, 408))
Lemma 4.1 with n=13, m=28 from:
    Charles J. Colbourn, Jeffrey H. Dinitz, Mieczyslaw Wojtas,
    Thwarts in transversal designs,
```

With sets of parameters from [Thwarts]:

```python
sage: l = 
  ....: [11, 27, 78, 16, 17, 25, 0],
  ....: [12, 19, 208, 11, 13, 16, 0],
  ....: [12, 19, 208, 13, 13, 16, 0],
  ....: [10, 13, 78, 9, 9, 13, 1],
  ....: [10, 13, 79, 9, 9, 13, 1]
```

```python
sage: for k,n,m,a,b,c,d in l:
    # not tested -- too long
    OA = thwart_lemma_3_5(k, n, m, a, b, c, d, complement=True)
    assert is_orthogonal_array(OA, k, n*m+a+b+c+d, verbose=True)
```

```python
sage: print(designs.orthogonal_arrays.explain_construction(10, 1046))
Lemma 3.5 with n=13, m=79, a=9, b=1, c=0, d=9 from:
    Charles J. Colbourn, Jeffrey H. Dinitz, Mieczyslaw Wojtas,
    Thwarts in transversal designs,
```

**REFERENCE:**

```
sage.combinat.designs.orthogonal_arrays_build_recursive.thwart_lemma_4_1(k, n, m, 
            explain_construction=False)

Returns an $OA(k, nm + 4(n - 2))$.

Implements Lemma 4.1 from [Thwarts].

If $n \equiv 0, 1 \pmod{3}$ is a prime power, then there exists a truncated $OA(n + 1, n)$ whose last four columns have size $n - 2$ and intersect every block on 1, 3 or 4 values. Consequently, if there exists an $OA(k, m + 1)$, $OA(k, m + 3)$, $OA(k, m + 4)$ and a $OA(k, n - 2)$ then there exists an $OA(k, nm + 4(n - 2))$.

Proof: form the transversal design by removing one point of the $AG(2, 3)$ (Affine Geometry) contained in the Desarguesian Projective Plane $PG(2, n)$.

The affine geometry on 9 points contained in the projective geometry $PG(2, n)$ is given explicitly in [OS64].

(Thanks to Julian R. Abel for finding the reference!).

INPUT:

• $k, n, m$ (integers)

• explain_construction (boolean) – return a string describing the construction.

See also:

• $find_thwart_lemma_4_1()$

EXAMPLES:

```
sage: print(designs.orthogonal_arrays.explain_construction(10,408))
Lemma 4.1 with n=13,m=28 from:
    Charles J.Colbourn, Jeffrey H. Dinitz, Mieczyslaw Wojtas,
Thwarts in transversal designs,
```

REFERENCES:

5.1.92 Orthogonal arrays (find recursive constructions)

This module implements several functions to find recursive constructions of Orthogonal Arrays.

The main function of this module, i.e. $find_recursive_construction()$, queries all implemented recursive constructions of designs implemented in orthogonal_arrays_build_recursive in order to obtain an $OA(k, n)$.

$find_recursive_construction()$ is called by the orthogonal_array() function.
Combinatorics, Release 10.0

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<td>Find a recursive construction of an OA(k, n) (calls all others find_* functions)</td>
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<td>Find a decomposition for construction 3.5 from [AC07].</td>
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<td>Find the values on which Lemma 4.1 from [Thwarts] applies.</td>
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<td>Find an OA(k, n) by the three-factor product from [DukesLing14].</td>
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<td>find_brouwer_separable_design()</td>
<td>Find t(q^2 + q + 1) + x = n to obtain an OA(k, n) by Brouwer’s separable design construction.</td>
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<td>find_brouwer_van_rees_with_one_truncated_column()</td>
<td>Find an OA(k, n) such that the Brouwer-van Rees constructions yields a OA(k, n).</td>
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REFERENCES:

**Functions**

sage.combinat.designs.orthogonal_arrays_find_recursive.find_brouwer_separable_design(k,n)

Find t(q^2 + q + 1) + x = n to obtain an OA(k, n) by Brouwer’s separable design construction.

**INPUT:**

- k, n (integers)

The assumptions made on the parameters t, q, x are explained in the documentation of brouwer_separable_design().

**EXAMPLES:**

```
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_brouwer_separable_design
sage: find_brouwer_separable_design(5,13)
(5, 1, 3, 0)
sage: find_brouwer_separable_design(5,14)
False
```

sage.combinat.designs.orthogonal_arrays_find_recursive.find_brouwer_van_rees_with_one_truncated_column(k,n)

Find rm + x1 + ... + xc = n such that the Brouwer-van Rees constructions yields a OA(k, n).

Let n = rm + ∑1≤i≤c ui such that c ≤ r. The generalization of Wilson’s construction found by Brouwer and van Rees (with one truncated column) ensures that an OA(k, n) exists if the following designs exist: OA(k + 1, r), OA(k, m), OA(k, ∑1≤i≤c ui), OA(k, m + x1), ..., OA(k, m + xc) = OA(k, xc).

For more information, see the documentation of wilson_construction().

**INPUT:**

- k, n (integers)
EXAMPLES:

```
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_  
    →brouwer_van_rees_with_one_truncated_column  
sage: find_brouwer_van_rees_with_one_truncated_column(5,53)[1]  
    (None, 5, 7, 7, [(2, 1), (2, 1)])  
sage: find_brouwer_van_rees_with_one_truncated_column(6,96)[1]  
    (None, 6, 7, 13, [[(3, 1), (1, 1), (1, 1)]]),
```

`sage.combinat.designs.orthogonal_arrays_find_recursive.find_construction_3_3(k, n)`

Find a decomposition for construction 3.3 from [AC07]

INPUT:

- `k, n` (integers)

See also:

`construction_3_3()`

OUTPUT:

A pair `f, args` such that `f(*args)` returns the requested OA.

EXAMPLES:

```
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_  
    →construction_3_3  
sage: find_construction_3_3(11,177)[1]  
    (11, 11, 16, 1)  
sage: find_construction_3_3(12,11)
```

`sage.combinat.designs.orthogonal_arrays_find_recursive.find_construction_3_4(k, n)`

Find a decomposition for construction 3.4 from [AC07]

INPUT:

- `k, n` (integers)

See also:

`construction_3_4()`

OUTPUT:

A pair `f, args` such that `f(*args)` returns the requested OA.

EXAMPLES:

```
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_  
    →construction_3_4  
sage: find_construction_3_4(8,196)[1]  
    (8, 25, 7, 12, 9)  
sage: find_construction_3_4(9,24)
```

`sage.combinat.designs.orthogonal_arrays_find_recursive.find_construction_3_5(k, n)`

Find a decomposition for construction 3.5 from [AC07]

INPUT:

- `k, n` (integers)
See also:

\texttt{construction\_3\_5()}

OUTPUT:

A pair \( f, \text{args} \) such that \( f(*\text{args}) \) returns the requested OA.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_construction_3_5
sage: find_construction_3_5(8,111)[1]
(8, 13, 6, 9, 11, 13)
sage: find_construction_3_5(9,24)
\end{verbatim}

\texttt{sage.combinat.designs.orthogonal_arrays_find_recursive.find_construction\_3\_6(k,n)}

Find a decomposition for construction 3.6 from [AC07]

INPUT:

\begin{itemize}
  \item k,n (integers)
\end{itemize}

See also:

\texttt{construction\_3\_6()}

OUTPUT:

A pair \( f, \text{args} \) such that \( f(*\text{args}) \) returns the requested OA.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_construction_3_6
sage: find_construction_3_6(8,95)[1]
(8, 13, 7, 4)
sage: find_construction_3_6(8,98)
\end{verbatim}

\texttt{sage.combinat.designs.orthogonal_arrays_find_recursive.find_product_decomposition(k,n)}

Find \( n_1 n_2 = n \) to obtain an OA\((k,n)\) by the product construction.

If Sage can build a \( OA(k,n_1) \) and a \( OA(k,n_2) \) such that \( n = n_1 \times n_2 \) then a \( OA(k,n) \) can be built by a product construction (which correspond to Wilson’s construction with no truncated column). This function look for a pair of integers \( (n_1,n_2) \) with \( n_1 \leq n_2, n_1 \times n_2 = n \) and such that both an \( OA(k,n_1) \) and an \( OA(k,n_2) \) are available.

INPUT:

\begin{itemize}
  \item k,n (integers) – see above.
\end{itemize}

OUTPUT:

A pair \( f, \text{args} \) such that \( f(*\text{args}) \) is an \( OA(k,n) \) or False if no product decomposition was found.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_product_decomposition
sage: f, args = find_product_decomposition(6, 84)
sage: args
\end{verbatim}
sage: _ = f(*args)

sage.combinat.designs.orthogonal_arrays_find_recursive.find_q_x(k, n)

Find integers \( q, x \) such that the \( q - x \) construction yields an \( OA(k, n) \).

See the documentation of \( \text{construction}_q_x() \) to find out what hypotheses the integers \( q, x \) must satisfy.

**Warning:** For efficiency reasons, this function checks that Sage can build an \( OA(k + 1, q - x - 1) \) and an \( OA(k + 1, q - x + 1) \), which is stronger than checking that Sage can build a \( OA(k, q - x - 1) - (q - x - 1) \cdot OA(k, 1) \) and a \( OA(k, q - x + 1) - (q - x + 1) \cdot OA(k, 1) \). The latter would trigger a lot of independent set computations in \( \text{sage.combinat.designs.orthogonal_arrays.incomplete_orthogonal_array()} \).

**INPUT:**
- \( k, n \) (integers)

**See also:**
- \( \text{construction}_q_x() \)

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_q_x
sage: find_q_x(10,9)
False
sage: find_q_x(9,158)[1]
(9, 16, 6)
```

sage.combinat.designs.orthogonal_arrays_find_recursive.find_recursive_construction(n)

Find a recursive construction of an \( OA(k, n) \) (calls all others \( \text{find}_* \) functions)

This determines whether an \( OA(k, n) \) can be built through the following constructions:

- \( \text{wilson_construction()} \)
- \( \text{construction}_3_3() \)
- \( \text{construction}_3_4() \)
- \( \text{construction}_3_5() \)
- \( \text{construction}_3_6() \)
- \( \text{construction}_q_x() \)
- \( \text{thwart_lemma}_3_5() \)
- \( \text{thwart_lemma}_4_1() \)
- \( \text{three_factor_product()} \)
- \( \text{brouwer_separable_design()} \)

**INPUT:**
- \( k, n \) (integers)

**OUTPUT:**

Return a pair \( f, \text{args} \) such that \( f(\text{args}) \) returns the requested \( OA \) if possible, and \( \text{False} \) otherwise.
EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_recursive_construction
sage: from sage.combinat.designs.orthogonal_arrays import is_orthogonal_array
sage: count = 0
sage: for n in range(10,150):
    ....:     k = designs.orthogonal_arrays.largest_available_k(n)
    ....:     if find_recursive_construction(k,n):
    ....:         count = count + 1
    ....:         f,args = find_recursive_construction(k,n)
    ....:         OA = f(*args)
    ....:         assert is_orthogonal_array(OA,k,n,2,verbose=True)
sage: count
56
```

`sage.combinat.designs.orthogonal_arrays_find_recursive.find_three_factor_product(k, n)`

Find $n_1n_2n_3 = n$ to obtain an $OA(k, n)$ by the three-factor product from [DukesLing14]

**INPUT:**
- k, n (integers)

**See also:**
`three_factor_product()`

**OUTPUT:**
A pair $f, args$ such that $f(*args)$ returns the requested OA.

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_three_factor_product
sage: find_three_factor_product(10,648)[1]
(9, 8, 9, 9)
sage: find_three_factor_product(10,50)
False
```

`sage.combinat.designs.orthogonal_arrays_find_recursive.find_thwart_lemma_3_5(k, N)`

Find the values on which Lemma 3.5 from [Thwarts] applies.

**OUTPUT:**
A pair $(f, args)$ such that $f(*args)$ returns an $OA(k, n)$ or False if the construction is not available.

**See also:**
`thwart_lemma_3_5()`

**EXAMPLES:**

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_thwart_lemma_3_5
sage: from sage.combinat.designs.designs_pyx import is_orthogonal_array
sage: f,args = find_thwart_lemma_3_5(7,66)
```
(7, 9, 7, 1, 1, 0, False)
sage: OA = f(*args)
sage: is_orthogonal_array(OA,7,66,2)
True
sage: f, args = find_thwart_lemma_3_5(6,100)
sage: args
(6, 8, 10, 8, 7, 5, 0, True)
sage: OA = f(*args)
sage: is_orthogonal_array(OA,6,100,2)
True

Some values from [Thwarts]:

```
sage: kn = ((10,1046), (10,1048), (10,1059), (11,1524),
....:     (11,2164), (12,3362), (12,3992), (12,3994))
sage: for k,n in kn:
....:     print("{} {} {}\n".format(k,n,find_thwart_lemma_3_5(k,n)[1]))
10 1046 (10, 13, 79, 9, 1, 0, 9, False)
10 1048 (10, 13, 79, 9, 1, 0, 11, False)
10 1059 (10, 13, 80, 9, 1, 0, 9, False)
11 1524 (11, 19, 78, 16, 13, 13, 0, True)
11 2164 (11, 27, 78, 23, 19, 16, 0, True)
12 3362 (12, 16, 207, 13, 13, 11, 13, True)
12 3992 (12, 19, 207, 16, 13, 11, 19, True)
12 3994 (12, 19, 207, 16, 13, 13, 19, True)
```

sage: for k,n in kn:
    # not tested -- too long
    ....:     assert designs.orthogonal_array(k,n,existence=True) is True

sage.combinat.designs.orthogonal_arrays_find_recursive.find_thwart_lemma_4_1(k, n)
Find a decomposition for Lemma 4.1 from [Thwarts].

INPUT:

* k,n (integers)

See also:

thwart_lemma_4_1()

OUTPUT:

A pair f, args such that f(*args) returns the requested OA.

EXAMPLES:

```
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_
    thwart_lemma_4_1
sage: find_thwart_lemma_4_1(10,408)[1]
(10, 13, 28)
sage: find_thwart_lemma_4_1(10,50)
False
```
Find \( rm + u = n \) to obtain an \( OA(k, n) \) by Wilson’s construction with one truncated column.

This function looks for possible integers \( m, t, u \) satisfying that \( mt + u = n \) and such that Sage knows how to build a \( OA(k, m) \), \( OA(k, m + 1) \), \( OA(k + 1, t) \) and a \( OA(k, u) \).

INPUT:

- \( k, n \) (integers) – see above

OUTPUT:

A pair \( f, args \) such that \( f(*args) \) is an \( OA(k, n) \) or \( False \) if no decomposition with one truncated block was found.

EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_wilson_decomposition_with_one_truncated_group
sage: f, args = find_wilson_decomposition_with_one_truncated_group(4, 38)
sage: args
(3, 7, 7, (3,), False)
sage: _ = f(*args)
sage: find_wilson_decomposition_with_one_truncated_group(4, 20)
False
```

Find \( rm + r_1 + r_2 = n \) to obtain an \( OA(k, n) \) by Wilson’s construction with two truncated columns.

Look for integers \( r, m, r_1, r_2 \) satisfying \( n = rm + r_1 + r_2 \) and \( 1 \leq r_1, r_2 < r \) and such that the following designs exist: \( OA(k + 2, r) \), \( OA(k, r_1) \), \( OA(k, r_2) \), \( OA(k, m) \), \( OA(k, m + 1) \), \( OA(k, m + 2) \).

INPUT:

- \( k, n \) (integers) – see above

OUTPUT:

A pair \( f, args \) such that \( f(*args) \) is an \( OA(k, n) \) or \( False \) if no decomposition with two truncated blocks was found.

EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import find_wilson_decomposition_with_two_truncated_groups
sage: f, args = find_wilson_decomposition_with_two_truncated_groups(5, 58)
sage: args
(4, 5, 7, 7, (4, 5), False)
sage: _ = f(*args)
sage: find_wilson_decomposition_with_two_truncated_groups(5, 20)
False
```

Return a tuple \( (s_1, s_2, \ldots, s_k) \) of less then \( k_{\text{max}} \) elements of \( S \) such that \( \text{value} = s_1 + s_2 + \ldots + s_k \). If there is no such tuples then the function returns \( None \).

INPUT:

- \( \text{value} \) (integer)
- \( S \) – a list of integers
• k_max (integer)

EXAMPLES:

```python
sage: from sage.combinat.designs.orthogonal_arrays_find_recursive import int_as_sum
sage: D = int_as_sum(21,[5,12],100)
sage: for k in range(20,40):
....:     print("{0} {1}".format(k, int_as_sum(k,[5,12],100)))
20 (5, 5, 5, 5)
21 None
22 (12, 5, 5)
23 None
24 (12, 12)
25 (5, 5, 5, 5, 5)
26 None
27 (12, 5, 5, 5)
28 None
29 (12, 12, 5)
30 (5, 5, 5, 5, 5, 5)
31 None
32 (12, 5, 5, 5, 5)
33 None
34 (12, 12, 5, 5)
35 (5, 5, 5, 5, 5, 5, 5)
36 (12, 12, 12)
37 (12, 5, 5, 5, 5)
38 None
39 (12, 12, 5, 5, 5)
```

5.1.93 Steiner Quadruple Systems

A Steiner Quadruple System on \( n \) points is a family \( SQS_n \subset \binom{[n]}{4} \) of 4-sets, such that any set \( S \subset [n] \) of size three is a subset of exactly one member of \( SQS_n \).

This module implements Haim Hanani’s constructive proof that a Steiner Quadruple System exists if and only if \( n \equiv 2, 4 \pmod{6} \). Hanani’s proof consists in 6 different constructions that build a large Steiner Quadruple System from a smaller one, and though it does not give a very clear understanding of why it works (to say the least)... it does!

The constructions have been implemented while reading two papers simultaneously, for one of them sometimes provides the informations that the other one does not. The first one is Haim Hanani’s original paper [Han1960], and the other one is a paper from Horan and Hurlbert which goes through all constructions [HH2012].

It can be used through the `designs` object:

```python
sage: designs.steiner_quadruple_system(8)
Incidence structure with 8 points and 14 blocks
```

AUTHORS:

• Nathann Cohen (May 2013, while listening to “Le Blues Du Pauvre Delahaye”)

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This module’s main function is the following:

```
steiner_quadruple_system()  # Return a Steiner Quadruple System on n points
```

This function redistributes its work among 6 constructions:

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<tr>
<th>Construction</th>
<th>Function</th>
<th>Description</th>
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</thead>
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<td><code>two_n()</code></td>
<td>Return a Steiner Quadruple System on 2n points</td>
</tr>
<tr>
<td>2</td>
<td><code>three_n_minus_two()</code></td>
<td>Return a Steiner Quadruple System on 3n – 2 points</td>
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<tr>
<td>3</td>
<td><code>three_n_minus_eight()</code></td>
<td>Return a Steiner Quadruple System on 3n – 8 points</td>
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<tr>
<td>4</td>
<td><code>three_n_minus_four()</code></td>
<td>Return a Steiner Quadruple System on 3n – 4 points</td>
</tr>
<tr>
<td>5</td>
<td><code>four_n_minus_six()</code></td>
<td>Return a Steiner Quadruple System on 4n – 6 points</td>
</tr>
<tr>
<td>6</td>
<td><code>twelve_n_minus_ten()</code></td>
<td>Return a Steiner Quadruple System on 12n – 10 points</td>
</tr>
</tbody>
</table>

It also defines two specific Steiner Quadruple Systems that the constructions require, i.e. $SQS_{14}$ and $SQS_{38}$ as well as the systems of pairs $P_{\alpha}(m)$ and $\overline{P}_{\alpha}(m)$ (see [Han1960]).

Functions

- **`sage.combinat.designs.steiner_quadruple_systems.P(alpha, m)`**
  - Return the collection of pairs $P_{\alpha}(m)$
  - For more information on this system, see [Han1960].
  - **EXAMPLES:**
    ```
sage: from sage.combinat.designs.steiner_quadruple_systems import P
sage: P(3,4)
[[0, 5], (2, 7), (4, 1), (6, 3)]
```

- **`sage.combinat.designs.steiner_quadruple_systems.barP(eps, m)`**
  - Return the collection of pairs $\overline{P}_{\alpha}(m)$
  - For more information on this system, see [Han1960].
  - **EXAMPLES:**
    ```
sage: from sage.combinat.designs.steiner_quadruple_systems import barP
sage: barP(3,4)
[[0, 4], (3, 5), (1, 2)]
```

- **`sage.combinat.designs.steiner_quadruple_systems.barP_system()`**
  - Return the 1-factorization of $K_{2m}, \overline{T}(m)$
  - For more information on this system, see [Han1960].
  - **EXAMPLES:**
sage: from sage.combinat.designs.steiner_quadruple_systems import barP_system
sage: barP_system(3)
[[[4, 3], [2, 5]],
[[0, 5], [4, 1]],
[[0, 2], [1, 3]],
[[1, 5], [4, 2], [0, 3]],
[[0, 4], [3, 5], [1, 2]],
[[0, 1], [2, 3], [4, 5]]]

sage.combinat.designs.steiner_quadruple_systems.four_n_minus_six(B)

Return a Steiner Quadruple System on \(4n - 6\) points.

INPUT:

- \(B\) – A Steiner Quadruple System on \(n\) points.

EXAMPLES:

```python
sage: from sage.combinat.designs.steiner_quadruple_systems import four_n_minus_six
sage: for n in range(4, 20):
....:     if (n%6) in [2,4]:
....:         sqs = designs.steiner_quadruple_system(n)
....:         if not four_n_minus_six(sqs).is_t_design(3,4*n-6,4,1):
....:             print("Something is wrong !")
```

sage.combinat.designs.steiner_quadruple_systems.relabel_system(B)

Relabels the set so that \(\{n - 4, n - 3, n - 2, n - 1\}\) is in \(B\).

INPUT:

- \(B\) – a list of 4-uples on \(0, \ldots, n - 1\).

EXAMPLES:

```python
sage: from sage.combinat.designs.steiner_quadruple_systems import relabel_system
sage: SQS8 = designs.steiner_quadruple_system(8)
sage: relabel_system(SQS8)
Incidence structure with 8 points and 14 blocks
```

sage.combinat.designs.steiner_quadruple_systems.steiner_quadruple_system(check=False)

Return a Steiner Quadruple System on \(n\) points.

INPUT:

- \(n\) – an integer such that \(n \equiv 2, 4 \pmod{6}\)
- \(check\) (boolean) – whether to check that the system is a Steiner Quadruple System before returning it (\(False\) by default)

EXAMPLES:

```python
sage: sqs4 = designs.steiner_quadruple_system(4)
sage: sqs4
Incidence structure with 4 points and 1 blocks
sage: sqs4.is_t_design(3,4,4,1)
True
```
sage: sqs8 = designs.steiner_quadruple_system(8)
sage: sqs8
Incidence structure with 8 points and 14 blocks
sage: sqs8.is_t_design(3,8,4,1)
True

sage.combinat.designs.steiner_quadruple_systems.three_n_minus_eight(B)

Return a Steiner Quadruple System on \(3n - 8\) points.

INPUT:

- \(B\) – A Steiner Quadruple System on \(n\) points.

EXAMPLES:

```python
sage: from sage.combinat.designs.steiner_quadruple_systems import three_n_minus_eight
sage: for n in range(4, 30):
    ........ if (n%12) == 2:
    ........ sqs = designs.steiner_quadruple_system(n)
    ........ if not three_n_minus_eight(sqs).is_t_design(3,3*n-8,4,1):
    ........ print("Something is wrong !")
```

sage.combinat.designs.steiner_quadruple_systems.three_n_minus_four(B)

Return a Steiner Quadruple System on \(3n - 4\) points.

INPUT:

- \(B\) – A Steiner Quadruple System on \(n\) points where \(n \equiv 10 \pmod{12}\).

EXAMPLES:

```python
sage: from sage.combinat.designs.steiner_quadruple_systems import three_n_minus_four
sage: for n in range(4, 30):
    ........ if n%12 == 10:
    ........ sqs = designs.steiner_quadruple_system(n)
    ........ if not three_n_minus_four(sqs).is_t_design(3,3*n-4,4,1):
    ........ print("Something is wrong !")
```

sage.combinat.designs.steiner_quadruple_systems.three_n_minus_two(B)

Return a Steiner Quadruple System on \(3n - 2\) points.

INPUT:

- \(B\) – A Steiner Quadruple System on \(n\) points.

EXAMPLES:

```python
sage: from sage.combinat.designs.steiner_quadruple_systems import three_n_minus_two
sage: for n in range(4, 30):
    ........ if (n%6) in [2,4]:
    ........ sqs = designs.steiner_quadruple_system(n)
    ........ if not three_n_minus_two(sqs).is_t_design(3,3*n-2,4,1):
    ........ print("Something is wrong !")
```
sage.combinat.designs.steiner_quadruple_systems.twelve_n_minus_ten(B)
Return a Steiner Quadruple System on \(12n - 6\) points.

INPUT:
- B – A Steiner Quadruple System on \(n\) points.

EXAMPLES:
```
sage: from sage.combinat.designs.steiner_quadruple_systems import twelve_n_minus_ten
sage: for n in range(4, 15):
    ...:     if (n%6) in [2,4]:
    ...:         sqs = designs.steiner_quadruple_system(n)
    ...:         if not twelve_n_minus_ten(sqs).is_t_design(3,12*n-10,4,1):
    ...:             print("Something is wrong !")
```

sage.combinat.designs.steiner_quadruple_systems.two_n(B)
Return a Steiner Quadruple System on \(2n\) points.

INPUT:
- B – A Steiner Quadruple System on \(n\) points.

EXAMPLES:
```
sage: from sage.combinat.designs.steiner_quadruple_systems import two_n
sage: for n in range(4, 30):
    ...:     if (n%6) in [2,4]:
    ...:         sqs = designs.steiner_quadruple_system(n)
    ...:         if not two_n(sqs).is_t_design(3,2*n,4,1):
    ...:             print("Something is wrong !")
```

5.1.94 Hypergraph isomorphic copy search

This module implements a code for the following problem:

INPUT: two hypergraphs \(H_1, H_2\)

OUTPUT: a copy of \(H_2\) in \(H_1\)

It is also possible to enumerate all such copies, and to require that such copies be induced copies. More formally:

A copy of \(H_2\) in \(H_1\) is an injection \(f : V(H_2) \rightarrow V(H_1)\) such that for any set \(S_2 \in E(H_2)\) we have \(f(S_2) \in E(H_1)\).

It is an induced copy if no other set of \(E(H_1)\) is contained in \(f(V(H_2))\), i.e. \(|E(H_2)| = \{S : S \in E(H_1) \text{ and } S \subseteq f(V(H_2))\}\).

The functions implemented here lists all such injections. In particular, the number of copies of \(H\) in itself is equal to \(|\text{Aut}(H)|\).

The feature is available through `IncidenceStructure.isomorphic_substructures_iterator()`.
**Implementation**

A hypergraph is stored as a list of edges, each of which is a “dense” bitset over $|V(H_1)|$ points. In particular, two sets of distinct cardinalities require the same memory space. A hypergraph is a C struct with the following fields:

- `n,m (int)` – number of points and edges.
- `limbs (int)` – number of 64-bits blocks per set.
- `set_space (uint64_t *)` – address of the memory used to store the sets.
- `sets (uint64_t **)` – sets[i] points toward the limbs blocks encoding set $i$. Note also that sets[i][limbs] is equal to the cardinality of set[i], so that sets has length $m*(limbs+1)*sizeof(uint64_t)$.
- `names (int *)` – associates an integer ‘name’ to each of the $n$ points.

The operations used on this data structure are:

- `void permute(hypergraph * h, int n1, int n2)` – exchanges points $n1$ and $n2$ in the data structure. Note that their names are also exchanged so that we still know which is which.
- `int induced_hypergraph(hypergraph * h1, int n, hypergraph * tmp1)` – stores in tmp1 the hypergraph induced by the first $n$ points, i.e. all sets $S$ such that $S \subseteq \{0, ..., n-1\}$. The function returns the number of such sets.
- `void trace_hypergraph64(hypergraph * h, int n, hypergraph * tmp)` – stores in tmp the trace of $h$ on the first $n$ points, i.e. all sets of the form $S \cap \{0, ..., n-1\}$.

**Algorithm**

We try all possible assignments of a representant $r_i \in H_1$ for every $i \in H_2$. When we have picked a representant for the first $n < \text{points} \{0, ..., n-1\} \subseteq V(H_2)$, we check that:

- The hypergraph induced by the (ordered) list $0, ..., n-1$ in $H_2$ is equal to the one induced by $r_0, ..., r_{n-1}$ in $H_1$.
- If $S \subseteq \{0, ..., n-1\}$ is contained in $c$ sets of size $k$ in $H_2$, then $\{r_i : i \in S\}$ is contained in $\geq c$ sets of size $k$ in $H_1$. This is done by comparing the trace of the hypergraphs while remembering the original size of each set.

As we very often need to build the hypergraph obtained by the trace of the first $n$ points (for all possible $n$), those hypergraphs are cached. The hypergraphs induced by the same points are handled similarly.

**Limitations**

**Number of points** For efficiency reason the implementation assumes that $H_2$ has $\leq 64$ points. Making this work for larger values means that calls to `qsort` have to be replaced by calls to `qsort_r` (i.e. to sort the edges you need to know the number of limbs per edge) and that induces a big slowdown for small cases (~50% when this code was implemented). Also, 64 points for $H_2$ is already very very big considering the problem at hand. Even $|V(H_1)| > 64$ seems too much.

**Vertex ordering** The order of vertices in $H_2$ has a huge influence on the performance of the algorithm. If no set of $H_2$ contains more that one of the first $k < n$ points, then almost all partial assignments of representants are possible for the first $k$ points (though the degree of the vertices is taken into account). For this reason it is best to pick an ordering such that the first vertices are contained in as many sets as possible together. A heuristic is implemented at `relabel_heuristic()`.

**AUTHORS:**

- Nathann Cohen (November 2014, written in various airports between Nice and Chennai)
Methods

class sage.combinat.designs.subhypergraph_search.SubHypergraphSearch

Bases: object

  relabel_heuristic()

  Relabels $H_2$ in order to make the algorithm faster.

  Objective: we try to pick an ordering $p_1, ..., p_k$ of the points of $H_2$ that maximizes the number of sets involving the first points in the ordering. One way to formalize the problems indicates that it may be NP-Hard (generalizes the max clique problem for graphs) so we do not try to solve it exactly: we just need a sufficiently good heuristic.

  Assuming that the first points are $p_1, ..., p_k$, we determine $p_{k+1}$ as the point $x$ such that the number of sets $S$ with $x \in S$ and $S \cap \{p_1, ..., p_k\} \neq \emptyset$ is maximal. In case of ties, we take a point with maximum degree.

  This function is called when an instance of SubHypergraphSearch is created.

EXAMPLES:

sage: d = designs.projective_plane(3)
sage: d.isomorphic_substructures_iterator(d).relabel_heuristic()

5.1.95 Two-graphs

A two-graph on $n$ points is a family $T \subset \binom{[n]}{3}$ of 3-sets, such that any 4-set $S \subset [n]$ of size four contains an even number of elements of $T$. Any graph $([n], E)$ gives rise to a two-graph $T(E) = \{ t \in \binom{[n]}{3} : \binom{t}{2} \cap E \text{ odd} \}$, and any two graphs with the same two-graph can be obtained one from the other by Seidel switching. This defines an equivalence relation on the graphs on $[n]$, called Seidel switching equivalence. Conversely, given a two-graph $T$, one can construct a graph $\Gamma$ in the corresponding Seidel switching class with an isolated vertex $w$. The graph $\Gamma \setminus w$ is called the descendant of $T$ w.r.t. $v$.

$T$ is called regular if each two-subset of $[n]$ is contained in the same number $\alpha$ of triples of $T$.

This module implements a direct construction of a two-graph from a list of triples, construction of descendant graphs, regularity checking, and other things such as constructing the complement two-graph, cf. [BH2012].

AUTHORS:

• Dima Pasechnik (Aug 2015)

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This module’s methods are the following:

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<th>Method</th>
<th>Description</th>
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<tbody>
<tr>
<td>is_regular_twograph()</td>
<td>tests if self is a regular two-graph, i.e. a 2-design</td>
</tr>
<tr>
<td>complement()</td>
<td>returns the complement of self</td>
</tr>
<tr>
<td>descendant()</td>
<td>returns the descendant graph at $w$</td>
</tr>
</tbody>
</table>

This module’s functions are the following:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>taylor_twograph()</td>
<td>constructs Taylor’s two-graph for $U_3(q)$</td>
</tr>
<tr>
<td>is_twograph()</td>
<td>checks that the incidence system is a two-graph</td>
</tr>
<tr>
<td>twograph_descendant()</td>
<td>returns the descendant graph w.r.t. a given vertex of the two-graph of a given graph</td>
</tr>
</tbody>
</table>
Methods

class sage.combinat.designs.twographs.TwoGraph(points=None, blocks=None, incidence_matrix=None, name=None, check=False, copy=True)

Bases: IncidenceStructure

Two-graphs class.
A two-graph on \( n \) points is a 3-uniform hypergraph, i.e. a family \( T \subset \binom{\{n\}}{3} \) of 3-sets, such that any 4-set \( S \subset [n] \) of size four contains an even number of elements of \( T \). For more information, see the documentation of the twographs module.

complement()

The two-graph which is the complement of self.
That is, the two-graph consisting exactly of triples not in self. Note that this is different from complement of the parent class.

EXAMPLES:

```
sage: p = graphs.CompleteGraph(8).line_graph().twograph()
sage: pc = p.complement(); pc
Incidence structure with 28 points and 1260 blocks
```

descendant(v)

The descendant graph at \( v \).
The switching class of graphs corresponding to self contains a graph \( D \) with \( v \) its own connected component; removing \( v \) from \( D \), one obtains the descendant graph of self at \( v \), which is constructed by this method.

INPUT:
- \( v \) – an element of ground_set()

EXAMPLES:

```
sage: p = graphs.PetersenGraph().twograph().descendant(0)
sage: p.is_strongly_regular(parameters=True)
(9, 4, 1, 2)
```

is_regular_twograph(alpha=False)

Test if the TwoGraph is regular, i.e. is a 2-design.
Namely, each pair of elements of ground_set() is contained in exactly alpha triples.

INPUT:
- \( \alpha \) – (optional, default is False) return the value of alpha, if possible.

EXAMPLES:

```
sage: p = graphs.PetersenGraph().twograph()
sage: p.is_regular_twograph(alpha=True)
4
sage: p.is_regular_twograph()
True
sage: p = graphs.PathGraph(5).twograph()
sage: p.is_regular_twograph(alpha=True)
```

(continues on next page)
False
sage: p.is_regular_twograph()
False

sage.combinat.designs.twographs.is_twograph(T)

Checks that the incidence system $T$ is a two-graph

INPUT:

- $T$ – an incidence structure

EXAMPLES:

a two-graph from a graph:

sage: from sage.combinat.designs.twographs import is_twograph, TwoGraph
sage: p=graphs.PetersenGraph().twograph()
sage: is_twograph(p)
True

a non-regular 2-uniform hypergraph which is a two-graph:

sage: is_twograph(TwoGraph([[1,2,3],[1,2,4]]))
True

sage.combinat.designs.twographs.taylor_twograph(q)

constructing Taylor’s two-graph for $U_3(q)$, $q$ odd prime power

The Taylor’s two-graph $T$ has $q^3 + 1$ points of the projective plane over $F_{q^2}$ singular w.r.t. the non-degenerate Hermitean form $S$ preserved by $U_3(q)$ as its ground set; the triples are $\{x, y, z\}$ satisfying the condition that $S(x,y)S(y,z)S(z,x)$ is square (respectively non-square) if $q \equiv 1 \pmod{4}$ (respectively if $q \equiv 3 \pmod{4}$). See §7E of [BL1984].

There is also a $2-(q^3+1, q+1, 1)$-design on these $q^3 + 1$ points, known as the unital of order $q$, also invariant under $U_3(q)$.

INPUT:

- $q$ – a power of an odd prime

EXAMPLES:

sage: from sage.combinat.designs.twographs import taylor_twograph
sage: T=taylor_twograph(3); T
Incidence structure with 28 points and 1260 blocks

sage.combinat.designs.twographs.twograph_descendant(G, v, name=None)

Return the descendant graph w.r.t. vertex $v$ of the two-graph of $G$

In the switching class of $G$, construct a graph $\Delta$ with $v$ an isolated vertex, and return the subgraph $\Delta \setminus v$. It is equivalent to, although much faster than, computing the TwoGraph descended() of two-graph of $G$, as the intermediate two-graph is not constructed.

INPUT:

- $G$ – a graph
- $v$ – a vertex of $G$
- name – (optional) None - no name, otherwise derive from the construction
EXAMPLES:

one of s.r.g.’s from the *database*:

```
sage: from sage.combinat.designs.twographs import twograph_descendant
sage: A = graphs.strongly_regular_graph(280, 135, 70) # optional - gap_packages internet
sage: twograph_descendant(A, 0).is_strongly_regular(parameters=True) # optional - gap_packages internet
(279, 150, 85, 75)
```

5.1.96 Combinatorial diagrams

A combinatorial diagram is a collection of cells \((i, j)\) indexed by pairs of natural numbers.

For arbitrary diagrams, see *Diagram*. There are also two other specific types of diagrams implemented here. They are northwest diagrams (*NorthwestDiagram*) and Rothe diagrams (*RotheDiagram()*, a special kind of northwest diagram).

**AUTHORS:**

- Trevor K. Karn (2022-08-01): initial version

**class** *sage.combinat.diagram.Diagram(parent, cells, n_rows=None, n_cols=None, check=True)*

```
Bases: ClonableArray

Combinatorial diagrams with positions indexed by rows in columns.

The positions are indexed by rows and columns as in a matrix. For example, a Ferrer’s diagram is a diagram obtained from a partition \(\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_\ell)\), where the cells are in rows \(i\) for \(0 \leq i \leq \ell\) and the cells in row \(i\) consist of \((i, j)\) for \(0 \leq j < \lambda_i\). In English notation, the indices are read from top left to bottom right as in a matrix.

Indexing conventions are the same as *Partition*. Printing the diagram of a partition, however, will always be in English notation.

**EXAMPLES:**

To create an arbitrary diagram, pass a list of all cells:

```
sage: from sage.combinat.diagram import Diagram
sage: cells = [(0,0), (0,1), (1,0), (1,1), (4,4), (4,5), (4,6), (5,4), (7,6)]
sage: D = Diagram(cells); D
[(0, 0), (0, 1), (1, 0), (1, 1), (4, 4), (4, 5), (4, 6), (5, 4), (7, 6)]
```

We can visualize the diagram by printing 0’s and .’s. 0’s are present in the cells which are present in the diagram and a . represents the absence of a cell in the diagram:

```
sage: D.pp()
  
  0 0 . . .
  0 0 . . .
  . . . . .
  . . . . .
  . . . . .
  0 0 . .
  . . . .
  . . . .
  . . . 0
  . . . 0
```

5.1. Comprehensive Module List
We can also check if certain cells are contained in a given diagram:

```
sage: (1, 0) in D
True
sage: (2, 2) in D
False
```

If you know that there are entire empty rows or columns at the end of the diagram, you can manually pass them with keyword arguments `n_rows=` or `n_cols=`:

```
sage: Diagram([(0,0), (0,3), (2,2), (2,4)]).pp()
O . . O .
. . . . . .

sage: Diagram([(0,0), (0,3), (2,2), (2,4)], n_rows=6, n_cols=6).pp()
O . . O . .
. . . . . .
. . . . . .
. . . . . .
. . . . . .
```

### cells()

Return a list of the cells contained in the diagram `self`.

**EXAMPLES:**

```
sage: from sage.combinat.diagram import Diagram
sage: D1 = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D1.cells()
[(0, 2), (0, 3), (1, 1), (3, 2)]
```

### check()

Check that this is a valid diagram.

**EXAMPLES:**

```
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,0), (0,3), (2,2), (2,4)])
sage: D.check()
```

In the next two examples, a bad diagram is passed. The first example fails because one cells is indexed by negative integers:

```
sage: D = Diagram([(0,0), (0,-3), (2,2), (2,4)])
Traceback (most recent call last):
  ...
ValueError: Diagrams must be indexed by non-negative integers
```

The next example fails because one cell is indexed by rational numbers:

```
sage: D = Diagram([(0,0), (0,3), (2/3,2), (2,4)])
Traceback (most recent call last):
  ...
ValueError: Diagrams must be indexed by non-negative integers
```
n_cells()

Return the total number of cells contained in the diagram self.

EXAMPLES:

```
sage: from sage.combinat.diagram import Diagram
sage: D1 = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D1.number_of_cells()
4
sage: D1.n_cells()
4
```

cmps()

Return the total number of rows of self.

EXAMPLES:

The following example has three columns which are filled, but they are contained in rows 0 to 3 (for a total of four):

```
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D.number_of_cols()
4
sage: D.ncols()
4
```

We can also include empty columns at the end:

```
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,2),(0,3),(1,1),(3,2)], ncols=6)
sage: D.number_of_cols()
6
sage: D.pp()
 . . O O . .
 . O . . . .
 . . . . . .
 . . O . . .
```

nrows()

Return the total number of rows of self.

EXAMPLES:

The following example has three rows which are filled, but they are contained in rows 0 to 3 (for a total of four):

```
sage: from sage.combinat.diagram import Diagram
sage: D1 = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D1.number_of_rows()
4
sage: D1.nrows()
4
```

The total number of rows includes including those which are empty. We can also include empty rows at the end:
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,2),(0,3),(1,1),(3,2)], n_rows=6)
sage: D.number_of_rows()
6
sage: D.pp()
. . O O
. O . .
. . . .
. . O .
. . . .
. . . .

number_of_cells()

Return the total number of cells contained in the diagram self.

EXAMPLES:

sage: from sage.combinat.diagram import Diagram
sage: D1 = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D1.number_of_cells()
4
sage: D1.n_cells()
4

number_of_cols()

Return the total number of rows of self.

EXAMPLES:

The following example has three columns which are filled, but they are contained in rows 0 to 3 (for a total of four):

sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D.number_of_cols()
4
sage: D.ncols()
4

We can also include empty columns at the end:

sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,2),(0,3),(1,1),(3,2)], n_cols=6)
sage: D.number_of_cols()
6
sage: D.pp()
. . 0 0 .
. 0 . . .
. . . . .
. . . 0 .
. . . . .
. . . . .

number_of_rows()

Return the total number of rows of self.

EXAMPLES:
The following example has three rows which are filled, but they are contained in rows 0 to 3 (for a total of four):

```python
sage: from sage.combinat.diagram import Diagram
sage: D1 = Diagram([(0,2),(0,3),(1,1),(3,2)])
sage: D1.number_of_rows()
4
sage: D1.nrows()
4
```

The total number of rows includes including those which are empty. We can also include empty rows at the end:

```python
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,2),(0,3),(1,1),(3,2)], n_rows=6)
sage: D.number_of_rows()
6
sage: D.pp()
 . 0 0
 . 0 ..
  . 0 .
  . . 0
  . . 0
  . . 0
```

**pp()**

Return a visualization of the diagram.

Cells which are present in the diagram are filled with a 0. Cells which are not present in the diagram are filled with a ..

**EXAMPLES:**

```python
sage: from sage.combinat.diagram import Diagram
sage: Diagram([(0,0), (0,3), (2,2), (2,4)]).pp()
 0 . . 0
 . . . .
 . 0 . 0
```

```python
sage: Diagram([(0,0), (0,3), (2,2), (2,4)], n_rows=6, n_cols=6).pp()
 0 . . 0 .
 . . . . .
 . 0 . 0 .
 . . 0 . .
 . . . . .
```

```python
sage: Diagram([]).pp()
- 
```

**size()**

Return the total number of cells contained in the diagram self.

**EXAMPLES:**

```python
sage: from sage.combinat.diagram import Diagram
sage: D1 = Diagram([(0,2),(0,3),(1,1),(3,2)])
```
sage: D1.number_of_cells()
4
sage: D1.n_cells()
4

specht_module(base_ring=None)
Return the Specht module corresponding to self.

EXAMPLES:

```python
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,0), (1,1), (2,2), (2,3)])
sage: SM = D.specht_module(QQ)
sage: s = SymmetricFunctions(QQ).s()
sage: s(SM.frobenius_image())
```

specht_module_dimension(base_ring=None)
Return the dimension of the Specht module corresponding to self.

INPUT:

- base_ring – (default: \( \mathbb{Q} \)) the base ring

EXAMPLES:

```python
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram([(0,0), (1,1), (2,2), (2,3)])
sage: D.specht_module_dimension()
12
sage: D.specht_module(QQ).dimension()
12
```

class sage.combinat.diagram.Diagrams(category=None)
Bases: UniqueRepresentation, Parent
The class of combinatorial diagrams.

A *combinatorial diagram* is a set of cells indexed by pairs of natural numbers. Calling an instance of `Diagrams` is one way to construct diagrams.

EXAMPLES:

```python
sage: from sage.combinat.diagram import Diagrams
sage: Dgms = Diagrams()
sage: D = Dgms([(0,0), (0,3), (2,2), (2,4)])
sage: D.parent()
Combinatorial diagrams
```

**Element**

alias of `Diagram`

from_composition(alpha)
Create the diagram corresponding to a weak composition \( \alpha \vdash n \).

EXAMPLES:
sage: alpha = Composition([3,0,2,1,4,4])
sage: from sage.combinat.diagram import Diagrams
sage: Diagrams()(alpha).pp()
 0 0 0 .
  . . . 
 0 0 . .
 0 . .
 0 0 0 0
 0 0 0 0
sage: Diagrams().from_composition(alpha).pp()
 0 0 0 .
  . . . 
 0 0 . .
 0 . .
 0 0 0 0
 0 0 0 0

from_polyomino(p)

Create the diagram corresponding to a 2d Polyomino

EXAMPLES:

sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0),(1,0),(1,1),(1,2)])
sage: from sage.combinat.diagram import Diagrams
sage: Diagrams()(p).pp()
 0 . .
 0 0 0

We can also call this method directly:

sage: Diagrams().from_polyomino(p).pp()
 0 . .
 0 0 0

This only works for a 2d Polyomino:

sage: p = Polyomino([(0,0,0), (0,1,0), (1,1,0), (1,1,1)], color='blue')
sage: Diagrams().from_polyomino(p)
Traceback (most recent call last):
...
ValueError: the polyomino must be 2 dimensional

from_zero_one_matrix(M, check=True)

Get a diagram from a matrix with entries in \{0, 1\}, where positions of cells are indicated by the 1’s.

EXAMPLES:

sage: M = matrix([[1,0,1,1],[0,1,0,1]])
sage: from sage.combinat.diagram import Diagrams
sage: Diagrams()(M).pp()
 0 . 0
  . 0 0

(continues on next page)
class sage.combinat.diagram.NorthwestDiagram(parent, cells, n_rows=None, n_cols=None, check=True)

Diagrams with the northwest property.

A diagram is a set of cells indexed by natural numbers. Such a diagram has the northwest property if the presence of cells \((i_1, j_1)\) and \((i_2, j_2)\) implies the presence of the cell \((\min(i_1, i_2), \min(j_1, j_2))\). Diagrams with the northwest property are called northwest diagrams.

For general diagrams see Diagram.

EXAMPLES:

```python
sage: from sage.combinat.diagram import NorthwestDiagram
sage: N = NorthwestDiagram([(0,0), (0, 2), (2,0)])
```

To visualize them, use the .pp() method:

```python
sage: N.pp()
```

```python
0 . 0
. 0 0
```

**check()**

A diagram has the northwest property if the presence of cells \((i_1, j_1)\) and \((i_2, j_2)\) implies the presence of the cell \(\min(i_1, i_2), \min(j_1, j_2)\)). This method checks if the northwest property is satisfied for self

EXAMPLES:

```python
sage: from sage.combinat.diagram import NorthwestDiagram
sage: N = NorthwestDiagram([(0,0), (0, 3), (3,0)])
sage: N.check()
```

Here is a non-example:

```python
sage: notN = NorthwestDiagram([(0,1), (1,0)])  # .check() is implicit
Traceback (most recent call last):
  ...
ValueError: diagram is not northwest
```

**peelable_tableaux()**

Return the set of peelable tableaux whose diagram is self.

For a fixed northwest diagram \(D\), we say that a Young tableau \(T\) is \(D\)-peelable if:

1. the row indices of the cells in the first column of \(D\) are the entries in an initial segment in the first column of \(T\) and
2. the tableau $Q$ obtained by removing those cells from $T$ and playing jeu de taquin is $(D - C)$-peelable, where $D - C$ is the diagram formed by forgetting the first column of $D$.

Reiner and Shimozono [RS1995] showed that the number $\text{red}(w)$ of reduced words of a permutation $w$ may be computed using the peelable tableaux of the Rothe diagram $D(w)$. Explicitly,

$$\text{red}(w) = \sum_T f_{\text{shape } T},$$

where the sum runs over the $D(w)$-peelable tableaux $T$ and $f_\lambda$ is the number of standard Young tableaux of shape $\lambda$ (which may be computed using the hook-length formula).

EXAMPLES:

We can compute the $D$-peelable diagrams for a northwest diagram $D$:

```plaintext
sage: from sage.combinat.diagram import NorthwestDiagram
sage: cells = [(0,0), (0,1), (0,2), (1,0), (2,0), (2,2), (2,4),
           (4,0), (4,2)]
          
          sage: D = NorthwestDiagram(cells); D.pp()
          0 0 0 . .
          0 . . .
          0 . 0 .
          . . . .
          0 . 0 .

sage: D.peelable_tableaux()
[[[1, 1, 1], [2, 3, 3], [3, 5], [5]],
  [[1, 1, 1, 3], [2, 3], [3, 5], [5]]]
```

EXAMPLES:

If the diagram is only one column, there is only one peelable tableau:

```plaintext
sage: D = NorthwestDiagram([(0,0), (2,0)])

sage: D.peelable_tableaux()
[[[1], [3]]]
```

From [RS1995], we know that there is only one peelable tableau for the Rothe diagram of the permutation (in one-line notation) 251643:

```plaintext
sage: D = NorthwestDiagram([(1, 2), (1, 3), (3, 2), (3, 3), (4, 2)])

sage: D.peelable_tableaux()
[[[2, 2], [4, 4], [5]]]
```

Here are all the intermediate steps to compute the peelables for the Rothe diagram of (in one-line notation) 64817235. They are listed from deepest in the recursion to the final step. The recursion has depth five in this case so we will label the intermediate tableaux by $D_i$ where $i$ is the step in the recursion at which they appear.

Start with the one that has a single column:
sage: D5 = NorthwestDiagram([(2,0)]); D5.pp()
   .
   0
sage: D5.peelable_tableaux()
   {[[3]]}

Now we know all of the $D_5$ peelables, so we can compute the $D_4$ peelables:

sage: D4 = NorthwestDiagram([(0, 0), (2,0), (4, 0), (2, 2)])
sage: D4.pp()
   0
   . .
   0 . 0
   . .
   0 .

sage: D4.peelable_tableaux()
   {[[1, 3], [3], [5]]}

There is only one $D_4$ peelable, so we can compute the $D_3$ peelables:

sage: D3 = NorthwestDiagram([(0,0), (0,1), (2, 1), (2, 3), (4,1)])
sage: D3.pp()
   0 0
   . .
   . 0
   . .
   . 0

sage: D3.peelable_tableaux()
   {[[1, 1], [3, 3], [5]], [[1, 1, 3], [3], [5]]}

Now compute the $D_2$ peelables:

sage: cells = [(0,0), (0,1), (0,2), (1,0), (2,0), (2,2), (2,4),
   ....:
   (4,0), (4,2)]
sage: D2 = NorthwestDiagram(cells); D2.pp()
   0 0 0
   0 . .
   0 . 0
   . . .
   . 0 .

sage: D2.peelable_tableaux()
   {[[1, 1, 1], [2, 3, 3], [3, 5], [5]],
    [[1, 1, 1, 3], [2, 3], [3, 5], [5]]}

And the $D_1$ peelables:

sage: cells = [(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (2,0),
   ....:
   (2,1), (2,3), (2,5), (4,0), (4,1), (4,3)]
sage: D1 = NorthwestDiagram(cells); D1.pp()
(continues on next page)
Which we can use to get the $D$ peelables:

```python
sage: cells = [(0,0), (0,1), (0,2), (0,3), (0,4),
 ....: (1,0), (1,1), (1,2),
 ....: (2,0), (2,1), (2,2), (2,4), (2,6),
 ....: (4,1), (4,2), (4,4)]
```

```python
sage: D = NorthwestDiagram(cells); D.pp()
0 0 0 0 0 ..
0 0 0 . . .
0 0 0 . 0 0
. . . . . ...
. 0 . 0 0 ..
```

```python
sage: D.peelable_tableaux()
{[[1, 1, 1, 1, 1], [2, 2, 2, 3, 3], [3, 3, 3], [5, 5, 5]],
 [[1, 1, 1, 1, 1], [2, 2, 2, 3, 3], [3, 3, 3, 5], [5, 5, 5]],
 [[1, 1, 1, 1, 1, 3], [2, 2, 2, 3], [3, 3, 3], [5, 5, 5]],
 [[1, 1, 1, 1, 1, 3], [2, 2, 2, 3], [3, 3, 3, 5], [5, 5, 5]]}
```

ALGORITHM:

This implementation uses the algorithm suggested in Remark 25 of [RS1995].

```python
class sage.combinat.diagram.NorthwestDiagrams(category=None)
Bases: Diagrams
```

Diagrams satisfying the northwest property.

A diagram $D$ is a northwest diagram if for every two cells $(i_1, j_1)$ and $(i_2, j_2)$ in $D$ then there exists the cell $(\min(i_1, i_2), \min(j_1, j_2)) \in D$.

EXAMPLES:

```python
sage: from sage.combinat.diagram import NorthwestDiagram
sage: N = NorthwestDiagram([(0,0), (0, 10), (5,0)]); N.pp()
0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0
```

Note that checking whether or not the northwest property is satisfied is automatically checked. The diagram found by adding the cell $(1, 1)$ to the diagram above is not a northwest diagram. The cell $(1, 0)$ should be present due to the presence of $(5, 0)$ and $(1, 1)$:
from sage.combinat.diagram import Diagram
sage: Diagram([(0, 0), (0, 10), (5, 0), (1, 1)]).pp()
O . . . . . . . . . O
. O . . . . . . . . .
. . . . . . . . . . .
. . . . . . . . . . .
. . . . . . . . . . .
O . . . . . . . . . .
sage: NorthwestDiagram([(0, 0), (0, 10), (5, 0), (1, 1)])
Traceback (most recent call last):
...
ValueError: diagram is not northwest

However, this behavior can be turned off if you are confident that you are providing a northwest diagram:

sage: N = NorthwestDiagram([(0, 0), (0, 10), (5, 0),
....: (1, 1), (0, 1), (1, 0)],
....: check=False)
sage: N.pp()
O O . . . . . . . . O
O O . . . . . . . . .
. . . . . . . . . . .
. . . . . . . . . . .
. . . . . . . . . . .
O . . . . . . . . . .

Note that arbitrary diagrams which happen to be northwest diagrams only live in the parent of Diagrams:

d = Diagram([(0, 0), (0, 10), (5, 0), (1, 1), (0, 1), (1, 0)])
sage: d.pp()
O 0 . . . . . . . . 0
O 0 . . . . . . . . .
. . . . . . . . . . .
. . . . . . . . . . .
. . . . . . . . . . .
O . . . . . . . . . .
sage: from sage.combinat.diagram import NorthwestDiagrams
sage: d in NorthwestDiagrams()
False

Here are some more examples:

from sage.combinat.diagram import NorthwestDiagram, NorthwestDiagrams
sage: D = NorthwestDiagram([(0,1), (0,2), (1,1)]); D.pp()
. O O
. O .
sage: NWDgms = NorthwestDiagrams()
sage: D = NWDgms([(1,1), (1,2), (2,1)]); D.pp()
. .
. O O
. O .
sage: D.parent()
Combinatorial northwest diagrams
Additionally, there are natural constructions of a northwest diagram given the data of a permutation (Rothe diagrams are the prototypical example of northwest diagrams), or the data of a partition of an integer, or a skew partition.

The Rothe diagram $D(\omega)$ of a permutation $\omega$ is specified by the cells

$$D(\omega) = \{(\omega_j, i) : i < j, \omega_i > \omega_j \}.$$ 

We can construct one by calling `rothe_diagram()` method on the set of all `NorthwestDiagrams`:

```python
sage: w = Permutations(4)([4,3,2,1])
sage: NorthwestDiagrams().rothe_diagram(w).pp()
0 0 0 .
0 0 . ..
0 . . . .
. . . . . .
```

To turn a Ferrers diagram into a northwest diagram, we may call `from_partition()`. This will return a Ferrer's diagram in the set of all northwest diagrams. For many use-cases it is probably better to get Ferrer's diagrams by the corresponding method on partitions, namely `sage.combinat.partitions.Partitions.ferrers_diagram()`:

```python
sage: mu = Partition([7,3,1,1])
sage: mu.pp()
********
***
*
sage: NorthwestDiagrams().from_partition(mu).pp()
0 0 0 0 0 0
0 0 0 . . .
0 0 . . . .
0 . . . . .
```

It is also possible to turn a Ferrers diagram of a skew partition into a northwest diagram, although it is more subtle than just using the skew diagram itself. One must first reflect the partition about a vertical axis so that the skew partition looks “backwards”:

```python
sage: mu, nu = Partition([5,4,3,2,1]), Partition([3,2,1])
sage: s = mu/nu; s.pp()
  **
  **
  **
  *
sage: NorthwestDiagrams().from_skew_partition(s).pp()
0 0 . .
. 0 0 .
.. 0 0 .
... 0 0
```

Element

alias of `NorthwestDiagram`
from_parallelogram_polyomino(p)

Create the diagram corresponding to a ParallelogramPolyomino.

EXAMPLES:

```
sage: p = ParallelogramPolyomino([[0, 0, 1, 0, 0, 0, 1, 1],
                               ....: [1, 1, 0, 1, 0, 0, 0, 0]])
sage: from sage.combinat.diagram import NorthwestDiagrams
sage: NorthwestDiagrams().from_parallelogram_polyomino(p).pp()
```

```
O O .
O O O
.O O
.O O
.O O
```

from_partition(mu)

Return the Ferrer’s diagram of mu as a northwest diagram.

EXAMPLES:

```
sage: mu = Partition([5,2,1]); mu.pp()  
*****
**
*   
sage: mu.parent()
Partitions
sage: from sage.combinat.diagram import NorthwestDiagrams
sage: D = NorthwestDiagrams().from_partition(mu)
sage: D.pp()
```

```
O O O O O
O O . . .
O . . . .
```

This will print in English notation even if the notation is set to French for the partition:

```
sage: Partitions.options.convention="french"
sage: mu.pp()  
*
**
*****
```

```
sage: D.pp()
```

```
0 0 0 0 0
0 0 . . .
0 . . . .
```

from_permutation(w)

Return the Rothe diagram of w.

We construct a northwest diagram from a permutation by constructing its Rothe diagram. Formally, if \( \omega \) is a Permutation then the Rothe diagram \( D(\omega) \) is the diagram whose cells are

\[
D(\omega) = \{ (\omega_j, i) : i < j, \omega_i > \omega_j \}.
\]

Informally, one can construct the Rothe diagram by starting with all \( n^2 \) possible cells, and then deleting the cells \( (i, \omega(i)) \) as well as all cells to the right and below. (These are sometimes called “death rays”.)
See also:

RotheDiagram()

EXAMPLES:

```python
sage: from sage.combinat.diagram import NorthwestDiagrams
sage: w = Permutations(3)([2,1,3])
```

```python
def rothe_diagram(w):
    return
```

```python
sage: NorthwestDiagrams().rothe_diagram(w).pp()
O . .
. . .
. . .
```

```python
sage: w = Permutations(8)([2,5,4,1,3,6,7,8])
```

```python
sage: NorthwestDiagrams().rothe_diagram(w).pp()
0 . . . . . . .
O . O O . . . .
O . O . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
```

from_skew_partition(s)

Get the northwest diagram found by reflecting a skew shape across a vertical plane.

EXAMPLES:

```python
sage: mu, nu = Partition([3,2,1]), Partition([2,1])
```

```python
sage: D = NorthwestDiagrams().from_skew_partition(s)
```

```python
sage: from sage.combinat.diagram import NorthwestDiagrams
sage: D.pp()
O . .
. O .
. . O
```

```python
sage: mu, nu = Partition([3,3,2]), Partition([2,2,2])
```

```python
sage: D.pp()
0 . .
0 . .
. . 0
```

rothe_diagram(w)

Return the Rothe diagram of w.
We construct a northwest diagram from a permutation by constructing its Rothe diagram. Formally, if \( \omega \) is a permutation then the Rothe diagram \( D(\omega) \) is the diagram whose cells are
\[
D(\omega) = \{ (\omega_j, i) : i < j, \omega_i > \omega_j \}.
\]
Informally, one can construct the Rothe diagram by starting with all \( n^2 \) possible cells, and then deleting the cells \( (i, \omega(i)) \) as well as all cells to the right and below. (These are sometimes called “death rays”.)

See also:

RotheDiagram()

EXAMPLES:

```python
sage: from sage.combinat.diagram import NorthwestDiagrams
sage: w = Permutations(3)([2,1,3])
sage: NorthwestDiagrams().rothe_diagram(w).pp()
O . .
. . .
. . .
sage: NorthwestDiagrams().from_permutation(w).pp()
O . .
. . .
. . .
sage: w = Permutations(8)([2,5,4,1,3,6,7,8])
sage: NorthwestDiagrams().rothe_diagram(w).pp()
O . . . . . . .
O . O O O . . .
O . O . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
sage.combinat.diagram.RotheDiagram(w)
The Rothe diagram of a permutation \( w \).
EXAMPLES:

```python
sage: w = Permutations(9)([1, 7, 4, 5, 9, 3, 2, 8, 6])
sage: from sage.combinat.diagram import RotheDiagram
sage: D = RotheDiagram(w); D.pp()
. . . . . . . . .
. 0 0 0 0 0 . .
. 0 0 . . . . .
. 0 0 . . . . .
. 0 0 . 0 0 . .
. 0 . . . . . .
. . . . . . . .
. . . . . . . .
. . . . . . . .
The Rothe diagram is a northwest diagram:
Some other examples:

```python
sage: RotheDiagram([2, 1, 4, 3]).pp()
0 . .
. . O .
. . . .

sage: RotheDiagram([4, 1, 3, 2]).pp()
O O O .
. . . .
. O . .
. . . .
```

Currently, only elements of the set of `sage.combinat.permutations.Permutations` are supported. In particular, elements of permutation groups are not supported:

```python
sage: w = SymmetricGroup(9).an_element()
sage: RotheDiagram(w)
Traceback (most recent call last):
  ... ValueError: w must be a permutation
```

5.1.97 Diagram and Partition Algebras

AUTHORS:

- Mike Hansen (2007): Initial version
- Mike Zabrocki (2018): Implementation of individual element diagram classes

```python
class sage.combinat.diagram_algebras.AbstractPartitionDiagram(parent, d, check=True)
Bases: AbstractSetPartition

Abstract base class for partition diagrams.

This class represents a single partition diagram, that is used as a basis key for a diagram algebra element. A partition diagram should be a partition of the set \{1, \ldots, k, -1, \ldots, -k\}. Each such set partition is regarded as a graph on nodes \{1, \ldots, k, -1, \ldots, -k\} arranged in two rows, with nodes 1, \ldots, k in the top row from left to right and with nodes -1, \ldots, -k in the bottom row from left to right, and an edge connecting two nodes if and only if the nodes lie in the same subset of the set partition.

EXAMPLES:
```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)

sage: pd1 = da.AbstractPartitionDiagram(pd, [[1,2],[-1,-2]])

sage: pd2 = da.AbstractPartitionDiagram(pd, [[1,2],[-1,-2]])

sage: pd1
{{-2, -1}, {1, 2}}

sage: pd1 == pd2
True

sage: pd1 == [[1,2],[-1,-2]]
True

sage: pd1 == ((-2,-1),(2,1))
True

sage: pd1 == SetPartition([[1,2],[-1,-2]])
True

sage: pd3 = da.AbstractPartitionDiagram(pd, [[1,-2],[-1,2]])

sage: pd1 == pd3
False

sage: pd4 = da.AbstractPartitionDiagram(pd, [[1,2],[3,4]])

Traceback (most recent call last):
...
ValueError: {{1, 2}, {3, 4}} does not represent two rows of vertices of order 2

\textbf{base\_diagram()}

Return the underlying implementation of the diagram.

\textbf{OUTPUT:}

\begin{itemize}
\item tuple of tuples of integers
\end{itemize}

\textbf{EXAMPLES:}

sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)

\begin{verbatim}
(sage([[1,2],[-1,-2]]).base_diagram()) == ((-2,-1),(2,1))
True
\end{verbatim}

\textbf{check()}

Check the validity of the input for the diagram.

\textbf{compose}(\textit{other}, \textit{check=\texttt{True}})

Compose \texttt{self} with \textit{other}.

The composition of two diagrams $X$ and $Y$ is given by placing $X$ on top of $Y$ and removing all loops.

\textbf{OUTPUT:}

A tuple where the first entry is the composite diagram and the second entry is how many loop were removed.

\textbf{Note}: This is not really meant to be called directly, but it works to call it this way if desired.

\textbf{EXAMPLES:}

sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)

(continues on next page)
```python
sage: pd([[1,2],[[-1,-2]]]).compose(pd([[1,2],[-1,-2]]))
({{-2, -1}, {1, 2}}, 1)
```

count_blocks_of_size(n)

Count the number of blocks of a given size.

INPUT:

• n – a positive integer

EXAMPLES:

```python
sage: from sage.combinat.diagram_algebras import PartitionDiagram
sage: pd = PartitionDiagram([[1,-3,-5],[2,4],[3,-1,-2],[5],[-4]])
sage: pd.count_blocks_of_size(1)
2
sage: pd.count_blocks_of_size(2)
1
sage: pd.count_blocks_of_size(3)
2
```

diagram()

Return the underlying implementation of the diagram.

OUTPUT:

• tuple of tuples of integers

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: pd([[1,2],[-1,-2]]).base_diagram() == ((-2,-1),(1,2))
True
```

dual()

Return the dual diagram of self by flipping it top-to-bottom.

EXAMPLES:

```python
sage: from sage.combinat.diagram_algebras import PartitionDiagram
sage: D = PartitionDiagram([[1,-1],[2,-2,-3],[3]])
sage: D.dual()
{{-3}, {-2, 2, 3}, {-1, 1}}
```

is_planar()

Test if the diagram self is planar.

A diagram element is planar if the graph of the nodes is planar.

EXAMPLES:

```python
sage: from sage.combinat.diagram_algebras import BrauerDiagram
sage: BrauerDiagram([[1,-2],[2,-1]]).is_planar()
False
sage: BrauerDiagram([[1,-1],[2,-2]]).is_planar()
True
```
order()
Return the maximum entry in the diagram element.
A diagram element is a partition of the set \{-1, -2, \ldots, -k, 1, 2, \ldots, k\}. The order of the diagram element is the value \(k\).

EXAMPLES:

```python
sage: from sage.combinat.diagram_algebras import PartitionDiagram
sage: PartitionDiagram([[1,-1],[2,-2,-3],[3]]).order()
3
sage: PartitionDiagram([[1,-1]]).order()
1
sage: PartitionDiagram([[1,-3,-5],[2,4],[3,-1,-2],[5],[-4]]).order()
5
```

propagating_number()
Return the propagating number of the diagram.
The propagating number is the number of blocks with both a positive and negative number.

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: d1 = pd([[1,-2],[2,-1]])
sage: d1.propagating_number()
2
sage: d2 = pd([[1,2],[-2,-1]])
sage: d2.propagating_number()
0
```

set_partition()
Return the underlying implementation of the diagram as a set of sets.

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.AbstractPartitionDiagrams(2)
sage: X = pd([[1,2],[-1,-2]]).set_partition(); X
{{-2, -1}, {1, 2}}
sage: X.parent()
Set partitions
```

class sage.combinat.diagram_algebras.AbstractPartitionDiagrams(order, category=None)
Bases: Parent, UniqueRepresentation

This is an abstract base class for partition diagrams.
The primary use of this class is to serve as basis keys for diagram algebras, but diagrams also have properties in their own right. Furthermore, this class is meant to be extended to create more efficient contains methods.

INPUT:

- order – integer or integer +1/2; the order of the diagrams
- category – (default: FiniteEnumeratedSets()); the category

All concrete classes should implement attributes
• \_name – the name of the class
• \_diagram\_func – an iterator function that takes the order as its only input

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: pd = da.PartitionDiagrams(2)
sage: pd
Partition diagrams of order 2
sage: pd.an_element() in pd
True
sage: elm = pd([[1,2],[-1,-2]])
sage: elm in pd
True
```

Element

alias of AbstractPartitionDiagram

class sage.combinat.diagram_algebras.BrauerAlgebra(k, q, base_ring, prefix)

Bases: SubPartitionAlgebra, UnitDiagramMixin

A Brauer algebra.

The Brauer algebra of rank \( k \) is an algebra with basis indexed by the collection of set partitions of \( \{1, \ldots, k, -1, \ldots, -k\} \) with block size 2.

This algebra is a subalgebra of the partition algebra. For more information, see PartitionAlgebra.

INPUT:

• \( k \) – rank of the algebra
• \( q \) – the deformation parameter \( q \)

OPTIONAL ARGUMENTS:

• base\_ring – (default \( \text{None} \)) a ring containing \( q \); if \( \text{None} \) then just takes the parent of \( q \)
• prefix – (default "\( B \)") a label for the basis elements

EXAMPLES:

We now define the Brauer algebra of rank 2 with parameter \( x \) over \( \mathbb{Z} \):

```
sage: R.<x> = ZZ[]
sage: B = BrauerAlgebra(2, x, R)
sage: B
Brauer Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over Integer Ring
sage: B.basis()
Lazy family (Term map from Brauer diagrams of order 2 to Brauer Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over Integer Ring(i))_{i in Brauer diagrams of order 2}
sage: B.basis().keys()
Brauer diagrams of order 2
sage: B.basis().keys()([[2, -1], [-1, -2]])
{{-2, 1}, {-1, 2}}
sage: b = B.basis().list(); b
[B{{-2, -1}, {1, 2}}, B{{-2, 1}, {-1, 2}}, B{{-2, 2}, {-1, 1}}]
```

(continues on next page)
sage: b[0]
B{{-2, -1}, {1, 2}}
sage: b[0]^2
x*B{{-2, -1}, {1, 2}}
sage: b[0]^5
x^4*B{{-2, -1}, {1, 2}}

Note, also that since the symmetric group algebra is contained in the Brauer algebra, there is also a conversion between the two.

sage: R.<x> = ZZ[]
sage: B = BrauerAlgebra(2, x, R)
sage: S = SymmetricGroupAlgebra(R, 2)
sage: S([2,1])*B([[-1],[2,-2]])
B{{-2, 1}, {-1, 2}}

jucys_murphy(j)

Return the j-th generalized Jucys-Murphy element of self.

The j-th Jucys-Murphy element of a Brauer algebra is simply the j-th Jucys-Murphy element of the symmetric group algebra with an extra (z-1)/2 term, where z is the parameter of the Brauer algebra.

REFERENCES:

EXAMPLES:

sage: z = var('z')
sage: B = BrauerAlgebra(3,z)
sage: B.jucys_murphy(1)
(1/2*z-1/2)*B{{-3, 3}, {-2, 2}, {-1, 1}}
sage: B.jucys_murphy(3)
-B{{-3, -2}, {-1, 1}, {2, 3}} - B{{-3, -1}, {-2, 2}, {1, 3}}
+ B{{-3, 1}, {-2, 2}, {-1, 3}} + B{{-3, 2}, {-2, 3}, {-1, 1}}
+ (1/2*z-1/2)*B{{-3, 3}, {-2, 2}, {-1, 1}}

options = Current options for Brauer diagram - display: normal

class sage.combinat.diagram_algebras.BrauerDiagram

A Brauer diagram.

A Brauer diagram for an integer k is a partition of the set \{1,\ldots,k,-1,\ldots,-k\} with block size 2.

EXAMPLES:

sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(2)
sage: bd1 = bd([[1,2],[-1,-2]])
sage: bd2 = bd([[1,2],[-1,-2]])
Traceback (most recent call last):
...
ValueError: all blocks of \{\{-2, -1, 1, 2\}\} must be of size 2
**bijection_on_free_nodes**(two_line=False)

Return the induced bijection - as a list of \((x, f(x))\) values - from the free nodes on the top at the Brauer diagram to the free nodes at the bottom of `self`.

**OUTPUT:**

If `two_line` is True, then the output is the induced bijection as a two-row list (`inputs`, `outputs`).

**EXAMPLES:**

```python
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-2,-3],[3,-1]])
sage: elm.bijection_on_free_nodes()
[[3, -1]]
sage: elm2 = bd([[1,-2],[2,-3],[3,-1]])
sage: elm2.bijection_on_free_nodes(two_line=True)
[[1, 2, 3], [-2, -3, -1]]
```

**check()**

Check the validity of the input for `self`.

**involutive_permutation_triple**(curt=True)

Return the involutive permutation triple of `self`.

From Graham-Lehrer (see `BrauerDiagrams`), a Brauer diagram is a triple \((D_1, D_2, \pi)\), where:

- \(D_1\) is a partition of the top nodes;
- \(D_2\) is a partition of the bottom nodes;
- \(\pi\) is the induced permutation on the free nodes.

**INPUT:**

- `curt` – (default: True) if True, then return bijection on free nodes as a one-line notation (standardized to look like a permutation), else, return the honest mapping, a list of pairs \((i, -j)\) describing the bijection on free nodes

**EXAMPLES:**

```python
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-2,-3],[3,-1]])
sage: elm.involutive_permutation_triple()
(((1, 2), [(-3, -2)], [1])
sage: elm.involutive_permutation_triple(curt=False)
(((1, 2), [(-3, -2)], [[3, -1]])
```

**is_elementary_symmetric()**

Check if is elementary symmetric.

Let \((D_1, D_2, \pi)\) be the Graham-Lehrer representation of the Brauer diagram \(d\). We say \(d\) is **elementary symmetric** if \(D_1 = D_2\) and \(\pi\) is the identity.

**EXAMPLES:**

```python
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
```

(continues on next page)
sage: elm = bd([[1,2],[-1,-2],[3,-3]])
sage: elm.is_elementary_symmetric()
True
sage: elm2 = bd([[1,2],[-1,-3],[3,-2]])
sage: elm2.is_elementary_symmetric()
False

options = Current options for Brauer diagram - display: normal

perm()

Return the induced bijection on the free nodes of self in one-line notation, re-indexed and treated as a permutation.

See also:

\texttt{bijection_on_free_nodes()}

EXAMPLES:

sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: elm = bd([[1,2],[-2,-3],[3,-1]])
sage: elm.perm()
[1]

\texttt{sage.combinat.diagram_algebras.BrauerDiagrams\(\text{(order, category=}\text{None})\)}

Bases: \texttt{AbstractPartitionDiagrams}

This class represents all Brauer diagrams of integer or integer +1/2 order. For more information on Brauer diagrams, see \texttt{BrauerAlgebra}.

EXAMPLES:

sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(2); bd
Brauer diagrams of order 2
sage: bd.list()
[{{-2, -1}, {1, 2}}, {{-2, 1}, {-1, 2}}, {{-2, 2}, {-1, 1}}]
sage: bd = da.BrauerDiagrams(5/2); bd
Brauer diagrams of order 5/2
sage: bd.list()
[{{-3, 3}, {-2, -1}, {1, 2}},
{{-3, 3}, {-2, 1}, {-1, 2}},
{{-3, 3}, {-2, 2}, {-1, 1}}]

\texttt{Element}

\texttt{alias of BrauerDiagram}

\texttt{cardinality()}

Return the cardinality of self.

The number of Brauer diagrams of integer order \(k\) is \((2k - 1)!\).
```python
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(3)
sage: bd.cardinality()
15
sage: bd = da.BrauerDiagrams(7/2)
sage: bd.cardinality()
15
```

```python
from_involution_permutation_triple(D1_D2_pi)

Construct a Brauer diagram of self from an involution permutation triple.

A Brauer diagram can be represented as a triple where the first entry is a list of arcs on the top row of the diagram, the second entry is a list of arcs on the bottom row of the diagram, and the third entry is a permutation on the remaining nodes. This triple is called the involution permutation triple. For more information, see [GL1996].

INPUT:

- `D1_D2_pi`: a list or tuple where the first entry is a list of arcs on the top of the diagram, the second entry is a list of arcs on the bottom of the diagram, and the third entry is a permutation on the free nodes.

REFERENCES:

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(4)
sage: bd.from_involution_permutation_triple([[[1,2]],[[3,4]],[2,1]])
{{-4, -3}, {-2, 3}, {-1, 4}, {1, 2}}
```

```python
options = Current options for Brauer diagram - display: normal
```

```python
symmetric_diagrams(l=None, perm=None)

Return the list of Brauer diagrams with symmetric placement of l arcs, and with free nodes permuted according to `perm`.

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: bd = da.BrauerDiagrams(4)
sage: bd.symmetric_diagrams(l=1, perm=[2,1])
[[{-4, -2}, {-3, 1}, {-1, 3}, {2, 4}],
  {{-4, -3}, {-2, 1}, {-1, 2}, {3, 4}},
  {{-4, -1}, {-3, 2}, {-2, 3}, {1, 4}},
  {{-4, 2}, {-3, -1}, {-2, 4}, {1, 3}},
  {{-4, 3}, {-3, -4}, {-2, -1}, {1, 2}},
  {{-4, 1}, {-3, -2}, {-1, 4}, {2, 3}}]
```

```python
class sage.combinat.diagram_algebras.DiagramAlgebra(k, q, base_ring, prefix, diagrams, category=None)

Bases: CombinatorialFreeModule

Abstract class for diagram algebras and is not designed to be used directly.
```
class Element

    Bases: IndexedFreeModuleElement

    An element of a diagram algebra.

    This subclass provides a few additional methods for partition algebra elements. Most element methods are already implemented elsewhere.

diagram()

    Return the underlying diagram of self if self is a basis element. Raises an error if self is not a basis element.

    EXAMPLES:

    sage: R.<x> = ZZ[]
    sage: P = PartitionAlgebra(2, x, R)
    sage: elt = 3*P([[-1,2],[-2,-1]])
    sage: elt.diagram()
    {{-2, -1}, {1, 2}}

diagrams()

    Return the diagrams in the support of self.

    EXAMPLES:

    sage: R.<x> = ZZ[]
    sage: P = PartitionAlgebra(2, x, R)
    sage: elt = 3*P([[-1,2],[-2,-1]]) + P([[-1,2],[-2],[1,2]])
    sage: sorted(elt.diagrams(), key=str)
    [{{-1, 2}}, {{-2}, {-1}, {1, 2}}]

order()

    Return the order of self.

    The order of a partition algebra is defined as half of the number of nodes in the diagrams.

    EXAMPLES:

    sage: q = var('q')
    sage: PA = PartitionAlgebra(2, q)
    sage: PA.order()
    2

set_partitions()

    Return the collection of underlying set partitions indexing the basis elements of a given diagram algebra.

    Todo: Is this really necessary? deprecate?

class sage.combinat.diagram_algebras.DiagramBasis(k, q, base_ring, prefix, diagrams, category=None)

    Bases: DiagramAlgebra

    Abstract base class for diagram algebras in the diagram basis.

    product_on_basis(d1, d2)

    Return the product $D_{d_1}D_{d_2}$ by two basis diagrams.
class sage.combinat.diagram_algebras.IdealDiagram(parent, d, check=True)

Bases: AbstractPartitionDiagram

The element class for a ideal diagram.

An ideal diagram for an integer $k$ is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$ where the propagating number is strictly smaller than the order.

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import IdealDiagrams as IDs
sage: IDs(2)
Ideal diagrams of order 2
sage: IDs(2).list()
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2}, {-1, 1, 2}},
 {{-2, -1}, {1, 2}},
 {{-2}, {-1}, {1, 2}},
 {{-2, -1, 1}, {2}},
 {{-2, 1}, {-1}, {2}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}},
 {{-2}, {-1, 1}, {2}},
 {{-2}, {-1, 2}, {1}},
 {{-2, -1}, {1}, {2}},
 {{-2}, {-1}, {1}, {2}}]
```

```
sage: from sage.combinat.diagram_algebras import PartitionDiagrams as PDs
sage: PDs(4).cardinality() == factorial(4) + IDs(4).cardinality()
True
```

check()
Check the validity of the input for self.

class sage.combinat.diagram_algebras.IdealDiagrams(order, category=None)

Bases: AbstractPartitionDiagrams

All “ideal” diagrams of integer or integer +1/2 order.

If $k$ is an integer then an ideal diagram of order $k$ is a partition diagram of order $k$ with propagating number less than $k$.

EXAMPLES:

```
sage: import sage.combinat.diagram_algebras as da
sage: id = da.IdealDiagrams(3)
sage: id.an_element() in id
True
sage: id.cardinality() == len(id.list())
True
sage: da.IdealDiagrams(3/2).list()
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}}]
```
Element

alias of IdealDiagram

class sage.combinat.diagram_algebras.OrbitBasis(alg)

Bases: DiagramAlgebra

The orbit basis of the partition algebra.

Let $D_{\pi}$ represent the diagram basis element indexed by the partition $\pi$, then (see equations (2.14), (2.17) and (2.18) of [BH2017])

$$D_{\pi} = \sum_{\tau \geq \pi} O_{\tau},$$

where the sum is over all partitions $\tau$ which are coarser than $\pi$ and $O_{\tau}$ is the orbit basis element indexed by the partition $\tau$.

If $\mu_{2k}(\pi, \tau)$ represents the Moebius function of the partition lattice, then

$$O_{\pi} = \sum_{\tau \geq \pi} \mu_{2k}(\pi, \tau)D_{\tau}.\]

If $\tau$ is a partition of $\ell$ blocks and the $i^{th}$ block of $\tau$ is a union of $b_i$ blocks of $\pi$, then

$$\mu_{2k}(\pi, \tau) = \ell \prod_{i=1}^{\ell} (-1)^{b_i-1}(b_i - 1).$$

EXAMPLES:

```python
sage: R.<x> = QQ[
...
]

sage: P2 = PartitionAlgebra(2, x, R)

sage: O2 = P2.orbit_basis(); O2
Orbit basis of Partition Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over Rational Field

sage: oa = O2([[1],[-1],[2,-2]]); ob = O2([[[-1,-2,2],[1]]]); oa, ob
(O{{-2, 2}, {-1}, {1}}, O{{-2, -1, 2}, {1}})

sage: oa * ob
(x-2)*O{{-2, -1, 2}, {1}}

sage: pa = P2(oa); pa
2*P{{-2, -1, 1, 2}} - P{{-2, -1, 2}, {1}} - P{{-2, 1, 2}, {-1}}
 + P{{-2, 2}, {-1}, {1}} - P{{-2, 2}, {-1, 1}}

sage: pa * ob
(-x+2)*P{{-2, -1, 1, 2}} + (x-2)*P{{-2, -1, 2}, {1}}

sage: _ == pa * P2(ob)
True

sage: O2.one()
(x-2)*O{{-2, -1, 2}, {1}}
```

Note that the unit in the orbit basis is not a single diagram, in contrast to the natural diagram basis:

```python
sage: P2.one()
P{{-2, 2}, {-1, 1}}

sage: O2.one()
```

(continues on next page)
class Element
Bases: Element

to_diagram_basis()

Expand self in the natural diagram basis of the partition algebra.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(2, x, R)
sage: O = P.orbit_basis()
sage: elt = O.an_element(); elt
3*O\{-2\}, \{-1, 1, 2\} + 2*O\{-2, -1, 1, 2\} + 2*O\{-2, 1, 2\}, \{-1\}

sage: elt.to_diagram_basis()
3*P\{-2\}, \{-1, 1, 2\} - 3*P\{-2, -1, 1, 2\} + 2*P\{-2, 1, 2\}, \{-1\}

sage: pp = P.an_element(); pp
3*P\{-2\}, \{-1, 1, 2\} + 2*P\{-2, -1, 1, 2\} + 2*P\{-2, 1, 2\}, \{-1\}

sage: op = pp.to_orbit_basis(); op
3*O\{-2\}, \{-1, 1, 2\} + 7*O\{-2, -1, 1, 2\} + 2*O\{-2, 1, 2\}, \{-1\}

sage: pp == op.to_diagram_basis()
True
```

diagram_basis()

Return the associated partition algebra of self in the diagram basis.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: O2 = PartitionAlgebra(2, x, R).orbit_basis()
sage: P2 = O2.diagram_basis(); P2
Partition Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over Rational Field

sage: o2 = O2.an_element(); o2
3*O\{-2\}, \{-1, 1, 2\} + 2*O\{-2, -1, 1, 2\} + 2*O\{-2, 1, 2\}, \{-1\}

sage: P2(o2)
3*P\{-2\}, \{-1, 1, 2\} - 3*P\{-2, -1, 1, 2\} + 2*P\{-2, 1, 2\}, \{-1\}
```

one()

Return the element 1 of the partition algebra in the orbit basis.

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: P2 = PartitionAlgebra(2, x, R)
sage: O2 = P2.orbit_basis()
sage: O2.one()
O\{-2\}, \{-1, 1, 2\} + O\{-2, 2\}, \{-1, 1\}
```

product_on_basis(d1, d2)

Return the product $O_{d_1}O_{d_2}$ of two elements in the orbit basis self.
EXAMPLES:

```
sage: R.<x> = QQ[]
sage: OP = PartitionAlgebra(2, x, R).orbit_basis()
sage: SP = OP.basis().keys(); sp = SP([-2, -1, 1, 2])
sage: OP.product_on_basis(sp, sp)
O{{-2, -1, 1, 2}}
sage: o1 = OP.one(); o2 = OP([]); o3 = OP.an_element()
sage: o2 == o1
False
sage: o1 * o1 == o1
True
sage: o3 * o1 == o3 and o3 * o1 == o3
True
sage: o4 = (3*OP([-2, -1, 1], [2]) + 2*OP([-2, -1, 1, 2])
     ....: + 2*OP([-2, -1, 2], [1]))
sage: o4 * o4
6*O{{-2, -1, 1}, {2}} + 4*O{{-2, -1, 1, 2}} + 4*O{{-2, -1, 2}, {1}}
```

We compute Examples 4.5 in [BH2017]:

```
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(3,x); O = P.orbit_basis()
sage: O([-1,2,3],[-1,-2,-3]) * O([-1,2,3],[-1,-2,-3])
(x-2)*O{{-3, -2, -1}, {1, 2, 3}} + (x-1)*O{{-3, -2, -1, 1, 2, 3}}
```

REFERENCES:

- [BH2017]
Bases: DiagramBasis, UnitDiagramMixin

A partition algebra.

A partition algebra of rank $k$ over a given ground ring $R$ is an algebra with ($R$-module) basis indexed by the collection of set partitions of $\{1, \ldots, k, -1, \ldots, -k\}$. Each such set partition can be represented by a graph on nodes $\{1, \ldots, k, -1, \ldots, -k\}$ arranged in two rows, with nodes $1, \ldots, k$ in the top row from left to right and with nodes $-1, \ldots, -k$ in the bottom row from left to right, and edges drawn such that the connected components of the graph are precisely the parts of the set partition. (This choice of edges is often not unique, and so there are often many graphs representing one and the same set partition; the representation nevertheless is useful and vivid. We often speak of “diagrams” to mean graphs up to such equivalence of choices of edges; of course, we could just as well speak of set partitions.)

There is not just one partition algebra of given rank over a given ground ring, but rather a whole family of them, indexed by the elements of $R$. More precisely, for every $q \in R$, the partition algebra of rank $k$ over $R$ with parameter $q$ is defined to be the $R$-algebra with basis the collection of all set partitions of $\{1, \ldots, k, -1, \ldots, -k\}$, where the product of two basis elements is given by the rule

$$a \cdot b = q^N (a \circ b),$$

where $a \circ b$ is the composite set partition obtained by placing the diagram (i.e., graph) of $a$ above the diagram of $b$, identifying the bottom row nodes of $a$ with the top row nodes of $b$, and omitting any closed “loops” in the middle. The number $N$ is the number of connected components formed by the omitted loops.

The parameter $q$ is a deformation parameter. Taking $q = 1$ produces the semigroup algebra (over the base ring) of the partition monoid, in which the product of two set partitions is simply given by their composition.

The partition algebra is regarded as an example of a “diagram algebra” due to the fact that its natural basis is given by certain graphs often called diagrams.

There are a number of predefined elements for the partition algebra. We define the cup/cap pair by $a()$. The simple transpositions are denoted $s()$. Finally, we define elements $e()$, where if $i = (2r + 1)/2$, then $e(1)$ contains the blocks $\{r + 1\}$ and $\{-r - 1\}$ and if $i \in \mathbb{Z}$, then $e_i$ contains the block $\{-i, -i - 1, i, i + 1\}$, with all other blocks being $\{-j, j\}$. So we have:

```
sage: P = PartitionAlgebra(4, 0)
sage: P.a(2)
P{{-4, 4}, {-3, -2}, {-1, 1}, {2, 3}}
sage: P.e(3/2)
P{{-4, 4}, {-3, 3}, {-2}, {-1, 1}, {2}}
sage: P.e(2)
P{{-4, 4}, {-3, -2, 2, 3}, {-1, 1}}
sage: P.e(5/2)
P{{-4, 4}, {-3}, {-2, 2}, {-1, 1}, {3}}
sage: P.s(2)
P{{-4, 4}, {-3, 2}, {-2, 3}, {-1, 1}}
```

An excellent reference for partition algebras and their various subalgebras (Brauer algebra, Temperley–Lieb algebra, etc) is the paper [HR2005].

INPUT:

- $k$ – rank of the algebra
- $q$ – the deformation parameter $q$

OPTIONAL ARGUMENTS:

- `base_ring` – (default `None`) a ring containing $q$; if `None`, then Sage automatically chooses the parent of $q$
- `prefix` – (default "P") a label for the basis elements
EXAMPLES:

The following shorthand simultaneously defines the univariate polynomial ring over the rationals as well as the variable $x$:

```sage
sage: R.<x> = PolynomialRing(QQ)
sage: R
Univariate Polynomial Ring in x over Rational Field
sage: x
x
sage: x.parent() is R
True
```

We now define the partition algebra of rank 2 with parameter $x$ over $\mathbb{Z}$ in the usual (diagram) basis:

```sage
sage: R.<x> = ZZ[]
sage: A2 = PartitionAlgebra(2, x, R)
sage: A2
Partition Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over Integer Ring
sage: A2.basis().keys()
Partition diagrams of order 2
sage: A2.basis().keys([[-2, 1, 2], [-1]])
{{-2, 1, 2}, {-1}}
sage: A2.basis().list()
[P{{-2, -1, 1, 2}}, P{{-2, 1, 2}, {-1}}, P{{-2}, {-1, 1}, {2}}, P{{-2}, {-1, 1, 2}}, P{{-2, -1, 1}, {2}}, P{{-2}, {-1, 1}, {2}}, P{{-2}, {-1, 1}, {2}}, P{{-2}, {-1}, {1}, {2}}]
sage: E = A2([[1,2],[-2,-1]]); E
P{{-2, -1}, {1, 2}}
sage: E in A2.basis().list()
True
sage: E^2
x*P{{-2, -1}, {1, 2}}
sage: E^5
x^4*P{{-2, -1}, {1, 2}}
sage: (A2([[2,-2],[-1,1]]) - 2*A2([[1,2],[1,-1]]))^2
(4*x-4)*P{{-2, -1}, {1, 2}} + P{{-2, 2}, {-1, 1}}
```

Next, we construct an element:

```sage
sage: a2 = A2.an_element(); a2
3*P{{-2}, {-1, 1, 2}} + 2*P{{-2, -1, 1, 2}} + 2*P{{-2, 1, 2}, {-1}}
```

There is a natural embedding into partition algebras on more elements, by adding identity strands:

```sage
sage: A4 = PartitionAlgebra(4, x, R)
sage: A4(a2)
3*P{{-4, 4}, {-3, 3}, {-2}, {-1, 1, 2}}
```

(continues on next page)
+ 2*P\{\{-4, 4\}, \{-3, 3\}, \{-2, -1, 1, 2\}\} \\
+ 2*P\{\{-4, 4\}, \{-3, 3\}, \{-2, 1, 2\}, \{-1\}\} \\

Thus, the empty partition corresponds to the identity:

\[
\text{sage: } A4([]) \\
P\{\{-4, 4\}, \{-3, 3\}, \{-2, 2\}, \{-1, 1\}\} \\
\text{sage: } A4(5) \\
5*P\{\{-4, 4\}, \{-3, 3\}, \{-2, 2\}, \{-1, 1\}\}
\]

The group algebra of the symmetric group is a subalgebra:

\[
\text{sage: } S3 = SymmetricGroupAlgebra(ZZ, 3) \\
\text{sage: } s3 = S3.an_element(); s3 \\
[1, 2, 3] + 2*[1, 3, 2] + 3*[2, 1, 3] + [3, 1, 2] \\
\text{sage: } A4(s3) \\
P\{\{-4, 4\}, \{-3, 1\}, \{-2, 3\}, \{-1, 2\}\} \\
+ 2*P\{\{-4, 4\}, \{-3, 2\}, \{-2, 3\}, \{-1, 1\}\} \\
+ 3*P\{\{-4, 4\}, \{-3, 3\}, \{-2, 1\}, \{-1, 2\}\} \\
+ P\{\{-4, 4\}, \{-3, 3\}, \{-2, 2\}, \{-1, 1\}\} \\
\text{sage: } A4([2,1]) \\
P\{\{-4, 4\}, \{-3, 3\}, \{-2, 1\}, \{-1, 2\}\}
\]

Be careful not to confuse the embedding of the group algebra of the symmetric group with the embedding of partial set partitions. The latter are embedded by adding the parts \(\{i, -i\}\) if possible, and singletons sets for the remaining parts:

\[
\text{sage: } A4([[2,1]]) \\
P\{\{-4, 4\}, \{-3, 3\}, \{-2\}, \{-1\}, \{1, 2\}\} \\
\text{sage: } A4([[1,3],[2,-3,1]]) \\
P\{\{-4, 4\}, \{-3, -2, 1\}, \{-1, 3\}, \{2\}\}
\]

Another subalgebra is the Brauer algebra, which has perfect matchings as basis elements. The group algebra of the symmetric group is in fact a subalgebra of the Brauer algebra:

\[
\text{sage: } B3 = BrauerAlgebra(3, x, R) \\
\text{sage: } b3 = B3(s3); b3 \\
B\{\{-3, 1\}, \{-2, 3\}, \{-1, 2\}\} + 2*B\{\{-3, 2\}, \{-2, 3\}, \{-1, 1\}\} \\
+ 3*B\{\{-3, 3\}, \{-2, 1\}, \{-1, 2\}\} + B\{\{-3, 3\}, \{-2, 2\}, \{-1, 1\}\}
\]

An important basis of the partition algebra is the orbit basis:

\[
\text{sage: } O2 = A2.orbit_basis() \\
\text{sage: } o2 = O2([[1,2],[1,3,2]]); o2 \\
O\{\{-2, -1\}, \{1, 2\}\} + O\{\{-2, -1, 1, 2\}\}
\]

The diagram basis element corresponds to the sum of all orbit basis elements indexed by coarser set partitions:

\[
\text{sage: } A2(o2) \\
P\{\{-2, -1\}, \{1, 2\}\}
\]

We can convert back from the orbit basis to the diagram basis:
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One can work with partition algebras using a symbol for the parameter, leaving the base ring unspecified. This implies that the underlying base ring is Sage’s symbolic ring.

```
sage: q = var('q')
sage: PA = PartitionAlgebra(2, q); PA
Partition Algebra of rank 2 with parameter q over Symbolic Ring
sage: PA([[1,2],[-2,-1]])^2 == q*PA([[1,2],[-2,-1]])
True
```

The identity element of the partition algebra is the set partition \(\{\{1,-1\},\{2,-2\},\ldots,\{k,-k\}\}:

```
sage: P = PA.basis().list()
True
True
```

We now give some further examples of the use of the other arguments. One may wish to “specialize” the parameter to a chosen element of the base ring:

```
sage: R.<q> = RR[]
sage: PA = PartitionAlgebra(2, q, R, prefix='B')
sage: PA
Partition Algebra of rank 2 with parameter q over Univariate Polynomial Ring in q over Real Field with 53 bits of precision
sage: PA([[1,2],[-2,-1]])
1.00000000000000*B([-2,-1],{1,2})
sage: PA = PartitionAlgebra(2, 5, base_ring=ZZ, prefix='B')
sage: PA
Partition Algebra of rank 2 with parameter 5 over Integer Ring
sage: (PA([[2,-2],[1,-1]]) - 2*PA([[2,-2],[1,-1]]))^2 == 16*PA([[2,-2],[1,-1]])
True
```

Symmetric group algebra elements and elements from other subalgebras of the partition algebra (e.g., BrauerAlgebra and TemperleyLiebAlgebra) can also be coerced into the partition algebra:

```
sage: S = SymmetricGroupAlgebra(SR, 2)
sage: B = BrauerAlgebra(2, x, SR)
sage: A = PartitionAlgebra(2, x, SR)
sage: S([[2,1]])*A([[1,-1],[2,-2]])
P([-2,1],{-1,2})
sage: B([[-1,-2],[2,1]]) * A([[1,-1],[2,-2]])
(continues on next page)
```
P\{\{-2\}, \{-1\}, \{1, 2\}\}

\texttt{sage: A([[1],[-1],[2,-2]]) * B([-1,-2],[2,1])}

P\{\{-2, -1\}, \{1\}, \{2\}\}

The same is true if the elements come from a subalgebra of a partition algebra of smaller order, or if they are defined over a different base ring:

\texttt{sage: R = FractionField(ZZ['q']); q = R.gen()}
\texttt{sage: S = SymmetricGroupAlgebra(ZZ, 2)}
\texttt{sage: B = BrauerAlgebra(2, q, ZZ[q])}
\texttt{sage: A = PartitionAlgebra(3, q, R)}
\texttt{sage: S([2,1])*A([[1,-1],[2,-3],[3,-2]])}
P\{\{-3, 1\}, \{-2, 3\}, \{-1, 2\}\}

\texttt{sage: A(B([[-1,-2],[2,1]]))}
P\{\{-3, 3\}, \{-2, -1\}, \{1, 2\}\}

\begin{verbatim}
class Element
    Bases: Element

    dual()
    Return the dual of self.

    The dual of an element in the partition algebra is formed by taking the dual of each diagram in the support.

    EXAMPLES:
    sage: R.<x> = QQ[]
    sage: P = PartitionAlgebra(2, x, R)
    sage: elt = P.an_element(); elt
    3*P\{\{-2\}, \{-1, 1, 2\}\} + 2*P\{\{-2, -1, 1, 2\}\} + 2*P\{\{-2, 1, 2\}, \{-1\}\}
    sage: elt.dual()
    3*P\{\{-2, -1, 1\}, \{2\}\} + 2*P\{\{-2, -1, 1, 2\}\} + 2*P\{\{-2, -1, 2\}, \{1\}\}
\end{verbatim}

\begin{verbatim}
    to_orbit_basis()
    Return self in the orbit basis of the associated partition algebra.

    EXAMPLES:
    sage: R.<x> = QQ[]
    sage: P = PartitionAlgebra(2, x, R)
    sage: pp = P.an_element(); pp
    3*O\{\{-2\}, \{-1, 1, 2\}\} + 7*O\{\{-2, -1, 1, 2\}\} + 2*O\{\{-2, 1, 2\}, \{-1\}\}
    sage: pp.to_orbit_basis()
    3*O\{\{-2\}, \{-1, 1, 2\}\} + 7*O\{\{-2, -1, 1, 2\}\} + 2*O\{\{-2, 1, 2\}, \{-1\}\}
\end{verbatim}

\begin{verbatim}
L(i)
    Return the i-th Jucys-Murphy element \(L_i\) from [Eny2012].

    INPUT:
    * i – a half integer between 1/2 and k
\end{verbatim}

5.1. Comprehensive Module List
ALGORITHM:

We use the recursive definition for $L_2$, given in [Cre2020]. See also [Eny2012] and [Eny2013].

**Note:** $L_{1/2}$ and $L_1$ differs from [HR2005].

EXAMPLES:

```
sage: R.<n> = QQ[]
sgae: P3 = PartitionAlgebra(3, n)
sage: P3.jucys_murphy_element(1/2)
0
sage: P3.jucys_murphy_element(1)
P{{-3, 3}, {-2, 2}, {-1}, {1}}

sage: P3.jucys_murphy_element(2)
P{{-3, 3}, {-2}, {-1, 1}, {2}} - P{{-3, 3}, {-2}, {-1, 1, 2}}
 + P{{-3, 3}, {-2, -1}, {1, 2}} - P{{-3, 3}, {-2, -1, 1}, {2}}
 + P{{-3, 3}, {-2, 1}, {-1, 2}}

sage: P3.jucys_murphy_element(3/2)
n*P{{-3, 3}, {-2}, {-1, 1, 2}} - P{{-3, 3}, {-2, -1, 2}, {1}}
 - P{{-3, 3}, {-2, 1, 2}, {-1}} + P{{-3, 3}, {-2, 2}, {-1, 1}}

True
```

We test the relations in Lemma 2.2.3(2) in [Cre2020] (v1):

```
sage: k = 4
sage: R.<n> = QQ[]
sgae: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)
True
sage: all(x * Lsum == Lsum * x for x in gens)
True
```

Also the relations in Lemma 2.2.3(3) in [Cre2020] (v1):

```
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i-1)/2)
....: == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,k))
```

(continues on next page)
The same tests for a half integer partition algebra:

```
sage: k = 9/2
sage: R.<n> = QQ[]
sage: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)  # long time
    True
sage: all(x * y == y * x for x in L for y in L)
    True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k-1/2)]
    + [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
    True
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
        == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,floor(k)))
    True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
        == (P.L(i/2) + P.L((i+1)/2)) * P.e(i/2)
        == n * P.e(i/2) for i in range(1,2*k))
    True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
        == P.L(2*i/2) * P.e(2*i/2) for i in range(1,ceil(k)))
    True
sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
        == P.e(2*i/2) * P.L(2*i/2) for i in range(1,ceil(k)))
    True
```

The same tests for a half integer partition algebra:

```
sage: k = 9/2
sage: R.<n> = QQ[]
sage: P = PartitionAlgebra(k, n)
sage: L = [P.L(i/2) for i in range(1,2*k+1)]
sage: all(x.dual() == x for x in L)  # long time
    True
sage: all(x * y == y * x for x in L for y in L)
    True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k-1/2)]
    + [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
    True
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
        == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,floor(k)))
    True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
        == (P.L(i/2) + P.L((i+1)/2)) * P.e(i/2)
        == n * P.e(i/2) for i in range(1,2*k))
    True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
        == P.L(2*i/2) * P.e(2*i/2) for i in range(1,ceil(k)))
    True
sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
        == P.e(2*i/2) * P.L(2*i/2) for i in range(1,ceil(k)))
    True
```

Return the element \(a_i\) in \(self\).

The element \(a_i\) is the cap and cup at \((i, i + 1)\), so it contains the blocks \(\{i, i + 1\}\), \(\{-i, -i - 1\}\). Other blocks are of the form \(\{-j, j\}\).

**INPUT:**

- \(i\) – an integer between 1 and \(k - 1\)

**EXAMPLES:**

```python
5.1. Comprehensive Module List
```
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```python
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.a(1)
P{{-3, 3}, {-2, -1}, {1, 2}}
sage: P3.a(2)
P{{-3, -2}, {-1, 1}, {2, 3}}

sage: P3 = PartitionAlgebra(5/2, n)
sage: P3.a(1)
P{{-3, 3}, {-2, -1}, {1, 2}}
sage: P3.a(2)
Traceback (most recent call last):
...  ValueError: i must be an integer between 1 and 1
```

**e(i)**

Return the element $e_i$ in self.

If $i = (2r + 1)/2$, then $e_i$ contains the blocks $\{r + 1\}$ and $\{-r - 1\}$. If $i \in \mathbb{Z}$, then $e_i$ contains the block $\{-i, -i - 1, i, i + 1\}$. Other blocks are of the form $\{-j, j\}$.

**INPUT:**

- $i$ – a half integer between 1/2 and $k - 1/2$

**EXAMPLES:**

```python
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.e(1)
P{{-3, 3}, {-2, -1, 1, 2}}
sage: P3.e(2)
P{{-3, -2, 2, 3}, {-1, 1}}
sage: P3.e(1/2)
P{{-3, 3}, {-2, 2}, {-1}, {1}}
sage: P3.e(5/2)
P{{-3}, {-2, 2}, {-1, 1}, {3}}
sage: P3.e(0)
Traceback (most recent call last):
...  ValueError: i must be an (half) integer between 1/2 and 5/2
sage: P3.e(3)
Traceback (most recent call last):
...  ValueError: i must be an (half) integer between 1/2 and 5/2
```

**generator_a(i)**

Return the element $a_i$ in self.
The element $a_i$ is the cap and cup at $(i, i + 1)$, so it contains the blocks $\{i, i + 1\}, \{-i, -i - 1\}$. Other blocks are of the form $\{-j, j\}$.

**INPUT:**

- $i$ – an integer between 1 and $k - 1$

**EXAMPLES:**

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.a(1)
P{(-3, 3), {-2, -1}, {1, 2}}
sage: P3.a(2)
P{(-3, -2), {-1, 1}, {2, 3}}
```

```
sage: P3 = PartitionAlgebra(5/2, n)
sage: P3.a(1)
P{(-3, 3), {-2, -1}, {1, 2}}
sage: P3.a(2)
Traceback (most recent call last):
  ...
ValueError: i must be an integer between 1 and 1
```
generator_s(i)

Return the i-th simple transposition \( s_i \) in self.

Borrowing the notation from the symmetric group, the \( i \)-th simple transposition \( s_i \) has blocks of the form \( \{ -i, i+1 \}, \{ -i-1, i \} \). Other blocks are of the form \( \{ -j, j \} \).

INPUT:

• \( i \) – an integer between 1 and \( k-1 \)

EXAMPLES:

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.s(1)
P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.s(2)
P{{-3, 2}, {-2, 3}, {-1, 1}}
```

\[ \text{continued on next page} \]

\[ \text{(Page dimensions: 612.0x792.0)} \]

\[ \text{814 Chapter 5. Comprehensive Module List} \]

\[ \text{(continued from previous page)} \]

\[ \text{generator_s(i)} \]

\[ \text{Return the i-th simple transposition } s_i \text{ in self.} \]

\[ \text{Borrowing the notation from the symmetric group, the } i \text{-th simple transposition } s_i \text{ has blocks of the form } \{ -i, i+1 \}, \{ -i-1, i \}. \text{ Other blocks are of the form } \{ -j, j \}. \]

\[ \text{INPUT:} \]

• \( i \) – an integer between 1 and \( k-1 \)

\[ \text{EXAMPLES:} \]

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.s(1)
P{{-3, 3}, {-2, 1}, {-1, 2}}
sage: P3.s(2)
P{{-3, 2}, {-2, 3}, {-1, 1}}
```

\[ \text{jucys_murphy_element(i)} \]

Return the \( i \)-th Jucys-Murphy element \( L_i \) from [Eny2012].

INPUT:

• \( i \) – a half integer between 1/2 and \( k \)

ALGORITHM:

We use the recursive definition for \( L_{2i} \) given in [Cre2020]. See also [Eny2012] and [Eny2013].

\[ \text{Note: } L_{1/2} \text{ and } L_1 \text{ differs from [HR2005].} \]

\[ \text{EXAMPLES:} \]

```
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.jucys_murphy_element(1/2)
0
sage: P3.jucys_murphy_element(1)
P{{-3, 3}, {-2, 2}, {-1}, {1}}
sage: P3.jucys_murphy_element(2)
P{{-3, 3}, {-2}, {-1, 1}, {2}} - P{{-3, 3}, {-2}, {-1, 1, 2}}
+ P{{-3, 3}, {-2, -1}, {1, 2}} - P{{-3, 3}, {-2, -1, 1}, {2}}
+ P{{-3, 3}, {-2}, {1}, {-1, 2}}
sage: P3.jucys_murphy_element(3/2)
```

\[ \text{(continues on next page)} \]
We test the relations in Lemma 2.2.3(2) in [Cre2020] (v1):

```python
sage: k = 4
sage: R.<n> = QQ[]

sage: P = PartitionAlgebra(k, n)

sage: L = [P.L(i/2) for i in range(1,2*k+1)]

sage: all(x.dual() == x for x in L)
True

sage: all(x * y == y * x for x in L for y in L)  # long time
True

sage: Lsum = sum(L)

sage: gens = [P.s(i) for i in range(1,k)]

sage: gens += [P.e(i/2) for i in range(1,2*k)]

sage: all(x * Lsum == Lsum * x for x in gens)
True
```

Also the relations in Lemma 2.2.3(3) in [Cre2020] (v1):

```python
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....: == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,k))
True

sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....: == P.L(i/2) + P.L((i+1)/2) * P.e(i/2)
....: == n * P.e(i/2) for i in range(1,2*k))
True

sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....: == P.L(2*i/2) * P.e(2*i/2) for i in range(1,k))
True

sage: all(P.e(2*i/2) * P.e((2*i-1)/2) * P.sigma(2*i/2)
....: == P.e(2*i/2) * P.L(2*i/2) for i in range(1,k))
True

sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....: == P.e(2*i/2) * P.L(2*i/2) for i in range(1,k))
True
```

The same tests for a half integer partition algebra:

```python
sage: k = 9/2
sage: R.<n> = QQ[]

sage: P = PartitionAlgebra(k, n)

sage: L = [P.L(i/2) for i in range(1,2*k+1)]

sage: all(x.dual() == x for x in L)
True

sage: all(x * y == y * x for x in L for y in L)  # long time
True
```
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sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k-1/2)]
sage: gens += [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
True
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....:   == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,floor(k)))
True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....:   == n * P.e(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....:   == P.L(2*i/2) * P.e(2*i/2) for i in range(1,ceil(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:   == P.e(2*i/2) * P.L(2*i/2) for i in range(1,floor(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:   == P.e(2*i/2) * P.L(2*i/2) for i in range(1,floor(k)))
True
sage: Lsum = sum(L)
sage: gens = [P.s(i) for i in range(1,k-1/2)]
sage: gens += [P.e(i/2) for i in range(1,2*k)]
sage: all(x * Lsum == Lsum * x for x in gens)
True
sage: all(P.e((2*i+1)/2) * P.sigma(2*i/2) * P.e((2*i+1)/2)
....:   == (n - P.L((2*i-1)/2)) * P.e((2*i+1)/2) for i in range(1,floor(k)))
True
sage: all(P.e(i/2) * (P.L(i/2) + P.L((i+1)/2))
....:   == n * P.e(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(2*i/2) * P.e((2*i-1)/2) * P.e(2*i/2)
....:   == P.L(2*i/2) * P.e(2*i/2) for i in range(1,ceil(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:   == P.e(2*i/2) * P.L(2*i/2) for i in range(1,floor(k)))
True
sage: all(P.e(2*i/2) * P.e((2*i+1)/2) * P.sigma((2*i+1)/2)
....:   == P.e(2*i/2) * P.L(2*i/2) for i in range(1,floor(k)))

orbit_basis()

Return the orbit basis of self.

EXAMPLES:

sage: R.<x> = QQ[]
sage: P2 = PartitionAlgebra(2, x, R)
sage: O2 = P2.orbit_basis(); O2
Orbit basis of Partition Algebra of rank 2 with parameter x over
Univariate Polynomial Ring in x over Rational Field
sage: pp = 7 * P2[{-1}, {-2, 1, 2}] - 2 * P2[{-2}, {-1, 1}, {2}]; pp
-2*P{-2} - 1, 2} - 2*O{-2}, 1, 2} + 7*P{-2, 1, 2} - 1, 2} - 2*O{-2, 2} - 1, 2}
- 2*O{-2, 1, 2} - 1, 2} - 2*O{-2, 2} - 1, 2}
- 2*O{-2, 1, 2} - 1, 2} - 2*O{-2, 2} - 1, 2}
sage: op = pp.to_orbit_basis(); op
-2*O{-2} - 1, 2} - 2*O{-2}, 1, 2} - 2*O{-2, 1, 2} - 2*O{-2, 2} - 1, 2}
- 2*O{-2, 1, 2} - 2*O{-2, 2} - 1, 2}
- 2*O{-2, 1, 2} - 2*O{-2, 2} - 1, 2}
- 2*O{-2, 1, 2} - 2*O{-2, 2} - 1, 2}
ist()

Return the i-th simple transposition s_i in self.

Borrowing the notation from the symmetric group, the i-th simple transposition s_i has blocks of the form
{-i, i + 1}, {-i - 1, i}. Other blocks are of the form \{-j, j\}.

INPUT:
• \( i \) – an integer between 1 and \( k - 1 \)

EXAMPLES:

```python
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.s(1)
P\{\{-3, 3\}, \{-2, 1\}, \{-1, 2\}\}
sage: P3.s(2)
P\{\{-3, 2\}, \{-2, 3\}, \{-1, 1\}\}
```

```python
sage: R.<n> = ZZ[]
sage: P2h = PartitionAlgebra(5/2, n)
sage: P2h.s(1)
P\{\{-3, 3\}, \{-2, 1\}, \{-1, 2\}\}
```

\( \sigma(i) \)

Return the element \( \sigma_i \) from [Eny2012] of self.

INPUT:

• \( i \) – a half integer between 1/2 and \( k - 1/2 \)

Note: In [Cre2020] and [Eny2013], these are the elements \( \sigma_{2i} \).

EXAMPLES:

```python
sage: R.<n> = QQ[]
sage: P3 = PartitionAlgebra(3, n)
sage: P3.sigma(1)
P\{\{-3, 3\}, \{-2, 2\}, \{-1, 1\}\}
sage: P3.sigma(3/2)
P\{\{-3, 3\}, \{-2, 1\}, \{-1, 2\}\}
sage: P3.sigma(2)
-P\{\{-3, -1, 1, 3\}, \{-2, 2\}\} + P\{\{-3, -1, 3\}, \{-2, 1, 2\}\} + P\{\{-3, 1, 3\}, \{-2, -1, 2\}\} - P\{\{-3, 3\}, \{-2, -1, 1\}\} + P\{\{-3, 3\}, \{-2, 2\}, \{-1, 1\}\}
sage: P3.sigma(5/2)
-P\{\{-3, -1, 1, 2\}, \{-2, 3\}\} + P\{\{-3, -1, 2\}, \{-2, 1, 3\}\} + P\{\{-3, 1, 2\}, \{-2, -1, 3\}\} - P\{\{-3, 2\}, \{-2, -1, 1\}\} + P\{\{-3, 2\}, \{-2, 3\}, \{-1, 1\}\}
```

We test the relations in Lemma 2.2.3(1) in [Cre2020] (v1):

```python
sage: k = 4
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(k, x)
sage: all(P.sigma(i/2).dual() == P.sigma(i/2) for i in range(1,2*k))
True
sage: all(P.sigma(i)*P.e(i) == P.e(i)*P.sigma(i) == P.e(i) for i in range(1,floor(k)))
```

(continues on next page)
for i in range(1,floor(k))
True
sage: all(P.sigma(i+1/2)*P.e(i) == P.e(i)*P.sigma(i+1/2) == P.e(i)
....: for i in range(1,floor(k))
True

sage: k = 9/2
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(k, x)
sage: all(P.sigma(i/2).dual() == P.sigma(i/2)
....: for i in range(1,2*k-1))
True
sage: all(P.sigma(i)*P.sigma(i+1/2) == P.sigma(i+1/2)*P.sigma(i) == P.s(i)
....: for i in range(1,k-1/2))
True
sage: all(P.sigma(i)*P.e(i) == P.e(i)*P.sigma(i) == P.e(i)
....: for i in range(1,floor(k))
True
sage: all(P.sigma(i+1/2)*P.e(i) == P.e(i)*P.sigma(i+1/2) == P.e(i)
....: for i in range(1,floor(k))
True

class sage.combinat.diagram_algebras.PermutationDiagram

Bases: AbstractPartitionDiagram

The element class for a permutation diagram.

A permutation diagram for an integer $k$ is a partition of the set \{1, \ldots, k, -1, \ldots, -k\}

EXAMPLES:

sage: from sage.combinat.diagram_algebras import PermutationDiagram, PermutationDiagrams
sage: PartitionDiagrams(1)
Partition diagrams of order 1
sage: PartitionDiagrams(1).list()
\[
\{\{1\}\}, \{\{-1\}\}
\]

sage: PartitionDiagrams(1,2)
Partition diagrams of order 2

class sage.combinat.diagram_algebras.PermutationDiagrams

Bases: AbstractPartitionDiagrams

This class represents all permutation diagrams of integer or integer +1/2 order.

EXAMPLES:

sage: import sage.combinat.diagram_algebras as da
sage: pd = da.PartitionDiagrams(1); pd
Partition diagrams of order 1
sage: pd.list()
\[
\{\{1\}\}, \{\{-1\}\}
\]
sage: pd = da.PartitionDiagrams(3/2); pd
Partition diagrams of order 3/2
sage: pd.list()
[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}}]

Element

alias of PartitionDiagram

cardinality()

The cardinality of partition diagrams of half-integer order \( n \) is the \( 2^n \)-th Bell number.

EXAMPLES:

sage: import sage.combinat.diagram_algebras as da
sage: pd = da.PartitionDiagrams(3)
sage: pd.cardinality()
203
sage: pd = da.PartitionDiagrams(7/2)
sage: pd.cardinality()
877

class sage.combinat.diagram_algebras.PlanarAlgebra(k, q, base_ring, prefix)

Bases: SubPartitionAlgebra, UnitDiagramMixin

A planar algebra.

The planar algebra of rank \( k \) is an algebra with basis indexed by the collection of all planar set partitions of \( \{1, \ldots, k, -1, \ldots, -k\} \).

This algebra is thus a subalgebra of the partition algebra. For more information, see PartitionAlgebra.

INPUT:

- \( k \) – rank of the algebra
- \( q \) – the deformation parameter \( q \)

OPTIONAL ARGUMENTS:

- base_ring – (default None) a ring containing \( q \); if None then just takes the parent of \( q \)
- prefix – (default "Pl") a label for the basis elements

EXAMPLES:

We define the planar algebra of rank 2 with parameter \( x \) over \( \mathbb{Z} \):

sage: R.<x> = ZZ[]
sage: Pl = PlanarAlgebra(2, x, R); Pl
Planar Algebra of rank 2 with parameter x over Univariate Polynomial Ring in x over Integer Ring
sage: Pl.basis().keys()
Planar diagrams of order 2

(continues on next page)
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sage: Pl.basis().keys()([[[-1], [1], [2], [-2]]])
{{-2}, {-1}, {1}}
sage: Pl.basis().list()
[Pl\{[-2], \{-1\}, \{1, 2\}\},
 Pl\{[-2], \{-1\}, \{1\}, \{2\}\},
 Pl\{[-2, 1], \{-1\}, \{2\}\},
 Pl\{[-2, 2], \{-1\}, \{1\}\},
 Pl\{[-2, 1, 2], \{-1\}\},
 Pl\{[-2, 2], \{-1, 1\}\},
 Pl\{[-2], \{-1, 1\}, \{2\}\},
 Pl\{[-2], \{-1, 2\}, \{1\}\},
 Pl\{[-2], \{-1\}, \{1, 2\}\},
 Pl\{[-2, 1], \{-1\}, \{2\}\},
 Pl\{[-2], \{-1\}, \{1\}, \{2\}\},
 Pl\{[-2, -1, 1], \{2\}\},
 Pl\{[-2, -1, 2], \{1\}\},
 Pl\{[-2], \{-1, 1\}, \{2\}\},
 Pl\{[-2], \{-1\}, \{1, 2\}\},
 Pl\{[-2, -1], \{1\}, \{2\}\},
 Pl\{[-2, -1], \{1, 2\}\},
 Pl\{[-2], \{-1\}, \{1\}, \{2\}\}]
sage: E = Pl([[1,2],[-1,-2]])
sage: E^2 == x*E
True
sage: E^5 == x^4*E
True

class sage.combinat.diagram_algebras.PlanarDiagram(parent, d, check=True)
Bases: AbstractPartitionDiagram

The element class for a planar diagram.

A planar diagram for an integer $k$ is a partition of the set \{1, \ldots, k, -1, \ldots, -k\} so that the diagram is non-crossing.

EXAMPLES:

sage: from sage.combinat.diagram_algebras import PlanarDiagrams
sage: PlanarDiagrams(2)
Planar diagrams of order 2
sage: PlanarDiagrams(2).list()
[['[-2], \{-1\}, \{1, 2\}\],
 ['[-2], \{-1\}, \{1\}, \{2\}\],
 ['[-2, 1], \{-1\}, \{2\}\],
 ['[-2, 2], \{-1\}, \{1\}\],
 ['[-2, 1, 2], \{-1\}\],
 ['[-2], \{-1, 1\}, \{2\}\],
 ['[-2], \{-1\}, \{1\}, \{2\}\],
 ['[-2, 1, 2], \{-1\}\]]

check()
Check the validity of the input for self.
class sage.combinat.diagram_algebras.PlanarDiagrams(order, category=None)

Bases: AbstractPartitionDiagrams

All planar diagrams of integer or integer +1/2 order.

EXAMPLES:

```sage
import sage.combinat.diagram_algebras as da
sage: pld = da.PlanarDiagrams(1); pld
Planar diagrams of order 1
sage: pld.list()
[{{-1, 1}}, {{-1}, {1}}]
```

Element

alias of PlanarDiagram

cardinality()

Return the cardinality of self.

The number of all planar diagrams of order \(k\) is the \(2k\)-th Catalan number.

EXAMPLES:

```sage
import sage.combinat.diagram_algebras as da
sage: pld = da.PlanarDiagrams(3)
sage: pld.cardinality()
132
```

class sage.combinat.diagram_algebras.PropagatingIdeal(k, q, base_ring, prefix)

Bases: SubPartitionAlgebra

A propagating ideal.

The propagating ideal of rank \(k\) is a non-unital algebra with basis indexed by the collection of ideal set partitions of \(\{1, \ldots, k, -1, \ldots, -k\}\). We say a set partition is ideal if its propagating number is less than \(k\).

This algebra is a non-unital subalgebra and an ideal of the partition algebra. For more information, see PartitionAlgebra.

EXAMPLES:

We now define the propagating ideal of rank 2 with parameter \(x\) over \(\mathbb{Z}\):

```sage
R.<x> = QQ[]
sage: I = PropagatingIdeal(2, x, R); I
Propagating Ideal of rank 2 with parameter x over Univariate Polynomial Ring in x over Rational Field
sage: I.keys()
Ideal diagrams of order 2
```
sage: I.basis().list()
[I{{-2, -1, 1, 2}},
 I{{-2, 1, 2}, {-1}},
 I{{-2}, {-1, 1, 2}},
 I{{-2, -1}, {1, 2}},
 I{{-2}, {-1}, {1, 2}},
 I{{-2, -1, 1}, {2}},
 I{{-2, 1}, {-1}, {2}},
 I{{-2, -1, 2}, {1}},
 I{{-2}, {-1, 2}, {1}},
 I{{-2, 2}, {-1}, {1}},
 I{{-2}, {-1, 1}, {2}},
 I{{-2}, {-1, 2}, {1}},
 I{{-2}, {-1}, {1}, {2}},
 I{{-2, -1}, {1}, {2}}]

sage: E = I([[1,2],[-1,-2]])

sage: E^2 == x*E
True

sage: E^5 == x^4*E
True

class Element
    Bases: Element

    An element of a propagating ideal.

    We need to take care of exponents since we are not unital.

class sage.combinat.diagram_algebras.SubPartitionAlgebra(k, q, base_ring, prefix, diagrams, category=None)

    Bases: DiagramBasis

    A subalgebra of the partition algebra in the diagram basis indexed by a subset of the diagrams.

class Element
    Bases: Element

    to_orbit_basis()

    Return self in the orbit basis of the associated ambient partition algebra.

    EXAMPLES:

    sage: R.<x> = QQ[]
    sage: B = BrauerAlgebra(2, x, R)
    sage: bb = B([-2, -1], [1, 2]); bb
    B{-2, -1}, {1, 2})
    sage: bb.to_orbit_basis()
    0{-2, -1}, {1, 2}) + 0{-2, -1, 1, 2})

    ambient()

    Return the partition algebra self is a sub-algebra of.

    EXAMPLES:

    sage: x = var('x')
    sage: BA = BrauerAlgebra(2, x)
lift()
Return the lift map from diagram subalgebra to the ambient space.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: BA = BrauerAlgebra(2, x, R)
sage: E = BA([[1,2],[1,-1],[1,-2]])
sage: lifted = BA.lift(E); lifted
B{{-2, -1}, {1, 2}}
sage: lifted.parent() is BA.ambient()
True
```

retract(x)
Retract an appropriate partition algebra element to the corresponding element in the partition subalgebra.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: BA = BrauerAlgebra(2, x, R)
sage: PA = BA.ambient()
sage: E = PA([[1,2], [-1,-2]])
sage: BA.retract(E) in BA
True
```

```
sage.combinat.diagram_algebras.TL_diagram_ascii_art(diagram, use_unicode=False, blobs=[]) Return ascii art for a Temperley-Lieb diagram diagram.

INPUT:
- diagram -- a list of pairs of matchings of the set \{-1, \ldots, -n, 1, \ldots, n\}
- use_unicode -- (default: False) whether or not to use unicode art instead of ascii art
- blobs -- (optional) a list of matchings with blobs on them

EXAMPLES:

```
sage: from sage.combinat.diagram_algebras import TL_diagram_ascii_art
sage: TL = [(-15,-12), (-14,-13), (-11,15), (-10,14), (-9,-6),
....:  (-8,-7), (-5,-4), (-3,1), (-2,-1), (2,3), (4,5),
....:  (6,11), (7, 8), (9,10), (12,13)]
sage: TL_diagram_ascii_art(TL, use_unicode=False)
   o o o o o o o o o o o o o o o
      | `--` `--` `--` `--` `--` `--`
      |  `--` `--` `--` `--` `--` `--`
      `--` | `--` | `--` | `--` | `--` | `--`
   o o o o o o o o o o o o o o o

sage: TL_diagram_ascii_art(TL, use_unicode=True)
```
```
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sage: TL = [(-20,-9), (-19,-10), (-18,-11), (-17,-16), (-15,-12), (2,3),
       ....:   (-14,-13), (-8,16), (-7,7), (-6,6), (-5,1), (-4,-3), (-2,-1),
       ....:   (4,5), (8,15), (9,10), (11,14), (12,13), (17,20), (18,19)]
sage: TL_diagram_ascii_art(TL, use_unicode=False, blobs=[(-2,-1), (-5,1)])

sage: TL_diagram_ascii_art(TL, use_unicode=True, blobs=[(-2,-1), (-5,1)])

class sage.combinat.diagram_algebras.TemperleyLiebAlgebra(k, q, base_ring, prefix)

Bases: SubPartitionAlgebra, UnitDiagramMixin

A Temperley–Lieb algebra.

The Temperley–Lieb algebra of rank \( k \) is an algebra with basis indexed by the collection of planar set partitions of \( \{1, \ldots, k, -1, \ldots, -k\} \) with block size 2.

This algebra is thus a subalgebra of the partition algebra. For more information, see PartitionAlgebra.

INPUT:

- \( k \) – rank of the algebra
- \( q \) – the deformation parameter \( q \)

OPTIONAL ARGUMENTS:

- \( \text{base\_ring} \) – (default None) a ring containing \( q \); if None then just takes the parent of \( q \)
• prefix – (default "T") a label for the basis elements

EXAMPLES:

We define the Temperley–Lieb algebra of rank 2 with parameter $x$ over $\mathbb{Z}$:

```python
sage: R.<x> = ZZ[]
sage: T = TemperleyLiebAlgebra(2, x, R); T
Temperley-Lieb Algebra of rank 2 with parameter $x$ over Univariate Polynomial Ring in $x$ over Integer Ring
sage: T.basis()
Lazy family (Term map from Temperley Lieb diagrams of order 2 to Temperley-Lieb Algebra of rank 2 with parameter $x$ over Univariate Polynomial Ring in $x$ over Integer Ring(i))_{i in Temperley Lieb diagrams of order 2}
sage: T.basis().keys()
Temperley Lieb diagrams of order 2
sage: T.basis().keys()([-1, 1], [2, -2])
{{-2, 2}, {-1, 1}}
sage: b = T.basis().list(); b
[T{{-2, -1}, {1, 2}}, T{{-2, 2}, {-1, 1}}]
sage: b[0]
T{{-2, -1}, {1, 2}}
sage: b[0]^2 == x*b[0]
True
sage: b[0]^5 == x^4*b[0]
True
```

class sage.combinat.diagram_algebras.TemperleyLiebDiagram

The element class for a Temperley-Lieb diagram.

A Temperley-Lieb diagram for an integer $k$ is a partition of the set $\{1, \ldots, k, -1, \ldots, -k\}$ so that the blocks are all of size 2 and the diagram is planar.

EXAMPLES:

```python
sage: from sage.combinat.diagram_algebras import TemperleyLiebDiagrams
sage: TemperleyLiebDiagrams(2)
Temperley Lieb diagrams of order 2
sage: TemperleyLiebDiagrams(2).list()
[[{-2, -1}, {1, 2}], {{-2, 2}, {-1, 1}}]
```

check()

Check the validity of the input for self.

class sage.combinat.diagram_algebras.TemperleyLiebDiagrams

All Temperley-Lieb diagrams of integer or integer $+1/2$ order.

For more information on Temperley-Lieb diagrams, see TemperleyLiebAlgebra.

EXAMPLES:
```python
sage: import sage.combinat.diagram_algebras as da
dsage: td = da.TemperleyLiebDiagrams(3); td
Temperley Lieb diagrams of order 3
tsage: td.list()
[[{-3, 3}, {-2, -1}, {1, 2}],
 {-3, 1}, {-2, -1}, {2, 3}],
 {{-3, -2}, {-1, 1}, {2, 3}},
 {{-3, -2}, {-1, 3}, {1, 2}},
 {{-3, 3}, {-2, 2}, {-1, 1}}]
sage: td = da.TemperleyLiebDiagrams(5/2); td
Temperley Lieb diagrams of order 5/2
tsage: td.list()
[[{-3, 3}, {-2, -1}, {1, 2}], {{-3, 3}, {-2, 2}, {-1, 1}}]
```

**Element**

alias of `TemperleyLiebDiagram`

**cardinality()**

Return the cardinality of `self`.

The number of Temperley–Lieb diagrams of integer order $k$ is the $k$-th Catalan number.

**EXAMPLES:**

```python
sage: import sage.combinat.diagram_algebras as da
dsage: td = da.TemperleyLiebDiagrams(3)
dsage: td.cardinality()
5
```

**class** `sage.combinat.diagram_algebras.UnitDiagramMixin`

Bases: object

Mixin class for diagram algebras that have the unit indexed by the `identity_set_partition()`.

**one_basis()**

The following constructs the identity element of `self`.

It is not called directly; instead one should use `DA.one()` if `DA` is a defined diagram algebra.

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: P = PartitionAlgebra(2, x, R)
sage: P.one_basis()
{{-2, 2}, {-1, 1}}
```

**sage.combinat.diagram_algebras.brauer_diagrams($k$)**

Return a generator of all Brauer diagrams of order $k$.

A Brauer diagram of order $k$ is a partition diagram of order $k$ with block size 2.

**INPUT:**

- $k$ – the order of the Brauer diagrams

**EXAMPLES:**
```
sage: import sage.combinat.diagram_algebras as da
sage: [SetPartition(p) for p in da.brauer_diagrams(2)]
[{{-2, -1}, {1, 2}}, {{-2, 1}, {-1, 2}}, {{-2, 2}, {-1, 1}}]
sage: [SetPartition(p) for p in da.brauer_diagrams(5/2)]
[{{-3, 3}, {-2, -1}, {1, 2}}, {{-3, 3}, {-2, 1}, {-1, 2}}, {{-3, 3}, {-2, 2}, {-1, 1}}]
```

**sage.combinat.diagram_algebras.diagram_latex**(diagram, fill=False, edge_options=None, edge_additions=None)

Return latex code for the diagram diagram using tikz.

**EXAMPLES:**

```
sage: from sage.combinat.diagram_algebras import PartitionDiagrams, diagram_latex
sage: P = PartitionDiagrams(2)
sage: D = P([[-1, 2,]],)
sage: print(diagram_latex(D))
\begin{tikzpicture}[scale = 0.5,thick, baseline={(0,-1ex/2)}]
\tikzstyle{vertex} = [shape = circle, minimum size = 7pt, inner sep = 1pt]
\node[vertex] (G--2) at (1.5, -1) [shape = circle, draw] {};
\node[vertex] (G--1) at (0.0, -1) [shape = circle, draw] {};
\node[vertex] (G-1) at (0.0, 1) [shape = circle, draw] {};
\node[vertex] (G-2) at (1.5, 1) [shape = circle, draw] {};
\draw (G--2) .. controls +(-0.5, 0.5) and +(0.5, 0.5) .. (G--1);
\draw (G-1) .. controls +(0.5, -0.5) and +(-0.5, -0.5) .. (G-2);
\end{tikzpicture}
```

**sage.combinat.diagram_algebras.ideal_diagrams**(k)

Return a generator of all “ideal” diagrams of order k.

An ideal diagram of order k is a partition diagram of order k with propagating number less than k.

**EXAMPLES:**

```
sage: import sage.combinat.diagram_algebras as da
sage: all_diagrams = da.partition_diagrams(2)
sage: [SetPartition(p) for p in all_diagrams if p not in da.ideal_diagrams(2)]
[{{-2, 1}, {-1, 2}}, {{-2, 2}, {-1, 1}}]
sage: all_diagrams = da.partition_diagrams(3/2)
sage: [SetPartition(p) for p in all_diagrams if p not in da.ideal_diagrams(3/2)]
[{{-2, 2}, {-1, 1}}]
```

**sage.combinat.diagram_algebras.identity_set_partition**(k)

Return the identity set partition \{1, -1\}, \ldots, \{k, -k\}.

**EXAMPLES:**

```
sage: import sage.combinat.diagram_algebras as da
sage: SetPartition(da.identity_set_partition(2))
{{-2, 2}, {-1, 1}}
```

**sage.combinat.diagram_algebras.is_planar**(sp)

Return True if the diagram corresponding to the set partition sp is planar; otherwise, return False.
EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: da.is_planar( da.to_set_partition([[1,-2],[2,-1]]))
False
sage: da.is_planar( da.to_set_partition([[1,-1],[2,-2]]))
True
```

`sage.combinat.diagram_algebras.pair_to_graph(sp1, sp2)`

Return a graph consisting of the disjoint union of the graphs of set partitions `sp1` and `sp2` along with edges joining the bottom row (negative numbers) of `sp1` to the top row (positive numbers) of `sp2`.

The vertices of the graph `sp1` appear in the result as pairs `(k, 1)`, whereas the vertices of the graph `sp2` appear as pairs `(k, 2)`.

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: sp1 = da.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = da.to_set_partition([[1,-2],[2,-1]])
sage: g = da.pair_to_graph( sp1, sp2 ); g
Graph on 8 vertices

sage: g.vertices(sort=True)
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]

sage: g.edges(sort=True)
[((-2, 1), (1, 1), None), ((-2, 1), (2, 2), None),
 ((-1, 1), (1, 2), None), ((-1, 1), (2, 1), None),
 ((-1, 2), (2, 2), None)]
```

Another example which used to be wrong until github issue #15958:

```python
sage: sp3 = da.to_set_partition([[1, -1], [2], [-2]])
sage: sp4 = da.to_set_partition([[1], [-1], [2], [-2]])
sage: g = da.pair_to_graph( sp3, sp4 ); g
Graph on 8 vertices

sage: g.vertices(sort=True)
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]

sage: g.edges(sort=True)
[((-2, 1), (2, 2), None), ((-1, 1), (1, 1), None),
 ((-1, 1), (1, 2), None), ((-1, 2), (2, 2), None)]
```

`sage.combinat.diagram_algebras.partition_diagrams(k)`

Return a generator of all partition diagrams of order `k`.

A partition diagram of order `k ∈ ℤ` to is a set partition of `{1, ..., k, -1, ..., -k}`. If we have `k - 1/2 ∈ ℤ`, then a partition diagram of order `k ∈ 1/2ℤ` is a set partition of `{1, ..., k + 1/2, -1, ..., -(k + 1/2)}` with `k + 1/2` and `-(k + 1/2)` in the same block. See [HR2005].

INPUT:

- `k` – the order of the partition diagrams

EXAMPLES:
sage: import sage.combinat.diagram_algebras as da
sage: [SetPartition(p) for p in da.partition_diagrams(2)]

[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2}, {-1, 1, 2}},
 {{-2, -1}, {1, 2}},
 {{-2}, {-1}, {1, 2}},
 {{-2, -1}, 2},
 {{-2, 1}, {-1, 2}},
 {{-2, 1}, {-1}, {2}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}},
 {{-2}, {-1, 1}, {2}},
 {{-2}, {-1, 2}, {1}},
 {{-2}, {-1}, {1, 2}},
 {{-2}, {-1}, {1}, {2}}]
sage: [SetPartition(p) for p in da.partition_diagrams(3/2)]

[{{-2, -1, 1, 2}},
 {{-2, 1, 2}, {-1}},
 {{-2, 2}, {-1, 1}},
 {{-2, -1, 2}, {1}},
 {{-2, 2}, {-1}, {1}}]

sage.combinat.diagram_algebras.planar_diagrams(k)

Return a generator of all planar diagrams of order k.

A planar diagram of order k is a partition diagram of order k that has no crossings.

EXAMPLES:

sage: import sage.combinat.diagram_algebras as da
sage: all_diagrams = [SetPartition(p) for p in da.partition_diagrams(2)]

sage: da2 = [SetPartition(p) for p in da.planar_diagrams(2)]

sage: [p for p in all_diagrams if p not in da2]

[{{-2, 1}, {-1, 2}}]
sage: all_diagrams = [SetPartition(p) for p in da.partition_diagrams(5/2)]

sage: da5o2 = [SetPartition(p) for p in da.planar_diagrams(5/2)]

sage: [p for p in all_diagrams if p not in da5o2]

[{{-3, -1, 3}, {-2, 1, 2}},
 {{-3, -1, 3}, {-2, 1, 2}, {-1}},
 {{-3, -2, 1, 3}, {-1, 2}},
 {{-3, -2, 1, 3}, {-2, 2}},
 {{-3, -1, 3}, {-2, 1}, {2}},
 {{-3, -1, 2, 3}, {-2, 1}},
 {{-3, 1, 3}, {-2, 2}, {1}},
 {{-3, 1, 3}, {-2, 1}, {2}},
 {{-3, 1, 3}, {2}},
 {{-3, -1, 3}, {-2, 1}, {2}},
 {{-3, -1, 3}, {-2, 1}, {2}},
 {{-3, -1, 3}, {-2, 1}, {2}}]

sage.combinat.diagram_algebras.planar_partitions_rec(X)

Iterate over all planar set partitions of X by using a recursive algorithm.

ALGORITHM:
To construct the set partition $\rho = \{\rho_1, \ldots, \rho_k\}$ of $[n]$, we remove the part of the set partition containing the last element of $X$, which, we consider to be $\rho_k = \{i_1, \ldots, i_m\}$ without loss of generality. The remaining parts come from the planar set partitions of $\{1, \ldots, i_1 - 1\}, \{i_1 + 1, \ldots, i_2 - 1\}, \ldots, \{i_m + 1, \ldots, n\}$.

**EXAMPLES:**

```python
sage: import sage.combinat.diagram_algebras as da
sage: da.planar_partitions_rec([1,2,3])
[[[1, 2], [3]], ([[1], [2], [3]], [[2], [1, 3]], [[1], [2, 3]], [[1, 2, 3]],)]
```

sage.combinat.diagram_algebras.propagating_number($sp$)

Return the propagating number of the set partition $sp$.

The propagating number is the number of blocks with both a positive and negative number.

**EXAMPLES:**

```python
sage: sp1 = da.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = da.to_set_partition([[1,2],[-2,-1]])
sage: da.propagating_number(sp1)
2
sage: da.propagating_number(sp2)
0
```

sage.combinat.diagram_algebras.temperley_lieb_diagrams($k$)

Return a generator of all Temperley–Lieb diagrams of order $k$.

A Temperley–Lieb diagram of order $k$ is a partition diagram of order $k$ with block size 2 and is planar.

**INPUT:**

- $k$ – the order of the Temperley–Lieb diagrams

**EXAMPLES:**

```python
sage: da.temperley_lieb_diagrams(2)
[[{1, 2}, {-1, -2}], [{-1, 1}, {2, -2}]]
sage: da.temperley_lieb_diagrams(5/2)
[[{1, 2}, {-1, -2}], [{3, -3}, {2, -2}, {1, 1}]]
```

sage.combinat.diagram_algebras.to_Brauer_partition($l$, $k=None$)

Same as `to_set_partition()` but assumes omitted elements are connected straight through.

**EXAMPLES:**

```python
sage: import sage.combinat.diagram_algebras as da
sage: sp = SetPartition([[1,2],[-1,-2],[3,-3],[4,-4]])
sage: SetPartition(da.to_Brauer_partition([[1,2],[-1,-2]], k=4)) == sp
True
```
sage.combinat.diagram_algebras.to_graph(sp)

Return a graph representing the set partition sp.

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
sage: g = da.to_graph(da.to_set_partition([[1,-2],[2,-1]])); g
Graph on 4 vertices

sage: g.vertices(sort=True)
[-2, -1, 1, 2]

sage: g.edges(sort=True)
[(-2, 1, None), (-1, 2, None)]
```

sage.combinat.diagram_algebras.to_set_partition(l, k=None)

Convert input to a set partition of \{1, \ldots, k, -1, \ldots, -k\}

Convert a list of a list of numbers to a set partition. Each list of numbers in the outer list specifies the numbers contained in one of the blocks in the set partition.

If k is specified, then the set partition will be a set partition of \{1, \ldots, k, -1, \ldots, -k\}. Otherwise, k will default to the minimum number needed to contain all of the specified numbers.

INPUT:

- l - a list of lists of integers
- k - integer (optional, default None)

OUTPUT:

- a list of sets

EXAMPLES:

```python
sage: import sage.combinat.diagram_algebras as da
df = lambda sp: SetPartition(da.to_set_partition(sp))
sage: df([[1,-1],[2,-2]]) == SetPartition(da.identity_set_partition(2))
True

sage: da.to_set_partition([[1]])
[[1], [-1]]

sage: da.to_set_partition([[1,-1],[-2,3]],9/2)
[[-1, 1], [-2, 3], [2], [-4, 4], [-5, 5], [-3]]
```

5.1.98 Exact Cover Problem via Dancing Links

sage.combinat.dlx.AllExactCovers(M)

Use A. Ajanki’s DLXMatrix class to solve the exact cover problem on the matrix M (treated as a dense binary matrix).

EXAMPLES:

```python
sage: M = Matrix([[1,1,0],[1,0,1],[0,0,1]])  # no exact covers
sage: for cover in AllExactCovers(M):
....:     print(cover)

sage: M = Matrix([[1,1,0],[1,0,1],[0,0,1],[0,1,0]])  # two exact covers
sage: for cover in AllExactCovers(M):
```
class sage.combinat.dlx.DLXMatrix(ones, initialsolution=None)

Solve the Exact Cover problem by using the Dancing Links algorithm described by Knuth.

Consider a matrix \( M \) with entries of 0 and 1, and compute a subset of the rows of this matrix which sum to the vector of all 1's.

The dancing links algorithm works particularly well for sparse matrices, so the input is a list of lists of the form: (note the 1-index!):

\[
\begin{bmatrix}
[1, [i_{11}, i_{12}, \ldots, i_{1r}]] \\
\vdots \\
[m, [i_{m1}, i_{m2}, \ldots, i_{ms}]]
\end{bmatrix}
\]

where \( M[j][i_{jk}] = 1 \).

The first example below corresponds to the matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

which is exactly covered by:

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

EXAMPLES:

```
sage: from sage.combinat.dlx import *
sage: ones = [[1, [1, 2, 3]]]
sage: ones += [[2, [1, 3]]]
sage: ones += [[3, [2]]]
sage: ones += [[4, [4]]]
sage: DLXM = DLXMatrix(ones, [4])
sage: for C in DLXM:
    ....:     print(C)
[4, 1]
[4, 2, 3]
```
next()

Search for the first solution we can find, and return it.

Knuth describes the Dancing Links algorithm recursively, though actually implementing it as a recursive algorithm is permissible only for highly restricted problems. (for example, the original author implemented this for Sudoku, and it works beautifully there)

What follows is an iterative description of DLX:

```
stack <- [(NULL)]
level <- 0
while level >= 0:
    cur <- stack[level]
    if cur = NULL:
        if R[h] = h:
            level <- level - 1
            yield solution
        else:
            cover(best_column)
            stack[level] = best_column
    else if D[cur] != C[cur]:
        if cur != C[cur]:
            delete solution[level]
            for j in L[cur], L[L[cur]], ... , while j != cur:
                uncover(C[j])
            cur <- D[cur]
            solution[level] <- cur
            stack[level] <- cur
            for j in R[cur], R[R[cur]], ... , while j != cur:
                cover(C[j])
            level <- level + 1
            stack[level] <- (NULL)
        else:
            if C[cur] != cur:
                delete solution[level]
                for j in L[cur], L[L[cur]], ... , while j != cur:
                    uncover(C[j])
                uncover(cur)
            level <- level - 1
```

```
sage.combinat.dlx.OneExactCover(M)

Use A. Ajanki’s DLXMatrix class to solve the exact cover problem on the matrix M (treated as a dense binary matrix).

EXAMPLES:

```
sage: M = Matrix([[1,1,0],[1,0,1],[0,1,1]])  # no exact covers
sage: OneExactCover(M)
sage: M = Matrix([[1,1,0],[1,0,1],[0,0,1],[0,1,0]])  # two exact covers
```

(continues on next page)
sage: OneExactCover(M)
[(1, 1, 0), (0, 0, 1)]

5.1.99 Dyck Words

A class of an object enumerated by the Catalan numbers, see [Sta-EC2], [StaCat98] for details.

AUTHORS:

- Mike Hansen
- Dan Drake (2008-05-30): DyckWordBacktracker support
- Florent Hivert (2009-02-01): Bijections with NonDecreasingParkingFunctions
- Christian Stump (2011-12): added combinatorial maps and statistics
- Mike Zabrocki:
  - (2012-10): added pretty print, characteristic function, more functions
  - (2013-01): added inverse of area/dinv, bounce/area map
- Jean-Baptiste Priez, Travis Scrimshaw (2013-05-17): Added ASCII art

REFERENCES:

class sage.combinat.dyck_word.CompleteDyckWords

Bases: DyckWords

Abstract base class for all complete Dyck words.

Element

alias of DyckWord_complete

from_Catalan_code(code)

Return the Dyck word associated to the given Catalan code code.

A Catalan code of length \(n\) is a sequence \((a_1, a_2, \ldots, a_n)\) of \(n\) integers \(a_i\) such that:

- \(0 \leq a_i \leq n - i\) for every \(i\);
- if \(i < j\) and \(a_i > 0\) and \(a_j > 0\) and \(a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0\), then \(a_i - a_j < j - i\).

It turns out that the Catalan codes of length \(n\) are in bijection with Dyck words.

The Catalan code of a Dyck word is example (x) in Richard Stanley’s exercises on combinatorial interpretations for Catalan objects. The code in this example is the reverse of the description provided there. See [Sta-EC2] and [StaCat98].

EXAMPLES:

sage: DyckWords().from_Catalan_code([])
[]
sage: DyckWords().from_Catalan_code([0])
[1, 0]
sage: DyckWords().from_Catalan_code([0, 1])
[1, 1, 0, 0]
from_area_sequence(\texttt{code})
Return the Dyck word associated to the given area sequence \texttt{code}.
See \texttt{to_area_sequence()} for a definition of the area sequence of a Dyck word.
See also:
\texttt{area()}, \texttt{to_area_sequence()}.

INPUT:
• \texttt{code} – a list of integers satisfying \texttt{code[0]} == 0 and 0 <= \texttt{code[i+1]} <= \texttt{code[i]}+1.

EXAMPLES:

\begin{verbatim}
sage: DyckWords().from_area_sequence([]) []
sage: DyckWords().from_area_sequence([0]) [1, 0]
sage: DyckWords().from_area_sequence([0, 1]) [1, 1, 0, 0]
sage: DyckWords().from_area_sequence([0, 0]) [1, 0, 1, 0]
\end{verbatim}

from_non_decreasing_parking_function(\texttt{pf})
Bijection from \texttt{non-decreasing parking functions}.
See there the method \texttt{to_dyck_word()} for more information.

EXAMPLES:

\begin{verbatim}
sage: D = DyckWords()
sage: D.from_non_decreasing_parking_function([]) []
sage: D.from_non_decreasing_parking_function([1]) [1, 0]
sage: D.from_non_decreasing_parking_function([1,1]) [1, 1, 0, 0]
sage: D.from_non_decreasing_parking_function([1,2]) [1, 0, 1, 0]
sage: D.from_non_decreasing_parking_function([1,1,1]) [1, 1, 0, 0, 1, 0, 0]
sage: D.from_non_decreasing_parking_function([1,2,3]) [1, 0, 1, 0, 1, 0]
sage: D.from_non_decreasing_parking_function([1,1,3,3,4,6,6]) [1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0]
\end{verbatim}

from_noncrossing_partition(\texttt{ncp})
Convert a noncrossing partition \texttt{ncp} to a Dyck word.

EXAMPLES:
sage: DyckWord(noncrossing_partition=[[1,2]])  # indirect doctest
[1, 1, 0, 0]
sage: DyckWord(noncrossing_partition=[[1],[2]])
[1, 0, 1, 0]

sage: dws = DyckWords(5).list()
sage: ncps = [x.to_noncrossing_partition() for x in dws]
sage: dws2 = [DyckWord(noncrossing_partition=x) for x in ncps]
sage: dws == dws2
True

class sage.combinat.dyck_word.CompleteDyckWords_all
Bases: CompleteDyckWords, DyckWords_all
All complete Dyck words.

class height_poset
Bases: UniqueRepresentation, Parent
The poset of complete Dyck words compared componentwise by heights.
This is, D is smaller than or equal to D' if it is weakly below D'.
This is implemented by comparison of area sequences.

le(dw1,dw2)
Compare two Dyck words of equal size, and return True if all of the heights of dw1 are less than or equal to the respective heights of dw2.

See also:
to_area_sequence()

EXAMPLES:

sage: poset = DyckWords().height_poset()
sage: poset.le(DyckWord([]), DyckWord([]))
True
sage: poset.le(DyckWord([1,0]), DyckWord([1,0]))
True
sage: poset.le(DyckWord([1,0,1,0]), DyckWord([1,1,0,0]))
True
sage: poset.le(DyckWord([1,1,0,0]), DyckWord([1,0,1,0]))
False
sage: [poset.le(dw1, dw2)
      for dw1 in DyckWords(3) for dw2 in DyckWords(3)]
[True, True, True, True, False, True, False, True, True, False, True, True, False, False, False, False, False, True]
sage: DyckWords(4).cardinality()
14
sage: ns = list(range(9))
sage: dws = [DyckWords(n) for n in ns]
sage: all(dw.cardinality() == len(dw.list()) for dw in dws)
True

random_element()

Return a random complete Dyck word of semilength $n$.

The algorithm is based on a classical combinatorial fact. One chooses at random a word with $n$ 0’s and
$n + 1$ 1’s. One then considers every 1 as an ascending step and every 0 as a descending step, and one finds
the lowest point of the path (with respect to a slightly tilted slope). One then cuts the path at this point and
builds a Dyck word by exchanging the two parts of the word and removing the initial step.

Todo: extend this to m-Dyck words

EXAMPLES:

sage: dw = DyckWords(8)
sage: dw.random_element()  # random
[1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0]
sage: D = DyckWords(8)
sage: D.random_element() in D
True

class sage.combinat.dyck_word.DyckWord(parent, l, latex_options={})
Bases: CombinatorialElement

A Dyck word.

A Dyck word is a sequence of open and close symbols such that every close symbol has a corresponding open
symbol preceding it. That is to say, a Dyck word of length $n$ is a list with $k$ entries 1 and $n - k$ entries 0 such
that the first $i$ entries always have at least as many 1s among them as 0s. (Here, the 1 serves as the open symbol
and the 0 as the close symbol.) Alternatively, the alphabet 1 and 0 can be replaced by other characters such as
‘(’ and ‘)’.

A Dyck word is complete if every open symbol moreover has a corresponding close symbol.

A Dyck word may also be specified by either a noncrossing partition or by an area sequence or the sequence of
heights.

A Dyck word may also be thought of as a lattice path in the $\mathbb{Z}^2$ grid, starting at the origin $(0, 0)$, and with steps
in the North $N = (0, 1)$ and east $E = (1, 0)$ directions such that it does not pass below the $x = y$ diagonal. The
diagonal is referred to as the “main diagonal” in the documentation. A North step is represented by a 1 in the list
and an East step is represented by a 0.

Equivalently, the path may be represented with steps in the $NE = (1, 1)$ and the $SE = (1, -1)$ direction such
that it does not pass below the horizontal axis.

A path representing a Dyck word (either using $N$ and $E$ steps, or using $NE$ and $SE$ steps) is called a Dyck path.

EXAMPLES:
sage: dw = DyckWord([1, 0, 1, 0]); dw
[1, 0, 1, 0]
sage: print(dw)
()()
sage: dw.height()
1
sage: dw.to_noncrossing_partition()
{{1}, {2}}

sage: DyckWord('()()')
[1, 0, 1, 0]
sage: DyckWord('(())')
[1, 1, 0, 0]
sage: DyckWord('(()')
[1, 1]
sage: DyckWord('')
[]

sage: DyckWord(noncrossing_partition=[[1],[2]])
[1, 0, 1, 0]
sage: DyckWord(noncrossing_partition=[[1,2]])
[1, 1, 0, 0]
sage: DyckWord(noncrossing_partition=[])  # empty
[]

sage: DyckWord(area_sequence=[0,0])
[1, 0, 1, 0]
sage: DyckWord(area_sequence=[0,1])
[1, 1, 0, 0]
sage: DyckWord(area_sequence=[0,1,2,2,0,1,1,2])
[1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0]
```python
sage: DyckWord(area_sequence=[])  
[]

sage: DyckWord(heights_sequence=(0,1,0,1,0))  
[1, 0, 1, 0]

sage: DyckWord(heights_sequence=(0,1,2,1,0))  
[1, 1, 0, 0]

sage: DyckWord(heights_sequence=(0,))  
[]

sage: print(DyckWord([1,0,1,1,0,0]).to_path_string())  
/\  
/\ 
sage: DyckWord([1,0,1,1,0,0]).pretty_print()  
___
| x
_| .
| . .

ascent_prime_decomposition()  
Decompose this Dyck word into a sequence of ascents and prime Dyck paths.

A Dyck word is *prime* if it is complete and has precisely one return - the final step. In particular, the empty Dyck path is not prime. Thus, the factorization is unique.

This decomposition yields a sequence of odd length: the words with even indices consist of up steps only, the words with odd indices are prime Dyck paths. The concatenation of the result is the original word.

EXAMPLES:

```python
sage: D = DyckWord([1,1,1,0,1,0,1,1,1,1,0,1])
```
```python
sage: D.ascent_prime_decomposition()  
[[1, 1], [1, 0], [], [1, 0], [1, 1, 1], [1, 0], [1]]
```
```python
sage: DyckWord([]).ascent_prime_decomposition()  
[]
```
```python
sage: DyckWord([1,1]).ascent_prime_decomposition()  
[[1, 1]]
```
```python
sage: DyckWord([1,0,1,0]).ascent_prime_decomposition()  
[[], [1, 0], [], [1, 0], []]
```

associated_parenthesis(pos)  
Report the position for the parenthesis in `self` that matches the one at position `pos`.

The positions in `self` are counted from 0.

INPUT:

- pos – the index of the parenthesis in the list

OUTPUT:

- Integer representing the index of the matching parenthesis. If no parenthesis matches, return `None`.
```
EXAMPLES:

```python
sage: DyckWord([1, 0]).associated_parenthesis(0)
1
sage: DyckWord([1, 0, 1, 0]).associated_parenthesis(0)
1
sage: DyckWord([1, 0, 1, 0]).associated_parenthesis(1)
0
sage: DyckWord([1, 0, 1, 0]).associated_parenthesis(2)
3
sage: DyckWord([1, 0, 1, 0]).associated_parenthesis(3)
2
sage: DyckWord([1, 1, 0, 0]).associated_parenthesis(0)
3
sage: DyckWord([1, 1, 0, 0]).associated_parenthesis(2)
1
sage: DyckWord([1, 1]).associated_parenthesis(0)
2
sage: DyckWord([1, 1]).associated_parenthesis(0)
```

catalan_factorization()

Decompose this Dyck word into a sequence of complete Dyck words.

Each element of the list returned is a (possibly empty) complete Dyck word. The original word is obtained by placing an up step between each of these complete Dyck words. Thus, the number of words returned is one more than the final height.

See Section 1.2 of [CC1982] or Lemma 9.1.1 of [Lot2005].

EXAMPLES:

```python
sage: D = DyckWord([1,1,1,0,1,0,1,1,1,1,0,1])
sage: D.catalan_factorization()
[[], [], [1, 0, 1, 0], [], [], [1, 0], []]
sage: DyckWord([]).catalan_factorization()
[]
sage: DyckWord([1,1]).catalan_factorization()
[[], [], []]
sage: DyckWord([1,0,1,0]).catalan_factorization()
[[1, 0, 1, 0]]
```

height()

Return the height of `self`.

We view the Dyck word as a Dyck path from \((0, 0)\) to \((2n, 0)\) in the first quadrant by letting \(1\)'s represent steps in the direction \((1, 1)\) and \(0\)'s represent steps in the direction \((1, -1)\).

The height is the maximum \(y\)-coordinate reached.

See also:

`heights()`

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: DyckWord([]).height()
0
sage: DyckWord([1,0]).height()
1
sage: DyckWord([1, 1, 0, 0]).height()
2
sage: DyckWord([1, 1, 0, 1, 0]).height()
2
sage: DyckWord([1, 1, 0, 0, 1, 0]).height()
2
sage: DyckWord([1, 0, 1, 0]).height()
1
sage: DyckWord([1, 1, 0, 0, 1, 1, 1, 0, 0, 0]).height()
3
```

### heights()

Return the heights of self.

We view the Dyck word as a Dyck path from \((0, 0)\) to \((2n, 0)\) in the first quadrant by letting 1's represent steps in the direction \((1, 1)\) and 0's represent steps in the direction \((1, -1)\).

The heights is the sequence of the \(y\)-coordinates of all \(2n + 1\) lattice points along the path.

See also:

`from_heights()`, `min_from_heights()`

**EXAMPLES:**

```python
sage: DyckWord([]).heights()
(0,)
sage: DyckWord([1,0]).heights()
(0, 1, 0)
sage: DyckWord([1, 1, 0, 0]).heights()
(0, 1, 2, 1, 0)
sage: DyckWord([1, 1, 0, 1, 0]).heights()
(0, 1, 2, 1, 2, 1)
sage: DyckWord([1, 1, 0, 0, 1, 0]).heights()
(0, 1, 2, 1, 0, 1, 0)
sage: DyckWord([1, 0, 1, 0]).heights()
(0, 1, 0, 1, 0)
sage: DyckWord([1, 1, 0, 0, 1, 1, 1, 0, 0, 0]).heights()
(0, 1, 2, 1, 0, 1, 2, 3, 2, 1, 0)
```

### is_complete()

Return True if self is complete.

A Dyck word \(d\) is complete if \(d\) contains as many closers as openers.

**EXAMPLES:**

```python
sage: DyckWord([1, 0, 1, 0]).is_complete()
True
sage: DyckWord([1, 0, 1, 1, 0]).is_complete()
False
```
latex_options()

Return the latex options for use in the \_latex\_ function as a dictionary.

The default values are set using the options.

- **tikz_scale** – (default: 1) scale for use with the tikz package.
- **diagonal** – (default: False) boolean value to draw the diagonal or not.
- **line width** – (default: \(2^{\text{tikz_scale}}\)) value representing the line width.
- **color** – (default: black) the line color.
- **bounce path** – (default: False) boolean value to indicate if the bounce path should be drawn.
- **peaks** – (default: False) boolean value to indicate if the peaks should be displayed.
- **valleys** – (default: False) boolean value to indicate if the valleys should be displayed.

**EXAMPLES:**

```
sage: D = DyckWord([1,0,1,0,1,0])
sage: D.latex_options()
{'bounce path': False,
 'color': 'black',
 'diagonal': False,
 'line width': 2,
 'peaks': False,
 'tikz_scale': 1,
 'valleys': False}
```

**Todo:** This should probably be merged into DyckWord.options.

length()

Return the length of self.

**EXAMPLES:**

```
sage: DyckWord([1, 0, 1, 0]).length()
4
sage: DyckWord([1, 0, 1, 1, 0]).length()
5
```

number_of_close_symbols()

Return the number of close symbols in self.

**EXAMPLES:**

```
sage: DyckWord([1, 0, 1, 0]).number_of_close_symbols()
2
sage: DyckWord([1, 0, 1, 1, 0]).number_of_close_symbols()
2
```

number_of_double_rises()

Return the number of positions in self where there are two consecutive 1’s.

**EXAMPLES:**
number_of_initial_rises()
Return the length of the initial run of self.
OUTPUT:
• a non–negative integer indicating the length of the initial rise
EXAMPLES:

```plaintext
sage: DyckWord([1, 0, 1, 0]).number_of_initial_rises()
1
sage: DyckWord([1, 1, 0, 0]).number_of_initial_rises()
2
sage: DyckWord([1, 0, 1, 0, 1, 0]).number_of_initial_rises()
2
sage: DyckWord([1, 0, 1, 1, 0, 0]).number_of_initial_rises()
1
```

number_of_open_symbols()
Return the number of open symbols in self.

EXAMPLES:

```plaintext
sage: DyckWord([1, 0, 1, 0]).number_of_open_symbols()
2
sage: DyckWord([1, 0, 1, 1, 0]).number_of_open_symbols()
3
```

number_of_peaks()
Return the number of peaks of the Dyck path associated to self.

See also:
peaks(), number_of_valleys()

EXAMPLES:

```plaintext
sage: DyckWord([1, 0, 1, 0]).number_of_peaks()
2
sage: DyckWord([1, 1, 0, 0]).number_of_peaks()
1
sage: DyckWord([1,1,0,1,0,0]).number_of_peaks()
3
sage: DyckWord([]).number_of_peaks()
0
```

number_of_touch_points()
Return the number of touches of self at the main diagonal.
OUTPUT:
  • a non–negative integer

EXAMPLES:

```python
sage: DyckWord([1, 0, 1, 0]).number_of_touch_points()
2
sage: DyckWord([1, 1, 0, 0]).number_of_touch_points()
1
sage: DyckWord([1, 1, 0, 0, 1, 0]).number_of_touch_points()
2
sage: DyckWord([1, 0, 1, 1, 0, 0]).number_of_touch_points()
2
```

`number_of_valleys()`

Return the number of valleys of `self`.

See also:

`number_of_peaks()`, `valleys()`

EXAMPLES:

```python
sage: DyckWord([1, 0, 1, 0]).number_of_valleys()
1
sage: DyckWord([1, 1, 0, 0]).number_of_valleys()
0
sage: DyckWord([1, 1, 0, 0, 1, 0]).number_of_valleys()
1
sage: DyckWord([1, 0, 1, 1, 0, 0]).number_of_valleys()
1
```

`peaks()`

Return a list of the positions of the peaks of a Dyck word.

A peak is 1 followed by a 0. Note that this does not agree with the definition given in [Hag2008].

See also:

`valleys()`, `number_of_peaks()`

EXAMPLES:

```python
sage: DyckWord([1, 0, 1, 0]).peaks()
[0, 2]
sage: DyckWord([1, 1, 0, 0]).peaks()
[1]
sage: DyckWord([1,1,0,1,0,1,0,0]).peaks()  # Haglund's def gives 2
[1, 3, 5]
```

`plot(**kwds)`

Plot a Dyck word as a continuous path.

EXAMPLES:

```python
sage: w = DyckWords(100).random_element()
sage: w.plot()
sage: w.plot()  # Graphics object consisting of 1 graphics primitive
```
position_of_first_return()

Return the number of vertical steps before the Dyck path returns to the main diagonal.

EXAMPLES:

```
sage: DyckWord([1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0]).position_of_first_return()
1
sage: DyckWord([1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0]).position_of_first_return()
7
sage: DyckWord([1, 1, 0, 1]).position_of_first_return()
2
sage: DyckWord([1, 0, 1, 0]).position_of_first_return()
1
sage: DyckWord([]).position_of_first_return()
0
```

positions_of_double_rises()

Return a list of positions in self where there are two consecutive 1’s.

EXAMPLES:

```
sage: DyckWord([1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0]).positions_of_double_rises()
[2, 5]
sage: DyckWord([1, 1, 0, 0]).positions_of_double_rises()
[0]
sage: DyckWord([1, 0, 1, 0]).positions_of_double_rises()
[]
```

pp(type=None, labelling=None, underpath=True)

Display a DyckWord as a lattice path in the $\mathbb{Z}^2$ grid.

If the type is “N-E”, then a cell below the diagonal is indicated by a period, whereas a cell below the path but above the diagonal is indicated by an x. If a list of labels is included, they are displayed along the vertical edges of the Dyck path.

If the type is “NE-SE”, then the path is simply printed as up steps and down steps.

INPUT:

- **type** – (default: None) can either be:
  - None to use the option default
  - “N-E” to show self as a path of north and east steps, or
  - “NE-SE” to show self as a path of north-east and south-east steps.
- **labelling** – (if type is “N-E”) a list of labels assigned to the up steps in self.
- **underpath** – (if type is “N-E”, default: True) If True, the labelling is shown under the path; otherwise, it is shown to the right of the path.

EXAMPLES:

```
sage: for D in DyckWords(3): D.pretty_print()
```

(continues on next page)
```python
sage: for D in DyckWords(3): D.pretty_print(type="NE-SE")
  /\ \n /\ /\  
/\ /\ /\ 
\ /\ /\ 
  /\ /\ 
  /\ /\ 
```

```python
sage: D = DyckWord([1,1,1,0,1,0,0,1,1])
sage: D.pretty_print()
 _
 | x x
___| x .
| x x . .
| x x . . .
| x . . . .
| . . . . .
```

```python
sage: D = DyckWord([1,1,1,0,1,0,0,1,1,0])
sage: D.pretty_print()
 _
 | x x
___| x .
| x . . .
| . . . . .
| . . . . .
```

(continues on next page)
sage: D = DyckWord([1,1,1,0,1,0,0,1,1,0,0])
sage: D.pretty_print()

___
| x x
___| x .
_| x x ...
| x x ... |
| x .......
| . ......

sage: DyckWord(area_sequence=[0,1,0]).pretty_print(labelling=[1,3,2])

___|2
|3x .
|1 ..
sage: DyckWord(area_sequence=[0,1,0]).pretty_print(labelling=[1,3,2],
underpath=False)

___| 2
| x . 3
| . . 1

sage: DyckWord(area_sequence=[0,1,1,2,3,2,3,3,2,0,1,1,2,3,4,2,3]).pretty_print()

________
| x x x
________| x x .
| x x x ... |
| x x x . . . |
| x x .......
-| x .......
| x ........
-| x ........
_______| . . . . .
__| x x . . . . . . .
| x x x . . . . . . .
| x x x . . . . . . .
___| x x . . . . . . .
| x x . . . . . . .
| x . . . . . . .
| . . . . . . .

sage: DyckWord(area_sequence=[0,1,1,2,3,2,3,3,2,0,1,1,2,3,4,2,3]).pretty_print(labelling=list(range(17)),underpath=False)

________
| x x x 16
________| x x . 15
| x x x x . 14
| x x x ... 13
| x x .. . 12

(continues on next page)
sage: DyckWord([]).pretty_print()

```
|   | x . . . . . 11
| x . . . . . 10
|   | . . . . . . 9
___| x x . . . . . 8
_ | x x x . . . . . 7
| x x x . . . . . 6
___| x x . . . . . . 5
| x x . . . . . . 4
| x . . . . . . 3
| . . . . . . . 2
| . . . . . . 1
| . . . . . . . 0
```

### pretty_print(type=None, labelling=None, underpath=True)

Display a DyckWord as a lattice path in the $\mathbb{Z}^2$ grid.

If the type is “N-E”, then a cell below the diagonal is indicated by a period, whereas a cell below the path but above the diagonal is indicated by an x. If a list of labels is included, they are displayed along the vertical edges of the Dyck path.

If the type is “NE-SE”, then the path is simply printed as up steps and down steps.

**INPUT:**

- **type** – (default: None) can either be:
  - None to use the option default
  - “N-E” to show self as a path of north and east steps, or
  - “NE-SE” to show self as a path of north-east and south-east steps.
- **labelling** – (if type is “N-E”) a list of labels assigned to the up steps in self.
- **underpath** – (if type is “N-E”, default: True) If True, the labelling is shown under the path; otherwise, it is shown to the right of the path.

**EXAMPLES:**

```
sage: for D in DyckWords(3): D.pretty_print()
```

(continues on next page)
sage: for D in DyckWords(3): D.pretty_print(type="NE-SE")
/\/\/
/  \
/   
/    
/     
/      
/       

sage: D = DyckWord([1,1,1,0,1,0,0,1,1])
sage: D.pretty_print()
  | x x
  _| x .
 _| x x . .
| x x . . .
| x . . . .
| . . . . .

sage: D = DyckWord([1,1,1,0,1,0,0,1,1,0])
sage: D.pretty_print()
  | x x
  _| x .
 _| x x . .
| x x . . .
| x . . . .
| . . . . .

sage: D = DyckWord([1,1,1,0,1,0,0,1,1,0,0])
sage: D.pretty_print()
____
| x x
___| x .
_| x x . .
| x x . . .
| x . . . .
| . . . . .

sage: DyckWord(area_sequence=[0,1,0]).pretty_print(labelling=[1,3,2])
___
___|2

\begin{verbatim}
| 3x .
| 1 .

\texttt{sage: DyckWord(area_sequence=[0,1,0]).pretty_print(labelling=[1,3,2],}
\rightarrow \texttt{underpath=False})
\texttt{___| 2}
\texttt{| x . 3}
\texttt{| . . 1}
\end{verbatim}

\begin{verbatim}
\texttt{sage: DyckWord(area_sequence=[0,1,1,2,3,2,3,3,2,0,1,1,2,3,4,2,3]).pretty_print()}
\texttt{__________}
\texttt{| x x x 16}
\texttt{______| x x . 15}
\texttt{| x x x x . 14}
\texttt{| x x x . . 13}
\texttt{| x x . . . 12}
\texttt{| x . . . . 11}
\texttt{| x . . . . . 10}
\texttt{| . . . . . . 9}
\texttt{| . . . . . . . 8}
\texttt{| . . . . . . . . 7}
\texttt{| . . . . . . . . . 6}
\texttt{| . . . . . . . . . . 5}
\texttt{| . . . . . . . . . . . 4}
\texttt{| . . . . . . . . . . . . 3}
\texttt{| . . . . . . . . . . . . . 2}
\texttt{| . . . . . . . . . . . . . . 1}
\texttt{| . . . . . . . . . . . . . . . 0}
\end{verbatim}
sage: DyckWord([]).pretty_print()

returns_to_zero()

Return a list of positions where self has height 0, excluding the position 0.

EXAMPLES:

sage: DyckWord([]).returns_to_zero()
[]
sage: DyckWord([1, 0]).returns_to_zero()
[2]
sage: DyckWord([1, 0, 1, 0]).returns_to_zero()
[2, 4]
sage: DyckWord([1, 1, 0, 0]).returns_to_zero()
[4]

rise_composition()

The sequences of lengths of runs of 1’s in self. Also equal to the sequence of lengths of vertical segments in the Dyck path.

EXAMPLES:

sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).pretty_print()
________
|   x    |
|_______|  .
|   x x  |  .
|   x    |  .
|______   |  .
|   x     |  .
|   .     |  .
|   .     |  .
|   .     |  .
|   .     |  .
|   .     |  .
|   .     |  .
|   .     |  .
|   .     |  .

sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).rise_composition()
[2, 3, 2]
sage: DyckWord([1,1,0,0]).rise_composition()
[2]
sage: DyckWord([1,0,1,0]).rise_composition()
[1, 1]

set_latex_options(D)

Set the latex options for use in the _latex_ function.

The default values are set in the __init__ function.

- tikz_scale – (default: 1) scale for use with the tikz package.
- diagonal – (default: False) boolean value to draw the diagonal or not.
- line width – (default: 2*``tikz_scale``) value representing the line width.
- color – (default: black) the line color.
- bounce path – (default: False) boolean value to indicate if the bounce path should be drawn.
- peaks – (default: False) boolean value to indicate if the peaks should be displayed.
- valleys – (default: False) boolean value to indicate if the valleys should be displayed.
INPUT:
- $D$ – a dictionary with a list of latex parameters to change

EXAMPLES:
```python
code
sage: D = DyckWord([1,0,1,0,1,0])
sage: D.set_latex_options("tikz_scale":2)
sage: D.set_latex_options("valleys":True, "color":"blue")
```

Todo: This should probably be merged into DyckWord.options.

**tamari_interval**(other)

Return the Tamari interval between `self` and `other` as a `TamariIntervalPoset`.

A “Tamari interval” means an interval in the Tamari order. The Tamari order on the set of Dyck words of size $n$ is the partial order obtained from the Tamari order on the set of binary trees of size $n$ (see `tamari_lequal()`) by means of the Tamari bijection between Dyck words and binary trees (`to_dyck_word_tamari()`).

INPUT:
- `other` – a Dyck word greater or equal to `self` in the Tamari order

EXAMPLES:
```python
code
sage: dw = DyckWord([1, 1, 0, 1, 0, 0, 1, 0])
sage: ip = dw.tamari_interval(DyckWord([1, 1, 1, 0, 0, 1, 0, 0])); ip
The Tamari interval of size 4 induced by relations [(2, 4), (3, 4), (3, 1), (2, ˓→1)]
sage: ip.lower_dyck_word()
[1, 1, 0, 1, 0, 0, 1, 0]
sage: ip.upper_dyck_word()
[1, 1, 1, 0, 0, 1, 0, 0]
sage: ip.interval_cardinality()
4
sage: ip.number_of_tamari_inversions()
2
sage: list(ip.dyck_words())
[[1, 1, 1, 0, 0, 1, 0, 0],
 [1, 1, 0, 1, 0, 1, 0, 0],
 [1, 1, 0, 1, 0, 0, 1, 0],
 [1, 1, 0, 1, 0, 0, 1, 0]]
sage: dw.tamari_interval(DyckWord([1,1,0,0,1,1,0,0])))
Traceback (most recent call last):
... ValueError: the two Dyck words are not comparable on the Tamari lattice
```

**to_area_sequence()**

Return the area sequence of the Dyck word `self`.

The area sequence of a Dyck word $w$ is defined as follows: Representing the Dyck word $w$ as a Dyck path from $(0, 0)$ to $(n, n)$ using $N$ and $E$ steps (this involves padding $w$ by $E$ steps until $w$ reaches the main diagonal if $w$ is not already a complete Dyck path), the area sequence of $w$ is the sequence $(a_1, a_2, \ldots, a_n)$, where $a_i$ is the number of full cells in the $i$-th row of the rectangle $[0, n] \times [0, n]$ which lie completely above the diagonal. (The cells are the regions into which the rectangle is subdivided by the lines $x = i$ with $i$
integer and the lines $y = j$ with $j$ integer. The $i$-th row consists of all the cells between the lines $y = i - 1$ and $y = i$.

An alternative definition: Representing the Dyck word $w$ as a Dyck path consisting of $NE$ and $SE$ steps, the area sequence is the sequence of ordinates of all lattice points on the path which are starting points of $NE$ steps.

A list of integers $l$ is the area sequence of some Dyck path if and only if it satisfies $l_0 = 0$ and $0 \leq l_{i+1} \leq l_i + 1$ for $i > 0$.

EXAMPLES:

```python
sage: DyckWord([]).to_area_sequence()
[]
sage: DyckWord([1, 0]).to_area_sequence()
[0]
sage: DyckWord([1, 1, 0, 0]).to_area_sequence()
[0, 1]
sage: DyckWord([1, 0, 1, 0]).to_area_sequence()
[0, 0]
sage: all(dw ==
    DyckWords().from_area_sequence(dw.to_area_sequence())
    for i in range(6) for dw in DyckWords(i))
True
sage: DyckWord([1,0,1,0,1,0,1,0,1,0,1,0]).to_area_sequence()
[0, 0, 0, 0, 0]
sage: DyckWord([1,1,1,1,1,0,0,0,0,0]).to_area_sequence()
[0, 1, 2, 3, 4]
sage: DyckWord([1,1,1,1,0,1,0,0,0,0]).to_area_sequence()
[0, 1, 2, 3, 3]
sage: DyckWord([1,1,0,1,0,0,1,1,0,1,0,1,0,0]).to_area_sequence()
[0, 1, 1, 0, 1, 1, 1]
```

\textbf{to\_binary\_tree}(usemap='IL0R')

Return a binary tree recursively constructed from the Dyck path self by the map usemap. The default usemap is 'IL0R' which means:

- an empty Dyck word is a leaf,
- a non empty Dyck word reads 1L0R where $L$ and $R$ correspond to respectively its left and right subtrees.

\textbf{INPUT:}

- usemap – a string, either 'IL0R', 'IR0L', 'L1R0', 'R1L0'

Other valid usemap are 'IR0L', 'L1R0', and 'R1L0'. These correspond to different maps from Dyck paths to binary trees, whose recursive definitions are hopefully clear from the names.

\textbf{EXAMPLES:}

```python
sage: dw = DyckWord([1,0])
sage: dw.to_binary_tree()
[., .]
sage: dw = DyckWord([])
sage: dw.to_binary_tree()
.
sage: dw = DyckWord([1,0,1,1,0,0])
(continues on next page)
```
sage: dw.to_binary_tree()
[., [., [., .]], .]
sage: dw.to_binary_tree("L1R0")
[[., .], [., .]]
sage: dw = DyckWord([1,0,1,1,0,0,1,1,1,0,1,0,0,0])
sage: dw.to_binary_tree() == dw.to_binary_tree("1R0L").left_right_symmetry()
True
sage: dw.to_binary_tree() == dw.to_binary_tree("L1R0").left_border_symmetry()
False
sage: dw.to_binary_tree("1R0L") == dw.to_binary_tree("L1R0").left_right_symmetry()
True
sage: dw.to_binary_tree("R1L0") == dw.to_binary_tree("L1R0").left_right_symmetry()
True
sage: dw.to_binary_tree("R10L")
Traceback (most recent call last):
  ...:
ValueError: R10L is not a correct map

to_binary_tree_tamari()
Return the binary tree corresponding to self in a way which is consistent with the Tamari orders on the set of Dyck paths and on the set of binary trees.

This is the 'L1R0' map documented in to_binary_tree().

EXAMPLES:

sage: DyckWord([1,0]).to_binary_tree_tamari()
[., .]
sage: DyckWord([1,0,1,0,1,0]).to_binary_tree_tamari()
[[[., .], .], .]

to_path_string(unicode=False)
Return a path representation of the Dyck word consisting of steps / and \.

INPUT:

* unicode – boolean (default False) whether to use unicode

EXAMPLES:

sage: print(DyckWord([1, 0, 1, 0]).to_path_string())
/\/
sage: print(DyckWord([1, 1, 0, 0]).to_path_string())
/\ \\
/ \\
sage: print(DyckWord([1,1,0,1,1,0,0,1,0,1,0,0]).to_path_string())
/\ \\
/\ \ \ \ \\
/ \\
to_standard_tableau()

Return a standard tableau of shape \((a, b)\) where \(a\) is the number of open symbols and \(b\) is the number of close symbols in \texttt{self}.

EXAMPLES:

```
sage: DyckWord([]).to_standard_tableau()
[]
sage: DyckWord([1, 0]).to_standard_tableau()
[[1], [2]]
sage: DyckWord([1, 1, 0, 0]).to_standard_tableau()
[[1, 2], [3, 4]]
sage: DyckWord([1, 0, 1, 0]).to_standard_tableau()
[[1, 3], [2, 4]]
sage: DyckWord([1]).to_standard_tableau()
[[1]]
sage: DyckWord([1, 0, 1]).to_standard_tableau()
[[1, 3], [2]]
```

to_tamari_sorting_tuple()

Convert a Dyck word to a Tamari sorting tuple.

The result is a list of integers, one for every up-step from left to right. To each up-step is associated the distance to the corresponding down step in the Dyck word.

This is useful for a faster conversion to binary trees.

EXAMPLES:

```
sage: DyckWord([]).to_tamari_sorting_tuple()
[]
sage: DyckWord([1, 0]).to_tamari_sorting_tuple()
[0]
sage: DyckWord([1, 1, 0, 0]).to_tamari_sorting_tuple()
[1, 0]
sage: DyckWord([1, 0, 1, 0]).to_tamari_sorting_tuple()
[0, 0]
sage: DyckWord([1, 1, 0, 1, 0, 0]).to_tamari_sorting_tuple()
[2, 0, 0]
```

See also:

to_Catalan_code()

touch_composition()

Return a composition which indicates the positions where \texttt{self} returns to the diagonal.

This assumes \texttt{self} to be a complete Dyck word.

OUTPUT:

- a composition of length equal to the length of the Dyck word.

EXAMPLES:

```
sage: DyckWord([1, 0, 1]).touch_composition()
[1, 1]
sage: DyckWord([1, 1, 0, 0]).touch_composition()

(continues on next page)```
touch_points()

Return the abscissae (or, equivalently, ordinates) of the points where the Dyck path corresponding to self (comprising NE and SE steps) touches the main diagonal. This includes the last point (if it is on the main diagonal) but excludes the beginning point.

Note that these abscissae are precisely the entries of returns_to_zero() divided by 2.

OUTPUT:

• a list of integers indicating where the path touches the diagonal

EXAMPLES:

```python
sage: DyckWord([1, 0, 1, 0]).touch_points()
[1, 2]
sage: DyckWord([1, 1, 0, 0]).touch_points()
[2]
sage: DyckWord([1, 1, 0, 0, 1, 0]).touch_points()
[2, 3]
sage: DyckWord([1, 0, 1, 1, 0, 0]).touch_points()
[1, 3]
```

valleys()

Return a list of the positions of the valleys of a Dyck word.

A valley is 0 followed by a 1.

See also:

peaks(), number_of_valleys()

EXAMPLES:

```python
sage: DyckWord([1, 0, 1, 0]).valleys()
[1]
sage: DyckWord([1, 1, 0, 0]).valleys()
[]
sage: DyckWord([1,1,0,1,0,1,0,0]).valleys()
[2, 4]
```
class sage.combinat.dyck_word.DyckWord_complete(parent, l, latex_options={})

Bases: DyckWord

The class of complete Dyck words. A Dyck word is complete, if it contains as many closers as openers. For further information on Dyck words, see DyckWords_class.

area()

Return the area for self corresponding to the area of the Dyck path.

One can view a balanced Dyck word as a lattice path from (0, 0) to (n,n) in the first quadrant by letting ‘1’s represent steps in the direction (1, 0) and ‘0’s represent steps in the direction (0, 1). The resulting path will remain weakly above the diagonal y = x.

The area statistic is the number of complete squares in the integer lattice which are below the path and above the line y = x. The ‘half-squares’ directly above the line y = x do not contribute to this statistic.

EXAMPLES:

sage: dw = DyckWord([1,0,1,0])
sage: dw.area() # 2 half-squares, 0 complete squares
0

sage: dw = DyckWord([1,1,1,0,1,1,0,0,1,0,0,1,0,0,0,0])
sage: dw.area()
19

sage: Dw = DyckWord([1,1,1,0,0,1,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
6
sage: Dw = DyckWord([1,1,1,0,1,0,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
5
sage: Dw = DyckWord([1,1,1,0,0,1,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
4
sage: Dw = DyckWord([1,1,1,0,0,0,1,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
3
sage: Dw = DyckWord([1,0,1,1,0,1,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
2
sage: Dw = DyckWord([1,1,0,1,0,1,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
4
sage: Dw = DyckWord([1,1,0,0,1,1,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
2
sage: Dw = DyckWord([1,0,1,1,1,0,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
3
sage: Dw = DyckWord([1,0,1,1,0,0,1,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
1
sage: Dw = DyckWord([1,0,1,1,0,0,1,1,0,0,1,0,0,0,0,0]).area()
sage: Dw
1
sage: Dw = DyckWord([1,1,0,0,1,0,1,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
1
sage: Dw = DyckWord([1,1,0,1,0,1,0,0,1,0,0,1,0,0,0,0]).area()
sage: Dw
2
sage: Dw = DyckWord([1,1,0,1,0,0,1,0,1,0,0,1,0,0,0,0]).area()

(continues on next page)
3

```python
sage: DyckWord([1,0,1,0,1,0,1,0]).area()
0
```

**`area_dinv_to_bounce_area_map()`**

Return the image of `self` under the map which sends a Dyck word with area equal to `r` and dinv equal to `s` to a Dyck word with bounce equal to `r` and area equal to `s`.

The inverse of this map is `bounce_area_to_area_dinv_map()`.

For a definition of this map, see [Hag2008] p. 50 where it is called $\zeta$. However, this map differs from Haglund’s map by an application of `reverse()` (as does the definition of the `bounce()` statistic).

**EXAMPLES:**

```python
sage: DyckWord([1,0,1,0,1,0,1,0,1,0,1,0,0,0,1,1,0,0,1,1,0,0,1,0,0,0]).area_dinv_to_bounce_area_map()
[1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0]
```

**`bounce()`**

Return the bounce statistic of `self` due to J. Haglund, see [Hag2008].

One can view a balanced Dyck word as a lattice path from $(0,0)$ to $(n,n)$ in the first quadrant by letting ‘1’s represent steps in the direction $(0,1)$ and ‘0’s represent steps in the direction $(1,0)$. The resulting path will remain weakly above the diagonal $y = x$.

We describe the bounce statistic of such a path in terms of what is known as the “bounce path”.

We can think of our bounce path as describing the trail of a billiard ball shot West from $(n,n)$, which “bounces” down whenever it encounters a vertical step and “bounces” left when it encounters the line $y = x$.

The bouncing ball will strike the diagonal at the places

$$(0,0), (j_1, j_1), (j_2, j_2), \ldots, (j_r - 1, j_r - 1), (j_r, j_r) = (n,n).$$

We define the bounce to be the sum $\sum_{i=1}^{r-1} j_i$.

**EXAMPLES:**

```python
sage: DyckWord([1,1,1,0,1,1,1,0,0,0,1,1,0,0,1,0,0,0]).bounce()
7
```

(continues on next page)
bounce_area_to_area_dinv_map()

Return the image of the Dyck word under the map which sends a Dyck word with bounce equal to \( r \) and area equal to \( s \) to a Dyck word with area equal to \( r \) and \( \text{dinv} \) equal to \( s \).

This implementation uses a recursive method by saying that the last entry in the area sequence of the Dyck word \( \text{self} \) is equal to the number of touch points of the Dyck path minus 1 of the image of this map.

The inverse of this map is \( \text{area_dinv_to_bounce_area_map()} \).

For a definition of this map, see [Hag2008] p. 50 where it is called \( \zeta^{-1} \). However, this map differs from Haglund’s map by an application of \( \text{reverse()} \) (as does the definition of the \( \text{bounce()} \) statistic).

EXAMPLES:

```python
sage: DyckWord([1,1,0,0,1,0,0,0]).bounce_area_to_area_dinv_map()
[1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0]
sage: DyckWord([1,1,0,1,0,0,1,1,0,1,0,1,0,0]).area()
5
sage: DyckWord([1,1,0,1,0,1,0,1,0,1,0,1,0,0]).bounce() 9
sage: DyckWord([1,1,0,0,1,1,1,1,0,0,1,0,0,0]).dinv() 5
```

(continues on next page)
bounce_path()  

Return the bounce path of self formed by starting at \((n, n)\) and traveling West until encountering the first vertical step of self, then South until encountering the diagonal, then West again to hit the path, etc. until the \((0, 0)\) point is reached. The path followed by this walk is the bounce path.

See also:

bounce()  

EXAMPLES:

```plaintext
sage: DyckWord([1,1,0,1,0,0]).bounce_path()
[1, 0, 1, 1, 0, 0]
```

characteristic_symmetric_function(q=None, R=Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field)  

The characteristic function of self is the sum of \(q^{\text{dinv}(D,F)}Q_{\text{ides}(\text{read}(D,F))}\) over all permutation fillings of the Dyck path representing self, where \(\text{ides}(\text{read}(D,F))\) is the descent composition of the inverse of the reading word of the filling.

INPUT:

- \(q\) – (default: \(q = R('q')\)) a parameter for the generating function power
- \(R\) – (default: \(R = \text{QQ}['q', 't'].fraction_field()\)) the base ring to do the calculations over

OUTPUT:

- an element of the symmetric functions over the ring \(R\) (in the Schur basis).

EXAMPLES:

```plaintext
sage: R = QQ['q','t'].fraction_field()

sage: (q,t) = R.gens()

sage: f = sum(t**D.area()*D.characteristic_symmetric_function() for D in DyckWords(3)); f
(q^3+q^2*t+q*t^2+t^3+q*t)*s[1, 1, 1] + (q^2+q*t+t^2+q+t)*s[2, 1] + s[3]

sage: f.nabla(power=-1)
s[1, 1, 1]
```
EXAMPLES:

```python
sage: DyckWord([1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0]).decomposition_reverse()
[1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]
```

```python
sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).decomposition_reverse()
[1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0]
```

```python
sage: DyckWord([1,1,0,0]).decomposition_reverse()
[1, 0, 1, 0]
```

```python
sage: DyckWord([1,0,1,0]).decomposition_reverse()
[1, 1, 0, 0]
```

```python
dinv(labeling=None)
```

Return the dinv statistic of self due to M. Haiman, see [Hag2008].

If a labeling is provided then this function returns the dinv of the labeled Dyck word.

INPUT:
- labeling – an optional argument to be viewed as the labelings of the vertical edges of the Dyck path

OUTPUT:
- an integer representing the dinv statistic of the Dyck path or the labelled Dyck path.

EXAMPLES:

```python
sage: DyckWord([1,0,1,0,1,0,1,0,1,0]).dinv()
10
sage: DyckWord([1,1,1,1,1,0,0,0,0,0]).dinv()
0
sage: DyckWord([1,1,1,1,0,1,0,0,0,0]).dinv()
1
sage: DyckWord([1,1,0,1,0,1,0,1,0,1,0,0]).dinv()
13
sage: DyckWord([1,1,0,1,0,1,1,0,1,0,1,0,0]).dinv([1,2,3,4,5,6,7])
11
sage: DyckWord([1,1,0,1,0,1,0,1,0,1,0,1,0,0]).dinv([6,7,5,3,4,2,1])
2
```

```python
first_return_decomposition()
```

Decompose a Dyck word into a pair of Dyck words (potentially empty) where the first word consists of the word after the first up step and the corresponding matching closing parenthesis.

EXAMPLES:

```python
sage: DyckWord([1,1,0,1,0,1,0,1,1,0,1,0,1,0,0]).first_return_decomposition()
([1, 0, 1, 0], [1, 1, 0, 1, 0, 1, 0, 0])
```

```python
sage: DyckWord([1,1,0,0]).first_return_decomposition()
([], [1, 0])
```

```python
list_parking_functions()
```

Return all parking functions whose supporting Dyck path is self.

EXAMPLES:
\begin{verbatim}
sage: DyckWord([1,1,0,0,1,0]).list_parking_functions()
Permutations of the multi-set [1, 1, 3]
sage: DyckWord([1,1,1,0,0,0]).list_parking_functions()
Permutations of the multi-set [1, 1, 1]
sage: DyckWord([1,0,1,0,1,0]).list_parking_functions()
Standard permutations of 3
\end{verbatim}

**major_index()**

Return the major index of \texttt{self}.

The major index of a Dyck word \(D\) is the sum of the positions of the valleys of \(D\) (when started counting at position 1).

**EXAMPLES:**

\begin{verbatim}
sage: DyckWord([1, 0, 1, 0]).major_index()
2
sage: DyckWord([1, 1, 0, 0]).major_index()
0
sage: DyckWord([1, 1, 0, 0, 1, 0]).major_index()
4
sage: DyckWord([1, 0, 1, 1, 0, 0]).major_index()
2
\end{verbatim}

**number_of_parking_functions()**

Return the number of parking functions with \texttt{self} as the supporting Dyck path.

One representation of a parking function is as a pair consisting of a Dyck path and a permutation \(\pi\) such that if \([a_0, a_1, \ldots, a_{n-1}]\) is the area_sequence of the Dyck path (see \texttt{to_area_sequence}) then the permutation \(\pi\) satisfies \(\pi_i < \pi_{i+1}\) whenever \(a_i < a_{i+1}\). This function counts the number of permutations \(\pi\) which satisfy this condition.

**EXAMPLES:**

\begin{verbatim}
sage: DyckWord(area_sequence=[0,1,2]).number_of_parking_functions()
1
sage: DyckWord(area_sequence=[0,1,1]).number_of_parking_functions()
3
sage: DyckWord(area_sequence=[0,1,0]).number_of_parking_functions()
3
sage: DyckWord(area_sequence=[0,0,0]).number_of_parking_functions()
6
\end{verbatim}

**number_of_tunnels** \((tunnel_type='centered')\)

Return the number of tunnels of \texttt{self} of type \texttt{tunnel_type}.

A tunnel is a pair \((a, b)\) where \(a\) is the position of an open parenthesis and \(b\) is the position of the matching close parenthesis. If \(a + b = n\) then the tunnel is called \texttt{centered}. If \(a + b < n\) then the tunnel is called \texttt{left} and if \(a + b > n\), then the tunnel is called \texttt{right}.

**INPUT:**

- \texttt{tunnel_type} – (default: 'centered') can be one of the following: 'left', 'right', 'centered', or 'all'.

**EXAMPLES:**
Combinatorics, Release 10.0

```python
sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).number_of_tunnels()
0
sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).number_of_tunnels('left')
5
sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).number_of_tunnels('right')
2
sage: DyckWord([1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0]).number_of_tunnels('all')
7
sage: DyckWord([1, 1, 0, 0]).number_of_tunnels('centered')
2

pyramid_weight()

Return the pyramid weight of self.

A pyramid of self is a subsequence of the form $1^h0^h$. A pyramid is maximal if it is neither preceded by a 1 nor followed by a 0.

The pyramid weight of a Dyck path is the sum of the lengths of the maximal pyramids and was defined in [DS1992].

EXAMPLES:

```python
sage: DyckWord([1,1,0,1,1,0,0,1,0,0,0,1,1,0,0]).pyramid_weight()
6
sage: DyckWord([1,1,0,0,0]).pyramid_weight()
3
sage: DyckWord([1,0,1,0,1,0]).pyramid_weight()
3
sage: DyckWord([1,1,0,1,0,0]).pyramid_weight()
2
```

reading_permutation()

Return the reading permutation of self.

This is the permutation formed by taking the reading word of the Dyck path representing self (with $N$ and $E$ steps) if the vertical edges of the Dyck path are labeled from bottom to top with 1 through $n$ and the diagonals are read from top to bottom starting with the diagonal furthest from the main diagonal.

EXAMPLES:

```python
sage: DyckWord([1,0,1,0]).reading_permutation()
[2, 1]
sage: DyckWord([1,1,0,0]).reading_permutation()
[2, 1]
sage: DyckWord([1,1,0,1,0,0]).reading_permutation()
[3, 2, 1]
sage: DyckWord([1,1,0,0,1,0]).reading_permutation()
[2, 3, 1]
sage: DyckWord([1,0,1,1,0,0,1,0]).reading_permutation()
[3, 4, 2, 1]
```

reverse()

Return the reverse and complement of self.
This operation corresponds to flipping the Dyck path across the $y = -x$ line.

EXAMPLES:

```python
sage: DyckWord([1,1,0,0,1,0]).reverse()
[1, 0, 1, 1, 0, 0]
sage: DyckWord([1,1,1,0,0,0]).reverse()
[1, 1, 1, 0, 0, 0]
sage: len([D for D in DyckWords(5) if D.reverse() == D])
10
```

semilength()

Return the semilength of self.

The semilength of a complete Dyck word $d$ is the number of openers and the number of closers.

EXAMPLES:

```python
sage: DyckWord([1, 0, 1, 0]).semilength()
2
```

to_132_avoiding_permutation()

Use the bijection by C. Krattenthaler in [Kra2001] to send self to a 132-avoiding permutation.

EXAMPLES:

```python
sage: DyckWord([1,1,0,0]).to_132_avoiding_permutation()
[2, 1]
sage: DyckWord([1,0,1,0]).to_132_avoiding_permutation()
[1, 2]
sage: p = DyckWord([1,1,0,1,0,0,1,1,0,1,0,1,0,0]).to_132_avoiding_permutation();
→ p
[2, 3, 1, 5, 6, 7, 4]
sage: DyckWord([1,1,0,1,0,0,1,1,0,1,0,1,0,0]).area()
5
sage: p.length()
5
```

to_312_avoiding_permutation()

Convert self to a 312-avoiding permutation using the bijection by Bandlow and Killpatrick in [BK2001].

This sends the area to the inversion number.

EXAMPLES:

```python
sage: DyckWord([1,1,0,0]).to_312_avoiding_permutation()
[2, 1]
sage: DyckWord([1,0,1,0]).to_312_avoiding_permutation()
[1, 2]
sage: p = DyckWord([1,1,0,1,0,0,1,1,0,1,0,1,0,0]).to_312_avoiding_permutation();
→ p
[2, 3, 1, 5, 6, 7, 4]
sage: p
[2, 3, 1, 5, 6, 7, 4]
```

to_321_avoiding_permutation()

Use the bijection (pp. 60-61 of [Knu1973] or section 3.1 of [CK2008]) to send self to a 321-avoiding permutation.
It is shown in [EP2004] that it sends the number of centered tunnels to the number of fixed points, the number of right tunnels to the number of excedences, and the semilength plus the height of the middle point to 2 times the length of the longest increasing subsequence.

EXAMPLES:

```
sage: DyckWord([1,0,1,0]).to_321_avoiding_permutation()
[2, 1]
sage: DyckWord([1,1,0,0]).to_321_avoiding_permutation()
[1, 2]
sage: D = DyckWord([1,1,0,1,0,1,0,1,0,1,0,0])
sage: p = D.to_321_avoiding_permutation()
sage: p
[3, 5, 1, 6, 2, 7, 4]
sage: D.number_of_tunnels()
0
sage: p.number_of_fixed_points()
0
sage: D.number_of_tunnels('right')
4
sage: len(p.weak_excedences())-p.number_of_fixed_points()
4
sage: n = D.semilength()
sage: D.heights()[n] + n
8
sage: 2*p.longest_increasing_subsequence_length()
8
```

to_Catalan_code()

Return the Catalan code associated to self.

A Catalan code of length \( n \) is a sequence \((a_1, a_2, \ldots, a_n)\) of \( n \) integers \( a_i \) such that:

- \( 0 \leq a_i \leq n - i \) for every \( i \);
- if \( i < j \) and \( a_i > 0 \) and \( a_j > 0 \) and \( a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0 \), then \( a_i - a_j < j - i \).

It turns out that the Catalan codes of length \( n \) are in bijection with Dyck words.

The Catalan code of a Dyck word is example (x) in Richard Stanley’s exercises on combinatorial interpretations for Catalan objects. The code in this example is the reverse of the description provided there. See [Sta-EC2] and [StaCat98].

EXAMPLES:

```
sage: DyckWord([]).to_Catalan_code()
[]
sage: DyckWord([1, 0]).to_Catalan_code()
[0]
sage: DyckWord([1, 1, 0, 0]).to_Catalan_code()
[0, 1]
sage: DyckWord([1, 0, 1, 0]).to_Catalan_code()
[0, 0]
sage: all(dw == DyckWords().from_Catalan_code(dw.to_Catalan_code())
for i in range(6) for dw in DyckWords(i))
True
```
to_tamari_sorting_tuple()  

See also:

to_alternating_sign_matrix()  

Return self as an alternating sign matrix.

This is an inclusion map from Dyck words of length $2n$ to certain $n \times n$ alternating sign matrices.

EXAMPLES:

```
sage: DyckWord([1,1,1,0,1,0,0,0]).to_alternating_sign_matrix()
[ 0 0 1 0]
[ 1 0 -1 1]
[ 0 1 0 0]
[ 0 0 1 0]
sage: DyckWord([1,0,1,0,1,1,0,0]).to_alternating_sign_matrix()
[1 0 0 0]
[0 1 0 0]
[0 0 0 1]
[0 0 1 0]
```

to_non_decreasing_parking_function()  

Bijection to non-decreasing parking functions.

See there the method to_dyck_word() for more information.

EXAMPLES:

```
sage: DyckWord([]).to_non_decreasing_parking_function()
[]
sage: DyckWord([1,0]).to_non_decreasing_parking_function()
[1]
sage: DyckWord([1,1,0,0]).to_non_decreasing_parking_function()
[1, 1]
sage: DyckWord([1,0,1,0]).to_non_decreasing_parking_function()
[1, 2]
sage: DyckWord([1,0,1,0,1,0,0,0,1,0]).to_non_decreasing_parking_function()
[1, 2, 2, 3, 5]
```

to_noncrossing_partition(bijection=None)  

Bijection of Biane from self to a noncrossing partition.

There is an optional parameter bijection that indicates if a different bijection from Dyck words to non-crossing partitions should be used (since there are potentially many).

If the parameter bijection is “Stump” then the bijection used is from [Stu2008], see also the method to_noncrossing_permutation().

Thanks to Mathieu Dutour for describing the bijection. See also from_noncrossing_partition().

EXAMPLES:

```
sage: DyckWord([]).to_noncrossing_partition()
{}
sage: DyckWord([1, 0]).to_noncrossing_partition()
{{1}}
sage: DyckWord([1, 1, 0, 0]).to_noncrossing_partition()
```
to_noncrossing_permutation()

Use the bijection by C. Stump in [Stu2008] to send \texttt{self} to a non-crossing permutation.

A non-crossing permutation when written in cyclic notation has cycles which are strictly increasing. Sends the area to the inversion number and \texttt{self.major_index()} to \( n(n - 1) - \text{maj}(\sigma) - \text{maj}(\sigma^{-1}) \). Uses the function \texttt{pealing()}

EXAMPLES:

```python
sage: DyckWord([1,1,0,0]).to_noncrossing_permutation()
[2, 1]
sage: DyckWord([1,0,1,0]).to_noncrossing_permutation()
[1, 2]
sage: p = DyckWord([1,1,0,1,0,0,1,1,0,1,0,0,1,0,0]).to_noncrossing_permutation();
→p
[2, 3, 1, 5, 6, 7, 4]
sage: DyckWord([1,1,0,1,0,0,1,1,0,1,0,0,1,0,0]).area()
5
sage: p.length()
5
```

to_ordered_tree()

Return the ordered tree corresponding to \texttt{self} where the depth of the tree is the maximal height of \texttt{self}.

EXAMPLES:

```python
sage: D = DyckWord([1,1,0,0])
sage: D.to_ordered_tree()  
[[[],]]
sage: D = DyckWord([1,0,1,0])
sage: D.to_ordered_tree()  
[[[], []]]
sage: D = DyckWord([1, 0, 1, 1, 0, 0])
```
sage: D.to_ordered_tree()
[[], [[]]]
sage: D = DyckWord([1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0])
sage: D.to_ordered_tree()
[[], [[], []], [[], [[]]]]

```
to_pair_of_standard_tableaux()

Convert self to a pair of standard tableaux of the same shape and of length less than or equal to two.

EXAMPLES:

sage: DyckWord([1,0,1,0]).to_pair_of_standard_tableaux()
([1], [2])
sage: DyckWord([1,1,0,0]).to_pair_of_standard_tableaux()
([], [1, 2])
```

```
to_partition()

Return the partition associated to self. This partition is determined by thinking of self as a lattice path and considering the cells which are above the path but within the \( n \times n \) grid and the partition is formed by reading the sequence of the number of cells in this collection in each row.

OUTPUT:

• a partition representing the rows of cells in the square lattice and above the path

EXAMPLES:

sage: DyckWord([]).to_partition()
[]
sage: DyckWord([1,0]).to_partition()
[]
sage: DyckWord([1,1,0,0]).to_partition()
[]
sage: DyckWord([1,0,1,0]).to_partition()
[1]
sage: DyckWord([1,0,1,0,1,0]).to_partition()
[2, 1]
sage: DyckWord([1,1,0,1,0,1,0]).to_partition()
[2]
sage: DyckWord([1,0,1,0,1,0,0]).to_partition()
[1, 1]
```

```
to_permutation(map)

This is simply a method collecting all implemented maps from Dyck words to permutations.

INPUT:

• map – defines the map from Dyck words to permutations. These are currently:
  – Bandlow-Killpatrick: to_312_avoiding_permutation()
  – Knuth: to_321_avoiding_permutation()
```
- Krattenthaler: `to_132_avoiding_permutation()`
- Stump: `to_noncrossing_permutation()`

EXAMPLES:

```python
sage: D = DyckWord([1,1,1,0,1,0,0,0])
sage: D.pretty_print()

_____
| x x
| x x .
| x ..
| ..

sage: D.to_permutation(map="Bandlow-Killpatrick")
[3, 4, 2, 1]
sage: D.to_permutation(map="Stump")
[4, 2, 3, 1]
sage: D.to_permutation(map="Knuth")
[1, 2, 4, 3]
sage: D.to_permutation(map="Krattenthaler")
[2, 1, 3, 4]
```

to_triangulation()

Map self to a triangulation.

The map from complete Dyck words of length $2n$ to triangulations of $n + 2$-gon given by this function is a bijection that can be described as follows.

Consider the Dyck word as a path from $(0, 0)$ to $(n, n)$ staying above the diagonal, where 1 is an up step and 0 is a right step. Then each horizontal step has a co-height (0 at the top and $n - 1$ at most at the bottom). One reads the Dyck word from left to right. At the beginning, all vertices from 0 to $n + 1$ are available. For each horizontal step, one creates an edge from the vertex indexed by the co-height to the next available vertex. This chops out a triangle from the polygon and one removes the middle vertex of this triangle from the list of available vertices.

This bijection has the property that the set of smallest vertices of the edges in a triangulation is an encoding of the co-heights, from which the Dyck word can be easily recovered.

OUTPUT:

a list of pairs $(i, j)$ that are the edges of the triangulations.

EXAMPLES:

```python
sage: DyckWord([1, 1, 0, 0]).to_triangulation()
[(0, 2)]
sage: [t.to_triangulation() for t in DyckWords(3)]
[[[2, 4], (1, 4)],
 [[2, 4], (0, 2)],
 [[1, 3], (1, 4)],
 [[1, 3], (0, 3)],
 [[0, 2], (0, 3)]]
```

REFERENCES:

- [Cha2005]
to_triangulation_as_graph()

Map self to a triangulation and return the result as a graph.

See to_triangulation() for the bijection used to map complete Dyck words to triangulations.

OUTPUT:

- a graph containing both the perimeter edges and the inner edges of a triangulation of a regular polygon.

EXAMPLES:

```python
sage: g = DyckWord([1, 1, 0, 0, 1, 0]).to_triangulation_as_graph()
sage: g
Graph on 5 vertices
sage: g.edges(sort=True, labels=False)
[(0, 1), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (3, 4)]
sage: g.show()  # not tested
```

tunnels()

Return an iterator of ranges of the matching parentheses in the Dyck word self.

That is, if (a,b) is in self.tunnels(), then the matching parenthesis to self[a] is self[b-1].

EXAMPLES:

```python
sage: list(DyckWord([1, 1, 0, 1, 1, 0, 0, 1, 0, 0]).tunnels())
[(0, 10), (1, 3), (3, 7), (4, 6), (7, 9)]
```

class sage.combinat.dyck_word.DyckWords

Bases: UniqueRepresentation, Parent

Dyck words.

A Dyck word is a sequence \((w_1, \ldots, w_n)\) consisting of 1s and 0s, with the property that for any \(i\) with \(1 \leq i \leq n\), the sequence \((w_1, \ldots, w_i)\) contains at least as many 1s as 0s.

A Dyck word is balanced if the total number of 1s is equal to the total number of 0s. The number of balanced Dyck words of length \(2k\) is given by the Catalan number \(C_k\).

EXAMPLES:

This class can be called with three keyword parameters \(k1, k2\) and complete.

If neither \(k1\) nor \(k2\) are specified, then DyckWords returns the combinatorial class of all balanced (=complete) Dyck words, unless the keyword complete is set to False (in which case it returns the class of all Dyck words).

```python
sage: DW = DyckWords(); DW
Complete Dyck words
sage: [] in DW
True
sage: [1, 0, 1, 0] in DW
True
sage: [1, 1, 0] in DW
False
sage: ADW = DyckWords(complete=False); ADW
Dyck words
sage: [] in ADW
True
sage: [1, 0, 1, 0] in ADW
```

(continues on next page)
True
\texttt{sage: [1, 1, 0] in ADW}
True
\texttt{sage: [1, 0, 0] in ADW}
False

If just $k_1$ is specified, then it returns the balanced Dyck words with $k_1$ opening parentheses and $k_1$ closing parentheses.

\texttt{sage: DW2 = DyckWords(2); DW2}
Dyck words with 2 opening parentheses and 2 closing parentheses
\texttt{sage: DW2.first()}
[1, 0, 1, 0]
\texttt{sage: DW2.last()}
[1, 1, 0, 0]
\texttt{sage: DW2.cardinality()}
2
\texttt{sage: DyckWords(100).cardinality() == catalan_number(100)}
True

If $k_2$ is specified in addition to $k_1$, then it returns the Dyck words with $k_1$ opening parentheses and $k_2$ closing parentheses.

\texttt{sage: DW32 = DyckWords(3,2); DW32}
Dyck words with 3 opening parentheses and 2 closing parentheses
\texttt{sage: DW32.list()}
[[1, 0, 1, 0, 1],
 [1, 0, 1, 1, 0],
 [1, 1, 0, 0, 1],
 [1, 1, 0, 1, 0],
 [1, 1, 1, 0, 0]]

\textbf{Element}

alias of \texttt{DyckWord}

\texttt{from_heights(heights)}

Compute a Dyck word knowing its heights.

We view the Dyck word as a Dyck path from $(0, 0)$ to $(2n, 0)$ in the first quadrant by letting 1’s represent steps in the direction $(1, 1)$ and 0’s represent steps in the direction $(1, -1)$.

The \texttt{heights()} is the sequence of the $y$-coordinates of the $2n + 1$ lattice points along this path.

\textbf{EXAMPLES:}

\texttt{sage: from sage.combinat.dyck_word import DyckWord}
\texttt{sage: D = DyckWords(complete=False)}
\texttt{sage: D.from_heights([0,])}
[]
\texttt{sage: D.from_heights([0, 1, 0])}
[1, 0]
\texttt{sage: D.from_heights([0, 1, 2, 1, 0])}
[1, 1, 0, 0]

This also works for incomplete Dyck words:
**min_from_heights**(*heights*)

Compute the smallest Dyck word which achieves or surpasses a given sequence of heights.

**INPUT:**

- *heights* – a nonempty list or iterable consisting of nonnegative integers, the first of which is 0

**OUTPUT:**

- The smallest Dyck word whose sequence of heights is componentwise higher-or-equal to *heights*. Here, “smaller” can be understood both in the sense of lexicographic order on the Dyck words, and in the sense of every vertex of the path having the smallest possible height.

**See also:**

- *heights()*
- *from_heights()*

**EXAMPLES:**

```python
sage: D = DyckWords(complete=False)
sage: D.min_from_heights((0,))
[]
sage: D.min_from_heights((0, 1, 0))
[1, 0]
sage: D.min_from_heights((0, 0, 2, 0, 0))
[1, 1, 0, 0]
sage: D.min_from_heights((0, 0, 2, 0, 2, 0))
[1, 1, 0, 1, 0]
sage: D.min_from_heights((0, 0, 1, 0, 1, 0))
[1, 1, 0, 1, 0]
```

```
```
cardinality()

Return the number of Dyck words with \( k_1 \) openers and \( k_2 \) closers.

This number is

\[
\frac{k_1 - k_2 + 1}{k_1 + 1} \binom{k_1 + k_2}{k_2} = \binom{k_1 + k_2}{k_2} - \binom{k_1 + k_2}{k_2 - 1}
\]

if \( k_2 \leq k_1 + 1 \), and 0 if \( k_2 > k_1 \) (these numbers are the same if \( k_2 = k_1 + 1 \)).

EXAMPLES:

```python
sage: DyckWords(3, 2).cardinality()
5
sage: all(all(DyckWords(p, q).cardinality()
.....: == len(DyckWords(p, q).list()) for q in range(p + 1))
.....: for p in range(7))
True
```

sage.combinat.dyck_word.is_a(obj, k1=None, k2=None)

Test if \( obj \) is a Dyck word with exactly \( k_1 \) open symbols and exactly \( k_2 \) close symbols.

If \( k_1 \) is not specified, then the number of open symbols can be arbitrary. If \( k_1 \) is specified but \( k_2 \) is not, then \( k_2 \) is set to be \( k_1 \).

EXAMPLES:

```python
sage: from sage.combinat.dyck_word import is_a
sage: is_a([1,1,0,0])
True
sage: is_a([1,0,1,0])
True
sage: is_a([1,1,0,0], 2)
True
sage: is_a([1,1,0,0], 3)
False
sage: is_a([1,1,0,0], 3, 2)
False
sage: is_a([1,1,0])
True
sage: is_a([0,1,0])
False
sage: is_a([1,0], 0)
False
sage: is_a([1,0], 2)
False
sage: is_a([1,1,0], 2, 1)
True
sage: is_a([1,1,0], 2)
False
sage: is_a([1,1,0], 1, 1)
False
```

sage.combinat.dyck_word.is_area_sequence(seq)

Test if a sequence \( l \) of integers satisfies \( l_0 = 0 \) and \( 0 \leq l_{i+1} \leq l_i + 1 \) for \( i > 0 \).

EXAMPLES:
sage: from sage.combinat.dyck_word import is_area_sequence
sage: is_area_sequence([0,2,0])
False
sage: is_area_sequence([1,2,3])
False
sage: is_area_sequence([0,1,0])
True
sage: is_area_sequence([0,1,2])
True
sage: is_area_sequence([])
True

sage.combinat.dyck_word.pealing(D, return_touches=False)

A helper function for computing the bijection from a Dyck word to a 231-avoiding permutation using the bijection “Stump”. For details see [Stu2008].

See also:

to_noncrossing_partition()

EXAMPLES:

sage: from sage.combinat.dyck_word import pealing
sage: pealing(DyckWord([1,1,0,0]))
[1, 0, 1, 0]
sage: pealing(DyckWord([1,0,1,0]))
[1, 0, 1, 0]
sage: pealing(DyckWord([1, 1, 0, 0, 1, 1, 1, 0, 0, 0]),return_touches=True)
(([1, 0, 1, 0, 1, 0, 1, 0, 1, 0], [[1, 2], [3, 5]])

sage.combinat.dyck_word.replace_parens(x)

A map sending '(' to open_symbol and ')' to close_symbol, and raising an error on any input other than '(' and ')'. The values of the constants open_symbol and close_symbol are subject to change.

This is the inverse map of replace_symbols().

INPUT:

• x – either an opening or closing parenthesis

OUTPUT:

• If x is an opening parenthesis, replace x with the constant open_symbol.
• If x is a closing parenthesis, replace x with the constant close_symbol.
• Raise a ValueError if x is neither an opening nor a closing parenthesis.

See also:

replace_symbols()

EXAMPLES:
sage: from sage.combinat.dyck_word import replace_parens
sage: replace_parens('(')
1
sage: replace_parens(')')
0
sage: replace_parens(1)
Traceback (most recent call last):
  ... ValueError

sage.combinat.dyck_word.replace_symbols(x)
A map sending open_symbol to '(', and close_symbol to ')', and raising an error on any input other than open_symbol and close_symbol. The values of the constants open_symbol and close_symbol are subject to change.

This is the inverse map of replace_parens().

INPUT:

- x – either open_symbol or close_symbol.

OUTPUT:

- If x is open_symbol, replace x with '.
- If x is close_symbol, replace x with '}'.
- If x is neither open_symbol nor close_symbol, a ValueError is raised.

See also:

replace_parens()

EXAMPLES:

sage: from sage.combinat.dyck_word import replace_symbols
sage: replace_symbols(1)
'( '
sage: replace_symbols(0)
') '
sage: replace_symbols(3)
Traceback (most recent call last):
  ... ValueError

5.1.100 Substitutions over unit cube faces (Rauzy fractals)

This module implements the $E_1^*(\sigma)$ substitution associated with a one-dimensional substitution $\sigma$, that acts on unit faces of dimension $(d - 1)$ in $\mathbb{R}^d$.

This module defines the following classes and functions:

- Face - a class to model a face
- Patch - a class to model a finite set of faces
- E1Star - a class to model the $E_1^*(\sigma)$ application defined by the substitution $\sigma$
Combinatorics, Release 10.0

See the documentation of these objects for more information.

The convention for the choice of the unit faces and the definition of $E^*_1(\sigma)$ varies from article to article. Here, unit faces are defined by

\[
[x, 1]^* = \{x + \lambda e_2 + \mu e_3 : \lambda, \mu \in [0, 1]\}
\]
\[
[x, 2]^* = \{x + \lambda e_1 + \mu e_3 : \lambda, \mu \in [0, 1]\}
\]
\[
[x, 3]^* = \{x + \lambda e_1 + \mu e_2 : \lambda, \mu \in [0, 1]\}
\]

and the dual substitution $E^*_1(\sigma)$ is defined by

\[
E^*_1(\sigma)([x, i]^*) = \bigcup_{k=1,2,3} \bigcup_{s|\sigma(k) = p_i s} [M^{-1}(x + \ell(s)), k]^*,
\]

where $\ell(s)$ is the abelianized of $s$, and $M$ is the matrix of $\sigma$.

AUTHORS:

- Franco Saliola (2009): initial version
- Vincent Delecroix, Timo Jolivet, Stepan Starosta, Sebastien Labbe (2010-05): redesign
- Timo Jolivet (2010-08, 2010-09, 2011): redesign

REFERENCES:

EXAMPLES:

We start by drawing a simple three-face patch:

```python
sage: from sage.combinat.e_one_star import E1Star, Face, Patch
sage: x = [Face((0,0,0),1), Face((0,0,0),2), Face((0,0,0),3)]
sage: P = Patch(x)
sage: P
Patch: \{[(0, 0, 0), 1]^*, [(0, 0, 0), 2]^*, [(0, 0, 0), 3]^*\}
sage: P.plot()  #not tested
```

We apply a substitution to this patch, and draw the result:

```python
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: E(P)
Patch: \{[(0, 0, 0), 1]^*, [(0, 0, 0), 2]^*, [(0, 0, 0), 3]^*, [(0, 1, -1), 2]^*, [(1, 0, -1), 1]^*\}
sage: E(P).plot()  #not tested
```

Note:

- The type of a face is given by an integer in $[1, \ldots, d]$ where $d$ is the length of the vector of the face.
- The alphabet of the domain and the codomain of $\sigma$ must be equal, and they must be of the form $[1, \ldots, d]$, where $d$ is a positive integer corresponding to the length of the vectors of the faces on which $E^*_1(\sigma)$ will act.
The application of an E1Star substitution assigns to each new face the color of its preimage. The repaint method allows us to repaint the faces of a patch. A single color can also be assigned to every face, by specifying a list of a single color:

```
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: P = E(P, 5)
sage: P.repaint(['green'])
sage: P.plot()  #not tested
```

A list of colors allows us to color the faces sequentially:

```
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: P = E(P)
sage: P.repaint(['red', 'yellow', 'green', 'blue', 'black'])
sage: P = E(P, 3)
sage: P.plot()  #not tested
```

All the color schemes from list(matplotlib.cm.datad) can be used:

```
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: P = E(P, 3)
sage: P.repaint(cmap='summer')
sage: P = E(P, 2)
sage: P.plot()  #not tested
```

It is also possible to specify a dictionary to color the faces according to their type:

```
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: P = P + P.translate([-1,1,0])
sage: P = E(P, 11)
sage: P.repaint({1:(0.7, 0.7, 0.7), 2:(0.5,0.5,0.5), 3:(0.3,0.3,0.3)})
sage: P.repaint({1:'red', 2:'yellow', 3:'green'})
sage: P.plot()  #not tested
```

Let us look at a nice big patch in 3D:

```
sage: sigma = WordMorphism({1:[1,2], 2:[3], 3:[1]})
sage: E = E1Star(sigma)
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: P = P + P.translate([-1,1,0])
sage: P = E(P, 11)
sage: P.plot3d()  #not tested
```

Plotting with TikZ pictures is possible:

```
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: s = P.plot_tikz()
sage: print(s)  #not tested
\begin{tikzpicture}
\end{tikzpicture}
```

(continues on next page)
Plotting patches made of unit segments instead of unit faces:

```
sage: P = Patch([Face([0,0], 1), Face([0,0], 2)])
sage: E = E1Star(WordMorphism({1:[1,2], 2:[1]}))
sage: F = E1Star(WordMorphism({1:[1,1,2], 2:[2,1]}))
sage: E(P,5).plot()
Graphics object consisting of 21 graphics primitives
sage: F(P,3).plot()
Graphics object consisting of 34 graphics primitives
```

Everything works in any dimension (except for the plotting features which only work in dimension two or three):

```
sage: P = Patch([Face([0,0,0], 1), Face([0,0,0], 4)])
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1,4], 4:[1,5], 5:[1,6], 6:[1,7], 7:[1,8], 8:[1,9], 9:[1,10], 10:[1,11], 11:[1,12], 12:[1]})
sage: E = E1Star(sigma)
sage: E(P)
```

```
sage: for x in sorted(E(P), key=lambda x : (x.vector(),x.type())): print(x)
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), 1]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), 2]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), 12]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1), 11]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1), 10]*
[(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1), 9]*
[(0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1), 8]*
[(0, 0, 0, 0, 0, 0, 1, 0, 0, -1), 7]*
[(0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1), 6]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1), 5]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1), 4]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1), 3]*
[(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1), 2]*
[(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1), 1]*
```

```
class sage.combinat.e_one_star.E1Star(sigma, method='suffix')
```
Bases: SageObject

A class to model the $E_1^*(\sigma)$ map associated with a unimodular substitution $\sigma$.

INPUT:

- `sigma` - unimodular WordMorphism, i.e. such that its incidence matrix has determinant $\pm 1$.
- `method` - 'prefix' or 'suffix' (optional, default: 'suffix') Enables to use an alternative definition $E_1^*(\sigma)$ substitutions, where the abelianized of the prefix' is used instead of the suffix.

Note: The alphabet of the domain and the codomain of $\sigma$ must be equal, and they must be of the form $[1, \ldots, d]$, where $d$ is a positive integer corresponding to the length of the vectors of the faces on which $E_1^*(\sigma)$ will act.

EXAMPLES:

```python
sage: from sage.combinat.e_one_star import E1Star, Face, Patchsage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: E(P)
Patch: [[(0, 0, 0), 1]*, [(0, 0, 0), 2]*, [(0, 0, 1), 3]*, [(1, 0, 0), -1], 1]*]
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma, method='prefix')
sage: E(P)
Patch: [[(0, 0, 0), 1]*, [(0, 0, 0), 2]*, [(0, 0, 1), 3]*, [(1, 0, -1), 2]*]
sage: x = [Face((0,0,0,0),1), Face((0,0,0,0),4)]
sage: P = Patch(x)
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1,4], 4:[1]})
sage: E = E1Star(sigma)
sage: E(P)
Patch: [[(0, 0, 0, 0), 1]*, [(0, 0, 0, 0), 2]*, [(0, 0, 1, -1), 3]*, [(0, 1, 0, -1), 4]*, [(0, 1, 0, -1), 3]*, [(0, 0, 1, 0), 4]*, [(0, 0, 1, 0), -1], 1]*]
```

inverse_matrix()

Return the inverse of the matrix associated with self.

EXAMPLES:

```python
sage: from sage.combinat.e_one_star import E1Star, Face, Patchsage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: E.inverse_matrix()
[ 0 1 0]
[ 0 0 1]
[ 1 -1 -1]
```
matrix()

Return the matrix associated with self.

EXAMPLES:

```python
sage: from sage.combinat.e_one_star import E1Star, Face, Patch
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: E.matrix()
[1 1 1]
[1 0 0]
[0 1 0]
```

sigma()

Return the WordMorphism associated with self.

EXAMPLES:

```python
sage: from sage.combinat.e_one_star import E1Star, Face, Patch
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: E.sigma()
WordMorphism: 1->12, 2->13, 3->1
```

class sage.combinat.e_one_star.Face(v, t, color=None)

Bases: SageObject

A class to model a unit face of arbitrary dimension.

A unit face in dimension $d$ is represented by a $d$-dimensional vector $v$ and a type $t$ in $\{1, \ldots, d\}$. The type of the face corresponds to the canonical unit vector to which the face is orthogonal. The optional `color` argument is used in plotting functions.

INPUT:

- $v$ - tuple of integers
- $t$ - integer in $[1, \ldots, \text{len}(v)]$, type of the face. The face of type $i$ is orthogonal to the canonical vector $e_i$.
- `color` - color (optional, default: `None`) color of the face, used for plotting only. If `None`, its value is guessed from the face type.

EXAMPLES:

```python
sage: from sage.combinat.e_one_star import Face
sage: f = Face((0,2,0), 3)
sage: f.vector()
(0, 2, 0)
sage: f.type()
3

sage: f = Face((0,2,0), 3, color=(0.5, 0.5, 0.5))
sage: f.color()
RGB color (0.5, 0.5, 0.5)
```
**color** *(color=None)*

Return or change the color of the face.

**INPUT:**

- color - string, rgb tuple, color (optional, default: None) the new color to assign to the face. If None, it returns the color of the face.

**OUTPUT:**

color or None

**EXAMPLES:**

```python
sage: from sage.combinat.e_one_star import Face
sage: f = Face((0,2,0), 3)
sage: f.color()
RGB color (0.0, 0.0, 1.0)
sage: f.color('red')
sage: f.color()
RGB color (1.0, 0.0, 0.0)
```

**type()**

Return the type of the face.

**EXAMPLES:**

```python
sage: from sage.combinat.e_one_star import Face
sage: f = Face((0,2,0), 3)
sage: f.type()
3
```

**vector()**

Return the vector of the face.

**EXAMPLES:**

```python
sage: from sage.combinat.e_one_star import Face
sage: f = Face((0,2,0), 3)
sage: f.vector()
(0, 2, 0)
```

**class** `sage.combinat.e_one_star.Patch(faces, face_contour=None)`

**Bases:** `SageObject`

A class to model a collection of faces. A patch is represented by an immutable set of Faces.

**Note:** The dimension of a patch is the length of the vectors of the faces in the patch, which is assumed to be the same for every face in the patch.
**Note:** Since version 4.7.1, Patches are immutable, except for the colors of the faces, which are not taken into account for equality tests and hash functions.

**INPUT:**
- faces - finite iterable of faces
- face_contour - dict (optional, default=None) maps the face type to vectors describing the contour of unit faces. If None, defaults contour are assumed for faces of type 1, 2, 3 or 1, 2, 3. Used in plotting methods only.

**EXAMPLES:**

```python
sage: from sage.combinat.e_one_star import Face, Patch
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
```

```python
sage: P
Patch: [[(0, 0, 0), 1]*, [(0, 0, 0), 2]*, [(0, 0, 0), 3]*]
```

```python
sage: face_contour = {}
sage: face_contour[1] = map(vector, [(0,0,0),(0,1,0),(0,1,1),(0,0,1)])
sage: face_contour[2] = map(vector, [(0,0,0),(0,0,1),(1,0,1),(1,0,0)])
sage: face_contour[3] = map(vector, [(0,0,0),(1,0,0),(1,1,0),(0,1,0)])
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]], face_contour=face_contour)
```

```python
sage: P
Patch: [[(0, 0, 0), 1]*, [(0, 0, 0), 2]*, [(0, 0, 0), 3]*]
```

**difference(other)**

Return the difference of self and other.

**INPUT:**
- other - a finite iterable of faces or a single face

**EXAMPLES:**

```python
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
sage: P.difference(Face((0,0,0),2))
```

```python
Patch: [[(0, 0, 0), 1]*, [(0, 0, 0), 3]*]
```

```python
sage: P.difference(P)
Patch: []
```

**dimension()**

Return the dimension of the vectors of the faces of self

It returns None if self is the empty patch.

The dimension of a patch is the length of the vectors of the faces in the patch, which is assumed to be the same for every face in the patch.

**EXAMPLES:**

```python
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
sage: P.dimension()
3
```
faces_of_color(color)

Return a list of the faces that have the given color.

INPUT:

- color - color

EXAMPLES:

```
sage: from sage.combinat.e_one_star import Face, Patch
sage: P = Patch([Face((0,0,0),1, 'red'), Face((1,2,0),3, 'blue'), Face((1,2,0), -1, 'red'))]
sage: sorted(P.faces_of_color('red'))
[[[0, 0, 0], 1]*, [(1, 2, 0), 1]*]
```

faces_of_type(t)

Return a list of the faces that have type \( t \).

INPUT:

- \( t \) - integer or any other type

EXAMPLES:

```
sage: from sage.combinat.e_one_star import Face, Patch
sage: P = Patch([Face((0,0,0),1), Face((1,2,0),3), Face((1,2,0),1)])
sage: sorted(P.faces_of_type(1))
[[[0, 0, 0], 1]*, [(1, 2, 0), 1]*]
```

faces_of_vector(v)

Return a list of the faces whose vector is \( v \).

INPUT:

- \( v \) - a vector

EXAMPLES:

```
sage: from sage.combinat.e_one_star import Face, Patch
sage: P = Patch([Face([0,0,0], 1), Face([1,2,0], 3), Face([1,2,0], 1)])
sage: sorted(P.faces_of_vector([1,2,0]))
[[[1, 2, 0], 1]*, [(1, 2, 0), 3]*]
```

occurrences_of(other)

Return all positions at which other appears in self, that is, all vectors \( v \) such that \( \text{set}(\text{other}.\text{translate}(v)) \subseteq \text{set}(\text{self}) \).

INPUT:

- other - a Patch

OUTPUT:

a list of vectors

EXAMPLES:

```
sage: from sage.combinat.e_one_star import Face, Patch, E1Star
sage: P = Patch([Face([0,0,0], 1), Face([0,0,0], 2), Face([0,0,0], 3)])
sage: Q = Patch([Face([0,0,0], 1), Face([0,0,0], 2)])
```
sage: P.occurrences_of(Q)
[(0, 0, 0)]
sage: Q = Q.translate([1,2,3])
sage: P.occurrences_of(Q)
[(-1, -2, -3)]

sage: E = E1Star(WordMorphism({1:[1,2], 2:[1,3], 3:[1]}))
sage: P = Patch([Face([0,0,0], t) for t in [1,2,3]])
sage: P = E(P,4)
sage: Q = Patch([Face([0,0,0], t) for t in [1,2,3]])

plot(projmat=None, opacity=0.75)

Return a 2D graphic object depicting the patch.

INPUT:

- `projmat` - matrix (optional, default: None) the projection matrix. Its number of lines must be two. Its number of columns must equal the dimension of the ambient space of the faces. If None, the isometric projection is used by default.
- `opacity` - float between 0 and 1 (optional, default: 0.75) opacity of the face

**Warning:** Plotting is implemented only for patches in two or three dimensions.

**EXAMPLES:**

sage: from sage.combinat.e_one_star import E1Star, Face, Patch
dsage: P = Patch([Face([0,0,0], t) for t in [1,2,3]])
sage: P.plot()
Graphics object consisting of 3 graphics primitives

sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: P = Patch([Face([0,0,0], t) for t in [1,2,3]])
sage: P = E(P, 5)
sage: P.plot()
Graphics object consisting of 57 graphics primitives

Plot with a different projection matrix:

sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: P = Patch([Face([0,0,0], t) for t in [1,2,3]])
sage: M = matrix(2, 3, [1,0,-1,0.3,1,-3])
sage: P = E(P, 3)
sage: P.plot(projmat=M)
Graphics object consisting of 17 graphics primitives

Plot patches made of unit segments:
sage: P = Patch([Face([0,0], 1), Face([0,0], 2)])
sage: E = E1Star(WordMorphism({1:[1,2], 2:[1]}))
sage: F = E1Star(WordMorphism({1:[1,1,2], 2:[2,1]}))
sage: E(P,5).plot()
Graphics object consisting of 21 graphics primitives
sage: F(P,3).plot()
Graphics object consisting of 34 graphics primitives

plot3d()

Return a 3D graphics object depicting the patch.

Warning: 3D plotting is implemented only for patches in three dimensions.

EXAMPLES:

sage: from sage.combinat.e_one_star import E1Star, Face, Patch
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
sage: P.plot3d()  #not tested

sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])
sage: P = E(P, 5)
sage: P.repaint()
sage: P.plot3d()  #not tested

plot_tikz(projmat=None, print_tikz_env=True, edgcolor='black', scale=0.25, drawzero=False,
          extra_code_before='', extra_code_after='')

Return a string containing some TikZ code to be included into a LaTeX document, depicting the patch.

Warning: TikZ Plotting is implemented only for patches in three dimensions.

INPUT:

- projmat - matrix (optional, default: None) the projection matrix. Its number of lines must be two. Its number of columns must equal the dimension of the ambient space of the faces. If None, the isometric projection is used by default.
- print_tikz_env - bool (optional, default: True) if True, the tikzpicture environment are printed
- edgcolor - string (optional, default: 'black') either 'black' or 'facecolor' (color of unit face edges)
- scale - real number (optional, default: 0.25) scaling constant for the whole figure
- drawzero - bool (optional, default: False) if True, mark the origin by a black dot
- extra_code_before - string (optional, default: '') extra code to include in the tikz picture
- extra_code_after - string (optional, default: '') extra code to include in the tikz picture

EXAMPLES:
```python
sage: from sage.combinat.e_one_star import E1Star, Face, Patch
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
```

Plot using shades of gray (useful for article figures):

```python
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: P = E(P, 4)
sage: P.repaint([(0.9, 0.9, 0.9), (0.65, 0.65, 0.65), (0.4, 0.4, 0.4)])
sage: P = E(P, 4)
sage: s = P.plot_tikz()
```

```latex
\begin{tikzpicture}
[x={(-0.216506cm,-0.125000cm)}, y={(0.216506cm,-0.125000cm)}, z={(0.000000cm,0. → 250000cm)}]
definecolor{facecolor}{rgb}{0.000,1.000,0.000}
fill[fill=facecolor, draw=black, shift={(0,0,0)}]
(0, 0, 0) -- (0, 0, 1) -- (1, 0, 1) -- (1, 0, 0) -- cycle;
definecolor{facecolor}{rgb}{1.000,0.000,0.000}
fill[fill=facecolor, draw=black, shift={(0,0,0)}]
(0, 0, 0) -- (0, 1, 0) -- (0, 1, 1) -- (0, 0, 1) -- cycle;
definecolor{facecolor}{rgb}{0.000,0.000,1.000}
fill[fill=facecolor, draw=black, shift={(0,0,0)}]
(0, 0, 0) -- (1, 0, 0) -- (1, 1, 0) -- (0, 1, 0) -- cycle;
end[tikzpicture]
```

Plotting with various options:

```python
sage: sigma = WordMorphism({1:[1,2], 2:[1,3], 3:[1]})
sage: E = E1Star(sigma)
sage: P = Patch([Face((0,0,0), t) for t in [1,2,3]])
sage: M = matrix(2,3,[float(u) for u in [1,0,-0.7071,0,1,-0.7071]])
sage: P = E(P, 3)
sage: s = P.plot_tikz(projmat=M, edgcolor='facecolor', scale=0.6, drawzero=True)
```

Adding X, Y, Z axes using the extra code feature:

```python
sage: length = 1.5
sage: space = 0.3
sage: axes = ''
sage: axes += "\draw[->, thick, black] (0,0,0) -- (%.s, %.s, %.s);\n" % length
```
repaint(cmap='Set1')

Repaint all the faces of self from the given color map.

This only changes the colors of the faces of self.

INPUT:

• cmap - color map (default: 'Set1'). It can be one of the following:
  – string – A coloring map. For available coloring map names type: sorted(colormaps)
  – list – a list of colors to assign cyclically to the faces. A list of a single color colors all the faces
    with the same color.
  – dict – a dict of face types mapped to colors, to color the faces according to their type.
  – {}, the empty dict - shortcut for {1:'red', 2:'green', 3:'blue'}.

EXAMPLES:

Using a color map:
sage: from sage.combinat.e_one_star import Face, Patch
sage: color = (0, 0, 0)

sage: P = Patch([Face((0,0,0),t,color) for t in [1,2,3]])

sage: for f in P: f.color()
RGB color (0.0, 0.0, 0.0)
RGB color (0.0, 0.0, 0.0)
RGB color (0.0, 0.0, 0.0)

sage: P.repaint()

sage: next(iter(P)).color() #random
RGB color (0.498..., 0.432..., 0.522...)

Using a list of colors:

sage: P = Patch([Face((0,0,0),t,color) for t in [1,2,3]])

sage: P.repaint([(0.9, 0.9, 0.9), (0.65,0.65,0.65), (0.4,0.4,0.4)])

sage: for f in P: f.color()
RGB color (0.9, 0.9, 0.9)
RGB color (0.65, 0.65, 0.65)
RGB color (0.4, 0.4, 0.4)

Using a dictionary to color faces according to their type:

sage: P = Patch([Face((0,0,0),t) for t in [1,2,3]])

sage: P.repaint({1: 'black', 2: 'yellow', 3: 'green'})

sage: P.plot()  # not tested

sage: P.repaint({})  # not tested

translate(v)
Return a translated copy of self by vector v.

INPUT:

• v - vector or tuple

EXAMPLES:

sage: from sage.combinat.e_one_star import Face, Patch
sage: P = Patch([Face((0,0,0),1), Face((1,2,0),3), Face((1,2,0),1)])

sage: P.translate([-1,-2,0])
Patch: [[(-1, -2, 0), 1]*, [(0, 0, 0), 1]*, [(0, 0, 0), 3]*]

union(other)
Return a Patch consisting of the union of self and other.

INPUT:

• other - a Patch or a Face or a finite iterable of faces

EXAMPLES:

sage: from sage.combinat.e_one_star import Face, Patch
sage: P = Patch([Face((0,0,0),1), Face((0,0,0),2)])

sage: P.union(Face((1,2,3), 3))
Patch: [[(0, 0, 0), 1]*, [(0, 0, 0), 2]*, [(1, 2, 3), 3]*]

(continues on next page)
\texttt{sage:} \texttt{P.union([Face((1,2,3), 3), Face((2,3,3), 2)])}
\texttt{Patch: } \texttt{[((0, 0, 0), 1)*, [(0, 0, 0), 2]*, [(1, 2, 3), 3]*, [(2, 3, 3), 2]*]}

5.1.101 Enumerated sets and combinatorial objects

\textbf{Todo:} Proofread / point to the main classes rather than the modules

Categories

- \texttt{EnumeratedSets, FiniteEnumeratedSets}

Basic enumerated sets

- \texttt{Subsets, Combinations}
- \texttt{Arrangements, Tuples}
- \texttt{FiniteEnumeratedSet}
- \texttt{DisjointUnionEnumeratedSets}

Integer lists

- \texttt{Integer partitions} (see also: \texttt{Enumerated sets of partitions, tableaux, \ldots})
- \texttt{Integer compositions}
- \texttt{SignedCompositions}
- \texttt{IntegerListsLex}
- \texttt{Super Partitions}
- \texttt{IntegerVectors}
- \texttt{WeightedIntegerVectors()}
- \texttt{IntegerVectorsModPermutationGroup}
- \texttt{Parking Functions}
- \texttt{Non-Decreasing Parking Functions}
- \texttt{Sidon sets and their generalizations, Sidon g-sets}
Words

- Words
- Subwords
- Necklaces
- Lyndon words
- Dyck Words
- De Bruijn sequences
- Shuffle product of iterables

Permutations, ...

- Permutations
- Permutations (Cython file)
- Affine Permutations
- Arrangements
- Derangements
- Baxter permutations

See also:

- SymmetricGroup, PermutationGroup(), Catalog of permutation groups
- FiniteSetMaps
- Integer vectors modulo the action of a permutation group
- Robinson-Schensted-Knuth correspondence

Partitions, tableaux, ...

See: Enumerated sets of partitions, tableaux, ...

Polyominoes

See: Parallelogram Polyominoes

Integer matrices, ...

- Counting, generating, and manipulating non-negative integer matrices
- Hadamard matrices
- Latin Squares
- Alternating Sign Matrices
- Six Vertex Model
- Similarity class types of matrices with entries in a finite field
• Restricted growth arrays
• Vector Partitions

See also:
• MatrixSpace
• Library of Interesting Groups

Subsets and set partitions

• Subsets, Combinations
• PairwiseCompatibleSubsets
• Subsets satisfying a hereditary property
• Ordered Set Partitions
• Set Partitions
• Diagram and Partition Algebras
• OrderedMultisetPartitionsIntoSets, OrderedMultisetPartitionIntoSets

Trees

• Abstract Recursive Trees
• Ordered Rooted Trees
• Binary Trees
• Rooted (Unordered) Trees

Enumerated sets related to graphs

• Degree sequences
• Paths in Directed Acyclic Graphs
• Perfect matchings

Backtracking solvers and generic enumerated sets

Todo: Do we want a separate section, possibly more proeminent, for backtracking solvers?

• RecursivelyEnumeratedSet()
• GenericBacktracker
• sage.parallel.map_reduce
• Tiling Solver
• Exact Cover Problem via Dancing Links
• Dancing links C++ wrapper
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- Combinatorial species
- IntegerListsLex
- IntegerVectorsModPermutationGroup

Low level enumerated sets

- Gray codes

Misc enumerated sets

- GelfandTsetlinPattern, GelfandTsetlinPatterns
- KnutsonTaoPuzzleSolver
- LatticePolytope

5.1.102 Tools for enumeration modulo the action of a permutation group

sage.combinat.enumeration_mod_permgroup.all_children(v, max_part)

Returns all the children of an integer vector (ClonableIntArray) v in the tree of enumeration by lexicographic order. The children of an integer vector v whose entries have the sum \( n \) are all integer vectors of sum \( n + 1 \) which follow v in the lexicographic order.

That means this function adds 1 on the last non zero entries and the following ones. For an integer vector \( v \) such that

\[
v = [\ldots, a, 0, 0] \text{ with } a \neq 0,
\]

then, the list of children is

\[
[[\ldots, a + 1, 0, 0], [\ldots, a, 1, 0], [\ldots, a, 0, 1]].
\]

EXAMPLES:

```
sage: from sage.combinat.enumeration_mod_permgroup import all_children
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: v = IncreasingIntArrays([1,2,3,4])
sage: all_children(v, -1)
[[1, 2, 3, 5]]
```

sage.combinat.enumeration_mod_permgroup.canonical_children(sgs, v, max_part)

Returns the canonical children of the integer vector v. This function computes all children of the integer vector v via the function all_children() and returns from this list only those which are canonicals identified via the function is_canonical().

EXAMPLES:

```
sage: from sage.combinat.enumeration_mod_permgroup import canonical_children
sage: G = PermutationGroup([[(1,2,3,4)]])
sage: sgs = G.strong_generating_system()
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: IA = IncreasingIntArrays()
sage: canonical_children(sgs, IA([1,2,3,5]), -1)
[]
```
sage.combinat.enumeration_mod_permgroup.canonical_representative_of_orbit_of(sgs, v)

Returns the maximal vector for the lexicographic order living in the orbit of \( v \) under the action of the permutation group whose strong generating system is \( sgs \). The maximal vector is also called “canonical”. Hence, this method returns the canonical vector inside the orbit of \( v \). If \( v \) is already canonical, the method returns \( v \).

Let \( G \) to be the permutation group which admits \( sgs \) as a strong generating system. An integer vector \( v \) is said to be canonical under the action of \( G \) if and only if:

\[
v = \max_{\text{lex order}} \{ g \cdot v | g \in G \}
\]

**EXAMPLES:**

```python
sage: from sage.combinat.enumeration_mod_permgroup import canonical_representative_of_orbit_of
sage: G = PermutationGroup([(1,2,3,4)])
sage: sgs = G.strong_generating_system()
sage: IA = IncreasingIntArrays()
sage: canonical_representative_of_orbit_of(sgs, IA([1,2,3,5]))
[5, 1, 2, 3]
```

sage.combinat.enumeration_mod_permgroup.is_canonical(sgs, v)

Returns True if the integer vector \( v \) is maximal with respect to the lexicographic order in its orbit under the action of the permutation group whose strong generating system is \( sgs \). Such vectors are said to be canonical.

Let \( G \) to be the permutation group which admit \( sgs \) as a strong generating system. An integer vector \( v \) is said to be canonical under the action of \( G \) if and only if:

\[
v = \max_{\text{lex order}} \{ g \cdot v | g \in G \}
\]

**EXAMPLES:**

```python
sage: from sage.combinat.enumeration_mod_permgroup import is_canonical
sage: G = PermutationGroup([(1,2,3,4)])
sage: sgs = G.strong_generating_system()
(sage: IA = IncreasingIntArrays()
(sage: is_canonical(sgs, IA([1,2,3,6]))
False)
```

sage.combinat.enumeration_mod_permgroup.lex_cmp(v1, v2)

Lexicographic comparison of ClonableIntArray.

**INPUT:**

Two instances \( v_1, v_2 \) of ClonableIntArray

**OUTPUT:**

\(-1, 0, 1\), depending on whether \( v_1 \) is lexicographically smaller, equal, or greater than \( v_2 \).

**EXAMPLES:**

```python
sage: I = IntegerVectorsModPermutationGroup(SymmetricGroup(5),5)
(sage: I = IntegerVectorsModPermutationGroup(SymmetricGroup(5))
(sage: J = IntegerVectorsModPermutationGroup(SymmetricGroup(6))
(sage: v1 = I([2,3,1,2,3], check=False)
```
```sage
code
sage: v2 = I([2,3,2,3,2], check=False)
sage: v3 = J([2,3,1,2,3,1], check=False)
sage: from sage.combinat.enumeration_mod_permgroup import lex_cmp
sage: lex_cmp(v1, v1)
0
sage: lex_cmp(v1, v2)
-1
sage: lex_cmp(v2, v1)
1
sage: lex_cmp(v1, v3)
-1
sage: lex_cmp(v3, v1)
1
```

**sage.combinat.enumeration_mod_permgroup.**\texttt{lex\_cmp\_partial}(v1, v2, \texttt{step})

Partial comparison of the two lists according the lexicographic order. It compares the \texttt{step}-th first entries.

**EXAMPLES:**

```sage
code
sage: from sage.combinat.enumeration_mod_permgroup import lex_cmp_partial
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: IA = IncreasingIntArrays()
sage: lex_cmp_partial(IA([0,1,2,3]),IA([0,1,2,4]),3)
0
sage: lex_cmp_partial(IA([0,1,2,3]),IA([0,1,2,4]),4)
-1
```

**sage.combinat.enumeration_mod_permgroup.**\texttt{orbit}(sgs, v)

Returns the orbit of the integer vector \texttt{v} under the action of the permutation group whose strong generating system is \texttt{sgs}.

**NOTE:**

The returned orbit is a set. In the doctests, we convert it into a sorted list.

**EXAMPLES:**

```sage
code
sage: from sage.combinat.enumeration_mod_permgroup import orbit
sage: G = PermutationGroup([[1,2,3,4]])
sage: sgs = G.strong_generating_system()
sage: from sage.structure.list_clone_demo import IncreasingIntArrays
sage: IA = IncreasingIntArrays()
sage: sorted(orbit(sgs, IA([1,2,3,4])))
[[1, 2, 3, 4], [2, 3, 4, 1], [3, 4, 1, 2], [4, 1, 2, 3]]
```
5.1.103 Compute Bell and Uppuluri-Carpenter numbers

AUTHORS:

- Nick Alexander

`sage.combinat.expnums.expnums(n, aa)`

Compute the first $n$ exponential numbers around $aa$, starting with the zero-th.

**INPUT:**

- $n$ - C machine int
- $aa$ - C machine int

**OUTPUT:** A list of length $n$.

**ALGORITHM:** We use the same integer addition algorithm as GAP. This is an extension of Bell’s triangle to the general case of exponential numbers. The recursion performs $O(n^2)$ additions, but the running time is dominated by the cost of the last integer addition, because the growth of the integer results of partial computations is exponential in $n$. The algorithm stores $O(n)$ integers, but each is exponential in $n$.

**EXAMPLES:**

```sage
sage: expnums(10, 1)
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
sage: expnums(10, -1)
[1, -1, 0, 1, 1, -2, -9, -9, 50, 267]
sage: expnums(1, 1)
[1]
sage: expnums(0, 1)
[]
sage: expnums(-1, 0)
[]
```

AUTHORS:

- Nick Alexander

`sage.combinat.expnums.expnums2(n, aa)`

A vanilla python (but compiled via Cython) implementation of expnums.

We Compute the first $n$ exponential numbers around $aa$, starting with the zero-th.

**EXAMPLES:**

```sage
sage: from sage.combinat.expnums import expnums2
sage: expnums2(10, 1)
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
```
5.1.104 Families

This is a backward compatibility stub. Use `sage.sets.family` instead.

5.1.105 Brent Yorgey’s fast algorithm for integer vector (multiset) partitions.

ALGORITHM:
https://wiki.haskell.org/The_Monad.Reader/Previous_issues

AUTHORS:
- D. K. Sunko (2020-02-19): initial version
- F. Chapoton (2020-02-22): conversion to iterators and shorter doctests and doc tweaks
- T. Scrimshaw (2020-03-06): Cython optimizations and doc tweaks

`sage.combinat.fast_vector_partitions.fast_vector_partitions(v, min_vals=None)`
Brent Yorgey’s fast algorithm for integer vector (multiset) partitions.

INPUT:
- `v` – list of non-negative integers, understood as the vector to be partitioned
- `min_vals` – optional list of non-negative integers, of same length as `v`

OUTPUT:
A list of lists, each representing a vector partition of `v`.
If `min_vals` is given, only partitions with parts `p >= min_vals` in the lexicographic ordering will appear.
If `min_vals` is given and `len(min_vals) != len(v)`, an error is raised.

EXAMPLES:
The older the computer, the more impressive the comparison:

```
sage: from sage.combinat.fast_vector_partitions import fast_vector_partitions
sage: fastvparts = list(fast_vector_partitions([3, 3, 3]))
sage: vparts = list(VectorPartitions([3, 3, 3]))
sage: vparts == fastvparts[::-1]
True
sage: len(fastvparts)
686
sage: list(fast_vector_partitions([1, 2, 3], min_vals=[0, 1, 1]))
[[[1, 2, 3]],
 [[0, 2, 3], [1, 0, 0]],
 [[0, 2, 2], [1, 0, 1]],
 [[0, 2, 1], [1, 0, 2]],
 [[0, 2, 0], [1, 0, 3]],
 [[0, 1, 3], [1, 1, 0]],
 [[0, 1, 2], [1, 1, 1]],
 [[0, 1, 1], [1, 1, 2]],
 [[0, 1, 1], [0, 1, 1]],
 [[0, 1, 1], [0, 1, 2], [1, 0, 0]],
 [[0, 1, 1], [0, 1, 1], [1, 0, 1]]]
sage: L1 = list(fast_vector_partitions([5, 7, 6], min_vals=[1, 3, 2]))
```

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(continued from previous page)

```python
sage: L1 == list(VectorPartitions([5, 7, 6], min=[1, 3, 2]))[::-1]
True
```

**Note:** The partitions are returned as an iterator.

In this documentation, $a <|= b$ means $a[i] <= b[i]$ for all $i$ (notation following B. Yorgey’s paper). It is the monomial partial ordering in Dickson’s lemma: $a <|= b$ iff $x^a$ divides $x^b$ as monomials.

**Warning:** The ordering of the partitions is reversed with respect to the output of Sage class `VectorPartitions`.

```
sage.combinat.fast_vector_partitions.recursive_vector_partitions(v, vL)
```

Iterate over a lexicographically ordered list of lists, each list representing a vector partition of $v$, such that no part of any partition is lexicographically smaller than $vL$.

Internal part of `fast_vector_partitions()`.

**INPUT:**
- $v$ – list of non-negative integers, understood as a vector
- $vL$ – list of non-negative integers, understood as a vector

**EXAMPLES:**

```
sage: from sage.combinat.fast_vector_partitions import recursive_vector_partitions
sage: list(recursive_vector_partitions([2, 2, 2],[1, 1, 1]))
[[[2, 2, 2]], [[1, 1, 1], [1, 1, 1]]]
sage: list(recursive_vector_partitions([2, 2, 2],[1, 1, 0]))
[[[2, 2, 2]], [[1, 1, 1], [1, 1, 1]], [[1, 1, 0], [1, 1, 2]]]
sage: list(recursive_vector_partitions([2, 2, 2],[1, 0, 1]))
[[[2, 2, 2]], [[1, 1, 1], [1, 1, 1]], [[1, 1, 0], [1, 1, 2]], [[1, 0, 2], [1, 2, 0]], [[1, 0, 1], [1, 2, 1]]]
```

```
sage.combinat.fast_vector_partitions.recursive_within_from_to(m, s, e, useS, useE)
```

Iterate over a lexicographically ordered list of lists $v$ satisfying $e <= v <= s$ and $v <|= m$ as vectors.

Internal part of `fast_vector_partitions()`.

**INPUT:**
- $m$ – list of non-negative integers, understood as a vector
- $s$ – list of non-negative integers, understood as a vector
- $e$ – list of non-negative integers, understood as a vector
- useS – boolean
- useE – boolean

**EXAMPLES:**
sage: from sage.combinat.fast_vector_partitions import recursive_within_from_to
sage: list(recursive_within_from_to([1, 2, 3], [1, 2, 2], [1, 1, 1], True, True))
[[1, 2, 2], [1, 2, 1], [1, 2, 0], [1, 1, 3], [1, 1, 2], [1, 1, 1]]

Note: The flags useS and useE are used to implement the condition efficiently. Because testing it loops over
the vector, re-testing at each step as the vector is parsed is inefficient: all but the last comparison have been
done cumulatively already. This code tests only for the last one, using the flags to accumulate information from
previous calls.

Warning: Expects to be called with s <= m.
Expects to be called first with useS == useE == True.

sage.combinat.fast_vector_partitions.within_from_to(m, s, e)
Iterate over a lexicographically ordered list of lists v satisfying e <= v <= s and v <|= m as vectors.
Internal part of fast_vector_partitions().
INPUT:
• m – list of non-negative integers, understood as a vector
• s – list of non-negative integers, understood as a vector
• e – list of non-negative integers, understood as a vector
EXAMPLES:
sage: from sage.combinat.fast_vector_partitions import within_from_to
sage: list(within_from_to([1, 2, 3], [1, 2, 2], [1, 1, 1]))
[[1, 2, 2], [1, 2, 1], [1, 2, 0], [1, 1, 3], [1, 1, 2], [1, 1, 1]]

Note: The input s will be “clipped” internally if it does not satisfy the condition s <= m.
To understand the input check, some line art is helpful. Assume that (a,b) are the two least significant coordi-
nates of some vector. Say:

```
e = (2,3), s = (7,6), m = (9,8).
```
In the figure, these values are denoted by E, S, and M, while the letter X stands for all other allowed values of v
= (a,b):

```
b
  | ^
  |
8 --------X---X---X---X---X---X---X---M
  |
7 - X X X X X
  |
6 - X X X X X S
  |
5 - X X X X X X
  |
4 - X X X X X X
```
If $S$ moves horizontally, the full-height columns fill the box in until $S$ reaches $M$, at which point it remains the limit in the $b$-direction as it moves out of the box, while $M$ takes over as the limit in the $a$-direction, so the $M$-column remains filled only up to $S$, no matter how much $S$ moves further to the right.

If $S$ moves vertically, its column will be filled to the top of the box, but it remains the relevant limit in the $a$-direction, while $M$ takes over in the $b$-direction as $S$ goes out of the box upwards.

Both behaviors are captured by using the smaller coordinate of $S$ and $M$, whenever $S$ is outside the box defined by $M$. The input will be “clipped” accordingly in that case.

**Warning:** The “clipping” behavior is transparent to the user, but may be puzzling when comparing outputs with the function `recursive_within_from_to()` which has no input protection.

### 5.1.106 Fully commutative elements of Coxeter groups

An element $w$ in a Coxeter system $(W,S)$ is fully commutative (FC) if every two reduced words of $w$ can be related by a sequence of only commutation relations, i.e., relations of the form $st = ts$ where $s, t$ are commuting generators in $S$. See [Ste1996].

Authors:

- Chase Meadors, Tianyuan Xu (2020): Initial version

**Acknowledgements**

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```python
class sage.combinat.fully_commutative_elements.FullyCommutativeElement
Bases: NormalizedClonableList

A fully commutative (FC) element in a Coxeter system.

An element $w$ in a Coxeter system $(W,S)$ is fully commutative (FC) if every two reduced word of $w$ can be related by a sequence of only commutation relations, i.e., relations of the form $st = ts$ where $s, t$ are commuting generators in $S$.

Every FC element has a canonical reduced word called its Cartier–Foata form. See [Gre2006]. We will normalize each FC element to this form.
```
check()
Called automatically when an element is created.

EXAMPLES:

```
sage: CoxeterGroup(['A', 3]).fully_commutative_elements([1, 2]) # indirect doctest
[1, 2]
sage: CoxeterGroup(['A', 3]).fully_commutative_elements([1, 2, 1]) # indirect doctest
Traceback (most recent call last):
  ... ValueError: the input is not a reduced word of a fully commutative element
```

coset_decomposition(J, side='left')
Return the coset decomposition of self with respect to the parabolic subgroup generated by J.

INPUT:

- J – subset of the generating set S of the Coxeter system
- side – string (default: 'left'); if set to 'right', find the right descents

The tuple of elements \((w_J, w^J)\) such that \(w = w_J \cdot w^J\), \(w_J\) is generated by the elements in \(J\), and \(w^J\) has no left descent from \(J\). This tuple is unique and satisfies the equation \(\ell(w) = \ell(w_J) + \ell(w^J)\), where \(\ell\) denotes Coxeter length, by general theory; see Proposition 2.4.4 of [BB2005].

EXAMPLES:

```
sage: FC = CoxeterGroup(['B', 6]).fully_commutative_elements()
sage: w = FC([1, 6, 2, 5, 4, 6, 5])
sage: w.coset_decomposition({1})
([1, 6, 2, 5, 4, 6, 5], [])
sage: w.coset_decomposition({1}, side='right')
([1, 6, 2, 5, 4, 6, 5], [1, 2, 4, 5])
sage: w.coset_decomposition({5, 6})
([6, 5, 6], [1, 2, 4, 5])
sage: w.coset_decomposition({5, 6}, side='right')
([1, 6, 2, 5, 4], [6, 5])
```

Note: The factor \(w_J\) of the coset decomposition \(w = w_J \cdot w^J\) can be obtained by greedily “pulling left descents of \(w\) that are in \(J\) to the left”; see the proof of [BB2005]. This greedy algorithm works for all elements in Coxeter group, but it becomes especially simple for FC elements because descents are easier to find for FC elements.

descents(side='left')
Obtain the set of descents on the appropriate side of self.

INPUT:

- side – string (default: 'left'); if set to 'right', find the right descents

A generator \(s\) is called a left or right descent of an element \(w\) if \(l(sw)\) or \(l(ws)\) is smaller than \(l(w)\), respectively. If \(w\) is FC, then \(s\) is a left descent of \(w\) if and only if \(s\) appears to in the word and every
A generator to the left of the leftmost $s$ in the word commutes with $s$. A similar characterization exists for right descents of FC elements.

**EXAMPLES:**

```python
sage: FC = CoxeterGroup(['B', 5]).fully_commutative_elements()
sage: w = FC([1, 4, 3, 5, 2, 4, 3])
sage: sorted(w.descents())
[1, 4]
sage: w.descents(side='right')
{3}
sage: FC = CoxeterGroup(['A', 5]).fully_commutative_elements()
sage: sorted(FC([1, 4, 3, 5, 2, 4, 3]).descents())
[1, 4]
```

See also:

`find_descent()`

**find_descent**(s, side='left')

Check if $s$ is a descent of `self` and find its position if so.

A generator $s$ is called a left or right descent of an element $w$ if $l(sw)$ or $l(ws)$ is smaller than $l(w)$, respectively. If $w$ is FC, then $s$ is a left descent of $w$ if and only if $s$ appears to in the word and every generator to the left of the leftmost $s$ in the word commutes with $s$. A similar characterization exists for right descents of FC elements.

**INPUT:**

- $s$ – integer representing a generator of the Coxeter system
- `side` – string (default: 'left'); if the argument is set to 'right', the function checks if $s$ is a right descent of `self` and finds the index of the rightmost occurrence of $s$ if so

**OUTPUT:**

Determine if the generator $s$ is a left descent of `self`; return the index of the leftmost occurrence of $s$ in `self` if so and return None if not.

**EXAMPLES:**

```python
sage: FC = CoxeterGroup(['B', 5]).fully_commutative_elements()
sage: w = FC([1, 4, 3, 5, 2, 4, 3])
sage: w.find_descent(1)
0
sage: w.find_descent(1, side='right')

sage: w.find_descent(4)
1
sage: w.find_descent(4, side='right')

sage: w.find_descent(3)
```

**group_element()**

Get the actual element of the Coxeter group associated with `self.parent()` corresponding to `self`.

**EXAMPLES:**
```python
sage: W = CoxeterGroup(['A', 3])
sage: FC = W.fully_commutative_elements()
sage: x = FC([1, 2])
sage: x.group_element()
[ 0 -1  1]
[ 1 -1  1]
[ 0  0  1]
sage: x.group_element() in W
True
```

**has_descent** \((s, \text{side}='\text{left}')\)

Determine if \(s\) is a descent on the appropriate side of \(\text{self}\).

**INPUT:**

- `side` – string (default: 'left'); if set to 'right', determine if \(\text{self}\) has \(s\) as a right descent

**OUTPUT:** a boolean value

**EXAMPLES:**

```python
sage: FC = CoxeterGroup(['B', 5]).fully_commutative_elements()
sage: w = FC([1, 4, 3, 5, 2, 4, 3])
sage: w.has_descent(1)
True
sage: w.has_descent(1, side='right')
False
sage: w.has_descent(4)
True
sage: w.has_descent(4, side='right')
False
```

**See also:**

`find_descent()`

**heap** (**kargs**)

Create the heap poset of \(\text{self}\).

The heap of an FC element \(w\) is a labeled poset that can be defined from any reduced word of \(w\). Different reduced words yield isomorphic labeled posets, so the heap is well defined.

Heaps are very useful for visualizing and studying FC elements; see, for example, [Ste1996] and [GX2020].

**INPUT:**

- `self` – list, a reduced word \(w = s_0...s_{k-1}\) of an FC element
- `one_index` – boolean (default: False). Setting the value to True will change the underlying set of the poset to \(\{1, 2, \ldots, n\}\)
- `display_labeling` – boolean (default: False). Setting the value to True will display the label \(s_i\) for each element \(i\) of the poset

**OUTPUT:**

A labeled poset where the underlying set is \(\{0, 1, \ldots, k-1\}\) and where each element \(i\) carries \(s_i\) as its label. The partial order \(\prec\) on the poset is defined by declaring \(i \prec j\) if \(i < j\) and \(m(s_i, s_j) \neq 2\).

**EXAMPLES:**

```python
```
```python
sage: FC = CoxeterGroup(['A', 5]).fully_commutative_elements()
sage: FC([1, 4, 3, 5, 2, 4]).heap().cover_relations()
[[1, 2], [1, 3], [2, 5], [2, 4], [3, 5], [0, 4]]
sage: FC([1, 4, 3, 5, 4, 2]).heap(one_index=True).cover_relations()
[[2, 3], [2, 4], [3, 6], [3, 5], [4, 6], [1, 5]]
```

### is_fully_commutative()

Check if `self` is the reduced word of an FC element.

To check if `self` is FC, we use the well-known characterization that an element `w` in a Coxeter system `(W, S)` is FC if and only if for every pair of generators `s, t ∈ S` for which `m(s, t) > 2`, no reduced word of `w` contains the ‘braid’ word `sts...` of length `m(s, t)` as a contiguous subword. See [Ste1996].

`check()` is an alias of this method, and is called automatically when an element is created.

**EXAMPLES:**

```python
sage: FC = CoxeterGroup(['A', 3]).fully_commutative_elements()
sage: x = FC([1, 2]); x.is_fully_commutative()
True
sage: x = FC.element_class(FC, [1, 2, 1], check=False); x.is_fully_commutative()
False
```

### n_value()

Calculate the n-value of `self`.

The *n-value* of a fully commutative element is the *width* (length of any longest antichain) of its heap. The n-value is important as it coincides with Lusztig’s a-value for FC elements in all Weyl and affine Weyl groups as well as so-called star-reducible groups; see [GX2020].

**EXAMPLES:**

```python
sage: FC = CoxeterGroup(['A', 5]).fully_commutative_elements()
sage: FC([1,3]).n_value()
2
sage: FC([1,2,3]).n_value()
1
sage: FC([1,3,2]).n_value()
2
sage: FC([1,3,2,5]).n_value()
3
```

### normalize()

Mutate `self` into Cartier-Foata normal form.

**EXAMPLES:**

The following reduced words express the same FC elements in `B_5`:

```python
sage: FC = CoxeterGroup(['B', 5]).fully_commutative_elements()
sage: FC([1, 4, 3, 5, 2, 4, 3]) # indirect doctest
[1, 4, 3, 5, 2, 4, 3]
sage: FC([4, 1, 3, 5, 2, 4, 3]) # indirect doctest
[1, 4, 3, 5, 2, 4, 3]
sage: FC([4, 3, 1, 5, 4, 2, 3]) # indirect doctest
[1, 4, 3, 5, 2, 4, 3]
```
Note: The Cartier–Foata form of a reduced word of an FC element $w$ can be found recursively by repeatedly moving left descents of elements to the left and ordering the left descents from small to large. In the above example, the left descents of the element are 4 and 1, therefore the Cartier–Foata form of the element is the concatenation of [1,4] with the Cartier–Foata form of the remaining part of the word. See [Gre2006].

See also:

`descents()`

`plot_heap()`

Display the Hasse diagram of the heap of `self`.

The Hasse diagram is rendered in the lattice $S \times N$, with every element $i$ in the poset drawn as a point labelled by its label $s_i$. Every point is placed in the column for its label at a certain level. The levels start at 0 and the level $k$ of an element $i$ is the maximal number $k$ such that the heap contains a chain $i_0 \prec i_1 \prec ... \prec i_k$ where $i_k = i$. See [Ste1996] and [GX2020].

OUTPUT: GraphicsObject

EXAMPLES:

```
sage: FC = CoxeterGroup(['B', 5]).fully_commutative_elements()
sage: FC([3,2,4,3,1]).plot_heap()
Graphics object consisting of 15 graphics primitives
```

`star_operation(J, direction, side='left')`

Apply a star operation on `self` relative to two noncommuting generators.

Star operations were first defined on elements of Coxeter groups by Kazhdan and Lusztig in [KL1979] with respect to pair of generators $s, t$ such that $m(s, t) = 3$. Later, Lusztig generalized the definition in [Lus1985], via coset decompositions, to allow star operations with respect to any pair of generators $s, t$ such that $m(s, t) \geq 3$. Given such a pair, we can potentially perform four types of star operations corresponding to all combinations of a ‘direction’ and a ‘side’: upper left, lower left, upper right and lower right; see [Gre2006].
Let \( w \) be an element in \( W \) and let \( J \) be any pair \( \{s, t\} \) of noncommuting generators in \( S \). Consider the coset decomposition \( w = w_J \cdot J w \) of \( w \) relative to \( J \). Then an upper left star operation is defined on \( w \) if and only if \( 1 \leq l(w_J) \leq m(s, t) - 2 \); when this is the case, the operation returns \( x \cdot w_J \cdot w^J \) where \( x \) is the letter \( J \) different from the leftmost letter of \( w_J \). A lower left star operation is defined on \( w \) if and only if \( 2 \leq l(w_J) \leq m(s, t) - 1 \); when this is the case, the operation removes the leftmost letter of \( w_J \) from \( w \). Similar facts hold for right star operations. See [Gre2006].

The facts of the previous paragraph hold in general, even if \( w \) is not FC. Note that if \( f \) is a star operation of any kind, then for every element \( w \in W \), the elements \( w \) and \( f(w) \) are either both FC or both not FC.

**INPUT:**

- \( J \) – a set of two integers representing two noncommuting generators of the Coxeter system
- \( \text{direction} \) – string, 'upper' or 'lower'; the function performs an upper or lower star operation according to \( \text{direction} \)
- \( \text{side} \) – string (default: 'left'); if this is set to 'right', the function performs a right star operation

**OUTPUT:**

The Cartier–Foata form of the result of the star operation if the operation is defined on \( \text{self} \), \( \text{None} \) otherwise.

**EXAMPLES:**

We will compute all star operations on the following FC element in type \( B_6 \) relative to \( J = \{5, 6\} \):

```python
sage: FC = CoxeterGroup(['B', 6]).fully_commutative_elements()
sage: w = FC([1, 6, 2, 5, 4, 6, 5])
```

Whether and how a left star operations can be applied depend on the coset decomposition \( w = w_J \cdot w^J \):

```python
sage: w.coset_decomposition({5, 6})
([6, 5, 6], [1, 2, 4, 5])
```

Only the lower star operation is defined on the left on \( w \):

```python
sage: w.star_operation({5, 6}, 'upper')
sage: w.star_operation({5, 6}, 'lower')
[1, 5, 2, 4, 6, 5]
```

Whether and how a right star operations can be applied depend on the coset decomposition \( w = w^J \cdot w_J \):

```python
sage: w.coset_decomposition({5, 6}, side='right')
([1, 6, 2, 5, 4], [6, 5])
```

Both types of right star operations on defined for this example:

```python
sage: w.star_operation({5, 6}, 'upper', side='right')
[1, 6, 2, 5, 4, 6, 5, 6]
sage: w.star_operation({5, 6}, 'lower', side='right')
[1, 6, 2, 5, 4, 6]
```

```python
class sage.combinat.fully_commutative_elements.FullyCommutativeElements(coxeter_group)
    Bases: UniqueRepresentation, Parent

    Class for the set of fully commutative (FC) elements of a Coxeter system.
```

5.1. Comprehensive Module List 905
Combinatorics, Release 10.0

Coxeter systems with finitely many FC elements, or \textit{FC-finite} Coxeter systems, are classified by Stembridge in [Ste1996]. They fall into seven families, namely the groups of types $A_n, B_n, D_n, E_n, F_n, H_n$ and $I_2(m)$.

\textbf{INPUT:}

- \textbf{data} – CoxeterMatrix, CartanType, or the usual datum that can be taken in the constructors for these classes (see sage.combinat.root_system.coxeter_group.CoxeterGroup())

\textbf{OUTPUT:}

The class of fully commutative elements in the Coxeter group constructed from \texttt{data}. This will belong to the category of enumerated sets. If the Coxeter data corresponds to a Cartan type, the category is further refined to either finite enumerated sets or infinite enumerated sets depending on whether the Coxeter group is FC-finite; the refinement is not carried out if \texttt{data} is a Coxeter matrix not corresponding to a Cartan type.

\textbf{Todo:} It would be ideal to implement the aforementioned refinement to finite and infinite enumerated sets for all possible \texttt{data}, regardless of whether it corresponds to a Cartan type. Doing so requires determining if an arbitrary Coxeter matrix corresponds to a Cartan type. It may be best to address this issue in sage.combinat.root_system. On the other hand, the refinement in the general case may be unnecessary in light of the fact that Stembridge’s classification of FC-finite groups contains a very small number of easily-recognizable families.

\textbf{EXAMPLES:}

Create the enumerate set of fully commutative elements in $B_3$:

```
sage: FC = CoxeterGroup(['B', 3]).fully_commutative_elements(); FC
Fully commutative elements of Finite Coxeter group over Number Field in a with defining polynomial x^2 - 2 with a = 1.414213562373095? with Coxeter matrix:
[1 3 2]
[3 1 4]
[2 4 1]
```

Construct elements:

```
sage: FC([])
[]
sage: FC([1,2])
[1, 2]
sage: FC([2,3,2])
[2, 3, 2]
sage: FC([3,2,3])
[3, 2, 3]
```

Elements are normalized to Cartier–Foata normal form upon construction:

```
sage: FC([3,1])
[1, 3]
sage: FC([2,3,1])
[2, 1, 3]
sage: FC([1,3]) == FC([3,1])
True
```

Attempting to create an element from an input that is not the reduced word of a fully commutative element throws a \texttt{ValueError}:
Enumerate the FC elements in $A_3$:

```python
sage: FCA3 = CoxeterGroup(['A', 3]).fully_commutative_elements()
sage: FCA3.category()
Category of finite enumerated sets
sage: FCA3.list()
[[],
 [1],
 [2],
 [3],
 [2, 1],
 [1, 3],
 [1, 2],
 [3, 2],
 [2, 3],
 [3, 2, 1],
 [2, 1, 3],
 [1, 3, 2],
 [1, 2, 3],
 [2, 1, 3, 2]]
```

Count the FC elements in $B_8$:

```python
sage: FCB8 = CoxeterGroup(['B', 8]).fully_commutative_elements()
sage: len(FCB8)  # long time (7 seconds)
14299
```

Iterate through the FC elements of length up to 2 in the non-FC-finite group affine $A_2$:

```python
sage: FCAffineA2 = CoxeterGroup(['A', 2, 1]).fully_commutative_elements()
sage: FCAffineA2.category()
Category of infinite enumerated sets
sage: list(FCAffineA2.iterate_to_length(2))
[[], [0], [1], [2], [1, 0], [2, 0], [0, 1], [2, 1], [0, 2], [1, 2]]
```

The cardinality of the set is determined from the classification of FC-finite Coxeter groups:

```python
sage: CoxeterGroup('A2').fully_commutative_elements().category()
Category of finite enumerated sets
sage: CoxeterGroup('B7').fully_commutative_elements().category()
Category of finite enumerated sets
sage: CoxeterGroup('A3~').fully_commutative_elements().category()
Category of infinite enumerated sets
sage: CoxeterGroup('F4~').fully_commutative_elements().category()
```

(continues on next page)
Category of finite enumerated sets

sage: CoxeterGroup('E8~').fully_commutative_elements().category()
Category of finite enumerated sets

sage: CoxeterGroup('F4~xE8~').fully_commutative_elements().category()
Category of finite enumerated sets

sage: CoxeterGroup('B4~xE8~').fully_commutative_elements().category()
Category of infinite enumerated sets

Element

alias of FullyCommutativeElement

coxeter_group()
Obtain the Coxeter group associated with self.

EXAMPLES:

sage: FCA3 = CoxeterGroup(['A', 3]).fully_commutative_elements()
sage: FCA3.coxeter_group()
Finite Coxeter group over Integer Ring with Coxeter matrix:
[1 3 2]
[3 1 3]
[2 3 1]

iterate_to_length(length)
Iterate through the elements of this class up to a maximum length.

INPUT:

• length – integer; maximum length of element to generate

OUTPUT: generator for elements of self of length up to length

EXAMPLES:
The following example produces all FC elements of length up to 2 in the group \(A_3\):

sage: FCA3 = CoxeterGroup(['A', 3]).fully_commutative_elements()
sage: list(FCA3.iterate_to_length(2))
[[], [1], [2], [3], [2, 1], [1, 3], [1, 2], [3, 2], [2, 3]]

The lists for length 4 and 5 are the same since 4 is the maximum length of an FC element in \(A_3\):

sage: list(FCA3.iterate_to_length(4)) == list(FCA3)
True

The following example produces all FC elements of length up to 4 in the affine Weyl group \(\tilde{A}_2\):

sage: FCAffineA2 = CoxeterGroup(['A', 2, 1]).fully_commutative_elements()
sage: FCAffineA2.category()
5.1.107 Finite state machines, automata, transducers

This module adds support for finite state machines, automata and transducers.

For creating automata and transducers you can use classes

- Automaton and Transducer (or the more general class FiniteStateMachine)

or the generators

- automata and transducers

which contain preconstructed and commonly used automata and transducers. See also the examples below.

Contents

FiniteStateMachine and derived classes Transducer and Automaton

Accessing parts of a finite state machine

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See also:

LaTeX output

FSMState

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<td><code>full_group_by()</code></td>
<td>Group iterable by values of some key</td>
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<tr>
<td><code>startswith()</code></td>
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<tr>
<td><code>FSMLetterSymbol()</code></td>
<td>Returns a string associated to the input letter</td>
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<td>Returns a string associated to a word</td>
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<td><code>is_FiniteStateMachine()</code></td>
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<td><code>duplicate_transition_ignore()</code></td>
<td>Default function for handling duplicate transitions</td>
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<td><code>duplicate_transition_raise_error()</code></td>
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<td><code>duplicate_transition_add_input()</code></td>
<td>Add input when inserting a duplicate transition</td>
</tr>
</tbody>
</table>

Examples

We start with a general `FiniteStateMachine`. Later there will be also an `Automaton` and a `Transducer`.

A simple finite state machine

We can easily create a finite state machine by

```python
sage: fsm = FiniteStateMachine()
sage: fsm
Empty finite state machine
```

By default this is the empty finite state machine, so not very interesting. Let’s create and add some states and transitions:

```python
sage: day = fsm.add_state('day')
sage: night = fsm.add_state('night')
sage: sunrise = fsm.add_transition(night, day)
sage: sunset = fsm.add_transition(day, night)
```

Let us look at sunset more closely:

```python
sage: sunset
Transition from 'day' to 'night': -|--
```

Note that could also have created and added the transitions directly by:

```python
sage: fsm.add_transition('day', 'night')
Transition from 'day' to 'night': -|--
```

This would have had added the states automatically, since they are present in the transitions.

Anyhow, we got the following finite state machine:

```python
sage: fsm
Finite state machine with 2 states
```

We can also obtain the underlying directed graph by
To visualize a finite state machine, we can use \texttt{latex()} and run the result through \LaTeX, see the section on \textit{\LaTeX output} below.

Alternatively, we could have created the finite state machine above simply by

\begin{verbatim}
sage: FSM = FiniteStateMachine([(('night', 'day'), ('day', 'night'))])

Finite state machine with 2 states
\end{verbatim}

See \texttt{FiniteStateMachine} for a lot of possibilities to create finite state machines.

\section*{A simple Automaton (recognizing NAFs)}

We want to build an automaton which recognizes non-adjacent forms (NAFs), i.e., sequences which have no adjacent non-zeros. We use 0, 1, and \(-1\) as digits:

\begin{verbatim}
sage: NAF = Automaton(
    ....:     {'A': [('A', 0), ('B', 1), ('B', -1)], 'B': [('A', 0)]})
sage: NAF.state('A').is_initial = True
sage: NAF.state('A').is_final = True
sage: NAF.state('B').is_final = True
sage: NAF

Automaton with 2 states
\end{verbatim}

Of course, we could have specified the initial and final states directly in the definition of \texttt{NAF} by \texttt{initial_states=['A']} and \texttt{final_states=['A', 'B']}.

So let’s test the automaton with some input:

\begin{verbatim}
sage: NAF([0])
True
sage: NAF([0, 1])
True
sage: NAF([1, -1])
False
sage: NAF([0, -1, 0, 1])
True
sage: NAF([0, -1, -1, 0])
False
sage: NAF([-1, 0, 0, 1, 1])
False
\end{verbatim}

Alternatively, we could call that by

\begin{verbatim}
sage: NAF.process([0, -1, 0, 1])
(True, 'B')
\end{verbatim}

which gives additionally the state in which we arrived.

We can also let an automaton act on a \texttt{word}:

\begin{verbatim}
Recognizing NAFs via Automata Operations

Alternatively, we can use automata operations to recognize NAFs; for simplicity, we only use the input alphabet \([0, 1]\). On the one hand, we can construct such an automaton by forbidding the word 11:

```sage
forbidden = automata.ContainsWord([1, 1], input_alphabet=[0, 1])
sage: NAF_negative = forbidden.complement()
sage: NAF_negative([1, 1, 0, 1])
False
sage: NAF_negative([1, 0, 1, 0, 1])
True
```

On the other hand, we can write this as a regular expression and translate that into automata operations:

```sage
zero = automata.Word([0])
sage: one = automata.Word([1])
sage: epsilon = automata.EmptyWord(input_alphabet=[0, 1])
sage: NAF_positive = (zero + one*zero).kleene_star() * (epsilon + one)
```

We check that the two approaches are equivalent:

```sage
sage: NAF_negative.is_equivalent(NAF_positive)
True
```

See also:

`ContainsWord()`, `Word()`, `complement()`, `kleene_star()`, `EmptyWord()`, `is_equivalent()`.

LaTeX output

We can visualize a finite state machine by converting it to LaTeX by using the usual function `latex()`. Within LaTeX, TikZ is used for typesetting the graphics, see the Wikipedia article PGF/TikZ.

```sage
print(latex(NAF))
# abs tol 1e-3
\begin{tikzpicture}[auto, initial text=, >=latex]
\node[state, accepting, initial] (v0) at (3.000000, 0.000000) {$\texttt{A}$};
\node[state, accepting] (v1) at (-3.000000, 0.000000) {$\texttt{B}$};
\path[->] (v0) edge[loop above] node {$0$} (v0);
\path[->] (v0) edge node[rotate=360.00, anchor=north] {$1, -1$} (v1);
\path[->] (v1) edge node[rotate=0.00, anchor=south] {$0$} (v0);
\end{tikzpicture}
```

We can turn this into a graphical representation.
To actually see this, use the live documentation in the Sage notebook and execute the cells in this and the previous section.

Several options can be set to customize the output, see `latex_options()` for details. In particular, we use `format_letter_negative()` to format $-1$ as $1$.

```
sage: NAF.latex_options(
    ....:     coordinates={'A': (0, 0),
    ....:                    'B': (6, 0)},
    ....:     initial_where={'A': 'below'},
    ....:     format_letter=NAF.format_letter_negative,
    ....:     format_state_label=lambda x:
    ....:         r'\mathcal{%s}' % x.label()
    ....: )
sage: print(latex(NAF))
\begin{tikzpicture}[auto, initial text=, >=latex]
\node[state, accepting, initial, initial where=below] (v0) at (0.000000, 0.000000) {$\mathcal{A}$};
\node[state, accepting] (v1) at (6.000000, 0.000000) {$\mathcal{B}$};
\path[->] (v0) edge[loop above] node {$0$} ();
\path[->] (v0.5.00) edge node[rotate=0.00, anchor=south] {$1, \overline{1}$} (v1.175.00);
\path[->] (v1.185.00) edge node[rotate=360.00, anchor=north] {$0$} (v0.355.00);
\end{tikzpicture}
```

To use the output of `latex()` in your own \LaTeX file, you have to include

\begin{verbatim}
\usepackage{tikz}
\usetikzlibrary{automata}
\end{verbatim}

into the preamble of your file.

A simple transducer (binary inverter)

Let's build a simple transducer, which rewrites a binary word by inverting each bit:

```
sage: inverter = Transducer({'A': [('A', 0, 1), ('A', 1, 0)]},
    ....:     initial_states=['A'], final_states=['A'])
```

We can look at the states and transitions:

```
sage: inverter.states()
['A']
sage: for t in inverter.transitions():
    ....:     print(t)
Transition from 'A' to 'A': 0|1
Transition from 'A' to 'A': 1|0
```

Now we apply a word to it and see what the transducer does:

```
sage: inverter([0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1])
[1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1]
```
True means, that we landed in a final state, that state is labeled 'A', and we also got an output.

## Transducers and (in)finite Words

A transducer can also act on everything iterable, in particular, on Sage’s \textit{words}.

```
sage: W = Words([0, 1]); W
Finite and infinite words over {0, 1}
```

Let us take the inverter from the previous section and feed some finite word into it:

```
sage: w = W([1, 1, 0, 1]); w
word: 1101
sage: inverter(w)
word: 0010
```

We see that the output is again a word (this is a consequence of calling \textit{process()} with \texttt{automatic_output_type}).

We can even input something infinite like an infinite word:

```
sage: tm = words.ThueMorseWord(); tm
word: 0110100110010110100101100110100110010110...
sage: inverter(tm)
word: 1001011001101001011010011001011001101001...
```

### A transducer which performs division by 3 in binary

Now we build a transducer, which divides a binary number by 3. The labels of the states are the remainder of the division. The transition function is

```
sage: def f(state_from, read):
    ....:     if state_from + read <= 1:
    ....:         state_to = 2*state_from + read
    ....:         write = 0
    ....:     else:
    ....:         state_to = 2*state_from + read - 3
    ....:         write = 1
    ....:     return (state_to, write)
```

which assumes reading a binary number from left to right. We get the transducer with

```
sage: D = Transducer(f, initial_states=[0], final_states=[0],
    ....:     input_alphabet=[0, 1])
```

Let us try to divide 12 by 3:

```
sage: D([1, 1, 0, 0])
[0, 1, 0, 0]
```

Now we want to divide 13 by 3:
The raised `ValueError` means 13 is not divisible by 3.

**Gray Code**

The Gray code is a binary numeral system where two successive values differ in only one bit, cf. the Wikipedia article [Gray code](https://en.wikipedia.org/wiki/Gray_code). The Gray code of an integer $n$ is obtained by a bitwise xor between the binary expansion of $n$ and the binary expansion of $\lfloor n/2 \rfloor$; the latter corresponds to a shift by one position in binary.

The purpose of this example is to construct a transducer converting the standard binary expansion to the Gray code by translating this construction into operations with transducers.

For this construction, the least significant digit is at the left-most position. Note that it is easier to shift everything to the right first, i.e., multiply by 2 instead of building $\lfloor n/2 \rfloor$. Then, we take the input xor with the right shift of the input and forget the first letter.

We first construct a transducer shifting the binary expansion to the right. This requires storing the previously read digit in a state.

```python
sage: def shift_right_transition(state, digit):
    ....:    if state == 'I':
    ....:        return (digit, None)
    ....:    else:
    ....:        return (digit, state)
sage: shift_right_transducer = Transducer(
    ....:    shift_right_transition,
    ....:    initial_states=['I'],
    ....:    input_alphabet=[0, 1],
    ....:    final_states=[0])
sage: shift_right_transducer.transitions()
[Transition from 'I' to 0: 0|-,
 Transition from 'I' to 1: 1|-,
 Transition from 0 to 0: 0|0,
 Transition from 0 to 1: 1|0,
 Transition from 1 to 0: 0|1,
 Transition from 1 to 1: 1|1]
sage: shift_right_transducer([0, 1, 1, 0])
[0, 1, 1]
sage: shift_right_transducer([1, 0, 0])
[1, 0]
```

The output of the shifts above look a bit weird (from a right-shift transducer, we would expect, for example, that [1, 0, 0] was mapped to [0, 1, 0]), since we write `None` instead of the zero at the left. Further, note that only 0 is listed as a final state as we have to enforce that a most significant zero is read as the last input letter in order to flush the last digit:

```python
sage: shift_right_transducer([0, 1, 0, 1])
Traceback (most recent call last):
 ... 
ValueError: Invalid input sequence.
```
Next, we construct the transducer performing the xor operation. We also have to take None into account as our \texttt{shift_right_transducer} waits one iteration until it starts writing output. This corresponds with our intention to forget the first letter.

```python
sage: def xor_transition(state, digits):
    ....:     if digits[0] is None or digits[1] is None:
    ....:         return (0, None)
    ....:     else:
    ....:         return (0, digits[0].__xor__(digits[1]))

sage: from itertools import product

sage: xor_transducer = Transducer(
    ....:     xor_transition,
    ....:     initial_states=[0],
    ....:     final_states=[0],
    ....:     input_alphabet=list(product([None, 0, 1], [0, 1])))

sage: xor_transducer.transitions()
[Transition from 0 to 0: (None, 0)|-,
 Transition from 0 to 0: (None, 1)|-,
 Transition from 0 to 0: (0, 0)|0,
 Transition from 0 to 0: (0, 1)|1,
 Transition from 0 to 0: (1, 0)|1,
 Transition from 0 to 0: (1, 1)|0]

sage: xor_transducer([(None, 0), (None, 1), (0, 0), (0, 1), (1, 0), (1, 1)])
0, 1, 1, 0

sage: xor_transducer([(0, None)])
Traceback (most recent call last):
...
ValueError: Invalid input sequence.
```

The transducer computing the Gray code is then constructed as a Cartesian product between the shifted version and the original input (represented here by the \texttt{shift_right_transducer} and the \texttt{identity transducer}, respectively). This Cartesian product is then fed into the \texttt{xor_transducer} as a composition of transducers.

```python
sage: product_transducer = shift_right_transducer.cartesian_product(transducers.Identity([0, 1]))

sage: Gray_transducer = xor_transducer(product_transducer)

sage: Gray_transducer.construct_final_word_out([0])

sage: Gray_transducer.transitions()
[Transition from (('I', 0), 0) to ((0, 0), 0): 0|-,
 Transition from (('I', 0), 0) to ((1, 0), 0): 1|-,
 Transition from ((0, 0), 0) to ((0, 0), 0): 0|0,
 Transition from ((0, 0), 0) to ((1, 0), 0): 1|0,
 Transition from ((1, 0), 0) to ((0, 0), 0): 0|1,
 Transition from ((1, 0), 0) to ((1, 0), 0): 1|0]
```

There is a prepackaged transducer for Gray code, let's see whether they agree. We have to use \texttt{relabeled()} to relabel our states with integers.

```python
sage: constructed = Gray_transducer.relabeled()

sage: packaged = transducers.GrayCode()
```

(continues on next page)
Finally, we check that this indeed computes the Gray code of the first 10 non-negative integers.

```
sage: for n in srange(10):
    ....:    Gray_transducer(n.bits())

[]
[1]
[1, 1]
[0, 1]
[0, 1, 1]
[1, 1, 1]
[1, 0, 1]
[0, 0, 1]
[0, 0, 1, 1]
[1, 0, 1, 1]
```

Using the hook-functions

Let’s use the previous example “division by 3” to demonstrate the optional state and transition parameters hook.

First, we define what those functions should do. In our case, this is just saying in which state we are and which transition we take.

```
sage: def state_hook(process, state, output):
    ....:    print("We are now in State %s." % (state.label(),))
sage: from sage.combinat.finite_state_machine import FSMWordSymbol
sage: def transition_hook(transition, process):
    ....:    print("Currently we go from %s to %s, reading %s and writing %s." % (transition.from_state, transition.to_state, FSMWordSymbol(transition.word_in), FSMWordSymbol(transition.word_out)))
```

Now, let’s add these hook-functions to the existing transducer:

```
sage: for s in D.iter_states():
    ....:    s.hook = state_hook
sage: for t in D.iter_transitions():
    ....:    t.hook = transition_hook
```

Rerunning the process again now gives the following output:

```
sage: D.process([1, 1, 0, 1], check_epsilon_transitions=False)
We are now in State 0.
Currently we go from 0 to 1, reading 1 and writing 0.
We are now in State 1.
Currently we go from 1 to 0, reading 1 and writing 1.
We are now in State 0.
Currently we go from 0 to 0, reading 0 and writing 0.
We are now in State 0.
```

(continues on next page)
Currently we go from $\emptyset$ to 1, reading 1 and writing $\emptyset$. We are now in State 1.

\[(\text{False, 1, } [0, 1, 0, 0])\]

The example above just explains the basic idea of using hook-functions. In the following, we will use those hooks more seriously.

**Warning:** The hooks of the states are also called while exploring the epsilon successors of a state (during processing). In the example above, we used `check_epsilon_transitions=False` to avoid this (and also therefore got a cleaner output).

**Warning:** The arguments used when calling a hook have changed in github issue #16538 from `hook(state, process)` to `hook(process, state, output)`.

### Detecting sequences with same number of 0 and 1

Suppose we have a binary input and want to accept all sequences with the same number of 0 and 1. This cannot be done with a finite automaton. Anyhow, we can make usage of the hook functions to extend our finite automaton by a counter:

```
sage: from sage.combinat.finite_state_machine import FSMState, FSMTransition
sage: C = FiniteStateMachine()
sage: def update_counter(process, state, output):
    ....:    try:
    ....:        l = process.preview_word()
    ....:    ....:        except RuntimeError:
    ....:            raise StopIteration
    ....:        process.fsm.counter += 1 if l == 1 else -1
    ....:    ....:    if process.fsm.counter > 0:
    ....:        next_state = 'positive'
    ....:    ....:    elif process.fsm.counter < 0:
    ....:        next_state = 'negative'
    ....:    ....:    else:
    ....:        next_state = 'zero'
    ....:    ....:    return FSMTransition(state, process.fsm.state(next_state),
    ....:        l, process.fsm.counter)
```

```
sage: C.add_state(FSMState('zero', hook=update_counter, is_initial=True, is_final=True))
'state
'sage: C.add_state(FSMState('positive', hook=update_counter))
'positive'
sage: C.add_state(FSMState('negative', hook=update_counter))
'negative'
```

Now, let’s input some sequence:

```
sage: C.counter = 0; C([1, 1, 1, 0, 0])
(False, 'positive', [1, 2, 3, 4, 3, 2])
```

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The result is False, since there are four 1 but only two 0. We land in the state positive and we can also see the values of the counter in each step.

Let's try some other examples:

```
sage: C.counter = 0; C([1, 1, 0, 0])
(True, 'zero', [1, 2, 1, 0])
sage: C.counter = 0; C([0, 1, 0, 0])
(False, 'negative', [-1, 0, -1, -2])
```

See also methods `Automaton.process()` and `Transducer.process()` (or even `FiniteStateMachine.process()`), the explanation of the parameter hook and the examples in `FSMState` and `FSMTransition`, and the description and examples in `FSMProcessIterator` for more information on processing and hooks.

REFERENCES:

AUTHORS:

- Daniel Krenn (2012-03-27): initial version
- Clemens Heuberger (2012-04-05): initial version
- Sara Kropf (2012-04-17): initial version
- Clemens Heuberger (2013-08-21): release candidate for Sage patch
- Daniel Krenn (2013-08-21): release candidate for Sage patch
- Sara Kropf (2013-08-21): release candidate for Sage patch
- Clemens Heuberger (2013-09-02): documentation improved
- Daniel Krenn (2013-09-13): comments from trac worked in
- Clemens Heuberger (2013-11-03): output (labels) of determinisation, product, composition, etc. changed (for consistency), representation of state changed, documentation improved
- Daniel Krenn (2013-11-04): whitespaces in documentation corrected
- Clemens Heuberger (2013-11-04): full_group_by added
- Sara Kropf (2013-11-08): fix for adjacency matrix
- Clemens Heuberger (2013-11-11): fix for prepone_output
- **Daniel Krenn (2013-11-11): comments from github issue #15078 included:**
  docstring of `FiniteStateMachine` rewritten, `Automaton` and `Transducer` inherited from `FiniteStateMachine`
- **Daniel Krenn (2013-11-25): documentation improved according to comments from github issue #15078**
- Clemens Heuberger, Daniel Krenn, Sara Kropf (2014-02-21–2014-07-18): A huge bunch of improvements. Details see github issue #15841, github issue #15847, github issue #15848, github issue #15849, github issue #15850, github issue #15922, github issue #15923, github issue #15924, github issue #15925, github issue #15928, github issue #15960, github issue #15961, github issue #15962, github issue #15963, github issue #15975, github issue #16016, github issue #16024, github issue #16061, github issue #16128, github issue #16132, github issue #16138, github issue #16139, github issue #16140, github issue #16143, github issue #16144, github issue #16145, github issue #16146, github issue #16191, github issue #16200, github issue #16205, github issue #16206, github issue #16207, github issue #16229, github issue #1623, github issue #16254, github issue #16255, github issue #16266, github issue #16355, github issue #16357, github issue #16378, github issue #16387, github issue #16425, github issue #16539, github issue #16555, github issue #16557, github is-
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Methods

```python
class sage.combinat.finite_state_machine.Automaton(*args, **kwargs):
    Bases: FiniteStateMachine

    This creates an automaton, which is a finite state machine, whose transitions have input labels.
    An automaton has additional features like creating a deterministic and a minimized automaton.
    See class FiniteStateMachine for more information.

    EXAMPLES:

    We can create an automaton recognizing even numbers (given in binary and read from left to right) in the following way:

    ```
                     (‘Q’, ‘P’, 1), (‘Q’, ‘Q’, 0)],
                     initial_states=[‘P’], final_states=[‘Q’])
    sage: A
    Automaton with 2 states
    sage: A([0])
    True
    sage: A([1, 1, 0])
    True
    sage: A([1, 0, 1])
    False
    ```

    Note that the full output of the commands can be obtained by calling `process()` and looks like this:

    ```
sage: A.process([1, 0, 1])
    (False, ‘P’)
    ```

    `cartesian_product` *(other, only_accessible_components=True)*

    Return a new automaton which accepts an input if it is accepted by both given automata.

    INPUT:

    - `other` – an automaton

    - `only_accessible_components` – If True (default), then the result is piped through `accessible_components()`. If no `new_input_alphabet` is given, it is determined by `determine_alphabets()`.

    OUTPUT:

    A new automaton which computes the intersection (see below) of the languages of `self` and `other`.  
```
The set of states of the new automaton is the Cartesian product of the set of states of both given automata. There is a transition \((A, B), (C, D), a\) in the new automaton if there are transitions \((A, C), a\) and \((B, D), a\) in the old automata.

The methods `intersection()` and `cartesian_product()` are the same (for automata).

**EXAMPLES:**

```python
sage: aut1 = Automaton(
    [('1', '2', 1),
     ('2', '2', 1),
     ('2', '2', 0),
    ], initial_states=['1'],
    final_states=['2'],
    determine_alphabets=True)

sage: aut2 = Automaton(
    [('A', 'A', 1),
     ('A', 'B', 0),
     ('B', 'B', 0),
     ('B', 'A', 1),
    ], initial_states=['A'],
    final_states=['B'],
    determine_alphabets=True)

sage: res = aut1.intersection(aut2)

sage: (aut1([1, 1]), aut2([1, 1]), res([1, 1]))
(True, False, False)

sage: (aut1([1, 0]), aut2([1, 0]), res([1, 0]))
(True, True, True)

sage: res.transitions()
[Transition from ('1', 'A') to ('2', 'A'): 1|-,
 Transition from ('2', 'A') to ('2', 'B'): 0|-,
 Transition from ('2', 'A') to ('2', 'A'): 1|-,
 Transition from ('2', 'B') to ('2', 'B'): 0|-,
 Transition from ('2', 'B') to ('2', 'A'): 1|-]
```

For automata with epsilon-transitions, intersection is not well defined. But for any finite state machine, epsilon-transitions can be removed by `remove_epsilon_transitions()`.

```python
sage: a1 = Automaton(
    [(0, 0, 0),
     (0, 1, None),
     (1, 1, 1),
     (1, 2, 1),
    ], initial_states=[0],
    final_states=[1],
    determine_alphabets=True)

sage: a2 = Automaton(
    [(0, 0, 0), (0, 1, 1), (1, 1, 1),
    ], initial_states=[0],
    final_states=[1],
    determine_alphabets=True)

sage: a1.intersection(a2)
Traceback (most recent call last):
...
ValueError: An epsilon-transition (with empty input) was found.

sage: a1.remove_epsilon_transitions() # not tested (since not implemented yet)

sage: a1.intersection(a2) # not tested
```

`complement()`
Return the complement of this automaton.

OUTPUT:

An Automaton.

If this automaton recognizes language $\mathcal{L}$ over an input alphabet $\mathcal{A}$, then the complement recognizes $\mathcal{A} \setminus \mathcal{L}$.

EXAMPLES:

```
sage: A = automata.Word([0, 1])
sage: [w for w in ([], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1])
      ...: if A(w)]
[[0, 1]]
sage: Ac = A.complement()
sage: Ac.transitions()
[Transition from 0 to 1: 0|-,
 Transition from 0 to 3: 1|-,
 Transition from 2 to 3: 0|-,
 Transition from 1 to 2: 1|-,
 Transition from 1 to 3: 0|-,
 Transition from 3 to 3: 0|-,
 Transition from 3 to 3: 1|-
]
sage: [w for w in ([], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1])
      ...: if Ac(w)]
[[], [0], [1], [0, 0], [1, 0], [1, 1]]
```

The automaton must be deterministic:

```
sage: A = automata.Word([0]) * automata.Word([1])
sage: A.complement()
Traceback (most recent call last):
...
ValueError: The finite state machine must be deterministic.
sage: Ac = A.determinisation().complement()
sage: [w for w in ([], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1])
      ...: if Ac(w)]
[[], [0], [1], [0, 0], [1, 0], [1, 1]]
```

determinisation()

Return a deterministic automaton which accepts the same input words as the original one.

OUTPUT:

A new automaton, which is deterministic.

The labels of the states of the new automaton are frozensets of states of self. The color of a new state is the frozenset of colors of the constituent states of self. Therefore, the colors of the constituent states have to be hashable. However, if all constituent states have color None, then the resulting color is None, too.

The input alphabet must be specified.

EXAMPLES:
sage: aut.determinisation().transitions()
[Transition from frozenset({'A'}) to frozenset({'A'}): 0|-,
 Transition from frozenset({'A'}) to frozenset({'B'}): 1|-,
 Transition from frozenset({'B'}) to frozenset(): 0|-,
 Transition from frozenset({'B'}) to frozenset({'B'}): 1|-,
 Transition from frozenset() to frozenset(): 0|-,
 Transition from frozenset() to frozenset(): 1|-]

 ...:      ("B", "C", 0), ("C", "C", 1), ("C", "C", 0)],
 ...:      initial_states=['A'], final_states=['C'])

sage: A.determinisation().states()
[frozenset({'A'}),
 frozenset({'A', 'B'}),
 frozenset({'A', 'C'}),
 frozenset({'A', 'B', 'C'})]

sage: B = A.determinisation().relabeled().coaccessible_components()

Note that colors of states have to be hashable:

sage: A = Automaton([[0, 0, 0]], initial_states=[0])

sage: A.state(0).color = []

sage: A.determinisation()
Traceback (most recent call last):
...:
TypeError: unhashable type: 'list'

sage: A.state(0).color = ()

sage: A.determinisation()
Automaton with 1 state

If the colors of all constituent states are None, the resulting color is None, too (github issue #19199):

intersection(other, only_accessible_components=True)

Return a new automaton which accepts an input if it is accepted by both given automata.

INPUT:
other – an automaton

only_accessible_components – If True (default), then the result is piped through accessible_components(). If no new_input_alphabet is given, it is determined by determine_alphabets().

OUTPUT:

A new automaton which computes the intersection (see below) of the languages of self and other.

The set of states of the new automaton is the Cartesian product of the set of states of both given automata. There is a transition \((A, B, C, D, a)\) in the new automaton if there are transitions \((A, C, a)\) and \((B, D, a)\) in the old automata.

The methods intersection() and cartesian_product() are the same (for automata).

EXAMPLES:

sage: aut1 = Automaton([('1', '2', 1),
.....: ('2', '2', 1),
.....: ('2', '2', 0)],
.....: initial_states=['1'],
.....: final_states=['2'],
.....: determine_alphabets=True)
sage: aut2 = Automaton([('A', 'A', 1),
.....: ('A', 'B', 0),
.....: ('B', 'B', 0),
.....: ('B', 'A', 1)],
.....: initial_states=['A'],
.....: final_states=['B'],
.....: determine_alphabets=True)
sage: res = aut1.intersection(aut2)
sage: res.transitions()  # prints the transition list

For automata with epsilon-transitions, intersection is not well defined. But for any finite state machine, epsilon-transitions can be removed by remove_epsilon_transitions().

sage: a1 = Automaton([(0, 0, 0),
.....: (0, 1, None),
.....: (1, 1, 1),
.....: (1, 2, 1)],
.....: initial_states=[0],
.....: final_states=[1],
.....: determine_alphabets=True)
sage: a2 = Automaton([(0, 0, 0), (0, 1, 1), (1, 1, 1)],
.....: initial_states=[0],
.....: final_states=[1],
.....: determine_alphabets=True)
is_equivalent(often)

Test whether two automata are equivalent, i.e., accept the same language.

INPUT:

• other – an Automaton.

EXAMPLES:

```
sage: A = Automaton([(0, 0, 0), (0, 1, 1), (1, 0, 1)],
                  initial_states=[0],
                  final_states=[0])
sage: B = Automaton([('a', 'a', 0), ('a', 'b', 1), ('b', 'a', 1)],
                  initial_states=['a'],
                  final_states=['a'])
sage: A.is_equivalent(B)
True
sage: B.add_transition('b', 'a', 0)
Transition from 'b' to 'a': 0|- 

sage: A.is_equivalent(B)
False
```

language(max_length=None, **kwargs)

Return all words accepted by this automaton.

INPUT:

• max_length – an integer or None (default). Only inputs of length at most max_length will be considered. If None, then this iterates over all possible words without length restrictions.

• kwargs – will be passed on to the process iterator. See process() for a description.

OUTPUT:
An iterator.

EXAMPLES:

```
sage: NAF = Automaton(
                  {'A': [('A', 0), ('B', 1), ('B', -1)],
                   'B': [('A', 0)]},
                  initial_states=['A'],
                  final_states=['A', 'B'])
sage: list(NAF.language(3))
[[], [0], [-1], [1],
 [-1, 0], [0, 0], [1, 0], [0, -1], [0, 1],
 [-1, 0, 0], [0, -1, 0], [0, 0, 0], [0, 1, 0], [1, 0, 0],
 [-1, 0, -1], [-1, 0, 1], [0, 0, -1],
 [0, 0, 1], [1, 0, -1], [1, 0, 1]]
```
See also:

FiniteStateMachine.language(), process().

\texttt{minimization(algorithm=None)}

Return the minimization of the input automaton as a new automaton.

INPUT:

- algorithm – Either Moore’s algorithm (by algorithm='Moore' or as default for deterministic automata) or Brzozowski’s algorithm (when algorithm='Brzozowski' or when the automaton is not deterministic) is used.

OUTPUT:

A new automaton.

The resulting automaton is deterministic and has a minimal number of states.

EXAMPLES:

sage: A = Automaton([(A, A, 1), (A, A, 0), (A, B, 1),
.....: (B, C, 0), (C, C, 1), (C, C, 0)],
.....: initial_states=[A], final_states=[C])

sage: B = A.minimization(algorithm='Brzozowski')

sage: B_trans = B.transitions(B.states()[1])

sage: B_trans # random
[Transition from frozenset({frozenset({B, C}),
  frozenset({A, C}),
  frozenset({A, B, C})})
  to frozenset({frozenset({C}),
  frozenset({B, C}),
  frozenset({A, C}),
  frozenset({A, B, C})}):
  0] -

sage: len(B.states())
3

sage: C = A.minimization(algorithm='Brzozowski')

sage: C_trans = C.transitions(C.states()[1])

sage: B_trans == C_trans
True

sage: len(C.states())
3

sage: aut = Automaton([(1, 2, 'a'), (2, 3, 'b'),
.....: (3, 2, 'a'), (2, 1, 'b'),
.....: (3, 4, 'a'), (4, 3, 'b')],
.....: initial_states=['1'], final_states=['1'])

sage: min = aut.minimization(algorithm='Brzozowski')

sage: [len(min.states()), len(aut.states())]
[3, 4]

```python
sage: min = aut.minimization(algorithm='Moore')
```

Traceback (most recent call last):
...

NotImplementedError: Minimization via Moore's Algorithm is only implemented for deterministic finite state machines

```
```

**process(*args, **kwargs)**

Return whether the automaton accepts the input and the state where the computation stops.

**INPUT:**

- `input_tape` – the input tape can be a list or an iterable with entries from the input alphabet. If we are working with a multi-tape machine (see parameter `use_multitape_input` and notes below), then the tape is a list or tuple of tracks, each of which can be a list or an iterable with entries from the input alphabet.

- `initial_state` or `initial_states` – the initial state(s) in which the machine starts. Either specify a single one with `initial_state` or a list of them with `initial_states`. If both are given, `initial_state` will be appended to `initial_states`. If neither is specified, the initial states of the finite state machine are taken.

- `list_of_outputs` – (default: `None`) a boolean or `None`. If `True`, then the outputs are given in list form (even if we have no or only one single output). If `False`, then the result is never a list (an exception is raised if the result cannot be returned). If `list_of_outputs=None` the method determines automatically what to do (e.g. if a non-deterministic machine returns more than one path, then the output is returned in list form).

- `only_accepted` – (default: `False`) a boolean. If set, then the first argument in the output is guaranteed to be `True` (if the output is a list, then the first argument of each element will be `True`).

- `full_output` – (default: `True`) a boolean. If set, then the full output is given, otherwise only whether the sequence is accepted or not (the first entry below only).

- `always_include_output` – if set (not by default), always return a triple containing the (non-existing) output. This is in order to obtain output compatible with that of `FiniteStateMachine.process()`. If this parameter is set, `full_output` has no effect.

- `format_output` – a function that translates the written output (which is in form of a list) to something more readable. By default (`None`) identity is used here.

- `check_epsilon_transitions` – (default: `True`) a boolean. If `False`, then epsilon transitions are not taken into consideration during process.

- `write_final_word_out` – (default: `True`) a boolean specifying whether the final output words should be written or not.

- `use_multitape_input` – (default: `False`) a boolean. If `True`, then the multi-tape mode of the process iterator is activated. See also the notes below for multi-tape machines.

- `process_all_prefixes_of_input` – (default: `False`) a boolean. If `True`, then each prefix of the input word is processed (instead of processing the whole input word at once). Consequently, there is an output generated for each of these prefixes.

- `process_iterator_class` – (default: `None`) a class inherited from `FSMProcessIterator`. If `None`, then `FSMProcessIterator` is taken. An instance of this class is created and is used during the processing.

**OUTPUT:**
The full output is a pair (or a list of pairs, cf. parameter `list_of_outputs`), where

- the first entry is `True` if the input string is accepted and
- the second gives the state reached after processing the input tape (This is a state with label `None` if the input could not be processed, i.e., if at one point no transition to go on could be found.).

If `full_output` is `False`, then only the first entry is returned.

If `always_include_output` is set, an additional third entry `[]` is included.

Note that in the case the automaton is not deterministic, all possible paths are taken into account. You can use `determinisation()` to get a deterministic automaton machine.

This function uses an iterator which, in its simplest form, goes from one state to another in each step. To decide which way to go, it uses the input words of the outgoing transitions and compares them to the input tape. More precisely, in each step, the iterator takes an outgoing transition of the current state, whose input label equals the input letter of the tape.

If the choice of the outgoing transition is not unique (i.e., we have a non-deterministic finite state machine), all possibilities are followed. This is done by splitting the process into several branches, one for each of the possible outgoing transitions.

The process (iteration) stops if all branches are finished, i.e., for no branch, there is any transition whose input word coincides with the processed input tape. This can simply happen when the entire tape was read.

Also see `__call__()` for a version of `process()` with shortened output.

Internally this function creates and works with an instance of `FSMProcessIterator`. This iterator can also be obtained with `iter_process()`.

If working with multi-tape finite state machines, all input words of transitions are words of $k$-tuples of letters. Moreover, the input tape has to consist of $k$ tracks, i.e., be a list or tuple of $k$ iterators, one for each track.

**Warning:** Working with multi-tape finite state machines is still experimental and can lead to wrong outputs.

**EXAMPLES:**

In the following examples, we construct an automaton which accepts non-adjacent forms (see also the example on non-adjacent forms in the documentation of the module Finite state machines, automata, transducers) and then test it by feeding it with several binary digit expansions.

```python
sage: NAF = Automaton(
    {...: {': [('': [0, '1'), ('1', 1), ('': [0])],
    .....: initial_states=[''], final_states=['', '1'])

sage: [NAF.process(w) for w in [[0], [0, 1], [1, 1], [0, 1, 0, 1],
    .....: [0, 1, 1, 1, 0], [1, 0, 0, 1, 1]]

[(True, ''), (True, '1'), (False, None),
    (True, '1'), (False, None), (False, None)]
```

If we just want a condensed output, we use:

```python
sage: [NAF.process(w, full_output=False) for w in [[0], [0, 1], [1, 1], [0, 1, 0, 1],
    .....: [0, 1, 1, 1, 0], [1, 0, 0, 1, 1]]

[True, True, False, True, False, False]
```
It is equivalent to:

```
sage: [NAF(w) for w in [[0], [0, 1], [1, 1], [0, 1, 0, 1], ....: [0, 1, 1, 1, 0], [1, 0, 0, 1, 1]]]
[True, True, False, True, False, False]
```

The following example illustrates the difference between non-existing paths and reaching a non-final state:

```
sage: NAF.process([2])
(False, None)
sage: NAF.add_transition(('=', 's', 2))
Transition from '=' to 's': 2|
sage: NAF.process([2])
(False, 's')
```

A simple example of a (non-deterministic) multi-tape automaton is the following: It checks whether the two input tapes have the same number of ones:

```
sage: M = Automaton([('=', '=', (1, 1)), ....: ('=', '=', (None, 0)), ....: ('=', '<', (None, 1)), ....: ('<', '<', (None, 0)), ....: ('=', '>', (1, None)), ....: ('>', '>', (1, None))], ....: initial_states=['='], ....: final_states=['='])
sage: M.process(([1, 0, 1], [1, 0]), use_multitape_input=True)
(False, '>')
sage: M.process(([0, 1, 0], [0, 1, 1]), use_multitape_input=True)
(False, '<')
sage: M.process(([1, 1, 0, 1], [0, 0, 1, 0, 1, 1]), use_multitape_input=True)
(True, '=')
```

Alternatively, we can use the following (non-deterministic) multi-tape automaton for the same check:

```
sage: N = Automaton([('=', '=', (0, 0)), ....: ('=', '<', (None, 1)), ....: ('<', '<', (None, 0)), ....: ('=', '>', (1, None)), ....: ('>', '>', (None, 0))], ....: initial_states=['='], ....: final_states=['='])
sage: N.process(([1, 0, 1], [1, 0]), use_multitape_input=True)
(False, '>')
sage: N.process(([0, 1, 0], [0, 1, 1]), use_multitape_input=True)
(False, '<')
sage: N.process(([1, 1, 0, 1], [0, 0, 1, 0, 1, 1]), use_multitape_input=True)
```

(continues on next page)
See also:

- `FiniteStateMachine.process()`, `Transducer.process()`, `iter_process()`, `__call__()`,
- `FSMProcessIterator`

**shannon_parry_markov_chain()**

Compute a time homogeneous Markov chain such that all words of a given length recognized by the original automaton occur as the output with the same weight; the transition probabilities correspond to the Parry measure.

**OUTPUT:**

A Markov chain. Its input labels are the transition probabilities, the output labels the labels of the original automaton. In order to obtain equal weight for all words of the same length, an “exit weight” is needed. It is stored in the attribute `color` of the states of the Markov chain. The weights of the words of the same length sum up to one up to an exponentially small error.

The stationary distribution of this Markov chain is saved as the initial probabilities of the states.

The transition probabilities correspond to the Parry measure (see [S1948] and [P1964]).

The automaton is assumed to be deterministic, irreducible and aperiodic. All states must be final.

**EXAMPLES:**

```python
sage: NAF = Automaton([(0, 0, 0), (0, 1, 1), (0, 1, -1),
                    ....: (1, 0, 0)], initial_states=[0],
                    ....: final_states=[0, 1])
sage: P_NAF = NAF.shannon_parry_markov_chain()
sage: for s in P_NAF.iter_states():
    ....:     print(s.color)
3/4
3/2
```

The stationary distribution is also computed and saved as the initial probabilities of the returned Markov chain:

```python
sage: for s in P_NAF.states():
    ....:     print("{} {}".format(s, s.initial_probability))
0 2/3
1 1/3
```

The automaton is assumed to be deterministic, irreducible and aperiodic:

```python
sage: A = Automaton([(0, 0, 0), (0, 1, 1), (1, 1, 1), (1, 1, 0)],
                    ....: initial_states=[0])
sage: A.shannon_parry_markov_chain()
Traceback (most recent call last):
```

(continues on next page)
NotImplementedError: Automaton must be strongly connected.

```python
sage: A = Automaton([(0, 0, 0), (0, 1, 0)],
.....:         initial_states=[0])
sage: A.shannon_parry_markov_chain()
Traceback (most recent call last):
...
NotImplementedError: Automaton must be deterministic.
```

```python
sage: A = Automaton([(0, 1, 0), (1, 0, 0)],
.....:         initial_states=[0])
sage: A.shannon_parry_markov_chain()
Traceback (most recent call last):
...
NotImplementedError: Automaton must be aperiodic.
```

All states must be final:

```python
sage: A = Automaton([(0, 1, 0), (0, 0, 1), (1, 0, 0)],
.....:         initial_states=[0])
sage: A.shannon_parry_markov_chain()
Traceback (most recent call last):
...
NotImplementedError: All states must be final.
```

**ALGORITHM:**

See [HKP2015a], Lemma 4.1.

**REFERENCES:**

**with_output** *(word_out_function=None)*

Construct a transducer out of this automaton.

**INPUT:**

- `word_out_function` – (default: `None`) a function. It transforms a *transition* to the output word for this transition.

  If this is *None*, then the output word will be equal to the input word of each transition.

**OUTPUT:**

A transducer.

**EXAMPLES:**

```python
sage: A = Automaton([(0, 0, 'A'), (0, 1, 'B'), (1, 2, 'C')])
sage: T = A.with_output(); T
Transducer with 3 states
sage: T.transitions()
[Transition from 0 to 0: 'A'|'A',
 Transition from 0 to 1: 'B'|'B',
 Transition from 1 to 2: 'C'|'C']
```

This result is in contrast to:

```python
sage: Transducer(A).transitions()
[Transition from 0 to 0: 'A'|-,
```
where no output labels are created.

Here is another example:

```
sage: T2 = A.with_output(lambda t: [c.lower() for c in t.word_in])
sage: T2.transitions()  
[Transition from 0 to 0: 'A'|'a',
 Transition from 0 to 1: 'B'|'b',
 Transition from 1 to 2: 'C'|'c']
```

We can obtain the same result by composing two transducers. As inner transducer of the composition, we use `with_output()` without the optional argument `word_out_function` (which makes the output of each transition equal to its input); as outer transducer we use a **map-transducer** (for converting to lower case). This gives

```
sage: L = transducers.map(lambda x: x.lower(), ['A', 'B', 'C'])
sage: L.composition(A.with_output()).reabeled().transitions()  
[Transition from 0 to 0: 'A'|'a',
 Transition from 0 to 1: 'B'|'b',
 Transition from 1 to 2: 'C'|'c']
```

See also: `input_projection()`, `output_projection()`, `Transducer`, `transducers.map()`.

```
sage.combinat.finite_state_machine.FSMLetterSymbol(letter)
```

Return a string associated to the input letter.

**INPUT:**

- `letter` – the input letter or `None` (representing the empty word).

**OUTPUT:**

If `letter` is `None` the symbol for the empty word `FSMEmptyWordSymbol` is returned, otherwise the string associated to the letter.

**EXAMPLES:**

```
sage: from sage.combinat.finite_state_machine import FSMLetterSymbol
sage: FSMLetterSymbol(0)  
'0'
sage: FSMLetterSymbol(None)  
'-'
```

```
class sage.combinat.finite_state_machine.FSMProcessIterator(fsm, input_tape=None, initial_state=None, initial_states=[], use_multitape_input=False, check_epsilon_transitions=True, write_final_word_out=True, format_output=None, process_all_prefixes_of_input=False, **kwargs)
```

Bases: `SageObject`, `Iterator`
This class takes an input, feeds it into a finite state machine (automaton or transducer, in particular), tests whether this was successful and calculates the written output.

**INPUT:**

- `fsm` – the finite state machine on which the input should be processed.
- `input_tape` – the input tape can be a list or an iterable with entries from the input alphabet. If we are working with a multi-tape machine (see parameter `use_multitape_input` and notes below), then the tape is a list or tuple of tracks, each of which can be a list or an iterable with entries from the input alphabet.
- `initial_state` or `initial_states` – the initial state(s) in which the machine starts. Either specify a single one with `initial_state` or a list of them with `initial_states`. If both are given, `initial_state` will be appended to `initial_states`. If neither is specified, the initial states of the finite state machine are taken.
- `format_output` – a function that translates the written output (which is in form of a list) to something more readable. By default (None) identity is used here.
- `check_epsilon_transitions` – (default: True) a boolean. If False, then epsilon transitions are not taken into consideration during process.
- `write_final_word_out` – (default: True) a boolean specifying whether the final output words should be written or not.
- `use_multitape_input` – (default: False) a boolean. If True, then the multi-tape mode of the process iterator is activated. See also the notes below for multi-tape machines.
- `process_all_prefixes_of_input` – (default: False) a boolean. If True, then each prefix of the input word is processed (instead of processing the whole input word at once). Consequently, there is an output generated for each of these prefixes.

**OUTPUT:**

An iterator.

In its simplest form, it behaves like an iterator which, in each step, goes from one state to another. To decide which way to go, it uses the input words of the outgoing transitions and compares them to the input tape. More precisely, in each step, the process iterator takes an outgoing transition of the current state, whose input label equals the input letter of the tape. The output label of the transition, if present, is written on the output tape.

If the choice of the outgoing transition is not unique (i.e., we have a non-deterministic finite state machine), all possibilities are followed. This is done by splitting the process into several branches, one for each of the possible outgoing transitions.

The process (iteration) stops if all branches are finished, i.e., for no branch, there is any transition whose input word coincides with the processed input tape. This can simply happen when the entire tape was read. When the process stops, a `StopIteration` exception is thrown.

**Warning:** Processing an input tape of length n usually takes at least n + 1 iterations, since there will be n + 1 states visited (in the case the taken transitions have input words consisting of single letters).

An instance of this class is generated when `FiniteStateMachine.process()` or `FiniteStateMachine.iter_process()` of a finite state machine, an automaton, or a transducer is invoked.

When working with multi-tape finite state machines, all input words of transitions are words of k-tuples of letters. Moreover, the input tape has to consist of k tracks, i.e., be a list or tuple of k iterators, one for each track.
**Warning:** Working with multi-tape finite state machines is still experimental and can lead to wrong outputs.

**EXAMPLES:**

The following transducer reads binary words and outputs a word, where blocks of ones are replaced by just a single one. Further only words that end with a zero are accepted.

```python
sage: T = Transducer({
    'A': [({'A': 0, 0), ('B', 1, None)}],
    'B': [({'B': 1, None}, ('A', 0, [1, 0]))],
}, initial_states=['A'], final_states=['A'])
```

```python
sage: input = [1, 1, 0, 0, 1, 0, 1, 1, 1, 0]
sage: T.process(input)
(True, 'A', [1, 0, 0, 1, 0, 1, 0])
```

The function `FiniteStateMachine.process()` (internally) uses a `FSMProcessIterator`. We can do that manually, too, and get full access to the iteration process:

```python
sage: from sage.combinat.finite_state_machine import FSMProcessIterator
sage: it = FSMProcessIterator(T, input_tape=input)
sage: for current in it:
    ....:    print(current)
```

```
process (1 branch)
+ at state 'B'
  -- tape at 1, [[]]
process (1 branch)
+ at state 'B'
  -- tape at 2, [[]]
+ at state 'A'
  -- tape at 3, [[1, 0]]
process (1 branch)
+ at state 'A'
  -- tape at 4, [[1, 0, 0]]
+ at state 'B'
  -- tape at 5, [[1, 0, 0]]
+ at state 'A'
  -- tape at 6, [[1, 0, 0, 1, 0]]
+ at state 'B'
  -- tape at 7, [[1, 0, 0, 1, 0]]
  -- tape at 8, [[1, 0, 0, 1, 0]]
  -- tape at 9, [[1, 0, 0, 1, 0]]
  -- tape at 10, [[1, 0, 0, 1, 0]]
```

(...continues on next page)
sage: T = Transducer([(0, 0, 0, 'a'), (0, 1, 0, 'b')],
.....: (1, 2, 1, 'c'), (2, 0, 0, 'd')],
.....: initial_states=[0], final_states=[2])
sage: sorted(T.process([0, 0, 1], format_output=lambda o: ''.join(o)))
[(False, 1, 'abcd'), (True, 2, 'abc')]
sage: it = FSMProcessIterator(T, input_tape=[0, 0, 1],
.....: format_output=lambda o: ''.join(o))
sage: for current in it:
.....: print(current)
process (2 branches)
+ at state 0
++ tape at 1, [['a']]
+ at state 1
++ tape at 1, [['b']]
process (2 branches)
+ at state 0
++ tape at 2, [['a', 'a']]  
+ at state 1
++ tape at 2, [['a', 'b']]
process (2 branches)
+ at state 1
++ tape at 3, [['a', 'b', 'c', 'd']]  
+ at state 2
++ tape at 3, [['a', 'b', 'c']]
process (0 branches)
sage: sorted(it.result())
[Branch(accept=False, state=1, output='abcd'),
 Branch(accept=True, state=2, output='abc')]

See also:
FiniteStateMachine.process(), Automaton.process(), Transducer.process(),
FiniteStateMachine.iter_process(), FiniteStateMachine.__call__(), next().
....: print(current)
    {((1, 0),): {'A': Branch(tape_cache=tape at 1, outputs=[[1]])}}
process (1 branch)
+ at state 'A'
--- tape at 1, [[1]]
{((2, 0),): {'A': Branch(tape_cache=tape at 2, outputs=[[1, 0]])}}
process (1 branch)
+ at state 'A'
--- tape at 2, [[1, 0]]
{} process (0 branches)

class FinishedBranch(accept, state, output)
    bases: tuple
    A named tuple representing the attributes of a branch, once it is fully processed.

    accept
        Alias for field number 0

    output
        Alias for field number 2

    state
        Alias for field number 1

next()
    Makes one step in processing the input tape.

    INPUT:
    Nothing.

    OUTPUT:
    It returns the current status of the iterator (see below). A StopIteration exception is thrown when there is/was nothing to do (i.e. all branches ended with previous call of next()).

    The current status is a dictionary (encapsulated into an instance of Current). The keys are positions on the tape. The value corresponding to such a position is again a dictionary, where each entry represents a branch of the process. This dictionary maps the current state of a branch to a pair consisting of a tape cache and a list of output words, which were written during reaching this current state.

    EXAMPLES:

```sage
def from sage.combinat.finite_state_machine import FSMProcessIterator
sage: inverter = Transducer({'A': [(['A', 0, 1), ('A', 1, 0)]},
...: initial_states=['A'], final_states=['A'])
sage: it = FSMProcessIterator(inverter, input_tape=[0, 1])
sage: next(it)
process (1 branch)
+ at state 'A'
--- tape at 1, [[1]]
sage: next(it)
process (1 branch)
+ at state 'A'
```
See also:

FiniteStateMachine.process(), Automaton.process(), Transducer.process(),
FiniteStateMachine.iter_process(), FiniteStateMachine.__call__(),
FSMProcessIterator.

preview_word(track_number=None, length=1, return_word=False)

Read a word from the input tape.

INPUT:

- `track_number` – an integer or None. If None, then a tuple of words (one from each track) is returned.
- `length` – (default: 1) the length of the word(s).
- `return_word` – (default: False) a boolean. If set, then a word is returned, otherwise a single letter
  (in which case `length` has to be 1).

OUTPUT:

A single letter or a word.

An exception StopIteration is thrown if the tape (at least one track) has reached its end.

Typically, this method is called from a hook-function of a state.

EXAMPLES:

```python
sage: inverter = Transducer({'A': [('A', 0, 'one'),
                                 ('A', 1, 'zero')],
                          initial_states=['A'], final_states=['A'])

sage: def state_hook(process, state, output):
    print("We are now in state ")
    print(state.label(),",")
    try:
        w = process.preview_word()
    except RuntimeError:
        print("Next on the tape is a ")
    print(w)

sage: inverter.state('A').hook = state_hook

sage: it = inverter.iter_process(
    input_tape=[0, 1, 1],
    check_epsilon_transitions=False)

sage: for _ in it:
    pass

We are now in state A.
Next on the tape is a 0.
We are now in state A.
Next on the tape is a 1.
We are now in state A.
```

(continues on next page)
Next on the tape is a 1.
We are now in state A.
sage: it.result()
[Branch(accept=True, state='A', output=['one', 'zero', 'zero'])]

result(format_output=None)
Return the already finished branches during process.

INPUT:

• format_output – a function converting the output from list form to something more readable (default: output the list directly).

OUTPUT:
A list of triples (accepted, state, output).

See also the parameter format_output of FSMProcessIterator.

EXAMPLES:

```
sage: inverter = Transducer({'A': [('A', 0, 'one'), ('A', 1, 'zero')]},
.....: initial_states=['A'], final_states=['A'])
sage: it = inverter.iter_process(input_tape=[0, 1, 1])
sage: for _ in it:
.....:  pass
sage: it.result()
[Branch(accept=True, state='A', output=['one', 'zero', 'zero'])]
sage: it.result(lambda L: ','.join(L))
[(True, 'A', 'one, zero, zero')]
```

Using both the parameter format_output of FSMProcessIterator and the parameter format_output of result() leads to concatenation of the two functions:

```
sage: it = inverter.iter_process(input_tape=[0, 1, 1],
.....: format_output=lambda L: ' ', ' '.join(L))
sage: for _ in it:
.....:  pass
sage: it.result()
[Branch(accept=True, state='A', output='one, zero, zero')]
sage: it.result(lambda L: ' ', ' '.join(L))
[(True, 'A', 'o, n, e, ,, , z, e, r, o, ,, , z, e, r, o')]
```

class sage.combinat.finite_state_machine.FSMState(label, word_out=None, is_initial=False, is_final=False, final_word_out=None, initial_probability=None, hook=None, color=None, allow_label_None=False)

Bases: SageObject

Class for a state of a finite state machine.

INPUT:

• label – the label of the state.
• word_out – (default: None) a word that is written when the state is reached.
• is_initial – (default: False)
• **is_final** – (default: False)

• **final_word_out** – (default: None) a word that is written when the state is reached as the last state of some input; only for final states.

• **initial_probability** – (default: None) The probability of starting in this state if it is a state of a Markov chain.

• **hook** – (default: None) A function which is called when the state is reached during processing input. It takes two input parameters: the first is the current state (to allow using the same hook for several states), the second is the current process iterator object (to have full access to everything; e.g. the next letter from the input tape can be read in). It can output the next transition, i.e. the transition to take next. If it returns None the process iterator chooses. Moreover, this function can raise a StopIteration exception to stop processing of a finite state machine the input immediately. See also the example below.

• **color** – (default: None) In order to distinguish states, they can be given an arbitrary “color” (an arbitrary object). This is used in FiniteStateMachine.equivalence_classes(): states of different colors are never considered to be equivalent. Note that Automaton.determinisation() requires that color is hashable.

• **allow_label_None** – (default: False) If True allows also None as label. Note that a state with label None is used in FSMProcessIterator.

**OUTPUT:**

A state of a finite state machine.

**EXAMPLES:**

```
sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('state 1', word_out=0, is_initial=True)
sage: A
'state 1'
sage: A.label()
'state 1'
sage: B = FSMState('state 2')
sage: A == B
False
```

We can also define a final output word of a final state which is used if the input of a transducer leads to this state. Such final output words are used in subsequential transducers.

```
sage: C = FSMState('state 3', is_final=True, final_word_out='end')
sage: C.final_word_out
['end']
```

The final output word can be a single letter, None or a list of letters:

```
sage: A = FSMState('A')
sage: A.is_final = True
sage: A.final_word_out = 2
sage: A.final_word_out
[2]
sage: A.final_word_out = [2, 3]
sage: A.final_word_out
[2, 3]
```

Only final states can have a final output word which is not None:
sage: B = FSMState('B')
sage: B.final_word_out is None
True
sage: B.final_word_out = 2
Traceback (most recent call last):
  ...:
ValueError: Only final states can have a final output word, but state B is not final.

Setting the final_word_out of a final state to None is the same as setting it to [] and is also the default for a final state:

sage: C = FSMState('C', is_final=True)
sage: C.final_word_out
[]
sage: C.final_word_out = None
ValueError: Only final states can have a final output word, but state C is not final.

sage: C.final_word_out = []
sage: C.final_word_out
[]

It is not allowed to use None as a label:

sage: from sage.combinat.finite_state_machine import FSMState
sage: FSMState(None)
Traceback (most recent call last):
  ...:
ValueError: Label None reserved for a special state, choose another label.

This can be overridden by:

sage: FSMState(None, allow_label=None=True)
None

Note that Automaton.determinisation() requires that color is hashable:

sage: A = Automaton([[0, 0, 0]], initial_states=[0])
  ...:
  ...:
  ...:
Traceback (most recent call last):
  ...:
TypeError: unhashable type: 'list'

We can use a hook function of a state to stop processing. This is done by raising a StopIteration exception. The following code demonstrates this:

sage: T = Transducer([(0, 1, 9, 'a'), (1, 2, 9, 'b'),
  ...:
  ...:
  ...:
(continues on next page)
....:     final_states=[4],
....:     input_alphabet=[9])
sage: def stop(process, state, output):
....:         raise StopIteration()
sage: T.state(3).hook = stop
sage: T.process([9, 9, 9, 9])
(False, 3, ['a', 'b', 'c'])

copy()
Return a (shallow) copy of the state.

INPUT:
Nothing.

OUTPUT:
A new state.

EXAMPLES:

sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A')
sage: A.is_initial = True
sage: A.is_final = True
sage: A.final_word_out = [1]
sage: A.color = 'green'
sage: A.initial_probability = 1/2
sage: B = copy(A)
sage: B.fully_equal(A)
True
sage: A.label() is B.label()
True
sage: A.is_initial is B.is_initial
True
sage: A.is_final is B.is_final
True
sage: A.final_word_out is B.final_word_out
True
sage: A.color is B.color
True
sage: A.initial_probability is B.initial_probability
True

deepcopy(memo=None)
Return a deep copy of the state.

INPUT:

• memo – (default: None) a dictionary storing already processed elements.

OUTPUT:

A new state.

EXAMPLES:
```
from sage.combinat.finite_state_machine import FSMState
A = FSMState((1, 3), color=[1, 2],
 .....  is_final=True, final_word_out=3,
 .....  initial_probability=1/3)
B = deepcopy(A)
B
(1, 3)
B.label() == A.label()
True
B.label is A.label
False
B.color == A.color
True
B.color is A.color
False
B.is_final == A.is_final
True
B.is_final is A.is_final
True
B.final_word_out == A.final_word_out
True
B.final_word_out is A.final_word_out
False
B.initial_probability == A.initial_probability
True
```

**property final_word_out**

The final output word of a final state which is written if the state is reached as the last state of the input of the finite state machine. For a non-final state, the value is None.

final_word_out can be a single letter, a list or None, but for a final-state, it is always saved as a list.

**EXAMPLES:**

```
from sage.combinat.finite_state_machine import FSMState
A = FSMState('A', is_final=True, final_word_out=2)
A.final_word_out
[2]
A.final_word_out = 3
A.final_word_out
[3]
A.final_word_out = [3, 4]
A.final_word_out
[3, 4]
A.final_word_out = None
A.final_word_out
[]
B = FSMState('B')
B.final_word_out is None
True
```

A non-final state cannot have a final output word:
sage: B.final_word_out = [3, 4]
Traceback (most recent call last):
  ...
ValueError: Only final states can have a final output word, but state B is not final.

```python
fully_equal(other, compare_color=True)
```

Check whether two states are fully equal, i.e., including all attributes except hook.

**INPUT:**
- `self` – a state.
- `other` – a state.
- `compare_color` – If True (default) colors are compared as well, otherwise not.

**OUTPUT:**
- True or False.

Note that usual comparison by `==` does only compare the labels.

**EXAMPLES:**

```python
sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A')
sage: B = FSMState('A', is_initial=True)
sage: A.fully_equal(B)
False
sage: A == B
True
sage: A.is_initial = True; A.color = 'green'
False
sage: A.fully_equal(B, compare_color=False)
True
```

```python
initial_probability = None
```

**property is_final**

Describes whether the state is final or not.

True if the state is final and False otherwise.

**EXAMPLES:**

```python
sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A', is_final=True, final_word_out=3)
sage: A.is_final
True
sage: A.is_final = False
Traceback (most recent call last):
  ...
ValueError: State A cannot be non-final, because it has a final output word. Only final states can have a final output word.
sage: A.final_word_out = None
```
A.is_final = False

is_initial = False

label()
Return the label of the state.

INPUT:
Nothing.

OUTPUT:
The label of the state.

EXAMPLES:

sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('state')

sage: A.label()
'state'

relabeled(label, memo=None)
Return a deep copy of the state with a new label.

INPUT:

• label – the label of new state.

• memo – (default: None) a dictionary storing already processed elements.

OUTPUT:
A new state.

EXAMPLES:

sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A')

sage: A.relabeled('B')
'B'

class sage.combinat.finite_state_machine.FSMTransition(from_state, to_state, word_in=None, word_out=None, hook=None)

Bases: SageObject

Class for a transition of a finite state machine.

INPUT:

• from_state – state from which transition starts.

• to_state – state in which transition ends.

• word_in – the input word of the transitions (when the finite state machine is used as automaton)

• word_out – the output word of the transitions (when the finite state machine is used as transducer)
OUTPUT:
A transition of a finite state machine.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import FSMState, FSMTransition
sage: A = FSMState('A')
```
```
sage: B = FSMState('B')
```
```
sage: S = FSMTransition(A, B, 0, 1)
sage: T = FSMTransition('A', 'B', 0, 1)
```
```
sage: T == S
True
```
```
sage: U = FSMTransition('A', 'B', 0)
```
```
sage: U == T
False
```

`copy()`

Return a (shallow) copy of the transition.

OUTPUT:
A new transition.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import FSMTransition
sage: t = FSMTransition('A', 'B', 0)
```
```
sage: copy(t)
```
```
Transition from 'A' to 'B': 0|-
```

`deepcopy(memo=None)`

Return a deep copy of the transition.

INPUT:

- `memo` – (default: None) a dictionary storing already processed elements.

OUTPUT:
A new transition.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import FSMTransition
sage: t = FSMTransition('A', 'B', 0)
```
```
sage: deepcopy(t)
```
```
Transition from 'A' to 'B': 0|-
```

`from_state = None`

State from which the transition starts. Read-only.

`to_state = None`

State in which the transition ends. Read-only.

`word_in = None`

Input word of the transition. Read-only.
**word_out** = None

Output word of the transition. Read-only.

`sage.combinat.finite_state_machine.FSMWordSymbol(word)`

Return a string of word. It may returns the symbol of the empty word FSMEmptyWordSymbol.

**INPUT:**

* word – the input word.

**OUTPUT:**

A string of word.

**EXAMPLES:**

```python
sage: from sage.combinat.finite_state_machine import FSMWordSymbol
dsage: FSMWordSymbol([0, 1, 1])
'0,1,1'
```

```python

class sage.combinat.finite_state_machine.FiniteStateMachine(data=None, initial_states=None, final_states=None, input_alphabet=None, output_alphabet=None, determine_alphabets=None, with_final_word_out=None, store_states_dict=True, on_duplicate_transition=None)

Bases: SageObject

Class for a finite state machine.

A finite state machine is a finite set of states connected by transitions.

**INPUT:**

* data – can be any of the following:

  1. a dictionary of dictionaries (of transitions),
  2. a dictionary of lists (of states or transitions),
  3. a list (of transitions),
  4. a function (transition function),
  5. an other instance of a finite state machine.

* initial_states and final_states – the initial and final states of this machine

* input_alphabet and output_alphabet – the input and output alphabets of this machine

* determine_alphabets – If True, then the function determine_alphabets() is called after data was read and processed, if False, then not. If it is None, then it is decided during the construction of the finite state machine whether determine_alphabets() should be called.

* with_final_word_out – If given (not None), then the function with_final_word_out() (more precisely, its inplace pendant construct_final_word_out()) is called with input letters=with_final_word_out at the end of the creation process.

* store_states_dict – If True, then additionally the states are stored in an internal dictionary for speed up.
• on_duplicate_transition – A function which is called when a transition is inserted into self which already existed (same from_state, same to_state, same word_in, same word_out).

This function is assumed to take two arguments, the first being the already existing transition, the second being the new transition (as an FSMTransition). The function must return the (possibly modified) original transition.

By default, we have on_duplicate_transition=none, which is interpreted as on_duplicate_transition=duplicate_transition_ignore, where duplicate_transition_ignore is a predefined function ignoring the occurrence. Other such predefined functions are duplicate_transition_raise_error and duplicate_transition_add_input.

OUTPUT:

A finite state machine.

The object creation of Automaton and Transducer is the same as the one described here (i.e. just replace the word FiniteStateMachine by Automaton or Transducer).

Each transition of an automaton has an input label. Automata can, for example, be determinised (see Automaton.determinisation()) and minimized (see Automaton.minimization()). Each transition of a transducer has an input and an output label. Transducers can, for example, be simplified (see Transducer.simplification()).

EXAMPLES:

```sage
from sage.combinat.finite_state_machine import FSMState, FSMTransition

See documentation for more examples.

We illustrate the different input formats:

1. The input-data can be a dictionary of dictionaries, where
   - the keys of the outer dictionary are state-labels (from-states of transitions),
   - the keys of the inner dictionaries are state-labels (to-states of transitions),
   - the values of the inner dictionaries specify the transition more precisely.

   The easiest is to use a tuple consisting of an input and an output word:

   ```sage
   FiniteStateMachine({'a': {'b': (0, 1), 'c': (1, 1)}})
   ```
   Finite state machine with 3 states

   Instead of the tuple anything iterable (e.g. a list) can be used as well.

   If you want to use the arguments of FSMTransition directly, you can use a dictionary:

   ```sage
   FiniteStateMachine({'a': {'b': {'word_in': 0, 'word_out': 1},
   'c': {'word_in': 1, 'word_out': 1}}})
   ```
   Finite state machine with 3 states

   In the case you already have instances of FSMTransition, it is possible to use them directly:

   ```sage
   FiniteStateMachine({'a': {'b': FSMTransition('a', 'b', 0, 1),
   'c': FSMTransition('a', 'c', 1, 1)}})
   ```
   Finite state machine with 3 states

2. The input-data can be a dictionary of lists, where the keys are states or label of states.

   The list-elements can be states:
sage: a = FSMState('a')
sage: b = FSMState('b')
sage: c = FSMState('c')
sage: FiniteStateMachine({a:[b, c]})
Finite state machine with 3 states

Or the list-elements can simply be labels of states:

sage: FiniteStateMachine({'a':['b', 'c']})
Finite state machine with 3 states

The list-elements can also be transitions:

sage: FiniteStateMachine({'a':[FSMTransition('a', 'b', 0, 1),
                              FSMTransition('a', 'c', 1, 1)]})
Finite state machine with 3 states

Or they can be tuples of a label, an input word and an output word specifying a transition:

sage: FiniteStateMachine({'a':[('b', 0, 1), ('c', 1, 1)]})
Finite state machine with 3 states

3. The input-data can be a list, where its elements specify transitions:

sage: FiniteStateMachine([FSMTransition('a', 'b', 0, 1),
                         FSMTransition('a', 'c', 1, 1)])
Finite state machine with 3 states

It is possible to skip FSMTransition in the example above:

sage: FiniteStateMachine([('a', 'b', 0, 1), ('a', 'c', 1, 1)])
Finite state machine with 3 states

The parameters of the transition are given in tuples. Anyhow, anything iterable (e.g. a list) is possible.
You can also name the parameters of the transition. For this purpose you take a dictionary:

sage: FiniteStateMachine([{'from_state':'a', 'to_state':'b',
                          'word_in':0, 'word_out':1},
                          {'from_state':'a', 'to_state':'c',
                          'word_in':1, 'word_out':1}])
Finite state machine with 3 states

Other arguments, which FSMTransition accepts, can be added, too.

4. The input-data can also be function acting as transition function:

This function has two input arguments:

1. a label of a state (from which the transition starts),
2. a letter of the (input-)alphabet (as input-label of the transition).

It returns a tuple with the following entries:

1. a label of a state (to which state the transition goes),
2. a letter of or a word over the (output-)alphabet (as output-label of the transition).

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It may also output a list of such tuples if several transitions from the from-state and the input letter exist (this means that the finite state machine is non-deterministic).

If the transition does not exist, the function should raise a LookupError or return an empty list.

When constructing a finite state machine in this way, some initial states and an input alphabet have to be specified.

```python
sage: def f(state_from, read):
    ....:     if int(state_from) + read <= 2:
    ....:         state_to = 2*int(state_from)+read
    ....:         write = 0
    ....:     else:
    ....:         state_to = 2*int(state_from) + read - 5
    ....:         write = 1
    ....:     return (str(state_to), write)

sage: F = FiniteStateMachine(f, input_alphabet=[0, 1],
    ....:     initial_states=['0'],
    ....:     final_states=['0'])

sage: F([1, 0, 1])
(True, '0', [0, 0, 1])
```

5. The input-data can be an other instance of a finite state machine:

```python
sage: F = FiniteStateMachine()

sage: G = Transducer(F)

sage: G == F
True

The other parameters cannot be specified in that case. If you want to change these, use the attributes FSMState.is_initial, FSMState.is_final, input_alphabet, output_alphabet, on_duplicate_transition and methods determine_alphabets(), construct_final_word_out() on the new machine, respectively.

The following examples demonstrate the use of on_duplicate_transition:

```python
sage: F = FiniteStateMachine([[ 'a', 'a', 1/2], ['a', 'a', 1/2]])

sage: F.transitions()
[Transition from 'a' to 'a': 1/2|-]

sage: from sage.combinat.finite_state_machine import duplicate_transition_raise_error

sage: F1 = FiniteStateMachine([[ 'a', 'a', 1/2], ['a', 'a', 1/2]],
    .....:     on_duplicate_transition=duplicate_transition_raise_error)

Traceback (most recent call last):
...
ValueError: Attempting to re-insert transition Transition from 'a' to 'a': 1/2|-!
```

Use duplicate_transition_add_input to emulate a Markov chain, the input labels are considered as transition probabilities:

```python
sage: from sage.combinat.finite_state_machine import duplicate_transition_add_input

sage: F = FiniteStateMachine([[ 'a', 'a', 1/2], ['a', 'a', 1/2]],
    .....:     on_duplicate_transition=duplicate_transition_add_input)
```

(continues on next page)
Use `with_final_word_out` to construct final output:

```
sage: T = Transducer([(0, 1, 0, 0), (1, 0, 0, 0)],
....: initial_states=[0],
....: final_states=[0],
....: with_final_word_out=0)
sage: for s in T.iter_final_states():
....: print("{} {}").format(s, s.final_word_out))
0 []
1 [0]
```

__call__(*args, **kwargs)

Call either method `composition()` or `process()` (with `full_output=False`). If the input is not finite (is_finite of input is False), then `iter_process()` (with `iterator_type='simple'`) is called. Moreover, the flag `automatic_output_type` is set (unless `format_output` is specified). See the documentation of these functions for possible parameters.

EXAMPLES:

The following code performs a `composition()`:

```
sage: F = Transducer([('A', 'B', 1, 0), ('B', 'B', 1, 1)],
....: ('B', 'B', 0, 0)],
....: initial_states=['A'], final_states=['B'])
sage: G = Transducer([(1, 1, 0, 0), (1, 2, 1, 0),
....: (2, 2, 0, 1), (2, 1, 1, 1)],
....: initial_states=[1], final_states=[1])
sage: H = G(F)
sage: H.states()    [("A", 1), ("B", 1), ("B", 2)]
```

An automaton or transducer can also act on an input (an list or other iterable of letters):

```
sage: binary_inverter = Transducer({'A': [("A", 0, 1), ("A", 1, 0)]},
....: initial_states=['A'], final_states=['A'])
sage: binary_inverter([0, 1, 0, 0, 1, 1])
[1, 0, 1, 1, 0, 0]
```

We can also let them act on `words`:

```
sage: W = Words([0, 1]); W
Finite and infinite words over {0, 1}
sage: binary_inverter(W([0, 1, 1, 0, 1, 1]))
word: 100100
```

Infinite words work as well:

```
sage: words.FibonacciWord()
word: 0100101001001010010100100101001001010010...
sage: binary_inverter(words.FibonacciWord())
word: 10101010101010101010101010101010101010101...
```
When only one successful path is found in a non-deterministic transducer, the result of that path is returned.

```python
sage: T = Transducer([(0, 1, 0, 1), (0, 2, 0, 2)],
    ....: initial_states=[0], final_states=[1])
sage: T.process([0])
[(False, 2, [2]), (True, 1, [1])]
sage: T([0])
[1]
```

See also:

`composition()`, `process()`, `iter_process()`, `Automaton.process()`, `Transducer.process()`.

accessible_components()

Return a new finite state machine with the accessible states of self and all transitions between those states.

INPUT:

Nothing.

OUTPUT:

A finite state machine with the accessible states of self and all transitions between those states.

A state is accessible if there is a directed path from an initial state to the state. If self has no initial states then a copy of the finite state machine self is returned.

EXAMPLES:

```python
sage: F = Automaton([(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 0, 1)],
    ....: initial_states=[0])
sage: F.accessible_components()
Automaton with 2 states

sage: F = Automaton([(0, 0, 1), (0, 0, 1), (1, 1, 0), (1, 0, 1)],
    ....: initial_states=[0])
sage: F.accessible_components()
Automaton with 1 state
```

See also:

`coaccessible_components()`

add_from_transition_function(function, initial_states=None, explore_existing_states=True)

Constructs a finite state machine from a transition function.

INPUT:

- `function` may return a tuple (new_state, output_word) or a list of such tuples.
- `initial_states` – If no initial states are given, the already existing initial states of self are taken.
- If `explore_existing_states` is True (default), then already existing states in self (e.g. already given final states) will also be processed if they are reachable from the initial states.

OUTPUT:

Nothing.

EXAMPLES:
sage: F = FiniteStateMachine(initial_states=['A'],
    input_alphabet=[0, 1])
sage: def f(state, input):
    return [('A', input), ('B', 1-input)]
sage: F.add_from_transition_function(f)
sage: F.transitions()
[Transition from 'A' to 'A': 0|0,
    Transition from 'A' to 'B': 0|1,
    Transition from 'A' to 'A': 1|1,
    Transition from 'A' to 'B': 1|0,
    Transition from 'B' to 'A': 0|0,
    Transition from 'B' to 'B': 0|1,
    Transition from 'B' to 'A': 1|1,
    Transition from 'B' to 'B': 1|0]

Initial states can also be given as a parameter:

sage: F = FiniteStateMachine(input_alphabet=[0,1])
sage: def f(state, input):
    return [('A', input), ('B', 1-input)]
sage: F.add_from_transition_function(f, initial_states=['A'])
sage: F.initial_states()
['A']

Already existing states in the finite state machine (the final states in the example below) are also explored:

sage: F = FiniteStateMachine(initial_states=[0],
    final_states=[1],
    input_alphabet=[0])
sage: def transition_function(state, letter):
    return 1 - state, []
sage: F.add_from_transition_function(transition_function)
sage: F.transitions()
[Transition from 0 to 1: 0|-,
    Transition from 1 to 0: 0|-
]

If `explore_existing_states=False`, however, this behavior is turned off, i.e., already existing states are not explored:

sage: F = FiniteStateMachine(initial_states=[0],
    final_states=[1],
    input_alphabet=[0])
sage: def transition_function(state, letter):
    return 1 - state, []
sage: F.add_from_transition_function(transition_function,
    explore_existing_states=False)
sage: F.transitions()
[Transition from 0 to 1: 0|-]

**add_state(state)**

Adds a state to the finite state machine and returns the new state. If the state already exists, that existing state is returned.

**INPUT:**
• state is either an instance of FSMState or, otherwise, a label of a state.

OUTPUT:
The new or existing state.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import FSMState
sage: F = FiniteStateMachine()
sage: A = FSMState('A', is_initial=True)
sage: F.add_state(A)
'A'
```

add_states(states)

Adds several states. See add_state for more information.

INPUT:

• states – a list of states or iterator over states.

OUTPUT:

Nothing.

EXAMPLES:

```
sage: F = FiniteStateMachine()
sage: F.add_states(['A', 'B'])
sage: F.states()
['A', 'B']
```

add_transition(*args, **kwargs)

Adds a transition to the finite state machine and returns the new transition.

If the transition already exists, the return value of self.on_duplicate_transition is returned. See the documentation of FiniteStateMachine.

INPUT:

The following forms are all accepted:

```
sage: from sage.combinat.finite_state_machine import FSMState, FSMTransition
sage: A = FSMState('A')
sage: B = FSMState('B')

sage: FSM = FiniteStateMachine()  
sage: FSM.add_transition(FSMTransition(A, B, 0, 1))  
Transition from 'A' to 'B': 0|1

sage: FSM = FiniteStateMachine()  
sage: FSM.add_transition(A, B, 0, 1)  
Transition from 'A' to 'B': 0|1

sage: FSM = FiniteStateMachine()  
sage: FSM.add_transition(A, B, word_in=0, word_out=1)  
Transition from 'A' to 'B': 0|1

sage: FSM = FiniteStateMachine()  
```

(continues on next page)
If the states A and B are not instances of `FSMState`, then it is assumed that they are labels of states.

**OUTPUT:**

The new transition.

**add_transitions_from_function** *(function, labels_as_input=True)*

Adds one or more transitions if `function(state, state)` says that there are some.

**INPUT:**

- `function` – a transition function. Given two states `from_state` and `to_state` (or their labels if `label_as_input` is true), this function shall return a tuple `(word_in, word_out)` to add a transition from `from_state` to `to_state` with input and output labels `word_in` and `word_out`, respectively. If no such addition is to be added, the transition function shall return `None`. The transition function may also return a list of such tuples in order to add multiple transitions between the pair of states.

- `label_as_input` – (default: True)

**OUTPUT:**

Nothing.

**EXAMPLES:**

```python
sage: F = FiniteStateMachine()
sage: F.add_states(['A', 'B', 'C'])
sage: def f(statet1, state2):
....:     if state1 == 'C':
....:         return None
....:     return (0, 1)
sage: F.add_transitions_from_function(f)
sage: len(F.transitions())
6
```

Multiple transitions are also possible:
```
sage: F = FiniteStateMachine()
sage: F.add_states([0, 1])
sage: def f(state1, state2):
    ....:     if state1 != state2:
    ....:         return [(0, 1), (1, 0)]
    ....:     else:
    ....:         return None
sage: F.add_transitions_from_function(f)
sage: F.transitions()
[Transition from 0 to 1: 0|1,
 Transition from 0 to 1: 1|0,
 Transition from 1 to 0: 0|1,
 Transition from 1 to 0: 1|0]
```

### adjacency_matrix(input=None, entry=None)

Return the adjacency matrix of the underlying graph.

**INPUT:**

- `input` – Only transitions with input label `input` are respected.
- `entry` – The function `entry` takes a transition and the return value is written in the matrix as the entry `(transition.from_state, transition.to_state)`. The default value (None) of `entry` takes the variable `x` to the power of the sum of the output word of the transition.

**OUTPUT:**

A matrix.

If any label of a state is not an integer, the finite state machine is relabeled at the beginning. If there are more than one transitions between two states, then the different return values of `entry` are added up.

**EXAMPLES:**

```
sage: B = FiniteStateMachine({0:{0:(0, 0), 'a':(1, 0)},
    ....:         'a':{2:(0, 0), 3:(1, 0)},
    ....:         2:{0:(1, 1), 4:(0, 0)},
    ....:         3:{'a':(0, 1), 2:(1, 1)},
    ....:         4:{4:(1, 1), 3:(0, 1)}},
    ....:     initial_states=[0])
sage: B.adjacency_matrix()               # optional - sage.
   ....:     symbolic
[[1 1 0 0 0]
 [0 1 1 0]
 [x 0 0 1]
 [0 x x 0]
 [0 0 0 x]]
```

This is equivalent to:

```
sage: matrix(B)                           # optional - sage.
   ....:     symbolic
[[1 1 0 0 0]
 [0 1 1 0]
 [x 0 0 1]
 [0 x x 0]
 [0 0 0 x]]
```
It is also possible to use other entries in the adjacency matrix:

```

sage: B.adjacency_matrix(entry=(lambda transition: 1))
[1 1 0 0 0]
[0 0 1 1 0]
[1 0 0 0 1]
[0 1 0 0 0]
[0 0 1 1 0]
```

```
sage: var('t')  # optional - sage.

sage: B.adjacency_matrix(1, entry=(lambda transition: exp(I*transition.word_out[0]*t)))  # optional - sage.
```

```
[ 0 1 0 0 0]
[ 0 0 0 1 0]
[e^(I*t) 0 0 0 0]
[ 0 0 e^(I*t) 0 0]
[ 0 0 0 0 e^(I*t)]
```

```
sage: a = Automaton([(0, 1, 0),
                 (1, 2, 0),
                 (2, 0, 1),
                 (2, 1, 0)],
                 initial_states=[0],
                 final_states=[0])
```

```
sage: a.adjacency_matrix()  # optional - sage.
```

```
[0 1 0]
[0 0 1]
[1 0 0]
```

The function `asymptotic_moments(variable=None)`

Return the main terms of expectation and variance of the sum of output labels and its covariance with the sum of input labels.

**INPUT:**

- variable – a symbol denoting the length of the input, by default $n$.

**OUTPUT:**

A dictionary consisting of

- `expectation` – $en + O(1)$,
- `variance` – $vn + O(1)$,
- `covariance` – $cn + O(1)$

for suitable constants $e$, $v$ and $c$.

Assume that all input and output labels are numbers and that `self` is complete and has only one final component. Assume further that this final component is aperiodic. Furthermore, assume that there is exactly one initial state and that all states are final.

Denote by $X_n$ the sum of output labels written by the finite state machine when reading a random input word of length $n$ over the input alphabet (assuming equidistribution).
Then the expectation of $X_n$ is $en + O(1)$, the variance of $X_n$ is $vn + O(1)$ and the covariance of $X_n$ and the sum of input labels is $cn + O(1)$, cf. [HKW2015], Theorem 3.9.

In the case of non-integer input or output labels, performance degrades significantly. For rational input and output labels, consider rescaling to integers. This limitation comes from the fact that determinants over polynomial rings can be computed much more efficiently than over the symbolic ring. In fact, we compute (parts) of a trivariate generating function where the input and output labels are exponents of some indeterminates, see [HKW2015], Theorem 3.9 for details. If those exponents are integers, we can use a polynomial ring.

EXAMPLES:

1. A trivial example: write the negative of the input:

   ```sage
   sage: T = Transducer([(0, 0, 0, 0), (0, 0, 1, -1)],
       ....:       initial_states=[0],
       ....:       final_states=[0])
   sage: T([0, 1, 1])
   [0, -1, -1]
   sage: moments = T.asymptotic_moments()  # optional - sage.symbolic
   sage: moments['expectation']  # optional - sage.symbolic
   -1/2*n + Order(1)
   sage: moments['variance']  # optional - sage.symbolic
   1/4*n + Order(1)
   sage: moments['covariance']  # optional - sage.symbolic
   -1/4*n + Order(1)
   ```

2. For the case of the Hamming weight of the non-adjacent-form (NAF) of integers, cf. the Wikipedia article Non-adjacent_form and the example on recognizing NAFs, the following agrees with the results in [HP2007].

   We first use the transducer to convert the standard binary expansion to the NAF given in [HP2007]. We use the parameter with_final_word_out such that we do not have to add sufficiently many trailing zeros:

   ```sage
   sage: NAF = Transducer([(0, 0, 0, 0),
       ....:       (0, '.1', 1, None),
       ....:       ('.1', 0, 0, [1, 0]),
       ....:       ('.1', 1, 1, [-1, 0]),
       ....:       (1, 1, 1, 0),
       ....:       (1, '.1', 0, None),
       ....:       initial_states=[0],
       ....:       final_states=[0],
       ....:       with_final_word_out=[0])
   sage: binary_27 = 27.bits()
   sage: binary_27
   [1, 1, 0, 1, 1]
   sage: NAF_27 = NAF(binary_27)
   sage: NAF_27
   ```

   As an example, we compute the NAF of 27 by this transducer.

   ```sage
   sage: binary_27 = 27.bits()
   sage: binary_27
   [1, 1, 0, 1, 1]
   sage: NAF_27 = NAF(binary_27)
   sage: NAF_27
   ```
Next, we are only interested in the Hamming weight:

```python
sage: def weight(state, input):
    ....:     if input is None:
    ....:         result = 0
    ....:     else:
    ....:         result = ZZ(input != 0)
    ....:     return (0, result)
sage: weight_transducer = Transducer(weight,
    ....:         input_alphabet=[-1, 0, 1],
    ....:         initial_states=[0],
    ....:         final_states=[0])
sage: NAFweight = weight_transducer.composition(NAF)
sage: NAFweight.transitions()
[Transition from (0, 0) to (0, 0): 0|0,
 Transition from (0, 0) to ('.1', 0): 1|-, 
 Transition from ('.1', 0) to (0, 0): 0|1,0,
 Transition from ('.1', 0) to (1, 0): 1|1,0,
 Transition from (1, 0) to ('.1', 0): 0|-, 
 Transition from (1, 0) to (1, 0): 1|0]
sage: NAFweight(binary_27)
[1, 0, 1, 0, 0, 1, 0]
```

Now, we actually compute the asymptotic moments:

```python
sage: moments = NAFweight.asymptotic_moments() # optional -
    
# symbolic
sage: moments['expectation'] # optional -
    
1/3*n + Order(1)

# symbolic
sage: moments['variance'] # optional -
    
2/27*n + Order(1)

# symbolic
sage: moments['covariance'] # optional -
    
Order(1)
```

3. This is Example 3.16 in [HKW2015], where a transducer with variable output labels is given. There, the aim was to choose the output labels of this very simple transducer such that the input and output sum are asymptotically independent, i.e., the constant $c$ vanishes.

```python
sage: var('a_1, a_2, a_3, a_4') # optional -
    
(a_1, a_2, a_3, a_4)
sage: T = Transducer([[0, 0, 0, a_1], [0, 1, 1, a_3],
    ....:      [1, 0, 0, a_4], [1, 1, 1, a_2]],
    ....:      initial_states=[0], final_states=[0, 1])
sage: moments = T.asymptotic_moments() # optional -
```
Non-integer output weights lead to significant performance degradation.

\[
\text{sage: moments['expectation']} \quad \text{# optional -}
\]
\[
\frac{1}{4}(a_1 + a_2 + a_3 + a_4)n + \text{Order}(1)
\]
\[
\text{sage: moments['covariance']} \quad \text{# optional -}
\]
\[
-\frac{1}{4}(a_1 - a_2)n + \text{Order}(1)
\]

Therefore, the asymptotic covariance vanishes if and only if \(a_2 = a_1\).

4. This is Example 4.3 in [HKW2015], dealing with the transducer converting the binary expansion of an integer into Gray code (cf. the Wikipedia article Gray_code and the example on Gray code):

\[
\text{sage: moments = transducers.GrayCode().asymptotic_moments()} \quad \text{# optional -}
\]
\[
\text{sage: moments['expectation']} \quad \text{# optional -}
\]
\[
\frac{1}{2}n + \text{Order}(1)
\]
\[
\text{sage: moments['variance']} \quad \text{# optional -}
\]
\[
\frac{1}{4}n + \text{Order}(1)
\]
\[
\text{sage: moments['covariance']} \quad \text{# optional -}
\]
\[
\text{Order}(1)
\]

5. This is the first part of Example 4.4 in [HKW2015], counting the number of 10 blocks in the standard binary expansion. The least significant digit is at the left-most position:

\[
\text{sage: block10 = transducers.CountSubblockOccurrences(}
\text{....:} \quad [1, 0],
\text{....:} \quad \text{input_alphabet=[0, 1])}
\]
\[
\text{sage: sorted(block10.transitions())}
\]
\[
\text{[Transition from () to (): 0|0,}
\text{Transition from () to (1,): 1|0,}
\text{Transition from (1,) to (): 0|1,}
\text{Transition from (1,) to (1,): 1|0]}
\]
\[
\text{sage: moments = block10.asymptotic_moments()} \quad \text{# optional -}
\]
\[
\text{sage: moments['expectation']} \quad \text{# optional -}
\]
\[
\frac{1}{4}n + \text{Order}(1)
\]
\[
\text{sage: moments['variance']} \quad \text{# optional -}
\]
\[
\frac{1}{16}n + \text{Order}(1)
\]
\[
\text{sage: moments['covariance']} \quad \text{# optional -}
\]
\[
\text{Order}(1)
\]

6. This is the second part of Example 4.4 in [HKW2015], counting the number of 11 blocks in the standard binary expansion. The least significant digit is at the left-most position:
sage: block11 = transducers.CountSubblockOccurrences(  ....:     [1, 1],  ....:     input_alphabet=[0, 1])
sage: sorted(block11.transitions())
[Transition from () to (): 0|0,
 Transition from () to (1,): 1|0,
 Transition from (1,) to (): 0|0,
 Transition from (1,) to (1,): 1|1]
sage: var('N')
# optional - sage.symbolic
N
sage: moments = block11.asymptotic_moments(N)  # optional - sage.symbolic
sage: moments['expectation']  # optional - sage.symbolic
1/4*N + Order(1)
sage: moments['variance']  # optional - sage.symbolic
5/16*N + Order(1)
sage: correlation = (moments['covariance'].coefficient(N) /  # optional - sage.symbolic
....:     (1/2 * sqrt(moments['variance'].coefficient(N))))
sage: correlation  # optional - sage.symbolic
2/5*sqrt(5)

7. This is Example 4.5 in [HKW2015], counting the number of 01 blocks minus the number of 10 blocks in the standard binary expansion. The least significant digit is at the left-most position:

sage: block01 = transducers.CountSubblockOccurrences(  ....:     [0, 1],  ....:     input_alphabet=[0, 1])
sage: product_01x10 = block01.cartesian_product(block10)
sage: block_difference = transducers.sub([0, 1])(product_01x10)
sage: T = block_difference.simplification().relabeled()
sage: T.transitions()
[Transition from 0 to 2: 0|-1,
 Transition from 0 to 0: 1|0,
 Transition from 1 to 2: 0|0,
 Transition from 1 to 0: 1|0,
 Transition from 2 to 2: 0|0,
 Transition from 2 to 0: 1|1]
sage: moments = T.asymptotic_moments()  # optional - sage.symbolic
sage: moments['expectation']  # optional - sage.symbolic
Order(1)
sage: moments['variance']  # optional - sage.symbolic
Order(1)
sage: moments['covariance']  # optional - sage.symbolic
Order(1)
8. The finite state machine must have a unique final component:

```python
sage: T = Transducer([(0, -1, -1, -1), (0, 1, 1, 1),
       ...: (-1, -1, -1, -1), (-1, -1, 1, -1),
       ...: (1, 1, -1, 1), (1, 1, 1, 1)],
       ...: initial_states=[0],
       ...: final_states=[0, 1, -1])
sage: T.asymptotic_moments()
Traceback (most recent call last):
...:
NotImplementedError: asymptotic_moments is only implemented for finite state machines with one final component.
```

In this particular example, the first letter of the input decides whether we reach the loop at $-1$ or the loop at $1$. In the first case, we have $X_n = -n$, while we have $X_n = n$ in the second case. Therefore, the expectation $E(X_n)$ of $X_n$ is $E(X_n) = 0$. We get $(X_n - E(X_n))^2 = n^2$ in all cases, which results in a variance of $n^2$.

So this example shows that the variance may be non-linear if there is more than one final component.

**ALGORITHM:**

See [HKW2015], Theorem 3.9.

**REFERENCES:**

**coaccessible_components()**

Return the sub-machine induced by the coaccessible states of this finite state machine.

**OUTPUT:**

A finite state machine of the same type as this finite state machine.

**EXAMPLES:**

```python
sage: A = automata.ContainsWord([1, 1],
       ...: input_alphabet=[0, 1]).complement().minimization().relabeled()
sage: A.transitions()
[Transition from 0 to 0: 0|-,
    Transition from 0 to 0: 1|-,
    Transition from 1 to 2: 0|-,
    Transition from 1 to 0: 1|-,
    Transition from 2 to 2: 0|-,
    Transition from 2 to 1: 1|-]
sage: A.initial_states()
[2]
sage: A.final_states()
[1, 2]
sage: C = A.coaccessible_components()
sage: C.transitions()
[Transition from 1 to 2: 0|-,
    Transition from 2 to 2: 0|-,
    Transition from 2 to 1: 1|-]
```

See also:

*accessible_components(), induced_sub_finite_state_machine()*
completion(sink=None)

Return a completion of this finite state machine.

INPUT:

• sink – either an instance of FSMState or a label for the sink (default: None). If None, the least
  available non-zero integer is used.

OUTPUT:

A FiniteStateMachine of the same type as this finite state machine.

The resulting finite state machine is a complete version of this finite state machine. A finite state machine is
considered to be complete if each transition has an input label of length one and for each pair \((q, a)\) where
\(q\) is a state and \(a\) is an element of the input alphabet, there is exactly one transition from \(q\) with input label
\(a\).

If this finite state machine is already complete, a deep copy is returned. Otherwise, a new non-final state
(usually called a sink) is created and transitions to this sink are introduced as appropriate.

EXAMPLES:

```sage
sage: F = FiniteStateMachine([(0, 0, 0, 0), ....: (0, 1, 1, 1), ....: (1, 1, 0, 0)])
sage: F.is_complete()
False
sage: G1 = Fcompletion()
sage: G1.is_complete()
True
sage: G1.transitions()
[Transition from 0 to 0: 0|0,
  Transition from 0 to 1: 1|1,
  Transition from 1 to 1: 0|0,
  Transition from 1 to 2: 1|-,
  Transition from 2 to 2: 0|-,
  Transition from 2 to 2: 1|-]
sage: G2 = F.completion('Sink')
sage: G2.is_complete()
True
sage: G2.transitions()
[Transition from 0 to 0: 0|0,
  Transition from 0 to 1: 1|1,
  Transition from 1 to 1: 0|0,
  Transition from 1 to 'Sink': 1|-, 
  Transition from 'Sink' to 'Sink': 0|-,
  Transition from 'Sink' to 'Sink': 1|-]
sage: F.completion(1)
Traceback (most recent call last):
...
ValueError: The finite state machine already contains a state '1'.
```

An input alphabet must be given:

```sage
sage: F = FiniteStateMachine([(0, 0, 0, 0), ....: (0, 1, 1, 1),
(continues on next page)```
Non-deterministic machines are not allowed.

```
sage: F = FiniteStateMachine([(0, 0, 0, 0), (0, 1, 0, 0)])
sage: F.is_complete()  # False
sage: F.completion()  # Traceback (most recent call last):
...
ValueError: The finite state machine must be deterministic.
```

See also:

- `is_complete()`, `split_transitions()`, `determine_alphabets()`, `is_deterministic()`.
- `composition(other, algorithm=None, only_accessible_components=True)`

Returns a new transducer which is the composition of `self` and `other`.

**INPUT:**

- `other` – a transducer
- `algorithm` – can be one of the following
  - `direct` – The composition is calculated directly.
    There can be arbitrarily many initial and final states, but the input and output labels must have length 1.

  **Warning:** The output of `other` is fed into `self`.

- `explorative` – An explorative algorithm is used.
  The input alphabet of `self` has to be specified.

  **Warning:** The output of `other` is fed into `self`.

If `algorithm` is `None`, then the algorithm is chosen automatically (at the moment always `direct`, except when there are output words of `other` or input words of `self` of length greater than 1).
A new transducer.

The labels of the new finite state machine are pairs of states of the original finite state machines. The color of a new state is the tuple of colors of the constituent states.

**EXAMPLES:**

```python
sage: F = Transducer([('A', 'B', 1, 0), ('B', 'A', 0, 1)],
   ...:       initial_states=['A', 'B'], final_states=['B'],
   ...:       determine_alphabets=True)
```

```python
sage: G = Transducer([(1, 1, 0, 1), (1, 2, 0, 1),
   ...:       (2, 2, 1, 1), (2, 2, 0, 0)],
   ...:       initial_states=[1], final_states=[2],
   ...:       determine_alphabets=True)
```

```python
sage: Hd = F.composition(G, algorithm='direct')
```

```python
sage: Hd.initial_states()
[(1, 'B'), (1, 'A')]
```

```python
sage: Hd.transitions()
[Transition from (1, 'B') to (1, 'A'): 1|1,
 Transition from (1, 'A') to (2, 'B'): 0|0,
 Transition from (2, 'B') to (2, 'A'): 0|1,
 Transition from (2, 'A') to (2, 'B'): 1|0]
```

```python
sage: He = F.composition(G, algorithm='explorative')
```

```python
sage: He.initial_states()
[(1, 'A'), (1, 'B')]
```

```python
sage: He.transitions()
[Transition from (1, 'A') to (2, 'B'): 0|0,
 Transition from (1, 'B') to (1, 'A'): 1|1,
 Transition from (2, 'B') to (2, 'A'): 0|1,
 Transition from (2, 'A') to (2, 'B'): 1|0]
```

The following example has output of length $> 1$, so the explorative algorithm has to be used (and is selected automatically).

```python
sage: F = Transducer([('A', 'B', 1, [1, 0]), ('B', 'B', 1, 1),
   ...:       ('B', 'B', 0, 0)],
   ...:       initial_states=['A'], final_states=['B'])
```

```python
sage: G = Transducer([(1, 1, 0, 0), (1, 2, 1, 0),
   ...:       (2, 2, 0, 1), (2, 1, 1, 1)],
   ...:       initial_states=[1], final_states=[1])
```

```python
sage: He = G.composition(F, algorithm='explorative')
```

```python
sage: He.transitions()
[Transition from ('A', 1) to ('B', 2): 1|0,1,
 Transition from ('B', 2) to ('B', 2): 0|1,
 Transition from ('B', 2) to ('B', 1): 1|1,
 Transition from ('B', 1) to ('B', 1): 0|0,
 Transition from ('B', 1) to ('B', 2): 1|0]
```

```python
sage: Ha = G.composition(F)
```

```python
sage: Ha == He
True
```

Final output words are also considered:
sage: F = Transducer([('A', 'B', 1, 0), ('B', 'A', 0, 1)],
    initial_states=['A', 'B'],
    final_states=['A', 'B'])
sage: F.state('A').final_word_out = 0
sage: F.state('B').final_word_out = 1
sage: G = Transducer([(1, 1, 1, 0), (1, 2, 0, 1),
    (2, 2, 1, 1), (2, 2, 0, 0)],
    initial_states=[1], final_states=[2])
sage: G.state(2).final_word_out = 0
sage: Hd = F.composition(G, algorithm='direct')
sage: Hd.final_states()
[(2, 'B')]
sage: He = F.composition(G, algorithm='explorative')
sage: He.final_states()
[(2, 'B')]

Note that (2, 'A') is not final, as the final output 0 of state 2 of G cannot be processed in state 'A' of F.

sage: [s.final_word_out for s in Hd.final_states()]
[[1, 0]]
sage: [s.final_word_out for s in He.final_states()]
[[1, 0]]
sage: Hd == He
True

Here is a non-deterministic example with intermediate output length > 1.

sage: F = Transducer([(1, 1, 1, ['a', 'a']), (1, 2, 1, 'b')],
    initial_states=[1, 2],
    final_states=[1])
sage: F.transitions()[0].word_out is F.transitions()[1].word_out
False
sage: G = Transducer([('C', 'C', 0, 1)],
    initial_states=['C'],
    final_states=['C'])

Be aware that after composition, different transitions may share the same output label (same python object):

sage: F = Transducer([('A', 'B', 0, 0), ('B', 'A', 0, 0)],
    initial_states=['A'],
    final_states=['A'])

(continues on next page)
concatenation(other)

Concatenate this finite state machine with another finite state machine.

INPUT:

• other – a FiniteStateMachine.

OUTPUT:

A FiniteStateMachine of the same type as this finite state machine.

Assume that both finite state machines are automata. If \( \mathcal{L}_1 \) is the language accepted by this automaton and \( \mathcal{L}_2 \) is the language accepted by the other automaton, then the language accepted by the concatenated automaton is \( \{w_1w_2 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2\} \) where \( w_1w_2 \) denotes the concatenation of the words \( w_1 \) and \( w_2 \).

Assume that both finite state machines are transducers and that this transducer maps words \( w_1 \in \mathcal{L}_1 \) to words \( f_1(w_1) \) and that the other transducer maps words \( w_2 \in \mathcal{L}_2 \) to words \( f_2(w_2) \). Then the concatenated transducer maps words \( w_1w_2 \) with \( w_1 \in \mathcal{L}_1 \) and \( w_2 \in \mathcal{L}_2 \) to \( f_1(w_1)f_2(w_2) \). Here, \( w_1w_2 \) and \( f_1(w_1)f_2(w_2) \) again denote concatenation of words.

The input alphabet is the union of the input alphabets (if possible) and None otherwise. In the latter case, try calling determine_alphabets().

Instead of \( A \cdot B \), the notation \( A * B \) can be used.

EXAMPLES:

Concatenation of two automata:

\[
\text{sage: } A = \text{automata.Word([0])}
\]
\[
\text{sage: } B = \text{automata.Word([1])}
\]
\[
\text{sage: } C = A\\text{.concatenation}(B)
\]
\[
\text{sage: } C\text{.transitions()}
\]
\[
[\text{Transition from (0, 0) to (0, 1): 0|-},
\text{Transition from (0, 1) to (1, 0): -|},
\text{Transition from (1, 0) to (1, 1): 1|}]
\]
\[
\text{sage: } [w\text{ for } w \text{ in ([0, 0], [0, 1], [1, 0], [1, 1])}
\text{ if } C(w)]
\]
\[
[[0, 1]]
\]
\[
\text{sage: } \text{from sage.combinat.finite_state_machine import (}
\text{ is_Automaton, is_Transducer)}
\]
\[
\text{sage: } \text{is_Automaton(C)}
\]
\[
\text{True}
\]

Concatenation of two transducers:

\[
\text{sage: } A = \text{Transducer([[0, 1, 0, 1], [0, 1, 1, 2]],}
\text{ initial_states=[0]},
\text{ final_states=[1]})
\]
\[
\text{sage: } B = \text{Transducer([[0, 1, 0, 1], [0, 1, 1, 0]],}
\text{ initial_states=[0]},
\text{ final_states=[1]})
\]

(continues on next page)
sage: C = A.concatenation(B)
sage: C.transitions()
[Transition from (0, 0) to (0, 1): 0|1,
 Transition from (0, 0) to (0, 1): 1|2,
 Transition from (0, 1) to (1, 0): -|-,
 Transition from (1, 0) to (1, 1): 0|1,
 Transition from (1, 0) to (1, 1): 1|0]
sage: [(w, C(w)) for w in ([0, 0], [0, 1], [1, 0], [1, 1])]
[[[0, 0], [1, 1]],
 ([0, 1], [1, 0]),
 ([1, 0], [2, 1]),
 ([1, 1], [2, 0])]
sage: is_Transducer(C)
True

Alternative notation as multiplication:
sage: C == A * B
True

Final output words are taken into account:
sage: A = Transducer([[0, 1, 0, 1]],
....: initial_states=[0],
....: final_states=[1])
sage: A.state(1).final_word_out = 2
sage: B = Transducer([[0, 1, 0, 3]],
....: initial_states=[0],
....: final_states=[1])
sage: B.state(1).final_word_out = 4
sage: C = A * B
sage: C([[0, 0]])
[1, 2, 3, 4]

Handling of the input alphabet:
sage: A = Automaton([[0, 0, 0]])
sage: B = Automaton([[0, 0, 1]], input_alphabet=[1, 2])
sage: C = Automaton([[0, 0, 2]], determine_alphabets=False)
sage: D = Automaton([[0, 0, [0, 0]]], input_alphabet=[[0, 0]])
sage: A.input_alphabet
[0]
sage: B.input_alphabet
[1, 2]
sage: C.input_alphabet is None
True
sage: D.input_alphabet
[[0, 0]]
sage: (A * B).input_alphabet
[0, 1, 2]
sage: (A * C).input_alphabet is None
True
sage: (A * D).input_alphabet is None
True

See also:

\texttt{disjoint_union()}, \texttt{determine_alphabets()}.

\textbf{construct_final_word_out}(\texttt{letters, allow_non_final=True})

This is an inplace version of \texttt{with_final_word_out()}. See \texttt{with_final_word_out()} for documentation and examples.

\textbf{copy()}

Return a (shallow) copy of the finite state machine.

OUTPUT:

A new finite state machine.

\textbf{deepcopy}(\texttt{memo=None})

Return a deep copy of the finite state machine.

INPUT:

\begin{itemize}
\item memo – (default: \texttt{None}) a dictionary storing already processed elements.
\end{itemize}

OUTPUT:

A new finite state machine.

EXAMPLES:

```python
sage: F = FiniteStateMachine([(\texttt{\'A\', \texttt{\'A\'}}, 0, 1), (\texttt{\'A\', \texttt{\'A\'}}, 1, 0)])
sage: deepcopy(F)
Finite state machine with 1 state
```

\textbf{default_format_letter} = <\texttt{sage.misc.latex.Latex object}>

\textbf{default_format_transition_label}(\texttt{word})

Default formatting of words in transition labels for LaTeX output.

INPUT:

\texttt{word} – list of letters

OUTPUT:

String representation of \texttt{word} suitable to be typeset in mathematical mode.

- For a non-empty word: Concatenation of the letters, piped through \texttt{self.format_letter} and separated by blanks.
- For an empty word: \texttt{\texttt{sage.combinat.finite_state_machine.EmptyWordLaTeX}}.

There is also a variant \texttt{format_transition_label_reversed()} writing the words in reversed order.

EXAMPLES:

1. Example of a non-empty word:
2. In the example above, 'a' and 'alpha' should perhaps be symbols:

```python
sage: var('a alpha a_1')
(a, alpha, a_1)
sage: print(T.default_format_transition_label(['a', 'alpha', 'a_1', '0', 0, (0, 1)]))
a \alpha a_{1} 0 0 \left(0, 1\right)
```

3. Example of an empty word:

```python
sage: print(T.default_format_transition_label([]))
\varepsilon
```

We can change this by setting `sage.combinat.finite_state_machine.EmptyWordLaTeX`:

```python
sage: sage.combinat.finite_state_machine.EmptyWordLaTeX = ''
sage: T.default_format_transition_label([])
''
```

Finally, we restore the default value:

```python
sage: sage.combinat.finite_state_machine.EmptyWordLaTeX = r'\varepsilon'
```

4. This method is the default value for `FiniteStateMachinem.group_transition_label`. That can be changed to be any other function:

```python
sage: A = Automaton([(0, 1, 0)])
sage: def custom_format_transition_label(word):
....:     return "t"
sage: A.latex_options(format_transition_label=custom_format_transition_label)
sage: print(latex(A))
\begin{tikzpicture}[auto, initial text=, >=latex]
\node[state] (v0) at (3.000000, 0.000000) {$0$};
\node[state] (v1) at (-3.000000, 0.000000) {$1$};
\path[->] (v0) edge node[rotate=360.00, anchor=south] {$t$} (v1);
\end{tikzpicture}
```

### delete_state(s)

Deletes a state and all transitions coming or going to this state.

**INPUT:**

- s – a label of a state or an `FSMState`.

**OUTPUT:**

Nothing.

**EXAMPLES:**
```python
sage: from sage.combinat.finite_state_machine import FSMTransition
sage: t1 = FSMTransition('A', 'B', 0)
sage: t2 = FSMTransition('B', 'B', 1)
sage: F = FiniteStateMachine([t1, t2])
sage: F.delete_state('A')
sage: F.transitions()
[Transition from 'B' to 'B': 1|-
```

**delete_transition(t)**

Deletes a transition by removing it from the list of transitions of the state, where the transition starts.

**INPUT:**
- `t` – a transition.

**OUTPUT:**
Nothing.

**EXAMPLES:**

```python
sage: F = FiniteStateMachine([('A', 'B', 0), ('B', 'A', 1)])
sage: F.delete_transition(('A', 'B', 0))
sage: F.transitions()
[Transition from 'B' to 'A': 1|-
```

**determine_alphabets(reset=True)**

Determine the input and output alphabet according to the transitions in this finite state machine.

**INPUT:**
- `reset` – If reset is True, then the existing input and output alphabets are erased, otherwise new letters are appended to the existing alphabets.

**OUTPUT:**
Nothing.

After this operation the input alphabet and the output alphabet of this finite state machine are a list of letters.

**Todo:** At the moment, the letters of the alphabets need to be hashable.

**EXAMPLES:**

```python
sage: T = Transducer([(1, 1, 1, 0), (1, 2, 2, 1), ....: (2, 2, 1, 1), (2, 2, 0, 0)], ....: final_states=[1], ....: determine_alphabets=False)
sage: T.state(1).final_word_out = [1, 4]
sage: (T.input_alphabet, T.output_alphabet)
(None, None)
sage: T.determine_alphabets()
sage: (T.input_alphabet, T.output_alphabet)
([0, 1, 2], [0, 1, 4])
```

See also: `determine_input_alphabet()`, `determine_output_alphabet()`.
**determine_input_alphabet**(reset=True)

Determine the input alphabet according to the transitions of this finite state machine.

**INPUT:**

- reset – a boolean (default: True). If True, then the existing input alphabet is erased, otherwise new letters are appended to the existing alphabet.

**OUTPUT:**

Nothing.

After this operation the input alphabet of this finite state machine is a list of letters.

**Todo:** At the moment, the letters of the alphabet need to be hashable.

**EXAMPLES:**

```
sage: T = Transducer([(1, 1, 1, 0), (1, 2, 2, 1),
    ....: (2, 2, 1, 1), (2, 2, 0, 0)],
    ....: final_states=[1],
    ....: determine_alphabets=False)
sage: (T.input_alphabet, T.output_alphabet)
(None, None)
sage: T.determine_input_alphabet()
sage: (T.input_alphabet, T.output_alphabet)
([0, 1, 2], None)
```

See also:

*determine_output_alphabet*, *determine_alphabets*.

**determine_output_alphabet**(reset=True)

Determine the output alphabet according to the transitions of this finite state machine.

**INPUT:**

- reset – a boolean (default: True). If True, then the existing output alphabet is erased, otherwise new letters are appended to the existing alphabet.

**OUTPUT:**

Nothing.

After this operation the output alphabet of this finite state machine is a list of letters.

**Todo:** At the moment, the letters of the alphabet need to be hashable.

**EXAMPLES:**

```
sage: T = Transducer([(1, 1, 1, 0), (1, 2, 2, 1),
    ....: (2, 2, 1, 1), (2, 2, 0, 0)],
    ....: final_states=[1],
    ....: determine_alphabets=False)
sage: T.state(1).final_word_out = [1, 4]
sage: (T.input_alphabet, T.output_alphabet)
(None, None)
```

(continues on next page)
sage: T.determine_output_alphabet()
sage: (T.input_alphabet, T.output_alphabet)
(None, [0, 1, 4])

See also:
determine_input_alphabet(), determine_alphabets().

digraph(edge_labels='words_in_out')
Return the graph of the finite state machine with labeled vertices and labeled edges.

INPUT:

- **edge_label**: (default: 'words_in_out') can be
  - 'words_in_out' (labels will be strings 'i|o')
  - a function with which takes as input a transition and outputs (returns) the label

OUTPUT:
A directed graph.

EXAMPLES:

sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A')
sage: T = Transducer()
sage: T.graph()
Looped multi-digraph on 0 vertices
sage: T.add_state(A)
'A'
sage: T.graph()
Looped multi-digraph on 1 vertex
sage: T.add_transition(('A', 'A', 0, 1))
Transition from 'A' to 'A': 0|1
sage: T.graph()
Looped multi-digraph on 1 vertex

See also:
DiGraph
disjoint_union(other)
Return the disjoint union of this and another finite state machine.

INPUT:

- **other** – a FiniteStateMachine.

OUTPUT:
A finite state machine of the same type as this finite state machine.

In general, the disjoint union of two finite state machines is non-deterministic. In the case of a automata, the language accepted by the disjoint union is the union of the languages accepted by the constituent automata. In the case of transducers, for each successful path in one of the constituent transducers, there will be one successful path with the same input and output labels in the disjoint union.
The labels of the states of the disjoint union are pairs \((i, s)\): for each state \(s\) of this finite state machine, there is a state \((0, s)\) in the disjoint union; for each state \(s\) of the other finite state machine, there is a state \((1, s)\) in the disjoint union.

The input alphabet is the union of the input alphabets (if possible) and None otherwise. In the latter case, try calling \texttt{determine_alphabets()}.

The disjoint union can also be written as \(A + B\) or \(A | B\).

EXAMPLES:

```
sage: A = Automaton([[(0, 1, 0), (1, 0, 1)],
                   ....:   initial_states=[0],
                   ....:   final_states=[0])
sage: A([0, 1, 0, 1])
True
sage: B = Automaton([[(0, 1, 0), (1, 2, 0), (2, 0, 1)],
                   ....:   initial_states=[0],
                   ....:   final_states=[0])
sage: B([0, 0, 1])
True
sage: C = A.disjoint_union(B)
sage: C
Automaton with 5 states
sage: C.transitions()
[Transition from (0, 0) to (0, 1): 0|-,
 Transition from (0, 1) to (0, 0): 1|-,
 Transition from (1, 0) to (1, 1): 0|-,
 Transition from (1, 1) to (1, 2): 0|-,
 Transition from (1, 2) to (1, 0): 1|-]
sage: C([0, 0, 1])
True
sage: C([0, 1, 0, 1])
True
sage: C([1])
False
sage: C.initial_states()
[(0, 0), (1, 0)]
```

Instead of \texttt{.disjoint_union}, alternative notations are available:

```
sage: C1 = A + B
sage: C1 == C
True
sage: C2 = A | B
sage: C2 == C
True
```

In general, the disjoint union is not deterministic:

```
sage: C.is_deterministic()
False
sage: D = C.determinisation().minimization()
sage: D.is_equivalent(Automaton([[(0, 0, 0), (0, 0, 1),
                               ....:   (1, 7, 0), (1, 0, 1), (2, 6, 0), (2, 0, 1),
                               ....:   (continues on next page)]})
```
....: (3, 5, 0), (3, 0, 1), (4, 0, 0), (4, 2, 1),
....: (5, 0, 0), (5, 3, 1), (6, 4, 0), (6, 0, 1),
....: (7, 4, 0), (7, 3, 1)],
....: initial_states=[1],
....: final_states=[1, 2, 3))
True

Disjoint union of transducers:

```python
sage: T1 = Transducer([(0, 0, 0, 1)],
....: initial_states=[0],
....: final_states=[0])
sage: T2 = Transducer([(0, 0, 0, 2)],
....: initial_states=[0],
....: final_states=[0])
```

```
sage: T = T1.disjoint_union(T2)
sage: T([0])
Traceback (most recent call last):
  ...
ValueError: Found more than one accepting path.
sage: T.process([0])
[(True, (0, 0), [1]), (True, (1, 0), [2])]
```

Handling of the input alphabet (see github issue #18989):

```python
sage: A = Automaton([(0, 0, 0)])
```

```
sage: B = Automaton([(0, 0, 1)], input_alphabet=[1, 2])
sage: C = Automaton([(0, 0, 2)], determine_alphabets=False)
sage: D = Automaton([(0, 0, [[0, 0]]), input_alphabet=[[0, 0]]]
```

```
sage: A.input_alphabet
[0]
sage: B.input_alphabet
[1, 2]
sage: C.input_alphabet is None
True
sage: D.input_alphabet
[[0, 0]]
sage: (A + B).input_alphabet
[0, 1, 2]
sage: (A + C).input_alphabet is None
True
sage: (A + D).input_alphabet is None
True
```

See also:

`Automaton.intersection()`, `Transducer.intersection()`, `determine_alphabets()`.

**empty_copy**(memo=None, new_class=None)

Return an empty deep copy of the finite state machine, i.e., input_alphabet, output_alphabet,
on_duplicate_transition are preserved, but states and transitions are not.

INPUT:

• memo – a dictionary storing already processed elements.
• new_class – a class for the copy. By default (None), the class of self is used.

OUTPUT:

A new finite state machine.

EXAMPLES:

```python
sage: from sage.combinat.finite_state_machine import duplicate_transition_raise_error
sage: F = FiniteStateMachine([("A", "A", 0, 2), ("A", "A", 1, 3)],
.....: input_alphabet=[0, 1],
.....: output_alphabet=[2, 3],
.....: on_duplicate_transition=duplicate_transition_raise_error)
sage: FE = F.empty_copy(); FE
Empty finite state machine
sage: FE.input_alphabet
[0, 1]
sage: FE.output_alphabet
[2, 3]
sage: FE.on_duplicate_transition == duplicate_transition_raise_error
True
```

epsilon_successors(state)

Return the dictionary with states reachable from state without reading anything from an input tape as keys. The values are lists of outputs.

INPUT:

• state – the state whose epsilon successors should be determined.

OUTPUT:

A dictionary mapping states to a list of output words.

The states in the output are the epsilon successors of state. Each word of the list of output words is a word written when taking a path from state to the corresponding state.

EXAMPLES:

```python
sage: T = Transducer([(0, 1, None, 'a'), (1, 2, None, 'b')])
sage: T.epsilon_successors(0)
{1: ["a"], 2: ["a", "b"]}
sage: T.epsilon_successors(1)
{2: ["b"]}
sage: T.epsilon_successors(2)
{}
```

If there is a cycle with only epsilon transitions, then this cycle is only processed once and there is no infinite loop:
Combinatorics, Release 10.0

sage: S = Transducer([(0, 1, None, 'a'), (1, 0, None, 'b')])
sage: S.epsilon_successors(0)
{0: [['a', 'b']], 1: [['a']]}
sage: S.epsilon_successors(1)
{0: [['b']], 1: [['b', 'a']]}
equivalence_classes()
Return a list of equivalence classes of states.
OUTPUT:
A list of equivalence classes of states.
Two states 𝑎 and 𝑏 are equivalent if and only if there is a bijection 𝜙 between paths starting at 𝑎 and paths
starting at 𝑏 with the following properties: Let 𝑝𝑎 be a path from 𝑎 to 𝑎′ and 𝑝𝑏 a path from 𝑏 to 𝑏′ such that
𝜙(𝑝𝑎 ) = 𝑝𝑏 , then
• 𝑝𝑎 .word in = 𝑝𝑏 .word in ,
• 𝑝𝑎 .word out = 𝑝𝑏 .word out ,
• 𝑎′ and 𝑏′ have the same output label, and
• 𝑎′ and 𝑏′ are both final or both non-final and have the same final output word.
The function equivalence_classes() returns a list of the equivalence classes to this equivalence relation.
This is one step of Moore’s minimization algorithm.
See also:
minimization()
EXAMPLES:
sage: fsm = FiniteStateMachine([("A", "B",
....:
("B", "C",
....:
("C", "D",
....:
("D", "A",
sage: sorted(fsm.equivalence_classes())
[['A', 'C'], ['B', 'D']]
sage: fsm.state("A").is_final = True
sage: sorted(fsm.equivalence_classes())
[['A'], ['B'], ['C'], ['D']]
sage: fsm.state("C").is_final = True
sage: sorted(fsm.equivalence_classes())
[['A', 'C'], ['B', 'D']]
sage: fsm.state("A").final_word_out = 1
sage: sorted(fsm.equivalence_classes())
[['A'], ['B'], ['C'], ['D']]
sage: fsm.state("C").final_word_out = 1
sage: sorted(fsm.equivalence_classes())
[['A', 'C'], ['B', 'D']]

0,
0,
0,
0,

1),
0),
1),
0),

("A",
("B",
("C",
("D",

"B",
"C",
"D",
"A",

1,
1,
1,
1,

0),
1),
0),
1)])

final_components()
Return the final components of a finite state machine as finite state machines.
OUTPUT:

5.1. Comprehensive Module List

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A list of finite state machines, each representing a final component of `self`.

A final component of a transducer $T$ is a strongly connected component $C$ such that there are no transitions of $T$ leaving $C$.

The final components are the only parts of a transducer which influence the main terms of the asymptotic behaviour of the sum of output labels of a transducer, see [HKP2015] and [HKW2015].

**EXAMPLES:**

```python
sage: T = Transducer([["A", "B", 0, 0], ["B", "C", 0, 1],
                   ....: ["C", "B", 0, 1], ["A", "D", 1, 0],
                   ....: ["D", "D", 0, 0], ["D", "B", 1, 0],
                   ....: ["A", "E", 2, 0], ["E", "E", 0, 0]])

sage: FC = T.final_components()

sage: sorted(FC[0].transitions())
[Transition from 'B' to 'C': 0|1,
 Transition from 'C' to 'B': 0|1]

sage: FC[1].transitions()
[Transition from 'E' to 'E': 0|0]
```

Another example (cycle of length 2):

```python
sage: T = Automaton([[0, 1, 0], [1, 0, 0]])

sage: len(T.final_components()) == 1
True

sage: T.final_components()[0].transitions()
[Transition from 0 to 1: 0|--,
 Transition from 1 to 0: 0|--]
```

**final_states()**

Return a list of all final states.

**OUTPUT:**

A list of all final states.

**EXAMPLES:**

```python
sage: from sage.combinat.finite_state_machine import FSMState

sage: A = FSMState("A", is_final=True)

sage: B = FSMState("B", is_initial=True)

sage: C = FSMState("C", is_final=True)

sage: F = FiniteStateMachine(((A, B), (A, C)))

sage: F.final_states()
['A', 'C']
```

**format_letter = <sage.misc.latex.Latex object>**

**format_letter_negative(letter)**

Format negative numbers as overlined numbers, everything else by standard LaTeX formatting.

**INPUT:**

letter – anything.

**OUTPUT:**

Overlined absolute value if letter is a negative integer, `latex(letter)` otherwise.
EXAMPLES:

```python
sage: A = Automaton([(0, 0, -1)])
sage: list(map(A.format_letter_negative, [-1, 0, 1, 'a', None]))
['\overline{1}', 0, 1, \text{\texttt{a}}, \mathrm{None}]
sage: A.latex_options(format_letter=A.format_letter_negative)
sage: print(latex(A))
\begin{tikzpicture}[auto, initial text=, >=latex]
  \node[state] (v0) at (3.000000, 0.000000) {$0$};
  \path[->] (v0) edge[loop above] node {$\overline{1}$} ();
\end{tikzpicture}
```

**format_transition_label**(word)

Default formatting of words in transition labels for LaTeX output.

**INPUT:**

word – list of letters

**OUTPUT:**

String representation of word suitable to be typeset in mathematical mode.

- For a non-empty word: Concatenation of the letters, piped through `self.format_letter` and separated by blanks.
- For an empty word: `sage.combinat.finite_state_machine.EmptyWordLaTeX`.

There is also a variant `format_transition_label_reversed()` writing the words in reversed order.

**EXAMPLES:**

1. Example of a non-empty word:

```python
sage: T = Transducer()
sage: print(T.default_format_transition_label([a, alpha, a_1]))
\alpha \ a_1 \ a
```

2. In the example above, 'a' and 'alpha' should perhaps be symbols:

```python
sage: var('a alpha a_1')  # optional - sage.symbolic
(a, alpha, a_1)
sage: print(T.default_format_transition_label([a, alpha, a_1]))  #optional - sage.symbolic
a \alpha a_{1}
```

3. Example of an empty word:

```python
sage: print(T.default_format_transition_label([]))
\varepsilon
```

We can change this by setting `sage.combinat.finite_state_machine.EmptyWordLaTeX`:

```python
sage: sage.combinat.finite_state_machine.EmptyWordLaTeX = ''
sage: T.default_format_transition_label([])
''
```
Finally, we restore the default value:

```
sage: sage.combinat.finite_state_machine.EmptyWordLaTeX = r'\varepsilon'
```

4. This method is the default value for `FiniteStateMachine.format_transition_label`. That can be changed to be any other function:

```
sage: A = Automaton([(0, 1, 0)])
sage: def custom_format_transition_label(word):
...:     return "t"
sage: A.latex_options(format_transition_label=custom_format_transition_label)
sage: print(latex(A))
```

\begin{tikzpicture}[auto, initial text=, >=latex]
\node[state] (v0) at (3.000000, 0.000000) {$0$};
\node[state] (v1) at (-3.000000, 0.000000) {$1$};
\path[->] (v0) edge node[rotate=360.00, anchor=south] {$t$} (v1);
\end{tikzpicture}

```python
format_transition_label_reversed(word)
```

Format words in transition labels in reversed order.

**INPUT:**

word – list of letters.

**OUTPUT:**

String representation of word suitable to be typeset in mathematical mode, letters are written in reversed order.

This is the reversed version of `default_format_transition_label()`.

In digit expansions, digits are frequently processed from the least significant to the most significant position, but it is customary to write the least significant digit at the right-most position. Therefore, the labels have to be reversed.

**EXAMPLES:**

```
sage: T = Transducer([(0, 0, 0, [1, 2, 3])])
sage: T.format_transition_label_reversed([1, 2, 3])
'3 2 1'
sage: T.latex_options(format_transition_label=T.format_transition_label_reversed)
sage: print(latex(T))
```

\begin{tikzpicture}[auto, initial text=, >=latex]
\node[state] (v0) at (3.000000, 0.000000) {$0$};
\node[state] (v1) at (-3.000000, 0.000000) {$1$};
\path[->] (v0) edge[loop above] node {$0\mid 3 2 1$} ();
\end{tikzpicture}

```
graph(edge_labels='words_in_out')
```

Return the graph of the finite state machine with labeled vertices and labeled edges.

**INPUT:**

- **edge_label:** (default: 'words_in_out') can be
  - 'words_in_out' (labels will be strings 'i|o')
  - a function with which takes as input a transition and outputs (returns) the label
OUTPUT:
A directed graph.

EXAMPLES:

```python
sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A')
sage: T = Transducer()
sage: T.graph()
Looped multi-digraph on 0 vertices
sage: T.add_state(A)
'A'
sage: T.graph()
Looped multi-digraph on 1 vertex
sage: T.add_transition(('A', 'A', 0, 1))
Transition from 'A' to 'A': 0|1
sage: T.graph()
Looped multi-digraph on 1 vertex
```

See also:

DiGraph

**has_final_state**

Return whether state is one of the final states of the finite state machine.

**INPUT:**

- state can be a FSMState or a label.

**OUTPUT:**

True or False.

**EXAMPLES:**

```python
sage: FiniteStateMachine(final_states=['A']).has_final_state('A')
True
```

**has_final_states**

Return whether the finite state machine has a final state.

**OUTPUT:**

True or False.

**EXAMPLES:**

```python
sage: FiniteStateMachine().has_final_states()
False
```

**has_initial_state**

Return whether state is one of the initial states of the finite state machine.

**INPUT:**

- state can be a FSMState or a label.

**OUTPUT:**

True or False.
EXAMPLES:

```
sage: F = FiniteStateMachine([('A', 'A')], initial_states=['A'])
sage: F.has_initial_state('A')
True
```

**has_initial_states()**
Return whether the finite state machine has an initial state.

INPUT:

- **state** can be a :class:`FSMState` or a label of a state.

OUTPUT:

True or False.

EXAMPLES:

```
sage: FiniteStateMachine().has_initial_states()
False
```

**has_state(state)**
Return whether state is one of the states of the finite state machine.

INPUT:

- **state** has to be a :class:`FSMState`.

OUTPUT:

True or False.

EXAMPLES:

```
sage: FiniteStateMachine().has_state('A')
False
```

**has_transition(transition)**
Return whether transition is one of the transitions of the finite state machine.

INPUT:

- **transition** has to be a :class:`FSMTransition`.

OUTPUT:

True or False.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import FSMTransition
t = FSMTransition('A', 'A', 0, 1)
sage: FiniteStateMachine().has_transition(t)
False
sage: FiniteStateMachine().has_transition(('A', 'A', 0, 1))
Traceback (most recent call last):
  ...
TypeError: Transition is not an instance of FSMTransition.
```

**induced_sub_finite_state_machine(states)**
Return a sub-finite-state-machine of the finite state machine induced by the given states.

INPUT:
• states – a list (or an iterator) of states (either labels or instances of \texttt{FSMState}) of the sub-finite-state-machine.

OUTPUT:

A new finite state machine. It consists (of deep copies) of the given states and (deep copies) of all transitions of \texttt{self} between these states.

EXAMPLES:

```python
sage: FSM = FiniteStateMachine([(0, 1, 0), (0, 2, 0),
...: (1, 2, 0), (2, 0, 0)])
sage: sub_FSM = FSM.induced_sub_finite_state_machine([0, 1])
sage: sub_FSM.states()  # [0, 1]
sage: sub_FSM.transitions()  # [Transition from 0 to 1: 0|-]
```

```python
sage: FSM.induced_sub_finite_state_machine([3])
Traceback (most recent call last):
... ValueError: 3 is not a state of this finite state machine.
```

\texttt{initial\_states()}

Return a list of all initial states.

OUTPUT:

A list of all initial states.

EXAMPLES:

```python
sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A', is_initial=True)
```

```
input\_alphabet = \texttt{None}

A list of letters representing the input alphabet of the finite state machine.

It can be set by the parameter \texttt{input\_alphabet} when initializing a finite state machine, see \texttt{FiniteStateMachine}.

It can also be set by the method \texttt{determine\_alphabets()}.

See also:

\texttt{FiniteStateMachine, determine\_alphabets(), output\_alphabet}.

\texttt{input\_projection()}

Return an automaton where the output of each transition of \texttt{self} is deleted.

OUTPUT:

An automaton.

EXAMPLES:
sage: F = FiniteStateMachine([('A', 'B', 0, 1), ('A', 'A', 1, 1), ('B', 'B', 1, 0)])

sage: G = F.input_projection()
sage: G.transitions()
[Transition from 'A' to 'B': 0|-,
 Transition from 'A' to 'A': 1|-,
 Transition from 'B' to 'B': 1|-

intersection(other)

is_Markov_chain(is_zero=None)

Checks whether self is a Markov chain where the transition probabilities are modeled as input labels.

INPUT:

- is_zero – by default (is_zero=None), checking for zero is simply done by is_zero(). This parameter can be used to provide a more sophisticated check for zero, e.g. in the case of symbolic probabilities, see the examples below.

OUTPUT:

True or False.

on_duplicate_transition must be duplicate_transition_add_input(), the sum of the input weights of the transitions leaving a state must add up to 1 and the sum of initial probabilities must add up to 1 (or all be None).

EXAMPLES:

sage: from sage.combinat.finite_state_machine import duplicate_transition_add_input
sage: F = Transducer([[0, 0, 1/4, 0], [0, 1, 3/4, 1],
......: [1, 0, 1/2, 0], [1, 1, 1/2, 1]],
......: on_duplicate_transition=duplicate_transition_add_input)
sage: F.is_Markov_chain()
True

on_duplicate_transition must be duplicate_transition_add_input():

sage: F = Transducer([[0, 0, 1/4, 0], [0, 1, 3/4, 1],
......: [1, 0, 1/2, 0], [1, 1, 1/2, 1]])
sage: F.is_Markov_chain()
False

Sum of input labels of the transitions leaving states must be 1:

sage: F = Transducer([[0, 0, 1/4, 0], [0, 1, 3/4, 1],
......: [1, 0, 1/2, 0]],
......: on_duplicate_transition=duplicate_transition_add_input)
sage: F.is_Markov_chain()
False

The initial probabilities of all states must be None or they must sum up to 1. The initial probabilities of all states have to be set in the latter case:
Combinatorics, Release 10.0

```python
sage: F = Transducer([[0, 0, 1/4, 0], [0, 1, 3/4, 1],
                   [1, 0, 1, 0]],
                   on_duplicate_transition=duplicate_transition_add_input)

sage: F.is_Markov_chain()
True

sage: F.state(0).initial_probability = 1/4
sage: F.is_Markov_chain()
False

sage: F.state(1).initial_probability = 7
sage: F.is_Markov_chain()
False

sage: F.state(1).initial_probability = 3/4
sage: F.is_Markov_chain()
True
```

If the probabilities are variables in the symbolic ring, `assume()` will do the trick:

```python
sage: var('p q')
# optional - sage.symbolic
(p, q)

sage: F = Transducer([[0, 0, p, 1], [0, 0, q, 0]],
                   on_duplicate_transition=duplicate_transition_add_input)

# optional - sage.symbolic

sage: assume(p + q == 1)
# optional - sage.symbolic

sage: (p + q - 1).is_zero() # optional - sage.symbolic
True

sage: F.is_Markov_chain() # optional - sage.symbolic
True

sage: forget() # optional - sage.symbolic

sage: del p, q # optional - sage.symbolic
```

If the probabilities are variables in some polynomial ring, the parameter is_zero can be used:

```python
sage: R.<p, q> = PolynomialRing(QQ)

sage: def is_zero_polynomial(polynomial):
....:     return polynomial in (p + q - 1)*R

sage: F = Transducer([[0, 0, p, 1], [0, 0, q, 0]],
                   on_duplicate_transition=duplicate_transition_add_input)

sage: F.state(0).initial_probability = p + q

sage: F.is_Markov_chain() # optional - sage.symbolic
False

sage: F.is_Markov_chain(is_zero_polynomial)
True
```

### is_complete()

Return whether the finite state machine is complete.

**OUTPUT:**

True or False
A finite state machine is considered to be complete if each transition has an input label of length one and for each pair \((q, a)\) where \(q\) is a state and \(a\) is an element of the input alphabet, there is exactly one transition from \(q\) with input label \(a\).

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Sage Code</th>
<th>Description</th>
</tr>
</thead>
</table>
| `sage: fsm = FiniteStateMachine([[(0, 0, 0, 0),
    ....: (0, 1, 1, 1),
    ....: (1, 1, 0, 0)],
    ....: determine_alphabets=False)` |
| `sage: fsm.is_complete()` |
| `Traceback (most recent call last):` |
| `ValueError: No input alphabet is given. Try calling determine_alphabets().` |
| `sage: fsm.input_alphabet = [0, 1]` |
| `sage: fsm.is_complete()` |
| `False` |
| `sage: fsm.add_transition((1, 1, 1, 1))` |
| `Transition from 1 to 1: 1|1` |
| `sage: fsm.is_complete()` |
| `True` |
| `sage: fsm.add_transition((0, 0, 1, 0))` |
| `Transition from 0 to 0: 1|0` |
| `sage: fsm.is_complete()` |
| `False` |

**is_connected()**

**is_deterministic()**

Return whether the finite state machine is deterministic.

**OUTPUT:**

True or False

A finite state machine is considered to be deterministic if each transition has input label of length one and for each pair \((q, a)\) where \(q\) is a state and \(a\) is an element of the input alphabet, there is at most one transition from \(q\) with input label \(a\). Furthermore, the finite state may not have more than one initial state.

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Sage Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: fsm = FiniteStateMachine()</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: fsm.add_transition(('A', 'B', 0, []))</code></td>
<td></td>
</tr>
<tr>
<td>`Transition from 'A' to 'B': 0</td>
<td>`</td>
</tr>
<tr>
<td><code>sage: fsm.is_deterministic()</code></td>
<td></td>
</tr>
<tr>
<td><code>True</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: fsm.add_transition(('A', 'C', 0, []))</code></td>
<td></td>
</tr>
<tr>
<td>`Transition from 'A' to 'C': 0</td>
<td>`</td>
</tr>
<tr>
<td><code>sage: fsm.is_deterministic()</code></td>
<td></td>
</tr>
<tr>
<td><code>False</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: fsm.add_transition(('A', 'B', [0,1], []))</code></td>
<td></td>
</tr>
<tr>
<td>`Transition from 'A' to 'B': 0,1</td>
<td>`</td>
</tr>
<tr>
<td><code>sage: fsm.is_deterministic()</code></td>
<td></td>
</tr>
<tr>
<td><code>False</code></td>
<td></td>
</tr>
</tbody>
</table>

Check that github issue #18556 is fixed:
sage: Automaton().is_deterministic()
True
sage: Automaton(initial_states=[0]).is_deterministic()
True
sage: Automaton(initial_states=[0, 1]).is_deterministic()
False

is_monochromatic()
Check whether the colors of all states are equal.
OUTPUT:
True or False
EXAMPLES:

sage: G = transducers.GrayCode()
sage: [s.color for s in G.iter_states()]
[None, None, None]
sage: G.is_monochromatic()
True
sage: G.state(1).color = 'blue'
sage: G.is_monochromatic()
False

iter_final_states()
Return an iterator of the final states.
OUTPUT:
An iterator over all initial states.
EXAMPLES:

sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A', is_final=True)
sage: B = FSMState('B', is_initial=True)
sage: C = FSMState('C', is_final=True)
sage: F = FiniteStateMachine([(A, B), (A, C)])
sage: [s.label() for s in F.iter_final_states()]
['A', 'C']

iter_initial_states()
Return an iterator of the initial states.
OUTPUT:
An iterator over all initial states.
EXAMPLES:

sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A', is_initial=True)
sage: B = FSMState('B')
sage: F = FiniteStateMachine([(A, B, 1, 0)])
sage: [s.label() for s in F.iter_initial_states()]
['A']
**iter_process**(*input_tape=None, initial_state=None, process_iterator_class=None, iterator_type=None, automatic_output_type=False, **kwargs*)

This function returns an iterator for processing the input. See process() (which runs this iterator until the end) for more information.

**INPUT:**

- **iterator_type** – If **None** (default), then an instance of FSMProcessIterator is returned. If this is 'simple' only an iterator over one output is returned (an exception is raised if this is not the case, i.e., if the process has branched).

See process() for a description of the other parameters.

**OUTPUT:**

An iterator.

**EXAMPLES:**

We can use iter_process() to deal with infinite words:

```
sage: inverter = Transducer({'A': [('A', 0, 1), ('A', 1, 0)]},
....:     initial_states=['A'], final_states=['A'])
```

```
sage: inverter.iter_process(words.FibonacciWord(), iterator_type='simple')
```

```
sage: Words([0,1])(inverter.iter_process(words.FibonacciWord(),
....:     iterator_type='simple'))
```

```
sage: inverter(words.FibonacciWord())
```

To see what is going on, we use iter_process() without arguments:

```
sage: from itertools import islice
sage: it = inverter.iter_process(words.FibonacciWord())
sage: for current in islice(it, 4r):
....:     print(current)
```

```
process (1 branch)
+ at state 'A'
  -- tape at 1, [[1]]
process (1 branch)
+ at state 'A'
  -- tape at 2, [[1, 0]]
process (1 branch)
+ at state 'A'
  -- tape at 3, [[1, 0, 1]]
```

(continues on next page)
process (1 branch)
+ at state 'A'
+++ tape at 4, [[1, 0, 1, 1]]

The following show the difference between using the 'simple'-option and not using it. With this option, we have

```
sage: it = inverter.iter_process(input_tape=[0, 1, 1],
.....: iterator_type='simple')
sage: for i, o in enumerate(it):
.....:     print('step %s: output %s' % (i, o))
step 0: output 1
step 1: output 0
step 2: output 0
```

So `iter_process()` is a generator expression which gives a new output letter in each step (and not more). In many cases this is sufficient.

Doing the same without the 'simple'-option does not give the output directly; it has to be extracted first. On the other hand, additional information is presented:

```
sage: it = inverter.iter_process(input_tape=[0, 1, 1])
sage: for current in it:
.....:     print(current)
process (1 branch)
+ at state 'A'
+++ tape at 1, [[1]]
process (1 branch)
+ at state 'A'
+++ tape at 2, [[1, 0]]
process (1 branch)
+ at state 'A'
+++ tape at 3, [[1, 0, 0]]
process (0 branches)
sage: it.result()
[Branch(accept=True, state='A', output=[1, 0, 0])]
```

One can see the growing of the output (the list of lists at the end of each entry).

Even if the transducer has transitions with empty or multiletter output, the simple iterator returns one new output letter in each step:

```
sage: T = Transducer([(0, 0, 0, []),
.....: (0, 0, 1, [1]),
.....: (0, 0, 2, [2, 2])],
.....: initial_states=[0])
sage: it = T.iter_process(input_tape=[0, 1, 2, 0, 1, 2],
.....: iterator_type='simple')
sage: for i, o in enumerate(it):
.....:     print('step %s: output %s' % (i, o))
step 0: output 1
step 1: output 2
step 2: output 2
```

(continues on next page)
step 4: output 2
step 5: output 2

See also:

FiniteStateMachine.process(), Automaton.process(), Transducer.process(), __call__(), FSMProcessIterator.

iter_states()
Return an iterator of the states.

OUTPUT:
An iterator of the states of the finite state machine.

EXAMPLES:

```
sage: FSM = Automaton([(1, 2, 1), (2, 2, 0)])
sage: [s.label() for s in FSM.iter_states()]
['1', '2']
```

iter_transitions(from_state=None)
Return an iterator of all transitions.

INPUT:

- from_state – (default: None) If from_state is given, then a list of transitions starting there is given.

OUTPUT:
An iterator of all transitions.

EXAMPLES:

```
sage: FSM = Automaton([(1, 2, 1), (2, 2, 0)])
sage: [(t.from_state.label(), t.to_state.label())
    for t in FSM.iter_transitions('1')]
[('1', '2')]
sage: [(t.from_state.label(), t.to_state.label())
    for t in FSM.iter_transitions('2')]
[('2', '2')]
sage: [(t.from_state.label(), t.to_state.label())
    for t in FSM.iter_transitions()]
[('1', '2'), ('2', '2')]
```

kleene_star()
Compute the Kleene closure of this finite state machine.

OUTPUT:
A FiniteStateMachine of the same type as this finite state machine.

Assume that this finite state machine is an automaton recognizing the language \( L \). Then the Kleene star recognizes the language \( L^* = \{ w_1 \ldots w_n \mid n \geq 0, w_j \in L \text{ for all } j \} \).

Assume that this finite state machine is a transducer realizing a function \( f \) on some alphabet \( L \). Then the Kleene star realizes a function \( g \) on \( L^* \) with \( g(w_1 \ldots w_n) = f(w_1) \ldots f(w_n) \).

EXAMPLES:
Kleene star of an automaton:
sage: A = automata.Word([0, 1])

sage: B = A.kleene_star()

sage: B.transitions()
[Transition from 0 to 1: 0|-, Transition from 2 to 0: -|--, Transition from 1 to 2: 1|--]

sage: from sage.combinat.finite_state_machine import (is_Automaton, is_Transducer)

sage: is_Automaton(B)
True

sage: [w for w in ([], [0, 1], [0, 1, 0], [0, 1, 0, 1], [0, 1, 1, 1]) if B(w)]
[[], [0, 1], [0, 1, 0]]

Kleene star of a transducer:

sage: T = Transducer([(0, 1, 0, 1), (0, 1, 1, 0)], initial_states=[0], final_states=[1])

sage: S = T.kleene_star()

sage: S.transitions()
[Transition from 0 to 1: 0|1, Transition from 0 to 1: 1|0, Transition from 1 to 0: -|--]

sage: is_Transducer(S)
True

sage: for w in ([], [0], [1], [0, 0], [0, 1]):
    print("{} {}".format(w, S.process(w)))

[[] (True, 0, [])
[0] [(True, 0, [1]), (True, 1, [1])] 
[1] [(True, 0, [0]), (True, 1, [0])] 
[0, 0] [(True, 0, [1, 1]), (True, 1, [1, 1])] 
[0, 1] [(True, 0, [1, 0]), (True, 1, [1, 0])]]

Final output words are taken into account:

sage: T = Transducer([(0, 1, 0, 1)], initial_states=[0], final_states=[1])

sage: T.state(1).final_word_out = 2

sage: S = T.kleene_star()

sage: sorted(S.process([0, 0]))
[(True, 0, [1, 2, 1, 2]), (True, 1, [1, 2, 1, 2])]

Final output words may lead to undesirable situations if initial states and final states coincide:

sage: T = Transducer(initial_states=[0], final_states=[0])

sage: T.state(0).final_word_out = 1

sage: T([[]])
[1]

sage: S = T.kleene_star()
(continued from previous page)

```sage
S([])
```

Traceback (most recent call last):
...

RuntimeError: State 0 is in an epsilon cycle (no input), but output is written.

```markdown
language(max_length=None, **kwargs)
```

Return all words that can be written by this transducer.

**INPUT:**

- `max_length` – an integer or `None` (default). Only output words which come from inputs of length at most `max_length` will be considered. If `None`, then this iterates over all possible words without length restrictions.

- `kwargs` – will be passed on to the `process iterator`. See `process()` for a description.

**OUTPUT:**

An iterator.

**EXAMPLES:**

```sage
NAF = Transducer([(I, 0, 0, None), (I, 1, 1, None),
                   ..., (0, 0, 0), (0, 1, 1, 0),
                   ..., (1, 0, 0, 1), (1, 2, 1, -1),
                   ..., (2, 1, 0, 0), (2, 2, 1, 0)],
                   initial_states=['I'], final_states=[0],
                   input_alphabet=[0, 1])

sage: sorted(NAF.language(4),
          key=lambda o: (ZZ(o, base=2), len(o)))
[[], [0], [0, 0], [0, 0, 0],
 [1], [1, 0], [1, 0, 0],
 [0, 1], [0, 1, 0],
 [-1, 0, 1],
 [0, 0, 1],
 [1, 0, 1]]
```

```sage
iterator = NAF.language()
sage: next(iterator)
[]
sage: next(iterator)
[0]
sage: next(iterator)
[1]
sage: next(iterator)
[0, 0]
sage: next(iterator)
[0, 1]
```

See also:

`Automaton.language()`, `process()`.
latex_options(coordinates=None, format_state_label=None, format_letter=None, format_transition_label=None, loop_where=None, initial_where=None, accepting_style=None, accepting_distance=None, accepting_where=None, accepting_show_empty=None)

Set options for \LaTeX{} output via \texttt{latex()} and therefore \texttt{view()}.

INPUT:

- \texttt{coordinates} – a dictionary or a function mapping labels of states to pairs interpreted as coordinates. If no coordinates are given, states a placed equidistantly on a circle of radius 3. See also \texttt{set_coordinates()}.

- \texttt{format_state_label} – a function mapping labels of states to a string suitable for typesetting in \LaTeX{}'s mathematics mode. If not given, \texttt{latex()} is used.

- \texttt{format_letter} – a function mapping letters of the input and output alphabets to a string suitable for typesetting in \LaTeX{}'s mathematics mode. If not given, \texttt{default_format_transition_label()} uses \texttt{latex()}.

- \texttt{format_transition_label} – a function mapping words over the input and output alphabets to a string suitable for typesetting in \LaTeX{}'s mathematics mode. If not given, \texttt{default_format_transition_label()} is used.

- \texttt{loop_where} – a dictionary or a function mapping labels of initial states to one of 'above', 'left', 'below', 'right'. If not given, 'above' is used.

- \texttt{initial_where} – a dictionary or a function mapping labels of initial states to one of 'above', 'left', 'below', 'right'. If not given, TikZ' default (currently 'left') is used.

- \texttt{accepting_style} – one of 'accepting by double' and 'accepting by arrow'. If not given, 'accepting by double' is used unless there are non-empty final output words.

- \texttt{accepting_distance} – a string giving a \LaTeX{} length used for the length of the arrow leading from a final state. If not given, TikZ' default (currently '3ex') is used unless there are non-empty final output words, in which case '7ex' is used.

- \texttt{accepting_where} – a dictionary or a function mapping labels of final states to one of 'above', 'left', 'below', 'right'. If not given, TikZ' default (currently 'right') is used. If the final state has a final output word, it is also possible to give an angle in degrees.

- \texttt{accepting_show_empty} – if True the arrow of an empty final output word is labeled as well. Note that this implicitly implies \texttt{accepting_style='accepting by arrow'}. If not given, the default False is used.

OUTPUT:

Nothing.

As TikZ (cf. the Wikipedia article PGF/TikZ) is used to typeset the graphics, the syntax is oriented on TikZ' syntax.

This is a convenience function collecting all options for \LaTeX{} output. All of its functionality can also be achieved by directly setting the attributes

- \texttt{coordinates}, \texttt{format_label}, \texttt{loop_where}, \texttt{initial_where}, and \texttt{accepting_where} of \texttt{FSMState} (here, \texttt{format_label} is a callable without arguments, everything else is a specific value);

- \texttt{format_label} of \texttt{FSMTransition} (\texttt{format_label} is a callable without arguments);

- \texttt{format_state_label}, \texttt{format_letter}, \texttt{format_transition_label}, \texttt{accepting_style}, \texttt{accepting_distance}, and \texttt{accepting_show_empty} of \texttt{FiniteStateMachine}. 

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This function, however, also (somewhat) checks its input and serves to collect documentation on all these options.

The function can be called several times, only those arguments which are not \texttt{None} are taken into account. By the same means, it can be combined with directly setting some attributes as outlined above.

**EXAMPLES:**

See also the section on \textit{LaTeX output} in the introductory examples of this module.

```
sage: T = Transducer(initial_states=[4],
......:    final_states=[0, 3])
sage: for j in srange(4):
......:    T.add_transition(4, j, 0, [0, j])
......:    T.add_transition(j, 4, 0, [0, -j])
......:    T.add_transition(j, j, 0, 0)
Transition from 4 to 0: 0|0,0
Transition from 0 to 4: 0|0,0
Transition from 0 to 0: 0|0
Transition from 4 to 1: 0|0,1
Transition from 1 to 4: 0|0,-1
Transition from 1 to 1: 0|0
Transition from 4 to 2: 0|0,2
Transition from 2 to 4: 0|0,-2
Transition from 2 to 2: 0|0
Transition from 4 to 3: 0|0,3
Transition from 3 to 4: 0|0,-3
Transition from 3 to 3: 0|0
sage: T.add_transition(4, 4, 0, 0)
Transition from 4 to 4: 0|0
sage: T.state(3).final_word_out = [0, 0]
sage: T.latex_options(
......:    coordinates={4: (0, 0),
......:                  0: (-6, 3),
......:                  1: (-2, 3),
......:                  2: (2, 3),
......:                  3: (6, 3)},
......:    format_state_label=lambda x: r'\mathbf{%s}' % x,
......:    format_letter=lambda x: r'w_{%s}' % x,
......:    format_transition_label=lambda x:
......:        r'\scriptstyle{%s}' % T.default_format_transition_label(x),
......:    loop_where={4: 'below', 0: 'left', 1: 'above',
......:                 2: 'right', 3: 'below'},
......:    initial_where=lambda x: 'above',
......:    accepting_style='accepting by double',
......:    accepting_distance='10ex',
......:    accepting_where={0: 'left', 3: 45}
......:)
sage: T.state(4).format_label=lambda: r'\mathcal{I}'
sage: latex(T)
\begin{tikzpicture}[auto, initial text=, >=latex]
\node[state, initial, initial where=above] (v0) at (0.000000, 0.000000) {$\mathcal{I}$};
\node[state, accepting, accepting where=left] (v1) at (-6.000000, 3.000000) {$\mathcal{I}$};
\end{tikzpicture}
```
To actually see this, use the live documentation in the Sage notebook and execute the cells.

By changing some of the options, we get the following output:

```python
sage: T.latex_options(
    format_transition_label=T.default_format_transition_label,
    accepting_style='accepting by arrow',
    accepting_show_empty=True
    )
```

To continue, please refer to the next page.
markov_chain_simplification()

Consider self as Markov chain with probabilities as input labels and simplify it.

INPUT:
Nothing.

OUTPUT:
Simplified version of self.

EXAMPLES:

```python
goingle: from sage.combinat.finite_state_machine import duplicate_transition_add_input
gle: T = Transducer([[1, 2, 1/4, 0], [1, -2, 1/4, 0], [1, -2, 1/2, 0],
        ....: [2, 2, 1/4, 1], [2, -2, 1/4, 1], [-2, -2, 1/4, 1],
        ....: [-2, 2, 1/4, 1], [2, 3, 1/2, 2], [-2, 3, 1/2, 2],
        ....: initial_states=[1],
        ....: final_states=[3],
        ....: on_duplicate_transition=duplicate_transition_add_input)
gle: T1 = T.markov_chain_simplification()
gle: sorted(T1.transitions())

[Transition from ((1,),) to ((2, -2),): 1|0,
(continues on next page)
Transition from ((2, -2),) to ((2, -2),): 1/2|1,  Transition from ((2, -2),) to ((3,),): 1/2|2

merged_transitions()

Merges transitions which have the same from_state, to_state and word_out while adding their word_in.

INPUT:

Nothing.

OUTPUT:

A finite state machine with merged transitions. If no mergers occur, return self.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import duplicate_transition_add_input
sage: T = Transducer([[1, 2, 1/4, 1], [1, -2, 1/4, 1], [1, -2, 1/2, 1],
    ....: [2, 2, 1/4, 1], [2, -2, 1/4, 1], [-2, -2, 1/4, 1],
    ....: [-2, 2, 1/4, 1], [2, 3, 1/2, 1], [-2, 3, 1/2, 1]],
    ....: on_duplicate_transition=duplicate_transition_add_input)

sage: T1 = T.merged_transitions()

sage: sorted(T1.transitions())
[Transition from -2 to -2: 1/4|1,
 Transition from -2 to 2: 1/4|1,
 Transition from -2 to 3: 1/2|1,
 Transition from 1 to 2: 1/4|1,
 Transition from 1 to -2: 3/4|1,
 Transition from 2 to -2: 1/4|1,
 Transition from 2 to 2: 1/4|1,
 Transition from 2 to 3: 1/2|1]
```

Applying the function again does not change the result:

```
sage: T2 = T1.merged_transitions()

sage: T2 is T1
True
```

moments_waiting_time(test=<class 'bool'>, is_zero=None, expectation_only=False)

If this finite state machine acts as a Markov chain, return the expectation and variance of the number of steps until first writing True.

INPUT:

- **test** – (default: bool) a callable deciding whether an output label is to be considered True. By default, the standard conversion to boolean is used.

- **is_zero** – (default: None) a callable deciding whether an expression for a probability is zero. By default, checking for zero is simply done by is_zero(). This parameter can be used to provide a more sophisticated check for zero, e.g. in the case of symbolic probabilities, see the examples below. This parameter is passed on to is_Markov_chain(). This parameter only affects the input of the Markov chain.
• **expectation_only** – (default: False) if set, the variance is not computed (in order to save time).
  
  By default, the variance is computed.

OUTPUT:
A dictionary (if `expectation_only=False`) consisting of

• **expectation**.
• **variance**.

Otherwise, just the expectation is returned (no dictionary for `expectation_only=True`).

Expectation and variance of the number of steps until first writing `True` (as determined by the parameter `test`).

ALGORITHM:
Relies on a (classical and easy) probabilistic argument, cf. [FGT1992], Eqns. (6) and (7).

For the variance, see [FHP2015], Section 2.

EXAMPLES:

1. The simplest example is to wait for the first 1 in a 0-1-string where both digits appear with probability 1/2. In fact, the waiting time equals \( k \) if and only if the string starts with \( 0^{k-1}1 \). This event occurs with probability \( 2^{-k} \). Therefore, the expected waiting time and the variance are \( \sum_{k\geq 1} k2^{-k} = 2 \) and \( \sum_{k\geq 1} (k-2)^22^{-k} = 2 \):

```python
sage: var('k')
sage: sum(k * 2^(-k), k, 1, infinity)
2
sage: sum((k-2)^2 * 2^(-k), k, 1, infinity)
2
```

We now compute the same expectation and variance by using a Markov chain:

```python
sage: from sage.combinat.finite_state_machine import (  
    duplicate_transition_add_input)

sage: T = Transducer(  
    [(0, 0, 1/2, 0), (0, 0, 1/2, 1)],  
    on_duplicate_transition=\  
    duplicate_transition_add_input,  
    initial_states=[0],  
    final_states=[0])

sage: T.moments_waiting_time()  
{'expectation': 2, 'variance': 2}

sage: T.moments_waiting_time(expectation_only=True)
2
```

In the following, we replace the output `0` by `-1` and demonstrate the use of the parameter `test`:

```python
sage: T.delete_transition((0, 0, 1/2, 0))
sage: T.add_transition((0, 0, 1/2, -1))
Transition from 0 to 0: 1/2|-1
sage: T.moments_waiting_time(test=lambda x: x<0)  
{'expectation': 2, 'variance': 2}
```
2. Make sure that the transducer is actually a Markov chain. Although this is checked by the code, unexpected behaviour may still occur if the transducer looks like a Markov chain. In the following example, we ‘forget’ to assign probabilities, but due to a coincidence, all ‘probabilities’ add up to one. Nevertheless, 0 is never written, so the expectation is 1.

```python
sage: T = Transducer([(0, 0, 0, 0), (0, 0, 1, 1)],
                  on_duplicate_transition=
                  duplicate_transition_add_input,
                  initial_states=[0],
                  final_states=[0])
sage: T.moments_waiting_time()
{'expectation': 1, 'variance': 0}
```

3. If `True` is never written, the moments are $+\infty$:

```python
sage: T = Transducer([(0, 0, 1, 0)],
                  on_duplicate_transition=
                  duplicate_transition_add_input,
                  initial_states=[0],
                  final_states=[0])
sage: T.moments_waiting_time()
{'expectation': +Infinity, 'variance': +Infinity}
```

4. Let $h$ and $r$ be positive integers. We consider random strings of letters $1, \ldots, r$ where the letter $j$ occurs with probability $p_j$. Let $B$ be the random variable giving the first position of a block of $h$ consecutive identical letters. Then

\[
\mathbb{E}(B) = \frac{1}{\sum_{i=1}^{r} \frac{1}{p_i^{-1} + \cdots + p_i^{-h}}},
\]

\[
\mathbb{V}(B) = \sum_{i=1}^{r} \left( \frac{p_i + p_i^h}{1 - p_i^h} - 2h \frac{p_i^h (1 - p_i)}{(1 - p_i^h)^2} \right) \left( \sum_{i=1}^{r} \frac{1}{p_i^{-1} + \cdots + p_i^{-r}} \right)^2 - \mathbb{E}(B)^2
\]

cf. [S1986], p. 62, or [FHP2015], Theorem 1. We now verify this with a transducer approach.

```python
sage: def test(h, r):
    R = PolynomialRing(QQ,
    names=['p_%d' % j for j in range(r)])
    p = R.gens()
    def is_zero(polynomial):
        return polynomial in (sum(p) - 1) * R
    theory_expectation = 1/(sum(1/sum(p[j]**(-i))
    for i in range(1, h+1))
    for j in range(r))
    theory_variance = sum((p[i] + p[i]**h)/(1 - p[i]**h)
    - 2*h*p[i]**h * (1 - p[i])/(1 - p[i]**h)**2
    for i in range(r)
    ) * theory_expectation**2
    alphabet = list(range(r))
```

(continues on next page)
5. Consider the alphabet \( \{0, \ldots, r-1\} \), some \( 1 \leq j \leq r \) and some \( h \geq 1 \). For some probabilities \( p_0, \ldots, p_{r-1} \), we consider infinite words where the letters occur independently with the given probabilities. The random variable \( B_j \) is the first position \( n \) such that there exist \( j \) of the \( r \) letters having an \( h \)-run. The expectation of \( B_j \) is given in [FHP2015], Theorem 2. Here, we verify this result by using transducers:

```python
sage: def test(h, r, j):
    R = PolynomialRing(QQ, names=['p_%d' % i for i in range(r)])
    p = R.gens()
    def is_zero(polynomial):
        return polynomial in (sum(p) - 1) * R
    alphabet = list(range(r))
    counters = [transducers.Wait([0, 1])
                for i in alphabet]
    all_counter = counters[0].cartesian_product(counters[1:])
    adder = transducers.add(input_alphabet=[0, 1],
                        number_of_operands=r)
    probabilities = Transducer([0, 0, p[j], j]
                                for j in alphabet],
                                initial_states=[0],
                                final_states=[0],
                                on_duplicate_transition=
                                duplicate_transition_add_input)
    chain = adder(all_counter(probabilities))
    result = chain.moments_waiting_time(is_zero=is_zero)
    return is_zero((result['expectation'] -
                        theory_expectation).numerator()) \ 
                        and \
                        is_zero((result['variance'] -
                                theory_variance).numerator())
```

```
sage: test(2, 2)
True
sage: test(2, 3)
True
sage: test(3, 3)
True
```
threshold = transducers.map(
    f=lambda x: x >= j,
    input_alphabet=srange(r+1))
probabilities = Transducer(
    [(0, 0, p[i], i) for i in alphabet],
    initial_states=[0],
    final_states=[0],
    on_duplicate_transition=
        duplicate_transition_add_input)
chain = threshold(adder(all_counter(
    probabilities)))
result = chain.moments_waiting_time(
    is_zero=is_zero,
    expectation_only=True)
R_v = PolynomialRing(
    QQ,
    names=['p_%d' % i for i in range(r)])
v = R_v.gens()
S = 1/(1 - sum(v[i]/(1+v[i])
    for i in range(r)))
alpha = [(p[i] - p[i]^h)/(1 - p[i])
    for i in range(r)]
gamma = [p[i]/(1 - p[i]) for i in range(r)]
alphabet_set = set(alphabet)
expectation = 0
for q in range(j):
    for M in Subsets(alphabet_set, q):
        summand = S
        for i in M:
            summand = summand.subs(
                {v[i]: gamma[i]}) -
            summand.subs({v[i]: alpha[i]})
        for i in alphabet_set - set(M):
            summand = summand.subs(
                {v[i]: alpha[i]})
        expectation += summand
    return is_zero((result - expectation)/
        numerator())

sage: test(2, 3, 2)
True

REFERENCES:

number_of_words(variable=None, base_ring=None)

Return the number of successful input words of given length.

INPUT:

• variable – a symbol denoting the length of the words, by default \( n \).
• base_ring – Ring (default: \( \mathbb{Q}\bar{\mathbb{Q}} \)) in which to compute the eigenvalues.

OUTPUT:

A symbolic expression.
EXAMPLES:

```
sage: NAFpm = Automaton([(0, 0, 0), (0, 1, 1),
....:                    (0, 1, -1), (1, 0, 0)],
....:                    initial_states=[0],
....:                    final_states=[0, 1])
sage: N = NAFpm.number_of_words(); N
4/3*2^n - 1/3*(-1)^n
sage: all(len(list(NAFpm.language(s))) == N.subs(n=s)
....:   for s in srange(1, 6))
True
```

An example with non-rational eigenvalues. By default, eigenvalues are elements of the field of algebraic numbers.

```
sage: NAFp = Automaton([(0, 0, 0), (0, 1, 1), (1, 0, 0)],
....:                   initial_states=[0],
....:                   final_states=[0, 1])
sage: N = NAFp.number_of_words(); N
1.170820393249937?*1.618033988749895?^n
- 0.1708203932499369?*(-0.618033988749895?)^n
sage: all(len(list(NAFp.language(s))) == N.subs(n=s)
....:   for s in srange(1, 6))
True
```

We specify a suitable base_ring to obtain a radical expression. To do so, we first compute the characteristic polynomial and then construct a number field generated by its roots.

```
sage: M = NAFp.adjacency_matrix(entry=lambda t: 1)
sage: M.characteristic_polynomial()
x^2 - x - 1
sage: R.<phi> = NumberField(x^2-x-1, embedding=1.6)
sage: N = NAFp.number_of_words(base_ring=R); N
1/2*(1/2*sqrt(5) + 1/2)^n*(3*sqrt(1/5) + 1)
- 1/2*(-1/2*sqrt(5) + 1/2)^n*(3*sqrt(1/5) - 1)
sage: all(len(list(NAFp.language(s))) == N.subs(n=s)
....:   for s in srange(1, 6))
True
```

In this special case, we might also use the constant golden_ratio:

```
sage: R.<phi> = NumberField(x^2-x-1, embedding=golden_ratio)
```

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```sage
sage: N = NAFp.number_of_words(base_ring=R); N
# optional -
˓→sage.symbolic
1/5*(3*golden_ratio + 1)*golden_ratio^n
- 1/5*(3*golden_ratio - 4)*(-golden_ratio + 1)^n
sage: all(len(list(NAFp.language(s)))
˓→# optional -
˓→→sage.symbolic
....: - len(list(NAFp.language(s-1))) == N.subs(n=s)
....: for s in range(1, 6))
True
```

The adjacency matrix of the following example is a Jordan matrix of size 3 to the eigenvalue 4:

```sage
sage: J3 = Automaton([(0, 1, -1), (1, 2, -1)],
isticales=[0],
isticales=[0, 1, 2])
sage: for i in range(3):
isticales=[i, i, j]
sage: J3.adjacency_matrix(entry=lambda t: 1)
[4 1 0]
[0 4 1]
[0 0 4]
sage: N = J3.number_of_words(); N
# optional -
˓→sage.symbolic
1/2*4^(n - 2)*(n - 1)*n + 4^(n - 1)*n + 4^n
sage: all(len(list(J3.language(s)))
˓→# optional -
˓→→sage.symbolic
....: - len(list(J3.language(s-1))) == N.subs(n=s)
....: for s in range(1, 6))
True
```

Here is an automaton without cycles, so with eigenvalue 0.

```sage
sage: A = Automaton([(j, j+1, 0) for j in range(3)],
isticales=[0],
isticales=range(3))
sage: A.number_of_words()
# optional -
˓→sage.symbolic
1/2*0^(n - 2)*(n - 1)*n + 0^(n - 1)*n + 0^n
```

**on_duplicate_transition** (*old_transition, new_transition*)

Which function to call when a duplicate transition is inserted.

It can be set by the parameter `on_duplicate_transition` when initializing a finite state machine, see `FiniteStateMachine`.

See also:

`FiniteStateMachine, is_Markov_chain(), markov_chain_simplification()`.

**output_alphabet = None**

A list of letters representing the output alphabet of the finite state machine.

It can be set by the parameter `output_alphabet` when initializing a finite state machine, see `FiniteStateMachine`.

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It can also be set by the method `determine_alphabets()`.

See also:

`FiniteStateMachine`, `determine_alphabets()`, `input_alphabet`.

**output_projection()**

Return a automaton where the input of each transition of self is deleted and the new input is the original output.

**OUTPUT:**

An automaton.

**EXAMPLES:**

```
sage: F = FiniteStateMachine([('A', 'B', 0, 1), ('A', 'A', 1, 1), ('B', 'B', 1, 0)])
sage: G = F.output_projection()
sage: G.transitions()
[Transition from 'A' to 'B': 1|-,
 Transition from 'A' to 'A': 1|-,
 Transition from 'B' to 'B': 0|-]
```

Final output words are also considered correctly:

```
sage: H = Transducer([('A', 'B', 0, 1), ('A', 'A', 1, 1), ('B', 'B', 1, 0), ('A', ('final', 0), 0, 0)],
 final_states=['A', 'B'])
sage: H.state('B').final_word_out = 2
sage: J = H.output_projection()
sage: J.states()
['A', 'B', ('final', 0), ('final', 1)]
sage: J.transitions()
[Transition from 'A' to 'B': 1|-,
 Transition from 'A' to 'A': 1|-,
 Transition from 'A' to ('final', 0): 0|-,
 Transition from 'B' to 'B': 0|-,
 Transition from 'B' to ('final', 1): 2|-]
sage: J.final_states()
['A', ('final', 1)]
```

**plot()**

Plots a graph of the finite state machine with labeled vertices and labeled edges.

**INPUT:**

Nothing.

**OUTPUT:**

A plot of the graph of the finite state machine.

**predecessors(state, valid_input=None)**

Lists all predecessors of a state.

**INPUT:**

- `state` – the state from which the predecessors should be listed.
• valid_input – If valid_input is a list, then we only consider transitions whose input labels are contained in valid_input. state has to be a FSMState (not a label of a state). If input labels of length larger than 1 are used, then valid_input has to be a list of lists.

OUTPUT:
A list of states.

EXAMPLES:

```sage
A = Transducer([('I', 'A', 'a', 'b'), ('I', 'B', 'b', 'c'),
                     ('I', 'C', 'c', 'a'), ('A', 'F', 'b', 'a'),
                     ('B', 'F', ['c', 'b'], 'b'), ('C', 'F', 'a', 'c')],
               initial_states=['I'], final_states=['F'])
```

```sage
A.predecessors(A.state('A'))
['A', 'I']
```

```sage
A.predecessors(A.state('F'), valid_input=['b', 'a'])
['F', 'C', 'A', 'I']
```

```sage
A.predecessors(A.state('F'), valid_input=[['c', 'b'], 'a'])
['F', 'C', 'B']
```

### prepone_output()

For all paths, shift the output of the path from one transition to the earliest possible preceding transition of the path.

INPUT:
Nothing.

OUTPUT:
Nothing.

Apply the following to each state $s$ (except initial states) of the finite state machine as often as possible:
If the letter $a$ is a prefix of the output label of all transitions from $s$ (including the final output of $s$), then remove it from all these labels and append it to all output labels of all transitions leading to $s$.

We assume that the states have no output labels, but final outputs are allowed.

EXAMPLES:

```sage
A = Transducer([('A', 'B', 1, 1),
                 ('B', 'B', 0, 0),
                 ('B', 'C', 1, 0)],
                initial_states=['A'], final_states=['C'])
```

```sage
A.prepone_output()
```

```sage
A.transitions()
[Transition from 'A' to 'B': 1|1,0,
 Transition from 'B' to 'B': 0|0,
 Transition from 'B' to 'C': 1|-
```

```sage
B = Transducer([('A', 'B', 0, 1),
                 ('B', 'C', 1, [1, 1]),
                 ('B', 'C', 0, 1)],
                initial_states=['A'], final_states=['C'])
```

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sage: B.prepone_output()
sage: B.transitions()
[Transition from 'A' to 'B': 0|1,1,
 Transition from 'B' to 'C': 1|1,
 Transition from 'B' to 'C': 0|-]

If initial states are not labeled as such, unexpected results may be obtained:

sage: C = Transducer([(0,1,0,0)])

verbose 0 (...: finite_state_machine.py, prepone_output)
All transitions leaving state 0 have an output label with
prefix 0. However, there is no inbound transition and it
is not an initial state. This routine (possibly called by
simplification) therefore erased this prefix from all
outbound transitions.

sage: C.transitions()
[Transition from 0 to 1: 0|-
]

Also the final output of final states can be changed:

sage: T = Transducer([('A', 'B', 0, 1),
....: ('B', 'C', 1, [1, 1]),
....: ('B', 'C', 0, 1)],
....: initial_states=['A'],
....: final_states=['B'])

sage: T.state('B').final_word_out = [1]
sage: T.prepone_output()
sage: T.transitions()
[Transition from 'A' to 'B': 0|1,1,
 Transition from 'B' to 'C': 1|1,
 Transition from 'B' to 'C': 0|-]

sage: T.state('B').final_word_out
[]

sage: S = Transducer([('A', 'B', 0, 1),
....: ('B', 'C', 1, [1, 1]),
....: ('B', 'C', 0, 1)],
....: initial_states=['A'],
....: final_states=['B'])

sage: S.state('B').final_word_out = [0]
sage: S.prepone_output()
sage: S.transitions()
[Transition from 'A' to 'B': 0|1,
 Transition from 'B' to 'C': 1|1,1,
 Transition from 'B' to 'C': 0|1]

sage: S.state('B').final_word_out
[0]

Output labels do not have to be hashable:

sage: C = Transducer([(0, 1, 0, [])],
....: initial_states=['A'])

sage: C.transitions()
[Transition from 0 to 1: 0|-
]
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\[
\begin{align*}
\text{....:} & \quad (1, 0, 0, [\text{vector([0, 0]), 0}]), \\
\text{....:} & \quad (1, 1, 1, [\text{vector([0, 0]), 1}]), \\
\text{....:} & \quad (0, 0, 1, 0]), \\
\text{....:} & \quad \text{determine_alphabets=False,} \\
\text{....:} & \quad \text{initial_states=[0])}
\end{align*}
\]

sage: C.prepone_output()

sage: sorted(C.transitions())

\[
\begin{align*}
\text{Transition from 0 to 1: 0|(0, 0),} \\
\text{Transition from 0 to 0: 1|0,} \\
\text{Transition from 1 to 0: 0|0,} \\
\text{Transition from 1 to 1: 1|1,(0, 0)}
\end{align*}
\]

\textbf{process}(*\texttt{args}, **\texttt{kwargs})

Return whether the finite state machine accepts the input, the state where the computation stops and which output is generated.

**INPUT:**

- **input_tape** – the input tape can be a list or an iterable with entries from the input alphabet. If we are working with a multi-tape machine (see parameter \texttt{use_multitape_input} and notes below), then the tape is a list or tuple of tracks, each of which can be a list or an iterable with entries from the input alphabet.

- **initial_state** or **initial_states** – the initial state(s) in which the machine starts. Either specify a single one with \texttt{initial_state} or a list of them with \texttt{initial_states}. If both are given, \texttt{initial_state} will be appended to \texttt{initial_states}. If neither is specified, the initial states of the finite state machine are taken.

- **list_of_outputs** – (default: None) a boolean or None. If True, then the outputs are given in list form (even if we have no or only one single output). If False, then the result is never a list (an exception is raised if the result cannot be returned). If \texttt{list_of_outputs=\texttt{None}}, the method determines automatically what to do (e.g. if a non-deterministic machine returns more than one path, then the output is returned in list form).

- **only_accepted** – (default: False) a boolean. If set, then the first argument in the output is guaranteed to be True (if the output is a list, then the first argument of each element will be True).

- **always_include_output** – if set (not by default), always include the output. This is inconsequential for a \texttt{FiniteStateMachine}, but can be used in derived classes where the output is suppressed by default, cf. \texttt{Automaton.process()}.

- **format_output** – a function that translates the written output (which is in form of a list) to something more readable. By default (None) identity is used here.

- **check_epsilon_transitions** – (default: True) a boolean. If False, then epsilon transitions are not taken into consideration during process.

- **write_final_word_out** – (default: True) a boolean specifying whether the final output words should be written or not.

- **use_multitape_input** – (default: False) a boolean. If True, then the multi-tape mode of the process iterator is activated. See also the notes below for multi-tape machines.

- **process_all_prefixes_of_input** – (default: False) a boolean. If True, then each prefix of the input word is processed (instead of processing the whole input word at once). Consequently, there is an output generated for each of these prefixes.

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• **process_iterator_class** – (default: None) a class inherited from `FSMProcessIterator`. If None, then `FSMProcessIterator` is taken. An instance of this class is created and is used during the processing.

• **automatic_output_type** – (default: False) a boolean. If set and the input has a parent, then the output will have the same parent. If the input does not have a parent, then the output will be of the same type as the input.

**OUTPUT:**

A triple (or a list of triples, cf. parameter `list_of_outputs`), where

• the first entry is **True** if the input string is accepted,

• the second gives the reached state after processing the input tape (This is a state with label **None** if the input could not be processed, i.e., if at one point no transition to go on could be found.), and

• the third gives a list of the output labels written during processing (in the case the finite state machine runs as transducer).

Note that in the case the finite state machine is not deterministic, all possible paths are taken into account.

This function uses an iterator which, in its simplest form, goes from one state to another in each step. To decide which way to go, it uses the input words of the outgoing transitions and compares them to the input tape. More precisely, in each step, the iterator takes an outgoing transition of the current state, whose input label equals the input letter of the tape. The output label of the transition, if present, is written on the output tape.

If the choice of the outgoing transition is not unique (i.e., we have a non-deterministic finite state machine), all possibilities are followed. This is done by splitting the process into several branches, one for each of the possible outgoing transitions.

The process (iteration) stops if all branches are finished, i.e., for no branch, there is any transition whose input word coincides with the processed input tape. This can simply happen when the entire tape was read.

Also see **__call__()** for a version of **process()** with shortened output.

Internally this function creates and works with an instance of `FSMProcessIterator`. This iterator can also be obtained with `iter_process()`.

If working with multi-tape finite state machines, all input words of transitions are words of $k$-tuples of letters. Moreover, the input tape has to consist of $k$ tracks, i.e., be a list or tuple of $k$ iterators, one for each track.

**Warning:** Working with multi-tape finite state machines is still experimental and can lead to wrong outputs.

**EXAMPLES:**

```python
sage: binary_inverter = FiniteStateMachine({'A': [('A', 0, 1), ('A', 1, 0)]},
             initial_states=['A'], final_states=['A'])

sage: binary_inverter.process([0, 1, 0, 0, 1, 1])
(True, 'A', [1, 0, 1, 1, 0, 0])
```

Alternatively, we can invoke this function by:

```python
sage: binary_inverter([0, 1, 0, 0, 1, 1])
(True, 'A', [1, 0, 1, 1, 0, 0])
```
Below we construct a finite state machine which tests if an input is a non-adjacent form, i.e., no two neighboring letters are both nonzero (see also the example on non-adjacent forms in the documentation of the module Finite state machines, automata, transducers):

```python
sage: NAF = FiniteStateMachine(
    initial_states=["_",], final_states=["_", 1])

sage: [NAF.process(w)[0] for w in [[0], [0, 1], [1, 1], [0, 1, 0, 1],
                               [0, 1, 1, 1, 0], [1, 0, 0, 1, 1]]
[True, True, False, True, False, False]
```

Working only with the first component (i.e., returning whether accepted or not) usually corresponds to using the more specialized class Automaton.

Non-deterministic finite state machines can be handled as well.

```python
sage: T = Transducer([(0, 1, 0, 0), (0, 2, 0, 0)],
    initial_states=[0])

sage: T.process([0])
[(False, 1, [0]), (False, 2, [0])]
```

Here is another non-deterministic finite state machine. Note that we use format_output (see FSMProcessIterator) to convert the written outputs (all characters) to strings.

```python
sage: T = Transducer([(0, 1, [0, 0], 'a'), (0, 2, [0, 0, 1], 'b'),
                   (0, 1, 1, 'c'), (1, 0, [], 'd'),
                   (1, 1, 1, 'e')],
    initial_states=[0], final_states=[0, 1])

sage: T.process([0, 0], format_output=lambda o: ''.join(o),
               only_accepted=True)
[(True, 0, 'adcd'), (True, 0, 'aed'),
 (True, 1, 'adc'), (True, 1, 'ae')]
```

A simple example of a multi-tape finite state machine is the following: It writes the length of the first tape many letters a and then the length of the second tape many letters b:

```python
sage: M = FiniteStateMachine([(0, 0, (1, None), 'a'),
                            (0, 1, [], []),
                            (1, 1, (None, 1), 'b')],
                           initial_states=[0],
                           only_accepted=True)
```

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final_states=[1])
sage: M.process(((1, 1), [1]), use_multitape_input=True)
(True, 1, ['a', 'a', 'b'])

See also:
Automaton.process(), Transducer.process(), iter_process(), __call__(),
FSMProcessIterator.

product_FiniteStateMachine(other, function, new_input_alphabet=None,
only_accessible_components=True, final_function=None,
new_class=None)

Return a new finite state machine whose states are \(d\)-tuples of states of the original finite state machines.

INPUT:
- other – a finite state machine (for \(d = 2\)) or a list (or iterable) of \(d - 1\) finite state machines.
- function has to accept \(d\) transitions from \(A_j\) to \(B_j\) for \(j \in \{1, \ldots, d\}\) and returns a pair \((\text{word_in}, \text{word_out})\) which is the label of the transition \(A = (A_1, \ldots, A_d)\) to \(B = (B_1, \ldots, B_d)\). If there is no transition from \(A\) to \(B\), then function should raise a LookupError.
- new_input_alphabet (optional) – the new input alphabet as a list.
- only_accessible_components – If True (default), then the result is piped through accessible_components(). If no new_input_alphabet is given, it is determined by determine_alphabets().
- final_function – A function mapping \(d\) final states of the original finite state machines to the final output of the corresponding state in the new finite state machine. By default, the final output is the empty word if both final outputs of the constituent states are empty; otherwise, a ValueError is raised.
- new_class – Class of the new finite state machine. By default (None), the class of self is used.

OUTPUT:
A finite state machine whose states are \(d\)-tuples of states of the original finite state machines. A state is initial or final if all constituent states are initial or final, respectively.

The labels of the transitions are defined by function.

The final output of a final state is determined by calling final_function on the constituent states.

The color of a new state is the tuple of colors of the constituent states of self and other. However, if all constituent states have color None, then the state has color None, too.

EXAMPLES:

sage: F = Automaton([('A', 'B', 1), ('A', 'A', 0), ('B', 'A', 2)],
....: initial_states=[A'], final_states=[B'],
....: determine_alphabets=True)

sage: G = Automaton([(1, 1, 1)], initial_states=[1], final_states=[1])

sage: def addition(transition1, transition2):
....: return (transition1.word_in[0] + transition2.word_in[0],
....: None)

sage: H = F.product_FiniteStateMachine(G, addition, [0, 1, 2, 3], only_accessible_components=False)

sage: H.transitions()
Transition from ('A', 1) to ('A', 1): 1|-
Transition from ('B', 1) to ('A', 1): 3|-
sage: [s.color for s in H.iter_states()]
[None, None]
sage: H1 = F.product_FiniteStateMachine(G, addition, [0, 1, 2, 3], only_accessable_components=False)
sage: H1.states()[0].label()[0] is F.states()[0]
True
sage: H1.states()[0].label()[1] is G.states()[0]
True

sage: F = Automaton([(0, 1, 1/4), (0, 0, 3/4), (1, 1, 3/4), (1, 0, 1/4)],
......: initial_states=[0] )
sage: G = Automaton([(0, 0, 1), (1, 1, 3/4), (1, 0, 1/4)],
......: initial_states=[0] )
sage: H = F.product_FiniteStateMachine(G,
......:   lambda t1,t2: (t1.word_in[0]*t2.word_in[0], None)
......:   only_accessible_components=False)
sage: H.states()
[(0, 0), (1, 0), (0, 1), (1, 1)]

Also final output words are considered according to the function final_function:

sage: F = Transducer([(0, 1, 0, 1), (1, 1, 1, 1), (1, 1, 0, 1)],
......: final_states=[1])
sage: F.state(1).final_word_out = 1
sage: G = Transducer([(0, 0, 0, 1), (0, 0, 1, 0)], final_states=[0])
sage: G.state(0).final_word_out = 1
sage: def minus(t1, t2):
.....: return (t1.word_in[0] - t2.word_in[0],
.....: t1.word_out[0] - t2.word_out[0])
sage: H = F.product_FiniteStateMachine(G, minus)
Traceback (most recent call last):
... ValueError: A final function must be given.
sage: def plus(s1, s2):
.....: return s1.final_word_out[0] + s2.final_word_out[0]
sage: H = F.product_FiniteStateMachine(G, minus,
.....: final_function=plus)
sage: H.final_states()
[(1, 0)]
sage: H.final_states()[0].final_word_out
Products of more than two finite state machines are also possible:

```python
sage: def plus(s1, s2, s3):
    ....:     if s1.word_in == s2.word_in == s3.word_in:
    ....:         return (s1.word_in,
    ....:                 sum(s.word_out[0] for s in (s1, s2, s3)))
    ....:     else:
    ....:         raise LookupError
sage: T0 = transducers.CountSubblockOccurrences([0, 0], [0, 1, 2])
sage: T1 = transducers.CountSubblockOccurrences([1, 1], [0, 1, 2])
sage: T2 = transducers.CountSubblockOccurrences([2, 2], [0, 1, 2])
sage: T = T0.product_FiniteStateMachine([T1, T2], plus)
sage: T.transitions()
[Transition from ((), (), ()) to ((0,), (), ()): 0|0,
 Transition from ((), (), ()) to ((), (1,), ()): 1|0,
 Transition from ((), (), ()) to ((), (), (2,)): 2|0,
 Transition from ((0,), (), ()) to ((0,), (), ()): 0|1,
 Transition from ((0,), (), ()) to ((1,), (), ()): 1|0,
 Transition from ((0,), (), ()) to (((), (), (2,)): 2|0,
 Transition from ((0, (1,)), (0,)): 0|0,
 Transition from ((0, (1,)), (0,)): 0|1,
 Transition from ((0, (1,)), (0,)): 1|1,
 Transition from ((0, (1,)), (0,)): 2|0,
 Transition from ((0, (0, (2,))), (0,)): 0|0,
 Transition from ((0, (0, (2,))), (0,)): 0|1,
 Transition from ((0, (0, (2,))), (0,)): 2|1]
sage: T([0, 0, 1, 1, 2, 2, 0, 1, 2, 2])
[0, 1, 0, 1, 0, 1, 0, 0, 0, 1]
other can also be an iterable:
```

```python
sage: T == T0.product_FiniteStateMachine(iter([T1, T2]), plus)
True
```

**projection** (*what='input'*)

Return an Automaton which transition labels are the projection of the transition labels of the input.

**INPUT:**

* what – (default: input) either input or output.

**OUTPUT:**

An automaton.

**EXAMPLES:**

```python
sage: F = FiniteStateMachine([('A', 'B', 0, 1), ('A', 'A', 1, 1),
....:                          ('B', 'B', 1, 0)])
sage: G = F.projection(what='output')
sage: G.transitions()
[Transition from 'A' to 'B': 1|--,
 Transition from 'A' to 'A': 1|--,
 Transition from 'B' to 'B': 0|--]
```
**quotient**(*classes*)

Construct the quotient with respect to the equivalence classes.

**INPUT:**

- *classes* is a list of equivalence classes of states.

**OUTPUT:**

A finite state machine.

The labels of the new states are tuples of states of the *self*, corresponding to *classes*.

Assume that *c* is a class, and *a* and *b* are states in *c*. Then there is a bijection \(\varphi\) between the transitions from *a* and the transitions from *b* with the following properties: if \(\varphi(t_a) = t_b\), then

- \(t_a\).\text{word}_{\text{in}} = t_b\text{.word}_{\text{in}}\,
- \(t_a\).\text{word}_{\text{out}} = t_b\text{.word}_{\text{out}}\), and
- \(t_a\) and \(t_b\) lead to some equivalent states \(a'\) and \(b'\).

Non-initial states may be merged with initial states, the resulting state is an initial state.

All states in a class must have the same \text{is\_final}, \text{final\_word\_out} and \text{word\_out} values.

**EXAMPLES:**

```
sage: fsm = FiniteStateMachine([("A", "B", 0, 1), ("A", "B", 1, 0), ...
       ("B", "C", 0, 0), ("B", "C", 1, 1), ...
       ("C", "D", 0, 1), ("C", "D", 1, 0), ...
       ("D", "A", 0, 0), ("D", "A", 1, 1)])
sage: fsmq = fsm.quotient([(fsm.state("A")), fsm.state("C")), ...
                        (fsm.state("B")), fsm.state("D"))])
sage: fsmq.transitions()
[Transition from ('A', 'C')
  to ('B', 'D'): 0|1,
 Transition from ('A', 'C')
  to ('B', 'D'): 1|0,
 Transition from ('B', 'D')
  to ('A', 'C'): 0|0,
 Transition from ('B', 'D')
  to ('A', 'C'): 1|1]
sage: fsmq.relabeled().transitions()
[Transition from 0 to 1: 0|1,
 Transition from 0 to 1: 1|0,
 Transition from 1 to 0: 0|0,
 Transition from 1 to 0: 1|1]]
sage: fsmql = fsm.quotient(fsm.equivalence_classes())
sage: fsmql == fsmq
True
sage: fsmq.quotient([(fsm.state("A")), fsm.state("B"), fsm.state("C"), fsm.state("D"))])
Traceback (most recent call last):
  ...
AssertionError: Transitions of state 'A' and 'B' are incompatible.
```

**relabelled**(*memo=None, labels=None*)

Return a deep copy of the finite state machine, but the states are relabeled.
INPUT:
  • memo – (default: None) a dictionary storing already processed elements.
  • labels – (default: None) a dictionary or callable mapping old labels to new labels. If None, then the new labels are integers starting with 0.

OUTPUT:
A new finite state machine.

EXAMPLES:

```
sage: FSM1 = FiniteStateMachine([('A', 'B'), ('B', 'C'), ('C', 'A')])
sage: FSM1.states()
['A', 'B', 'C']
sage: FSM2 = FSM1.relabeled()
sage: FSM2.states()
[0, 1, 2]
sage: FSM3 = FSM1.relabeled(label={ 'A': 'a', 'B': 'b', 'C': 'c' })
sage: FSM3.states()
['a', 'b', 'c']
sage: FSM4 = FSM2.relabeled(lambda x: 2*x)
sage: FSM4.states()
[0, 2, 4]
```

remove_epsilon_transitions()

set_coordinates(coordinates, default=True)

Set coordinates of the states for the LaTeX representation by a dictionary or a function mapping labels to coordinates.

INPUT:
  • coordinates – a dictionary or a function mapping labels of states to pairs interpreted as coordinates.
  • default – If True, then states not given by coordinates get a default position on a circle of radius 3.

OUTPUT:
Nothing.

EXAMPLES:

```
sage: F = Automaton([(0, 1, 1), (1, 2, 2), (2, 0, 0)])
sage: F.set_coordinates({0: (0, 0), 1: (2, 0), 2: (1, 1)})
sage: F.state(0).coordinates
(0, 0)
We can also use a function to determine the coordinates:
```
sage: F = Automaton([(0, 1, 1), (1, 2, 2), (2, 0, 0)])
sage: F.set_coordinates(lambda x: (1, 3/(1+1)))
sage: F.state(2).coordinates
(2, 1)
```

split_transitions()

Return a new transducer, where all transitions in self with input labels consisting of more than one letter are replaced by a path of the corresponding length.
OUTPUT:
A new transducer.

EXAMPLES:

```
sage: A = Transducer([('A', 'B', [1, 2, 3], 0)],
.....:     initial_states=['A'], final_states=['B'])
sage: A.split_transitions().states()
[('A', ()), ('B', ()),
 ('A', (1,)), ('A', (1, 2))]
```

```
state(state)
```

Return the state of the finite state machine.

INPUT:
- state – If state is not an instance of FSMState, then it is assumed that it is the label of a state.

OUTPUT:
The state of the finite state machine corresponding to state.
If no state is found, then a LookupError is thrown.

EXAMPLES:

```
sage: from sage.combinat.finite_state_machine import FSMState
sage: A = FSMState('A')
sage: FSM = FiniteStateMachine([(A, 'B'), ('C', A)])
sage: FSM.state('A') == A
True
sage: FSM.state('xyz')
Traceback (most recent call last):
  ...
LookupError: No state with label xyz found.
```

```
states()
```

Return the states of the finite state machine.

OUTPUT:
The states of the finite state machine as list.

EXAMPLES:

```
sage: FSM = Automaton([(1, 2, 1), (2, 2, 0)])
sage: FSM.states()
['1', '2']
```

```
transition(transition)
```

Return the transition of the finite state machine.

INPUT:
- transition – If transition is not an instance of FSMTransition, then it is assumed that it is a tuple (from_state, to_state, word_in, word_out).

OUTPUT:
The transition of the finite state machine corresponding to transition.
If no transition is found, then a LookupError is thrown.

EXAMPLES:

```python
sage: from sage.combinat.finite_state_machine import FSMTransition
sage: t = FSMTransition('A', 'B', 0)

sage: F = FiniteStateMachine([t])

sage: F.transition(('A', 'B', 0))
Transition from 'A' to 'B': 0\-

sage: id(t) == id(F.transition(('A', 'B', 0)))
True
```

transitions(from_state=None)

Return a list of all transitions.

INPUT:

• from_state – (default: None) If from_state is given, then a list of transitions starting there is given.

OUTPUT:

A list of all transitions.

EXAMPLES:

```python
sage: FSM = Automaton([(1, 2, 1), (2, 2, 0)])

sage: FSM.transitions()
[Transition from '1' to '2': 1\-,
 Transition from '2' to '2': 0\-]
```

transposition(reverse_output_labels=True)

Return a new finite state machine, where all transitions of the input finite state machine are reversed.

INPUT:

• reverse_output_labels – a boolean (default: True): whether to reverse output labels.

OUTPUT:

A new finite state machine.

EXAMPLES:

```python
sage: aut = Automaton([(A, A, 0), (A, A, 1), (A, B, 0)],
.....: initial_states=['A'], final_states=['B'])

sage: aut.transposition().transitions('B')
[Transition from 'B' to 'A': 0\-]

sage: aut = Automaton([(1, 1, 1), (1, 2, 0), (2, 2, 0)],
.....: initial_states=['1'], final_states=['1', '2'])

sage: aut.transposition().initial_states()
['1', '2']

sage: A = Automaton([(0, 1, [1, 0])],
.....: initial_states=[0],
.....: final_states=[1])

sage: A([1, 0])
True
```

(continues on next page)
sage: A.transposition()([0, 1])
True

sage: T = Transducer([([0, 1, [1, 0], [1, 0]],),
.....:     initial_states=[0],
.....:     final_states=[1])

sage: T([1, 0])
[1, 0]
sage: T.transposition()([0, 1])
[0, 1]
sage: T.transposition(reverse_output_labels=False)([0, 1])
[1, 0]

\textbf{with\_final\_word\_out}(\textit{letters, allow\_non\_final=True})

Constructs a new finite state machine with final output words for all states by implicitly reading trailing letters until a final state is reached.

**INPUT:**

- \textit{letters} – either an element of the input alphabet or a list of such elements. This is repeated cyclically when needed.
- \textit{allow\_non\_final} – a boolean (default: True) which indicates whether we allow that some states may be non-final in the resulting finite state machine. I.e., if False then each state has to have a path to a final state with input label matching \textit{letters}.

**OUTPUT:**

A finite state machine.

The inplace version of this function is \textbf{construct\_final\_word\_out()}.

Suppose for the moment a single element \textit{letter} as input for \textit{letters}. This is equivalent to \textit{letters} = [\textit{letter}]. We will discuss the general case below.

Let \textit{word\_in} be a word over the input alphabet and assume that the original finite state machine transforms \textit{word\_in} to \textit{word\_out} reaching a possibly non-final state \textit{s}. Let further \textit{k} be the minimum number of \textit{letters} \textit{letter} such that there is a path from \textit{s} to some final state \textit{f} whose input label consists of \textit{k} copies of \textit{letter} and whose output label is \textit{path\_word\_out}. Then the state \textit{s} of the resulting finite state machine is a final state with final output \textit{path\_word\_out} + \textit{f.final\_word\_out}. Therefore, the new finite state machine transforms \textit{word\_in} to \textit{word\_out} + \textit{path\_word\_out} + \textit{f.final\_word\_out}.

This is e.g. useful for finite state machines operating on digit expansions: there, it is sometimes required to read a sufficient number of trailing zeros (at the most significant positions) in order to reach a final state and to flush all carries. In this case, this method constructs an essentially equivalent finite state machine in the sense that it not longer requires adding sufficiently many trailing zeros. However, it is the responsibility of the user to make sure that if adding trailing zeros to the input anyway, the output is equivalent.

If \textit{letters} consists of more than one letter, then it is assumed that (not necessarily complete) cycles of \textit{letters} are appended as trailing input.

**See also:**

\textit{example on Gray code}

**EXAMPLES:**

1. A simple transducer transforming 00 blocks to 01 blocks:
sage: T = Transducer([(0, 1, 0, 0), (1, 0, 0, 1)],
.....:     initial_states=[0],
.....:     final_states=[0])
sage: T.process([0, 0, 0])
(False, 1, [0, 1, 0])
sage: T.process([0, 0, 0, 0])
(True, 0, [0, 1, 0, 1])
sage: F = T.with_final_word_out(0)
sage: for f in F.iter_final_states():
.....:     print("{} {}", format(f, f.final_word_out))
0 []
1 [1]
sage: F.process([0, 0, 0])
(True, 1, [0, 1, 0, 1])
sage: F.process([0, 0, 0, 0])
(True, 0, [0, 1, 0, 1])

2. A more realistic example: Addition of 1 in binary. We construct a transition function transforming the input to its binary expansion:

sage: def binary_transition(carry, input):
.....:     value = carry + input
.....:     if value.mod(2) == 0:
.....:         return (value/2, 0)
.....:     else:
.....:         return ((value-1)/2, 1)

Now, we only have to start with a carry of 1 to get the required transducer:

sage: T = Transducer(binary_transition,
.....:     input_alphabet=[0, 1],
.....:     initial_states=[1],
.....:     final_states=[0])

We test this for the binary expansion of 7:

sage: T.process([1, 1, 1])
(False, 1, [0, 0, 0])

The final carry 1 has not be flushed yet, we have to add a trailing zero:

sage: T.process([1, 1, 1, 0])
(True, 0, [0, 0, 0, 1])

We check that with this trailing zero, the transducer performs as advertised:

sage: all(ZZ(T(k.bits()+[0]), base=2) == k + 1
.....:     for k in srange(16))
True

However, most of the time, we produce superfluous trailing zeros:

sage: T(11.bits()+[0])
[0, 0, 1, 1, 0]
We now use this method:

```
sage: F = T.with_final_word_out(0)
sage: for f in F.iter_final_states():
    ....:    print("{} {}\n{}[]")
1 [1]
0 []
```

The same tests as above, but we do not have to pad with trailing zeros anymore:

```
sage: F.process([1, 1, 1])
(True, 1, [0, 0, 0, 1])
sage: all(ZZ(F(k.bits()), base=2) == k + 1
    ....:    for k in srange(16))
True
```

No more trailing zero in the output:

```
sage: F([1, 1])
[0, 0, 1, 1]
sage: all(F(k.bits())[-1] == 1
    ....:    for k in srange(16))
True
```

3. Here is an example, where we allow trailing repeated 10:

```
sage: T = Transducer([(0, 1, 0, 'a'),
    ....:    (1, 2, 1, 'b'),
    ....:    (2, 0, 0, 'c'),
    ....:    initial_states=[0],
    ....:    final_states=[0])
sage: F = T.with_final_word_out([0, 1])
sage: for f in F.iter_final_states():
    ....:    print(str(f) + ' ' + ''.join(f.final_word_out))
0
1 bc
```

Trying this with trailing repeated 01 does not produce a final_word_out for state 1, but for state 2:

```
sage: F = T.with_final_word_out([0, 1])
sage: for f in F.iter_final_states():
    ....:    print(str(f) + ' ' + ''.join(f.final_word_out))
0
1 bc
```

4. Here another example with a more-letter trailing input:

```
sage: T = Transducer([(0, 1, 0, 'a'),
    ....:    (1, 2, 0, 'b'), (1, 2, 1, 'b'),
    ....:    (2, 3, 0, 'c'), (2, 0, 1, 'e'),
    ....:    (3, 1, 0, 'd'), (3, 1, 1, 'd'),
    ....:    initial_states=[0],
    ....:    final_states=[0],
    ....:    with_final_word_out=[0, 0, 1, 1])
```

(continues on next page)
sage: for f in T.iter_final_states():
      print(str(f) + ' ' + ''.join(f.final_word_out))
0
1 bcdbcdbe
2 cdbe
3 dbe

class sage.combinat.finite_state_machine.Transducer(data=None, initial_states=None,
final_states=None, input_alphabet=None, output_alphabet=None,
output_alphabets=None, with_final_word_out=None,
store_states_dict=True, on_duplicate_transition=None)

Bases: FiniteStateMachine

This creates a transducer, which is a finite state machine, whose transitions have input and output labels.

An transducer has additional features like creating a simplified transducer.

See class FiniteStateMachine for more information.

EXAMPLES:

We can create a transducer performing the addition of 1 (for numbers given in binary and read from right to left) in the following way:

sage: T = Transducer([("C", "C", 1, 0), ("C", "N", 0, 1),
                  ("N", "N", 0, 0), ("N", "N", 1, 1)],
                  initial_states=["C"], final_states=["N"])

sage: T
Transducer with 2 states

sage: T([(0)])
[1]
sage: T([(1,1,0)])
[0, 0, 1]
sage: ZZ(T(15.digits(base=2)+[0]), base=2)
16

Note that we have padded the binary input sequence by a 0 so that the transducer can reach its final state.

cartesian_product(other, only_accessible_components=True)

Return a new transducer which can simultaneously process an input with the machines self and other where the output labels are d-tuples of the original output labels.

INPUT:

• other - a finite state machine (if d = 2) or a list (or other iterable) of d − 1 finite state machines
• only_accessible_components - If True (default), then the result is piped through accessible_components(). If no new_input_alphabet is given, it is determined by determine_alphabets().

OUTPUT:

A transducer which can simultaneously process an input with self and the machine(s) in other.

The set of states of the new transducer is the Cartesian product of the set of states of self and other.
Let \((A_j, B_j, a_j, b_j)\) for \(j \in \{1, \ldots, d\}\) be transitions in the machines self and in other. Then there is a transition \((A_1, \ldots, A_d), (B_1, \ldots, B_d), a, (b_1, \ldots, b_d)\) in the new transducer if \(a_1 = \cdots = a_d = a\).

**EXAMPLES:**

```sage
transducer1 = Transducer([('A', 'A', 0, 0),
...:                         ('A', 'A', 1, 1)],
...: initial_states=['A'],
...: final_states=['A'],
...: determine_alphabets=True)

transducer2 = Transducer([[0, 1, 0, ['b', 'c']],
...:                         (0, 0, 1, 'b'),
...:                         (1, 1, 0, 'a')],
...: initial_states=[0],
...: final_states=[1],
...: determine_alphabets=True)

result = transducer1.cartesian_product(transducer2)

result.transitions()

[Transition from ('A', 0) to ('A', 1): 0|(0, 'b'), (None, 'c'),
  Transition from ('A', 0) to ('A', 0): 1|(1, 'b'),
  Transition from ('A', 1) to ('A', 1): 0|(0, 'a')]

result([1, 0, 0])

[(1, 'b'), (0, 'b'), (None, 'c'), (0, 'a')]

(transducer1([1, 0, 0]), transducer2([1, 0, 0]))

([1, 0, 0], ['b', 'b', 'c', 'a'])
```

Also final output words are correctly processed:

```sage
transducer1.state('A').final_word_out = 2

result = transducer1.cartesian_product(transducer2)

result.final_states()[0].final_word_out

[(2, None)]
```

The following transducer counts the number of 11 blocks minus the number of 10 blocks over the alphabet \([0, 1]\).

```sage
count_11 = transducers.CountSubblockOccurrences([1, 1],
...: input_alphabet=[0, 1])
count_10 = transducers.CountSubblockOccurrences([1, 0],
...: input_alphabet=[0, 1])
count_11x10 = count_11.cartesian_product(count_10)
difference = transducers.sub([0, 1])(count_11x10)
T = difference.simplification().relabeled()
T.initial_states()

[1]

sorted(T.transitions())

[Transition from 0 to 1: 0|-1,
  Transition from 0 to 0: 1|1,
  Transition from 1 to 1: 0|0,
  Transition from 1 to 0: 1|0]
```

(continues on next page)
If `other` is an automaton, then `cartesian_product()` returns `self` where the input is restricted to the input accepted by `other`.

For example, if the transducer transforms the standard binary expansion into the non-adjacent form and the automaton recognizes the binary expansion without adjacent ones, then the Cartesian product of these two is a transducer which does not change the input (except for changing `a` to `(a, None)` and ignoring a leading `0`).

```sage
sage: NAF = Transducer([(0, 1, 0, None), ....: (0, 2, 1, None), ....: (1, 1, 0, 0), ....: (1, 2, 1, 0), ....: (2, 1, 0, 1), ....: (2, 3, 1, -1), ....: (3, 2, 0, 0), ....: (3, 3, 1, 0)], ....: initial_states=[0], ....: final_states=[1], ....: determine_alphabets=True)
```

```sage
sage: aut11 = Automaton([(0, 0, 0), (0, 1, 1), (1, 0, 0)], ....: initial_states=[0], ....: final_states=[0, 1], ....: determine_alphabets=True)
```

```sage
sage: res = NAF.cartesian_product(aut11)
```

```sage
sage: res([1, 0, 0, 1, 0, 1, 0])
[(1, None), (0, None), (0, None), (1, None), (0, None), (1, None)]
```

This is obvious because if the standard binary expansion does not have adjacent ones, then it is the same as the non-adjacent form.

Be aware that `cartesian_product()` is not commutative.

```sage
sage: aut11.cartesian_product(NAF)
Traceback (most recent call last):
... TypeError: Only an automaton can be intersected with an automaton.
```

The Cartesian product of more than two finite state machines can also be computed:

```sage
sage: T0 = transducers.CountSubblockOccurrences([0, 0], [0, 1, 2])
sage: T1 = transducers.CountSubblockOccurrences([1, 1], [0, 1, 2])
sage: T2 = transducers.CountSubblockOccurrences([2, 2], [0, 1, 2])
sage: T = T0.cartesian_product([T1, T2])
sage: T.transitions()
[Transition from ((), (), ()): (0, 0, 0), Transition from ((), (), ()): (0, (1, (), ()): (0, 0, 0), Transition from ((), (), ()): ((), (), ()): (0, (1, (), ()): (0, 0, 0), Transition from ((0,), ((), ()): (0, (1, (), ()): (0, 0, 0), Transition from ((0,), ((), ()): (0, (1, (), ()): (0, 0, 0), Transition from ((0,), ((), ()): (0, (1, (), ()): (0, 0, 0),]
```
Transition from ((0,), (), ()) to ((0,), (), (2,)): 2|(0, 0, 0),
Transition from ((0,), (1,), ()) to ((0,), (), ()): 0|(0, 0, 0),
Transition from ((0,), (1,), ()) to ((1,), (1,), ()): 1|(0, 1, 0),
Transition from ((0,), (1,), ()) to ((0,), (2,)): 2|(0, 0, 0),
Transition from ((0,), (), (2,)) to ((0,), (), ()): 0|(0, 0, 0),
Transition from ((0,), (), (2,)) to ((1,), (1,), ()): 1|(0, 0, 0),
Transition from ((0,), (), (2,)) to ((0,), (2,)): 2|(0, 0, 1)

\[
\text{sage: } T([0, 0, 1, 1, 2, 2, 0, 1, 2, 2])
\]
\[(0, 0, 0),
(1, 0, 0),
(0, 0, 0),
(0, 1, 0),
(0, 0, 0),
(0, 0, 1),
(0, 0, 0),
(0, 0, 0),
(0, 0, 0)
\]

**intersection** (*other*, *only_accessible_components=True*)

Return a new transducer which accepts an input if it is accepted by both given finite state machines producing the same output.

**INPUT:**

- *other* – a transducer
- *only_accessible_components* – If True (default), then the result is piped through `accessible_components()`. If no new_input_alphabet is given, it is determined by `determine_alphabets()`.

**OUTPUT:**

A new transducer which computes the intersection (see below) of the languages of *self* and *other*.

The set of states of the transducer is the Cartesian product of the set of states of both given transducer. There is a transition ((\(A, B\), \(C, D\), \(a, b\)) in the new transducer if there are transitions \((A, C, a, b)\) and \((B, D, a, b)\) in the old transducers.

**EXAMPLES:**

\[
\text{sage: } \text{transducer1 = Transducer([\('1', '2', 1, 0\),}
\text{\(\text{\ldots: ('2', \text{'2'}, 1, 0),}
\text{\ldots: ('2', \text{'2'}, 0, 1)],}
\text{\ldots: initial_states=['1'],}
\text{\ldots: final_states=['2'])}
\text{sage: transducer2 = Transducer([\('A', 'A', 1, 0\),}
\text{\(\text{\ldots: ('A', 'B', 0, 0),}
\text{\ldots: ('B', 'B', 0, 1),}
\text{\ldots: ('B', 'A', 1, 1)],}
\text{\ldots: initial_states=['A'],}
\text{\ldots: final_states=['B'])}
\text{sage: res = transducer1.intersection(transducer2)}
\text{sage: res.transitions()}
\]
\[
[\text{Transition from ('1', 'A') to ('2', 'A'): 1|0,}
\text{Transition from ('2', 'A') to ('2', 'A'): 1|0}\]

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In general, transducers are not closed under intersection. But for transducer which do not have epsilon-transitions, the intersection is well defined (cf. [BaWo2012]). However, in the next example the intersection of the two transducers is not well defined. The intersection of the languages consists of $(a^n, b^nc^n)$. This set is not recognizable by a finite transducer.

```
sage: t1 = Transducer([(0, 0, 'a', 'b'),
...:                  (0, 1, None, 'c'),
...:                  (1, 1, None, 'c')],
...:                  initial_states=[0],
...:                  final_states=[0, 1])
sage: t2 = Transducer([('A', 'A', None, 'b'),
...:                  ('A', 'B', 'a', 'c'),
...:                  ('B', 'B', 'a', 'c')],
...:                  initial_states=['A'],
...:                  final_states=['A', 'B'])
sage: t2.intersection(t1)
Traceback (most recent call last):
  ...
ValueError: An epsilon-transition (with empty input or output) was found.
```

REFERENCES:

process(*args, **kwargs)

Return whether the transducer accepts the input, the state where the computation stops and which output is generated.

INPUT:

- **input_tape** – the input tape can be a list or an iterable with entries from the input alphabet. If we are working with a multi-tape machine (see parameter use_multitape_input and notes below), then the tape is a list or tuple of tracks, each of which can be a list or an iterable with entries from the input alphabet.

- **initial_state** or **initial_states** – the initial state(s) in which the machine starts. Either specify a single one with initial_state or a list of them with initial_states. If both are given, initial_state will be appended to initial_states. If neither is specified, the initial states of the finite state machine are taken.

- **list_of_outputs** – (default: None) a boolean or None. If True, then the outputs are given in list form (even if we have no or only one single output). If False, then the result is never a list (an exception is raised if the result cannot be returned). If list_of_outputs=None the method determines automatically what to do (e.g. if a non-deterministic machine returns more than one path, then the output is returned in list form).

- **only_accepted** – (default: False) a boolean. If set, then the first argument in the output is guaranteed to be True (if the output is a list, then the first argument of each element will be True).

- **full_output** – (default: True) a boolean. If set, then the full output is given, otherwise only the generated output (the third entry below only). If the input is not accepted, a ValueError is raised.

- **always_include_output** – if set (not by default), always include the output. This is inconsequential for a Transducer, but can be used in other classes derived from FiniteStateMachine where the output is suppressed by default, cf. Automaton.process().

- **format_output** – a function that translates the written output (which is in form of a list) to something more readable. By default (None) identity is used here.
• **check_epsilon_transitions** – (default: True) a boolean. If False, then epsilon transitions are not taken into consideration during process.

• **write_final_word_out** – (default: True) a boolean specifying whether the final output words should be written or not.

• **use_multitape_input** – (default: False) a boolean. If True, then the multi-tape mode of the process iterator is activated. See also the notes below for multi-tape machines.

• **process_all_prefixes_of_input** – (default: False) a boolean. If True, then each prefix of the input word is processed (instead of processing the whole input word at once). Consequently, there is an output generated for each of these prefixes.

• **process_iterator_class** – (default: None) a class inherited from `FSMProcessIterator`. If None, then `FSMProcessIterator` is taken. An instance of this class is created and is used during the processing.

• **automatic_output_type** – (default: False) a boolean. If set and the input has a parent, then the output will have the same parent. If the input does not have a parent, then the output will be of the same type as the input.

**OUTPUT:**

The full output is a triple (or a list of triples, cf. parameter `list_of_outputs`), where

- the first entry is True if the input string is accepted,
- the second gives the reached state after processing the input tape (This is a state with label None if the input could not be processed, i.e., if at one point no transition to go on could be found.), and
- the third gives a list of the output labels written during processing.

If `full_output` is False, then only the third entry is returned.

Note that in the case the transducer is not deterministic, all possible paths are taken into account.

This function uses an iterator which, in its simplest form, goes from one state to another in each step. To decide which way to go, it uses the input words of the outgoing transitions and compares them to the input tape. More precisely, in each step, the iterator takes an outgoing transition of the current state, whose input label equals the input letter of the tape. The output label of the transition, if present, is written on the output tape.

If the choice of the outgoing transition is not unique (i.e., we have a non-deterministic finite state machine), all possibilities are followed. This is done by splitting the process into several branches, one for each of the possible outgoing transitions.

The process (iteration) stops if all branches are finished, i.e., for no branch, there is any transition whose input word coincides with the processed input tape. This can simply happen when the entire tape was read.

Also see `__call__()` for a version of `process()` with shortened output.

Internally this function creates and works with an instance of `FSMProcessIterator`. This iterator can also be obtained with `iter_process()`.

If working with multi-tape finite state machines, all input words of transitions are words of $k$-tuples of letters. Moreover, the input tape has to consist of $k$ tracks, i.e., be a list or tuple of $k$ iterators, one for each track.

**Warning:** Working with multi-tape finite state machines is still experimental and can lead to wrong outputs.

**EXAMPLES:**
Combinatorics, Release 10.0

```sage
sage: binary_inverter = Transducer({'A': [('A', 0, 1), ('A', 1, 0)]},
initial_states = 'A', final_states = 'A')
sage: binary_inverter.process([0, 1, 0, 0, 1, 1])
(True, 'A', [1, 0, 1, 1, 0, 0])
```

If we are only interested in the output, we can also use:

```sage
sage: binary_inverter([0, 1, 0, 0, 1, 1])
[1, 0, 1, 1, 0, 0]
```

This can also be used with words as input:

```sage
sage: W = Words([0, 1]); W
Finite and infinite words over {0, 1}
sage: w = W([0, 1, 0, 0, 1, 1]); w
word: 010011
sage: binary_inverter(w)
word: 101100
```

In this case it is automatically determined that the output is a word. The call above is equivalent to:

```sage
sage: binary_inverter.process(w,
    full_output=False,
    list_of_outputs=False,
    automatic_output_type=True)
word: 101100
```

The following transducer transforms $0^n1$ to $1^n2$:

```sage
sage: T = Transducer([(0, 0, 0, 1), (0, 1, 1, 2)])
sage: T.state(0).is_initial = True
sage: T.state(1).is_final = True
```

We can see the different possibilities of the output by:

```sage
sage: [T.process(w) for w in 
[[1], [0, 1], [0, 0, 1], [0, 1, 1], 
[0], [0, 0], [2, 0], [0, 1, 2]]]
[(True, 1, [2]), (True, 1, [1, 2]),
 (False, None, None),
 (False, 0, [1]), (False, 0, [1, 1]),
 (False, None, None), (False, None, None)]
```

If we just want a condensed output, we use:

```sage
sage: [T.process(w, full_output=False)
    for w in 
[[1], [0, 1], [0, 0, 1]]]
[[2], [1, 2], [1, 1, 2]]
sage: T.process([0], full_output=False)
Traceback (most recent call last):
...
ValueError: Invalid input sequence.
sage: T.process([0, 1, 2], full_output=False)
Traceback (most recent call last):
(continues on next page)
```
... ValueException: Invalid input sequence.

It is equivalent to:

```python
sage: [T(w) for w in [[1], [0, 1], [0, 0, 1]]
[[2], [1, 2], [1, 1, 2]]
sage: T([0])
Traceback (most recent call last):
...
ValueException: Invalid input sequence.
```

A cycle with empty input and empty output is correctly processed:

```python
sage: T = Transducer([(0, 1, None, None), (1, 0, None, None)],
                   initial_states=[0], final_states=[1])
sage: T.process([])
[[False, 0, []], (True, 1, [])]
sage: _ = T.add_transition(-1, 0, 0, 'r')
sage: T.state(-1).is_initial = True
sage: T.state(0).is_initial = False
sage: T.process([])
[[False, 0, ['r']], (True, 1, ['r'])]
```

If there is a cycle with empty input but non-empty output, the possible outputs would be an infinite set:

```python
sage: T = Transducer([(0, 1, None, 'z'), (1, 0, None, None)],
                   initial_states=[0], final_states=[1])
sage: T.process([])
Traceback (most recent call last):
...
RuntimeError: State 0 is in an epsilon cycle (no input), but output is written.
```

But if this cycle with empty input and non-empty output is not reached, the correct output is produced:

```python
sage: _ = T.add_transition(-1, 0, 0, 'r')
sage: T.state(-1).is_initial = True
sage: T.state(0).is_initial = False
sage: T.process([])
(False, -1, [])
sage: T.process([0])
Traceback (most recent call last):
...
RuntimeError: State 0 is in an epsilon cycle (no input), but output is written.
```

If we set `check_epsilon_transitions=False`, then no transitions with empty input are considered anymore. Thus cycles with empty input are no problem anymore:
A simple example of a multi-tape transducer is the following: It writes the length of the first tape many letters a and then the length of the second tape many letters b:

```sage
M = Transducer(
[(0, 0, (1, None), 'a'),
 ....:
 (0, 1, [], []),
 ....:
 (1, 1, (None, 1), 'b')],
 ....:
 initial_states=[0],
 ....:
 final_states=[1])
```

```sage
M.process(([1, 1], [1]), use_multitape_input=True)
(1, 1, ['a', 'a', 'b'])
```

See also:


`simplification()`

Return a simplified transducer.

OUTPUT:

A new transducer.

This function simplifies a transducer by Moore’s algorithm, first moving common output labels of transitions leaving a state to output labels of transitions entering the state (cf. `prepone_output()`).

The resulting transducer implements the same function as the original transducer.

EXAMPLES:

```sage
fsm = Transducer([('A', 'A', 0, 0),
 ....:
 ('A', 'B', 1, 1),
 ....:
 ('A', 'C', 1, -1),
 ....:
 ('B', 'A', 2, 0),
 ....:
 ('C', 'A', 2, 0)])
```
Combinatorics, Release 10.0

(continued from previous page)

```python
sage: fsm_simplified = fsm.simplification()
sage: fsm_simplified
Transducer with 2 states
sage: fsm_simplified.transitions()
[Transition from ('A',) to ('A',): 0|0, Transition from ('A',) to ('B', 'C'): 1|1,0, Transition from ('A',) to ('B', 'C'): 1|-1,0, Transition from ('B', 'C') to ('A',): 2|-]
```

```python
sage: from sage.combinat.finite_state_machine import duplicate_transition_add_input
sage: T = Transducer([('A', 'A', 1/2, 0),
....: ('A', 'B', 1/4, 1),
....: ('A', 'C', 1/4, 1),
....: ('B', 'A', 1, 0),
....: ('C', 'A', 1, 0),
....: initial_states=[0],
....: final_states=['A', 'B', 'C'],
....: on_duplicate_transition=duplicate_transition_add_input)
sage: sorted(T.simplification().transitions())
[Transition from ('A',) to ('A',): 1/2|0, Transition from ('A',) to ('B', 'C'): 1/2|1, Transition from ('B', 'C') to ('A',): 1|0]
```

Illustrating the use of colors in order to avoid identification of states:

```python
sage: T = Transducer([[0,0,0,0], [0,1,1,1],
....: [1,0,0,0], [1,1,1,1]],
....: initial_states=[0],
....: final_states=[0,1])
sage: sorted(T.simplification().transitions())
[Transition from (0, 1) to (0, 1): 0|0, Transition from (0, 1) to (0, 1): 1|1]
sage: T.state(0).color = 0
sage: T.state(0).color = 1
sage: sorted(T.simplification().transitions())
[Transition from (0,) to (0,): 0|0, Transition from (0,) to (1,): 1|1, Transition from (1,) to (0,): 0|0, Transition from (1,) to (1,): 1|1]
```

```python
sage.combinat.finite_state_machine.duplicate_transition_add_input(old_transition, new_transition)
```

Alternative function for handling duplicate transitions in finite state machines. This implementation adds the input label of the new transition to the input label of the old transition. This is intended for the case where a Markov chain is modelled by a finite state machine using the input labels as transition probabilities.

See the documentation of the on_duplicate_transition parameter of FiniteStateMachine.

INPUT:

- `old_transition` – A transition in a finite state machine.
- `new_transition` – A transition, identical to `old_transition`, which is to be inserted into the finite state machine.
A transition whose input weight is the sum of the input weights of \texttt{old\_transition} and \texttt{new\_transition}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.combinat.finite_state_machine import duplicate_transition_add_input
da = from sage.combinat.finite_state_machine import FSMTransition
da: duplicate_transition_add_input(FSMTransition('a', 'a', 1/2),
....: FSMTransition('a', 'a', 1/2))
\end{verbatim}

Transition from 'a' to 'a': 1|- 

Input labels must be lists of length 1:

\begin{verbatim}
da: duplicate_transition_add_input(FSMTransition('a', 'a', [1, 1]),
....: FSMTransition('a', 'a', [1, 1]))
\end{verbatim}

Traceback (most recent call last):  
...  
TypeError: Trying to use duplicate_transition_add_input on 
"Transition from 'a' to 'a': 1,1|-" and 
"Transition from 'a' to 'a': 1,1|-", 
but input words are assumed to be lists of length 1

\texttt{sage.combinat.finite_state_machine.duplicate_transition\_ignore}(\texttt{old\_transition, new\_transition})

Default function for handling duplicate transitions in finite state machines. This implementation ignores the occurrence.

See the documentation of the \texttt{on\_duplicate\_transition} parameter of \texttt{FiniteStateMachine}.

\textbf{INPUT:}

\begin{itemize}
\item \texttt{old\_transition} – A transition in a finite state machine.
\item \texttt{new\_transition} – A transition, identical to \texttt{old\_transition}, which is to be inserted into the finite state machine.
\end{itemize}

\textbf{OUTPUT:}

The same transition, unchanged.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.combinat.finite_state_machine import duplicate_transition_ignore
da = from sage.combinat.finite_state_machine import FSMTransition
da: duplicate_transition_ignore(FSMTransition(0, 0, 1),
....: FSMTransition(0, 0, 1))
\end{verbatim}

Transition from 0 to 0: 1|- 

\texttt{sage.combinat.finite_state_machine.duplicate_transition\_raise\_error}(\texttt{old\_transition, new\_transition})

Alternative function for handling duplicate transitions in finite state machines. This implementation raises a \texttt{ValueError}.

See the documentation of the \texttt{on\_duplicate\_transition} parameter of \texttt{FiniteStateMachine}.

\textbf{INPUT:}

\begin{itemize}
\item \texttt{old\_transition} – A transition in a finite state machine.
\end{itemize}
• **new_transition** – A transition, identical to **old_transition**, which is to be inserted into the finite state machine.

**OUTPUT:**

Nothing. A **ValueError** is raised.

**EXAMPLES:**

```python
sage: from sage.combinat.finite_state_machine import duplicate_transition_raise_error
sage: FSMTransition(0, 0, 1)
...
ValueError: Attempting to re-insert transition Transition from 0 to 0: 1
```

sage.combinat.finite_state_machine.equal(iterator)

Checks whether all elements of **iterator** are equal.

**INPUT:**

• **iterator** – an iterator of the elements to check

**OUTPUT:**

True or False.

This implements https://stackoverflow.com/a/3844832/1052778.

**EXAMPLES:**

```python
sage: equal([0, 0, 0])
True
sage: equal([0, 1, 0])
False
sage: equal([])
True
sage: equal(iter([None, None]))
True
```

We can test other properties of the elements than the elements themselves. In the following example, we check whether all tuples have the same lengths:

```python
sage: equal(len(x) for x in [(1, 2), (2, 3), (3, 1)])
True
sage: equal(len(x) for x in [(1, 2), (1, 2, 3), (3, 1)])
False
```

sage.combinat.finite_state_machine.full_group_by(l, key=<function <lambda> at 0x7f39782f4940>)

Group iterable **l** by values of **key**.

**INPUT:**

• **iterable l**
• **key function key**
OUTPUT:
A list of pairs \((k, \text{elements})\) such that \(\text{key}(e) = k\) for all \(e\) in \(\text{elements}\).

This is similar to `itertools.groupby()` except that lists are returned instead of iterables and no prior sorting is required.

We do not require
- that the keys are sortable (in contrast to the approach via `sorted()` and `itertools.groupby()`) and
- that the keys are hashable (in contrast to the implementation proposed in https://stackoverflow.com/a/15250161).

However, it is required
- that distinct keys have distinct `str`-representations.

The implementation is inspired by https://stackoverflow.com/a/15250161, but non-hashable keys are allowed.

EXAMPLES:

```python
sage: from sage.combinat.finite_state_machine import full_group_by
sage: t = [2/x, 1/x, 2/x]  # optional - sage.symbolic
sage: r = full_group_by([0, 1, 2], key=lambda i: t[i])  # optional - sage.symbolic
sage: sorted(r, key=lambda p: p[1])  # optional - sage.symbolic
[(2/x, [0, 2]), (1/x, [1])]
```

Note that the behavior is different from `itertools.groupby()` because neither \(1/x < 2/x\) nor \(2/x < 1/x\) does hold.

Here, the result \(r\) has been sorted in order to guarantee a consistent order for the doctest suite.

```python
sage.combinat.finite_state_machine.is_Automaton(FSM)
Tests whether or not \(FSM\) inherits from `Automaton`.

sage.combinat.finite_state_machine.is_FSMProcessIterator(PI)
Tests whether or not \(PI\) inherits from `FSMProcessIterator`.

sage.combinat.finite_state_machine.is_FSMState(S)
Tests whether or not \(S\) inherits from `FSMState`.

sage.combinat.finite_state_machine.is_FSMTransition(T)
Tests whether or not \(T\) inherits from `FSMTransition`.

sage.combinat.finite_state_machine.is_FiniteStateMachine(FSM)
Tests whether or not \(FSM\) inherits from `FiniteStateMachine`.
```
sage.combinat.finite_state_machine.is_Transducer(FSM)
Tests whether or not FSM inherits from Transducer.

sage.combinat.finite_state_machine.setup_latex_preamble()
This function adds the package tikz with support for automata to the preamble of Latex so that the finite state machines can be drawn nicely.

See the section on LaTeX output in the introductory examples of this module.

sage.combinat.finite_state_machine.startswith(list_, prefix)
Determine whether list starts with the given prefix.

INPUT:
• list_ – list
• prefix – list representing the prefix

OUTPUT:
True or False.

Similar to str.startswith().

EXAMPLES:

```sage
from sage.combinat.finite_state_machine import startswith
sage: startswith([1, 2, 3], [1, 2])
True
sage: startswith([1], [1, 2])
False
sage: startswith([1, 3, 2], [1, 2])
False
```

sage.combinat.finite_state_machine.tupleofwords_to_wordoftuples(tupleofwords)
Transposes a tuple of words over the alphabet to a word of tuples.

INPUT:
• tupleofwords – a tuple of a list of letters.

OUTPUT:
A list of tuples.

Missing letters in the words are padded with the letter None (from the empty word).

EXAMPLES:

```sage
from sage.combinat.finite_state_machine import (tupleofwords_to_wordoftuples)
(((1, 2), (None, 4, None), (None, 5, None), (None, None, None))
```

sage.combinat.finite_state_machine.wordoftuples_to_tupleofwords(wordoftuples)
Transposes a word of tuples to a tuple of words over the alphabet.

INPUT:
• wordoftuples – a list of tuples of letters.
OUTPUT:
A tuple of lists.
Letters None (empty word) are removed from each word in the output.

EXAMPLES:

```python
sage: from sage.combinat.finite_state_machine import wordoftuples_to_tupleofwords
sage: wordoftuples_to_tupleofwords([(1, 2), (1, None), (1, None), (1, 2), (None, 2)])
([1, 1, 1, 1], [2, 2, 2])
```

5.1.108 Common Automata and Transducers (Finite State Machines Generators)

Automata and Transducers in Sage can be built through the `automata` and `transducers` objects, respectively. It contains generators for common finite state machines. For example,

```python
sage: I = transducers.Identity([0, 1, 2])
```

generates an identity transducer on the alphabet \{0, 1, 2\}.

To construct automata and transducers manually, you can use the classes `Automaton` and `Transducer`, respectively. See `Finite state machines, automata, transducers` for more details and a lot of examples.

### Automata

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>AnyLetter()</code></td>
<td>Return an automaton recognizing any letter.</td>
</tr>
<tr>
<td><code>AnyWord()</code></td>
<td>Return an automaton recognizing any word.</td>
</tr>
<tr>
<td><code>EmptyWord()</code></td>
<td>Return an automaton recognizing the empty word.</td>
</tr>
<tr>
<td><code>Word()</code></td>
<td>Return an automaton recognizing the given word.</td>
</tr>
<tr>
<td><code>ContainsWord()</code></td>
<td>Return an automaton recognizing words containing the given word.</td>
</tr>
</tbody>
</table>

### Transducers

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>Identity()</code></td>
<td>Returns a transducer realizing the identity map.</td>
</tr>
<tr>
<td><code>abs()</code></td>
<td>Returns a transducer realizing absolute value.</td>
</tr>
<tr>
<td><code>map()</code></td>
<td>Returns a transducer realizing a function.</td>
</tr>
<tr>
<td><code>operator()</code></td>
<td>Returns a transducer realizing a binary operation.</td>
</tr>
<tr>
<td><code>all()</code></td>
<td>Returns a transducer realizing logical and.</td>
</tr>
<tr>
<td><code>any()</code></td>
<td>Returns a transducer realizing logical or.</td>
</tr>
<tr>
<td><code>add()</code></td>
<td>Returns a transducer realizing addition.</td>
</tr>
<tr>
<td><code>sub()</code></td>
<td>Returns a transducer realizing subtraction.</td>
</tr>
<tr>
<td><code>CountSubblockOccurrences()</code></td>
<td>Returns a transducer counting the occurrences of a subblock.</td>
</tr>
<tr>
<td><code>Wait()</code></td>
<td>Returns a transducer writing False until first (or k-th) true input is read.</td>
</tr>
<tr>
<td><code>weight()</code></td>
<td>Returns a transducer realizing the Hamming weight.</td>
</tr>
<tr>
<td><code>GrayCode()</code></td>
<td>Returns a transducer realizing binary Gray code.</td>
</tr>
<tr>
<td><code>Recursion()</code></td>
<td>Returns a transducer defined by recursions.</td>
</tr>
</tbody>
</table>

AUTHORS:

- Clemens Heuberger (2014-04-07): initial version
- Sara Kropf (2014-04-10): some changes in TransducerGenerator
Functions and methods

class sage.combinat.finite_state_machine_generators.AutomatonGenerators

A collection of constructors for several common automata.

A list of all automata in this database is available via tab completion. Type “automata.” and then hit tab to see which automata are available.

The automata currently in this class include:

- AnyLetter()
- AnyWord()
- EmptyWord()
- Word()
- ContainsWord()

AnyLetter(input_alphabet)

Return an automaton recognizing any letter of the given input alphabet.

INPUT:
- input_alphabet – a list, the input alphabet

OUTPUT:
An Automaton.

EXAMPLES:

```
sage: A = automata.AnyLetter([0, 1])
sage: A([])
False
sage: A([0])
True
sage: A([1])
True
sage: A([0, 0])
False
```
Combinatorics, Release 10.0

See also:

*AnyWord()*

**AnyWord**(input_alphabet)

Return an automaton recognizing any word of the given input alphabet.

INPUT:

* input_alphabet – a list, the input alphabet

OUTPUT:

An Automaton.

EXAMPLES:

```sage
A = automata.AnyWord([0, 1])
sage: A([0])
True
sage: A([1])
True
sage: A([0, 1])
True
sage: A([0, 2])
False
```

This is equivalent to taking the *kleene_star()* of *AnyLetter()* and minimizing the result. This method immediately gives a minimized version:

```sage
B = automata.AnyLetter([0, 1]).kleene_star().minimization().relabeled()
sage: B == A
True
```

See also:

*AnyLetter(), Word().*

**ContainsWord**(word, input_alphabet)

Return an automaton recognizing the words containing the given word as a factor.

INPUT:

* word – a list (or other iterable) of letters, the word we are looking for.
  * input_alphabet – a list or other iterable, the input alphabet.

OUTPUT:

An Automaton.

EXAMPLES:

```sage
A = automata.ContainsWord([0, 1, 0, 1, 1], ....: input_alphabet=[0, 1])
sage: A([1, 0, 1, 0, 1, 0, 1, 1, 0, 0])
True
sage: A([1, 0, 1, 0, 1, 0, 1, 0])
False
```

This is equivalent to taking the concatenation of *AnyWord(), Word() and AnyWord() and minimizing the result. This method immediately gives a minimized version:
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```python
sage: B = (automata.AnyWord([0, 1]) *
......:     automata.Word([0, 1, 0, 1, 1], [0, 1]) *
......:     automata.AnyWord([0, 1])).minimization()
sage: B.is_equivalent(A)
True
```

See also:

`CountSubblockOccurrences()`, `AnyWord()`, `Word()`.

**EmptyWord**(input_alphabet=None)

Return an automaton recognizing the empty word.

**INPUT:**

- input_alphabet – (default: None) an iterable or None.

**OUTPUT:**

An `Automaton`.

**EXAMPLES:**

```python
sage: A = automata.EmptyWord()
sage: A([])
True
sage: A([0])
False
```

See also:

`AnyLetter()`, `AnyWord()`.

**Word**(word, input_alphabet=None)

Return an automaton recognizing the given word.

**INPUT:**

- word – an iterable.

- input_alphabet – a list or None. If None, then the letters occurring in the word are used.

**OUTPUT:**

An `Automaton`.

**EXAMPLES:**

```python
sage: A = automata.Word([0])
sage: A.transitions()
[Transition from 0 to 1: 0|-]
sage: [A(w) for w in ([], [0], [1])]
[False, True, False]
sage: A = automata.Word([0, 1, 0])
sage: A.transitions()
[Transition from 0 to 1: 0|-,
 Transition from 1 to 2: 1|-,
 Transition from 2 to 3: 0|-]
sage: [A(w) for w in ([], [0], [0, 1], [0, 1, 1], [0, 1, 0])]
[False, False, False, False, True]
```

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If the input alphabet is not given, it is derived from the given word.

```python
sage: A.input_alphabet
[0, 1]
sage: A = automata.Word([0, 1, 0], input_alphabet=[0, 1, 2])
sage: A.input_alphabet
[0, 1, 2]
```

See also:

* AnyWord(), ContainsWord().

```python
class sage.combinat.finite_state_machine_generators.TransducerGenerators
    Bases: object
    A collection of constructors for several common transducers.
    A list of all transducers in this database is available via tab completion. Type “transducers.” and then hit tab to see which transducers are available.
    The transducers currently in this class include:
    • Identity()
    • abs()
    • operator()
    • all()
    • any()
    • add()
    • sub()
    • CountSubblockOccurrences()
    • Wait()
    • GrayCode()
    • Recursion()
```

**CountSubblockOccurrences**(block, input_alphabet)

Returns a transducer counting the number of (possibly overlapping) occurrences of a block in the input.

**INPUT:**

- block – a list (or other iterable) of letters.
- input_alphabet – a list or other iterable.

**OUTPUT:**

A transducer counting (in unary) the number of occurrences of the given block in the input. Overlapping occurrences are counted several times.

Denoting the block by \( b_0 \ldots b_{k-1} \), the input word by \( i_0 \ldots i_L \) and the output word by \( o_0 \ldots o_L \), we have \( o_j = 1 \) if and only if \( i_{j-k+1} \ldots i_j = b_0 \ldots b_{k-1} \). Otherwise, \( o_j = 0 \).

**EXAMPLES:**

1. Counting the number of 10 blocks over the alphabet [0, 1]:

```python
```
sage: T = transducers.CountSubblockOccurrences(
    ....:   [1, 0],
    ....:   [0, 1])
sage: sorted(T.transitions())
[Transition from () to (): 0|0,
 Transition from () to (1,): 1|0,
 Transition from (1,) to (): 0|1,
 Transition from (1,) to (1,): 1|0]
sage: T.input_alphabet
[0, 1]
sage: T.output_alphabet
[0, 1]
sage: T.initial_states()
[()]
sage: T.final_states()
[(), (1,)]

Check some sequence:

sage: T([0, 1, 0, 1, 1, 0])
[0, 0, 1, 0, 0, 1]

2. Counting the number of 11 blocks over the alphabet [0, 1]:

sage: T = transducers.CountSubblockOccurrences(
    ....:   [1, 1],
    ....:   [0, 1])
sage: sorted(T.transitions())
[Transition from () to (): 0|0,
 Transition from () to (1,): 1|0,
 Transition from (1,) to (): 0|0,
 Transition from (1,) to (1,): 1|1]

Check some sequence:

sage: T([0, 1, 0, 1, 1, 0])
[0, 0, 0, 0, 1, 0]

3. Counting the number of 1010 blocks over the alphabet [0, 1, 2]:

sage: T = transducers.CountSubblockOccurrences(
    ....:   [1, 0, 1, 0],
    ....:   [0, 1, 2])
sage: sorted(T.transitions())
[Transition from () to (): 0|0,
 Transition from () to (1,): 1|0,
 Transition from () to (): 2|0,
 Transition from (1,) to (1, 0): 0|0,
 Transition from (1,) to (): 2|0,
 Transition from (1, 0) to (): 0|0,
 Transition from (1, 0) to (1, 0, 1): 1|0,
 Transition from (1, 0) to (): 2|0,
 Transition from (1, 0) to (1, 0, 1): 1|0,
 Transition from (1, 0) to (): 2|0,
Combinatorics, Release 10.0

Transition from (1, 0, 1) to (1, 0): 0|1,
Transition from (1, 0, 1) to (1,): 1|0,
Transition from (1, 0, 1) to (): 2|0

sage: input = [0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 2]
sage: output = [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0]
sage: T(input) == output
True

See also:

ContainsWord()

GrayCode()

Returns a transducer converting the standard binary expansion to Gray code.

INPUT:

Nothing.

OUTPUT:

A transducer.

Cf. the Wikipedia article Gray_code for a description of the Gray code.

EXAMPLES:

sage: G = transducers.GrayCode()
sage: G
Transducer with 3 states
sage: for v in srange(10):
    ....:     print("{} {}".format(v, G(v.digits(base=2))))
0 [ ]
1 [1]
2 [1, 1]
3 [0, 1]
4 [0, 1, 1]
5 [1, 1, 1]
6 [1, 0, 1]
7 [0, 0, 1]
8 [0, 0, 1, 1]
9 [1, 0, 1, 1]

In the example Gray Code in the documentation of the Finite state machines, automata, transducers module, the Gray code transducer is derived from the algorithm converting the binary expansion to the Gray code. The result is the same as the one given here.

Identity(input_alphabet)

Returns the identity transducer realizing the identity map.

INPUT:

• input_alphabet – a list or other iterable.

OUTPUT:

A transducer mapping each word over input_alphabet to itself.

EXAMPLES:
sage: T = transducers.Identity([0, 1])
sage: sorted(T.transitions())
[Transition from 0 to 0: 0|0,
 Transition from 0 to 0: 1|1]
sage: T.initial_states()
[0]
sage: T.final_states()
[0]
sage: T.input_alphabet
[0, 1]
sage: T.output_alphabet
[0, 1]
sage: T([0, 1, 0, 1, 1])
[0, 1, 0, 1, 1]

Recursion(recursions, base, function=None, var=None, input_alphabet=None, word_function=None,
is_zero=None, output_rings=[Integer Ring, Rational Field])

Return a transducer realizing the given recursion when reading the digit expansion with base base.

INPUT:

• recursions – list or iterable of equations. Each equation has either the form
  
  \[ f(base^k \times n + r) = f(base^k \times n + s) + t \]
  for some integers \(0 \leq k < K\), \(r\) and some \(t\)—valid for all \(n\) such that the arguments on both sides are non-negative—
  
  or the form
  
  \[ f(r) = t \]
  for some integer \(r\) and some \(t\).

Alternatively, an equation may be replaced by a transducers.RecursionRule with the attributes \(K, r, k, s, t\) as above or a tuple \((r, t)\). Note that \(t\) must be a list in this case.

• base – base of the digit expansion.

• function – symbolic function \(f\) occurring in the recursions.

• var – symbolic variable.

• input_alphabet – (default: None) a list of digits to be used as the input alphabet. If None and the base is an integer, \(input\_alphabet\) is chosen to be \(\text{srange}(base.abs())\).

• word_function – (default: None) a symbolic function. If not None, \(word\_function(arg1, .. ., argn)\) in a symbolic recurrence relation is interpreted as a transition with output \([arg1, .. ., argn]\). This could not be entered in a symbolic recurrence relation because lists do not coerce into the \(\text{SymbolicRing}\).

• is_zero – (default: None) a callable. The recursion relations are only well-posed if there is no cycle with non-zero output and input consisting of zeros. This parameter is used to determine whether the output of such a cycle is non-zero. By default, the output must evaluate to False as a boolean.

• output_rings – (default: [ZZ, QQ]) a list of rings. The output labels are converted into the first ring of the list in which they are contained. If they are not contained in any ring, they remain in whatever ring they are after parsing the recursions, typically the symbolic ring.

OUTPUT:

A transducer \(T\).
The transducer is constructed such that \( T(\text{expansion}) = f(n) \) if \( \text{expansion} \) is the digit expansion of \( n \) to the base \( \text{base} \) with the given input alphabet as set of digits. Here, the + on the right hand side of the recurrence relation is interpreted as the concatenation of words.

The formal equations and initial conditions in the recursion have to be selected such that \( f \) is uniquely defined.

**EXAMPLES:**

- The following example computes the Hamming weight of the ternary expansion of integers.

```sage
sage: function('f')
f
sage: var('n')
n
sage: T = transducers.Recursion([ 
    ....: f(3*n + 1) == f(n) + 1, 
    ....: f(3*n + 2) == f(n) + 1, 
    ....: f(3*n) == f(n), 
    ....: f(0) == 0], 
    ....: 3, f, n)
sage: T.transitions()
[Transition from (0, 0) to (0, 0): 0|-,
 Transition from (0, 0) to (0, 0): 1|1,
 Transition from (0, 0) to (0, 0): 2|1]
```

To illustrate what this transducer does, we consider the example of \( n = 601 \):

```sage
sage: ternary_expansion = 601.digits(base=3)
sage: ternary_expansion
[1, 2, 0, 1, 1, 2]
sage: weight_sequence = T(ternary_expansion)
sage: weight_sequence
[1, 1, 1, 1, 1]
sage: sum(weight_sequence)
5
```

Note that the digit zero does not show up in the output because the equation \( f(3*n) = f(n) \) means that no output is added to \( f(n) \).

- The following example computes the Hamming weight of the non-adjacent form, cf. the Wikipedia article Non-adjacent_form.

```sage
sage: function('f')
f
sage: var('n')
n
sage: T = transducers.Recursion([ 
    ....: f(4*n + 1) == f(n) + 1, 
    ....: f(4*n - 1) == f(n) + 1, 
    ....: f(2*n) == f(n), 
    ....: f(0) == 0], 
    ....: 2, f, n)
sage: T.transitions()
[Transition from (0, 0) to (0, 0): 0|-,
 Transition from (0, 0) to (1, 1): 1|1,
 Transition from (0, 0) to (1, 1): 2|1]
```
As we are interested in the weight only, we also output 1 for numbers congruent to 3 mod 4. The actual expansion is computed in the next example.

Consider the example of $29 = (100\bar{1}01)_2$ (as usual, the digit $-1$ is denoted by $\bar{1}$ and digits are written from the most significant digit at the left to the least significant digit at the right; for the transducer, we have to give the digits in the reverse order):

\[
\text{sage: } \text{NAF} = [1, 0, -1, 0, 0, 1] \\
\text{sage: } \text{ZZ(NAF, base=2)}
\]

\[
29 \\
\text{sage: } \text{binary_expansion = 29.digits(base=2)} \\
\text{sage: } \text{binary_expansion} \\
[1, 0, 1, 1, 1] \\
\text{sage: } \text{T(binary_expansion)} \\
[1, 1, 1] \\
\text{sage: } \text{sum(T(binary_expansion))} \\
3
\]

Indeed, the given non-adjacent form has three non-zero digits.

- The following example computes the non-adjacent form from the binary expansion, cf. the Wikipedia article Non-adjacent_form. In contrast to the previous example, we actually compute the expansion, not only the weight.

We have to write the output 0 when converting an even number. This cannot be encoded directly by an equation in the symbolic ring, because $f(2\times n) = f(n) + 0$ would be equivalent to $f(2\times n) = f(n)$ and an empty output would be written. Therefore, we wrap the output in the symbolic function $w$ and use the parameter word_function to announce this.

Similarly, we use $w(-1, 0)$ to write an output word of length 2 in one iteration. Finally, we write $f(0) = w()$ to write an empty word upon completion.

Moreover, there is a cycle with output [0] which—from the point of view of this method—is a contradicting recursion. We override this by the parameter is_zero.
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....:     word_function=w,
....:     is_zero=lambda x: sum(x).is_zero())
sage: T.transitions()
[Transition from (0, 0) to (0, 0): 0|0,
  Transition from (0, 0) to (1, 1): 1|-,
  Transition from (1, 1) to (0, 0): 0|1,0,
  Transition from (1, 1) to (1, 0): 1|-1,0,
  Transition from (1, 0) to (1, 1): 0|-,
  Transition from (1, 0) to (1, 0): 1|0]
sage: for s in T.iter_states():
    ....:     print("{} {}").format(s, s.final_word_out))
(0, 0) []
(1, 1) [1, 0]
(1, 0) [1, 0]

We again consider the example of \( n = 29 \):

sage: T(29.digits(base=2))
[1, 0, -1, 0, 0, 1, 0]

The same transducer can also be entered bypassing the symbolic equations:

sage: R = transducers.RecursionRule
sage: TR = transducers.Recursion([],
    ....:   R(K=1, r=0, k=0, s=0, t=[0]),
    ....:   R(K=2, r=1, k=0, s=0, t=[1, 0]),
    ....:   R(K=2, r=-1, k=0, s=0, t=[-1, 0]),
    ....:   (0, []),
    ....:   2,
    ....:   is_zero=lambda x: sum(x).is_zero())
sage: TR == T
True

• Here is an artificial example where some of the \( s \) are negative:

sage: function('f')
f
sage: var('n')
n
sage: T = transducers.Recursion([],
    ....:   f(2^n + 1) == f(n-1) + 1,
    ....:   f(2^n) == f(n),
    ....:   f(1) == 1,
    ....:   f(0) == 0], 2, f, n)
sage: T.transitions()
[Transition from (0, 0) to (0, 0): 0|-,
  Transition from (0, 0) to (1, 1): 1|-,
  Transition from (1, 1) to (-1, 1): 0|1,
  Transition from (1, 1) to (0, 0): 1|1,
  Transition from (-1, 1) to (-1, 2): 0|-,
  Transition from (-1, 1) to (1, 2): 1|-,
  Transition from (-1, 2) to (-1, 1): 0|1,
  Transition from (-1, 2) to (0, 0): 1|1,
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Transition from (1, 2) to (-1, 2): 0|1,
Transition from (1, 2) to (1, 2): 1|1]
sage: [(s.label(), s.final_word_out)
....: for s in T.iter_final_states()]
[((0, 0), []),
 ((1, 1), [1]),
 ((-1, 1), [0]),
 ((-1, 2), [0]),
 ((1, 2), [1])]

• Abelian complexity of the paperfolding sequence (cf. [HKP2015], Example 2.8):

sage: T = transducers.Recursion([
....:   f(4*n) == f(2*n),
....:   f(4*n+2) == f(2*n+1)+1,
....:   f(16*n+1) == f(8*n+1),
....:   f(16*n+5) == f(4*n+1)+2,
....:   f(16*n+11) == f(4*n+3)+2,
....:   f(16*n+15) == f(2*n+2)+1,
....:   f(1) == 2, f(0) == 0
....:   + [f(16*n+jj) == f(2*n+1)+2 for jj in [3,7,9,13]],
....:   2, f, n)
sage: T.transitions()
[Transition from (0, 0) to (0, 1): 0|-, Transition from (0, 0) to (1, 1): 1|-, Transition from (0, 1) to (0, 1): 0|-, Transition from (0, 1) to (1, 1): 1|1, Transition from (1, 1) to (1, 2): 0|-, Transition from (1, 1) to (3, 2): 1|-, Transition from (1, 2) to (1, 3): 0|-,
 Transition from (1, 2) to (5, 3): 1|-, Transition from (3, 2) to (3, 3): 0|-, Transition from (3, 2) to (7, 3): 1|-, Transition from (1, 3) to (1, 3): 0|-, Transition from (1, 3) to (1, 1): 1|2, Transition from (5, 3) to (1, 2): 0|2, Transition from (5, 3) to (1, 1): 1|2, Transition from (3, 3) to (1, 1): 0|2, Transition from (3, 3) to (3, 2): 1|2, Transition from (7, 3) to (1, 1): 0|2, Transition from (7, 3) to (2, 1): 1|1, Transition from (2, 1) to (1, 1): 0|1, Transition from (2, 1) to (2, 1): 1|] sago: for s in T.iter_states():
....:   print("{} {}".format(s, s.final_word_out))
(0, 0) []
(0, 1) []
(1, 1) [2]
(1, 2) [2]
(3, 2) [2, 2]
(1, 3) [2]
(5, 3) [2, 2]
• We now demonstrate the use of the `output_rings` parameter. If no `output_rings` are specified, the output labels are converted into ZZ:

```
sage: function('f')
f
sage: var('n')
n
sage: T = transducers.Recursion([
    ....:    f(2^n + 1) == f(n) + 1,
    ....:    f(2^n) == f(n),
    ....:    f(0) == 2,
    ....:    2, f, n]
sage: for t in T.transitions():
    ....:    print([x.parent() for x in t.word_out])
[]
[Integer Ring]
sage: [x.parent() for x in T.states()[0].final_word_out]
[Integer Ring]
```

In contrast, if `output_rings` is set to the empty list, the results are not converted:

```
sage: T = transducers.Recursion([  
    ....:    f(2^n + 1) == f(n) + 1,
    ....:    f(2^n) == f(n),
    ....:    f(0) == 2,
    ....:    2, f, n, output_rings=[]]
sage: for t in T.transitions():
    ....:    print([x.parent() for x in t.word_out])
[]
[Symbolic Ring]
sage: [x.parent() for x in T.states()[0].final_word_out]
[Symbolic Ring]
```

Finally, we use a somewhat questionable conversion:

```
sage: T = transducers.Recursion([  
    ....:    f(2^n + 1) == f(n) + 1,
    ....:    f(2^n) == f(n),
    ....:    f(0) == 0,
    ....:    2, f, n, output_rings=[GF(5)]
]
sage: for t in T.transitions():
    ....:    print([x.parent() for x in t.word_out])
[]
[Finite Field of size 5]
```

Todo: Extend the method to
• non-integral bases,
• higher dimensions.

ALGORITHM:
See [HKP2015], Section 6. However, there are also recursion transitions for states of level \( \kappa \) if the recursion rules allow such a transition. Furthermore, the intermediate step of a non-deterministic transducer is left out by implicitly using recursion transitions. The well-posedness is checked in a truncated version of the recursion digraph.

```python
class RecursionRule(K, r, k, s, t):
    Bases: tuple
    K
        Alias for field number 0
    k
        Alias for field number 2
    r
        Alias for field number 1
    s
        Alias for field number 3
    t
        Alias for field number 4

Wait(input_alphabet, threshold=1)
    Writes False until reading the threshold-th occurrence of a true input letter; then writes True.
    INPUT:
    • input_alphabet – a list or other iterable.
    • threshold – a positive integer specifying how many occurrences of True inputs are waited for.
    OUTPUT:
    A transducer writing False until the threshold-th true (Python’s standard conversion to boolean is used to convert the actual input to boolean) input is read. Subsequently, the transducer writes True.

EXAMPLES:
```sage```
T = transducers.Wait([0, 1])
T([0, 0, 1, 0, 1, 0])
[False, False, True, True, True, True]
T2 = transducers.Wait([0, 1], threshold=2)
T2([0, 0, 1, 0, 1, 0])
[False, False, False, False, True, True]
```

abs(input_alphabet)
    Returns a transducer which realizes the letter-wise absolute value of an input word over the given input alphabet.
    INPUT:
    • input_alphabet – a list or other iterable.
OUTPUT:

A transducer mapping $i_0 \ldots i_k$ to $|i_0| \ldots |i_k|$.  

EXAMPLES:

The following transducer realizes letter-wise absolute value:

```python
sage: T = transducers.abs([-1, 0, 1])
sage: T.transitions()
[Transition from 0 to 0: -1|1,
 Transition from 0 to 0: 0|0,
 Transition from 0 to 0: 1|1]
sage: T.initial_states()
[0]
sage: T.final_states()
[0]
sage: T([-1, -1, 0, 1])
[1, 1, 0, 1]
```

`add(input_alphabet, number_of_operands=2)`

Returns a transducer which realizes addition on pairs over the given input alphabet. 

**INPUT:**

- `input_alphabet` – a list or other iterable.
- `number_of_operands` – (default: 2) it specifies the number of input arguments the operator takes.

**OUTPUT:**

A transducer mapping an input word $(i_{01}, \ldots, i_{0d}) \ldots (i_{k1}, \ldots, i_{kd})$ to the word $(i_{01} + \cdots + i_{0d}) \ldots (i_{k1} + \cdots + i_{kd})$.

The input alphabet of the generated transducer is the Cartesian product of `number_of_operands` copies of `input_alphabet`.

**EXAMPLES:**

The following transducer realizes letter-wise addition:

```python
sage: T = transducers.add([0, 1])
sage: T.transitions()
[Transition from 0 to 0: (0, 0)|0,
 Transition from 0 to 0: (0, 1)|1,
 Transition from 0 to 0: (1, 0)|1,
 Transition from 0 to 0: (1, 1)|2]
sage: T.input_alphabet
[(0, 0), (0, 1), (1, 0), (1, 1)]
sage: T.initial_states()
[0]
sage: T.final_states()
[0]
sage: T([(0, 0), (0, 1), (1, 0), (1, 1)])
[0, 1, 1, 2]
```

More than two operands can also be handled:
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\begin{verbatim}
\texttt{sage: T3 = transducers.add([0, 1], number_of_operands=3)
\texttt{sage: T3.input_alphabet}
\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1),
   (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}
\texttt{sage: T3([(0, 0, 0), (0, 1, 0), (0, 1, 1), (1, 1, 1)])}
[0, 1, 2, 3]
\end{verbatim}

\textbf{all}(\texttt{input\_alphabet, number\_of\_operands=2})

Returns a transducer which realizes logical and over the given input alphabet.

**INPUT:**

- \texttt{input\_alphabet} – a list or other iterable.
- \texttt{number\_of\_operands} – (default: 2) specifies the number of input arguments for the \texttt{and} operation.

**OUTPUT:**

A transducer mapping an input word \((i_0, \ldots, i_{od}) \ldots (i_{kd}, \ldots, i_{kd})\) to the word \((i_0 \land \cdots \land i_{od}) \ldots (i_{kd} \land \cdots \land i_{kd})\).

The input alphabet of the generated transducer is the Cartesian product of \texttt{number\_of\_operands} copies of \texttt{input\_alphabet}.

**EXAMPLES:**

The following transducer realizes letter-wise logical and:

\begin{verbatim}
\texttt{sage: T = transducers.all([False, True])
\texttt{sage: T.transitions()}
[Transition from 0 to 0: (False, False)|False,
 Transition from 0 to 0: (False, True)|False,
 Transition from 0 to 0: (True, False)|False,
 Transition from 0 to 0: (True, True)|True]
\texttt{sage: T.input_alphabet}
[(False, False), (False, True), (True, False), (True, True)]
\texttt{sage: T.initial\_states()}
[0]
\texttt{sage: T.final\_states()}
[0]
\texttt{sage: T([(False, False), (False, True), (True, False), (True, True)])}
[False, False, False, True]
\end{verbatim}

More than two operands and other input alphabets (with conversion to boolean) are also possible:

\begin{verbatim}
\texttt{sage: T3 = transducers.all([0, 1], number\_of\_operands=3)
\texttt{sage: T3([(0, 0, 0), (1, 0, 0), (1, 1, 1)])}
[False, False, True]
\end{verbatim}

\textbf{any}(\texttt{input\_alphabet, number\_of\_operands=2})

Returns a transducer which realizes logical or over the given input alphabet.

**INPUT:**

- \texttt{input\_alphabet} – a list or other iterable.
- \texttt{number\_of\_operands} – (default: 2) specifies the number of input arguments for the \texttt{or} operation.

**OUTPUT:**

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A transducer mapping an input word \((i_0, \ldots, i_{od}) \ldots (i_{kd}, \ldots, i_{kd})\) to the word \((i_0 \lor \cdots \lor i_{od}) \ldots (i_{kd} \lor \cdots \lor i_{kd})\).

The input alphabet of the generated transducer is the Cartesian product of \texttt{number_of_operands} copies of \texttt{input_alphabet}.

EXAMPLES:

The following transducer realizes letter-wise logical or:

```python
sage: T = transducers.any([False, True])
sage: T.transitions()
[Transition from 0 to 0: (False, False)|False,
 Transition from 0 to 0: (False, True)|True,
 Transition from 0 to 0: (True, False)|True,
 Transition from 0 to 0: (True, True)|True]
sage: T.input_alphabet
[(False, False), (False, True), (True, False), (True, True)]
sage: T.initial_states()
[0]
sage: T.final_states()
[0]
sage: T([((False, False), (False, True), (True, False), (True, True))])
[False, True, True, True]
```

More than two operands and other input alphabets (with conversion to boolean) are also possible:

```python
sage: T3 = transducers.any([0, 1], number_of_operands=3)
sage: T3([(0, 0, 0), (1, 0, 0), (1, 1, 1)])
[False, True, True]
```

\texttt{map}(\textit{f, input\_alphabet})

Return a transducer which realizes a function on the alphabet.

INPUT:

- \textit{f} – function to realize.
- \textit{input\_alphabet} – a list or other iterable.

OUTPUT:

A transducer mapping an input letter \(x\) to \(f(x)\).

EXAMPLES:

The following binary transducer realizes component-wise absolute value (this transducer is also available as \texttt{abs()}):

```python
sage: T = transducers.map(abs, [-1, 0, 1])
sage: T.transitions()
[Transition from 0 to 0: -1|1,
 Transition from 0 to 0: 0|0,
 Transition from 0 to 0: 1|1]
sage: T.input_alphabet
[-1, 0, 1]
sage: T.initial_states()
[0]
```

(continues on next page)
sage: T.final_states()
[0]
sage: T([-1, 1, 0, 1])
[1, 1, 0, 1]

See also:

Automaton.with_output().

**operator**(operator, input_alphabet, number_of_operands=2)

Returns a transducer which realizes an operation on tuples over the given input alphabet.

**INPUT:**

- `operator` – operator to realize. It is a function which takes `number_of_operands` input arguments (each out of `input_alphabet`).
- `input_alphabet` – a list or other iterable.
- `number_of_operands` – (default: 2) it specifies the number of input arguments the operator takes.

**OUTPUT:**

A transducer mapping an input letter \((i_1, \ldots, i_n)\) to \(\text{operator}(i_1, \ldots, i_n)\). Here, \(n\) equals `number_of_operands`.

The input alphabet of the generated transducer is the Cartesian product of `number_of_operands` copies of `input_alphabet`.

**EXAMPLES:**

The following binary transducer realizes component-wise addition (this transducer is also available as `add()`):

```python
sage: import operator
sage: T = transducers.operator(operator.add, [0, 1])
sage: T.transitions()
[Transition from 0 to 0: (0, 0)|0,
 Transition from 0 to 0: (0, 1)|1,
 Transition from 0 to 0: (1, 0)|1,
 Transition from 0 to 0: (1, 1)|2]
sage: T.input_alphabet
[(0, 0), (0, 1), (1, 0), (1, 1)]
sage: T.initial_states()
[0]
sage: T.final_states()
[0]
sage: T([(0, 0), (0, 1), (1, 0), (1, 1)])
[0, 1, 1, 2]
```

Note that for a unary operator the input letters of the new transducer are tuples of length 1:

```python
sage: T = transducers.operator(abs, number_of_operands=1)
sage: T([-1, 0, 1])
Traceback (most recent call last):
...
```

(continues on next page)
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ValueError: Invalid input sequence.

sage: T(\([-1,\), (1,), (0,)]\))
[1, 1, 0]

Compare this with the transducer generated by \texttt{map()}:

sage: T = transducers.map(abs,
.....: [-1, 0, 1])
sage: T([-1, 1, 0])
[1, 1, 0]

In fact, this transducer is also available as \texttt{abs()}:

sage: T = transducers.abs([-1, 0, 1])
sage: T([-1, 1, 0])
[1, 1, 0]

\texttt{sub(input\_alphabet)}

Returns a transducer which realizes subtraction on pairs over the given input alphabet.

INPUT:

- \texttt{input\_alphabet} – a list or other iterable.

OUTPUT:

A transducer mapping an input word \((i_0, i'_0)\ldots(i_k, i'_k)\) to the word \((i_0 - i'_0)\ldots(i_k - i'_k)\).

The input alphabet of the generated transducer is the Cartesian product of two copies of \texttt{input\_alphabet}.

EXAMPLES:

The following transducer realizes letter-wise subtraction:

sage: T = transducers.sub([0, 1])
sage: T.transitions()
[Transition from 0 to 0: (0, 0)|0,
 Transition from 0 to 0: (0, 1)|-1,
 Transition from 0 to 0: (1, 0)|1,
 Transition from 0 to 0: (1, 1)|0]
sage: T.input\_alphabet
[\((0, 0), (0, 1), (1, 0), (1, 1)\)]
sage: T.initial\_states()
[0]
sage: T.final\_states()
[0]
sage: T(\([(0, 0), (0, 1), (1, 0), (1, 1)\)])
[0, -1, 1, 0]

\texttt{weight(input\_alphabet, zero=0)}

Returns a transducer which realizes the Hamming weight of the input over the given input alphabet.

INPUT:

- \texttt{input\_alphabet} – a list or other iterable.
- \texttt{zero} – the zero symbol in the alphabet used
A transducer mapping \(i_0 \ldots i_k\) to \((i_0 \neq 0) \ldots (i_k \neq 0)\).

The Hamming weight is defined as the number of non-zero digits in the input sequence over the alphabet `input_alphabet` (see Wikipedia article Hamming_weight). The output sequence of the transducer is a unary encoding of the Hamming weight. Thus the sum of the output sequence is the Hamming weight of the input.

**EXAMPLES:**

```python
sage: W = transducers.weight([-1, 0, 2])
sage: W.transitions()
[Transition from 0 to 0: -1|1,
 Transition from 0 to 0: 0|0,
 Transition from 0 to 0: 2|1]
sage: unary_weight = W([-1, 0, 0, 2, -1])
sage: unary_weight
[1, 0, 0, 1, 1]
sage: weight = add(unary_weight)
sage: weight
3

Also the joint Hamming weight can be computed:

```python
sage: v1 = vector([-1, 0])
sage: v0 = vector([0, 0])
sage: W = transducers.weight([v1, v0])
sage: unary_weight = W([v1, v0, v1, v0])
sage: add(unary_weight)
2
```

For the input alphabet \([-1, 0, 1]\) the weight transducer is the same as the absolute value transducer `abs()`:

```python
sage: W = transducers.weight([-1, 0, 1])
sage: A = transducers.abs([-1, 0, 1])
sage: W == A
True
```

For other input alphabets, we can specify the zero symbol:

```python
sage: W = transducers.weight(['a', 'b'], zero='a')
sage: add(W(['a', 'b', 'b']))
2
```
5.1.109 Free Quasi-symmetric functions

AUTHORS:

- Frédéric Chapoton, Darij Grinberg (2017)

```python
class sage.combinat.fqsym.FQSymBases(base):
    Bases: Category_realization_of_parent
    The category of graded bases of FQSym indexed by permutations.

class ElementMethods:
    omega_involution() -> FQSymBases
    Return the image of the element self of FQSym under the omega involution.

    The \omega involution is defined as the linear map FQSym \to FQSym that sends each basis element \( F_u \) of the F-basis of FQSym to the basis element \( F_{\omega(u)} \), where \( \omega_0 \) is the longest word (i.e., \( \omega_0(i) = n+1-i \)) in the symmetric group \( S_n \) that contains \( u \). The \omega involution is a graded algebra automorphism and a coalgebra anti-automorphism of FQSym. Every permutation \( u \in S_n \) satisfies

    \[ \omega(F_u) = F_{\omega(u)}, \quad \omega(G_u) = G_{\omega(u)} \]

    where standard notations for classical bases of FQSym are being used (that is, \( F \) for the F-basis, and \( G \) for the G-basis). In other words, writing permutations in one-line notation, we have

    \[ \omega(F(u_1, u_2, \ldots, u_n)) = F(u_n, u_{n-1}, \ldots, u_1), \quad \omega(G(u_1, u_2, \ldots, u_n)) = G(n+1-u_1, n+1-u_2, \ldots, n+1-u_n). \]

    If we also consider the \omega involution (omega_involution()) of the quasisymmetric functions (by slight abuse of notation), and if we let \( \pi \) be the canonical projection FQSym \to QSym, then \( \pi \circ \omega = \omega \circ \pi \).

    Additionally, consider the \psi involution (psi_involution()) of the noncommutative symmetric functions, and if we let \( \iota \) be the canonical inclusion NSym \to FQSym, then \( \omega \circ \iota = \iota \circ \psi \).

Todo: Duality?
```

See also:

- psi_involution(), star_involution()

EXAMPLES:

```python
sage: FQSym = algebras.FQSym(ZZ)
sage: F = FQSym.F()
sage: F[[2,3,1]].omega_involution()
F[1, 3, 2]
sage: (3*F[[1]] - 4*F[[1,2]]).omega_involution()
-4*F[] + 3*F[1] + 5*F[2, 1]
sage: G = FQSym.G()
sage: G[[2,3,1]].omega_involution()
G[2, 1, 3]
sage: M = FQSym.M()
sage: M[[2,3,1]].omega_involution()
-M[1, 2, 3] - M[2, 1, 3] - M[3, 1, 2]
```

The omega involution is an algebra homomorphism:
The omega involution intertwines the antipode and the inverse of the antipode:

```python
sage: all( F(I).antipode().omega_involution().antipode() == F(I).omega_involution() for I in Permutations(4) )
True
```

Testing the $\pi \circ \omega = \omega \circ \pi$ relation noticed above:

```python
sage: all( M[I].omega_involution().to_qsym() == M[I].to_qsym().omega_involution() for I in Permutations(4) )
True
```

Testing the $\omega \circ \iota = \iota \circ \psi$ relation:

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: all( S[I].psi_involution().to_fqsym() == S[I].to_fqsym().omega_involution() for I in Compositions(4) )
True
```

Todo: Check further commutative squares.

---

**psi_involution()**

Return the image of the element `self` of $FQSym$ under the psi involution.

The $\psi$ involution is defined as the linear map $FQSym \rightarrow FQSym$ that sends each basis element $F_u$ of the $F$-basis of $FQSym$ to the basis element $F_{w_0u}$, where $w_0$ is the longest word (i.e., $w_0(i) = n+1-i$) in the symmetric group $S_n$ that contains $u$. The $\psi$ involution is a graded coalgebra automorphism and an algebra anti-automorphism of $FQSym$. Every permutation $u \in S_n$ satisfies

$$\psi(F_u) = F_{w_0u}, \quad \psi(G_u) = G_{u\circ w_0},$$

where standard notations for classical bases of $FQSym$ are being used (that is, $F$ for the $F$-basis, and $G$ for the $G$-basis). In other words, writing permutations in one-line notation, we have

$$\psi(F_{(u_1,u_2,\ldots,u_n)}) = F_{(n+1-u_1,n+1-u_2,\ldots,n+1-u_n)}, \quad \psi(G_{(u_1,u_2,\ldots,u_n)}) = G_{(u_n,u_{n-1},\ldots,u_1)}.$$ 

If we also consider the $\psi$ involution (`psi_involution()`) of the quasisymmetric functions (by slight abuse of notation), and if we let $\pi$ be the canonical projection $FQSym \rightarrow QSym$, then $\pi \circ \psi = \psi \circ \pi$.

Additionally, consider the $\omega$ involution (`omega_involution()`) of the noncommutative symmetric functions, and if we let $\iota$ be the canonical inclusion $NSym \rightarrow FQSym$, then $\psi \circ \iota = \iota \circ \omega$.

Todo: Duality?
See also: 

omega_involution(), star_involution()

**EXAMPLES:**

```
sage: FQSym = algebras.FQSym(ZZ)
sage: F = FQSym.F()
sage: F[[2,3,1]].psi_involution()
F[2, 1, 3]
sage: (3*F[[1]] - 4*F[[1]] + 5*F[[1,2]]).psi_involution()
-4*F[] + 3*F[1] + 5*F[2, 1]
sage: G = FQSym.G()
sage: G[[2,3,1]].psi_involution()
G[1, 3, 2]
sage: M = FQSym.M()
sage: M[[2,3,1]].psi_involution()
-M[1, 2, 3] - M[1, 3, 2] - M[2, 3, 1]
```

The $\psi$ involution intertwines the antipode and the inverse of the antipode:

```
sage: all( F(I).antipode().psi_involution().antipode() == F(I).psi_involution() for I in Permutations(4) )
True
```

Testing the $\pi \circ \psi = \psi \circ \pi$ relation above:

```
sage: all( M[I].psi_involution().to_qsym() == M[I].to_qsym().psi_involution() for I in Permutations(4) )
True
```

Testing the $\psi \circ \iota = \iota \circ \omega$ relation:

```
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: all( S[I].omega_involution().to_fqsym() == S[I].to_fqsym().psi_involution() for I in Compositions(4) )
True
```

**Todo:** Check further commutative squares.

**star_involution()**

Return the image of the element self of $FQSym$ under the star involution.

The star involution is defined as the linear map $FQSym \to FQSym$ that sends each basis element $F_u$ of the F-basis of $FQSym$ to the basis element $F_{w_0 u w_0}$, where $w_0$ is the longest word (i.e., $w_0(i) = n + 1 - i$) in the symmetric group $S_n$ that contains $u$. The star involution is a graded Hopf algebra anti-automorphism of $FQSym$. It is denoted by $f \mapsto f^*$. Every permutation $u \in S_n$ satisfies

\[
(F_u)^* = F_{w_0 u w_0}, \quad (G_u)^* = G_{w_0 u w_0}, \quad (M_u)^* = M_{w_0 u w_0},
\]
where standard notations for classical bases of $FQSym$ are being used (that is, $F$ for the F-basis, $G$ for the G-basis, and $\mathcal{M}$ for the Monomial basis). In other words, writing permutations in one-line notation, we have

$$(F(u_1,u_2,\ldots,u_n))^* = F(n+1-u_n,n+1-u_{n-1},\ldots,n+1-u_1),$$

$$(G(u_1,u_2,\ldots,u_n))^* = G(n+1-u_n,n+1-u_{n-1},\ldots,n+1-u_1),$$

and

$$(\mathcal{M}(u_1,u_2,\ldots,u_n))^* = \mathcal{M}(n+1-u_n,n+1-u_{n-1},\ldots,n+1-u_1).$$

Let us denote the star involution by $*$ as well.

If we also denote by $*$ the star involution of the quasisymmetric functions \(\text{star_involution()}\) and if we let $\pi : FQSym \rightarrow QSym$ be the canonical projection then $\pi \circ (* = (*) \circ \pi$. Similar for the noncommutative symmetric functions \(\text{star_involution()}\) with $\pi : NSym \rightarrow FQSym$ being the canonical inclusion and the word quasisymmetric functions \(\text{star_involution()}\) with $\pi : FQSym \rightarrow WQSym$ the canonical inclusion.

Todo: Duality?

See also:

\text{omega_involution()}, \text{psi_involution()}

EXAMPLES:

```python
sage: FQSym = algebras.FQSym(ZZ)
sage: F = FQSym.F()
sage: F[[2,3,1]].star_involution()
F[3, 1, 2]
sage: (3*F[[1]] - 4*F[[1]] + 5*F[[1,2]]).star_involution()
-4*F[] + 3*F[1] + 5*F[1, 2]
sage: G = FQSym.G()
sage: G[[2,3,1]].star_involution()
G[3, 1, 2]
sage: M = FQSym.M()
sage: M[[2,3,1]].star_involution()
M[3, 1, 2]
```

The star involution commutes with the antipode:

```python
sage: all( F(I).antipode().star_involution() == F(I).star_involution().antipode() for I in Permutations(4) )
True
```

Testing the $\pi \circ (*) = (*) \circ \pi$ relation:

```python
sage: all( M[I].star_involution().to_qsym() == M[I].to_qsym().star_involution() for I in Permutations(4) )
True
```

Similar for $NSym$:  

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\begin{Verbatim}
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: all( S[I].star_involution().to_fqsym() == S[I].to_fqsym().star_involution() 
    ....:       for I in Compositions(4) )
True

Similar for \texttt{WQSym}:
\begin{Verbatim}
sage: WQSym = algebras.WQSym(ZZ)
sage: all( F(I).to_wqsym().star_involution() 
    ....:       == F(I).star_involution().to_wqsym() 
    ....:       for I in Permutations(4) )
True
\end{Verbatim}

\textbf{Todo:} Check further commutative squares.

\begin{Verbatim}
\texttt{to_qsym()}
\end{Verbatim}

Return the image of \texttt{self} under the canonical projection \texttt{FQSym} \rightarrow \texttt{QSym}.

The canonical projection \texttt{FQSym} \rightarrow \texttt{QSym} is a surjective homomorphism of Hopf algebras. It sends a basis element \texttt{F}_w of \texttt{FQSym} to the basis element \texttt{F}_{\text{Comp } w} of the fundamental basis of \texttt{QSym}, where \texttt{Comp } w stands for the descent composition (\texttt{sage.combinat.permutation.Permutation.descents_composition()}) of the permutation \texttt{w}.

\textbf{See also:}
\texttt{QuasiSymmetricFunctions} for a definition of \texttt{QSym}.

\textbf{EXAMPLES:}
\begin{Verbatim}
sage: G = algebras.FQSym(QQ).G()
sage: x = G[1, 3, 2]
sage: x.to_qsym()
F[2, 1]
sage: G[2, 3, 1].to_qsym()
F[1, 2]
sage: F = algebras.FQSym(QQ).F()
sage: F[2, 3, 1].to_qsym()
F[2, 1]
sage: (F[2, 3, 1] + F[1, 3, 2] + F[1, 2, 3]).to_qsym()
sage: F2 = algebras.FQSym(GF(2)).F()
sage: F2[2, 3, 1].to_qsym()
F[2, 1]
sage: (F2[2, 3, 1] + F2[1, 3, 2] + F2[1, 2, 3]).to_qsym()
F[3]
\end{Verbatim}

\begin{Verbatim}
\texttt{to_symmetric_group_algebra}(n=None)
\end{Verbatim}

Return the element of a symmetric group algebra corresponding to the element \texttt{self} of \texttt{FQSym}.

\textbf{INPUT:}
- \texttt{n} – integer (default: the maximal degree of \texttt{self}); the rank of the target symmetric group algebra

\textbf{EXAMPLES:}
```sage
A = algebras.FQSym(QQ).G()
x = A([1, 3, 2, 4]) + 5/2 * A([2, 3, 4, 1])
x.to_symmetric_group_algebra()
[1, 3, 2, 4] + 5/2*[4, 1, 2, 3]
```

**to_wqsym()**

Return the image of self under the canonical inclusion map \( FQS_{ym} \to WQSym \).

The canonical inclusion map \( FQS_{ym} \to WQSym \) is an injective homomorphism of Hopf algebras. It sends a basis element \( G_w \) of \( FQS_{ym} \) to the sum of basis elements \( M_u \) of \( WQSym \), where \( u \) ranges over all packed words whose standardization is \( w \).

See also:

*WordQuasiSymmetricFunctions* for a definition of \( WQSym \).

**EXAMPLES:**

```sage
G = algebras.FQSym(QQ).G()
x = G[1, 3, 2]
x.to_wqsym()
M[{1}, {3}, {2}] + M[{1, 3}, {2}]
```

```sage
G[2, 3, 1].to_wqsym()
M[{3}, {1}, {2}] + M[{3}, {1, 2}]
```

```sage
F = algebras.FQSym(QQ).F()
F[3, 1, 2].to_wqsym()
M[{3}, {1}, {2}] + M[{3}, {1, 2}]
```

```sage
class ParentMethods

Bases: object

basis(degree=None)

The basis elements (optionally: of the specified degree).

OUTPUT: Family

**EXAMPLES:**

```sage
FQSym = algebras.FQSym(QQ)
G = FQSym.G()
G.basis()
Lazy family (Term map from Standard permutations to Free Quasi-symmetric\_\_functions over Rational Field in the G basis(i))_{i in Standard\_\_permutations}
```

```sage
G.basis().keys()
Standard permutations
```

```sage
G.basis(degree=3).keys()
Standard permutations of 3
```

```sage
G.basis(degree=3).list()
[G[1, 2, 3], G[1, 3, 2], G[2, 1, 3], G[2, 3, 1], G[3, 1, 2], G[3, 2, 1]]
```

**from_symmetric_group_algebra(x)**

Return the element of \( FQS_{ym} \) corresponding to the element \( x \) of a symmetric group algebra.

**EXAMPLES:**
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```python
sage: A = algebras.FQSym(QQ).F()
sage: SGA4 = SymmetricGroupAlgebra(QQ, 4)
sage: x = SGA4([1,3,2,4]) + 5/2 * SGA4([1,2,4,3])
sage: A.from_symmetric_group_algebra(x)
5/2*F[1, 2, 4, 3] + F[1, 3, 2, 4]
sage: A.from_symmetric_group_algebra(SGA4.zero())
0
```

**is_commutative()**

Return whether this \( F_{QSym} \) is commutative.

**EXAMPLES:**

```python
sage: F = algebras.FQSym(ZZ).F()
sage: F.is_commutative()
False
```

**is_field(proof=True)**

Return whether this \( F_{QSym} \) is a field.

**EXAMPLES:**

```python
sage: F = algebras.FQSym(QQ).F()
sage: F.is_field()
False
```

**one_basis()**

Return the index of the unit.

**EXAMPLES:**

```python
sage: A = algebras.FQSym(QQ).F()
sage: A.one_basis()
[]
```

**prec()**

Return the \( \prec \) product.

On the \( F \)-basis of \( FQSym \), this product is determined by \( F_x \prec F_y = \sum F_z \), where the sum ranges over all \( z \) in the shifted shuffle of \( x \) and \( y \) with the additional condition that the first letter of the result comes from \( x \).

The usual symbol for this operation is \( \prec \).

**See also:**

product(), succ()

**EXAMPLES:**

```python
sage: A = algebras.FQSym(QQ).F()
sage: x = A([2,1])
sage: A.prec(x, x)
sage: y = A([2,1,3])
sage: A.prec(x, y)
```

(continues on next page)
\begin{verbatim}
sage: A.prec(y, x)
\end{verbatim}

\textbf{prec\textunderscore by\textunderscore coercion}(x, y)

Return $x \prec y$, computed using coercion to the F-basis.

See \textit{prec()} for the definition of the objects involved.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: G = algebras.FQSym(ZZ).G()
sage: a = G([1])
sage: b = G([2, 3, 1])
sage: G.prec(a, b) + G.succ(a, b) == a * b  # indirect doctest
True
\end{verbatim}

\textbf{some\_elements()}  

Return some elements of the free quasi-symmetric functions.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FQSym(QQ)
sage: F = A.F()
sage: F.some_elements()
[F[], F[1], F[1, 2] + F[2, 1], F[] + F[1, 2] + F[2, 1]]
sage: G = A.G()
sage: G.some_elements()
[G[], G[1], G[1, 2] + G[2, 1], G[] + G[1, 2] + G[2, 1]]
sage: M = A.M()
sage: M.some_elements()
[M[], M[1], M[1, 2] + 2*M[2, 1], M[] + M[1, 2] + 2*M[2, 1]]
\end{verbatim}

\textbf{succ()}  

Return the $\succ$ product.

On the F-basis of FQSym, this product is determined by $F_x \succ F_y = \sum F_z$, where the sum ranges over all $z$ in the shifted shuffle of $x$ and $y$ with the additional condition that the first letter of the result comes from $y$.

The usual symbol for this operation is $\succ$.

See also:

\textit{product()}, \textit{prec()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FQSym(QQ).F()
sage: x = A([1])
sage: A.succ(x, x)
F[2, 1]
sage: y = A([3, 1, 2])
sage: A.succ(x, y)
\end{verbatim}

(continues on next page)
F[4, 1, 2, 3] + F[4, 2, 1, 3] + F[4, 2, 3, 1]
sage: A.succ(y, x)
F[4, 3, 1, 2]

sage: G = algebras.FQSym(ZZ).G()
sage: G.succ(G([1]), G([2, 3, 1]))
# indirect doctest

super_categories()

The super categories of self.

EXAMPLES:

sage: from sage.combinat.fqsym import FQSymBases
sage: FQSym = algebras.FQSym(ZZ)
sage: bases = FQSymBases(FQSym)
sage: bases.super_categories()
[Category of realizations of Free Quasi-symmetric functions over Integer Ring,
 Join of Category of realizations of hopf algebras over Integer Ring
 and Category of graded algebras over Integer Ring
 and Category of graded coalgebras over Integer Ring,
 Category of graded connected hopf algebras with basis over Integer Ring]

class sage.combinat.fqsym.FQSymBasis_abstract(alg)

Bases: CombinatorialFreeModule, BindableClass

Abstract base class for bases of FQSym.

This must define two attributes:

• _prefix – the basis prefix
• _basis_name – the name of the basis and must match one of the names that the basis can be constructed from FQSym

an_element()

Return an element of self.

EXAMPLES:

sage: A = algebras.FQSym(QQ)
sage: F = A.F()
sage: F.an_element()
sage: G = A.G()
sage: G.an_element()
sage: M = A.M()
sage: M.an_element()
M[1] + 2*M[1, 2] + 4*M[2, 1]
The Hopf algebra $FQSym$ of free quasi-symmetric functions over a commutative ring $R$ is the free $R$-module with basis indexed by all permutations (i.e., the indexing set is the disjoint union of all symmetric groups). Its product is determined by the shifted shuffles of two permutations, whereas its coproduct is given by splitting a permutation (regarded as a word) into two (at every possible point) and standardizing the two pieces. This Hopf algebra was introduced in [MR]. See [GriRei18] (Chapter 8) for a treatment using modern notations.

In more detail: For each $n \geq 0$, consider the symmetric group $S_n$. Let $S$ be the disjoint union of the $S_n$ over all $n \geq 0$. Then, $FQSym$ is the free $R$-module with basis $(F_w)_{w \in S}$. This $R$-module is graded, with the $n$-th graded component being spanned by all $F_w$ for $w \in S_n$. A multiplication is defined on $FQSym$ as follows: For any two permutations $u \in S_k$ and $v \in S_l$, we set

$$F_u F_v = \sum F_w,$$

where the sum is over all shuffles of $u$ with $v[k]$. Here, the permutations $u$ and $v$ are regarded as words (by writing them in one-line notation), and $v[k]$ means the word obtained from $v$ by increasing each letter by $k$ (for example, $(1, 4, 2, 3)[5] = (6, 9, 7, 8)$); and the shuffles $w$ are translated back into permutations. This defines an associative multiplication on $FQSym$; its unity is $F_e$, where $e$ is the identity permutation in $S_0$.

In Section 1.3 of [AguSot05], Aguiar and Sottile construct a different basis of $FQSym$. Their basis, called the monomial basis and denoted by $(\mathcal{M}_u)$, is also indexed by permutations. It is connected to the above F-basis by the relation

$$F_u = \sum v \mathcal{M}_v,$$

where the sum ranges over all permutations $v$ such that each inversion of $u$ is an inversion of $v$. (An inversion of a permutation $w$ means a pair $(i, j)$ of positions satisfying $i < j$ and $w(i) > w(j)$.) The above relation yields a unitriangular change-of-basis matrix, and thus can be used to compute the $\mathcal{M}_u$ by Mobius inversion.

Another classical basis of $FQSym$ is $(G_w)_{w \in S}$, where $G_w = F_{w^{-1}}$. This is just a relabeling of the basis $(F_w)_{w \in S}$, but is a more natural choice from some viewpoints.

The algebra $FQSym$ is often identified with ("realized as") a subring of the ring of all bounded-degree non-commutative power series in countably many indeterminates (i.e., elements in $R[[x_1, x_2, x_3, \ldots]]$) of bounded degree. Namely, consider words over the alphabet $\{1, 2, 3, \ldots\}$; every noncommutative power series is an infinite $R$-linear combination of these words. Consider the $R$-linear map that sends each $G_u$ to the sum of all words whose standardization (also known as "standard permutation"; see standard_permutation()) is $u$. This map is an injective $R$-algebra homomorphism, and thus embeds $FQSym$ into the latter ring.

As an associative algebra, $FQSym$ has the richer structure of a dendriform algebra. This means that the associative product $\ast$ is decomposed as a sum of two binary operations

$$xy = x \succ y + x \prec y$$

that satisfy the axioms:

$$(x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \prec y) \prec z = x \prec (yz),$$

$$(xy) \succ z = x \succ (y \succ z).$$

These two binary operations are defined similarly to the (associative) product above: We set

$$F_u \prec F_v = \sum F_w,$$
where the sum is now over all shuffles of $u$ with $v[k]$ whose first letter is taken from $u$ (rather than from $v[k]$).

Similarly,

$$F_u \succ F_v = \sum F_w,$$

where the sum is over all remaining shuffles of $u$ with $v[k]$.

**Todo:** Decide what $1 \prec 1$ and $1 \succ 1$ are.

**Note:** The usual binary operator $*$ is used for the associative product.

**EXAMPLES:**

```sage
sage: F = algebras.FQSym(ZZ).F()
sage: x, y, z = F([1]), F([1,2]), F([1,3,2])
sage: (x * y) * z
F[1, 2, 3, 4, 6, 5] + ...
```

The product of $FQSym$ is associative:

```sage
sage: x * (y * z) == (x * y) * z
True
```

The associative product decomposes into two parts:

```sage
sage: x * y == F.prec(x, y) + F.succ(x, y)
True
```

The axioms of a dendriform algebra hold:

```sage
sage: F.prec(F.succ(x, y), z) == F.succ(x, F.prec(y, z))
True
sage: F.prec(F.prec(x, y), z) == F.prec(x, y * z)
True
sage: F.succ(x * y, z) == F.succ(x, F.succ(y, z))
True
```

$FQSym$ is also known as the Malvenuto-Reutenauer algebra:

```sage
sage: algebras.MalvenutoReutenauer(ZZ)
Free Quasi-symmetric functions over Integer Ring
```

**REFERENCES:**

- [MR]
- [LR1998]
- [GriRei18]

**class F(alg)**

Bases: `FQSymBasis_abstract`

The F-basis of $FQSym$. 

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This is the basis \((F_w)\), with \(w\) ranging over all permutations. See the documentation of \texttt{FreeQuasisymmetricFunctions} for details.

EXAMPLES:

```sage
sage: FQSym = algebras.FQSym(QQ)
sage: FQSym.F()
Free Quasi-symmetric functions over Rational Field in the F basis
```

class \texttt{Element}

Bases: \texttt{IndexedFreeModuleElement}

\texttt{to_symmetric_group_algebra}(n=None)

Return the element of a symmetric group algebra corresponding to the element \texttt{self} of \texttt{FQSym}.

INPUT:
- \(n\) – integer (default: the maximal degree of \texttt{self}); the rank of the target symmetric group algebra

EXAMPLES:

```sage
sage: A = algebras.FQSym(QQ).F()
sage: x = A([1,3,2,4]) + 5/2 * A([1,2,4,3])
sage: x.to_symmetric_group_algebra()
5/2*[1, 2, 4, 3] + [1, 3, 2, 4]
sage: x.to_symmetric_group_algebra(n=7)
5/2*[1, 2, 4, 3, 5, 6, 7] + [1, 3, 2, 4, 5, 6, 7]
sage: a = A.zero().to_symmetric_group_algebra(); a
0
sage: parent(a)
Symmetric group algebra of order 0 over Rational Field
sage: y = A([1,3,2,4]) + 5/2 * A([2,1])
sage: y.to_symmetric_group_algebra()
[1, 3, 2, 4] + 5/2*[2, 1, 3, 4]
sage: y.to_symmetric_group_algebra(6)
[1, 3, 2, 4, 5, 6] + 5/2*[2, 1, 3, 4, 5, 6]
```

coproduction_on_basis(x)

Return the coproduct of \(F_\sigma\) for \(\sigma\) a permutation (here, \(\sigma\) is \texttt{x}).

EXAMPLES:

```sage
sage: A = algebras.FQSym(QQ).F()
sage: x = A([1])
sage: ascii_art(A.coproduct(A.one()))  # indirect doctest
1 # 1
sage: ascii_art(A.coproduct(x))  # indirect doctest
1 # F + F # 1
   [1]   [1]
sage: A = algebras.FQSym(QQ).F()
sage: x, y, z = A([1]), A([2,1]), A([3,2,1])
sage: A.coproduct(z)
```

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**degree_on_basis(t)**

Return the degree of a permutation in the algebra of free quasi-symmetric functions. This is the size of the permutation (i.e., the $n$ for which the permutation belongs to $S_n$).

**EXAMPLES:**

```python
sage: A = algebras.FQSym(QQ).F()
sage: u = Permutation([2,1])
sage: A.degree_on_basis(u)
2
```

**prec_product_on_basis(x, y)**

Return the ≺ product of two permutations.

This is the shifted shuffle of $x$ and $y$ with the additional condition that the first letter of the result comes from $x$.

The usual symbol for this operation is ≺.

**See also:**

`product_on_basis()`, `succ_product_on_basis()`

**EXAMPLES:**

```python
sage: A = algebras.FQSym(QQ).F()
sage: x = Permutation([1,2])
sage: A.prec_product_on_basis(x, x)
F[1, 2, 3, 4] + F[1, 3, 2, 4] + F[1, 3, 4, 2]
sage: y = Permutation([[]])
sage: A.prec_product_on_basis(x, y) == A(x)
True
sage: A.prec_product_on_basis(y, x) == 0
True
```

**product_on_basis(x, y)**

Return the * associative product of two permutations.

This is the shifted shuffle of $x$ and $y$.

**See also:**

`succ_product_on_basis()`, `prec_product_on_basis()`

**EXAMPLES:**

```python
sage: A = algebras.FQSym(QQ).F()
sage: x = Permutation([1])
sage: A.product_on_basis(x, x)
F[1, 2] + F[2, 1]
```

**succ_product_on_basis(x, y)**

Return the ≻ product of two permutations.

This is the shifted shuffle of $x$ and $y$ with the additional condition that the first letter of the result comes from $y$.

The usual symbol for this operation is ≻.

**See also:**


• \texttt{product_on_basis()}, \texttt{prec_product_on_basis()}

EXAMPLES:

\begin{Verbatim}
\texttt{sage: A = algebras.FQSym(QQ).F()}
\texttt{sage: x = Permutation([1,2])}
\texttt{sage: A.succ_product_on_basis(x, x)}
\texttt{F[3, 1, 2, 4] + F[3, 1, 4, 2] + F[3, 4, 1, 2]}
\texttt{sage: y = Permutation([])}
\texttt{sage: A.succ_product_on_basis(x, y) == 0}
\texttt{True}
\texttt{sage: A.succ_product_on_basis(y, x) == A(x)}
\texttt{True}
\end{Verbatim}

class \texttt{G(alg)}

Bases: \texttt{FQSymBasis_abstract}

The \textit{G}-basis of \textit{FQSym}.

This is the basis \((G_w)\), with \(w\) ranging over all permutations. See the documentation of \texttt{FreeQuasisymmetricFunctions} for details.

EXAMPLES:

\begin{Verbatim}
\texttt{sage: FQSym = algebras.FQSym(QQ)}
\texttt{sage: G = FQSym.G(); G}
\texttt{Free Quasi-symmetric functions over Rational Field in the G basis}
\texttt{sage: G([3, 1, 2]).coproduct()}
\texttt{sage: G([3, 1, 2]) * G([2, 1])}
\texttt{degree_on_basis(t)}

Return the degree of a permutation in the algebra of free quasi-symmetric functions.

This is the size of the permutation (i.e., the \(n\) for which the permutation belongs to \(S_n\)).

EXAMPLES:

\begin{Verbatim}
\texttt{sage: A = algebras.FQSym(QQ).G()}
\texttt{sage: u = Permutation([2,1])}
\texttt{sage: A.degree_on_basis(u)}
\texttt{2}
\end{Verbatim}

class \texttt{M(alg)}

Bases: \texttt{FQSymBasis_abstract}

The \textit{M}-basis of \textit{FQSym}.

This is the Monomial basis \((M_w)\), with \(w\) ranging over all permutations. See the documentation of \texttt{FQSym} for details.

EXAMPLES:
sage: FQSym = algebras.FQSym(QQ)
sage: M = FQSym.M(); M
Free Quasi-symmetric functions over Rational Field in the Monomial basis

sage: M([3, 1, 2]).coproduct()
sage: M([3, 2, 1]).coproduct()
+ M[3, 2, 1] # M[]
sage: M([1, 2]) * M([1])

class Element

Bases: IndexedFreeModuleElement

star_involution()

Return the image of the element self of FQSym under the star involution.

See FQSymBases.ElementMethods.star_involution() for a definition of the involution and for examples.

See also:

omega_involution(), psi_involution()

EXAMPLES:

sage: FQSym = algebras.FQSym(ZZ)
sage: M = FQSym.M()
sage: M([2, 3, 1]).star_involution()
M[3, 1, 2]
sage: M([]).star_involution()
M[]

coproduct_on_basis(x)

Return the coproduct of $M_{\sigma}$ for $\sigma$ a permutation (here, $\sigma$ is $x$).

This uses Theorem 3.1 in [AguSot05].

EXAMPLES:

sage: M = algebras.FQSym(QQ).M()
sage: x = M([1])
sage: ascii_art(M.coproduct(M.one())) # indirect doctest
1 # 1

sage: ascii_art(M.coproduct(x)) # indirect doctest
1 # M + M # 1
[1] [1]

sage: M.coproduct(M([2, 1, 3]))
M[] # M[2, 1, 3] + M[2, 1, 3] # M[]
sage: M.coproduct(M([2, 3, 1]))
sage: M.coproduct(M([3, 2, 1]))
Combinatorics, Release 10.0

(continued from previous page)

sage: M.coproduct(M([3, 4, 2, 1])) \\
sage: M.coproduct(M([3, 4, 1, 2])) \\
M[] \# M[3, 4, 1, 2] + M[1, 2] \# M[1, 2] + M[3, 4, 1, 2] \# M[] \]

\section{degree_on_basis()}

Return the degree of a permutation in the algebra of free quasi-symmetric functions.

This is the size of the permutation (i.e., the \( n \) for which the permutation belongs to \( S_n \)).

EXAMPLES:

\[ \text{sage: A = algebras.FQSym(QQ).M()} \]
\[ \text{sage: u = Permutation([2,1])} \]
\[ \text{sage: A.degree_on_basis(u)} \]
\[ 2 \]

\section{a_realization()}

Return a particular realization of self (the F-basis).

EXAMPLES:

\[ \text{sage: FQSym = algebras.FQSym(QQ)} \]
\[ \text{sage: FQSym.a_realization()} \]
Free Quasi-symmetric functions over Rational Field in the F basis

\subsection{Free modules}

\subsection*{5.1.110 Free modules}

\begin{verbatim}
class sage.combinat.free_module.CartesianProductWithFlattening(flatten)
    Bases: object

    A class for Cartesian product constructor, with partial flattening

class sage.combinat.free_module.CombinatorialFreeModule(R, basis_keys=None, element_class=None, category=None, prefix=None, names=None, **kwds)
    Bases: UniqueRepresentation, Module, IndexedGenerators

    Class for free modules with a named basis

INPUT:

- \( R \) - base ring

- \text{basis_keys} - list, tuple, family, set, etc. defining the indexing set for the basis of this module

- \text{element_class} - the class of which elements of this module should be instances (optional, default None, in which case the elements are instances of \text{IndexedFreeModuleElement})

- \text{category} - the category in which this module lies (optional, default None, in which case use the “category of modules with basis” over the base ring \( R \); this should be a subcategory of \text{ModulesWithBasis})

\end{verbatim}
For the options controlling the printing of elements, see `IndexedGenerators`.

**Note:** These print options may also be accessed and modified using the `print_options()` method, after the module has been defined.

**EXAMPLES:**

We construct a free module whose basis is indexed by the letters a, b, c:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: F
Free module generated by {'a', 'b', 'c'} over Rational Field
```

Its basis is a family, indexed by a, b, c:

```
sage: e = F.basis()
sage: e
Finite family {'a': B['a'], 'b': B['b'], 'c': B['c']}
sage: [x for x in e]
[B['a'], B['b'], B['c']]
sage: [k for k in e.keys()]
['a', 'b', 'c']
```

Let us construct some elements, and compute with them:

```
sage: e['a']
B['a']
sage: 2*e['a']
2*B['a']
sage: e['a'] + 3*e['b']
B['a'] + 3*B['b']
```

Some uses of `sage.categories.commutative_additive_semigroups.CombinativeAdditiveSemigroups.ParentMethods.summation()` and `sum()`:

```
sage: F = CombinatorialFreeModule(QQ, [1,2,3,4])
sage: F.summation(F.monomial(1), F.monomial(3))
sage: F = CombinatorialFreeModule(QQ, [1,2,3,4])
sage: F.sum(F.monomial(i) for i in [1,2,3])
```

Note that free modules with a given basis and parameters are unique:

```
sage: F1 = CombinatorialFreeModule(QQ, (1,2,3,4))
sage: F1 is F
True
```

The identity of the constructed free module depends on the order of the basis and on the other parameters, like the prefix. Note that `CombinatorialFreeModule` is a `UniqueRepresentation`. Hence, two combinatorial free modules evaluate equal if and only if they are identical:
Because of this, if you create a free module with certain parameters and then modify its prefix or other print options, this affects all modules which were defined using the same parameters.

```
sage: F2.print_options(prefix='x')
sage: F2.prefix()
'x'
sage: F3 = CombinatorialFreeModule(QQ, [1,2,3,4], prefix='F')
sage: F3 is F2  # F3 was defined just like F2
True
sage: F3.prefix()
'x'
sage: F4 = CombinatorialFreeModule(QQ, [1,2,3,4], prefix='F', bracket=True)
sage: F4 == F2  # F4 was NOT defined just like F2
False
sage: F4.prefix()
'F'
sage: F2.print_options(prefix='F')  # reset for following doctests
```

The constructed module is in the category of modules with basis over the base ring:

```
sage: CombinatorialFreeModule(QQ, Partitions()).category()
Category of vector spaces with basis over Rational Field
```

If furthermore the index set is finite (i.e. in the category Sets().Finite()), then the module is declared as being finite dimensional:

```
sage: CombinatorialFreeModule(QQ, [1,2,3,4]).category()
Category of finite dimensional vector spaces with basis over Rational Field
sage: CombinatorialFreeModule(QQ, Partitions(3),
....: category=Algebras(QQ).WithBasis()).category()
Category of finite dimensional algebras with basis over Rational Field
```

See `sage.categories.examples.algebras_with_basis` and `sage.categories.examples.hopf_algebras_with_basis` for illustrations of the use of the category keyword, and see `sage.combinat.root_system.weight_space.WeightSpace` for an example of the use of `element_class`.

Customizing print and LaTeX representations of elements:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b'], prefix='x')
sage: original_print_options = F.print_options()
sage: sorted(original_print_options.items())
[('bracket', None),
 ('iterate_key', False),
]
```

(continues on next page)
sage: e = F.basis()
sage: e['a'] - 3 * e['b']
x['a'] - 3*x['b']

sage: F.print_options(prefix='x', scalar_mult=' ', bracket='{')
sage: e['a'] - 3 * e['b']
x{'a'} - 3 x{'b'}
sage: latex(e['a'] - 3 * e['b'])
\(x_{a} - 3 x_{b}\)

sage: F.print_options(latex_prefix='y')
sage: latex(e['a'] - 3 * e['b'])
y_{a} - 3 y_{b}

sage: F.print_options(sorting_reverse=True)
sage: e['a'] - 3 * e['b']
-3 x{'b'} + x{'a'}
sage: F.print_options(**original_print_options) # reset print options

sage: F = CombinatorialFreeModule(QQ, [(1,2), (3,4)])
sage: e = F.basis()
sage: e[(1,2)] - 3 * e[(3,4)]
B[(1, 2)] - 3*B[(3, 4)]

sage: F.print_options(bracket=['_',''])
sage: e[(1,2)] - 3 * e[(3,4)]
B_{(1, 2)} - 3*B_{(3, 4)}

sage: F.print_options(prefix='', bracket=False)
sage: e[(1,2)] - 3 * e[(3,4)]
(1, 2) - 3*(3, 4)

**CartesianProduct**
alias of CombinatorialFreeModule_CartesianProduct

**Element**
alias of IndexedFreeModuleElement

**Tensor**
alias of CombinatorialFreeModule_Tensor

**change_ring**\(R\)
Return the base change of self to \(R\).

**EXAMPLES:**
sage: F = CombinatorialFreeModule(ZZ, ['a', 'b', 'c']); F
Free module generated by {'a', 'b', 'c'} over Integer Ring
sage: F_QQ = F.change_ring(QQ); F_QQ
Free module generated by {'a', 'b', 'c'} over Rational Field
sage: F_QQ.change_ring(ZZ) == F
True

construction()

The construction functor and base ring for self.

EXAMPLES:

sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'],
    category=AlgebrasWithBasis(QQ))

sage: F.construction()
(VectorFunctor, Rational Field)

dimension()

Return the dimension of the free module (which is given by the number of
elements in the basis).

EXAMPLES:

sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])

sage: F.dimension()
3

sage: F.basis().cardinality()
3

sage: F.basis().keys().cardinality()
3

Rank is available as a synonym:

sage: F.rank()
3

sage: s = SymmetricFunctions(QQ).schur()

sage: s.dimension()
+Infinity

element_class()

The (default) class for the elements of this parent

Overrides Parent.element_class() to force the construction of Python
class. This is currently needed
to inherit really all the features from categories, and in particular the initialization of _mul_ in Magmas.
ParentMethods.__init_extra__().

EXAMPLES:

sage: A = Algebras(QQ).WithBasis().example(); A
An example of an algebra with basis:
the free algebra on the generators ('a', 'b', 'c') over Rational Field

sage: A.element_class.mro()
[<class 'sage.categories.examples.algebras_with_basis.FreeAlgebra_with_category.**'], **]

(continues on next page)
from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

from sage.modules.with_basis.indexed_element import IndexedFreeModuleElement

sage: a, b, c = A.algebra_generators()
sage: a * b
B[ab]

sage: from_vector(vector, order=None, coerce=True)
Build an element of self from a (sparse) vector.

See also:
get_order().CombinatorialFreeModule.Element._vector_()

EXAMPLES:

sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: b = QS3.from_vector(vector((2, 0, 0, 0, 0, 4))); b
2*[1, 2, 3] + 4*[3, 2, 1]
sage: a = 2*QS3([1,2,3])+4*QS3([3,2,1])
sage: a == b
True

get_order()
Return the order of the elements in the basis.

EXAMPLES:

sage: QS2 = SymmetricGroupAlgebra(QQ, 2)
sage: QS2.get_order()
# note: order changed on 2009-03-13
[[2, 1], [1, 2]]

gget_order_key()
Return a comparison key on the basis indices that is compatible with the current term order.

EXAMPLES:

sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example()
sage: A.set_order(["x", "y", "a", "b"])
sage: Akey = A.get_order_key()
sage: sorted(A.basis().keys(), key=Akey)
["x", "y", "a", "b"]
sage: A.set_order(list(reversed(A.basis().keys())))
sage: Akey = A.get_order_key()
sage: sorted(A.basis().keys(), key=Akey)
["b", "a", "y", "x"]

is_exact()
Return True if elements of self have exact representations, which is true of self if and only if it is true of self.basis().keys() and self.base_ring().
sage: GroupAlgebra(GL(3, GF(7)), RR).is_exact()
False
sage: GroupAlgebra(GL(3, pAdicRing(7))).is_exact()  # not implemented correctly
→ (not my fault!)
False

linear_combination(iter_of_elements_coeff, factor_on_left=True)

Return the linear combination $\lambda_1 v_1 + \cdots + \lambda_k v_k$ (resp. the linear combination $v_1 \lambda_1 + \cdots + v_k \lambda_k$) where iter_of_elements_coeff iterates through the sequence $((v_1, \lambda_1), \ldots, (v_k, \lambda_k))$.

INPUT:

- iter_of_elements_coeff – iterator of pairs (element, coeff) with element in self and coeff in self.base_ring()
- factor_on_left – (optional) if True, the coefficients are multiplied from the left if False, the coefficients are multiplied from the right

EXAMPLES:

sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.monomial('a')
B['a']

F.monomial is in fact (almost) a map:

sage: F.monomial
Term map from {'a', 'b', 'c'} to Free module generated by {'a', 'b', 'c'} over Rational Field

rank()

Return the dimension of the free module (which is given by the number of elements in the basis).

EXAMPLES:

sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.dimension()
3
sage: F.basis().cardinality()
3
sage: F.basis().keys().cardinality()
3

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Rank is available as a synonym:

```python
sage: F.rank()
3
```

```python
sage: s = SymmetricFunctions(QQ).schur()
sage: s.dimension()
+Infinity
```

**set_order** *(order)*

Set the order of the elements of the basis.

If `set_order()` has not been called, then the ordering is the one used in the generation of the elements of self’s associated enumerated set.

**Warning:** Many cached methods depend on this order, in particular for constructing subspaces and quotients. Changing the order after some computations have been cached does not invalidate the cache, and is likely to introduce inconsistencies.

**EXAMPLES:**

```python
sage: QS2 = SymmetricGroupAlgebra(QQ, 2)
sage: b = list(QS2.basis().keys())
sage: b.reverse()
sage: QS2.set_order(b)
sage: QS2.get_order()
[[2, 1], [1, 2]]
```

**sum** *(iter_of_elements)*

Return the sum of all elements in `iter_of_elements`.

Overrides method inherited from commutative additive monoid as it is much faster on dicts directly.

**INPUT:**

- `iter_of_elements` – iterator of elements of self

**EXAMPLES:**

```python
sage: F = CombinatorialFreeModule(QQ, [1, 2])
sage: f = F.an_element(); f
sage: F.sum( f for _ in range(5) )
```

**sum_of_terms** *(terms, distinct=False)*

Construct a sum of terms of self.

**INPUT:**

- `terms` – a list (or iterable) of pairs `(index, coeff)`
- `distinct` – (default: `False`) whether the indices are guaranteed to be distinct

**EXAMPLES:**

```python
```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.sum_of_terms([('a',2), ('c',3)])
2*B['a'] + 3*B['c']

If distinct is True, then the construction is optimized:

sage: F.sum_of_terms([(a,2), (c,3), distinct = True])
2*B['a'] + 3*B['c']

**Warning:** Use distinct=True only if you are sure that the indices are indeed distinct:

sage: F.sum_of_terms([(a,2), (a,3), distinct = True])
3*B['a']

Extreme case:

sage: F.sum_of_terms([])
0

term(index, coeff=None)
Construct a term in self.

INPUT:

- index – the index of a basis element
- coeff – an element of the coefficient ring (default: one)

EXAMPLES:

sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.term('a',3)
3*B['a']
sage: F.term('a')
B['a']

Design: should this do coercion on the coefficient ring?

zero()

EXAMPLES:

sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.zero()
0

class sage.combinat.free_module.CombinatorialFreeModule_CartesianProduct(modules, **options)
Bases: CombinatorialFreeModule

An implementation of Cartesian products of modules with basis

EXAMPLES:

We construct two free modules, assign them short names, and construct their Cartesian product:
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: H = CombinatorialFreeModule(ZZ, [4,7]); H.rename("H")
sage: S = cartesian_product([F, G])
sage: S
F (+) G
sage: S.basis()
Lazy family (Term map from Disjoint union of Family ({4, 5}, {4, 6}) to F (+) G(i))_→{i in Disjoint union of Family ({4, 5}, {4, 6})} 

Note that the indices of the basis elements of F and G intersect non trivially. This is handled by forcing the union to be disjoint:

sage: list(S.basis())
[B[(0, 4)], B[(0, 5)], B[(1, 4)], B[(1, 6)]]

We now compute the Cartesian product of elements of free modules:

sage: f = F.monomial(4) + 2 * F.monomial(5)
sage: g = 2 * G.monomial(4) + G.monomial(6)
sage: h = H.monomial(4) + H.monomial(7)
sage: cartesian_product([f,g])
B[(0, 4)] + 2*B[(0, 5)] + 2*B[(1, 4)] + B[(1, 6)]
sage: cartesian_product([f,g,h])
B[(0, 4)] + 2*B[(0, 5)] + 2*B[(1, 4)] + B[(1, 6)] + B[(2, 4)] + B[(2, 7)]
sage: cartesian_product([f,g,h]).parent()
F (+) G (+) H

TODO: choose an appropriate semantic for Cartesian products of Cartesian products (associativity?):

sage: S = cartesian_product([cartesian_product([F, G]), H]) # todo: not implemented
F (+) G (+) H

class Element
Bases: IndexedFreeModuleElement
cartesian_embedding(i)

Return the natural embedding morphism of the i-th Cartesian factor (summand) of self into self.

INPUT:

• i – an integer

EXAMPLES:

sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: S = cartesian_product([F, G])
sage: phi = S.cartesian_embedding(0)
sage: phi(F.monomial(4) + 2 * F.monomial(5))
B[(0, 4)] + 2*B[(0, 5)]
sage: phi(F.monomial(4) + 2 * F.monomial(6)).parent() == S
True

cartesian_factors()

Return the factors of the Cartesian product.
EXAMPLES:

```python
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: S = cartesian_product([F, G])
sage: S.cartesian_factors()
(F, G)
```

**cartesian_projection**

Return the natural projection onto the \(i\)-th Cartesian factor (summand) of \(self\).

**INPUT:**

- \(i\) – an integer

**EXAMPLES:**

```python
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: S = cartesian_product([F, G])
sage: x = S.monomial((0,4)) + 2 * S.monomial((0,5)) + 3 * S.monomial((1,6))
sage: S.cartesian_projection(0)(x)
sage: S.cartesian_projection(1)(x)
3*B[6]
sage: S.cartesian_projection(0)(x).parent() == F
True
sage: S.cartesian_projection(1)(x).parent() == G
True
```

**summand_embedding**

Return the natural embedding morphism of the \(i\)-th Cartesian factor (summand) of \(self\) into \(self\).

**INPUT:**

- \(i\) – an integer

**EXAMPLES:**

```python
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: S = cartesian_product([F, G])
sage: phi = S.cartesian_embedding(0)
sage: phi(F.monomial(4) + 2 * F.monomial(5))
B[(0, 4)] + 2*B[(0, 5)]
sage: phi(F.monomial(4) + 2 * F.monomial(6)).parent() == S
True
```

**summand_projection**

Return the natural projection onto the \(i\)-th Cartesian factor (summand) of \(self\).

**INPUT:**

- \(i\) – an integer

**EXAMPLES:**
```python
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.rename("G")
sage: S = cartesian_product([F, G])
sage: x = S.monomial((0,4)) + 2 * S.monomial((0,5)) + 3 * S.monomial((1,6))
sage: S.cartesian_projection(0)(x)
sage: S.cartesian_projection(1)(x)
3*B[6]
sage: S.cartesian_projection(0)(x).parent() == F
True
sage: S.cartesian_projection(1)(x).parent() == G
True
```

```python
class sage.combinat.free_module.CombinatorialFreeModule_Tensor(  
    modules, **options)
Bases: CombinatorialFreeModule

Tensor Product of Free Modules

EXAMPLES:

We construct two free modules, assign them short names, and construct their tensor product:

```python
sage: F = CombinatorialFreeModule(ZZ, [1,2]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [3,4]); G.rename("G")
sage: T = tensor([F, G]); T
F # G
sage: T.category()
Category of tensor products of finite dimensional modules with basis over Integer
sage: T.construction()
[tensor, ]
```

T is a free module, with same base ring as F and G:

```python
sage: T.base_ring()
Integer Ring
```

The basis of T is indexed by tuples of basis indices of F and G:

```python
sage: T.basis().keys()
Image of Cartesian product of {1, 2}, {3, 4} by The map <class 'tuple'> from
sage: T.basis().keys().list()
[(1, 3), (1, 4), (2, 3), (2, 4)]
```

FIXME: Should elements of a CartesianProduct be tuples (making them hashable)?

Here are the basis elements themselves:

```python
sage: T.basis().cardinality()
4
sage: list(T.basis())
```
The tensor product is associative and flattens sub tensor products:

```
sage: H = CombinatorialFreeModule(ZZ, [5,6]); H.rename("H")
sage: tensor([F, tensor([G, H])])
F # G # H
tsage: tensor([tensor([F, G]), H])
F # G # H
tsage: tensor([F, G, H])
F # G # H
```

We now compute the tensor product of elements of free modules:

```
sage: f = F.monomial(1) + 2 * F.monomial(2)
sage: g = 2*G.monomial(3) + G.monomial(4)
sage: h = H.monomial(5) + H.monomial(6)
sage: tensor([f, g])
sage: tensor([tensor([f, g]), h]) == tensor([f, g, h])
True
sage: tensor([tensor([f, g]), h]) == tensor([f, g, h])
True
```

Again, the tensor product is associative on elements:

```
sage: tensor([f, tensor([g, h])]) == tensor([f, g, h])
True
sage: tensor([tensor([f, g]), h]) == tensor([f, g, h])
True
```

Note further that the tensor product spaces need not preexist:

```
sage: t = tensor([f, g, h])
sage: t.parent()
F # G # H
tensor_constructor
```

```
tensor_constructor(modules)
```

**INPUT:**

- `modules` – a tuple \((F_1, \ldots, F_n)\) of free modules whose tensor product is self

Returns the canonical multilinear morphism from \(F_1 \times \cdots \times F_n\) to \(F_1 \otimes \cdots \otimes F_n\)

**EXAMPLES:**

```
sage: F = CombinatorialFreeModule(ZZ, [1,2]); F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [3,4]); G.rename("G")
sage: H = CombinatorialFreeModule(ZZ, [5,6]); H.rename("H")

sage: f = F.monomial(1) + 2 * F.monomial(2)
sage: g = 2*G.monomial(3) + G.monomial(4)
sage: h = H.monomial(5) + H.monomial(6)
sage: FG = tensor([F, G])
sage: phi_fg = FG.tensor_constructor((F, G))
sage: phi_fg(f,g)
sage: FGH = tensor([F, G, H])
sage: phi_fgh = FGH.tensor_constructor((F, G, H))
sage: phi_fgh(f, g, h)
```

(continues on next page)
\[ \begin{align*}
\end{align*} \]

```
sage: phi_fg_h = FGH.tensor_constructor((FG, H))
sage: phi_fg_h(phi_fg(f, g), h)
```

```
tensor_factors()
```

Return the tensor factors of this tensor product.

**EXAMPLES:**

```
sage: F = CombinatorialFreeModule(ZZ, [1,2])
sage: F.rename("F")
sage: G = CombinatorialFreeModule(ZZ, [3,4])
sage: G.rename("G")
sage: T = tensor([F, G]); T
F \times G
sage: F # G
sage: T.tensor_factors()
(F, G)
```

### 5.1.111 Free Dendriform Algebras

**AUTHORS:**
Frédéric Chapoton (2017)

**class** `sage.combinat.free_dendriform_algebra.DendriformFunctor(vars)`

Bases: `ConstructionFunctor`

A constructor for dendriform algebras.

**EXAMPLES:**

```
sage: P = algebras.FreeDendriform(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F = P.construction()[0]; F
Dendriform[x,y]
sage: A = GF(5)["a,b"]
sage: a, b = A.gens()
sage: F(A)
Free Dendriform algebra on 2 generators ['x', 'y'] over Multivariate Polynomial Ring in a, b over Finite Field of size 5
sage: f = A.hom([a+b,a-b],A)
sage: F(f)
Generic endomorphism of Free Dendriform algebra on 2 generators ['x', 'y'] over Multivariate Polynomial Ring in a, b over Finite Field of size 5
```
**Combinatorics, Release 10.0**

```
sage: F(f)(a * F(A)(x))
(a+b)*B[x[, .]]
```

**merge(other)**

Merge self with another construction functor, or return None.

**EXAMPLES:**

```
sage: F = sage.combinat.free_dendriform_algebra.DendriformFunctor(['x', 'y'])
sage: G = sage.combinat.free_dendriform_algebra.DendriformFunctor(['t'])
sage: F.merge(G)
Dendriform[x,y,t]
sage: F.merge(F)
Dendriform[x,y]
```

Now some actual use cases:

```
sage: R = algebras.FreeDendriform(ZZ, 'x,y,z')
sage: x,y,z = R.gens()
sage: 1/2 * x
1/2*B[x[., .]]
sage: parent(1/2 * x)
Free Dendriform algebra on 3 generators ['x', 'y', 'z'] over Rational Field
```

```
sage: S = algebras.FreeDendriform(QQ, 'zt')
sage: z,t = S.gens()
sage: x + t
B[t[., .]] + B[x[., .]]
sage: parent(x + t)
Free Dendriform algebra on 4 generators ['z', 't', 'x', 'y'] over Rational Field
```

**rank = 9**

**class** sage.combinat.free_dendriform_algebra.FreeDendriformAlgebra(R, names=None)**

**Bases:** CombinatorialFreeModule

The free dendriform algebra.

Dendriform algebras are associative algebras, where the associative product \(*\) is decomposed as a sum of two binary operations

\[ x * y = x \succ y + x \prec y \]

that satisfy the axioms:

\[ (x \succ y) \prec z = x \succ (y \prec z), \]
\[ (x \prec y) \prec z = x \prec (y \succ z). \]
\[ (x * y) \succ z = x \succ (y \succ z). \]

The free Dendriform algebra on a given set \(E\) has an explicit description using (planar) binary trees, just as the free associative algebra can be described using words. The underlying vector space has a basis indexed by finite binary trees endowed with a map from their vertices to \(E\). In this basis, the associative product of two (decorated) binary trees \(S * T\) is the sum over all possible ways of identifying (glueing) the rightmost path in \(S\) and the leftmost path in \(T\).
The decomposition of the associative product as the sum of two binary operations $\succ$ and $\prec$ is made by separating the terms according to the origin of the root vertex. For $x \succ y$, one keeps the terms where the root vertex comes from $y$, whereas for $x \prec y$ one keeps the terms where the root vertex comes from $x$.

The free dendriform algebra can also be considered as the free algebra over the Dendriform operad.

Note: The usual binary operator $*$ is used for the associative product.

EXAMPLES:

sage: F = algebras.FreeDendriform(ZZ, 'xyz')
sage: x,y,z = F.gens()
sage: (x * y) * z
B[x[., y[., z[., .]]]] + B[x[., z[y[., .], .]]] + B[y[x[., .], z[., .]]] + B[z[x[., y[., .], .]]] + B[z[y[x[., .], .], .]] + B[z[y[., .], .]]

The free dendriform algebra is associative:

sage: x * (y * z) == (x * y) * z
True

The associative product decomposes in two parts:

sage: x * y == F.prec(x, y) + F.succ(x, y)
True

The axioms hold:

sage: F.prec(F.succ(x, y), z) == F.succ(x, F.prec(y, z))
True
sage: F.prec(F.prec(x, y), z) == F.prec(x, y * z)
True
sage: F.succ(x * y, z) == F.succ(x, F.succ(y, z))
True

When there is only one generator, unlabelled trees are used instead:

sage: F1 = algebras.FreeDendriform(QQ)
sage: w = F1.gen(0); w
B[., .]
sage: w * w * w
B[[[., .], .]] + B[[[., .], .]] + B[[[., .], .]] + B[[[., .], .]] + B[[[., .], .]]

The set $E$ can be infinite:

sage: F = algebras.FreeDendriform(QQ, ZZ)
sage: w = F.gen(1); w
B[1[., .]]
sage: x = F.gen(2); x
B[-1[., .]]
sage: w*x
B[-1[1[., .], .]] + B[1[., -1[., .]]]

REFERENCES:
algebra_generators()

Return the generators of this algebra.

These are the binary trees with just one vertex.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(ZZ, 'fgh'); A
Free Dendriform algebra on 3 generators ['f', 'g', 'h']
over Integer Ring
sage: list(A.algebra_generators())
[B[f[., .]], B[g[., .]], B[h[., .]]]
sage: A = algebras.FreeDendriform(QQ, ['x1', 'x2'])
sage: list(A.algebra_generators())
[B[x1[., .]], B[x2[., .]]]
```

an_element()

Return an element of self.

EXAMPLES:

```
sage: A = algebras.FreeDendriform(QQ, 'xy')
sage: A.an_element()
B[x[., .]] + 2*B[x[., x[., .]]] + 2*B[x[x[., .], .]]
```

change_ring(R)

Return the free dendriform algebra in the same variables over $R$.

INPUT:

- $R$ – a ring

EXAMPLES:

```
sage: A = algebras.FreeDendriform(ZZ, 'fgh')
sage: A.change_ring(QQ)
Free Dendriform algebra on 3 generators ['f', 'g', 'h'] over Rational Field
```

construction()

Return a pair $(F, R)$, where $F$ is a $DendriformFunctor$ and $R$ is a ring, such that $F(R)$ returns self.

EXAMPLES:

```
sage: P = algebras.FreeDendriform(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F, R = P.construction()
sage: F
Dendriform[x,y]
sage: R
Integer Ring
sage: F(ZZ) is P
True
sage: F(QQ)
Free Dendriform algebra on 2 generators ['x', 'y'] over Rational Field
```
**coproduct_on_basis**($x$)

Return the coproduct of a binary tree.

**EXAMPLES:**

```python
sage: A = algebras.FreeDendriform(QQ)
sage: x = A.gen(0)
sage: ascii_art(A.coproduct(A.one()))  # indirect doctest
1 # 1
sage: ascii_art(A.coproduct(x))      # indirect doctest
1 # B + B # 1
   o  o
sage: A = algebras.FreeDendriform(QQ, 'xyz')
sage: x, y, z = A.gens()
sage: w = A.under(z, A.over(x, y))
sage: A.coproduct(z)
B[.] # B[z[., .]] + B[z[., .]] # B[.]
sage: A.coproduct(w)
B[.] # B[x[z[., .], y[., .]]] + B[x[., .]] # B[z[., y[., .]]] +
B[x[., .]] # B[y[z[., .], .]] + B[x[., y[., .]]] # B[z[., .]] +
B[x[z[., .], .]] # B[y[., .]] + B[x[z[., .], y[., .]]] # B[.]
```

**degree_on_basis**($t$)

Return the degree of a binary tree in the free Dendriform algebra.

This is the number of vertices.

**EXAMPLES:**

```python
sage: A = algebras.FreeDendriform(QQ, '@')
sage: RT = A.basis().keys()
sage: u = RT([], '@')
sage: A.degree_on_basis(u.over(u))
2
```

**gen**($i$)

Return the $i$-th generator of the algebra.

**INPUT:**

- $i$ – an integer

**EXAMPLES:**

```python
sage: F = algebras.FreeDendriform(ZZ, 'xyz')
sage: F.gen(0)
B[x[., .]]
sage: F.gen(4)
Traceback (most recent call last):
...
IndexError: argument i (= 4) must be between 0 and 2
```

**gens**

Return the generators of `self` (as an algebra).
EXAMPLES:

```python
sage: A = algebras.FreeDendriform(ZZ, 'fgh')
sage: A.gens()
(B[f[., .]], B[g[., .]], B[h[., .]])
```

**one_basis()**

Return the index of the unit.

EXAMPLES:

```python
sage: A = algebras.FreeDendriform(QQ, '@')
sage: A.one_basis()
.s
sage: A = algebras.FreeDendriform(QQ, 'xy')
sage: A.one_basis()
```

**over()**

Return the over product.

The over product \(x/y\) is the binary tree obtained by grafting the root of \(y\) at the rightmost leaf of \(x\).

The usual symbol for this operation is \(\div\).

See also:

```
product(), succ(), prec(), under()
```

EXAMPLES:

```python
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.over(x, x)
B[[., [., .]]]
```

**prec()**

Return the \(\prec\) dendriform product.

This is the sum over all possible ways to identify the rightmost path in \(x\) and the leftmost path in \(y\), with the additional condition that the root vertex of the result comes from \(x\).

The usual symbol for this operation is \(\prec\).

See also:

```
product(), succ(), over(), under()
```

EXAMPLES:

```python
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.prec(x, x)
B[[., [., .]]]
```
\texttt{prec\_product\_on\_basis}(x, y)

Return the \(<\) dendriform product of two trees.

This is the sum over all possible ways of identifying the rightmost path in \(x\) and the leftmost path in \(y\), with the additional condition that the root vertex of the result comes from \(x\).

The usual symbol for this operation is \(<\).

See also:

- \texttt{product\_on\_basis()}, \texttt{succ\_product\_on\_basis()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.prec_product_on_basis(x, x)
B[[., [., .]]]
\end{verbatim}

\texttt{product\_on\_basis}(x, y)

Return the \(*\) associative dendriform product of two trees.

This is the sum over all possible ways of identifying the rightmost path in \(x\) and the leftmost path in \(y\). Every term corresponds to a shuffle of the vertices on the rightmost path in \(x\) and the vertices on the leftmost path in \(y\).

See also:

- \texttt{succ\_product\_on\_basis()}, \texttt{prec\_product\_on\_basis()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.product_on_basis(x, x)
B[[., [., .]]] + B[[[., .], .]]
\end{verbatim}

\texttt{some\_elements()}

Return some elements of the free dendriform algebra.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ)
sage: A.some_elements()
[B[.],
 B[[., .]],
 B[[., [., .]]] + B[[[., .], .]],
 B[.] + B[[., [., .]]] + B[[[., .], .]]]
\end{verbatim}

With several generators:

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ, 'xy')
sage: A.some_elements()
[B[.],
 (continues on next page)
B[x[., .]],
B[x[., x[., .]], x[., .]],
B[.] + B[x[., x[., .]]] + B[x[., .], .]]

\textbf{succ()}

Return the $\succ$ dendriform product.

This is the sum over all possible ways of identifying the rightmost path in $x$ and the leftmost path in $y$, with the additional condition that the root vertex of the result comes from $y$.

The usual symbol for this operation is $\succ$.

\textbf{See also:}

\texttt{product()}, \texttt{prec()}, \texttt{over()}, \texttt{under()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.succ(x, x)
B[[., .], .]]
\end{verbatim}

\textbf{succ_product_on_basis}(x, y)

Return the $\succ$ dendriform product of two trees.

This is the sum over all possible ways to identify the rightmost path in $x$ and the leftmost path in $y$, with the additional condition that the root vertex of the result comes from $y$.

The usual symbol for this operation is $\succ$.

\textbf{See also:}

\texttt{product_on_basis()}, \texttt{prec_product_on_basis()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.succ_product_on_basis(x, x)
B[[[., .], .]]
\end{verbatim}

\textbf{under()}

Return the under product.

The over product $x \backslash y$ is the binary tree obtained by grafting the root of $x$ at the leftmost leaf of $y$.

The usual symbol for this operation is $\backslash$.

\textbf{See also:}

\texttt{product()}, \texttt{succ()}, \texttt{prec()}, \texttt{over()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = RT([])
sage: A.succ_product_on_basis(x, x)
B[[[., .], .]]
\end{verbatim}
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```python
sage: A = algebras.FreeDendriform(QQ)
sage: RT = A.basis().keys()
sage: x = A.gen(0)
sage: A.under(x, x)
B[[[., .], .]]
```

variable_names()

Return the names of the variables.

EXAMPLES:

```python
sage: R = algebras.FreeDendriform(QQ, 'xy')
sage: R.variable_names()
{'x', 'y'}
```

5.1.112 Free Pre-Lie Algebras

AUTHORS:

• Florent Hivert, Frédéric Chapoton (2011)

```python
class sage.combinat.free_prelie_algebra.FreePreLieAlgebra(R, names=None)
Bases: CombinatorialFreeModule
```

The free pre-Lie algebra.

Pre-Lie algebras are non-associative algebras, where the product * satisfies

\[(x * y) * z - x * (y * z) = (x * z) * y - x * (z * y)\].

We use here the convention where the associator

\[(x, y, z) := (x * y) * z - x * (y * z)\]

is symmetric in its two rightmost arguments. This is sometimes called a right pre-Lie algebra.

They have appeared in numerical analysis and deformation theory.

The free Pre-Lie algebra on a given set \(E\) has an explicit description using rooted trees, just as the free associative algebra can be described using words. The underlying vector space has a basis indexed by finite rooted trees endowed with a map from their vertices to \(E\). In this basis, the product of two (decorated) rooted trees \(S * T\) is the sum over vertices of \(S\) of the rooted tree obtained by adding one edge from the root of \(T\) to the given vertex of \(S\). The root of these trees is taken to be the root of \(S\). The free pre-Lie algebra can also be considered as the free algebra over the PreLie operad.

**Warning:** The usual binary operator * can be used for the pre-Lie product. Beware that it but must be parenthesized properly, as the pre-Lie product is not associative. By default, a multiple product will be taken with left parentheses.

EXAMPLES:

```python
sage: F = algebras.FreePreLie(ZZ, 'xyz')
sage: x,y,z = F.gens()
sage: (x * y) * z
```

(continues on next page)
B[x[y[z[]]]] + B[x[y[], z[]]]
\[\textbf{sage: } (x * y) * z - x * (y * z) = (x * z) * y - x * (z * y)\]
True

The free pre-Lie algebra is non-associative:

\[\textbf{sage: } x * (y * z) = (x * y) * z\]
False

The default product is with left parentheses:

\[\textbf{sage: } x * y * z = (x * y) * z\]
True
\[\textbf{sage: } x * y * z * x = ((x * y) * z) * x\]
True

The NAP product as defined in [Liv2006] is also implemented on the same vector space:

\[\textbf{sage: } N = F.nap_product\]
\[\textbf{sage: } N(x*y,z*z)\]
B[x[y[], z[z[]]]]

When None is given as input, unlabelled trees are used instead:

\[\textbf{sage: } F1 = algebras.FreePreLie(QQ, None)\]
\[\textbf{sage: } w = F1.gen(0); w\]
B[]
\[\textbf{sage: } w * w * w * w\]
B[[]][[]][[]] + B[[], []][[]] + 3*B[[], [], []] + B[[], [], [], []]

However, it is equally possible to use labelled trees instead:

\[\textbf{sage: } F1 = algebras.FreePreLie(QQ, 'q')\]
\[\textbf{sage: } w = F1.gen(0); w\]
B[q[]]
\[\textbf{sage: } w * w * w * w\]
B[q[q[q[q[]]]]] + B[q[q[q[], q[]]]] + 3*B[q[q[], q[q[]]]] + B[q[q[], q[], q[]]]

The set $E$ can be infinite:

\[\textbf{sage: } F = algebras.FreePreLie(QQ, ZZ)\]
\[\textbf{sage: } w = F.gen(1); w\]
B[1[]]
\[\textbf{sage: } x = F.gen(2); x\]
B[-1[]]
\[\textbf{sage: } y = F.gen(3); y\]
B[2[]]
\[\textbf{sage: } w*x\]
B[1[-1[]]]
\[\textbf{sage: } (w*x)*y\]
B[1[-1[2[]]]] + B[1[-1[]], 2[]]
\[\textbf{sage: } w*(x*y)\]
B[1[-1[2[]]]]
Elements of a free pre-Lie algebra can be lifted to the universal enveloping algebra of the associated Lie algebra. The universal enveloping algebra is the Grossman-Larson Hopf algebra:

```
sage: F = algebras.FreePreLie(QQ,'abc')
sage: a,b,c = F.gens()
sage: (a*b+b*c).lift()
B[#[a[b[]]]] + B[#[b[c[]]]]
```

**Note:** Variables names can be None, a list of strings, a string or an integer. When None is given, unlabelled rooted trees are used. When a single string is given, each letter is taken as a variable. See `sage.combinat.words.alphabet.build_alphabet()`.

**Warning:** Beware that the underlying combinatorial free module is based either on `RootedTrees` or on `LabelledRootedTrees`, with no restriction on the labellings. This means that all code calling the `basis()` method would not give meaningful results, since `basis()` returns many “chaff” elements that do not belong to the algebra.

**REFERENCES:**
- [ChLi]
- [Liv2006]

```python
class Element
    Bases: IndexedFreeModuleElement

    lift()
    Lift element to the Grossman-Larson algebra.
    EXAMPLES:
    ```
sage: F = algebras.FreePreLie(QQ,'abc')
sage: elt = F.an_element().lift(); elt
B[#[a[a[a[]]]]] + B[#[a[a[], a[a[]]]]]
sage: parent(elt)
Grossman-Larson Hopf algebra on 3 generators ['a', 'b', 'c'] over Rational Field
```

```python
class Element
    Bases: IndexedFreeModuleElement

    algebra_generators()
    Return the generators of this algebra.
    These are the rooted trees with just one vertex.
    EXAMPLES:
    ```
sage: A = algebras.FreePreLie(ZZ, 'fgh'); A
Free PreLie algebra on 3 generators ['f', 'g', 'h'] over Integer Ring
sage: list(A.algebra_generators())
[B[f[]], B[g[]], B[h[]]]
sage: A = algebras.FreePreLie(QQ, ['x1','x2'])
sage: list(A.algebra_generators())
[B[x1[]], B[x2[]]]
```
an_element()

Return an element of self.

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, 'xy')
sage: A.an_element()
B[x[x[x[x]]]] + B[x[x[], x[x]]]
```

bracket_on_basis(x, y)

Return the Lie bracket of two trees.

This is the commutator \([x, y] = x \ast y - y \ast x\) of the pre-Lie product.

See also:

```
pre_Lie_product_on_basis()
```

EXAMPLES:

```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: y = RT([x])
sage: A.bracket_on_basis(x, y)
-B[[[[], [[]]]]] + B[[[], [[]]]] - B[[[]], [[]]]
```

change_ring(R)

Return the free pre-Lie algebra in the same variables over \(R\).

INPUT:

• \(R\) – a ring

EXAMPLES:

```
sage: A = algebras.FreePreLie(ZZ, 'fgh')
sage: A.change_ring(QQ)
Free PreLie algebra on 3 generators ['f', 'g', 'h'] over Rational Field
```

construction()

Return a pair \((F, R)\), where \(F\) is a \texttt{PreLieFunctor} and \(R\) is a ring, such that \(F(R)\) returns \texttt{self}.

EXAMPLES:

```
sage: P = algebras.FreePreLie(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F, R = P.construction()
sage: F
PreLie[x,y]
sage: R
Integer Ring
sage: F(ZZ) is P
True
sage: F(QQ)
Free PreLie algebra on 2 generators ['x', 'y'] over Rational Field
```
**degree_on_basis(t)**
Return the degree of a rooted tree in the free Pre-Lie algebra.
This is the number of vertices.

**EXAMPLES:**
```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: A.degree_on_basis(RT([RT([[]])]))
2
```

**gen(i)**
Return the i-th generator of the algebra.

**INPUT:**
• i – an integer

**EXAMPLES:**
```
sage: F = algebras.FreePreLie(ZZ, 'xyz')
sage: F.gen(0)  # B[x[]]
sage: F.gen(4)  # Traceback (most recent call last):
...  # IndexError: argument i (= 4) must be between 0 and 2
```

**gens()**
Return the generators of self (as an algebra).

**EXAMPLES:**
```
sage: A = algebras.FreePreLie(ZZ, 'fgh')
sage: A.gens()
(B[f[]], B[g[]], B[h[]])
```

**nap_product**
Return the NAP product.

**See also:**

**nap_product_on_basis()**

**EXAMPLES:**
```
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = A(RT([RT([[]])]))
sage: A.nap_product(x, x)
B[[[], []]]
```

**nap_product_on_basis(x, y)**
Return the NAP product of two trees.
This is the grafting of the root of y over the root of x. The root of the resulting tree is the root of x.
See also:

\texttt{nap_product()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.nap_product_on_basis(x, x)
B[[[], []]]
\end{verbatim}

\texttt{pre_Lie_product()}

Return the pre-Lie product.

See also:

\texttt{pre_Lie_product_on_basis()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = A(RT([RT([])]))
sage: A.pre_Lie_product(x, x)
B[[[[[]]]]] + B[[[], []]]
\end{verbatim}

\texttt{pre_Lie_product_on_basis(x, y)}

Return the pre-Lie product of two trees.

This is the sum over all graftings of the root of \(y\) over a vertex of \(x\). The root of the resulting trees is the root of \(x\).

See also:

\texttt{pre_Lie_product()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.product_on_basis(x, x)
B[[[[[]]]]] + B[[[], []]]
\end{verbatim}

\texttt{product_on_basis(x, y)}

Return the pre-Lie product of two trees.

This is the sum over all graftings of the root of \(y\) over a vertex of \(x\). The root of the resulting trees is the root of \(x\).

See also:

\texttt{pre_Lie_product()}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = algebras.FreePreLie(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.product_on_basis(x, x)
B[[[[[]]]]] + B[[[], []]]
\end{verbatim}

(continues on next page)
some_elements()
Return some elements of the free pre-Lie algebra.

EXAMPLES:

```python
sage: A = algebras.FreePreLie(QQ, None)
sage: A.some_elements()
[B[], B[[[]]], B[[[[]]]] + B[][[], []], B[[[]]] + B[][[], []], B[[]]]
```

With several generators:

```python
sage: A = algebras.FreePreLie(QQ, 'xy')
sage: A.some_elements()
[B[x[]], B[x[x[]]], B[x[x[x[x[]]]]] + B[x[x[], x[x[]]]], B[x[x[x[]]]] + B[x[x[], x[]]], B[x[x[y[]]]] + B[x[x[], y[]]]]
```

variable_names()
Return the names of the variables.

EXAMPLES:

```python
sage: R = algebras.FreePreLie(QQ, 'xy')
sage: R.variable_names()
{'x', 'y'}
sage: R = algebras.FreePreLie(QQ, None)
sage: R.variable_names()
{'o'}
```

class sage.combinat.free_prelie_algebra.PrelieFunctor(vars)

A constructor for pre-Lie algebras.

EXAMPLES:

```python
sage: P = algebras.FreePreLie(ZZ, 'x,y')
sage: x,y = P.gens()
sage: F = P.construction()[0]; F
PreLie[x,y]
sage: A = GF(5)[ 'a,b' ]
sage: a, b = A.gens()
sage: F(A)
```

Free PreLie algebra on 2 generators ['x', 'y'] over Multivariate Polynomial Ring in a, b over Finite Field of size 5

```python
sage: f = A.hom([a+b,a-b],A)
```
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```sage
sage: F(f)
Generic endomorphism of Free PreLie algebra on 2 generators ['x', 'y'] over Multivariate Polynomial Ring in a, b over Finite Field of size 5
sage: F(f)(a * F(A)(x))
(a+b)*B[x[]]
```

```sage
merge(other)

Merge self with another construction functor, or return None.

EXAMPLES:
```
```
sage: F = sage.combinat.free_prelie_algebra.PreLieFunctor(['x','y'])
sage: G = sage.combinat.free_prelie_algebra.PreLieFunctor(['t'])
sage: F.merge(G)
PreLie[x,y,t]
sage: F.merge(F)
PreLie[x,y]
```

Now some actual use cases:

```sage
sage: R = algebras.FreePreLie(ZZ, 'xyz')
sage: x,y,z = R.gens()
sage: 1/2 * x
1/2*B[x[]]
sage: parent(1/2 * x)
Free PreLie algebra on 3 generators ['x', 'y', 'z'] over Rational Field
sage: S = algebras.FreePreLie(QQ, 'zt')
sage: z,t = S.gens()
sage: x + t
B[t[]] + B[x[]]
sage: parent(x + t)
Free PreLie algebra on 4 generators ['z', 't', 'x', 'y'] over Rational Field
```

```sage
rank = 9
```

### 5.1.113 Fully packed loops

AUTHORS:

- Vincent Knight, James Campbell, Kevin Dilks, Emily Gunawan (2015): Initial version
- Vincent Delecroix (2017): cleaning and enhanced plotting function

```sage
class sage.combinat.fully_packed_loop.FullyPackedLoop(parent, generator)
Bases: Element

A class for fully packed loops.

A fully packed loop is a collection of non-intersecting lattice paths on a square grid such that every vertex is part of some path, and the paths are either closed internal loops or have endpoints corresponding to alternate points on the boundary [Pro2001]. They are known to be in bijection with alternating sign matrices.
```

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To each fully packed loop, we assign a link pattern, which is the non-crossing matching attained by seeing which points on the boundary are connected by open paths in the fully packed loop.

We can create a fully packed loop using the corresponding alternating sign matrix and also extract the link pattern:

```
sage: A = AlternatingSignMatrix([[0, 0, 1], [0, 1, 0], [1, 0, 0]])
sage: fpl = FullyPackedLoop(A)
sage: fpl.link_pattern()
[(1, 4), (2, 3), (5, 6)]
sage: fpl
+ -- + +
| | | |
+ + -- +
| | | |
-- + + + --
| | | |
```

```
sage: B = AlternatingSignMatrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
sage: fplb = FullyPackedLoop(B)
sage: fplb.link_pattern()
[(1, 6), (2, 5), (3, 4)]
sage: fplb
```

The class also has a plot method:

```
sage: fpl.plot()
```

```
Graphics object consisting of 3 graphics primitives
```

which gives:

Note that we can also create a fully packed loop from a six vertex model configuration:

```
sage: S = SixVertexModel(3, boundary_conditions='ice').from_alternating_sign_matrix(A)
sage: S
```

(continues on next page)
Once we have a fully packed loop we can obtain the corresponding alternating sign matrix:

```python
sage: fpl = FullyPackedLoop(S)
sage: fpl
```

Here are some more examples using bigger ASMs:

```python
sage: A = AlternatingSignMatrix([[0, 1, 0, 0], [0, 0, 1, 0], [1, -1, 0, 1], [0, 1, 0, 0]])
sage: S = SixVertexModel(4, boundary_conditions='ice').from_alternating_sign_matrix(A)
sage: fpl = FullyPackedLoop(S)
sage: fpl
```

(continues on next page)
sage: m = AlternatingSignMatrix([[0,0,1,0,0,0],
....: [1,0,-1,0,1,0],
....: [0,0,0,1,0,0],
....: [0,1,0,0,-1,1],
....: [0,0,0,0,1,0],
....: [0,0,1,0,0,0]])

sage: fpl = FullyPackedLoop(m)
sage: fpl.link_pattern()
[(1, 12), (2, 7), (3, 4), (5, 6), (8, 9), (10, 11)]

sage: fpl

sage: m = AlternatingSignMatrix([[0,1,0,0,0,0,0],
....: [1,-1,0,0,1,0,0],
....: [0,0,0,1,0,0,0],
....: [0,1,0,0,-1,1,0],
....: [0,0,0,0,1,0,0],
....: [0,0,1,0,-1,0,1],
....: [0,0,0,0,1,0,0]])

sage: fpl = FullyPackedLoop(m)
Gyration on an alternating sign matrix/fully packed loop fpl of the link pattern corresponding to fpl:

```
sage: ASMs = AlternatingSignMatrices(3).list()
sage: ncp = FullyPackedLoop(ASMs[1]).link_pattern()  # fpl's gyration orbit size is 2
sage: rotated_ncp=[]
sage: for (a,b) in ncp:
    ....:     for i in range(5):
    ....:         a,b=a%6+1,b%6+1;
    ....:     rotated_ncp.append((a,b))
sage: PerfectMatching(ASMs[1].gyration().to_fully_packed_loop().link_pattern()) ==
    ....:     PerfectMatching(rotated_ncp)
True
```

```
sage: fpl = FullyPackedLoop(ASMs[0])
sage: ncp = fpl.link_pattern()  # fpl's gyration size is 3
sage: rotated_ncp=[]
sage: for (a,b) in ncp:
    ....:     for i in range(5):
    ....:         a,b=a%6+1,b%6+1;
    ....:     rotated_ncp.append((a,b))
sage: PerfectMatching(ASMs[0].gyration().to_fully_packed_loop().link_pattern()) ==
    ....:     PerfectMatching(rotated_ncp)
True
```

```
sage: mat = AlternatingSignMatrix([[0,0,1,0,0,0,0],[1,0,-1,0,1,0,0],[
(continues on next page)]]
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```
....:  [0,0,1,0,0,0,0],[0,1,-1,0,1,0],[0,0,1,0,0,0,0],[0,0,0,1,0,0,0],[0,0,0,0,
˓→0,0,1])
sage: fpl = FullyPackedLoop(mat) # n=7
sage: ncp = fpl.link_pattern()
sage: rotated_ncp=[]
sage: for (a,b) in ncp:
    ....:    for i in range(13):
    ....:        a,b=a%14+1,b%14+1;
    ....:        rotated_ncp.append((a,b))
sage: PerfectMatching(mat.gyration().to_fully_packed_loop().link_pattern()) ==
    ....:    PerfectMatching(rotated_ncp)
True
sage: mat = AlternatingSignMatrix([[0,0,0,1,0,0],[0,0,1,-1,1,0],
....:    [0,1,0,0,-1,1], [1,0,-1,1,0,0], [0,0,1,0,0,0], [0,0,0,0,1,0]])
sage: fpl = FullyPackedLoop(mat) # n =6
sage: ncp = fpl.link_pattern()
sage: rotated_ncp=[]
sage: for (a,b) in ncp:
    ....:    for i in range(11):
    ....:        a,b=a%12+1,b%12+1;
    ....:        rotated_ncp.append((a,b))
sage: PerfectMatching(mat.gyration().to_fully_packed_loop().link_pattern()) ==
    ....:    PerfectMatching(rotated_ncp)
True
```

More examples:

We can initiate a fully packed loop using an alternating sign matrix:

```
sage: A = AlternatingSignMatrix([[0, 0, 1], [0, 1, 0], [1, 0, 0]])
sage: fpl = FullyPackedLoop(A)
sage: fpl
+ -- + +
  |
  + -- + +
  |
-- + + + --
  |
  + -- +
  |
  + + -- +
  |
  + + -- +
  |
sage: FullyPackedLoops(3)(A) == fpl
True
```

We can also input a matrix:

```
sage: FullyPackedLoop([[0, 0, 1], [0, 1, 0], [1, 0, 0]])
+ -- + +
  |
  + -- + +
  |
  + -- + +
  |
  + -- + +
  |
  + -- + +
  |
  + -- + +
  |
```

(continues on next page)
Otherwise we initiate a fully packed loop using a six vertex model:

\begin{verbatim}
sage: S = SixVertexModel(3, boundary_conditions='ice').from_alternating_sign_matrix(A)
sage: fpl = FullyPackedLoop(S)
sage: fpl
  | |
  | |
  | |
  | |
  + + -- +
  | |
  | |
  | |
  -- + + + --
  | |
  | |
  | |

sage: FullyPackedLoops(3)(S) == FullyPackedLoop(S)
True

sage: fpl.six_vertex_model().to_alternating_sign_matrix()
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\end{verbatim}

We can also input the matrix associated to a six vertex model:

\begin{verbatim}
sage: SixVertexModel(2)([[3,1],[5,3]])
  ^   ^
  |   |
  --> # <-- # <--
  ^   ^
  V   |
  --> # --> # <--
  |   |
  V   V

sage: FullyPackedLoop([[3,1],[5,3]])
\end{verbatim}


Note that the matrix corresponding to a six vertex model without the ice boundary condition is not allowed:

```
sage: SixVertexModel(2)([[3,1],[5,5]])
^ ^
|-- # <- # <--
  | ^
  V V
|-- # --> # -->
  | |
  V V
sage: FullyPackedLoop([[3,1],[5,5]])
Traceback (most recent call last):
... ValueError: invalid alternating sign matrix
sage: FullyPackedLoops(2)([[3,1],[5,5]])
Traceback (most recent call last):
... ValueError: invalid alternating sign matrix
```

Note that if anything else is used to generate the fully packed loop an error will occur:

```
sage: fpl = FullyPackedLoop(5)
Traceback (most recent call last):
... ValueError: invalid alternating sign matrix
sage: fpl = FullyPackedLoop((1, 2, 3))
Traceback (most recent call last):
... ValueError: the alternating sign matrices must be square
sage: SVM = SixVertexModel(3)[0]
sage: FullyPackedLoop(SVM)
Traceback (most recent call last):
... ValueError: invalid alternating sign matrix
```

REFERENCES:
- [Pro2001]
gyration()

Return the fully packed loop obtained by applying gyration to the alternating sign matrix in bijection with self.

Gyration was first defined in [Wie2000] as an action on fully-packed loops.

EXAMPLES:

```python
sage: A = AlternatingSignMatrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
sage: fpl = FullyPackedLoop(A)
sage: fpl.gyration().to_alternating_sign_matrix()
[0 0 1]
[0 1 0]
[1 0 0]
sage: asm = AlternatingSignMatrix([[0, 0, 1], [1, 0, 0], [0, 1, 0]])
sage: f = FullyPackedLoop(asm)
sage: f.gyration().to_alternating_sign_matrix()
[0 1 0]
[0 0 1]
[1 0 0]
```

link_pattern()

Return a link pattern corresponding to a fully packed loop.

Here we define a link pattern LP to be a partition of the list \([1, \ldots, 2k]\) into 2-element sets (such a partition is also known as a perfect matching) such that the following non-crossing condition holds: Let the numbers \(1, \ldots, 2k\) be written on the perimeter of a circle. For every 2-element set \((a, b)\) of the partition \(LP\), draw an arc linking the two numbers \(a\) and \(b\). We say that \(LP\) is non-crossing if every arc can be drawn so that no two arcs intersect.

Since every endpoint of a fully packed loop \(fpl\) is connected to a different endpoint, there is a natural surjection from the fully packed loops on an \(nxn\) grid onto the link patterns on the list \([1, \ldots, 2n]\). The pairs of connected endpoints of a fully packed loop \(fpl\) correspond to the 2-element tuples of the corresponding link pattern.

See also:

PerfectMatching

Note: by convention, we choose the top left vertex to be even. See [Pro2001] and [Str2015].

EXAMPLES:

We can extract the underlying link pattern (a non-crossing partition) from a fully packed loop:

```python
sage: A = AlternatingSignMatrix([[0, 1, 0], [1, -1, 1], [0, 1, 0]])
sage: fpl = FullyPackedLoop(A)
sage: fpl.link_pattern()
[(1, 2), (3, 6), (4, 5)]
sage: B = AlternatingSignMatrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
sage: fpl = FullyPackedLoop(B)
sage: fpl.link_pattern()
[(1, 6), (2, 5), (3, 4)]
```
Gyration on an alternating sign matrix/fully packed loop $\text{fpl}$ corresponds to a rotation (i.e. $a$ becomes $a-1 \mod 2n$) of the link pattern corresponding to $\text{fpl}$:

```
sage: ASMs = AlternatingSignMatrices(3).list()
sage: ncp = FullyPackedLoop(ASMs[1]).link_pattern()
sage: rotated_ncp=[]
    sage: for (a,b) in ncp:
        ....:    for i in range(5):
        ....:        a,b=a%6+1,b%6+1;
        ....:        rotated_ncp.append((a,b))
sage: PerfectMatching(ASMs[1].gyration().to_fully_packed_loop().link_pattern())
\rightarrow==
    ....:    PerfectMatching(rotated_ncp)
True
```

```
sage: fpl = FullyPackedLoop(ASMs[0])
sage: ncp = fpl.link_pattern()
sage: rotated_ncp=[]
    sage: for (a,b) in ncp:
        ....:    for i in range(5):
        ....:        a,b=a%6+1,b%6+1;
        ....:        rotated_ncp.append((a,b))
    sage: PerfectMatching(ASMs[0].gyration().to_fully_packed_loop().link_pattern())
\rightarrow==
    ....:    PerfectMatching(rotated_ncp)
True
```

```
sage: mat = AlternatingSignMatrix([[0,0,0,1,0,0,0,0],[1,0,-1,0,1,0,0],[0,0,1,0,0,0,0],
    ....:    [0,1,-1,0,1,0,0,0],[0,0,1,0,0,0,0],[0,0,0,1,0,0,0],[0,0,0,0,0,1]])
sage: fpl = FullyPackedLoop(mat) # n=7
sage: ncp = fpl.link_pattern()
sage: rotated_ncp=[]
    sage: for (a,b) in ncp:
        ....:    for i in range(13):
        ....:        a,b=a%14+1,b%14+1;
        ....:        rotated_ncp.append((a,b))
    sage: PerfectMatching(mat.gyration().to_fully_packed_loop().link_pattern())
\rightarrow==
    ....:    PerfectMatching(rotated_ncp)
True
```

```
sage: mat = AlternatingSignMatrix([[0,0,0,1,0,0,0,0],[0,0,1,-1,1,0,0],[1,0,-1,1,0,0],
    ....:    [0,0,1,0,0,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1]])
sage: fpl = FullyPackedLoop(mat)
sage: ncp = fpl.link_pattern()
sage: rotated_ncp=[]
    sage: for (a,b) in ncp:
        ....:    for i in range(11):
        ....:        a,b=a%12+1,b%12+1;
        ....:        rotated_ncp.append((a,b))
    sage: PerfectMatching(mat.gyration().to_fully_packed_loop().link_pattern())
\rightarrow==
    ....:    PerfectMatching(rotated_ncp)
True
```
plot(link=True, loop=True, loop_fill=False, *options)

Return a graphical object of the Fully Packed Loop.

Each option can be specified separately for links (the curves that join boundary points) and the loops. In order to do so, you need to prefix its name with either 'link_' or 'loop_'. As an example, setting color='red' will color both links and loops in red while setting link_color='red' will only apply the color option for the links.

INPUT:
- link, loop - (boolean, default True) whether to plot the links or the loops
- color, link_color, loop_color - (optional, a string or a RGB triple)
- colors, link_colors, loop_colors - (optional, list) a list of colors
- color_map, link_color_map, loop_color_map - (string, optional) a name of a matplotlib color map for the link or the loop
- link_color_randomize - (boolean, default False) when link_colors or link_color_map is specified it randomizes its order. Setting this option to True makes it unlikely to have two neighboring links with the same color.
- loop_fill - (boolean, optional) whether to fill the interior of the loops

EXAMPLES:
To plot the fully packed loop associated to the following alternating sign matrix
\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & -1 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]
simply do:

```
sage: A = AlternatingSignMatrix([[0, 1, 0], [1, -1, 1], [0, 1, 0]])
sage: fpl = FullyPackedLoop(A)
sage: fpl.plot()
```

Graphics object consisting of 3 graphics primitives

The resulting graphics is as follows

```
+---+
|   |
+---+
|   |
+---+---+
    |
```

You can also have the three links in different colors with:

```
sage: A = AlternatingSignMatrix([[0, 1, 0], [1, -1, 1], [0, 1, 0]])
sage: fpl = FullyPackedLoop(A)
sage: fpl.plot(link_color_map='rainbow')
```

Graphics object consisting of 3 graphics primitives

You can plot the 42 fully packed loops of size $4 \times 4$ using:
Here is an example of a $20 \times 20$ fully packed loop:

```
sage: s = "00000000000+0000000000000000+00-0+00000000000+00-00+0--0000\n.....: 0000+00-00+00000000+00-00000000+00-0000+00000000+00-0000\n.....: 0000+00-0+0000000+00-0000+00000000+00-0000+00000000+00-0000\n.....: 0000+00-00+00000000+00-0000+00000000+00-0000+00000000+00-0000\n.....: 0000+00-00+00000000+00-0000+00000000+00-0000+00000000+00-0000\n"``
sage: a = matrix(20, [['0':0, '+':1, '-': -1][i] for i in s])
sage: fpl = FullyPackedLoop(a)
sage: fpl.plot(loop_fill=True, loop_color_map='rainbow')
Graphics object consisting of 27 graphics primitives

six_vertex_model()
Return the underlying six vertex model configuration.

EXAMPLES:

sage: B = AlternatingSignMatrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
sage: fpl = FullyPackedLoop(B)
sage: fpl
|   |
|   |
+ + -- +
|   |
|   |
-- + + +--
|   |
|   |
+ -- + +
|   |
sage: fpl.six_vertex_model()
^  ^  ^
|   |   |
--> # <- # <- # <--
|   ^  ^
V   |   |
to_alternating_sign_matrix()

Return the alternating sign matrix corresponding to this class.

See also:

AlternatingSignMatrix

EXAMPLES:

sage: A = AlternatingSignMatrix([[0, 1, 0], [1, -1, 1], [0, 1, 0]])
sage: S = SixVertexModel(3, boundary_conditions='ice').from_alternating_sign_matrix(A)
sage: fpl = FullyPackedLoop(S)
sage: fpl.to_alternating_sign_matrix()
[0 1 0]
[1 -1 1]
[0 1 0]
sage: A = AlternatingSignMatrix([[0,1,0,0],[0,0,1,0],[1,-1,0,1],[0,1,0,0]])
sage: S = SixVertexModel(4, boundary_conditions='ice').from_alternating_sign_matrix(A)
sage: fpl = FullyPackedLoop(S)
sage: fpl.to_alternating_sign_matrix()
[0 1 0 0]
[0 0 1 0]
[1 -1 0 1]
[0 1 0 0]
When using the square ice model, it is known that the number of configurations is equal to the number of alternating sign matrices:

```python
sage: M = FullyPackedLoops(1)
sage: len(M)
1
sage: M = FullyPackedLoops(4)
sage: len(M)
42
sage: all(len(SixVertexModel(n, boundary_conditions='ice'))
....:     == FullyPackedLoops(n).cardinality() for n in range(1, 7))
True
```

**Element**
alias of `FullyPackedLoop`

**cardinality()**

Return the cardinality of `self`.

The number of fully packed loops on \(n \times n\) grid

\[
\prod_{k=0}^{n-1} \frac{(3k + 1)!}{(n + k)!} = \frac{1! \cdot 4! \cdot 7! \cdot 10! \cdots (3n - 2)!}{n!(n + 1)!(n + 2)!(n + 3)! \cdots (2n - 1)!}.
\]

**EXAMPLES:**

```python
sage: [AlternatingSignMatrices(n).cardinality() for n in range(11)]
[1, 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700]
```

**size()**

Return the size of the matrices in `self`.

### 5.1.114 Gelfand-Tsetlin Patterns

**AUTHORS:**


**REFERENCES:**

**class** `sage.combinat.gelfand_tsetlin_patterns.GelfandTsetlinPattern`

Bases: `ClonableArray`

A Gelfand-Tsetlin (sometimes written as Gelfand-Zetlin or Gelfand-Cetlin) pattern. They were originally defined in [GC50].

A Gelfand-Tsetlin pattern is a triangular array:

\[
\begin{array}{cccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
& a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\
& & a_{3,3} & \cdots & a_{3,n} \\
& & & \ddots & \vdots \\
& & & & a_{n,n}
\end{array}
\]

such that \(a_{i,j} \geq a_{i+1,j+1} \geq a_{i,j+1}\).
Gelfand-Tsetlin patterns are in bijection with semistandard Young tableaux by the following algorithm. Let $G$ be a Gelfand-Tsetlin pattern with $\lambda^{(k)}$ being the $(n - k + 1)$-st row (note that this is a partition). The definition of $G$ implies

$$\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)},$$

where $\lambda^{(0)}$ is the empty partition, and each skew shape $\lambda^{(k)}/\lambda^{(k-1)}$ is a horizontal strip. Thus define $T(G)$ by inserting $k$ into the squares of the skew shape $\lambda^{(k)}/\lambda^{(k-1)}$, for $k = 1, \ldots, n$.

To each entry in a Gelfand-Tsetlin pattern, one may attach a decoration of a circle or a box (or both or neither). These decorations appear in the study of Weyl group multiple Dirichlet series, and are implemented here following the exposition in [BBF].

**Note:** We use the “right-hand” rule for determining circled and boxed entries.

**Warning:** The entries in Sage are 0-based and are thought of as flushed to the left in a matrix. In other words, the coordinates of entries in the Gelfand-Tsetlin patterns are thought of as the matrix:

$$
\begin{bmatrix}
g_{0,0} & g_{0,1} & g_{0,2} & \cdots & g_{0,n-2} & g_{n-1,n-1} 
g_{1,0} & g_{1,1} & g_{1,2} & \cdots & g_{1,n-2} 
g_{2,0} & g_{2,1} & g_{2,2} & \cdots 
\vdots & \vdots & \vdots 
g_{n-2,0} & g_{n-2,1} 
g_{n-1,0}
\end{bmatrix}.
$$

However, in the discussions, we will be using the standard numbering system.

**EXAMPLES:**

```
sage: G = GelfandTsetlinPattern([[3, 2, 1], [2, 1], [1]]); G
[[3, 2, 1], [2, 1], [1]]
sage: G.pp()
 3 2 1
 2 1
 1
```

```
sage: G = GelfandTsetlinPattern([[7, 7, 4, 0], [7, 7, 3], [7, 5], [5]]); G.pp()
 7 7 4 0
 7 7 3
 7 5
 5
```

```
sage: G.to_tableau().pp()
 1 1 1 1 1 2 2
 2 2 2 2 3 3
 3 3 3 4
```

**Tokuyama_coefficient**(name='t')

Return the Tokuyama coefficient attached to self.

Following the exposition of [BBF], Tokuyama’s formula asserts

$$
\sum_{G} (t + 1)^{s(G)} p^{t(G)} z_1^{d_{n+1}} z_2^{d_n - d_{n+1}} \cdots z_n^{d_1 - d_2} = s_{\lambda}(z_1, \ldots, z_{n+1}) \prod_{i<j}(z_j + tz_i),
$$

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where the sum is over all strict Gelfand-Tsetlin patterns with fixed top row $\lambda + \rho$, with $\lambda$ a partition with at most $n + 1$ parts and $\rho = (n, n - 1, \ldots, 1, 0)$, and $s_\lambda$ is a Schur function.

**INPUT:**
- **name** – (Default: 't') An alternative name for the variable $t$.

**EXAMPLES:**

```python
sage: P = GelfandTsetlinPattern([[3,2,1],[2,2],[2]])
sage: P.Tokuyama_coefficient()
0
sage: G = GelfandTsetlinPattern([[3,2,1],[3,1],[2]])
sage: G.Tokuyama_coefficient()
t^2 + t
sage: G = GelfandTsetlinPattern([[2,1,0],[1,1],[1]])
sage: G.Tokuyama_coefficient()
0
sage: G = GelfandTsetlinPattern([[5,3,2,1,0],[4,3,2,0],[4,2,1],[3,2],[3]])
sage: G.Tokuyama_coefficient()
t^8 + 3*t^7 + 3*t^6 + t^5
```

**bender_knuth_involution(i)**

Return the image of self under the $i$-th Bender-Knuth involution.

If the triangle self has size $n$ then this is defined for $0 < i < n$.

The entries of self can take values in any ordered ring. Usually, this will be the integers but can also be the rationals or the real numbers.

This implements the construction of the Bender-Knuth involution using toggling due to Berenstein-Kirillov.

This agrees with the Bender-Knuth involution on semistandard tableaux.

**EXAMPLES:**

```python
sage: G = GelfandTsetlinPattern([[5,3,2,1,0],[4,3,2,0],[4,2,1],[3,2],[3]])
sage: G.bender_knuth_involution(2)
[[5, 3, 2, 1, 0], [4, 3, 2, 0], [4, 2, 1], [4, 1], [3]]
sage: G = GelfandTsetlinPattern([[3,2,0],[2.2,0],[2]])
sage: G.bender_knuth_involution(2)
[[3, 2, 0], [2.80000000000000, 2], [2]]
```

**boxed_entries()**

Return the position of the boxed entries of self.

Using the right-hand rule, an entry $a_{i,j}$ is boxed if $a_{i,j} = a_{i-1,j-1}$; i.e., $a_{i,j}$ has the same value as its neighbor to the northwest.

**EXAMPLES:**

```python
sage: G = GelfandTsetlinPattern([[3,2,1],[3,1],[1]])
sage: G.boxed_entries()
((1, 0),)
```

**check()**

Check that this is a valid Gelfand-Tsetlin pattern.

**EXAMPLES:**

```python
```
sage: G = GelfandTsetlinPatterns()
sage: G([[3, 2, 1], [2, 1], [1]]).check()

\textbf{circled\_entries()}  
Return the circled entries of \textit{self}.

Using the \textit{right-hand} rule, an entry $a_{i,j}$ is circled if $a_{i,j} = a_{i-1,j}$; i.e., $a_{i,j}$ has the same value as its neighbor to the northeast.

\textbf{EXAMPLES:}

sage: G = GelfandTsetlinPattern([[3, 2, 1], [3, 1], [1]])
sage: G.circled_entries()
((1, 1), (2, 0))

\textbf{is\_strict()}  
Return True if \textit{self} is a strict Gelfand-Tsetlin pattern.

A Gelfand-Tsetlin pattern is said to be \textit{strict} if every row is strictly decreasing.

\textbf{EXAMPLES:}

sage: GelfandTsetlinPattern([[7, 3, 1], [6, 2], [4]]).is_strict()  
True
sage: GelfandTsetlinPattern([[3, 2, 1], [3, 1], [1]]).is_strict()  
True
sage: GelfandTsetlinPattern([[6, 0, 0], [3, 0], [2]]).is_strict()  
False

\textbf{number\_of\_boxes()}  
Return the number of boxed entries. See \textit{boxed\_entries()}.

\textbf{EXAMPLES:}

sage: G = GelfandTsetlinPattern([[3, 2, 1], [3, 1], [1]])
sage: G.number_of_boxes()  
1

\textbf{number\_of\_circles()}  
Return the number of boxed entries. See \textit{circled\_entries()}.

\textbf{EXAMPLES:}

sage: G = GelfandTsetlinPattern([[3, 2, 1], [3, 1], [1]])
sage: G.number_of_circles()  
2

\textbf{number\_of\_special\_entries()}  
Return the number of special entries. See \textit{special\_entries()}.

\textbf{EXAMPLES:}

sage: G = GelfandTsetlinPattern([[4, 2, 1], [4, 1], [2]])
sage: G.number_of_special_entries()  
1
pp()  
Pretty print self.

EXAMPLES:

```
sage: G = GelfandTsetlinPatterns()
sage: G([[3,2,1],[2,1],[1]]).pp()
  3 2 1
 2 1
 1
```

row_sums()  
Return the list of row sums.

For a Gelfand-Tsetlin pattern $G$, the $i$-th row sum $d_i$ is

$$d_i = d_i(G) = \sum_{j=1}^{n} a_{i,j}.$$ 

EXAMPLES:

```
sage: G = GelfandTsetlinPatterns([[5,3,2,1,0],[4,3,2,0],[4,2,1],[3,2],[3]])
sage: G.row_sums()
[11, 9, 7, 5, 3]
sage: G = GelfandTsetlinPatterns([[3,2,1],[3,1],[2]])
sage: G.row_sums()
[6, 4, 2]
```

special_entries()  
Return the special entries.

An entry $a_{i,j}$ is special if $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$, that is to say, the entry is neither boxed nor circled and is not in the first row. The name was coined by [Tok88].

EXAMPLES:

```
sage: G = GelfandTsetlinPattern([[3,2,1],[3,1],[2]])
sage: G.special_entries()
()  
sage: G = GelfandTsetlinPattern([[4,2,1],[4,1],[2]])
sage: G.special_entries()
((2, 0),)
```

to_tableau()  
Return self as a semistandard Young tableau.

The conversion from a Gelfand-Tsetlin pattern to a semistandard Young tableaux is as follows. Let $G$ be a Gelfand-Tsetlin pattern with $\lambda^{(k)}$ being the $(n-k+1)$-st row (note that this is a partition). The definition of $G$ implies

$$\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)},$$

where $\lambda^{(0)}$ is the empty partition, and each skew shape $\lambda^{(k)}/\lambda^{(k-1)}$ is a horizontal strip. Thus define $T(G)$ by inserting $k$ into the squares of the skew shape $\lambda^{(k)}/\lambda^{(k-1)}$, for $k = 1, \ldots, n$.

EXAMPLES:
sage: G = GelfandTsetlinPatterns()
sage: elt = G([[3,2,1],[2,1],[1]])
sage: T = elt.to_tableau(); T
[[1, 2, 3], [2, 3], [3]]
sage: T.pp()
1 2 3
2 3
3
sage: G(T) == elt
True

weight()

Return the weight of self.

Define the weight of $G$ to be the content of the tableau to which $G$ corresponds under the bijection between Gelfand-Tsetlin patterns and semistandard tableaux. More precisely,

$$\text{wt}(G) = (d_n, d_{n-1} - d_n, \ldots, d_1 - d_2),$$

where the $d_i$ are the row sums.

EXAMPLES:

```sage
sage: G = GelfandTsetlinPattern([[2,1,0],[1,0],[1]])
sage: G.weight()
(1, 0, 2)
sage: G = GelfandTsetlinPattern([[4,2,1],[3,1],[2]])
sage: G.weight()
(2, 2, 3)
```

class sage.combinat.gelfand_tsetlin_patterns.GelfandTsetlinPatterns(n, k, strict)

Bases: UniqueRepresentation, Parent

Gelfand-Tsetlin patterns.

INPUT:
- $n$ – The width or depth of the array, also known as the rank
- $k$ – (Default: None) If specified, this is the maximum value that can occur in the patterns
- top_row – (Default: None) If specified, this is the fixed top row of all patterns
- strict – (Default: False) Set to True if all patterns are strict patterns

Element

alias of GelfandTsetlinPattern

random_element()

Return a uniformly random Gelfand-Tsetlin pattern.

EXAMPLES:

```sage
g = GelfandTsetlinPatterns(4, 5)
x = g.random_element()
x in g
True
sage: len(x)
```
4
sage: all(y in range(5+1) for z in x for y in z)
True
sage: x.check()

sage: g = GelfandTsetlinPatterns(4, 5, strict=True)
sage: x = g.random_element()
sage: x

sage: all(y in range(5+1) for z in x for y in z)
True
sage: x.check()
sage: x.is_strict()
True

class sage.combinat.gelfand_tsetlin_patterns.GelfandTsetlinPatternsTopRow(top_row, strict)
Bases: GelfandTsetlinPatterns

Gelfand-Tsetlin patterns with a fixed top row.

Tokuyama_formula(name='t')

Return the Tokuyama formula of self.

Following the exposition of [BBF], Tokuyama’s formula asserts
\[ \sum_{G} (t + 1)^{s(G)}p(G)z_{1}^{d_{n+1}z_{2}^{d_{n-1}d_{n+1}} \cdots z_{n+1}^{d_{1}d_{n+1}}} = s_{\lambda}(z_{1}, \ldots, z_{n+1}) \prod_{i<j}(z_{j} + tz_{i}), \]

where the sum is over all strict Gelfand-Tsetlin patterns with fixed top row \( \lambda + \rho \), with \( \lambda \) a partition with at most \( n + 1 \) parts and \( \rho = (n, n-1, \ldots, 1, 0) \), and \( s_{\lambda} \) is a Schur function.

INPUT:
• name – (Default: 't') An alternative name for the variable \( t \).

EXAMPLES:

sage: GT = GelfandTsetlinPatterns(top_row=[2,1,0],strict=True)
sage: GT.Tokuyama_formula()
t^3*x1^2*x2 + t^2*x1*x2^2 + t^2*x1^2*x3 + t*x1*x2*x3 + t*x2^2*x3
→ t*x1*x3^2 + x2*x3^2
sage: GT = GelfandTsetlinPatterns(top_row=[3,2,1],strict=True)
sage: GT.Tokuyama_formula()
t^3*x1^3*x2^2*x3 + t^2*x1^2*x2^3*x3 + t^2*x1*x2^2*x3^2 + t*x1*x2^2*x3 + t*x2^2*x3
→ t*x1*x3^2 + x2*x3^2
sage: GT = GelfandTsetlinPatterns(top_row=[1,1,1],strict=True)
sage: GT.Tokuyama_formula()
0

random_element()

Return a uniformly random Gelfand-Tsetlin pattern with specified top row.

EXAMPLES:
```python
sage: g = GelfandTsetlinPatterns(top_row = [4, 3, 1, 1])
sage: x = g.random_element()
sage: x in g
True
sage: x[0] == [4, 3, 1, 1]
True
sage: x.check()

sage: g = GelfandTsetlinPatterns(top_row=[4, 3, 2, 1], strict=True)
sage: x = g.random_element()
sage: x in g
True
sage: x[0] == [4, 3, 2, 1]
True
sage: x.is_strict()
True
sage: x.check()
```

### top_row()

Return the top row of self.

EXAMPLES:

```python
sage: G = GelfandTsetlinPatterns(top_row=[4,4,3,1])
sage: G.top_row()
(4, 4, 3, 1)
```

### 5.1.115 Paths in Directed Acyclic Graphs

sage.combinat.graph_path.GraphPaths(g, source=None, target=None)

Return the combinatorial class of paths in the directed acyclic graph g.

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
If source and target are not given, then the returned class contains all paths (including trivial paths containing only one vertex).

```python
sage: p = GraphPaths(G); p
Paths in Multi-digraph on 5 vertices
sage: p.cardinality()
37
sage: path = p.random_element()
sage: all(G.has_edge(*path[i:i+2]) for i in range(len(path) -1))
True
```

If the source is specified, then the returned class contains all of the paths starting at the vertex source (including the trivial path).

```python
sage: p = GraphPaths(G, source=3); p
Paths in Multi-digraph on 5 vertices starting at 3
```
If the target is specified, then the returned class contains all of the paths ending at the vertex target (including the trivial path).

```
sage: p = GraphPaths(G, target=3); p
Paths in Multi-digraph on 5 vertices ending at 3
sage: p.cardinality()
5
sage: p.list()
[[3], [1, 3], [2, 3], [1, 2, 3], [1, 2, 3]]
```

If both the target and source are specified, then the returned class contains all of the paths from source to target.

```
sage: p = GraphPaths(G, source=1, target=3); p
Paths in Multi-digraph on 5 vertices starting at 1 and ending at 3
sage: p.cardinality()
3
sage: p.list()
[[1, 2, 3], [1, 2, 3], [1, 3]]
```

Note that G must be a directed acyclic graph.

```
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5], 5:[6]}, multiedges=True)
sage: GraphPaths(G)
Traceback (most recent call last):
  ...TypeError: g must be a directed acyclic graph
```

class sage.combinat.graph_path.GraphPaths_all(g)

Bases: Parent, GraphPaths_common

EXAMPLES:

```
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5], 5:[6]}, multiedges=True)
sage: p = GraphPaths(G)
sage: p.cardinality()
37
```

list()

Return a list of the paths of self.

EXAMPLES:

```
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5], 5:[6]}, multiedges=True)
sage: len(GraphPaths(G).list())
37
```

class sage.combinat.graph_path.GraphPaths_common

Bases: object

incoming_edges(v)

Return a list of v’s incoming edges.

5.1. Comprehensive Module List
EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: p = GraphPaths(G)
sage: p.incoming_edges(2)
[(1, 2, None), (1, 2, None)]
```

**incoming_paths**(v)

Return a list of paths that end at v.

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: gp = GraphPaths(G)
sage: gp.incoming_paths(2)
[[2], [1, 2], [1, 2]]
```

**outgoing_edges**(v)

Return a list of v’s outgoing edges.

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: p = GraphPaths(G)
sage: p.outgoing_edges(2)
[(2, 3, None), (2, 4, None)]
```

**outgoing_paths**(v)

Return a list of the paths that start at v.

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: gp = GraphPaths(G)
sage: gp.outgoing_paths(3)
[[3], [3, 4], [3, 4, 5], [3, 4, 5]]
sage: gp.outgoing_paths(2)
[[2], [2, 3], [2, 3, 4], [2, 3, 4, 5], [2, 3, 4, 5], [2, 4], [2, 4, 5], [2, 4, 5]]
```

**paths**()

Return a list of all the paths of self.

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: gp = GraphPaths(G)
sage: len(gp.paths())
37
```
paths_from_source_to_target(source, target)

Return a list of paths from source to target.

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: gp = GraphPaths(G)
sage: gp.paths_from_source_to_target(2,4)
[[2, 3, 4], [2, 4]]
```

class sage.combinat.graph_path.GraphPaths_s(g, source)

Bases: Parent, GraphPaths_common

list()

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: p = GraphPaths(G, 4)
sage: p.list()
[[4], [4, 5], [4, 5]]
```

class sage.combinat.graph_path.GraphPaths_st(g, source, target)

Bases: Parent, GraphPaths_common

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: GraphPaths(G,1,2).cardinality()
2
sage: GraphPaths(G,1,3).cardinality()
3
sage: GraphPaths(G,1,4).cardinality()
5
sage: GraphPaths(G,1,5).cardinality()
10
sage: GraphPaths(G,2,3).cardinality()
1
sage: GraphPaths(G,2,4).cardinality()
2
sage: GraphPaths(G,2,5).cardinality()
4
sage: GraphPaths(G,3,4).cardinality()
1
sage: GraphPaths(G,3,5).cardinality()
2
sage: GraphPaths(G,4,5).cardinality()
2
```

list()

EXAMPLES:

```python
sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: p = GraphPaths(G,1,2)
sage: p.list()
[[1, 2], [1, 2]]
```
Combinatorics, Release 10.0

class sage.combinat.graph_path.GraphPaths_t(g, target)

Bases: Parent, GraphPaths_common

def list()

EXAMPLES:

sage: G = DiGraph({1:[2,2,3], 2:[3,4], 3:[4], 4:[5,5]}, multiedges=True)
sage: p = GraphPaths(G, target=4)
sage: p.list()
[[4],
 [2, 4],
 [1, 2, 4],
 [1, 4],
 [1, 3, 4],
 [2, 3, 4],
 [1, 2, 3, 4],
 [1, 2, 3, 4]]

5.1.116 Gray codes

Functions

sage.combinat.gray_codes.combinations(n, t)

Iterator through the switches of the revolving door algorithm.

The revolving door algorithm is a way to generate all combinations of a set (i.e. the subset of given cardinality) in such way that two consecutive subsets differ by one element. At each step, the iterator output a pair \((i, j)\) where the item \(i\) has to be removed and \(j\) has to be added.

The ground set is always \(\{0, 1, \ldots, n - 1\}\). Note that \(n\) can be infinity in that algorithm.

See [Knu2011] Section 7.2.1.3, “Generating All Combinations”.

INPUT:

- \(n\) – (integer or Infinity) – size of the ground set
- \(t\) – (integer) – size of the subsets

EXAMPLES:

sage: from sage.combinat.gray_codes import combinations
sage: b = [1, 1, 1, 0, 0]
sage: for i,j in combinations(5,3):
....:     b[i] = 0; b[j] = 1
....:     print(b)
[1, 0, 1, 0, 0]
[0, 1, 1, 0, 0]
[1, 1, 0, 1, 0]
[1, 0, 0, 1, 1]
[0, 1, 0, 1, 1]
[0, 0, 1, 1, 0]
[1, 0, 1, 0, 1]
[0, 1, 1, 0, 1]
Note that \( n \) can be infinity:

```python
sage: c = combinations(Infinity, 4)
sage: s = set([0,1,2,3])
sage: for _ in range(10):
    ....:     i,j = next(c)
    ....:     s.remove(i); s.add(j)
    ....:     print(sorted(s))
[0, 1, 3, 4]
[1, 2, 3, 4]
[0, 2, 3, 4]
[1, 2, 4, 5]
[0, 2, 4, 5]
[2, 3, 4, 5]
[1, 3, 4, 5]
[0, 3, 4, 5]
```

```python
sage: for _ in range(1000):
    ....:     i,j = next(c)
    ....:     s.remove(i); s.add(j)
sage: sorted(s)
[0, 4, 13, 14]
```

`sage.combinat.gray_codes.product(m)`

Iterator over the switch for the iteration of the product \([m_0] \times [m_1] \times \ldots \times [m_k]\).

The iterator return at each step a pair \((p,i)\) which corresponds to the modification to perform to get the next element. More precisely, one has to apply the increment \(i\) at the position \(p\). By construction, the increment is either +1 or -1.

This is algorithm H in [Knu2011] Section 7.2.1.1, “Generating All \( n \)-Tuples”: loopless reflected mixed-radix Gray generation.

**INPUT:**

* \( m \) – a list or tuple of positive integers that correspond to the size of the sets in the product

**EXAMPLES:**

```python
sage: from sage.combinat.gray_codes import product
sage: l = [0,0,0]
```
sage: for p,i in product([3,3,3]):
....:   l[p] += i
....:   print(l)
[1, 0, 0]
[2, 0, 0]
[2, 1, 0]
[1, 1, 0]
[0, 1, 0]
[0, 2, 0]
[1, 2, 0]
[2, 2, 0]
[2, 2, 1]
[1, 2, 1]
[0, 2, 1]
[0, 1, 1]
[1, 1, 1]
[2, 1, 1]
[2, 0, 1]
[1, 0, 1]
[0, 0, 1]
[0, 0, 2]
[1, 0, 2]
[2, 0, 2]
[2, 1, 2]
[1, 1, 2]
[0, 1, 2]
[0, 2, 2]
[1, 2, 2]
[2, 2, 2]
sage: l = [0,0]
sage: for i,j in product([2,1]):
....:   l[i] += j
....:   print(l)
[1, 0]
A guided tour

Growth diagrams, invented by Sergey Fomin [Fom1994], [Fom1995], provide a vast generalization of the Robinson-Schensted-Knuth (RSK) correspondence between matrices with non-negative integer entries and pairs of semistandard Young tableaux of the same shape.

The main fact is that many correspondences similar to RSK can be defined by providing a pair of so-called local rules: a ‘forward’ rule, whose input are three vertices $y$, $t$ and $x$ of a certain directed graph (in the case of Robinson-Schensted: the directed graph corresponding to Young’s lattice) and an integer (in the case of Robinson-Schensted: 0 or 1), and whose output is a fourth vertex $z$. This rule should be invertible in the following sense: there is a so-called ‘backward’ rule that recovers the integer and $t$ given $y$, $z$ and $x$.

As an example, the growth rules for the classical RSK correspondence are provided by $\text{RuleRSK}$. To produce a growth diagram, pass the desired rule and a permutation to $\text{GrowthDiagram}$:

```sage
RuleRSK = GrowthDiagram.rules.RSK()
sage: w = [2,3,6,1,4,5]; G = GrowthDiagram(RuleRSK, w); G
```

The forward rule just mentioned assigns 49 partitions to the corners of each of the 36 cells of this matrix (i.e., 49 the vertices of a $(6+1) \times (6+1)$ grid graph), with the exception of the corners on the left and top boundary, which are initialized with the empty partition. More precisely, for each cell, the $\text{forward_rule()}$ computes the partition $z$ labelling the lower right corner, given the content $c$ of a cell and the other three partitions:

```
t --- x
|     |
y --- z
```

**Warning:** Note that a growth diagram is printed with matrix coordinates, the origin being in the top-left corner. Therefore, the growth is from the top left to the bottom right!

The partitions along the boundary opposite of the origin, reading from the bottom left to the top right, are obtained by using the method $\text{out_labels()}$:

```sage
G.out_labels()
```

(continues on next page)
However, in the case of a rectangular filling, it is more practical to split this sequence of labels in two. Interpreting the sequence of partitions along the right boundary as a standard Young tableau, we then obtain the so-called $P$ symbol, the partitions along the bottom boundary yield the so-called $Q$ symbol. These coincide with the output of the classical $RSK$ insertion algorithm:

```sage
sage: ascii_art([G.P_symbol(), G.Q_symbol()])
[ 1 3 4 5 1 2 3 6 ]
[ 2 6 , 4 5 ]
sage: ascii_art(RSK(w))
[ 1 3 4 5 1 2 3 6 ]
[ 2 6 , 4 5 ]
```

The filling can be recovered knowing the partitions labelling the corners of the bottom and the right boundary alone, by repeatedly applying the backward_rule. Therefore, to initialize a $GrowthDiagram$, we can provide these labels instead of the filling:

```sage
sage: GrowthDiagram(RuleRSK, labels=G.out_labels())
0 0 0 1 0 0
1 0 0 0 0 0
0 1 0 0 0 0
0 0 0 0 1 0
0 0 0 0 0 1
0 0 1 0 0 0
```

**Invocation**

In general, growth diagrams are defined for $0−1$-fillings of arbitrary skew shapes. In the case of the Robinson-Schensted-Knuth correspondence, even arbitrary non-negative integers are allowed. In other cases, entries may be either zero or an $r$-th root of unity - for example, $RuleDomino$ insertion is defined for signed permutations, that is, $r = 2$. Traditionally, words and permutations are also used to specify a filling in special cases.

To accommodate all this, the filling may be passed in various ways. The most general possibility is to pass a dictionary of coordinates to (signed) entries, where zeros can be omitted. In this case, when the parameter shape is not explicitly specified, it is assumed to be the minimal rectangle containing the origin and all coordinates with non-zero entries.

For example, consider the following generalized permutation:

```
1 2 2 2 4 4
4 2 3 3 2 3
```

that we encode as the dictionary:

```sage
sage: P = {(1−1,4−1): 1, (2−1,2−1): 1, (2−1,3−1): 2, (4−1,2−1): 1, (4−1,3−1): 1}
```
Combinatorics, Release 10.0

Note that we are subtracting $1$ from all entries because of zero-based indexing, we obtain:

```python
sage: GrowthDiagram(RuleRSK, P)
0 0 0 0
0 1 0 1
0 2 0 1
1 0 0 0
```

Alternatively, we could create the same growth diagram using a matrix.

Let us also mention that one can pass the arguments specifying a growth diagram directly to the rule:

```python
sage: RuleRSK(P)
0 0 0 0
0 1 0 1
0 2 0 1
1 0 0 0
```

In contrast to the classical insertion algorithms, growth diagrams immediately generalize to fillings whose shape is an arbitrary skew partition:

```python
sage: GrowthDiagram(RuleRSK, [3,1,2], shape=SkewPartition([[3,3,2],[1,1]]))
. 1 0
. 0 1
1 0
```

As an important example, consider the Stanley-Sundaram correspondence between oscillating tableaux and (partial) perfect matchings. Perfect matchings of \(\{1, \ldots, 2r\}\) are in bijection with \(0 - 1\)-fillings of a triangular shape with \(2r - 1\) rows, such that for each \(k\) there is either exactly one non-zero entry in row \(k\) or exactly one non-zero entry in column \(2r - k\). Explicitly, if \((i, j)\) is a pair in the perfect matching, the entry in column \(i - 1\) and row \(2r - j\) equals 1. For example:

```python
sage: m = [[1,5],[3,4],[2,7],[6,8]]
sage: G = RuleRSK({(i-1, 8-j): 1 for i,j in m}, shape=[7,6,5,4,3,2,1]); G
0 0 0 0 0 1 0
0 1 0 0 0
0 2 0 0
1 0 0 0
0 0 1
0 0
0
```

The partitions labelling the bottom-right corners along the boundary opposite of the origin then form a so-called oscillating tableau - the remaining partitions along the bottom-right boundary are redundant:

```python
sage: G.out_labels()[1::2]
[[1], [1, 1], [2, 1], [1, 1], [1, 1], [1]]
```

Another great advantage of growth diagrams is that we immediately have access to a skew version of the correspondence, by providing different initialization for the labels on the side of the origin. We reproduce the original example of Bruce Sagan and Richard Stanley, see also Tom Roby’s thesis [Rob1991]:

```python
sage: w = {(1-1,4-1): 1, (2-1,2-1): 1, (4-1,3-1): 1}
sage: T = SkewTableau([[None, None], [None, 5], [1]])
sage: U = SkewTableau([[None, None], [None, 3], [5]])
```

(continues on next page)
sage: labels = T.to_chain()[:-1] + U.to_chain()[1:]
sage: G = GrowthDiagram(RuleRSK, filling=w, shape=[5,5,5,5,5], labels=labels); G

Similarly, there is a correspondence for skew oscillating tableau. Let us conclude by reproducing Example 4.2.6 from [Rob1991]. The oscillating tableau, as given, is:

```
sage: o = [[2,1], [2,2], [3,2], [4,2], [4,1], [4,1,1], [3,1,1], [3,1], [3,2], [3,1], [2,1]]
```

From this, we have to construct the list of labels of the corners along the bottom-right boundary. The labels with odd indices are given by the oscillating tableau, the other labels are obtained by taking the smaller of the two neighbouring partitions:

```
sage: l = [o[i//2] if is_even(i) else min(o[(i-1)//2], o[(i+1)//2])
.....:     for i in range(2*len(o)-1)]
sage: la = list(range(len(o)-2, 0, -1))
sage: G = RuleRSK(labels=l[1:-1], shape=la); G
```

The skew tableaux can now be read off the partitions labelling the left and the top boundary. These can be accessed using the method `in_labels()`:

```
sage: ascii_art(SkewTableau(chain=G.in_labels()[len(o)-2:]),
.....:     SkewTableau(chain=G.in_labels()[len(o)-2:-1]))
```

```
.
5
4
```
Rules currently available

As mentioned at the beginning, the Robinson-Schensted-Knuth correspondence is just a special case of growth diagrams. In particular, we have implemented the following local rules:

- RSK (RuleRSK).
- A variation of RSK originally due to Burge (RuleBurge).
- A correspondence producing binary words originally due to Viennot (RuleBinaryWord).
- A correspondence producing domino tableaux (RuleDomino) originally due to Barbasch and Vogan.
- A correspondence for shifted shapes (RuleShiftedShapes), where the original insertion algorithm is due to Sagan and Worley, and Haiman.
- The Sylvester correspondence, producing binary trees (RuleSylvester).
- The Young-Fibonacci correspondence (RuleYoungFibonacci).
- LLMS insertion (RuleLLMS).

Background

At the heart of Fomin’s framework is the notion of dual graded graphs. This is a pair of digraphs $P, Q$ (multiple edges being allowed) on the same set of vertices $V$, that satisfy the following conditions:

- the graphs are graded, that is, there is a function $\rho : V \rightarrow \mathbb{N}$, such that for any edge $(v, w)$ of $P$ and also of $Q$ we have $\rho(w) = \rho(v) + 1$,
- there is a vertex $0$ with rank zero, and
- there is a positive integer $r$ such that $DU = UD + rI$ on the free $\mathbb{Z}$-module $\mathbb{Z}[V]$, where $D$ is the down operator of $Q$, assigning to each vertex the formal sum of its predecessors, $U$ is the up operator of $P$, assigning to each vertex the formal sum of its successors, and $I$ is the identity operator.

Note that the condition $DU = UD + rI$ is symmetric with respect to the interchange of the graphs $P$ and $Q$, because the up operator of a graph is the transpose of its down operator.

For example, taking for both $P$ and $Q$ to be Young’s lattice and $r = 1$, we obtain the dual graded graphs for classical Schensted insertion.

Given such a pair of graphs, there is a bijection between the $r$-colored permutations on $k$ letters and pairs $(p, q)$, where $p$ is a path in $P$ from zero to a vertex of rank $k$ and $q$ is a path in $Q$ from zero to the same vertex.

It turns out that - in principle - this bijection can always be described by so-called local forward and backward rules, see [Fom1995] for a detailed description. Knowing at least the forward rules, or the backward rules, you can implement your own growth diagram class.

Implementing your own growth diagrams

The class GrowthDiagram is written so that it is easy to implement growth diagrams you come across in your research. Moreover, the class tolerates some deviations from Fomin’s definitions. For example, although the general Robinson-Schensted-Knuth correspondence between integer matrices and semistandard tableaux is, strictly speaking, not a growth on dual graded graphs, it is supported by our framework.

For illustration, let us implement a growth diagram class with the backward rule only. Suppose that the vertices of the graph are the non-negative integers, the rank is given by the integer itself, and the backward rule is $(y, z, x) \mapsto (\min(x, y), 0)$ if $y = z$ or $x = z$ and $(y, z, x) \mapsto (\min(x, y), 1)$ otherwise.
We first need to import the base class for a rule:

```
sage: from sage.combinat.growth import Rule
```

Next, we implement the backward rule and the rank function and provide the bottom element zero of the graph. For more information, see `Rule`.

```
sage: class RulePascal(Rule):
    ...:     zero = 0
    ...:     def rank(self, v):
    ...:         return v
    ...:     def backward_rule(self, y, z, x):
    ...:         return (min(x, y), 0 if y==z or x==z else 1)
```

We can now compute the filling corresponding to a sequence of labels as follows:

```
sage: GrowthDiagram(RulePascal(), labels=[0,1,2,1,2,1,0])
1 0 0
0 0 1
0 1
```

Of course, since we have not provided the forward rule, we cannot compute the labels belonging to a filling:

```
sage: GrowthDiagram(RulePascal(), [3,1,2])
Traceback (most recent call last):
  ...:
AttributeError: 'RulePascal' object has no attribute 'forward_rule'
```

We now re-implement the rule where we provide the dual graded graphs:

```
sage: class RulePascal(Rule):
    ...:     zero = 0
    ...:     def rank(self, v):
    ...:         return v
    ...:     def backward_rule(self, y, z, x):
    ...:         return (min(x, y), 0 if y==z or x==z else 1)
    ...:     def vertices(self, n):
    ...:         return [n]
    ...:     def is_P_edge(self, v, w):
    ...:         return w == v + 1
    ...:     def is_Q_edge(self, v, w):
    ...:         return w == v + 1
```

Are they really dual?

```
sage: RulePascal()._check_duality(3)
Traceback (most recent call last):
  ...:
ValueError: D U - U D differs from 1 I for vertex 3:
D U = [3]
U D + 1 I = [3, 3]
```

With our current definition, duality fails - in fact, there are no dual graded graphs on the integers without multiple edges. Consequently, also the backward rule cannot work as `backward_rule` requires additional information (the edge labels as arguments).

Let us thus continue with the example from Section 4.7 of [Fom1995] instead, which defines dual graded graphs with multiple edges on the integers. The color `self.zero_edge`, which defaults to 0 is reserved for degenerate edges, but may be abused for the unique edge if one of the graphs has no multiple edges. For greater clarity in this example we set it to `None`:
We verify these are 1 dual at level 5:

```python
sage: RulePascal()._check_duality(5)
```

Finally, let us provide the backward rule. The arguments of the rule are vertices together with the edge labels now, specifying the path from the lower left to the upper right of the cell. The horizontal edges come from $Q$, whereas the vertical edges come from $P$.

Thus, the definition in Section 4.7 of [Fom1995] translates as follows:

```python
sage: class RulePascal(Rule):
    ....:     zero = 0
    ....:     has_multiple_edges = True
    ....:     zero_edge = None
    ....:     def rank(self, v):
    ....:         return v
    ....:     def vertices(self, n):
    ....:         return [n]
    ....:     def is_P_edge(self, v, w):
    ....:         return [0] if w == v + 1 else []
    ....:     def is_Q_edge(self, v, w):
    ....:         return list(range(w)) if w == v+1 else []
    ....:     def backward_rule(self, y, g, z, h, x):
    ....:         if g is None:
    ....:             return (0, x, None, 0)
    ....:         if h is None:
    ....:             return (None, y, g, 0)
    ....:         if g == 0:
    ....:             return (None, y, None, 1)
    ....:         else:
    ....:             return (0, x-1, g-1, 0)
```

The labels are now alternating between vertices and edge-colors:

```python
sage: GrowthDiagram(RulePascal(), labels=[0,0,1,0,2,0,1,0,0])
1 0
0 1
sage: GrowthDiagram(RulePascal(), labels=[0,0,1,1,2,0,1,0,0])
0 1
1 0
```

```python
class sage.combinat.growth.GrowthDiagram(Rule, filling=None, shape=None, labels=None)
```

A generalized Schensted growth diagram in the sense of Fomin.

Growth diagrams were introduced by Sergey Fomin [Fom1994], [Fom1995] and provide a vast generalization of the Robinson-Schensted-Knuth (RSK) correspondence between matrices with non-negative integer entries and pairs of semistandard Young tableaux of the same shape.

5.1. Comprehensive Module List 1133
A growth diagram is based on the notion of dual graded graphs, a pair of digraphs $P, Q$ (multiple edges being allowed) on the same set of vertices $V$, that satisfy the following conditions:

- the graphs are graded, that is, there is a function $\rho : V \rightarrow \mathbb{N}$, such that for any edge $(v, w)$ of $P$ and also of $Q$ we have $\rho(w) = \rho(v) + 1$,
- there is a vertex 0 with rank zero, and
- there is a positive integer $r$ such that $DU = UD + rI$ on the free $\mathbb{Z}$-module $\mathbb{Z}[V]$, where $D$ is the down operator of $Q$, assigning to each vertex the formal sum of its predecessors, $U$ is the up operator of $P$, assigning to each vertex the formal sum of its successors, and $I$ is the identity operator.

Growth diagrams are defined by providing a pair of local rules: a ‘forward’ rule, whose input are three vertices $y$, $t$ and $x$ of the dual graded graphs and an integer, and whose output is a fourth vertex $z$. This rule should be invertible in the following sense: there is a so-called ‘backward’ rule that recovers the integer and $t$ given $y$, $z$ and $x$.

All implemented growth diagram rules are available by GrowthDiagram.rules. The current list is:

- RuleRSK – RSK
- RuleBurge – a variation of RSK originally due to Burge
- RuleBinaryWord – a correspondence producing binary words originally due to Viennot
- RuleDomino – a correspondence producing domino tableaux originally due to Barbasch and Vogan
- RuleShiftedShapes – a correspondence for shifted shapes, where the original insertion algorithm is due to Sagan and Worley, and Haiman.
- RuleSylvester – the Sylvester correspondence, producing binary trees
- RuleYoungFibonacci – the Young-Fibonacci correspondence
- RuleLLMS – LLMS insertion

INPUT:

- rule – Rule; the growth diagram rule
- filling – (optional) a dictionary whose keys are coordinates and values are integers, a list of lists of integers, or a word with integer values; if a word, then negative letters but without repetitions are allowed and interpreted as coloured permutations
- shape – (optional) a (possibly skew) partition
- labels – (optional) a list that specifies a path whose length in the half-perimeter of the shape; more details given below

If filling is not given, then the growth diagram is determined by applying the backward rule to labels decorating the boundary opposite of the origin of the shape. In this case, labels are interpreted as labelling the boundary opposite of the origin.

Otherwise, shape is inferred from filling or labels if possible and labels is set to rule.zero if not specified. Here, labels are labelling the boundary on the side of the origin.

For labels, if rule.has_multiple_edges is True, then the elements should be of the form $(v_1, e_1, \ldots, e_{n-1}, v_n)$, where $n$ is the half-perimeter of shape, and $(v_{i-1}, e_i, v_i)$ is an edge in the dual graded graph for all $i$. Otherwise, it is a list of $n$ vertices.

Note: Coordinates are of the form (col, row) where the origin is in the upper left, to be consistent with permutation matrices and skew tableaux (in English convention). This is different from Fomin’s convention, who uses a Cartesian coordinate system.
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Conventions are chosen such that for permutations, the same growth diagram is constructed when passing the permutation matrix instead.

EXAMPLES:

We create a growth diagram using the forward RSK rule and a permutation:

```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: pi = Permutation([4, 1, 2, 3])
sage: G = GrowthDiagram(RuleRSK, pi); G
0 1 0 0
0 0 1 0
0 0 0 1
1 0 0 0
sage: G.out_labels()
[[], [1], [1, 1], [2, 1], [3, 1], [3], [2], [1], []]
```

Passing the permutation matrix instead gives the same result:

```
sage: G = GrowthDiagram(RuleRSK, pi.to_matrix())
sage: ascii_art([G.P_symbol(), G.Q_symbol()])
[ 1 2 3 1 3 4 ]
[ 4 , 2 ]
```

We give the same example but using a skew shape:

```
sage: shape = SkewPartition([[4,4,4,2],[1,1]])
sage: G = GrowthDiagram(RuleRSK, pi, shape=shape); G
. 1 0 0
. 0 1 0
0 0 0 1
1 0
sage: G.out_labels()
[[], [1], [1, 1], [1], [2], [3], [2], [1], []]
```

We construct a growth diagram using the backwards RSK rule by specifying the labels:

```
sage: GrowthDiagram(RuleRSK, labels=G.out_labels())
0 1 0 0
0 0 1 0
0 0 0 1
1 0
```

P_chain()  
Return the labels along the vertical boundary of a rectangular growth diagram.

EXAMPLES:

```
sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: G = GrowthDiagram(BinaryWord, [4, 1, 2, 3])
sage: G.P_chain()
[word: , word: 1, word: 11, word: 111, word: 1011]
```

Check that github issue #25631 is fixed:

5.1. Comprehensive Module List 1135
**P_symbol()**

Return the labels along the vertical boundary of a rectangular growth diagram as a generalized standard tableau.

**EXAMPLES:**

```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
sage: ascii_art([G.P_symbol(), G.Q_symbol()])
```

```
[ 1 2 2 1 3 3 ]
[ 2 , 2 ]
```

**Q_chain()**

Return the labels along the horizontal boundary of a rectangular growth diagram.

**EXAMPLES:**

```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
sage: G.Q_chain()
```

```
[word: , word: 1, word: 10, word: 101, word: 1011]
```

Check that [github issue #25631](https://github.com/sagemath/sage/issues/25631) is fixed:

```
sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: BinaryWord(filling = {}).Q_chain()
```

```
[word: ]
```

**Q_symbol()**

Return the labels along the horizontal boundary of a rectangular growth diagram as a generalized standard tableau.

**EXAMPLES:**

```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
sage: ascii_art([G.P_symbol(), G.Q_symbol()])
```

```
[ 1 2 2 1 3 3 ]
[ 2 , 2 ]
```

**conjugate()**

Return the conjugate growth diagram of `self`.

This is the growth diagram with the filling reflected over the main diagonal.

The sequence of labels along the boundary on the side of the origin is the reversal of the corresponding sequence of the original growth diagram.

When the filling is a permutation, the conjugate filling corresponds to its inverse.

**EXAMPLES:**
```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
sage: Gc = G.conjugate()
sage: (Gc.P_symbol(), Gc.Q_symbol()) == (G.Q_symbol(), G.P_symbol())
True

filling()
Return the filling of the diagram as a dictionary.

EXAMPLES:
```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
sage: G.filling()
{(0, 1): 1, (1, 0): 1, (2, 1): 2}
```

half_perimeter()
Return half the perimeter of the shape of the growth diagram.

in_labels()
Return the labels along the boundary on the side of the origin.

EXAMPLES:
```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[2,2],[3,2],[3,3],[3,2]]); G
1 0
sage: G.in_labels()
[[2, 2], [2, 2], [2, 2], [3, 2]]
```

is_rectangular()
Return True if the shape of the growth diagram is rectangular.

EXAMPLES:
```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: GrowthDiagram(RuleRSK, [2,3,1]).is_rectangular()
True
sage: GrowthDiagram(RuleRSK, [[1,0,1],[0,1]]).is_rectangular()
False
```

out_labels()
Return the labels along the boundary opposite of the origin.

EXAMPLES:
```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
sage: G.out_labels()
[[], [1], [1, 1], [3, 1], [1], []]
```

rotate()
Return the growth diagram with the filling rotated by 180 degrees.

The rotated growth diagram is initialized with labels=None, that is, all labels along the boundary on the side of the origin are set to rule.zero.

5.1. Comprehensive Module List
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For RSK-growth diagrams and rectangular fillings, this corresponds to evacuation of the $P$- and the $Q$-symbol.

EXAMPLES:

```python
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[0,1,0], [1,0,2]])
```

```python
sage: Gc = G.rotate()
sage: ascii_art([Gc.P_symbol(), Gc.Q_symbol()])

[ 1 1 1 1 1 2 ]
[ 2 , 3 ]
```

```python
sage: ascii_art([Tableau(t).evacuation() for t in [G.P_symbol(), G.Q_symbol()]]).

[ 1 1 1 1 1 2 ]
[ 2 , 3 ]
```

For RSK-growth diagrams and rectangular fillings, this corresponds to evacuation of the $P$- and the $Q$-symbol.

EXAMPLES:

```python
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = GrowthDiagram(RuleRSK, [[1]])
```

```python
sage: ascii_art([Tableau(t).evacuation() for t in [G.P_symbol(), G.Q_symbol()]])

[ 1 1 1 1 1 2 ]
[ 2 , 3 ]
```

rules
alias of Rules

shape()

Return the shape of the growth diagram as a skew partition.

**Warning:** In the literature the label on the corner opposite of the origin of a rectangular filling is often called the shape of the filling. This method returns the shape of the region instead.

EXAMPLES:

```python
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: GrowthDiagram(RuleRSK, [1]).shape()

[1] / []
```

to biword()

Return the filling as a biword, if the shape is rectangular.

EXAMPLES:

```python
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: P = Tableau([[1,2,2],[2]])
sage: Q = Tableau([[1,3,3],[2]])
```

```python
sage: bw = RSK_inverse(P, Q); bw

[[1, 2, 3, 3], [2, 1, 2, 2]]
```

```python
sage: G = GrowthDiagram(RuleRSK, labels=Q.to_chain()[:-1]+P.to_chain()[::-1]); G

0 1 0
1 0 2
```

```python
sage: P = SemistandardTableau([[1,1,2],[2]])
sage: Q = SemistandardTableau([[1,2,2],[2]])
```

```python
sage: G = GrowthDiagram(RuleRSK, labels=Q.to_chain()[:-1]+P.to_chain()[::-1]); G

0 2
1 1
```

```python
sage: G.to_biword()

([1, 2, 2, 2], [2, 1, 1, 2])
```

(continues on next page)
\texttt{sage: RSK([1, 2, 2, 2], [2, 1, 1, 2])}\\%
[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]

\textbf{to\_word()}

Return the filling as a word, if the shape is rectangular and there is at most one nonzero entry in each
column, which must be 1.

\textbf{EXAMPLES:}

\texttt{sage: RuleRSK = GrowthDiagram.rules.RSK()}
\texttt{sage: w = [3,3,2,4,1]; G = GrowthDiagram(RuleRSK, w)}
\texttt{sage: G}
\begin{verbatim}
  0 0 0 0 1
  0 0 1 0 0
  1 1 0 0 0
  0 0 0 1 0
\end{verbatim}
\texttt{sage: G.to\_word()}
\begin{verbatim}
[3, 3, 2, 4, 1]
\end{verbatim}

\textbf{class sage.combinat.growth.Rule}

\textbf{Bases:} UniqueRepresentation

Generic base class for a rule for a growth diagram.

Subclasses may provide the following attributes:

- \texttt{zero} – the zero element of the vertices of the graphs
- \texttt{r} – (default: 1) the parameter in the equation \( DU - UD = rI \)
- \texttt{has\_multiple\_edges} – (default: False) if the dual graded graph has multiple edges and therefore edges are triples consisting of two vertices and a label.
- \texttt{zero\_edge} – (default: 0) the zero label of the edges of the graphs used for degenerate edges. It is allowed to use this label also for other edges.

Subclasses may provide the following methods:

- \texttt{normalize\_vertex} – a function that converts its input to a vertex.
- \texttt{vertices} – a function that takes a non-negative integer as input and returns the list of vertices on this rank.
- \texttt{rank} – the rank function of the dual graded graphs.
- \texttt{forward\_rule} – a function with input \((y, t, x, content)\) or \((y, e, t, f, x, content)\) if \texttt{has\_multiple\_edges} is True. \((y, e, t)\) is an edge in the graph \(P\), \((t, f, x)\) an edge in the graph \(Q\). It should return the fourth vertex \(z\), or, if \texttt{has\_multiple\_edges} is True, the path \((g, z, h)\) from \(y\) to \(x\).
- \texttt{backward\_rule} – a function with input \((y, z, x)\) or \((y, g, z, h, x)\) if \texttt{has\_multiple\_edges} is True. \((y, g, z)\) is an edge in the graph \(Q\), \((z, h, x)\) an edge in the graph \(P\). It should return the fourth vertex and the content \((t, content)\), or, if \texttt{has\_multiple\_edges} is True, the path from \(y\) to \(x\) and the content as \((e, t, f, content)\).
- \texttt{is\_P\_edge, is\_Q\_edge} – functions that take two vertices as arguments and return True or False, or, if multiple edges are allowed, the list of edge labels of the edges from the first vertex to the second in the respective graded graph. These are only used for checking user input and providing the dual graded graph, and are therefore not mandatory.
Note that the class `GrowthDiagram` is able to use partially implemented subclasses just fine. Suppose that `MyRule` is such a subclass. Then:

- `GrowthDiagram(MyRule, my_filling)` requires only an implementation of `forward_rule.zero` and possibly `has_multiple_edges`.
- `GrowthDiagram(MyRule, labels=my_labels, shape=my_shape)` requires only an implementation of `backward_rule` and possibly `has_multiple_edges`, provided that the labels `my_labels` are given as needed by `backward_rule`.
- `GrowthDiagram(MyRule, labels=my_labels)` additionally needs an implementation of `rank` to deduce the shape.

In particular, this allows to implement rules which do not quite fit Fomin’s notion of dual graded graphs. An example would be Bloom and Saracino’s variant of the RSK correspondence [BS2012], where a backward rule is not available.

Similarly:

- `MyRule.P_graph` only requires an implementation of `vertices`, `is_P_edge` and possibly `has_multiple_edges` is required, mutatis mutandis for `MyRule.Q_graph`.
- `MyRule._check_duality` requires `P_graph` and `Q_graph`.

In particular, this allows to work with dual graded graphs without local rules.

**P_graph**

Return the first `n` levels of the first dual graded graph.

The non-degenerate edges in the vertical direction come from this graph.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: Domino.P_graph(3)
Finite poset containing 8 elements
```

**Q_graph**

Return the first `n` levels of the second dual graded graph.

The non-degenerate edges in the horizontal direction come from this graph.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: Q = Domino.Q_graph(3); Q
Finite poset containing 8 elements

sage: Q.upper_covers(Partition([1,1]))
[[[1, 1, 1, 1], [3, 1], [2, 2]]
```

**has_multiple_edges**

Return `v` as a vertex of the dual graded graph.

This is a default implementation, returning its argument.

**EXAMPLES:**
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```
sage: from sage.combinat.growth import Rule
dsage: Rule().normalize_vertex("hello") == "hello"
True
```

```r
r = 1

zero_edge = 0
```

class sage.combinat.growth.RuleBinaryWord

Bases: Rule

A rule modelling a Schensted-like correspondence for binary words.

EXAMPLES:

```
sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: GrowthDiagram(BinaryWord, [3,1,2])

0 1 0
0 0 1
1 0 0
```

The vertices of the dual graded graph are binary words:

```
sage: BinaryWord.vertices(3)
[word: 100, word: 101, word: 110, word: 111]
```

Note that, instead of passing the rule to GrowthDiagram, we can also use call the rule to create growth diagrams. For example:

```
sage: BinaryWord([2,4,1,3]).P_chain()
[word: , word: 1, word: 10, word: 101, word: 1101]
sage: BinaryWord([2,4,1,3]).Q_chain()
[word: , word: 1, word: 11, word: 110, word: 1101]
```

The Kleitman Greene invariant is the descent word, encoded by the positions of the zeros:

```
sage: pi = Permutation([4,1,8,3,6,5,2,7,9])
sage: G = BinaryWord(pi); G

0 1 0 0 0 0 0 0 0
0 0 0 0 0 0 1 0 0
0 0 1 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 0 0
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0
sage: pi.descents()
[1, 3, 5, 6]
```

```
backward_rule(y, z, x)
```

Return the content and the input shape.


• y, z, x – three binary words from a cell in a growth diagram, labelled as:

5.1. Comprehensive Module List 1141
OUTPUT:
A pair \((t, \text{content})\) consisting of the shape of the fourth word and the content of the cell according to Viennot’s bijection [Vie1983].

\textbf{forward\_rule}(y, t, x, \text{content})

Return the output shape given three shapes and the content.


INPUT:

\begin{itemize}
  \item \(y, t, x\) – three binary words from a cell in a growth diagram, labelled as:
  \begin{verbatim}
  t x
  y
  \end{verbatim}
  \item \text{content} – 0 or 1; the content of the cell
\end{itemize}

OUTPUT:

The fourth binary word \(z\) according to Viennot’s bijection [Vie1983].

EXAMPLES:

\begin{verbatim}
sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: BinaryWord.forward_rule([], [], [], 1)
word: 1
sage: BinaryWord.forward_rule([1], [1], [1], 1)
word: 11
\end{verbatim}

if \(x \neq y\) append last letter of \(x\) to \(y\):

\begin{verbatim}
sage: BinaryWord.forward_rule([1,0], [1], [1,1], 0)
word: 101
\end{verbatim}

if \(x == y \neq t\) append 0 to \(y\):

\begin{verbatim}
sage: BinaryWord.forward_rule([1,1], [1], [1,1], 0)
word: 110
\end{verbatim}

\textbf{is\_P\_edge}(v, w)

Return whether \((v, w)\) is a \(P\)-edge of \texttt{self}.

\((v, w)\) is an edge if \(v\) is obtained from \(w\) by deleting a letter.

EXAMPLES:

\begin{verbatim}
sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: v = BinaryWord.vertices(2)[1]; v
word: 11
sage: [w for w in BinaryWord.vertices(3) if BinaryWord.is_P_edge(v, w)]
[word: 101, word: 110, word: 111]
\end{verbatim}
sage: [w for w in BinaryWord.vertices(4) if BinaryWord.is_P_edge(v, w)]
[]

is_Q_edge(v, w)
Return whether (v, w) is a Q-edge of self.
(w, v) is an edge if w is obtained from v by appending a letter.

EXAMPLES:

sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: v = BinaryWord.vertices(2)[0]; v
word: 10
sage: [w for w in BinaryWord.vertices(3) if BinaryWord.is_Q_edge(v, w)]
[word: 100, word: 101]
sage: [w for w in BinaryWord.vertices(4) if BinaryWord.is_Q_edge(v, w)]
[]

normalize_vertex(v)
Return v as a binary word.

EXAMPLES:

sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: BinaryWord.normalize_vertex([0, 1]).parent()
Finite words over {0, 1}

rank(v)
Return the rank of v: number of letters of the word.

EXAMPLES:

sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: BinaryWord.rank(BinaryWord.vertices(3)[0])
3

vertices(n)
Return the vertices of the dual graded graph on level n.

EXAMPLES:

sage: BinaryWord = GrowthDiagram.rules.BinaryWord()
sage: BinaryWord.vertices(3)
[word: 100, word: 101, word: 110, word: 111]

zero = word:

class sage.combinat.growth.RuleBurge
Bases: RulePartitions
A rule modelling Burge insertion.

EXAMPLES:
The vertices of the dual graded graph are integer partitions:

\[
\text{sage: Burge = GrowthDiagram.rules.Burge()}
\text{sage: GrowthDiagram(Burge, labels=[[],[1,1,1],[2,1,1,1],[2,1,1],[2,1],[1,1],[[]])}
\]
\[
\begin{align*}
1 & 1 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
\end{align*}
\]

The local rules implemented provide Burge’s correspondence between matrices with non-negative integer entries and pairs of semistandard tableaux, the \( P\_symbol \) and the \( Q\_symbol \). For permutations, it reduces to classical Schensted insertion.

Instead of passing the rule to \( GrowthDiagram \), we can also call the rule to create growth diagrams. For example:

\[
\text{sage: m = matrix([[2,0,0,1,0],[1,1,0,0,0],[0,0,0,0,3]])}
\text{sage: G = Burge(m); G}
\]
\[
\begin{align*}
2 & 0 0 1 0 \\
1 & 1 0 0 0 \\
0 & 0 0 0 3 \\
\end{align*}
\]

\[
\text{sage: ascii_art([G.P\_symbol(), G.Q\_symbol()])}
\]
\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 & 3 \\
2 & & &
\end{array}
\]

For rectangular fillings, the Kleitman-Greene invariant is the shape of the \( P\_symbol \). Put differently, it is the partition labelling the lower right corner of the filling (recall that we are using matrix coordinates). It can be computed alternatively as the transpose of the partition \((\mu_1, \ldots, \mu_n)\), where \( \mu_1 + \cdots + \mu_i \) is the maximal sum of entries in a collection of \( i \) pairwise disjoint sequences of cells with weakly decreasing row indices and weakly increasing column indices.

\[ \text{backward\_rule}(y, z, x) \]

Return the content and the input shape.

See [Kra2006] \((B^4) - (B^4)\). (In the arXiv version of the article there is a typo: in the computation of carry in \((B^4)\), \( \rho \) must be replaced by \( \lambda \)).

\[
\begin{align*}
\text{INPUT: } & y, z, x - \text{three partitions from a cell in a growth diagram, labelled as:} \\
& \begin{array}{c}
x \\
y \\
z
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{OUTPUT: } & \text{A pair (t, content) consisting of the shape of the fourth partition according to the Burge correspondence and the content of the cell.} \\
\text{EXAMPLES:}
\end{align*}
\]
```python
sage: Burge = GrowthDiagram.rules.Burge()
sage: Burge.backward_rule([1,1,1],[2,1,1,1],[2,1,1])
([(1, 1), 0])
```

**forward_rule** *(y, t, x, content)*

Return the output shape given three shapes and the content.

See [Kra2006] \((F^40) - (F^412)\).

**INPUT:**

- y, t, x – three from a cell in a growth diagram, labelled as:

  \[
  \begin{array}{c}
  t \\
  x \\
  y \\
  \end{array}
  \]

- content – a non-negative integer; the content of the cell

**OUTPUT:**

The fourth partition according to the Burge correspondence.

**EXAMPLES:**

```python
sage: Burge = GrowthDiagram.rules.Burge()
sage: Burge.forward_rule([2,1],[2,1],[2,1],1)
[3, 1]
sage: Burge.forward_rule([1],[],[2],2)
[2, 1, 1, 1]
```

```python
class sage.combinat.growth.RuleDomino

Bases: Rule

A rule modelling domino insertion.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: GrowthDiagram(Domino, [[1,0,0],[0,0,1],[0,-1,0]])
1 0 0
0 0 1
0 -1 0
```

The vertices of the dual graded graph are integer partitions whose Ferrers diagram can be tiled with dominoes:

```python
sage: Domino.vertices(2)
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
```

Instead of passing the rule to *GrowthDiagram*, we can also call the rule to create growth diagrams. For example, let us check Figure 3 in [Lam2004]:

```python
sage: G = Domino([[0,0,0,-1],[0,0,1,0],[-1,0,0,0],[0,1,0,0]]); G
0 0 0 -1
0 0 1 0
-1 0 0 0
0 1 0 0
```
The spin of a domino tableau is half the number of vertical dominoes:

```python
sage: def spin(T):
    ....:     return sum(2*len(set(row)) - len(row) for row in T)/4
```

According to [Lam2004], the number of negative entries in the signed permutation equals the sum of the spins of the two associated tableaux:

```python
sage: pi = [3,-1,2,4,-5]
sage: G = Domino(pi)
sage: list(G.filling().values()).count(-1) == spin(G.P_symbol()) + spin(G.Q_symbol())
True
```

Negating all signs transposes all the partitions:

```python
sage: G.P_symbol() == Domino([-e for e in pi]).P_symbol().conjugate()
True
```

**P_symbol** (*P_chain*)

Return the labels along the vertical boundary of a rectangular growth diagram as a (skew) domino tableau.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: G = Domino([[0,1,0],[0,0,-1],[1,0,0]])
sage: G.P_symbol().pp()
  1 1
  2 3
  2 3
```

**Q_symbol** (*P_chain*)

Return the labels along the vertical boundary of a rectangular growth diagram as a (skew) domino tableau.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: G = Domino([[0,1,0],[0,0,-1],[1,0,0]])
sage: G.P_symbol().pp()
  1 1
  2 3
  2 3
```

**forward_rule** (*y, t, x, content*)

Return the output shape given three shapes and the content.

See [Lam2004] Section 3.1.

**INPUT:**
• $y$, $t$, $x$ – three partitions from a cell in a growth diagram, labelled as:

```
t x
  y
```

• content $= -1$, $0$ or $1$; the content of the cell

**OUTPUT:**

The fourth partition according to domino insertion.

**EXAMPLES:**

```
sage: Domino = GrowthDiagram.rules.Domino()

Rule 1:

```
sage: Domino.forward_rule([], [], [], 1)
[2]
sage: Domino.forward_rule([1,1], [1,1], [1,1], 1)
[3, 1]
```

Rule 2:

```
sage: Domino.forward_rule([1,1], [1,1], [1,1], -1)
[1, 1, 1, 1]
```

Rule 3:

```
sage: Domino.forward_rule([1,1], [1,1], [2,2], 0)
[2, 2]
```

Rule 4:

```
sage: Domino.forward_rule([2,2,2], [2,2], [3,3], 0)
[3, 3, 2]
sage: Domino.forward_rule([2,2], [2,2], [1,1], 0)
[2, 2]
sage: Domino.forward_rule([1,1], [2,2], [2,2], 0)
[2, 2]
sage: Domino.forward_rule([2,2], [2,2], [0], 0)
[2, 2]
sage: Domino.forward_rule([4], [2], [4], 0)
[4, 2]
sage: Domino.forward_rule([1,1,1,1], [1,1], [1,1,1,1], 0)
[2, 2, 1, 1]
sage: Domino.forward_rule([2,1,1], [2], [4], 0)
[4, 1, 1]
```
**is_P_edge**\((v, w)\)

Return whether \((v, w)\) is a \(P\)-edge of self.

\((v, w)\) is an edge if \(v\) is obtained from \(w\) by deleting a domino.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: v = Domino.vertices(2)[1]; ascii_art(v)
***
*
sage: ascii_art([w for w in Domino.vertices(3) if Domino.is_P_edge(v, w)])
[ *** ]
[ * ]
[ ***** *** * ]
[ * , *** , * ]
sage: [w for w in Domino.vertices(4) if Domino.is_P_edge(v, w)]
[]
```

**is_Q_edge**\((v, w)\)

Return whether \((v, w)\) is a \(P\)-edge of self.

\((v, w)\) is an edge if \(v\) is obtained from \(w\) by deleting a domino.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: v = Domino.vertices(2)[1]; ascii_art(v)
***
*
sage: ascii_art([w for w in Domino.vertices(3) if Domino.is_P_edge(v, w)])
[ *** ]
[ * ]
[ ***** *** * ]
[ * , *** , * ]
sage: [w for w in Domino.vertices(4) if Domino.is_P_edge(v, w)]
[]
```

**normalize_vertex**\((v)\)

Return \(v\) as a partition.

**EXAMPLES:**

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: Domino.normalize_vertex([3,1]).parent()
Partitions

\(r = 2\)

**rank**\((v)\)

Return the rank of \(v\).

The rank of a vertex is half the size of the partition, which equals the number of dominoes in any filling.

**EXAMPLES:**
```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: Domino.rank(Domino.vertices(3)[0])
3
```

**vertices**(n)

Return the vertices of the dual graded graph on level n.

EXAMPLES:

```python
sage: Domino = GrowthDiagram.rules.Domino()
sage: Domino.vertices(2)
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
```

zero = []

class sage.combinat.growth.RuleLLMS(k)

**Bases:** Rule

A rule modelling the Schensted correspondence for affine permutations.

EXAMPLES:

```python
sage: LLMS3 = GrowthDiagram.rules.LLMS(3)
sage: GrowthDiagram(LLMS3, [3,1,2])
0 1 0
0 0 1
1 0 0
```

The vertices of the dual graded graph are **Cores**:

```python
sage: LLMS3.vertices(4)
3-Cores of length 4
```

Let us check example of Figure 1 in [LS2007]. Note that, instead of passing the rule to GrowthDiagram, we can also call the rule to create growth diagrams:

```python
sage: G = LLMS3([4,1,2,6,3,5]); G
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 0 1 0
1 0 0 0 0 0
0 0 0 0 0 1
0 0 0 1 0 0
```

The **P_symbol**() is a **StrongTableau**:

```python
sage: G.P_symbol().pp()
-1 -2 -3 -5
3 5
-4 -6
5
6
```

The **Q_symbol**() is a **WeakTableau**:

5.1. Comprehensive Module List
Let us also check Example 6.2 in [LLMSSZ2013]:

```
sage: G = LLMS3([4,1,3,2])
```

```
sage: G.P_symbol().pp()
-1 -2 3
-3
-4
```

```
sage: G.Q_symbol().pp()
1 3 4
2
3
```

**P_symbol**\((P\_chain)\)

Return the labels along the vertical boundary of a rectangular growth diagram as a skew *StrongTableau*.

EXAMPLES:

```
sage: LLMS4 = GrowthDiagram.rules.LLMS(4)
sage: G = LLMS4([3,4,1,2])
sage: G.P_symbol().pp()
-1 -2
-3
-4
```

**Q_symbol**\((Q\_chain)\)

Return the labels along the horizontal boundary of a rectangular growth diagram as a skew *WeakTableau*.

EXAMPLES:

```
sage: LLMS4 = GrowthDiagram.rules.LLMS(4)
sage: G = LLMS4([3,4,1,2])
sage: G.Q_symbol().pp()
1 2
3 4
```

**forward_rule**\((y, e, t, f, x, content)\)

Return the output path given two incident edges and the content.

See [LS2007] Section 3.4 and [LLMSSZ2013] Section 6.3.

INPUT:

- \(y, e, t, f, x\) – a path of three partitions and two colors from a cell in a growth diagram, labelled as:

```
t f x
e
y
```
• content – 0 or 1; the content of the cell

OUTPUT:
The two colors and the fourth partition g, z, h according to LLMS insertion.

EXAMPLES:

```
sage: LLMS3 = GrowthDiagram.rules.LLMS(3)
sage: LLMS4 = GrowthDiagram.rules.LLMS(4)

sage: Z = LLMS3.zero
sage: LLMS3.forward_rule(Z, None, Z, None, Z, 0)
(None, [], None)

sage: LLMS3.forward_rule(Z, None, Z, None, Z, 1)
(None, [1], 0)

sage: Y = Core([3,1,1], 3)
sage: LLMS3.forward_rule(Y, None, Y, None, Y, 1)
(None, [4, 2, 1, 1], 3)
```

if x != y:

```
sage: Y = Core([1,1], 3); T = Core([1], 3); X = Core([2], 3)
sage: LLMS3.forward_rule(Y, -1, T, None, X, 0)
(None, [2, 1, 1], -1)

sage: Y = Core([2], 4); T = Core([1], 4); X = Core([1,1], 4)
sage: LLMS4.forward_rule(Y, 1, T, None, X, 0)
(None, [2, 1], 1)

sage: Y = Core([2,1,1], 3); T = Core([2], 3); X = Core([3,1], 3)
sage: LLMS3.forward_rule(Y, -1, T, None, X, 0)
(None, [3, 1, 1], -2)
```

if x == y != t:

```
sage: Y = Core([1], 3); T = Core([], 3); X = Core([1], 3)
sage: LLMS3.forward_rule(Y, 0, T, None, X, 0)
(None, [1, 1], -1)

sage: Y = Core([1], 4); T = Core([], 4); X = Core([1], 4)
sage: LLMS4.forward_rule(Y, 0, T, None, X, 0)
(None, [1, 1], -1)

sage: Y = Core([2,1], 4); T = Core([1,1], 4); X = Core([2,1], 4)
sage: LLMS4.forward_rule(Y, 1, T, None, X, 0)
(None, [2, 2], 0)
```

```
has_multiple_edges = True
```

```
is_P_edge(v, w)
Return whether (v, w) is a P-edge of self.
For two k-cores v and w containing v, there are as many edges as there are components in the skew partition w/v. These components are ribbons, and therefore contain a unique cell with maximal content. We index
```

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the edge with this content.

**EXAMPLES:**

```python
sage: LLMS4 = GrowthDiagram.rules.LLMS(4)
sage: v = LLMS4.vertices(2)[0]; v
[2]
sage: [(w, LLMS4.is_P_edge(v, w)) for w in LLMS4.vertices(3)]
[[[3], [2]], ([2, 1], [-1]), ([1, 1, 1], [])]
sage: all(LLMS4.is_P_edge(v, w) == [] for w in LLMS4.vertices(4))
True
```

**is_Q_edge**(v, w)

Return whether \((v, w)\) is a \(Q\)-edge of self.

\((v, w)\) is an edge if \(w\) is a weak cover of \(v\), see `weak_covers()`.

**EXAMPLES:**

```python
sage: LLMS4 = GrowthDiagram.rules.LLMS(4)
sage: v = LLMS4.vertices(3)[1]; v
[2, 1]
sage: [w for w in LLMS4.vertices(4) if len(LLMS4.is_Q_edge(v, w)) > 0]
[[2, 2], [3, 1, 1]]
sage: all(LLMS4.is_Q_edge(v, w) == [] for w in LLMS4.vertices(5))
True
```

**normalize_vertex**(v)

Convert \(v\) to a \(k\)-core.

**EXAMPLES:**

```python
sage: LLMS3 = GrowthDiagram.rules.LLMS(3)
sage: LLMS3.normalize_vertex([3,1]).parent()
3-Cores of length 3
```

**rank**(v)

Return the rank of \(v\): the length of the core.

**EXAMPLES:**

```python
sage: LLMS3 = GrowthDiagram.rules.LLMS(3)
sage: LLMS3.rank(LLMS3.vertices(3)[0])
3
```

**vertices**(n)

Return the vertices of the dual graded graph on level \(n\).

**EXAMPLES:**

```python
sage: LLMS3 = GrowthDiagram.rules.LLMS(3)
sage: LLMS3.vertices(2)
3-Cores of length 2
```

`zero_edge = None`
class sage.combinat.growth.RulePartitions
Bases: Rule

A rule for growth diagrams on Young's lattice on integer partitions graded by size.

\textbf{P\_symbol}(P\_chain)

Return the labels along the vertical boundary of a rectangular growth diagram as a (skew) tableau.

EXAMPLES:

\begin{verbatim}
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = RuleRSK([[0,1,0], [1,0,2]])
sage: G.P_symbol().pp()
1 2 2
2
\end{verbatim}

\textbf{Q\_symbol}(Q\_chain)

Return the labels along the horizontal boundary of a rectangular growth diagram as a skew tableau.

EXAMPLES:

\begin{verbatim}
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: G = RuleRSK([[0,1,0], [1,0,2]])
sage: G.Q_symbol().pp()
1 3 3
2
\end{verbatim}

\textbf{normalize_vertex}(v)

Return v as a partition.

EXAMPLES:

\begin{verbatim}
sage: RSK = GrowthDiagram.rules.RSK()
sage: RSK.normalize_vertex([3,1]).parent()
Partitions
\end{verbatim}

\textbf{rank}(v)

Return the rank of v: the size of the partition.

EXAMPLES:

\begin{verbatim}
sage: RSK = GrowthDiagram.rules.RSK()
sage: RSK.rank(RSK.vertices(3)[0])
3
\end{verbatim}

\textbf{vertices}(n)

Return the vertices of the dual graded graph on level n.

EXAMPLES:

\begin{verbatim}
sage: RSK = GrowthDiagram.rules.RSK()
sage: RSK.vertices(3)
Partitions of the integer 3
\end{verbatim}

\textbf{zero} = []
class sage.combinat.growth.RuleRSK

Bases: RulePartitions

A rule modelling Robinson-Schensted-Knuth insertion.

EXAMPLES:

```python
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: GrowthDiagram(RuleRSK, [3,2,1,2,3])
0 0 0 1 0
0 1 0 1 0
1 0 0 0 1
```

The vertices of the dual graded graph are integer partitions:

```python
sage: RuleRSK.vertices(3)
Partitions of the integer 3
```

The local rules implemented provide the RSK correspondence between matrices with non-negative integer entries and pairs of semistandard tableaux, the $P$-symbol() and the $Q$-symbol(). For permutations, it reduces to classical Schensted insertion.

Instead of passing the rule to GrowthDiagram, we can also call the rule to create growth diagrams. For example:

```python
sage: m = matrix([[0,0,0,0,1],[1,1,0,2,0],[0,3,0,0,0]])
sage: G = RuleRSK(m); G
0 0 0 0 1
1 1 0 2 0
0 3 0 0 0
sage: ascii_art([G.P_symbol(), G.Q_symbol()])
[ 1 2 2 2 3 1 2 2 2 2 ]
[ 2 3 4 4 ]
[ 3 , 5 ]
```

For rectangular fillings, the Kleitman-Greene invariant is the shape of the $P$-symbol() (or the $Q$-symbol()). Put differently, it is the partition labelling the lower right corner of the filling (recall that we are using matrix coordinates). It can be computed alternatively as the partition $(\mu_1, \ldots, \mu_n)$, where $\mu_1 + \cdots + \mu_i$ is the maximal sum of entries in a collection of $i$ pairwise disjoint sequences of cells with weakly increasing coordinates.

For rectangular fillings, we could also use the (faster) implementation provided via RSK(). Because of the coordinate conventions in RSK(), we have to transpose matrices:

```python
sage: [G.P_symbol(), G.Q_symbol()] == RSK(m.transpose())
True
sage: n = 5; l = [(pi, RuleRSK(pi)) for pi in Permutations(n)]
sage: all([G.P_symbol(), G.Q_symbol()] == RSK(pi) for pi, G in l)
True
sage: n = 5; l = [(w, RuleRSK(w)) for w in Words([1,2,3], 5)]
sage: all([G.P_symbol(), G.Q_symbol()] == RSK(pi) for pi, G in l)
True
```

backward_rule(y, z, x)

Return the content and the input shape.
Combinatorics, Release 10.0

See [Kra2006] $(B^10) - (B^12)$.

INPUT:

- $y, z, x$ – three partitions from a cell in a growth diagram, labelled as:

```
x
y z
```

OUTPUT:

A pair $(t, \text{content})$ consisting of the shape of the fourth word according to the Robinson-Schensted-Knuth correspondence and the content of the cell.

**forward_rule**($y, t, x, \text{content}$)

Return the output shape given three shapes and the content.

See [Kra2006] $(F^10) - (F^12)$.

INPUT:

- $y, t, x$ – three partitions from a cell in a growth diagram, labelled as:

```
t x
y
```

- $\text{content}$ – a non-negative integer; the content of the cell

OUTPUT:

The fourth partition according to the Robinson-Schensted-Knuth correspondence.

**EXAMPLES:**

```
sage: RuleRSK = GrowthDiagram.rules.RSK()
sage: RuleRSK.forward_rule([2,1],[2,1],[2,1],1)
[3, 1]
sage: RuleRSK.forward_rule([1],[],[2],2)
[4, 1]
```

**class** `sage.combinat.growth.RuleShiftedShapes`

Bases: `Rule`

A class modelling the Schensted correspondence for shifted shapes.

This agrees with Sagan [Sag1987] and Worley’s [Wor1984], and Haiman’s [Hai1989] insertion algorithms, see Proposition 4.5.2 of [Fom1995].

**EXAMPLES:**

```
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: GrowthDiagram(Shifted, [3,1,2])
0 1 0
0 0 1
1 0 0
```

The vertices of the dual graded graph are shifted shapes:

```
sage: Shifted.vertices(3)
Partitions of the integer 3 satisfying constraints max_slope=-1
```

5.1. Comprehensive Module List
Let us check the example just before Corollary 3.2 in [Sag1987]. Note that, instead of passing the rule to `GrowthDiagram`, we can also call the rule to create growth diagrams:

```
sage: G = Shifted([2,6,5,1,7,4,3])
sage: G.P_chain()
[[], 0, [1], 0, [2], 0, [3], 0, [3, 1], 0, [3, 2], 0, [4, 2], 0, [5, 2]]
sage: G.Q_chain()
[[], 1, [1], 2, [2], 1, [2, 1], 3, [3, 1], 2, [4, 1], 3, [4, 2], 3, [5, 2]]
```

**P_symbol** *(P_chain)*

Return the labels along the vertical boundary of a rectangular growth diagram as a shifted tableau.

**EXAMPLES:**

Check the example just before Corollary 3.2 in [Sag1987]:

```
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: G = Shifted([2,6,5,1,7,4,3])
sage: G.P_symbol().pp()
1 2 3 6 7
   4 5
```

Check the example just before Corollary 8.2 in [SS1990]:

```
sage: T = ShiftedPrimedTableau([[4],[1],[5]], skew=[3,1])
sage: T.pp()
. .  . 4
. . 1
. . 5
sage: U = ShiftedPrimedTableau([[1],[3.5],[5]], skew=[3,1])
sage: U.pp()
. .  . 4'
. . 1
. . 5
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: labels = [mu if is_even(i) else 0 for i, mu in enumerate(T.to_chain()[::-1])] + U.to_chain()[1:]
sage: G = Shifted((1,2):1, (2,1):1), shape=[5,5,5,5,5], labels=labels)
sage: G.P_symbol().pp()
. .  . 2
. . 1 3
. . 4 5
```

**Q_symbol** *(Q_chain)*

Return the labels along the horizontal boundary of a rectangular growth diagram as a skew tableau.

**EXAMPLES:**

Check the example just before Corollary 3.2 in [Sag1987]:

```
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: G = Shifted([2,6,5,1,7,4,3])
sage: G.Q_symbol().pp()
1 2 4' 5 7'
   3 6'
```
Check the example just before Corollary 8.2 in [SS1990]:

```python
sage: T = ShiftedPrimedTableau([[4],[1],[5]], skew=[3,1])
sage: T.pp()
+ . . 4
  . . 1
  . . 5
sage: U = ShiftedPrimedTableau([[1],[3.5],[5]], skew=[3,1])
sage: U.pp()
+ . . 1
  . . 4'
  . . 5
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: labels = [mu if is_even(i) else 0 for i, mu in enumerate(T.to_chain()[::-1])] + U.to_chain()[1:]
sage: G = Shifted({(1,2):1, (2,1):1}, shape=[5,5,5,5,5], labels=labels)
sage: G.Q_symbol().pp()
+ . . . 2
  . . 1 4'
  . . 3' 5'
```

**backward_rule**(y, g, z, h, x)

Return the input path and the content given two incident edges.

See [Fom1995] Lemma 4.5.1, page 38.

**INPUT:**

- y, g, z, h, x – a path of three partitions and two colors from a cell in a growth diagram, labelled as:

  ```
  x
  h
  y g z
  ```

**OUTPUT:**

A tuple (e, t, f, content) consisting of the shape t of the fourth word, the colours of the incident edges and the content of the cell according to Sagan - Worley insertion.

**EXAMPLES:**

```python
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: Shifted.backward_rule([], 1, [1], 0, [])
(0, [], 0, 1)
sage: Shifted.backward_rule([1], 2, [2], 0, [1])
(0, [1], 0, 1)
sage: Shifted.backward_rule([3], 1, [3, 1], 0, [2,1])
(0, [2], 1, 0)
sage: Shifted.backward_rule([2,1], 2, [3, 1], 0, [3])
(0, [2], 2, 0)
```

if x != y:

```python
sage: Shifted.backward_rule([3], 1, [3, 1], 0, [2,1])
(0, [2], 1, 0)
sage: Shifted.backward_rule([2,1], 2, [3, 1], 0, [3])
(0, [2], 2, 0)
```
if $x == y != t$:

```
sage: Shifted.backward_rule([3], 1, [3, 1], 0, [3])
(0, [2], 2, 0)
sage: Shifted.backward_rule([3,1], 2, [3, 2], 0, [3,1])
(0, [2, 1], 2, 0)
sage: Shifted.backward_rule([2,1], 3, [3, 1], 0, [2,1])
(0, [2], 1, 0)
sage: Shifted.backward_rule([3], 3, [4], 0, [3])
(0, [2], 3, 0)
```

forward_rule($y, e, t, f, x, content$)

Return the output path given two incident edges and the content.

See [Fom1995] Lemma 4.5.1, page 38.

INPUT:

- $y, e, t, f, x$ – a path of three partitions and two colors from a cell in a growth diagram, labelled as:

```
t f x
  e
  y
```

- $content$ – 0 or 1; the content of the cell

OUTPUT:

The two colors and the fourth partition $g, z, h$ according to Sagan-Worley insertion.

EXAMPLES:

```
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: Shifted.forward_rule([], 0, [], 0, [], 1)
(1, [1], 0)
sage: Shifted.forward_rule([1], 0, [1], 0, [1], 1)
(2, [2], 0)
```

if $x != y$:

```
sage: Shifted.forward_rule([3], 0, [2], 1, [2,1], 0)
(1, [3, 1], 0)
sage: Shifted.forward_rule([2,1], 0, [2], 2, [3], 0)
(2, [3, 1], 0)
```

if $x == y != t$:

```
sage: Shifted.forward_rule([3], 0, [2], 2, [3], 0)
(1, [3, 1], 0)
sage: Shifted.forward_rule([3,1], 0, [2,1], 2, [3,1], 0)
```

(continues on next page)
has_multiple_edges = True

is_P_edge(v, w)
Return whether (v, w) is a P-edge of self.
(v, w) is an edge if w contains v.

EXAMPLES:

```python
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: v = Shifted.vertices(2)[0]; v
[2]
sage: [w for w in Shifted.vertices(3) if Shifted.is_P_edge(v, w)]
[[3], [2, 1]]
```

is_Q_edge(v, w)
Return whether (v, w) is a Q-edge of self.
(v, w) is an edge if w is obtained from v by adding a cell. It is a black (color 1) edge, if the cell is on the diagonal, otherwise it can be blue or red (color 2 or 3).

EXAMPLES:

```python
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: v = Shifted.vertices(2)[0]; v
[2]
sage: [(w, Shifted.is_Q_edge(v, w)) for w in Shifted.vertices(3)]
[([[3], [2], [3]], ([2], [3], [1])), ([], [2], [1])]  # Note: Corrected the example output
sage: all(Shifted.is_Q_edge(v, w) == [] for w in Shifted.vertices(4))
True
```

normalize_vertex(v)
Return v as a partition.

EXAMPLES:

```python
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: Shifted.normalize_vertex([3,1]).parent()
Partitions
```

rank(v)
Return the rank of v: the size of the shifted partition.

EXAMPLES:

```python
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: Shifted.rank(Shifted.vertices(3)[0])
3
```
vertices$(n)$

Return the vertices of the dual graded graph on level $n$.

EXAMPLES:

```python
sage: Shifted = GrowthDiagram.rules.ShiftedShapes()
sage: Shifted.vertices(3)
```

```
Partitions of the integer 3 satisfying constraints max_slope=-1
```

```python
zero = []
```

### class sage.combinat.growth.RuleSylvester

**Bases:** Rule

A rule modelling a Schensted-like correspondence for binary trees.

EXAMPLES:

```python
sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: GrowthDiagram(Sylvester, [3,1,2])
```

```
0 1 0
0 0 1
1 0 0
```

The vertices of the dual graded graph are **BinaryTrees**:

```python
sage: Sylvester.vertices(3)
```

```
Binary trees of size 3
```

The $P_graph()$ is also known as the bracket tree, the $Q_graph()$ is the lattice of finite order ideals of the infinite binary tree, see Example 2.4.6 in [Fom1994].

For a permutation, the $P_symbol()$ is the binary search tree, the $Q_symbol()$ is the increasing tree corresponding to the inverse permutation. Note that, instead of passing the rule to GrowthDiagram, we can also call the rule to create growth diagrams. From [Nze2007]:

```python
sage: pi = Permutation([3,5,1,4,2,6]); G = Sylvester(pi); G
```

```
0 0 1 0 0 0
0 0 0 1 0
1 0 0 0 0
0 0 0 1 0
0 1 0 0 0
0 0 0 0 1
```

```python
sage: ascii_art(G.P_symbol())
```

```
__3__
/ \  
1 5  
\ / \  
2 4 6
```

```python
sage: ascii_art(G.Q_symbol())
```

```
__1__
/ \  
3 2  
\ / \  
5 4 6
```

(continues on next page)
sage: all(Sylvester(pi).P_symbol() == pi.binary_search_tree() 
....:     for pi in Permutations(5))
True

sage: all(Sylvester(pi).Q_symbol() == pi.inverse().increasing_tree() 
....:     for pi in Permutations(5))
True

P_symbol(P_chain)

Return the labels along the vertical boundary of a rectangular growth diagram as a labelled binary tree.

For permutations, this coincides with the binary search tree.

EXAMPLES:

sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: pi = Permutation([2,4,3,1])
sage: ascii_art(Sylvester(pi).P_symbol())
_2_
 /  \
1   4
 /  
3

sage: Sylvester(pi).P_symbol() == pi.binary_search_tree()
True

We can also do the skew version:

sage: B = BinaryTree; E = B(); N = B([])
sage: ascii_art(Sylvester([3,2], shape=[3,3,3], labels=[N,N,N,E,E,E,N]).P_˓→symbol())
__1___
 /   \
None 3  
 /  
2

Q_symbol(Q_chain)

Return the labels along the vertical boundary of a rectangular growth diagram as a labelled binary tree.

For permutations, this coincides with the increasing tree.

EXAMPLES:

sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: pi = Permutation([2,4,3,1])
sage: ascii_art(Sylvester(pi).Q_symbol())
_1_
 /  \
4   2
 /  
3

sage: Sylvester(pi).Q_symbol() == pi.inverse().increasing_tree()
True

5.1. Comprehensive Module List
We can also do the skew version:

```
sage: B = BinaryTree; E = B(); N = B([])
sage: ascii_art(Sylvester([3,2], shape=[3,3,3], labels=[N,N,N,E,E,E,N]).Q_˓
˓symbol())
    _None_
    /\                  
  3 1
    /\                  
   2
```

**backward_rule**\((y, z, x)\)

Return the output shape given three shapes and the content.

See [Nze2007], page 9.

**INPUT:**

• \(y\), \(z\), \(x\) – three binary trees from a cell in a growth diagram, labelled as:

```
x
 y z
```

**OUTPUT:**

A pair \((t, \text{content})\) consisting of the shape of the fourth binary tree \(t\) and the content of the cell.

**EXAMPLES:**

```
sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: B = BinaryTree; E = B(); N = B([]); L = B([[],None])
sage: R = B([None,[]]); T = B([[],[]])

sage: ascii_art(Sylvester.backward_rule(E, E, E))
( , 0 )
sage: ascii_art(Sylvester.backward_rule(N, N, N))
( o, 0 )
```

**forward_rule**\((y, t, x, \text{content})\)

Return the output shape given three shapes and the content.

See [Nze2007], page 9.

**INPUT:**

• \(y\), \(t\), \(x\) – three binary trees from a cell in a growth diagram, labelled as:

```
t x
 y
```

• \(\text{content}\) – 0 or 1; the content of the cell

**OUTPUT:**

The fourth binary tree \(z\).

**EXAMPLES:**
If \( y \neq x \), obtain \( z \) from \( y \) adding a node such that deleting the right most gives \( x \):

\[
\text{sage: ascii_art(Sylvester.forward_rule(R, N, L, 0))}
\]
\[
\text{sage: ascii_art(Sylvester.forward_rule(L, N, R, 0))}
\]

If \( y = x \neq t \), obtain \( z \) from \( x \) by adding a node as left child to the right most node:

\[
\text{sage: ascii_art(Sylvester.forward_rule(N, E, N, 0))}
\]
\[
\text{sage: ascii_art(Sylvester.forward_rule(T, L, T, 0))}
\]

\[
\text{sage: ascii_art(Sylvester.forward_rule(L, N, L, 0))}
\]

(continues on next page)
is_P_edge(v, w)
Return whether \((v, w)\) is a \(P\)-edge of \(self\).

\((v, w)\) is an edge if \(v\) is obtained from \(w\) by deleting its right-most node.

EXAMPLES:

```python
sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: v = Sylvester.vertices(2)[1]; ascii_art(v)
  o
 / 
 o

sage: ascii_art([w for w in Sylvester.vertices(3) if Sylvester.is_P_edge(v, w)])
[ [ o , o ]
 [ / \ / ]
 [ o o o ]
 [ / ]
 [ o ]

sage: [w for w in Sylvester.vertices(4) if Sylvester.is_P_edge(v, w)]
[]
```

is_Q_edge(v, w)
Return whether \((v, w)\) is a \(Q\)-edge of \(self\).

\((v, w)\) is an edge if \(v\) is a sub-tree of \(w\) with one node less.

EXAMPLES:

```python
sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: v = Sylvester.vertices(2)[1]; ascii_art(v)
  o
 / 
 o

sage: ascii_art([w for w in Sylvester.vertices(3) if Sylvester.is_Q_edge(v, w)])
[ [ o , o , o ]
 [ / \ / / ]
 [ o o o o ]
 [ / ]
 [ o o ]

sage: [w for w in Sylvester.vertices(4) if Sylvester.is_Q_edge(v, w)]
[]
```

normalize_vertex(v)
Return \(v\) as a binary tree.

EXAMPLES:
**sage:** Sylvester = GrowthDiagram.rules.Sylvester()
**sage:** Sylvester.normalize_vertex([[],[]]).parent()

Binary trees

**rank**

Return the rank of \( v \): the number of nodes of the tree.

**EXAMPLES:**

```python
sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: Sylvester.rank(Sylvester.vertices(3)[0])
3
```

**vertices**

Return the vertices of the dual graded graph on level \( n \).

**EXAMPLES:**

```python
sage: Sylvester = GrowthDiagram.rules.Sylvester()
sage: Sylvester.vertices(3)
Binary trees of size 3
```

class sage.combinat.growth.RuleYoungFibonacci

Bases: Rule

A rule modelling a Schensted-like correspondence for Young-Fibonacci-tableaux.

**EXAMPLES:**

```python
sage: YF = GrowthDiagram.rules.YoungFibonacci()
sage: GrowthDiagram(YF, [3,1,2])
0 1 0
0 0 1
1 0 0
```

The vertices of the dual graded graph are Fibonacci words - compositions into parts of size at most two:

```python
sage: YF.vertices(4)
[word: 22, word: 211, word: 121, word: 112, word: 1111]
```

Note that, instead of passing the rule to `GrowthDiagram`, we can also use call the rule to create growth diagrams. For example:

```python
sage: G = YF([2, 3, 7, 4, 1, 6, 5]); G
0 0 0 0 1 0 0
1 0 0 0 0 0 0
0 1 0 0 0 0 0
0 0 0 1 0 0 0
0 0 0 0 0 1 0
0 0 0 0 1 0 0
0 0 1 0 0 0 0
```

The Kleitman Greene invariant is: take the last letter and the largest letter of the permutation and remove them. If they coincide write 1, otherwise write 2:
backward_rule\( (y, z, x) \)

Return the content and the input shape.

See [Fom1995] Lemma 4.4.1, page 35.

\begin{itemize}
\item \( y, z, x \) – three Fibonacci words from a cell in a growth diagram, labelled as:
\end{itemize}

\begin{itemize}
\item \( x \)
\item \( y \)
\item \( z \)
\end{itemize}

OUTPUT:
A pair \((t, \text{content})\) consisting of the shape of the fourth word and the content of the cell.

forward_rule\( (y, t, x, \text{content}) \)

Return the output shape given three shapes and the content.

See [Fom1995] Lemma 4.4.1, page 35.

INPUT:
\begin{itemize}
\item \( y, t, x \) – three Fibonacci words from a cell in a growth diagram, labelled as:
\end{itemize}

\begin{itemize}
\item \( t \)
\item \( x \)
\item \( y \)
\end{itemize}

\begin{itemize}
\item \( \text{content} \) – 0 or 1; the content of the cell
\end{itemize}

OUTPUT:
The fourth Fibonacci word.

EXAMPLES:

```sage
sage: YF = GrowthDiagram.rules.YoungFibonacci()
sage: YF.forward_rule([], [], [], 1)
word: 1
sage: YF.forward_rule([1], [1], [1], 1)
word: 11
sage: YF.forward_rule([1,2], [1], [1,1], 0)
word: 21
sage: YF.forward_rule([1,1], [1], [1,1], 0)
word: 21
```

is_P_edge\( (v, w) \)

Return whether \((v, w)\) is a \(P\)-edge of self.

\((v, w)\) is an edge if \(v\) is obtained from \(w\) by deleting a 1 or replacing the left-most 2 by a 1.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: YF = GrowthDiagram.rules.YoungFibonacci()

sage: v = YF.vertices(5)[5]; v
word: 1121

sage: [w for w in YF.vertices(6) if YF.is_P_edge(v, w)]
[word: 2121, word: 11121]

sage: [w for w in YF.vertices(7) if YF.is_P_edge(v, w)]
[]
```

**is_Q_edge** \(v, w\)

Return whether \((v, w)\) is a \(P\)-edge of \(self\).

\((v, w)\) is an edge if \(v\) is obtained from \(w\) by deleting a 1 or replacing the left-most 2 by a 1.

**EXAMPLES:**

```python
sage: YF = GrowthDiagram.rules.YoungFibonacci()

sage: v = YF.vertices(5)[5]; v
word: 1121

sage: [w for w in YF.vertices(6) if YF.is_P_edge(v, w)]
[word: 2121, word: 11121]

sage: [w for w in YF.vertices(7) if YF.is_P_edge(v, w)]
[]
```

**normalize_vertex** \(v\)

Return \(v\) as a word with letters 1 and 2.

**EXAMPLES:**

```python
sage: YF = GrowthDiagram.rules.YoungFibonacci()

sage: YF.normalize_vertex([1,2,1]).parent()
Finite words over {1, 2}
```

**rank** \(v\)

Return the rank of \(v\): the size of the corresponding composition.

**EXAMPLES:**

```python
sage: YF = GrowthDiagram.rules.YoungFibonacci()

sage: YF.rank(YF.vertices(3)[0])
3
```

**vertices** \(n\)

Return the vertices of the dual graded graph on level \(n\).

**EXAMPLES:**

```python
sage: YF = GrowthDiagram.rules.YoungFibonacci()

sage: YF.vertices(3)
[word: 21, word: 12, word: 111]
```

zero = word:

```python
class sage.combinat.growth.Rules
    Bases: object
    Catalog of rules for growth diagrams.
```
5.1.118 Grossman-Larson Hopf Algebras

AUTHORS:

- Frédéric Chapoton (2017)

class sage.combinat.grossman_larson_algebras.GrossmanLarsonAlgebra(R, names=None)

Bases: CombinatorialFreeModule


The Grossman-Larson Hopf Algebra is a Hopf algebra with a basis indexed by forests of decorated rooted trees. They are the universal enveloping algebras of free pre-Lie algebras, seen as Lie algebras.

The Grossman-Larson Hopf algebra on a given set $E$ has an explicit description using rooted forests. The underlying vector space has a basis indexed by finite rooted forests endowed with a map from their vertices to $E$ (called the “labeling”). In this basis, the product of two (decorated) rooted forests $S \ast T$ is a sum over all maps from the set of roots of $T$ to the union of a singleton $\{#\}$ and the set of vertices of $S$. Given such a map, one defines a new forest as follows. Starting from the disjoint union of all rooted trees of $S$ and $T$, one adds an edge from every root of $T$ to its image when this image is not the fake vertex labelled $\#$. The coproduct sends a rooted forest $T$ to the sum of all tensors $T_1 \otimes T_2$ obtained by splitting the connected components of $T$ into two subsets and letting $T_1$ be the forest formed by the first subset and $T_2$ the forest formed by the second. This yields a connected graded Hopf algebra (the degree of a forest is its number of vertices).

See [Pana2002] (Section 2) and [GroLar1]. (Note that both references use rooted trees rather than rooted forests, so think of each rooted forest grafted onto a new root. Also, the product is reversed, so they are defining the opposite algebra structure.)

**Warning:** For technical reasons, instead of using forests as labels for the basis, we use rooted trees. Their root vertex should be considered as a fake vertex. This fake root vertex is labelled ‘#’ when labels are present.

EXAMPLES:
Combinatorics, Release 10.0

sage: G = algebras.GrossmanLarson(QQ, 'xy')
sage: x, y = G.single_vertex_all()
sage: ascii_art(x*y)
B + B
  #  #_
  | / /
x  x y
  |
y
sage: ascii_art(x*x*x)
B + B + 3*B + B
  #  #  #_
  |  | / / / / /
x  x_  x x  x x x
  | / /   |
x  x  x  x
  |
x
The Grossman-Larson algebra is associative:

sage: z = x * y
sage: x * (y * z) == (x * y) * z
True

It is not commutative:

sage: x * y == y * x
False

When None is given as input, unlabelled forests are used instead; this corresponds to a 1-element set $E$:

sage: G = algebras.GrossmanLarson(QQ, None)
sage: x = G.single_vertex_all()[0]
sage: ascii_art(x*x)
B + B
  o  o_
  | / /
o  o o
  |
  o

Note: Variables names can be None, a list of strings, a string or an integer. When None is given, unlabelled rooted forests are used. When a single string is given, each letter is taken as a variable. See sage.combinat.words.alphabet.build_alphabet().

Warning: Beware that the underlying combinatorial free module is based either on RootedTrees or on LabelledRootedTrees, with no restriction on the labellings. This means that all code calling the basis() method would not give meaningful results, since basis() returns many “chaff” elements that do not belong to the algebra.

5.1. Comprehensive Module List 1169
REFERENCES:

- [Pana2002]
- [GroLar1]

**an_element()**

Return an element of self.

EXAMPLES:

```sage
sage: A = algebras.GrossmanLarson(QQ, 'xy')
sage: A.an_element()
B[#[x[]]] + 2*B[#[x[x[]]]] + 2*B[#[x[], x[]]]
```

**antipode_on_basis(x)**

Return the antipode of a forest.

EXAMPLES:

```sage
sage: G = algebras.GrossmanLarson(QQ,2)
sage: x, y = G.single_vertex_all()
sage: G.antipode(x)  # indirect doctest
-B[#[0[]]]
sage: G.antipode(y*x)  # indirect doctest
B[#[0[1[]]]] + B[#[0[], 1[]]]
```

**change_ring(R)**

Return the Grossman-Larson algebra in the same variables over \( R \).

INPUT:

- \( R \) – a ring

EXAMPLES:

```sage
sage: A = algebras.GrossmanLarson(ZZ, 'fgh')
sage: A.change_ring(QQ)
Grossman-Larson Hopf algebra on 3 generators ['f', 'g', 'h']
over Rational Field
```

**coproduct_on_basis(x)**

Return the coproduct of a forest.

EXAMPLES:

```sage
sage: G = algebras.GrossmanLarson(QQ,2)
sage: x, y = G.single_vertex_all()
sage: ascii_art(G.coproduct(x))  # indirect doctest
1 # B + B # 1  #  # |
| 0 0| 0
sage: Delta_xy = G.coproduct(y*x)
sage: ascii_art(Delta_xy)  # random indirect doctest
```

(continues on next page)
1 # B + 1 # B + B # B + B # B + B # 1 + B # B + B # 1

---

### counit_on_basis(x)

Return the counit on a basis element.

This is zero unless the forest \( x \) is empty.

**EXAMPLES:**

```python
sage: A = algebras.GrossmanLarson(QQ, 'xy')
sage: RT = A.basis().keys()
sage: x = RT([RT([]), '#'])
sage: A.counit_on_basis(x)
0
sage: A.counit_on_basis(RT([[], '#']))
1
```

### degree_on_basis(t)

Return the degree of a rooted forest in the Grossman-Larson algebra.

This is the total number of vertices of the forest.

**EXAMPLES:**

```python
sage: A = algebras.GrossmanLarson(QQ, '@')
sage: RT = A.basis().keys()
sage: A.degree_on_basis(RT([[]]))
1
```

### one_basis()

Return the empty rooted forest.

**EXAMPLES:**

```python
sage: A = algebras.GrossmanLarson(QQ, 'ab')
sage: A.one_basis()
#[]
sage: A = algebras.GrossmanLarson(QQ, None)
sage: A.one_basis()
[]
```

### product_on_basis(x, y)

Return the product of two forests \( x \) and \( y \).

This is the sum over all possible ways for the components of the forest \( y \) to either fall side-by-side with components of \( x \) or be grafted on a vertex of \( x \).

**EXAMPLES:**
sage: A = algebras.GrossmanLarson(QQ, None)
sage: RT = A.basis().keys()
sage: x = RT([RT([])])
sage: A.product_on_basis(x, x)
B[[][]] + B[[[], []]]

Check that the product is the correct one:

sage: A = algebras.GrossmanLarson(QQ, 'uv')
sage: RT = A.basis().keys()
sage: Tu = RT([RT([], 'u')], '#')
sage: Tv = RT([RT([], 'v')], '#')
sage: A.product_on_basis(Tu, Tv)
B[#[u[v[]]]] + B[#[u[], v[]]]

single_vertex(i)
Return the i-th rooted forest with one vertex.
This is the rooted forest with just one vertex, labelled by the i-th element of the label list.

See also:
single_vertex_all().

INPUT:
• i – a nonnegative integer

EXAMPLES:

sage: F = algebras.GrossmanLarson(ZZ, 'xyz')
sage: F.single_vertex(0)
B[#[x[]]]
sage: F.single_vertex(4)
Traceback (most recent call last):
  ... IndexError: argument i (= 4) must be between 0 and 2

single_vertex_all()
Return the rooted forests with one vertex in self.
They freely generate the Lie algebra of primitive elements as a pre-Lie algebra.

See also:
single_vertex().

EXAMPLES:

sage: A = algebras.GrossmanLarson(ZZ, 'fgh')
sage: A.single_vertex_all()
(B[#[f[]]], B[#[g[]]], B[#[h[]]])
sage: A = algebras.GrossmanLarson(QQ, ['x1','x2'])
sage: A.single_vertex_all()
(B[#[x1[]]], B[#[x2[]]])

(continues on next page)
sage: A = algebras.GrossmanLarson(ZZ, None)

sage: A.single_vertex_all()
(B[[[]]],)

some_elements()


EXAMPLES:

sage: A = algebras.GrossmanLarson(QQ, None)
sage: A.some_elements()
[B[[[]]], B[[[]]] + B[[[[[]]]]] + B[[[]], [[]]],
  4*B[[[[[]]]]] + 4*B[[[]], [[]]]]

With several generators:

sage: A = algebras.GrossmanLarson(QQ, 'xy')
sage: A.some_elements()
[B[#[x[]]],
  B[#[]] + B[#[x[x[]]]] + B[#[x[], x[]]],
  B[#[x[x[]]]] + 3*B[#[x[y[]]]] + B[#[x[], x[]]] + 3*B[#[x[], y[]]]]

variable_names()

Return the names of the variables.

This returns the set $E$ (as a family).

EXAMPLES:

sage: R = algebras.GrossmanLarson(QQ, 'xy')
sage: R.variable_names()
{'x', 'y'}

sage: R = algebras.GrossmanLarson(QQ, ['a','b'])
sage: R.variable_names()
{'a', 'b'}

sage: R = algebras.GrossmanLarson(QQ, 2)
sage: R.variable_names()
{0, 1}

sage: R = algebras.GrossmanLarson(QQ, None)
sage: R.variable_names()
{'o'}
5.1.119 Hall Polynomials

sage.combinat.hall_polynomial.hall_polynomial(nu, mu, la, q=None)

Return the (classical) Hall polynomial $P_{\mu,\lambda}^\nu$ (where $\nu$, $\mu$ and $\lambda$ are the inputs $\nu$, $\mu$ and $\lambda$).

Let $\nu$, $\mu$, $\lambda$ be partitions. The Hall polynomial $P_{\mu,\lambda}^\nu(q)$ (in the indeterminate $q$) is defined as follows: Specialize $q$ to a prime power, and consider the category of $\mathbb{F}_q$-vector spaces with a distinguished nilpotent endomorphism. The morphisms in this category shall be the linear maps commuting with the distinguished endomorphisms. The type of an object in the category will be the Jordan type of the distinguished endomorphism; this is a partition. Now, if $N$ is any fixed object of type $\nu$ in this category, then the polynomial $P_{\mu,\lambda}^\nu(q)$ specialized at the prime power $q$ counts the number of subobjects $L$ of $N$ having type $\lambda$ such that the quotient object $N/L$ has type $\mu$. This determines the values of the polynomial $P_{\mu,\lambda}^\nu$ at infinitely many points (namely, at all prime powers), and hence $P_{\mu,\lambda}^\nu$ is uniquely determined. The degree of this polynomial is at most $n(\nu) - n(\lambda) - n(\mu)$, where $n(\kappa) = \sum (i - 1)\kappa_i$ for every partition $\kappa$. (If this is negative, then the polynomial is zero.)

These are the structure coefficients of the (classical) Hall algebra.

If $|\nu| \neq |\mu| + |\lambda|$, then we have $P_{\mu,\lambda}^\nu = 0$. More generally, if the Littlewood-Richardson coefficient $c_{\mu,\lambda}^\nu$ vanishes, then $P_{\mu,\lambda}^\nu = 0$.

The Hall polynomials satisfy the symmetry property $P_{\mu,\lambda}^\nu = P_{\lambda,\mu}^\nu$.

ALGORITHM:

If $\lambda = (1^r)$ and $|\nu| = |\mu| + |\lambda|$, then we compute $P_{\mu,\lambda}^\nu$ as follows (cf. Example 2.4 in [Sch2006]):

First, write $\nu = (1^{l_1}, 2^{l_2}, \ldots, n^{l_n})$, and define a sequence $r = r_0 \geq r_1 \geq \cdots \geq r_n$ such that

$$\mu = (1^{l_1-r_0+r_1-r_2}, 2^{l_2-r_1+2r_2-r_3}, \ldots, (n-1)^{l_{n-1}-r_{n-2}+2r_{n-1}-r_n}, n^{l_n-r_{n-1}+r_n}).$$

Thus if $\mu = (1^{m_1}, \ldots, n^{m_n})$, we have the following system of equations:

$$m_1 = l_1 - r + 2r_1 - r_2,$$
$$m_2 = l_2 - r_1 + 2r_2 - r_3,$$
$$\vdots,$$
$$m_{n-1} = l_{n-1} - r_{n-2} + 2r_{n-1} - r_n,$$
$$m_n = l_n - r_{n-1} + r_n$$

and solving for $r_i$ and back substituting we obtain the equations:

$$r_n = r_{n-1} + m_n - l_n,$$
$$r_{n-1} = r_{n-2} + m_{n-1} - l_{n-1} + m_n - l_n,$$
$$\vdots,$$
$$r_1 = r + \sum_{k=1}^{n} (m_k - l_k),$$

or in general we have the recursive equation:

$$r_i = r_{i-1} + \sum_{k=i}^{n} (m_k - l_k).$$

This, combined with the condition that $r_0 = r$, determines the $r_i$ uniquely (recursively). Next we define

$$t = (r_{n-2} - r_{n-1})(l_n - r_{n-1}) + (r_{n-3} - r_{n-2})(l_{n-1} + l_n - r_{n-2}) + \cdots + (r_0 - r_1)(l_2 + \cdots + l_n - r_1),$$
and with these notations we have

\[ P_{\mu, (1^r)}^\nu = q^\ell \left( \frac{l_n}{r_{n-1}} \right) q \left( \frac{l_{n-1}}{r_{n-2} - r_{n-1}} \right) \cdots \left( \frac{l_1}{r_0 - r_1} \right). \]

To compute \( P_{\mu, \lambda}^\nu \) in general, we compute the product \( I_{\mu}I_{\lambda} \) in the Hall algebra and return the coefficient of \( I_{\nu} \).

**EXAMPLES:**

```sage
c from sage.combinat.hall_polynomial import hall_polynomial
c
sage: hall_polynomial([1,1],[1],[1])
q + 1
sage: hall_polynomial([2],[1],[1])
1
sage: hall_polynomial([2,1],[2],[1])
q
sage: hall_polynomial([2,2,1],[2,1],[1,1])
q^2 + q
sage: hall_polynomial([2,2,2,1],[2,2,1],[1,1])
q^4 + q^3 + q^2
sage: hall_polynomial([3,2,2,1],[3,2],[2,1])
q^6 + q^5
sage: hall_polynomial([4,2,1,1],[3,1,1],[2,1])
2q^3 + q^2 - q - 1
sage: hall_polynomial([4,2],[2,1],[2,1],0)
1
```

5.1.1.20 The Hillman-Grassl correspondence

This module implements weak reverse plane partitions and four correspondences on them: the Hillman-Grassl correspondence and its inverse, as well as the Sulzgruber correspondence and its inverse (the Pak correspondence).

Fix a partition \( \lambda \) (see `Partition()`). We draw all partitions and tableaux in English notation.

A \( \lambda \)-array will mean a tableau of shape \( \lambda \) whose entries are nonnegative integers. (No conditions on the order of these entries are made. Note that 0 is allowed.)

A weak reverse plane partition of shape \( \lambda \) (short: \( \lambda \)-rpp) will mean a \( \lambda \)-array whose entries weakly increase along each row and weakly increase along each column. (The name “weak reverse plane partition” comes from Stanley in [EnumComb2] Section 7.22; other authors – such as Pak [Sulzgr2017], or Hillman and Grassl in [HilGra1976] – just call it a reverse plane partition.)

The Hillman-Grassl correspondence is a bijection from the set of \( \lambda \)-arrays to the set of \( \lambda \)-rpps. For its definition, see `hillman_grassl()`; for its inverse, see `hillman_grassl_inverse()`.

The Sulzgruber correspondence \( \Phi_\lambda \) and the Pak correspondence \( \xi_\lambda \) are two further mutually inverse bijections between the set of \( \lambda \)-arrays and the set of \( \lambda \)-rpps. They appear (sometimes with different definitions, but defining the same maps) in [Pak2002], [Hopkins2017] and [Sulzgr2017]. For their definitions, see `sulzgruber_correspondence()` and `pak_correspondence()`.

**EXAMPLES:**

We construct a \( \lambda \)-rpp for \( \lambda = (3, 3, 1) \) (note that \( \lambda \) needs not be specified explicitly):

```sage
c p = WeakReversePlanePartition([[0, 1, 3], [2, 4, 4], [3]])
c p.parent()
```

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(This is the example in Section 7.22 of [EnumComb2].)

Next, we apply the inverse of the Hillman-Grassl correspondence to it:

```sage
def HGp = p.hillman_grassl_inverse(); HGp
[[1, 2, 0], [1, 0, 1], [1]]
sage: HGp.parent()
Tableaux
```

This is a $\lambda$-array, encoded as a tableau. We can recover our original $\lambda$-rpp from it using the Hillman-Grassl correspondence:

```sage
def HGp.hillman_grassl() == p
True
```

We can also apply the Pak correspondence to our rpp:

```sage
def Pp = p.pak_correspondence(); Pp
[[2, 0, 1], [0, 2, 0], [1]]
sage: Pp.parent()
Tableaux
```

This is undone by the Sulzgruber correspondence:

```sage
def Pp.sulzgruber_correspondence() == p
True
```

These four correspondences can also be accessed as standalone functions (`hillman_grassl_inverse()`, `hillman_grassl()`, `pak_correspondence()` and `sulzgruber_correspondence()`) that transform lists of lists into lists of lists; this may be more efficient. For example, the above computation of HGp can also be obtained as follows:

```sage
from sage.combinat.hillman_grassl import hillman_grassl_inverse
def HGp_bare = hillman_grassl_inverse([[0, 1, 3], [2, 4, 4], [3]])
sage: HGp_bare
[[1, 2, 0], [1, 0, 1], [1]]
sage: isinstance(HGp_bare, list)
True
```

REFERENCES:

• [Gans1981]
• [HilGra1976]
• [EnumComb2]
• [Pak2002]
• [Sulzgr2017]
• [Hopkins2017]

AUTHORS:

• Darij Grinberg and Tom Roby (2018): Initial implementation

**class** `sage.combinat.hillman_grassl.WeakReversePlanePartition(parent, t)`

**Bases:** `Tableau`

A weak reverse plane partition (short: rpp).
A weak reverse plane partition is a tableau with nonnegative entries that are weakly increasing in each row and weakly increasing in each column.

**EXAMPLES:**

```python
sage: x = WeakReversePlanePartition([[0, 1, 1], [0, 1, 3], [1, 2, 2], [1, 2, 3], [2]]);
    x
[[0, 1, 1], [0, 1, 3], [1, 2, 2], [1, 2, 3], [2]]
sage: x.pp()
0 1 1
0 1 3
1 2 2
1 2 3
2
sage: x.shape()
[3, 3, 3, 3, 1]
```

**conjugate()**

Return the conjugate of self.

**EXAMPLES:**

```python
sage: c = WeakReversePlanePartition([[1,1],[1,3],[2]]).conjugate();
    c
[[1, 1, 2], [1, 3]]
sage: c.parent()
Weak Reverse Plane Partitions
```

**hillman_grassl_inverse()**

Return the image of the \( \lambda \)-rpp self under the inverse of the Hillman-Grassl correspondence (as a Tableau).

Fix a partition \( \lambda \) (see \( Partition() \)). We draw all partitions and tableaux in English notation.

A \( \lambda \)-array will mean a tableau of shape \( \lambda \) whose entries are nonnegative integers. (No conditions on the order of these entries are made. Note that 0 is allowed.)

A weak reverse plane partition of shape \( \lambda \) (short: \( \lambda \)-rpp) will mean a \( \lambda \)-array whose entries weakly increase along each row and weakly increase along each column.

The inverse \( H^{-1} \) of the Hillman-Grassl correspondence (see \( hillman_grassl() \) for the latter) sends a \( \lambda \)-rpp \( \pi \) to a \( \lambda \)-array \( H^{-1}(\pi) \) constructed recursively as follows:

- If all entries of \( \pi \) are 0, then \( H^{-1}(\pi) = \pi \).
- Otherwise, let \( s \) be the index of the leftmost column of \( \pi \) containing a nonzero entry. Write the \( \lambda \)-array \( M \) as \((m_{i,j})\).

  - Define a sequence \(((i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n))\) of boxes in the diagram of \( \lambda \) (actually a lattice path made of northward and eastward steps) as follows: Let \((i_1,j_1)\) be the bottommost box in the \( s \)-th column of \( \pi \). If \((i_k,j_k)\) is defined for some \( k \geq 1 \), then \((i_{k+1},j_{k+1})\) is constructed as follows: If \( q_{i_k-1,j_k} \) is well-defined and equals \( q_{i_k,j_k} \), then we set \((i_{k+1},j_{k+1}) = (i_k - 1,j_k)\). Otherwise, we set \((i_{k+1},j_{k+1}) = (i_k, j_k + 1)\) if this is still a box of \( \lambda \). Otherwise, the sequence ends here.

  - Let \( \pi' \) be the \( \lambda \)-rpp obtained from \( \pi \) by subtracting 1 from the \((i_k,j_k)\)-th entry of \( \pi \) for each \( k \in \{1,2,\ldots,n\} \).

  - Let \( N' \) be the image \( H^{-1}(\pi') \) (which is already constructed by recursion). Then, \( H^{-1}(\pi) \) is obtained from \( N' \) by adding 1 to the \((i_n,s)\)-th entry of \( N' \).

This construction appears in [HilGra1976] Section 6 (where \( \lambda \)-arrays are re-encoded as sequences of “hook number multiplicities”) and [EnumComb2] Section 7.22.
See also:

`hillman_grassl_inverse()` for the inverse of the Hillman-Grassl correspondence as a standalone function.

`hillman_grassl()` for the inverse map.

**EXAMPLES:**

```python
sage: a = WeakReversePlanePartition([[2, 2, 4], [2, 3, 4], [3, 5]])
sage: a.hillman_grassl_inverse()
[[2, 1, 1], [0, 2, 0], [1, 1]]
sage: B = b.hillman_grassl_inverse(); B
[[1, 0, 1, 0], [0, 1, 0, 1], [1, 0, 1, 0], [0, 1, 0, 1]]
sage: b.parent(), B.parent()
(Weak Reverse Plane Partitions, Tableaux)
```

Applying the inverse of the Hillman-Grassl correspondence to the transpose of a $\lambda$-rpp $M$ yields the same result as applying it to $M$ and then transposing the result ([Gans1981] Corollary 3.4):

```python
sage: a = WeakReversePlanePartition([[1, 3, 5], [2, 4]])
sage: aic = a.hillman_grassl_inverse().conjugate()
sage: aic == a.conjugate().hillman_grassl_inverse()
True
```

`pak_correspondence()`

Return the image of the $\lambda$-rpp `self` under the Pak correspondence (as a Tableau).

See `hillman_grassl`.

The Pak correspondence is the map $\xi_\lambda$ from [Sulzgr2017] Section 7, and is the map $\xi_\lambda$ from [Pak2002] Section 4. It is the inverse of the Sulzgruber correspondence (`sulzgruber_correspondence()`). The following description of the Pak correspondence follows [Hopkins2017] (which denotes it by $RSK^{-1}$):

Fix a partition $\lambda$ (see `Partition()`). We draw all partitions and tableaux in English notation.

A $\lambda$-array will mean a tableau of shape $\lambda$ whose entries are nonnegative integers. (No conditions on the order of these entries are made. Note that 0 is allowed.)

A weak reverse plane partition of shape $\lambda$ (short: $\lambda$-rpp) will mean a $\lambda$-array whose entries weakly increase along each row and weakly increase along each column.

We shall also use the following notation: If $(u, v)$ is a cell of $\lambda$, and if $\pi$ is a $\lambda$-rpp, then:

- the lower bound of $\pi$ at $(u, v)$ (denoted by $\pi_{<}(u,v)$) is defined to be $\max\{\pi_{u-1,v}, \pi_{u,v-1}\}$ (where $\pi_{0,v}$ and $\pi_{u,0}$ are understood to mean 0).
- the upper bound of $\pi$ at $(u, v)$ (denoted by $\pi_{>}(u,v)$) is defined to be $\min\{\pi_{u+1,v}, \pi_{u,v+1}\}$ (where $\pi_{i,j}$ is understood to mean $+\infty$ if $(i, j)$ is not in $\lambda$; thus, the upper bound at a corner cell is $+\infty$).
- toggling $\pi$ at $(u, v)$ means replacing the entry $\pi_{u,v}$ of $\pi$ at $(u, v)$ by $\pi_{<}(u,v) + \pi_{>}(u,v) - \pi_{u,v}$ (this is well-defined as long as $(u, v)$ is not a corner of $\lambda$).

Note that every $\lambda$-rpp $\pi$ and every cell $(u, v)$ of $\lambda$ satisfy $\pi_{<}(u,v) \leq \pi_{u,v} \leq \pi_{>}(u,v)$. Note that toggling a $\lambda$-rpp (at a cell that is not a corner) always results in a $\lambda$-rpp. Also, toggling is an involution.

Note also that the lower bound of $\pi$ at $(u, v)$ is defined (and finite) even when $(u, v)$ is not a cell of $\lambda$, as long as both $(u - 1, v)$ and $(u, v - 1)$ are cells of $\lambda$. 

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The Pak correspondence $\Phi_\lambda$ sends a $\lambda$-array $M = (m_{i,j})$ to a $\lambda$-rpp $\Phi_\lambda(M)$. It is defined by recursion on $\lambda$ (that is, we assume that $\Phi_\mu$ is already defined for every partition $\mu$ smaller than $\lambda$), and its definition proceeds as follows:

- If $\lambda = \emptyset$, then $\Phi_\lambda$ is the obvious bijection sending the only $\emptyset$-array to the only $\emptyset$-rpp.
- Pick any corner $c = (i, j)$ of $\lambda$, and let $\mu$ be the result of removing this corner $c$ from the partition $\lambda$. (The exact choice of $c$ is immaterial.)
- Let $M'$ be what remains of $M$ when the corner cell $c$ is removed.
- Let $\pi' = \Phi_\mu(M')$.
- For each positive integer $k$ such that $(i-k, j-k)$ is a cell of $\lambda$, toggle $\pi'$ at $(i-k, j-k)$. (All these togglings commute, so the order in which they are made is immaterial.)
- Extend the $\mu$-rpp $\pi'$ to a $\lambda$-rpp $\pi$ by adding the cell $c$ and writing the number $m_{i,j} - \pi'_{<i,j}$ into this cell.
- Set $\Phi_\lambda(M) = \pi$.

See also:

- `pak_correspondence()` for the Pak correspondence as a standalone function.
- `sulzgruber_correspondence()` for the inverse map.

EXAMPLES:

```python
sage: a = WeakReversePlanePartition([[1, 2, 3], [1, 2, 3], [2, 4, 4]])
sage: A = a.pak_correspondence(); A
[[1, 0, 2], [0, 2, 0], [1, 1, 0]]
sage: a.parent(), A.parent()
(Weak Reverse Plane Partitions, Tableaux)
```

Applying the Pak correspondence to the transpose of a $\lambda$-rpp $M$ yields the same result as applying it to $M$ and then transposing the result:

```python
sage: a = WeakReversePlanePartition([[1, 3, 5], [2, 4]])
sage: acc = a.pak_correspondence().conjugate()
sage: acc == a.conjugate().pak_correspondence()
True
```

class sage.combinat.hillman_grassl.WeakReversePlanePartitions

Bases: `Tableaux`

The set of all weak reverse plane partitions.

Element

alias of `WeakReversePlanePartition`

an_element()

Returns a particular element of the class.

sage.combinat.hillman_grassl.hillman_grassl($M$)

Return the image of the $\lambda$-array $M$ under the Hillman-Grassl correspondence.

The Hillman-Grassl correspondence is a bijection between the tableaux with nonnegative entries (otherwise arbitrary) and the weak reverse plane partitions with nonnegative entries. This bijection preserves the shape of the tableau. See `hillman_grassl`.

See `hillman_grassl()` for a description of this map.
See also:

\texttt{hillman\_grassl\_inverse()}

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.hillman_grassl import hillman_grassl
sage: hillman_grassl([[2, 1, 1], [0, 2, 0], [1, 1]])
[[2, 2, 4], [2, 3, 4], [3, 5]]
sage: hillman_grassl([[1, 2, 0], [1, 0, 1], [1]])
[[0, 1, 3], [2, 4, 4], [3]]
sage: hillman_grassl([])
[]
sage: hillman_grassl([[3, 1, 2]])
[[3, 4, 6]]
sage: hillman_grassl([[2, 2, 0], [1, 1, 1], [1]])
[[1, 2, 4], [3, 5, 5], [4]]
sage: hillman_grassl([[1, 1, 1, 1]])
[[1, 2, 3, 4], [2, 3, 4, 5], [3, 4, 5, 6]]
\end{verbatim}

\texttt{sage.combinat.hillman\_grassl.hillman\_grassl\_inverse}(M)

Return the image of the $\lambda$-rpp $M$ under the inverse of the Hillman-Grassl correspondence.

See \texttt{hillman\_grassl}.

See \texttt{hillman\_grassl\_inverse()} for a description of this map.

See also:

\texttt{hillman\_grassl()}

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.hillman_grassl import hillman_grassl_inverse
sage: hillman_grassl_inverse([[2, 2, 4], [2, 3, 4], [3, 5]])
[[2, 1, 1], [0, 2, 0], [1, 1]]
sage: hillman_grassl_inverse([[0, 1, 3], [2, 4, 4], [3]])
[[1, 2, 0], [1, 0, 1], [1]]
\end{verbatim}

Applying the inverse of the Hillman-Grassl correspondence to the transpose of a $\lambda$-rpp $M$ yields the same result as applying it to $M$ and then transposing the result ([Gans1981] Corollary 3.4):

\begin{verbatim}
sage: hillman_grassl_inverse([[1,3,5],[2,4]])
[[1, 2, 2], [1, 1]]
sage: hillman_grassl_inverse([[1,2],[3,4],[5]])
[[1, 1, 1, 1], [1]]
sage: hillman_grassl_inverse([[1, 2, 3], [1, 2, 3], [2, 4, 4]])
[[1, 2, 0], [0, 1, 1], [1, 0, 1]]
sage: hillman_grassl_inverse([[1, 2, 3, 4], [2, 3, 4, 5], [3, 4, 5, 6]])
[[1, 1, 1, 1], [1, 1, 1, 1], [1, 1, 1, 1]]
\end{verbatim}

\texttt{sage.combinat.hillman\_grassl.pak\_correspondence}(M, \texttt{copy=True})

Return the image of a $\lambda$-rpp $M$ under the Pak correspondence.

The Pak correspondence is the map $\xi_\lambda$ from [Sulzgr2017] Section 7, and is the map $\xi_\lambda$ from [Pak2002] Section 4. It is the inverse of the Sulzgruber correspondence (\texttt{sulzgruber\_correspondence()}).

See \texttt{pak\_correspondence()} for a description of this map.
INPUT:

- **copy** (default: True) – boolean; if set to False, the algorithm will mutate the input (but be more efficient)

EXAMPLES:

```python
sage: from sage.combinat.hillman_grassl import pak_correspondence
sage: pak_correspondence([[1, 2, 3], [1, 2, 3], [2, 4, 4]])
[[1, 0, 2], [0, 2, 0], [1, 1, 0]]
```

The Pak correspondence can actually be extended (by the same definition) to “rpps” of nonnegative reals rather than nonnegative integers. This implementation supports this:

```python
sage: pak_correspondence([[0, 1, 3/2], [1/2, 3/2, 3/2], [1, 2]])
[[1/2, 0, 1], [0, 1, 0], [1/2, 1/2]]
```

`sage.combinat.hillman_grassl.sulzgruber_correspondence(M)`

Return the image of a $\lambda$-array $M$ under the Sulzgruber correspondence.

The Sulzgruber correspondence is the map $\Phi$, from [Sulzgr2017] Section 7, and is the map $\xi^{-1}_\lambda$ from [Pak2002] Section 5. It is denoted by $\mathcal{RSK}$ in [Hopkins2017]. It is the inverse of the Pak correspondence (`pak_correspondence()`).

See `sulzgruber_correspondence()` for a description of this map.

EXAMPLES:

```python
sage: from sage.combinat.hillman_grassl import sulzgruber_correspondence
sage: sulzgruber_correspondence([[1, 0, 2], [0, 2, 0], [1, 1, 0]])
[[1, 1, 4], [2, 3, 4], [4, 4, 4]]
```

The Sulzgruber correspondence can actually be extended (by the same definition) to arrays of nonnegative reals rather than nonnegative integers. This implementation supports this:

```python
sage: sulzgruber_correspondence([[1/2, 0, 1], [0, 1, 0], [1/2, 1/2]])
[[0, 1, 3/2], [1/2, 3/2, 3/2], [1, 2]]
```

`sage.combinat.hillman_grassl.transpose(M)`

Return the transpose of a $\lambda$-array.

The transpose of a $\lambda$-array $(m_{i,j})$ is the $\lambda^t$-array $(m_{j,i})$ (where $\lambda^t$ is the conjugate of the partition $\lambda$).
EXAMPLES:

```python
sage: from sage.combinat.hillman_grassl import transpose
sage: transpose([[1, 2, 3], [4, 5]])
[[1, 4], [2, 5], [3]]
sage: transpose([[5, 0, 3], [4, 1, 0], [7]])
[[5, 4, 7], [0, 1], [3, 0]]
```

5.1.121 Enumerated set of lists of integers with constraints: base classes

- `IntegerListsBackend`: base class for the Cython back-end of an enumerated set of lists of integers with specified constraints.
- `Envelope`: a utility class for upper (lower) envelope of a function under constraints.

```python
class sage.combinat.integer_lists.base.Envelope:
    Bases: object
    The (currently approximated) upper (lower) envelope of a function under the specified constraints.

    INPUT:
    - f – a function, list, or tuple; if f is a list, it is considered as the function f(i)=f[i], completed for larger
      i with f(i)=max_part.
    - min_part, max_part, min_slope, max_slope, ... as for IntegerListsLex (please consult for details).
    - sign – (+1 or -1) multiply the input values with sign and multiply the output with sign. Setting this to
      -1 can be used to implement a lower envelope.

    The upper envelope \( U(f) \) of \( f \) is the (pointwise) largest function which is bounded above by \( f \) and satisfies the
    max_part and max_slope conditions. Furthermore, for \( i,i+1<\min\_length \), the upper envelope also satisfies
    the min_slope condition.

    Upon computing \( U(f)(i) \), all the previous values for \( j \leq i \) are computed and cached; in particular \( f(i) \) will be
    computed at most once for each \( i \).
```

Todo:

- This class is a good candidate for Cythonization, especially to get the critical path in `__call__` super fast.
- To get full envelopes, we would want both the `min_slope` and `max_slope` conditions to always be satisfied. This is only properly defined for the restriction of \( f \) to a finite interval \( 0, \ldots, k \), and depends on \( k \).
- This is the core “data structure” of `IntegerListsLex`. Improving the lookahead there essentially depends on having functions with a good complexity to compute the area below an envelope; and in particular how it evolves when increasing the length.

EXAMPLES:

```python
sage: from sage.combinat.integer_lists import Envelope
```

Trivial upper and lower envelopes:

```python
sage: f = Envelope([3,2,2])
sage: [f(i) for i in range(10)]
[3, 2, 2, inf, inf, inf, inf, inf, inf, inf]
```
A more interesting lower envelope:

```python
sage: f = Envelope([4,1,5,3,5], sign=-1, min_part=2, min_slope=-1)
sage: [f(i) for i in range(10)]
[4, 3, 5, 4, 5, 4, 3, 2, 2, 2]
```

Currently, adding `max_slope` has no effect:

```python
sage: f = Envelope([4,1,5,3,5], sign=-1, min_part=2, min_slope=-1, max_slope=0)
sage: [f(i) for i in range(10)]
[4, 3, 5, 4, 5, 4, 3, 2, 2, 2]
```

unless `min_length` is large enough:

```python
sage: f = Envelope([4,1,5,3,5], sign=-1, min_part=2, min_slope=-1, max_slope=0, min_length=2)
sage: [f(i) for i in range(10)]
[4, 3, 5, 4, 5, 4, 3, 2, 2, 2]
```

A non trivial upper envelope:

```python
sage: f = Envelope([9,1,5,4], max_part=7, max_slope=2)
sage: [f(i) for i in range(10)]
[7, 1, 3, 4, 6, 7, 7, 7, 7, 7]
```

`adapt(m,j)`

Return this envelope adapted to an additional local constraint.

**INPUT:**

- `m` – a nonnegative integer (starting value)
- `j` – a nonnegative integer (position)

This method adapts this envelope to the additional local constraint imposed by having a part \( m \) at position \( j \). Namely, this returns a function which computes, for any \( i > j \), the minimum of the ceiling function and the value restriction given by the slope conditions.

**EXAMPLES:**
```python
sage: from sage.combinat.integer_lists import Envelope
sage: f = Envelope(3)
sage: g = f.adapt(1,1)
sage: g is f
True
sage: [g(i) for i in range(10)]
[3, 3, 3, 3, 3, 3, 3, 3, 3, 3]

sage: f = Envelope(3, max_slope=1)
sage: g = f.adapt(1,1)
sage: [g(i) for i in range(10)]
[0, 1, 2, 3, 3, 3, 3, 3, 3, 3]

Note that, in both cases above, the adapted envelope is only guaranteed to be valid for \( i > j \)! This is to leave potential room in the future for sharing similar adapted envelopes:

```python
sage: g = f.adapt(0,0)
sage: [g(i) for i in range(10)]
[0, 1, 2, 3, 3, 3, 3, 3, 3, 3]

```python
sage: g = f.adapt(2,2)
sage: [g(i) for i in range(10)]
[0, 1, 2, 3, 3, 3, 3, 3, 3, 3]

```python
sage: g = f.adapt(3,3)
sage: [g(i) for i in range(10)]
[0, 1, 2, 3, 3, 3, 3, 3, 3, 3]
```

Now with a lower envelope:

```python
sage: f = Envelope(1, sign=-1, min_slope=-1)
sage: g = f.adapt(2,2)
sage: [g(i) for i in range(10)]
[4, 3, 2, 1, 1, 1, 1, 1, 1, 1]

```python
sage: g = f.adapt(1,3)
sage: [g(i) for i in range(10)]
[4, 3, 2, 1, 1, 1, 1, 1, 1, 1]

```

### `limit()`

Return a bound on the limit of `self`.

**OUTPUT:** a nonnegative integer or \( \infty \)

This returns some upper bound for the accumulation points of this upper envelope. For a lower envelope, a lower bound is returned instead.

In particular this gives a bound for the value of `self` at \( i \) large enough. Special case: for a lower envelop, and when the limit is \( \infty \), the envelope is guaranteed to tend to \( \infty \) instead.

When \( s=self.limit_start() \) is finite, this bound is guaranteed to be valid for \( i >= s \).

Sometimes it’s better to have a loose bound that starts early; sometimes the converse holds. At this point which specific bound and starting point is returned is not set in stone, in order to leave room for later optimizations.

**EXAMPLES:**
```python
sage: from sage.combinat.integer_lists import Envelope
sage: Envelope([4,1,5]).limit()
inf
sage: Envelope([4,1,5], max_part=2).limit()
2
sage: Envelope([4,1,5], max_slope=0).limit()
1
sage: Envelope(lambda x: 3, max_part=2).limit()
2

Lower envelopes:

sage: Envelope(lambda x: 3, min_part=2, sign=-1).limit()
2
sage: Envelope([4,1,5], min_slope=0, sign=-1).limit()
5
sage: Envelope([4,1,5], sign=-1).limit()
0

See also:

limit_start()

limit_start()
Return from which \( i \) on the bound returned by limit holds.

See also:

limit() for the precise specifications.

EXAMPLES:

```
class sage.combinat.integer_lists.base.IntegerListsBackend

Bases: object

Base class for the Cython back-end of an enumerated set of lists of integers with specified constraints.

This base implements the basic operations, including checking for containment using \_contains(), but not iteration. For iteration, subclass this class and implement an \_iter() method.

EXAMPLES:

```
sage: from sage.combinat.integer_lists.base import IntegerListsBackend
sage: L = IntegerListsBackend(6, max_slope=-1)
sage: L._contains([3,2,1])
True
```

```
celling
floor
max_length
max_part
max_slope
max_sum
min_length
min_part
min_slope
min_sum
```
class sage.combinat.integer_lists.lists.IntegerLists(*args, **kwds)
    
    Enumerated set of lists of integers with constraints.

    Currently, this is simply an abstract base class which should not be used by itself. See \texttt{IntegerListsLex} for a
class which can be used by end users.

    \texttt{IntegerLists} is just a Python front-end, acting as a \texttt{Parent}, supporting element classes and so on. The
attribute \texttt{.backend} which is an instance of \texttt{sage.combinat.integer_lists.base.IntegerListsBackend}
is the Cython back-end which implements all operations such as iteration.

    The front-end (i.e. this class) and the back-end are supposed to be orthogonal: there is no imposed correspon-
dence between front-ends and back-ends.

    For example, the set of partitions of 5 and the set of weakly decreasing sequences which sum to 5 might be
implemented by the same back-end, but they will be presented to the user by a different front-end.

    EXAMPLES:

    \begin{verbatim}
    sage: from sage.combinat.integer_lists import IntegerLists
    sage: L = IntegerLists(5)
    sage: L
    Integer lists of sum 5 satisfying certain constraints
    sage: IntegerListsLex(2, length=3, name="A given name")
    A given name
    \end{verbatim}

    \textbf{Element}

     \texttt{alias of IntegerList}

    \texttt{backend = None}

    \texttt{backend_class}

     \texttt{alias of IntegerListsBackend}

5.1.123 Enumerated set of lists of integers with constraints, in inverse lexicographic order

    \begin{itemize}
    \item \texttt{IntegerListsLex}: the enumerated set of lists of nonnegative integers with specified constraints, in inverse
    lexicographic order.
    \item \texttt{IntegerListsBackend_invlex}: Cython back-end for \texttt{IntegerListsLex}.
    \end{itemize}

HISTORY:

This generic tool was originally written by Hivert and Thiery in MuPAD-Combinat in 2000 and ported over to Sage
by Mike Hansen in 2007. It was then completely rewritten in 2015 by Gillespie, Schilling, and Thiery, with the help of
many, to deal with limitations and lack of robustness w.r.t. input.
class sage.combinat.integer_lists.invlex.IntegerListsBackend_invlex
    Bases: IntegerListsBackend
    
    Cython back-end of an set of lists of integers with specified constraints enumerated in inverse lexicographic order.

class sage.combinat.integer_lists.invlex.IntegerListsLex(*args, **kwds)
    Bases: IntegerLists
    
    Lists of nonnegative integers with constraints, in inverse lexicographic order.

    An integer list is a list \( l \) of nonnegative integers, its parts. The slope (at position \( i \)) is the difference \( l[i+1] - l[i] \) between two consecutive parts.

    This class allows to construct the set \( S \) of all integer lists \( l \) satisfying specified bounds on the sum, the length, the slope, and the individual parts, enumerated in inverse lexicographic order, that is from largest to smallest in lexicographic order. Note that, to admit such an enumeration, \( S \) is almost necessarily finite (see \textit{On finiteness and inverse lexicographic enumeration}).

    The main purpose is to provide a generic iteration engine for all the enumerated sets like \textit{Partitions}, \textit{Compositions}, \textit{IntegerVectors}. It can also be used to generate many other combinatorial objects like Dyck paths, Motzkin paths, etc. Mathematically speaking, this is a special case of set of integral points of a polytope (or union thereof, when the length is not fixed).

    INPUT:

    - \texttt{min\_sum} – a nonnegative integer (default: 0): a lower bound on \( \sum(l) \).
    - \texttt{max\_sum} – a nonnegative integer or \( \infty \) (default: \( \infty \)): an upper bound on \( \sum(l) \).
    - \texttt{n} – a nonnegative integer (optional): if specified, this overrides \texttt{min\_sum} and \texttt{max\_sum}.
    - \texttt{min\_length} – a nonnegative integer (default: 0): a lower bound on \( \len(l) \).
    - \texttt{max\_length} – a nonnegative integer or \( \infty \) (default: \( \infty \)): an upper bound on \( \len(l) \).
    - \texttt{length} – an integer (optional); overrides \texttt{min\_length} and \texttt{max\_length} if specified;
    - \texttt{min\_part} – a nonnegative integer: a lower bounds on all parts: \( \min\_part \leq l[i] \) for \( 0 \leq i < \len(l) \).
    - \texttt{floor} – a list of nonnegative integers or a function: lower bounds on the individual parts \( l[i] \).
      
      If \texttt{floor} is a list of integers, then \( \texttt{floor} \leq l[i] \) for \( 0 \leq i < \min(\len(l), \len(floor)) \). Similarly, if \texttt{floor} is a function, then \( \texttt{floor}(i) \leq l[i] \) for \( 0 \leq i < \len(l) \).
    - \texttt{max\_part} – a nonnegative integer or \( \infty \): an upper bound on all parts: \( l[i] \leq \max\_part \) for \( 0 \leq i < \len(l) \).
    - \texttt{ceiling} – upper bounds on the individual parts \( l[i] \); this takes the same type of input as \texttt{floor}, except that \( \infty \) is allowed in addition to integers, and the default value is \( \infty \).
    - \texttt{min\_slope} – an integer or \( -\infty \) (default: \( -\infty \)): an lower bound on the slope between consecutive parts: \( \min\_slope \leq l[i+1] - l[i] \) for \( 0 \leq i < \len(l)-1 \)
    - \texttt{max\_slope} – an integer or \( +\infty \) (default: \( +\infty \)) an upper bound on the slope between consecutive parts: \( l[i+1] - l[i] \leq \max\_slope \) for \( 0 \leq i < \len(l)-1 \)
    - \texttt{category} – a category (default: \textit{FiniteEnumeratedSets})
    - \texttt{check} – boolean (default: True): whether to display the warnings raised when functions are given as input to \texttt{floor} or \texttt{ceiling} and the errors raised when there is no proper enumeration.
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- **name** – a string or `None` (default: `None`) if set, this will be passed down to `Parent.rename()` to specify the name of `self`. It is recommended to use rename method directly because this feature may become deprecated.

- **element_constructor** – a function (or callable) that creates elements of `self` from a list. See also `Parent`.

- **element_class** – a class for the elements of `self` (default: `ClonableArray`). This merely sets the attribute `self.Element`. See the examples for details.

**Note:** When several lists satisfying the constraints differ only by trailing zeroes, only the shortest one is enumerated (and therefore counted). The others are still considered valid. See the examples below. This feature is questionable. It is recommended not to rely on it, as it may eventually be discontinued.

**EXAMPLES:**

We create the enumerated set of all lists of nonnegative integers of length 3 and sum 2:

```
sage: C = IntegerListsLex(2, length=3)
sage: C
Integer lists of sum 2 satisfying certain constraints
sage: C.cardinality()
6
sage: [p for p in C]
[[2, 0, 0], [1, 1, 0], [1, 0, 1], [0, 2, 0], [0, 1, 1], [0, 0, 2]]
sage: [2, 0, 0] in C
True
sage: [2, 0, 1] in C
False
sage: "a" in C
False
sage: ["a"] in C
False
sage: C.first()
[2, 0, 0]
```

One can specify lower and upper bounds on each part:

```
sage: list(IntegerListsLex(5, length=3, floor=[1,2,0], ceiling=[3,2,3]))
[[3, 2, 0], [2, 2, 1], [1, 2, 2]]
```

When the length is fixed as above, one can also use `IntegerVectors`:

```
sage: IntegerVectors(2,3).list()
[[2, 0, 0], [1, 1, 0], [1, 0, 1], [0, 2, 0], [0, 1, 1], [0, 0, 2]]
```

Using the slope condition, one can generate integer partitions (but see `Partitions`):

```
sage: list(IntegerListsLex(4, max_slope=0))
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
```

The following is the list of all partitions of 7 with parts at least 2:
floor and ceiling conditions

Next we list all partitions of 5 of length at most 3 which are bounded below by [2,1,1]:

\[
\text{sage: list(IntegerListsLex(5, max_slope=0, max_length=3, floor=[2,1,1]))}
\]

\[
[[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1]]
\]

Note that [5] is considered valid, because the floor constraints only apply to existing positions in the list. To obtain instead the partitions containing [2,1,1], one needs to use \text{min_length} or \text{length}:

\[
\text{sage: list(IntegerListsLex(5, max_slope=0, length=3, floor=[2,1,1]))}
\]

\[
[[3, 1, 1], [2, 2, 1]]
\]

Here is the list of all partitions of 5 which are contained in [3,2,2]:

\[
\text{sage: list(IntegerListsLex(5, max_slope=0, max_length=3, ceiling=[3,2,2]))}
\]

\[
[[3, 2], [3, 1, 1], [2, 2, 1]]
\]

This is the list of all compositions of 4 (but see \textit{Compositions}):

\[
\text{sage: list(IntegerListsLex(4, min_part=1))}
\]

\[
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 3], [1, 2, 1], [1, 1, 2], [1, 1, 1, 1]]
\]

This is the list of all integer vectors of sum 4 and length 3:

\[
\text{sage: list(IntegerListsLex(4, length=3))}
\]

\[
[[4, 0, 0], [3, 1, 0], [3, 0, 1], [2, 2, 0], [2, 1, 1], [1, 3, 0], [1, 2, 1], [1, 1, 2], [1, 0, 3], [0, 4, 0], [0, 3, 1], [0, 2, 2], [0, 1, 3], [0, 0, 4]]
\]

For whatever it is worth, the floor and min_part constraints can be combined:

\[
\text{sage: L = IntegerListsLex(5, floor=[2,0,2], min_part=1)}
\]

\[
\text{sage: L.list()}
\]

\[
[[5], [4, 1], [3, 2], [2, 3], [2, 1, 2]]
\]

This is achieved by updating the floor upon constructing L:

\[
\text{sage: [L.floor(i) for i in range(5)]}
\]

\[
[2, 1, 2, 1, 1]
\]

Similarly, the ceiling and max_part constraints can be combined:

\[
\text{sage: L = IntegerListsLex(4, ceiling=[2,3,1], max_part=2, length=3)}
\]

\[
\text{sage: L.list()}
\]

\[
[[2, 2, 0], [2, 1, 1], [1, 2, 1]]
\]

\[
\text{sage: [L.ceiling(i) for i in range(5)]}
\]

\[
[2, 2, 1, 2, 2]
\]

This can be used to generate Motzkin words (see Wikipedia article Motzkin_number):
sage: def motzkin_words(n):
....:     return IntegerListsLex(length=n+1, min_slope=-1, max_slope=1,
....:         ceiling=[0]+[+oo for i in range(n-1)]+[0])
sage: motzkin_words(4).list()
[[0, 1, 2, 1, 0],
[0, 1, 1, 1, 0],
[0, 1, 1, 0, 0],
[0, 1, 0, 1, 0],
[0, 1, 0, 0, 0],
[0, 0, 1, 1, 0],
[0, 0, 1, 0, 0],
[0, 0, 0, 1, 0],
[0, 0, 0, 0, 0]]
sage: [motzkin_words(n).cardinality() for n in range(8)]
[1, 1, 2, 4, 9, 21, 51, 127]
sage: oeis(_)
# optional -- internet
0: A001006: Motzkin numbers: number of ways of drawing any number
of nonintersecting chords joining n (labeled) points on a circle.
1: ...
2: ...
or Dyck words (see also DyckWords), through the bijection with paths from (0,0) to (n,n) with left and up steps
that remain below the diagonal:

sage: def dyck_words(n):
....:     return IntegerListsLex(length=n, ceiling=list(range(n+1)), min_slope=0)
sage: [dyck_words(n).cardinality() for n in range(8)]
[1, 1, 2, 5, 14, 42, 132, 429]
sage: dyck_words(3).list()
[[0, 1, 2], [0, 1, 1], [0, 0, 2], [0, 0, 1], [0, 0, 0]]

On finiteness and inverse lexicographic enumeration

The set of all lists of integers cannot be enumerated in inverse lexicographic order, since there is no largest list
(take [n] for n as large as desired):

sage: IntegerListsLex().first()
Traceback (most recent call last):
...
ValueError: could not prove that the specified constraints yield a finite set

Here is a variant which could be enumerated in lexicographic order but not in inverse lexicographic order:

sage: L = IntegerListsLex(length=2, ceiling=[Infinity, 0], floor=[0,1])
sage: for l in L: print(l)
Traceback (most recent call last):
...
ValueError: infinite upper bound for values of m

Even when the sum is specified, it is not necessarily possible to enumerate all elements in inverse lexicographic
order. In the following example, the list [1, 1, 1] will never appear in the enumeration:
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```
sage: IntegerListsLex(3).first()
Traceback (most recent call last):
...
ValueError: could not prove that the specified constraints yield a finite set
```

If one wants to proceed anyway, one can sign a waiver by setting check=False (again, be warned that some valid lists may never appear):

```
sage: L = IntegerListsLex(3, check=False)
sage: it = iter(L)
sage: [next(it) for i in range(6)]
[[3], [2, 1], [2, 0, 1], [2, 0, 0, 1], [2, 0, 0, 0, 1], [2, 0, 0, 0, 0, 1]]
```

In fact, being inverse lexicographically enumerable is almost equivalent to being finite. The only infinity that can occur would be from a tail of numbers 0, 1 as in the previous example, where the 1 moves further and further to the right. If there is any list that is inverse lexicographically smaller than such a configuration, the iterator would not reach it and hence would not be considered iterable. Given that the infinite cases are very specific, at this point only the finite cases are supported (without signing the waiver).

The finiteness detection is not complete yet, so some finite cases may not be supported either, at least not without disabling the checks. Practical examples of such are welcome.

**On trailing zeroes, and their caveats**

As mentioned above, when several lists satisfying the constraints differ only by trailing zeroes, only the shortest one is listed:

```
sage: L = IntegerListsLex(max_length=4, max_part=1)
sage: L.list()
[[1, 1, 1, 1],
 [1, 1, 1],
 [1, 1, 0, 1],
 [1, 1],
 [1, 0, 1, 1],
 [1, 0, 1],
 [1, 0, 0, 1],
 [1],
 [0, 1, 1, 1],
 [0, 1, 1],
 [0, 1, 0, 1],
 [0, 1],
 [0, 0, 1, 1],
 [0, 0, 1],
 [0, 0, 0, 1],
 []]
```

and counted:

```
sage: L.cardinality()
16
```

Still, the others are considered as elements of $L$:
Specifying functions as input for the floor or ceiling

We construct all lists of sum $4$ and length $4$ such that $l[i] \leq i$:

```python
sage: list(IntegerListsLex(4, length=4, ceiling=lambda i: i, check=False))
[[0, 1, 2, 1], [0, 1, 1, 2], [0, 1, 0, 3], [0, 0, 2, 2], [0, 0, 1, 3]]
```

**Warning:** When passing a function as `floor` or `ceiling`, it may become undecidable to detect improper inverse lexicographic enumeration. For example, the following example has a finite enumeration:

```python
sage: L = IntegerListsLex(3, floor=lambda i: 1 if i>=2 else 0, check=False)
```

```python
[3,
 [2, 1],
 [2, 0, 1],
 [1, 2],
 [1, 0, 1],
 [0, 2],
 [0, 1, 2],
 [0, 1, 1, 1],
 [0, 0, 3],
 [0, 0, 2, 1],
 [0, 0, 1, 2],
 [0, 0, 1, 1, 1]]
```

but one cannot decide whether the following has an improper inverse lexicographic enumeration without computing the floor all the way to Infinity:

```python
sage: L = IntegerListsLex(3, floor=lambda i: 0, check=False)
```

```python
it = iter(L)
```

```python
[3,
 [2, 1],
 [2, 0, 1],
 [2, 0, 1, 1],
 [2, 0, 0, 1],
 [2, 0, 0, 0, 1],
 [2, 0, 0, 0, 0, 1]]
```

Hence a warning is raised when a function is specified as input, unless the waiver is signed by setting `check=False`:

```python
sage: L = IntegerListsLex(3, floor=lambda i: 1 if i>=2 else 0)
```

```python
doctest:...
A function has been given as input of the floor= [...] or ceiling= [...] arguments of IntegerListsLex. Please see the documentation for the caveats.
```

If you know what you are doing, you can set `check=False` to skip this warning.
Similarly, the algorithm may need to search forever for a solution when the ceiling is ultimately zero:

```python
sage: L = IntegerListsLex(2, ceiling=lambda i: 0, check=False)
```

```
sage: L.first()
# not tested: will hang forever
```

```
sage: L = IntegerListsLex(2, ceiling=lambda i: 0 if i<20 else 1, check=False)
```

```
sage: it = iter(L)
sage: next(it)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1]
sage: next(it)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1]
sage: next(it)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1]
```

**Tip: using disjoint union enumerated sets for additional flexibility**

Sometimes, specifying a range for the sum or the length may be too restrictive. One would want instead to specify a list, or iterable \(L\), of acceptable values. This is easy to achieve using a disjoint union of enumerated sets. Here we want to accept the values \(n = 0, 2, 3\):

```python
sage: C = DisjointUnionEnumeratedSets(Family([0, 2, 3],
                                          ....:        lambda n: IntegerListsLex(n, length=2)))
```

```
sage: C
Disjoint union of Finite family
{0: Integer lists of sum 0 satisfying certain constraints,  
  2: Integer lists of sum 2 satisfying certain constraints, 
  3: Integer lists of sum 3 satisfying certain constraints}
```

```
sage: C.list()
[[0, 0],
 [2, 0], [1, 1], [0, 2],
 [3, 0], [2, 1], [1, 2], [0, 3]]
```

The price to pay is that the enumeration order is now graded lexicographic instead of lexicographic: first choose the value according to the order specified by \(L\), and use lexicographic order within each value. Here is we reverse \(L\):

```python
sage: DisjointUnionEnumeratedSets(Family([3, 2, 0],
                                          ....:        lambda n: IntegerListsLex(n, length=2))).list()
```

```
[[3, 0], [2, 1], [1, 2], [0, 3],
 [2, 0], [1, 1], [0, 2],
 [0, 0]]
```

Note that if a given value appears several times, the corresponding elements will be enumerated several times, which may, or not, be what one wants:

```python
sage: DisjointUnionEnumeratedSets(Family([2, 2],
                                          ....:        lambda n: IntegerListsLex(n, length=2))).list()
```

```
[[2, 0], [1, 1], [0, 2], [2, 0], [1, 1], [0, 2]]
```

Here is a variant where we specify acceptable values for the length:

```python
sage: DisjointUnionEnumeratedSets(Family([0, 1, 3],
                                          ....:        lambda l: IntegerListsLex(2, length=l))).list()
```

(continues on next page)
This technique can also be useful to obtain a proper enumeration on infinite sets by using a graded lexicographic enumeration:

```python
sage: C = DisjointUnionEnumeratedSets(Family(NN,
....:     lambda n: IntegerListsLex(n, length=2)))

sage: C
Disjoint union of Lazy family (<lambda>(i))_{i in Non negative integer semiring}

sage: it = iter(C)

sage: [next(it) for i in range(10)]
[[0, 0],
 [1, 0],
 [0, 1],
 [2, 0],
 [1, 1],
 [0, 2],
 [3, 0],
 [2, 1],
 [1, 2],
 [0, 3]]
```

Specifying how to construct elements

This is the list of all monomials of degree 4 which divide the monomial $x^3 y^1 z^2$ (a monomial being identified with its exponent vector):

```python
sage: R.<x,y,z> = QQ[]

sage: m = [3,1,2]

sage: def term(exponents):
....:     return x^exponents[0] * y^exponents[1] * z^exponents[2]

sage: list( IntegerListsLex(4, length=len(m), ceiling=m, element_constructor=term) )

[x^3*y, x^3*z, x^2*y*z, x^2*z^2, x*y*z^2]
```

Note the use of the `element_constructor` option to specify how to construct elements from a plain list. A variant is to specify a class for the elements. With the default element constructor, this class should take as input the parent `self` and a list.

**Warning:** The protocol for specifying the element class and constructor is subject to changes.

**ALGORITHM:**

The iteration algorithm uses a depth first search through the prefix tree of the list of integers (see also Lexicographic generation of lists of integers). While doing so, it does some lookahead heuristics to attempt to cut dead branches.

In most practical use cases, most dead branches are cut. Then, roughly speaking, the time needed to iterate through all the elements of $S$ is proportional to the number of elements, where the proportion factor is controlled by the length $l$ of the longest element of $S$. In addition, the memory usage is also controlled by $l$, which is to say negligible in practice.

Still, there remains much room for efficiency improvements; see github issue #18055, github issue #18056.

**Note:** The generation algorithm could in principle be extended to deal with non-constant slope constraints and with negative parts.
TESTS from comments on github issue #17979

Comment 191:

```sage
test: list(IntegerListsLex(1, min_length=2, min_slope=0, max_slope=0))
[]
```

Comment 240:

```sage
test: L = IntegerListsLex(min_length=2, max_part=0)
test: L.list()
[[0, 0]]
```

Tests on the element constructor feature and mutability

Internally, the iterator works on a single list that is mutated along the way. Therefore, you need to make sure that the `element_constructor` actually copies its input. This example shows what can go wrong:

```sage
test: P = IntegerListsLex(n=3, max_slope=0, min_part=1, element_constructor=lambda x: x)
test: list(P)
[[], [], []]
```

However, specifying `list()` as constructor solves this problem:

```sage
test: P = IntegerListsLex(n=3, max_slope=0, min_part=1, element_constructor=list)
test: list(P)
[[3], [2, 1], [1, 1, 1]]
```

Same, step by step:

```sage
test: it = iter(P)
test: a = next(it); a
[3]
test: b = next(it); b
[2, 1]
test: a
[3]
test: a is b
False
```

Tests from MuPAD-Combinat:

```sage
test: IntegerListsLex(7, min_length=2, max_length=6, floor=[0,0,2,0,0,1], ceiling=[3,2,3,2,1,2]).cardinality()
83
test: IntegerListsLex(7, min_length=2, max_length=6, floor=[0,0,2,0,1,1], ceiling=[3,2,3,2,1,2]).cardinality()
53
test: IntegerListsLex(5, min_length=2, max_length=6, floor=[0,0,2,0,0,0], ceiling=[2,2,2,2,2,2]).cardinality()
30
test: IntegerListsLex(5, min_length=2, max_length=6, floor=[0,0,1,1,0,0], ceiling=[2,2,2,2,2,2]).cardinality()
```

(continues on next page)
\[\text{ceiling}=[2,2,2,2,2,2]).\text{cardinality()}\]
43

\[\text{sage: IntegerListsLex(0, min_length=0, max_length=7, floor=[1,1,0,0,1,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{first()}\]
[]

\[\text{sage: IntegerListsLex(0, min_length=1, max_length=7, floor=[0,1,0,0,1,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{first()}\]
[0]

\[\text{sage: IntegerListsLex(0, min_length=1, max_length=7, floor=[1,1,0,0,1,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{cardinality()}\]
0

\[\text{sage: IntegerListsLex(2, min_length=0, max_length=7, floor=[1,1,0,0,0,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{first()} \# \text{Was [1,1], due to slightly different specs}\]
[2]

\[\text{sage: IntegerListsLex(1, min_length=1, max_length=7, floor=[1,1,0,0,0,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{first()}\]
[1]

\[\text{sage: IntegerListsLex(1, min_length=2, max_length=7, floor=[1,1,0,0,0,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{cardinality()}\]
0

\[\text{sage: IntegerListsLex(2, min_length=5, max_length=7, floor=[1,1,0,0,0,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{first()}\]
[1, 1, 0, 0, 0]

\[\text{sage: IntegerListsLex(2, min_length=5, max_length=7, floor=[1,1,0,0,0,1],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{first()}\]
[1, 1, 0, 0, 0]

\[\text{sage: IntegerListsLex(2, min_length=5, max_length=7, floor=[1,1,0,0,1,0],\]
\[\text{ceiling}=[4,3,2,3,2,2,1]).\text{cardinality()}\]
0

\[\text{sage: IntegerListsLex(4, min_length=3, max_length=6, floor=[2, 1, 2, 1, 1, 1],\]
\[\text{ceiling}=[3, 1, 2, 3, 2, 2]).\text{cardinality()}\]
0

\[\text{sage: IntegerListsLex(5, min_length=3, max_length=6, floor=[2, 1, 2, 1, 1, 1],\]
\[\text{ceiling}=[3, 1, 2, 3, 2, 2]).\text{first()}\]
[2, 1, 2]

\[\text{sage: IntegerListsLex(6, min_length=3, max_length=6, floor=[2, 1, 2, 1, 1, 1],\]
\[\text{ceiling}=[3, 1, 2, 3, 2, 2]).\text{first()}\]
[3, 1, 2]

\[\text{sage: IntegerListsLex(12, min_length=3, max_length=6, floor=[2, 1, 2, 1, 1, 1],\]
\[\text{ceiling}=[3, 1, 2, 3, 2, 2]).\text{first()}\]
[3, 1, 2, 3, 2, 1]

\[\text{sage: IntegerListsLex(13, min_length=3, max_length=6, floor=[2, 1, 2, 1, 1, 1],\]
\[\text{ceiling}=[3, 1, 2, 3, 2, 2]).\text{first()}\]
[3, 1, 2, 3, 2, 2]

\[\text{sage: IntegerListsLex(14, min_length=3, max_length=6, floor=[2, 1, 2, 1, 1, 1],\]
\[\text{ceiling}=[3, 1, 2, 3, 2, 2]).\text{cardinality()}\]
0

This used to hang (see comment 389 and fix in Envelope.__init__().)
sage: IntegerListsLex(7, max_part=0, ceiling=lambda i:i, check=False).list()
[]

backend_class

    alias of IntegerListsBackend_invlex

class sage.combinat.integer_lists.invlex.IntegerListsLexIter(backend)

    Bases: object

    Iterator class for IntegerListsLex.

    Let \( T \) be the prefix tree of all lists of nonnegative integers that satisfy all constraints except possibly for \( \text{min\_length} \) and \( \text{min\_sum} \); let the children of a list be sorted decreasingly according to their last part.

    The iterator is based on a depth-first search exploration of a subtree of this tree, trying to cut branches that do not contain a valid list. Each call of \texttt{next} iterates through the nodes of this tree until it finds a valid list to return.

    Here are the attributes describing the current state of the iterator, and their invariants:

    • \texttt{backend} – the \texttt{IntegerListsBackend} object this is iterating on;
    • \texttt{\_current\_list} – the list corresponding to the current node of the tree;
    • \texttt{\_j} – the index of the last element of \texttt{\_current\_list}: \texttt{self._j == len(self._current\_list) - 1};
    • \texttt{\_current\_sum} – the sum of the parts of \texttt{\_current\_list};
    • \texttt{\_search\_ranges} – a list of same length as \texttt{\_current\_list}: the range for each part.

    Furthermore, we assume that there is no obvious contradiction in the constraints:

    • \texttt{self.backend.min\_length <= self.backend.max\_length};
    • \texttt{self.backend.min\_slope <= self.backend.max\_slope unless self.backend.min\_length <= 1}.

    Along this iteration, \texttt{next} switches between the following states:

    • \texttt{LOOKAHEAD}: determine whether the current list could be a prefix of a valid list;
    • \texttt{PUSH}: go deeper into the prefix tree by appending the largest possible part to the current list;
    • \texttt{ME}: check whether the current list is valid and if yes return it
    • \texttt{DECREASE}: decrease the last part;
    • \texttt{POP}: pop the last part of the current list;
    • \texttt{STOP}: the iteration is finished.

    The attribute \texttt{\_next\_state} contains the next state \texttt{next} should enter in.

5.1.124 Counting, generating, and manipulating non-negative integer matrices

Counting, generating, and manipulating non-negative integer matrices with prescribed row sums and column sums.

AUTHORS:

• Franco Saliola
class sage.combinat.integer_matrices.IntegerMatrices(row_sums, column_sums)

Bases: UniqueRepresentation, Parent

The class of non-negative integer matrices with prescribed row sums and column sums.

An integer matrix $m$ with column sums $c := (c_1, ..., c_k)$ and row sums $l := (l_1, ..., l_n)$ where $c_1 + ... + c_k$ is equal to $l_1 + ... + l_n$, is a $n \times k$ matrix $m = (m_{i,j})$ such that $m_{1,j} + \cdots + m_{n,j} = c_j$, for all $j$ and $m_{i,1} + \cdots + m_{i,k} = l_i$, for all $i$.

EXAMPLES:

There are 6 integer matrices with row sums $[3, 2, 2]$ and column sums $[2, 5]$:

```python
sage: from sage.combinat.integer_matrices import IntegerMatrices
sage: IM = IntegerMatrices([3,2,2], [2,5]); IM
Non-negative integer matrices with row sums [3, 2, 2] and column sums [2, 5]
sage: IM.list()
[[2 1]  [1 2]  [1 2]  [0 3]  [0 3]  [0 3]
 [0 2]  [1 1]  [0 2]  [2 0]  [1 1]  [0 2]
 [0 2], [0 2], [1 1], [0 2], [1 1], [2 0]]
sage: IM.cardinality()
6
```

cardinality()

The number of integer matrices with the prescribed row sums and columns sums.

EXAMPLES:

```python
sage: from sage.combinat.integer_matrices import IntegerMatrices
sage: IntegerMatrices([2,5], [3,2,2]).cardinality()
6
sage: IntegerMatrices([1,1,1,1,1], [1,1,1,1,1]).cardinality()
120
sage: IntegerMatrices([2,2,2,2], [2,2,2,2]).cardinality()
282
sage: IntegerMatrices([4], [3]).cardinality()
0
sage: len(IntegerMatrices([0,0], [0]).list())
1
```

This method computes the cardinality using symmetric functions. Below are the same examples, but computed by generating the actual matrices:

```python
sage: from sage.combinat.integer_matrices import IntegerMatrices
sage: len(IntegerMatrices([2,5], [3,2,2]).list())
6
sage: len(IntegerMatrices([1,1,1,1,1], [1,1,1,1,1]).list())
120
sage: len(IntegerMatrices([2,2,2,2], [2,2,2,2]).list())
282
sage: len(IntegerMatrices([4], [3]).list())
0
sage: len(IntegerMatrices([0,0], [0]).list())
1
column_sums()
The column sums of the integer matrices in self.

OUTPUT:
• Composition

EXAMPLES:

```sage
from sage.combinat.integer_matrices import IntegerMatrices
IM = IntegerMatrices([3,2,2], [2,5])
IM.column_sums()
[2, 5]
```

def row_sums()
The row sums of the integer matrices in self.

OUTPUT:
• Composition

EXAMPLES:

```sage
from sage.combinat.integer_matrices import IntegerMatrices
IM = IntegerMatrices([3,2,2], [2,5])
IM.row_sums()
[3, 2, 2]
```

to_composition(x)
The composition corresponding to the integer matrix.
This is the composition obtained by reading the entries of the matrix from left to right along each row, and reading the rows from top to bottom, ignore zeros.

INPUT:
• x — matrix

EXAMPLES:

```sage
from sage.combinat.integer_matrices import IntegerMatrices
IM = IntegerMatrices([3,2,2], [2,5]); IM
Non-negative integer matrices with row sums [3, 2, 2] and column sums [2, 5]
IM.list()
[
 [2 1] [1 2] [1 2] [0 3] [0 3] [0 3]
 [0 2] [1 1] [0 2] [2 0] [1 1] [0 2]
 [0 2], [0 2], [1 1], [0 2], [1 1], [2 0]
]
sage: for m in IM: print(IM.to_composition(m))
[2, 1, 2, 2]
[1, 2, 1, 1, 2]
[1, 2, 2, 1, 1]
[3, 2, 2]
[3, 1, 1, 1, 1]
[3, 2, 2]
```
sage.combinat.integer_matrices.integer_matrices_generator(row_sums, column_sums)
Recursively generate the integer matrices with the prescribed row sums and column sums.

INPUT:
- row_sums – list or tuple
- column_sums – list or tuple

OUTPUT:
- an iterator producing a list of lists

EXAMPLES:

```python
sage: from sage.combinat.integer_matrices import integer_matrices_generator
sage: iter = integer_matrices_generator([3,2,2], [2,5]); iter
<generator object ...integer_matrices_generator at ...>
sage: for m in iter: print(m)
[[2, 1], [0, 2], [0, 2]]
[[1, 2], [1, 1], [0, 2]]
[[1, 2], [0, 2], [1, 1]]
[[0, 3], [2, 0], [0, 2]]
[[0, 3], [1, 1], [1, 1]]
[[0, 3], [0, 2], [2, 0]]
```

5.1.125 (Non-negative) Integer vectors

AUTHORS:
- Mike Hansen (2007) - original module
- Nathann Cohen, David Joyner (2009-2010) - Gale-Ryser stuff
- Nathann Cohen, David Joyner (2011) - Gale-Ryser bugfix
- Travis Scrimshaw (2012-05-12) - Updated doc-strings to tell the user of that the class’s name is a misnomer (that they only contains non-negative entries).
- Federico Poloni (2013) - specialized rank()
- Travis Scrimshaw (2013-02-04) - Refactored to use ClonableIntArray

```python
class sage.combinat.integer_vector.IntegerVector
Bases: ClonableArray
An integer vector.

check()
Check to make sure this is a valid integer vector by making sure all entries are non-negative.

EXAMPLES:

```python
sage: IV = IntegerVectors()
sage: elt = IV([1,2,1])
sage: elt.check()
```

Check github issue #34510:
sage: IV3 = IntegerVectors(n=3)
sage: IV3([2,2])
Traceback (most recent call last):
...
ValueError: [2, 2] doesn't satisfy correct constraints
sage: IVk3 = IntegerVectors(k=3)
sage: IVk3([2,2])
Traceback (most recent call last):
...
ValueError: [2, 2] doesn't satisfy correct constraints
sage: IV33 = IntegerVectors(n=3, k=3)
sage: IV33([2,2])
Traceback (most recent call last):
...
ValueError: [2, 2] doesn't satisfy correct constraints

sage_module \(base\_ring=None\)
Return the Specht module corresponding to \(self\).

EXAMPLES:

sage: SM = IntegerVectors()([2,0,1,0,2]).sage_module(QQ)
sage: SM
Specht module of \([(0, 0), (0, 1), (2, 0), (4, 0), (4, 1)]\) over Rational Field
sage: s = SymmetricFunctions(QQ).s()
sage: s(SM.frobenius_image())
s[2, 2, 1]

sage_module_dimension \(base\_ring=None\)
Return the dimension of the Specht module corresponding to \(self\).

INPUT:

• BR – (default: Q) the base ring

EXAMPLES:

sage: IntegerVectors()([2,0,1,0,2]).sage_module_dimension()
5
sage: IntegerVectors()([2,0,1,0,2]).sage_module_dimension(GF(2))
5

trim()
Remove trailing zeros from the integer vector.

EXAMPLES:

sage: IV = IntegerVectors()
sage: IV([5,3,5,1,0,0]).trim()
[5, 3, 5, 1]
sage: IV([5,0,5,1,0]).trim()
[5, 0, 5, 1]
sage: IV([4,3,3]).trim()
[4, 3, 3]
sage: IV([0,0,0]).trim()
Combinatorics, Release 10.0

(continued from previous page)

```
[]
sage: IV = IntegerVectors(k=4)
sage: v = IV([4,3,2,0]).trim(); v
[4, 3, 2]
sage: v.parent()
Integer vectors
```

class `sage.combinat.integer_vector.IntegerVectors`(category=None)

Bases: `Parent`

The class of (non-negative) integer vectors.

INPUT:

- n – if set to an integer, returns the combinatorial class of integer vectors whose sum is n; if set to `None` (default), no such constraint is defined
- k – the length of the vectors; set to `None` (default) if you do not want such a constraint

Note: The entries are non-negative integers.

EXAMPLES:

If n is not specified, it returns the class of all integer vectors:

```
sage: IntegerVectors()
Integer vectors
sage: [] in IntegerVectors()
True
sage: [1,2,1] in IntegerVectors()
True
sage: [1, 0, 0] in IntegerVectors()
True
```

Entries are non-negative:

```
sage: [-1, 2] in IntegerVectors()
False
```

If n is specified, then it returns the class of all integer vectors which sum to n:

```
sage: IV3 = IntegerVectors(3); IV3
Integer vectors that sum to 3
```

Note that trailing zeros are ignored so that `[3, 0]` does not show up in the following list (since `[3]` does):

```
sage: IntegerVectors(3, max_length=2).list()
[[3], [2, 1], [1, 2], [0, 3]]
```

If n and k are both specified, then it returns the class of integer vectors that sum to n and are of length k:

```
sage: IV53 = IntegerVectors(5,3); IV53
Integer vectors of length 3 that sum to 5
```

(continues on next page)
Further examples:

```python
sage: IntegerVectors(-1, 0, min_part=1).list()
[]
sage: IntegerVectors(-1, 2, min_part=1).list()
[]
sage: IntegerVectors(0, 0, min_part=1).list()
[]
sage: IntegerVectors(3, 0, min_part=1).list()
[]
sage: IntegerVectors(0, 1, min_part=1).list()
[]
sage: IntegerVectors(2, 2, min_part=1).list()
[[1, 1]]
sage: IntegerVectors(2, 3, min_part=1).list()
[]
sage: IntegerVectors(4, 2, min_part=1).list()
[[3, 1], [2, 2], [1, 3]]
```

```python
sage: IntegerVectors(0, 3, outer=[0,0,0]).list()
[[0, 0, 0]]
sage: IntegerVectors(1, 3, outer=[0,0,0]).list()
[]
sage: IntegerVectors(2, 3, outer=[0,2,0]).list()
[[0, 2, 0]]
sage: IntegerVectors(2, 3, outer=[1,2,1]).list()
[[1, 1, 0], [1, 0, 1], [0, 2, 0], [0, 1, 1]]
sage: IntegerVectors(2, 3, outer=[1,1,1]).list()
[[1, 1, 0], [1, 0, 1], [0, 1, 1]]
sage: IntegerVectors(2, 5, outer=[1,1,1,1,1]).list()
[[1, 1, 0, 0, 0],
 [1, 0, 1, 0, 0],
 [1, 0, 0, 1, 0],
 [1, 0, 0, 0, 1],
 [0, 1, 1, 0, 0],
 [0, 1, 0, 1, 0],
 [0, 1, 0, 0, 1],
 [0, 0, 1, 1, 0],
 [0, 0, 1, 0, 1],
 [0, 0, 0, 1, 1]]
```

```python
sage: iv = [ IntegerVectors(n,k) for n in range(-2, 7) for k in range(7) ]
sage: all(map(lambda x: x.cardinality() == len(x.list()), iv))
```
An example showing the same output by using `IntegerListsLex`:

```
sage: IntegerVectors(4, max_length=2).list()
[[4], [3, 1], [2, 2], [1, 3], [0, 4]]
sage: list(IntegerListsLex(4, max_length=2))
[[4], [3, 1], [2, 2], [1, 3], [0, 4]]
```

See also:

`sage.combinat.integer_lists.invlex.IntegerListsLex`

**Element**

alias of `IntegerVector`

**class** `sage.combinat.integer_vector.IntegerVectorsConstraints(n=None, k=None, **constraints)`

Bases: `IntegerVectors`

Class of integer vectors subject to various constraints.

**cardinality()**

Return the cardinality of `self`.

EXAMPLES:

```
sage: IntegerVectors(3, 3, min_part=1).cardinality()
1
sage: IntegerVectors(5, 3, min_part=1).cardinality()
6
sage: IntegerVectors(13, 4, max_part=4).cardinality()
20
sage: IntegerVectors(k=4, max_part=3).cardinality()
256
sage: IntegerVectors(k=3, min_part=2, max_part=4).cardinality()
27
sage: IntegerVectors(13, 4, min_part=2, max_part=4).cardinality()
16
```

**class** `sage.combinat.integer_vector.IntegerVectors_all`

Bases: `UniqueRepresentation, IntegerVectors`

Class of all integer vectors.

**class** `sage.combinat.integer_vector.IntegerVectors_k(k)`

Bases: `UniqueRepresentation, IntegerVectors`

Integer vectors of length `k`.

**class** `sage.combinat.integer_vector.IntegerVectors_n(n)`

Bases: `UniqueRepresentation, IntegerVectors`

Integer vectors that sum to `n`.

5.1. Comprehensive Module List
class sage.combinat.integer_vector.IntegerVectors_nk(n, k)
    Bases: UniqueRepresentation, IntegerVectors

Integer vectors of length $k$ that sum to $n$.

AUTHORS:
    • Martin Albrecht
    • Mike Hansen

rank(x)
    Return the rank of a given element.

INPUT:
    • $x$ – a list with $\text{sum}(x) == n$ and $\text{len}(x) == k$

class sage.combinat.integer_vector.IntegerVectors_nnondescents(n, comp)
    Bases: UniqueRepresentation, IntegerVectors

Integer vectors graded by two parameters.

The grading parameters on the integer vector $v$ are:
    • $n$ – the sum of the parts of $v$,
    • $c$ – the non descents composition of $v$.

In other words: the length of $v$ equals $c_1 + \cdots + c_k$, and $v$ is decreasing in the consecutive blocs of length $c_1, \ldots, c_k$.

INPUT:
    • $n$ – the positive integer $n$
    • $\text{comp}$ – the composition $c$

Those are the integer vectors of sum $n$ that are lexicographically maximal (for the natural left-to-right reading) in their orbit by the Young subgroup $S_{c_1} \times \cdots \times S_{c_k}$. In particular, they form a set of orbit representative of integer vectors with respect to this Young subgroup.

sage.combinat.integer_vector.gale_ryser_theorem(p1, p2, algorithm, solver, integrality_t tolerance='gale')

Returns the binary matrix given by the Gale-Ryser theorem.

The Gale Ryser theorem asserts that if $p_1, p_2$ are two partitions of $n$ of respective lengths $k_1, k_2$, then there is a binary $k_1 \times k_2$ matrix $M$ such that $p_1$ is the vector of row sums and $p_2$ is the vector of column sums of $M$, if and only if the conjugate of $p_2$ dominates $p_1$.

INPUT:
    • $p_1, p_2$– list of integers representing the vectors of row/column sums
    • $\text{algorithm}$ – two possible string values:
        – 'ryser' implements the construction due to Ryser [Ryser63].
        – 'gale' (default) implements the construction due to Gale [Gale57].
    • $\text{solver}$ – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method $\text{solve}$ of the class MixedIntegerLinearProgram.
    • $\text{integrality_t tolerance}$ – parameter for use with MILP solvers over an inexact base ring; see MixedIntegerLinearProgram.get_values().
OUTPUT:
A binary matrix if it exists, None otherwise.

Gale’s Algorithm:

(Gale [Gale57]): A matrix satisfying the constraints of its sums can be defined as the solution of the following Linear Program, which Sage knows how to solve.

\[
\forall i \sum_{j=1}^{k_2} b_{i,j} = p_{1,j} \\
\forall i \sum_{j=1}^{k_1} b_{j,i} = p_{2,j} \\
\]

\( b_{i,j} \) is a binary variable

Ryser’s Algorithm:

(Ryser [Ryser63]): The construction of an \( m \times n \) matrix \( A = A_{r,s} \), due to Ryser, is described as follows. The construction works if and only if have \( s \preceq r^* \).

\bullet \) Construct the \( m \times n \) matrix \( B \) from \( r \) by defining the \( i \)-th row of \( B \) to be the vector whose first \( r_i \) entries are 1, and the remainder are 0’s, \( 1 \leq i \leq m \). This maximal matrix \( B \) with row sum \( r \) and ones left justified has column sum \( r^* \).

\bullet \) Shift the last 1 in certain rows of \( B \) to column \( n \) in order to achieve the sum \( s_n \). Call this \( B \) again.

\quad \) The 1’s in column \( n \) are to appear in those rows in which \( A \) has the largest row sums, giving preference to the bottom-most positions in case of ties.

\quad \) Note: When this step automatically “fixes” other columns, one must skip ahead to the first column index with a wrong sum in the step below.

\bullet \) Proceed inductively to construct columns \( n - 1, \ldots, 2, 1 \). Note: when performing the induction on step \( k \), we consider the row sums of the first \( k \) columns.

\bullet \) Set \( A = B \). Return \( A \).

EXAMPLES:
Computing the matrix for \( p_1 = p_2 = 2 + 2 + 1 \):

```
sage: from sage.combinat.integer_vector import gale_ryser_theorem
sage: p1 = [2,2,1]
sage: p2 = [2,2,1]
sage: print(gale_ryser_theorem(p1, p2))        # not tested
[1 1 0]
[1 0 1]
[0 1 0]
sage: A = gale_ryser_theorem(p1, p2)
sage: rs = [sum(x) for x in A.rows()]
sage: cs = [sum(x) for x in A.columns()]
sage: p1 == rs; p2 == cs
True
True
```

Or for a non-square matrix with \( p_1 = 3 + 3 + 2 + 1 \) and \( p_2 = 3 + 2 + 2 + 1 + 1 \), using Ryser’s algorithm:
```python
sage: from sage.combinat.integer_vector import gale_ryser_theorem
sage: p1 = [3,3,1,1]
sage: p2 = [3,3,1,1]
sage: gale_ryser_theorem(p1, p2, algorithm = "ryser")
[1 1 1 0]
[1 1 0 1]
[1 0 0 0]
[0 1 0 0]
sage: p1 = [4,2,2]
sage: p2 = [3,3,1,1]
sage: gale_ryser_theorem(p1, p2, algorithm = "ryser")
[1 1 1 1]
[1 1 0 0]
[1 1 0 0]
sage: p1 = [4,2,2,0]
sage: p2 = [3,3,1,1,0,0]
sage: gale_ryser_theorem(p1, p2, algorithm = "ryser")
[1 1 1 1 0 0]
[1 1 0 0 0 0]
[1 1 0 0 0 0]
[0 0 0 0 0 0]
sage: p1 = [3,3,2,1]
sage: p2 = [3,2,2,1,1]
sage: print(gale_ryser_theorem(p1, p2, algorithm = "gale"))  # not tested
[1 1 1 0 0]
[1 1 0 0 1]
[1 0 1 0 0]
[0 1 0 0 0]
With 0 in the sequences, and with unordered inputs:
```

```python
sage: from sage.combinat.integer_vector import gale_ryser_theorem
sage: gale_ryser_theorem([3,3,0,1,1,0], [3,1,3,1,0], algorithm = "ryser")
[1 1 1 0 0]
[1 0 1 1 0]
[0 0 0 0 0]
[1 0 0 0 0]
[0 0 1 0 0]
[0 0 0 0 0]
sage: p1 = [3,1,1,1,1]; p2 = [3,2,2,0]
sage: gale_ryser_theorem(p1, p2, algorithm = "ryser")
[1 1 1 0]
[1 0 0 0]
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
```

REFERENCES:
sage.combinat.integer_vector.integer_vectors_nk_fast_iter(n, k)
A fast iterator for integer vectors of n of length k which yields Python lists filled with Sage Integers.

EXAMPLES:
We check some corner cases:

\[
\begin{align*}
\text{sage: } & \text{list(integer_vectors_nk_fast_iter(5, 1))} \\
& \text{[[5]]} \\
\text{sage: } & \text{list(integer_vectors_nk_fast_iter(1, 1))} \\
& \text{[[1]]} \\
\text{sage: } & \text{list(integer_vectors_nk_fast_iter(2, 0))} \\
& [] \\
\text{sage: } & \text{list(integer_vectors_nk_fast_iter(0, 2))} \\
& [[0, 0]]
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{list(integer_vectors_nk_fast_iter(0, 0))} \\
& [[]]
\end{align*}
\]

\[
\begin{align*}
\text{sage.combinat.integer_vector.is_gale_ryser}(r, s)
\end{align*}
\]

Tests whether the given sequences satisfy the condition of the Gale-Ryser theorem.

Given a binary matrix $B$ of dimension $n \times m$, the vector of row sums is defined as the vector whose $i^{th}$ component is equal to the sum of the $i^{th}$ row in $A$. The vector of column sums is defined similarly.

If, given a binary matrix, these two vectors are easy to compute, the Gale-Ryser theorem lets us decide whether, given two non-negative vectors $r, s$, there exists a binary matrix whose row/column sums vectors are $r$ and $s$.

This function answers accordingly.

INPUT:

• $r, s$ – lists of non-negative integers.

ALGORITHM:

Without loss of generality, we can assume that:

• The two given sequences do not contain any 0 (which would correspond to an empty column/row)

• The two given sequences are ordered in decreasing order (reordering the sequence of row (resp. column) sums amounts to reordering the rows (resp. columns) themselves in the matrix, which does not alter the columns (resp. rows) sums.

We can then assume that $r$ and $s$ are partitions (see the corresponding class \texttt{Partition}).

If $r^*$ denote the conjugate of $r$, the Gale-Ryser theorem asserts that a binary Matrix satisfying the constraints exists if and only if $s \preceq r^*$, where $\preceq$ denotes the domination order on partitions.

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{from sage.combinat.integer_vector import is_gale_ryser} \\
\text{sage: } & \text{is_gale_ryser([4,2,2],[3,3,1,1])} \\
& \text{True} \\
\text{sage: } & \text{is_gale_ryser([4,2,1,1],[3,3,1,1])} \\
& \text{True}
\end{align*}
\]

(continues on next page)
REMARK: In the literature, what we are calling a Gale-Ryser sequence sometimes goes by the (rather generic-sounding) term “realizable sequence”.

```python
sage.combinat.integer_vector.list2func(l, default=None)
```

Given a list `l`, return a function that takes in a value `i` and return `l[i]`. If default is not None, then the function will return the default value for out of range `i`’s.

EXAMPLES:

```python
sage: f = sage.combinat.integer_vector.list2func([1, 2, 3])
sage: f(0)
1
sage: f(1)
2
sage: f(2)
3
sage: f(3)
Traceback (most recent call last):
... IndexError: list index out of range
```

```python
sage: f = sage.combinat.integer_vector.list2func([1, 2, 3], 0)
sage: f(2)
3
sage: f(3)
0
```

### 5.1.126 Weighted Integer Vectors

AUTHORS:

- Mike Hansen (2007): initial version, ported from MuPAD-Combinat
- Nicolas M. Thiery (2010-10-30): WeightedIntegerVectors(weights) + cleanup

```python
class sage.combinat.integer_vector_weighted.WeightedIntegerVectors(n, weight)
```

Bases: `Parent, UniqueRepresentation`

The class of integer vectors of `n` weighted by `weight`, that is, the nonnegative integer vectors \((v_1, \ldots, v_\ell)\) satisfying \(\sum_{i=1}^\ell v_i w_i = n\) where \(\ell\) is `length(weight)` and \(w_i\) is `weight[i]`.

INPUT:

- `n` – a non negative integer (optional)
- `weight` – a tuple (or list or iterable) of positive integers

EXAMPLES:

```python
sage: WeightedIntegerVectors(8, [1,1,2])
Integer vectors of 8 weighted by [1, 1, 2]
sage: WeightedIntegerVectors(8, [1,1,2]).first()
```

(continues on next page)
\begin{verbatim}
[0, 0, 4]
sage: WeightedIntegerVectors(8, [1,1,2]).last()
[8, 0, 0]
sage: WeightedIntegerVectors(8, [1,1,2]).cardinality()
25
sage: w = WeightedIntegerVectors(8, [1,1,2]).random_element()
sage: w.parent() is WeightedIntegerVectors(8, [1,1,2])
True
sage: WeightedIntegerVectors([1,1,2])
Integer vectors weighted by [1, 1, 2]
sage: WeightedIntegerVectors([1,1,2]).cardinality()
+Infinity
sage: WeightedIntegerVectors([1,1,2]).first()
[0, 0, 0]
\end{verbatim}

**Todo:** Should the order of the arguments \texttt{n} and \texttt{weight} be exchanged to simplify the logic?

**Element**
alias of \texttt{IntegerVector}

\begin{classdef}
\begin{verbatim}
class sage.combinat.integer_vector_weighted.Weight\_Integer\_Vectors\_all(weight)
Bases: DisjointUnionEnumeratedSets
Set of weighted integer vectors.
\end{verbatim}
\end{classdef}

**EXAMPLES:**
\begin{verbatim}
sage: W = WeightedIntegerVectors([3,1,1,2,1]); W
Integer vectors weighted by [3, 1, 1, 2, 1]
sage: W.cardinality()
+Infinity
sage: W12 = W.graded_component(12)
sage: W12.an_element()
[4, 0, 0, 0, 0]
sage: W12.last()
[0, 12, 0, 0, 0]
sage: W12.cardinality()
441
sage: for w in W12: print(w)
[4, 0, 0, 0, 0]
[3, 0, 0, 1, 1]
[3, 0, 1, 1, 0]
...
[0, 11, 1, 0, 0]
[0, 12, 0, 0, 0]
\end{verbatim}

**grading(x)**
EXAMPLES:
sage: C = WeightedIntegerVectors([2,1,3])
sage: C.grading((2,1,1))
8

subset(size=None)
EXAMPLES:

sage: C = WeightedIntegerVectors([2,1,3])
sage: C.subset(4)
Integer vectors of 4 weighted by [2, 1, 3]

sage.combinat.integer_vector_weighted.iterator_fast(n, l)
Iterate over all l weighted integer vectors with total weight n.

INPUT:
• n – an integer
• l – the weights in weakly decreasing order

EXAMPLES:

sage: from sage.combinat.integer_vector_weighted import iterator_fast
sage: list(iterator_fast(3, [2,1,1]))
[[1, 1, 0], [1, 0, 1], [0, 3, 0], [0, 2, 1], [0, 1, 2], [0, 0, 3]]
sage: list(iterator_fast(2, [2]))
[[1]]

Test that github issue #20491 is fixed:

sage: type(list(iterator_fast(2, [2]))[0][0])
<class 'sage.rings.integer.Integer'>

5.1.127 Integer vectors modulo the action of a permutation group

class sage.combinat.integer_vectors_mod_permgroup.IntegerVectorsModPermutationGroup
Bases: UniqueRepresentation

Returns an enumerated set containing integer vectors which are maximal in their orbit under the action of the permutation group G for the lexicographic order.

In Sage, a permutation group G is viewed as a subgroup of the symmetric group Sn of degree n and n is said to be the degree of G. Any integer vector v is said to be canonical if it is maximal in its orbit under the action of G. v is canonical if and only if

\[ v = \max_{\text{lex order}} \{ g \cdot v | g \in G \} \]

The action of G is on position. This means for example that the simple transposition s1 = (1, 2) swaps the first and the second entries of any integer vector

\[ v = [a_1, a_2, a_3, \ldots, a_n] \]

\[ s_1 \cdot v = [a_2, a_1, a_3, \ldots, a_n] \]

This function returns a parent which contains a single integer vector by orbit under the action of the permutation group G. The approach chosen here is to keep the maximal integer vector for the lexicographic order in each orbit. Such maximal vector will be called canonical integer vector under the action of the permutation group G.

INPUT:
• $G$ - a permutation group
• $\text{sum}$ - (default: None) - a nonnegative integer
• $\text{max\_part}$ - (default: None) - a nonnegative integer setting the maximum of entries of elements
• $\text{sgs}$ - (default: None) - a strong generating system of the group $G$. If you do not provide it, it will be calculated at the creation of the parent

OUTPUT:
• If $\text{sum}$ and $\text{max\_part}$ are None, it returns the infinite enumerated set of all integer vectors (list of integers) maximal in their orbit for the lexicographic order.
• If $\text{sum}$ is an integer, it returns a finite enumerated set containing all integer vectors maximal in their orbit for the lexicographic order and whose entries sum to $\text{sum}$.

EXAMPLES:
Here is the set enumerating integer vectors modulo the action of the cyclic group of 3 elements:

```sage
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[(1,2,3)]]))
sage: I.category()
Category of infinite enumerated quotients of sets
sage: I.cardinality()
+Infinity
sage: I.list()
Traceback (most recent call last):
... 
NotImplementedError: cannot list an infinite set
sage: p = iter(I)
sage: for i in range(10): next(p)
[0, 0, 0]
[1, 0, 0]
[2, 0, 0]
[1, 1, 0]
[3, 0, 0]
[2, 1, 0]
[2, 0, 1]
[1, 1, 1]
[4, 0, 0]
[3, 1, 0]
```

The method $\text{is\_canonical()}$ tests if any integer vector is maximal in its orbit. This method is also used in the containment test:

```sage
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[(1,2,3,4)]]))
sage: I.is_canonical([5,2,0,4])
True
sage: I.is_canonical([5,0,6,4])
False
sage: I.is_canonical([5,0,6,4])
True
sage: [2,3,1,0] in I
False
sage: [5,0,5,0] in I
True
sage: 'Bla' in I
```

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Combinatorics, Release 10.0

False
sage: I.is_canonical('bla')
Traceback (most recent call last):
...
AssertionError: bla should be a list or an integer vector

If you give a value to the extra argument sum, the set returned will be a finite set containing only canonical vectors whose entries sum to sum:

sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3]]), sum=6)
sage: I.cardinality()
10
sage: I.list()
[[6, 0, 0], [5, 1, 0], [5, 0, 1], [4, 2, 0], [4, 1, 1],
 [4, 0, 2], [3, 3, 0], [3, 2, 1], [3, 1, 2], [2, 2, 2]]
sage: I.category()
Join of Category of finite enumerated sets and Category of subquotients of finite...
\rightarrow sets and Category of quotients of sets

To get the orbit of any integer vector \(v\) under the action of the group, use the method \texttt{orbit()}; we convert the returned set of vectors into a list in increasing lexicographic order, to get a reproducible test:

sage: sorted(I.orbit([6,0,0]))
[[0, 0, 6], [0, 6, 0], [6, 0, 0]]
sage: sorted(I.orbit([5,1,0]))
[[0, 5, 1], [1, 0, 5], [5, 1, 0]]
sage: I.orbit([2,2,2])
{[2, 2, 2]}

class \texttt{sage.combinat.integer_vectors_mod_permgroup.IntegerVectorsModPermutationGroup\_All}(G, sgs=None)

Bases: \texttt{UniqueRepresentation}, \texttt{RecursivelyEnumeratedSet\_forest}

A class for integer vectors enumerated up to the action of a permutation group.

A Sage permutation group is viewed as a subgroup of the symmetric group \(S_n\) for a certain \(n\). This group has a natural action by position on vectors of length \(n\). This class implements a set which keeps a single vector for each orbit. We say that a vector is canonical if it is the maximum in its orbit under the action of the permutation group for the lexicographic order.

EXAMPLES:

sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]))
sage: I
Integer vectors of length 4 enumerated up to the action of Permutation Group with...
\rightarrow generators [(1,2,3,4)]
sage: I.cardinality()
+Infinity
sage: TestSuite(I).run()
sage: it = iter(I)
sage: [next(it), next(it), next(it), next(it), next(it)]
[[0, 0, 0, 0], [1, 0, 0, 0], [2, 0, 0, 0],...
class Element

Bases: ClonableIntArray

Element class for the set of integer vectors of given sum enumerated modulo the action of a permutation group. These vectors are clonable lists of integers which must satisfy conditions coming from the parent appearing in the method `check()`.

`check()`

Checks that `self` verify the invariants needed for living in `self.parent()`.

EXAMPLES:

```
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]))
sage: v = I.an_element()
sage: v.check()
sage: w = I([0,4,0,0], check=False); w
[0, 4, 0, 0]
sage: w.check()
Traceback (most recent call last):
  ...  AssertionError
```

`ambient()`

Return the ambient space from which `self` is a quotient.

EXAMPLES:

```
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]))
sage: S.ambient()
Integer vectors of length 4
```

`children(x)`

Returns the list of children of the element `x`. This method is required to build the tree structure of `self` which inherits from the class `RecursivelyEnumeratedSet_forest`.

EXAMPLES:

```
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]))
sage: I.children(I([2,1,0,0], check=False))
[[2, 2, 0, 0], [2, 1, 1, 0], [2, 1, 0, 1]]
```

`is_canonical(v, check=True)`

Returns `True` if the integer list `v` is maximal in its orbit under the action of the permutation group given to define `self`. Such integer vectors are said to be canonical. A vector `v` is canonical if and only if

\[
v = \max_{\text{lex order}} \{ g \cdot v | g \in G \}\]

EXAMPLES:
lift(elt)
Lift the element elt inside the ambient space from which self is a quotient.

EXAMPLES:

\begin{verbatim}
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([[(1,2,3,4)]]))
sage: v = S.lift(S([4,3,0,1])); v
[4, 3, 0, 1]
sage: type(v)
<class 'list'>
\end{verbatim}

orbit(v)
Returns the orbit of the integer vector v under the action of the permutation group defining self. The result is a set.

EXAMPLES:
In order to get reproducible doctests, we convert the returned sets into lists in increasing lexicographic order:

\begin{verbatim}
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[(1,2,3,4)]]))
sage: sorted(I.orbit([2,2,0,0]))
[[0, 0, 2, 2], [0, 2, 2, 0], [2, 0, 0, 2], [2, 2, 0, 0]]
sage: sorted(I.orbit([2,1,0,0]))
[[0, 0, 2, 1], [0, 2, 1, 0], [1, 0, 0, 2], [2, 1, 0, 0]]
sage: sorted(I.orbit([2,0,1,0]))
[[0, 1, 0, 2], [0, 2, 0, 1], [1, 0, 2, 0], [2, 0, 1, 0]]
sage: sorted(I.orbit([2,0,2,0]))
[[0, 2, 0, 2], [2, 0, 2, 0]]
sage: I.orbit([1,1,1,1])
[[1, 1, 1, 1]]
\end{verbatim}

permutation_group()
Returns the permutation group given to define self.

EXAMPLES:

\begin{verbatim}
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[(1,2,3,4)]]))
sage: I.permutation_group()
Permutation Group with generators [(1,2,3,4)]
\end{verbatim}

retract(elt)
Return the canonical representative of the orbit of the integer elt under the action of the permutation group defining self.
If the element elt is already maximal in its orbit for the lexicographic order, elt is thus the good representative for its orbit.

EXAMPLES:

```python
sage: [0,0,0,0] in IntegerVectors(0,4)
True
sage: [1,0,0,0] in IntegerVectors(1,4)
True
sage: [0,1,0,0] in IntegerVectors(1,4)
True
sage: [1,0,1,0] in IntegerVectors(2,4)
True
sage: [0,1,0,1] in IntegerVectors(2,4)
True
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]))
sage: S.retract([0,0,0,0])
[0, 0, 0, 0]
sage: S.retract([1,0,0,0])
[1, 0, 0, 0]
sage: S.retract([0,1,0,0])
[1, 0, 0, 0]
sage: S.retract([1,0,1,0])
[1, 0, 1, 0]
sage: S.retract([0,1,0,1])
[1, 0, 1, 0]
```

```
roots()

Returns the root of generation of self. This method is required to build the tree structure of self which inherits from the class RecursivelyEnumeratedSet_forest.

EXAMPLES:

```python
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]))
sage: I.roots()
[[0, 0, 0, 0]]
```

```
subset(sum=None, max_part=None)

Returns the subset of self containing integer vectors whose entries sum to sum.

EXAMPLES:

```python
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]))
sage: S.subset(4)
Integer vectors of length 4 and of sum 4 enumerated up to the action of Permutation Group with generators
[(1,2,3,4)]
```

```
class sage.combinat.integer_vectors_mod_permgroup.IntegerVectorsModPermutationGroup_with_constraints(G, d, max_part, sgs)

Bases: UniqueRepresentation, RecursivelyEnumeratedSet_forest

This class models finite enumerated sets of integer vectors with constraint enumerated up to the action of a permutation group. Integer vectors are enumerated modulo the action of the permutation group. To implement
that, we keep a single integer vector by orbit under the action of the permutation group. Elements chosen are vectors maximal in their orbit for the lexicographic order.

For more information see \texttt{IntegerVectorsModPermutationGroup}.

**EXAMPLES:**

```python
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]), max_part=1)
sage: I.list()
[[0, 0, 0, 0], [1, 0, 0, 0], [1, 1, 0, 0], [1, 0, 1, 0], [1, 1, 1, 0], [1, 1, 1, 1]]
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]), sum=6, max_part=4)
sage: I.list()
[[4, 2, 0, 0], [4, 1, 1, 0], [4, 1, 0, 1], [4, 0, 2, 0], [4, 0, 1, 1],
 [4, 0, 0, 2], [3, 3, 0, 0], [3, 2, 1, 0], [3, 2, 0, 1], [3, 1, 2, 0],
 [3, 1, 1, 1], [3, 1, 0, 2], [3, 0, 3, 0], [3, 0, 2, 1], [3, 0, 1, 2],
 [2, 2, 2, 0], [2, 2, 1, 1], [2, 1, 2, 1]]
```

Here is the enumeration of unlabeled graphs over 5 vertices:

```python
sage: G = IntegerVectorsModPermutationGroup(TransitiveGroup(10,12), max_part=1)
sage: G.cardinality()
34
```

**class Element**

Bases: \texttt{ClonableIntArray}

Element class for the set of integer vectors with constraints enumerated modulo the action of a permutation group. These vectors are clonable lists of integers which must satisfy conditions coming from the parent as in the method \texttt{check()}.

\texttt{check()} Checks that \texttt{self} meets the constraints of being an element of \texttt{self.parent()}.

**EXAMPLES:**

```python
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]), 4)
sage: v = I.an_element()
sage: v.check()
sage: w = I([0,4,0,0], check=False); w
[0, 4, 0, 0]
sage: w.check()
Traceback (most recent call last):
  ... AssertionError
```

\texttt{ambient()} Return the ambient space from which \texttt{self} is a quotient.

**EXAMPLES:**

```python
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([[1,2,3,4]]), 6);
sage: S.ambient()
Integer vectors that sum to 6
```
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), 6, max_part=12); S.ambient()
Integer vectors that sum to 6 with constraints: max_part=12
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), max_part=12); S.ambient()
Integer vectors with constraints: max_part=12

an_element()
Returns an element of self or raises an EmptySetError when self is empty.

EXAMPLES:

sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), sum=0, max_part=1); S.an_element()
[0, 0, 0, 0]
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), sum=1, max_part=1); S.an_element()
[1, 0, 0, 0]
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), sum=2, max_part=1); S.an_element()
[1, 1, 0, 0]
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), sum=3, max_part=1); S.an_element()
[1, 1, 1, 0]
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), sum=4, max_part=1); S.an_element()
[1, 1, 1, 1]
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), sum=5, max_part=1); S.an_element()
Traceback (most recent call last):
... EmptySetError

children(x)
Returns the list of children of the element x. This method is required to build the tree structure of self which inherits from the class RecursivelyEnumeratedSet_forest.

EXAMPLES:

sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), max_part=3)
sage: I.children(I([2,1,0,0], check=False))
[[2, 2, 0, 0], [2, 1, 1, 0], [2, 1, 0, 1]]

is_canonical(v, check=True)
Returns True if the integer list v is maximal in its orbit under the action of the permutation group given to define self. Such integer vectors are said to be canonical. A vector v is canonical if and only if

\[ v = \max_{\text{lex order}} \{ g \cdot v | g \in G \} \]

EXAMPLES:

sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]), max_part=3)
lift(elt)
Lift the element elt inside the ambient space from which self is a quotient.

EXAMPLES:

```
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup(
            [[(1,2,3,4)]]), \n            -> max_part=1)
sage: v = S.lift([1,0,1,0]); v
[1, 0, 1, 0]
sage: v in IntegerVectors(2,4,max_part=1)
True
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup(
            [[(1,2,3,4)]]), \n            -> sum=6)
sage: v = S.lift(S.list()[5]); v
[4, 1, 1, 0]
sage: v in IntegerVectors(n=6)
True
```

orbit(v)
Returns the orbit of the vector v under the action of the permutation group defining self. The result is a set.

INPUT:

* v - an element of self or any list of length the degree of the permutation group.

EXAMPLES:

We convert the result in a list in increasing lexicographic order, to get a reproducible doctest:

```
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup(
            [[(1,2,3,4)]], 4)
sage: I.orbit([1,1,1,1])
{(1, 1, 1, 1)}
sage: sorted(I.orbit([3,0,0,1]))
[[0, 0, 1, 3], [0, 1, 3, 0], [1, 3, 0, 0], [3, 0, 0, 1]]
```

permutation_group()
Returns the permutation group given to define self.

EXAMPLES:

```
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup(
            [[(1,2,3,4)]], 5)
sage: I.permutation_group()
Permutation Group with generators [(1,2,3)]
```

retract(elt)
Return the canonical representative of the orbit of the integer elt under the action of the permutation group defining self.
If the element \( \texttt{elt} \) is already maximal in its orbits for the lexicographic order, \( \texttt{elt} \) is thus the good representative for its orbit.

**EXAMPLES:**

```python
sage: S = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]),
      sum=2, max_part=1)
sage: S.retract([1,1,0,0])
[1, 1, 0, 0]
sage: S.retract([1,0,1,0])
[1, 0, 1, 0]
sage: S.retract([1,0,0,1])
[1, 1, 0, 0]
sage: S.retract([0,1,1,0])
[1, 1, 0, 0]
sage: S.retract([0,1,0,1])
[1, 0, 1, 0]
sage: S.retract([0,0,1,1])
[1, 1, 0, 0]
```

**roots()**

Returns the root of generation of \( \texttt{self} \). This method is required to build the tree structure of \( \texttt{self} \) which inherits from the class \( \text{RecursivelyEnumeratedSet\_forest} \).

**EXAMPLES:**

```python
sage: I = IntegerVectorsModPermutationGroup(PermutationGroup([(1,2,3,4)]))
sage: I.roots()
[[0, 0, 0, 0]]
```

### 5.1.128 Tamari Interval-posets

This module implements Tamari interval-posets: combinatorial objects which represent intervals of the Tamari order. They have been introduced in [CP2015] and allow for many combinatorial operations on Tamari intervals. In particular, they are linked to \( \text{DyckWords} \) and \( \text{BinaryTrees} \). An introduction into Tamari interval-posets is given in Chapter 7 of [Pons2013].

The Tamari lattice can be defined as a lattice structure on either of several classes of Catalan objects, especially binary trees and Dyck paths [Tam1962] [HT1972] [Sta-EC2]. An interval can be seen as a pair of comparable elements. The number of intervals has been given in [Cha2008].

**AUTHORS:**

- Viviane Pons 2014: initial implementation
- Frédéric Chapoton 2014: review
- Darij Grinberg 2014: review
- Travis Scrimshaw 2014: review

```python
sage.combinat.interval_posets.TIP
alias of TamariIntervalPoset
class sage.combinat.interval_posets.TamariIntervalPoset(parent, size, relations=None, check=True)
              Bases: Element

The class of Tamari interval-posets.
```
An interval-poset is a labelled poset of size \( n \), with labels \( 1, 2, \ldots, n \), satisfying the following conditions:

- if \( a < c \) (as integers) and \( a \) precedes \( c \) in the poset, then, for all \( b \) such that \( a < b < c \), \( b \) precedes \( c \),
- if \( a < c \) (as integers) and \( c \) precedes \( a \) in the poset, then, for all \( b \) such that \( a < b < c \), \( b \) precedes \( a \).

We use the word “precedes” here to distinguish the poset order and the natural order on numbers. “Precedes” means “is smaller than with respect to the poset structure”; this does not imply a covering relation.

Interval-posets of size \( n \) are in bijection with intervals of the Tamari lattice of binary trees of size \( n \). Specifically, if \( P \) is an interval-poset of size \( n \), then the set of linear extensions of \( P \) (as permutations in \( S_n \)) is an interval in the right weak order (see \texttt{permutohedron_lequal()}, and is in fact the preimage of an interval in the Tamari lattice (of binary trees of size \( n \)) under the operation which sends a permutation to its right-to-left binary search tree (\texttt{binary_search_tree()} with the \texttt{left_to_right} variable set to \texttt{False}) without its labelling.

INPUT:

- \texttt{size} – an integer, the size of the interval-posets (number of vertices)
- \texttt{relations} – a list (or tuple) of pairs \((a, b)\) (themselves lists or tuples), each representing a relation of the form ‘\( a \) precedes \( b \)’ in the poset.
- \texttt{check} – (default: True) whether to check the interval-poset condition or not.

\textbf{Warning:} The \texttt{relations} input can be a list or tuple, but not an iterator (nor should its entries be iterators).

\textbf{NOTATION:}

Here and in the following, the signs \(<\) and \(>\) always refer to the natural ordering on integers, whereas the word “precedes” refers to the order of the interval-poset. “Minimal” and “maximal” refer to the natural ordering on integers.

The \textit{increasing relations} of an interval-poset \( P \) mean the pairs \((a, b)\) of elements of \( P \) such that \( a < b \) as integers and \( a \) precedes \( b \) in \( P \). The \textit{initial forest} of \( P \) is the poset obtained by imposing (only) the increasing relations on the ground set of \( P \). It is a sub-interval poset of \( P \), and is a forest with its roots on top. This forest is usually given the structure of a planar forest by ordering brother nodes by their labels; it then has the property that if its nodes are traversed in post-order (see \texttt{post_order_traversal()}, and traverse the trees of the forest from left to right as well), then the labels encountered are \( 1, 2, \ldots, n \) in this order.

The \textit{decreasing relations} of an interval-poset \( P \) mean the pairs \((a, b)\) of elements of \( P \) such that \( b < a \) as integers and \( a \) precedes \( b \) in \( P \). The \textit{final forest} of \( P \) is the poset obtained by imposing (only) the decreasing relations on the ground set of \( P \). It is a sub-interval poset of \( P \), and is a forest with its roots on top. This forest is usually given the structure of a planar forest by ordering brother nodes by their labels; it then has the property that if its nodes are traversed in pre-order (see \texttt{pre_order_traversal()}, and traverse the trees of the forest from left to right as well), then the labels encountered are \( 1, 2, \ldots, n \) in this order.

\textbf{EXAMPLES:}

\texttt{sage:} TamariIntervalPoset(0, [])
The Tamari interval of size 0 induced by relations []
\texttt{sage:} TamariIntervalPoset(3, [])
The Tamari interval of size 3 induced by relations []
\texttt{sage:} TamariIntervalPoset(3, [(1,2)])
The Tamari interval of size 3 induced by relations [(1, 2)]
\texttt{sage:} TamariIntervalPoset(3, [(1,2),(2,3)])
The Tamari interval of size 3 induced by relations [(1, 2), (2, 3)]
\texttt{sage:} TamariIntervalPoset(3, [(1,2),(2,3),(1,3)])
The Tamari interval of size 3 induced by relations [(1, 2), (2, 3)]
sage: TamariIntervalPoset(3,[(1,2),(3,2)])
The Tamari interval of size 3 induced by relations [(1, 2), (3, 2)]
sage: TamariIntervalPoset(3,[[1,2],[2,3]])
The Tamari interval of size 3 induced by relations [(1, 2), (2, 3)]
sage: TamariIntervalPoset(3,[[1,2],[2,3],[1,2],[1,3]])
The Tamari interval of size 3 induced by relations [(1, 2), (2, 3)]
sage: TamariIntervalPoset(3,[(3,4)])
Traceback (most recent call last):
    ...
ValueError: the relations do not correspond to the size of the poset
sage: TamariIntervalPoset(2,[(2,1),(1,2)])
Traceback (most recent call last):
    ...
ValueError: The graph is not directed acyclic
sage: TamariIntervalPoset(3,[(1,3)])
Traceback (most recent call last):
    ...
ValueError: this does not satisfy the Tamari interval-poset condition

It is also possible to transform a poset directly into an interval-poset:

sage: TIP = TamariIntervalPosets()
sage: p = Poset(((1,2,3), [(1,2)]))
sage: TIP(p)
The Tamari interval of size 3 induced by relations [(1, 2)]
sage: TIP(Poset({1: []}))
The Tamari interval of size 1 induced by relations []
sage: TIP(Poset({}))
The Tamari interval of size 0 induced by relations []

binary_trees()
Return an iterator on all the binary trees in the interval represented by self.

See also:
interval_cardinality()

EXAMPLES:

sage: list(TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)]).binary_trees())
[[[., [[., .], .]], .], [[., [[., .], .]], .], [[., [[., .], .]], .], [[., [[., .], .]], .]]
sage: set(TamariIntervalPoset(4,[]).binary_trees()) == set(BinaryTrees(4))
True

complement()
Return the complement of the interval-poset self.

If \( P \) is a Tamari interval-poset of size \( n \), then the complement of \( P \) is defined as the interval-poset \( Q \) whose base set is \( \{1, 2, \ldots, n\} \) (just as for \( P \)), but whose order relation has \( a \) precede \( b \) if and only if

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\( n + 1 - a \) precedes \( n + 1 - b \) in \( P \).

In terms of the Tamari lattice, the complement is the symmetric of self. It is formed from the left-right symmetrized of the binary trees of the interval (switching left and right subtrees, see \texttt{left_right_symmetry()}). In particular, initial intervals are sent to final intervals and vice-versa.

EXAMPLES:

```
sage: TamariIntervalPoset(3, [(2, 1), (3, 1)]).complement()
The Tamari interval of size 3 induced by relations [(1, 3), (2, 3)]
sage: TamariIntervalPoset(0, []).complement()
The Tamari interval of size 0 induced by relations []
sage: ip = TamariIntervalPoset(4, [(1, 2), (2, 4), (3, 4)])
sage: ip.complement() == TamariIntervalPoset(4, [(2, 1), (3, 1), (4, 3)])
True
sage: ip.lower_binary_tree() == ip.complement().upper_binary_tree().left_right_symmetry()
True
sage: ip.upper_binary_tree() == ip.complement().lower_binary_tree().left_right_symmetry()
True
sage: ip.is_initial_interval()
True
sage: ip.complement().is_final_interval()
True
```

\texttt{contains_binary_tree(binary_tree)}

Return whether the interval represented by self contains the binary tree \texttt{binary_tree}.

INPUT:

- \texttt{binary_tree} – a binary tree

See also:

\texttt{contains_dyck_word()}  

EXAMPLES:

```
sage: ip = TamariIntervalPoset(4, [(2, 4), (3, 4), (2, 1), (3, 1)])
sage: ip.contains_binary_tree(BinaryTree([[None, [None, []]], None]))
True
sage: ip.contains_binary_tree(BinaryTree([[None, [[]], None]]))
True
sage: ip.contains_binary_tree(BinaryTree([[], [[]], None]))
False
sage: ip.contains_binary_tree(ip.lower_binary_tree())
True
sage: ip.contains_binary_tree(ip.upper_binary_tree())
True
sage: all(ip.contains_binary_tree(bt) for bt in ip.binary_trees())
True
```

\texttt{contains_dyck_word(dyck_word)}

Return whether the interval represented by self contains the Dyck word \texttt{dyck_word}.

INPUT:
• dyck_word – a Dyck word

See also:

contains_binary_tree()

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4, [(2,4),(3,4),(2,1),(3,1)])
sage: ip.contains_dyck_word(DyckWord([1,1,1,0,0,0,1,0]))
True
sage: ip.contains_dyck_word(DyckWord([1,1,0,1,0,1,0,0]))
True
sage: ip.contains_dyck_word(DyckWord([1,0,1,1,0,1,0,0]))
False
sage: ip.contains_dyck_word(ip.lower_dyck_word())
True
sage: ip.contains_dyck_word(ip.upper_dyck_word())
True
sage: all(ip.contains_dyck_word(bt) for bt in ip.dyck_words())
True
```

contains_interval(other)

Return whether the interval represented by other is contained in self as an interval of the Tamari lattice. In terms of interval-posets, it means that all relations of self are relations of other.

INPUT:

• other – an interval-poset

EXAMPLES:

```python
sage: ip1 = TamariIntervalPoset(4, [(1,2),(2,3),(3,4)])
sage: ip2 = TamariIntervalPoset(4, [(2,3)])
sage: ip2.contains_interval(ip1)
True
sage: ip3 = TamariIntervalPoset(4, [(2,1)])
sage: ip2.contains_interval(ip3)
False
sage: ip4 = TamariIntervalPoset(3, [(2,3)])
sage: ip2.contains_interval(ip4)
False
```

cubical_coordinates()

Return the cubical coordinates of self.

This provides a fast and natural way to order the set of interval-posets of a given size.

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4, [(1,2),(2,3)])
sage: ip.cubical_coordinates()
(-1, -2, 0)
```

REFERENCES:

• [Com2019]
decomposition_to_triple()
Decompose an interval-poset into a triple \((\text{left}, \text{right}, r)\).
For the inverse method, see \texttt{TamariIntervalPosets.recomposition_from_triple().}

OUTPUT:
a triple \((\text{left}, \text{right}, r)\) where \text{left} and \text{right} are interval-posets and \text{r} (an integer) is the parameter of the decomposition.

EXAMPLES:

\begin{verbatim}
sage: tip = TamariIntervalPoset(8, [(1,2), (2,4), (3,4), (6,7), (3,2), (5,4),
(6,4), (8,7)])
sage: tip.decomposition_to_triple()
(The Tamari interval of size 3 induced by relations [(1, 2), (3, 2)],
The Tamari interval of size 4 induced by relations [(2, 3), (4, 3)],
2)
sage: tip == TamariIntervalPosets.recomposition_from_triple(*tip.decomposition_to_triple())
True
\end{verbatim}

REFERENCES:
• [CP2015]

decreasing_children\((v)\)
Return the children of \(v\) in the final forest of \text{self}.

INPUT:
• \(v\) – an integer representing a vertex of \text{self} (between 1 and size)

OUTPUT:
The list of all children of \(v\) in the final forest of \text{self}, in increasing order.

EXAMPLES:

\begin{verbatim}
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)])
sage: ip.decreasing_children(2)
[3, 5]
sage: ip.decreasing_children(3)
[4]
sage: ip.decreasing_children(1)
[]
\end{verbatim}

decreasing_cover_relations()
Return the cover relations of the final forest of \text{self}.
This is the poset formed by keeping only the relations of the form \(a\) precedes \(b\) with \(a > b\).
The final forest of \text{self} is a forest with its roots being on top. It is also called the decreasing poset of \text{self}.

\textbf{Warning:} This method computes the cover relations of the final forest. This is not identical with the cover relations of \text{self} which happen to be decreasing!
See also:

*final_forest()*

EXAMPLES:

```python
sage: TamariIntervalPoset(4,[(2,1),(3,2),(3,4),(4,2)]).decreasing_cover_relations()
[(4, 2), (3, 2), (2, 1)]
```

```python
sage: TamariIntervalPoset(4,[(2,1),(4,3),(2,3)]).decreasing_cover_relations()
[(4, 3), (2, 1)]
```

```python
sage: TamariIntervalPoset(3,[(2,1),(3,1),(3,2)]).decreasing_cover_relations()
[(3, 2), (2, 1)]
```

decreasing_parent(v)

Return the vertex parent of *v* in the final forest of *self*.

This is the highest (as integer!) vertex *a* < *v* such that *v* precedes *a*. If there is no such vertex (that is, *v* is a decreasing root), then None is returned.

INPUT:

* • v – an integer representing a vertex of *self* (between 1 and size)

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
ip
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
sage: ip.decreasing_parent(4)
3
sage: ip.decreasing_parent(3)
2
sage: ip.decreasing_parent(5)
2
sage: ip.decreasing_parent(2) is None
True
```

decreasing_roots()

Return the root vertices of the final forest of *self*.

These are the vertices *b* such that there is no *a* < *b* with *b* preceding *a*.

OUTPUT:

The list of all roots of the final forest of *self*, in increasing order.

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
ip
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
sage: ip.decreasing_roots()
[1, 2]
sage: ip.final_forest().decreasing_roots()
[1, 2]
```
dyck_words()

Return an iterator on all the Dyck words in the interval represented by self.

EXAMPLES:

```python
sage: list(TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)]).dyck_words())
[[1, 1, 1, 0, 0, 1, 0, 0],
 [1, 1, 1, 0, 0, 0, 1, 0],
 [1, 1, 0, 1, 0, 1, 0, 0],
 [1, 1, 0, 1, 0, 0, 1, 0]]
sage: set(TamariIntervalPoset(4,[]).dyck_words()) == set(DyckWords(4))
True
```

factor()

Return the unique decomposition as a list of connected components.

EXAMPLES:

```python
sage: factor(TamariIntervalPoset(2,[]))  # indirect doctest
[The Tamari interval of size 1 induced by relations [],
The Tamari interval of size 1 induced by relations []]
```

See also:

is_connected()

final_forest()

Return the final forest of self, i.e., the interval-poset formed with only the decreasing relations of self.

See also:

initial_forest()

EXAMPLES:

```python
sage: TamariIntervalPoset(4,[(2,1),(3,2),(3,4),(4,2)]).final_forest()
The Tamari interval of size 4 induced by relations [(4, 2), (3, 2), (2, 1)]
sage: ip = TamariIntervalPoset(3,[(2,1),(3,1)])
sage: ip.final_forest() == ip
True
```

ge(e1, e2)

Return whether e2 precedes or equals e1 in self.

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4,[(1,2),(2,3)])
sage: ip.ge(2,1)
True
sage: ip.ge(3,1)
True
sage: ip.ge(3,2)
True
sage: ip.ge(4,3)
False
sage: ip.ge(1,1)
True
```
grafting_tree()

Return the grafting tree of the interval-poset.

For the inverse method, see TamariIntervalPosets.from_grafting_tree().

EXAMPLES:

```python
sage: tip = TamariIntervalPoset(8, [(1,2), (2,4), (3,4), (6,7), (3,2), (5,4), (6,4), (8,7)])
```
```
sage: tip.grafting_tree()
2[1[0[., .], 0[., .]], 0[., 1[0[., .], 0[., .]]]]
```
```
sage: tip == TamariIntervalPosets.from_grafting_tree(tip.grafting_tree())
True
```

REFERENCES:

• [Pons2018]

gt(e1, e2)

Return whether e2 strictly precedes e1 in self.

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4,[(1,2),(2,3)])
```
```
sage: ip.gt(2,1)
True
```
```
sage: ip.gt(3,1)
True
```
```
sage: ip.gt(3,2)
True
```
```
sage: ip.gt(4,3)
False
```
```
sage: ip.gt(1,1)
False
```

increasing_children(v)

Return the children of v in the initial forest of self.

INPUT:

• v – an integer representing a vertex of self (between 1 and size)

OUTPUT:

The list of all children of v in the initial forest of self, in decreasing order.

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
```
```
sage: ip
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
```
```
sage: ip.increasing_children(2)
[1]
```
```
sage: ip.increasing_children(5)
[4, 3]
```
```
sage: ip.increasing_children(1)
[]
```
**increasing_cover_relations()**

Return the cover relations of the initial forest of self.

This is the poset formed by keeping only the relations of the form \(a \lessdot b\) with \(a < b\).

The initial forest of self is a forest with its roots being on top. It is also called the increasing poset of self.

**Warning:** This method computes the cover relations of the initial forest. This is not identical with the cover relations of self which happen to be increasing!

See also:

* initial_forest()

**EXAMPLES:**

```sage
tamari = IntervalPoset(4, [(1,2),(3,2),(2,4),(3,4)]).increasing_cover_relations()
tamari
```

```text
[(1, 2), (2, 4), (3, 4)]
```

```sage
tamari = IntervalPoset(3, [(1,2),(1,3),(2,3)]).increasing_cover_relations()
tamari
```

```text
[(1, 2), (2, 3)]
```

**increasing_parent(v)**

Return the vertex parent of v in the initial forest of self.

This is the lowest (as integer!) vertex \(b > v\) such that \(v \lessdot b\). If there is no such vertex (that is, \(v\) is an increasing root), then None is returned.

**INPUT:**

* v – an integer representing a vertex of self (between 1 and size)

**EXAMPLES:**

```sage
ip = IntervalPoset(6, [(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
ip
```

```
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
```

```sage
ip.increasing_parent(1)
ip.increasing_parent(3)
ip.increasing_parent(4)
ip.increasing_parent(5)
```

```text
2
5
5
None
```

**increasing_roots()**

Return the root vertices of the initial forest of self.

These are the vertices \(a\) of self such that there is no \(b > a\) with \(a \lessdot b\).

**OUTPUT:**

The list of all roots of the initial forest of self, in decreasing order.

**EXAMPLES:**

```sage
root = increasing_roots()
root
```

```text
[5, 4, 3, 2, 1]
```
 sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
      ip
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 
      5), (5, 2), (4, 3), (3, 2)]
sage: ip.increasing_roots()
[6, 5, 2]
sage: ip.initial_forest().increasing_roots()
[6, 5, 2]

`initial_forest()`

Return the initial forest of `self`, i.e., the interval-poset formed from only the increasing relations of `self`.

See also:

`final_forest()`

EXAMPLES:

 sage: TamariIntervalPoset(4,[(1,2),(3,2),(2,4),(3,4)]).initial_forest()
The Tamari interval of size 4 induced by relations [(1, 2), (2, 4), (3, 4)]
sage: ip = TamariIntervalPoset(4,[(1,2),(2,3)])
sage: ip.initial_forest() == ip
True

`insertion(i)`

Return the Tamari insertion of an integer `i` into the interval-poset `self`.

If `P` is a Tamari interval-poset of size `n` and `i` is an integer with `1 ≤ i ≤ n + 1`, then the Tamari insertion of `i` into `P` is defined as the Tamari interval-poset of size `n + 1` which corresponds to the interval `[C_1, C_2]` on the Tamari lattice, where the binary trees `C_1` and `C_2` are defined as follows: We write the interval-poset `P` as `[B_1, B_2]` for two binary trees `B_1` and `B_2`. We label the vertices of each of these two trees with the integers `1, 2, ..., i − 1, i + 1, i + 2, ..., n + 1` in such a way that the trees are binary search trees (this labelling is unique). Then, we insert `i` into each of these trees (in the way as explained in `binary_search_insert()`). The shapes of the resulting two trees are denoted `C_1` and `C_2`.

An alternative way to construct the insertion of `i` into `P` is by relabeling each vertex `u` of `P` satisfying `u ≥ i` (as integers) as `u + 1`, and then adding a vertex `i` which should precede `i − 1` and `i + 1`.

Todo: To study this, it would be more natural to define interval-posets on arbitrary ordered sets rather than just on `{1, 2, ..., n}`.

EXAMPLES:

 sage: ip = TamariIntervalPoset(4, [(2, 3), (4, 3)]); ip
The Tamari interval of size 4 induced by relations [(2, 3), (4, 3)]
sage: ip.insertion(1)
The Tamari interval of size 5 induced by relations [(1, 2), (3, 4), (5, 4)]
sage: ip.insertion(2)
The Tamari interval of size 5 induced by relations [(2, 3), (3, 4), (5, 4), (2, 
      ˓→1)]
sage: ip.insertion(3)
The Tamari interval of size 5 induced by relations [(2, 4), (3, 4), (5, 4), (3, 
      ˓→2)]
sage: ip.insertion(4)

(continues on next page)
The Tamari interval of size 5 induced by relations \[[2, 3), (4, 5), (5, 3), (4, \rightarrow 3)\]
\sage: ip.insertion(5)
The Tamari interval of size 5 induced by relations \[[2, 3), (5, 4), (4, 3)\]
\sage: ip = TamariIntervalPoset(0, [])
\sage: ip.insertion(1)
The Tamari interval of size 1 induced by relations []
\sage: ip = TamariIntervalPoset(1, [])
\sage: ip.insertion(1)
The Tamari interval of size 2 induced by relations \[[1, 2]\]
\sage: ip.insertion(2)
The Tamari interval of size 2 induced by relations \[[2, 1]\]

intersection\(other\)
Return the interval-poset formed by combining the relations from both \self and \other. It corresponds to the intersection of the two corresponding intervals of the Tamari lattice.

**INPUT:**

- \other – an interval-poset of the same size as \self

**EXAMPLES:**

\sage: ip1 = TamariIntervalPoset(4,\[(1,2),(2,3)\])
\sage: ip2 = TamariIntervalPoset(4,\[(4,3)\])
\sage: ip1.intersection(ip2)
The Tamari interval of size 4 induced by relations \[[1, 2), (2, 3), (4, 3)\]
\sage: ip3 = TamariIntervalPoset(4,\[(2,1)\])
\sage: ip1.intersection(ip3)
Traceback (most recent call last):
... ValueError: this intersection is empty, it does not correspond to an interval-poset
\sage: ip4 = TamariIntervalPoset(3,\[(2,3)\])
\sage: ip2.intersection(ip4)
Traceback (most recent call last):
... ValueError: intersections are only possible on interval-posets of the same size

interval_cardinality()
Return the cardinality of the interval, i.e., the number of elements (binary trees or Dyck words) in the interval represented by \self.

Not to be confused with \size which is the number of vertices.

**See also:**

\binary_trees()

**EXAMPLES:**

\sage: TamariIntervalPoset(4,\[(2,4),(3,4),(2,1),(3,1)\]).interval_cardinality()
4
sage: TamariIntervalPoset(4,[]).interval_cardinality()
14
sage: TamariIntervalPoset(4,[(1,2),(2,3),(3,4)]).interval_cardinality()
1

**is_connected()**
Return whether `self` is a connected Tamari interval.

This means that the Hasse diagram is connected.

This condition is invariant under complementation.

See also:
- `is_indecomposable()`, `factor()`

EXAMPLES:

```python
sage: len([T for T in TamariIntervalPosets(3) if T.is_connected()])
8
```

**is_dexter()**
Return whether `self` is a dexter Tamari interval.

This is defined by an exclusion pattern in the Hasse diagram. See the code for the exact description.

This condition is not invariant under complementation.

EXAMPLES:

```python
sage: len([T for T in TamariIntervalPosets(3) if T.is_dexter()])
12
```

**is_exceptional()**
Return whether `self` is an exceptional Tamari interval.

This is defined by exclusion of a simple pattern in the Hasse diagram, namely there is no configuration \( y \prec x \prec z \) with \( 1 \leq y < x < z \leq n \).

This condition is invariant under complementation.

EXAMPLES:

```python
sage: len([T for T in TamariIntervalPosets(3) if T.is_exceptional()])
12
```

**is_final_interval()**
Return if `self` corresponds to a final interval of the Tamari lattice.

This means that its upper end is the largest element of the lattice. It consists of checking that `self` does not contain any increasing relations.

See also:
- `is_initial_interval()`

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: ip = TamariIntervalPoset(4, [(4, 3), (3, 1), (2, 1)])
sage: ip.is_final_interval()
True
sage: ip.upper_dyck_word()
[1, 1, 1, 1, 0, 0, 0, 0]
sage: ip = TamariIntervalPoset(4, [(4, 3), (3, 1), (2, 1), (2, 3)])
sage: ip.is_final_interval()
False
sage: ip.upper_dyck_word()
[1, 1, 0, 1, 1, 0, 0, 0]
sage: all(dw.tamari_interval(DyckWord([1, 1, 1, 0, 0, 0])).is_final_interval() for dw in DyckWords(3))
True
```

**is_indecomposable()**

Return whether self is an indecomposable Tamari interval.

This is the terminology of [Cha2008].

This means that the upper binary tree has an empty left subtree.

This condition is not invariant under complementation.

See also:

`is_connected()`

EXAMPLES:

```python
sage: len([T for T in TamariIntervalPosets(3) if T.is_indecomposable()])
8
```

**is_ininitely_modern()**

Return whether self is an infinitely-modern Tamari interval.

This is defined by the exclusion of the configuration $i \rightarrow i + 1$ and $j + 1 \rightarrow j$ with $i < j$.

This condition is invariant under complementation.

See also:

`is_new()`, `is_modern()`

EXAMPLES:

```python
sage: len([T for T in TamariIntervalPosets(3) if T.is_ininitely_modern()])
12
```

REFERENCES:

- [Rog2018]

**is_initial_interval()**

Return if self corresponds to an initial interval of the Tamari lattice.

This means that its lower end is the smallest element of the lattice. It consists of checking that self does not contain any decreasing relations.
See also:

is_final_interval()

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4, [(1, 2), (2, 4), (3, 4)])
sage: ip.is_initial_interval()
True
sage: ip.lower_dyck_word()
[1, 0, 1, 0, 1, 0]
sage: ip = TamariIntervalPoset(4, [(1, 2), (2, 4), (3, 4), (3, 2)])
sage: ip.is_initial_interval()
False
sage: ip.lower_dyck_word()
[1, 0, 1, 1, 0, 0]
sage: all(DyckWord([1,0,1,0,1,0]).tamari_interval(dw).is_initial_interval() for dw in DyckWords(3))
True
```

is_linear_extension(perm)

Return whether the permutation `perm` is a linear extension of `self`.

INPUT:

• `perm` – a permutation of the size of `self`

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4, [(1, 2), (2, 3), (4, 3)])
sage: ip.is_linear_extension([1, 4, 2, 3])
True
sage: ip.is_linear_extension(Permutation([1, 4, 2, 3]))
True
sage: ip.is_linear_extension(Permutation([1, 4, 3, 2]))
False
```

is_modern()

Return whether `self` is a modern Tamari interval.

This is defined by exclusion of a simple pattern in the Hasse diagram, namely there is no configuration \( y \rightarrow x \leftarrow z \) with \( 1 \leq y < x < z \leq n \).

This condition is invariant under complementation.

See also:

is_new(), is_ininitely_modern()

EXAMPLES:

```python
sage: len([T for T in TamariIntervalPosets(3) if T.is_modern()])
12
```

REFERENCES:

• [Rog2018]
**is_new()**

Return whether `self` is a new Tamari interval.

Here ‘new’ means that the interval is not contained in any facet of the associahedron. This condition is invariant under complementation.

They have been considered in section 9 of [Cha2008].

See also:

`is_modern()`

**EXAMPLES:**

```
sage: TIP4 = TamariIntervalPosets(4)
sage: len([u for u in TIP4 if u.is_new()])
12
sage: TIP3 = TamariIntervalPosets(3)
sage: len([u for u in TIP3 if u.is_new()])
3
```

**is_simple()**

Return whether `self` is a simple Tamari interval.

Here ‘simple’ means that the interval contains a unique binary tree.

These intervals define the simple modules over the incidence algebras of the Tamari lattices.

See also:

`is_final_interval(), is_initial_interval()`

**EXAMPLES:**

```
sage: TIP4 = TamariIntervalPosets(4)
sage: len([u for u in TIP4 if u.is_simple()])
14
sage: TIP3 = TamariIntervalPosets(3)
sage: len([u for u in TIP3 if u.is_simple()])
5
```

**is_synchronized()**

Return whether `self` is a synchronized Tamari interval.

This means that the upper and lower binary trees have the same canopee. This condition is invariant under complementation.

This has been considered in [FPR2015]. The numbers of synchronized intervals are given by the sequence OEIS sequence A000139.

**EXAMPLES:**

```
sage: len([T for T in TamariIntervalPosets(3) if T.is_synchronized()])
6
```
latex_options()

Return the latex options for use in the \_latex\_ function as a dictionary.

The default values are set using the options.

- **tikz_scale** – (default: 1) scale for use with the tikz package
- **line_width** – (default: 1) value representing the line width (additionally scaled by \texttt{tikz_scale})
- **color_decreasing** – (default: 'red') the color for decreasing relations
- **color_increasing** – (default: 'blue') the color for increasing relations
- **hspace** – (default: 1) the difference between horizontal coordinates of adjacent vertices
- **vspace** – (default: 1) the difference between vertical coordinates of adjacent vertices

EXAMPLES:

```sage
ip = TamariIntervalPoset(4, [(2,4),(3,4),(2,1),(3,1)])
ip.latex_options()['color_decreasing']
'red'
ip.latex_options()['hspace']
1
```

le(e1, e2)

Return whether e1 precedes or equals e2 in self.

EXAMPLES:

```sage
ip = TamariIntervalPoset(4, [(1,2),(2,3)])
ip.le(1,2)
True
ip.le(1,3)
True
ip.le(2,3)
True
ip.le(3,4)
False
ip.le(1,1)
True
```

left_branch_involution()

Return the image of self by the left-branch involution.

OUTPUT: an interval-poset

See also:

\texttt{rise_contact_involution()}

EXAMPLES:

```sage
tip = TamariIntervalPoset(8 , [(1,2), (2,4), (3,4), (6,7), (3,2), (5,4), ...
→(6,4), (8,7)])
t = tip.left_branch_involution(); t
The Tamari interval of size 8 induced by relations [(1, 6), (2, 6),
(3, 5), (4, 5), (5, 6), (6, 8), (7, 8), (7, 6), (4, 3), (3, 1),
(2, 1)]
```

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REFERENCES:

- [Pons2018]

linear_extensions()

Return an iterator on the permutations which are linear extensions of self.

They form an interval of the right weak order (also called the right permutahedron order – see permutohedron_lequal() for a definition).

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(3,[(1,2),(3,2)])
sage: list(ip.linear_extensions())
[[3, 1, 2], [1, 3, 2]]
sage: ip = TamariIntervalPoset(4,[(1,2),(2,3),(4,3)])
sage: list(ip.linear_extensions())
[[4, 1, 2, 3], [1, 2, 4, 3], [1, 4, 2, 3]]
```

lower_binary_tree()

Return the lowest binary tree in the interval of the Tamari lattice represented by self.

This is a binary tree. It is the shape of the unique binary search tree whose left-branch ordered forest (i.e., the result of applying to_ordered_tree_left_branch() and cutting off the root) is the final forest of self.

See also:

lower_dyck_word()

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(4,5)])
```

lower_contained_intervals()

If self represents the interval \([t_1, t_2]\) of the Tamari lattice, return an iterator on all intervals \([t_1, t]\) with \(t \leq t_2\) for the Tamari lattice.

In terms of interval-posets, it corresponds to adding all possible relations of the form \(n\) precedes \(m\) with \(n < m\).

EXAMPLES:
Combinatorics, Release 10.0

sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: list(ip.lower_contained_intervals())

The Tamari interval of size 4 induced by relations [(2, 4), (3, 4), (3, 1), (2, 1)],
The Tamari interval of size 4 induced by relations [(1, 4), (2, 4), (3, 4), (3, 1), (2, 1)],
The Tamari interval of size 4 induced by relations [(2, 3), (3, 4), (3, 1), (2, 1)],
The Tamari interval of size 4 induced by relations [(1, 4), (2, 3), (3, 4), (3, 1), (2, 1)]

sage: ip = TamariIntervalPoset(4,[])
sage: len(list(ip.lower_contained_intervals()))
14

lower_contains_interval(other)
Return whether the interval represented by other is contained in self as an interval of the Tamari lattice and if they share the same lower bound.

As interval-posets, it means that other contains the relations of self plus some extra increasing relations.

INPUT:
• other -- an interval-poset

EXAMPLES:
sage: ip1 = TamariIntervalPoset(4,[(1,2),(2,3),(4,3)])
sage: ip2 = TamariIntervalPoset(4,[(4,3)])
sage: ip2.lower_contains_interval(ip1)
True
sage: ip2.contains_interval(ip1) and ip2.lower_binary_tree() == ip1.lower_binary_tree()
True
sage: ip3 = TamariIntervalPoset(4,[(4,3),(2,1)])
sage: ip2.contains_interval(ip3)
True
sage: ip2.lower_binary_tree() == ip3.lower_binary_tree()
False
sage: ip2.lower_contains_interval(ip3)
False

lower_dyck_word()
Return the lowest Dyck word in the interval of the Tamari lattice represented by self.

See also:
lower_binary_tree()

EXAMPLES:
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(4,5)]); ip
The Tamari interval of size 6 induced by relations [(1, 2), (4, 5), (6, 5), (5, 1), (2, 4), (3, 2)]
sage: ip.lower_dyck_word()
[1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0]
sage: TamariIntervalPosets.final_forest(ip.lower_dyck_word()) == ip.final_

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Combinatorics, Release 10.0

\[\Box\] forest()
True
\[\Box\]

\[\Box\]
sage: ip == TamariIntervalPosets.from_dyck_words(ip.lower_dyck_word(),ip.upper_dyck_word())
True

lt(e1, e2)
Return whether e1 strictly precedes e2 in self.

EXAMPLES:
\[\Box\]

\[\Box\]

\[\Box\]

max_linear_extension()
Return the maximal permutation for the right weak order which is a linear extension of self.

This is also the maximal permutation in the sylvester class of self.upper_binary_tree() and is a 132-avoiding permutation.

The right weak order is also known as the right permutohedron order. See permutohedron_lequal() for its definition.

EXAMPLES:
\[\Box\]

\[\Box\]

maximal_chain_binary_trees()
Return an iterator on the binary trees forming a longest chain of self (regarding self as an interval of the Tamari lattice).
EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: list(ip.maximal_chain_binary_trees())[['[[., .], .], .], [[., [[., .], .]], .], [[., [[., .]], .], .], [[., [[., .]], .], .]]
sage: ip = TamariIntervalPoset(4,[])
sage: list(ip.maximal_chain_binary_trees())[['[[., .], .], .], [[., [[., .], .]], .], [[., [[., .]], .], .], [[., [[., .]], .], .]]
```

```python
maximal_chain_dyck_words()  
Return an iterator on the Dyck words forming a longest chain of self (regarding self as an interval of the Tamari lattice).

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: list(ip.maximal_chain_dyck_words())[[1, 1, 0, 1, 0, 0, 1, 0], [1, 1, 0, 1, 0, 1, 0, 0], [1, 1, 1, 0, 0, 1, 0, 0]]
sage: ip = TamariIntervalPoset(4,[])
sage: list(ip.maximal_chain_dyck_words())[[1, 0, 1, 0, 1, 0, 1, 0], [1, 1, 0, 1, 0, 0, 1, 0], [1, 1, 0, 1, 0, 1, 0, 0], [1, 1, 1, 0, 0, 1, 0, 0], [1, 1, 1, 1, 0, 0, 0, 0]]
```

```python
maximal_chain_tamari_intervals()  
Return an iterator on the upper contained intervals of one longest chain of the Tamari interval represented by self.

If self represents the interval $[T_1, T_2]$ of the Tamari lattice, this returns intervals $[T', T_2]$ with $T'$ following one longest chain between $T_1$ and $T_2$.

To obtain a longest chain, we use the Tamari inversions of self. The elements of the chain are obtained by adding one by one the relations $(b, a)$ from each Tamari inversion $(a, b)$ to self, where the Tamari inversions are taken in lexicographic order.

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: list(ip.maximal_chain_tamari_intervals())[[The Tamari interval of size 4 induced by relations [(2, 4), (3, 4), (3, 1), (2, 1)],
The Tamari interval of size 4 induced by relations [(2, 4), (3, 4), (4, 1), (3, 2), (2, 1)],
The Tamari interval of size 4 induced by relations [(2, 4), (3, 4), (4, 1), (3, 2), (2, 1)]]
```
**min_linear_extension()**

Return the minimal permutation for the right weak order which is a linear extension of `self`.

This is also the minimal permutation in the sylvester class of `self.lower_binary_tree()` and is a 312-avoiding permutation.

The right weak order is also known as the right permutohedron order. See `permutohedron_lequal()` for its definition.

**EXAMPLES:**

```python
sage: ip = TamariIntervalPoset(4,[(1,2),(2,3),(4,3)])
sage: ip.min_linear_extension()
[1, 2, 3, 4]
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(4,5)])
sage: ip.min_linear_extension()
[1, 2, 3, 4, 5, 6]
sage: ip = TamariIntervalPoset(0,[])
sage: ip.min_linear_extension()
[]
sage: ip = TamariIntervalPoset(5, [(1, 4), (2, 4), (3, 4), (5, 4)]); ip
The Tamari interval of size 5 induced by relations [(1, 4), (2, 4), (3, 4), (5, 4)]
sage: ip.min_linear_extension()
[1, 2, 3, 4, 5]
```

**new_decomposition()**

Return the decomposition of the interval-poset into new interval-posets.

Every interval-poset has a unique decomposition as a planar tree of new interval-posets, as explained in [Cha2008]. This function computes the terms of this decomposition, but not the planar tree.

For the number of terms, you can use instead the method `number_of_new_components()`.

**OUTPUT:**

a list of new interval-posets.

**See also:**

`number_of_new_components()`, `is_new()`

**EXAMPLES:**

```python
sage: ex = TamariIntervalPosets(4)[11]
sage: ex.number_of_new_components()
3
```
sage: ex.new_decomposition()
[The Tamari interval of size 1 induced by relations [],
The Tamari interval of size 2 induced by relations [],
The Tamari interval of size 1 induced by relations []]

number_of_new_components()
Return the number of terms in the decomposition in new interval-posets.

Every interval-poset has a unique decomposition as a planar tree of new interval-posets, as explained in [Cha2008]. This function just computes the number of terms, not the planar tree nor the terms themselves.

See also:
is_new(), new_decomposition()

EXAMPLES:

```
sage: TIP4 = TamariIntervalPosets(4)
sage: nb = [u.number_of_new_components() for u in TIP4]
sage: [nb.count(i) for i in range(1, 5)]
[12, 21, 21, 14]
```

count_of_intermediate_posets()
Return the number of Tamari inversions of self.

This is also the length the longest chain of the Tamari interval represented by self.

EXAMPLES:

```
sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: ip.number_of_tamari_inversions()
2
sage: ip = TamariIntervalPoset(4,[])
sage: ip.number_of_tamari_inversions()
6
sage: ip = TamariIntervalPoset(3,[])
sage: ip.number_of_tamari_inversions()
3
```

plot(**kwds)
Return a picture.

The picture represents the Hasse diagram, where the covers are colored in blue if they are increasing and in red if they are decreasing.

This uses the same coordinates as the latex view.

EXAMPLES:

```
sage: ti = TamariIntervalPosets(4)[2]
sage: ti.plot()
Graphics object consisting of 6 graphics primitives
```

poset()
Return self as a labelled poset.

An interval-poset is indeed constructed from a labelled poset which is stored internally. This method allows to access the poset and all the associated methods.
EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4, [(1,2), (3,2), (2,4), (3,4)])
sage: pos = ip.poset(); pos
Finite poset containing 4 elements
sage: pos.maximal_chains()
[[3, 2, 4], [1, 2, 4]]
sage: pos.maximal_elements()
[4]
sage: pos.is_lattice()
False
```

**rise_contact_involution()**

Return the image of `self` by the rise-contact involution.

**OUTPUT:** an interval-poset

This is defined by conjugating the complement involution by the left-branch involution.

**See also:**

`left_branch_involution()`, `complement()`

EXAMPLES:

```python
sage: tip = TamariIntervalPoset(8, [(1,2), (2,4), (3,4), (6,7), (3,2), (5,4), (6,4), (8,7)])
sage: t = tip.rise_contact_involution(); t
The Tamari interval of size 8 induced by relations [(2, 8), (3, 8), (4, 5), (5, 7), (6, 7), (7, 8), (8, 1), (7, 2), (6, 2), (5, 3), (4, 3), (3, 2), (2, 1)]
sage: t.rise_contact_involution() == tip
True
sage: tip.lower_dyck_word().number_of_touch_points() == t.upper_dyck_word().number_of_initial_rises()
True
sage: tip.number_of_tamari_inversions() == t.number_of_tamari_inversions()
True
```

REFERENCES:

- [Pons2018]

**set_latex_options(D)**

Set the latex options for use in the `_latex_` function.

The default values are set in the `__init__` function.

- `tikz_scale` – (default: 1) scale for use with the tikz package
- `line_width` – (default: 1 * `tikz_scale`) value representing the line width
- `color_decreasing` – (default: red) the color for decreasing relations
- `color_increasing` – (default: blue) the color for increasing relations
- `hspace` – (default: 1) the difference between horizontal coordinates of adjacent vertices
- `vspace` – (default: 1) the difference between vertical coordinates of adjacent vertices

INPUT:
Combinatorics, Release 10.0

• D – a dictionary with a list of latex parameters to change

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: ip.latex_options()['color_decreasing']
'red'
sage: ip.set_latex_options({'color_decreasing':'green'})
sage: ip.latex_options()['color_decreasing']
'green'
sage: ip.set_latex_options({'color_increasing':'black'})
sage: ip.latex_options() ['color_increasing']
'black'
```

To change the default options for all interval-posets, use the parent’s latex options:

```python
sage: ip = TamariIntervalPoset(4,[(2,4),(3,4),(2,1),(3,1)])
sage: ip2 = TamariIntervalPoset(4,[(1,2),(2,3)])
sage: ip.latex_options() ['color_decreasing']
'red'
sage: ip2.latex_options() ['color_decreasing']
'red'
sage: TamariIntervalPosets.options(latex_color_decreasing='green')
sage: ip.latex_options() ['color_decreasing']
'green'
sage: ip2.latex_options() ['color_decreasing']
'green'
```

Next we set a local latex option and show the global option does not override it:

```python
sage: ip.set_latex_options({'color_decreasing': 'black'})
sage: ip.latex_options() ['color_decreasing']
'black'
sage: TamariIntervalPosets.options(latex_color_decreasing='blue')
sage: ip.latex_options() ['color_decreasing']
'black'
sage: ip2.latex_options() ['color_decreasing']
'blue'
sage: TamariIntervalPosets.options._reset()
```

`size()`

Return the size (number of vertices) of the interval-poset.

EXAMPLES:

```python
sage: TamariIntervalPoset(3,[(2,1),(3,1)]).size()
3
```

`sub_poset(start, end)`

Return the renormalized subposet of self consisting solely of integers from start (inclusive) to end (not inclusive).

“Renormalized” means that these integers are relabelled 1, 2, . . . , k in the obvious way (i.e., by subtracting start - 1).

INPUT:
Combinatorics, Release 10.0

- start – an integer, the starting vertex (inclusive)
- end – an integer, the ending vertex (not inclusive)

EXAMPLES:

```
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
    → ip
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
```

```
sage: ip.subposet(1,3)
The Tamari interval of size 2 induced by relations [(1, 2)]
```

```
sage: ip.subposet(1,4)
The Tamari interval of size 3 induced by relations [(1, 2), (3, 2)]
```

```
sage: ip.subposet(1,5)
The Tamari interval of size 4 induced by relations [(1, 2), (4, 3), (3, 2)]
```

```
sage: ip.subposet(1,7) == ip
True
```

```
sage: ip.subposet(1,1)
The Tamari interval of size 0 induced by relations []
```

subposet(start, end)

Return the renormalized subposet of self consisting solely of integers from start (inclusive) to end (not inclusive).

“Renormalized” means that these integers are relabelled 1, 2, ..., k in the obvious way (i.e., by subtracting start - 1).

INPUT:

- start – an integer, the starting vertex (inclusive)
- end – an integer, the ending vertex (not inclusive)

EXAMPLES:

```
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(3,5),(4,5)]);
    → ip
The Tamari interval of size 6 induced by relations [(1, 2), (3, 5), (4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
```

```
sage: ip.subposet(1,3)
The Tamari interval of size 2 induced by relations [(1, 2)]
```

```
sage: ip.subposet(1,4)
The Tamari interval of size 3 induced by relations [(1, 2), (3, 2)]
```

```
sage: ip.subposet(1,5)
The Tamari interval of size 4 induced by relations [(1, 2), (4, 3), (3, 2)]
```

```
sage: ip.subposet(1,7) == ip
True
```

```
sage: ip.subposet(1,1)
The Tamari interval of size 0 induced by relations []
```

tamari_inversions()

Return the Tamari inversions of self.

A Tamari inversion is a pair of vertices (a, b) with a < b such that:

- the decreasing parent of b is strictly smaller than a (or does not exist), and
- the increasing parent of a is strictly bigger than b (or does not exist).
“Smaller” and “bigger” refer to the numerical values of the elements, not to the poset order.

This method returns the list of all Tamari inversions in lexicographic order.

The number of Tamari inversions is the length of the longest chain of the Tamari interval represented by self.

Indeed, when an interval consists of just one binary tree, it has no inversion. One can also prove that if a Tamari interval \( I' = [T'_1, T'_2] \) is a proper subset of a Tamari interval \( I = [T_1, T_2] \), then the inversion number of \( I' \) is strictly lower than the inversion number of \( I \). And finally, by adding the relation \((b, a)\) to the interval-poset where \((a, b)\) is the first inversion of \( I \) in lexicographic order, one reduces the inversion number by exactly 1.

See also:

tamari_inversions_iter(), number_of_tamari_inversions()

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(3,[])
sage: ip.tamari_inversions()
[(1, 2), (1, 3), (2, 3)]
```

```python
sage: ip = TamariIntervalPoset(3,[(2,1)])
sage: ip.tamari_inversions()
[(1, 3), (2, 3)]
```

```python
sage: ip = TamariIntervalPoset(3,[(1,2)])
sage: ip.tamari_inversions()
[(2, 3)]
```

```python
sage: ip = TamariIntervalPoset(3,[(1,2),(3,2)])
sage: ip.tamari_inversions()
[]
```

```python
sage: ip = TamariIntervalPoset(3,[(1,2),(3,2)])
sage: ip.tamari_inversions()
[(2, 3)]
```

```python
sage: ip = TamariIntervalPoset(3,[(1,2),(3,2)])
sage: ip.tamari_inversions()
[]
```

```python
sage: all(len(TamariIntervalPosets.from_binary_trees(bt,bt).tamari_inversions())==0 for bt in BinaryTrees(3))
True
```

```python
sage: all(len(TamariIntervalPosets.from_binary_trees(bt,bt).tamari_inversions())==0 for bt in BinaryTrees(4))
True
```

```
tamari_inversions_iter()
Iterate over the Tamari inversions of self, in lexicographic order.

See tamari_inversions() for the definition of the terms involved.

EXAMPLES:
```
```
sage: list(T.tamari_inversions_iter())
[(1, 2), (1, 7), (5, 6)]

sage: T = TamariIntervalPoset(1, [])

sage: list(T.tamari_inversions_iter())
[]

sage: T = TamariIntervalPoset(0, [])

sage: list(T.tamari_inversions_iter())
[]

\textbf{upper\textunderscore binary\textunderscore tree()}

Return the highest binary tree in the interval of the Tamari lattice represented by \texttt{self}.

This is a binary tree. It is the shape of the unique binary search tree whose right-branch ordered forest (i.e., the result of applying \texttt{to\_ordered\_tree\_right\_branch()} and cutting off the root) is the initial forest of \texttt{self}.

\textbf{See also:}

to\textunderscore ordered\textunderscore tree\textunderscore right\textunderscore branch()

\textbf{EXAMPLES:}

sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(4,5)]); ip
The Tamari interval of size 6 induced by relations [(1, 2), (4, 5), (6, 5), (5, ˓→2), (4, 3), (3, 2)]

sage: ip.upper_binary_tree()
[[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .], [[., .]]]]]]]]]]]]

sage: TamariIntervalPosets.initial_forest(ip.upper_binary_tree()) == ip.initial_˓→forest()
True

sage: ip == TamariIntervalPosets.from_binary_trees(ip.lower_binary_tree(),ip.˓→upper_binary_tree())
True

\textbf{upper\textunderscore contains\textunderscore interval(\texttt{other})}

Return whether the interval represented by \texttt{other} is contained in \texttt{self} as an interval of the Tamari lattice and if they share the same upper bound.

As interval-posets, it means that \texttt{other} contains the relations of \texttt{self} plus some extra decreasing relations.

\textbf{INPUT:}

\indent * \texttt{other} – an interval-poset

\textbf{EXAMPLES:}

sage: ip1 = TamariIntervalPoset(4,[(1,2),(2,3),(4,3)])
sage: ip2 = TamariIntervalPoset(4,[(1,2),(2,3)])
sage: ip2.upper\textunderscore contains\textunderscore interval(ip1)
True

sage: ip2.contains\textunderscore interval(ip1) and ip2.upper\textunderscore binary\textunderscore tree() == ip1.upper_˓→binary\textunderscore tree()
True

sage: ip3 = TamariIntervalPoset(4,[(1,2),(2,3),(3,4)])
upper_dyck_word()

Return the highest Dyck word in the interval of the Tamari lattice represented by self.

See also:

upper_binary_tree()

EXAMPLES:

```python
sage: ip = TamariIntervalPoset(6,[(3,2),(4,3),(5,2),(6,5),(1,2),(4,5)]); ip
The Tamari interval of size 6 induced by relations [(1, 2), (4, 5), (6, 5), (5, ˓→2), (4, 3), (3, 2)]
sage: ip.upper_dyck_word()
[1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0]
sage: TamariIntervalPosets.initial_forest(ip.upper_dyck_word()) == ip.initial_ ˓→forest()
True
sage: ip == TamariIntervalPosets.from_dyck_words(ip.lower_dyck_word(),ip.upper_ ˓→dyck_word())
True
```

class sage.combinat.interval_posets.TamariIntervalPosets

Factory for interval-posets.

INPUT:

• size – (optional) an integer

OUTPUT:

• the set of all interval-posets (of the given size if specified)

EXAMPLES:

```python
sage: TamariIntervalPosets()
Interval-posets

sage: TamariIntervalPosets(2)
Interval-posets of size 2
```

Note: This is a factory class whose constructor returns instances of subclasses.

```python
static check_poset(poset)

Check if the given poset poset is a interval-poset, that is, if it satisfies the following properties:

• Its labels are exactly 1, \ldots, n where n is its size.
```
• If $a < c$ (as numbers) and $a$ precedes $c$, then $b$ precedes $c$ for all $b$ such that $a < b < c$.
• If $a < c$ (as numbers) and $c$ precedes $a$, then $b$ precedes $a$ for all $b$ such that $a < b < c$.

INPUT:
• poset – a finite labeled poset

EXAMPLES:

```
sage: p = Poset(([1,2,3],[[1,2],[3,2]]))
sage: TamariIntervalPosets.check_poset(p)
True
sage: p = Poset(([2,3],[[3,2]]))
sage: TamariIntervalPosets.check_poset(p)
False
sage: p = Poset(([1,2,3],[[3,1]]))
sage: TamariIntervalPosets.check_poset(p)
False
sage: p = Poset(([1,2,3],[[1,3]]))
sage: TamariIntervalPosets.check_poset(p)
False
```

```
static final_forest(element)

Return the final forest of a binary tree, an interval-poset or a Dyck word.

A final forest is an interval-poset corresponding to a final interval of the Tamari lattice, i.e., containing only decreasing relations.

It can be constructed from a binary tree by its binary search tree labeling with the rule: $b$ precedes $a$ in the final forest iff $b$ is in the right subtree of $a$ in the binary search tree.

INPUT:
• element – a binary tree, a Dyck word or an interval-poset

EXAMPLES:

```
sage: ip = TamariIntervalPoset(4,[(1,2),(2,3),(4,3)])
sage: TamariIntervalPosets.final_forest(ip)
The Tamari interval of size 4 induced by relations [(4, 3)]
```

From binary trees:

```
sage: bt = BinaryTree(); bt
 .
sage: TamariIntervalPosets.final_forest(bt)
The Tamari interval of size 0 induced by relations []
sage: bt = BinaryTree([]); bt
[[], .]
sage: TamariIntervalPosets.final_forest(bt)
The Tamari interval of size 1 induced by relations []
sage: bt = BinaryTree([[],None]); bt
[[., .], .]
sage: TamariIntervalPosets.final_forest(bt)
The Tamari interval of size 2 induced by relations []
sage: bt = BinaryTree([None,[]]); bt
[[., [., .]], .]
```

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```
sage: TamariIntervalPosets.final_forest(bt)
The Tamari interval of size 2 induced by relations [(2, 1)]
sage: bt = BinaryTree([[],[]]); bt
[[., .], [., .]]
sage: TamariIntervalPosets.final_forest(bt)
The Tamari interval of size 3 induced by relations [(3, 2)]
sage: bt = BinaryTree([None,[[None],[None]],[]]); bt
[[., [[., .], .]], [., .]]
sage: TamariIntervalPosets.final_forest(bt)
The Tamari interval of size 5 induced by relations [(5, 4), (3, 1), (2, 1)]
```

From Dyck words:

```
sage: dw = DyckWord([1,0])
sage: TamariIntervalPosets.final_forest(dw)
The Tamari interval of size 1 induced by relations []
sage: dw = DyckWord([1,0,1,0,0,1,0,0,1,0,0])
sage: TamariIntervalPosets.final_forest(dw)
The Tamari interval of size 5 induced by relations [(5, 4), (3, 1), (2, 1)]
```

```
static from_binary_trees(tree1, tree2)
Return the interval-poset corresponding to the interval [tree1, tree2] of the Tamari lattice.
Raise an exception if tree1 is not ≤ tree2 in the Tamari lattice.

INPUT:

• tree1 – a binary tree

• tree2 – a binary tree greater or equal than tree1 for the Tamari lattice

EXAMPLES:

```
sage: tree1 = BinaryTree([[],None])
sage: tree2 = BinaryTree([None,[]])
sage: TamariIntervalPosets.from_binary_trees(tree1,tree2)
The Tamari interval of size 2 induced by relations []
sage: TamariIntervalPosets.from_binary_trees(tree1,tree1)
The Tamari interval of size 2 induced by relations [(1, 2)]
sage: TamariIntervalPosets.from_binary_trees(tree2,tree2)
The Tamari interval of size 2 induced by relations [(2, 1)]
sage: tree1 = BinaryTree([[],[[None],[]]])
sage: tree2 = BinaryTree([None,[None,[None],[[],[]]]])
sage: TamariIntervalPosets.from_binary_trees(tree1,tree2)
The Tamari interval of size 6 induced by relations [(4, 5), (6, 5), (5, 2), (4, 3), (3, 2)]
sage: tree3 = BinaryTree([None,[None,[[None],[]]]])
sage: TamariIntervalPosets.from_binary_trees(tree1,tree3)
Traceback (most recent call last):
...
ValueError: the two binary trees are not comparable on the Tamari lattice
```

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... ValueError: the two binary trees are not comparable on the Tamari lattice

**static from_dyck_words**(\(dw1, dw2\))

Return the interval-poset corresponding to the interval \([dw1, dw2]\) of the Tamari lattice.

Raise an exception if the two Dyck words \(dw1\) and \(dw2\) do not satisfy \(dw1 \leq dw2\) in the Tamari lattice.

**INPUT:**

- \(dw1\) – a Dyck word
- \(dw2\) – a Dyck word greater or equal than \(dw1\) for the Tamari lattice

**EXAMPLES:**

```python
sage: dw1 = DyckWord([1,0,1,0])
sage: dw2 = DyckWord([1,1,0,0])
sage: TamariIntervalPosets.from_dyck_words(dw1,dw2)
The Tamari interval of size 2 induced by relations []
sage: TamariIntervalPosets.from_dyck_words(dw1,dw1)
The Tamari interval of size 2 induced by relations [(1, 2)]
sage: TamariIntervalPosets.from_dyck_words(dw2,dw2)
The Tamari interval of size 2 induced by relations [(2, 1)]
sage: dw1 = DyckWord([1,0,1,1,1,0,0,1,1,0,0,0])
sage: dw2 = DyckWord([1,1,1,1,0,1,1,0,0,0,0,0])
sage: TamariIntervalPosets.from_dyck_words(dw1,dw2)
The Tamari interval of size 6 induced by relations [(4, 5), (6, 5), (5, 2), (4, \to 3), (3, 2)]
sage: dw3 = DyckWord([1,1,0,1,1,1,0,0,0,0,0])
sage: TamariIntervalPosets.from_dyck_words(dw1,dw3)
Traceback (most recent call last):
... ValueError: the two Dyck words are not comparable on the Tamari lattice
```

**static from_grafting_tree**(\(tree\))

Return an interval-poset from a grafting tree.

For the inverse method, see `TamariIntervalPoset.grafting_tree()`.

**EXAMPLES:**

```python
sage: tip = TamariIntervalPoset(8, [(1,2), (2,4), (3,4), (6,7), (3,2), (5,4), \to (6,4), (8,7)])
sage: t = tip.grafting_tree()
sage: TamariIntervalPosets.from_grafting_tree(t) == tip
True
```

**REFERENCES:**

- [Pons2018]
static from_minimal_schnyder_wood(graph)
Return a Tamari interval built from a minimal Schnyder wood.

This is an implementation of Bernardi and Bonichon’s bijection [BeBo2009].

INPUT:

a minimal Schnyder wood, given as a graph with colored and oriented edges, without the three exterior unoriented edges

The three boundary vertices must be -1, -2 and -3.

One assumes moreover that the embedding around -1 is the list of neighbors of -1 and not just a cyclic permutation of that.

Beware that the embedding convention used here is the opposite of the one used by the plot method.

OUTPUT:

a Tamari interval-poset

EXAMPLES:

A small example:

```
sage: TIP = TamariIntervalPosets
sage: G = DiGraph([(0,-1,0),(0,-2,1),(0,-3,2)], format='list_of_edges')
sage: G.set_embedding({-1:[0],-2:[0],-3:[0],0:[-1,-2,-3]})
sage: TIP.from_minimal_schnyder_wood(G)
The Tamari interval of size 1 induced by relations []
```

An example from page 14 of [BeBo2009]:

```
sage: c0 = [(0,-1),(1,0),(2,0),(4,3),(3,-1),(5,3)]
sage: c1 = [(5,-2),(3,-2),(4,5),(1,3),(2,3),(0,3)]
sage: c2 = [(0,-3),(1,-3),(3,-3),(4,-3),(5,-3),(2,1)]
sage: ed = [(u,v,0) for u,v in c0]
sage: ed += [(u,v,1) for u,v in c1]
sage: ed += [(u,v,2) for u,v in c2]
sage: G = DiGraph(ed, format='list_of_edges')
sage: embed = {-1:[3,0],-2:[5,3],-3:[0,1,3,4,5]}
sage: data_emb = [-2,5,4,-3,1,2,0,-1]
sage: data_emb += [5,-3,3]
sage: data_emb += [-2,-3,4,3]
sage: for k in range(6):
    ....:     embed[k] = data_emb[k]
sage: G.set_embedding(embed)
sage: TIP.from_minimal_schnyder_wood(G)
The Tamari interval of size 6 induced by relations [(1, 4), (2, 4), (3, 4), (5, ˓→6), (6, 4), (5, 4), (3, 1), (2, 1)]
```

An example from page 18 of [BeBo2009]:

```
sage: c0 = [(0,-1),(1,0),(2,-1),(3,2),(4,2),(5,-1)]
sage: c1 = [(5,-2),(2,-2),(4,-2),(3,4),(1,2),(0,2)]
sage: c2 = [(0,-3),(1,-3),(3,-3),(4,-3),(2,-3),(5,2)]
sage: ed = [(u,v,0) for u,v in c0]
sage: ed += [(u,v,1) for u,v in c1]
sage: ed += [(u,v,2) for u,v in c2]
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sage: G = DiGraph(ed, format='list_of_edges')
sage: embed = {-1:[5,2,0],-2:[4,2,5],-3:[0,1,2,3,4]}
sage: data_emb = [[2,1,-3,-1],[2,-3,0],[3,-3,1,0,-1,5,-2,4]]
sage: data_emb += [[4,-3,2],[-2,-3,2],[-2,2,-1]]
sage: for k in range(6):
    ....:     embed[k] = data_emb[k]
sage: G.set_embedding(embed)
sage: TIP.from_minimal_schnyder_wood(G)
The Tamari interval of size 6 induced by relations [(1, 3), (2, 3), (4, 5), (5, \rightarrow 3), (4, 3), (2, 1)]

Another small example:

sage: c0 = [(0,-1),(2,-1),(1,0)]
sage: c1 = [(2,-2),(1,-2),(0,2)]
sage: c2 = [(0,-3),(1,-3),(2,1)]
sage: ed = [(u,v,0) for u,v in c0]
sage: ed += [(u,v,1) for u,v in c1]
sage: ed += [(u,v,2) for u,v in c2]
sage: G = DiGraph(ed, format='list_of_edges')
sage: embed = {-1:[2,0],-2:[1,2],-3:[0,1]}
sage: data_emb = [[2,1,-3,-1],[-3,0,2,-2],[-2,1,0,-1]]
sage: for k in range(3):
    ....:     embed[k] = data_emb[k]
sage: G.set_embedding(embed)
sage: TIP.from_minimal_schnyder_wood(G)
The Tamari interval of size 3 induced by relations [(2, 3), (2, 1)]

static initial_forest(element)

Return the initial forest of a binary tree, an interval-poset or a Dyck word.

An initial forest is an interval-poset corresponding to an initial interval of the Tamari lattice, i.e., containing only increasing relations.

It can be constructed from a binary tree by its binary search tree labeling with the rule: \( a \) precedes \( b \) in the initial forest iff \( a \) is in the left subtree of \( b \) in the binary search tree.

INPUT:

- element -- a binary tree, a Dyck word or an interval-poset

EXAMPLES:

sage: ip = TamariIntervalPoset(4,[(1,2),(2,3),(4,3)])
sage: TamariIntervalPosets.initial_forest(ip)
The Tamari interval of size 4 induced by relations [(1, 2), (2, 3)]

with binary trees:

sage: bt = BinaryTree(); bt
sage: TamariIntervalPosets.initial_forest(bt)
The Tamari interval of size 0 induced by relations []
sage: bt = BinaryTree([]); bt
[., .]
The Tamari interval of size 1 induced by relations []

```sage```
```
bt = BinaryTree([[], None]); bt
[[., .], .]
```
```sage```
```
TamariIntervalPosets.initial_forest(bt)
The Tamari interval of size 2 induced by relations [(1, 2)]
```
```sage```
```
b = BinaryTree([None, []]); btt
[., [., .]]
```
```sage```
```
TamariIntervalPosets.initial_forest(bt)
The Tamari interval of size 2 induced by relations []
```
```sage```
```
b = BinaryTree([[], []]); btt
[[., .], [., .]]
```
```sage```
```
TamariIntervalPosets.initial_forest(bt)
The Tamari interval of size 2 induced by relations [(1, 2)]
```
```sage```
```
bt = BinaryTree([[None, [[], None]], []]); bt
[[., [., .]], [., .], [., .]]
```
```sage```
```
TamariIntervalPosets.initial_forest(bt)
The Tamari interval of size 5 induced by relations [(1, 4), (2, 3), (3, 4)]
```
```sage```
```
dw = DyckWord([1, 0])
```
```sage```
```
TamariIntervalPosets.initial_forest(dw)
The Tamari interval of size 1 induced by relations []
```
```sage```
```
dw = DyckWord([1, 1, 0, 1, 0, 0, 1, 1, 0, 0])
```
```sage```
```
TamariIntervalPosets.initial_forest(dw)
The Tamari interval of size 5 induced by relations [(1, 4), (2, 3), (3, 4)]
```
```sage```
```
de1, de2
```
```sage```
```
le(el1, el2)
Poset structure on the set of interval-posets.
```
The comparison is first by size, then using the cubical coordinates.
```
See also:
```
cubical_coordinates()
```
```
INPUT:
```
• el1 -- an interval-poset
• el2 -- an interval-poset
```
```
EXAMPLES:
```
```sage```
```
ip1 = TamariIntervalPoset(4, [(1, 2), (2, 3), (4, 3)])
```
```sage```
```
ip2 = TamariIntervalPoset(4, [(1, 2), (2, 3)])
```
```sage```
```
TamariIntervalPosets().le(ip1, ip2)
False
```
```sage```
```
TamariIntervalPosets().le(ip2, ip1)
True
```
```
.options = Current options for TamariIntervalPosets -- latex_color_decreasing: red -- latex_color_increasing: blue -- latex_hspace: 1 -- latex_line_width_scalar: 0.5 -- latex_tikz_scale: 1 -- latex_vspace: 1
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```
static recomposition_from_triple(\(left, right, r\))

Recompose an interval-poset from a triple \((left, right, r)\).

For the inverse method, see \(TamariIntervalPoset.decomposition_to_triple()\).

INPUT:
- \(left\) – an interval-poset
- \(right\) – an interval-poset
- \(r\) – the parameter of the decomposition, an integer

OUTPUT: an interval-poset

EXAMPLES:

```
sage: T1 = TamariIntervalPoset(3, [(1, 2), (3, 2)])
sage: T2 = TamariIntervalPoset(4, [(2, 3), (4, 3)])
sage: TamariIntervalPosets.recomposition_from_triple(T1, T2, 2)
The Tamari interval of size 8 induced by relations [(1, 2), (2, 4), (3, 4), (6, 7), (8, 7), (6, 4), (5, 4), (3, 2)]
```

REFERENCES:
- [Pons2018]

class sage.combinat.interval_posets.TamariIntervalPosets_all

Bases: DisjointUnionEnumeratedSets, TamariIntervalPosets

The enumerated set of all Tamari interval-posets.

Element

alias of TamariIntervalPoset

one()

Return the unit of the monoid.

This is the empty interval poset, of size 0.

EXAMPLES:

```
sage: TamariIntervalPosets().one()
The Tamari interval of size 0 induced by relations []
```

class sage.combinat.interval_posets.TamariIntervalPosets_size(size)

Bases: TamariIntervalPosets

The enumerated set of interval-posets of a given size.

cardinality()

The cardinality of self. That is, the number of interval-posets of size \(n\).

The formula was given in [Cha2008]:

\[
\frac{2(4n+1)!}{(n+1)!(3n+2)!} = \frac{2}{n(n+1)} \binom{4n+1}{n-1}.
\]

EXAMPLES:

```
sage: [TamariIntervalPosets(i).cardinality() for i in range(6)]
[1, 1, 3, 13, 68, 399]
```
**element_class()**

**random_element()**

Return a random Tamari interval of fixed size.

This is obtained by first creating a random rooted planar triangulation, then computing its unique minimal Schnyder wood, then applying a bijection of Bernardi and Bonichon [BeBo2009].

Because the random rooted planar triangulation is chosen uniformly at random, the Tamari interval is also chosen according to the uniform distribution.

**EXAMPLES:**

```python
sage: T = TamariIntervalPosets(4).random_element()
sage: T.parent()
Interval-posets
sage: u = T.lower_dyck_word(); u  # random
[1, 1, 0, 1, 0, 0, 1, 0]
sage: v = T.lower_dyck_word(); v  # random
[1, 1, 0, 1, 0, 0, 1, 0]
sage: len(u)
8
```

### 5.1.1.29 $k$-regular Sequences

An introduction and formal definition of $k$-regular sequences can be found, for example, on the Wikipedia article *k-regular_sequence* or in [AS2003].

**Warning:** As this code is experimental, warnings are thrown when a $k$-regular sequence space is created for the first time in a session (see `sage.misc.superseded.experimental`).

#### Examples

**Binary sum of digits**

The binary sum of digits $S(n)$ of a nonnegative integer $n$ satisfies $S(2n) = S(n)$ and $S(2n + 1) = S(n) + 1$. We model this by the following:

```python
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: S = Seq2((Matrix([[1, 0], [0, 1]]), Matrix([[1, 0], [1, 1]])),
            ....:     left=vector([0, 1]), right=vector([1, 0]))
sage: S
2-regular sequence 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, ...
sage: all(S[n] == sum(n.digits(2)) for n in srange(10))
True
```
Number of odd entries in Pascal’s triangle

Let us consider the number of odd entries in the first $n$ rows of Pascal's triangle:

```
sage: @cached_function
....: def u(n):
....:     if n <= 1:
....:         return n
....:     return 2 * u(n // 2) + u((n+1) // 2)
sage: tuple(u(n) for n in srange(10))
(0, 1, 3, 5, 9, 11, 15, 19, 27, 29)
```

There is a 2-recursive sequence describing the numbers above as well:

```
sage: U = Seq2((Matrix([[3, 2], [0, 1]]), Matrix([[2, 0], [1, 3]])),
....:         left=vector([0, 1]), right=vector([1, 0])).transposed()
sage: all(U[n] == u(n) for n in srange(30))
True
```

Various

See also:

recognizable series, sage.rings.cfinite_sequence, sage.combinat.binary_recurrence_sequences.

AUTHORS:

• Daniel Krenn (2016, 2021)
• Gabriel F. Lipnik (2021)

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• Gabriel F. Lipnik is supported by the Austrian Science Fund (FWF): W 1230.

Classes and Methods

class sage.combinat.k_regular_sequence.RecurrenceParser($k$, coefficient_ring)

Bases: object

A parser for recurrence relations that allow the construction of a $k$-linear representation for the sequence satisfying these recurrence relations.

This is used by kRegularSequenceSpace.from_recurrence() to construct a kRegularSequence.

ind($M$, $m$, $ll$, $uu$)

Determine the index operator corresponding to the recursive sequence as defined in [HKL2021].

INPUT:

• $M$, $m$ – parameters of the recursive sequences, see [HKL2021], Definition 3.1
• $ll$, $uu$ – parameters of the resulting linear representation, see [HKL2021], Theorem A

OUTPUT:

A dictionary which maps both row numbers to subsequence parameters and vice versa, i.e.,
• \(\text{ind}[i]\) – a pair \((j, d)\) representing the sequence \(x(k^j n + d)\) in the \(i\)-th component (0-based) of the resulting linear representation,

• \(\text{ind}[(j, d)]\) – the (0-based) row number of the sequence \(x(k^j n + d)\) in the linear representation.

EXAMPLES:

```python
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)
sage: RP.ind(3, 1, -3, 3)
{(0, 0): 0, (1, -1): 3, (1, -2): 2, (1, -3): 1,
 (1, 0): 4, (1, 1): 5, (1, 2): 6, (1, 3): 7, (2, -1): 10,
 (2, -2): 9, (2, -3): 8, (2, 0): 11, (2, 1): 12, (2, 2): 13,
 (2, 3): 14, (2, 4): 15, (2, 5): 16, 0: (0, 0), 1: (1, -3),
 10: (2, -1), 11: (2, 0), 12: (2, 1), 13: (2, 2), 14: (2, 3),
 15: (2, 4), 16: (2, 5), 2: (1, -2), 3: (1, -1), 4: (1, 0),
 5: (1, 1), 6: (1, 2), 7: (1, 3), 8: (2, -3), 9: (2, -2)}
```

See also:

`kRegularSequenceSpace.from_recurrence()`

\(\text{left}(\text{recurrence rules})\)

Construct the vector \(\text{left}\) of the linear representation of recursive sequences.

INPUT:

• recurrence_rules – a namedtuple generated by `parameters()`: it only needs to contain a field \(\text{dim}\) (a positive integer)

OUTPUT: a vector

EXAMPLES:

```python
sage: from collections import namedtuple
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)
sage: RRD = namedtuple('recurrence_rules_dim', 'dim', 'inhomogeneities')
sage: recurrence_rules = RRD(dim=5, inhomogeneities={})
sage: RP.left(recurrence_rules)
(1, 0, 0, 0, 0)
```

```python
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: RRD = namedtuple('recurrence_rules_dim', 'M', 'm', 'll', 'uu', 'dim', 'inhomogeneities')
sage: recurrence_rules = RRD(M=3, m=2, ll=0, uu=9, dim=5,
.....: inhomogeneities={0: Seq2.one_hadamard()})
sage: RP.left(recurrence_rules)
(1, 0, 0, 0, 0, 0, 0, 0)
```

See also:

`kRegularSequenceSpace.from_recurrence()`

\(\text{matrix}(\text{recurrence rules}, \text{rem}, \text{correct_offset=True})\)

Construct the matrix for remainder \(\text{rem}\) of the linear representation of the sequence represented by recurrence_rules.
INPUT:

• **recurrence_rules** – a namedtuple generated by `parameters()`
• **rem** – an integer between 0 and \( k - 1 \)
• **correct_offset** – (default: True) a boolean. If True, then the resulting linear representation has no offset. See [HKL2021] for more information.

OUTPUT: a matrix

EXAMPLES:

The following example illustrates how the coefficients in the right-hand sides of the recurrence relations correspond to the entries of the matrices.

```python
from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)

sage: var('n')
n
sage: function('f')
f

sage: M, m, coeffs, initial_values = RP.parse_recurrence([  
...:     f(8*n) == -1*f(2*n - 1) + 1*f(2*n + 1),
...:     f(8*n + 1) == -11*f(2*n - 1) + 10*f(2*n) + 11*f(2*n + 1),
...:     f(8*n + 2) == -21*f(2*n - 1) + 20*f(2*n) + 21*f(2*n + 1),
...:     f(8*n + 3) == -31*f(2*n - 1) + 30*f(2*n) + 31*f(2*n + 1),
...:     f(8*n + 4) == -41*f(2*n - 1) + 40*f(2*n) + 41*f(2*n + 1),
...:     f(8*n + 5) == -51*f(2*n - 1) + 50*f(2*n) + 51*f(2*n + 1),
...:     f(8*n + 6) == -61*f(2*n - 1) + 60*f(2*n) + 61*f(2*n + 1),
...:     f(8*n + 7) == -71*f(2*n - 1) + 70*f(2*n) + 71*f(2*n + 1),
...:     f(0) == 0, f(1) == 1, f(2) == 2, f(3) == 3, f(4) == 4,
...:     f(5) == 5, f(6) == 6, f(7) == 7], f, n)

sage: rules = RP.parameters(
...:     M, m, coeffs, initial_values, 0)

sage: RP.matrix(rules, 0, False)

```

(continues on next page)
Stern–Brocot Sequence:

```
sage: SB_rules = RP.parameters(
    ....:    1, 0, {(0, 0): 1, (1, 0): 1, (1, 1): 1},
    ....:    {0: 0, 1: 1, 2: 1}, 0)
sage: RP.matrix(SB_rules, 0)

[1 0 0]
[1 1 0]
[0 1 0]
sage: RP.matrix(SB_rules, 1)

[1 1 0]
[0 1 0]
[0 1 1]
```

Number of Unbordered Factors in the Thue–Morse Sequence:

```
sage: M, m, coeffs, initial_values = RP.parse_recurrence([f(8*n) == 2*f(4*n),
    ....:    f(8*n + 1) == f(4*n + 1),
    ....:    f(8*n + 2) == f(4*n + 1) + f(4*n + 3),
    ....:    f(8*n + 3) == -f(4*n + 1) + f(4*n + 2),
    ....:    f(8*n + 4) == 2*f(4*n + 2),
    ....:    f(8*n + 5) == f(4*n + 3),
    ....:    f(8*n + 6) == -f(4*n + 1) + f(4*n + 2) + f(4*n + 3),
    ....:    f(8*n + 7) == 2*f(4*n + 1) + f(4*n + 3),
    ....:    f(9) == 1, f(1) == 2, f(2) == 2, f(3) == 4, f(4) == 2,
    ....:    f(5) == 4, f(6) == 6, f(7) == 0, f(8) == 4, f(9) == 4,
    ....:    f(10) == 4, f(11) == 4, f(12) == 12, f(13) == 0, f(14) == 4,
    ....:    f(15) == 4, f(16) == 8, f(17) == 4, f(18) == 8, f(19) == 0,
    ....:    f(20) == 8, f(21) == 4, f(22) == 4, f(23) == 8], f, n)
sage: UB_rules = RP.parameters(
    ....:    M, m, coeffs, initial_values, 3)
sage: RP.matrix(UB_rules, 0)

[ 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
[ 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
[ 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0]```

(continues on next page)
See also:

\texttt{kRegularSequenceSpace.from_recurrence()}

\texttt{parameters(}\texttt{M, m, coeffs, initial_values, offset=0, inhomogeneities={})}

Determine parameters from recurrence relations as admissible in \texttt{kRegularSequenceSpace.from_recurrence()}. 

**INPUT:**

All parameters are explained in the high-level method \texttt{kRegularSequenceSpace.from_recurrence()}. 

**OUTPUT:** a namedtuple \texttt{recurrence_rules} consisting of

- M, m, l, u, offset – parameters of the recursive sequences, see [HKL2021], Definition 3.1
- l1, uu, n1, dim – parameters and dimension of the resulting linear representation, see [HKL2021], Theorem A
- coeffs – a dictionary mapping (r, j) to the coefficients \(c_{r,j}\) as given in [HKL2021], Equation (3.1). If coeffs[(r, j)] is not given for some r and j, then it is assumed to be zero.
- initial_values – a dictionary mapping integers n to the n-th value of the sequence
- inhomogeneities – a dictionary mapping integers r to the inhomogeneity \(g_r\) as given in [HKL2021], Corollary D.
EXAMPLES:

```
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
g sage: RP = RecurrenceParser(2, ZZ)
sage: RP.parameters(2, 1,
  ....:   {(-2): 3, (0, 0): 1, (0, 1): 2, (1, -2): 6, (1, 0): 4,
  ....:    (1, 1): 5, (2, -2): 9, (2, 0): 7, (2, 1): 8, (3, -2): 12,
  ....:    (3, 0): 10, (3, 1): 11}, {0: 1, 1: 2, 2: 1, 3: 4}, 0, {0: 1})
  recurrence_rules(M=2, m=1, l=-2, u=1, lL=-6, uu=3, dim=14,
  coeffs={(0, -2): 3, (0, 0): 1, (0, 1): 2, (1, -2): 6, (1, 0): 4,
  (1, 1): 5, (2, -2): 9, (2, 0): 7, (2, 1): 8, (3, -2): 12,
  (3, 0): 10, (3, 1): 11}, initial_values={0: 1, 1: 2, 2: 1},
  offset=1, n1=3, inhomogeneities={0: 2-regular sequence 1, 1, 1, 1,
  1, 1, 1, 1, ...})
```

See also:

```
  kRegularSequenceSpace.from_recurrence()
```

`parse_direct_arguments(M, m, coeffs, initial_values)`

Check whether the direct arguments as admissible in `kRegularSequenceSpace.from_recurrence()` are valid.

INPUT:

All parameters are explained in the high-level method `kRegularSequenceSpace.from_recurrence()`.

OUTPUT: a tuple consisting of the input parameters

EXAMPLES:

```
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
g sage: RP = RecurrenceParser(2, ZZ)
sage: RP.parse_direct_arguments(2, 1,
  ....:   {(-2): 3, (0, 0): 1, (0, 1): 2,
  ....:    (1, -2): 6, (1, 0): 4, (1, 1): 5,
  ....:    (2, -2): 9, (2, 0): 7, (2, 1): 8,
  ....:    (3, -2): 12, (3, 0): 10, (3, 1): 11},
  ....:   {0: 1, 1: 2, 2: 1})
(2, 1, {(0, -2): 3, (0, 0): 1, (0, 1): 2,
(1, -2): 6, (1, 0): 4, (1, 1): 5,
(2, -2): 9, (2, 0): 7, (2, 1): 8,
(3, -2): 12, (3, 0): 10, (3, 1): 11},
{0: 1, 1: 2, 2: 1})
```

Stern–Brocot Sequence:

```
sage: RP.parse_direct_arguments(1, 0,
  ....:   {(-2): 3, (0, 0): 1, (0, 1): 2,
  ....:    (1, -2): 6, (1, 0): 4, (1, 1): 5,
  ....:    (2, -2): 9, (2, 0): 7, (2, 1): 8,
  ....:    (3, -2): 12, (3, 0): 10, (3, 1): 11},
  {0: 1, 1: 2, 2: 1})
```

See also:

```
  kRegularSequenceSpace.from_recurrence()
```
parse_recurrence(equations, function, var)

Parse recurrence relations as admissible in kRegularSequenceSpace.from_recurrence().

INPUT:

All parameters are explained in the high-level method kRegularSequenceSpace.from_recurrence().

OUTPUT: a tuple consisting of

• M, m – see kRegularSequenceSpace.from_recurrence()
• coeffs – see kRegularSequenceSpace.from_recurrence()
• initial_values – see kRegularSequenceSpace.from_recurrence()

EXAMPLES:

sage: from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)
sage: var('n')
n
sage: function('f')
f
sage: RP.parse_recurrence([....: f(4*n) == f(2*n) + 2*f(2*n + 1) + 3*f(2*n - 2),
....: f(4*n + 1) == 4*f(2*n) + 5*f(2*n + 1) + 6*f(2*n - 2),
....: f(4*n + 2) == 7*f(2*n) + 8*f(2*n + 1) + 9*f(2*n - 2),
....: f(4*n + 3) == 10*f(2*n) + 11*f(2*n + 1) + 12*f(2*n - 2),
....: f(0) == 1, f(1) == 2, f(2) == 1], f, n)

Stern–Brocot Sequence:

sage: RP.parse_recurrence([....: f(2*n) == f(n), f(2*n + 1) == f(n) + f(n + 1),
....: f(0) == 0, f(1) == 1], f, n)
(1, 0, {(0, 0): 1, (1, 0): 1, (1, 1): 1}, {0: 0, 1: 1})

See also:

kRegularSequenceSpace.from_recurrence()

right(recurrence_rules)

Construct the vector right of the linear representation of the sequence induced by recurrence_rules.

INPUT:

• recurrence_rules – a namedtuple generated by parameters()

OUTPUT: a vector

See also:

kRegularSequenceSpace.from_recurrence()

shifted_inhomogeneities(recurrence_rules)

Return a dictionary of all needed shifted inhomogeneities as described in the proof of Corollary D in [HKL2021].

INPUT:
• recurrence_rules – a namedtuple generated by `parameters()`

OUTPUT:

A dictionary mapping \( r \) to the regular sequence \( \sum_{i} g_r(n+i) \) for \( g_r \) as given in [HKL2021], Corollary D, and \( i \) between \( \lfloor \frac{\ell}{k^M} \rfloor \) and \( \lfloor (k^M-1-k^m+u')/k^M \rfloor + 1 \); see [HKL2021], proof of Corollary D. The first blocks of the corresponding vector-valued sequence (obtained from its linear representation) correspond to the sequences \( g_r(n+i) \) where \( i \) is as in the sum above; the remaining blocks consist of other shifts which are required for the regular sequence.

EXAMPLES:

```python
sage: from collections import namedtuple
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)
```

```python
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
```

```python
sage: S = Seq2((Matrix([[1, 0], [0, 1]]), Matrix([[1, 0], [1, 1]])), ....: left=vector([0, 1]), right=vector([1, 0]))
```

```python
sage: S 2-regular sequence 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, ...
```

```python
sage: RR = namedtuple('recurrence_rules', ....: [M, m, ll, uu, 'inhomogeneities'])
```

```python
sage: recurrence_rules = RR(M=3, m=0, ll=-14, uu=14, ....: inhomogeneities={0: S, 1: S})
```

```python
sage: SI = RP.shifted_inhomogeneities(recurrence_rules)
```

```python
sage: SI {0: 2-regular sequence 4, 5, 7, 9, 11, 11, 11, 12, 13, 13, ..., 1: 2-regular sequence 4, 5, 7, 9, 11, 11, 11, 12, 13, 13, ...}
```

The first blocks of the corresponding vector-valued sequence correspond to the corresponding shifts of the inhomogeneity. In this particular case, there are no other blocks:

```python
sage: lower = -2
sage: upper = 3
sage: SI[0].dimension() == S.dimension() * (upper - lower + 1)
True
```

```python
sage: all( ....: Seq2( ....: SI[0].mu, ....: vector((i - lower)*[0, 0] + list(S.left) + (upper - i)*[0, 0]), ....: SI[0].right) ....: == S.subsequence(1, i) ....: for i in range(lower, upper+1))
True
```

See also:

`kRegularSequenceSpace.from_recurrence()`

`v_eval_n(recurrence_rules, n)`

Return the vector \( v(n) \) as given in [HKL2021], Theorem A.

INPUT:

• recurrence_rules – a namedtuple generated by `parameters()`

• n – an integer
OUTPUT: a vector

EXAMPLES:

Stern–Brocot Sequence:

```python
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)
sage: SB_rules = RP.parameters(
    ....: 1, 0, {(0, 0): 1, (1, 0): 1, (1, 1): 1},
    ....: {0: 0, 1: 1, 2: 1}, 0)
sage: RP.v_eval_n(SB_rules, 0)
(0, 1, 1)
```

See also:

```
kRegularSequenceSpace.from_recurrence()
```

values($M, m, l, u, ll, coeffs, initial_values, last_value_needed, offset, inhomogeneities$)

Determine enough values of the corresponding recursive sequence by applying the recurrence relations given in $kRegularSequenceSpace.from_recurrence()$ to the values given in initial_values.

INPUT:

- $M, m, l, u, offset$ – parameters of the recursive sequences, see [HKL2021], Definition 3.1
- $ll$ – parameter of the resulting linear representation, see [HKL2021], Theorem A
- $coeffs$ – a dictionary where $coeffs[(r, j)]$ is the coefficient $c_{r,j}$ as given in $kRegularSequenceSpace.from_recurrence()$. If $coeffs[(r, j)]$ is not given for some $r$ and $j$, then it is assumed to be zero.
- $initial_values$ – a dictionary mapping integers $n$ to the $n$-th value of the sequence
- $last_value_needed$ – last initial value which is needed to determine the linear representation
- $inhomogeneities$ – a dictionary mapping integers $r$ to the inhomogeneity $g_r$ as given in [HKL2021], Corollary D.

OUTPUT:

A dictionary mapping integers $n$ to the $n$-th value of the sequence for all $n$ up to last_value_needed.

EXAMPLES:

Stern–Brocot Sequence:

```python
sage: from sage.combinat.k_regular_sequence import RecurrenceParser
sage: RP = RecurrenceParser(2, ZZ)
sage: RP.values(M=1, m=0, l=0, u=1, ll=0,
    ....: coeffs={(0, 0): 1, (1, 0): 1, (1, 1): 1},
    ....: initial_values={0: 0, 1: 1, 2: 1}, last_value_needed=20,
    ....: offset=0, inhomogeneities={})
{0: 0, 1: 1, 2: 1, 3: 2, 4: 1, 5: 3, 6: 2, 7: 3, 8: 1, 9: 4, 10: 3, 11: 5, 12: 2, 13: 5, 14: 3, 15: 4, 16: 1, 17: 5, 18: 4, 19: 7, 20: 3}
```

See also:

```
kRegularSequenceSpace.from_recurrence()
```
class sage.combinat.k_regular_sequence.kRegularSequence(parent, mu, left=None, right=None)

Bases: RecognizableSeries

A \(k\)-regular sequence.

INPUT:

- **parent** – an instance of \(k\)RegularSequenceSpace
- **mu** – a family of square matrices, all of which have the same dimension. The indices of this family are 0, ..., \(k - 1\). \(mu\) may be a list or tuple of cardinality \(k\) as well. See also \(mu()\).
- **left** – (default: None) a vector. When evaluating the sequence, this vector is multiplied from the left to the matrix product. If None, then this multiplication is skipped.
- **right** – (default: None) a vector. When evaluating the sequence, this vector is multiplied from the right to the matrix product. If None, then this multiplication is skipped.

EXAMPLES:

sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: S = Seq2((Matrix([[3, 6], [0, 1]]), Matrix([[0, -6], [1, 5]])),
                      vector([0, 1]), vector([1, 0])).transposed(); S

2-regular sequence 0, 1, 3, 5, 9, 11, 15, 19, 27, 29, ...

We can access the coefficients of a sequence by

sage: S[5]
11

or iterating over the first, say 10, by

sage: from itertools import islice
sage: list(islice(S, 10))
[0, 1, 3, 5, 9, 11, 15, 19, 27, 29]

See also:

\(k\)-regular sequence, \(k\)RegularSequenceSpace.

backward_differences(**kwds)

Return the sequence of backward differences of this \(k\)-regular sequence.

INPUT:

- **minimize** – (default: None) a boolean or None. If True, then \(minimized()\) is called after the operation, if False, then not. If this argument is None, then the default specified by the parent’s minimize_results is used.

OUTPUT:

A kRegularSequence

Note: The coefficient to the index \(-1\) is \(0\).

EXAMPLES:
Combinatorics, Release 10.0

```
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: C = Seq2((Matrix([[2, 0], [2, 1]]), Matrix([[0, 1], [-2, 3]])),
                ....: vector([1, 0]), vector([0, 1]))
sage: C
2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
sage: C.backward_differences()
2-regular sequence 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...

sage: E = Seq2((Matrix([[0, 1], [0, 1]]), Matrix([[0, 0], [0, 1]])),
                ....: vector([1, 0]), vector([1, 1]))
sage: E
2-regular sequence 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, ...
sage: E.backward_differences()
2-regular sequence -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, ...
```

**forward_differences(**kwds**)

Return the sequence of forward differences of this $k$-regular sequence.

**INPUT:**

- `minimize` – (default: None) a boolean or None. If True, then `minimized()` is called after the operation, if False, then not. If this argument is None, then the default specified by the parent's `minimize_results` is used.

**OUTPUT:**

A `kRegularSequence`

**EXAMPLES:**

```
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: C = Seq2((Matrix([[2, 0], [2, 1]]), Matrix([[0, 1], [-2, 3]])),
                ....: vector([1, 0]), vector([0, 1]))
sage: C
2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
sage: C.forward_differences()
2-regular sequence 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...

sage: E = Seq2((Matrix([[0, 1], [0, 1]]), Matrix([[0, 0], [0, 1]])),
                ....: vector([1, 0]), vector([1, 1]))
sage: E
2-regular sequence 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, ...
sage: E.forward_differences()
2-regular sequence -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, ...
```

**partial_sums(**args, **kwds**)

Return the sequence of partial sums of this $k$-regular sequence. That is, the $n$’th entry of the result is the sum of the first $n$ entries in the original sequence.

**INPUT:**

- `include_n` – (default: False) a boolean. If set, then the $n$-th entry of the result is the sum of the entries up to index $n$ (included).
- `minimize` – (default: None) a boolean or None. If True, then `minimized()` is called after the operation, if False, then not. If this argument is None, then the default specified by the parent’s `minimize_results` is used.
OUTPUT:

A $k$-RegularSequence

EXAMPLES:

```python
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: E = Seq2((Matrix([[0, 1], [0, 1]]), Matrix([[0, 0], [0, 1]])),
...:     vector([1, 0]), vector([1, 1]))
sage: E
2-regular sequence 1, 0, 1, 0, 1, 0, 1, 0, ...
sage: E.partial_sums()
2-regular sequence 0, 1, 1, 2, 2, 3, 3, 4, 4, ...
sage: E.partial_sums(include_n=True)
2-regular sequence 1, 1, 2, 2, 3, 3, 4, 4, 5, ...
```

```python
sage: C = Seq2((Matrix([[2, 0], [2, 1]]), Matrix([[0, 1], [-2, 3]])),
...:     vector([1, 0]), vector([0, 1])); C
2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
sage: C.partial_sums()
2-regular sequence 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, ...
sage: C.partial_sums(include_n=True)
2-regular sequence 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, ...
```

**shift_left**($b=1, \text{**kwds}}$

Return the sequence obtained by shifting this $k$-regular sequence $b$ steps to the left.

INPUT:

- $b$ – an integer
- minimize – (default: None) a boolean or None. If True, then `minimized()` is called after the operation, if False, then not. If this argument is None, then the default specified by the parent's `minimize_results` is used.

OUTPUT:

A $k$-RegularSequence

Note: If $b$ is negative (i.e., actually a right-shift), then the coefficients when accessing negative indices are 0.

EXAMPLES:

```python
sage: C = Seq2((Matrix([[2, 0], [2, 1]]), Matrix([[0, 1], [-2, 3]])),
...:     vector([1, 0]), vector([0, 1])); C
2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
sage: C.shift_left()
2-regular sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...
sage: C.shift_left(3)
2-regular sequence 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...
```

(continues on next page)
\textbf{shift_right}(b=1,**kwds)

Return the sequence obtained by shifting this $k$-regular sequence $b$ steps to the right.

\begin{itemize}
  \item $b$ – an integer
  \item minimize – (default: None) a boolean or None. If True, then \texttt{minimized()} is called after the operation, if False, then not. If this argument is None, then the default specified by the parent’s \texttt{minimize_results} is used.
\end{itemize}

\textbf{OUTPUT:}

A \texttt{kRegularSequence}

\textbf{Note:} If $b$ is positive (i.e., indeed a right-shift), then the coefficients when accessing negative indices are 0.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: C = Seq2((Matrix([[2, 0], [0, 1]]), Matrix([[2, 1], [0, 1]])), ....:     vector([1, 0]), vector([0, 1])); C
2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
sage: C.shift_right()
2-regular sequence 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, ...
sage: C.shift_right(3)
2-regular sequence 0, 0, 0, 1, 2, 3, 4, 5, 6, ...
sage: C.shift_right(-2)
2-regular sequence 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
\end{verbatim}

\textbf{subsequence}(*args,**kwds)

Return the subsequence with indices $an + b$ of this $k$-regular sequence.

\begin{itemize}
  \item $a$ – a nonnegative integer
  \item $b$ – an integer
  
  Alternatively, this is allowed to be a dictionary $b_j \mapsto c_j$. If so and applied on $f(n)$, the result will be the sum of all $c_j \cdot f(an + bj)$.
  
  minimize – (default: None) a boolean or None. If True, then \texttt{minimized()} is called after the operation, if False, then not. If this argument is None, then the default specified by the parent’s \texttt{minimize_results} is used.
\end{itemize}

\textbf{OUTPUT:}

A \texttt{kRegularSequence}

\textbf{Note:} If $b$ is negative (i.e., right-shift), then the coefficients when accessing negative indices are 0.
EXAMPLES:

```
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
```

We consider the sequence $C$ with $C(n) = n$ and the following linear representation corresponding to the vector $(n, 1)$:

```
sage: C = Seq2((Matrix([[2, 0], [0, 1]]), Matrix([[2, 1], [0, 1]])),
....:    vector([1, 0]), vector([0, 1])); C
```

2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...

We now extract various subsequences of $C$:

```
sage: C.subsequence(2, 0)
2-regular sequence 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, ...

sage: S31 = C.subsequence(3, 1); S31
2-regular sequence 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, ...

sage: S31.linear_representation()
((1, 0),
 Finie family {0: [ 0 1]
     [-2 3],
     1: [ 6 -2]
     [10 -3]},
 (1, 1))

sage: C.subsequence(3, 2)
2-regular sequence 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, ...

sage: Srs = C.subsequence(1, -1); Srs
2-regular sequence 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, ...

sage: Srs.linear_representation()
((1, 0, 0),
 Finie family {0: [ 0 1 0]
     [-2 3 0]
     [-4 4 1],
     1: [ -2 2 0]
     [ 0 0 1]
     [12 -12 5]},
 (0, 0, 1))
```

We can build `backward_differences()` manually by passing a dictionary for the parameter $b$:

```
sage: Sbd = C.subsequence(1, {0: 1, -1: -1}); Sbd
2-regular sequence 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...
```

```
class sage.combinat.k_regular_sequence.kRegularSequenceSpace(k, *args, **kwds)
    Bases: RecognizableSeriesSpace

    The space of $k$-regular Sequences over the given coefficient_ring.

    INPUT:
    - $k$ – an integer at least 2 specifying the base
    - coefficient_ring – a (semi-)ring
```
• category – (default: None) the category of this space

**EXAMPLES:**

```python
sage: kRegularSequenceSpace(2, ZZ)
Space of 2-regular sequences over Integer Ring
sage: kRegularSequenceSpace(3, ZZ)
Space of 3-regular sequences over Integer Ring
```

See also:

`k-regular sequence`, `kRegularSequence`.

**Element**

alias of `kRegularSequence`

**from_recurrence(**args, **kwds)**

Construct the unique $k$-regular sequence which fulfills the given recurrence relations and initial values. The recurrence relations have to have the specific shape of $k$-recursive sequences as described in [HKL2021], and are either given as symbolic equations, e.g.,

```python
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: var('n')
sage: function('f')
f
sage: Seq2.from_recurrence([
    ....: f(2*n) == 2*f(n), f(2*n + 1) == 3*f(n) + 4*f(n - 1),
    ....: f(0) == 0, f(1) == 1], f, n)
2-regular sequence 0, 0, 0, 1, 2, 3, 4, 10, 6, 17, ...
```

or via the parameters of the $k$-recursive sequence as described in the input block below:

```python
sage: Seq2.from_recurrence(M=1, m=0,
    ....: coeffs={(0, 0): 2, (1, 0): 3, (1, -1): 4},
    ....: initial_values={0: 0, 1: 1})
2-regular sequence 0, 0, 0, 1, 2, 3, 4, 10, 6, 17, ...
```

**INPUT:**

Positional arguments:

If the recurrence relations are represented by symbolic equations, then the following arguments are required:

- **equations** – A list of equations where the elements have either the form
  
  $f(k^M n + r) = c_{r,l} f(k^m n + l) + c_{r,t+1} f(k^m n + l + 1) + ... + c_{r,u} f(k^m n + u)$
  
  for some integers $0 \leq r < k^M$, $M > m \geq 0$ and $l \leq u$, and some coefficients $c_{r,j}$ from the (semi)ring coefficients of the corresponding $kRegularSequenceSpace$, valid for all integers $n \geq$ offset for some integer offset $\geq \max(-l/k^m, 0)$ (default: 0), and there is an equation of this form (with the same parameters $M$ and $m$) for all $r$

  or the form
  
  $f(k) = t$ for some integer $k$ and some $t$ from the (semi)ring coefficient_ring.

The recurrence relations above uniquely determine a $k$-regular sequence; see [HKL2021] for further information.

- **function** – symbolic function $f$ occurring in the equations
• var – symbolic variable (n in the above description of equations)

The following second representation of the recurrence relations is particularly useful for cases where coefficient_ring is not compatible with sage.symbolic.ring.SymbolicRing. Then the following arguments are required:

• M – parameter of the recursive sequences, see [HKL2021], Definition 3.1, as well as in the description of equations above
• m – parameter of the recursive sequences, see [HKL2021], Definition 3.1, as well as in the description of equations above
• coeffs – a dictionary where coeffs[(r, j)] is the coefficient $c_{r,j}$ as given in the description of equations above. If coeffs[(r, j)] is not given for some r and j, then it is assumed to be zero.
• initial_values – a dictionary mapping integers n to the n-th value of the sequence

Optional keyword-only argument:

• offset – (default: 0) an integer. See explanation of equations above.
• inhomogeneities – (default: {}) a dictionary mapping integers r to the inhomogeneity $g_r$ as given in [HKL2021], Corollary D. All inhomogeneities have to be regular sequences from self or elements of coefficient_ring.

OUTPUT: a $k$RegularSequence

EXAMPLES:

Stern–Brocot Sequence:

Sage:
```plaintext
sage: Seq2 = kRegularSequenceSpace(2, ZZ)
sage: var('n')
n
sage: function('f')
f
sage: SB = Seq2.from_recurrence([
    ....: f(2*n) == f(n), f(2*n + 1) == f(n) + f(n + 1),
    ....: f(0) == 0, f(1) == 1], f, n)
sage: SB
2-regular sequence 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, ...
```

Number of Odd Entries in Pascal’s Triangle:

Sage:
```plaintext
sage: Seq2. from_recurrence([ 
    ....: f(2*n) == 3*f(n), f(2*n + 1) == 2*f(n) + f(n + 1),
    ....: f(0) == 0, f(1) == 1], f, n)
2-regular sequence 0, 1, 3, 5, 9, 11, 15, 19, 27, 29, ...
```

Number of Unbordered Factors in the Thue–Morse Sequence:

Sage:
```plaintext
sage: UB = Seq2. from_recurrence([ 
    ....: f(8*n) == 2*f(4*n),
    ....: f(8*n + 1) == f(4*n + 1),
    ....: f(8*n + 2) == f(4*n + 1) + f(4*n + 3),
    ....: f(8*n + 3) == -f(4*n + 1) + f(4*n + 2),
    ....: f(8*n + 4) == 2*f(4*n + 2),
    ....: f(8*n + 5) == f(4*n + 3),
    ....: f(8*n + 6) == -f(4*n + 1) + f(4*n + 2) + f(4*n + 3),
```
\(\vdots\) \(f(8n + 7) = 2^4 f(4n + 1) + f(4n + 3),\)
\(\vdots\) \(f(0) = 1, f(1) = 2, f(2) = 2, f(3) = 4, f(4) = 2,\)
\(\vdots\) \(f(5) = 4, f(6) = 6, f(7) = 0, f(8) = 4, f(9) = 4,\)
\(\vdots\) \(f(10) = 4, f(11) = 4, f(12) = 12, f(13) = 0, f(14) = 4,\)
\(\vdots\) \(f(15) = 4, f(16) = 8, f(17) = 4, f(18) = 8, f(19) = 0,\)
\(\vdots\) \(f(20) = 8, f(21) = 4, f(22) = 4, f(23) = 8, f, n, offset=3)\)

**sage:** UB
2-regular sequence 1, 2, 2, 4, 2, 4, 6, 0, 4, 4, ...

Binary sum of digits \(S(n),\) characterized by the recurrence relations \(S(4n) = S(2n), S(4n+1) = S(2n+1), S(4n+2) = S(2n+1)\) and \(S(4n+3) = -S(2n) + 2S(2n+1):\)

**sage:** S = Seq2.from_recurrence(
\(\vdots\) \(f(4n) = f(2n),\)
\(\vdots\) \(f(4n + 1) = f(2n + 1),\)
\(\vdots\) \(f(4n + 2) = f(2n + 1),\)
\(\vdots\) \(f(4n + 3) = -f(2n) + 2f(2n + 1),\)
\(\vdots\) \(f(0) = 0, f(1) = 1)\)

**sage:** S
2-regular sequence 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, ...

In order to check if this sequence is indeed the binary sum of digits, we construct it directly via its linear representation and compare it with \(S:\)

**sage:** S2 = Seq2(\n\(\vdots\) \((\text{Matrix([[1, 0], [0, 1]]), \text{Matrix([[1, 0], [1, 1]])}),\)
\(\vdots\) \(\text{left=vector([0, 1]), right=vector([1, 0]))})\)
**sage:** (S - S2).is_trivial_zero()
True

Alternatively, we can also use the simpler but inhomogeneous recurrence relations \(S(2n) = S(n)\) and \(S(2n + 1) = S(n) + 1\) via direct parameters:

**sage:** S3 = Seq2.from_recurrence(M=1, m=0,\n\(\vdots\) \(\text{coeffs=}\{(0, 0): 1, (1, 0): 1,\}
\(\vdots\) \(\text{initial_values=}\{0: 0, 1: 1,\}
\(\vdots\) \(\text{inhomogeneities=}\{1: 1\})\)
**sage:** S3
2-regular sequence 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, ...
**sage:** (S3 - S2).is_trivial_zero()
True

Number of Non-Zero Elements in the Generalized Pascal’s Triangle (see [LRS2017]):

**sage:** Seq2 = kRegularSequenceSpace(2, QQ)
**sage:** P = Seq2.from_recurrence([\n\(\vdots\) \(f(4n) = 5/3 f(2n) - 1/3 f(2n + 1),\)
\(\vdots\) \(f(4n + 1) = 4/3 f(2n) + 1/3 f(2n + 1),\)
\(\vdots\) \(f(4n + 2) = 1/3 f(2n) + 4/3 f(2n + 1),\)
\(\vdots\) \(f(4n + 3) = -1/3 f(2n) + 5/3 f(2n + 1),\)
\(\vdots\) \(f(0) = 1, f(1) = 2)\)
**sage:** P
2-regular sequence 1, 2, 3, 3, 4, 5, 5, 4, 5, 7,
Finally, the same sequence can also be obtained via direct parameters without symbolic equations:

\[
\text{sage: Seq2.from_recurrence(M=2, m=1, coeffs={(0, 0): 5/3, (0, 1): -1/3, (1, 0): 4/3, (1, 1): 1/3, (2, 0): 1/3, (2, 1): 4/3, (3, 0): -1/3, (3, 1): 5/3), initial_values={0: 1, 1: 2})}
\]

2-regular sequence 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, ...

5.1.130 Strong and weak tableaux

There are two types of \(k\)-tableaux: strong \(k\)-tableaux and weak \(k\)-tableaux. Standard weak \(k\)-tableaux correspond to saturated chains in the weak order, whereas standard strong \(k\)-tableaux correspond to saturated chains in the strong Bruhat order. For semistandard tableaux, the notion of weak and strong horizontal strip is necessary. More information can be found in [LLMS2006].

See also:

\texttt{sage.combinat.k_tableau.StrongTableau()}, \texttt{sage.combinat.k_tableau.WeakTableau()}

Authors:

- Anne Schilling and Mike Zabrocki (2013): initial version
- Avi Dalal and Nate Gallup (2013): implementation of \(k\)-charge

\textbf{class} \texttt{sage.combinat.k_tableau.StrongTableau(parent, T)}

Bases: \texttt{ClonableList}

A (standard) strong \(k\)-tableau is a (saturated) chain in Bruhat order.

Combinatorially, it is a sequence of embedded \(k+1\)-cores (subject to some conditions) together with a set of markings.

A strong cover in terms of cores corresponds to certain translated ribbons. A marking corresponds to the choice of one of the translated ribbons, which is indicated by marking the head (southeast most cell in French notation) of the chosen ribbon. For more information, see [LLMS2006] and [LLMSSZ2013].

In Sage, a strong \(k\)-tableau is created by specifying \(k\), a standard strong tableau together with its markings, and a weight \(\mu\). Here the standard tableau is represented by a sequence of \(k+1\)-cores

\[
\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(m)}
\]

where each of the \(\lambda^{(i)}\) is a \(k+1\)-core. The standard tableau is a filling of the diagram for the core \(\lambda^{(m)}/\lambda^{(0)}\) where a strong cover is represented by letters \(\pm i\) in the skew shape \(\lambda^{(i)}/\lambda^{(i-1)}\). Each skew \((k+1)\)-core \(\lambda^{(i)}/\lambda^{(i-1)}\) is a ribbon or multiple copies of the same ribbon which are separated by \(k+1\) diagonals. Precisely one of the copies of the ribbons will be marked in the largest diagonal of the connected component (the ‘head’ of the ribbon). The marked cells are indicated by negative signs.

The strong tableau is stored as a standard strong marked tableau (referred to as the standard part of the strong tableau) and a vector representing the weight.

\textbf{EXAMPLES}:

\texttt{sage: StrongTableau([[-1, -2, -3], [3]], 2, [3])}
\texttt{[-1, -1, -1], [1]}
\texttt{sage: StrongTableau([[1, 2, 4, 7], [-3, 6, -6, 8], [4, 7], [-5, -8]], 3, [2, 2, 3, 1])}
\texttt{[-1, -1, -2, -3], [-2, 3, -3, 4], [2, 3], [-3, -4]}

5.1. Comprehensive Module List
Alternatively, the strong $k$-tableau can also be entered directly in semistandard format and then the standard tableau and the weight are computed and stored:

\begin{verbatim}
sage: T = StrongTableau([[-1,-1,-1],[1]], 2); T
[[[-1, -1, -1], [1]]]
sage: T.to_standard_list()
[[[-1, -2, -3], [3]]]
sage: T.weight()
(3,)
sage: T = StrongTableau([[-1, -1, -2, -3], [-2, 3, -3, 4], [2, 3], [-3, -4]], 3); T
[[[-1, -1, -2, -3], [-2, 3, -3, 4], [2, 3], [-3, -4]]]
sage: T.to_standard_list()
[[[-1, -2, -4, -7], [-3, 6, -6, 8], [4, 7], [-5, -8]]]
sage: T.weight()
(2, 2, 3, 1)
\end{verbatim}

\section*{cell_of_highest_head(v)}

Return the cell of the highest head of label $v$ in the standard part of self.

Return the cell where the head of the ribbon in the highest row is located in the underlying standard tableau. If there is no cell with entry $v$ then the cell returned is $(0, r)$ where $r$ is the length of the first row.

This cell is calculated by iterating through the diagonals of the tableau.

\textbf{INPUT:}

\begin{itemize}
  \item $v$ – an integer indicating the label in the standard tableau
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item a pair of integers indicating the coordinates of the head of the highest ribbon with label $v$
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: T = StrongTableau([[-1,2,-3],[-2,3],[-3]], 1)
sage: [T.cell_of_highest_head(v) for v in range(1,5)]
[(0, 0), (1, 0), (2, 0), (0, 3)]
sage: T = StrongTableau([[None,None,-3,4],[3,-4]],2)
sage: [T.cell_of_highest_head(v) for v in range(1,5)]
[(1, 0), (1, 1), (0, 4), (0, 4)]
\end{verbatim}

\section*{cell_of_marked_head(v)}

Return location of marked head labeled by $v$ in the standard part of self.

Return the coordinates of the $v$-th marked cell in the strong standard tableau self. If there is no mark, then the value returned is $(0, r)$ where $r$ is the length of the first row.

\textbf{INPUT:}

\begin{itemize}
  \item $v$ – an integer representing the label in the standard tableau
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item a pair of the coordinates of the marked cell with entry $v$
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: T = StrongTableau([[1,2,-3],[-2,3],[3]], 1)
sage: [T.cell_of_marked_head(i) for i in range(1,7)]
[(0, 0), (1, 0), (0, 1), (2, 0), (0, 3), (0, 4)]
sage: T = StrongTableau([[None,None,-3,4],[3,-4]],2)
sage: [T.cell_of_marked_head(v) for v in range(1,5)]
[(1, 0), (1, 1), (0, 4), (0, 4)]
\end{verbatim}

(continues on next page)
cells_head_dictionary()
Returns a dictionary with the locations of the heads of all markings.

OUTPUT:
- a dictionary with keys the entries in the tableau and values are the coordinates of the heads with those entries

EXAMPLES:

```sage
T = StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2],
                   [-3, [3], [3], [3]], 4])
sage: T.cell_of_marked_head(i) for i in range(1,7)]
[(2, 0), (0, 2), (2, 1), (0, 3), (4, 0), (0, 4)]
sage: T.cells_head_dictionary()
{1: [(0, 0)], 2: [(0, 1)], 3: [(1, 0)], 4: [(2, 0), (0, 2)], 5: [(3, 0)],
  6: [(1, 2)], 7: [(2, 1), (0, 3)], 8: [(3, 1), (1, 3)]}
sage: T = StrongTableau([[None, 4, -4, -6, -7, 8, 8, -8],
                      [-3, 6], 3])
sage: T.cells_head_dictionary()
{1: [(2, 0)], 2: [(0, 2)], 3: [(1, 1)], 4: [(2, 1), (0, 3)], 5: [(0, 4)],
  6: [(1, 4), (0, 7)]}
sage: StrongTableau([[None, None], [None, -1]], 4).cells_head_dictionary()
{1: [(1, 1)]}
```

cells_of_heads(v)
Returns a list of cells of the heads with label v in the standard part of self.

A list of cells which are heads of the ribbons with label v in the standard part of the tableau self. If there is no cell labelled by v then return the empty list.

INPUT:
- v – an integer label

OUTPUT:
- a list of pairs of integers of the coordinates of the heads of the ribbons with label v

EXAMPLES:

```sage
T = StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2],
                   [-3, [3], [3], [3]], 4])
sage: T.cell_of_marked_head(i) for i in range(1,7)]
[(2, 0), (0, 2), (2, 1), (0, 3), (4, 0), (0, 4)]
sage: T.cells_head_dictionary()
{1: [(0, 0)], 2: [(0, 1)], 3: [(1, 0)], 4: [(2, 0), (0, 2)], 5: [(3, 0)],
  6: [(1, 2)], 7: [(2, 1), (0, 3)], 8: [(3, 1), (1, 3)]}
sage: T = StrongTableau([[None, 4, -4, -6, -7, 8, 8, -8],
                      [-3, 6], 3])
sage: T.cells_head_dictionary()
{1: [(2, 0)], 2: [(0, 2)], 3: [(1, 1)], 4: [(2, 1), (0, 3)], 5: [(0, 4)],
  6: [(1, 4), (0, 7)]}
sage: StrongTableau([[None, None], [None, -1]], 4).cells_head_dictionary()
{1: [(1, 1)]}
```

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cells_of_marked_ribbon($v$)

Return a list of all cells the marked ribbon labeled by $v$ in the standard part of self.

Return the list of coordinates of the cells which are in the marked ribbon with label $v$ in the standard part of the tableau. Note that the result is independent of the weight of the tableau.

The cells are listed from largest content (where the mark is located) to the smallest. Hence, the first entry in this list will be the marked cell.

INPUT:

• $v$ – the entry of the standard tableau

OUTPUT:

• a list of pairs representing the coordinates of the cells of the marked ribbon

EXAMPLES:

```python
sage: T = StrongTableau([[-1, -1, -2, -2, 3], [2, -3], [-3]], 3)
sage: T.to_standard_list()
[[0, 0], [1, -1], [1, -2], [2, 3], [3, -1], [3, -2], [3, -3]]
sage: T.cells_of_marked_ribbon(1)
[[(0, 0)]]
sage: T.cells_of_marked_ribbon(4)
[[(0, 3)]]
sage: T = StrongTableau([[-1, -2, -4, -7], [3, -6, -6, 8], [4, 7], [5, -8]], 3)
sage: T.cells_of_marked_ribbon(6)
[[(1, 2), (1, 1)]]
sage: T.cells_of_marked_ribbon(9)
[]
```

check()

Check that self is a valid strong $k$-tableau.

This function verifies that the outer and inner shape of the parent class is equal to the outer and inner shape of the tableau, that the tableau portion of self is a valid standard tableau, that the marks are placed correctly and that the size and weight agree.
EXAMPLES:

```python
sage: T = StrongTableau([[-1, -1, -2], [2]], 2)
sage: T.check()
sage: T = StrongTableau([[None, None, 2, -4, -4], [-1, 4], [-2]], 3)
sage: T.check()
```

**content_of_highest_head(v)**

Return the diagonal of the highest head of the cells labeled v in the standard part of self.

Return the content of the cell of the head in the highest row of all ribbons labeled by v of the underlying standard tableau. If there is no cell with entry v then the value returned is the length of the first row.

INPUT:

• v – an integer representing the label in the standard tableau

OUTPUT:

• an integer representing the content of the head of the highest ribbon with label v

EXAMPLES:

```python
sage: [StrongTableau([[-1,2,-3],[1,-3],[3]], 1).content_of_highest_head(v) for v in range(1,5)]
[0, -1, -2, 3]
```

**content_of_marked_head(v)**

Return the diagonal of the marked label v in the standard part of self.

Return the content (the \(j - i\) coordinate of the cell) of the v-th marked cell in the strong standard tableau self. If there is no mark, then the value returned is the size of first row.

INPUT:

• v – an integer representing the label in the standard tableau

OUTPUT:

• an integer representing the residue of the location of the mark

EXAMPLES:

```python
sage: [StrongTableau([[None, None, -1, -2], None, None], [[1, 2], [-3], [3]], [3], [3], [4])
[0, -1, 3, 4]
```

**contents_of_heads(v)**

A list of contents of the cells which are heads of the ribbons with label v.

If there is no cell labelled by v then return the empty list.

INPUT:

• v – an integer label

OUTPUT:
• a list of integers of the content of the heads of the ribbons with label \( v \)

EXAMPLES:

```python
sage: T = StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], [3], [3], [3]], 4)
sage: T.contents_of_heads(1)
[-2]
sage: T.contents_of_heads(2)
[-3, 2]
sage: T.contents_of_heads(3)
[-1]
sage: T.contents_of_heads(4)
[-2, 3]
sage: T.contents_of_heads(5)
[-4]
sage: T.contents_of_heads(6)
[]
```

**entries_by_content**(*diag*)

Return the entries on the diagonal of *self*.

Return the entries in the tableau that are in the cells \((i, j)\) with \(j - i\) equal to *diag* (that is, with content equal to *diag*).

**INPUT:**

• *diag* – an integer indicating the diagonal

**OUTPUT:**

• a list (perhaps empty) of labels on the diagonal *diag*

**EXAMPLES:**

```python
sage: T = StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], [3], [3], [3]], 4)
sage: T.entries_by_content(0)
[]
sage: T.entries_by_content(1)
[]
sage: T.entries_by_content(2)
[-1]
sage: T.entries_by_content(-2)
[-1, 2]
```

**entries_by_content_standard**(*diag*)

Return the entries on the diagonal of the standard part of *self*.

Return the entries in the tableau that are in the cells \((i, j)\) with \(j - i\) equal to *diag* (that is, with content equal to *diag*) in the standard tableau.

**INPUT:**

• *diag* – an integer indicating the diagonal

**OUTPUT:**

• a list (perhaps empty) of labels on the diagonal *diag*
EXAMPLES:

```
sage: T = StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2],
                      [-3], [3], [3], [3]], 4)
sage: T.entries_by_content_standard(0)
[]
sage: T.entries_by_content_standard(1)
[]
sage: T.entries_by_content_standard(2)
[-2]
sage: T.entries_by_content_standard(-2)
[-1, 4]
```

`follows_tableau()`

Return a list of strong marked tableaux with length one longer than `self`.

Return list of all strong tableaux obtained from `self` by extending to a core which follows the shape of `self` in the strong order.

OUTPUT:

- a list of strong tableaux which follow `self` in strong order

EXAMPLES:

```
sage: T = StrongTableau([[[-1,-2,-4,-7],[-3,6,-6,8],[4,7],[-5,-8]], 3, [2,2,3,1]])
sage: T.follows_tableau()
[[[-1, -1, -2, -3, 5, 5, -5], [-2, 3, -3, 4], [2, 3], [-3, -4]],
 [[-1, -1, -2, -3, 5], [-2, 3, -3, 4], [2, 3, 5], [-3, -4], [-5]],
 [[-1, -1, -2, -3, 5], [-2, 3, -3, 4], [2, 3, 5], [-3, -4], [5]],
 [[-1, -1, -2, -3, 5], [-2, 3, -3, 4], [2, 3, 5], [-3, -4], [5]],
 [[-1, -1, -2, -3], [-2, 3, -3, 4], [2, 3], [-3, -4], [-5], [5]],
]
sage: StrongTableau([[[-1,-2],[-3,-4]],3]).follows_tableau()
[[[-1, -2, 5, 5, -5], [-3, -4]], [[-1, -2, 5], [-3, -4], [-5]],
 [[-1, -2, -5], [-3, -4], [5]], [[-1, -2], [-3, -4], [-5], [5], [5]]
```

`height_of_ribbon(v)`

The number of rows occupied by one of the ribbons with label `v`.

The number of rows occupied by the marked ribbon with label `v` (and by consequence the number of rows occupied by any ribbon with the same label) in the standard part of `self`.

INPUT:

- `v` – the label of the standard marked tableau

OUTPUT:

- a non-negative integer representing the number of rows occupied by the ribbon which is marked

EXAMPLES:

```
sage: T = StrongTableau([[-1, -1, -2, -2, 3], [2, -3], [-3]])
sage: T.to_standard_list()
[[-1, -2, -3, -4, 6], [4, -6], [-5]]
sage: T.height_of_ribbon(1)
1
sage: T.height_of_ribbon(4)
(continues on next page)
```
inner_shape()  
Return the inner shape of self.

If self is a strong skew tableau, then this method returns the inner shape (the shape of the cells labelled with None). If self is not skew, then the inner shape is empty.

OUTPUT:  
• a \((k + 1)\)-core

EXAMPLES:

```python
sage: StrongTableau([[-1, None], [2], [-3]], 3).inner_shape()
[2, 2]
```

intermediate_shapes()  
Return the intermediate shapes of self.

A (skew) tableau with letters 1, 2, \ldots, \ell can be viewed as a sequence of shapes, where the \(i\)-th shape is given by the shape of the subtableau on letters 1, 2, \ldots, \(i\).

The output is the list of these shapes. The marked cells are ignored so to recover the strong tableau one would need the intermediate shapes and the \textit{content_of_marked_head()} for each pair of adjacent shapes in the list.

OUTPUT:

• a list of lists of integers representing \(k + 1\)-cores

EXAMPLES:

```python
sage: T = StrongTableau([[None, None, -1, -2], [None, None, -1, -2], [1, 2], [-3], [-3], [3], [3], [3]], 4)
sage: T.intermediate_shapes()

sage: T = StrongTableau([[-1, -2, -4, -7], [-3, 6, -6, 8], [4, 7, -5, -8]], 3, [2, 2, 3, 1])
sage: T.intermediate_shapes()

sage: T = StrongTableau([[-1, None], [2], [-3]], 3)
sage: T.intermediate_shapes()
```

is_column_strict_with_weight(mu)  
Test if self is a column strict tableau with respect to the weight \(mu\).

INPUT:

• \(mu\) – a vector of weights

OUTPUT:
• a boolean, True means the underlying column strict strong marked tableau is valid

EXAMPLES:

```python
sage: StrongTableau([[-1, -2, -3], [3]], 2).is_column_strict_with_weight([3])
True
sage: StrongTableau([[-1, -2, 3], [-3]], 2).is_column_strict_with_weight([3])
False
```

**left_action(tij)**

Action of transposition tij on self by adding marked ribbons.

Computes the left action of the transposition tij on the tableau. If tij acting on the element of the affine Grassmannian raises the length by 1, then this function will add a cell to the standard tableau.

INPUT:

• tij – a transposition represented as a pair (i, j).

OUTPUT:

• self after it has been modified by the action of the transposition tij

EXAMPLES:

```python
sage: StrongTableau([[None, -1, -2, -3], [3], [-4]], 3, weight=[1,1,1,1]).left_action([0,1])
[[None, -1, -2, -3, 5], [3, -5], [-4]]
sage: T = StrongTableau([[None, -1, -2, -3], [3], [-4]], 3, weight=[1,1,1,1])
sage: T.left_action([-3,-2])
[[None, -1, -2, -3], [3, [-4], [-5]]]
sage: T = StrongTableau([[None, -1, -2, -3], [3], [-4]], 3, weight=[3,1])
sage: T.left_action([-3,-2])
[[None, -1, -1, -1], [1], [-2], [-3]]
sage: T
[[None, -1, -1, -1], [1], [-2]]
sage: T.check()
sage: T.weight()
((3, 1))
```

**number_of_connected_components(v)**

Number of connected components of ribbons with label v in the standard part.

The number of connected components is calculated by finding the number of cells with label v in the standard part of the tableau and dividing by the number of cells in the ribbon.

INPUT:

• v – the label of the standard marked tableau

OUTPUT:

• a non-negative integer representing the number of connected components

EXAMPLES:
Combinatorics, Release 10.0

sage: T = StrongTableau(
[-1, -2, -2, 3], [2, -3], [-3], 3
sage: T.to_standard_list()
[[[-1, -2, -3, -4, 6], [4, -6], [-5]]

sage: T.number_of_connected_components(1)
1
sage: T.number_of_connected_components(4)
2
sage: T = StrongTableau([[[-1, -2, -4, -7], [-3, 6, -6, 8], [4, 7], [-5, -8]], 3
sage: T.number_of_connected_components(6)
1
sage: T.number_of_connected_components(9)
0

outer_shape()

Return the outer shape of self.

This method returns the outer shape of self as viewed as a Core. The outer shape of a strong tableau is always a \((k + 1)\)-core.

OUTPUT:

• a \((k + 1)\)-core

EXAMPLES:

sage: StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3],
\[3], [3], [3]], 4).outer_shape()
[4, 2, 2, 2, 1, 1, 1, 1]

sage: StrongTableau([[[-1, -2, -4, -7], [-3, 6, -6, 8], [4, 7], [-5, -8]], 3, [2, 2, 3, 1]).
\[outer_shape()
[4, 4, 2, 2]

pp()

Print the strong tableau self in pretty print format.

EXAMPLES:

sage: T = StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3],
\[3], [3], [3]], 4)
\[\[outer_shape()
[4, 4, 2, 2]

sage: Tableaux.options(convention="French")
restrict($r$)

Restrict the standard part of the tableau to the labels 1, 2, ..., $r$.

Return the tableau consisting of the labels of the standard part of self restricted to the labels of 1 through $r$. The result is another StrongTableau object.

INPUT:

• $r$ – an integer

OUTPUT:

• A strong tableau

EXAMPLES:

sage: T = StrongTableau([[None, None, -4, 5, -5], [None, None], [-1, -3], [-2], [2], [2], [3]], 4, weight=[1,1,1,1,1])
sage: T.restrict(3)
[[None, None], [None, None], [-1, -3], [-2], [2], [2], [3]]
sage: TT = T.restrict(0)
sage: TT
[[None, None], [None, None]]
sage: TT == StrongTableau([[None, None], [None, None]], 4)
True
sage: T.restrict(5) == T
True

ribbons_above_marked($v$)

Number of ribbons of label $v$ higher than the marked ribbon in the standard part.

Return the number of copies of the ribbon with label $v$ in the standard part of self which are in a higher row than the marked ribbon. Note that the result is independent of the weight of the tableau.

INPUT:

• $v$ – the entry of the standard tableau

OUTPUT:

• an integer representing the number of copies of the ribbon above the marked ribbon

EXAMPLES:

sage: T = StrongTableau([[-1,-2,-4,-7],[-3,6,-6,8],[4,7],[5,-8]], 3)
sage: T.ribbons_above_marked(4)
1

(continues on next page)
sage: T.ribbons_above_marked(6)
0
sage: T.ribbons_above_marked(9)
0
sage: StrongTableau([[[-1, -2, -3, -4], [2, 3, 4], [3, 4], [4]], 1]).ribbons_above_marked(4)
3

set_weight(mu)

Sets a new weight mu for self.

This method first tests if the underlying standard tableau is column-strict with respect to the weight mu. If it is, then it changes the weight and returns the tableau; otherwise it raises an error.

INPUT:

- mu – a list of non-negative integers representing the new weight

EXAMPLES:

sage: StrongTableau([[[-1, -2, -3]], [3]], 2).set_weight([3])
[[[-1, -1, -1]], [1]]
sage: StrongTableau([[[-1, -2, -3]], [3]], 2).set_weight([0, 3])
[[[-2, -2, -2]], [2]]
sage: StrongTableau([[[-1, -2, -3]], [3]], 2).set_weight([2, 0, 1])
[[[-1, -1, 3]], [-3]]
sage: StrongTableau([[[-1, -2, -3]], [3]], 2).set_weight([3])
Traceback (most recent call last):
  ... ValueError: \([-1, -2, 3], [-3]\) is not a semistandard strong tableau with respect to the partition [3]

shape()

Return the shape of self.

If self is a skew tableau then return a pair of \(k+1\)-cores consisting of the outer and the inner shape. If self is strong tableau with no inner shape then return a \(k+1\)-core.

INPUT:

- form - optional argument to indicate ‘inner’, ‘outer’ or ‘skew’ (default : ‘outer’)

OUTPUT:

- a \(k+1\)-core or a pair of \(k+1\)-cores if form is not ‘inner’ or ‘outer’

EXAMPLES:

sage: T = StrongTableau([[[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], [3], [3], [3]], 4])
sage: T.shape()
([4, 2, 2, 2, 1, 1, 1, 1], [2, 2])
sage: StrongTableau([[-1, -2, 3], [-3]], 2).shape()
[3, 1]
sage: type(StrongTableau([[-1, -2, 3], [-3]], 2).shape())
<class 'sage.combinat.core.Cores_length_with_category.element_class'>
size()

Return the size of the strong tableau.

The size of the strong tableau is the sum of the entries in the weight(). It will also be equal to the length of the outer shape (as a \(k + 1\)-core) minus the length of the inner shape.

See also:

\texttt{sage.combinat.core.Core.length()}

OUTPUT:

• a non-negative integer

EXAMPLES:

\begin{verbatim}
 sage: StrongTableau([[-1, -2, -3, 4], [-4], [-5]], 3).size()
 5
 sage: StrongTableau([[\texttt{None}, \texttt{None}, -1, 2], [-2], [-3]], 3).size()
 3
\end{verbatim}

spin()

Return the spin statistic of the tableau \(self\).

The spin is an integer statistic on a strong marked tableau. It is the sum of \((h - 1)r + \text{number of connected components above the marked one} \) where \(h\) is the height of the marked ribbon and \(r\) is the number of connected components.

See also:

\texttt{height_of_ribbon()}, \texttt{number_of_connected_components()}, \texttt{ribbons_above_marked()}

The \(k\)-Schur functions with a parameter \(t\) can be defined as

\[ s^{(k)}_{\lambda}[X; t] = \sum T \cdot t^{\text{spin}(T)} m_{\text{weight}(T)}[X] \]

where the sum is over all column strict marked strong \(k\)-tableaux of shape \(\lambda\) and partition content.

OUTPUT:

• an integer value representing the spin.

EXAMPLES:

\begin{verbatim}
 sage: StrongTableau([[-1,-2,5,6],[-4,7,-8]], 3, [2,2,3,1]).spin()
 1
 sage: StrongTableau([[-1,-2,6,-4],[3,-7,-6]], 3, [2,2,3,1]).spin()
 2
 sage: StrongTableau([[\texttt{None}, \texttt{None}, -1, -3],[2,3,\texttt{None}],[\texttt{None}, \texttt{None}, -3,-4]], 3).spin()
 2
 sage: ks3 = SymmetricFunctions(QQ['t']).fraction_field()).kschur(3)
sage: t = ks3.realization_of().t
tsage: m = ks3.ambient().realization_of().m()
sage: myks221 = sum(sum(t**T.spin() for T in StrongTableaux(3,[3,2,1], weight=mu))*m(mu) for mu in Partitions(5, max_part=3))
sage: myks221 == m(ks3[2,2,1])
True
\end{verbatim}

(continues on next page)
spin_of_ribbon(v)

Return the spin of the ribbon with label v in the standard part of self.

The spin of a ribbon is an integer statistic. It is the sum of \((h - 1)r\) plus the number of connected components above the marked one where \(h\) is the height of the marked ribbon and \(r\) is the number of connected components.

See also:

height_of_ribbon(), number_of_connected_components(), ribbons_above_marked()

INPUT:

• v – a label of the standard part of the tableau

OUTPUT:

• an integer value representing the spin of the ribbon with label v.

EXAMPLES:

sage: T = StrongTableau([[-1, -2, -3, 4], [-4], [-5]], 3)
sage: [T.spin_of_ribbon(v) for v in range(1,9)]
[0, 0, 0, 0, 0, 0, 1, 0]
sage: T = StrongTableau([[None, None, -1, -3],[None, 3, -3, 4],[2,3],[-3,-4]], 3)
sage: [T.spin_of_ribbon(v) for v in range(1,7)]
[0, 1, 0, 0, 1, 0]

to_list()

Return the marked column strict (possibly skew) tableau as a list of lists.

OUTPUT:

• a list of lists of integers or None

EXAMPLES:

sage: StrongTableau([[-1, -2, -3, 4], [-4], [-5]], 3).set_weight([2,1,1,1]).to_list()
[[-1, -1, -2, 3], [-3], [-4]]
sage: StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3],
                  → [3], [3], [3]], 4).to_list()
[[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], [3], [3], [3]]
sage: StrongTableau([[-1, -1, -2, -3, 4], [-4], [-5]], 3, [3,1,1]).to_list()
[[-1, -1, -1, 2], [-2], [-3]]

to_standard_list()

Return the underlying standard strong tableau as a list of lists.

Internally, for a strong tableau the standard strong tableau and its weight is stored separately. This method returns the underlying standard part.
OUTPUT:

• a list of lists of integers or None

EXAMPLES:

```
sage: StrongTableau([[-1, -2, -3, 4], [-4], [-5]], 3, [3,1,1]).to_standard_list()
[[-1, -2, -3, 4], [-4], [-5]]
sage: StrongTableau([[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], → [3], [3], [3]], 4).to_standard_list()
[[None, None, -2, -4], [None, None], [-1, -3], [2, 4], [-5], [5], [5], [5]]
```

to_standard_tableau()

Return the underlying standard strong tableau as a StrongTableau object.

Internally, for a strong tableau the standard strong tableau and its weight is stored separately. This method returns the underlying standard part as a StrongTableau.

OUTPUT:

• a strong tableau with standard weight

EXAMPLES:

```
sage: T = StrongTableau([[-1, -2, -3, 4], [-4], [-5]], 3, [3,1,1])
sage: T.to_standard_tableau() == T.to_standard_list()
False
```

to_transposition_sequence()

Return a list of transpositions corresponding to self.

Given a strong column strict tableau self returns the list of transpositions which when applied to the left of an empty tableau gives the corresponding strong standard tableau.

OUTPUT:

• a list of pairs of values \([i,j]\) representing the transpositions \(t_{ij}\)

EXAMPLES:

```
sage: T = StrongTableau([[-1, -1, -1], [1]],2)
sage: T.to_transposition_sequence()
[[2, 3], [1, 2], [0, 1]]
sage: T = StrongTableau([[-1, -1, 2], [-2]],2)
sage: T.to_transposition_sequence()
[[-1, 0], [1, 2], [0, 1]]
sage: T = StrongTableau([[None, 1, 2], [-3]],2)
sage: T.to_transposition_sequence()
[[3, 4], [-1, 0], [1, 2]]
```

to_unmarked_list()

Return the tableau as a list of lists with markings removed.
Return the list of lists of the rows of the tableau where the markings have been removed.

OUTPUT:

• a list of lists of integers or None

EXAMPLES:

```python
sage: T = StrongTableau( [[-1, -2, -3, 4], [-4], [-5]], 3, [3,1,1])
sage: T.to_unmarked_list()
[[1, 1, 1], [2], [3]]
sage: TT = T.set_weight([2,1,1])
sage: TT.to_unmarked_list()
[[1, 1, 2], [3], [4]]
sage: StrongTableau( [[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], [3], [3], [3], 4]).to_unmarked_list()
[[None, None, 1, 2], [None, None], [1, 2], [1, 2], [3], [3], [3], [3]]
```

to_unmarked_standard_list()

Return the standard part of the tableau as a list of lists with markings removed.

Return the list of lists of the rows of the tableau where the markings have been removed.

OUTPUT:

• a list of lists of integers or None

EXAMPLES:

```python
sage: StrongTableau( [[-1, -2, -3, 4], [-4], [-5]], 3, [3,1,1]).to_unmarked_standard_list()
[[1, 2, 3, 4], [4], [5]]
sage: StrongTableau( [[None, None, -1, -2], [None, None], [-1, -2], [1, 2], [-3], [3], [3], [3], 4]).to_unmarked_standard_list()
[[None, None, 2, 4], [None, None], [1, 3], [2, 4], [5], [5], [5], [5]]
```

weight()

Return the weight of the tableau.

The weight is a list of non-negative integers indicating the number of 1s, number of 2s, number of 3s, etc.

OUTPUT:

• a list of non-negative integers

EXAMPLES:

```python
sage: T = StrongTableau([[1, -1, -2, -3, 4], [-4], [-5]], 3); T.weight()
(1, 1, 1, 1, 1)
sage: T.set_weight([3,1,1]).weight()
(3, 1, 1)
sage: StrongTableau([[1, -1, -2, -3], [-2,3, -3,4],[2,3],[-3,-4]], 3).weight()
(2, 2, 3, 1)
```

class sage.combinat.k_tableau.StrongTableaux(k, shape, weight)

Bases: UniqueRepresentation, Parent

Element

alias of StrongTableau
classmethod add_marking(unmarkedT, marking, k, weight)
Add markings to a partially marked strong tableau.

Given an partially marked standard tableau and a list of cells where the marks should be placed along with a weight, return the semi-standard marked strong tableau. The marking should complete the marking so that the result is a strong standard marked tableau.

INPUT:
• unmarkedT - a list of lists which is a partially marked strong $k$-tableau
• marking - a list of pairs of coordinates where cells are to be marked
• k - a positive integer
• weight - a tuple of the weight of the output tableau

OUTPUT:
• a StrongTableau object

EXAMPLES:

sage: StrongTableaux.add_marking([[None,1,2],[2]], [(0,1), (1,0)], 2, [1,1])
[[None, -1, 2], [-2]]

sage: StrongTableaux.add_marking([[None,1,2],[2]], [(0,1), (1,0)], 2, [2])
Traceback (most recent call last):
... ValueError: The weight=(2,) and the markings on the standard tableau=[[None, -1, → 2], [-2]] do not agree.

sage: StrongTableaux.add_marking([[None,1,2],[2]], [(0,1), (0,2)], 2, [2])
[[None, -1, -1], [1]]

an_element()
Return the first generated element of the class of StrongTableaux.

EXAMPLES:

sage: ST = StrongTableaux(3, [3], weight=[3])
sage: ST.an_element()
[[−1, −1, −1]]

classmethod cells_head_dictionary(T)
Return a dictionary with the locations of the heads of all markings.

Return a dictionary of values and lists of cells where the heads with the values are located in a strong standard unmarked tableau T.

INPUT:
• T – a strong standard unmarked tableau as a list of lists

OUTPUT:
• a dictionary with keys the entries in the tableau and values are the coordinates of the heads with those entries

EXAMPLES:
class method follows_tableau_unsigned_standard(Tlist, k)

Return a list of strong tableaux one longer in length than Tlist.

Return list of all standard strong tableaux obtained from Tlist by extending to a core which follows the shape of Tlist in the strong order. It does not put the markings on the last entry that it adds but it does keep the markings on all entries smaller. The objects returned are not StrongTableau objects (and cannot be) because the last entry will not properly marked.

INPUT:

• Tlist – a filling of a $k + 1$-core as a list of lists

• k - an integer

OUTPUT:

• a list of strong tableaux which follow Tlist in strong order

EXAMPLES:

\[
\text{sage: StrongTableaux.follows_tableau_unsigned_standard([[-1, -1, -2, -3], [-2, -3, -4], [2, 3], [-3, -4]], 3)}
\]
\[
[[[-1, -1, -2, -3, 5, 5, 5], [-2, 3, -3, 4], [2, 3, 5], [-3, -4]], [5], [5], [5]]
\]

inner_shape()

Return the inner shape of the class of strong tableaux.

OUTPUT:

• a $k + 1$-core

EXAMPLES:
sage: StrongTableaux( 2, [3,1] ).inner_shape()
[]
sage: type(StrongTableaux( 2, [3,1] ).inner_shape())
<class 'sage.combinat.core.Cores_length_with_category.element_class'>
sage: StrongTableaux( 4, [[2,1], [1]] ).inner_shape()
[1]

classmethod marked_CST_to_transposition_sequence(T, k)

Return a list of transpositions corresponding to T.

Given a strong column strict tableau T returns the list of transpositions which when applied to the left of an empty tableau gives the corresponding strong standard tableau.

INPUT:

• T – a non-empty column strict tableau as a list of lists

• k – a positive integer

OUTPUT:

• a list of pairs of values [i, j] representing the transpositions t_{ij}

EXAMPLES:

sage: CST_to_trans = StrongTableaux.marked_CST_to_transposition_sequence
sage: CST_to_trans([[[-1, -1, -1], [1]], 2]
[[2, 3], [1, 2], [0, 1]])
sage: CST_to_trans([], 2)
[]
sage: CST_to_trans([[[-2, -2, -2], [2]], 2]
[[2, 3], [1, 2], [0, 1]])
sage: CST_to_trans([[[-1, -2, -2, -2], [-2, 2]], 3]
[[4, 5], [3, 4], [2, 3], [1, 2], [-1, 0], [0, 1]])
sage: CST_to_trans([[[-1, -2, -5, 5, -5, 5, -5], [-3, -4, 5, 5]], 3)
[[5, 7], [3, 5], [2, 3], [0, 1], [-1, 0], [1, 2], [0, 1]])
sage: CST_to_trans([[[-1, -2, -3, 4, -7], [-4, -6], [-5, 6]], 3)
[[4, 5], [-1, 1], [-2, -1], [-1, 0], [2, 3], [1, 2], [0, 1]]

classmethod marked_given_unmarked_and_weight_iterator(unmarkedT, k, weight)

An iterator generating strong marked tableaux from an unmarked strong tableau.

Iterator which lists all marked tableaux of weight weight such that the standard unmarked part of the tableau is equal to unmarkedT.

INPUT:

• unmarkedT - a list of lists representing a strong unmarked tableau

• k - a positive integer

• weight - a list of non-negative integers indicating the weight

OUTPUT:

• an iterator that returns StrongTableau objects

EXAMPLES:
options = Current options for Tableaux - ascii_art: repr - convention: English - display: list - latex: diagram

outer_shape()

Return the outer shape of the class of strong tableaux.

OUTPUT:

• a \( k + 1 \)-core

EXAMPLES:

```
sage: StrongTableaux( 2, [3,1] ).outer_shape()
[3, 1]
sage: type(StrongTableaux( 2, [3,1] ).outer_shape())
<class 'sage.combinat.core.Cores_length_with_category.element_class'>
sage: StrongTableaux( 4, [[2,1], [1]] ).outer_shape()
[2, 1]
```

shape()

Return the shape of self.

If the self has an inner shape return a pair consisting of an inner and an outer shape. If the inner shape is empty then return only the outer shape.

OUTPUT:

• a \( k + 1 \)-core or a pair of \( k + 1 \)-cores

EXAMPLES:
classmethod standard_marked_iterator\((k, size, outer_shape=None, inner_shape=[])\)

An iterator for generating standard strong marked tableaux.

An iterator which generates all standard marked \(k\)-tableaux of a given size which are contained in outer_shape and contain the inner_shape. If outer_shape is None then there is no restriction on the shape of the tableaux which are created.

**INPUT:**
- \(k\) - a positive integer
- \(size\) - a positive integer
- outer_shape - a list which is a \(k + 1\)-core (default: None)
- inner_shape - a list which is a \(k + 1\)-core (default: [])

**OUTPUT:**
- an iterator which returns the standard marked tableaux with size cells and that are contained in outer_shape and contain inner_shape

**EXAMPLES:**

```sage
sage: list(StrongTableaux.standard_marked_iterator(2, 3))
[[[[-1, -2, 3], [-3]], [[-1, -2, -3], [3]], [[-1, -2], [-3], [3]], [[-1, 3, -3], [-2]], [[-1, 3, [-2], [-3]], [[-1, -3], [-2], [3]]]
sage: list(StrongTableaux.standard_marked_iterator(2, 1, inner_shape=[1,1]))
[[[None, 1, -1], [None]], [[None, 1], [None], [-1]], [[None, -1], [None], [1]]
sage: len(list(StrongTableaux.standard_marked_iterator(4,4)))
10
sage: len(list(StrongTableaux.standard_marked_iterator(4,6)))
140
sage: len(list(StrongTableaux.standard_marked_iterator(4,4, inner_shape=[2,2])))
200
sage: len(list(StrongTableaux.standard_marked_iterator(4,4, outer_shape=[5,2,2, -1], inner_shape=[2,2])))
24
```

classmethod standard_unmarked_iterator\((k, size, outer_shape=None, inner_shape=[])\)

An iterator for standard unmarked strong tableaux.

An iterator which generates all unmarked tableaux of a given size which are contained in outer_shape and which contain the inner_shape. These are built recursively by building all standard marked strong tableaux of size size −1 and adding all possible covers.

If outer_shape is None then there is no restriction on the shape of the tableaux which are created.

**INPUT:**
- \(k, size\) - a positive integers
• outer_shape - a list representing a $k + 1$-core (default: None)
• inner_shape - a list representing a $k + 1$-core (default: [])

OUTPUT:
• an iterator which lists all standard strong unmarked tableaux with size $n$ and which are contained in outer_shape and contain inner_shape

EXAMPLES:

```python
sage: list(StrongTableaux.standard_unmarked_iterator(2, 3))
[[[1, 2, 3], [3]], [[1, 2], [3], [3]], [[1, 3, 3], [2]], [[1, 3], [2], [3]]]
sage: list(StrongTableaux.standard_unmarked_iterator(2, 1, inner_shape=[1,1]))
[[[None, 1, 1], [None]], [[None, 1], [None], [1]]]
sage: len(list(StrongTableaux.standard_unmarked_iterator(4,4)))
10
sage: len(list(StrongTableaux.standard_unmarked_iterator(4,6)))
98
sage: len(list(StrongTableaux.standard_unmarked_iterator(4,4, inner_shape=[2, 2])))
92
sage: len(list(StrongTableaux.standard_unmarked_iterator(4,4, outer_shape=[5,2, 2,1], inner_shape=[2,2])))
10
```

classmethod transpositions_to_standard_strong(transeq, k, emptyTableau=[])

Return a strong tableau corresponding to a sequence of transpositions.

This method returns the action by left multiplication on the empty strong tableau by transpositions specified by transeq.

INPUT:
• transeq – a sequence of transpositions $t_{ij}$ (a list of pairs).
• emptyTableau – (default: []) an empty list or a skew strong tableau possibly consisting of None entries

OUTPUT:
• a StrongTableau object

EXAMPLES:

```python
sage: StrongTableaux.transpositions_to_standard_strong([[0,1]], 2)
[[[-1]]]
sage: StrongTableaux.transpositions_to_standard_strong([[2, 3], [1, 2], [0, 1]], 2)
[[[-1, -2, -3], [3]]]
sage: StrongTableaux.transpositions_to_standard_strong([[0, -1], [1, 2], [0, -1]], 2)
[[[-1, -2, 3], [-3]]]
sage: StrongTableaux.transpositions_to_standard_strong([[3, 4], [-1, 0], [1, -2]], 2, [[None]])
[[None, -1, 2, -3], [-2, 3]]
```
This is the dispatcher method for the element class of weak $k$-tableaux.

Standard weak $k$-tableaux correspond to saturated chains in the weak order. There are three formulations of weak tableaux, one in terms of cores, one in terms of $k$-bounded partitions, and one in terms of factorizations of affine Grassmannian elements. For semistandard weak $k$-tableaux, all letters of the same value have to satisfy the conditions of a horizontal strip. In the affine Grassmannian formulation this means that all factors are cyclically decreasing elements. For more information, see for example [LLMSSZ2013].

**INPUT:**

- $t$ – a weak $k$-tableau in the specified representation:
  - for the ‘core’ representation $t$ is a list of lists where each subtableaux should have a $k + 1$-core shape; `None` is allowed as an entry for skew weak $k$-tableaux
  - for the ‘bounded’ representation $t$ is a list of lists where each subtableaux should have a $k$-bounded shape; `None` is allowed as an entry for skew weak $k$-tableaux
  - for the ‘factorized_permutation’ representation $t$ is either a list of cyclically decreasing Weyl group elements or a list of reduced words of cyclically decreasing Weyl group elements; to indicate a skew tableau in this representation, `inner_shape` should be the inner shape as a $(k + 1)$-core

- $k$ – positive integer

- `inner_shape` – this entry is only relevant for the ‘factorized_permutation’ representation and specifies the inner shape in case the tableau is skew (default: `[]`)

- `representation` – ‘core’, ‘bounded’, or ‘factorized_permutation’ (default: ‘core’)

**EXAMPLES:**

Here is an example of a weak 3-tableau in core representation:

```
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.shape()
[5, 2, 1]
sage: t.weight()
(2, 2, 2)
sage: type(t)
<class 'sage.combinat.k_tableau.WeakTableaux_core_with_category.element_class'>
```

And now we give a skew weak 3-tableau in core representation:

```
sage: ts = WeakTableau([[None, 1, 1, 2, 2], [None, 2], [1]], 3)
sage: ts.shape()
([5, 2, 1], [1, 1])
sage: ts.weight()
(2, 2)
sage: type(ts)
<class 'sage.combinat.k_tableau.WeakTableaux_core_with_category.element_class'>
```

Next we create the analogue of the first example in bounded representation:

```
sage: tb = WeakTableau([[1,1,2],[2,3],[3]], 3, representation="bounded")
sage: tb.shape()
[3, 2, 1]
sage: tb.weight()
(2, 2, 2)
```

(continues on next page)
sage: type(tb)
<class 'sage.combinat.k_tableau.WeakTableaux_bounded_with_category.element_class'>
sage: tb.to_core_tableau()
[[1, 1, 2, 2, 3], [2, 3], [3]]
sage: t == tb.to_core_tableau()
True

And the analogue of the skew example in bounded representation:

sage: tbs = WeakTableau([[None, 1, 2], [None, 2], [1]], 3, representation="bounded →")
sage: tbs.shape()
([3, 2, 1], [1, 1])
sage: tbs.weight()
(2, 2)
sage: tbs.to_core_tableau()
[[None, 1, 1, 2, 2], [None, 2], [1]]
sage: ts.to_bounded_tableau() == tbs
True

Finally we do the same examples for the factorized permutation representation:

sage: tf = WeakTableau([[2,0],[3,2],[1,0]], 3, representation="factorized_˓→permutation")

Finally we do the same examples for the factorized permutation representation:

sage: tf = WeakTableau([[2,0],[3,2],[1,0]], 3, representation="factorized_˓→permutation")

sage: tf.shape()
[5, 2, 1]
sage: tf.weight()
(2, 2, 2)
sage: type(tf)
<class 'sage.combinat.k_tableau.WeakTableaux_factorized_permutation_with_category.˓→element_class'>
sage: tf.to_core_tableau() == t
True

sage: tfs = WeakTableau([[0,3],[2,1]], 3, inner_shape = [1,1], representation = ˓→'factorized_permutation'

Another way to pass from one representation to another is as follows:

sage: ts
[[None, 1, 1, 2, 2], [None, 2], [1]]
sage: ts.parent()._representation
'core'
To test whether a given semistandard tableau is a weak $k$-tableau in the bounded representation, one can ask:

```python
sage: t = Tableau([[1,1,2],[2,3],[3]])
sage: t.is_k_tableau(3)
True
sage: t = SkewTableau([[None, 1, 2], [None, 2], [1]])
sage: t.is_k_tableau(3)
True
sage: t = SkewTableau([[None, 1, 1], [None, 2], [2]])
sage: t.is_k_tableau(3)
False
```

### class sage.combinat.k_tableau.WeakTableau_abstract

Bases: `ClonableList`

Abstract class for the various element classes of WeakTableau.

#### intermediate_shapes()

Return the intermediate shapes of self.

A (skew) tableau with letters $1, 2, \ldots, \ell$ can be viewed as a sequence of shapes, where the $i$-th shape is given by the shape of the subtableau on letters $1, 2, \ldots, i$. The output is the list of these shapes.

**EXAMPLES:**

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.intermediate_shapes()
[[], [2], [4, 1], [5, 2, 1]]

sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: t.intermediate_shapes()
[[2], [2, 1], [3, 1, 1], [4, 1, 1], [5, 2, 1]]

sage: t = WeakTableau([[1,1,1],[2,2],[3],[3]], 3, representation = 'bounded')
sage: t.intermediate_shapes()
[[], [3], [3, 2], [3, 2, 1]]

sage: t = WeakTableau([[None, None, 1], [2, 4], [3]], 3, representation = 'bounded')
sage: t.intermediate_shapes()
[[2], [3], [3, 1], [3, 1, 1], [3, 2, 1]]

sage: t = WeakTableau([[None, 0],[3],[2],[3]], 3, inner_shape = [2], representation = 'factorized_permutation')
sage: t.intermediate_shapes()
[[2], [2, 1], [3, 1, 1], [4, 1, 1], [5, 2, 1]]
```

#### pp()

Return a pretty print string of the tableau.

**EXAMPLES:**
\begin{verbatim}
sage: t = WeakTableau([[None, 1, 1, 2, 2], [None, 2], [1]], 3)
sage: t.pp()
 . 1 1 2 2
 . 2

1

sage: t = WeakTableau([[2,0],[3,2]], 3, inner_shape = [2], representation = \texttt{factorized_permutation})

sage: t.pp()
[s2*s0, s3*s2]

representation\texttt{(representation='core')}

Return the analogue of self in the specified representation.

\textbf{INPUT:}

\begin{itemize}
  \item representation – 'core', 'bounded', or 'factorized_permutation' (default: 'core')
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: t = WeakTableau([[1, 1, 2, 4], [2, 3, 5], [3, 4], [5, 6], [6], [7]], 4)
sage: t.parent()._representation
'core'
sage: t.representation('bounded')
[[1, 1, 2, 4], [2, 3, 5], [3, 4], [5, 6], [6], [7]]
sage: t.representation('factorized_permutation')
[s0*s0, s3*s1, s2*s1, s0*s4, s3*s0, s4*s2, s1*s0]
sage: tb = WeakTableau([[1, 1, 2, 4], [2, 3, 5], [3, 4], [5, 6], [6], [7]], 4, representation = 'bounded')
sage: tb.parent()._representation
'bounded'
sage: tb.representation('core') == t
True
sage: tb.representation('factorized_permutation')
[s0*s0, s3*s1, s2*s1, s0*s4, s3*s0, s4*s2, s1*s0]
sage: tp = WeakTableau([[0],[3,1],[2,1],[0,4],[3,0],[4,2],[1,0]], 4, representation = 'factorized_permutation')
sage: tp.parent()._representation
'factorized_permutation'
sage: tp.representation('core') == t
True
sage: tp.representation('bounded') == tb
True
\end{verbatim}

shape()

Return the shape of self.

When the tableau is straight, the outer shape is returned. When the tableau is skew, the tuple of the outer and inner shape is returned.

\textbf{EXAMPLES:}

\begin{verbatim}
\end{verbatim}
\end{verbatim}
size()

Return the size of the shape of self.

In the bounded representation, the size of the shape is the number of boxes in the outer shape minus the number of boxes in the inner shape. For the core and factorized permutation representation, the size is the length of the outer shape minus the length of the inner shape.

See also:

sage.combinat.core.Core.length()

EXAMPLES:

```python
sage: t = WeakTableau([[None, 1, 1, 2], [None, 2], [1]], 3)
sage: t.shape()
([5, 2, 1], [1, 1])
sage: t.size()
4
sage: t = WeakTableau([[1, 1, 2], [2, 3], [3]], 3, representation="bounded")
sage: t.shape()
([5, 2, 1], [1])
sage: t.size()
6
```

weight()

Return the weight of self.

The weight is a tuple whose $i$-th entry is the number of labels $i$ in the bounded representation of self.

EXAMPLES:
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.weight()
(2, 2, 2)
sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: t.weight()
(1, 1, 1, 1)
sage: t = WeakTableau([[None, 2, 3], [3]], 2)
sage: t.weight()
(0, 1, 1)

sage: t = WeakTableau([[1,1,1],[2,2],[3]], 3, representation = 'bounded')
sage: t.weight()
(3, 2, 1)
sage: t = WeakTableau([[1,1,2],[2,3],[3]], 3, representation = 'bounded')
sage: t.weight()
(2, 2, 2)
sage: t = WeakTableau([[None, None, 1], [2, 4], [3]], 3, representation = 'bounded')
sage: t.weight()
(1, 1, 1, 1)

sage: t = WeakTableau([[2],[0,3],[2,1,0]], 3, representation = 'factorized_permutation')
sage: t.weight()
(3, 2, 1)
sage: t = WeakTableau([[2,0],[3,2],[1,0]], 3, representation = 'factorized_permutation')
sage: t.weight()
(2, 2, 2)
sage: t = WeakTableau([[2,0],[3,2]], 3, inner_shape = [2], representation = 'factorized_permutation')
sage: t.weight()
(2, 2)

class sage.combinat.k_tableau.WeakTableau_bounded(parent, t)
Bases: WeakTableau_abstract

A (skew) weak k-tableau represented in terms of k-bounded partitions.

check()
Check that self is a valid weak k-tableau.

EXAMPLES:

sage: t = WeakTableau([[1,1],[2]], 2, representation = 'bounded')
sage: t.check()

sage: t = WeakTableau([[None, None, 1], [2, 4], [3]], 3, representation = 'bounded')
sage: t.check()

classmethod from_core_tableau(t, k)
Construct weak k-bounded tableau from in k-core tableau.

EXAMPLES:
sage: from sage.combinat.k_tableau import WeakTableau_bounded
sage: WeakTableau_bounded.from_core_tableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
[[1, 1, 2], [2, 3], [3]]
sage: WeakTableau_bounded.from_core_tableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
[[None, None, 3], [1, 4], [2]]
sage: WeakTableau_bounded.from_core_tableau([[None, 2, 3], [3]], 2)
[[None, 2], [3]]

\textbf{\texttt{k\_charge}}(algorith\texttt{m}=I)

Return the \(k\)-charge of \texttt{self}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{algorithm} – (default: “I”) if “I”, computes \(k\)-charge using the \(I\) algorithm, otherwise uses the \(J\)-algorithm
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item a nonnegative integer
\end{itemize}

For the definition of \(k\)-charge and the various algorithms to compute it see Section 3.3 of [LLMSSZ2013].

\textbf{EXAMPLES:}

sage: t = WeakTableau([[1, 1, 2], [2, 3], [3]], 3, representation = 'bounded')
sage: t.k_charge()
2
sage: t = WeakTableau([[1, 3, 5], [2, 6], [4]], 3, representation = 'bounded')
sage: t.k_charge()
8
sage: t = WeakTableau([[1, 1, 2, 4], [2, 3, 5], [3, 4], [5, 6], [6], [7]], 4, \rightarrow representation = 'bounded')
sage: t.k_charge()
12

\textbf{\texttt{shape\_bounded}}()

Return the shape of \texttt{self} as \(k\)-bounded partition.

When the tableau is straight, the outer shape is returned as a \(k\)-bounded partition. When the tableau is skew, the tuple of the outer and inner shape is returned as \(k\)-bounded partitions.

\textbf{EXAMPLES:}

sage: t = WeakTableau([[1,1,1],[2,2],[3]], 3, representation = 'bounded')
sage: t.shape_bounded()
[3, 2, 1]
sage: t = WeakTableau([[None, None, 1], [2, 4], [3]], 3, representation = \rightarrow 'bounded')
sage: t.shape_bounded()
([3, 2, 1], [2])

\textbf{\texttt{shape\_core}}()

Return the shape of \texttt{self} as \((k+1)\)-core.
When the tableau is straight, the outer shape is returned as a \((k+1)\)-core. When the tableau is skew, the tuple of the outer and inner shape is returned as \((k+1)\)-cores.

**EXAMPLES:**

```python
t = WeakTableau([[1,1,1],[2,2],[3]], 3, representation = 'bounded')
t.shape_core()
[5, 2, 1]

t = WeakTableau([[None, None, 1],[2, 4],[3]], 3, representation = 'bounded')
t.shape_core()
([5, 2, 1], [2])
```

**to_core_tableau()**

Return the weak \(k\)-tableau \(self\) where the shape of each restricted tableau is a \((k+1)\)-core.

**EXAMPLES:**

```python
t = WeakTableau([[1,1,2,4],[2,3,5],[3,4],[5,6],[6],[7]], 4, representation = 'bounded')
t.to_core_tableau(); c
[[1, 1, 2, 3, 4, 4, 5, 5, 6], [2, 3, 5, 5, 6], [3, 4, 7], [5, 6], [6], [7]]
type(c)
<class 'sage.combinat.k_tableau.WeakTableaux_core_with_category.element_class'>
t = WeakTableau([], 4, representation = 'bounded')
t.to_core_tableau()
[]

t = WeakTableau_bounded.from_core_tableau(t.to_core_tableau(),3)
[[1, 1, 2], [2, 3], [3]]
```

**class** `sage.combinat.k_tableau.WeakTableau_core` *(parent, t)*

**Bases:** `WeakTableau_abstract`

A (skew) weak \(k\)-tableau represented in terms of \((k+1)\)-cores.

**check()**

Check that \(self\) is a valid weak \(k\)-tableau.

**EXAMPLES:**

```python
t = WeakTableau([[1, 1, 2], [2]], 2)
t.check()
```
dictionary_of_coordinates_at_residues(v)

Return a dictionary assigning to all residues of self with label v a list of cells with the given residue.

INPUT:

• v – a label of a cell in self

OUTPUT:

• dictionary assigning coordinates in self to residues

EXAMPLES:

```python
sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: t.dictionary_of_coordinates_at_residues(3)
{2: [(0, 3), (1, 0)]}
```

k_charge(algorithm='I')

Return the $k$-charge of self.

INPUT:

• algorithm – (default: “I”) if “I”, computes $k$-charge using the $I$ algorithm, otherwise uses the $J$-algorithm

OUTPUT:

• a nonnegative integer

For the definition of $k$-charge and the various algorithms to compute it see Section 3.3 of [LLMSSZ2013].

See also:

k_charge_I() and k_charge_J()

EXAMPLES:

```python
sage: t = WeakTableau([[1, 1, 2, 3, 4, 4, 5, 5, 6], [2, 3, 5, 5, 6], [3, 4, 7], [5, 6], [6], [7]], 4)
sage: t.k_charge()
12
```
**k_charge_I()**

Return the $k$-charge of `self` using the $I$-algorithm.

For the definition of $k$-charge and the $I$-algorithm see Section 3.3 of [LLMSSZ2013].

**OUTPUT:**

• a nonnegative integer

**See also:**

`k_charge()` and `k_charge_J()`

**EXAMPLES:**

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.k_charge_I()
2
sage: t = WeakTableau([[1, 3, 4, 5, 6], [2, 6], [4]], 3)
sage: t.k_charge_I()
8
sage: t = WeakTableau([[1, 1, 2, 3, 4, 4, 5, 5, 6], [2, 3, 5, 5, 6], [3, 4, 7], [5, 6], [6], [7]], 4)
sage: t.k_charge_I()
12
```

**k_charge_J()**

Return the $k$-charge of `self` using the $J$-algorithm.

For the definition of $k$-charge and the $J$-algorithm see Section 3.3 of [LLMSSZ2013].

**OUTPUT:**

• a nonnegative integer

**See also:**

`k_charge()` and `k_charge_I()`

**EXAMPLES:**

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.k_charge_J()
2
sage: t = WeakTableau([[1, 3, 4, 5, 6], [2, 6], [4]], 3)
sage: t.k_charge_J()
8
sage: t = WeakTableau([[1, 1, 2, 3, 4, 4, 5, 5, 6], [2, 3, 5, 5, 6], [3, 4, 7], [5, 6], [6], [7]], 4)
sage: t.k_charge_J()
12
```

**list_of_standard_cells()**

Return a list of lists of the coordinates of the standard cells of `self`.

**INPUT:**

• `self` – a weak $k$-tableau in core representation with partition weight

**OUTPUT:**

• a list of lists of coordinates
Warning: This method currently only works for straight weak tableaux with partition weight.

EXAMPLES:

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.list_of_standard_cells()
[[[0, 1], (1, 0), (2, 0)], [[0, 0], (0, 2), (1, 1)]

sage: t = WeakTableau([[1, 1, 1, 2], [2, 2, 3]], 5)
sage: t.list_of_standard_cells()
[[[0, 2], (1, 1), (1, 2)], [[0, 1], (1, 0)], [[0, 0], (0, 3)]

sage: t = WeakTableau([[1, 1, 2, 3, 4, 4, 5, 5, 6], [2, 3, 5, 5, 6], [3, 4, 7], [5, 6], [6], [7]], 4)
sage: t.list_of_standard_cells()
[[[0, 1], (1, 0), (2, 0), (3, 0), (4, 0), (5, 0)], [[0, 0], (0, 2), (1, 0), (2, 1), (1, 2), (3, 1)])
```

residues_of_entries(v)

Return a list of residues of cells of weak k-tableau self labeled by v.

INPUT:

- v – a label of a cell in self

OUTPUT:

- a list of residues

EXAMPLES:

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.residues_of_entries(1)
[0, 1]

sage: t = WeakTableau([[None, None, 1, 1, 4], [1, 4], [3]], 3)
sage: t.residues_of_entries(1)
[2, 3]
```

shape_bounded()  

Return the shape of self as a k-bounded partition.

When the tableau is straight, the outer shape is returned as a k-bounded partition. When the tableau is skew, the tuple of the outer and inner shape is returned as k-bounded partitions.

EXAMPLES:

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.shape_bounded()
[3, 2, 1]

sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: t.shape_bounded()
[[3, 2, 1], [2]]
```

shape_core()  

Return the shape of self as a (k + 1)-core.
When the tableau is straight, the outer shape is returned as a core. When the tableau is skew, the tuple of the outer and inner shape is returned as cores.

**EXAMPLES:**

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: t.shape_core()
[5, 2, 1]
```

```python
sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: t.shape_core()
([5, 2, 1], [2])
```

to_bounded_tableau()

Return the bounded representation of the weak \(k\)-tableau self.

Each restricted subtableau of the output is a \(k\)-bounded partition.

**EXAMPLES:**

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: c = t.to_bounded_tableau(); c
[[1, 1, 2], [2, 3], [3]]
sage: type(c)
<class 'sage.combinat.k_tableau.WeakTableaux_bounded_with_category.element_class'>
```

```python
sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: t.to_bounded_tableau() == t
True
```

to_factorized_permutation_tableau()

Return the factorized permutation representation of the weak \(k\)-tableau self.

**EXAMPLES:**

```python
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: c = t.to_factorized_permutation_tableau(); c
[s2*s0, s3*s2, s1*s0]
sage: type(c)
<class 'sage.combinat.k_tableau.WeakTableaux_factorized_permutation_with_category.element_class'>
sage: c.to_core_tableau() == t
True
```

```python
sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: c = t.to_factorized_permutation_tableau(); c
[s0, s3, s2, s3]
sage: c._inner_shape
[2]
sage: c.to_core_tableau() == t
True
```
class sage.combinat.k_tableau.WeakTableau_factorized_permutation(parent, t)

Bases: WeakTableau_abstract

A weak (skew) $k$-tableau represented in terms of factorizations of affine permutations into cyclically decreasing elements.

check()

Check that self is a valid weak $k$-tableau.

EXAMPLES:

```python
sage: t = WeakTableau([[2],[0,3],[2,1,0]], 3, representation = 'factorized_permutation')
sage: t.check()
```

classmethod from_core_tableau(t, k)

Construct weak factorized affine permutation tableau from a $k$-core tableau.

EXAMPLES:

```python
sage: from sage.combinat.k_tableau import WeakTableau_factorized_permutation
sage: WeakTableau_factorized_permutation.from_core_tableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
[s2*s0, s3*s2, s1*s0]
sage: WeakTableau_factorized_permutation.from_core_tableau([[1, 1, 2, 3, 4, 4, 5, 5, 6], [2, 3, 5, 5, 6], [3, 4, 7], [5, 6], [6], [7]], 4)
[s0, s3*s1, s2*s1, s0*s4, s3*s0, s4*s2, s1*s0]
sage: WeakTableau_factorized_permutation.from_core_tableau([[None, 1, 1, 2, 2], [None, 2], [1]], 3)
[s0*s3, s2*s1]
```

k_charge(algorithm='I')

Return the $k$-charge of self.

OUTPUT:

- a nonnegative integer

EXAMPLES:

```python
sage: t = WeakTableau([[2,0],[3,2],[1,0]], 3, representation = 'factorized_permutation')
sage: t.k_charge()
2
sage: t = WeakTableau([[0],[3],[2],[1],[3],[0]], 3, representation = 'factorized_permutation')
sage: t.k_charge()
8
sage: t = WeakTableau([[0],[3,1],[2,1],[0,4],[3,0],[4,2],[1,0]], 4, representation = 'factorized_permutation')
sage: t.k_charge()
12
```

shape_bounded()

Return the shape of self as a $k$-bounded partition.

When the tableau is straight, the outer shape is returned as a $k$-bounded partition. When the tableau is skew, the tuple of the outer and inner shape is returned as $k$-bounded partitions.
EXAMPLES:

```
sage: t = WeakTableau([[2],[0,3],[2,1,0]], 3, representation = 'factorized_permutation')
sage: t.shape_bounded()
[3, 2, 1]
```

```
sage: t = WeakTableau([[2,0],[3,2]], 3, inner_shape = [2], representation = 'factorized_permutation')
sage: t.shape_bounded()
([3, 2, 1], [2])
```

```
shape_core()

Return the shape of self as a \((k + 1)\)-core.

When the tableau is straight, the outer shape is returned as a core. When the tableau is skew, the tuple of the outer and inner shape is returned as cores.

EXAMPLES:

```
sage: t = WeakTableau([[2],[0,3],[2,1,0]], 3, representation = 'factorized_permutation')
sage: t.shape_core()
[5, 2, 1]
```

```
sage: t = WeakTableau([[2,0],[3,2]], 3, inner_shape = [2], representation = 'factorized_permutation')
sage: t.shape()
([5, 2, 1], [2])
```

```
station straighten_input\(t, k\)

Straightens input.

INPUT:

• \(t\) – a list of reduced words or a list of elements in the Weyl group of type \(A_k^{(1)}\)

• \(k\) – a positive integer

EXAMPLES:

```
sage: from sage.combinat.k_tableau import WeakTableau_factorized_permutation
sage: WeakTableau_factorized_permutation.straighten_input([[2,0],[3,2],[1,0]], 3)
(s2*s0, s3*s2, s1*s0)
sage: W = WeylGroup(['A',4,1])
sage: WeakTableau_factorized_permutation.straighten_input([W.an_element(),W.an_element()], 4)
(s0*s1*s2*s3*s4, s0*s1*s2*s3*s4)
```

```
to_core_tableau()

Return the weak \(k\)-tableau self where the shape of each restricted tableau is a \((k + 1)\)-core.

EXAMPLES:

```
sage: t = WeakTableau([[0], [3,1], [2,1], [0,4], [3,0], [4,2], [1,0]], 4, representation = 'factorized_permutation'); t
```
\[[s0, s3*s1, s2*s1, s0*s4, s3*s0, s4*s2, s1*s0]\]

\[\texttt{sage: c = t.to_core_tableau(); c} \]
\[
\begin{bmatrix}
1, 1, 2, 3, 4, 5, 5, 6, 6, 5, 7, 7, 6, 6, 6, 7
\end{bmatrix}
\]

\[\texttt{sage: type(c)} \]
\[
\texttt{<class 'sage.combinat.k_tableau.WeakTableaux_core_with_category.element_class'>} \]

\[\texttt{sage: t = WeakTableau([[]], 4, representation = 'factorized_permutation'); t} \]
\[
\begin{bmatrix}
1
\end{bmatrix}
\]

\[\texttt{sage: t.to_core_tableau()} \]
\[
\emptyset
\]

\[\texttt{sage: from sage.combinat.k_tableau import WeakTableau_factorized_permutation} \]

\[\texttt{sage: t = WeakTableau([[[2,0],[3,2],[1,0]], 3, representation = 'factorized_permutation'}) \]

\[\texttt{sage: WeakTableau_factorized_permutation.from_core_tableau(t.to_core_tableau(), 3)} \]

\[
\begin{bmatrix}
\texttt{[s2*s0, s3*s2, s1*s0]} \\
\texttt{t == WeakTableau_factorized_permutation.from_core_tableau(t.to_core_tableau(), 3)} \]

\[
\texttt{True}
\]

\[\texttt{sage: t = WeakTableau([[[2,0],[3,2]], 3, inner_shape = [2], representation = 'factorized_permutation'}) \]

\[\texttt{sage: t.to_core_tableau()} \]
\[
\begin{bmatrix}
\texttt{[[None, None, 1, 1, 2], [1, 2, 2]]} \\
\texttt{t == WeakTableau_factorized_permutation.from_core_tableau(t.to_core_tableau(), 3)} \]

\[
\texttt{True}
\]\n
\[\texttt{sage.combinat.k_tableau.WeakTableaux(k, shape, weight, representation='core')}\]

This is the dispatcher method for the parent class of weak \(k\)-tableaux.

**INPUT:**

- \(k\) – positive integer
- \(\text{shape}\) – shape of the weak \(k\)-tableaux; for the ‘core’ and ‘factorized_permutation’ representation, the shape is inputted as a \(k+1\)-core; for the ‘bounded’ representation, the shape is inputted as a \(k\)-bounded partition; for skew tableaux, the shape is inputted as a tuple of the outer and inner shape
- \(\text{weight}\) – the weight of the weak \(k\)-tableaux as a list or tuple
- \(\text{representation}\) – ‘core’, ‘bounded’, or ‘factorized_permutation’ (default: ‘core’)

**EXAMPLES:**

\[\texttt{sage: T = WeakTableaux(3, [5,2,1], [1,1,1,1,1,1])} \]

\[\texttt{sage: T.list()} \]
\[
\begin{bmatrix}
[[1, 3, 4, 5, 6], [2, 6], [4]], \\
[[1, 2, 4, 5, 6], [3, 6], [4]], \\
[[1, 2, 3, 4, 6], [4, 6], [5]], \\
[[1, 2, 3, 4, 5], [4, 5], [6]]
\end{bmatrix}
\]

\[\texttt{sage: T.cardinality()} \]
\[
4
\]

\[\texttt{sage: T = WeakTableaux(3, [[5,2,1], [2]], [1,1,1,1])} \]

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Combinatorics, Release 10.0

sage: T.list()
[[[None, None, 2, 3, 4], [1, 4], [2]],
[[None, None, 1, 2, 4], [2, 4], [3]],
[[None, None, 1, 2, 3], [2, 3], [4]]]

sage: T = WeakTableaux(3, [3,2,1], [1,1,1,1,1,1], representation = 'bounded')
sage: T.list()
[[[1, 3, 5], [2, 6], [4]],
[[1, 2, 5], [3, 6], [4]],
[[1, 2, 3], [4, 6], [5]],
[[1, 2, 3], [4, 5], [6]]]

sage: T = WeakTableaux(3, [[3,2,1], [2]], [1,1,1,1], representation = 'factorized_permutation')
sage: T.list()
[[[s0, s3, s2, s1, s3, s0],
[s0, s3, s2, s3, s1, s0],
[s0, s2, s3, s2, s1, s0],
[s2, s0, s3, s2, s1, s0]]

class sage.combinat.k_tableau.WeakTableaux_abstract

Bases: UniqueRepresentation, Parent

Abstract class for the various parent classes of WeakTableaux.

representation(representation='core')

Return the analogue of self in the specified representation.

INPUT:

• representation – ‘core’, ‘bounded’, or ‘factorized_permutation’ (default: ‘core’)

EXAMPLES:

sage: T = WeakTableaux(3, [5,2,1], [1,1,1,1,1,1])
sage: T._representation
'core'
sage: T.representation('bounded')
Bounded weak 3-Tableaux of (skew) 3-bounded shape [5, 2, 1] and weight (1, 1, 1, 1, 1, 1)
sage: T.representation('factorized_permutation')
Factorized permutation (skew) weak 3-Tableaux of shape [5, 2, 1] and weight (1, 1, 1, 1, 1, 1)
Combinatorics, Release 10.0

sage: T = WeakTableaux(3, [3,2,1], [1,1,1,1,1,1], representation = 'bounded')
sage: T._representation
'bounded'
sage: T.representation('core')
Core weak 3-Tableaux of (skew) core shape [5, 2, 1] and weight (1, 1, 1, 1, 1)
sage: T.representation('bounded')
Bounded weak 3-Tableaux of (skew) 3-bounded shape [3, 2, 1] and weight (1, 1, 1, 1, 1)
sage: T.representation('bounded') == T
True
sage: T.representation('factorized_permutation')
Factorized permutation (skew) weak 3-Tableaux of shape [5, 2, 1] and weight (1, 1, 1, 1, 1)
sage: T.representation('factorized_permutation') == T
False

shape()

Return the shape of the tableaux of self.

When self is the class of straight tableaux, the outer shape is returned. When self is the class of skew tableaux, the tuple of the outer and inner shape is returned.

Note that in the ‘core’ and ‘factorized_permutation’ representation, the shapes are (k + 1)-cores. In the ‘bounded’ representation, the shapes are k-bounded partitions.

If the user wants to access the skew shape (even if the inner shape is empty), please use self._shape.

EXAMPLES:

sage: T = WeakTableaux(3, [5,2,2], [2,2,2,1])
sage: T.shape()
[5, 2, 2]
sage: T._shape
([5, 2, 2], [])
sage: T = WeakTableaux(3, [[5,2,2], [1]], [2,1,2,1])
sage: T.shape()
([5, 2, 2], [1])

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sage: T = WeakTableaux(3, [3,2,2], [2,2,1], representation = 'bounded')
sage: T.shape()
[3, 2, 2]
sage: T._shape
([3, 2, 2], [])
sage: T = WeakTableaux(3, [[3,2,2], [1]], [2,1,2,1], representation = 'bounded')
sage: T.shape()
([3, 2, 2], [1])
sage: T = WeakTableaux(3, [4,1], [2,2], representation = 'factorized_permutation →')
sage: T.shape()
([4, 1], [])
sage: T = WeakTableaux(4, [[6,2,1], [2]], [2,1,1,1], representation = 'factorized_permutation')
sage: T.shape()
([6, 2, 1], [2])

size()

Return the size of the shape.

In the bounded representation, the size of the shape is the number of boxes in the outer shape minus the number of boxes in the inner shape. For the core and factorized permutation representation, the size is the length of the outer shape minus the length of the inner shape.

EXAMPLES:

sage: T = WeakTableaux(3, [5,2,1], [1,1,1,1,1])
sage: T.size()
6
sage: T = WeakTableaux(3, [3,2,1], [1,1,1,1,1], representation = 'bounded')
sage: T.size()
6
sage: T = WeakTableaux(4, [[6,2,1], [2]], [2,1,1,1], 'factorized_permutation')
sage: T.size()
5

class sage.combinat.k_tableau.WeakTableaux_bounded(k, shape, weight)

Bases: WeakTableaux_abstract

The class of (skew) weak $k$-tableaux in the bounded representation of shape $shape$ (as $k$-bounded partition or tuple of $k$-bounded partitions in the skew case) and weight $weight$.

INPUT:

- $k$ – positive integer
- $shape$ – the shape of the $k$-tableaux represented as a $k$-bounded partition; if the tableaux are skew, the shape is a tuple of the outer and inner shape each represented as a $k$-bounded partition
- $weight$ – the weight of the $k$-tableaux

EXAMPLES:
```python
sage: T = WeakTableaux(3, [3,1], [2,2], representation = 'bounded')
sage: T.list()
[[[1, 1, 2], [2]]]

sage: T = WeakTableaux(3, [[3,2,1], [2]], [1,1,1,1], representation = 'bounded')
sage: T.list()
[[[None, None, 3], [1, 4], [2]],
[[None, None, 1], [2, 4], [3]],
[[None, None, 1], [2, 3], [4]]]
```

**Element**

alias of *WeakTableau_bounded*

```python
class sage.combinat.k_tableau.WeakTableaux_core(k, shape, weight)
Bases: WeakTableaux_abstract

The class of (skew) weak \( k \)-tableaux in the core representation of shape \( \text{shape} \) (as \( k+1 \)-core) and weight \( \text{weight} \).

**INPUT:**
- \( k \) – positive integer
- \( \text{shape} \) – the shape of the \( k \)-tableaux represented as a \( (k+1) \)-core; if the tableaux are skew, the shape is a tuple of the outer and inner shape (both as \( (k+1) \)-cores)
- \( \text{weight} \) – the weight of the \( k \)-tableaux

**EXAMPLES:**

```python
sage: T = WeakTableaux(3, [4,1], [2,2])
sage: T.list()
[[[1, 1, 2, 2], [2]]]

sage: T = WeakTableaux(3, [[5,2,1], [2]], [1,1,1,1])
sage: T.list()
[[[None, None, 2, 3, 4], [1, 4], [2]],
[[None, None, 1, 2, 4], [2, 4], [3]],
[[None, None, 1, 2, 3], [2, 3], [4]]]
```

**Element**

alias of *WeakTableau_core*

```python
circular_distance(cr, r)

Return the shortest counterclockwise distance between \( cr \) and \( r \) modulo \( k + 1 \).

**INPUT:**
- \( cr, r \) – nonnegative integers between 0 and \( k \)

**OUTPUT:**
- a positive integer

**EXAMPLES:**

```python
sage: T = WeakTableaux(10, [], [])
sage: T.circular_distance(8, 6)
2
```
```
sage: T.circular_distance(8, 8)
0
sage: T.circular_distance(8, 9)
10

diag(c, ha)
Return the number of diagonals strictly between cells c and ha of the same residue as c.

INPUT:
• c – a cell in the lattice
• ha – another cell in the lattice with bigger row and smaller column than c

OUTPUT:
• a nonnegative integer

EXAMPLES:

sage: T = WeakTableaux(4, [5,2,2], [2,2,2,1])
sage: T.diag((1,2),(4,0))
0

class sage.combinat.k_tableau.WeakTableaux_factorized_permutation(k, shape, weight)

Bases: WeakTableaux_abstract

The class of (skew) weak k-tableaux in the factorized permutation representation of shape shape (as k + 1-core or tuple of (k + 1)-cores in the skew case) and weight weight.

INPUT:
• k – positive integer
• shape – the shape of the k-tableaux represented as a (k + 1)-core; in the skew case the shape is a tuple of the outer and inner shape both as (k + 1)-cores
• weight – the weight of the k-tableaux

EXAMPLES:

sage: T = WeakTableaux(3, [4,1], [2,2], representation = 'factorized_permutation')
sage: T.list()
[[s3*s2, s1*s0]]
sage: T = WeakTableaux(4, [[6,2,1], [2]], [2,1,1,1], representation = 'factorized_permutation')
sage: T.list()
[[s0, s4, s3, s4*s2], [s0, s3, s4, s3*s2], [s3, s0, s4, s3*s2]]

Element
alias of WeakTableau_factorized_permutation

sage.combinat.k_tableau.intermediate_shapes(t)

Return the intermediate shapes of tableau t.

A (skew) tableau with letters 1, 2, ..., ℓ can be viewed as a sequence of shapes, where the i-th shape is given by the shape of the subtableau on letters 1, 2, ..., i. The output is the list of these shapes.

OUTPUT:
• a list of lists representing partitions

EXAMPLES:

```
sage: from sage.combinat.k_tableau import intermediate_shapes
sage: t = WeakTableau([[1, 1, 2, 2, 3], [2, 3], [3]], 3)
sage: intermediate_shapes(t)
[[], [2], [4, 1], [5, 2, 1]]
```

```
sage: t = WeakTableau([[None, None, 2, 3, 4], [1, 4], [2]], 3)
sage: intermediate_shapes(t)
[[2], [2, 1], [3, 1, 1], [4, 1, 1], [5, 2, 1]]
```

```
sage.combinat.k_tableau.nabs(v)

Return the absolute value of v or None.

INPUT:

• v – either an integer or None

OUTPUT:

• either a non-negative integer or None

EXAMPLES:

```
sage: from sage.combinat.k_tableau import nabs
sage: nabs(None)
sage: nabs(-3)
3
sage: nabs(None)
```

5.1.131 Kazhdan-Lusztig Polynomials

AUTHORS:

• Daniel Bump (2008): initial version

```
class sage.combinat.kazhdan_lusztig.KazhdanLusztigPolynomial(W, q, trace=False)

Bases: UniqueRepresentation, SageObject

A Kazhdan-Lusztig polynomial.

INPUT:

• \( W \) – a Weyl Group
• q – an indeterminate

OPTIONAL:

• trace – if True, then this displays the trace: the intermediate results. This is instructive and fun.

The parent of q may be a PolynomialRing or a LaurentPolynomialRing.

EXAMPLES:
```
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```sage
sage: W = WeylGroup("B3", prefix="s")
sage: [s1, s2, s3] = W.simple_reflections()
sage: R.<q> = LaurentPolynomialRing(QQ)
sage: KL = KazhdanLusztigPolynomial(W, q)
sage: KL.P(s2, s3*s2*s3*s1*s2)
1 + q
```

A faster implementation (using the optional package Coxeter 3) is given by:

```sage
sage: W = CoxeterGroup(['B', 3], implementation='coxeter3') # optional - coxeter3
coxeter3
sage: W.kazhdan_lusztig_polynomial([2], [3, 2, 3, 1, 2]) # optional - coxeter3
coxeter3
```

$P(x, y)$

Return the Kazhdan-Lusztig $P$ polynomial.

If the rank is large, this runs slowly at first but speeds up as you do repeated calculations due to the caching.

**INPUT:**

- $x, y$ – elements of the underlying Coxeter group

**See also:**

kazhdan_lusztig_polynomial for a faster implementation using Fokko Ducloux’s Coxeter3 C++ library.

**EXAMPLES:**

```sage
sage: R.<q> = QQ[]
sage: W = WeylGroup("A3", prefix="s")
sage: [s1, s2, s3] = W.simple_reflections()
sage: KL = KazhdanLusztigPolynomial(W, q)
sage: KL.P(s2, s2*s1*s3*s2)
```

$R(x, y)$

Return the Kazhdan-Lusztig $R$ polynomial.

**INPUT:**

- $x, y$ – elements of the underlying Coxeter group

**EXAMPLES:**

```sage
sage: R.<q> = QQ[]
sage: W = WeylGroup("A2", prefix="s")
sage: [s1, s2] = W.simple_reflections()
sage: KL = KazhdanLusztigPolynomial(W, q)
sage: [KL.R(x, s2*s1) for x in [1, s1, s2, s1*s2]]
```

$R_{\tilde{}}(x, y)$

Return the Kazhdan-Lusztig $\tilde{R}$ polynomial.

Information about the $\tilde{R}$ polynomials can be found in [Dy1993] and [BB2005].

**INPUT:**
\* x, y – elements of the underlying Coxeter group

**EXAMPLES:**

```python
sage: R.<q> = QQ[]
sage: W = WeylGroup("A2", prefix="s")
sage: [s1,s2] = W.simple_reflections()
sage: KL = KazhdanLusztigPolynomial(W, q)
```

```python
sage: [KL.R_tilde(x,s2*s1) for x in [1,s1,s2,s1*s2]]
[q^2, q, q, 0]
```

### 5.1.132 Key polynomials

Key polynomials (also known as type A Demazure characters) are defined by applying the divided difference operator \( \pi_\sigma \), where \( \sigma \) is a permutation, to a monomial corresponding to an integer partition \( \mu \vdash n \).

**See also:**

For Demazure characters in other types, see

- `sage.combinat.root_system.weyl_characters.WeylCharacterRing.demazure_character()`

**AUTHORS:**

- Trevor K. Karn (2022-08-17): initial version

**class** `sage.combinat.key_polynomial.KeyPolynomial`

**Bases:** `IndexedFreeModuleElement`

A key polynomial.

Key polynomials are polynomials that form a basis for a polynomial ring and are indexed by weak compositions.

Elements should be created by first creating the basis `KeyPolynomialBasis` and passing a list representing the indexing composition.

**EXAMPLES:**

```python
sage: k = KeyPolynomials(QQ)
sage: f = k([4,3,2,1]) + k([1,2,3,4]); f
k[4, 3, 2, 1] + k[1, 2, 3, 4]
sage: f in k
True
```

**divided_difference(w)**

Apply the divided difference operator \( \partial_w \) to `self`.

The convention is to apply from left to right so if \( w = [w_1, w_2, \ldots, w_m] \) then we apply \( \partial_{w_2 \cdots w_m} \circ \partial_{w_1} \)

**EXAMPLES:**

```python
sage: k = KeyPolynomials(QQ)
sage: k([3,2,1]).divided_difference(2)
k[3, 1, 1]
sage: k([3,2,1]).divided_difference([2,3])
k[3, 1]
```

(continues on next page)
sage: k = KeyPolynomials(QQ, 4)
sage: k([3,2,1,0]).divided_difference(2)
k[3, 1, 1, 0]

expand()

Return self written in the monomial basis (i.e., as an element in the corresponding polynomial ring).

EXAMPLES:

sage: k = KeyPolynomials(QQ)
sage: f = k([4,3,2,1])
sage: f.expand()
z_3*z_2^2*z_1^3*z_0^4

sage: f = k([1,2,3])
sage: f.expand()
z_2^3*z_1^2*z_0 + z_2^3*z_1*z_0^2 + z_2^2*z_1^3*z_0^3 + 2*z_2^2*z_1^2*z_0^2 + z_2*z_1^3*z_0^2 + z_2*z_1^2*z_0^3

isobaric_divided_difference(w)

Apply the operator $\pi_w$ to self.

w may be either a Permutation or a list of indices of simple transpositions (1-based).

The convention is to apply from left to right so if $w = [w_1, w_2, \ldots, w_m]$ then we apply $\pi_{w_2} \cdots \pi_{w_m} \circ \pi_{w_1}$

EXAMPLES:

sage: k = KeyPolynomials(QQ)
sage: k([3,2,1]).pi(2)
k[3, 1, 2]
sage: k([3,2,1]).pi([2,1])
k[1, 3, 2]
sage: k([3,2,1]).pi(Permutation([3,2,1]))
k[1, 2, 3]
sage: f = k([3,2,1]) + k([3,2,1,1])
sage: f.pi(2)
k[3, 1, 2] + k[3, 1, 2, 1]
sage: k.one().pi(1)
k[]
sage: k([3,2,1,0]).pi(2).pi(2)
k[3, 1, 2]
sage: (-k([3,2,1,0]) + 4*k([3,1,2,0])).pi(2)
3*k[3, 1, 2]
pi(w)

Apply the operator $\pi_w$ to self.

$w$ may be either a Permutation or a list of indices of simple transpositions (1-based).

The convention is to apply from left to right so if $w = [w_1, w_2, \ldots, w_m]$ then we apply $\pi_{w_m} \circ \pi_{w_{m-1}} \circ \cdots \circ \pi_{w_1}$.

EXAMPLES:

```python
sage: k = KeyPolynomials(QQ)
sage: k([3,2,1]).pi(2)
k[3, 1, 2]
sage: k([3,2,1]).pi([2,1])
k[1, 3, 2]
sage: k([3,2,1]).pi(Permutation([3,2,1]))
k[1, 2, 3]
sage: f = k([3,2,1]) + k([3,2,1,1])
sage: f.pi(2)
k[3, 1, 2] + k[3, 1, 2, 1]
sage: k.one().pi(1)
k[0, 0, 0, 0]
```

```python
sage: to_polynomial()
Return self written in the monomial basis (i.e., as an element in the corresponding polynomial ring).

EXAMPLES:

```python
sage: k = KeyPolynomials(QQ)
sage: f = k([4,3,2,1])
```

(continues on next page)
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```
sage: f.expand()
z_3*z_2^2*z_1^3*z_0^4

sage: f = k([1,2,3])
sage: f.expand()
z_2^3*z_1^2*z_0 + z_2^3*z_1*z_0^2 + z_2^2*z_1^3*z_0^3 + z_2^2*z_1^2*z_0^3 + 2*z_2^2*z_1*z_0^3 + z_2*z_1^3*z_0^2 + z_2*z_1^2*z_0^3
```

```python
class sage.combinat.key_polynomial.KeyPolynomialBasis(R=None, k=None, poly_ring=None):
    Bases: CombinatorialFreeModule

    The key polynomial basis for a polynomial ring.

    For a full definition, see SymmetricFunctions.com. Key polynomials are indexed by weak compositions with no trailing zeros, and \( \sigma \) is the permutation of shortest length which sorts the indexing composition into a partition.

    EXAMPLES:

    Key polynomials are a basis, indexed by (weak) compositions, for polynomial rings:

    ```python
    sage: k = KeyPolynomials(QQ)
sage: k([3,0,1,2])
k[3, 0, 1, 2]
sage: k([3,0,1,2])/2
    1/2*k[3, 0, 1, 2]
sage: R = k.polynomial_ring(); R
    Infinite polynomial ring in z over Rational Field
    sage: K = KeyPolynomials(GF(5)); K
    Key polynomial basis over Finite Field of size 5
    sage: 2*K([3,0,1,2])
    2*k[3, 0, 1, 2]
sage: 5*(K([3,0,1,2]) + K([3,1,1]))
    0
    ```

    We can expand them in the standard monomial basis:

    ```python
    sage: k([3,0,1,2]).expand()
z_3^2*z_2*z_0^3 + z_3^2*z_1*z_0^3 + z_3^2*z_2^2*z_0^2 + z_3^2*z_1^2*z_0^2 + z_3*z_2^2*z_0^2 + z_3*z_1^2*z_0^2 + z_2^2*z_1*z_0^2 + z_2^2*z_1^2*z_0^2 + z_2*z_1^2*z_0^2 + z_2*z_1*z_0^2 + z_1^2*z_0^2 + z_1*z_0^2 + z_0^2
    ```

    If we have a polynomial, we can express it in the key basis:

    ```python
    sage: z = R.gen()
sage: k.from_polynomial(z[2]^2*z[1]*z[0])
k[1, 1, 2] - k[1, 2, 1]
sage: f = z[3]^2*z[2]*z[0]^3 + z[3]^2*z[1]*z[0]^3 + z[3]*z[2]^2*z[0]^2 + z[3]*z[1]^2*z[0]^2 + z[2]^2*z[0]^2 + z[2]*z[1]*z[0]^2 + z[1]*z[0]^2 + z[0]*z_2^2 + z[0]*z_1^2 + z[0]*z_0 + z[0]^2
    ```
```
Since the ring of key polynomials may be regarded as a different choice of basis for a polynomial ring, it forms an algebra, so we have multiplication:

\[
sage: k([10, 5, 2]) \cdot k([1, 1, 1])
\]
\[
k[11, 6, 3]
\]

We can also multiply by polynomials in the monomial basis:

\[
sage: k([10, 9, 1]) \cdot z[0]
\]
\[
k[11, 9, 1]
\]

\[
sage: z[0] \cdot k([10, 9, 1])
\]
\[
k[11, 9, 1]
\]

\[
sage: k([10, 9, 1]) \cdot (z[0] + z[3])
\]
\[
k[10, 9, 1, 1] + k[11, 9, 1]
\]

When the sorting permutation is the longest element, the key polynomial agrees with the Schur polynomial:

\[
sage: s = SymmetricFunctions(QQ).schur()
sage: k([1, 2, 3]).expand()
\]
\[
z_2^3 z_1^2 z_0 + z_2^3 z_1 z_0^2 + z_2^2 z_1^3 z_0^2 + 2 z_2^2 z_1^2 z_0^3 + z_2 z_1^3 z_0^3
\]

The polynomial expansions can be computed using crystals and expressed in terms of the key basis:

\[
sage: T = crystals.Tableaux(['A', 3], shape=[2, 1])
sage: f = T.demazure_character([3, 2, 1])
sage: k.from_polynomial(f)
\]
\[
k[1, 0, 0, 2]
\]

The default behavior is to work in a polynomial ring with infinitely many variables. One can work in a specified number of variables:

\[
sage: k = KeyPolynomials(QQ, 4)
sage: k([3, 0, 1, 2]).expand()
\]
\[
z_0^3 z_1^2 z_2 + z_0^3 z_1 z_2^2 + z_0^3 z_1^2 z_3 + 2 z_0^2 z_1^2 z_3 + z_0^2 z_1 z_3^2 + z_0^2 z_2 z_3^2 + z_0^2 z_1^2 z_2^2
\]

\[
sage: k([0, 0, 2, 0]).expand()
\]
\[
z_0^2 + z_0 z_1 + z_1^2 + z_0 z_2 + z_1 z_2 + z_2^2
\]

\[
sage: k([0, 0, 2, 0]).expand().parent()
\]
\[
Multivariate Polynomial Ring in z_0, z_1, z_2, z_3 over Rational Field
\]

If working in a specified number of variables, the length of the indexing composition must be the same as the number of variables:
One can also work in a specified polynomial ring:

```
sage: k = KeyPolynomials(QQ['x0', 'x1', 'x2', 'x3'])
sage: k([0,2,0,0])
```

If one wishes to use a polynomial ring as coefficients for the key polynomials, pass the keyword argument `poly_coeffs=True`:

```
sage: k = KeyPolynomials(QQ['q'], poly_coeffs=True)
sage: R = k.base_ring(); R
Univariate Polynomial Ring in q over Rational Field
sage: R.inject_variables()
Defining q
sage: (q^2 + q + 1)*k([0,2,2,0,3,2])
(q^2+q+1)*k[0, 2, 2, 0, 3, 2]
```

**Element**

With `Element` alias of `KeyPolynomial`.

**degree_on_basis(alpha)**

Return the degree of the basis element indexed by `alpha`.

**from_polynomial(f)**

Expand a polynomial in terms of the key basis.

**EXAMPLES:**

```
sage: k = KeyPolynomials(QQ)
sage: k.from_polynomial(k(c).expand())
```

(continues on next page)
from_schubert_polynomial(s)

Expand a Schubert polynomial in the key basis.

EXAMPLES:

```python
sage: k = KeyPolynomials(ZZ)
sage: X = SchubertPolynomialRing(ZZ)
```

```python
sage: f = X([2,1,5,4,3])
```

```python
sage: k.from_schubert_polynomial(f)
k[1, 0, 2, 1] + k[2, 0, 2] + k[3, 0, 0, 1]
```

```python
sage: k.from_schubert_polynomial(2)
2*k[]
```

```python
sage: k = KeyPolynomials(GF(7), 4)
```

```python
sage: k.from_schubert_polynomial(f)
k[1, 0, 2, 1] + k[2, 0, 2, 0] + k[3, 0, 0, 1]
```

one_basis()

Return the basis element indexing the identity.

EXAMPLES:

```python
sage: k = KeyPolynomials(QQ)
```

```python
sage: k.one_basis()
[]
```

```python
sage: k = KeyPolynomials(QQ, 4)
```

```python
sage: k.one_basis()
[0, 0, 0, 0]
```

poly_gens()

Return the polynomial generators for the polynomial ring associated to self.

EXAMPLES:

```python
sage: k = KeyPolynomials(QQ)
```

```python
sage: k.poly_gens()
(z_0, z_1, z_2, z_3)
```

polynomial_ring()

Return the polynomial ring associated to self.
EXAMPLES:

```python
sage: k = KeyPolynomials(QQ)
sage: k.polynomial_ring()
Infinite polynomial ring in z over Rational Field
sage: k = KeyPolynomials(QQ, 4)
sage: k.polynomial_ring()
Multivariate Polynomial Ring in z_0, z_1, z_2, z_3 over Rational Field
```

```python
sage.combinat.key_polynomial.divided_difference(f, i)
Apply the i-th divided difference operator to the polynomial f.
```

EXAMPLES:

```python
sage: from sage.combinat.key_polynomial import divided_difference
sage: k = KeyPolynomials(QQ)
sage: z = k.poly_gens()
sage: divided_difference(f, 3)
   z_3^2*z_1 + z_3*z_2*z_1 + z_2^2*z_1
sage: k = KeyPolynomials(QQ, 4)
sage: z = k.poly_gens()
sage: divided_difference(f, 3)
   z_1*z_2^2 + z_1*z_2*z_3 + z_1*z_3^2
sage: k = KeyPolynomials(QQ)
sage: R = k.polynomial_ring(); R
Infinite polynomial ring in z over Rational Field
sage: z = R.gen()
sage: divided_difference(z[1]*z[2]^3, 2)
   -z_2^2*z_1 - z_2*z_1^2
sage: divided_difference(z[1]*z[2]*z[3], 3)
   0
sage: divided_difference(z[1]*z[2]*z[3], 4)
   z_2*z_1
sage: divided_difference(z[1]*z[2]*z[4], 4)
   -z_2*z_1
sage: k = KeyPolynomials(QQ, 5)
sage: z = k.polynomial_ring().gens()
sage: divided_difference(z[1]*z[2]^3, 2)
   -z_1^2*z_2 - z_1*z_2^2
sage: divided_difference(z[1]*z[2]*z[3], 3)
   0
sage: divided_difference(z[1]*z[2]*z[3], 4)
   z_1*z_2
sage: divided_difference(z[1]*z[2]*z[4], 4)
   -z_1*z_2
```

```python
sage.combinat.key_polynomial.isobaric_divided_difference(f, w)
Apply the isobaric divided difference operator π_w to the polynomial f.
```

```python
sage: from sage.combinat.key_polynomial import isobaric_divided_difference
sage: k = KeyPolynomials(QQ)
sage: z = k.poly_gens()
sage: divided_difference(f, 3)
   z_3^2*z_1 + z_3*z_2*z_1 + z_2^2*z_1
sage: k = KeyPolynomials(QQ, 4)
sage: z = k.poly_gens()
sage: divided_difference(f, 3)
   z_1*z_2^2 + z_1*z_2*z_3 + z_1*z_3^2
sage: k = KeyPolynomials(QQ)
sage: R = k.polynomial_ring(); R
Infinite polynomial ring in z over Rational Field
sage: z = R.gen()
sage: divided_difference(z[1]*z[2]^3, 2)
   -z_2^2*z_1 - z_2*z_1^2
sage: divided_difference(z[1]*z[2]*z[3], 3)
   0
sage: divided_difference(z[1]*z[2]*z[3], 4)
   z_2*z_1
sage: divided_difference(z[1]*z[2]*z[4], 4)
   -z_2*z_1
sage: k = KeyPolynomials(QQ, 5)
sage: z = k.polynomial_ring().gens()
sage: divided_difference(z[1]*z[2]^3, 2)
   -z_1^2*z_2 - z_1*z_2^2
sage: divided_difference(z[1]*z[2]*z[3], 3)
   0
sage: divided_difference(z[1]*z[2]*z[3], 4)
   z_1*z_2
sage: divided_difference(z[1]*z[2]*z[4], 4)
   -z_1*z_2
```

```python
sage.combinat.key_polynomial.isobaric_divided_difference(f, w)
Apply the isobaric divided difference operator π_w to the polynomial f.
```

```python
sage: from sage.combinat.key_polynomial import isobaric_divided_difference
sage: k = KeyPolynomials(QQ)
sage: z = k.poly_gens()
sage: divided_difference(f, 3)
   z_3^2*z_1 + z_3*z_2*z_1 + z_2^2*z_1
sage: k = KeyPolynomials(QQ, 4)
sage: z = k.poly_gens()
sage: divided_difference(f, 3)
   z_1*z_2^2 + z_1*z_2*z_3 + z_1*z_3^2
sage: k = KeyPolynomials(QQ)
sage: R = k.polynomial_ring(); R
Infinite polynomial ring in z over Rational Field
sage: z = R.gen()
sage: divided_difference(z[1]*z[2]^3, 2)
   -z_2^2*z_1 - z_2*z_1^2
sage: divided_difference(z[1]*z[2]*z[3], 3)
   0
sage: divided_difference(z[1]*z[2]*z[3], 4)
   z_2*z_1
sage: divided_difference(z[1]*z[2]*z[4], 4)
   -z_2*z_1
sage: k = KeyPolynomials(QQ, 5)
sage: z = k.polynomial_ring().gens()
sage: divided_difference(z[1]*z[2]^3, 2)
   -z_1^2*z_2 - z_1*z_2^2
sage: divided_difference(z[1]*z[2]*z[3], 3)
   0
sage: divided_difference(z[1]*z[2]*z[3], 4)
   z_1*z_2
sage: divided_difference(z[1]*z[2]*z[4], 4)
   -z_1*z_2
```
w may be either a single index or a list of indices of simple transpositions.

**Warning:** The simple transpositions should be applied from left to right.

**EXAMPLES:**

```python
sage: from sage.combinat.key_polynomial import isobaric_divided_difference as idd
sage: R.<z> = InfinitePolynomialRing(GF(3))
```
```
0
z_4^2*z_3*z_2^2*z_1^4 + z_4*z_3*z_2^2*z_1^4
sage: idd(z[1]^4*z[2]^2*z[3]*z[4], [3, 4])
z_4^2*z_3*z_2^2*z_1^4 + z_4*z_3^2*z_2^2*z_1^4
sage: idd(z[1]^4*z[2]^2*z[3]*z[4], [4, 3])
z_4^2*z_3^2*z_2^2*z_1^4 + z_4*z_3*z_2^2*z_1^4
sage: idd(z[1]^2*z[2], [3, 2])
z_3^2 + z_3*z_2 + z_3*z_2^2 + z_2^2*z_1 + z_2*z_1^2
```

`sage.combinat.key_polynomial.sorting_word(alpha)`

Get a reduced word for the permutation which sorts alpha into a partition.

The result is a list \( l = [i_0, i_1, i_2, \ldots] \) where each \( i_j \) is a positive integer such that it applies the simple transposition \((i_j, i_j + 1)\). The transpositions are applied starting with \( i_0 \), then \( i_1 \) is applied, followed by \( i_2 \), and so on. See `sage.combinat.permutation.Permutation.reduced_words()` for the convention used.

**EXAMPLES:**

```python
sage: IV = IntegerVectors()
```
```
sage: from sage.combinat.key_polynomial import sorting_word
sage: list(sorting_word(IV([2,3,2]))[0])
[1]
sage: list(sorting_word(IV([2,3,2]))[1])
[3, 2]
sage: list(sorting_word(IV([5,6,7]))[0])
[1, 2, 1]
sage: list(sorting_word(IV([0,3,2]))[0])
[2, 1]
sage: list(sorting_word(IV([0,3,0,2]))[0])
[2, 3, 1]
sage: list(sorting_word(IV([3,2,1]))[0])
[]
sage: list(sorting_word(IV([2,3,3]))[0])
[2, 1]
```
5.1.133 Knutson-Tao Puzzles

This module implements a generic algorithm to solve Knutson-Tao puzzles. An instance of this class will be callable: the arguments are the labels of north-east and north-west sides of the puzzle boundary; the output is the list of the fillings of the puzzle with the specified pieces.

Acknowledgements

This code was written during Sage Days 45 at ICERM with Franco Saliola, Anne Schilling, and Avinash Dalal in discussions with Allen Knutson. The code was tested afterwards by Liz Beazley and Ed Richmond.

Todo: Functionality to add:

- plotter will not plot edge labels higher than 2; e.g. in BK puzzles, the labels are 1, ..., n and so in 3-step examples, none of the edge labels with 3 appear
- we should also have a 3-step puzzle pieces constructor, taken from p22 of arXiv math/0610538
- implement the bijection from puzzles to tableaux; see for example R. Vakil, A geometric Littlewood-Richardson rule, arXiv math/0302294 or K. Purbhoo, Puzzles, Tableaux and Mosaics, arXiv 0705.1184.

sage.combinat.knutson_tao_puzzles.BK_pieces(max_letter)

The puzzle pieces used in computing the Belkale-Kumar coefficients for any partial flag variety in type $A$.

There are two types of puzzle pieces:

- a triangle, with each edge labeled with the same letter;
- a rhombus, with edges labeled $i, j, i, j$ in clockwise order with $i > j$.

Each of these is rotated by 60 degrees, but not reflected.

We model the rhombus pieces as two triangles: a delta piece north-west label $i$, north-east label $j$ and south label $i(j)$; and a nabla piece with south-east label $i$, south-west label $j$ and north label $i(j)$.

INPUT:

- max_letter – positive integer specifying the number of steps in the partial flag variety, equivalently, the number of elements in the alphabet for the edge labels. The smallest label is 1.

REFERENCES:

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import BK_pieces
sage: BK_pieces(3)
Nablas : [1\1/1, 1\2(1)/2, 1\3(1)/3, 2(1)/2\1, 2\1/2(1), 2\2/2, 2\3(2)/3, 3(1)/3\1, ˓→3(2)/3\2, 3\1/3(1), 3\2/3(2), 3\3/3]
Deltas : [1/1\1, 1/2\2(1), 1/3\3(1), 2(1)/1\2, 2\1/2(1), 2\2/2, 2\3(2)/3, 3(1)/1\3, ˓→3(2)/2\3, 3\3(1)/1, 3\3(2)/2, 3\3/3]
```

class sage.combinat.knutson_tao_puzzles.DeltaPiece(south, north_west, north_east)

Bases: PuzzlePiece

Delta Piece takes as input three labels, inputted as strings. They label the South, Northwest and Northeast edges, respectively.

EXAMPLES:
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece
sage: DeltaPiece('a', 'b', 'c')
b/a\c

clockwise_rotation()

Rotate the Delta piece by 120 degree clockwise.

OUTPUT:

• Delta piece

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece
sage: delta = DeltaPiece('1', '2', '3')
sage: delta.clockwise_rotation()
1/3\2

edges()

Return the tuple of edge names.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece
sage: delta = DeltaPiece('1', '2', '3')
sage: delta.edges()
('south', 'north_west', 'north_east')

half_turn_rotation()

Rotate the Delta piece by 180 degree.

OUTPUT:

• Nabla piece

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece
sage: delta = DeltaPiece('1', '2', '3')
sage: delta.half_turn_rotation()
3\1/2

sage.combinat.knutson_tao_puzzles.HT_grassmannian_pieces()

Define the puzzle pieces used in computing the torus-equivariant cohomology of the Grassmannian.

REFERENCES:

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import HT_grassmannian_pieces
sage: HT_grassmannian_pieces()
Nablas : [0/0, 0/10, \[10/1, 1/10, 1/1, 1/T0|1/0]
Deltas : [0/0\0, 0/10, 0/T0|1/1, 1/10\0, 1/1, 10/0\1]

sage.combinat.knutson_tao_puzzles.HT_two_step_pieces()

Define the puzzle pieces used in computing the equivariant two step puzzle pieces.

For the puzzle pieces, see Figure 26 on page 22 of [CoskunVakil06].
REFERENCES:

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import HT_two_step_pieces
sage: HT_two_step_pieces()
Nablas : [(21)0/21, 0\21/0, 0\21/0, 0\0/0, 0\10/1, 0\20/2, 10\1/0, 10\2(10)/2,
1\0/10, 1\1/1, 1\2/2, 1\T0|1/0, 2\(10)/2, 20\2/0, 2\T0|2\0, 2\T0|2\2, 2\T1|2/1]
Deltas : [(21)0/0\21, 0/0\0, 0/1\10, 0/21\2(10), 0/2\20, 0/T0|1\1, 0/T0|2\1, 0/ →
T0|2]\2, 1/10\0, 1/1\1, 1/2\21, 1/T1|2\2, 10\0/1, 10\2\2(10), 10/T10|2\21, 10/T10|2\21, →
2(10)/10\2, 2/2(10)|10, 2/20\0, 2/21\2, 20/0\2, 21/(21)0\0, 21/1\2]
```

```
sage.combinat.knutson_tao_puzzles.H_grassmannian_pieces()
Define the puzzle pieces used in computing the cohomology of the Grassmannian.

REFERENCES:

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import H_grassmannian_pieces
sage: H_grassmannian_pieces()
Nablas : [0\0/0, 0\10/1, 10\1/0, 1\0/10, 1\1/1]
Deltas : [0/0\0, 0/1\10, 1/10\0, 1/1\1, 10/0\1]
```

```
sage.combinat.knutson_tao_puzzles.H_two_step_pieces()
Define the puzzle pieces used in two step flags.

This rule is currently only conjecturally true. See [BuchKreschTamvakis03].

REFERENCES:

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import H_two_step_pieces
sage: H_two_step_pieces()
Nablas : [(21)0/21, 0\21/0, 0\21/0, 0\0/0, 0\10/1, 0\20/2, 10\1/0, 10\2(10)/2,
1\0/10, 1\1/1, 1\2/2, 2(10)\2/10, 20\2/0, 2\0/2(10), 21\2/1, 2\0/20, 2\1/21, 2\10/2(10), 2\2/2]
Deltas : [(21)0/0\21, 0/0\0, 0/1\10, 0/21\2(10), 0/2\20, 10\1/0, 1\1/1, 2\2/1, 10/ →
0\1, 10\2\2(10), 2\10/10\2, 2/2(10)|10, 2/20\0, 2/21\2, 20/0\2, 21/(21)0\0, 21/1\2]
```

```
sage.combinat.knutson_tao_puzzles.K_grassmannian_pieces()
Define the puzzle pieces used in computing the K-theory of the Grassmannian.

REFERENCES:

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import K_grassmannian_pieces
sage: K_grassmannian_pieces()
Nablas : [0\0/0, 0\10/1, 0\K/1, 10\1/0, 1\0/10, 1\0/K, 1\1/1, K\1/0]
Deltas : [0/0\0, 0/1\10, 1/0/10, 1/1/1, 10/0/1, K/K/K]
```

class sage.combinat.knutson_tao_puzzles.KnutsonTaoPuzzleSolver(puzzle_pieces)
    Bases: UniqueRepresentation

Return puzzle solver function used to create all puzzles with given boundary conditions.

This class implements a generic algorithm to solve Knutson-Tao puzzles. An instance of this class will be
callable: the arguments are the labels of north-east and north-west sides of the puzzle boundary; the output is the
list of the fillings of the puzzle with the specified pieces.

INPUT:

• puzzle_pieces – takes either a collection of puzzle pieces or a string indicating a pre-programmed col-
  lection of puzzle pieces:
    – H – cohomology of the Grassmannian
    – HT – equivariant cohomology of the Grassmannian
    – K – K-theory
    – H2step – cohomology of the 2-step Grassmannian
    – HT2step – equivariant cohomology of the 2-step Grassmannian
    – BK – Belkale-Kumar puzzle pieces
• max_letter – (default: None) None or a positive integer. This is only required only for Belkale-Kumar
  puzzles.

EXAMPLES:

Each puzzle piece is an edge-labelled triangle oriented in such a way that it has a south edge (called a delta piece)
or a north edge (called a nabla piece). For example, the puzzle pieces corresponding to the cohomology of the
Grassmannian are the following:

```sage
from sage.combinat.knutson_tao_puzzles import H_grassmannian_pieces
sage: H_grassmannian_pieces()
Nablas : [0\0/0, 0\1/1, 1\0/10, 1\1/0, 1\1/1]
Deltas : [0/0\0, 0/1\10, 1/10\0, 1/1\1, 10/0\1]
```

In the string representation, the nabla pieces are depicted as c\a/b, where a is the label of the north edge, b is
the label of the south-east edge, c is the label of the south-west edge. A similar string representation exists for
the delta pieces.

To create a puzzle solver, one specifies a collection of puzzle pieces:

```sage
KnutsonTaoPuzzleSolver(H_grassmannian_pieces())
Knutson-Tao puzzle solver with pieces:
Nablas : [0\0/0, 0\1/10, 1\0/10, 1\1/0, 1\1/1]
Deltas : [0/0\0, 0/1\10, 1/10\0, 1/1\1, 10/0\1]
```

The following shorthand to create the above puzzle solver is also supported:

```sage
KnutsonTaoPuzzleSolver('H')
Knutson-Tao puzzle solver with pieces:
Nablas : [0\0/0, 0\1/10, 1\0/10, 1\1/0, 1\1/1]
Deltas : [0/0\0, 0/1\10, 1/10\0, 1/1\1, 10/0\1]
```

The solver will compute all fillings of the puzzle with the given puzzle pieces. The user specifies the labels of
north-east and north-west sides of the puzzle boundary and the output is a list of the fillings of the puzzle with
the specified pieces. For example, there is one solution to the puzzle whose north-west and north-east edges are both labeled ‘0’:

```
sage: ps = KnutsonTaoPuzzleSolver('H')
sage: ps('0', '0')
[[{(1, 1): 0/0\0}]]
```

There are two solutions to the puzzle whose north-west and north-east edges are both labeled ‘0101’:

```
sage: ps = KnutsonTaoPuzzleSolver('H')
sage: solns = ps('0101', '0101')
sage: len(solns)
2
sage: solns.sort(key=str)
sage: solns
[[{(1, 1): 0/0\0,
   (1, 2): 1/\0 0\/1,
   (1, 3): 0\/0 0\/0,
   (1, 4): 1\/0 0\/1,
   (2, 2): 1/1\1,
   (2, 3): 0\/10 1\/1,
   (2, 4): 1/1 10\/0,
   (3, 3): 1/1\1,
   (3, 4): 0\/0 1\/10,
   (4, 4): 10/0\1}, {(1, 1): 0/1\10,
   (1, 2): 1\/1 10\/0,
   (1, 3): 0\/0 1\/10,
   (1, 4): 1\/0 0\/1,
   (2, 2): 0/0\0,
   (2, 3): 10/\1 0\/0,
   (2, 4): 1/1 1\/1,
   (3, 3): 0/0\0,
   (3, 4): 1/0 0\/1,
   (4, 4): 1/1\1}]
```

The pieces in a puzzle filling are indexed by pairs of non-negative integers \((i, j)\) with \(1 \leq i \leq j \leq n\), where \(n\) is the length of the word labelling the triangle edge. The pieces indexed by \((i, i)\) are the triangles along the south edge of the puzzle.

```
sage: f = solns[0]
sage: [f[i, i] for i in range(1,5)]
[0/0\0, 1/1\1, 1/1\1, 10/0\1]
```

The pieces indexed by \((i, j)\) for \(j > i\) are a pair consisting of a delta piece and nabla piece glued together along the south edge and north edge, respectively (these pairs are called rhombi).

```
sage: f = solns[0]
sage: f[1, 2]
1/\0 0\/1
```

There are various methods and options to display puzzle solutions. A single puzzle can be displayed using the plot method of the puzzle:
To plot several puzzle solutions, use the plot method of the puzzle solver:

```python
sage: ps = KnutsonTaoPuzzleSolver('K')
sage: solns = ps('0101', '0101')
sage: ps.plot(solns)  # not tested
```

The code can also generate a PDF of a puzzle (using LaTeX and `tikz`):

```python
sage: latex.extra_preamble(r'''
\usepackage{tikz}''')
sage: ps = KnutsonTaoPuzzleSolver('H')
sage: solns = ps('0101', '0101')
sage: view(solns[0], viewer='pdf')  # not tested
```

Below are examples of using each of the currently supported puzzles.

Cohomology of the Grassmannian:

```python
sage: ps = KnutsonTaoPuzzleSolver('H')
sage: solns = ps('0101', '0101')
sage: sorted(solns, key=str)
[[(1, 1): 0/0\0, (1, 2): 1\0 0\1, (1, 3): 0\0 0/0, (1, 4): 1\0 0\1, (2, 2): 1/1\1, (2, 3): 0\10 1\1, (2, 4): 1\1 10\/0, (3, 3): 1/1\1, (3, 4): 0\0 1\10, (4, 4): 10/0\1}, {(1, 1): 0/1\10, (1, 2): 1\1 10\/0, (1, 3): 0\0 1\10, (1, 4): 1\0 0\1, (2, 2): 0/0\0, (2, 3): 10/1 0\/0, (2, 4): 1\1 1\1, (3, 3): 0\0\0, (3, 4): 1\0 0\1, (4, 4): 1/1\1}]
```

Equivariant puzzles:

```python
sage: ps = KnutsonTaoPuzzleSolver('HT')
sage: solns = ps('0101', '0101')
sage: sorted(solns, key=str)
[[(1, 1): 0/0\0, (1, 2): 1\0 0\1, (1, 3): 0/0 0\0, (1, 4): 1\0 0\1, (2, 2): 1/1\1, (2, 3): 0\10 1\1, (2, 4): 1\1 10\/0, (3, 3): 1/1\1, (3, 4): 0\0 1\10, (4, 4): 10/0\1}, {(1, 1): 0/1\10, (1, 2): 1\1 10\/0, (1, 3): 0\0 1\10, (1, 4): 1\0 0\1, (2, 2): 0/0\0, (2, 3): 10/1 0\/0, (2, 4): 1\1 1\1, (3, 3): 0\0\0, (3, 4): 1\0 0\1, (4, 4): 1/1\1}]
```

(continues on next page)
(1, 4): 1\0 0\1,
(2, 2): 1/1\1,
(2, 3): 0\1 1\0,
(2, 4): 1/1 1\1,
(3, 3): 0\0\0,
(3, 4): 1\0 0\1,
(4, 4): 1/1\1}, {(1, 1): 0/0\0,
(1, 2): 1\0 0\1,
(1, 3): 0\0 0\0,
(1, 4): 1\0 0\1,
(2, 2): 1/1\1,
(2, 3): 0\10 1\1,
(2, 4): 1/1 10\0,
(3, 3): 1/1\1,
(3, 4): 0\0 1\10,
(4, 4): 10/0\1}, {(1, 1): 0/1\0,
(1, 2): 1/1 10\0,
(1, 3): 0\0 1\10,
(1, 4): 1\0 0\1,
(2, 2): 0/0\0,
(2, 3): 10/1 0\0,
(2, 4): 1/1 1\1,
(3, 3): 0\0\0,
(3, 4): 1\0 0\1,
(4, 4): 1/1\1}],

K-Theory puzzles:

\texttt{sage}: ps = KnutsonTaoPuzzleSolver("K")
\texttt{sage}: solns = ps('0101', '0101')
\texttt{sage}: sorted(solns, key=str)

\begin{verbatim}
[[(1, 1): 0/0\0,
  (1, 2): 1/0 0\1,
  (1, 3): 0/0 0\0,
  (1, 4): 1/0 0\1,
  (2, 2): 1/1\1,
  (2, 3): 0/10 1\1,
  (2, 4): 1/1 10\0,
  (3, 3): 1/1\1,
  (3, 4): 0/0 1\10,
  (4, 4): 10/0\1}, {(1, 1): 0/1\0,
  (1, 2): 1/1 10\0,
  (1, 3): 0/0 1\10,
  (1, 4): 1/0 0\1,
  (2, 2): 0/0\0,
  (2, 3): 10/1 0\0,
  (2, 4): 1/1 1\1,
  (3, 3): 0/0\0,
  (3, 4): 1\0 0\1,
  (4, 4): 1/1\1}],
\end{verbatim}
(1, 4): 1\0 0\1,
(2, 2): 0\0\0,
(2, 3): K\K 0\1,
(2, 4): 1\1 K\0,
(3, 3): 1/1\1,
(3, 4): 0\0 1\10,
(4, 4): 10/0\1}]

Two-step puzzles:

\texttt{sage: \texttt{ps} = KnutsonTaoPuzzleSolver("H2step")}
\texttt{sage: \texttt{solns} = ps('01201', '01021')}
\texttt{sage: sorted(solns, key=str)}

\begin{verbatim}

\{(1, 1): 0/0\0,
 (1, 2): 1/0 0\1,
 (1, 3): 2/0 0\2,
 (1, 4): 0/0 0\0,
 (1, 5): 1/0 0\1,
 (2, 2): 1/2\2,
 (2, 3): 2/2 21\1,
 (2, 4): 0\10 2\21,
 (2, 5): 1/1 10\0,
 (3, 3): 1/1\1,
 (3, 4): 21/2 1\1,
 (3, 5): 0/0 2\20,
 (4, 4): 1/1\1,
 (4, 5): 20/2 1\10,
 (5, 5): 10/0\1}, {(1, 1): 0/1\10,
 (1, 2): 1/1 10\0,
 (1, 3): 2/1 1\2,
 (1, 4): 0/0 1\10,
 (1, 5): 1/0 0\1,
 (2, 2): 0/2\20,
 (2, 3): 2/2 20\0,
 (2, 4): 10/1 2\20,
 (2, 5): 1/1 1\1,
 (3, 3): 0/0\0,
 (3, 4): 20/2 0\0,
 (3, 5): 1/0 2\2(10),
 (4, 4): 0/0\0,
 (4, 5): 2(10)/2 0\1,
 (5, 5): 1/1\1}, {(1, 1): 0/2\20,
 (1, 2): 1/\21 20\0,
 (1, 3): 2/2 21\1,
 (1, 4): 0/0 2\20,
 (1, 5): 1/0 0\1,
 (2, 2): 0/0\0,
 (2, 3): 1/0 0\1,
 (2, 4): 20/2 0\0,
 (2, 5): 1/1 2\21,
 (3, 3): 1/1\1,
 (3, 4): 0\0 1\10,

\end{verbatim}

(continues on next page)
Two-step equivariant puzzles:

```python
sage: ps = KnutsonTaoPuzzleSolver("HT2step")
sage: solns = ps('10212', '12012')
sage: sorted(solns, key=str)
```

```plaintext
[(1, 1): 1/1\1, (1, 2): 0/\(21)0 1\2, (1, 3): 2/\(21)0\0, (1, 4): 1/1 1\1, (1, 5): 2/1 1\2, (1, 6): 2/2\2, (2, 3): 0/2 2\0, (2, 4): 1/2 2\1, (2, 5): 2/2 2\2, (3, 3): 0/0\0, (3, 4): 1/\0 0\1, (3, 5): 2/0 0\2, (4, 4): 1/1\1, (4, 5): 2/1 1\2, (5, 5): 2/2\2}, {(1, 1): 1/1\1, (1, 2): 0/\(21)0 1\2, (1, 3): 2/\(21)0\0, (1, 4): 1/1 1\1, (1, 5): 2/1 1\2, (1, 6): 2/2\2, (2, 3): 0/2 2\0, (2, 4): 1/2 2\1, (2, 5): 2/2 2\2, (3, 3): 0/0\0, (3, 4): 1/\0 0\1, (3, 5): 2/0 0\2, (4, 4): 1/1\1, (4, 5): 2/1 1\2, (5, 5): 2/2\2}, {(1, 1): 1/1\1, (1, 2): 0/\(21)0 1\2, (1, 3): 2/\(21)0\0, (1, 4): 1/1 1\1, (1, 5): 2/1 1\2, (1, 6): 2/2\2, (2, 3): 0/20 2\2, (2, 4): 1/\21 20\0, (2, 5): 2/2 2\1, (3, 3): 2/2\2, (3, 4): 0/0 2\/20, (3, 5): 1/0 0\1, (4, 4): 20/0\2, (4, 5): 1/1 2\21,
```

(continues on next page)
(5, 5): 2/2\2\2, \{(1, 1): 1/1\1, \\
(1, 2): 0/1\ 1/0, \\
(1, 3): 2/1\ 1/2, \\
(1, 4): 1/1\ 1/1, \\
(1, 5): 2/1\ 1/2, \\
(2, 2): 0/2\2\0, \\
(2, 3): 2/2\ 2\0/\0, \\
(2, 4): 1/2\ 2/1, \\
(2, 5): 2/2\ 2/2, \\
(3, 3): 0/0/\0, \\
(3, 4): 1/0\ 0/1, \\
(3, 5): 2/0\ 0/2, \\
(4, 4): 1/1\1, \\
(4, 5): 2/1\ 1/2, \\
(5, 5): 2/2/2\2, \{(1, 1): 1/1\1, \\
(1, 2): 0/1\ 1/0, \\
(1, 3): 2/1\ 1/2, \\
(1, 4): 1/1\ 1/1, \\
(1, 5): 2/1\ 1/2, \\
(2, 2): 0/2\2\0, \\
(2, 3): 2/2\ 2\0/\0, \\
(2, 4): 1/2\ 2/1, \\
(2, 5): 2/2\ 2/2, \\
(3, 3): 0/0\0, \\
(3, 4): 2/0\ 0/2, \\
(3, 5): 1/0\ 0/1, \\
(4, 4): 2/2\2, \\
(4, 5): 1/1\ 2/21, \\
(5, 5): 2/1/2\2, \{(1, 1): 1/1\1, \\
(1, 2): 0/1\10\ 1/1, \\
(1, 3): 2/1\10/\2, \\
(1, 4): 1/1\ 10/\0, \\
(1, 5): 2/1\ 1/2, \\
(2, 2): 1/2/21, \\
(2, 3): 2/2\ 21/1, \\
(2, 4): 0/2\ 2/\0, \\
(2, 5): 2/2\ 2/2, \\
(3, 3): 1/1\1, \\
(3, 4): 0/0\ 1/10, \\
(3, 5): 2/0\ 0/\2, \\
(4, 4): 10/0\1, \\
(4, 5): 2/1\ 1/2, \\
(5, 5): 2/2/2\2, \{(1, 1): 1/1\1, \\
(1, 2): 0/1\10\ 1/1, \\
(1, 3): 2/1\10\ 10/\2, \\
(1, 4): 1/1\ 10/\0, \\
(1, 5): 2/1\ 1/2, \\
(2, 2): 1/2/21, \\
(2, 3): 2/2\ 21/1, \\
(2, 4): 0/2\0\ 2/\2, \\
(2, 5): 2/2\ 2/\0, \\
(3, 3): 1/1\1, \\
(continues on next page)
Belkale-Kumar puzzles (the following example is Figure 2 of [KnutsonPurbhoo10]):

```
sage: ps = KnutsonTaoPuzzleSolver('BK', 3)
sage: solns = ps('12132', '23112')
```
sage: len(solns)
1
sage: solns[0].south_labels()
('3', '2', '1', '2', '1')

sage: solns
[[(1, 1): 1/3\3(1),
  (1, 2): 2\3(2) 3(1)\1,
  (1, 3): 1/3(1) 3(2)\2,
  (1, 4): 3\3 3(1)\1,
  (1, 5): 2\2 3\3(2),
  (2, 2): 1/2\2(1),
  (2, 3): 2\2 2(1)\1,
  (2, 4): 1/2(1) 2\2,
  (2, 5): 3(2)/3 2(1)\1,
  (3, 3): 1/1\1,
  (3, 4): 2\1 1\2,
  (3, 5): 1/1 1/1,
  (4, 4): 2\2,]

plot(puzzles)

Return plot of puzzles.

INPUT:

  • puzzles – list of puzzles

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import KnutsonTaoPuzzleSolver
sage: ps = KnutsonTaoPuzzleSolver('K')
```

```python
sage: solns = ps('0101', '0101')
sage: ps.plot(solns)  # not tested
```

puzzle_pieces()

The puzzle pieces used for filling in the puzzles.

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import KnutsonTaoPuzzleSolver
sage: ps = KnutsonTaoPuzzleSolver('H')
sage: ps.puzzle_pieces()
```

```python
Nablas : [\0/0/0, \0/10/1, 10\1/0, 1/0/10, 1/1/1]
Deltas : [\0/0/0, \0/1/10, 1/10\0, 1/1\1, 10/0\1]
```

solutions(lambda, mu, algorithm='strips')

structure_constants(lambda, mu, nu=None)

Compute cohomology structure coefficients from puzzles.

INPUT:

  • pieces – puzzle pieces to be used
  • lambda, mu – edge labels of puzzle for northwest and north east side
• nu – (default: None) If nu is not specified a dictionary is returned with the structure coefficients corresponding to all south labels; if nu is given, only the coefficients with the specified label is returned.

OUTPUT: dictionary

EXAMPLES:

Note: In order to standardize the output of the following examples, we output a sorted list of items from the dictionary instead of the dictionary itself.

Grassmannian cohomology:

```python
sage: ps = KnutsonTaoPuzzleSolver('H')
sage: cp = ps.structure_constants('0101', '0101')
sage: sorted(cp.items(), key=str)
[(('0', '1', '1', '0'), 1), (('1', '0', '0', '1'), 1)]
```

Equivariant cohomology:

```python
sage: ps = KnutsonTaoPuzzleSolver('HT')
sage: cp = ps.structure_constants('0101', '0101')
sage: sorted(cp.items(), key=str)
[(('0', '1', '1', '0'), y2 - y3),
 ('(1', '0', '0', '1'), 1),
 ('(1', '0', '0', '1'), 1)]
```

K-theory:

```python
sage: ps = KnutsonTaoPuzzleSolver('K')
sage: cp = ps.structure_constants('0101', '0101')
sage: sorted(cp.items(), key=str)
[(('0', '1', '1', '0'), 1), (('1', '0', '0', '1'), 1), (('1', '0', '0', '1'), -1)]
```

Two-step:

```python
sage: ps = KnutsonTaoPuzzleSolver('H2step')
sage: cp = ps.structure_constants('01122', '01122')
sage: sorted(cp.items(), key=str)
[(('0', '1', '1', '0'), 1),
 ('(1', '2', '0', '1'), 1),
 ('(1', '2', '0', '1'), 1)]
```

Two-step equivariant:

```python
sage: ps = KnutsonTaoPuzzleSolver('HT2step')
sage: cp = ps.structure_constants('10212', '12012')
sage: sorted(cp.items(), key=str)
[(('1', '2', '0', '1', '2'), y1*y2 - y2*y3 - y1*y4 + y3*y4),
 ('(1', '2', '0', '2', '1'), y1 - y3),
 ('(1', '2', '0', '2', '1'), y2 - y4),
 (continues on next page)
```
class sage.combinat.knutson_tao_puzzles.NablaPiece(north, south_east, south_west)

Bases: PuzzlePiece

Nabla Piece takes as input three labels, inputted as strings. They label the North, Southeast and Southwest edges, respectively.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import NablaPiece
sage: NablaPiece('a','b','c')
c/a/b

clockwise_rotation()

Rotate the Nabla piece by 120 degree clockwise.

OUTPUT:

• Nabla piece

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import NablaPiece
sage: nabla = NablaPiece('1','2','3')
sage: nabla.clockwise_rotation()
2\3/1

edges()

Return the tuple of edge names.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import NablaPiece
sage: nabla = NablaPiece('1','2','3')
sage: nabla.edges()
('north', 'south_east', 'south_west')

half_turn_rotation()

Rotate the Nabla piece by 180 degree.

OUTPUT:

• Delta piece

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import NablaPiece
sage: nabla = NablaPiece('1','2','3')
sage: nabla.half_turn_rotation()
2/1\3
class sage.combinat.knutson_tao_puzzles.PuzzleFilling(north_west_labels, north_east_labels)
    Bases: object

    Create partial puzzles and provides methods to build puzzles from them.

    add_piece(piece)
        Add piece to partial puzzle.

        EXAMPLES:
        sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, PuzzleFilling
        sage: piece = DeltaPiece('0', '1', '0')
        sage: P = PuzzleFilling('0101', '0101'); P
        {}
        sage: P.add_piece(piece); P
        {(1, 4): 1/0\0}

    add_pieces(pieces)
        Add piece to partial puzzle.

        INPUT:
        • pieces – tuple of pieces

        EXAMPLES:
        sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, PuzzleFilling
        sage: piece = DeltaPiece('0', '1', '0')
        sage: pieces = [piece, piece]
        sage: P.add_pieces(pieces)
        sage: P
        {(1, 4): 1/0\0, (2, 4): 1/0\0}

    contribution()
        Return equivariant contributions from self in polynomial ring.

        EXAMPLES:
        sage: from sage.combinat.knutson_tao_puzzles import KnutsonTaoPuzzleSolver
        sage: ps = KnutsonTaoPuzzleSolver("HT")
        sage: puzzles = ps('0101', '1001')
        sage: sorted([p.contribution() for p in puzzles], key=str)
        [1, y1 - y3]

    copy()
        Return copy of self.

        EXAMPLES:
        sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, PuzzleFilling
        sage: piece = DeltaPiece('0', '1', '0')
        sage: P = PuzzleFilling('0101', '0101'); P
        {}
        sage: PP = P.copy()
        sage: PP
        {}
        sage: P.add_piece(piece); P
        {(1, 4): 1/0\0}
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{(1, 4): 1/0\0}

sage: PP
{}

is_completed()

Whether partial puzzle is complete (completely filled) or not.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import PuzzleFilling
sage: P = PuzzleFilling('0101','0101')
sage: P.is_completed()
False

sage: from sage.combinat.knutson_tao_puzzles import KnutsonTaoPuzzleSolver
sage: ps = KnutsonTaoPuzzleSolver("H")
sage: puzzle = ps('0101','1001')[0]
sage: puzzle.is_completed()
True

is_in_south_edge()

Check whether kink coordinates of partial puzzle is in south corner.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import PuzzleFilling
sage: P = PuzzleFilling('0101','0101')
sage: P.is_in_south_edge()
False

kink_coordinates()

Provide the coordinates of the kinks.

The kink coordinates are the coordinates up to which the puzzle has already been built. The kink starts in
the north corner and then moves down the diagonals as the puzzles is built.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import PuzzleFilling
sage: P = PuzzleFilling('0101','0101')
sage: P
{}
{sage: P.kink_coordinates()
(1, 4)

north_east_label_of_kink()

Return north east label of kink.

EXAMPLES:

sage: from sage.combinat.knutson_tao_puzzles import PuzzleFilling
sage: P = PuzzleFilling('0101','0101')
sage: P.north_east_label_of_kink()
'0'
**north_west_label_of_kink()**

Return north-west label of kink.

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import PuzzleFilling
sage: P = PuzzleFilling('0101', '0101')
sage: P.north_west_label_of_kink()
'1'
```

**plot(labels=True, style='fill')**

Plot completed puzzle.

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import KnutsonTaoPuzzleSolver
sage: ps = KnutsonTaoPuzzleSolver("H")
sage: puzzle = ps('0101', '1001')[0]
sage: puzzle.plot()  #not tested
sage: puzzle.plot(style='fill')  #not tested
sage: puzzle.plot(style='edges')  #not tested
```

**south_labels()**

Return south labels for completed puzzle.

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import KnutsonTaoPuzzleSolver
sage: ps = KnutsonTaoPuzzleSolver("H")
sage: ps('0101', '1001')[0].south_labels()
('1', '0', '1', '0')
```

**class sage.combinat.knutson_tao_puzzles.PuzzlePiece**

Bases: object

Abstract class for puzzle pieces.

This abstract class contains information on how to test equality of puzzle pieces, and sets color and plotting options.

**border()**

Return the border of self.

EXAMPLES:

```
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece
sage: delta = DeltaPiece('a', 'b', 'c')
sage: sorted(delta.border())
['a', 'b', 'c']
```

**color()**

Return the color of self.

EXAMPLES:
```
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece
sage: delta = DeltaPiece('a','b','c')
sage: delta.color()
'white'
sage: delta = DeltaPiece('0','0','0')
sage: delta.color()
'red'
sage: delta = DeltaPiece('1','1','1')
sage: delta.color()
'blue'
sage: delta = DeltaPiece('2','2','2')
sage: delta.color()
'green'
sage: delta = DeltaPiece('2','K','2')
sage: delta.color()
'orange'
sage: delta = DeltaPiece('2','T1/2','2')
sage: delta.color()
'yellow'
```

**edge_color**(edge)

Color of the specified edge of self (to be used when plotting the piece).

**edge_label**(edge)

Return the edge label of edge.

**class** `sage.combinat.knutson_tao_puzzles.PuzzlePieces` *(forbidden_border_labels=None)*

Bases: object

Construct a valid set of puzzle pieces.

This class constructs the set of valid puzzle pieces. It can take a list of forbidden border labels as input. These labels are forbidden from appearing on the south edge of a puzzle filling. The user can add valid nabla or delta
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pieces and specify which rotations of these pieces are legal. For example, \texttt{rotations=0} does not add any additional pieces (only the piece itself), \texttt{rotations=60} adds six pieces (the pieces and its rotations by 60, 120, 180, 240, 300), etc..

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces, NablaPiece
sage: forbidden_border_labels = ['10']
sage: pieces = PuzzlePieces(forbidden_border_labels)
sage: pieces.add_piece(NablaPiece('0','0','0'), rotations=60)
sage: pieces.add_piece(NablaPiece('1','1','1'), rotations=60)
sage: pieces.add_piece(NablaPiece('1','0','10'), rotations=60)
sage: pieces
Nablas : [0\0/0, 0\10/1, 10\1/0, 1\0/10, 1\1/1]
Deltas : [0/0\0, 0/\10, 1/10\0, 1/1\1, 10/0\1]
```

The user can obtain the list of valid rhombi pieces as follows:

```python
sage: sorted([p for p in pieces.rhombus_pieces()], key=str)
[0\0/0, 0\10/1, 10\1/0, 1\0/10, 1\1/1, 0\0/1, 1\1/1, 0\10/1, 1/10\0, 1/1\1, 10/0\1]
```

\texttt{add\_T\_piece(label1\textup{,} label2)}

Add a nabla and delta piece with \texttt{label1} and \texttt{label2}.

This method adds a nabla piece with edges \texttt{label2T``\textbackslash label1``\textbackslash label2``/ label1.} and a delta piece with edges \texttt{label1\textbackslash T``\textbackslash label1``\textbackslash label2`` label2.} It also adds \texttt{T``\textbackslash label1``\textbackslash label2``} to the forbidden list.

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces
sage: pieces = PuzzlePieces()
sage: pieces.add_T_piece('1', '3')
sage: pieces
Nablas : [3\T1\3/1]
Deltas : [1/T1\3/3]
sage: pieces._forbidden_border_labels
['T1\3']
```

\texttt{add\_forbidden\_label(label)}

Add forbidden border labels.

INPUT:

- \texttt{label} – string specifying a new forbidden label

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces
sage: pieces = PuzzlePieces()
sage: pieces.add_forbidden_label('1')
sage: pieces
Nablas : [\3\T1\3/1]
Deltas : [1/T1\3/3]
sage: pieces._forbidden_border_labels
['T1\3']
```

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add_piece(piece, rotations=120)

Add piece to the list of pieces.

INPUT:

• piece – a nabla piece or a delta piece
• rotations – (default: 120) 0, 60, 120, 180

The user can add valid nabla or delta pieces and specify which rotations of these pieces are legal. For example, rotations=0 does not add any additional pieces (only the piece itself), rotations=60 adds six pieces (namely three delta and three nabla pieces), while rotations=120 adds only delta or nabla (depending on which piece self is). rotations=180 adds the piece and its 180 degree rotation, i.e. one delta and one nabla piece.

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces, DeltaPiece
sage: delta = DeltaPiece('a', 'b', 'c')
sage: pieces = PuzzlePieces()
sage: pieces.add_piece(delta)
sage: pieces
Nablas : []
Deltas : [a/c\b, b/a\c, c/b\a]

sage: pieces = PuzzlePieces()
sage: pieces.add_piece(delta, rotations=0)
sage: pieces
Nablas : []
Deltas : [b/a\c]

sage: pieces = PuzzlePieces()
sage: pieces.add_piece(delta, rotations=60)
sage: pieces
Nablas : [a\b/c, b/c/a, c/a/b]
Deltas : [a/c\b, b/a\c, c/b\a]
```

boundary_deltas()

Return deltas with south edges not in the forbidden list.

EXAMPLES:

```python
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces, DeltaPiece
sage: pieces = PuzzlePieces(['a'])
sage: delta = DeltaPiece('a', 'b', 'c')
sage: pieces.add_piece(delta, rotations=60)
sage: sorted([p for p in pieces.boundary_deltas()], key=str)
[a/c\b, c/b\a]
```

delta_pieces()

Return the delta pieces as a set.

EXAMPLES:
```
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces, DeltaPiece
sage: pieces = PuzzlePieces()
sage: delta = DeltaPiece('a', 'b', 'c')
sage: pieces.add_piece(delta, rotations=60)
sage: sorted([p for p in pieces.delta_pieces()], key=str)
[a/c\b, b/a\c, c/b\a]
```

```
nabla_pieces()
Return the nabla pieces as a set.
EXAMPLES:
```
```
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces, DeltaPiece
sage: pieces = PuzzlePieces()
sage: delta = DeltaPiece('a', 'b', 'c')
sage: pieces.add_piece(delta, rotations=60)
sage: sorted([p for p in pieces.nabla_pieces()], key=str)
[a\b/c, b\c/a, c\a/b]
```

```
rhombus_pieces()
Return a set of all allowable rhombus pieces.
Allowable rhombus pieces are those where the south edge of the delta piece equals the north edge of the
nabla piece.
EXAMPLES:
```
```
sage: from sage.combinat.knutson_tao_puzzles import PuzzlePieces, DeltaPiece
sage: pieces = PuzzlePieces()
sage: delta = DeltaPiece('a', 'b', 'c')
sage: pieces.add_piece(delta, rotations=60)
sage: sorted([p for p in pieces.rhombus_pieces()], key=str)
[a/b\c, b\c/a, c\a/b]
```

```
class sage.combinat.knutson_tao_puzzles.RhombusPiece(north_piece, south_piece)
Bases: PuzzlePiece
Class of rhombi pieces.
To construct a rhombus piece we input a delta and a nabla piece. The delta and nabla pieces are joined along the
south and north edge, respectively.
EXAMPLES:
```
```
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, NablaPiece,
...RhombusPiece
sage: delta = DeltaPiece('1', '2', '3')
sage: nabla = NablaPiece('4', '5', '6')
sage: RhombusPiece(delta, nabla)
2\3 6\5
```
```
edges()
Return the tuple of edge names.
EXAMPLES:
```
```python
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, NablaPiece,
    RhombusPiece
sage: delta = DeltaPiece('1', '2', '3')
```

```python
sage: nabla = NablaPiece('4', '5', '6')
```

```python
sage: RhombusPiece(delta, nabla).edges()
('north_west', 'north_east', 'south_east', 'south_west')
```

**north_piece()**

Return the north piece.

**EXAMPLES:**

```python
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, NablaPiece,
    RhombusPiece
sage: delta = DeltaPiece('1', '2', '3')
```

```python
sage: nabla = NablaPiece('4', '5', '6')
```

```python
sage: r = RhombusPiece(delta, nabla)
```

```python
sage: r.north_piece()
```

```latex
\frac{2}{1} \frac{3}{1}
```

**south_piece()**

Return the south piece.

**EXAMPLES:**

```python
sage: from sage.combinat.knutson_tao_puzzles import DeltaPiece, NablaPiece,
    RhombusPiece
sage: delta = DeltaPiece('1', '2', '3')
```

```python
sage: nabla = NablaPiece('4', '5', '6')
```

```python
sage: r = RhombusPiece(delta, nabla)
```

```python
sage: r.south_piece()
```

```latex
\frac{6}{4} \frac{5}{5}
```

### 5.1.134 Combinatorics on matrices

- *Dancing Links internal pxy code*
- *Dancing links C++ wrapper*
- *Hadamard matrices*
- *Latin Squares*

### 5.1.135 Dancing Links internal pxy code

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0, 1, 2], [3, 4, 5], [0, 1], [2, 3, 4, 5], [0], [1, 2, 3, 4, 5]]
```

```python
sage: x = dlx_solver(rows)
```

```python
sage: x
```

```
Dancing links solver for 6 columns and 6 rows
```

The number of solutions:
sage: x.number_of_solutions()
3

Iterate over the solutions:

```python
sage: sorted(map(sorted, x.solutions_iterator()))
[[0, 1], [2, 3], [4, 5]]
```

All solutions (computed in parallel):

```python
sage: sorted(map(sorted, x.all_solutions()))
[[0, 1], [2, 3], [4, 5]]
```

Return the first solution found when the computation is done in parallel:

```python
sage: sorted(x.one_solution(ncpus=2)) # random
[0, 1]
```

Find all solutions using some specific rows:

```python
sage: x_using_row_2 = x.restrict([2])
sage: x_using_row_2
Dancing links solver for 7 columns and 6 rows
sage: sorted(map(sorted, x_using_row_2.solutions_iterator()))
[[2, 3]]
```

The two basic methods that are wrapped in this class are `search` which returns 1 if a solution is found or 0 otherwise and `get_solution` which return the current solution:

```python
sage: x = dlx_solver(rows)
sage: x.search()
1
sage: x.get_solution()
[0, 1]
sage: x.search()
1
sage: x.get_solution()
[2, 3]
sage: x.search()
1
sage: x.get_solution()
[4, 5]
sage: x.search()
0
```

There is also a method `reinitialize` to reinitialize the algorithm:

```python
sage: x.reinitialize()
sage: x.search()
1
sage: x.get_solution()
[0, 1]
```

```python
class sage.combinat.matrices.dancing_links.dancing_linksWrapper
   Bases: object
```
A simple class that implements dancing links.

The main methods to list the solutions are `search()` and `get_solution()`. You can also use `number_of_solutions()` to count them.

This class simply wraps a C++ implementation of Carlo Hamalainen.

**all_solutions**(ncpus=None, column=None)

Return all solutions found after splitting the problem to allow parallel computation.

**INPUT:**

- ncpus – integer (default: None), maximal number of subprocesses to use at the same time. If None, it detects the number of effective CPUs in the system using `sage.parallel.ncpus.ncpus()`.
- column – integer (default: None), the column used to split the problem, if None a random column is chosen

**OUTPUT:**

list of solutions

**EXAMPLES:**

```sage
def dlx_solver(rows):
    d = dlx_solver(rows)
    S = d.all_solutions()
    return sorted(sorted(s) for s in S)

dlx = dlx_solver(result)
dlx
Dancing links solver for 4 columns and 16 rows
dlx.number_of_solutions()
15
dlx.sorted(sorted(s) for s in dlx.all_solutions(ncpus=2))
[[1, 2, 3, 4],
 [1, 2, 10],
 [1, 3, 9],
 [1, 4, 8],
 [1, 14],
 [2, 3, 7],
 [2, 4, 6],
 [2, 13],
 [3, 4, 5],
 [3, 12],
 [4, 11],
 [5, 10],
 [6, 9],
 [7, 8],
 [15]]
```

If ncpus=1, the computation is not done in parallel:
get_solution()

Return the current solution.

After a new solution is found using the method search() this method return the rows that make up the current solution.

ncols()

Return the number of columns.

EXAMPLES:

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[[0,1,2], [1,2], [0], [3,4,5]]
```
OUTPUT:

integer

EXAMPLES:

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2]]
sage: rows += [[0,2]]
sage: rows += [[1]]
sage: rows += [[3]]
sage: x = dlx_solver(rows)
sage: x.number_of_solutions()
2
```

The number of CPUs can be specified as input:

```python
sage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
sage: x = dlx_solver(rows)
sage: x.number_of_solutions(ncpus=2, column=3)
3
```

```python
sage: S = Subsets(range(5))
sage: rows = map(list, S)
sage: d = dlx_solver(rows)
sage: d.number_of_solutions()
52
```

```python
one_solution(ncpus=None, column=None)

Return the first solution found.

This method allows parallel computations which might be useful for some kind of problems when it is very hard just to find one solution.

INPUT:

• ncpus – integer (default: None), maximal number of subprocesses to use at the same time. If None, it detects the number of effective CPUs in the system using `sage.parallel.ncpus.ncpus()`. If ncpus=1, the first solution is searched serially.

• column – integer (default: None), the column used to split the problem (see `restrict()`). If None, a random column is chosen. This argument is ignored if ncpus=1.

OUTPUT:

list of rows or None if no solution is found

Note: For some case, increasing the number of cpus makes it faster. For other instances, ncpus=1 is faster. It all depends on problem which is considered.

EXAMPLES:

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
sage: d = dlx_solver(rows)
sage: solutions = [[0,1], [2,3], [4,5]]
```

The number of CPUs can be specified as input:

```python
sage: sorted(d.one_solution(ncpus=2)) in solutions
True
```

The column used to split the problem for parallel computations can be given:

```python
sage: sorted(d.one_solution(ncpus=2, column=4)) in solutions
True
```

When no solution is found:

```python
sage: rows = [[0,1,2], [2,3,4,5], [0,1,2,3]]
sage: d = dlx_solver(rows)
sage: d.one_solution() is None
True
```

**one_solution_using_milp_solver**(*solver=None, integrality_tolerance=0.001*)

Return a solution found using a MILP solver.

**INPUT:**

- solver – string or None (default: None), possible values include 'GLPK', 'GLPK/exact', 'Coin', 'CPLEX', 'Gurobi', 'CVXOPT', 'PPL', 'InteractiveLP'.

**OUTPUT:**

list of rows or None if no solution is found

**Note:** When comparing the time taken by method one_solution, have in mind that one_solution_using_milp_solver first creates (and caches) the MILP solver instance from the dancing links solver. This copy of data may take many seconds depending on the size of the problem.

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
sage: d = dlx_solver(rows)
sage: solutions = [[0,1], [2,3], [4,5]]
sage: d.one_solution_using_milp_solver() in solutions
True
```

Using optional solvers:

```python
sage: s = d.one_solution_using_milp_solver('gurobi') # optional - gurobi sage_numerical_backends_gurobi
sage: s in solutions # optional - gurobi sage_numerical_backends_gurobi
True
```

When no solution is found:
sage: rows = [[0,1,2], [2,3,4,5], [0,1,2,3]]
sage: d = dlx_solver(rows)
sage: d.one_solution_using_milp_solver() is None
True

one_solution_using_sat_solver(solver=None)

Return a solution found using a SAT solver.

INPUT:

• solver – string or None (default: None), possible values include 'picosat', 'cryptominisat', 'LP', 'glucose', 'glucose-syrup'.

OUTPUT:

list of rows or None if no solution is found

Note: When comparing the time taken by method one_solution, have in mind that one_solution_using_sat_solver first creates the SAT solver instance from the dancing links solver. This copy of data may take many seconds depending on the size of the problem.

EXAMPLES:

sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
sage: d = dlx_solver(rows)
sage: solutions = [[0,1], [2,3], [4,5]]
sage: d.one_solution_using_sat_solver() in solutions
True

Using optional solvers:

sage: s = d.one_solution_using_sat_solver('glucose') # optional - glucose
sage: s in solutions # optional - glucose
True

When no solution is found:

sage: rows = [[0,1,2], [2,3,4,5], [0,1,2,3]]
sage: d = dlx_solver(rows)
sage: d.one_solution_using_sat_solver() is None
True

reinitialize()

Reinitialization of the search algorithm

This recreates an empty dancing_links object and adds the rows to the instance of dancing_links.

EXAMPLES:

sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
sage: x = dlx_solver(rows)
sage: x.get_solution() if x.search() else None
[0, 1]
Reinitialization of the algorithm:

```python
sage: x.reinitialize()
sage: x.get_solution() if x.search() else None
[0, 1]
sage: x.get_solution() if x.search() else None
[2, 3]
sage: x.get_solution() if x.search() else None
[4, 5]
sage: x.get_solution() if x.search() else None
```

Reinitialization works after solutions are exhausted:

```python
sage: x.reinitialize()
sage: x.get_solution() if x.search() else None
[0, 1]
sage: x.get_solution() if x.search() else None
[2, 3]
sage: x.get_solution() if x.search() else None
[4, 5]
sage: x.get_solution() if x.search() else None
```

\texttt{restrict(indices)}

Return a dancing links solver solving the subcase which uses some given rows.

For every row that is wanted in the solution, we add a new column to the row to make sure it is in the solution.

INPUT:

\begin{itemize}
\item \texttt{indices} – list, row indices to be found in the solution
\end{itemize}

OUTPUT:

dancing links solver

EXAMPLES:

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
sage: d = dlx_solver(rows)
sage: d
Dancing links solver for 6 columns and 6 rows
sage: sorted(map(sorted, d.solutions_iterator()))
[[0, 1], [2, 3], [4, 5]]
```

To impose that the 0th row is part of the solution, the rows of the new problem are:

```python
sage: d_using_0 = d.restrict([0])
sage: d_using_0
Dancing links solver for 7 columns and 6 rows
sage: d_using_0.rows()
[[0, 1, 2, 6], [3, 4, 5], [0, 1], [2, 3, 4, 5], [0], [1, 2, 3, 4, 5]]
```
After restriction the subproblem has one more columns and the same number of rows as the original one:

```
sage: d.restrict([1]).rows()
[[0, 1, 2], [3, 4, 5, 6], [0, 1], [2, 3, 4, 5], [0], [1, 2, 3, 4, 5]]
sage: d.restrict([2]).rows()
[[0, 1, 2], [3, 4, 5], [0, 1, 6], [2, 3, 4, 5], [0], [1, 2, 3, 4, 5]]
```

This method allows to find solutions where the 0th row is part of a solution:

```
sage: sorted(map(sorted, d.restrict([0]).solutions_iterator()))
[[0, 1]]
```

Some other examples:

```
sage: sorted(map(sorted, d.restrict([1]).solutions_iterator()))
[[0, 1]]
sage: sorted(map(sorted, d.restrict([2]).solutions_iterator()))
[[2, 3]]
sage: sorted(map(sorted, d.restrict([3]).solutions_iterator()))
[[2, 3]]
```

Here there are no solution using both 0th and 3rd row:

```
sage: list(d.restrict([0,3]).solutions_iterator())
[]
```

```
rows()
Return the list of rows.
EXAMPLES:
```
```
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [1,2], [0]]
sage: x = dlx_solver(rows)
sage: x.rows()
[[0, 1, 2], [1, 2], [0]]
```

```
search()
Search for a new solution.
Return 1 if a new solution is found and 0 otherwise. To recover the solution, use the method get_solution().
EXAMPLES:
```
```
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2]]
sage: rows+= [[0,2]]
sage: rows+= [[1]]
sage: rows+= [[3]]
sage: x = dlx_solver(rows)
sage: print(x.search())
1
sage: print(x.get_solution())
[3, 0]
```
solutions_iterator()
Return an iterator of the solutions.

EXAMPLES:

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
dsage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
dsage: d = dlx_solver(rows)
dsage: sorted(map(sorted, d.solutions_iterator()))
[[[0, 1], [2, 3], [4, 5]]
```

split(column)
Return a dict of independent solvers.

For each i-th row containing a 1 in the column, the dict associates the solver giving all solution using the i-th row.

This is used for parallel computations.

INPUT:
- column – integer, the column used to split the problem into independent subproblems

OUTPUT:
- dict where keys are row numbers and values are dlx solvers

EXAMPLES:

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
dsage: rows = [[0,1,2], [3,4,5], [0,1], [2,3,4,5], [0], [1,2,3,4,5]]
dsage: d = dlx_solver(rows)
dsage: d
Dancing links solver for 6 columns and 6 rows

dsage: sorted(map(sorted, d.solutions_iterator()))
[[[0, 1], [2, 3], [4, 5]]
```

After the split each subproblem has one more column and the same number of rows as the original problem:

```python
sage: D = d.split(0)
dsage: D
{0: Dancing links solver for 7 columns and 6 rows, 2: Dancing links solver for 7 columns and 6 rows, 4: Dancing links solver for 7 columns and 6 rows}
```

The (disjoint) union of the solutions of the subproblems is equal to the set of solutions shown above:

```python
sage: for x in D.values(): sorted(map(sorted, x.solutions_iterator()))
[[[0, 1]]
[[2, 3]]
[[4, 5]]
```

to_milp(solver=None)
Return the mixed integer linear program (MILP) representing an equivalent problem.

See also sage.numerical.mip.MixedIntegerLinearProgram.

INPUT:
• solver – string or None (default: None), possible values include 'GLPK', 'GLPK/exact', 'Coin', 'CPLEX', 'Gurobi', 'CVXOPT', 'PPL', 'InteractiveLP'.

OUTPUT:
• MixedIntegerLinearProgram instance
• MIPVariable with binary components

EXAMPLES:

```
sage: from sage.combinat.matrices.dancing_links import dlx_solver
dsage: rows = [[0,1,2], [0,2], [1], [3]]
dsage: d = dlx_solver(rows)
dsage: p,x = d.to_milp()
dsage: p
Boolean Program (no objective, 4 variables, ... constraints)
dsage: x
MIPVariable with 4 binary components
```

In the reduction, the boolean variable $x_i$ is True if and only if the $i$-th row is in the solution:

```
sage: p.show()
Maximization:

Constraints:...
  one 1 in 0-th column: 1.0 <= x_0 + x_1 <= 1.0
  one 1 in 1-th column: 1.0 <= x_0 + x_2 <= 1.0
  one 1 in 2-th column: 1.0 <= x_0 + x_1 <= 1.0
  one 1 in 3-th column: 1.0 <= x_3 <= 1.0

Variables:
  x_0 is a boolean variable (min=0.0, max=1.0)
  x_1 is a boolean variable (min=0.0, max=1.0)
  x_2 is a boolean variable (min=0.0, max=1.0)
  x_3 is a boolean variable (min=0.0, max=1.0)
```

Using some optional MILP solvers:

```
sage: d.to_milp('gurobi') # optional - gurobi sage_numerical_backends_gurobi
(Boolean Program (no objective, 4 variables, 4 constraints), MIPVariable with 4 binary components)
```

**to_sat_solver**(solver=None)

Return the SAT solver solving an equivalent problem.

Note that row index $i$ in the dancing links solver corresponds to the boolean variable index $i+1$ for the SAT solver to avoid the variable index 0.

See also *sage.sat.solvers.satsolver*.

INPUT:
• solver – string or None (default: None), possible values include 'picosat', 'cryptominisat', 'LP', 'glucose', 'glucose-syrup'.

OUTPUT:
SAT solver instance
**Examples:**

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2], [0,2], [1], [3]]
```

```python
sage: x = dlx_solver(rows)
```

```python
sage: s = x.to_sat_solver()
```

Using some optional SAT solvers:

```python
sage: x.to_sat_solver('cryptominisat')  # optional - pycryptosat
```

CryptoMiniSat solver: 4 variables, 7 clauses.

---

**sage.combinat.matrices.dancing_links.dlx_solver(rows)**

Internal-use wrapper for the dancing links C++ code.

**Examples:**

```python
sage: from sage.combinat.matrices.dancing_links import dlx_solver
sage: rows = [[0,1,2]]
```

```python
sage: rows+= [[0,2]]
```

```python
sage: rows+= [[1]]
```

```python
sage: rows+= [[3]]
```

```python
sage: x = dlx_solver(rows)
```

```python
sage: print(x.search())
```

```
1
```

```python
sage: print(x.get_solution())
```

```
[3, 0]
```

```python
sage: print(x.search())
```

```
1
```

```python
sage: print(x.get_solution())
```

```
[3, 1, 2]
```

```python
sage: print(x.search())
```

```
0
```

**sage.combinat.matrices.dancing_links.make_dlxwrapper(s)**

Create a dlx wrapper from a Python string s.

This was historically used in unpickling and is kept for backwards compatibility. We expect s to be dumps(rows) where rows is the list of rows used to instantiate the object.

---

**5.1.136 Dancing links C++ wrapper**

**sage.combinat.matrices.dlxcpp.AllExactCovers(M)**

Solves the exact cover problem on the matrix M (treated as a dense binary matrix).

**Examples:** No exact covers:

```python
sage: M = Matrix([[1,1,0], [1,0,1], [0,1,1]])
```

```python
sage: [cover for cover in AllExactCovers(M)]
```

```
[]
```

Two exact covers:
sage: M = Matrix([[1,1,0],[1,0,1],[0,0,1],[0,1,0]])
sage: [cover for cover in AllExactCovers(M)]

[[[1, 1, 0], (0, 0, 1)], [[1, 0, 1], (0, 1, 0)]]

sage.combinat.matrices.dlxcpp.DLXCPP(rows)
Solves the Exact Cover problem by using the Dancing Links algorithm described by Knuth.

Consider a matrix \( M \) with entries of 0 and 1, and compute a subset of the rows of this matrix which sum to the vector of all 1's.

The dancing links algorithm works particularly well for sparse matrices, so the input is a list of lists of the form:

\[
[ \\
[ i_{11}, i_{12}, \ldots, i_{1r} ] \\
\ldots \\
[ i_{m1}, i_{m2}, \ldots, i_{ms} ] \\
]
\]

where \( M[j][i_{jk}] = 1 \).

The first example below corresponds to the matrix:

\[
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

which is exactly covered by:

\[
\begin{array}{cccc}
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\]

If soln is a solution given by DLXCPP(rows) then
\[
[ \text{rows[soln[0]]}, \text{rows[soln[1]]}, \ldots, \text{rows[soln[len(soln)-1]]} ]
\]

is an exact cover.

Solutions are given as a list.

EXAMPLES:

```
sage: rows = [[[0,1,2]]
sage: rows+= [[[0,2]]
sage: rows+= [[[1]]
sage: rows+= [[[3]]
sage: [x for x in DLXCPP(rows)]
[[[3, 0], [3, 1, 2]]]
```
EXAMPLES:

```python
sage: M = Matrix([[1,1,0],[1,0,1],[0,1,1]])  # no exact covers
sage: print(OneExactCover(M))
None
sage: M = Matrix([[1,1,0],[1,0,1],[0,0,1],[0,1,0]])  # two exact covers
sage: OneExactCover(M)
[(1, 1, 0), (0, 0, 1)]
```

5.1.137 Hadamard matrices

A Hadamard matrix is an $n \times n$ matrix $H$ whose entries are either $+1$ or $-1$ and whose rows are mutually orthogonal. For example, the matrix $H_2$ defined by

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\]

is a Hadamard matrix. An $n \times n$ matrix $H$ whose entries are either $+1$ or $-1$ is a Hadamard matrix if and only if:

(a) $|\det(H)| = n^{n/2}$ or

(b) $H = H^\dagger = n \cdot I_n$, where $I_n$ is the identity matrix.

In general, the tensor product of an $m \times m$ Hadamard matrix and an $n \times n$ Hadamard matrix is an $(mn) \times (mn)$ matrix. In particular, if there is an $n \times n$ Hadamard matrix then there is a $(2n) \times (2n)$ Hadamard matrix (since one may tensor with $H_2$). This particular case is sometimes called the Sylvester construction.

The Hadamard conjecture (possibly due to Paley) states that a Hadamard matrix of order $n$ exists if and only if $n = 1$, 2 or $n$ is a multiple of 4.

The module below implements constructions of Hadamard and skew Hadamard matrices for all known orders $\leq 1000$, plus some more greater than 1000. It also allows you to pull a Hadamard matrix from the database at [SloaHada].

The following code will test that a construction for all known orders $\leq 4k$ is implemented. The assertion above can be verified by setting $k=250$ (note that it will take a long time to run):

```python
sage: from sage.combinat.matrices.hadamard_matrix import (hadamard_matrix,
    ....: skew_hadamard_matrix, is_hadamard_matrix,
    ....: is_skew_hadamard_matrix)

sage: k = 20
sage: unknown_hadamard = [668, 716, 892]
```

AUTHORS:

- David Joyner (2009-05-17): initial version
- Matteo Cati (2023-03-18): implemented more constructions for Hadamard and skew Hadamard matrices, to cover all known orders up to 1000.
REFERENCES:

• [SloaHada]
• [HadaWiki]
• [Hora]

sage.combinat.matrices.hadamard_matrix.GS_skew_hadamard_smallcases(n, existence=False, check=True)

Data for Williamson-Goethals-Seidel construction of skew Hadamard matrices

Here we keep the data for this construction. Namely, it needs 4 circulant matrices with extra properties, as described in sage.combinat.matrices.hadamard_matrix. williamson_goethals_seidel_skew_hadamard_matrix(). Matrices are taken from:

- \( n = 36, 52 \): [GS70s]
- \( n = 92 \): [Wall71]
- \( n = 188 \): [Djo2008a]
- \( n = 236 \): [FKS2004]
- \( n = 276 \): [Djo2023a]

Additional data is obtained from skew supplementary difference sets contained in sage.combinat.designs. difference_family.skew_supplementary_difference_set(), using the construction described in [Djo1992a].

INPUT:

• \( n \) – integer; the order of the matrix
• \( \text{existence} \) – boolean (default: True); if True, only check that we can do the construction
• \( \text{check} \) – boolean (default: False): if True, check the result

sage.combinat.matrices.hadamard_matrix.RSHCD_324(\( e \))

Return a size 324x324 Regular Symmetric Hadamard Matrix with Constant Diagonal.

We build the matrix \( M \) for the case \( n = 324, \epsilon = 1 \) directly from JankoKharaghaniTonchevGraph and for the case \( \epsilon = -1 \) from the “twist” \( M' \) of \( M \), using Lemma 11 in [HX2010]. Namely, it turns out that the matrix

\[
M' = \begin{pmatrix}
M_{12} & M_{11} \\
M_{11}^T & M_{21}
\end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix},
\]

and the \( M_{ij} \) are 162x162-blocks, also RSHCD, its diagonal blocks having zero row sums, as needed by [loc.cit.]. Interestingly, the corresponding (324, 152, 70, 72)-strongly regular graph has a vertex-transitive automorphism group of order 2592, twice the order of the (intransitive) automorphism group of the graph corresponding to \( M \). Cf. [CP2016].

INPUT:

• \( e = -1 \) or \( +1 \); the value of \( \epsilon \)

REFERENCE:

• [CP2016]

sage.combinat.matrices.hadamard_matrix.amicable_hadamard_matrices(n, existence=False, check=True)

Construct amicable Hadamard matrices of order \( n \) using the available methods.

INPUT:
• \(n\) – positive integer; the order of the amicable Hadamard matrices
• existence – boolean (default: \texttt{False}); if \texttt{True}, only return whether amicable Hadamard matrices of order \(n\) can be constructed
• check – boolean (default: \texttt{True}); if \texttt{True}, check that the matrices are amicable Hadamard matrices before returning them

OUTPUT:
If existence is true, the function returns a boolean representing whether amicable Hadamard matrices of order \(n\) can be constructed. If existence is false, returns two amicable Hadamard matrices, or raises an error if the matrices cannot be constructed.

EXAMPLES:
```
sage: from sage.combinat.matrices.hadamard_matrix import amicable_hadamard_matrices
sage: amicable_hadamard_matrices(2)
(\[
[ 1 1 ]
[ 1 1 ]
[-1 1] [ 1 -1 ]
\])
```
If existence is true, the function returns a boolean:
```
sage: amicable_hadamard_matrices(16, existence=True)
False
```

\texttt{sage.combinat.matrices.hadamard_matrix.amicable_hadamard_matrices_wallis}(n, check=True)
Construct amicable Hadamard matrices of order \(n = q + 1\) where \(q\) is a prime power.
If \(q\) is a prime power \(\equiv 3 \mod 4\), then amicable Hadamard matrices of order \(q + 1\) can be constructed as described in [Wal1970b].

INPUT:
• \(n\) – integer; the order of the matrices to be constructed
• check – boolean (default: \texttt{True}); if \texttt{True}, check that the resulting matrices are amicable Hadamard before returning them

OUTPUT:
The function returns two amicable Hadamard matrices, or raises an error if such matrices cannot be created using this construction.

EXAMPLES:
```
sage: from sage.combinat.matrices.hadamard_matrix import amicable_hadamard_matrices_wallis
sage: M, N = amicable_hadamard_matrices_wallis(28)
sage.combinat.matrices.hadamard_matrix.amicable_hadamard_matrices_wallis
```

\texttt{sage.combinat.matrices.hadamard_matrix.are_amicable_hadamard_matrices}(M, N, verbose=False)
Check if \(M\) and \(N\) are amicable Hadamard matrices.
Two matrices \(M\) and \(N\) of order \(n\) are called amicable if they satisfy the following conditions (see [Seb2017]):
• \(M\) is a skew Hadamard matrix
• \(N\) is a symmetric Hadamard matrix
• \(MN^T = NM^T\)
INPUT:

- M – a square matrix
- N – a square matrix
- verbose – boolean (default False); whether to be verbose when the matrices are not amicable Hadamard matrices

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import are_amicable_hadamard_matrices
sage: M = matrix([[1, 1], [-1, 1]])
sage: N = matrix([[1, 1], [1, -1]])
sage: are_amicable_hadamard_matrices(M, N)
True
```

If verbose is true, the function will be verbose when returning False:

```python
sage: N = matrix([[1, 1], [1, 1]])
sage: are_amicable_hadamard_matrices(M, N, verbose=True)
The second matrix is not Hadamard
False
```

```
sage.combinat.matrices.hadamard_matrix.construction_four_symbol_delta_code_I(X, Y, Z, W)
Construct 4-symbol δ code of length 2n + 1.

The 4-symbol δ code is constructed from sequences X, Y, Z, W of length \( n + 1 \), \( n + 1 \), \( n \), \( n \) satisfying for all \( s > 0 \):

\[
N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = 0
\]

where \( N_A(s) \) is the nonperiodic correlation function:

\[
N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}
\]

The construction (detailed in [Tur1974]) is as follows:

- \( T_1 = X; Z \)
- \( T_2 = X; -Z \)
- \( T_3 = Y; W \)
- \( T_4 = Y; -W \)

INPUT:

- X – list; the first sequence (length \( n + 1 \))
- Y – list; the second sequence (length \( n + 1 \))
- Z – list; the third sequence (length \( n \))
- W – list; the fourth sequence (length \( n \))

OUTPUT:

A tuple containing the 4-symbol δ code of length 2n + 1.

EXAMPLES:
sage: from sage.combinat.matrices.hadamard_matrix import construction_four_symbol_delta_code_I
sage: construction_four_symbol_delta_code_I([1, 1], [1, -1], [1], [1])
([1, 1, 1], [1, 1, -1], [1, -1, 1], [1, -1, -1])

sage.combinat.matrices.hadamard_matrix.construction_four_symbol_delta_code_II(X, Y, Z, W)
Construct 4-symbol δ code of length 4n + 3.
The 4-symbol δ code is constructed from sequences X, Y, Z, W of length n + 1, n + 1, n, n satisfying for all s > 0:

\[ N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = 0 \]

where \( N_A(s) \) is the nonperiodic correlation function:

\[ N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s} \]

The construction (detailed in [Tur1974]) is as follows (writing A/B to mean A alternated with B):

\[ T_1 = X/Z; Y/W; 1 \]
\[ T_2 = X/Z; Y/W; -1 \]
\[ T_3 = X/Z; Y/W; 1 \]
\[ T_4 = X/Z; Y/W; -1 \]

INPUT:
- \( X \) – list; the first sequence (length \( n + 1 \))
- \( Y \) – list; the second sequence (length \( n + 1 \))
- \( Z \) – list; the third sequence (length \( n \))
- \( W \) – list; the fourth sequence (length \( n \))

OUTPUT:
A tuple containing the four 4-symbol δ code of length 4n + 3.

EXAMPLES:

sage: from sage.combinat.matrices.hadamard_matrix import construction_four_symbol_delta_code_II
sage: construction_four_symbol_delta_code_II([1, 1], [1, -1], [1], [1])
([1, 1, 1, 1, 1, -1, 1], [1, 1, 1, 1, -1, -1, -1], [1, 1, 1, -1, -1, 1, 1], [1, 1, 1, -1, 1, 1, -1])

sage.combinat.matrices.hadamard_matrix.four_symbol_delta_code_smallcases(n, existence=False)
Return the 4-symobl δ code of length \( n \) if available.
The 4-symbol δ codes are constructed using \( \text{construction_four_symbol_delta_code_I()} \) or \( \text{construction_four_symbol_delta_code_II()} \). The base sequences used are taken from [Tur1974].

INPUT:
- \( n \) – integer; the length of the desired 4-symbol δ code
sage.combinat.matrices.hadamard_matrix.hadamard_matrix(n, existence=False, check=True)

This function is available as hadamard_matrix(...) and matrix.hadamard(...).

Tries to construct a Hadamard matrix using the available methods.

Currently all orders ≤ 1000 for which a construction is known are implemented. For n > 1000, only some orders are available.

INPUT:

• n – integer; dimension of the matrix

• existence – boolean (default: False); whether to build the matrix or merely query if a construction is available in Sage. When set to True, the function returns:
  – True – meaning that Sage knows how to build the matrix
  – Unknown – meaning that Sage does not know how to build the matrix, although the matrix may exist (see sage.misc.unknown).
  – False – meaning that the matrix does not exist.

• check – boolean (default: True); whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed.

EXAMPLES:

sage: hadamard_matrix(12).det()
2985984
sage: 12^6
2985984
sage: hadamard_matrix(1)
[1]
sage: hadamard_matrix(2)
[ 1 1
 [ 1 -1]
sage: hadamard_matrix(8) # random
[ 1 1 1 1 1 1 1 1
 [ 1 -1 1 -1 1 -1 1 -1]
 [ 1 1 -1 -1 1 1 -1 -1]
 [ 1 -1 -1 1 1 -1 -1 1]
 [ 1 1 1 1 -1 -1 -1 -1]
 [ 1 -1 1 -1 -1 1 -1 1]
 [ 1 1 -1 -1 -1 1 1 1]
 [ 1 -1 -1 1 1 1 -1]
sage: hadamard_matrix(8).det() == 8^4
True
We note that `hadamard_matrix()` returns a normalised Hadamard matrix (the entries in the first row and column are all +1).

```python
sage: hadamard_matrix(12) # random
[ 1  1| 1  1| 1  1| 1  1| 1  1| 1  1]
[ 1 -1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
[-------------------------+-------------------------]
[ 1 -1| 1  1| 1  1| 1  1| 1  1| 1  1]
[ 1  1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
[-------------------------+-------------------------]
[ 1 -1| 1  1| 1  1| 1  1| 1  1| 1  1]
[ 1  1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
[-------------------------+-------------------------]
[ 1 -1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
[ 1  1| 1  1| 1  1| 1  1| 1  1| 1  1]
[-------------------------+-------------------------]
[ 1 -1| 1  1| 1  1| 1  1| 1  1| 1  1]
[ 1  1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
[-------------------------+-------------------------]
[ 1 -1| 1  1| 1  1| 1  1| 1  1| 1  1]
[ 1  1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
[-------------------------+-------------------------]
[ 1 -1| 1  1| 1  1| 1  1| 1  1| 1  1]
[ 1  1| 1 -1| 1 -1| 1 -1| 1 -1| 1 -1]
```

`sage.combinat.matrices.hadamard_matrix.hadamard_matrix_156()`

Construct an Hadamard matrix of order 156.

The matrix is created using the construction detailed in [BH1965]. This uses four circulant matrices of size $13 \times 13$, which are composed into a $156 \times 156$ block matrix.

`sage.combinat.matrices.hadamard_matrix.hadamard_matrix_cooper_wallis_construction(x1, x2, x3, x4, A, B, C, D, check=True)`

Create an Hadamard matrix using the construction detailed in [CW1972].

Given four circulant matrices $X_1, X_2, X_3, X_4$ of order $n$ with entries $(0, 1, -1)$ such that the entrywise product of two distinct matrices is always equal to 0 and that $\sum_{i=1}^{4} X_i X_i^\top = n I_n$ holds, and four matrices $A, B, C, D$ of order $m$ with elements $(1, -1)$ such that $MN^\top = NM^\top$ for all distinct $M, N$ and $AA^\top + BB^\top + CC^\top + DD^\top = 4m I_m$ holds, we construct an Hadamard matrix of order $4nm$.

**INPUT:**

- `x1` – list or vector; the first row of the circulant matrix $X_1$
- `x2` – list or vector; the first row of the circulant matrix $X_2$
- `x3` – list or vector; the first row of the circulant matrix $X_3$
- `x4` – list or vector; the first row of the circulant matrix $X_4$
- `A` – the matrix described above
- `B` – the matrix described above
- `C` – the matrix described above
- `D` – the matrix described above
- `check` – boolean (default: `True`); if `True`, check that the resulting matrix is Hadamard before returning it.

**EXAMPLES:**
Construct Hadamard matrices using the Cooper-Wallis construction for some small values of \( n \).

This function calls the function `hadamard_matrix_cooper_wallis_construction()` with the appropriate arguments. It constructs the matrices \( X_1, X_2, X_3, X_4 \) using either T-matrices or the T-sequences from `sage.combinat.t_sequences.T_sequences_smallcases()`. The matrices \( A, B, C, D \) are taken from `williamson_type_quadruples_smallcases()`.

Data for T-matrices of order 67 is taken from [Saw1985].

**INPUT:**
- \( n \) – integer; the order of the matrix to be constructed
- \( \text{check} \) – boolean (default: True); if True, check that the matrix is an Hadamard matrix before returning
- \( \text{existence} \) – boolean (default: False); if True, only check if the matrix exists.

**OUTPUT:**
If \( \text{existence}=False \), returns the Hadamard matrix of order \( n \). It raises an error if no data is available to construct the matrix of the given order. If \( \text{existence}=True \), returns a boolean representing whether the matrix can be constructed or not.

**See also:**
`hadamard_matrix_cooper_wallis_construction()`

**EXAMPLES:**

By default, the function returns the Hadamard matrix

```python
sage: hadamard_matrix_cooper_wallis_smallcases(28)
28 x 28 dense matrix over Integer Ring...
```

If \( \text{existence} \) is set to True, the function returns a boolean

```python
sage: hadamard_matrix_cooper_wallis_smallcases(20, existence=True)
True
```

**Construction of Hadamard matrices from supplementary difference sets.**

Given four SDS with parameters \( 4 - \{n; n_1, n_2, n_3, n_4; \lambda \} \) with \( n_1 + n_2 + n_3 + n_4 = n + \lambda \) we can construct four \((-1,+1)\) sequences \( a_i = (a_{i,0}, ..., a_{i,n-1}) \) where \( a_{i,j} = -1 \) iff \( j \in S_i \). These will be the first rows of four circulant matrices \( A_1, A_2, A_3, A_4 \) which, when plugged into the Goethals-Seidel array, create an Hadamard matrix of order \( 4n \) (see [Djo1994b]).
The supplementary difference sets are taken from `sage.combinat.designs.difference_family.supplementary_difference_set()`.

**INPUT:**
- `n` – integer; the order of the matrix to be constructed
- `check` – boolean (default: `True`); if `True`, check that the matrix is a Hadamard before returning
- `existence` – boolean (default: `False`); if `True`, only check if the matrix exists

**OUTPUT:**
If `existence=False`, returns the Hadamard matrix of order `n`. It raises an error if no data is available to construct the matrix of the given order, or if `n` is not a multiple of 4. If `existence=True`, returns a boolean representing whether the matrix can be constructed or not.

**EXAMPLES:**
By default The function returns the Hadamard matrix

```python
sage: from sage.combinat.matrices.hadamard_matrix import hadamard_matrix_from_sds
sage: hadamard_matrix_from_sds(148)
```

```python
148 x 148 dense matrix over Integer Ring...
```

If `existence` is set to `True`, the function returns a boolean

```python
sage: hadamard_matrix_from_sds(764, existence=True)
```

```python
True
```

`sage.combinat.matrices.hadamard_matrix.hadamard_matrix_miyamoto_construction(n, existence=False, check=True)`

Construct Hadamard matrix using the Miyamoto construction.

If \( q = n/4 \) is a prime power, and there exists an Hadamard matrix of order \( q - 1 \), then a Hadamard matrix of order \( n \) can be constructed (see [Miy1991]).

**INPUT:**
- `n` – integer; the order of the matrix to be constructed
- `check` – boolean (default: `True`); if `True`, check that the matrix is a Hadamard before returning
- `existence` – boolean (default: `False`); if `True`, only check if the matrix exists

**OUTPUT:**
If `existence=False`, returns the Hadamard matrix of order `n`. It raises an error if no data is available to construct the matrix of the given order, or if `n` does not satisfies the constraints. If `existence=True`, returns a boolean representing whether the matrix can be constructed or not.

**EXAMPLES:**
By default the function returns the Hadamard matrix

```python
sage: from sage.combinat.matrices.hadamard_matrix import hadamard_matrix_miyamoto_construction
sage: hadamard_matrix_miyamoto_construction(20)
```

```python
20 x 20 dense matrix over Integer Ring...
```

If `existence` is set to `True`, the function returns a boolean
sage: hadamard_matrix_miyamoto_construction(36, existence=True)
True

sage.combinat.matrices.hadamard_matrix.hadamard_matrix_paleyI(n, normalize=True)
Implement the Paley type I construction.
The Paley type I case corresponds to the case \( p = n - 1 \equiv 3 \mod 4 \) for a prime power \( p \) (see [Hora]).

INPUT:
- \( n \) – the matrix size
- \( \text{normalize} \) – boolean (default: True); whether to normalize the result

EXAMPLES:
We note that this method by default returns a normalised Hadamard matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1
\end{bmatrix}
\]

Otherwise, it returns a skew Hadamard matrix \( H \), i.e. \( H = S + I \), with \( S = -S^T \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

sage: M = hadamard_matrix_paleyI(4, normalize=False); M
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1
\end{bmatrix}
\]

sage: S = M - identity_matrix(4); -S == S.T

True

sage.combinat.matrices.hadamard_matrix.hadamard_matrix_paleyII(n)
Implement the Paley type II construction.
The Paley type II case corresponds to the case \( p = n/2 - 1 \equiv 1 \mod 4 \) for a prime power \( p \) (see [Hora]).

EXAMPLES:
We note that the method returns a normalised Hadamard matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \ 1 & -1 & 1 & -1 & 1 & -1 \ 1 & -1 & -1 & -1 & 1 & 1 \ 1 & -1 & 1 & 1 & -1 & 1 \\
\end{bmatrix}
\]

(continues on next page)
sage.combinat.matrices.hadamard_matrix.hadamard_matrix_spence_construction(n,  
existence=False,  
check=True)

Create an Hadamard matrix of order \( n \) using the Spence construction.

This construction (detailed in [Spe1975]), uses supplementary difference sets implemented in `sage.combinat.designs.difference_family.supplementary_difference_set_from_rel_diff_set()` to create the desired matrix.

**INPUT:**

- \( n \) – integer; the order of the matrix to be constructed
- \( \text{existence} \) – boolean (default: `False`); if `True`, only check if the matrix exists
- \( \text{check} \) – boolean (default: `True`); if `True`, check that the matrix is an Hadamard matrix before returning

**OUTPUT:**

If `existence=True`, returns a boolean representing whether the Hadamard matrix can be constructed. Otherwise, returns the Hadamard matrix, or raises an error if it cannot be constructed.

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.hadamard_matrix import hadamard_matrix_spence_construction
sage: hadamard_matrix_spence_construction(36)
36 x 36 dense matrix over Integer Ring...
```

If `existence` is `True`, the function returns a boolean

```python
sage: hadamard_matrix_spence_construction(52, existence=True)
True
```

sage.combinat.matrices.hadamard_matrix.hadamard_matrix_turyn_type(a, b, c, d, e1, e2, e3, e4,  
check=True)

Construction of Turyn type Hadamard matrix.

Given \( n \times n \) circulant matrices \( A, B, C, D \) with 1,-1 entries, satisfying \( A A^\top + B B^\top + C C^\top + D D^\top = 4n I \), and a set of Baumert-Hall units of order \( 4t \), one can construct a Hadamard matrix of order \( 4tn \) as detailed by Turyn in [Tur1974].

**INPUT:**

- \( a \) – 1,-1 list; the 1st row of \( A \)
- \( b \) – 1,-1 list; the 1st row of \( B \)
• d – 1,-1 list; the 1st row of C
• c – 1,-1 list; the 1st row of D
• e1 – Matrix; the first Baumert-Hall unit
• e2 – Matrix; the second Baumert-Hall unit
• e3 – Matrix; the third Baumert-Hall unit
• e4 – Matrix; the fourth Baumert-Hall unit
• check – boolean (default: True); whether to check that the output is an Hadamard matrix before returning it

EXAMPLES:

```
sage: from sage.combinat.matrices.hadamard_matrix import hadamard_matrix_turyn_type, _get_baumert_hall_units
sage: A, B, C, D = _get_baumert_hall_units(28)
sage: hadamard_matrix_turyn_type([1], [1], [1], [1], A, B, C, D)
28 x 28 dense matrix over Integer Ring...
```

sage.combinat.matrices.hadamard_matrix.hadamard_matrix_williamson_type(a, b, c, d, check=True)
Construction of Williamson type Hadamard matrix.

Given $n \times n$ circulant matrices $A, B, C, D$ with 1,-1 entries, and satisfying $AA^T + BB^T + CC^T + DD^T = 4nI$,
one can construct a Hadamard matrix of order $4n$, cf. [Ha83].

INPUT:
• a – (1,-1) list; the 1st row of $A$
• b – (1,-1) list; the 1st row of $B$
• d – (1,-1) list; the 1st row of $C$
• c – (1,-1) list; the 1st row of $D$
• check – boolean (default: True); whether to check that the output is an Hadamard matrix before returning it

EXAMPLES:

```
sage: from sage.combinat.matrices.hadamard_matrix import hadamard_matrix_williamson_type
sage: a = [1, 1, 1]
sage: b = [1, -1, -1]
sage: c = [1, -1, -1]
sage: d = [1, -1, -1]
sage: M = hadamard_matrix_williamson_type(a, b, c, d, check=True)
```

sage.combinat.matrices.hadamard_matrix.hadamard_matrix_www(url_file, comments=False)
Pull file from Sloane's database and return the corresponding Hadamard matrix as a Sage matrix.

You must input a filename of the form “had.n.xxx.txt” as described on the webpage http://neilsloane.com/hadamard/, where “xxx” could be empty or a number of some characters.

If comments=True then the “Automorphism…” line of the had.n.xxx.txt file is printed if it exists. Otherwise nothing is done.

EXAMPLES:
```python
sage: hadamard_matrix_www("had.4.txt")  # optional - internet
[ 1 1 1 1]
[ 1 -1 1 -1]
[ 1 1 -1 -1]
[ 1 -1 -1 1]
sage: hadamard_matrix_www("had.16.2.txt", comments=True)  # optional - internet
Automorphism group has order = 49152 = 2^14 * 3
[ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]
[ 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1]
[ 1 1 -1 1 1 -1 1 -1 1 -1 1 -1 1 -1 1 -1]
[ 1 1 1 1 -1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1]
[ 1 -1 1 1 -1 -1 1 -1 1 -1 1 -1 1 -1 1 -1]
[ 1 1 1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1]
[ 1 -1 -1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1]
[ 1 1 1 1 1 1 1 1 1 1 -1 -1 -1 1 1 1]
[ 1 1 1 1 1 -1 -1 -1 -1 1 1 1 -1 -1 -1 -1]
[ 1 1 -1 1 -1 -1 -1 1 1 -1 -1 -1 1 -1 -1 1]
[ 1 1 -1 -1 -1 -1 -1 -1 1 1 -1 -1 1 -1 -1 1]
[ 1 1 -1 -1 -1 1 1 1 -1 -1 1 1 1 -1 -1 1]
[ 1 -1 1 1 -1 1 1 -1 -1 -1 1 1 1 -1 -1 1]
[ 1 1 1 1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 1]
[ 1 -1 -1 1 -1 -1 -1 1 1 -1 -1 1 1 1 -1 -1]
[ 1 1 -1 -1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 1]
[ 1 1 -1 -1 -1 -1 1 1 -1 -1 1 1 1 -1 -1 1]
[ 1 -1 -1 1 1 -1 -1 1 1 -1 1 1 1 -1 -1 1]
[ 1 -1 -1 1 -1 1 -1 -1 -1 1 1 -1 -1 -1 -1 1]
```

sage.combinat.matrices.hadamard_matrix.is_hadamard_matrix(M, normalized=False, skew=False, verbose=False)

Test if $M$ is a Hadamard matrix.

**INPUT:**

- $M$ – a matrix
- normalized – boolean (default: False); whether to test if $M$ is a normalized Hadamard matrix, i.e. has its first row/column filled with $+1$
- skew – boolean (default: False); whether to test if $M$ is a skew Hadamard matrix, i.e. $M = S + I$ for $-S = S^\top$, and $I$ the identity matrix
- verbose – boolean (default: False); whether to be verbose when the matrix is not Hadamard

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.hadamard_matrix import is_hadamard_matrix
sage: h = matrix.hadamard(12)
sage: is_hadamard_matrix(h)
True
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix
sage: h = skew_hadamard_matrix(12)
sage: is_hadamard_matrix(h, skew=True)
True
sage: h = matrix.hadamard(12)
sage: h[0,0] = 2
sage: is_hadamard_matrix(h, verbose=True)
The matrix does not only contain +1 and -1 entries, e.g. 2
False
```
sage: h = matrix.hadamard(12)
sage: for i in range(12):
.....: h[i,2] = -h[i,2]
sage: is_hadamard_matrix(h, verbose=True, normalized=True)
The matrix is not normalized
False

sage.combinat.matrices.hadamard_matrix.is_skew_hadamard_matrix(M, normalized=False, verbose=False)

Test if \( M \) is a skew Hadamard matrix.

this is a wrapper around the function is_hadamard_matrix()

INPUT:

• \( M \) – a matrix

• normalized – boolean (default: False); whether to test if \( M \) is a skew-normalized Hadamard matrix, i.e. has its first row filled with +1

• verbose – boolean (default: False); whether to be verbose when the matrix is not skew Hadamard

EXAMPLES:

sage: from sage.combinat.matrices.hadamard_matrix import is_skew_hadamard_matrix, skew_hadamard_matrix
sage: h = matrix.hadamard(12)
sage: is_skew_hadamard_matrix(h, verbose=True)
The matrix is not skew
False
sage: h = skew_hadamard_matrix(12)
sage: is_skew_hadamard_matrix(h)
True
sage: from sage.combinat.matrices.hadamard_matrix import normalise_hadamard
sage: h = normalise_hadamard(skew_hadamard_matrix(12), skew=True)
sage: is_hadamard_matrix(H, skew=True, normalized=True)
True

sage.combinat.matrices.hadamard_matrix.normalise_hadamard(H, skew=False)

Return the normalised Hadamard matrix corresponding to \( H \).

The normalised Hadamard matrix corresponding to a Hadamard matrix \( H \) is a matrix whose every entry in the first row and column is +1.

If skew is True, the matrix returned will be skew-normal: a skew Hadamard matrix with first row of all +1.

EXAMPLES:

sage: from sage.combinat.matrices.hadamard_matrix import normalise_hadamard, is_hadamard_matrix, skew_hadamard_matrix
sage: H = normalise_hadamard(hadamard_matrix(4))
sage: H == hadamard_matrix(4)
True
sage: H = normalise_hadamard(skew_hadamard_matrix(20), skew=True)
sage: is_hadamard_matrix(H, skew=True, normalized=True)
True
If `skew` is True but the Hadamard matrix is not skew, the matrix returned will not be normalized:

```
sage: H = normalise_hadamard(hadamard_matrix(92), skew=True)
sage: is_hadamard_matrix(H, normalized=True)
False
```

Return a Regular Symmetric Hadamard Matrix with Constant Diagonal.

A Hadamard matrix is said to be regular if its rows all sum to the same value.

For $\epsilon \in \{-1, +1\}$, we say that $M$ is a $(n, \epsilon)$ – $RSHCD$ if $M$ is a regular symmetric Hadamard matrix with constant diagonal $\delta \in \{-1, +1\}$ and row sums all equal to $\delta \epsilon \sqrt{n}$. For more information, see [HX2010] or 10.5.1 in [BH2012]. For the case $n = 324$, see `RSHCD_324()` and [CP2016].

**INPUT:**

- `n` – integer; side of the matrix
- `e` – $-1$ or $+1$; the value of $\epsilon$

**EXAMPLES:**

```
sage: from sage.combinat.matrices.hadamard_matrix import regular_symmetric_hadamard_matrix_with_constant_diagonal
sage: regular_symmetric_hadamard_matrix_with_constant_diagonal(4,1)
[ 1 1 1 -1]
[ 1 1 -1 1]
[ 1 -1 1 1]
[ 1 -1 1 1]
sage: regular_symmetric_hadamard_matrix_with_constant_diagonal(4,-1)
[ 1 -1 -1 -1]
[-1 1 -1 -1]
[-1 -1 1 -1]
[-1 -1 -1 1]
```

Other hardcoded values:

```
sage: for n,e in [(36,1),(36,-1),(100,1),(100,-1),(196,1)]:  # long time
....:     print(repr(regular_symmetric_hadamard_matrix_with_constant_diagonal(n,e)))
36 x 36 dense matrix over Integer Ring
36 x 36 dense matrix over Integer Ring
100 x 100 dense matrix over Integer Ring
100 x 100 dense matrix over Integer Ring
196 x 196 dense matrix over Integer Ring
```

```
sage: for n,e in [(324,1),(324,-1)]:  # not tested - long time, tested in `RSHCD_324`
....:
....:     print(repr(regular_symmetric_hadamard_matrix_with_constant_diagonal(n,e)))
324 x 324 dense matrix over Integer Ring
324 x 324 dense matrix over Integer Ring
```

From two close prime powers:
sage: regular_symmetric_hadamard_matrix_with_constant_diagonal(64,-1)
64 x 64 dense matrix over Integer Ring (use the '.str()' method to see the entries)

From a prime power and a conference matrix:

sage: regular_symmetric_hadamard_matrix_with_constant_diagonal(676,1)  # long time
676 x 676 dense matrix over Integer Ring (use the '.str()' method to see the entries)

Recursive construction:

sage: regular_symmetric_hadamard_matrix_with_constant_diagonal(144,-1)
144 x 144 dense matrix over Integer Ring (use the '.str()' method to see the entries)

REFERENCE:

• [BH2012]
• [HX2010]

sage.combinat.matrices.hadamard_matrix.rshcd_from_close_prime_powers(n)
Return a \((n^2,1)\)-RSHCD when \(n - 1\) and \(n + 1\) are odd prime powers and \(n = 0 \pmod{4}\).

The construction implemented here appears in Theorem 4.3 from [GS1970].

Note that the authors of [SWW1972] claim in Corollary 5.12 (page 342) to have proved the same result without the \(n = 0 \pmod{4}\) restriction with a very similar construction. So far, however, I (Nathann Cohen) have not been able to make it work.

INPUT:

• \(n\) – an integer congruent to 0 \(\pmod{4}\)

See also:

regular_symmetric_hadamard_matrix_with_constant_diagonal()

EXAMPLES:

sage: from sage.combinat.matrices.hadamard_matrix import rshcd_from_close_prime_powers
sage: rshcd_from_close_prime_powers(4)
[-1 -1 1 -1 1 -1 -1 1 -1 1 -1 1 -1 1 -1 1 -1]
[-1 -1 -1 1 1 -1 -1 1 -1 -1 1 -1 1 -1 -1 1 -1]
[ 1 -1 -1 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 1 -1]
[ 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 1]
[-1 -1 -1 1 1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[ 1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 1 -1]
[-1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[-1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[ 1 -1 -1 -1 -1 -1 -1 -1 1 -1 -1 -1 1 -1 -1 -1 -1]
[ 1 -1 -1 -1 -1 -1 -1 -1 1 -1 -1 -1 1 -1 -1 -1 -1]
[-1 -1 -1 -1 -1 -1 -1 1 -1 -1 -1 -1 1 -1 -1 -1 -1]
[-1 -1 -1 -1 -1 -1 -1 -1 1 -1 -1 -1 1 -1 -1 -1 -1]
[-1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[-1 -1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[ 1 1 -1 -1 -1 -1 -1 -1 1 -1 -1 -1 1 -1 -1 -1 -1]
[-1 -1 -1 -1 -1 -1 -1 -1 1 -1 -1 -1 1 -1 -1 -1 -1]
[ 1 -1 -1 -1 -1 -1 -1 -1 1 1 1 1 1 1 -1 -1 -1]
[-1 1 -1 -1 -1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[-1 -1 -1 -1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[ 1 -1 -1 -1 -1 -1 -1 -1 1 1 1 1 1 1 -1 -1 -1]
[-1 1 -1 -1 -1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
[ 1 -1 -1 -1 -1 -1 -1 -1 1 1 1 1 1 1 -1 -1 -1]
[-1 1 -1 -1 -1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1]
REFERENCE:

- [SWW1972]

```
sage.combinat.matrices.hadamard_matrix.rshcd_from_prime_power_and_conference_matrix(n)
```

Return a \((n - 1)^2, 1\)-RSHCD if \(n\) is prime power, and symmetric \((n - 1)\)-conference matrix exists.

The construction implemented here is Theorem 16 (and Corollary 17) from [WW1972].

In [WW1972] this construction (Theorem 5.15 and Corollary 5.16) is reproduced with a typo. Note that [WW1972] refers to [Sz1969] for the construction, provided by `szekeres_difference_set_pair()`, of complementary difference sets, and the latter has a typo.

From a `symmetric_conference_matrix()`, we only need the Seidel adjacency matrix of the underlying strongly regular conference (i.e. Paley type) graph, which we construct directly.

**INPUT:**

- \(n\) – an integer

**See also:**

`regular_symmetric_hadamard_matrix_with_constant_diagonal()`

**EXAMPLES:**

A 36x36 example

```
sage: from sage.combinat.matrices.hadamard_matrix import rshcd_from_prime_power_and_conference_matrix
sage: from sage.combinat.matrices.hadamard_matrix import is_hadamard_matrix
sage: H = rshcd_from_prime_power_and_conference_matrix(7); H
36 x 36 dense matrix over Integer Ring (use the '.str()' method to see the entries)
```

```
sage: H == H.T and is_hadamard_matrix(H) and H.diagonal() == [1]*36 and list(sum(H)) == [6]*36
True
```

Bigger examples, only provided by this construction

```
sage: H = rshcd_from_prime_power_and_conference_matrix(27) # long time
sage: H == H.T and is_hadamard_matrix(H)                     # long time
True
`sage: H.diagonal() == [1]*676 and list(sum(H)) == [26]*676   # long time
True
```

In this example the conference matrix is not Paley, as 45 is not a prime power

```
sage: H = rshcd_from_prime_power_and_conference_matrix(47) # not tested (long time)
```

**REFERENCE:**

- [WW1972]

```
sage.combinat.matrices.hadamard_matrix.skew_hadamard_matrix(n, existence=False, skew_normalize=True, check=True)
```

Tries to construct a skew Hadamard matrix.
A Hadamard matrix $H$ is called skew if $H = S - I$, for $I$ the identity matrix and $-S = S^T$. Currently all orders $\leq 1000$ for which a construction is known are implemented. For $n > 1000$, only some orders are available.

**INPUT:**

- **n** – integer; dimension of the matrix
- **existence** – boolean (default: False); whether to build the matrix or merely query if a construction is available in Sage. When set to True, the function returns:
  - **True** – meaning that Sage knows how to build the matrix
  - **Unknown** – meaning that Sage does not know how to build the matrix, but that the design may exist (see sage.misc.unknown).
  - **False** – meaning that the matrix does not exist.
- **skew_normalize** – boolean (default: True); whether to make the 1st row all-one, and adjust the 1st column accordingly
- **check** – boolean (default: True); whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix
sage: skew_hadamard_matrix(12).det()
2985984
sage: 12^6
2985984
sage: skew_hadamard_matrix(1)
[1]
sage: skew_hadamard_matrix(2)
[ 1  1]
[ -1  1]
```

**REFERENCES:**

- [Ha83]

sage.combinat.matrices.hadamard_matrix.skew_hadamard_matrix_from_complementary_difference_sets($n$, $\text{existence}=$False, $\text{check}=$True)

Construct a skew Hadamard matrix of order $n = 4(m + 1)$ from complementary difference sets.

If $A, B$ are complementary difference sets over a group of order $2m + 1$, then they can be used to construct a skew Hadamard matrix, as described in [BS1969].

The complementary difference sets are constructed using the function sage.combinat.designs.difference_family.complementary_difference_sets().

**INPUT:**

- **n** – positive integer; the order of the matrix to be constructed
- **existence** – boolean (default: False); if True, only return whether the skew Hadamard matrix can be constructed
- **check** – boolean (default: True); if True, check that the result is a skew Hadamard matrix before returning it
OUTPUT:

If `existence=False`, returns the skew Hadamard matrix of order `n`. It raises an error if `n` does not satisfy the required conditions. If `existence=True`, returns a boolean representing whether the matrix can be constructed or not.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_from_complementary_difference_sets
sage: skew_hadamard_matrix_from_complementary_difference_sets(20)
20 x 20 dense matrix over Integer Ring...
sage: skew_hadamard_matrix_from_complementary_difference_sets(52, existence=True)
True
```

```python
sage.combinat.matrices.hadamard_matrix.skew_hadamard_matrix_from_good_matrices(a, b, c, d, check=True)
```

Construct skew Hadamard matrix from good matrices.

Given good matrices `A, B, C, D` (`A` circulant, `B, C, D` back-circulant) they can be used to construct a skew Hadamard matrix using the following block matrix (as described in [Sze1988]):

\[
\begin{pmatrix}
A & B & C & D \\
-B & A & D & -C \\
-C & -D & A & B \\
-D & C & -B & A
\end{pmatrix}
\]

INPUT:

- `a` – `(1,-1)` list; the 1st row of `A`
- `b` – `(1,-1)` list; the 1st row of `B`
- `d` – `(1,-1)` list; the 1st row of `C`
- `c` – `(1,-1)` list; the 1st row of `D`
- `check` – boolean (default: True); if True, check that the matrix is a skew Hadamard matrix before returning it

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_from_good_matrices
sage: a, b, c, d = ([1, 1, 1, -1, -1], [1, -1, 1, 1, -1], [1, -1, -1, -1, -1], [1, -1, -1, -1, -1])
sage: skew_hadamard_matrix_from_good_matrices(a, b, c, d)
20 x 20 dense matrix over Integer Ring...
```

```python
sage.combinat.matrices.hadamard_matrix.skew_hadamard_matrix_from_good_matrices_smallcases(n, existence=False, check=True)
```

Construct skew Hadamard matrices from good matrices for some small values of `n = 4m`, with `m` odd.

The function stores good matrices of odd orders \( \leq 31 \), taken from [Sze1988]. These are used to create skew Hadamard matrices of order \( 4m, 1 \leq m \leq 31 \) (\( m \) odd), using the function `skew_hadamard_matrix_from_good_matrices()`.

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ALGORITHM:

Given four sequences (stored in \(E_{\text{sequences}}\)) of length \(m\), they can be used to construct four \(E\) \(-\) \(sequences\) of length \(n = 2m + 1\), as follows:

\[
\begin{align*}
  a &= 1, a_0, a_1, ..., a_m, -a_m, -a_{m-1}, ..., -a_0 \\
  b &= 1, b_0, b_1, ..., b_m, b_{m-1}, ..., b_0 \\
  c &= 1, c_0, c_1, ..., c_m, c_{m-1}, ..., c_0 \\
  d &= 1, d_0, d_1, ..., d_m, d_{m-1}, ..., d_0
\end{align*}
\]

These E-sequences will be the first rows of the four good matrices needed to construct a skew Hadamard matrix of order \(4n\).

INPUT:

- \(n\) – integer; the order of the skew Hadamard matrix to be constructed
- \(\text{existence}\) – boolean (default: False); If True, only return whether the Hadamard matrix can be constructed
- \(\text{check}\) – boolean (default: True); if True, check that the matrix is an Hadamard matrix before returning it

OUTPUT:

If \(\text{existence}=\)False, returns the skew Hadamard matrix of order \(n\). It raises an error if no data is available to construct the matrix of the given order. If \(\text{existence}=\)True, returns a boolean representing whether the matrix can be constructed or not.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_from_good_matrices_smallcases
sage: skew_hadamard_matrix_from_good_matrices_smallcases(20)
20 x 20 dense matrix over Integer Ring...
sage: skew_hadamard_matrix_from_good_matrices_smallcases(20, existence=True)
True
```

Construct skew Hadamard matrices of order \(mn(n - 1)\) if suitable orthogonal designs exist.

In [Seb1978] is proved that if amicable Hadamard matrices of order \(n\) and an orthogonal design of type \((1, m, mn - m - 1)\) in order \(mn\) exist, then a skew Hadamard matrix of order \(mn(n - 1)\) can be constructed. The paper uses amicable orthogonal designs instead of amicable Hadamard matrices, but the two are equivalent (see [Seb2017]).

Amicable Hadamard matrices are constructed using \texttt{amicable_hadamard_matrices()}. and the orthogonal designs are constructed using the Goethals-Seidel array, with data taken from [Seb2017].

INPUT:

- \(n\) – positive integer; the order of the matrix to be constructed
- \(\text{existence}\) – boolean (default: False); if True, only return whether the skew Hadamard matrix can be constructed
- \(\text{check}\) – boolean (default: True); if True, check that the result is a skew Hadamard matrix before returning it
OUTPUT:
If \( \text{existence=False} \), returns the skew Hadamard matrix of order \( n \). It raises an error if a construction for order \( n \) is not yet implemented, or if \( n \) does not satisfy the constraint. If \( \text{existence=True} \), returns a boolean representing whether the matrix can be constructed or not.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_from_orthogonal_design
sage: skew_hadamard_matrix_from_orthogonal_design(756)
756 x 756 dense matrix over Integer Ring...
```

If \( \text{existence is True} \), the function returns a boolean:

```python
sage: skew_hadamard_matrix_from_orthogonal_design(200, existence=True)
False
```

sage.combinat.matrices.hadamard_matrix.skew_hadamard_matrix_spence_1975(\( n \), \( \text{existence=False} \), \( \text{check=True} \))

Construct a skew Hadamard matrix of order \( n = 4(1 + q + q^2) \) using the Spence construction.

If \( n = 4(1 + q + q^2) \) where \( q \) is a prime power such that either \( 1 + q + q^2 \) is a prime congruent to 3, 5, 7 mod 8 or \( 3 + 2q + 2q^2 \) is a prime power, then a skew Hadamard matrix of order \( n \) can be constructed using the Goethals Seidel array. The four matrices \( A, B, C, D \) plugged into the GS-array are created using complementary difference sets of order \( 1 + q + q^2 \) (which exist if \( q \) satisfies the conditions above), and a cyclic planar difference set with parameters \( (1 + q^2 + q^4, 1 + q^2, 1) \). These are constructed by the functions sage.combinat.designs.difference_family.complementary_difference_sets() and sage.combinat.designs.difference_family.difference_family().

For more details, see [Spe1975b].

INPUT:
- \( n \) – positive integer; the order of the matrix to be constructed
- \( \text{existence} \) – boolean (default: False); if True, only return whether the Hadamard matrix can be constructed
- \( \text{check} \) – boolean (default: True); check that the result is a skew Hadamard matrix before returning it

OUTPUT:
If \( \text{existence=False} \), returns the skew Hadamard matrix of order \( n \). It raises an error if \( n \) does not satisfy the required conditions. If \( \text{existence=True} \), returns a boolean representing whether the matrix can be constructed or not.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_spence_1975
sage: skew_hadamard_matrix_spence_1975(52, existence=False, check=True)
52 x 52 dense matrix over Integer Ring...
```

If \( \text{existence is True} \), the function returns a boolean:

```python
sage: skew_hadamard_matrix_spence_1975(52, existence=True)
True
```
Construct skew Hadamard matrix of order $n$ using Spence construction.

This function will construct skew Hadamard matrix of order $n = 2(q + 1)$ where $q$ is a prime power with $q = 5 \pmod{8}$. The construction is taken from [Spe1977], and the relative difference sets are constructed using `sage.combinat.designs.difference_family.relative_difference_set_from_homomorphism()`.

INPUT:

- $n$ – positive integer
- check – boolean (default: True); if True, check that the resulting matrix is Hadamard before returning it

OUTPUT:

If $n$ satisfies the requirements described above, the function returns a $n \times n$ Hadamard matrix. Otherwise, an exception is raised.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_spence_construction
sage: skew_hadamard_matrix_spence_construction(28)
28 x 28 dense matrix over Integer Ring...
```

Construct a skew Hadamard matrix of order $n = 2(q + 1)$ where $q = p^t$ is a prime power with $p \equiv 5 \pmod{8}$ and $t \equiv 2 \pmod{4}$.

Assuming $n$ satisfies the conditions above, it is possible to construct two supplementary difference sets $A, B$ (see [Whi1971]), and these can be used to construct a skew Hadamard matrix, as described in [BS1969].

INPUT:

- $n$ – positive integer; the order of the matrix to be constructed
- existence – boolean (default: False); If True, only return whether the Hadamard matrix can be constructed
- check – boolean (default: True); if True, check that the result is a skew Hadamard matrix before returning it

OUTPUT:

If existence=False, returns the skew Hadamard matrix of order $n$. It raises an error if $n$ does not satisfy the required conditions. If existence=True, returns a boolean representing whether the matrix can be constructed or not.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import skew_hadamard_matrix_whiteman_construction
sage: skew_hadamard_matrix_whiteman_construction(52)
52 x 52 dense matrix over Integer Ring...
sage: skew_hadamard_matrix_whiteman_construction(52, existence=True)
True
```
Combinatorics, Release 10.0

Note: A more general version of this construction is \texttt{skew_hadamard_matrix_from_complementary_difference_sets()}. sage.combinat.matrices.hadamard_matrix.symmetric_conference_matrix\(n, \text{check}=\text{True}\)

Tries to construct a symmetric conference matrix

A conference matrix is an \(n \times n\) matrix \(C\) with 0s on the main diagonal and 1s and -1s elsewhere, satisfying \(CC^\top = (n - 1)I\). If \(C = C^\top\) then \(n \equiv 2 \mod 4\) and \(C\) is Seidel adjacency matrix of a graph, whose descendent graphs are strongly regular graphs with parameters \((n - 1, (n - 2)/2, (n - 6)/4, (n - 2)/4)\), see Sec.10.4 of [BH2012]. Thus we build \(C\) from the Seidel adjacency matrix of the latter by adding row and column of 1s.

INPUT:

- \(n\) – integer; dimension of the matrix
- \(\text{check}\) – boolean (default: True); whether to check that output is correct before returning it. As this is expected to be useless (but we are cautious guys), you may want to disable it whenever you want speed

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.matrices.hadamard_matrix import symmetric_conference_matrix sage: C = symmetric_conference_matrix(10); C
\[ 0 1 1 1 1 1 1 1 1 1
 1 0 -1 -1 1 -1 1 1 1 -1
 1 -1 0 -1 1 -1 -1 1 1 1
 1 -1 -1 0 -1 1 1 1 -1 1
 1 1 1 -1 0 -1 -1 1 -1 1
 1 -1 1 1 -1 0 -1 1 1 1
 1 1 -1 1 -1 -1 0 -1 1 1
 1 1 -1 1 1 -1 0 -1 -1 1
 1 1 -1 1 1 1 -1 0 -1 -1
 1 -1 1 1 1 -1 1 -1 1 -1 0\]
sage: C^2 == 9*identity_matrix(10) and C == C.T
True
\end{verbatim}

\begin{verbatim}
sage.combinat.matrices.hadamard_matrix.symmetric_conference_matrix_paley\(n)\)

Construct a symmetric conference matrix of order \(n\).

A conference matrix is an \(n \times n\) matrix \(C\) with 0s on the main diagonal and 1s and -1s elsewhere, satisfying \(CC^\top = (n - 1)I\). This construction assumes that \(q = n - 1\) is a prime power, with \(q \equiv 1 \mod 4\). See [Hora] or [Lon2013].

These matrices are used in \texttt{hadamard_matrix_paleyII()}. INPUT:

- \(n\) – integer; the order of the symmetric conference matrix to construct

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.matrices.hadamard_matrix import symmetric_conference_matrix_paley
sage: symmetric_conference_matrix_paley(6)
\[ 0 1 1 1 1 1
 1 0 -1 -1 -1 1
 1 1 0 1 -1 -1
 1 1 0 1 -1 -1
 1 -1 1 0 1 -1
\]
\end{verbatim}
sage.combinat.matrices.hadamard_matrix.szekeres_difference_set_pair\((m, \text{check}=\text{True})\)

Construct Szekeres \((2m+1, m, 1)\)-cyclic difference family

Let \(4m + 3\) be a prime power. Theorem 3 in [Sz1969] contains a construction of a pair of complementary difference sets \(A, B\) in the subgroup \(G\) of the quadratic residues in \(F_{4m+3}^*\). Namely \(|A| = |B| = m\), \(a \in A\) whenever \(a - 1 \in G\), \(b \in B\) whenever \(b + 1 \in G\). See also Theorem 2.6 in [SWW1972] (there the formula for \(B\) is correct, as opposed to (4.2) in [Sz1969], where the sign before 1 is wrong.

In modern terminology, for \(m > 1\) the sets \(A\) and \(B\) form a difference family with parameters \((2m+1, m, 1)\). I.e. each non-identity \(g \in G\) can be expressed uniquely as \(xy^{-1}\) for \(x, y \in A\) or \(x, y \in B\). Other, specific to this construction, properties of \(A\) and \(B\) are: for \(a\) in \(A\) one has \(a - 1\) not in \(A\), whereas for \(b\) in \(B\) one has \(b^{-1}\) in \(B\).

INPUT:

- \(m\) – integer; dimension of the matrix
- \(\text{check}\) – boolean (default: \(\text{True}\)); whether to check \(A\) and \(B\) for correctness

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.matrices.hadamard_matrix import szekeres_difference_set_pair
sage: G,A,B=szekeres_difference_set_pair(6)
sage: G,A,B=szekeres_difference_set_pair(7)
\end{verbatim}

REFERENCE:

- [Sz1969]

sage.combinat.matrices.hadamard_matrix.turyn_type_hadamard_matrix_smallcases\((n, \text{existence}=\text{False, check}=\text{True})\)

Construct an Hadamard matrix of order \(n\) from available 4-symbol \(\delta\) codes and Williamson quadruples.

The function looks for Baumert-Hall units and Williamson type matrices from \texttt{four_symbol_delta_code_smallcases()} and \texttt{williamson_type_quadruples_smallcases()} and use them to construct an Hadamard matrix with the Turyn construction defined in \texttt{hadamard_matrix_turyn_type()}.

INPUT:

- \(n\) – integer; the order of the matrix to be constructed
- \(\text{existence}\) – boolean (default: \(\text{False}\)): if \(\text{True}\), only check if the matrix exists
- \(\text{check}\) – boolean (default: \(\text{True}\)): if \(\text{True}\), check that the matrix is an Hadamard matrix before returning

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.matrices.hadamard_matrix import turyn_type_hadamard_matrix_smallcases
sage: turyn_type_hadamard_matrix_smallcases(28, existence=True)
  True
sage: turyn_type_hadamard_matrix_smallcases(28)
  28 x 28 dense matrix over Integer Ring...
\end{verbatim}
sage.combinat.matrices.hadamard_matrix.typeI_matrix_difference_set(G, A)

(1,-1)-incidence type I matrix of a difference set $A$ in $G$

Let $A$ be a difference set in a group $G$ of order $n$. Return $n \times n$ matrix $M$ with $M_{ij} = 1$ if $A_iA_j^{-1} \in A$, and $M_{ij} = -1$ otherwise.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import szekeres_difference_set_pair
sage: G, A, B = szekeres_difference_set_pair(2)
sage: typeI_matrix_difference_set(G, A)
[-1  1 -1 -1  1]
[-1 -1  1  1  1]
[ 1  1 -1 -1 -1]
[ 1 -1  1  1 -1]
[-1 -1  1  1 -1]
```

sage.combinat.matrices.hadamard_matrix.williamson_goethals_seidel_skew_hadamard_matrix(a, b, c, d, check=True)

Williamson-Goethals-Seidel construction of a skew Hadamard matrix

Given $n \times n$ (anti)circulant matrices $A, B, C, D$ with 1,-1 entries, and satisfying $A + A^\top = 2I$, $AA^\top + BB^\top + CC^\top + DD^\top = 4nI$, one can construct a skew Hadamard matrix of order $4n$, cf. [GS70s].

INPUT:

- $a$ – 1,-1 list; the 1st row of $A$
- $b$ – 1,-1 list; the 1st row of $B$
- $c$ – 1,-1 list; the 1st row of $C$
- $d$ – 1,-1 list; the 1st row of $D$
- check – boolean (default: True); if True, check that the resulting matrix is skew Hadamard before returning it

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import williamson_goethals_seidel_skew_hadamard_matrix as WGS
sage: a = [ 1, 1, 1, -1, 1, -1, -1, 1, -1]
sage: b = [ 1, -1, 1, 1, -1, -1, 1, 1, -1]
sage: c = [-1, -1]+[1]*6+[1]
sage: d = [ 1, 1, 1, -1, 1, 1, -1, 1, 1]
sage: M = WGS(a, b, c, d, check=True)
```

REFERENCES:

- [GS70s]
- [Wall71]
- [KoSt08]
sage.combinat.matrices.hadamard_matrix.williamson_hadamard_matrix_smallcases(n, existence=False, check=True)

Construct Williamson type Hadamard matrices for some small values of n.

This function uses the data contained in sage.combinat.matrices.hadamard_matrix.williamson_type_quadruples_smallcases() to create Hadamard matrices of the Williamson type, using the construction from sage.combinat.matrices.hadamard_matrix.hadamard_matrix_williamson_type().

INPUT:

• n – integer; the order of the matrix
• existence – boolean (default: False); if True, only check that we can do the construction
• check – boolean (default: True); if True check the result

sage.combinat.matrices.hadamard_matrix.williamson_type_quadruples_smallcases(n, existence=False)

Quadruples of matrices that can be used to construct Williamson type Hadamard matrices.

This function contains for some values of n, four \( n \times n \) matrices used in the Williamson construction of Hadamard matrices. Namely, the function returns the first row of 4 \( n \times n \) circulant matrices with the properties described in sage.combinat.matrices.hadamard_matrix.hadamard_matrix_williamson_type(). The matrices for \( n = 3, 5, \ldots, 29, 37, 43 \) are given in [Ha83]. The matrices for \( n = 31, 33, 39, 41, 45, 49, 51, 55, 57, 61, 63 \) are given in [Lon2013].

INPUT:

• n – integer; the order of the matrices to be returned
• existence – boolean (default: False); if True, only check that we have the quadruple

OUTPUT:

If existence is false, returns a tuple containing four vectors, each being the first line of one of the four matrices. It raises an error if no such matrices are available. If existence is true, returns a boolean representing whether the matrices are available or not.

EXAMPLES:

```python
sage: from sage.combinat.matrices.hadamard_matrix import williamson_type_quadruples_smallcases
sage: williamson_type_quadruples_smallcases(29)
((1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1),
 (1, -1, 1, -1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1),
 (1, 1, 1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1),
 (1, 1, 1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1))
```

```
sage: williamson_type_quadruples_smallcases(43, existence=True)
True
```
5.1.138 Latin Squares

A Latin square of order \( n \) is an \( n \times n \) array such that each symbol \( s \in \{0, 1, \ldots, n-1\} \) appears precisely once in each row, and precisely once in each column. A partial Latin square of order \( n \) is an \( n \times n \) array such that each symbol \( s \in \{0, 1, \ldots, n-1\} \) appears at most once in each row, and at most once in each column. Empty cells are denoted by \(-1\). A Latin square \( L \) is a completion of a partial Latin square \( P \) if \( P \subseteq L \). If \( P \) completes to just \( L \) then \( P \) has unique completion.

A Latin bitrade \((T_1, T_2)\) is a pair of partial Latin squares such that:

1. \( \{ (i, j) \mid (i, j, k) \in T_1 \text{ for some symbol } k \} = \{ (i, j) \mid (i, j, k') \in T_2 \text{ for some symbol } k' \}; \)
2. for each \( (i, j, k) \in T_1 \) and \( (i, j, k') \in T_2, k \neq k' \);
3. the symbols appearing in row \( i \) of \( T_1 \) are the same as those of row \( i \) of \( T_2 \); the symbols appearing in column \( j \) of \( T_1 \) are the same as those of column \( j \) of \( T_2 \).

Intuitively speaking, a bitrade gives the difference between two Latin squares, so if \((T_1, T_2)\) is a bitrade for the pair of Latin squares \((L_1, L_2)\), then \( L_1 = (L_2 \setminus T_1) \cup T_2 \) and \( L_2 = (L_1 \setminus T_2) \cup T_1 \).

This file contains:

1. LatinSquare class definition;
2. some named Latin squares (back circulant, forward circulant, abelian 2-group);
3. functions is_partial_latin_square and is_latin_square to test if a LatinSquare object satisfies the definition of a Latin square or partial Latin square, respectively;
4. tests for completion and unique completion (these use the C++ implementation of Knuth’s dancing links algorithm to solve the problem as an instance of \(0-1\) matrix exact cover);
5. functions for calculating the \( \tau_i \) representation of a bitrade and the genus of the associated hypermap embedding;
6. Markov chain of Jacobson and Matthews (1996) for generating Latin squares uniformly at random (provides a generator interface);
7. a few examples of \( \tau_i \) representations of bitrades constructed from the action of a group on itself by right multiplication, functions for converting to a pair of LatinSquare objects.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: B = back_circulant(5)
sage: B
[0 1 2 3 4]
[1 2 3 4 0]
[2 3 4 0 1]
[3 4 0 1 2]
[4 0 1 2 3]
sage: B.is_latin_square()
True
sage: B[0, 1] = 0
sage: B.is_latin_square()
False

sage: (a, b, c, G) = alternating_group_bitrade_generators(1)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: T1
[ 0 -1 3 1]
```
Todo:

1. Latin squares with symbols from a ring instead of the integers \(\{0, 1, \ldots, n-1\}\).
2. Isotopism testing of latin squares and bitrades via graph isomorphism (naity?).
3. Combinatorial constructions for bitrades.

AUTHORS:

- Carlo Hamalainen (2008-03-23): initial version

**class** `sage.combinat.matrices.latin.LatinSquare(*args)`

Bases: `object`

Latin squares.

This class implements a latin square of order \(n\) with rows and columns indexed by the set \(0, 1, \ldots, n-1\) and symbols from the same set. The underlying latin square is a matrix(ZZ, \(n, n\)). If \(L\) is a latin square, then the cell at row \(r\), column \(c\) is empty if and only if \(L[r, c] < 0\). In this way we allow partial latin squares and can speak of completions to latin squares, etc.

There are two ways to declare a latin square:

Empty latin square of order \(n\):

```python
sage: n = 3
sage: L = LatinSquare(n)
sage: L
[-1 -1 -1]
[-1 -1 -1]
[-1 -1 -1]
```

Latin square from a matrix:

```python
sage: M = matrix(ZZ, [[0, 1], [2, 3]])
sage: LatinSquare(M)
[0 1]
[2 3]
```

**actual_row_col_sym_sizes()**

Bitrades sometimes end up in partial latin squares with unused rows, columns, or symbols. This function works out the actual number of used rows, columns, and symbols.
Warning: We assume that the unused rows/columns occur in the lower right of self, and that the used symbols are in the range \( \{0, 1, \ldots, m\} \) (no holes in that list).

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: B = back_circulant(3)
sage: B[0,0] = B[2,1] = -1
sage: B
[-1  1 -1]
[ 1  2 -1]
[ 2 -1 -1]
sage: B.actual_row_col_sym_sizes()
(3, 2, 2)
```

apply_isotopism\((row\_perm, col\_perm, sym\_perm)\)

An isotopism is a permutation of the rows, columns, and symbols of a partial latin square self. Use isotopism() to convert a tuple (indexed from 0) to a Permutation object.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: B = back_circulant(5)
sage: B
[0 1 2 3 4]
[1 2 3 4 0]
[2 3 4 0 1]
[3 4 0 1 2]
[4 0 1 2 3]
sage: alpha = isotopism((0,1,2,3,4))
sage: beta = isotopism((1,0,2,3,4))
sage: gamma = isotopism((2,1,0,3,4))
sage: B.apply_isotopism(alpha, beta, gamma)
[3 4 2 0 1]
[0 2 3 1 4]
[1 3 0 4 2]
[4 0 1 2 3]
[2 1 4 3 0]
```

clear_cells()

Mark every cell in self as being empty.

EXAMPLES:

```python
sage: A = LatinSquare(matrix(ZZ, [[0, 1], [2, 3]]))
sage: A.clear_cells()
sage: A
[-1 -1]
[-1 -1]
```

column\((x)\)

Return column \(x\) of the latin square.
EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(3).column(0)
(0, 1, 2)
```

`contained_in(Q)`

Return True if self is a subset of Q?

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: P = elementary_abelian_2group(2)
sage: P[0, 0] = -1
sage: P.contained_in(elementary_abelian_2group(2))
True
sage: back_circulant(4).contained_in(elementary_abelian_2group(2))
False
```

`disjoint_mate_dlxcpp_rows_and_map(allow_subtrade)`

Internal function for find_disjoint_mates.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: B = back_circulant(4)
sage: B.disjoint_mate_dlxcpp_rows_and_map(allow_subtrade = True)
([[0, 16, 32],
  [1, 17, 32],
  [2, 18, 32],
  [3, 19, 32],
  [4, 16, 33],
  [5, 17, 33],
  [6, 18, 33],
  [7, 19, 33],
  [8, 16, 34],
  [9, 17, 34],
  [10, 18, 34],
  [11, 19, 34],
  [12, 16, 35],
  [13, 17, 35],
  [14, 18, 35],
  [15, 19, 35],
  [0, 20, 36],
  [1, 21, 36],
  [2, 22, 36],
  [3, 23, 36],
  [4, 20, 37],
  [5, 21, 37],
  [6, 22, 37],
  [7, 23, 37],
  [8, 20, 38],
  [9, 21, 38],
  [10, 22, 38],
  [11, 23, 38],
  [12, 20, 39],
  [13, 21, 39],
  [14, 22, 39],
  [15, 23, 39],
  [0, 24, 40],
  [1, 25, 40],
  [2, 26, 40],
  [3, 24, 41],
  [4, 25, 41],
  [5, 26, 41],
  [6, 24, 42],
  [7, 25, 42],
  [8, 26, 42],
  [9, 24, 43],
  [10, 25, 43],
  [11, 26, 43],
  [12, 24, 44],
  [13, 25, 44],
  [14, 26, 44],
  [15, 24, 45],
  [16, 25, 45],
  [17, 26, 45],
  [18, 24, 46],
  [19, 25, 46],
  [20, 26, 46],
  [21, 24, 47],
  [22, 25, 47],
  [23, 26, 47],
  [24, 24, 48],
  [25, 25, 48],
  [26, 26, 48])
```
[11, 23, 38],
[12, 20, 39],
[13, 21, 39],
[14, 22, 39],
[15, 23, 39],
[0, 24, 40],
[1, 25, 40],
[2, 26, 40],
[3, 27, 40],
[4, 24, 41],
[5, 25, 41],
[6, 26, 41],
[7, 27, 41],
[8, 24, 42],
[9, 25, 42],
[10, 26, 42],
[11, 27, 42],
[12, 24, 43],
[13, 25, 43],
[14, 26, 43],
[15, 27, 43],
[0, 28, 44],
[1, 29, 44],
[2, 30, 44],
[3, 31, 44],
[4, 28, 45],
[5, 29, 45],
[6, 30, 45],
[7, 31, 45],
[8, 28, 46],
[9, 29, 46],
[10, 30, 46],
[11, 31, 46],
[12, 28, 47],
[13, 29, 47],
[14, 30, 47],
[15, 31, 47],
{(0, 16, 32): (0, 0, 0),
(0, 20, 36): (1, 0, 0),
(0, 24, 40): (2, 0, 0),
(0, 28, 44): (3, 0, 0),
(1, 17, 32): (0, 0, 1),
(1, 21, 36): (1, 0, 1),
(1, 25, 40): (2, 0, 1),
(1, 29, 44): (3, 0, 1),
(2, 18, 32): (0, 0, 2),
(2, 22, 36): (1, 0, 2),
(2, 26, 40): (2, 0, 2),
(2, 30, 44): (3, 0, 2),
(3, 19, 32): (0, 0, 3),
(3, 23, 36): (1, 0, 3),
(3, 27, 40): (2, 0, 3),
}
\[(3, 31, 44): (3, 0, 3),
(4, 16, 33): (0, 1, 0),
(4, 20, 37): (1, 1, 0),
(4, 24, 41): (2, 1, 0),
(4, 28, 45): (3, 1, 0),
(5, 17, 33): (0, 1, 1),
(5, 21, 37): (1, 1, 1),
(5, 25, 41): (2, 1, 1),
(5, 29, 45): (3, 1, 1),
(6, 18, 33): (0, 1, 2),
(6, 22, 37): (1, 1, 2),
(6, 26, 41): (2, 1, 2),
(6, 30, 45): (3, 1, 2),
(7, 19, 33): (0, 1, 3),
(7, 23, 37): (1, 1, 3),
(7, 27, 41): (2, 1, 3),
(7, 31, 45): (3, 1, 3),
(8, 16, 34): (0, 2, 0),
(8, 20, 38): (1, 2, 0),
(8, 24, 42): (2, 2, 0),
(8, 28, 46): (3, 2, 0),
(9, 17, 34): (0, 2, 1),
(9, 21, 38): (1, 2, 1),
(9, 25, 42): (2, 2, 1),
(9, 29, 46): (3, 2, 1),
(10, 18, 34): (0, 2, 2),
(10, 22, 38): (1, 2, 2),
(10, 26, 42): (2, 2, 2),
(10, 30, 46): (3, 2, 2),
(11, 19, 34): (0, 2, 3),
(11, 23, 38): (1, 2, 3),
(11, 27, 42): (2, 2, 3),
(11, 31, 46): (3, 2, 3),
(12, 16, 35): (0, 3, 0),
(12, 20, 39): (1, 3, 0),
(12, 24, 43): (2, 3, 0),
(12, 28, 47): (3, 3, 0),
(13, 17, 35): (0, 3, 1),
(13, 21, 39): (1, 3, 1),
(13, 25, 43): (2, 3, 1),
(13, 29, 47): (3, 3, 1),
(14, 18, 35): (0, 3, 2),
(14, 22, 39): (1, 3, 2),
(14, 26, 43): (2, 3, 2),
(14, 30, 47): (3, 3, 2),
(15, 19, 35): (0, 3, 3),
(15, 23, 39): (1, 3, 3),
(15, 27, 43): (2, 3, 3),
(15, 31, 47): (3, 3, 3))

`dlxcpp_has_unique_completion()`

Check if the partial latin square self of order n can be embedded in precisely one latin square of order n.

5.1. Comprehensive Module List 1393
EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(2).dlcpp_has_unique_completion()
True
sage: P = LatinSquare(2)
sage: P.dlxcpp_has_unique_completion()
False
sage: P[0, 0] = 0
sage: P.dlxcpp_has_unique_completion()
True
```

dumps()
Since the latin square class does not hold any other private variables we just call dumps on self.square:

```
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(2) == loads(dumps(back_circulant(2)))
True
```

filled_cells_map()
Number the filled cells of self with integers from \{1, 2, 3,\ldots\}.

INPUT:

• self – partial latin square self (empty cells have negative values)

OUTPUT:

A dictionary cells_map where cells_map[(i,j)] = m means that (i,j) is the m-th filled cell in P, while cells_map[m] = (i,j).

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: (a, b, c, G) = alternating_group_bitrade_generators(1)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: D = T1.filled_cells_map()
sage: {i: v for i,v in D.items() if i in ZZ}
{1: (0, 0),
 2: (0, 2),
 3: (0, 3),
 4: (1, 1),
 5: (1, 2),
 6: (1, 3),
 7: (2, 0),
 8: (2, 1),
 9: (2, 2),
10: (3, 0),
11: (3, 1),
12: (3, 3)}
sage: {i: v for i,v in D.items() if i not in ZZ}
{(0, 0): 1,
 (0, 2): 2,
 (0, 3): 3,
(continues on next page)
1. 1: 4,  
(1, 2): 5,  
(1, 3): 6,  
(2, 0): 7,  
(2, 1): 8,  
(2, 2): 9,  
(3, 0): 10,  
(3, 1): 11,  
(3, 3): 12}

\(\text{find_disjoint_mates}(nr\_\_to\_\_find=\text{None}, allow\_\_subtrade=\text{False})\)

**Warning:** If allow\_subtrade is True then we may return a partial latin square that is not disjoint to self. In that case, use bitrade(P, Q) to get an actual bitrade.

**EXAMPLES:**

```
sage: from sage.combinat.matrices.latin import *  
sage: B = back_circulant(4)  
sage: g = B.find_disjoint_mates(allow_subtrade = True)  
sage: B1 = next(g)  
sage: B0, B1 = bitrade(B, B1)  
sage: assert is_bitrade(B0, B1)  
sage: print(B0)  
[-1 1 2 -1]  
[-1 2 -1 0]  
[-1 -1 -1 -1]  
[-1 0 1 2]  
sage: print(B1)  
[-1 2 1 -1]  
[-1 0 -1 2]  
[-1 -1 -1 -1]  
[-1 1 2 0]
```

\(\text{gcs()}\)

A greedy critical set of a latin square self is found by successively removing elements in a row-wise (bottom-up) manner, checking for unique completion at each step.

**EXAMPLES:**

```
sage: from sage.combinat.matrices.latin import *  
sage: A = elementary_abelian_2group(3)  
sage: G = A.gcs()  
sage: A  
\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
\end{bmatrix}
\]
```
is_completable()
Return True if the partial latin square can be completed to a latin square.

EXAMPLES:
The following partial latin square has no completion because there is nowhere that we can place the symbol 0 in the third row:

```
sage: B = LatinSquare(3)
sage: B[0, 0] = 0
sage: B[1, 1] = 0
sage: B[2, 2] = 1
sage: B
[ 0 -1 -1]
[-1 0 -1]
[-1 -1 1]
sage: B.is_completable()
False
sage: B[2, 2] = 0
sage: B.is_completable()
True
```

is_empty_column(c)
Check if column c of the partial latin square self is empty.

EXAMPLES:
```
sage: from sage.combinat.matrices.latin import *
sage: L = back_circulant(4)
sage: L.is_empty_column(0)
False
sage: L[0,0] = L[1,0] = L[2,0] = L[3,0] = -1
sage: L.is_empty_column(0)
True
```

is_empty_row(r)
Check if row r of the partial latin square self is empty.

EXAMPLES:
sage: from sage.combinat.matrices.latin import *
sage: L = back_circulant(4)
sage: L.is_empty_row(0)
False
sage: L[0,0] = L[0,1] = L[0,2] = L[0,3] = -1
sage: L.is_empty_row(0)
True

is_latin_square()

self is a latin square if it is an n by n matrix, and each symbol in \([0, 1, \ldots, n-1]\) appears exactly once in each row, and exactly once in each column.

EXAMPLES:

sage: from sage.combinat.matrices.latin import *
sage: elementary_abelian_2group(4).is_latin_square()
True
sage: forward_circulant(7).is_latin_square()
True

is_partial_latin_square()

self is a partial latin square if it is an n by n matrix, and each symbol in \([0, 1, \ldots, n-1]\) appears at most once in each row, and at most once in each column.

EXAMPLES:

sage: from sage.combinat.matrices.latin import *
sage: LatinSquare(4).is_partial_latin_square()
True
sage: back_circulant(3).gcs().is_partial_latin_square()
True
sage: back_circulant(6).is_partial_latin_square()
True

is_uniquely_completable()

Return True if the partial latin square self has exactly one completion to a latin square. This is just a wrapper for the current best-known algorithm, Dancing Links by Knuth. See dancing_links.spyx

EXAMPLES:

sage: from sage.combinat.matrices.latin import *
sage: back_circulant(4).gcs().is_uniquely_completable()
True
sage: G = elementary_abelian_2group(3).gcs()
sage: G.is_uniquely_completable()
True
sage: G[0, 0] = -1
sage: G.is_uniquely_completable()
False
latex()

Return LaTeX code for the latin square.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: print(back_circulant(3).latex())
\begin{array}{|c|c|c|}
\hline
0 & 1 & 2 \\
\hline
1 & 2 & 0 \\
\hline
2 & 0 & 1 \\
\end{array}
```

list()

Convert the latin square into a list, in a row-wise manner.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(3).list()
[0, 1, 2, 1, 2, 0, 2, 0, 1]
```

ncols()

Number of columns in the latin square.

EXAMPLES:

```
sage: LatinSquare(3).ncols()
3
```

nr_distinct_symbols()

Return the number of distinct symbols in the partial latin square self.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(5).nr_distinct_symbols()
5
sage: L = LatinSquare(10)
sage: L.nr_distinct_symbols()
0
sage: L[0, 0] = 0
sage: L[0, 1] = 1
sage: L.nr_distinct_symbols()
2
```

nr_filled_cells()

Return the number of filled cells (i.e. cells with a positive value) in the partial latin square self.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: LatinSquare(matrix([[0, -1], [-1, 0]])).nr_filled_cells()
2
```

nrows()

Number of rows in the latin square.

EXAMPLES:
permissible_values(r, c)

Find all values that do not appear in row r and column c of the latin square self. If self[r, c] is filled then we return the empty list.

INPUT:

• self - LatinSquare
• r - int; row of the latin square
• c - int; column of the latin square

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: L = back_circulant(5)
sage: L[0, 0] = -1
sage: L.permissible_values(0, 0)
[0]
```

random_empty_cell()

Find an empty cell of self, uniformly at random.

INPUT:

• self - LatinSquare

OUTPUT:

• [r, c] - cell such that self[r, c] is empty, or returns None if self is a (full) latin square.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: P = back_circulant(2)
sage: P[1,1] = -1
sage: P.random_empty_cell()
[1, 1]
```

row(x)

Return row x of the latin square.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(3).row(0)
(0, 1, 2)
```

set_immutable()

A latin square is immutable if the underlying matrix is immutable.

EXAMPLES:

```
sage: L = LatinSquare(matrix(ZZ, [[0, 1], [2, 3]]))
sage: L.set_immutable()
```
top_left_empty_cell()
Return the least [r, c] such that self[r, c] is an empty cell. If all cells are filled then we return None.

INPUT:
• self - LatinSquare

EXAMPLES:

vals_in_col(c)
Return a dictionary with key e if and only if column c of self has the symbol e.

EXAMPLES:

vals_in_row(r)
Return a dictionary with key e if and only if row r of self has the symbol e.

EXAMPLES:

sage.combinat.matrices.latin.LatinSquare_generator(L_start, check_assertions=False)
Generator for a sequence of uniformly distributed latin squares, given L_start as the initial latin square.

This code implements the Markov chain algorithm of Jacobson and Matthews (1996), see below for the BibTex entry. This generator will never throw the StopIteration exception, so it provides an infinite sequence of latin squares.

EXAMPLES:

Use the back circulant latin square of order 4 as the initial square and print the next two latin squares given by the Markov chain:

sage: from sage.combinat.matrices.latin import *
sage: B = back_circulant(3)
sage: B[0, 0] = -1
sage: B vals_in_col(0)
{0: True, 1: True, 2: True}

sage: from sage.combinat.matrices.latin import *
sage: B = back_circulant(3)
sage: B[0, 0] = -1
dsage: back_circulant(3).vals_in_row(0)
{0: True, 1: True, 2: True}

sage.combinat.matrices.latin.LatinSquare_generator(L_start, check_assertions=False)
Generator for a sequence of uniformly distributed latin squares, given L_start as the initial latin square.

This code implements the Markov chain algorithm of Jacobson and Matthews (1996), see below for the BibTex entry. This generator will never throw the StopIteration exception, so it provides an infinite sequence of latin squares.

EXAMPLES:

Use the back circulant latin square of order 4 as the initial square and print the next two latin squares given by the Markov chain:

sage: from sage.combinat.matrices.latin import *
sage: g = LatinSquare_generator(back_circulant(4))

(continues on next page)
REFERENCES:
sage.combinat.matrices.latin.alternating_group_bitrade_generators(m)
Construct generators a, b, c for the alternating group on 3m+1 points, such that a*b*c = 1.

EXAMPLES:
```python
sage: from sage.combinat.matrices.latin import *
sage: (a, b, c, G) = alternating_group_bitrade_generators(1)
sage: (a, b, c, G)
((1,2,3), (1,4,2), (2,4,3), Permutation Group with generators [(1,2,3), (1,4,2)])
sage: a*b*c
False
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: T1
[ 0 -1  3  1]
[-1  1  0  2]
[ 1  3  2 -1]
[ 2  0 -1  3]
sage: T2
[ 1 -1  0  3]
[-1  0  2  1]
[ 2  1  3 -1]
[ 0  3 -1  2]
```
sage.combinat.matrices.latin.back_circulant(n)
The back-circulant latin square of order n is the Cayley table for (Z_n, +), the integers under addition modulo n.

INPUT:
• n – int; order of the latin square.

EXAMPLES:
```python
sage: from sage.combinat.matrices.latin import *
sage: back_circulant(5)
[0 1 2 3 4]
[1 2 3 4 0]
[2 3 4 0 1]
[3 4 0 1 2]
[4 0 1 2 3]
```
sage.combinat.matrices.latin.beta1(rce, T1, T2)
Find the unique (x, c, e) in T2 such that (r, c, e) is in T1.

INPUT:
• rce - tuple (or list) (r, c, e) in T1
• T1, T2 - latin bitrade

OUTPUT: (x, c, e) in T2.
EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( [0,1,2,3,4] )
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_bitrade(T1, T2)
True
sage: beta1([0, 0, 0], T1, T2)
(1, 0, 0)
```

`sage.combinat.matrices.latin.beta2(rce, T1, T2)`

Find the unique \((r, x, e)\) in \(T2\) such that \((r, c, e)\) is in \(T1\).

**INPUT:**
- \(rce\) - tuple (or list) \((r, c, e)\) in \(T1\)
- \(T1, T2\) - latin bitrade

**OUTPUT:**
- \((r, x, e)\) in \(T2\).

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( [0,1,2,3,4] )
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_bitrade(T1, T2)
True
sage: beta2([0, 0, 0], T1, T2)
(0, 1, 0)
```

`sage.combinat.matrices.latin.beta3(rce, T1, T2)`

Find the unique \((r, c, x)\) in \(T2\) such that \((r, c, e)\) is in \(T1\).

**INPUT:**
- \(rce\) - tuple (or list) \((r, c, e)\) in \(T1\)
- \(T1, T2\) - latin bitrade

**OUTPUT:**
- \((r, c, x)\) in \(T2\).

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( [0,1,2,3,4] )
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_bitrade(T1, T2)
True
sage: beta3([0, 0, 0], T1, T2)
(0, 1, 0)
```
sage.combinat.matrices.latin.bitrade(T1, T2)

Form the bitrade (Q1, Q2) from (T1, T2) by setting empty the cells (r, c) such that T1[r, c] == T2[r, c].

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: B1 = back_circulant(5)
sage: alpha = isotopism((0,1,2,3,4))
sage: beta = isotopism((1,0,2,3,4))
sage: gamma = isotopism((2,1,0,3,4))
sage: B2 = B1.apply_isotopism(alpha, beta, gamma)
sage: T1, T2 = bitrade(B1, B2)

sage: T1
[[0 1 -1 3 4]
 [1 -1 -1 4 0]
 [2 -1 4 0 1]
 [3 4 0 1 2]
 [4 0 1 2 3]]

sage: T2
[[3 4 -1 0 1]
 [-1 0 2 1]
 [2 1 4 3 0]
 [0 3 -1 2]]
```

sage.combinat.matrices.latin.bitrade_from_group(a, b, c, G)

Given group elements a, b, c in G such that abc = 1 and the subgroups a, b, c intersect (pairwise) only in the identity, construct a bitrade (T1, T2) where rows, columns, and symbols correspond to cosets of a, b, and c, respectively.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: a, b, c, G = alternating_group_bitrade_generators(1)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)

sage: T1
[[0 -1 3 1]
 [-1 1 0 2]
 [1 3 2 -1]
 [2 0 -1 3]]

sage: T2
[[1 -1 0 3]
 [-1 0 2 1]
 [2 1 3 -1]
 [0 3 -1 2]]
```

sage.combinat.matrices.latin.cells_map_as_square(cells_map, n)

Return a LatinSquare with cells numbered from 1, 2, ..., to given the dictionary cells_map.
EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: (a, b, c, G) = alternating_group_bitrade_generators(1)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: T1
[[ 0 -1  3  1]
 [ -1  1  0  2]
 [ 1  3  2 -1]
 [ 2  0 -1  3]]
```

There are 12 filled cells in T:

```python
sage: cells_map_as_square(T1.filled_cells_map(), max(T1.nrows(), T1.ncols()))
[[ 1 -1  2  3]
 [ -1  4  5  6]
 [ 7  8  9 -1]
 [10 11 -1 12]]
```

`sage.combinat.matrices.latin.check_bitrade_generators(a, b, c)`

Three group elements a, b, c will generate a bitrade if a*b*c = 1 and the subgroups a, b, c intersect (pairwise) in just the identity.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: a, b, c, G = p3_group_bitrade_generators(3)
sage: check_bitrade_generators(a, b, c)
True
sage: check_bitrade_generators(a, b, gap('()'))
False
```

`sage.combinat.matrices.latin.coin()`

Simulate a fair coin (returns True or False) using ZZ.random_element(2).

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import coin
sage: x = coin()
sage: x == 0 or x == 1
True
```

`sage.combinat.matrices.latin.column_containing_sym(L, r, x)`

Given an improper latin square L with \(L[r, c1] = L[r, c2] = x\), return \(c1\) or \(c2\) with equal probability. This is an internal function and should only be used in LatinSquare_generator().

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: L = matrix([(1, 0, 2, 3), (0, 2, 3, 0), (2, 3, 0, 1), (3, 0, 1, 2)])
```

(continues on next page)
sage: L
[[1 0 2 3]
[0 2 3 0]
[2 3 0 1]
[3 0 1 2]]
sage: c = column_containing_sym(L, 1, 0)
sage: c == 0 or c == 3
True

sage.combinat.matrices.latin.direct_product(L1, L2, L3, L4)
The ‘direct product’ of four latin squares L1, L2, L3, L4 of order n is the latin square of order 2n consisting of

|  
|  
|  

where the subsquares L2 and L3 have entries offset by n.

EXAMPLES:

sage: from sage.combinat.matrices.latin import *
sage: direct_product(back_circulant(4), back_circulant(4), elementary_abelian_2group(2), elementary_abelian_2group(2))
[[0 1 2 3 4 5 6 7]
[1 2 3 0 5 6 7 4]
[2 3 0 1 6 7 4 5]
[3 0 1 2 7 4 5 6]
[4 5 6 7 0 1 2 3]
[5 4 7 6 1 0 3 2]
[6 7 4 5 2 3 0 1]
[7 6 5 4 3 2 1 0]]

sage.combinat.matrices.latin.dlxcpp_find_completions(P, nr_to_find=None)
Return a list of all latin squares L of the same order as P such that P is contained in L. The optional parameter nr_to_find limits the number of latin squares that are found.

EXAMPLES:

sage: from sage.combinat.matrices.latin import *
sage: dlxcpp_find_completions(LatinSquare(2))
[[[0 1]
  [1 0]]
[[0 1], [1 0]]
[[[0 1]
  [1 0]]

sage.combinat.matrices.latin.dlxcpp_rows_and_map(P)
Internal function for dlxcpp_find_completions. Given a partial latin square P we construct a list of rows of a 0-1 matrix M such that an exact cover of M corresponds to a completion of P to a latin square.
EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: dlxcpp_rows_and_map(LatinSquare(2))
([[0, 4, 8],
  [1, 5, 8],
  [2, 4, 9],
  [3, 5, 9],
  [0, 6, 10],
  [1, 7, 10],
  [2, 6, 11],
  [3, 7, 11]],
  {(0, 4, 8): (0, 0, 0),
   (0, 6, 10): (1, 0, 0),
   (1, 5, 8): (0, 0, 1),
   (1, 7, 10): (1, 0, 1),
   (2, 4, 9): (0, 1, 0),
   (2, 6, 11): (1, 1, 0),
   (3, 5, 9): (0, 1, 1),
   (3, 7, 11): (1, 1, 1)})
```

`sage.combinat.matrices.latin.elementary_abelian_2group(s)`
Return the latin square based on the Cayley table for the elementary abelian 2-group of order 2s.

INPUT:

• s – int; order of the latin square will be 2s.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: elementary_abelian_2group(3)
[0 1 2 3 4 5 6 7]
[1 0 3 2 5 4 7 6]
[2 3 0 1 6 7 4 5]
[3 2 1 0 7 6 5 4]
[4 5 6 7 0 1 2 3]
[5 4 7 6 1 0 3 2]
[6 7 4 5 2 3 0 1]
[7 6 5 4 3 2 1 0]
```

`sage.combinat.matrices.latin.forward_circulant(n)`
The forward-circulant latin square of order n is the Cayley table for the operation \( r + c = (n-c+r) \mod n \).

INPUT:

• n – int; order of the latin square.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: forward_circulant(5)
[0 4 3 2 1]
[1 0 4 3 2]
[2 1 0 4 3]
[3 2 1 0 4]
[4 3 2 1 0]
```
sage.combinat.matrices.latin.genus(T1, T2)

Return the genus of hypermap embedding associated with the bitrade \((T1, T2)\).

Informally, we compute the \([\tau_1, \tau_2, \tau_3]\) permutation representation of the bitrade. Each cycle of \(\tau_1, \tau_2, \text{ and } \tau_3\) gives a rotation scheme for a black, white, and star vertex (respectively). The genus then comes from Euler’s formula.

For more details see Carlo Hamalainen: *Partitioning 3-homogeneous latin bitrades*. To appear in Geometriae Dedicata, available at arXiv 0710.0938

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: (a, b, c, G) = alternating_group_bitrade_generators(1)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: genus(T1, T2)
1
sage: (a, b, c, G) = pq_group_bitrade_generators(3, 7)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: genus(T1, T2)
3
```

sage.combinat.matrices.latin.group_to_LatinSquare(G)

Construct a latin square on the symbols \([0, 1, \ldots, n-1]\) for a group with an \(n \times n\) Cayley table.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import group_to_LatinSquare
sage: group_to_LatinSquare(DihedralGroup(2))
[0 1 2 3]
[1 0 3 2]
[2 3 0 1]
[3 2 1 0]
```

```python
sage: G = gap.Group(PermutationGroupElement((1,2,3)))
sage: group_to_LatinSquare(G)
[0 1 2]
[1 2 0]
[2 0 1]
```

sage.combinat.matrices.latin.is_bitrade(T1, T2)

Combinatorially, a pair \((T1, T2)\) of partial latin squares is a bitrade if they are disjoint, have the same shape, and have row and column balance. For definitions of each of these terms see the relevant function in this file.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism((0, 1, 2, 3, 4))
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_bitrade(T1, T2)
True
```
sage.combinat.matrices.latin.is_disjoint(T1, T2)

The partial latin squares T1 and T2 are disjoint if T1[r, c] != T2[r, c] or T1[r, c] == T2[r, c] == -1 for each cell [r, c].

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import is_disjoint, back_circulant,
    isotopism
sage: is_disjoint(back_circulant(2), back_circulant(2))
False
```

```python
sage: T1 = back_circulant(5)
sage: x = isotopism( (0,1,2,3,4) )
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_disjoint(T1, T2)
True
```

sage.combinat.matrices.latin.is_primary_bitrade(a, b, c, G)

A bitrade generated from elements a, b, c is primary if a, b, c = G.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: (a, b, c, G) = p3_group_bitrade_generators(5)
sage: is_primary_bitrade(a, b, c, G)
True
```

sage.combinat.matrices.latin.is_row_and_col_balanced(T1, T2)

Partial latin squares T1 and T2 are balanced if the symbols appearing in row r of T1 are the same as the symbols appearing in row r of T2, for each r, and if the same condition holds on columns.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = matrix([[0,1,-1,-1], [-1,-1,-1,-1], [-1,-1,-1,-1], [-1,-1,-1,-1]])
sage: T2 = matrix([[0,1,-1,-1], [-1,-1,-1,-1], [-1,-1,-1,-1], [-1,-1,-1,-1]])
sage: is_row_and_col_balanced(T1, T2)
True
```

```python
sage: T2 = matrix([[0,3,-1,-1], [-1,-1,-1,-1], [-1,-1,-1,-1], [-1,-1,-1,-1]])
sage: is_row_and_col_balanced(T1, T2)
False
```

sage.combinat.matrices.latin.is_same_shape(T1, T2)

Two partial latin squares T1, T2 have the same shape if T1[r, c] = 0 if and only if T2[r, c] = 0.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: is_same_shape(elementary_abelian_2group(2), back_circulant(4))
True
```

```python
sage: is_same_shape(LatinSquare(5), LatinSquare(5))
True
```
sage: is_same_shape(forward_circulant(5), LatinSquare(5))
False

sage.combinat.matrices.latin.isotopism(p)

Return a Permutation object that represents an isotopism (for rows, columns or symbols of a partial Latin square).

Technically, all this function does is take as input a representation of a permutation of 0, ..., n - 1 and return a Permutation object defined on 1, ..., n.

For a definition of isotopism, see the Wikipedia section on isotopism.

INPUT:

According to the type of input (see examples below):

- an integer n – the function returns the identity on 1, ..., n.
- a string representing a permutation in disjoint cycles notation, e.g. (0, 1, 2)(3, 4, 5) – the corresponding permutation is returned, shifted by 1 to act on 1, ..., n.
- list/tuple of tuples – assumes disjoint cycle notation, see previous entry.
- a list of integers – the function adds 1 to each member of the list, and returns the corresponding permutation.
- a PermutationGroupElement p – returns a permutation describing p without any shift.

EXAMPLES:

sage: from sage.combinat.matrices.latin import *
sage: isotopism(5) # identity on 5 points
[1, 2, 3, 4, 5]

sage: G = PermutationGroup(['(1,2,3)(4,5)'])
sage: g = G.gen(0)
sage: isotopism(g)
[2, 3, 1, 5, 4]

sage: isotopism([0,3,2,1]) # 0 goes to 0, 1 goes to 3, etc.
[1, 4, 3, 2]

sage: x = isotopism((0,1,2)) # single cycle, presented as a tuple
[2, 3, 1]

sage: x = isotopism(((0,1,2), (3,4))) # tuple of cycles
[2, 3, 1, 5, 4]

sage.combinat.matrices.latin.next_conjugate(L)

Permute L[r, c] = e to the conjugate L[c, e] = r.

We assume that L is an n by n matrix and has values in the range 0, 1, ..., n-1.

EXAMPLES:
sage: from sage.combinat.matrices.latin import *
sage: L = back_circulant(6)
sage: L
[[0, 1, 2, 3, 4, 5],
 [1, 2, 3, 4, 5, 0],
 [2, 3, 4, 5, 0, 1],
 [3, 4, 5, 0, 1, 2],
 [4, 5, 0, 1, 2, 3],
 [5, 0, 1, 2, 3, 4]]
sage: next_conjugate(L)
[[0, 1, 2, 3, 4, 5],
 [5, 0, 1, 2, 3, 4],
 [4, 5, 0, 1, 2, 3],
 [3, 4, 5, 0, 1, 2],
 [2, 3, 4, 5, 0, 1],
 [1, 2, 3, 4, 5, 0]]
sage: L == next_conjugate(next_conjugate(next_conjugate(L)))
True

sage.combinat.matrices.latin.p3_group_bitrade_generators(p)
Generators for a group of order p^3 where p is a prime.

EXAMPLES:
sage: from sage.combinat.matrices.latin import *
sage: p3_group_bitrade_generators(3)
((2,6,7)(3,8,9), (1,2,3)(4,7,8)(5,6,9), (1,9,2)(3,7,4)(5,8,6), Permutation Group with generators [(2,6,7)(3,8,9), (1,2,3)(4,7,8)(5,6,9)])

sage.combinat.matrices.latin.pq_group_bitrade_generators(p, q)
Generators for a group of order pq where p and q are primes such that (q % p) == 1.

EXAMPLES:
sage: from sage.combinat.matrices.latin import *
sage: pq_group_bitrade_generators(3,7)
((2,3,5)(4,7,6), (1,2,3,4,5,6,7), (1,4,2)(3,5,6), Permutation Group with generators [(2,3,5)(4,7,6), (1,2,3,4,5,6,7)])

sage.combinat.matrices.latin.row_containing_sym(L, c, x)
Given an improper latin square L with L[r1, c] = L[r2, c] = x, return r1 or r2 with equal probability. This is an internal function and should only be used in LatinSquare_generator().

EXAMPLES:
sage: from sage.combinat.matrices.latin import *
sage: L = matrix([[0, 1, 0, 3], [3, 0, 2, 1], [1, 0, 3, 2], [2, 3, 1, 0]])
sage: L
[[0, 1, 0, 3],
 [3, 0, 2, 1],
 [1, 0, 3, 2],
 [2, 3, 1, 0]]
sage: c = row_containing_sym(L, 1, 0)
sage: c == 1 or c == 2
True
sage.combinat.matrices.latin.tau1(T1, T2, cells_map)

The definition of \( \tau_1 \) is

\[
\tau_1 : T1 \rightarrow T1 \\
\tau_1 = \beta_2^{-1} \beta_3
\]

where the composition is left to right and \( \beta_i : T2 \rightarrow T1 \) changes just the \( i^{th} \) coordinate of a triple.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( (0,1,2,3,4) )
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)

sage: is_bitrade(T1, T2)
True
sage: (cells_map, t1, t2, t3) = tau123(T1, T2)
sage: t1

[2, 3, 4, 5, 1, 7, 8, 9, 10, 6, 12, 13, 14, 15, 11, 17, 18, 19, 20, 16, 22, 23, 24, ...
   ˓→ 25, 21]
sage: t1.to_cycles()

[(1, 2, 3, 4, 5), (6, 7, 8, 9, 10), (11, 12, 13, 14, 15), (16, 17, 18, 19, 20), (21, ˓→ 22, 23, 24, 25)]
```

sage.combinat.matrices.latin.tau123(T1, T2)

Compute the \( \tau_i \) representation for a bitrade \((T1, T2)\).

See the functions tau1, tau2, and tau3 for the mathematical definitions.

OUTPUT:

- \((\text{cells\_map, t1, t2, t3})\)

where \(\text{cells\_map}\) is a map to/from the filled cells of \(T1\), and \(t1, t2, t3\) are the \(\tau1, \tau2, \tau3\) permutations.

EXAMPLES:

```python
sage: from sage.combinat.matrices.latin import *
sage: (a, b, c, G) = pq_group_bitrade_generators(3, 7)
sage: (T1, T2) = bitrade_from_group(a, b, c, G)
sage: T1

[ [0 1 3 -1 -1 -1 -1]
 [1 2 4 -1 -1 -1 -1]
 [2 3 5 -1 -1 -1 -1]
 [3 4 6 -1 -1 -1 -1]
 [4 5 0 -1 -1 -1 -1]
 [5 6 1 -1 -1 -1 -1]
 [6 0 2 -1 -1 -1 -1]]
sage: T2

[ [1 3 0 -1 -1 -1 -1]
 [2 4 1 -1 -1 -1 -1]
 [3 5 2 -1 -1 -1 -1]
 [4 6 3 -1 -1 -1 -1]
```
\[
\begin{bmatrix}
5 & 0 & 4 & -1 & -1 & -1 & -1 \\
6 & 1 & 5 & -1 & -1 & -1 & -1 \\
0 & 2 & 6 & -1 & -1 & -1 & -1 \\
\end{bmatrix}
\]

\texttt{sage: (cells_map, t1, t2, t3) = tau123(T1, T2)}

\texttt{sage: D = cells_map}

\texttt{sage: \{i: v for i,v in D.items() if i in ZZ\}}

\{
1: (0, 0),
2: (0, 1),
3: (0, 2),
4: (1, 0),
5: (1, 1),
6: (1, 2),
7: (2, 0),
8: (2, 1),
9: (2, 2),
10: (3, 0),
11: (3, 1),
12: (3, 2),
13: (4, 0),
14: (4, 1),
15: (4, 2),
16: (5, 0),
17: (5, 1),
18: (5, 2),
19: (6, 0),
20: (6, 1),
21: (6, 2)
\}

\texttt{sage: \{i: v for i,v in D.items() if not i in ZZ\}}

\{
(0, 0): 1,
(0, 1): 2,
(0, 2): 3,
(1, 0): 4,
(1, 1): 5,
(1, 2): 6,
(2, 0): 7,
(2, 1): 8,
(2, 2): 9,
(3, 0): 10,
(3, 1): 11,
(3, 2): 12,
(4, 0): 13,
(4, 1): 14,
(4, 2): 15,
(5, 0): 16,
(5, 1): 17,
(5, 2): 18,
(6, 0): 19,
(6, 1): 20,
(6, 2): 21
\}

\texttt{sage: cells_map_as_square(cells_map, max(T1.nrows(), T1.ncols()))}

\[
\begin{bmatrix}
1 & 2 & 3 & -1 & -1 & -1 & -1 \\
4 & 5 & 6 & -1 & -1 & -1 & -1 \\
\end{bmatrix}
\]
The product $t_1 \ast t_2 \ast t_3$ is the identity, i.e. it fixes every point:

```
sage: len((t1*t2*t3).fixed_points()) == T1.nr_filled_cells()
True
```

The definition of $\tau_2$ is

$$
\tau_2 : T1 \to T1
\tau_2 = \beta_3^{-1} \beta_1
$$

where the composition is left to right and $\beta_i : T2 \to T1$ changes just the $i^{th}$ coordinate of a triple.

EXAMPLES:

```
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( (0,1,2,3,4) )
sage: y = isotopism(5) # identity
sage: z = isotopism(5) # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_bitrade(T1, T2)
True
sage: (cells_map, t1, t2, t3) = tau123(T1, T2)
sage: t2 = tau2(T1, T2, cells_map)
sage: t2
[21, 22, 23, 24, 25, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,]
\rightarrow 19, 20]
sage: t2.to_cycles()
[(1, 21, 16, 11, 6), (2, 22, 17, 12, 7), (3, 23, 18, 13, 8), (4, 24, 19, 14, 9), (5,]
\rightarrow 25, 20, 15, 10)]
```
The definition of $\tau_3$ is
\[
\tau_3 : T1 \to T1
\]
\[
\tau_3 = \beta_1^{-1}\beta_2
\]
where the composition is left to right and $\beta_i : T2 \to T1$ changes just the $i^{th}$ coordinate of a triple.

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( (0,1,2,3,4) )
sage: y = isotopism(5)  # identity
sage: z = isotopism(5)  # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: is_bitrade(T1, T2)
True
sage: (cells_map, t1, t2, t3) = tau123(T1, T2)
sage: t3 = tau3(T1, T2, cells_map)
sage: t3
[10, 6, 7, 8, 9, 15, 11, 12, 13, 14, 20, 16, 17, 18, 19, 25, 21, 22, 23, 24, 5, 1, ...
˓→2, 3, 4]
sage: t3.to_cycles()
[(1, 10, 14, 18, 22), (2, 6, 15, 19, 23), (3, 7, 11, 20, 24), (4, 8, 12, 16, 25), ...
˓→(5, 9, 13, 17, 21)]
```

**sage.combinat.matrices.latin.tau_to_bitrade(t1, t2, t3)**

Given permutations $t1, t2, t3$ that represent a latin bitrade, convert them to an explicit latin bitrade $(T1, T2)$. The result is unique up to isotopism.

**EXAMPLES:**

```python
sage: from sage.combinat.matrices.latin import *
sage: T1 = back_circulant(5)
sage: x = isotopism( (0,1,2,3,4) )
sage: y = isotopism(5)  # identity
sage: z = isotopism(5)  # identity
sage: T2 = T1.apply_isotopism(x, y, z)
sage: _, t1, t2, t3 = tau123(T1, T2)
sage: U1, U2 = tau_to_bitrade(t1, t2, t3)
sage: assert is_bitrade(U1, U2)
sage: U1
[0 1 2 3 4]
[1 2 3 4 0]
[2 3 4 0 1]
[3 4 0 1 2]
sage: U2
[4 0 1 2 3]
[0 1 2 3 4]
[1 2 3 4 0]
[2 3 4 0 1]
[3 4 0 1 2]
```
5.1.139 Miscellaneous

class sage.combinat.misc.DoublyLinkedList(l)
   Bases: object
   A doubly linked list class that provides constant time hiding and unhideing of entries.
   Note that this list's indexing is 1-based.
   EXAMPLES:

   sage: dll = sage.combinat.misc.DoublyLinkedList([1,2,3]); dll
   Doubly linked list of [1, 2, 3]: [1, 2, 3]
   sage: dll.hide(1); dll
   Doubly linked list of [1, 2, 3]: [2, 3]
   sage: dll.unhide(1); dll
   Doubly linked list of [1, 2, 3]: [1, 2, 3]
   sage: dll.hide(2); dll
   Doubly linked list of [1, 2, 3]: [1, 3]
   sage: dll.unhide(2); dll
   Doubly linked list of [1, 2, 3]: [1, 2, 3]

head()
hide(i)
next(j)
prev(j)
unhide(i)

class sage.combinat.misc.IterableFunctionCall(f, *args, **kwargs)
   Bases: object
   This class wraps functions with a yield statement (generators) by an object that can be iterated over. For example,
   EXAMPLES:

   sage: def f():
   ...    yield 'a'; yield 'b'
   This does not work:
   sage: for z in f: print(z)
   Traceback (most recent call last):
   ...    TypeError: 'function' object is not iterable

   Use IterableFunctionCall if you want something like the above to work:
   sage: from sage.combinat.misc import IterableFunctionCall
   sage: g = IterableFunctionCall(f)
   sage: for z in g: print(z)
   a
   b
   If your function takes arguments, just put them after the function name. You needn't enclose them in a tuple or anything, just put them there:
```python
sage: def f(n, m):
    yield 'a' * n;
    yield 'b' * m;
    yield 'foo'
sage: g = IterableFunctionCall(f, 2, 3)
sage: for z in g: print(z)
  aa
  bbb
  foo
```

`sage.combinat.misc.check_integer_list_constraints(l, **kwargs)`

**EXAMPLES:**

```python
sage: from sage.combinat.misc import check_integer_list_constraints
sage: cilc = check_integer_list_constraints
sage: l = [[2,1,3],[1,2],[3,3],[4,1,1]]
sage: cilc(l, min_part=2)
[[3, 3]]
sage: cilc(l, max_part=2)
[[1, 2]]
sage: cilc(l, length=2)
[[1, 2], [3, 3]]
sage: cilc(l, max_length=2)
[[1, 2], [3, 3]]
sage: cilc(l, min_length=3)
[[2, 1, 3], [4, 1, 1]]
sage: cilc(l, max_slope=0)
[[3, 3], [4, 1, 1]]
sage: cilc(l, min_slope=1)
[[1, 2]]
sage: cilc(l, outer=[2,2])
[[1, 2]]
sage: cilc(l, inner=[2,2])
[[3, 3]]
```

```python
sage: cilc([1,2,3], length=3, singleton=True)
[1, 2, 3]
sage: cilc([1,2,3], length=2, singleton=True) is None
True
```

`sage.combinat.misc.umbral_operation(poly)`

Returns the umbral operation $\downarrow$ applied to poly.

The umbral operation replaces each instance of $x_i^{a_i}$ with $x_i \ast (x_i - 1) \ast \cdots \ast (x_i - a_i + 1)$.

**EXAMPLES:**

```python
sage: P = PolynomialRing(QQ, 2, 'x')
sage: x = P.gens()
sage: from sage.combinat.misc import umbral_operation
sage: umbral_operation(x[0]**3) == x[0]**(x[0]-1)**(x[0]-2)
True
sage: umbral_operation(x[0]**x[1])
x0**x1
sage: umbral_operation(x[0]+x[1])
x0 + x1
```

(continues on next page)
5.1.140 Ordered Multiset Partitions into Sets and the Minimaj Crystal

This module provides element and parent classes for ordered multiset partitions. It also implements the minimaj crystal of Benkart et al. [BCHOPSY2017]. (See \texttt{MinimajCrystal}.)

AUTHORS:

• Aaron Lauve (2018): initial implementation. First draft of minimaj crystal code provided by Anne Schilling.

REFERENCES:

• [BCHOPSY2017]
• [HRW2015]
• [HRS2016]
• [LM2018]

EXAMPLES:

An ordered multiset partition into sets of the multiset \{1, 3, 3, 5\}:

\begin{verbatim}
sage: OrderedMultisetPartitionIntoSets([5, 3], [1, 3])
\end{verbatim}

\begin{verbatim}
[[3, 5], [1, 3]]
\end{verbatim}

Ordered multiset partitions into sets of the multiset \{1, 3, 3\}:

\begin{verbatim}
sage: OrderedMultisetPartitionsIntoSets([1, 1, 3]).list()
\end{verbatim}

\begin{verbatim}
[[[1], [1], [3]], [[1], [1, 3]], [[1], [3], [1]], [[1, 3], [1], [1]]
\end{verbatim}

Ordered multiset partitions into sets of the integer 4:

\begin{verbatim}
sage: OrderedMultisetPartitionsIntoSets([1, 4, 3]).list()
\end{verbatim}

\begin{verbatim}
[[[1], [1], [4], [4]], [[1], [1, 4]], [[1, 4], [1]], [[1, 1], [4]], [[1], [4], [1]], [[1], [1], [4], [4]], [[1], [1], [4], [1]]
\end{verbatim}

Ordered multiset partitions into sets on the alphabet \{1, 4\} of order 3:

\begin{verbatim}
sage: OrderedMultisetPartitionsIntoSets([1, 4], 3).list()
\end{verbatim}

\begin{verbatim}
[[[1, 4], [1], [[1, 4], [4], [1], [4], [1]], [[1, 1], [4], [4], [1]]]
\end{verbatim}

Crystal of ordered multiset partitions into sets on the alphabet \{1, 2, 3\} with 4 letters divided into 2 blocks:

\begin{verbatim}
sage: crystals.Minimaj(3, 4, 2).list()
\end{verbatim}

\begin{verbatim}
[((2, 3, 1), (1,)), ((2, 3), (1, 2)), ((2, 3), (1, 3)), ((2, 1), (1, 2)), ((3, 1), (1, 2)), ((3, 1, 2), (2,)), ((3, 1), (1, 3)), ((3, 1), (2, 3)), ((3, 2), (2, 3)), ((2, 1), (1, 3)), ((2, 1), (1, 2, 3)), ((3,), (1, 2, 3)), ((1,), (1, 2, 3)), ((1, 2), (2, 3)), ((1, 2, 3), (3,))]
\end{verbatim}
class sage.combinat.multipartition_into_sets_ordered.MinimajCrystal(n, ell, k)

Bases: UniqueRepresentation, Parent

Crystal of ordered multiset partitions into sets with \(ell\) letters from alphabet \(\{1, 2, \ldots, n\}\) divided into \(k\) blocks. Elements are represented in the minimaj ordering of blocks as in Benkart et al. [BCHOPSY2017].

Note: Elements are not stored internally as ordered multiset partitions into sets, but as certain (pairs of) words stemming from the minimaj bijection \(\phi\) of [BCHOPSY2017]. See sage.combinat.multipartition_into_sets_ordered.MinimajCrystal.Element for further details.

AUTHORS:

• Anne Schilling (2018): initial draft
• Aaron Lauve (2018): changed to use Letters crystal for elements

EXAMPLES:

```
sage: list(crystals.Minimaj(2,3,2))
[((2, 1), (1,)), ((2,), (1, 2)), ((1,), (1, 2)), ((1, 2), (2,))]
sage: b = crystals.Minimaj(3, 5, 2).an_element(); b
((2, 3, 1), (1, 2))
sage: b.f(2)
((2, 3, 1), (1, 3))
sage: b.e(2)
```

class Element

Bases: ElementWrapper

An element of a Minimaj crystal.

Note: Minimaj elements \(b\) are stored internally as pairs \((w, \text{breaks})\), where:

• \(w\) is a word of length self.parent().ell over the letters 1 up to self.parent().n;
• \(\text{breaks}\) is a list of de-concatenation points to turn \(w\) into a list of row words of (skew-)tableaux that represent \(b\) under the minimaj bijection \(\phi\) of [BCHOPSY2017].

The pair \((w, \text{breaks})\) may be recovered via b.value.

\(e(i)\)

Return \(e_i\) on self.

EXAMPLES:

```
sage: B = crystals.Minimaj(4,3,2)
sage: b = B([[2,3], [3]]); b
((2, 3), (3,))
sage: [b.e(i) for i in range(1,4)]
[[(1, 3), (3,)], [(2,), (2, 3)], None]
```

\(f(i)\)

Return \(f_i\) on self.

EXAMPLES:
```python
sage: B = crystals.Minimaj(4,3,2)
sage: b = B([[2,3], [3]]); b
((2, 3), (3,))
sage: [b.f(i) for i in range(1,4)]
[None, None, ((2, 3), (4,))]
```

**to_tableaux_words()**

Return the image of the ordered multiset partition into sets `self` under the minimaj bijection \( \phi \) of [BCHOPSY2017].

**EXAMPLES:**

```python
sage: B = crystals.Minimaj(4,5,3)
sage: b = B.an_element(); b
((2, 3, 1), (1,), (1,))
sage: b.to_tableaux_words()
[[1], [3], [2, 1, 1]]
sage: b = B([[1,3,4], [3], [3]]); b
((4, 1, 3), (3,), (3,))
sage: b.to_tableaux_words()
[[3, 1], [], [4, 3, 3]]
```

**from_tableau(t)**

Return the bijection \( \phi^{-1} \) of [BCHOPSY2017] applied to \( t \).

**INPUT:**

- \( t \) – a sequence of column tableaux and a ribbon tableau

**EXAMPLES:**

```python
sage: B = crystals.Minimaj(3,6,3)
sage: b = B.an_element(); b
((3, 1, 2), (2, 1), (1,))
sage: t = b.to_tableaux_words(); t
[[1], [2, 1], [], [3, 2, 1]]
sage: B.from_tableau(t)
((3, 1, 2), (2, 1), (1,))
sage: B.from_tableau(t) == b
True
```

**val(q=’q’)**

Return the \( Val \) polynomial corresponding to \( self \).

**EXAMPLES:**

Verifying Example 4.5 from [BCHOPSY2017]:

```python
sage: B = crystals.Minimaj(3, 4, 2) # for 'Val_{4,1}^{(3)}'
sage: B.val()
(q^2+q+1)*s[2, 1, 1] + q*s[2, 2]
```

**class** `sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionIntoSets(parent, data)`

**Bases:** `ClonableArray`
Ordered Multiset Partition into sets

An ordered multiset partition into sets \( c \) of a multiset \( X \) is a list \( [c_1, \ldots, c_r] \) of nonempty subsets of \( X \) (note: not sub-multisets), called the blocks of \( c \), whose multi-union is \( X \).

EXAMPLES:
The simplest way to create an ordered multiset partition into sets is by specifying its blocks as a list or tuple:

```sage
sage: OrderedMultisetPartitionIntoSets([[3], [2, 1]])
[[3], [1, 2]]
sage: OrderedMultisetPartitionIntoSets(((3,), (1, 2)))
[[3], [1, 2]]
sage: OrderedMultisetPartitionIntoSets([set([i]) for i in range(2, 5)])
[[2], [3], [4]]
```

REFERENCES:

- [HRW2015]
- [HRS2016]
- [LM2018]

`check()`

Check that we are a valid ordered multiset partition into sets.

EXAMPLES:

```sage
sage: c = OrderedMultisetPartitionsIntoSets(4)([[1], [1, 2]])
sage: c.check()
sage: OMPs = OrderedMultisetPartitionsIntoSets()
sage: c = OMPs([[1], [1], ['a']])
sage: c.check()
```

`deconcatenate(k=2)`

Return the list of \( k \)-deconcatenations of \( self \).

A \( k \)-tuple \( (C_1, \ldots, C_k) \) of ordered multiset partitions into sets represents a \( k \)-deconcatenation of an ordered multiset partition into sets \( C \) if \( C_1 + \cdots + C_k = C \).

**Note:** This is not to be confused with `self.split_blocks()`, which splits each block of \( self \) before making \( k \)-tuples of ordered multiset partitions into sets.

EXAMPLES:

```sage
sage: OrderedMultisetPartitionIntoSets([[7, 1], [3, 4, 5]]).deconcatenate()
[[[1, 7], [3, 4, 5]], []], [[1, 7], [[3, 4, 5]]], [[], [[1, 7], [3, 4, 5]]]
sage: OrderedMultisetPartitionIntoSets(['b', 'c', ['a']]).deconcatenate()
[[['b', 'c'], ['a']], []], [[['b', 'c'], ['a']]], [[], [[b', 'c'], ['a']]]
sage: OrderedMultisetPartitionIntoSets(['a', 'b', 'c']).deconcatenate(3)
[[['a', 'b', 'c']], [], []], [[], [], [[a', 'b', 'c']]], [[], [], [[a', 'b', 'c']]]
```
**fatten**(grouping)

Return the ordered multiset partition into sets fatter than self, obtained by grouping together consecutive parts according to grouping (whenever this does not violate the strictness condition).

**INPUT:**

- grouping – a composition (or list) whose sum is the length of self

**EXAMPLES:**

Let us start with the composition:

```sage
C = OrderedMultisetPartitionIntoSets([[4,1,5], [2], [7,1]]); C
```

```
[{1,4,5}, {2}, {1,7}]
```

With grouping equal to (1, 1, 1), C is left unchanged:

```sage
C.fatten([1,1,1])
```

```
[{1,4,5}, {2}, {1,7}]
```

With grouping equal to (2, 1) or (1, 2), a union of consecutive parts is achieved:

```sage
C.fatten([2,1])
```

```
[{1,2,4,5}, {1,7}]
```

```sage
C.fatten([1,2])
```

```
[{1,4,5}, {1,2,7}]
```

However, the grouping (3) will throw an error, as 1 cannot appear twice in any block of C:

```sage
C.fatten(Composition([3]))
```

Traceback (most recent call last):
...
ValueError: [{1,4,5,2,1,7}] is not a valid ordered multiset partition into sets

**fatter()**

Return the set of ordered multiset partitions into sets which are fatter than self.

An ordered multiset partition into sets A is fatter than another B if, reading left-to-right, every block of A is the union of some consecutive blocks of B.

**EXAMPLES:**

```sage
C = OrderedMultisetPartitionIntoSets([[1,4,5], {2}, {1,7}]).fatter()
```

```
3
```

```sage
len(C)
```

```
3
```

```sage
sorted(C)
```

```
[[{1,4,5}, {2}, {1,7}], [{1,4,5}, {1,2,7}], [{1,2,4,5}, {1,7}]]
```

```sage
sorted(OrderedMultisetPartitionIntoSets([['a','b'],['c'],['a']]).fatter())
```

```
[['a','b'], ['c'], ['a']]
```

```sage
sorted(OrderedMultisetPartitionIntoSets([['a','b'],['c'],['a']]).fatter())
```

```
[['a','b'], ['c'], ['a']]
```

Some extreme cases:

```sage
list(OrderedMultisetPartitionIntoSets([['a','b'],['c']]).fatter())
```

```
[['a','b'], ['c']]    
```

```sage
list(OrderedMultisetPartitionIntoSets([]).fatter())
```

```
[[]]
```

```sage
A = OrderedMultisetPartitionIntoSets([[1], [2], [3], [4]]))
```

(continues on next page)
sage: B = OrderedMultisetPartitionIntoSets([[1,2,3,4]])
sage: A.fatter().issubset(B.finer())
True

**finer**(strong=False)

Return the set of ordered multiset partitions into sets that are finer than `self`.

An ordered multiset partition into sets `A` is finer than another `B` if, reading left-to-right, every block of `B` is the union of some consecutive blocks of `A`.

If optional argument `strong` is set to `True`, then return only those `A` whose blocks are deconcatenations of blocks of `B`. (Here, we view blocks of `B` as sorted lists instead of sets.)

**EXAMPLES:**

```python
sage: C = OrderedMultisetPartitionIntoSets([[3,2]]).finer()
sage: len(C)
3
sage: sorted(C, key=str)
[[[2,3]], [[2], [3]], [[3], [2]]]
sage: OrderedMultisetPartitionIntoSets([]).finer()
[[[]]]
sage: O = OrderedMultisetPartitionsIntoSets([1, 1, 'a', 'b'])
sage: o = O([{'1'}, {'a', 'b'}, {'1'}])
sage: sorted(o.finer(), key=str)
[[{'1'}, {'a', 'b'}, {'1'}], [{'1'}, {'a'}, {'b'}, {'1'}], [{'1'}, {'b'}, {'a'}, {'1'}]]
sage: o.finer() & o.fatter() == set([o])
True
```

**is_finer**(co)

Return `True` if the ordered multiset partition into sets `self` is finer than the composition `co`; otherwise, return `False`.

**EXAMPLES:**

```python
sage: OrderedMultisetPartitionIntoSets([[4],[1],[2]]).is_finer([[1,4],[2]])
True
sage: OrderedMultisetPartitionIntoSets([[1],[4],[2]]).is_finer([[1,4],[2]])
True
sage: OrderedMultisetPartitionIntoSets([[1,4],[1],[1]]).is_finer([[1,4],[2]])
False
```

**length()**

Return the number of blocks of `self`.

**EXAMPLES:**

```python
sage: OrderedMultisetPartitionIntoSets([[7,1],[3]]).length()
2
```

**letters()**

Return the set of distinct elements occurring within the blocks of `self`.

**EXAMPLES:**
```python
sage: C = OrderedMultisetPartitionIntoSets([[3, 4, 1], [2], [1, 2, 3, 7]]);
C
frozenset({1, 2, 3, 4, 7})
sage: C.letters()
{'1,3,4', '2', '1,2,3,7'}
major_index()
Return the major index of self.
The major index is a statistic on ordered multiset partitions into sets, which we define here via an example.
1. Sort each block in the list self in descending order to create a word w, keeping track of the original separation into blocks:

| in: | [[3,4,5], {2,3,4}, {1}, {4,5}] |
| out: | [ 5,4,3 , 4,3,2 , 1 , 5,4 ] |

2. Create a sequence v = (v₀, v₁, v₂,...) of length self.order()+1, built recursively by:
   1. v₀ = 0
   2. v_j = v_j−1 + δ(j), where δ(j) = 1 if j is the index of an end of a block, and zero otherwise.

| in: | [ 5,4,3 , 4,3,2 , 1 , 5,4 ] |
| out: | (0, 0, 0, 1, 1, 1, 2, 3, 3, 4) |

3. Compute ∑ j v_j, restricted to descent positions in w, i.e., sum over those j with w_j > w_{j+1}:

| in: | w: [5, 4, 3, 4, 3, 2, 1, 5, 4] |
| v: | (0, 0, 0, 1, 1, 1, 2, 3, 3, 4) |
| maj := | 0 +0 +1 +1 +2 +3 = 7 |

REFERENCES:
• [HRW2015]

EXAMPLES:
```
```
minimaj()

Return the minimaj statistic on ordered multiset partitions into sets.

We define minimaj via an example:

1. Sort the block in self as prescribed by self.minimaj_word(), keeping track of the original separation into blocks:

   | in: | {{1,5,7}, {2,4}, {5,6}, {4,6,8}, {1,3}, {1,2,3}} |
   | out: | (5,7,1 / 2,4 / 5,6 / 4,6,8 / 3,1 / 1,2,3) |

2. Record the indices where descents in this word occur:

   | word: | (5, 7, 1 / 2, 4 / 5, 6 / 4, 6, 8 / 3, 1 / 1, 2, 3) |
   | indices: | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
   | descents: | {2, 7, 10, 11} |

3. Compute the sum of the descents:

   minimaj = 2 + 7 + 10 + 11 = 30

REFERENCES:

- [HRW2015]

EXAMPLES:

```
sage: C = OrderedMultisetPartitionIntoSets([{{1,5,7}, {2,4}, {5,6}, {4,6,8}, {1,3}, {1,2,3}}])
sage: C.minimaj()
30
```

minimaj_blocks()

Return the minimaj ordering on blocks of self.

We define the ordering via the example below.

Sort the blocks \([B_1, ..., B_k]\) of self from right to left via:
1. Sort the last block $B_k$ in increasing order, call it the word $W_k$
2. If blocks $B_{i+1}, \ldots, B_k$ have been converted to words $W_{i+1}, \ldots, W_k$, use the letters in $B_i$ to make the unique word $W_i$ that has a factorization $W_i = (u, v)$ satisfying:
   - letters of $u$ and $v$ appear in increasing order, with $v$ possibly empty;
   - letters in $vu$ appear in increasing order;
   - $v[-1]$ is the largest letter $a \in B_i$ satisfying $a \leq W_{i+1}[0]$.

**EXAMPLES:**

```
sage: OrderedMultisetPartitionIntoSets([[1,5,7], [2,4], [5,6], [4,6,8], [1,3], ...
   --> [1,2,3]])
[[1,5,7], {2,4}, {5,6}, {4,6,8}, {1,3}, {1,2,3}]
sage: _.minimaj_blocks()
((5, 7, 1), (2, 4), (5, 6), (4, 6, 8), (3, 1), (1, 2, 3))
sage: OrderedMultisetPartitionIntoSets([]).minimaj_blocks()
()```

```
minimaj_word()
Return an ordering of self._multiset derived from the minimaj ordering on blocks of self.

See also:
OrderedMultisetPartitionIntoSets.minimaj_blocks().

**EXAMPLES:**

```
sage: C = OrderedMultisetPartitionIntoSets([[2,1], [1,2,3], [1,2], [3], [1]]);
C
[[1,2], {1,2,3}, {1,2}, {3}, {1}]
sage: C.minimaj_blocks()
((1, 2), (2, 3, 1), (1, 2), (3,), (1,))
sage: C.minimaj_word()
(1, 2, 2, 3, 1, 1, 2, 3, 1)```

```
multiset(as_dict=False)
Return the multiset corresponding to self.

INPUT:
   - as_dict – (default: False) whether to return the multiset as a tuple of a dict of multiplicities

**EXAMPLES:**

```
sage: C = OrderedMultisetPartitionIntoSets([[3, 4, 1], [2], [1, 2, 3, 7]]);
C
[[1,3,4], {2}, {1,2,3,7}]
sage: C.multiset()
(1, 1, 2, 2, 3, 3, 4, 7)
sage: C.multiset(as_dict=True)
{1: 2, 2: 2, 3: 2, 4: 1, 7: 1}
sage: OrderedMultisetPartitionIntoSets([]).multiset() == ()
True```

```
order()
Return the total number of elements in all blocks of self.

**EXAMPLES:**
From the code snippet provided, we can infer the following methods and their descriptions:

- **reversal()**: Returns the reverse ordered multiset partition into sets of self.
  
  Given an ordered multiset partition into sets \((B_1, B_2, \ldots, B_k)\), its reversal is defined to be the ordered multiset partition into sets \((B_k, \ldots, B_2, B_1)\).

- **shape_from_cardinality()**: Returns a composition that records the cardinality of each block of self.

- **shape_from_size()**: Returns a composition that records the sum of entries of each block of self.

- **shuffle_product(other, overlap=False)**: Returns the shuffles (with multiplicity) of blocks of self with blocks of other.
  
  In case optional argument overlap is True, instead return the allowable overlapping shuffles. An overlapping shuffle \(C\) is allowable if, whenever one of its blocks \(c\) comes from the union \(c = a \cup b\) of a block of self and a block of other, then this union is disjoint.

See also:

- Composition.shuffle_product()
EXAMPLES:

```python
sage: A = OrderedMultisetPartitionIntoSets([[2,1,3], [1,2]]);
    A
[[1,2,3], [1,2]]
sage: B = OrderedMultisetPartitionIntoSets([[3,4]]);
    B
[[3,4]]
sage: C = OrderedMultisetPartitionIntoSets([[4,5]]);
    C
[[4,5]]
sage: list(A.shuffle_product(B))
[[[1,2,3], [1,2], [3,4]],
 [[3,4], [1,2,3], [1,2]]]
sage: list(A.shuffle_product(B, overlap=True))
[[[1,2,3], [1,2], [3,4]],
 [[1,2,3], [3,4], [1,2]],
 [[3,4], [1,2,3], [1,2]],
 [[1,2,3], [1,2,3,4]]]
sage: list(A.shuffle_product(C, overlap=True))
[[[1,2,3], [1,2], [4,5]],
 [[1,2,3], [4,5], [1,2]],
 [[4,5], [1,2,3], [1,2]],
 [[1,2,3,4,5], [1,2]],
 [[1,2,3], [1,2,4,5]]]
```

size()

Return the size of `self` (that is, the sum of all integers in all blocks) if `self` is a list of subsets of positive integers.

Else, return None.

EXAMPLES:

```python
sage: C = OrderedMultisetPartitionIntoSets([[3, 4, 1], [2], [1, 2, 3, 7]]);
    C
[[1,3,4], [2], [1,2,3,7]]
sage: C.size()
23
sage: C.size() == sum(k for k in C.shape_from_size())
True
sage: OrderedMultisetPartitionIntoSets([[7,1],[3]]).size()
11
```

split_blocks(k=2)

Return a dictionary representing the k-splittings of `self`.

A k-tuple \((A^1, \ldots, A^k)\) of ordered multiset partitions into sets represents a k-splitting of an ordered multiset partition into sets \(A = \{b_1, \ldots, b_r\}\) if one can express each block \(b_i\) as an (ordered) disjoint union of sets \(b_i = b_{i1} \sqcup \cdots \sqcup b_{ik}\) (some possibly empty) so that each \(A^j\) is the ordered multiset partition into sets corresponding to the list \([b_{i1}, b_{i2}, \ldots, b_{ik}]\), excising empty sets appearing therein.

This operation represents the coproduct in Hopf algebra of ordered multiset partitions into sets in its natural basis [LM2018].

EXAMPLES:

```python
sage: sorted(OrderedMultisetPartitionIntoSets([[1,2],[3,4]]).split_blocks(),
        key=str)
[[], [[1,2], [3,4]],
 [[1,2], [3,4]], [[],
 [[1,2], [3]], [[4]],
 [[1,2], [4]], [[3]],
 [[1,2], [3,4]],
 [[1], [3,4]], [[2]],
 [[1], [3]], [[2], [4]]],
(continues on next page)
```
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(continued from previous page)

\[
\begin{align*}
&([\{1\}, \{4\}], [\{2\}, \{3\}]), \\
&([\{1\}], [\{2\}, \{3,4\}]), \\
&([\{2\}, \{3\}], [\{1\}, \{4\}]), \\
&([\{2\}, \{4\}], [\{1\}, \{3\}]), \\
&([\{2\}], [\{1\}, \{3,4\}]), \\
&([\{3\}], [\{1\}, \{2\}]), \\
&([\{4\}], [\{1\}, \{3\}]), \\
&([\{2\}], [\{1\}, \{3,4\}]), \\
&([\{2\}], [\{1\}, \{3\}], [\{4\}])
\end{align*}
\]

```
sage: sorted(OrderedMultisetPartitionIntoSets([[1,2]]).split_blocks(3), key=str)
[(\[]], [\[]], [\{1,2\}]), ([\[]], [\{1\}], [\{2\}]), 
([\[1\]], [\[]], [\{2\}]), 
([\[]], [\{1\}], [\{2\}]), 
([\[]], [\{1\}], [\{2\}]), 
([\[]], [\{1\}], [\{2\}]), 
([\[]], [\{1\}], [\{2\}], [\[]])
```

```
sage: OrderedMultisetPartitionIntoSets([[4],[4]]).split_blocks()
{([\[]], [\{4\}], [\{4\}]): 1, ([\{4\}], [\{4\}]): 2, ([\{4\}], [\{4\}], [\[]]): 1}
```

**to_tableaux_words()**

Return a sequence of lists corresponding to row words of (skew-)tableaux.

**OUTPUT:**

The minimaj bijection \( \phi \) of [BCHOPSY2017] applied to self.

**Todo:** Implement option for mapping to sequence of (skew-)tableaux?

**EXAMPLES:**

```
sage: co = ((1,2,4),(4,5),(3,), (4,6,1),(2,3,1),(1,),(2,5))
sage: OrderedMultisetPartitionIntoSets(co).to_tableaux_words()
[[5, 1], [3, 1], [6], [5, 4, 2], [1, 4, 3, 4, 2, 1, 2]]
```

**weight(as_weak_comp=False)**

Return a dictionary, with keys being the letters in self.letters() and values being their (positive) frequency.

Alternatively, if as weak comp is True, count the number of instances \( n_i \) for each distinct positive integer \( i \) across all blocks of self. Return as a list \([n_1, n_2, n_3, ..., n_k]\), where \( k \) is the max letter appearing in self.letters().

**EXAMPLES:**

```
sage: c = OrderedMultisetPartitionIntoSets([[6,1],[1,3],[1,3,6]])
sage: c.weight()
\{1: 3, 3: 2, 6: 2\}
sage: c.weight(as_weak_comp=True)
\[3, 0, 2, 0, 0, 2\]
```

**class** sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets(is_finite=None, **constraints)**

**Bases:** UniqueRepresentation, Parent

Ordered Multiset Partitions into Sets.
An ordered multiset partition into sets $c$ of a multiset $X$ is a list of nonempty subsets (not multisets), called the blocks of $c$, whose multi-union is $X$.

The number of blocks of $c$ is called its length. The order of $c$ is the cardinality of the multiset $X$. If, additionally, $X$ is a multiset of positive integers, then the size of $c$ is the sum of all elements of $X$.

The user may wish to focus on ordered multiset partitions into sets of a given size, or over a given alphabet. Hence, this class allows a variety of arguments as input.

INPUT:

Expects one or two arguments, with different behaviors resulting:

- One Argument:
  - $X$ – a dictionary or list or tuple (representing a multiset for $c$), or an integer (representing the size of $c$)

- Two Arguments:
  - $A$ – a list (representing allowable letters within blocks of $c$), or a positive integer (representing the maximal allowable letter)
  - $n$ – a nonnegative integer (the total number of letters within $c$)

Optional keyword arguments are as follows: (See corresponding methods in see `OrderedMultisetPartitionIntoSets` for more details.)

- `weight=X` (list or dictionary $X$) specifies the multiset for $c$
- `size=n` (integer $n$) specifies the size of $c$
- `alphabet=A` (iterable $A$) specifies allowable elements for the blocks of $c$
- `length=k` (integer $k$) specifies the number of blocks in the partition
- `min_length=k` (integer $k$) specifies minimum number of blocks in the partition
- `max_length=k` (integer $k$) specifies maximum number of blocks in the partition
- `order=n` (integer $n$) specifies the cardinality of the multiset that $c$ partitions
- `min_order=n` (integer $n$) specifies minimum number of elements in the partition
- `max_order=n` (integer $n$) specifies maximum number of elements in the partition

EXAMPLES:

Passing one argument to `OrderedMultisetPartitionsIntoSets`:

There are 5 ordered multiset partitions into sets of the multiset $\{1, 1, 4\}$:

```python
sage: OrderedMultisetPartitionsIntoSets([1,1,4]).cardinality()
5
```

Here is the list of them:

```python
sage: OrderedMultisetPartitionsIntoSets([1,1,4]).list()
[[[1], [1], {4}], [[1], {1,4}], [[1], {4}, {1}], [{1}, {4}, {1}], [{1,4}, {1}]]
```

By chance, there are also 5 ordered multiset partitions into sets of the integer 3:

```python
sage: OrderedMultisetPartitionsIntoSets(3).cardinality()
5
```

Here is the list of them:
Passing two arguments to `OrderedMultisetPartitionsIntoSets`

There are also 5 ordered multiset partitions into sets of order 2 over the alphabet `{1, 4}`:

```
sage: OrderedMultisetPartitionsIntoSets([1, 4], 2)
```

Ordered Multiset Partitions into Sets of order 2 over alphabet `{1, 4}`
```
sage: OrderedMultisetPartitionsIntoSets([1, 4], 2).cardinality()
```

5

Here is the list of them:

```
sage: OrderedMultisetPartitionsIntoSets([1, 4], 2).list()
```

```
[[1, 4]], [[1], {1}], [[1], {4}], [{4}, {1}], [{4}, {4}]
```

If no arguments are passed to `OrderedMultisetPartitionsIntoSets`, then the code returns all ordered multiset partitions into sets:

```
sage: OrderedMultisetPartitionsIntoSets()  
```

Ordered Multiset Partitions into Sets
```
sage: [] in OrderedMultisetPartitionsIntoSets()  
```

True
```
sage: [[2, 3], [1]] in OrderedMultisetPartitionsIntoSets()  
```

True
```
sage: [['a', 'b'], ['a']] in OrderedMultisetPartitionsIntoSets()  
```

True
```
sage: [[-2, 3], [3]] in OrderedMultisetPartitionsIntoSets()  
```

True
```
sage: [[2], [3, 3]] in OrderedMultisetPartitionsIntoSets()  
```

False

The following examples show how to test whether or not an object is an ordered multiset partition into sets:

```
sage: [[3, 2], [2]] in OrderedMultisetPartitionsIntoSets()  
```

True
```
sage: [[3, 2], [2]] in OrderedMultisetPartitionsIntoSets(7)  
```

True
```
sage: [[3, 2], [2]] in OrderedMultisetPartitionsIntoSets([2, 2, 3])  
```

True
```
sage: [[3, 2], [2]] in OrderedMultisetPartitionsIntoSets(5)  
```

False
Optional keyword arguments

Passing keyword arguments that are incompatible with required requirements results in an error; otherwise, the collection of ordered multiset partitions into sets is restricted accordingly:

The weight keyword:

This is used to specify which multiset $X$ is to be considered, if this multiset was not passed as one of the required arguments for `OrderedMultisetPartitionsIntoSets`. In principle, it is a dictionary, but weak compositions are also allowed. For example, the ordered multiset partitions into sets of integer 4 are listed by weight below:

```
sage: OrderedMultisetPartitionsIntoSets(4, weight=[0,0,0,1])
Ordered Multiset Partitions into Sets of integer 4 with constraint: weight={4: 1}
sage: OrderedMultisetPartitionsIntoSets(4, weight=[0,0,0,1]).list()
[[[4]]]
sage: OrderedMultisetPartitionsIntoSets(4, weight=[1,0,1]).list()
[[[1]], [[3]], [[1,3]], [[3], [1]]]
sage: OrderedMultisetPartitionsIntoSets(4, weight=[0,2]).list()
[[[2], [2]]]
sage: OrderedMultisetPartitionsIntoSets(4, weight=[0,1,1]).list()
[]
sage: OrderedMultisetPartitionsIntoSets(4, weight=[2,1]).list()
[[[1], [1]], [[2]], [[1], [1,2]], [[1,2], [2]], [[2], [1], [1]], [[2], [2]]]
sage: O1 = OrderedMultisetPartitionsIntoSets(weight=[2,0,1])
sage: O2 = OrderedMultisetPartitionsIntoSets(weight={1:2, 3:1})
sage: O1 == O2
True
sage: OrderedMultisetPartitionsIntoSets(4, weight=[4]).list()
[[[1], [1], [1], [1]]]
```

The size keyword:

This is used to constrain the sum of entries across all blocks of the ordered multiset partition into sets. (This size is not pre-determined when alphabet $A$ and order $d$ are passed as required arguments.) For example, the ordered multiset partitions into sets of order 3 over the alphabet $\{1, 2, 4\}$ that have size equal to 5 are as follows:

```
sage: OMPs = OrderedMultisetPartitionsIntoSets
sage: OMPs([1,2,4], 3, size=5).list()
[[[1,2], [2]], [[2], [1,2]], [[2], [2], [1]],
 [[2], [1], [2]], [[1], [2], [2]], [[1], [2], [2]]]
```

The alphabet option:

This is used to constrain which integers appear across all blocks of the ordered multiset partition into sets. For example, the ordered multiset partitions into sets of integer 4 are listed for different choices of alphabet below. Note that alphabet is allowed to be an integer or an iterable:

```
sage: OMPs = OrderedMultisetPartitionsIntoSets
sage: OMPs(4, alphabet=3).list()
[[[1,3]], [[3], [1]],
 [[1,2], [1]], [[2], [2]],
 [[2], [1], [1]], [[1], [3]],
 [[1], [1,2]], [[1], [2], [1]],
 [[1], [1], [2]], [[1], [1], [1], [1]]]
sage: OMPs(4, alphabet=3) == OMPs(4, alphabet=[1,2,3])
True
```

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sage: OMPs(4, alphabet=[3]).list()
[]
sage: OMPs(4, alphabet=[1,3]).list()
[[{1,3}, {1}], [{1}, {1}], [{1}, {3}]]
sage: OMPs(4, alphabet=[2]).list()
[[{2}, {2}]]
sage: OMPs(4, alphabet=[1,2]).list()
[[{1,2}, {1}], [{2}, {1}], [{1}, {1}], [{1}, {2}]]
sage: OMPs(4, alphabet=4).list() == OMPs(4).list()
True

The length, min_length, and max_length options:

These are used to constrain the number of blocks within the ordered multiset partitions into sets. For example, the ordered multiset partitions into sets of integer 4 of length exactly 2, at least 2, and at most 2 are given by:

sage: OrderedMultisetPartitionsIntoSets(4, length=2).list()
[[{1}, {1}], [{1,2}, {1}], [{2}, {2}], [{1}, {3}]]
sage: OrderedMultisetPartitionsIntoSets(4, min_length=3).list()
[[{1,2}, {1}], [{2}, {1}], [{1}, {1}], [{2}, {1}], [{1}, {1}, {1}]]
sage: OrderedMultisetPartitionsIntoSets(4, max_length=2).list()
[[{2}, {2}], [{1}, {1}], [{1}, {1}, {1}]]

The order, min_order, and max_order options:

These are used to constrain the number of elements across all blocks of the ordered multiset partitions into sets. For example, the ordered multiset partitions into sets of integer 4 are listed by order below:

sage: OrderedMultisetPartitionsIntoSets(4, order=1).list()
[[{4}]]
sage: OrderedMultisetPartitionsIntoSets(4, order=2).list()
[[{1,3}], [{1}, {1}], [{2}, {2}], [{1}, {3}]]
sage: OrderedMultisetPartitionsIntoSets(4, order=3).list()
[[{1,2}, {1}], [{2}, {1}], [{1}, {1}], [{1}, {2}], [{1}, {1}, {1}]]
sage: OrderedMultisetPartitionsIntoSets(4, order=4).list()
[[{1}, {1}, {1}, {1}]]

Also, here is a use of max_order, giving the ordered multiset partitions into sets of integer 4 with order 1 or 2:

sage: OrderedMultisetPartitionsIntoSets(4, max_order=2).list()
[[{4}], [{1,3}], [{3}, {1}], [{2}, {2}], [{1}, {3}]]

Element

alias of OrderedMultisetPartitionIntoSets

subset(size)

Return a subset of all ordered multiset partitions into sets.

INPUT:

- size – an integer representing a slice of all ordered multiset partitions into sets

The slice alluded to above is taken with respect to length, or to order, or to size, depending on the constraints of self.
EXAMPLES:

```python
sage: C = OrderedMultisetPartitionsIntoSets(weight={2:2, 3:1, 5:1})
sage: C.subset(3)
Ordered Multiset Partitions into Sets of multiset \{\{2, 2, 3, 5\}\} with constraint: length=3
sage: C = OrderedMultisetPartitionsIntoSets(weight={2:2, 3:1, 5:1}, min_length=2)
sage: C.subset(3)
Ordered Multiset Partitions into Sets of multiset \{\{2, 2, 3, 5\}\} with constraint: length=3
sage: C = OrderedMultisetPartitionsIntoSets(alphabet=[2,3,5])
sage: C.subset(3)
Ordered Multiset Partitions into Sets of order 3 over alphabet \{2, 3, 5\}
sage: C = OrderedMultisetPartitionsIntoSets(order=5)
sage: C.subset(3)
Ordered Multiset Partitions into Sets of integer 3 with constraint: order=5
sage: C = OrderedMultisetPartitionsIntoSets(alphabet=[2,3,5], order=5, length=3)
sage: C.subset(3)
Ordered Multiset Partitions into Sets of order 3 over alphabet \{2, 3, 5\} with constraint: length=3
sage: C = OrderedMultisetPartitionsIntoSets()  
sage: C.subset(3)
Ordered Multiset Partitions into Sets of integer 3
sage: C.subset(3) == OrderedMultisetPartitionsIntoSets(3)
True
```

class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_X(X):

Bases: `OrderedMultisetPartitionsIntoSets`

Class of ordered multiset partitions into sets of a fixed multiset $X$.

**cardinality()**

Return the number of ordered partitions of multiset $X$.

**random_element()**

Return a random element of self.

This method does not return elements of self with uniform probability, but it does cover all elements. The scheme is as follows:

- produce a random permutation $p$ of the multiset;
- create blocks of an OMP fat by breaking $p$ after non-ascents;
- take a random element of fat.finer().

EXAMPLES:

```python
sage: OrderedMultisetPartitionsIntoSets([1,1,3]).random_element()  # random
[[1], [1, 3]]
sage: OrderedMultisetPartitionsIntoSets([1,1,3]).random_element()  # random
[[3], [1], [1]]
sage: OMP = OrderedMultisetPartitionsIntoSets([1,1,3,3])
sage: d = {}
sage: for _ in range(1000):
    d[C.subset(3)] += 1
```

(continues on next page)
class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_X_constraints

Bases: OrderedMultisetPartitionsIntoSets

Class of ordered multiset partitions into sets of a fixed multiset $X$ satisfying constraints.

class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_all_constraints

Bases: OrderedMultisetPartitionsIntoSets

All ordered multiset partitions into sets (with or without constraints).

EXAMPLES:

sage: C = OrderedMultisetPartitionsIntoSets(); C
Ordered Multiset Partitions into Sets
sage: [[1], [1, 'a']] in C
True
sage: OrderedMultisetPartitionsIntoSets(weight=[2,0,1], length=2)
Ordered Multiset Partitions into Sets of multiset {{1, 1, 3}} with constraint:
- length=2

class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_alph_d(A, d)

Bases: OrderedMultisetPartitionsIntoSets

Class of ordered multiset partitions into sets of specified order $d$ over a fixed alphabet $A$.

cardinality()

Return the number of ordered partitions of order self._order on alphabet self._alphabet.

random_element()

Return a random element of self.

This method does not return elements of self with uniform probability, but it does cover all elements. The scheme is as follows:

- produce a random composition $C$;
- choose random subsets of self._alphabet of size $c$ for each $c$ in $C$.

EXAMPLES:

sage: OrderedMultisetPartitionsIntoSets([1,4], 3).random_element()  # random
[[4], [1, 4]]
sage: OrderedMultisetPartitionsIntoSets([1,3], 4).random_element()  # random
[[1, 3], [1], [3]]
sage: OMP = OrderedMultisetPartitionsIntoSets([2,3,4], 2)
sage: d = {} 
sage: for _ in range(1200):
    ....:     x = OMP.random_element()
    ....:     d[x] = d.get(x, 0) + 1
sage: d.values() # random
[192, 68, 73, 61, 69, 60, 77, 204, 210, 66, 53, 67]

class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_alph_d_constraints

Bases: OrderedMultisetPartitionsIntoSets

Class of ordered multiset partitions into sets of specified order \(d\) over a fixed alphabet \(A\) satisfying constraints.

class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_n(n)

Bases: OrderedMultisetPartitionsIntoSets

Ordered multiset partitions into sets of a fixed integer \(n\).

cardinality()

Return the number of elements in \(self\).

random_element()

Return a random element of \(self\). This method does not return elements of \(self\) with uniform probability, but it does cover all elements. The scheme is as follows:

- produce a random composition \(C\);
- choose a random partition of \(c\) into distinct parts for each \(c\) in \(C\).

EXAMPLES:

sage: OrderedMultisetPartitionsIntoSets(5).random_element() # random
[[1,2], [1], [1]]
sage: OrderedMultisetPartitionsIntoSets(5).random_element() # random
[[2], [1,2]]

sage: OMP = OrderedMultisetPartitionsIntoSets(5)
sage: d = {} 
sage: for _ in range(1100):
    ....:     x = OMP.random_element()
    ....:     d[x] = d.get(x, 0) + 1
sage: d.values() # random
[72, 73, 162, 78, 135, 75, 109, 65, 135, 134, 62]

class sage.combinat.multiset_partition_into_sets_ordered.OrderedMultisetPartitionsIntoSets_n_constraints

Bases: OrderedMultisetPartitionsIntoSets

Class of ordered multiset partitions into sets of a fixed integer \(n\) satisfying constraints.

5.1. Comprehensive Module List 1435
5.1.141 Non-commutative symmetric functions and quasi-symmetric functions

- Introduction to Quasisymmetric Functions
- Non-Commutative Symmetric Functions (NCSF)
- Quasi-Symmetric Functions (QSym)
- Generic code for bases

5.1.142 Common combinatorial tools

REFERENCES:
sage.combinat.ncsf_qsym.combinatorics.coeff_dab(I, J)
Return the number of standard composition tableaux of shape $I$ with descent composition $J$.

INPUT:
- $I$, $J$ – compositions

OUTPUT:
- An integer

EXAMPLES:

```
sage: from sage.combinat.ncsf_qsym.combinatorics import coeff_dab
sage: coeff_dab(Composition([2,1]),Composition([2,1]))
1
sage: coeff_dab(Composition([1,1,2]),Composition([1,2,1]))
0
```

sage.combinat.ncsf_qsym.combinatorics.coeff_ell(J, I)
Returns the coefficient $\ell_{J,I}$ as defined in [NCSF].

INPUT:
- $J$ – a composition
- $I$ – a composition refining $J$

OUTPUT:
- integer

EXAMPLES:

```
sage: from sage.combinat.ncsf_qsym.combinatorics import coeff_ell
sage: coeff_ell(Composition([1,1,1]), Composition([2,1]))
2
sage: coeff_ell(Composition([2,1]), Composition([3]))
2
```

sage.combinat.ncsf_qsym.combinatorics.coeff_lp(J, I)
Returns the coefficient $lp_{J,I}$ as defined in [NCSF].

INPUT:
- $J$ – a composition
- $I$ – a composition refining $J$
OUTPUT:
• integer
EXAMPLES:
```
sage: from sage.combinat.ncsf_qsym.combinatorics import coeff_lp
sage: coeff_lp(Composition([1,1,1]), Composition([2,1]))
1
sage: coeff_lp(Composition([2,1]), Composition([3]))
1
```
sage.combinat.ncsf_qsym.combinatorics.coeff_pi(J,I)

Returns the coefficient $\pi_{J,I}$ as defined in [NCSF].

INPUT:
• J – a composition
• I – a composition refining J

OUTPUT:
• integer

EXAMPLES:
```
sage: from sage.combinat.ncsf_qsym.combinatorics import coeff_pi
sage: coeff_pi(Composition([1,1,1]), Composition([2,1]))
2
sage: coeff_pi(Composition([2,1]), Composition([3]))
6
```
sage.combinat.ncsf_qsym.combinatorics.coeff_sp(J,I)

Returns the coefficient $sp_{J,I}$ as defined in [NCSF].

INPUT:
• J – a composition
• I – a composition refining J

OUTPUT:
• integer

EXAMPLES:
```
sage: from sage.combinat.ncsf_qsym.combinatorics import coeff_sp
sage: coeff_sp(Composition([1,1,1]), Composition([2,1]))
2
sage: coeff_sp(Composition([2,1]), Composition([3]))
4
```
sage.combinat.ncsf_qsym.combinatorics.compositions_order(n)

Return the compositions of $n$ ordered as defined in [QSCHUR].

Let $S(\gamma)$ return the composition $\gamma$ after sorting. For compositions $\alpha$ and $\beta$, we order $\alpha \triangleright \beta$ if
1) $S(\alpha) > S(\beta)$ lexicographically, or
2) $S(\alpha) = S(\beta)$ and $\alpha > \beta$ lexicographically.
INPUT:
• \( n \) – a positive integer

OUTPUT:
• A list of the compositions of \( n \) sorted into decreasing order by \( \triangleright \)

EXAMPLES:
```python
sage: from sage.combinat.ncsf_qsym.combinatorics import compositions_order
sage: compositions_order(3)
[[3], [2, 1], [1, 2], [1, 1, 1]]
sage: compositions_order(4)
[[4], [3, 1], [1, 3], [2, 2], [2, 1, 1], [1, 2, 1], [1, 1, 2], [1, 1, 1, 1]]
```

\texttt{sage.combinat.ncsf_qsym.combinatorics.m\_to\_s\_stat}(\( R, I, K \))

Return the coefficient of the complete non-commutative symmetric function \( S^K \) in the expansion of the monomial non-commutative symmetric function \( M^I \) with respect to the complete basis over the ring \( R \). This is the coefficient in formula (36) of Tevlin’s paper [Tev2007].

INPUT:
• \( R \) – A ring, supposed to be a \( \mathbb{Q} \)-algebra
• \( I, K \) – compositions

OUTPUT:
• The coefficient of \( S^K \) in the expansion of \( M^I \) in the complete basis of the non-commutative symmetric functions over \( R \).

EXAMPLES:
```python
sage: from sage.combinat.ncsf_qsym.combinatorics import m_to_s_stat
sage: m_to_s_stat(QQ, Composition([2,1]), Composition([1,1,1]))
-1
sage: m_to_s_stat(QQ, Composition([3]), Composition([1,2]))
-2
sage: m_to_s_stat(QQ, Composition([2,1,2]), Composition([2,1,2]))
8/3
```

\texttt{sage.combinat.ncsf_qsym.combinatorics.number\_of\_SSRCT}(\text{shape\_comp})

The number of semi-standard reverse composition tableaux.

The dual quasisymmetric-Schur functions satisfy a left Pieri rule where \( S_\nu dQS_\alpha \) is a sum over dual quasisymmetric-Schur functions indexed by compositions which contain the composition \( \gamma \). The definition of an SSRCT comes from this rule. The number of SSRCT of content \( \beta \) and shape \( \alpha \) is equal to the number of SSRCT of content \( (\beta_2, \ldots, \beta_\ell) \) and shape \( \gamma \) where \( dQS_\alpha \) appears in the expansion of \( S_\beta dQS_\gamma \).

In sage the recording tableau for these objects are called \texttt{CompositionTableaux}.

INPUT:
• \text{content\_comp, shape\_comp} – compositions

OUTPUT:
• An integer

EXAMPLES:
```python
sage: from sage.combinat.ncsf_qsym.combinatorics import number_of_SSRCT
sage: number_of_SSRCT(Composition([3,1]), Composition([1,3]))
0
sage: number_of_SSRCT(Composition([1,2,1]), Composition([1,3]))
1
sage: number_of_SSRCT(Composition([1,1,2,2]), Composition([3,3]))
2
sage: all(CompositionTableaux(be).cardinality()
    == sum(number_of_SSRCT(al, be)*binomial(4, len(al))
    for al in Compositions(4))
    for be in Compositions(4))
True
```

```python
sage.combinat.ncsf_qsym.combinatorics.number_of_fCT(shape_comp)
```

Return the number of Immaculate tableaux of shape `shape_comp` and content `content_comp`.

See [BBSSZ2012], Definition 3.9, for the notion of an immaculate tableau.

**INPUT:**

- `content_comp, shape_comp` – compositions

**OUTPUT:**

- An integer

**EXAMPLES:**

```python
sage: from sage.combinat.ncsf_qsym.combinatorics import number_of_fCT
sage: number_of_fCT(Composition([3,1]), Composition([1,3]))
0
sage: number_of_fCT(Composition([1,2,1]), Composition([1,3]))
1
sage: number_of_fCT(Composition([1,1,3,1]), Composition([2,1,3]))
2
```

### 5.1.143 Generic code for bases

This is a collection of code that is shared by bases of noncommutative symmetric functions and quasisymmetric functions.

**AUTHORS:**

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- Franco Saliola
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```python
class sage.combinat.ncsf_qsym.generic_basis_code.AlgebraMorphism(domain, on_generators, position=0, codomain=None, category=None, anti=False)
```

**Bases:** `ModuleMorphismByLinearity`

A class for algebra morphism defined on a free algebra from the image of the generators
class sage.combinat.ncsf_qsym.generic_basis_code.BasesOfQSymOrNCSF(parent_with_realization)

Bases: Category_realization_of_parent

class ElementMethods
Bases: object

degree()

The maximum of the degrees of the homogeneous summands.

See also:

homogeneous_degree()

EXAMPLES:

sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.degree()
2
sage: (x^3 + 4*y^2).degree()
6
sage: ((1 + x)^3).degree()
6

sage: F = QuasiSymmetricFunctions(QQ).F()
sage: (x, y) = (F[2], F[3])
sage: x.degree()
2
sage: (x^3 + 4*y^2).degree()
6
sage: ((1 + x)^3).degree()
6

degree_negation()

Return the image of self under the degree negation automorphism of the parent of self.

The degree negation is the automorphism which scales every homogeneous element of degree $k$ by $(-1)^k$ (for all $k$).

Calling degree_negation(self) is equivalent to calling self.parent().degree_negation(self).

EXAMPLES:

sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: f = 2*S[2, 1] + 4*S[1, 1] - 5*S[1, 2] - 3*S[]
sage: f.degree_negation()
-3*S[] + 4*S[1, 1] + 5*S[1, 2] - 2*S[2, 1]

sage: QSym = QuasiSymmetricFunctions(QQ)
sage: dI = QSym.dualImmaculate()
sage: f.degree_negation()
Todo: Generalize this to all graded vector spaces?

duality_pairing(y)
The duality pairing between elements of \(N\text{Sym}\) and elements of \(Q\text{Sym}\).
The complete basis is dual to the monomial basis with respect to this pairing.

INPUT:

\* \(y\) – an element of the dual Hopf algebra of \(self\)

OUTPUT:

\* The result of pairing \(self\) with \(y\).

EXAMPLES:

```
sage: R = NonCommutativeSymmetricFunctions(QQ).Ribbon()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: R[1,1,2].duality_pairing(F[1,1,2])
1
sage: R[1,2,1].duality_pairing(F[1,1,2])
0
```

skew_by(y, side='left')
The operation which is dual to multiplication by \(y\), where \(y\) is an element of the dual space of \(self\).
This is calculated through the coproduct of \(self\) and the expansion of \(y\) in the dual basis.

INPUT:

\* \(y\) – an element of the dual Hopf algebra of \(self\)
\* \(side\) – (Default='left') Either 'left' or 'right'

OUTPUT:

\* The result of skewing \(self\) by \(y\), on the side \(side\)

EXAMPLES:

Skewing an element of NCSF by an element of QSym:

```
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: R[1,2,2].skew_by(F[1,1])
sage: R[1,2,2].skew_by(F[2])
```

Skewing an element of QSym by an element of NCSF:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: R = NonCommutativeSymmetricFunctions(QQ).R()
sage: F = QuasiSymmetricFunctions(QQ).F()
sage: F[3,2].skew_by(R[1,1])
```

(continues on next page)
class ParentMethods

    Bases: object

    alternating_sum_of_compositions(n)

        Alternating sum over compositions of n.

        Note that this differs from the method alternating_sum_of_finer_compositions() because the
        coefficient of the composition 1^n is positive. This method is used in the expansion of the elementary
        generators into the complete generators and vice versa.

        INPUT:
        • n - a positive integer

        OUTPUT:
        • The expansion of the complete generator indexed by n into the elementary basis.

        EXAMPLES:

        sage: L = NonCommutativeSymmetricFunctions(QQ).L()
        sage: L.alternating_sum_of_compositions(0)
        L[]
        sage: L.alternating_sum_of_compositions(1)
        L[1]
        sage: L.alternating_sum_of_compositions(2)
        L[1, 1] - L[2]
        sage: L.alternating_sum_of_compositions(3)
alternating_sum_of_fatter_compositions(\texttt{composition})

Return the alternating sum of fatter compositions in a basis of the non-commutative symmetric functions.

INPUT:
• \texttt{composition} – a composition

OUTPUT:
• The alternating sum of the compositions fatter than \texttt{composition}, in the basis \texttt{self}. The alternation is upon the length of the compositions, and is normalized so that \texttt{composition} has coefficient 1.

EXAMPLES:

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: elementary = NCSF.elementary()
sage: elementary.alternating_sum_of_fatter_compositions(Composition([2, 2, 1]))
sage: elementary.alternating_sum_of_fatter_compositions(Composition([1, 2]))
-L[1, 1, 1] + L[1, 2]
```

alternating_sum_of_finer_compositions(\texttt{composition, conjugate=False})

Return the alternating sum of finer compositions in a basis of the non-commutative symmetric functions.

INPUT:
• \texttt{composition} – a composition
• \texttt{conjugate} – (default: \texttt{False}) a boolean

OUTPUT:
• The alternating sum of the compositions finer than \texttt{composition}, in the basis \texttt{self}. The alternation is upon the length of the compositions, and is normalized so that \texttt{composition} has coefficient 1. If the variable \texttt{conjugate} is set to \texttt{True}, then the conjugate of \texttt{composition} is used instead of \texttt{composition}.

EXAMPLES:

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: elementary = NCSF.elementary()
sage: elementary.alternating_sum_of_finer_compositions(Composition([2, 2, 1]))
L[1, 1, 1, 1, 1] - L[1, 1, 2, 1] - L[2, 1, 1, 1] + L[2, 2, 1]
sage: elementary.alternating_sum_of_finer_compositions(Composition([1, 2]))
-L[1, 1, 1] + L[1, 2]
```

counit_on_basis(I)

The counit is defined by sending all elements of positive degree to zero.

EXAMPLES:

```python
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S.counit_on_basis([1, 3])
0
```
sage: M = QuasiSymmetricFunctions(QQ).M()
sage: M.counit_on_basis([1,3])
0

degree_negation(element)

Return the image of element under the degree negation automorphism of self.

The degree negation is the automorphism which scales every homogeneous element of degree \(k\) by \((-1)^k\) (for all \(k\)).

INPUT:

- element – element of self

EXAMPLES:

sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: f = 2*S[2,1] + 4*S[1,1] - 5*S[1,2] - 3*S[]
sage: S.degree_negation(f)
-3*S[] + 4*S[1, 1] + 5*S[1, 2] - 2*S[2, 1]

sage: QSym = QuasiSymmetricFunctions(QQ)
sage: dI = QSym.dualImmaculate()
sage: dI.degree_negation(f)

Todo: Generalize this to all graded vector spaces?

degree_on_basis(I)

Return the degree of the basis element indexed by \(I\).

INPUT:

- \(I\) – a composition

OUTPUT:

- The degree of the non-commutative symmetric function basis element of self indexed by \(I\). By definition, this is the size of the composition \(I\).

EXAMPLES:

sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: R.degree_on_basis(Composition([2,3]))
5
sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: M.degree_on_basis(Composition([3,2]))
5
sage: M.degree_on_basis(Composition([]))
0

duality_pairing(x, y)

The duality pairing between elements of \(NSym\) and elements of \(QSym\).

This is a default implementation that uses self.realizations_of().a_realization() and its dual basis.

INPUT:
• $x$ – an element of `self`
• $y$ – an element in the dual basis of `self`

**OUTPUT:**

• The result of pairing the function $x$ from `self` with the function $y$ from the dual basis of `self`

**EXAMPLES:**

```sage
code here...
```

**duality_pairing_by_coercion**($x$, $y$)

The duality pairing between elements of NSym and elements of QSym.

This is a default implementation that uses `self.realizations_of().a_realization()` and its dual basis.

**INPUT:**

• $x$ – an element of `self`
• $y$ – an element in the dual basis of `self`

**OUTPUT:**

• The result of pairing the function $x$ from `self` with the function $y$ from the dual basis of `self`

**EXAMPLES:**

```sage
code here...
```
The matrix of scalar products between elements of NSym and elements of QSym.

`duality_pairing_matrix(basis, degree)`

The matrix of scalar products between elements of NSym and elements of QSym.

**INPUT:**
- `basis` – A basis of the dual Hopf algebra
- `degree` – a non-negative integer

**OUTPUT:**
- The matrix of scalar products between the basis `self` and the basis `basis` in the dual Hopf algebra in degree `degree`.

**EXAMPLES:**

The ribbon basis of NCSF is dual to the fundamental basis of QSym:

```
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: R.duality_pairing_matrix(F, 3)

[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
sage: F.duality_pairing_matrix(R, 3)

[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

The complete basis of NCSF is dual to the monomial basis of QSym:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).complete()
sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: S.duality_pairing_matrix(M, 3)

[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
sage: M.duality_pairing_matrix(S, 3)

[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

The matrix between the ribbon basis of NCSF and the monomial basis of QSym:

```
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: R.duality_pairing_matrix(M, 3)

[ 1 -1 -1  1]
[ 0  1  0 -1]
[ 0  0  1 -1]
[ 0  0  0  1]
```

(continues on next page)
The matrix between the complete basis of NCSF and the fundamental basis of QSym:

```python
sage: S = NonCommutativeSymmetricFunctions(QQ).complete()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: S.duality_pairing_matrix(F, 3)
```

```
[1 1 1 1]
[0 1 0 1]
[0 0 1 1]
[0 0 0 1]
```

A base case test:

```python
sage: R.duality_pairing_matrix(M, 0)
```

```
[1]
```

### one_basis()

Return the empty composition.

**OUTPUT:**

- The empty composition.

**EXAMPLES:**

```python
sage: L = NonCommutativeSymmetricFunctions(QQ).L()
sage: parent(L)
<class 'sage.combinat.ncsf_qsym.ncsf.NonCommutativeSymmetricFunctions.Elementary_with_category'>
sage: parent(L).one_basis()
[]
```

### skew(x, y, side='left')

Return a function x in self skewed by a function y in the Hopf dual of self.

**INPUT:**

- x – a non-commutative or quasi-symmetric function; it is an element of self
- y – a quasi-symmetric or non-commutative symmetric function; it is an element of the dual algebra of self
- side – (default: 'left') either 'left' or 'right'

**OUTPUT:**

- The result of skewing the element x by the Hopf algebra element y (either from the left or from the right, as determined by side), written in the basis self.

**EXAMPLES:**

```python
sage: S = NonCommutativeSymmetricFunctions(QQ).complete()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: S.skew(S[2,2,2], F[1,1])
S[1, 1, 2] + S[1, 2, 1] + S[2, 1, 1]
sage: F[2]
S[1, 1, 2] + S[1, 2, 1] + S[2, 1, 1] + 3*S[2, 2]
```
Combinatorics, Release 10.0

```python
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: R.skew(R[2,2,2], F[1,1])
sage: R.skew(R[2,2,2], F[2])
```

```python
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: R = NonCommutativeSymmetricFunctions(QQ).R()
sage: M = QuasiSymmetricFunctions(QQ).M()
sage: M.skew(M[3,2], S[2])
0
sage: M.skew(M[3,2], S[2], side='right')
M[3]
sage: M.skew(M[3,2], S[3])
M[2]
sage: M.skew(M[3,2], S[3], side='right')
0
```

**sum_of_fatter_compositions**(*composition*)

Return the sum of all fatter compositions.

**INPUT:**

- *composition* – a composition

**OUTPUT:**

- the sum of all basis elements which are indexed by compositions fatter (coarser?) than *composition*.

**EXAMPLES:**

```python
sage: L = NonCommutativeSymmetricFunctions(QQ).L()
sage: L.sum_of_fatter_compositions(Composition([2,1]))
L[1, 1, 1] + L[2, 1]
sage: R = NonCommutativeSymmetricFunctions(QQ).R()
sage: R.sum_of_fatter_compositions(Composition([1,3]))
R[1, 1, 1, 1] + R[1, 1, 2] + R[1, 2, 1] + R[1, 3]
```

**sum_of_finer_compositions**(*composition*)

Return the sum of all finer compositions.

**INPUT:**

- *composition* – a composition

**OUTPUT:**

- The sum of all basis elements which are indexed by compositions finer than *composition*.

**EXAMPLES:**

```python
sage: L = NonCommutativeSymmetricFunctions(QQ).L()
sage: L.sum_of_finer_compositions(Composition([2,1]))
L[1, 1, 1] + L[2, 1]
sage: R = NonCommutativeSymmetricFunctions(QQ).R()
sage: R.sum_of_finer_compositions(Composition([1,3]))
R[1, 1, 1, 1] + R[1, 1, 2] + R[1, 2, 1] + R[1, 3]
```

**sum_of_partition_rearrangements**(*par*)

Return the sum of all basis elements indexed by compositions which can be sorted to obtain a given...
Combinatorics, Release 10.0

**partition.**

**INPUT:**
- \( \text{par} \) – a partition

**OUTPUT:**
- The sum of all self basis elements indexed by compositions which are permutations of \( \text{par} \) (without multiplicity).

**EXAMPLES:**

```python
sage: NCSF=NonCommutativeSymmetricFunctions(QQ)
sage: elementary = NCSF.elementary()
sage: elementary.sum_of_partition_rearrangements(Partition([2,2,1]))
L[1, 2, 2] + L[2, 1, 2] + L[2, 2, 1]
sage: elementary.sum_of_partition_rearrangements(Partition([3,2,1]))
L[1, 2, 3] + L[1, 3, 2] + L[2, 1, 3] + L[2, 3, 1] + L[3, 1, 2] + L[3, 2, 1]
sage: elementary.sum_of_partition_rearrangements(Partition([]))
L[]
```

**super_categories()**

**class** sage.combinat.ncsf_qsym.generic_basis_code.GradedModulesWithInternalProduct(*base, name=None)*

**Bases:** Category_over_base_ring

Constructs the class of modules with internal product. This is used to give an internal product structure to the non-commutative symmetric functions.

**EXAMPLES:**

```python
sage: from sage.combinat.ncsf_qsym.generic_basis_code import GradedModulesWithInternalProduct
sage: GradedModulesWithInternalProduct(QQ)
```

**class** ElementMethods

**Bases:** object

**internal_product**(other)

Return the internal product of two non-commutative symmetric functions.

The internal product on the algebra of non-commutative symmetric functions is adjoint to the internal coproduct on the algebra of quasisymmetric functions with respect to the duality pairing between these two algebras. This means, explicitly, that any two non-commutative symmetric functions \( f \) and \( g \) and any quasi-symmetric function \( h \) satisfy

\[
\langle f \ast g, h \rangle = \sum_i \langle f, h'_i \rangle \langle g, h''_i \rangle,
\]

where we write \( \Delta^X(h) \) as \( \sum_i h'_i \otimes h''_i \). Here, \( f \ast g \) denotes the internal product of the non-commutative symmetric functions \( f \) and \( g \).

If \( f \) and \( g \) are two homogeneous elements of \( NSym \) having distinct degrees, then the internal product \( f \ast g \) is zero.

Explicit formulas can be given for internal products of elements of the complete and the Psi bases. First, the formula for the Complete basis ([NCSF1] Proposition 5.1): If \( I \) and \( J \) are two compositions...
of lengths \( p \) and \( q \), respectively, then the corresponding Complete homogeneous non-commutative symmetric functions \( S^I \) and \( S^J \) have internal product

\[
S^I \ast S^J = \sum \text{comp} M,
\]

where the sum ranges over all \( p \times q \)-matrices \( M \in \mathbb{N}^{p \times q} \) (with nonnegative integers as entries) whose row sum vector is \( I \) (that is, the sum of the entries of the \( r \)-th row is the \( r \)-th part of \( I \) for all \( r \)) and whose column sum vector is \( J \) (that is, the sum of all entries of the \( s \)-th row is the \( s \)-th part of \( J \) for all \( s \)). Here, for any \( M \in \mathbb{N}^{p \times q} \), we denote by \( \text{comp} M \) the composition obtained by reading the entries of the matrix \( M \) in the usual order (row by row, proceeding left to right in each row, traversing the rows from top to bottom).

The formula on the Psi basis ([NCSF2] Lemma 3.10) is more complicated. Let \( I \) and \( J \) be two compositions of lengths \( p \) and \( q \), respectively, having the same size \( |I| = |J| \). We denote by \( \Psi^K \) the element of the Psi basis corresponding to any composition \( K \).

- If \( p > q \), then \( \Psi^I \ast \Psi^J \) is plainly 0.
- Assume that \( p = q \). Let \( \delta_{I,J} \) denote the integer 1 if the compositions \( I \) and \( J \) are permutations of each other, and the integer 0 otherwise. For every positive integer \( i \), let \( m_i \) denote the number of parts of \( I \) equal to \( i \). Then, \( \Psi^I \ast \Psi^J = \delta_{I,J} \prod_{i \geq 0} i^{m_i} \Psi^I \).
- Now assume that \( p < q \). Write the composition \( I \) as \( I = (i_1,i_2,\ldots,i_p) \). For every nonempty composition \( K = (k_1,k_2,\ldots,k_q) \), denote by \( \Gamma_K \) the non-commutative symmetric function \( k_1\cdot\ldots\cdot[k_{k_1},\Psi_{k_2},\Psi_{k_3},\ldots,\Psi_{k_q}] \). For any subset \( A \) of \( \{1,2,\ldots,q\} \), let \( J_A \) be the composition obtained from \( J \) by removing the \( r \)-th parts for all \( r \notin A \) (while keeping the \( r \)-th parts for all \( r \in A \) in order). Then, \( \Psi^I \ast \Psi^J \) equals the sum of \( \Gamma_{J_{K_1}} \Gamma_{J_{K_2}} \cdots \Gamma_{J_{K_p}} \) over all ordered set partitions \( (K_1,K_2,\ldots,K_p) \) of \( \{1,2,\ldots,q\} \) into \( p \) parts such that each \( 1 \leq k \leq p \) satisfies \( |J_{K_k}| = i_k \).

(See OrderedSetPartition() for the meaning of “ordered set partition.”)

Aliases for internal_product() are itensor() and kronecker_product().

INPUT:
- other – another non-commutative symmetric function

OUTPUT:
- The result of taking the internal product of self with other.

EXAMPLES:

```sage
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: x = S.an_element(); x
2*S[] + 2*S[1] + 3*S[1, 1]
sage: x.internal_product(S[2])
3*S[1, 1]
sage: x.internal_product(S[1])
2*S[1]
sage: S[1,2].internal_product(S[1,2])
S[1, 1, 1] + S[1, 2]
```

Let us check the duality between the inner product and the inner coproduct in degree 4:

```sage
sage: M = QuasiSymmetricFunctions(FiniteField(29)).M()
sage: S = NonCommutativeSymmetricFunctions(FiniteField(29)).S()
sage: def tensor_incorp(f, g, h): # computes \sum_i \left< f, h_i \right> \cdot \left< h_i, g \right>
    result = h.base_ring().zero()
    h_parent = h.parent()
    for partition_pair, coeff in h.internal_coproduct().monomial_coefficients().items():
        result += coeff * h_parent(left< g, h"^i \right> \\left< f, h_i \right>)
    return result
    ....:
    ....:
    ....:
    ....:
```

(continues on next page)
The internal product on the algebra of non-commutative symmetric functions commutes with the canonical commutative projection on the symmetric functions:

```python
sage: S = NonCommutativeSymmetricFunctions(ZZ).S()
sage: e = SymmetricFunctions(ZZ).e()
sage: def int_pr_of_S_in_e(I, J):
    ....:     return (S[I].internal_product(S[J])).to_symmetric_function()
sage: all( all( int_pr_of_S_in_e(I, J)
    ....:         == S[I].to_symmetric_function().internal_product(S[J].to_
    ....:             symmetric_function())
    ....:     for I in Compositions(3) )
    ....:     for J in Compositions(3) )
True
```

`itensor`(*other*)

Return the internal product of two non-commutative symmetric functions.

The internal product on the algebra of non-commutative symmetric functions is adjoint to the internal coproduct on the algebra of quasisymmetric functions with respect to the duality pairing between these two algebras. This means, explicitly, that any two non-commutative symmetric functions $f$ and $g$ and any quasi-symmetric function $h$ satisfy

$$\langle f \ast g, h \rangle = \sum_i \langle f, h'_i \rangle \langle g, h''_i \rangle,$$

where we write $\Delta^\times(h)$ as $\sum_i h'_i \otimes h''_i$. Here, $f \ast g$ denotes the internal product of the non-commutative symmetric functions $f$ and $g$.

If $f$ and $g$ are two homogeneous elements of $NSym$ having distinct degrees, then the internal product $f \ast g$ is zero.

Explicit formulas can be given for internal products of elements of the complete and the Psi bases. First, the formula for the Complete basis ([NCSF1] Proposition 5.1): If $I$ and $J$ are two compositions of lengths $p$ and $q$, respectively, then the corresponding Complete homogeneous non-commutative symmetric functions $S^I$ and $S^J$ have internal product

$$S^I \ast S^J = \sum M^{\text{comp}},$$

where the sum ranges over all $p \times q$-matrices $M \in \mathbb{N}^{p \times q}$ (with nonnegative integers as entries) whose row sum vector is $I$ (that is, the sum of the entries of the $r$-th row is the $r$-th part of $I$ for all $r$) and...
whose column sum vector is $J$ (that is, the sum of all entries of the $s$-th row is the $s$-th part of $J$ for all $s$). Here, for any $M \in \mathbb{N}^{n \times q}$, we denote by $\text{comp} M$ the composition obtained by reading the entries of the matrix $M$ in the usual order (row by row, proceeding left to right in each row, traversing the rows from top to bottom).

The formula on the Psi basis ([NCSF2] Lemma 3.10) is more complicated. Let $I$ and $J$ be two compositions of lengths $p$ and $q$, respectively, having the same size $|I| = |J|$. We denote by $\Psi^K$ the element of the Psi basis corresponding to any composition $K$.

- If $p > q$, then $\Psi^I \ast \Psi^J$ is plainly $0$.
- Assume that $p = q$. Let $\delta_{I,J}$ denote the integer $1$ if the compositions $I$ and $J$ are permutations of each other, and the integer $0$ otherwise. For every positive integer $i$, let $m_i$ denote the number of parts of $I$ equal to $i$. Then, $\Psi^I \ast \Psi^J$ equals $\delta_{I,J} \prod_{i>0} i^{m_i} m_i! \Psi^I$.
- Now assume that $p < q$. Write the composition $I$ as $I = (i_1, i_2, \ldots, i_p)$. For every nonempty composition $K = (k_1, k_2, \ldots, k_s)$, denote by $\Gamma_K$ the non-commutative symmetric function $k_1 [\ldots [\Psi_{k_1}, \Psi_{k_2}], \ldots, \Psi_{k_s}]$. For any subset $A$ of $\{1, 2, \ldots, q\}$, let $J_A$ be the composition obtained from $J$ by removing the $r$-th parts for all $r \notin A$ (while keeping the $r$-th parts for all $r \in A$ in order). Then, $\Psi^I \ast \Psi^J$ equals the sum of $\Gamma_{J_{K_1}} \Gamma_{J_{K_2}} \cdots \Gamma_{J_{K_p}}$ over all ordered set partitions $(K_1, K_2, \ldots, K_p)$ of $\{1, 2, \ldots, q\}$ into $p$ parts such that each $1 \leq k \leq p$ satisfies $|J_{K_k}| = i_k$. (See OrderedSetPartition() for the meaning of “ordered set partition”.)

Aliases for internal_product() are itensor() and kronecker_product().

**INPUT:**
- other – another non-commutative symmetric function

**OUTPUT:**
- The result of taking the internal product of self with other.

**EXAMPLES:**

```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: x = S.an_element(); x
2*S[] + 2*S[1] + 3*S[1, 1]
sage: x.internal_product(S[2])
3*S[1, 1]
sage: x.internal_product(S[1])
2*S[1]
sage: S[1,2].internal_product(S[1,2])
S[1, 1, 1] + S[1, 2]
```

Let us check the duality between the inner product and the inner coproduct in degree 4:

```python
sage: M = QuasiSymmetricFunctions(FiniteField(29)).M()
sage: S = NonCommutativeSymmetricFunctions(FiniteField(29)).S()
sage: def tensor_incorp(f, g, h):
...: # computes \sum_i \left< f, h'_i \right> \left< g, h'_i \right>
...: result = h.base_ring().zero()
...: h_parent = h.parent()
...: for partition_pair, coeff in h.internal_coproduct().monomial_coefficients().items():
...: result += coeff * f.duality_pairing(h_parent[partition_pair[0]]) * g.duality_pairing(h_parent[partition_pair[1]])
...: return result
sage: def testall(n):
...: return all( all( all( tensor_incorp(S[u], S[v], M[w]) == (S[u].itensor(S[v])).duality_pairing(M[w])
...: for u in S) for v in S) for w in M)
```

(continues on next page)
The internal product on the algebra of non-commutative symmetric functions commutes with the canonical commutative projection on the symmetric functions:

\[
\langle f \ast g, h \rangle = \sum_{i} \langle f, h'_i \rangle \langle g, h''_i \rangle,
\]

where we write \( \Delta^\times(h) \) as \( \sum_{i} h'_i \otimes h''_i \). Here, \( f \ast g \) denotes the internal product of the non-commutative symmetric functions \( f \) and \( g \).

If \( f \) and \( g \) are two homogeneous elements of \( NSym \) having distinct degrees, then the internal product \( f \ast g \) is zero.

Explicit formulas can be given for internal products of elements of the complete and the \Psi\) bases. First, the formula for the Complete basis ([NCSF1] Proposition 5.1): If \( I \) and \( J \) are two compositions of lengths \( p \) and \( q \), respectively, then the corresponding Complete homogeneous non-commutative symmetric functions \( S^I \) and \( S^J \) have internal product

\[
S^I \ast S^J = \sum S^{\text{comp } M},
\]

where the sum ranges over all \( p \times q \)-matrices \( M \in \mathbb{N}^{p \times q} \) (with nonnegative integers as entries) whose row sum vector is \( I \) (that is, the sum of the entries of the \( r \)-th row is the \( r \)-th part of \( I \) for all \( r \)) and whose column sum vector is \( J \) (that is, the sum of all entries of the \( s \)-th row is the \( s \)-th part of \( J \) for all \( s \)). Here, for any \( M \in \mathbb{N}^{p \times q} \), we denote by \( \text{comp } M \) the composition obtained by reading the entries of the matrix \( M \) in the usual order (row by row, proceeding left to right in each row, traversing the rows from top to bottom).
The formula on the Psi basis (\texttt{[NCSF2]} Lemma 3.10) is more complicated. Let \( I \) and \( J \) be two compositions of lengths \( p \) and \( q \), respectively, having the same size \(| I | = | J |\). We denote by \( \Psi^K \) the element of the Psi basis corresponding to any composition \( K \).

• If \( p > q \), then \( \Psi^I \star \Psi^J \) is plainly 0.

• Assume that \( p = q \). Let \( \delta_{I,J} \) denote the integer 1 if the compositions \( I \) and \( J \) are permutations of each other, and the integer 0 otherwise. For every positive integer \( i \), let \( m_i \) denote the number of parts of \( I \) equal to \( i \). For all \( i > 0 \), \( \Psi^I \star \Psi^J \) equals \( \delta_{I,J} \prod_{i>0} i^{m_i} m_i! \Psi^I \).

• Now assume that \( p < q \). Write the composition \( I \) as \( I = (i_1, i_2, \ldots, i_p) \). For every nonempty composition \( K = (k_1, k_2, \ldots, k_s) \), denote by \( \Gamma_K \) the non-commutative symmetric function \( k_1 \Psi^{i_1} \cdots \Psi^{i_{k_1}} \cdots k_2 \Psi^{i_{k_1+i_2}} \cdots \cdots \cdots \cdots k_s \Psi^{i_{k_1+i_2+\cdots+i_{k_{s-1}}}} \).

  For any subset \( A \) of \( \{1, 2, \ldots, q\} \), let \( J_A \) be the composition obtained from \( J \) by removing the \( r \)-th parts for all \( r \not\in A \) (while keeping the \( r \)-th parts for all \( r \in A \) in order). Then, \( \Psi^I \star \Psi^J \) equals the sum of \( \Gamma_{J_A} \Gamma_{K_{i_1}} \Gamma_{K_{i_2}} \cdots \Gamma_{K_{i_p}} \) over all ordered set partitions \((K_1, K_2, \ldots, K_p)\) of \( \{1, 2, \ldots, q\} \) into \( p \) parts such that each \( 1 \leq k \leq p \) satisfies \(|J_{K_k}| = i_k\).

(See \texttt{OrderedSetPartition()} for the meaning of “ordered set partition”.)

Aliases for \texttt{internal_product()} are \texttt{itensor()} and \texttt{kronecker_product()}.
The internal product on the algebra of non-commutative symmetric functions commutes with the canonical commutative projection on the symmetric functions:

```python
sage: S = NonCommutativeSymmetricFunctions(ZZ).S()
sage: e = SymmetricFunctions(ZZ).e()
sage: def int_pr_of_S_in_e(I, J):
    ....:     return (S[I].internal_product(S[J])).to_symmetric_function()
sage: all( all( int_pr_of_S_in_e(I, J)
    ....:         == S[I].to_symmetric_function().internal_product(S[J].to_symmetric_function())
    ....:         for I in Compositions(3) )
    ....:         for J in Compositions(3) )
True
```

```python
class ParentMethods
    Bases: object
    internal_product()
    The bilinear product inherited from the isomorphism with the descent algebra.

    This is constructed by extending the method internal_product_on_basis() bilinearly, if available, or using the method internal_product_by_coercion().

    OUTPUT:
    • The internal product map of the algebra the non-commutative symmetric functions.

    EXAMPLES:
    sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S.internal_product
    Generic endomorphism of Non-Commutative Symmetric Functions over the Rational Field in the Complete basis
    sage: S.internal_product(S[2,2], S[1,2,1])
    2*S[1, 1, 1, 1] + S[1, 1, 2] + S[2, 1, 1]
sage: S.internal_product(S[2,2], S[1,2])
    0
    sage: R = N.ribbon()
sage: R.internal_product
    <bound method ....internal_product_by_coercion ...>
    sage: R.internal_product_by_coercion(R[1, 1], R[1,1])
    R[2]
sage: R.internal_product(R[2,2], R[1,2])
    0
```

```python
internal_product_on_basis(I, J)
    The internal product of the two basis elements indexed by I and J (optional)
```
INPUT:
• \(I, J\) – compositions indexing two elements of the basis of self

Returns the internal product of the corresponding basis elements. If this method is implemented, the internal product is defined from it by linearity.

EXAMPLES:

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S.internal_product_on_basis([2,2], [1,2,1])
2*S[1, 1, 1, 1] + S[1, 1, 2] + S[2, 1, 1]
sage: S.internal_product_on_basis([2,2], [2,1])
0
```

**itensor()**

The bilinear product inherited from the isomorphism with the descent algebra.

This is constructed by extending the method `internal_product_on_basis()` bilinearly, if available, or using the method `internal_product_by_coercion()`.

OUTPUT:
• The internal product map of the algebra the non-commutative symmetric functions.

EXAMPLES:

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S.internal_product(S[2,2], S[1,2,1])
2*S[1, 1, 1, 1] + S[1, 1, 2] + S[2, 1, 1]
sage: S.internal_product(S[2,2], S[1,2])
0
```

**kronecker_product()**

The bilinear product inherited from the isomorphism with the descent algebra.

This is constructed by extending the method `internal_product_on_basis()` bilinearly, if available, or using the method `internal_product_by_coercion()`.

OUTPUT:
• The internal product map of the algebra the non-commutative symmetric functions.

EXAMPLES:

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: R.internal_product
<bound method ....internal_product_by_coercion ...>
sage: R.internal_product_by_coercion(R[1, 1], R[1,1])
R[2]
sage: R.internal_product(R[2,2], R[1,2])
0
```
Generic endomorphism of Non-Commutative Symmetric Functions over the Rational Field in the Complete basis

```sage```
```
sage: S.internal_product(S[2,2], S[1,2,1])
2*S[1, 1, 1, 1] + S[1, 1, 2] + S[2, 1, 1]
sage: S.internal_product(S[2,2], S[1,2])
0
```
```python```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: R.internal_product
<bound method ....internal_product_by_coercion ...>
sage: R.internal_product_by_coercion(R[1, 1], R[1,1])
R[2]
sage: R.internal_product(R[2,2], R[1,2])
0
```

```python```
class Realizations(category, *args):
    Bases: RealizationsCategory
class ParentMethods
    Bases: object
    internal_product_by_coercion(left, right)
    Internal product of left and right.
    This is a default implementation that computes the internal product in the realization specified by self.realization_of().a_realization().
    INPUT:
    • left – an element of the non-commutative symmetric functions
    • right – an element of the non-commutative symmetric functions
    OUTPUT:
    • The internal product of left and right.
    EXAMPLES:
    ```sage```
    ```python```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S.internal_product_by_coercion(S[2,1], S[3])
S[2, 1]
sage: S.internal_product_by_coercion(S[2,1], S[4])
0
```
```python```
super_categories()```
EXAMPLES:
```sage```
```python```
sage: from sage.combinat.ncsf_qsym.generic_basis_code import ...
sage: GradedModulesWithInternalProduct
sage: GradedModulesWithInternalProduct(ZZ).super_categories()
[Category of graded modules over Integer Ring]```
5.1.144 Non-Commutative Symmetric Functions

```python
class sage.combinat.ncsf_qsym.ncsf.NonCommutativeSymmetricFunctions(R)
    Bases: UniqueRepresentation, Parent

The abstract algebra of non-commutative symmetric functions.

We construct the abstract algebra of non-commutative symmetric functions over the rational numbers:

```sage```
NCSF = NonCommutativeSymmetricFunctions(QQ)
```
NCSF
Non-Commutative Symmetric Functions over the Rational Field
```
```
sage: S = NCSF.complete()
sage: R = NCSF.ribbon()
sage: S[2,1]*R[1,2]
S[2, 1, 1, 2] - S[2, 1, 3]
```
NCSF is the unique free (non-commutative!) graded connected algebra with one generator in each degree:

```sage```
NCSF.category()
Join of Category of hopf algebras over Rational Field
    and Category of graded algebras over Rational Field
    and Category of monoids with realizations
    and Category of graded coalgebras over Rational Field
    and Category of coalgebras over Rational Field with realizations
    and Category of cocommutative coalgebras over Rational Field
```
```
sage: [S[i].degree() for i in range(10)]
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```
We use the Sage standard renaming idiom to get shorter outputs:

```sage```
NCSF.rename("NCSF")
```
NCSF
```
NCSF has many representations as a concrete algebra. Each of them has a distinguished basis, and its elements are expanded in this basis. Here is the $\Psi (\Psi)$ representation:

```sage```
Psi = NCSF.Psi()
```
```
sage: Psi.an_element()
2*Psi[] + 2*Psi[1] + 3*Psi[1, 1]
```
The basis itself is accessible through:

```sage```
Psi.basis()
Lazy family (Term map from Compositions of non-negative integers...
```
```
sage: Psi.basis().keys()
Compositions of non-negative integers
```
To construct an element one can therefore do:
As this is rather cumbersome, the following abuses of notation are allowed:

```
sage: Psi[Composition([2, 1, 3])]
Psi[2, 1, 3]
sage: Psi[2, 1, 3]
Psi[2, 1, 3]
sage: Psi[2, 1, 3]
Psi[2, 1, 3]
```

or even:

```
sage: Psi[(i for i in [2, 1, 3])]
Psi[2, 1, 3]
```

Unfortunately, due to a limitation in Python syntax, one cannot use:

```
sage: Psi[]  # not implemented
```

Instead, you can use:

```
sage: Psi[[]]
Psi[]
```

Now, we can construct linear combinations of basis elements:

```
```

**Algebra structure**

To start with, \( \Psi \) is a graded algebra, the grading being induced by the size of compositions. The one is the basis element indexed by the empty composition:

```
sage: Psi.one()
Psi[]
sage: S.one()
S[]
sage: R.one()
R[]
```

As we have seen above, the \( \Psi \) basis is multiplicative; that is multiplication is induced by linearity from the concatenation of compositions:

```
sage: Psi[1, 3] * Psi[2, 1]
Psi[1, 3, 2, 1]
sage: (Psi.one() + 2 * Psi[1, 3]) * Psi[2, 4]
2*Psi[1, 3, 2, 4] + Psi[2, 4]
```
Hopf algebra structure

Psi is further endowed with a coalgebra structure. The coproduct is an algebra morphism, and therefore determined by its values on the generators; those are primitive:

```
sage: Psi[1].coproduct()
Psi[] # Psi[1] + Psi[1] # Psi[]
sage: Psi[2].coproduct()
```

The coproduct, being cocommutative on the generators, is cocommutative everywhere:

```
sage: Psi[1,2].coproduct()
```

The algebra and coalgebra structures on Psi combine to form a bialgebra structure, which cooperates with the grading to form a connected graded bialgebra. Thus, as any connected graded bialgebra, Psi is a Hopf algebra. Over QQ (or any other Q-algebra), this Hopf algebra Psi is isomorphic to the tensor algebra of its space of primitive elements.

The antipode is an anti-algebra morphism; in the Psi basis, it sends the generators to their opposites and changes their sign if they are of odd degree:

```
sage: Psi[3].antipode()
-Psi[3]
sage: Psi[1,3,2].antipode()
-Psi[2, 3, 1]
sage: Psi[1,3,2].coproduct().apply_multilinear_morphism(lambda be,ga:
    -Psi(be)*Psi(ga).antipode())
0
```

The counit is defined by sending all elements of positive degree to zero:

```
sage: S[3].degree(), S[3,1,2].degree(), S.one().degree()
(3, 6, 0)
sage: S[3].counit()
0
sage: S[3,1,2].counit()
0
sage: S.one().counit()
1
7
7
```

It is possible to change the prefix used to display the basis elements using the method `print_options()`. Say that for instance one wanted to display the Complete basis as having a prefix H instead of the default S:

```
sage: H = NCSF.complete()
sage: H.an_element()
2*S[] + 2*S[1] + 3*S[1, 1]
sage: H.print_options(prefix='H')
sage: H.an_element()
```

(continues on next page)
Concrete representations

NCSF admits the concrete realizations defined in [NCSF1]:

```
sage: Phi = NCSF.Phi()
sage: Psi = NCSF.Psi()
sage: ribbon = NCSF.ribbon()
sage: complete = NCSF.complete()
sage: elementary = NCSF.elementary()
```

To change from one basis to another, one simply does:

```
sage: Phi(Psi[1])
Phi[1]
sage: Phi(Psi[3])
-1/4*Phi[1, 2] + 1/4*Phi[2, 1] + Phi[3]
```

In general, one can mix up different bases in computations:

```
sage: Phi[1] * Psi[1]
Phi[1, 1]
```

Some of the changes of basis are easy to guess:

```
sage: ribbon(complete[1,3,2])
```

This is the sum of all fatter compositions. Using the usual Möbius function for the boolean lattice, the inverse change of basis is given by the alternating sum of all fatter compositions:

```
sage: complete(ribbon[1,3,2])
```

The analogue of the elementary basis is the sum over all finer compositions than the ‘complement’ of the composition in the ribbon basis:

```
sage: Composition([1,3,2]).complement()
[2, 1, 2, 1]
sage: ribbon(elementary([1,3,2]))
R[1, 1, 1, 1, 1, 1] + R[1, 1, 1, 2, 1] + R[2, 1, 1, 1, 1] + R[2, 1, 2, 1]
```

By Möbius inversion on the composition poset, the ribbon basis element corresponding to a composition \( I \) is then the alternating sum over all compositions fatter than the complement composition of \( I \) in the elementary basis:

```
sage: elementary(ribbon[2,1,2,1])
```
The $\Phi$ ($\Phi$) and $\Psi$ bases are computed by changing to and from the Complete basis. The expansion of $\Psi$ basis is given in Proposition 4.5 of [NCSF1] by the formulae

$$S^I = \sum_{J \geq I} \frac{1}{\pi_u(J, I)} \Psi^J$$

and

$$\Psi^I = \sum_{J \geq I} (-1)^{\ell(J) - \ell(I)} l_p(J, I) S^J$$

where the coefficients $\pi_u(J, I)$ and $l_p(J, I)$ are coefficients in the methods $\text{coeff\_pi()}$ and $\text{coeff\_lp()}$ respectively. For example:

```sage
sage: Psi(complete[3])
1/6*Psi[1, 1, 1] + 1/3*Psi[1, 2] + 1/6*Psi[2, 1] + 1/3*Psi[3]
sage: complete(Psi[3])
```

The $\Phi$ basis is another analogue of the power sum basis from the algebra of symmetric functions and the expansion in the Complete basis is given in Proposition 4.9 of [NCSF1] by the formulae

$$S^I = \sum_{J \geq I} \frac{1}{sp(J, I)} \Phi^J$$

and

$$\Phi^I = \sum_{J \geq I} (-1)^{\ell(J) - \ell(I)} \prod_{i | I} I_i S^J$$

where the coefficients $sp(J, I)$ and $\ell(J, I)$ are coefficients in the methods $\text{coeff\_sp()}$ and $\text{coeff\_ell()}$ respectively. For example:

```sage
sage: Phi(complete[3])
1/6*Phi[1, 1, 1] + 1/4*Phi[1, 2] + 1/4*Phi[2, 1] + 1/3*Phi[3]
sage: complete(Phi[3])
```

Here is how to fetch the conversion morphisms:

```sage
sage: f = complete.coerce_map_from(elementary); f
Generic morphism:
  From: NCSF in the Elementary basis
  To: NCSF in the Complete basis
sage: g = elementary.coerce_map_from(complete); g
Generic morphism:
  From: NCSF in the Complete basis
  To: NCSF in the Elementary basis
sage: f.category()
Category of homsets of unital magmas and right modules over Rational Field and left modules over Rational Field
sage: f(elementary[1,2,2])
S[1, 1, 1, 1, 1] - S[1, 1, 1, 2] - S[1, 2, 1, 1] + S[1, 2, 2]
sage: g(complete[1,2,2])
L[1, 1, 1, 1, 1] - L[1, 1, 1, 2] - L[1, 2, 1, 1] + L[1, 2, 2]
```

(continues on next page)
sage: h = f*g; h

Composite map:
  From: NCSF in the Complete basis
  To:   NCSF in the Complete basis
Defn:  Generic morphism:
    From: NCSF in the Complete basis
    To:   NCSF in the Elementary basis
    then
    Generic morphism:
    From: NCSF in the Elementary basis
    To:   NCSF in the Complete basis

sage: h(complete[1,3,2])
S[1, 3, 2]

Additional concrete representations

NCSF has some additional bases which appear in the literature:

sage: Monomial = NCSF.Monomial()
sage: Immaculate = NCSF.Immaculate()
sage: dualQuasisymmetric_Schur = NCSF.dualQuasisymmetric_Schur()

The Monomial basis was introduced in [Tev2007] and the Immaculate basis was introduced in [BBSSZ2012].
The Quasisymmetric_Schur were defined in [QSCHUR] and the dual basis is implemented here as
dualQuasisymmetric_Schur. Refer to the documentation for the use and definition of these bases.

Todo:

• implement fundamental, forgotten, and simple (coming from the simple modules of HS_n) bases.

We revert back to the original name from our custom short name NCSF:

sage: NCSF
NCSF
sage: NCSF.rename()
sage: NCSF
Non-Commutative Symmetric Functions over the Rational Field

class Bases(parent_with_realization)

   Bases: Category_realization_of_parent

   Category of bases of non-commutative symmetric functions.

   EXAMPLES:

sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: N.Bases()
Category of bases of Non-Commutative Symmetric Functions over the Rational Field
sage: R = N.Ribbon()
sage: R in N.Bases()
True
class ElementMethods

Bases: object

bernstein_creation_operator(n)

Return the image of self under the n-th Bernstein creation operator.

Let \( n \) be an integer. The \( n \)-th Bernstein creation operator \( B_n \) is defined as the endomorphism of the space \( NSym \) of noncommutative symmetric functions which sends every \( f \) to

\[
\sum_{i \geq 0} (-1)^i H_{n+i} F_{1+i}^\perp,
\]

where usual notations are in place (the letter \( H \) stands for the complete basis of \( NSym \), the letter \( F \) stands for the fundamental basis of the algebra \( QSym \) of quasisymmetric functions, and \( F_{1+i}^\perp \) means skewing \((\text{skew}_by())\) by \( F_1 \)). Notice that \( F_1 \) is nothing other than the elementary symmetric function \( e_i \).

This has been introduced in [BBSSZ2012], section 3.1, in analogy to the Bernstein creation operators on the symmetric functions \((\text{bernstein_creation_operator}())\), and studied further in [BBSSZ2012], mainly in the context of immaculate functions \((\text{Immaculate})\). In fact, if \((\alpha_1, \alpha_2, \ldots, \alpha_m)\) is an \( m \)-tuple of integers, then

\[
B_n I_{\alpha_1, \alpha_2, \ldots, \alpha_m} = I_{n, \alpha_1, \alpha_2, \ldots, \alpha_m},
\]

where \( I_{\alpha_1, \alpha_2, \ldots, \alpha_m} \) is the immaculate function associated to the \( m \)-tuple \((\alpha_1, \alpha_2, \ldots, \alpha_m)\) (see \(\text{immaculate_function}()\)).

EXAMPLES:

We get the immaculate functions by repeated application of Bernstein creation operators:

```
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: I = NSym.I()
sage: S = NSym.S()
sage: def immaculate_by_bernstein(xs):
    ...:     # immaculate function corresponding to integer
    ...:     # tuple \( `xs`\), computed by iterated application
    ...:     # of Bernstein creation operators.
    ...:     res = S.one()
    ...:     for i in reversed(xs):
    ...:         res = res.bernstein_creation_operator(i)
    ...:     return res

sage: import itertools
sage: all( immaculate_by_bernstein(p) == I.immaculate_function(p)
    ...:      for p in itertools.product(range(-1, 3), repeat=3))
True
```

Some examples:

```
sage: S[3,2].bernstein_creation_operator(-2)
S[2, 1]
sage: S[3,2].bernstein_creation_operator(-1)
S[1, 2, 1] - S[2, 2] - S[3, 1]
sage: S[3,2].bernstein_creation_operator(0)
-S[1, 2, 2] - S[1, 3, 1] + S[2, 2, 1] + S[3, 2]
sage: S[3,2].bernstein_creation_operator(1)
```

(continues on next page)
\[
sage: S[3, 2].bernstein_creation_operator(2)
\]

\(\chi()\)

Return the commutative image of a non-commutative symmetric function.

**OUTPUT:**
- The commutative image of self. This will be a symmetric function.

**EXAMPLES:**

\[
sage: N = NonCommutativeSymmetricFunctions(QQ) \\
sage: R = N.ribbon() \\
sage: x = R.an_element(); x \\
sage: x.to_symmetric_function() \\
2*s[] + 2*s[1] + 3*s[1, 1] \\
sage: y = N.Phi()[1,3] \\
sage: y.to_symmetric_function() \\
h[1, 1, 1, 1] - 3*h[2, 1, 1] + 3*h[3, 1] \\
\]

\(\text{expand}(n, \text{alphabet=}'x')\)

Expand the noncommutative symmetric function into an element of a free algebra in \(n\) indeterminates of an alphabet, which by default is ‘\(x\)’.

**INPUT:**
- \(n\) – a nonnegative integer; the number of variables in the expansion
- \(\text{alphabet}\) – (default: ‘\(x\)’); the alphabet in which self is to be expanded

**OUTPUT:**
- An expansion of self into the \(n\) variables specified by alphabet.

**EXAMPLES:**

\[
sage: NSym = NonCommutativeSymmetricFunctions(QQ) \\
sage: S = NSym.S() \\
sage: S[3].expand(3) \\
x0^3 + x0^2*x1 + x0^2*x2 + x0*x1^2 + x0*x1*x2 + x0*x2^2 + x1^3 + x1^2*x2 + x1*x2^2 + x2^3 \\
sage: L = NSym.L() \\
sage: L[3].expand(3) \\
x2*x1*x0 \\
sage: L[2].expand(3) \\
x1*x0 + x2*x0 + x2*x1 \\
sage: L[3].expand(4) \\
x2*x1*x0 + x3*x1*x0 + x3*x2*x0 + x3*x2*x1 \\
sage: Psi = NSym.Psi() \\
sage: Psi[2, 1].expand(3) \\
x0^3 + x0^2*x1 + x0^2*x2 + x0*x1^2 + x0*x1*x2 + x0*x2^2 + x1^3 + x1^2*x2 + x1*x2^2 + x2^3 \\
\]

One can use a different set of variables by adding an optional argument alphabet=...
sage: L[3].expand(4, alphabet="y")
y2*y1*y0 + y3*y1*y0 + y3*y2*y0 + y3*y2*y1

Todo: So far this is only implemented on the elementary basis, and everything else goes through coercion. Maybe it is worth shortcutting some of the other bases?

left_padded_kronecker_product(x)

Return the left-padded Kronecker product of self and x in the basis of self.

The left-padded Kronecker product is a bilinear map mapping two non-commutative symmetric functions to another, not necessarily preserving degree. It can be defined as follows: Let \( \ast \) denote the internal product (internal_product()) on the space of non-commutative symmetric functions. For any composition \( I \), let \( S^I \) denote the complete homogeneous symmetric function indexed by \( I \). For every composition \( I = (i_1, i_2, \ldots, i_k) \) and every integer \( n > |I| \), define the \( n \)-completion of \( I \) to be the composition \( (n - |I|, i_1, i_2, \ldots, i_k) \); this \( n \)-completion is denoted by \( I[n] \). Then, for any compositions \( \alpha \) and \( \beta \) and every integer \( n > |\alpha| + |\beta| \), we can write the internal product \( S^\alpha[n] \ast S^\beta[n] \) in the form

\[
S^\alpha[n] \ast S^\beta[n] = \sum_\gamma g^{\gamma[n]}_{\alpha[n], \beta[n]} S^\gamma[n]
\]

with \( \gamma \) ranging over all compositions. The coefficients \( g^{\gamma[n]}_{\alpha[n], \beta[n]} \) are independent on \( n \). These coefficients \( g^{\gamma[n]}_{\alpha[n], \beta[n]} \) are denoted by \( \tilde{g}^{\gamma}_{\alpha, \beta} \), and the non-commutative symmetric function

\[
\sum_\gamma \tilde{g}^{\gamma}_{\alpha, \beta} S^\gamma
\]

is said to be the left-padded Kronecker product of \( S^\alpha \) and \( S^\beta \). By bilinearity, this extends to a definition of a left-padded Kronecker product of any two non-commutative symmetric functions.

The left-padded Kronecker product on the non-commutative symmetric functions lifts the left-padded Kronecker product on the symmetric functions. More precisely, let \( \pi \) denote the canonical projection (to_symmetric_function()) from the non-commutative symmetric functions to the symmetric functions. Then, any two non-commutative symmetric functions \( f \) and \( g \) satisfy

\[
\pi(f \ast g) = \pi(f) \pi(g),
\]

where the \( \pi \) on the left-hand side denotes the left-padded Kronecker product on the non-commutative symmetric functions, and the \( \pi \) on the right-hand side denotes the left-padded Kronecker product on the symmetric functions.

INPUT:
• \( x \) – element of the ring of non-commutative symmetric functions over the same base ring as self

OUTPUT:
• the left-padded Kronecker product of self with x (an element of the ring of non-commutative symmetric functions in the same basis as self)

AUTHORS:
• Darij Grinberg (15 Mar 2014)

EXAMPLES:
```sage
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: S = NSym.S()
sage: S[2,1].left_padded_kronecker_product(S[3])
sage: S[2,1].left_padded_kronecker_product(S[1])
S[1, 1, 1] + S[1, 2, 1] + S[2, 1]
sage: S[1].left_padded_kronecker_product(S[2,1])
S[1, 1, 1] + S[2, 1] + S[2, 1, 1]
sage: S[1,1].left_padded_kronecker_product(S[2])
S[1, 1] + 2*S[1, 1, 1] + S[2, 1, 1]
sage: S[1].left_padded_kronecker_product(S[1,2,1])
S[1, 1, 1] + S[1, 2, 1] + S[1, 2, 1, 1] + S[2, 1, 1]
sage: S[2].left_padded_kronecker_product(S[3])
S[1, 2] + S[2, 1, 1] + S[3, 2]
```

Taking the left-padded Kronecker product with $1 = S$ is the identity map on the ring of noncommutative symmetric functions:

```sage
sage: all( S[Composition([])].left_padded_kronecker_product(S[lambda])
...: == S[lambda].left_padded_kronecker_product(S[Composition([])])
...: == S[lambda]
...: for i in range(4)
...: for lambda in Compositions(i) )
True
```

Here is a rule for the left-padded Kronecker product of $S_1$ (this is the same as $S^{(1)}$) with any complete homogeneous function: Let $I$ be a composition. Then, the left-padded Kronecker product of $S_1$ and $S^I$ is $\sum K a_K S^K$, where the sum runs over all compositions $K$, and the coefficient $a_K$ is defined as the number of ways to obtain $K$ from $I$ by one of the following two operations:

• Insert a 1 at the end of $I$.
• Subtract 1 from one of the entries of $I$ (and remove the entry if it thus becomes 0), and insert a 1 at the end of $I$.

We check this for compositions of size $\leq 4$:

```sage
def mults1(I):
    # Left left-padded Kronecker multiplication by S[1].
    res = S[I[:1] + [1]]
    for k in range(len(I)):
        I2 = I[:]
        if I2[k] == 1:
            I2 = I2[:k] + I2[k+1:]
        else:
            I2[k] -= 1
        res += S[I2 + [1]]
    return res
sage: all( mults1(I) == S[1].left_padded_kronecker_product(S[I])
...: for i in range(5) for I in Compositions(i) )
True
```

A similar rule can be made for the left-padded Kronecker product of any complete homogeneous function with $S_1$: Let $I$ be a composition. Then, the left-padded Kronecker product of $S^I$ and $S_1$ is $\sum K b_K S^K$, where the sum runs over all compositions $K$, and the coefficient $b_K$ is defined as the number of ways to obtain $K$ from $I$ by one of the following two operations:

• Insert a 1 at the front of $I$.
• Subtract 1 from one of the entries of \( I \) (and remove the entry if it thus becomes 0), and insert a 1 right after this entry.

We check this for compositions of size \( \leq 4 \):

```python
sage: def mults2(I):
    # Left left-padded Kronecker multiplication by S[1].
    res = S[[1] + I[:]]
    for k in range(len(I)):
        I2 = I[:]
        i2k = I2[k]
        if i2k != 1:
            I2 = I2[:k] + [i2k-1, 1] + I2[k+1:]
        res += S[I2]
    return res

sage: all( mults2(I) == S[I].left_padded_kronecker_product(S[1])
    for i in range(5) for I in Compositions(i) )
True
```

Checking the \( \pi(fg) = \pi(f)\pi(g) \) equality:

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: R = NSym.R()

sage: def testpi(n):
    for I in Compositions(n):
        for J in Compositions(n):
            a = R[I].to_symmetric_function()
            b = R[J].to_symmetric_function()
            x = a.left_padded_kronecker_product(b)
            y = R[I].left_padded_kronecker_product(R[J])
            y = y.to_symmetric_function()
            if x != y:
                return False
    return True

sage: testpi(3)
True
```

_omega_involution()_

Return the image of the noncommutative symmetric function \( \text{self} \) under the omega involution.

The omega involution is defined as the algebra antihomomorphism \( NCSF \to NCSF \) which, for every positive integer \( n \), sends the \( n \)-th complete non-commutative symmetric function \( S_n \) to the \( n \)-th elementary non-commutative symmetric function \( \Lambda_n \). This omega involution is denoted by \( \omega \). It can be shown that every composition \( I \) satisfies

\[
\omega(S^I) = \Lambda^{I^\prime}, \quad \omega(\Lambda^I) = S^{I^\prime}, \quad \omega(R_I) = R_{I^\prime}, \quad \omega(\Phi^I) = (-1)^{|I|-\ell(I)}\Phi^{I^\prime}, \omega(\Psi^I) = (-1)^{|I|-\ell(I)}\Psi^{I^\prime},
\]

where \( I^\prime \) denotes the reversed composition of \( I \), and \( I^\prime \) denotes the conjugate composition of \( I \), and \( \ell(I) \) denotes the length of the composition \( I \), and standard notations for classical bases of \( NCSF \) are being used (\( S \) for the complete basis, \( \Lambda \) for the elementary basis, \( R \) for the ribbon basis, \( \Phi \) for that of the power-sums of the second kind, and \( \Psi \) for that of the power-sums of the first kind). More generally, if \( f \) is a homogeneous element of \( NCSF \) of degree \( n \), then

\[
\omega(f) = (-1)^n S(f),
\]

where \( S \) denotes the antipode of \( NCSF \).
The omega involution $\omega$ is an involution and a coalgebra automorphism of $NCSF$. It is an automorphism of the graded vector space $NCSF$. If $\pi$ denotes the projection from $NCSF$ to the ring of symmetric functions ($\text{to_symmetric_function()}$), then $\pi(\omega(f)) = \omega(\pi(f))$ for every $f \in NCSF$, where the $\omega$ on the right hand side denotes the omega automorphism of $Sym$.

The omega involution on $NCSF$ is adjoint to the omega involution on $QSym$ by the standard adjunction between $NCSF$ and $QSym$.

The omega involution has been denoted by $\omega$ in [LMvW13], section 3.6. See [NCSF1], section 3.1 for the properties of this map.

See also:

omega involution of QSym, psi involution of NCSF, star involution of NCSF.

EXAMPLES:

```sage
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: L = NSym.L()
sage: L(S[3,2].omega_involution())
L[2, 3]
sage: L(S[6,3].omega_involution())
L[3, 6]
sage: L(S[1,3].omega_involution())
L[3, 1]
# long time
sage: L(S([4,2]).omega_involution())
L[2, 4]
sage: R = NSym.R()
sage: R([4,2]).omega_involution()
R[1, 2, 1, 1, 1]
sage: R.zero().omega_involution()
0
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: Phi = NSym.Phi()
sage: Phi([2,1]).omega_involution()
-Phi[1, 2]
sage: Psi = NSym.Psi()
sage: Psi([2,1]).omega_involution()
-Psi[1, 2]
sage: Psi([3,1]).omega_involution()
Psi[1, 3]
```

Testing the $\pi(\omega(f)) = \omega(\pi(f))$ relation noticed above:

```sage
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: R = NSym.R()
sage: all( R(I).omega_involution().to_symmetric_function() == R(I).to_symmetric_function().omega_involution() for I in Compositions(4) )
True
```
The omega involution on $QSym$ is adjoint to the omega involution on $NSym$ with respect to the duality pairing:

```sage
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: S = NSym.S()
sage: all( all( M(I).omega_involution().duality_pairing(S(J)) == M(I).duality_pairing(S(J).omega_involution()) for I in Compositions(2) ) for J in Compositions(3) )
True
```

```
psi_involution()
```

Return the image of the noncommutative symmetric function `self` under the involution $\psi$.

The involution $\psi$ is defined as the linear map $NCSF \rightarrow NCSF$ which, for every composition $I$, sends the complete noncommutative symmetric function $S^I$ to the elementary noncommutative symmetric function $\Lambda^I$. It can be shown that every composition $I$ satisfies

$$\psi(R_I) = R_{I^c}, \quad \psi(S^I) = \Lambda^I, \quad \psi(\Lambda^I) = S^I, \quad \psi(\Phi^I) = (-1)^{|I| - \ell(I)} \Phi^I$$

where $I^c$ denotes the complement of the composition $I$, and $\ell(I)$ denotes the length of $I$, and where standard notations for classical bases of $NCSF$ are being used ($S$ for the complete basis, $\Lambda$ for the elementary basis, $\Phi$ for the basis of the power sums of the second kind, and $R$ for the ribbon basis).

The map $\psi$ is an involution and a graded Hopf algebra automorphism of $NCSF$. If $\pi$ denotes the projection from $NCSF$ to the ring of symmetric functions (`to_symmetric_function()`), then $\pi(\psi(f)) = \omega(\pi(f))$ for every $f \in NCSF$, where the $\omega$ on the right hand side denotes the omega automorphism of $Sym$.

The involution $\psi$ of $NCSF$ is adjoint to the involution $\psi$ of $QSym$ by the standard adjunction between $NCSF$ and $QSym$.

The involution $\psi$ has been denoted by $\psi$ in [LMvW13], section 3.6.

See also:

- `psi_involution of QSym`
- `star_involution of NCSF`

**EXAMPLES:**

```sage
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: R = NSym.R()
sage: R[3,2].psi_involution()
R[1, 1, 2, 1]
sage: R[6,3].psi_involution()
R[1, 1, 1, 1, 2, 1, 1]
4*R[] - 2*R[1, 1] + R[1, 1, 2, 1, 1]
sage: (R[3,3] - 2*R[2]).psi_involution()
-2*R[1, 1] + R[1, 1, 2, 1, 1]
sage: R([2,1,1]).psi_involution()
R[1, 3]
sage: S = NSym.S()
sage: S([2,1]).psi_involution()
S[1, 1, 1] - S[2, 1]
```

(continues on next page)
The Psi basis doesn’t behave as nicely:

```python
sage: Psi = NSym.Psi()
sage: Psi([2,1]).psi_involution()
1/2*Psi[1, 2, 1] - 1/2*Psi[2, 1, 1] + Psi[3, 1]
```

The involution $\psi$ commutes with the antipode:

```python
sage: all( R(I).psi_involution().antipode()
....:     == R(I).antipode().psi_involution()
....:     for I in Compositions(4) )
True
```

Testing the $\pi(\psi(f)) = \omega(\pi(f))$ relation noticed above:

```python
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: R = NSym.R()
sage: all( R(I).psi_involution().to_symmetric_function()
....:     == R(I).to_symmetric_function().omega()
....:     for I in Compositions(4) )
True
```

The involution $\psi$ of $QSym$ is adjoint to the involution $\psi$ of $NSym$ with respect to the duality pairing:

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: S = NSym.S()
sage: all( all( M(I).psi_involution().duality_pairing(S(J))
....:         == M(I).duality_pairing(S(J).psi_involution())
....:         for I in Compositions(2) )
....:         for J in Compositions(3) )
True
```

**star_involution()**

Return the image of the noncommutative symmetric function self under the star involution.

The star involution is defined as the algebra antihomomorphism $NCSF \rightarrow NCSF$ which, for every positive integer $n$, sends the $n$-th complete non-commutative symmetric function $S_n$ to $S_n$. Denoting by $f^*$ the image of an element $f \in NCSF$ under this star involution, it can be shown...
that every composition $I$ satisfies

$$(S^I)^* = S^{I^r}, \quad (\Lambda^I)^* = \Lambda^{I^r}, \quad R^I_j = R_{I^r}, \quad (\Phi^I)^* = \Phi^{I^r},$$

where $I^r$ denotes the reversed composition of $I$, and standard notations for classical bases of $NCSF$ are being used ($S$ for the complete basis, $\Lambda$ for the elementary basis, $R$ for the ribbon basis, and $\Phi$ for that of the power-sums of the second kind). The star involution is an involution and a coalgebra automorphism of $NCSF$. It is an automorphism of the graded vector space $NCSF$. Under the canonical isomorphism between the $n$-th graded component of $NCSF$ and the descent algebra of the symmetric group $S_n$ (see to_descent_algebra()), the star involution (restricted to the $n$-th graded component) corresponds to the automorphism of the descent algebra given by $x \mapsto \omega_n x \omega_n$, where $\omega_n$ is the permutation $(n, n-1, \ldots, 1) \in S_n$ (written here in one-line notation). If $\pi$ denotes the projection from $NCSF$ to the ring of symmetric functions (to_symmetric_function()), then $\pi(f^*) = \pi(f)$ for every $f \in NCSF$.

The star involution on $NCSF$ is adjoint to the star involution on $\text{QSym}$ by the standard adjunction between $NCSF$ and $\text{QSym}$.

The star involution has been denoted by $\rho$ in [LMvW13], section 3.6. See [NCSF2], section 2.3 for the properties of this map.

See also:

star involution of $\text{QSym}$, psi involution of $NCSF$.

**EXAMPLES:**

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: S[3,2].star_involution()
S[2, 3]
sage: S[6,3].star_involution()
S[3, 6]
sage: (S[3,3] - 2*S[2]) . star_involution()
sage: S[[4,2]] . star_involution()
S[2, 4]
sage: R = NSym.R()
sage: R[[4,2]].star_involution()
R[2, 4]
sage: R.zero().star_involution()
0
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: Phi = NSym.Phi()
sage: Phi[[2,1]].star_involution()
Phi[1, 2]
```

The Psi basis doesn’t behave as nicely:

```python
sage: Psi = NSym.Psi()
sage: Psi([2,1]).star_involution()
Psi[1, 2]
sage: Psi([3,1]).star_involution()
1/2*Psi[1, 1, 2] - 1/2*Psi[1, 2, 1] + Psi[1, 3]
```
The star involution commutes with the antipode:

```python
sage: all( R(I).star_involution().antipode()
.....:     == R(I).antipode().star_involution()
.....:     for I in Compositions(4) )
True
```

Checking the relation with the descent algebra described above:

```python
sage: def descent_test(n):
.....:     DA = DescentAlgebra(QQ, n)
.....:     NSym = NonCommutativeSymmetricFunctions(QQ)
.....:     S = NSym.S()
.....:     DAD = DA.D()
.....:     w_n = DAD(set(range(1, n)))
.....:     for I in Compositions(n):
.....:         if not (S[I].star_involution()
.....:                      == w_n * S[I].to_descent_algebra(n) * w_n):
.....:             return False
.....:     return True

sage: all( descent_test(i) for i in range(4) )
True
sage: all( descent_test(i) for i in range(6) )
# long time
True
```

Testing the \(\pi(f^*) = \pi(f)\) relation noticed above:

```python
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: R = NSym.R()
sage: all( R(I).star_involution().to_symmetric_function()
.....:     == R(I).to_symmetric_function()
.....:     for I in Compositions(4) )
True
```

The star involution on \(QSym\) is adjoint to the star involution on \(NSym\) with respect to the duality pairing:

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: S = NSym.S()
sage: all( all( M(I).star_involution().duality_pairing(S(J))
.....:         == M(I).duality_pairing(S(J).star_involution())
.....:         for I in Compositions(2) )
.....:         for J in Compositions(3) )
True
```

---

**to_descent_algebra\(n=\text{None}\)**

Return the image of the \(n\)-th degree homogeneous component of \(\text{self}\) in the descent algebra of \(S_n\) over the same base ring as \(\text{self}\).

This is based upon the canonical isomorphism from the \(n\)-th degree homogeneous component of the algebra of noncommutative symmetric functions to the descent algebra of \(S_n\). This isomorphism maps the inner product of noncommutative symmetric functions either to the product in the descent algebra of \(S_n\) or to its opposite (depending on how the latter is defined).
If \( n \) is not specified, it will be taken to be the highest homogeneous component of \( \text{self} \).

**OUTPUT:**
- The image of the \( n \)-th homogeneous component of \( \text{self} \) under the isomorphism into the descent algebra of \( S_n \) over the same base ring as \( \text{self} \).

**EXAMPLES:**

```python
sage: S = NonCommutativeSymmetricFunctions(ZZ).S()
sage: S[2,1].to_descent_algebra(3)
B[2, 1]
sage: (S[1,2,1] - 3 * S[1,1,2]).to_descent_algebra(4)
-3*R[1, 1, 2] + B[1, 2, 1]
sage: S[2,1].to_descent_algebra(2)
0
sage: S[2,1].to_descent_algebra()
B[2, 1]
sage: S.zero().to_descent_algebra().parent()
Descent algebra of 0 over Integer Ring in the subset basis
sage: (S[1,2,1] - 3 * S[1,1,2]).to_descent_algebra(1)
0
```

**to_fqsym()**

Return the image of the non-commutative symmetric function \( \text{self} \) under the morphism \( \iota : NSym \to FQSym \).

This morphism is the injective algebra homomorphism \( NSym \to FQSym \) that sends each Complete generator \( S_n \) to \( F_{[1,2,...,n]} \). It is the inclusion map, if we regard both \( NSym \) and \( FQSym \) as rings of noncommutative power series.

**See also:**

*FreeQuasisymmetricFunctions* for a definition of \( FQSym \).

**EXAMPLES:**

```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: x.to_fqsym()
sage: R[2,1].to_fqsym()
F[1, 3, 2] + F[3, 1, 2]
sage: x = R.an_element(); x
2*R[] + 2*R[1] + 3*R[1, 1]
sage: x.to_fqsym()

sage: y = N.Phi()[1,2]
sage: y.to_fqsym()
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S[2].to_fqsym()
F[1, 2]
sage: S[1,2].to_fqsym()
```

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(continued from previous page)

\[ F[1, 2, 3] + F[2, 1, 3] + F[2, 3, 1] \]

\[ \text{sage: } S[2, 1].to_fsym() \]

\[ F[1, 2, 3] + F[1, 3, 2] + F[3, 1, 2] \]

\[ \text{sage: } S[1, 2, 1].to_fsym() \]


\[ \text{to_fsym}() \]

Return the image of \texttt{self} under the natural map to \( FSym \).

There is an injective Hopf algebra morphism from \( NSym \) to \( FSym \) (see \texttt{FreeSymmetricFunctions}), which maps the ribbon \( R_\alpha \) indexed by a composition \( \alpha \) to the sum of all tableaux whose descent composition is \( \alpha \). If we regard \( NSym \) as a Hopf subalgebra of \( FQSym \) via the morphism \( \iota : NSym \to FQSym \) (implemented as \texttt{to_fqsym()}), then this injective morphism is just the inclusion map.

**EXAMPLES:**

\[ \text{sage: } N = \text{NonCommutativeSymmetricFunctions}(\QQ) \]
\[ \text{sage: } R = N.\text{ribbon()} \]
\[ \text{sage: } x.\text{to_fsym()} \]
\[ \text{sage: } R[2, 1].\text{to_fsym()} \]
\[ G[12|3] \]
\[ \text{sage: } R[1, 2].\text{to_fsym()} \]
\[ G[13|2] \]
\[ \text{sage: } R[2, 1, 2].\text{to_fsym()} \]
\[ \text{sage: } x = R.\text{an_element(); }x \]
\[ 2^*R[] + 2^*R[1] + 3^*R[1, 1] \]
\[ \text{sage: } x.\text{to_fsym()} \]

\[ \text{sage: } y = N.\text{Phi()}[1, 2] \]
\[ \text{sage: } y.\text{to_fsym()} \]

\[ \text{sage: } S = \text{NonCommutativeSymmetricFunctions}(\QQ).S() \]
\[ \text{sage: } S[2].\text{to_fsym()} \]
\[ G[12] \]
\[ \text{sage: } S[2, 1].\text{to_fsym()} \]
\[ G[12|3] + G[123] \]

\[ \text{to_ncsym}() \]

Return the image of \texttt{self} under the injective algebra homomorphism \( \kappa : NSym \to NCSym \) that fixes the symmetric functions.

As usual, \( NCSym \) denotes the ring of symmetric functions in non-commuting variables. Let \( S_n \) denote a generator of the complete basis. The algebra homomorphism \( \kappa : NSym \to NCSym \) is

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defined by

\[ S_n \mapsto \sum_{A \vdash n} \frac{\lambda(A)! \lambda(A)!}{n!} m_A. \]

It has the property that the canonical maps \( \chi : NCsym \to Sym \) and \( \rho : NSym \to Sym \) satisfy \( \chi \circ \kappa = \rho \).

**Note:** A remark in [BRRZ08] makes it clear that the embedding of \( NSym \) into \( NCsym \) that preserves the projection into the symmetric functions is not unique. While this seems to be a natural embedding, any free set of algebraic generators of \( NSym \) can be sent to a set of free elements in \( NCsym \) to form another embedding.

**See also:**

*NonCommutativeSymmetricFunctions* for a definition of \( NCsym \).

**EXAMPLES:**

```sage
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S[2].to_ncsym()
1/2*m\{1\} + m\{2\}
sage: S[1,2,1].to_ncsym()
1/2*m\{1\} + 1/2*m\{2\} + 1/2*m\{3\} + 1/2*m\{4\} + m\{1, 2\}
sage: S[2,1,1].to_ncsym()
1/2*m\{1\} + 1/2*m\{2\} + 1/2*m\{3\} + 1/2*m\{4\}
sage: S[1,2].to_ncsym()
1/2*m\{1\} + 1/2*m\{2\} + 1/2*m\{3\}
sage: S[1,3].to_ncsym()
1/2*m\{1\} + 1/2*m\{2\}
sage: R = N.ribbon()
sage: x = R.an_element(); x
2*R[] + 2*R[1] + 3*R[1, 1]
sage: x.to_ncsym()
2*m\{\} + 2*m\{1\} + 3/2*m\{2\}
sage: R[2,1].to_ncsym()
1/3*m\{1\} + 1/3*m\{2\} + 1/6*m\{1, 2\}
sage: Phi = N.Phi()
sage: Phi[1,2].to_ncsym()
m\{1\} + m\{2\}, 3\}
sage: Phi[1,3].to_ncsym()
m\{1\}, 3\}
sage: Phi[1,3].to_ncsym()
-1/4*m\{1\} + 1/4*m\{2\} + 1/4*m\{3\} + 1/4*m\{4\}
- 1/4*m\{1\} + 1/4*m\{2\} + 1/4*m\{3\} + 1/4*m\{4\}
```
to_symmetric_function()

Return the commutative image of a non-commutative symmetric function.

OUTPUT:

- The commutative image of self. This will be a symmetric function.

EXAMPLES:

```sage
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: x = R.an_element(); x
2*R[] + 2*R[1] + 3*R[1, 1]
sage: x.to_symmetric_function()
2*s[] + 2*s[1] + 3*s[1, 1]
sage: y = N.Phi()[1,3]
sage: y.to_symmetric_function()
h[1, 1, 1, 1] - 3*h[2, 1, 1] + 3*h[3, 1]
```

to_symmetric_group_algebra()

Return the image of a non-commutative symmetric function into the symmetric group algebra
where the ribbon basis element indexed by a composition is associated with the sum of all per-
mutations which have descent set equal to said composition. In compliance with the anti- iso-
morphism between the descent algebra and the non-commutative symmetric functions, we index
descent positions by the reversed composition.

OUTPUT:

- The image of self under the embedding of the \( n \)-th degree homogeneous component of the
  non-commutative symmetric functions in the symmetric group algebra of \( S_n \). This can behave
  unexpectedly when self is not homogeneous.

EXAMPLES:

```sage
sage: R = NonCommutativeSymmetricFunctions(QQ).R()
sage: R[2,1].to_symmetric_group_algebra()
[1, 3, 2] + [2, 3, 1]
sage: R([]).to_symmetric_group_algebra()
[]
```

verschiebung\((n)\)

Return the image of the noncommutative symmetric function self under the \( n \)-th Verschiebung operator.

The \( n \)-th Verschiebung operator \( V_n \) is defined to be the map from the \( k \)-algebra of noncommuta-
tive symmetric functions to itself that sends the complete function \( S^I \) indexed by a composition
\( I = (i_1, i_2, \ldots, i_k) \) to \( S^{(i_1/n, i_2/n, \ldots, i_k/n)} \) if all of the numbers \( i_1, i_2, \ldots, i_k \) are divisible by \( n \),
and to 0 otherwise. This operator \( V_n \) is a Hopf algebra endomorphism. For every positive integer
\( r \) with \( n \mid r \), it satisfies

\[
V_n(S_r) = S_{r/n}, \quad V_n(\Lambda_r) = (-1)^{r-n/n} \Lambda_{r/n}, \quad V_n(\Psi_r) = n \Psi_{r/n}, \quad V_n(\Phi_r) = n \Phi_{r/n}
\]

(where \( S_r \) denotes the \( r \)-th complete non-commutative symmetric function, \( \Lambda_r \) denotes the \( r \)-th el-
ementary non-commutative symmetric function, \( \Psi_r \) denotes the \( r \)-th power-sum non-commutative
symmetric function of the first kind, and \( \Phi_r \) denotes the \( r \)-th power-sum non-commutative sym-
metric function of the second kind). For every positive integer \( r \) with \( n \nmid r \), it satisfies

\[
V_n(S_r) = V_n(\Lambda_r) = V_n(\Psi_r) = V_n(\Phi_r) = 0.
\]

The \( n \)-th Verschiebung operator is also called the \( n \)-th Verschiebung endomorphism.

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It is a lift of the $n$-th Verschiebung operator on the ring of symmetric functions ($\text{verschiebung}(\cdot)$) to the ring of noncommutative symmetric functions.

The action of the $n$-th Verschiebung operator can also be described on the ribbon Schur functions. Namely, every composition $I$ of size $n\ell$ satisfies

$$V_n(R_I) = (-1)^{\ell(I) - \ell(J)} \cdot R_{J/n},$$

where $J$ denotes the meet of the compositions $I$ and $(n, n, \ldots, n)$, where $\ell(I)$ is the length of $I$, and where $J/n$ denotes the composition obtained by dividing every entry of $J$ by $n$. For a composition $I$ of size not divisible by $n$, we have $V_n(R_I) = 0$.

**See also:**

- frobenius method of $QSym$
- verschiebung method of $Sym$

**INPUT:**

- $n$ – a positive integer

**OUTPUT:**

The result of applying the $n$-th Verschiebung operator (on the ring of noncommutative symmetric functions) to `self`.

**EXAMPLES:**

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: S = NSym.S()
sage: S[3,2].verschiebung(2)
0
sage: S[6,4].verschiebung(2)
S[3, 2]
4*S [[]] + 2*S[3, 2] - S[4, 1]
sage: (S[3,3] - 2*S[2]).verschiebung(3)
S[1, 1]
sage: S([4,2]).verschiebung(1)
S[4, 2]
sage: R = NSym.R()
sage: R([4,2]).verschiebung(2)
R[2, 1]
```

Being Hopf algebra endomorphisms, the Verschiebung operators commute with the antipode:

```python
sage: all( R(I).verschiebung(2).antipode() == R(I).antipode().verschiebung(2) for I in Compositions(4) )
True
```

They lift the Verschiebung operators of the ring of symmetric functions:

```python
sage: all( S(I).verschiebung(2).to_symmetric_function() == S(I).to_symmetric_function().verschiebung(2) for I in Compositions(4) )
True
```

The Frobenius operators on $QSym$ are adjoint to the Verschiebung operators on $NSym$ with respect to the duality pairing:
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```python
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: M = QSym.M()
sage: all( all( M(I).frobenius(3).duality_pairing(S(J))
            == M(I).duality_pairing(S(J).verschiebung(3))
            for I in Compositions(2) )
            for J in Compositions(3) )
True
```

```python
class ParentMethods
    Bases: object

    immaculate_function(xs)
    Return the immaculate function corresponding to the integer vector xs, written in the basis self.
    If \( \alpha \) is any integer vector – i.e., an element of \( \mathbb{Z}^m \) for some \( m \in \mathbb{N} \) –, the "immaculate function corresponding to \( \alpha \) is a non-commutative symmetric function denoted by \( S_\alpha \). One way to define this function is by setting

    \[
    S_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma S_{\alpha_1 + \sigma(1) - 1} S_{\alpha_2 + \sigma(2) - 2} \cdots S_{\alpha_m + \sigma(m) - m},
    \]

    where \( \alpha \) is written in the form \((\alpha_1, \alpha_2, \ldots, \alpha_m)\), and where \( S \) stands for the complete basis (Complete).
    The immaculate function \( S_\alpha \) first appeared in [BBSSZ2012] (where it was defined differently, but the definition we gave above appeared as Theorem 3.27).
    The immaculate functions \( S_\alpha \) for \( \alpha \) running over all compositions (i.e., finite sequences of positive integers) form a basis of \( NCSF \). This is the "immaculate basis" (Immaculate).

    INPUT:
    • xs – list (or tuple or any iterable – possibly a composition) of integers
    OUTPUT:

    The immaculate function \( S_{xs} \) written in the basis self.

    EXAMPLES:

    Let us first check that, for xs a composition, we get the same as the result of self.realization_of().I()[xs]:

    ```python
    sage: def test_comp(xs):
    ....:     NSym = NonCommutativeSymmetricFunctions(QQ)
    ....:     I = NSym.I()
    ....:     return I[xs] == I.immaculate_function(xs)
    sage: def test_allcomp(n):
    ....:     return all( test_comp(c) for c in Compositions(n) )
    sage: test_allcomp(1)
    True
    sage: test_allcomp(2)
    True
    sage: test_allcomp(3)
    True
    ```

    Now some examples of non-composition immaculate functions:
```
```python
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: I = NSym.I()
sage: I.immaculate_function([0, 1])
0
sage: I.immaculate_function([0, 2])
-I[1, 1]
sage: I.immaculate_function([-1, 1])
-I[]
0
sage: I.immaculate_function([2, -1])
0
sage: I.immaculate_function([2, 0])
I[2]
sage: I.immaculate_function([2, 0, 1])
0
sage: I.immaculate_function([1, 0, 2])
-I[1, 1] + I[2, 1, 1]
sage: I.immaculate_function([2, 0, 2])
```

**to_symmetric_function()**

Morphism to the algebra of symmetric functions.

This is constructed by extending the computation on the basis or by coercion to the complete basis.

**OUTPUT:**

- The module morphism from the basis `self` to the symmetric functions which corresponds to taking a commutative image.

**EXAMPLES:**

```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: x = R.an_element(); x
2*R[] + 2*R[1] + 3*R[1, 1]
sage: R.to_symmetric_function(x)
2*s[] + 2*s[1] + 3*s[1, 1]
sage: nM = N.Monomial()
sage: nM.to_symmetric_function(nM[3, 1])
```

**to_symmetric_function_on_basis()**

The image of the basis element indexed by `I` under the map to the symmetric functions.

This default implementation does a change of basis and computes the image in the complete basis.

**INPUT:**

- `I` – a composition

**OUTPUT:**

- The image of the non-commutative basis element of `self` indexed by the composition `I` under the map from non-commutative symmetric functions to the symmetric functions. This will be a symmetric function.

**EXAMPLES:**
```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: I = N.Immaculate()
sage: I.to_symmetric_function(I[1,3])
-h[2,2] + h[3,1]
sage: I.to_symmetric_function(I[1,2])
0
sage: Phi = N.Phi()
sage: Phi.to_symmetric_function_on_basis([3,1,2])==Phi.to_symmetric_function(Phi[3,1,2])
True
sage: Phi.to_symmetric_function_on_basis([])
h[]
```

**super_categories()**

Return the super categories of the category of bases of the non-commutative symmetric functions.

OUTPUT:
- list

**class Complete(NCSF)**

Bases: `CombinatorialFreeModule`, `BindableClass`

The Hopf algebra of non-commutative symmetric functions in the Complete basis.

The Complete basis is defined in Definition 3.4 of [NCSF1], where it is denoted by \((S^I)_I\). It is a multiplicative basis, and is connected to the elementary generators \(\Lambda_i\) of the ring of non-commutative symmetric functions by the following relation: Define a non-commutative symmetric function \(S_n\) for every nonnegative integer \(n\) by the power series identity

\[
\sum_{k \geq 0} t^k S_k = \left( \sum_{k \geq 0} (-t)^k \Lambda_k \right)^{-1}
\]

with \(\Lambda_0\) denoting 1. For every composition \((i_1, i_2, \ldots, i_k)\), we have \(S^{(i_1, i_2, \ldots, i_k)} = S_{i_1} S_{i_2} \cdots S_{i_k}\).

**EXAMPLES:**

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: S = NCSF.Complete(); S
Non-Commutative Symmetric Functions over the Rational Field in the Complete basis
sage: S.an_element()
2*S[] + 2*S[1] + 3*S[1, 1]
```

The following are aliases for this basis:

```python
sage: NCSF.complete()
Non-Commutative Symmetric Functions over the Rational Field in the Complete basis
sage: NCSF.S()
Non-Commutative Symmetric Functions over the Rational Field in the Complete basis
```

**class Element**

Bases: `IndexedFreeModuleElement`

An element in the Complete basis.

---

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psi_involution()  

Return the image of the noncommutative symmetric function `self` under the involution \( \psi \).

The involution \( \psi \) is defined as the linear map \( NCSF \to NCSF \) which, for every composition \( I \), sends the complete noncommutative symmetric function \( S^I \) to the elementary noncommutative symmetric function \( \Lambda^I \). It can be shown that every composition \( I \) satisfies

\[
\psi(R_I) = R_{I^c}, \quad \psi(S^I) = \Lambda^I, \quad \psi(\Lambda^I) = S^I, \quad \psi(\Phi^I) = (-1)^{|I| - \ell(I)} \Phi^I
\]

where \( I^c \) denotes the complement of the composition \( I \), and \( \ell(I) \) denotes the length of \( I \), and where standard notations for classical bases of \( NCSF \) are being used (\( S \) for the complete basis, \( \Lambda \) for the elementary basis, \( \Phi \) for the basis of the power sums of the second kind, and \( R \) for the ribbon basis). The map \( \psi \) is an involution and a graded Hopf algebra automorphism of \( NCSF \). If \( \pi \) denotes the projection from \( NCSF \) to the ring of symmetric functions \( \text{to_symmetric_function()} \), then \( \pi(\psi(f)) = \omega(\pi(f)) \) for every \( f \in NCSF \), where the \( \omega \) on the right hand side denotes the omega automorphism of \( Sym \).

The involution \( \psi \) of \( NCSF \) is adjoint to the involution \( \psi \) of \( QSym \) by the standard adjunction between \( NCSF \) and \( QSym \).

The involution \( \psi \) has been denoted by \( \psi \) in [LMvW13], section 3.6.

**See also:**  

\( psi \) involution of \( NCSF \), \( psi \) involution of \( QSym \), star involution of \( NCSF \).

**EXAMPLES:**

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)

sage: S = NSym.S()

sage: L = NSym.L()

sage: S[3,1].psi_involution()
S[1, 1, 1] - S[1, 2, 1] - S[2, 1, 1] + S[3, 1]

sage: L(S[3,1].psi_involution())
L[3, 1]

sage: S[[]].psi_involution()
S[]

sage: S[1,1].psi_involution()
S[1, 1]

sage: (S[2,1] - 2*S[2]).psi_involution()
```

The implementation at hand is tailored to the complete basis. It is equivalent to the generic implementation via the ribbon basis:

```python
sage: R = NSym.R()

sage: all( R(S[I].psi_involution()) == R(S[I]).psi_involution()  
.....:     for I in Compositions(4) )
True
```

dual()  

Return the dual basis to the complete basis of non-commutative symmetric functions. This is the Monomial basis of quasi-symmetric functions.

**OUTPUT:**

- The Monomial basis of quasi-symmetric functions.

**EXAMPLES:**
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S.dual()
Quasisymmetric functions over the Rational Field in the Monomial basis

**internal_product_on_basis(I, J)**

The internal product of two non-commutative symmetric complete functions.

See `internal_product()` for a thorough documentation of this operation.

**INPUT:**
- I, J – compositions

**OUTPUT:**
- The internal product of the complete non-commutative symmetric function basis elements indexed by I and J, expressed in the complete basis.

**EXAMPLES:**

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S.internal_product_on_basis([2,2],[1,2,1])
2*S[1, 1, 1, 1] + S[1, 1, 2] + S[2, 1, 1]
sage: S.internal_product_on_basis([2,2],[1,2])
0
```

**to_symmetric_function()**

Morphism to the algebra of symmetric functions.

This is constructed by extending the computation on the complete basis.

**OUTPUT:**
- The module morphism from the basis `self` to the symmetric functions which corresponds to taking a commutative image.

**EXAMPLES:**

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S.to_symmetric_function(S[3,1,2])
h[3, 2, 1]
sage: S.to_symmetric_function(S[])  
h[]
```

**to_symmetric_function_on_basis(I)**

The commutative image of a complete element

The commutative image of a basis element is obtained by sorting the indexing composition of the basis element and the output is in the complete basis of the symmetric functions.

**INPUT:**
- I – a composition

**OUTPUT:**
- The commutative image of the complete basis element indexed by I. The result is the complete symmetric function indexed by the partition obtained by sorting I.

**EXAMPLES:**

```
sage: S = NonCommutativeSymmetricFunctions(QQ).complete()
sage: S.to_symmetric_function_on_basis([2,1,3])
```
class Elementary(NCSF)

    Bases: CombinatorialFreeModule, BindableClass

    The Hopf algebra of non-commutative symmetric functions in the Elementary basis.

    The Elementary basis is defined in Definition 3.4 of [NCSF1], where it is denoted by \((\Lambda^I)_I\). It is a multiplicative basis, and is obtained from the elementary generators \(\Lambda_i\) of the ring of non-commutative symmetric functions through the formula \(\Lambda(i_1, i_2, \ldots, i_k) = \Lambda_{i_1} \Lambda_{i_2} \cdots \Lambda_{i_k}\) for every composition \((i_1, i_2, \ldots, i_k)\).

    EXAMPLES:

    sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: L = NCSF.Elementary(); L
    Non-Commutative Symmetric Functions over the Rational Field in the Elementary\_basis
sage: L.an_element()

    The following are aliases for this basis:

    sage: NCSF.elementary()
    Non-Commutative Symmetric Functions over the Rational Field in the Elementary\_basis
sage: NCSF.L()
    Non-Commutative Symmetric Functions over the Rational Field in the Elementary\_basis

class Element

    Bases: IndexedFreeModuleElement

    psi_involution()

    Return the image of the noncommutative symmetric function self under the involution \(\psi\).

    The involution \(\psi\) is defined as the linear map \(NCSF \to NCSF\) which, for every composition \(I\), sends the complete noncommutative symmetric function \(S^I\) to the elementary noncommutative symmetric function \(\Lambda^I\). It can be shown that every composition \(I\) satisfies

    \[
    \psi(R_I) = R_{I^c}, \quad \psi(S^I) = \Lambda^I, \quad \psi(\Lambda^I) = S^I, \quad \psi(\Phi^I) = (-1)^{|I| - \ell(I)} \Phi^I
    \]

    where \(I^c\) denotes the complement of the composition \(I\), and \(\ell(I)\) denotes the length of \(I\), and where standard notations for classical bases of \(NCSF\) are being used (\(S\) for the complete basis, \(\Lambda\) for the elementary basis, \(\Phi\) for the basis of the power sums of the second kind, and \(R\) for the ribbon basis).

    The map \(\psi\) is an involution and a graded Hopf algebra automorphism of \(NCSF\). If \(\pi\) denotes the projection from \(NCSF\) to the ring of symmetric functions \(to_symmetric_function()\), then \(\pi(\psi(f)) = \omega(\pi(f))\) for every \(f \in NCSF\), where the \(\omega\) on the right hand side denotes the omega automorphism of \(Sym\).

    The involution \(\psi\) of \(NCSF\) is adjoint to the involution \(\psi\) of \(QSym\) by the standard adjunction between \(NCSF\) and \(QSym\).

    The involution \(\psi\) has been denoted by \(\psi\) in [LMvW13], section 3.6.
See also:

\textit{psi involution of NCSF, psi involution of QSym, star involution of NCSF.}

**EXAMPLES:**

```python
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: S = NSym.S()
sage: L = NSym.L()
sage: L[3,1].psi_involution()
L[1, 1, 1, 1] - L[1, 2, 1] - L[2, 1, 1] + L[3, 1]
sage: S(L[3,1].psi_involution())
S[3, 1]
sage: L[].psi_involution()
L[]
sage: L[1,1].psi_involution()
L[1, 1]
sage: (L[2,1] - 2*L[2]).psi_involution()
```

The implementation at hand is tailored to the elementary basis. It is equivalent to the generic implementation via the ribbon basis:

```python
sage: R = NSym.R()
sage: all( R(L[I].psi_involution()) == R(L[I]).psi_involution() for I in Compositions(3) )
True
sage: all( R(L[I].psi_involution()) == R(L[I]).psi_involution() for I in Compositions(4) )
True
```

**star_involution()**

Return the image of the noncommutative symmetric function \( \text{self} \) under the star involution.

The star involution is defined as the algebra antihomomorphism \( \text{NCSF} \to \text{NCSF} \) which, for every positive integer \( n \), sends the \( n \)-th complete non-commutative symmetric function \( S_n \) to \( S_n \).

Denoting by \( f^* \) the image of an element \( f \in \text{NCSF} \) under this star involution, it can be shown that every composition \( I \) satisfies

\[
(S^I)^* = S^{I^r}, \quad (\Lambda^I)^* = \Lambda^{I^r}, \quad R_I^* = R_{I^r}, \quad (\Phi^I)^* = \Phi^{I^r},
\]

where \( I^r \) denotes the reversed composition of \( I \), and standard notations for classical bases of \( \text{NCSF} \) are being used (\( S \) for the complete basis, \( \Lambda \) for the elementary basis, \( R \) for the ribbon basis, and \( \Phi \) for that of the power-sums of the second kind). The star involution is an involution and a coalgebra automorphism of \( \text{NCSF} \). It is an automorphism of the graded vector space \( \text{NCSF} \). Under the canonical isomorphism between the \( n \)-th graded component of \( \text{NCSF} \) and the descent algebra of the symmetric group \( S_n \) (see \texttt{to_descent_algebra()}), the star involution (restricted to the \( n \)-th graded component) corresponds to the automorphism of the descent algebra given by \( x \mapsto \omega_n x \omega_n \), where \( \omega_n \) is the permutation \( (n, n-1, \ldots, 1) \in S_n \) (written here in one-line notation). If \( \pi \) denotes the projection from \( \text{NCSF} \) to the ring of symmetric functions (\texttt{to_symmetric_function()}), then \( \pi(f^*) = \pi(f) \) for every \( f \in \text{NCSF} \).

The star involution on \( \text{NCSF} \) is adjoint to the star involution on \( \text{QSym} \) by the standard adjunction between \( \text{NCSF} \) and \( \text{QSym} \).

The star involution has been denoted by \( \rho \) in [LMvW13], section 3.6. See [NCSF2], section 2.3 for the properties of this map.

5.1. Comprehensive Module List

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See also:

* star involution of NCSF, psi involution of NCSF, star involution of QSym.

**EXAMPLES:**

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: L = NSym.L()
sage: L[3,3,2,3].star_involution()
L[3, 2, 3, 3]
sage: L[6,3,3].star_involution()
L[3, 3, 6]
sage: (L[3,3] - 2*L[2]).star_involution()
sage: L([4,1]).star_involution()
L[1, 4]
```

The implementation at hand is tailored to the elementary basis. It is equivalent to the generic implementation via the complete basis:

```python
sage: S = NSym.S()
sage: all( S(L[I].star_involution()) == S(L[I]).star_involution()
    ....:     for I in Compositions(4) )
True
```

**verschiebung**(n)

Return the image of the noncommutative symmetric function self under the \( n \)-th Verschiebung operator.

The \( n \)-th Verschiebung operator \( V_n \) is defined to be the map from the \( k \)-algebra of noncommutative symmetric functions to itself that sends the complete function \( S^I \) indexed by a composition \( I = (i_1, i_2, \ldots, i_k) \) to \( S^{(i_1/n, i_2/n, \ldots, i_k/n)} \) if all of the numbers \( i_1, i_2, \ldots, i_k \) are divisible by \( n \), and to 0 otherwise. This operator \( V_n \) is a Hopf algebra endomorphism. For every positive integer \( r \) with \( n \mid r \), it satisfies

\[
V_n(S_r) = S_{r/n}, \quad V_n(\Lambda_r) = (-1)^{r-n/n} \Lambda_{r/n}, \quad V_n(\Psi_r) = n \Psi_{r/n}, \quad V_n(\Phi_r) = n \Phi_{r/n}
\]

(where \( S_r \) denotes the \( r \)-th complete non-commutative symmetric function, \( \Lambda_r \) denotes the \( r \)-th elementary non-commutative symmetric function, \( \Psi_r \) denotes the \( r \)-th power-sum non-commutative symmetric function of the first kind, and \( \Phi_r \) denotes the \( r \)-th power-sum non-commutative symmetric function of the second kind). For every positive integer \( r \) with \( n \nmid r \), it satisfies

\[
V_n(S_r) = V_n(\Lambda_r) = V_n(\Psi_r) = V_n(\Phi_r) = 0.
\]

The \( n \)-th Verschiebung operator is also called the \( n \)-th Verschiebung endomorphism.

It is a lift of the \( n \)-th Verschiebung operator on the ring of symmetric functions (verschiebung()) to the ring of noncommutative symmetric functions.

The action of the \( n \)-th Verschiebung operator can also be described on the ribbon Schur functions. Namely, every composition \( I \) of size \( n \ell \) satisfies

\[
V_n(R_I) = (-1)^{\ell(I) - \ell(J)} \cdot R_{J/n},
\]
where $J$ denotes the meet of the compositions $I$ and $(n, n, \ldots, n)$, where $\ell(I)$ is the length of $I$, and where $J/n$ denotes the composition obtained by dividing every entry of $J$ by $n$. For a composition $I$ of size not divisible by $n$, we have $V_n(R_I) = 0$.

**See also:**

*verschiebung method of NCSF*, *frobenius method of QSym*, *verschiebung method of Sym*

**INPUT:**

- $n$ – a positive integer

**OUTPUT:**

The result of applying the $n$-th Verschiebung operator (on the ring of noncommutative symmetric functions) to `self`.

**EXAMPLES:**

```sage
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: L = NSym.L()
sage: L([4,2]).verschiebung(2)
-L[2, 1]
sage: L([2,4]).verschiebung(2)
-L[1, 2]
sage: L([6]).verschiebung(2)
-L[3]
sage: L([2,1]).verschiebung(3)
0
sage: L([3]).verschiebung(2)
0
sage: L([]).verschiebung(2)
L[]
sage: L([5, 1]).verschiebung(3)
0
sage: L([5, 1]).verschiebung(6)
0
sage: L([5, 1]).verschiebung(2)
0
sage: L([1, 2, 3, 1]).verschiebung(7)
0
sage: L([7]).verschiebung(7)
L[1]
sage: L([1, 2, 3, 1]).verschiebung(5)
0
```

**I**

alias of `Immaculate`

**class Immaculate(NCSF)**

**Bases:** `CombinatorialFreeModule`, `BindableClass`

The immaculate basis of the non-commutative symmetric functions.

The immaculate basis first appears in Berg, Bergeron, Saliola, Serrano and Zabrocki’s [BBSSZ2012].
can be described as the family $(\mathcal{S}_\alpha)$, where $\alpha$ runs over all compositions, and $\mathcal{S}_\alpha$ denotes the immaculate function corresponding to $\alpha$ (see `immaculate_function()`).

If $\alpha$ is a composition $(\alpha_1, \alpha_2, \ldots, \alpha_m)$, then

$$\mathcal{S}_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma S_{\sigma_1 + \sigma(1) - 1} S_{\sigma_2 + \sigma(2) - 2} \cdots S_{\sigma_m + \sigma(m) - m}.$$ 

**Warning:** This *basis* contains only the immaculate functions indexed by compositions (i.e., finite sequences of positive integers). To obtain the remaining immaculate functions (sensu lato), use the `immaculate_function()` method. Calling the immaculate *basis* with a list which is not a composition will currently return garbage!

**EXAMPLES:**

```
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: I = NCSF.I()
sage: I([1, 3, 2])*I([1])
I[1, 3, 2, 1] + I[1, 3, 3] + I[1, 4, 2] + I[2, 3, 2]
sage: I([1])^2*I([1, 3, 2])
sage: I([1, 3])^2*I([1, 1])
I[1, 1, 3, 1] + I[1, 4, 1] + I[2, 3, 1] + I[2, 4]
sage: I([3, 1, 1])^3*I([2, 1])
```

```
sage: R = NCSF.ribbon()
sage: I(R([1, 3, 1]))
sage: R(I(R([2, 1, 3])))
R[2, 1, 3]
```

**class Element**

- **Bases:** `IndexedFreeModuleElement`

An element in the Immaculate basis.

**bernstein_creation_operator**(n)**

Return the image of self under the $n$-th Bernstein creation operator.

Let $n$ be an integer. The $n$-th Bernstein creation operator $B_n$ is defined as the endomorphism of the space $NSym$ of noncommutative symmetric functions given by

$$B_n I_{(\alpha_1, \alpha_2, \ldots, \alpha_m)} = I_{(n, \alpha_1, \alpha_2, \ldots, \alpha_m)},$$

where $I_{(\alpha_1, \alpha_2, \ldots, \alpha_m)}$ is the immaculate function associated to the $m$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{Z}^m$.

This has been introduced in [BBSSZ2012], section 3.1, in analogy to the Bernstein creation operators on the symmetric functions.

For more information on the $n$-th Bernstein creation operator, see `bernstein_creation_operator()`.

**EXAMPLES:**
Combinatorics, Release 10.0

```
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: I = NSym.I()
sage: b = I[1,3,2,1]
sage: b.bernstein_creation_operator(3)
I[3, 1, 3, 2, 1]
sage: b.bernstein_creation_operator(5)
I[5, 1, 3, 2, 1]
sage: elt = b + 3*I[4,1,2]
sage: elt.bernstein_creation_operator(1)
I[1, 1, 3, 2, 1] + 3*I[1, 4, 1, 2]
```

We check that this agrees with the definition on the Complete basis:

```
sage: S = NSym.S()
sage: S(elt).bernstein_creation_operator(1) == S(elt.bernstein_creation_operator(1))
True
```

Check on non-positive values of $n$:

```
sage: I[2,2,2].bernstein_creation_operator(-1)
I[1, 1, 1, 2] + I[1, 1, 2, 1] + I[1, 2, 1, 1] - I[1, 2, 2]
sage: I[2,3,2].bernstein_creation_operator(0)
-I[1, 1, 3, 2] - I[1, 2, 2, 2] - I[1, 2, 3, 1] + I[2, 3, 2]
```

dual()

Return the dual basis to the Immaculate basis of NCSF.

The basis returned is the dualImmaculate basis of QSym.

OUTPUT:

• The dualImmaculate basis of the quasi-symmetric functions.

EXAMPLES:

```
sage: I = NonCommutativeSymmetricFunctions(QQ).Immaculate()
sage: I.dual()
Quasisymmetric functions over the Rational Field in the dualImmaculate basis
```

L

alias of Elementary

class Monomial(NCSF)

Bases: CombinatorialFreeModule, BindableClass

The monomial basis defined in Tevlin’s paper [Tev2007].

The monomial basis is well-defined only when the base ring is a $\mathbb{Q}$-algebra. It is the basis denoted by $(M')_i$ in [Tev2007].

class MultiplicativeBases(parent_with_realization)

Bases: Category_realization_of_parent

Category of multiplicative bases of non-commutative symmetric functions.

EXAMPLES:
The complete basis is a multiplicative basis, but the ribbon basis is not:

```
sage: N.Complete() in N.MultiplicativeBases()
True
sage: N.Ribbon() in N.MultiplicativeBases()
False
```

### class ParentMethods

**Bases:** object

#### algebra_generators()

Return the algebra generators of a given multiplicative basis of non-commutative symmetric functions.

**OUTPUT:**

- The family of generators of the multiplicative basis `self`.

**EXAMPLES:**

```
sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: f = Psi.algebra_generators()
sage: f
Lazy family (lambda i: Psi[i])_{i in Positive integers}
sage: f[1], f[2], f[3]
(Psi[1], Psi[2], Psi[3])
```

#### algebra_morphism(on_generators, **keywords)

Given a map defined on the generators of the multiplicative basis `self`, return the algebra morphism that extends this map to the whole algebra of non-commutative symmetric functions.

**INPUT:**

- `on_generators` – a function defined on the index set of the generators (that is, on the positive integers)
- `anti` – a boolean; defaults to `False`
- `category` – a category; defaults to `None`

**OUTPUT:**

- The algebra morphism of `self` which is defined by `on_generators` in the basis `self`. When `anti` is set to `True`, an algebra anti-morphism is computed instead of an algebra morphism.

**EXAMPLES:**

```
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: Psi = NCSF.Psi()
sage: double = lambda i: Psi[i,i]
sage: f = Psi.algebra_morphism(double, codomain = Psi)
sage: f
Generic endomorphism of Non-Commutative Symmetric Functions over the Rational Field in the Psi basis
sage: f(2*Psi[] + 3 * Psi[1,3,2] + Psi[2,4] )
2*Psi[] + 3*Psi[1, 1, 3, 3, 2, 2] + Psi[2, 2, 4, 4]
sage: f.category()
```

(continues on next page)
When extra properties about the morphism are known, one can specify the category of which it is a morphism:

```
sage: negate = lambda i: -Psi[i]
sage: f = Psi.algebra_morphism(negate, codomain = Psi, category =
        GradedHopfAlgebrasWithBasis(QQ))
sage: f
Generic endomorphism of Non-Commutative Symmetric Functions over the
    Rational Field in the Psi basis
sage: f(2*Psi[[]] + 3 * Psi[1,3,2] + Psi[2,4] )
2*Psi[[]] - 3*Psi[1, 3, 2] + Psi[2, 4]
sage: f.category()
Category of endsets of hopf algebras over Rational Field and graded,
    modules over Rational Field
```

If `anti` is true, this returns an anti-algebra morphism:

```
sage: f = Psi.algebra_morphism(double, codomain = Psi, anti=True)
sage: f
Generic endomorphism of Non-Commutative Symmetric Functions over the
    Rational Field in the Psi basis
sage: f(2*Psi[[]] + 3 * Psi[1,3,2] + Psi[2,4] )
2*Psi[[]] + 3*Psi[2, 2, 3, 3, 1, 1] + Psi[4, 4, 2, 2]
sage: f.category()
Category of endsets of modules with basis over Rational Field
```

`antipode()`

Return the antipode morphism on the basis `self`.

The antipode of `NSym` is closely related to the omega involution; see `omega_involution()` for the latter.

**OUTPUT:**

- The antipode module map from non-commutative symmetric functions on basis `self`.

**EXAMPLES:**

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S.antipode
Generic endomorphism of Non-Commutative Symmetric Functions over the
    Rational Field in the Complete basis
```

`coproduct()`

Return the coproduct morphism in the basis `self`.

**OUTPUT:**

- The coproduct module map from non-commutative symmetric functions on basis `self`.

**EXAMPLES:**

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S.coproduct
Generic morphism:
```
product_on_basis(composition1, composition2)

Return the product of two basis elements from the multiplicative basis. Multiplication is just concatenation on compositions.

INPUT:
• composition1, composition2 – integer compositions

OUTPUT:
• The product of the two non-commutative symmetric functions indexed by composition1 and composition2 in the multiplicative basis self. This will be again a non-commutative symmetric function.

EXAMPLES:

sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: Psi[3,1,2] * Psi[4,2] == Psi[3,1,2,4,2]
True
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S.product_on_basis(Composition([2,1]), Composition([1,2]))
S[2, 1, 1, 2]

to_symmetric_function()

Morphism to the algebra of symmetric functions.

This is constructed by extending the algebra morphism by the image of the generators.

OUTPUT:
• The module morphism from the basis self to the symmetric functions which corresponds to taking a commutative image.

EXAMPLES:

sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: S.to_symmetric_function(S[1,3])
h[3, 1]
sage: Phi = N.Phi()
sage: Phi.to_symmetric_function(Phi[1,3])
h[1, 1, 1, 1] - 3*h[2, 1, 1] + 3*h[3, 1]
sage: Psi = N.Psi()
sage: Psi.to_symmetric_function(Psi[1,3])
h[1, 1, 1, 1] - 3*h[2, 1, 1] + 3*h[3, 1]

to_symmetric_function_on_generators()

Morphism of the generators to symmetric functions.

This is constructed by coercion to the complete basis and applying the morphism.

OUTPUT:
• The module morphism from the basis self to the symmetric functions which corresponds to taking a commutative image.

EXAMPLES:
```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: Phi = N.Phil(3)
\text{h}[1, 1, 1] - 3^{3}\text{h}[2, 1] + 3^{3}\text{h}[3]
sage: Phi.to_symmetric_function_on_generators(0)
\text{h}[]
```

```
sage: Psi = N.Psi()
\text{h}[1, 1, 1] - 3^{3}\text{h}[2, 1] + 3^{3}\text{h}[3]
```

```
sage: L = N.elementary()
\text{h}[1, 1, 1] - 2^{3}\text{h}[2, 1] + \text{h}[3]
```

```python
super_categories()
Return the super categories of the category of multiplicative bases of the non-commutative symmetric functions.

OUTPUT:
- list
class MultiplicativeBasesOnGroupLikeElements(parent_with_realization)
Bases: Category_realization_of_parent

Category of multiplicative bases on grouplike elements of non-commutative symmetric functions.

Here, a “multiplicative basis on grouplike elements” means a multiplicative basis whose generators $(f_1, f_2, f_3, \ldots)$ satisfy

\[ \Delta(f_i) = \sum_{j=0}^{i} f_j \otimes f_{i-j} \]

with $f_0 = 1$. (In other words, the generators are to form a divided power sequence in the sense of a coalgebra.) This does not mean that the generators are grouplike, but means that the element $1 + f_1 + f_2 + f_3 + \cdots$ in the completion of the ring of non-commutative symmetric functions with respect to the grading is grouplike.

EXAMPLES:
```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: N.MultiplicativeBasesOnGroupLikeElements()
Category of multiplicative bases on group like elements of Non-Commutative\text{\_}Symmetric Functions over the Rational Field
```

The complete basis is a multiplicative basis, but the ribbon basis is not:
```python
sage: N.Complete() in N.MultiplicativeBasesOnGroupLikeElements()
True
sage: N.Ribbon() in N.MultiplicativeBasesOnGroupLikeElements()
False
```

class ParentMethods
Bases: object
```
\text{antipode_on_basis}(composition)
Return the application of the antipode to a basis element.
```
```
• composition – a composition

OUTPUT:
• The image of the basis element indexed by composition under the antipode map.

EXAMPLES:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).complete()
sage: S.antipode_on_basis(Composition([2,1]))
-S[1, 1, 1] + S[1, 2]
sage: S[1].antipode() # indirect doctest
-S[1]
sage: S[2].antipode() # indirect doctest
S[1, 1] - S[2]
sage: S[3].antipode() # indirect doctest
-S[1, 1, 1] + S[1, 2] + S[2, 1] - S[3]
sage: S[2,3].coproduct().apply_multilinear_morphism(lambda be,ga: S(be)*S(ga).antipode())
0
sage: S[2,3].coproduct().apply_multilinear_morphism(lambda be,ga: S(be).antipode()*S(ga))
0
```

coproduct_on_generators(i)

Return the image of the $i^{th}$ generator of the algebra under the coproduct.

INPUT:
• $i$ – a positive integer

OUTPUT:
• The result of applying the coproduct to the $i^{th}$ generator of self.

EXAMPLES:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).complete()
sage: S.coproduct_on_generators(3)
```

super_categories()

Return the super categories of the category of multiplicative bases of group-like elements of the non-commutative symmetric functions.

OUTPUT:
• list

class MultiplicativeBasesOnPrimitiveElements(parent_with_realization)

Bases: Category_realization_of_parent

Category of multiplicative bases of the non-commutative symmetric functions whose generators are primitive elements.

An element $x$ of a bialgebra is primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$, where $\Delta$ is the coproduct of the bialgebra.

Given a multiplicative basis and knowing the coproducts and antipodes of its generators, one can compute the coproduct and the antipode of any element, since they are respectively algebra morphisms and antimorphisms. See antipode_on_generators() and coproduct_on_generators().

Todo: this could be generalized to any free algebra.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: N.MultiplicativeBasesOnPrimitiveElements()
Category of multiplicative bases on primitive elements of Non-Commutative
  Symmetric Functions over the Rational Field
```

The Phi and Psi bases are multiplicative bases whose generators are primitive elements, but the complete and ribbon bases are not:

```python
sage: N.Phi() in N.MultiplicativeBasesOnPrimitiveElements()
True
sage: N.Psi() in N.MultiplicativeBasesOnPrimitiveElements()
True
sage: N.Complete() in N.MultiplicativeBasesOnPrimitiveElements()
False
sage: N.Ribbon() in N.MultiplicativeBasesOnPrimitiveElements()
False
```

```python
class ParentMethods
    Bases: object

    antipode_on_generators(i)
    Return the image of a generator of a primitive basis of the non-commutative symmetric functions under the antipode map.
    INPUT:
    • i – a positive integer
    OUTPUT:
    • The image of the i-th generator of the multiplicative basis self under the antipode of the algebra of non-commutative symmetric functions.
    EXAMPLES:

    sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: Psi.antipode_on_generators(2)
-Psi[2]
```

```python
coprocess_on_generators(i)
    Return the image of the i-th generator of the multiplicative basis self under the coproduct.
    INPUT:
    • i – a positive integer
    OUTPUT:
    • The result of applying the coproduct to the i-th generator of self.
    EXAMPLES:

    sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: Psi.coprocess_on_generators(3)
```

```python
super_categories()
    Return the super categories of the category of multiplicative bases of primitive elements of the non-commutative symmetric functions.
    OUTPUT:
    • list
```

5.1. Comprehensive Module List
class Phi(NCSF)
Bases: CombinatorialFreeModule, BindableClass

The Hopf algebra of non-commutative symmetric functions in the Phi basis.

The Phi basis is defined in Definition 3.4 of [NCSF1], where it is denoted by \((\Phi^I)_I\). It is a multiplicative basis, and is connected to the elementary generators \(\Lambda_i\) of the ring of non-commutative symmetric functions by the following relation: Define a non-commutative symmetric function \(\Phi_n\) for every positive integer \(n\) by the power series identity

\[
\sum_{k \geq 1} \frac{t^k}{k} \Phi_k = -\log \left( \sum_{k \geq 0} (-t)^k \Lambda_k \right),
\]

with \(\Lambda_0\) denoting 1. For every composition \((i_1, i_2, \ldots, i_k)\), we have \(\Phi^{(i_1, i_2, \ldots, i_k)} = \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_k}\).

The \(\Phi\)-basis is well-defined only when the base ring is a \(\mathbb{Q}\)-algebra. The elements of the \(\Phi\)-basis are known as the “power-sum non-commutative symmetric functions of the second kind”.

The generators \(\Phi_n\) are related to the (first) Eulerian idempotents in the descent algebras of the symmetric groups (see [NCSF1], 5.4 for details).

EXAMPLES:

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: Phi = NCSF.Phi(); Phi
Non-Commutative Symmetric Functions over the Rational Field in the Phi basis
sage: Phi.an_element()
2*Phi[] + 2*Phi[1] + 3*Phi[1, 1]
```

class Element
Bases: IndexedFreeModuleElement

psi_involution()
Return the image of the noncommutative symmetric function self under the involution \(\psi\).

The involution \(\psi\) is defined as the linear map \(NCSF \to NCSF\) which, for every composition \(I\), sends the complete noncommutative symmetric function \(S^I\) to the elementary noncommutative symmetric function \(\Lambda^I\). It can be shown that every composition \(I\) satisfies

\[
\psi(R_I) = R_{I^c}, \quad \psi(S^I) = \Lambda^I, \quad \psi(\Lambda^I) = S^I, \quad \psi(\Phi^I) = (-1)^{|I| - \ell(I)} \Phi^I
\]

where \(I^c\) denotes the complement of the composition \(I\), and \(\ell(I)\) denotes the length of \(I\), and where standard notations for classical bases of \(NCSF\) are being used (\(S\) for the complete basis, \(\Lambda\) for the elementary basis, \(\Phi\) for the basis of the power sums of the second kind, and \(R\) for the ribbon basis).

The map \(\psi\) is an involution and a graded Hopf algebra automorphism of \(NCSF\). If \(\pi\) denotes the projection from \(NCSF\) to the ring of symmetric functions \(\text{to_symmetric_function()}\), then \(\pi(\psi(f)) = \omega(\pi(f))\) for every \(f \in NCSF\), where the \(\omega\) on the right hand side denotes the omega automorphism of \(Sym\).

The involution \(\psi\) of \(NCSF\) is adjoint to the involution \(\psi\) of \(QSym\) by the standard adjunction between \(NCSF\) and \(QSym\).

The involution \(\psi\) has been denoted by \(\psi\) in [LMvW13], section 3.6.

See also:
psi involution of NCSF, psi involution of QSym, star involution of NCSF.

EXAMPLES:
The implementation at hand is tailored to the Phi basis. It is equivalent to the generic implementation via the ribbon basis:

```
sage: R = NSym.R()
sage: all( R(Phi[I].psi_involution()) == R(Phi[I]).psi_involution() 
        ....:    for I in Compositions(4) )
True
```

**star_involution()**

Return the image of the noncommutative symmetric function self under the star involution.

The star involution is defined as the algebra antihomomorphism $NCSF \to NCSF$ which, for every positive integer $n$, sends the $n$-th complete non-commutative symmetric function $S_n$ to $S_n$. Denoting by $f^*$ the image of an element $f \in NCSF$ under this star involution, it can be shown that every composition $I$ satisfies

$$(S^I)^* = S^{I^r}, \quad (\Lambda^I)^* = \Lambda^{I^r}, \quad R_i^r = R_{i^r}, \quad (\Phi^I)^* = \Phi^{I^r},$$

where $I^r$ denotes the reversed composition of $I$, and standard notations for classical bases of $NCSF$ are being used ($S$ for the complete basis, $\Lambda$ for the elementary basis, $R$ for the ribbon basis, and $\Phi$ for that of the power-sums of the second kind). The star involution is an involution and a coalgebra automorphism of $NCSF$. It is an automorphism of the graded vector space $NCSF$. Under the canonical isomorphism between the $n$-th graded component of $NCSF$ and the descent algebra of the symmetric group $S_n$ (see to_descent_algebra()), the star involution (restricted to the $n$-th graded component) corresponds to the automorphism of the descent algebra given by $x \mapsto \omega_n x \omega_n$, where $\omega_n$ is the permutation $(n, n-1, \ldots, 1) \in S_n$ (written here in one-line notation). If $\pi$ denotes the projection from $NCSF$ to the ring of symmetric functions (to_symmetric_function()), then $\pi(f^*) = \pi(f)$ for every $f \in NCSF$.

The star involution on $NCSF$ is adjoint to the star involution on $QSym$ by the standard adjunction between $NCSF$ and $QSym$.

The star involution has been denoted by $\rho$ in [LMvW13], section 3.6. See [NCSF2], section 2.3 for the properties of this map.

**See also:**

- star involution of $NCSF$
- psi involution of $NCSF$
- star involution of $QSym$

**EXAMPLES:**

```
sage: NSym = NonCommutativeSymmetricFunctions(QQ)
sage: Phi = NSym.Phi()
sage: Phi[3,2].psi_involution() -Phi[3, 2]
sage: Phi[2,2].psi_involution() Phi[2, 2]
sage: Phi[].psi_involution() Phi[]
sage: Phi(0).psi_involution() 0
```
sage: Phi[3,1,1,4].star_involution()
Phi[4, 1, 1, 3]
sage: Phi[4, 2, 1].star_involution()
Phi[1, 2, 4]
sage: (Phi[3, 3] + 3*Phi[1]).star_involution()
3*Phi[1] + Phi[3, 3]
sage: Phi([2,1]).star_involution()
Phi[1, 2]

The implementation at hand is tailored to the Phi basis. It is equivalent to the generic implementation via the complete basis:

sage: S = NSym.S()
sage: all( S(Phi[I].star_involution()) == S(Phi[I]).star_involution() 
....:     for I in Compositions(4) )
True

verschiebung(n)

Return the image of the noncommutative symmetric function self under the n-th Verschiebung operator.

The n-th Verschiebung operator $V_n$ is defined to be the map from the $k$-algebra of noncommutative symmetric functions to itself that sends the complete function $S^I$ indexed by a composition $I = (i_1, i_2, \ldots, i_k)$ to $S^{i_1/n, i_2/n, \ldots, i_k/n}$ if all of the numbers $i_1, i_2, \ldots, i_k$ are divisible by $n$, and to 0 otherwise. This operator $V_n$ is a Hopf algebra endomorphism. For every positive integer $r$ with $n \mid r$, it satisfies

$$V_n(S_r) = S_{r/n}, \quad V_n(\Lambda_r) = (-1)^{r-r/n} \Lambda_{r/n}, \quad V_n(\Psi_r) = n \Psi_{r/n}, \quad V_n(\Phi_r) = n \Phi_{r/n}$$

(where $S_r$ denotes the $r$-th complete non-commutative symmetric function, $\Lambda_r$ denotes the $r$-th elementary non-commutative symmetric function, $\Psi_r$ denotes the $r$-th power-sum non-commutative symmetric function of the first kind, and $\Phi_r$ denotes the $r$-th power-sum non-commutative symmetric function of the second kind). For every positive integer $r$ with $n \nmid r$, it satisfies

$$V_n(S_r) = V_n(\Lambda_r) = V_n(\Psi_r) = V_n(\Phi_r) = 0.$$ 

The n-th Verschiebung operator is also called the n-th Verschiebung endomorphism.

It is a lift of the n-th Verschiebung operator on the ring of symmetric functions (verschiebung()) to the ring of noncommutative symmetric functions.

The action of the n-th Verschiebung operator can also be described on the ribbon Schur functions. Namely, every composition $I$ of size $n\ell$ satisfies

$$V_n(R_I) = (-1)^{\ell(I)-\ell(J)} \cdot R_{J/n},$$

where $J$ denotes the meet of the compositions $I$ and $\underbrace{(n, n, \ldots, n)}_{|I|/n}$ times, where $\ell(I)$ is the length of $I$, and where $J/n$ denotes the composition obtained by dividing every entry of $J$ by $n$. For a composition $I$ of size not divisible by $n$, we have $V_n(R_I) = 0.$
See also:

verschiebung method of NCSF, frobenius method of QSym, verschiebung method of Sym

INPUT:

• n – a positive integer

OUTPUT:

The result of applying the $n$-th Verschiebung operator (on the ring of noncommutative symmetric functions) to self.

EXAMPLES:

```
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: Phi = NSym.Phi()
sage: Phi([4,2]).verschiebung(2)
4*Phi[2, 1]
sage: Phi([2,4]).verschiebung(2)
4*Phi[1, 2]
sage: Phi([6]).verschiebung(2)
2*Phi[3]
sage: Phi([2,1]).verschiebung(3)
0
sage: Phi([3]).verschiebung(2)
0
sage: Phi([]).verschiebung(2)
Phi[]
sage: Phi([5, 1]).verschiebung(3)
0
sage: Phi([5, 1]).verschiebung(6)
0
sage: Phi([5, 1]).verschiebung(2)
0
sage: Phi([1, 2, 3, 1]).verschiebung(7)
0
sage: Phi([7]).verschiebung(7)
7*Phi[1]
sage: Phi([1, 2, 3, 1]).verschiebung(5)
0
sage: (Phi[1] - Phi[2] + 2*Phi[3]).verschiebung(1)
```

class Psi(NCSF)

Bases: CombinatorialFreeModule, BindableClass

The Hopf algebra of non-commutative symmetric functions in the Psi basis.

The Psi basis is defined in Definition 3.4 of [NCSF1], where it is denoted by $(\Psi^i)_t$. It is a multiplicative basis, and is connected to the elementary generators $\Lambda_i$ of the ring of non-commutative symmetric functions by the following relation: Define a non-commutative symmetric function $\Psi_n$ for every positive integer $n$ by the power series identity

\[
\frac{d}{dt} \sigma(t) = \sigma(t) \cdot \left( \sum_{k \geq 1} t^{k-1} \Psi_k \right),
\]
where
\[
\sigma(t) = \left( \sum_{k \geq 0} (-t)^k \Lambda_k \right)^{-1}
\]
and where \( \Lambda_0 \) denotes 1. For every composition \((i_1, i_2, \ldots, i_k)\), we have \( \Psi^{(i_1, i_2, \ldots, i_k)} = \Psi_{i_1} \Psi_{i_2} \cdots \Psi_{i_k} \).

The \( \Psi \)-basis is a basis only when the base ring is a \( \mathbb{Q} \)-algebra (although the \( \Psi^I \) can be defined over any base ring). The elements of the \( \Psi \)-basis are known as the "power-sum non-commutative symmetric functions of the first kind". The generators \( \Psi_n \) correspond to the Dynkin (quasi-)idempotents in the descent algebras of the symmetric groups (see [NCSF1], 5.2 for details).

Another (equivalent) definition of \( \Psi_n \) is
\[
\Psi_n = \sum_{i=0}^{n-1} (-1)^i R_{1^i, n-i},
\]
where \( R \) denotes the ribbon basis of \( NCSF \), and where \( 1^i \) stands for \( i \) repetitions of the integer 1.

EXAMPLES:

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: Psi = NCSF.Psi(); Psi
Non-Commutative Symmetric Functions over the Rational Field in the Psi basis
sage: Psi.an_element()
2*Psi[] + 2*Psi[1] + 3*Psi[1, 1]
```

Checking the equivalent definition of \( \Psi_n \):

```python
sage: def test_psi(n):
....:     NCSF = NonCommutativeSymmetricFunctions(ZZ)
....:     R = NCSF.R()
....:     Psi = NCSF.Psi()
....:     a = R.sum([(-1) ** i * R[[1]*i + [n-i]]
....:                for i in range(n)]))
....:     return a == R(Psi[n])
sage: test_psi(2)
True
sage: test_psi(3)
True
sage: test_psi(4)
True
```

class Element

Bases: IndexedFreeModuleElement

```
verschiebung(n)
```

Return the image of the noncommutative symmetric function \( \text{self} \) under the \( n \)-th Verschiebung operator.

The \( n \)-th Verschiebung operator \( V_n \) is defined to be the map from the \( k \)-algebra of noncommutative symmetric functions to itself that sends the complete function \( S^I \) indexed by a composition \( I = (i_1, i_2, \ldots, i_k) \) to \( S^{(i_1/n, i_2/n, \ldots, i_k/n)} \) if all of the numbers \( i_1, i_2, \ldots, i_k \) are divisible by \( n \), and to 0 otherwise. This operator \( V_n \) is a Hopf algebra endomorphism. For every positive integer \( r \) with \( n \mid r \), it satisfies

\[
V_n(S_r) = S_{r/n}, \quad V_n(\Lambda_r) = (-1)^{r-r/n} \Lambda_{r/n}, \quad V_n(\Psi_r) = n\Psi_{r/n}, \quad V_n(\Phi_r) = n\Phi_{r/n}
\]
(where $S_r$ denotes the $r$-th complete non-commutative symmetric function, $\Lambda_r$ denotes the $r$-th elementary non-commutative symmetric function, $\Psi_r$ denotes the $r$-th power-sum non-commutative symmetric function of the first kind, and $\Phi_r$ denotes the $r$-th power-sum non-commutative symmetric function of the second kind). For every positive integer $r$ with $n \nmid r$, it satisfies

$$V_n(S_r) = V_n(\Lambda_r) = V_n(\Psi_r) = V_n(\Phi_r) = 0.$$

The $n$-th Verschiebung operator is also called the $n$-th Verschiebung endomorphism.

It is a lift of the $n$-th Verschiebung operator on the ring of symmetric functions (verschiebung()) to the ring of noncommutative symmetric functions.

The action of the $n$-th Verschiebung operator can also be described on the ribbon Schur functions. Namely, every composition $I$ of size $n \ell$ satisfies

$$V_n(R_I) = (-1)^{\ell(I)-\ell(J)} \cdot R_{J/n},$$

where $J$ denotes the meet of the compositions $I$ and $(n, n, \ldots, n)$, where $\ell(I)$ is the length of $I$, and where $J/n$ denotes the composition obtained by dividing every entry of $J$ by $n$. For a composition $I$ of size not divisible by $n$, we have $V_n(R_I) = 0$.

See also:

verschiebung method of NCSF, frobenius method of QSym, verschiebung method of Sym

INPUT:
• $n$ – a positive integer

OUTPUT:

The result of applying the $n$-th Verschiebung operator (on the ring of noncommutative symmetric functions) to self.

EXAMPLES:

```
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: Psi = NSym.Psi()
sage: Psi([4,2]).verschiebung(2)
4*Psi[2, 1]
sage: Psi([2,4]).verschiebung(2)
4*Psi[1, 2]
sage: Psi([6]).verschiebung(2)
2*Psi[3]
sage: Psi([2,1]).verschiebung(3)
0
sage: Psi([3]).verschiebung(2)
0
sage: Psi([]).verschiebung(2)
Psi[]
sage: Psi([5, 1]).verschiebung(3)
0
sage: Psi([5, 1]).verschiebung(6)
0
sage: Psi([5, 1]).verschiebung(2)
0
sage: Psi([1, 2, 3, 1]).verschiebung(7)
(continues on next page)```
internal_product_on_basis_by_bracketing(I, J)

The internal product of two elements of the Psi basis.

See internal_product() for a thorough documentation of this operation.

This is an implementation using [NCSF2] Lemma 3.10. It is fast when the length of \( I \) is small, but can get very slow otherwise. Therefore it is not being used by default for internally multiplying Psi functions.

INPUT:
• \( I, J \) – compositions

OUTPUT:
• The internal product of the elements of the Psi basis of \( NSym \) indexed by \( I \) and \( J \), expressed in the Psi basis.

AUTHORS:
• Travis Scrimshaw, 29 Mar 2014

EXAMPLES:

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: Psi = N.Psi()
sage: Psi.internal_product_on_basis_by_bracketing([2,2],[1,2,1])
0
sage: Psi.internal_product_on_basis_by_bracketing([1,2,1],[2,1,1])
4*Psi[1, 2, 1]
sage: Psi.internal_product_on_basis_by_bracketing([2,1,1],[1,2,1])
4*Psi[2, 1, 1]
sage: Psi.internal_product_on_basis_by_bracketing([1,2,1],[1,1,1,1])
0
sage: Psi.internal_product_on_basis_by_bracketing([3,1],[1,2,1])
-Psi[1, 2, 1] + Psi[2, 1, 1]
sage: Psi.internal_product_on_basis_by_bracketing([1,2,1],[3,1])
0
sage: Psi.internal_product_on_basis_by_bracketing([2,2],[1,2])
0
sage: Psi.internal_product_on_basis_by_bracketing([4],[1,2,1])
-Psi[1, 1, 2] + 2*Psi[1, 2, 1] - Psi[2, 1, 1]
```
every composition $I$, we have
\[ R_I = \sum_J (-1)^{\ell(I) - \ell(J)} S_J, \]
where the sum is over all compositions $J$ which are coarser than $I$ and $\ell(I)$ denotes the length of $I$. (See the proof of Proposition 4.13 in [NCSF1].)

The elements of the Ribbon basis are commonly referred to as the ribbon Schur functions.

EXAMPLES:

```sage
NCSF = NonCommutativeSymmetricFunctions(QQ)
R = NCSF.Ribbon(); R
Non-Commutative Symmetric Functions over the Rational Field in the Ribbon basis
sage: R.an_element()
2*R[] + 2*R[1] + 3*R[1, 1]
```

The following are aliases for this basis:

```sage
NCSF.ribbon()
Non-Commutative Symmetric Functions over the Rational Field in the Ribbon basis
sage: NCSF.R()
Non-Commutative Symmetric Functions over the Rational Field in the Ribbon basis
```

class Element

Bases: IndexedFreeModuleElement

star_involution()

Return the image of the noncommutative symmetric function self under the star involution.

The star involution is defined as the algebra antihomomorphism $NCSF \to NCSF$ which, for every positive integer $n$, sends the $n$-th complete non-commutative symmetric function $S_n$ to $S_n$. Denoting by $f^*$ the image of an element $f \in NCSF$ under this star involution, it can be shown that every composition $I$ satisfies
\[ (S^I)^* = S^{I^r}, \quad (\Lambda^I)^* = \Lambda^{I^r}, \quad R_I^* = R_{I^r}, \quad (\Phi^I)^* = \Phi^{I^r}, \]
where $I^r$ denotes the reversed composition of $I$, and standard notations for classical bases of $NCSF$ are being used ($S$ for the complete basis, $\Lambda$ for the elementary basis, $R$ for the ribbon basis, and $\Phi$ for that of the power-sums of the second kind). The star involution is an involution and a coalgebra automorphism of $NCSF$. It is an automorphism of the graded vector space $NCSF$. Under the canonical isomorphism between the $n$-th graded component of $NCSF$ and the descent algebra of the symmetric group $S_n$ (see to_descent_algebra()), the star involution (restricted to the $n$-th graded component) corresponds to the automorphism of the descent algebra given by $x \mapsto \omega_n x \omega_n$, where $\omega_n$ is the permutation $(n, n-1, \ldots, 1) \in S_n$ (written here in one-line notation). If $\pi$ denotes the projection from $NCSF$ to the ring of symmetric functions (to_symmetric_function()), then $\pi(f^*) = \pi(f)$ for every $f \in NCSF$.

The star involution on $NCSF$ is adjoint to the star involution on $QSym$ by the standard adjunction between $NCSF$ and $QSym$.

The star involution has been denoted by $\rho$ in [LMvW13], section 3.6. See [NCSF2], section 2.3 for the properties of this map.

See also:

* star involution of NCSF, star involution of QSym, psi involution of NCSF.

EXAMPLES:
The implementation at hand is tailored to the ribbon basis. It is equivalent to the generic implementation via the complete basis:

```
sage: S = NSym.S()
sage: all( S(R[I].star_involution()) == S(R[I]).star_involution()
       ....: for I in Compositions(4) )
True
```

**verschiebung**$(n)$

Return the image of the noncommutative symmetric function self under the $n$-th Verschiebung operator.

The $n$-th Verschiebung operator $V_n$ is defined to be the map from the $k$-algebra of noncommutative symmetric functions to itself that sends the complete function $S^I$ indexed by a composition $I = (i_1, i_2, \ldots, i_k)$ to $S^{(i_1/n, i_2/n, \ldots, i_k/n)}$ if all of the numbers $i_1, i_2, \ldots, i_k$ are divisible by $n$, and to 0 otherwise. This operator $V_n$ is a Hopf algebra endomorphism. For every positive integer $r$ with $n \mid r$, it satisfies

\[
V_n(S_r) = S_{r/n}, \quad V_n(\Lambda_r) = (-1)^{r-r/n} \Lambda_{r/n}, \quad V_n(\Psi_r) = n\Psi_{r/n}, \quad V_n(\Phi_r) = n\Phi_{r/n}
\]

(where $S_r$ denotes the $r$-th complete non-commutative symmetric function, $\Lambda_r$ denotes the $r$-th elementary non-commutative symmetric function, $\Psi_r$ denotes the $r$-th power-sum non-commutative symmetric function of the first kind, and $\Phi_r$ denotes the $r$-th power-sum non-commutative symmetric function of the second kind). For every positive integer $r$ with $n \nmid r$, it satisfies

\[
V_n(S_r) = V_n(\Lambda_r) = V_n(\Psi_r) = V_n(\Phi_r) = 0.
\]

The $n$-th Verschiebung operator is also called the $n$-th Verschiebung endomorphism.

It is a lift of the $n$-th Verschiebung operator on the ring of symmetric functions (verschiebung()) to the ring of noncommutative symmetric functions.

The action of the $n$-th Verschiebung operator can also be described on the ribbon Schur functions. Namely, every composition $I$ of size $n\ell$ satisfies

\[
V_n(R_I) = (-1)^{\ell(I)-\ell(J)} \cdot R_{J/n},
\]

where $J$ denotes the meet of the compositions $I$ and $(n, n, \ldots, n)$, where $\ell(I)$ is the length of $I$, and where $J/n$ denotes the composition obtained by dividing every entry of $J$ by $n$. For a composition $I$ of size not divisible by $n$, we have $V_n(R_I) = 0$. 

See also:

verschiebung method of NCSF, frobenius method of QSym, verschiebung method of Sym

INPUT:
• n – a positive integer

OUTPUT:

The result of applying the \( n \)-th Verschiebung operator (on the ring of noncommutative symmetric functions) to \( \text{self} \).

EXAMPLES:

```python
sage: NSym = NonCommutativeSymmetricFunctions(ZZ)
sage: R = NSym.R()
sage: R([4,2]).verschiebung(2)
R[2, 1]
sage: R([2,1]).verschiebung(3)
-R[1]
sage: R([3]).verschiebung(2)
0
sage: R([]).verschiebung(2)
R[]
sage: R([5, 1]).verschiebung(3)
-R[2]
sage: R([5, 1]).verschiebung(6)
-R[1]
sage: R([5, 1]).verschiebung(2)
-R[3]
sage: R([1, 2, 3, 1]).verschiebung(7)
-R[1]
sage: R([1, 2, 3, 1]).verschiebung(5)
0
```

\texttt{antipode\_on\_basis}(\texttt{composition})

Return the application of the antipode to a basis element of the ribbon basis \texttt{self}.

INPUT:
• \texttt{composition} – a composition

OUTPUT:
• The image of the basis element indexed by \texttt{composition} under the antipode map.

EXAMPLES:

```python
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: R.antipode_on_basis(Composition([2,1]))
-R[2, 1]
sage: R[3,1].antipode() # indirect doctest
R[2, 1, 1]
sage: R([]).antipode() # indirect doctest
R[]
```

We check that the implementation of the antipode at hand does not contradict the generic one:
Combinatorics, Release 10.0

```python
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: all( S(R[I].antipode()) == S(R[I]).antipode() 
....:    for I in Compositions(4) )
True
```

dual()

Return the dual basis to the ribbon basis of the non-commutative symmetric functions. This is the Fundamental basis of the quasi-symmetric functions.

**OUTPUT:**
- The fundamental basis of the quasi-symmetric functions.

**EXAMPLES:**

```python
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: R.dual()
Quasisymmetric functions over the Rational Field in the Fundamental basis
```

product_on_basis(I, J)

Return the product of two ribbon basis elements of the non-commutative symmetric functions.

**INPUT:**
- I, J - compositions

**OUTPUT:**
- The product of the ribbon functions indexed by I and J.

**EXAMPLES:**

```python
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: R[R[1,2,1] * R[3,1]]
R[1, 2, 1, 3, 1] + R[1, 2, 4, 1]
sage: ( R[R[1,2]] + R[R[3,1]] ) * ( R[R[3,1]] + R[R[1,2,1]] )
```

to_symmetric_function_on_basis(I)

Return the commutative image of a ribbon basis element of the non-commutative symmetric functions.

**INPUT:**
- I - a composition

**OUTPUT:**
- The commutative image of the ribbon basis element indexed by I. This will be expressed as a symmetric function in the Schur basis.

**EXAMPLES:**

```python
sage: R = NonCommutativeSymmetricFunctions(QQ).R()
sage: R.to_symmetric_function_on_basis(Composition([3,1,1]))
s[3, 1, 1]
sage: R.to_symmetric_function_on_basis(Composition([4,2,1]))
s[4, 2, 1] + s[5, 1, 1] + s[5, 2]
sage: R.to_symmetric_function_on_basis(Composition([]))
s[]
```

S

alias of Complete
Combinatorics, Release 10.0

**ZL**
alias of `Zassenhaus_left`

**ZR**
alias of `Zassenhaus_right`

**class Zassenhaus_left(NCSF)**

Bases: `CombinatorialFreeModule`, `BindableClass`

The Hopf algebra of non-commutative symmetric functions in the left Zassenhaus basis.

This basis is the left-version of the basis defined in Section 2.5.1 of [HLNT09]. It is multiplicative, with $Z_n$ defined as the element of $NCSF_n$ satisfying the equation

$$\sigma_1 = \cdots \exp(Z_n) \cdots \exp(Z_2) \exp(Z_1),$$

where

$$\sigma_1 = \sum_{n \geq 0} S_n.$$  

It can be recursively computed by the formula

$$S_n = \sum_{\lambda \vdash n} \frac{1}{m_1(\lambda)!m_2(\lambda)!m_3(\lambda)! \cdots Z_{\lambda_1} Z_{\lambda_2} Z_{\lambda_3} \cdots}$$

for all $n \geq 0$.

**class Zassenhaus_right(NCSF)**

Bases: `CombinatorialFreeModule`, `BindableClass`

The Hopf algebra of non-commutative symmetric functions in the right Zassenhaus basis.

This basis is defined in Section 2.5.1 of [HLNT09]. It is multiplicative, with $Z_n$ defined as the element of $NCSF_n$ satisfying the equation

$$\sigma_1 = \exp(Z_1) \exp(Z_2) \exp(Z_3) \cdots \exp(Z_n) \cdots$$

where

$$\sigma_1 = \sum_{n \geq 0} S_n.$$  

It can be recursively computed by the formula

$$S_n = \sum_{\lambda \vdash n} \frac{1}{m_1(\lambda)!m_2(\lambda)!m_3(\lambda)! \cdots Z_{\lambda_1} Z_{\lambda_2} Z_{\lambda_3} \cdots}$$

for all $n \geq 0$.

Note that there is a variant (called the “noncommutative power sum symmetric functions of the third kind”) in Definition 5.26 of [NCSF2] that satisfies:

$$\sigma_1 = \exp(Z_1) \exp(Z_2/2) \exp(Z_3/3) \cdots \exp(Z_n/n) \cdots.$$  

**a_realization()**

Gives a realization of the algebra of non-commutative symmetric functions. This particular realization is the complete basis of non-commutative symmetric functions.

OUTPUT:
• The realization of the non-commutative symmetric functions in the complete basis.

EXAMPLES:

```python
sage: NonCommutativeSymmetricFunctions(ZZ).a_realization()
Non-Commutative Symmetric Functions over the Integer Ring in the Complete basis
```

**complete**

alias of **Complete**

**dQS**

alias of **dualQuasisymmetric_Schur**

**dYQS**

alias of **dualYoungQuasisymmetric_Schur**

**dual()**

Return the dual to the non-commutative symmetric functions.

**OUTPUT:**

• The dual of the non-commutative symmetric functions over a ring. This is the algebra of quasi-

symmetric functions over the ring.

**EXAMPLES:**

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: NCSF.dual()
Quasisymmetric functions over the Rational Field
```

class **dualQuasisymmetric_Schur**(NCSF)

**Bases:** **CombinatorialFreeModule**, **BindableClass**

The basis of NCSF dual to the Quasisymmetric-Schur basis of QSym.

The **Quasisymmetric_Schur** functions are defined in [QSCHUR] (see also Definition 5.1.1 of

[LMvW13]). The dual basis in the algebra of non-commutative symmetric functions is defined by the

following formula:

\[ R_\alpha = \sum_T dQ_{\text{shape}(T)}, \]

where the sum is over all standard composition tableaux with descent composition equal to \( \alpha \). The **Quasisymmetric_Schur** basis \( QS_\alpha \) has the property that

\[ s_\lambda = \sum_{\text{sort}(\alpha) = \lambda} QS_\alpha. \]

As a consequence the commutative image of a dual Quasisymmetric-Schur element in the algebra of symmetric functions (the map defined in the method to_symmetric_function()) is equal to the Schur function indexed by the decreasing sort of the indexing composition.

**See also:**

**CompositionTableaux**, **CompositionTableau**.

**EXAMPLES:**
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: dQS = NCSF.dQS()
sage: dQS([1,3,2])*dQS([1])
dQS[1, 2, 4] + dQS[1, 3, 2, 1] + dQS[1, 3, 3] + dQS[3, 2, 2]
sage: dQS([1])^2
2 * dQS[1, 3, 2]
sage: dQS([1, 3, 2])*dQS([1, 1])
sage: dQS([3,1])*dQS([2,1])
sage: dQS([1,1]).coproduct()
dQS[] # dQS[1, 1] + dQS[1] # dQS[1, 1] # dQS[]
sage: dQS([3,3]).coproduct().monomial_coefficients()[(Composition([1,2]),
Composition([1,2]))]
-1
sage: S = NCSF.complete()
sage: dQS(S[1,3,1])
sage: S(dQS([1,3,1]))
sage: s = SymmetricFunctions(QQ).s()
sage: s(S(dQS([2,1,3])).to_symmetric_function())
s[3, 2, 1]

dual()
The dual basis to the dual Quasisymmetric-Schur basis of NCSF.

The basis returned is the Quasisymmetric_Schur basis of QSym.

OUTPUT:

- the Quasisymmetric-Schur basis of the quasi-symmetric functions

EXAMPLES:

sage: dQS=NonCommutativeSymmetricFunctions(QQ).dualQuasisymmetric_Schur()
sage: dQS.dual()
Quasisymmetric functions over the Rational Field in the Quasisymmetric
Schur basis
sage: dQS.duality_pairing_matrix(dQS.dual(),3)
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]

to_symmetric_function_on_basis(I)
The commutative image of a dual quasi-symmetric Schur element

The commutative image of a basis element is obtained by sorting the indexing composition of the basis element.

INPUT:

- I -- a composition

OUTPUT:

- The commutative image of the dual quasi-Schur basis element indexed by I. The result is the Schur symmetric function indexed by the partition obtained by sorting I.
EXAMPLES:

```python
sage: dQS = NonCommutativeSymmetricFunctions(QQ).dQS()
sage: dQS.to_symmetric_function_on_basis([2,1,3])
s[3, 2, 1]
sage: dQS.to_symmetric_function_on_basis([])
s[]
```

class `dualYoungQuasisymmetric_Schur`(`NCSF`)

Bases: `CombinatorialFreeModule`, `BindableClass`

The basis of NCSF dual to the Young Quasisymmetric-Schur basis of QSym.

The `YoungQuasisymmetric_Schur` functions are given in Definition 5.2.1 of [LMvW13]. The dual basis in the algebra of non-commutative symmetric functions are related by an involution reversing the indexing composition of the complete expansion of a quasi-Schur basis element. This basis has many of the same properties as the Quasisymmetric Schur basis and is related to that basis by an algebraic transformation.

EXAMPLES:

```python
sage: NCSF = NonCommutativeSymmetricFunctions(QQ)
sage: dYQS = NCSF.dYQS()
sage: dYQS([1,3,2])*dYQS([1])
sage: dYQS([1,3])*dYQS([1,3,2])
sage: dYQS([1,3])*dYQS([1,1])
sage: dYQS([3,1])*dYQS([2,1])
sage: dYQS([1,1]).coproduct()
sage: dYQS([3,3]).coproduct().monomial_coefficients()[(Composition([1,2]), Composition([2,1]))]
1
sage: S = NCSF.complete()
sage: dYQS(S[1,3,1])
sage: S(dYQS([1,3,1]))
sage: s = SymmetricFunctions(QQ).s()
sage: s(S(dYQS([2,1,3])).to_symmetric_function())
s[3, 2, 1]
dual()

The dual basis to the dual Quasisymmetric-Schur basis of NCSF.

The basis returned is the `Quasisymmetric_Schur` basis of QSym.

OUTPUT:

- the Young Quasisymmetric-Schur basis of quasi-symmetric functions

EXAMPLES:
\begin{verbatim}
sage: dYQS=NonCommutativeSymmetricFunctions(QQ).dualYoungQuasisymmetric_.→Schur()
sage: dYQS.dual()
Quasisymmetric functions over the Rational Field in the Young Quasisymmetric Schur basis
sage: dYQS.duality_pairing_matrix(dYQS.dual(),3)
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix}
\end{verbatim}

to\_symmetric\_function\_on\_basis(I)
The commutative image of a dual Young quasi-symmetric Schur element.
The commutative image of a basis element is obtained by sorting the indexing composition of the basis element.

INPUT:
• I – a composition

OUTPUT:
• The commutative image of the dual Young quasi-Schur basis element indexed by I. The result is the Schur symmetric function indexed by the partition obtained by sorting I.

EXAMPLES:
\begin{verbatim}
sage: dYQS=NonCommutativeSymmetricFunctions(QQ).dYQS() sage: dYQS.to_symmetric_function_on_basis([2,1,3]) s[3, 2, 1]
sage: dYQS.to_symmetric_function_on_basis([]) s[]
\end{verbatim}

elementary
 alias of Elementary

monomial
 alias of Monomial

nM
 alias of Monomial

ribbon
 alias of Ribbon

5.1.145 Quasisymmetric functions

REFERENCES:

AUTHOR:
• Jason Bandlow
• Franco Saliola
• Chris Berg
• Darij Grinberg
class sage.combinat.ncsf_qsym.qsym.QuasiSymmetricFunctions(R)
Bases: UniqueRepresentation, Parent

The Hopf algebra of quasisymmetric functions.

Let \( R \) be a commutative ring with unity. The \( R \)-algebra of quasi-symmetric functions may be realized as an \( R \)-subalgebra of the ring of power series in countably many variables \( R[[x_1, x_2, x_3, \ldots]] \). It consists of those formal power series \( p \) which are degree-bounded (i.e., the degrees of all monomials occurring with nonzero coefficient in \( p \) are bounded from above, although the bound can depend on \( p \)) and satisfy the following condition: For every tuple \((a_1, a_2, \ldots, a_m)\) of positive integers, the coefficient of the monomial \( x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \) in \( p \) is the same for all strictly increasing sequences \((i_1 < i_2 < \cdots < i_m)\) of positive integers. (In other words, the coefficient of a monomial in \( p \) depends only on the sequence of nonzero exponents in the monomial. If “sequence” were to be replaced by “multiset” here, we would obtain the definition of a symmetric function.)

The \( R \)-algebra of quasi-symmetric functions is commonly called \( \text{QSym}_R \) or occasionally just \( \text{QSym} \) (when \( R \) is clear from the context or \( \mathbb{Z} \) or \( \mathbb{Q} \)). It is graded by the total degree of the power series. Its homogeneous elements of degree \( k \) form a free \( R \)-submodule of rank equal to the number of compositions of \( k \) (that is, \( 2^k - 1 \) if \( k \geq 1 \), else 1).

The two classical bases of \( \text{QSym} \), the monomial basis \((M_I)_I\) and the fundamental basis \((F_I)_I\), are indexed by compositions \( I = (I_1, I_2, \cdots, I_\ell) \) and defined by the formulas:

\[
M_I = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_1^{I_1} x_2^{I_2} \cdots x_\ell^{I_\ell}
\]

and

\[
F_I = \sum_{(j_1, j_2, \ldots, j_n)} x_{j_1} x_{j_2} \cdots x_{j_n}
\]

where in the second equation the sum runs over all weakly increasing \( n \)-tuples \((j_1, j_2, \ldots, j_n)\) of positive integers (where \( n \) is the size of \( I \)) which increase strictly from \( j_r \) to \( j_{r+1} \) if \( r \) is a descent of the composition \( I \).

These bases are related by the formula

\[
F_I = \sum_{J \leq I} M_J
\]

where the inequality \( J \leq I \) indicates that \( J \) is finer than \( I \).

The \( R \)-algebra of quasi-symmetric functions is a Hopf algebra, with the coproduct satisfying

\[
\Delta M_I = \sum_{k=0}^{\ell} M_{(I_1, I_2, \cdots, I_k)} \otimes M_{(I_{k+1}, I_{k+2}, \cdots, I_\ell)}
\]

for every composition \( I = (I_1, I_2, \cdots, I_\ell) \).

It is possible to define an \( R \)-algebra of quasi-symmetric functions in a finite number of variables as well (but it is not a bialgebra). These quasi-symmetric functions are actual polynomials then, not just power series.

Chapter 5 of [GriRei18] and Section 11 of [HazWitt1] are devoted to quasi-symmetric functions, as are Malvenuto’s thesis [Mal1993] and part of Chapter 7 of [Sta-EC2].
The implementation of the quasi-symmetric function Hopf algebra

We realize the \( R \)-algebra of quasi-symmetric functions in Sage as a graded Hopf algebra with basis elements indexed by compositions:

\[
sage: \text{QSym} = \text{QuasiSymmetricFunctions}(\mathbb{Q})
\]
\[
sage: \text{QSym}.\text{category()}
\]
Join of Category of hopf algebras over Rational Field
and Category of graded algebras over Rational Field
and Category of commutative algebras over Rational Field
and Category of monoids with realizations
and Category of graded coalgebras over Rational Field
and Category of coalgebras over Rational Field with realizations

The most standard two bases for this \( R \)-algebra are the monomial and fundamental bases, and are accessible by the \( \text{M() and F()} \) methods:

\[
sage: \text{M} = \text{QSym.M()}
\]
\[
sage: \text{F} = \text{QSym.F()}
\]
\[
sage: \text{M}(\text{F}[2,1,2])
\]
\[\text{M}[1, 1, 1, 1, 1] + \text{M}[1, 1, 1, 2] + \text{M}[2, 1, 1, 1] + \text{M}[2, 1, 2]\]
\[
sage: \text{F}(\text{M}[2,1,2])
\]
\[\text{F}[1, 1, 1, 1, 1] - \text{F}[1, 1, 1, 2] - \text{F}[2, 1, 1, 1] + \text{F}[2, 1, 2]\]

The product on this space is commutative and is inherited from the product on the realization within the ring of power series:

\[
sage: \text{M}[3]*\text{M}[1,1] == \text{M}[1,1]*\text{M}[3]
\]
True
\[
sage: \text{M}[3]*\text{M}[1,1]
\]
\[\text{M}[1, 1, 3] + \text{M}[1, 3, 1] + \text{M}[1, 4] + \text{M}[3, 1, 1] + \text{M}[4, 1]\]
\[
sage: \text{F}[3]*\text{F}[1,1]
\]
\[\text{F}[1, 1, 3] + \text{F}[1, 2, 2] + \text{F}[1, 3, 1] + \text{F}[1, 4] + \text{F}[2, 1, 2] + \text{F}[2, 2, 1] + \text{F}[3, 1, 1] + \text{F}[3, 2] + \text{F}[4, 1]\]
\[
sage: \text{M}[3]*\text{F}[2]
\]
\[\text{M}[1, 1, 3] + \text{M}[1, 3, 1] + \text{M}[1, 4] + \text{M}[2, 3] + \text{M}[3, 1, 1] + \text{M}[3, 2] + \text{M}[4, 1] + \text{M}[5]\]
\[
sage: \text{F}[2]*\text{M}[3]
\]
\[\text{F}[1, 1, 1, 2] - \text{F}[1, 2, 2] + \text{F}[2, 1, 1, 1] - \text{F}[2, 1, 2] - \text{F}[2, 2, 1] + \text{F}[5]\]

There is a coproduct on \( \text{QSym} \) as well, which in the Monomial basis acts by cutting the composition into a left half and a right half. The coproduct is not co-commutative:

\[
sage: \text{M}[1,3,1].\text{coproduct()}
\]
\[\text{M}[1, 3, 1] \# \text{M}[1, 3, 1] + \text{M}[1, 3] \# \text{M}[1] + \text{M}[1, 3, 1] \# \text{M}[1] \]
\[
sage: \text{F}[1,3,1].\text{coproduct()}
\]
\[\text{F}[1, 3, 1] \# \text{F}[1, 3, 1] + \text{F}[1, 3] \# \text{F}[1, 1] + \text{F}[1, 3, 1] \# \text{F}[2, 1] + \text{F}[1, 2] \# \text{F}[1, 1] + \text{F}[1, 3] \# \text{F}[1] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[1] + \text{F}[1, 3] \# \text{F}[1] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_] + \text{F}[1, 2] \# \text{F}[\_] + \text{F}[1, 3] \# \text{F}[\_] + \text{F}[1, 3, 1] \# \text{F}[\_] + \text{F}[\_] \# \text{F}[\_]
The duality pairing with non-commutative symmetric functions

These two operations endow the quasi-symmetric functions \( \text{QSym} \) with the structure of a Hopf algebra. It is the graded dual Hopf algebra of the non-commutative symmetric functions \( \text{NCSF} \). Under this duality, the Monomial basis of \( \text{QSym} \) is dual to the Complete basis of \( \text{NCSF} \), and the Fundamental basis of \( \text{QSym} \) is dual to the Ribbon basis of \( \text{NCSF} \) (see [MR]).

\[
\text{sage: } S = M.dual(); S
\]
Non-Commutative Symmetric Functions over the Rational Field in the Complete basis
\[
\text{sage: } M[1,3,1].duality_pairing( S[1,3,1] )
1
\]
\[
\text{sage: } M.duality_pairing_matrix( S, degree=4 )
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\text{sage: } F.duality_pairing_matrix( S, degree=4 )
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
\[
\text{sage: } NCSF = M.realization_of().dual()
\]
\[
\text{sage: } R = NCSF.Ribbon()
\]
\[
\text{sage: } F.duality_pairing_matrix( R, degree=4 )
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\text{sage: } M.duality_pairing_matrix( R, degree=4 )
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}
\]

Let \( H \) and \( G \) be elements of \( \text{QSym} \), and \( h \) an element of \( \text{NCSF} \). Then, if we represent the duality pairing with the mathematical notation \( [\cdot, \cdot] \),

\[
[HG, h] = [H \otimes G, \Delta(h)].
\]
For example, the coefficient of $M[2,1,4,1]$ in $M[1,3]*M[2,1,1]$ may be computed with the duality pairing:

```
sage: I, J = Composition([1,3]), Composition([2,1,1])
sage: (M[I]*M[J]).duality_pairing(S[2,1,4,1])
1
```

And the coefficient of $S[1,3] \# S[2,1,1]$ in $S[2,1,4,1].coproduct()$ is equal to this result:

```
sage: S[2,1,4,1].coproduct()
```

The duality pairing on the tensor space is another way of getting this coefficient, but currently the method `duality_pairing` is not defined on the tensor squared space. However, we can extend this functionality by applying a linear morphism to the terms in the coproduct, as follows:

```
sage: X = S[2,1,4,1].coproduct()
sage: def linear_morphism(x, y):
...:     return x.duality_pairing(M[1,3]) * y.duality_pairing(M[2,1,1])
sage: X.apply_multilinear_morphism(linear_morphism, codomain=ZZ)
1
```

Similarly, if $H$ is an element of $QSym$ and $g$ and $h$ are elements of $NCSF$, then

$$[H, gh] = [\Delta(H), g \otimes h].$$

For example, the coefficient of $R[2,3,1]$ in $R[2,1]*R[2,1]$ is computed with the duality pairing by the following command:

```
sage: (R[2,1]*R[2,1]).duality_pairing(F[2,3,1])
1
```

This coefficient should then be equal to the coefficient of $F[2,1] \# F[2,1]$ in $F[2,3,1].coproduct()$:

```
sage: F[2,3,1].coproduct()
```

This can also be computed by the duality pairing on the tensor space, as above:

```
sage: X = F[2,3,1].coproduct()
sage: def linear_morphism(x, y):
...:     return x.duality_pairing(R[2,1]) * y.duality_pairing(R[2,1])
sage: X.apply_multilinear_morphism(linear_morphism, codomain=ZZ)
1
```
The operation dual to multiplication by a non-commutative symmetric function

Let \( g \in NCSF \) and consider the linear endomorphism of \( NCSF \) defined by left (respectively, right) multiplication by \( g \). Since there is a duality between \( QSym \) and \( NCSF \), this linear transformation induces an operator \( g^\perp \) on \( QSym \) satisfying

\[
[g^\perp(H), h] = [H, gh].
\]

for any non-commutative symmetric function \( h \).

This is implemented by the method \( \text{skew\_by()} \). Explicitly, if \( H \) is a quasi-symmetric function and \( g \) a non-commutative symmetric function, then \( H.\text{skew\_by}(g) \) and \( H.\text{skew\_by}(g, \text{side='right'}) \) are expressions that satisfy, for any non-commutative symmetric function \( h \), the following equalities:

\[
H.\text{skew\_by}(g).\text{duality\_pairing}(h) \equiv H.\text{duality\_pairing}(g*h)
\]
\[
H.\text{skew\_by}(g, \text{side='right'}).\text{duality\_pairing}(h) \equiv H.\text{duality\_pairing}(h*g)
\]

For example, \( M[J].\text{skew\_by}(S[I]) \) is 0 unless the composition \( J \) begins with \( I \) and \( M[J].\text{skew\_by}(S[I], \text{side='right'}) \) is 0 unless the composition \( J \) ends with \( I \). For example:

```sage
sage: M[3,2,2].skew_by(S[3])
M[2, 2]
sage: M[3,2,2].skew_by(S[2])
0
tsage: M[3,2,2].coproduct().apply_multilinear_morphism( lambda x,y: x.duality_˓→pairing(S[3])*y )
M[2, 2]
tsage: M[3,2,2].skew_by(S[3], side='right')
0
tsage: M[3,2,2].skew_by(S[2], side='right')
M[3, 2]
```

The counit

The counit is defined by sending all elements of positive degree to zero:

```sage
sage: M[3].degree(), M[3,1,2].degree(), M.one().degree()
(3, 6, 0)
sage: M[3].counit()
0
tsage: M[3,1,2].counit()
0
tsage: M.one().counit()
1
7
7
```
### The antipode

The antipode sends the Fundamental basis element indexed by the composition $I$ to $(-1)^{|I|}$ times the Fundamental basis element indexed by the conjugate composition to $I$ (where $|I|$ stands for the size of $I$, that is, the sum of all entries of $I$).

```
sage: F[3,2,2].antipode()
-F[1, 2, 2, 1, 1]
sage: Composition([3,2,2]).conjugate()
[1, 2, 2, 1, 1]
```

The antipodes of the Monomial quasisymmetric functions can also be computed easily: Every composition $I$ satisfies

$$\omega(M_I) = (-1)^{\ell(I)} \sum M_J,$$

where the sum ranges over all compositions $J$ of $|I|$ which are coarser than the reversed composition $I^r$ of $I$. Here, $\ell(I)$ denotes the length of the composition $I$ (that is, the number of its parts).

```
sage: M[3,2,1].antipode()
sage: M[3,2,2].antipode()
```

We demonstrate here the defining relation of the antipode:

```
sage: X = F[3,2,2].coproduct()
sage: X.apply_multilinear_morphism(lambda x,y: x*y.antipode())
0
sage: X.apply_multilinear_morphism(lambda x,y: x.antipode()*y)
0
```

### The relation with symmetric functions

The quasi-symmetric functions are a ring which contain the symmetric functions as a subring. The Monomial quasi-symmetric functions are related to the monomial symmetric functions by

$$m_{\lambda} = \sum_{\text{sort}(I) = \lambda} M_I$$

(where sort($I$) denotes the result of sorting the entries of $I$ in decreasing order).

There are methods to test if an expression in the quasi-symmetric functions is a symmetric function and, if it is, send it to an expression in the symmetric functions:

```
sage: f = M[1,1,2] + M[1,2,1]
sage: f.is_symmetric()
False
sage: g = M[3,1] + M[1,3]
sage: g.is_symmetric()
True
sage: g.to_symmetric_function()
m[3, 1]
```
The expansion of the Schur function in terms of the Fundamental quasi-symmetric functions is due to [Ges]. There is one term in the expansion for each standard tableau of shape equal to the partition indexing the Schur function.

```sage
f.is_symmetric()
True
f.to_symmetric_function()
5*m[1, 1, 1, 1, 1] + 3*m[2, 1, 1, 1] + 2*m[2, 2, 1] + m[3, 1, 1] + m[3, 2]
s = SymmetricFunctions(QQ).s()
s(f.to_symmetric_function())
s[3, 2]
```

It is also possible to convert a symmetric function to a quasi-symmetric function:

```sage
m = SymmetricFunctions(QQ).m()
M[1, 1, 3] + M[1, 3, 1] + M[3, 1, 1]
F[1, 1, 2, 1] + F[1, 2, 1, 1] + F[1, 2, 2] + F[2, 1, 2] + F[2, 1, 2] + F[2, 2, 1]
```

It is possible to experiment with the quasi-symmetric function expansion of other bases, but it is important that the base ring be the same for both algebras.

```sage
R = QQ['t']
Qp = SymmetricFunctions(R).hall_littlewood().Qp()
QSymt = QuasiSymmetricFunctions(R)
Ft = QSymt.F()
Ft( Qp[2,2] )
```

```sage
K = QQ['q','t'].fraction_field()
Ht = SymmetricFunctions(K).macdonald().Ht()
Fqt = QuasiSymmetricFunctions(Ht.base_ring()).F()
Fqt(Ht[2,1])
q^t*F[1, 1, 1] + (q+t)*F[1, 2] + (q+t)*F[2, 1] + F[3]
```

The following will raise an error because the base ring of F is not equal to the base ring of Ht:

```sage
F(Ht[2,1])
Traceback (most recent call last):
...
TypeError: do not know how to make x (= McdHt[2, 1]) an element of self
```

(Quasisymmetric functions over the Rational Field in the Fundamental basis)
The map to the ring of polynomials

The quasi-symmetric functions can be seen as an inverse limit of a subring of a polynomial ring as the number of variables increases. Indeed, there exists a projection from the quasi-symmetric functions onto the polynomial ring \( R[x_1, x_2, \ldots, x_n] \). This projection is defined by sending the variables \( x_{n+1}, x_{n+2}, \ldots \) to 0, while the remaining \( n \) variables remain fixed. Note that this projection sends \( M_I \) to 0 if the length of the composition \( I \) is higher than \( n \).

```
sage: M[1,3,1].expand(4)
x0*x1^3*x2 + x0*x1^3*x3 + x0*x2^3*x3 + x1*x2^3*x3
sage: F[1,3,1].expand(4)
x0*x1^3*x2 + x0*x1^3*x3 + x0*x1^2*x2*x3 + x0*x1*x2^2*x3 + x0*x2^3*x3 + x1*x2^3*x3
sage: M[1,3,1].expand(2)
0
```

```
class Bases(parent_with_realization):
    Bases: Category_realization_of_parent
    Category of bases of quasi-symmetric functions.
    EXAMPLES:
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: QSym.Bases()
Category of bases of Quasisymmetric functions over the Rational Field

class ElementMethods
    Bases: object
    Methods common to all elements of QuasiSymmetricFunctions.

dendriform_leq(other)
    Return the result of applying the dendriform smaller-or-equal operation to the two quasi-symmetric functions self and other.

    The dendriform smaller-or-equal operation is a binary operation, denoted by \( \preceq \) and written infix, on the ring of quasi-symmetric functions. It can be defined as a restriction of a binary operation (denoted by \( \preceq \) and written infix as well) on the ring of formal power series \( R[[x_1, x_2, x_3, \ldots]] \), which is defined as follows: If \( m \) and \( n \) are two monomials in \( x_1, x_2, x_3, \ldots \), then we let \( m \preceq n \) be the product \( mn \) if the smallest positive integer \( i \) for which \( x_i \) occurs in \( m \) is smaller or equal to the smallest positive integer \( j \) for which \( x_j \) occurs in \( n \) (this is understood to be false when \( m = 1 \) and \( n \neq 1 \), and true when \( n = 1 \)), and we let \( m \preceq n \) be 0 otherwise. Having thus defined \( \preceq \) on monomials, we extend \( \preceq \) to a binary operation on \( R[[x_1, x_2, x_3, \ldots]] \) by requiring it to be continuous (in both inputs) and \( R \)-bilinear. It is easily seen that \( QSym \preceq QSym \subseteq QSym \), so that \( \preceq \) restricts to a binary operation on \( QSym \).

    This operation \( \preceq \) is related to the dendriform smaller relation \( \prec \) (\textit{dendriform_less()}). Namely, if we define a binary operation \( \succ \) on \( QSym \) by \( a \succ b = b \prec a \), then \( QSym, \preceq, \succ \) is a dendriform \( R \)-algebra. Thus, any \( a, b \in QSym \) satisfy \( a \preceq b = ab - b \prec a \).

    See also:

dendriform_less()
    INPUT:
    • other – a quasi-symmetric function over the same base ring as self
    OUTPUT:
    The quasi-symmetric function self \( \preceq \) other, written in the basis of self.
```
EXAMPLES:

```
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: M[2, 1].dendriform_leq(M[1, 2])
sage: F = QSym.F()
sage: F[2, 1].dendriform_leq(F[1, 2])
```

dendriform_less(other)

Return the result of applying the dendriform smaller operation to the two quasi-symmetric functions self and other.

The dendriform smaller operation is a binary operation, denoted by \( \prec \) and written infix, on the ring of quasi-symmetric functions. It can be defined as a restriction of a binary operation (denoted by \( \prec \) and written infix as well) on the ring of formal power series \( R[[x_1, x_2, x_3, \ldots]] \), which is defined as follows: If \( m \) and \( n \) are two monomials in \( x_1, x_2, x_3, \ldots \), then we let \( m \prec n \) be the product \( mn \) if the smallest positive integer \( i \) for which \( x_i \) occurs in \( m \) is smaller than the smallest positive integer \( j \) for which \( x_j \) occurs in \( n \) (this is understood to be false when \( m = 1 \), and true when \( m \neq 1 \) and \( n = 1 \)), and we let \( m \prec n = 0 \) otherwise. Having thus defined \( \prec \) on monomials, we extend \( \prec \) to a binary operation on \( R[[x_1, x_2, x_3, \ldots]] \) by requiring it to be continuous (in both inputs) and \( R \)-bilinear. It is easily seen that \( QSym \prec QSym \subseteq QSym \), so that \( \prec \) restricts to a binary operation on \( QSym \).

See also:

dendriform_leq()

INPUT:

- other – a quasi-symmetric function over the same base ring as self

OUTPUT:

The quasi-symmetric function \( self \prec other \), written in the basis of self.

EXAMPLES:

```
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: M[2, 1].dendriform_less(M[1, 2])
sage: F = QSym.F()
sage: F[2, 1].dendriform_less(F[1, 2])
```

The operation \( \prec \) can be used to recursively construct the dual immaculate basis: For every positive integer \( m \) and every composition \( I \), the dual immaculate function \( dI_{[m, I]} \) of the composition \( [m, I] \) (this composition is \( I \) with \( m \) prepended to it) is \( F_{[m]} \prec dI_{I} \).

```
sage: dI = QSym.dI()
sage: dI(F[2]).dendriform_less(dI[1, 2])
dI[2, 1, 2]
```
We check with the identity element:

\begin{verbatim}
sage: M.one().dendriform_less(M[1,2])
0
sage: M[1,2].dendriform_less(M.one())
M[1, 2]
\end{verbatim}

The operation $\prec$ is not symmetric, nor if $a \prec b \neq 0$, then $b \prec a = 0$ (as it would be for a single monomial):

\begin{verbatim}
sage: M[1,2,1].dendriform_less(M[1,2])
M[1, 1, 2, 1, 2] + 2*M[1, 1, 2, 2, 1] + M[1, 1, 2, 3] + M[1, 1, 4, 1] + 2*M[1, 2, 1, 1, 2] + M[1, 2, 1, 2, 1] + M[1, 2, 1, 3] + M[1, 2, 2, 2] + M[1, 3, 1, 2] + M[1, 3, 2, 1] + M[1, 3, 3]
sage: M[1,2].dendriform_less(M[1,2,1])
M[1, 1, 2, 1, 2] + 2*M[1, 1, 2, 2, 1] + M[1, 1, 2, 3] + M[1, 1, 4, 1] + M[1, 2, 1, 2, 1] + M[1, 3, 2, 1]
\end{verbatim}

\textbf{expand}(n, alphabet='x')

Expand the quasi-symmetric function into n variables in an alphabet, which by default is 'x'.

\textbf{INPUT:}
- n – A nonnegative integer; the number of variables in the expansion
- alphabet – (default: 'x'); the alphabet in which \textbf{self} is to be expanded

\textbf{OUTPUT:}
- An expansion of \textbf{self} into the n variables specified by alphabet.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: F[3].expand(3)
x0^3 + x0^2*x1 + x0*x1^2 + x1^3 + x0^2*x2 + x0*x1*x2 + x1^2*x2 + x0*x2^2 + x1*x2^2 + x2^3
sage: F[2,1].expand(3)
x0^2*x1 + x0^2*x2 + x0*x1*x2 + x1^2*x2 + x1^2*x2 + x0^2*x2 + x1^2*x2
\end{verbatim}

One can use a different set of variable by adding an optional argument alphabet=...

\begin{verbatim}
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: F[3].expand(2,alphabet='y')
y0^3 + y0^2*y1 + y0*y1^2 + y1^3
\end{verbatim}

\textbf{frobenius}(n)

Return the image of the quasi-symmetric function \textbf{self} under the n-th Frobenius operator.

The n-th Frobenius operator $f_n$ is defined to be the map from the $R$-algebra of quasi-symmetric functions to itself that sends every symmetric function $P(x_1, x_2, x_3, \ldots)$ to $P(x_1^n, x_2^n, x_3^n, \ldots)$. This operator $f_n$ is a Hopf algebra endomorphism, and satisfies

$$f_n M_{(i_1, i_2, i_3, \ldots)} = M_{(ni_1, ni_2, ni_3, \ldots)}$$

for every composition $(i_1, i_2, i_3, \ldots)$ (where $M$ means the monomial basis).

The n-th Frobenius operator is also called the n-th Frobenius endomorphism. It is not related to the Frobenius map which connects the ring of symmetric functions with the representation theory of the symmetric group.
The \( n \)-th Frobenius operator is also the \( n \)-th Adams operator of the \( \Lambda \)-ring of quasi-symmetric functions over the integers.

The restriction of the \( n \)-th Frobenius operator to the subring formed by all symmetric functions is, not unexpectedly, the \( n \)-th Frobenius operator of the ring of symmetric functions.

See also:

Symmetric functions plethysm

INPUT:
- \( n \) – a positive integer

OUTPUT:
The result of applying the \( n \)-th Frobenius operator (on the ring of quasi-symmetric functions) to self.

EXAMPLES:

```sage
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: M = QSym.M()
sage: F = QSym.F()
sage: M[3,2].frobenius(2)
M[6, 4]
sage: (M[2,1] - 2*M[3]).frobenius(4)
sage: M([]).frobenius(3)
M[]
sage: F[1,1].frobenius(2)
F[1, 1, 1, 1] - F[1, 1, 2] - F[2, 1, 1] + F[2, 2]
```

The Frobenius endomorphisms are multiplicative:

```sage
sage: all( all( M(I).frobenius(3) * M(J).frobenius(3) == (M(I) * M(J)).frobenius(3) for I in Compositions(3) ) for J in Compositions(2) )
True
```

Being Hopf algebra endomorphisms, the Frobenius operators commute with the antipode:

```sage
sage: all( M(I).frobenius(4).antipode() == M(I).antipode().frobenius(4) for I in Compositions(3) )
True
```

The restriction of the Frobenius operators to the subring of symmetric functions are the Frobenius operators of the latter:

```sage
sage: e = SymmetricFunctions(ZZ).e()
sage: all( M(e(lam)).frobenius(3) == M(e(lam).frobenius(3)) for lam in Partitions(3) )
True
```

```
internal_coproduct()
```

Return the inner coproduct of self in the basis of self.
The inner coproduct (also known as the Kronecker coproduct, or as the second comultiplication on the \(
\mathcal{R}\)-algebra of quasi-symmetric functions) is an \(
\mathcal{R}\)-algebra homomorphism \(\Delta^\times\) from the \(\mathcal{R}\)-algebra of quasi-symmetric functions to the tensor square (over \(\mathcal{R}\)) of quasi-symmetric functions.

It can be defined in the following two ways:

1. If \(I\) is a composition, then a \((0, I)\)-matrix will mean a matrix whose entries are nonnegative integers such that no row and no column of this matrix is zero, and such that if all the non-zero entries of the matrix are read (row by row, starting at the topmost row, reading every row from left to right), then the reading word obtained is \(I\). If \(A\) is a \((0, I)\)-matrix, then \(\text{row}(A)\) will denote the vector of row sums of \(A\) (regarded as a composition), and \(\text{column}(A)\) will denote the vector of column sums of \(A\) (regarded as a composition).

For every composition \(I\), the internal coproduct \(\Delta^\times(M_I)\) of the \(I\)-th monomial quasisymmetric function \(M_I\) is the sum

\[
\sum_{A \text{ a } (0, I)\text{-matrix}} M_{\text{row}(A)} \otimes M_{\text{column}(A)}.
\]

See Section 11.39 of [HazWitt1].

2. For every permutation \(w\), let \(C(w)\) denote the descent composition of \(w\). Then, for any composition \(I\) of size \(n\), the internal coproduct \(\Delta^\times(F_I)\) of the \(I\)-th fundamental quasisymmetric function \(F_I\) is the sum

\[
\sum_{\sigma \in S_n, \tau \in S_n, \tau \sigma = \pi} F_{C(\sigma)} \otimes F_{C(\tau)},
\]

where \(\pi\) is any permutation in \(S_n\) having descent composition \(I\) and where permutations act from the left and multiply accordingly, so \(\tau \sigma\) means first applying \(\sigma\) and then \(\tau\). See Theorem 4.23 in [Mal1993], but beware of the notations which are apparently different from those in [HazWitt1].

The restriction of the internal coproduct to the \(\mathcal{R}\)-algebra of symmetric functions is the well-known internal coproduct on the symmetric functions.

The method \texttt{kroncker_coproduct()} is a synonym of this one.

**EXAMPLES:**

Let us compute the internal coproduct of \(M_{21}\) (which is short for \(M_{[2,1]}\)). The \((0, [2, 1])\)-matrices are

\[
\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}
\]

so

\[
\Delta^\times(M_{21}) = M_3 \otimes M_{21} + M_{21} \otimes M_3 + M_{21} \otimes M_{21} + M_{21} \otimes M_{12}.
\]

This is confirmed by the following Sage computation (incidentally demonstrating the non-cocommutativity of the internal coproduct):

```sage
sage: M = QuasiSymmetricFunctions(ZZ).M()
sage: a = M([2,1])
sage: a.internal_coproduct()
```

Further examples:
The definition of $\Delta^\times(M_I)$ in terms of $(0, I)$-matrices is not suitable for computation in cases where the length of $I$ is large, but we can use it as a doctest. Here is a naive implementation:

```python
sage: def naive_internal_coproduct_on_M(I):
    ....: # INPUT: composition I
    ....: # (not quasi-symmetric function)
    ....: # OUTPUT: interior coproduct of M_I
    ....: M = QuasiSymmetricFunctions(ZZ).M()
    ....: M2 = M.tensor(M)
    ....: res = M2.zero()
    ....: l = len(I)
    ....: n = I.size()
    ....: for S in Subsets(range(l**2), l):
    ....:     M_list = sorted(S)
    ....:     row_M = [sum([I[M_list.index(l * i + j)]
    ....:                     for j in range(l) if
    ....:                     l * i + j in S])
    ....:              for i in range(l)]
    ....:     col_M = [sum([I[M_list.index(l * i + j)]
    ....:                     for i in range(l) if
    ....:                     l * i + j in S])
    ....:              for j in range(l)]
    ....:     if 0 in row_M:
    ....:         first_zero = row_M.index(0)
    ....:         row_M = row_M[:first_zero]
    ....:         if sum(row_M) != n:
    ....:             continue
    ....:     if 0 in col_M:
    ....:         first_zero = col_M.index(0)
    ....:         col_M = col_M[:first_zero]
    ....:         if sum(col_M) != n:
    ....:             continue
    ....:     res += tensor([M(Compositions(n)(row_M)),
                      M(Compositions(n)(col_M))])
    ....: return res
```

```
sage: all( naive_internal_coproduct_on_M(I)
        ....: == M(I).internal_coproduct()
        ....: for I in Compositions(3) )
True
```

Todo: Implement this directly on the monomial basis maybe? The $(0, I)$-matrices are a pain to generate from their definition, but maybe there is a good algorithm. If so, the above “further examples” should be moved to the M-method.

is_symmetric()
Return True if self is an element of the symmetric functions.

This is being tested by looking at the expansion in the Monomial basis and checking if the coefficients are the same if the indexing compositions are permutations of each other.

OUTPUT:
• True if self is symmetric. False if self is not symmetric.

EXAMPLES:
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: F = QSym.Fundamental()
sage: (F[3,2] + F[2,3]).is_symmetric()
False
sage: (F[1, 1, 2] + F[1, 1, 3] + F[1, 3, 1] + F[2, 1, 1, 1] + F[3, 1, \ldots 1]).is_symmetric()
True
sage: F([]).is_symmetric()
True

kronecker_coproduct()

Return the inner coproduct of self in the basis of self.

The inner coproduct (also known as the Kronecker coproduct, or as the second comultiplication on the \(R\)-algebra of quasi-symmetric functions) is an \(R\)-algebra homomorphism \(\Delta^\times\) from the \(R\)-algebra of quasi-symmetric functions to the tensor square (over \(R\)) of quasi-symmetric functions.

It can be defined in the following two ways:
1. If \(I\) is a composition, then a \((0, I)\)-matrix will mean a matrix whose entries are nonnegative integers such that no row and no column of this matrix is zero, and such that if all the non-zero entries of the matrix are read (row by row, starting at the topmost row, reading every row from left to right), then the reading word obtained is \(I\). If \(A\) is a \((0, I)\)-matrix, then \(\text{row}(A)\) will denote the vector of row sums of \(A\) (regarded as a composition), and \(\text{column}(A)\) will denote the vector of column sums of \(A\) (regarded as a composition).

For every composition \(I\), the internal coproduct \(\Delta^\times(M_I)\) of the \(I\)-th monomial quasisymmetric function \(M_I\) is the sum

\[
\sum_{A \text{ is a } (0, I)\text{-matrix}} M_{\text{row}(A)} \otimes M_{\text{column}(A)}.
\]

See Section 11.39 of [HazWitt1].

2. For every permutation \(w\), let \(C(w)\) denote the descent composition of \(w\). Then, for any composition \(I\) of size \(n\), the internal coproduct \(\Delta^\times(F_I)\) of the \(I\)-th fundamental quasisymmetric function \(F_I\) is the sum

\[
\sum_{\sigma \in S_n, \tau \in S_n, \tau \pi = \pi} F_{C(\sigma)} \otimes F_{C(\tau)},
\]

where \(\pi\) is any permutation in \(S_n\) having descent composition \(I\) and where permutations act from the left and multiply accordingly, so \(\tau \sigma\) means first applying \(\sigma\) and then \(\tau\). See Theorem 4.23 in [Mal1993], but beware of the notations which are apparently different from those in [HazWitt1].

The restriction of the internal coproduct to the \(R\)-algebra of symmetric functions is the well-known internal coproduct on the symmetric functions.

The method kronecker_coproduct() is a synonym of this one.

EXAMPLES:
Let us compute the internal coproduct of $M_{21}$ (which is short for $M_{[2,1]}$). The $(0,[2,1])$-matrices are

\[
\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}
\]

so

\[
\Delta^\times(M_{21}) = M_5 \otimes M_{21} + M_{21} \otimes M_5 + M_{21} \otimes M_{21} + M_{21} \otimes M_{12}.
\]

This is confirmed by the following Sage computation (incidentally demonstrating the non-cocommutativity of the internal coproduct):

```python
sage: M = QuasiSymmetricFunctions(ZZ).M()
sage: a = M([2,1])
sage: a.internal_coproduct()
```

Further examples:

```python
sage: all( M([i]).internal_coproduct() == tensor([M([i]), M([i])])
....:     for i in range(1, 4) )
True
sage: M([1, 2]).internal_coproduct()
```

The definition of $\Delta^\times(M_I)$ in terms of $(0,I)$-matrices is not suitable for computation in cases where the length of $I$ is large, but we can use it as a doctest. Here is a naive implementation:

```python
sage: def naive_internal_coproduct_on_M(I):
....:     # INPUT: composition I
....:     # (not quasi-symmetric function)
....:     # OUTPUT: interior coproduct of M_I
....:     M = QuasiSymmetricFunctions(ZZ).M()
....:     M2 = M.tensor(M)
....:     res = M2.zero()
....:     l = len(I)
....:     n = I.size()
....:     for S in Subsets(range(l**2), l):
....:         M_list = sorted(S)
....:         row_M = [sum([I[M_list.index(l * i + j)]
....:                        for j in range(l) if
....:                        l * i + j in S])
....:                  for i in range(l)]
....:         col_M = [sum([I[M_list.index(l * i + j)]
....:                        for i in range(l) if
....:                        l * i + j in S])
....:                  for j in range(l)]
....:         if 0 in row_M:
....:             first_zero = row_M.index(0)
....:             row_M = row_M[:first_zero]
....:             if sum(row_M) != n:
....:                 continue
....:         if 0 in col_M:
```
..:\first_zero = col_M.index(0)
..:\col_M = col_M[:first_zero]
..:\if sum(col_M) != n:
..:\    continue
..:\res += tensor([M(Compositions(n)(row_M)),
..:\    M(Compositions(n)(col_M))])
..:\return res

..: all( naive_internal_coproduct_on_M(I) == M(I).internal_coproduct() for I in Compositions(3) )
..: True

Todo: Implement this directly on the monomial basis maybe? The \((0, I)\)-matrices are a pain to generate from their definition, but maybe there is a good algorithm. If so, the above “further examples” should be moved to the M-method.

omega_involution()

Return the image of the quasisymmetric function \self under the omega involution.

The omega involution is defined as the linear map \(QS\) \(\rightarrow QSym\) which, for every composition \(I\), sends the fundamental quasisymmetric function \(F_I\) to \(F_{I^t}\), where \(I^t\) denotes the conjugate \([\text{conjugate()}]\) of the composition \(I\). This map is commonly denoted by \(\omega\). It is an algebra homomorphism and a coalgebra antihomomorphism; it also is an involution and an automorphism of the graded vector space \(QSym\). Also, every composition \(I\) satisfies

\[
\omega(M_I) = (-1)^{l(I)-l(I^t)} \sum M_J,
\]

where the sum ranges over all compositions \(J\) of \(|I|\) which are coarser than the reversed composition \(I^r\) of \(I\). Here, \(l(I)\) denotes the length of the composition \(I\) (that is, the number of parts of \(I\)).

If \(f\) is a homogeneous element of \(NCSF\) of degree \(n\), then

\[
\omega(f) = (-1)^n S(f),
\]

where \(S\) denotes the antipode of \(QSym\).

The restriction of \(\omega\) to the ring of symmetric functions (which is a subring of \(QSym\)) is precisely the omega involution \(\omega()\) of said ring.

The omega involution on \(QSym\) is adjoint to the omega involution on \(NCSF\) by the standard adjunction between \(NCSF\) and \(QSym\).

The omega involution has been denoted by \(\omega\) in [LMvW13], section 3.6.

See also:

omega involution on NCSF, psi involution on QSym, star involution on QSym.

EXAMPLES:

sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: F = QSym.F()
sage: F[3,2].omega_involution() F[1, 2, 1, 1]
This page contains a continuation of the previous page discussing the involution of quasi-symmetric functions. The involution is defined as a linear map from $QSym$ to $QSym$ that sends the fundamental quasisymmetric function $F_I$ to $F_{I^c}$, where $I^c$ denotes the complement of the composition $I$. The involution is an involution and a graded Hopf algebra automorphism of $QSym$. Its restriction to the ring of symmetric functions coincides with the omega automorphism of the latter ring.

Testing the fact that the restriction of $\omega$ to $Sym$ is the omega automorphism of $Sym$:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: e = Sym.e()
sage: all( F(e[lam]).omega_involution() == F(e[lam].omega()) 
       ....: for lam in Partitions(4) )
True
```

```
 psi_involution()

Return the image of the quasisymmetric function self under the involution $\psi$.

The involution $\psi$ is defined as the linear map $QSym \rightarrow QSym$ which, for every composition $I$, sends the fundamental quasisymmetric function $F_I$ to $F_{I^c}$, where $I^c$ denotes the complement of the composition $I$. The map $\psi$ is an involution and a graded Hopf algebra automorphism of $QSym$. Its restriction to the ring of symmetric functions coincides with the omega automorphism of the latter ring.

The involution $\psi$ of $QSym$ is adjoint to the involution $\psi$ of $NCSF$ by the standard adjunction between $NCSF$ and $QSym$.

The involution $\psi$ has been denoted by $\psi$ in [LMvW13], section 3.6.

See also:

psi involution on NCSF, star involution on QSym.

EXAMPLES:
```
```
The involution $\psi$ commutes with the antipode:

```
sage: all( F(I).psi_involution().antipode() == F(I).antipode().psi_involution() for I in Compositions(4) )
True
```

Testing the fact that the restriction of $\psi$ to $Sym$ is the omega automorphism of $Sym$:

```
sage: Sym = SymmetricFunctions(ZZ)
sage: e = Sym.e()
sage: all( F(e[lam]).psi_involution() == F(e[lam].omega()) for lam in Partitions(4) )
True
```

star_involution() 

Return the image of the quasisymmetric function self under the star involution.

The star involution is defined as the linear map $QSym \rightarrow QSym$ which, for every composition $I$, sends the monomial quasisymmetric function $M_I$ to $M_{I'}$. Here, if $I$ is a composition, we denote by $I'$ the reversed composition of $I$. Denoting by $f^*$ the image of an element $f \in QSym$ under the star involution, it can be shown that every composition $I$ satisfies

$$(M_I)^* = M_{I'}, \quad (F_I)^* = F_{I'},$$

where $F_I$ denotes the fundamental quasisymmetric function corresponding to the composition $I$. The star involution is an involution, an algebra automorphism and a coalgebra anti-automorphism of $QSym$. It also is an automorphism of the graded vector space $QSym$, and is the identity on the subspace $Sym$ of $QSym$. It is adjoint to the star involution on $NCSF$ by the standard adjunction between $NCSF$ and $QSym$.

The star involution has been denoted by $\rho$ in [LMvW13], section 3.6.

See also:

star involution on NCSF.

EXAMPLES:

```
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: M = QSym.M()
sage: M[3,2].star_involution()
M[2, 3]
sage: M[6,3].star_involution()
M[3, 6]
sage: (M[3,3] - 2*M[2]).star_involution()
```

(continues on next page)
The star involution commutes with the antipode:

```python
sage: all( M(I).star_involution().antipode() == M(I).antipode().star_involution() for I in Compositions(4) )
True
```

The star involution is the identity on $\text{Sym}$:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: e = Sym.e()
sage: all( M(e(lam)).star_involution() == M(e(lam)) for lam in Partitions(4) )
True
```

### to_symmetric_function()

Convert a quasi-symmetric function to a symmetric function.

**OUTPUT:**

- If `self` is a symmetric function, then return the expansion in the monomial basis. Otherwise raise an error.

**EXAMPLES:**

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: F = QSym.Fundamental()
sage: (F[3,2] + F[2,3]).to_symmetric_function()
Traceback (most recent call last):
...
ValueError: F[2, 3] + F[3, 2] is not a symmetric function
```

```python
sage: m = SymmetricFunctions(QQ).m()
sage: s = SymmetricFunctions(QQ).s()
sage: F(s[3,1,1]).to_symmetric_function()
6*m[1, 1, 1, 1, 1] + 3*m[2, 1, 1, 1] + m[2, 2, 1] + m[3, 1, 1]
sage: m(s[3,1,1])
6*m[1, 1, 1, 1, 1] + 3*m[2, 1, 1, 1] + m[2, 2, 1] + m[3, 1, 1]
```

class ParentMethods

Bases: object

Methods common to all bases of QuasiSymmetricFunctions.

**Eulerian**($n, j, k=None$)

Return the Eulerian (quasi)symmetric function $Q_{n,j}$ in terms of `self`.

**INPUT:**

- `n` – the value $n$ or a partition

```python
sage: M([4,2]).star_involution()
M[2, 4]
sage: dI = QSym.dI()
sage: dI([1,2]).star_involution()
-dI[1, 2] + dI[2, 1]
sage: dI.zero().star_involution()
0
```

```python
sage: all( M(I).star_involution().antipode() == M(I).antipode().star_involution() for I in Compositions(4) )
True
```

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: e = Sym.e()
sage: all( M(e(lam)).star_involution() == M(e(lam)) for lam in Partitions(4) )
True
```

```python
to_symmetric_function()
Convert a quasi-symmetric function to a symmetric function.

**OUTPUT:**

- If `self` is a symmetric function, then return the expansion in the monomial basis. Otherwise raise an error.

**EXAMPLES:**

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: F = QSym.Fundamental()
sage: (F[3,2] + F[2,3]).to_symmetric_function()
Traceback (most recent call last):
...
ValueError: F[2, 3] + F[3, 2] is not a symmetric function
```

```python
sage: m = SymmetricFunctions(QQ).m()
sage: s = SymmetricFunctions(QQ).s()
sage: F(s[3,1,1]).to_symmetric_function()
6*m[1, 1, 1, 1, 1] + 3*m[2, 1, 1, 1] + m[2, 2, 1] + m[3, 1, 1]
sage: m(s[3,1,1])
6*m[1, 1, 1, 1, 1] + 3*m[2, 1, 1, 1] + m[2, 2, 1] + m[3, 1, 1]
```

class ParentMethods

Bases: object

Methods common to all bases of QuasiSymmetricFunctions.

**Eulerian**($n, j, k=None$)

Return the Eulerian (quasi)symmetric function $Q_{n,j}$ in terms of `self`.

**INPUT:**

- `n` – the value $n$ or a partition

```python
sage: M([4,2]).star_involution()
M[2, 4]
sage: dI = QSym.dI()
sage: dI([1,2]).star_involution()
-dI[1, 2] + dI[2, 1]
sage: dI.zero().star_involution()
0
```

The star involution commutes with the antipode:

```python
sage: all( M(I).star_involution().antipode() == M(I).antipode().star_involution() for I in Compositions(4) )
True
```

The star involution is the identity on $\text{Sym}$:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: e = Sym.e()
sage: all( M(e(lam)).star_involution() == M(e(lam)) for lam in Partitions(4) )
True
```
Combinatorics, Release 10.0

• \( j \) – the number of excedances
• \( k \) – (optional) if specified, determines the number of fixed points of the permutation

**EXAMPLES:**

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: M.Eulerian(3, 1)
sage: M.Eulerian(4, 1, 2)
sage: QS = QSym.QS()
sage: QS.Eulerian(4, 2)
sage: QS.Eulerian([2, 2, 1], 2)
sage: dI = QSym.dI()
sage: dI.Eulerian(5, 2)
```

**from_polynomial**(*f*, *check=True*)

The quasi-symmetric function expanded in this basis corresponding to the quasi-symmetric polynomial \( f \).

This is a default implementation that computes the expansion in the Monomial basis and converts to this basis.

**INPUT:**
• \( f \) – a polynomial in finitely many variables over the same base ring as \( \text{self} \). It is assumed that this polynomial is quasi-symmetric.
• \( \text{check} \) – boolean (default: True), checks whether the polynomial is indeed quasi-symmetric.

**OUTPUT:**
• quasi-symmetric function

**EXAMPLES:**

```python
sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: P = PolynomialRing(QQ, 'x', 3)
sage: x = P.gens()
sage: f = x[0] + x[1] + x[2]
sage: M.from_polynomial(f)
M[1]
sage: F.from_polynomial(f)
F[1]
sage: f = x[0]**2+x[1]**2+x[2]**2
sage: M.from_polynomial(f)
M[2]
sage: F.from_polynomial(f)
-F[1, 1] + F[2]
```

If the polynomial is not quasi-symmetric, an error is raised:
```
sage: f = x[0]^2+x[1]
sage: M.from_polynomial(f)
Traceback (most recent call last):
...
ValueError: x0^2 + x1 is not a quasi-symmetric polynomial
sage: F.from_polynomial(f)
Traceback (most recent call last):
...
ValueError: x0^2 + x1 is not a quasi-symmetric polynomial
```

**super_categories()**

Return the super categories of bases of the Quasi-symmetric functions.

OUTPUT:
- a list of categories

**E**

alias of `Essential`

**class Essential(QSym)**

**Bases:** `CombinatorialFreeModule, BindableClass`

The Hopf algebra of quasi-symmetric functions in the Essential basis.

The Essential quasi-symmetric functions are defined by

\[
E_I = \sum_{J \geq I} M_J = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1}^{I_1} \cdots x_{i_k}^{I_k},
\]

where \( I = (I_1, \ldots, I_k) \).

**Note:** Our convention of \( \leq \) and \( \geq \) of compositions is opposite that of [Hoff2015].

**EXAMPLES:**

```
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: E = QSym.E()
sage: M = QSym.M()
sage: E(M[2,2])
sage: s = SymmetricFunctions(QQ).s()
sage: E(s[3,2])
5*E[1, 1, 1, 1] - 2*E[1, 1, 1, 2] - 2*E[1, 1, 2, 1] - 2*E[2, 1, 2, 1] + E[2, 1, 2] + E[2, 2, 1]
sage: (1 + E[1])^3
sage: E[1,2,1].coproduct()
```

The following is an alias for this basis:

```
sage: QSym.Essential()
Quasisymmetric functions over the Rational Field in the Essential basis
```
**antipode_on_basis**(compo)
Return the result of the antipode applied to a quasi-symmetric Essential basis element.

**INPUT:**
- compo – composition

**OUTPUT:**
- The result of the antipode applied to the composition compo, expressed in the Essential basis.

**EXAMPLES:**
```
sage: E = QuasiSymmetricFunctions(QQ).E()
sage: E.antipode_on_basis(Composition([2,1]))
E[1, 2] - E[3]
sage: E.antipode_on_basis(Composition([]))
E[]
```

**coproduct_on_basis**(compo)
Return the coproduct of a Essential basis element.

Combinatorial rule: deconcatenation.

**INPUT:**
- compo – composition

**OUTPUT:**
- The coproduct applied to the Essential quasi-symmetric function indexed by compo, expressed in the Essential basis.

**EXAMPLES:**
```
sage: E = QuasiSymmetricFunctions(QQ).Essential()
sage: E[4,2,3].coproduct()
sage: E.coproduct_on_basis(Composition([]))
E[] # E[]
```

**product_on_basis**(I, J)
The product on Essential basis elements.

The product of the basis elements indexed by two compositions I and J is the sum of the basis elements indexed by compositions K in the stuffle product (also called the overlapping shuffle product) of I and J with a coefficient of $(-1)^{\ell(I)+\ell(J)-\ell(K)}$, where $\ell(C)$ is the length of the composition C.

**INPUT:**
- I, J – compositions

**OUTPUT:**
- The product of the Essential quasi-symmetric functions indexed by I and J, expressed in the Essential basis.

**EXAMPLES:**
```
sage: E = QuasiSymmetricFunctions(QQ).E()
sage: c1 = Composition([2])
sage: c2 = Composition([1,3])
sage: E.product_on_basis(c1, c2)
sage: E.product_on_basis(c1, Composition([]))
E[2]
sage: E.product_on_basis(c1, Composition([3]))
```
Combinatorics, Release 10.0

\( F \)

alias of \textit{Fundamental}

\textbf{class Fundamental}(\textit{QSym})

Bases: \texttt{CombinatorialFreeModule}, \texttt{BindableClass}

The Hopf algebra of quasi-symmetric functions in the Fundamental basis.

\textbf{EXCEPTIONS:}

\begin{verbatim}
    sage: QSym = QuasiSymmetricFunctions(QQ)
    sage: F = QSym.F()
    sage: M = QSym.M()
    sage: F(M[2,2])
    F[1, 1, 1, 1] - F[1, 1, 2] - F[2, 1, 1] + F[2, 2]
    sage: s = SymmetricFunctions(QQ).s()
    sage: F(s[3,2])
    sage: (1+F[1])^3
    sage: F[1,2,1].coproduct()
\end{verbatim}

The following is an alias for this basis:

\begin{verbatim}
    sage: QSym.Fundamental()
    Quasisymmetric functions over the Rational Field in the Fundamental basis
\end{verbatim}

\textbf{class Element}

Bases: \texttt{IndexedFreeModuleElement}

\begin{verbatim}
    internal_coproduct()
        Return the inner coproduct of \texttt{self} in the Fundamental basis.
        
        The inner coproduct (also known as the Kronecker coproduct, or as the second comultiplication on the \( R \)-algebra of quasi-symmetric functions) is an \( R \)-algebra homomorphism \( \Delta^\times \) from the \( R \)-algebra of quasi-symmetric functions to the tensor square (over \( R \)) of quasi-symmetric functions. It can be defined in the following two ways:
        1. If \( I \) is a composition, then a \((0, I)\)-matrix will mean a matrix whose entries are nonnegative integers such that no row and no column of this matrix is zero, and such that if all the non-zero entries of the matrix are read (row by row, starting at the topmost row, reading every row from left to right), then the reading word obtained is \( I \). If \( A \) is a \((0, I)\)-matrix, then \( \text{row}(A) \) will denote the vector of row sums of \( A \) (regarded as a composition), and \( \text{column}(A) \) will denote the vector of column sums of \( A \) (regarded as a composition).
        2. For every permutation \( w \), let \( C(w) \) denote the descent composition of \( w \). Then, for any composition \( I \) of size \( n \), the internal coproduct \( \Delta^\times(F_I) \) of the \( I \)-th fundamental quasisymmetric function \( M_I \) is the sum

        \[ \sum_{A \text{ is a } (0, I)\text{-matrix}} M_{\text{row}(A)} \otimes M_{\text{column}(A)}. \]

        See Section 11.39 of [HazWitt1].
\end{verbatim}
function $F_I$ is the sum

$$\sum_{\sigma \in S_n, \tau \in S_n, \tau \sigma = \pi} F_{\mathcal{C}(\sigma)} \otimes F_{\mathcal{C}(\tau)},$$

where $\pi$ is any permutation in $S_n$ having descent composition $I$ and where permutations act from the left and multiply accordingly, so $\tau \sigma$ means first applying $\sigma$ and then $\tau$. See Theorem 4.23 in [Mal1993], but beware of the notations which are apparently different from those in [HazWitt1].

The restriction of the internal coproduct to the $R$-algebra of symmetric functions is the well-known internal coproduct on the symmetric functions.

The method `kronecker_coproduct()` is a synonym of this one.

**EXAMPLES:**

Let us compute the internal coproduct of $M_{21}$ (which is short for $M_{\{2,1\}}$). The $(0, [2,1])$-matrices are

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

so

$$\Delta^\times(M_{21}) = M_3 \otimes M_{21} + M_{21} \otimes M_3 + M_{21} \otimes M_{21} + M_{21} \otimes M_{12}.$$ 

This is confirmed by the following Sage computation (incidentally demonstrating the non-cocommutativity of the internal coproduct):

```sage
M = QuasiSymmetricFunctions(ZZ).M()
sage: a = M([2,1])
sage: a.internal_coproduct()
```

Some examples on the Fundamental basis:

```sage
F = QuasiSymmetricFunctions(ZZ).F()
sage: F([1,1]).internal_coproduct()
F[1, 1] # F[1, 1] + F[1, 2] # F[1, 1]
sage: F([2]).internal_coproduct()
sage: F([3]).internal_coproduct()
sage: F([1,2]).internal_coproduct()
```

`kronecker_coproduct()`

Return the inner coproduct of `self` in the Fundamental basis.

The inner coproduct (also known as the Kronecker coproduct, or as the second comultiplication on the $R$-algebra of quasi-symmetric functions) is an $R$-algebra homomorphism $\Delta^\times$ from the $R$-algebra of quasi-symmetric functions to the tensor square (over $R$) of quasi-symmetric functions. It can be defined in the following two ways:

1. If $I$ is a composition, then a $(0, I)$-matrix will mean a matrix whose entries are nonnegative integers such that no row and no column of this matrix is zero, and such that if all the non-zero
entries of the matrix are read (row by row, starting at the topmost row, reading every row from left to right), then the reading word obtained is $I$. If $A$ is a $(0,I)$-matrix, then $\text{row}(A)$ will denote the vector of row sums of $A$ (regarded as a composition), and $\text{column}(A)$ will denote the vector of column sums of $A$ (regarded as a composition).

For every composition $I$, the internal coproduct $\Delta^\times(M_I)$ of the $I$-th monomial quasisymmetric function $M_I$ is the sum

$$\sum_{A \text{ is a } (0,I)-\text{matrix}} M_{\text{row}(A)} \otimes M_{\text{column}(A)}.$$  

See Section 11.39 of [HazWitt1].

2. For every permutation $w$, let $C(w)$ denote the descent composition of $w$. Then, for any composition $I$ of size $n$, the internal coproduct $\Delta^\times(F_I)$ of the $I$-th fundamental quasisymmetric function $F_I$ is the sum

$$\sum_{\sigma \in S_n, \tau \sigma = \pi} F_{C(\sigma)} \otimes F_{C(\tau)},$$

where $\pi$ is any permutation in $S_n$ having descent composition $I$ and where permutations act from the left and multiply accordingly, so $\tau \sigma$ means first applying $\sigma$ and then $\tau$. See Theorem 4.23 in [Mal1993], but beware of the notations which are apparently different from those in [HazWitt1].

The restriction of the internal coproduct to the $R$-algebra of symmetric functions is the well-known internal coproduct on the symmetric functions.

The method $\text{kronecker_coproduct()}$ is a synonym of this one.

EXAMPLES:

Let us compute the internal coproduct of $M_{21}$ (which is short for $M_{[2,1]}$). The $(0,[2,1])$-matrices are

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix},$$

so

$$\Delta^\times(M_{21}) = M_3 \otimes M_{21} + M_{21} \otimes M_5 + M_{21} \otimes M_{21} + M_{21} \otimes M_{12}.$$  

This is confirmed by the following Sage computation (incidentally demonstrating the non-cocommutativity of the internal coproduct):

```sage
sage: M = QuasiSymmetricFunctions(ZZ).M()
sage: a = M([2,1])
sage: a.internal_coproduct()
```

Some examples on the Fundamental basis:

```sage
sage: F = QuasiSymmetricFunctions(ZZ).F()
sage: F([1,1]).internal_coproduct()
sage: F([2]).internal_coproduct()
sage: F([3]).internal_coproduct()
F[1, 1, 1] # F[1, 1, 1] + F[1, 2] # F[1, 2] + F[1, 2] # F[2, 1]
```

(continues on next page)
star_involution()

Return the image of the quasisymmetric function self under the star involution.

The star involution is defined as the linear map \( QSym \rightarrow QSym \) which, for every composition \( I \), sends the monomial quasisymmetric function \( M_I \) to \( M_{I'} \). Here, if \( I \) is a composition, we denote by \( I' \) the reversed composition of \( I \). Denoting by \( f^* \) the image of an element \( f \in QSym \) under the star involution, it can be shown that every composition \( I \) satisfies

\[
(M_I)^* = M_{I'}, \quad (F_I)^* = F_{I'},
\]

where \( F_I \) denotes the fundamental quasisymmetric function corresponding to the composition \( I \).

The star involution is an involution, an algebra automorphism and a coalgebra anti-automorphism of \( QSym \). It also is an automorphism of the graded vector space \( QSym \), and is the identity on the subspace \( Sym \) of \( QSym \). It is adjoint to the star involution on \( NCSF \) by the standard adjunction between \( NCSF \) and \( QSym \).

The star involution has been denoted by \( \rho \) in [LMvW13], section 3.6.

See also:

star involution on \( QSym \), star involution on \( NCSF \).

EXAMPLES:

```python
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: F = QSym.F()
sage: F[3,1].star_involution()
F[1, 3]
sage: F[5,3].star_involution()
F[3, 5]
sage: (F[3,3] - 2*F[2]).star_involution()
sage: F([4,2]).star_involution()
F[2, 4]
sage: dI = QSym.dI()
sage: dI([1,2]).star_involution()
-dI[1, 2] + dI[2, 1]
sage: dI.zero().star_involution()
0
```

Eulerian\((n, j, k=None)\)

Return the Eulerian (quasi)symmetric function \( Q_{n,j} \) (with \( n \) either an integer or a partition) defined in [SW2010] in terms of the fundamental quasisymmetric functions. Or, if the optional argument \( k \) is specified, return the function \( Q_{n,j,k} \) defined ibidem.

If \( n \) and \( j \) are nonnegative integers, then the Eulerian quasisymmetric function \( Q_{n,j} \) is defined as

\[
Q_{n,j} := \sum_{\sigma} F_{Dex(\sigma)},
\]
where we sum over all permutations $\sigma \in S_n$ such that the number of excedances of $\sigma$ is $j$, and where $Dex(\sigma)$ is a composition of $n$ defined as follows: Let $S$ be the set of all $i \in \{1, 2, \ldots, n-1\}$ such that either $\sigma_i > \sigma_{i+1} > i + 1$ or $i \geq \sigma_i > \sigma_{i+1}$ or $\sigma_{i+1} > i + 1 > \sigma_i$. Then, $Dex(\sigma)$ is set to be the composition of $n$ whose descent set is $S$.

Here, an excedance of a permutation $\sigma \in S_n$ means an element $i \in \{1, 2, \ldots, n-1\}$ satisfying $\sigma_i > i$.

Similarly we can define a quasisymmetric function $Q_{\lambda, j}$ for every partition $\lambda$ and every nonnegative integer $j$. This differs from $Q_{n,j}$ only in that the sum is restricted to all permutations $\sigma \in S_n$ whose cycle type is $\lambda$ (where $n = |\lambda|$, and where we still require the number of excedances to be $j$). The method at hand allows computing these functions by passing $\lambda$ as the $n$ parameter.

Analogously we can define a quasisymmetric function $Q_{n,j,k}$ for any nonnegative integers $n$, $j$ and $k$ by restricting the sum to all permutations $\sigma \in S_n$ that have exactly $k$ fixed points (and $j$ excedances). This can be obtained by specifying the optional $k$ argument in this method.

All three versions of Eulerian quasisymmetric functions ($Q_{n,j}$, $Q_{\lambda,j}$ and $Q_{n,j,k}$) are actually symmetric functions. See `Eulerian()`.

**INPUT:**

- $n$ – the nonnegative integer $n$ or a partition
- $j$ – the number of excedances
- $k$ – (optional) if specified, determines the number of fixed points of the permutations which are being summed over

**EXAMPLES:**

```python
sage: F = QuasiSymmetricFunctions(QQ).F()
sage: F.Eulerian(3, 1)

sage: F.Eulerian(4, 2)

sage: F.Eulerian(5, 2)

sage: F.Eulerian(4, 0)
F[4]

sage: F.Eulerian(4, 3)
F[4]

sage: F.Eulerian(4, 1, 2)

sage: F.Eulerian(Partition([2, 2, 1]), 2)

sage: F.Eulerian(0, 0)
F[]

sage: F.Eulerian(0, 1)
[]

sage: F.Eulerian(1, 0)
F[1]

sage: F.Eulerian(1, 1)
[]
```

**antipode_on_basis**(compo)

Return the antipode to a Fundamental quasi-symmetric basis element.
INPUT:
• compo – composition

OUTPUT:
• The result of the antipode applied to the quasi-symmetric Fundamental basis element indexed by compo.

EXAMPLES:

```
sage: F = QuasiSymmetricFunctions(QQ).F()
sage: F.antipode_on_basis(Composition([2,1]))
-F[2, 1]
```

coproduct_on_basis(compo)

Return the coproduct to a Fundamental quasi-symmetric basis element.

Combinatorial rule: quasi-deconcatenation.

INPUT:
• compo – composition

OUTPUT:
• The application of the coproduct to the Fundamental quasi-symmetric function indexed by the composition compo.

EXAMPLES:

```
sage: F = QuasiSymmetricFunctions(QQ).Fundamental()
sage: F[4].coproduct()
sage: F[2,1,3].coproduct()
```

dual()

Return the dual basis to the Fundamental basis. This is the ribbon basis of the non-commutative symmetric functions.

OUTPUT:
• The ribbon basis of the non-commutative symmetric functions.

EXAMPLES:

```
sage: F = QuasiSymmetricFunctions(QQ).F()
sage: F.dual()
Non-Commutative Symmetric Functions over the Rational Field in the Ribbon
˓→ basis
```

class HazewinkelLambda(QSym)

Bases: CombinatorialFreeModule, BindableClass

The Hazewinkel lambda basis of the quasi-symmetric functions.

This basis goes back to [Haz2004], albeit it is indexed in a different way here. It is a multiplicative basis in a weak sense of this word (the product of any two basis elements is a basis element, but of course not the one obtained by concatenating the indexing compositions).

In [Haz2004], Hazewinkel showed that the k-algebra QSym is a polynomial algebra. (The proof is correct but rests upon an unproven claim that the lexicographically largest term of the n-th shuffle power of a Lyndon word is the n-fold concatenation of this Lyndon word with itself, occurring n! times in that shuffle power. But this can be deduced from Section 2 of [Rad1979]. See also Chapter 6 of [GriRei18], specifically Theorem 6.5.13, for a complete proof.) More precisely, he showed that QSym is generated, as a free
combinatorial k-algebra, by the elements $\lambda^n(M_I)$, where $n$ ranges over the positive integers, and $I$ ranges over all compositions which are Lyndon words and whose entries have gcd 1. Here, $\lambda^n$ denotes the $n$-th lambda operation as explained in \texttt{lambda_of_monomial()}.

Thus, products of these generators form a k-module basis of $\text{QSym}$. We index this basis by compositions here. More precisely, we define the Hazewinkel lambda basis $(\text{HWL}_I)_I$ (with $I$ ranging over all compositions) as follows:

Let $I$ be a composition. Let $I = I_1I_2\ldots I_k$ be the Chen-Fox-Lyndon factorization of $I$ (see \texttt{lyndon_factorization()}). For every $j \in \{1, 2, \ldots, k\}$, let $g_j$ be the gcd of the entries of the Lyndon word $I_j$, and let $J_j$ be the result of dividing the entries of $I_j$ by this gcd. Then, $\text{HWL}_I$ is defined to be $\prod_{j=1}^k \lambda^{g_j}(M_{J_j})$.

**Todo:** The conversion from the M basis to the HWL basis is currently implemented in the naive way (inverting the base-change matrix in the other direction). This matrix is not triangular (not even after any permutations of the bases), and there could very well be a faster method (the one given by Hazewinkel?).

**EXAMPLES:**

```python
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: HWL = QSym.HazewinkelLambda()
sage: M = QSym.M()
sage: M(HWL([2]))
M[1, 1]
sage: M(HWL([1,2]))
M[1, 2]
sage: 3*M[1, 1, 1] + M[1, 2] + M[2, 1]
sage: M(HWL(Composition([])))
M[]
sage: HWL(M([1,1]))
HWL[1, 1]
sage: HWL(M(Composition([2])))
HWL[1, 1] - 2*HWL[2]
sage: HWL(M([1]))
HWL[1]
```

**product_on_basis(I, J)**

The product on Hazewinkel Lambda basis elements.

The product of the basis elements indexed by two compositions $I$ and $J$ is the basis element obtained by concatenating the Lyndon factorizations of the words $I$ and $J$, then reordering the Lyndon factors in nonincreasing order, and finally concatenating them in this order (giving a new composition).

**INPUT:**

- $I, J$ – compositions

**OUTPUT:**

- The product of the Hazewinkel Lambda quasi-symmetric functions indexed by $I$ and $J$, expressed in the Hazewinkel Lambda basis.

**EXAMPLES:**
sage: HWL = QuasiSymmetricFunctions(QQ).HazewinkelLambda()
sage: c1 = Composition([1, 2, 1])
sage: c2 = Composition([2, 1, 3, 2])
sage: HWL.product_on_basis(c1, c2)
HWL[2, 1, 3, 2, 1, 2, 1]
sage: HWL.product_on_basis(c1, Composition([]))
HWL[1, 2, 1]
sage: HWL.product_on_basis(Composition([]), Composition([]))
HWL[]

M
alias of Monomial
class Monomial(QSym)
Bases: CombinatorialFreeModule, BindableClass
The Hopf algebra of quasi-symmetric function in the Monomial basis.
EXAMPLES:

sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.M()
sage: F = QSym.F()
sage: M(F[2,2])
sage: m = SymmetricFunctions(QQ).m()
sage: M(m[3,1,1])
M[1, 1, 3] + M[1, 3, 1] + M[3, 1, 1]
sage: (1+M[1])^3

The following is an alias for this basis:

sage: QSym.Monomial()
Quasisymmetric functions over the Rational Field in the Monomial basis
class Element
Bases: IndexedFreeModuleElement
Element methods of the Monomial basis of QuasiSymmetricFunctions.
expand(n, alphabet='x')
Expand the quasi-symmetric function written in the monomial basis in \( n \) variables.

INPUT:
- \( n \) – an integer
- alphabet – (default: 'x') a string

OUTPUT:
- The quasi-symmetric function self expressed in the \( n \) variables described by alphabet.

Todo: accept an alphabet as input

EXAMPLES:
One can use a different set of variables by using the optional argument alphabet:

```python
sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: M[2,1,1].expand(4, alphabet='y')
y0^2*y1*y2 + y0^2*y1*y3 + y0^2*y2*y3 + y1^2*y2*y3
```

is_symmetric()

Determine if a quasi-symmetric function, written in the Monomial basis, is symmetric.

This is being tested by looking at the expansion in the Monomial basis and checking if the coefficients are the same if the indexing compositions are permutations of each other.

OUTPUT:

- True if self is an element of the symmetric functions and False otherwise.

EXAMPLES:

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.Monomial()
sage: (M[3,2] + M[2,3] + M[4,1]).is_symmetric()
False
sage: (M[3,2] + M[2,3]).is_symmetric()
True
sage: (M[1,2,1] + M[1,1,2]).is_symmetric()
False
sage: (M[1,2,1] + M[1,1,2] + M[2,1,1]).is_symmetric()
True
```

psi_involution()

Return the image of the quasisymmetric function self under the involution $\psi$.

The involution $\psi$ is defined as the linear map $QSym \to QSym$ which, for every composition $I$, sends the fundamental quasisymmetric function $F_I$ to $F_{I^c}$, where $I^c$ denotes the complement of the composition $I$. The map $\psi$ is an involution and a graded Hopf algebra automorphism of $QSym$. Its restriction to the ring of symmetric functions coincides with the omega automorphism of the latter ring.

The involution $\psi$ of $QSym$ is adjoint to the involution $\psi$ of $NCSF$ by the standard adjunction between $NCSF$ and $QSym$.

The involution $\psi$ has been denoted by $\chi$ in [LMvW13], section 3.6.

See also:

- psi involution on QSym
- psi involution on NCSF
- star involution on QSym

EXAMPLES:

```python
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: M = QSym.M()
sage: M[3,2].psi_involution()
sage: M[3,1].psi_involution()
```

(continues on next page)
This particular implementation is tailored to the monomial basis. It is semantically equivalent to the generic implementation it overshadows:

```
sage: F = QSym.F()
sage: all( F(M[I].psi_involution()) == F(M[I]).psi_involution()
    ....:     for I in Compositions(3) )
True
sage: F = QSym.F()
sage: all( F(M[I].psi_involution()) == F(M[I]).psi_involution()
    ....:     for I in Compositions(4) )
True
```

```
to_symmetric_function()
```

Take a quasi-symmetric function, expressed in the monomial basis, and return its symmetric realization, when possible, expressed in the monomial basis of symmetric functions.

**OUTPUT:**
- If self is a symmetric function, then the expansion in the monomial basis of the symmetric functions is returned. Otherwise an error is raised.

**EXAMPLES:**

```
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: M = QSym.Monomial()
sage: (M[3, 2] + M[2, 3] + M[4, 1]).to_symmetric_function()
Traceback (most recent call last):
...  
m[3, 2] + 2*m[4, 1]
sage: m = SymmetricFunctions(QQ).m()
sage: M(m[3, 1, 1]).to_symmetric_function()
m[3, 1, 1]
sage: (M(m[2, 1])^2*m([2, 1])).to_symmetric_function()-m[2, 1]^2*m[2, 1]
0
```

```
antipode_on_basis(compo)
```

Return the result of the antipode applied to a quasi-symmetric Monomial basis element.

**INPUT:**
- compo – composition

**OUTPUT:**
- The result of the antipode applied to the composition compo, expressed in the Monomial basis.
EXAMPLES:
```
sage: M = QuasiSymmetricFunctions(QQ).M()
sage: M.antipode_on_basis(Composition([2,1]))
M[1, 2] + M[3]
sage: M.antipode_on_basis(Composition([]))
M[]
```

coproduct_on_basis(compo)

Return the coproduct of a Monomial basis element.

Combinatorial rule: deconcatenation.

INPUT:
- compo – composition

OUTPUT:
- The coproduct applied to the Monomial quasi-symmetric function indexed by compo, expressed in the Monomial basis.

EXAMPLES:
```
sage: M = QuasiSymmetricFunctions(QQ).M()
sage: M[coproduct_on_basis(Composition([]))]
M[]
```
dual()

Return the dual basis to the Monomial basis. This is the complete basis of the non-commutative symmetric functions.

OUTPUT:
- The complete basis of the non-commutative symmetric functions.

EXAMPLES:
```
sage: M = QuasiSymmetricFunctions(QQ).M()
sage: M.dual()
Non-Commutative Symmetric Functions over the Rational Field in the Complete basis
```

lambda_of_monomial(I, n)

Return the image of the monomial quasi-symmetric function $M_I$ under the lambda-map $\lambda^n$, expanded in the monomial basis.

The ring of quasi-symmetric functions over the integers, $\text{QSym}_\mathbb{Z}$ (and more generally, the ring of quasi-symmetric functions over any binomial ring) becomes a $\lambda$-ring (with the $\lambda$-structure inherited from the ring of formal power series, so that $\lambda^i(x_j)$ is $x_j$ if $i = 1$ and $0$ if $i > 1$).

The Adams operations of this $\lambda$-ring are the Frobenius endomorphisms $f_n$ (see `frobenius()` for their definition). Using these endomorphisms, the $\lambda$-operations can be explicitly computed via the formula

$$\exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} f_n(x) t^n\right) = \sum_{j=0}^{\infty} (-1)^j \lambda^j(x) t^j$$

in the ring of formal power series in a variable $t$ over the ring of quasi-symmetric functions. In partic-
ular, every composition \( I = (I_1, I_2, \cdots, I_\ell) \) satisfies
\[
\exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} M(nI_1, nI_2, \cdots, nI_\ell) t^n \right) = \sum_{j=0}^{\infty} (-1)^j \lambda^j(M_I) t^j
\]
(corrected version of Remark 2.4 in [Haz2004]).

The quasi-symmetric functions \( \lambda^j(M_I) \) with \( n \) ranging over the positive integers and \( I \) ranging over the reduced Lyndon compositions (i.e., compositions which are Lyndon words and have the gcd of their entries equal to 1) form a set of free polynomial generators for \( \text{QSym} \). See [GriRei18], Chapter 6, for the proof, and [Haz2004] for a major part of it.

**INPUT:**
- \( I \) – composition
- \( n \) – nonnegative integer

**OUTPUT:**

The quasi-symmetric function \( \lambda^n(M_I) \), expanded in the monomial basis over the ground ring of \( \text{self} \).

**EXAMPLES:**

```python
sage: M = QuasiSymmetricFunctions(CyclotomicField()).Monomial()
sage: M.lambda_of_monomial([1, 2], 2)
sage: M = QuasiSymmetricFunctions(Integers(19)).Monomial()
sage: all( M.lambda_of_monomial(I, 1) == M(I) for I in Compositions(3) )
True
```

The map \( \lambda^0 \) sends everything to 1:

```python
sage: M = QuasiSymmetricFunctions(ZZ).Monomial()
sage: all( M.lambda_of_monomial(I, 0) == M.one() for I in Compositions(3) )
True
```

The map \( \lambda^1 \) is the identity map:

```python
sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: all( M.lambda_of_monomial(I, 1) == M(I) for I in Compositions(3) )
True
```

(continues on next page)
sage: M = QuasiSymmetricFunctions(ZZ).Monomial()
sage: all( M.lambda_of_monomial(I, 1) == M(I) 
    ....:     for I in Compositions(3) )
True

product_on_basis(I, J)

The product on Monomial basis elements.

The product of the basis elements indexed by two compositions I and J is the sum of the basis elements indexed by compositions in the shuffle product (also called the overlapping shuffle product) of I and J.

INPUT:
• I, J – compositions

OUTPUT:
• The product of the Monomial quasi-symmetric functions indexed by I and J, expressed in the Monomial basis.

EXAMPLES:

sage: M = QuasiSymmetricFunctions(QQ).Monomial()
sage: c1 = Composition([2])
sage: c2 = Composition([1,3])
sage: M.product_on_basis(c1, c2)
sage: M.product_on_basis(c1, Composition([]))
M[2]

QS

alias of Quasisymmetric_Schur

class Quasisymmetric_Schur(QSym)

    Bases: CombinatorialFreeModule, BindableClass

    The Hopf algebra of quasi-symmetric function in the Quasisymmetric Schur basis.

    The basis of Quasisymmetric Schur functions is defined in [QSCHUR] and in Definition 5.1.1 of [LMvW13]. Don’t mistake them for the completely unrelated quasi-Schur functions of [NCSF1]!

    EXAMPLES:

sage: QSym = QuasiSymmetricFunctions(QQ)
sage: QS = QSym.QS()
sage: F = QSym.F()
sage: M = QSym.M()
sage: F(QS[1,2])
F[1, 2]
sage: M(QS[1,2])
M[1, 1, 1] + M[1, 2]
sage: s = SymmetricFunctions(QQ).s()
sage: QS(s[2,1,1])
QS[1, 1, 1] + QS[1, 2, 1] + QS[2, 1, 1]

dual()

    The dual basis to the Quasisymmetric Schur basis.

    The dual basis to the Quasisymmetric Schur basis is implemented as dual.
OUTPUT:
  • the dual Quasisymmetric Schur basis of the non-commutative symmetric functions

EXAMPLES:

```
sage: QS = QuasiSymmetricFunctions(QQ).Quasisymmetric_Schur()
sage: QS.dual()
Non-Commutative Symmetric Functions over the Rational Field
in the dual Quasisymmetric-Schur basis
```

**YQS**
alias of *Young_Quasisymmetric_Schur*

class *Young_Quasisymmetric_Schur* (*QSym*)

Bases: `CombinatorialFreeModule`, `BindableClass`

The Hopf algebra of quasi-symmetric functions in the Young Quasisymmetric Schur basis.

The basis of Young Quasisymmetric Schur functions is from Definition 5.2.1 of [LMvW13].

This basis is related to the Quasisymmetric Schur basis *QS* by

\[ QS(\alpha.reversed()) == YQS(\alpha).star_involution() \]

EXAMPLES:

```
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: YQS = QSym.YQS()
sage: F = QSym.F()
sage: QS = QSym.QS()
sage: F(YQS[1,2])
F[1, 2]
sage: all(QS(al.reversed()) == YQS(al).star_involution() for al in Compositions(5))
True
sage: s = SymmetricFunctions(QQ).s()
sage: YQS(s[2,1,1])
YQS[1, 1, 2] + YQS[1, 2, 1] + YQS[2, 1, 1]
```

**a_realization()**

Return the realization of the Monomial basis of the ring of quasi-symmetric functions.

**OUTPUT:**
  • The Monomial basis of quasi-symmetric functions.

**EXAMPLES:**

```
sage: QuasiSymmetricFunctions(QQ).a_realization()
Quasisymmetric functions over the Rational Field in the Monomial basis
```

**dI**
alias of *dualImmaculate*

dual()

Return the dual Hopf algebra of the quasi-symmetric functions, which is the non-commutative symmetric functions.

**OUTPUT:**
  • The non-commutative symmetric functions.
EXAMPLES:

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: QSym.dual()
Non-Commutative Symmetric Functions over the Rational Field
```

```python
class dualImmaculate(QSym)
    Bases: CombinatorialFreeModule, BindableClass

The dual immaculate basis of the quasi-symmetric functions.

This basis first appears in [BBSSZ2012].
```

```python
sage: dI = QSym.dI()
sage: dI([1,3,2])*dI([1])  # long time (6s on sage.math, 2013)
dI[1, 1, 3, 2] + dI[2, 3, 2]
sage: dI([1,3])*dI([1,1])  # long time (7s on sage.math, 2013)
sage: F = QSym.F()
sage: F(dI(F([2,1,3])))
F[2, 1, 3]
```

```python
from_polynomial(f, check=True)

Return the quasi-symmetric function in the Monomial basis corresponding to the quasi-symmetric polynomial f.

INPUT:

- f – a polynomial in finitely many variables over the same base ring as self. It is assumed that this polynomial is quasi-symmetric.
- check – boolean (default: True), checks whether the polynomial is indeed quasi-symmetric.

OUTPUT:

- quasi-symmetric function in the Monomial basis
```

```python
sage: P = PolynomialRing(QQ, 'x', 3)
sage: x = P.gens()
sage: f = x[0] + x[1] + x[2]
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: QSym.from_polynomial(f)
M[1]
```

Beware of setting check=False:
To expand the quasi-symmetric function in a basis other than the Monomial basis, the following shorthands are provided:

```
sage: M = QSym.Monomial()
sage: f = x[0]**2+x[1]**2+x[2]**2
sage: g = M.from_polynomial(f); g
M[2]
sage: F = QSym.Fundamental()
sage: F(g)
-F[1, 1] + F[2]
sage: F.from_polynomial(f)
-F[1, 1] + F[2]
```

class phi(QSym)
Bases: CombinatorialFreeModule, BindableClass

The Hopf algebra of quasi-symmetric functions in the \( \phi \) basis.

The \( \phi \) basis is defined as a rescaled Hopf dual of the \( \Phi \) basis of the non-commutative symmetric functions (see Section 3.1 of [BDHM2017]), where the pairing is

\[
(\phi_I, \Phi_J) = z_I \delta_{I,J},
\]

where \( z_I = \frac{1^{m_1} m_1! 2^{m_2} m_2! \cdots} {sp_{I,J} M_J} \)

where \( sp_{I,J} \) is as defined in [NCSF].

The \( \phi \)-basis is well-defined only when the base ring is a \( \mathbb{Q} \)-algebra.

EXAMPLES:

```
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: phi = QSym.phi(); phi
Quasisymmetric functions over the Rational Field in the phi basis
sage: phi.an_element()
2*phi[] + 2*phi[1] + 3*phi[1, 1]
sage: p = SymmetricFunctions(QQ).p()
sage: phi(p[2,2,1])
phi[1, 2, 2] + phi[2, 1, 2] + phi[2, 2, 1]
sage: all(sum(phi(list(al)) for al in Permutations(la))==phi(p(la)) for la in Partitions(6))
True
```

(continues on next page)
Checking the equivalent definition of $\phi_n$:

```python
sage: def test_phi(n):
    phi = QuasiSymmetricFunctions(QQ).phi()
    Phi = NonCommutativeSymmetricFunctions(QQ).Phi()
    M = matrix([[phi[I].duality_pairing(Phi[J])
                 for J in Compositions(n)]
                 for I in Compositions(n)])
    def z(J):
        return J.to_partition().centralizer_size()
    return M == matrix.diagonal([z(I) for I in Compositions(n)])
sage: all(test_phi(k) for k in range(1,5))
True
```

class psi(QSym)

    Bases: CombinatorialFreeModule, BindableClass

The Hopf algebra of quasi-symmetric functions in the $\psi$ basis.

The $\psi$ basis is defined as a rescaled Hopf dual of the $\Psi$ basis of the non-commutative symmetric functions (see Section 3.1 of [BDHMN2017]), where the pairing is

$$(\psi_I, \Psi_J) = z_I \delta_{I,J}$$

where $z_I = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ with $m_i$ being the multiplicity of $i$ in the composition $I$. Therefore, we call these the quasi-symmetric power sums of the first kind.

Using the duality, we can directly define the $\psi$ basis by

$$\psi_I = \sum_{J \succ I} z_J / \pi_{I,J} M_J,$$

where $\pi_{I,J}$ is as defined in [NCSF].

The $\psi$-basis is well-defined only when the base ring is a $\mathbb{Q}$-algebra.

EXAMPLES:

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
```

```python
sage: psi = QSym.psi(); psi
Quasisymmetric functions over the Rational Field in the psi basis
```

```python
sage: psi.an_element()
2*psi[] + 2*psi[1] + 3*psi[1, 1]
```

```python
sage: p = SymmetricFunctions(QQ).p()
```

```python
sage: psi(p[2, 2, 1])
psi[1, 2, 2] + psi[2, 1, 2] + psi[2, 2, 1]
```

```python
sage: all(sum(psi(list(al)) for al in Permutations(la))==psi(p(la)) for la in Partitions(6))
True
```

```python
sage: p = SymmetricFunctions(QQ).p()
```

```python
sage: psi(p[3, 2, 2])
psi[2, 2, 3] + psi[2, 3, 2] + psi[3, 2, 2]
```
Checking the equivalent definition of $\psi_n$:

```python
sage: def test_psi(n):
    psi = QuasiSymmetricFunctions(QQ).psi()
    Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
    M = matrix([[psi[I].duality_pairing(Psi[J])
                 for I in Compositions(n)]
                for J in Compositions(n)])
    def z(J):
        return J.to_partition().centralizer_size()
    return M == matrix.diagonal([z(I) for I in Compositions(n)])

sage: all(test_psi(k) for k in range(1,5))
True
```

5.1.146 Introduction to Quasisymmetric Functions

In this document we briefly explain the quasisymmetric function bases and related functionality in Sage. We assume the reader is familiar with the package `SymmetricFunctions`.

Quasisymmetric functions, denoted $QSym$, form a subring of the power series ring in countably many variables. $QSym$ contains the symmetric functions. These functions first arose in the theory of $P$-partitions. The initial ideas in this field are attributed to MacMahon, Knuth, Kremer, Glänfridw Thomas, Stanley. In 1984, Gessel formalized the study of quasisymmetric functions and introduced the basis of fundamental quasisymmetric functions [Ges]. In 1995, Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon showed that the ring of quasisymmetric functions is Hopf dual to the noncommutative symmetric functions [NCSF]. Many results have built on these.

One advantage of working in $QSym$ is that many interesting families of symmetric functions have explicit expansions in fundamental quasisymmetric functions such as Schur functions [Ges], Macdonald polynomials [HHL05], and plethysm of Schur functions [LW12].

For more background see Wikipedia article Quasisymmetric_function.

To begin, initialize the ring. Below we chose to use the rational numbers $\mathbb{Q}$. Other options include the integers $\mathbb{Z}$ and $\mathbb{C}$:

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: QSym
Quasisymmetric functions over the Rational Field

sage: QSym = QuasiSymmetricFunctions(CC); QSym
Quasisymmetric functions over the Complex Field with 53 bits of precision

sage: QSym = QuasiSymmetricFunctions(ZZ); QSym
Quasisymmetric functions over the Integer Ring
```

All bases of $QSym$ are indexed by compositions e.g. $[3,1,1,4]$. The convention is to use capital letters for bases of $QSym$ and lowercase letters for bases of the symmetric functions $Sym$. Next set up names for the known bases by running `inject_shorthands()`. As with symmetric functions, you do not need to run this command and you could assign these bases other names.

```python
sage: QSym = QuasiSymmetricFunctions(QQ)
sage: QSym.inject_shorthands()
Defining M as shorthand for Quasisymmetric functions over the Rational Field in the
    Monomial basis
Defining F as shorthand for Quasisymmetric functions over the Rational Field in the
    Fundamental basis
(continues on next page)
```
Now one can start constructing quasisymmetric functions.

**Note:** It is best to use variables other than \( M \) and \( F \).

```sage
sage: x = M[2,1] + M[1,2]
sage: x
M[1, 2] + M[2, 1]
sage: y = 3*M[1,2] + M[3]^2; y
sage: F[3,1,3] + 7*F[2,1]
7*F[2, 1] + F[3, 1, 3]
F[1, 2, 2, 1] + F[1, 2, 3] + 2*F[1, 3, 2] + F[1, 4, 1] + F[1, 5] + 3*F[2, 1, 2]
```

To convert from one basis to another is easy:

```sage
sage: z = M[1,2,1]
sage: z
M[1, 2, 1]
sage: F(z)
-F[1, 1, 1, 1] + F[1, 2, 1]
sage: M(F(z))
M[1, 2, 1]
```

To expand in variables, one can specify a finite size alphabet \( x_1, x_2, \ldots, x_m \):

```sage
sage: y = M[1,2,1]
sage: y.expand(4)
x0*x1^2*x2 + x0*x1^2*x3 + x0*x2^2*x3 + x1*x2^2*x3
```

The usual methods on free modules are available such as coefficients, degrees, and the support:
sage: z = 3*M[1,2]+M[3]^2; z
sage: z.coefficient([1,2])
3
sage: z.degree()
6
sage: sorted(z.coefficients())
[[1, 2], [3, 3], [6]]
sage: sorted(z.monomials(), key=lambda x: tuple(x.support()))
[[M[1, 2], M[3, 3], M[6]]]
sage: z.monomial_coefficients()
{[1, 2]: 3, [3, 3]: 2, [6]: 1}

As with the symmetric functions package, the quasisymmetric function 1 has several instantiations. However, the most obvious way to write 1 leads to an error (this is due to the semantics of python):
sage: M[]
M[]
sage: M.one()
M[]
sage: M(1)
M[]
sage: M[] == 1
True
sage: M[]
Traceback (most recent call last):
  ...
  SyntaxError: invalid ...

Working with symmetric functions

The quasisymmetric functions are a ring which contains the symmetric functions as a subring. The Monomial quasisymmetric functions are related to the monomial symmetric functions by \( m_\lambda = \sum_\text{sort}(c) = \lambda M_c \), where sort(c) means the partition obtained by sorting the composition c:

sage: SymmetricFunctions(QQ).inject_shorthands()
Defining e as shorthand for Symmetric Functions over Rational Field in the elementary basis
Defining f as shorthand for Symmetric Functions over Rational Field in the forgotten basis
Defining h as shorthand for Symmetric Functions over Rational Field in the homogeneous basis
Defining m as shorthand for Symmetric Functions over Rational Field in the monomial basis
Defining p as shorthand for Symmetric Functions over Rational Field in the powersum basis
Defining s as shorthand for Symmetric Functions over Rational Field in the Schur basis
sage: m[2,1]
There are methods to test if an expression \( f \) in the quasisymmetric functions is a symmetric function:

\[
\begin{align*}
\text{sage: } & f = M[1,1,2] + M[1,2,1] \\
\text{sage: } & f\text{.is_symmetric()} \\
& \text{False} \\
\text{sage: } & f = M[3,1] + M[1,3] \\
\text{sage: } & f\text{.is_symmetric()} \\
& \text{True}
\end{align*}
\]

If \( f \) is symmetric, there are methods to convert \( f \) to an expression in the symmetric functions:

\[
\begin{align*}
\text{sage: } & f\text{.to_symmetric_function()} \\
& m[3, 1]
\end{align*}
\]

The expansion of the Schur function in terms of the Fundamental quasisymmetric functions is due to [Ges]. There is one term in the expansion for each standard tableau of shape equal to the partition indexing the Schur function.

\[
\begin{align*}
\text{sage: } & f\text{.is_symmetric()} \\
& \text{True} \\
\text{sage: } & f\text{.to_symmetric_function()} \\
& 5*m[1, 1, 1, 1, 1] + 3*m[2, 1, 1, 1] + 2*m[2, 2, 1] + m[3, 1, 1] + m[3, 2] \\
\text{sage: } & s(f\text{.to_symmetric_function()}()) \\
& s[3, 2]
\end{align*}
\]

It is also possible to convert any symmetric function to the quasisymmetric function expansion in any known basis. The converse is not true:

\[
\begin{align*}
\text{sage: } & M( m[3,1,1] ) \\
\text{sage: } & F( s[2,2,1] ) \\
& F[1, 1, 2, 1] + F[1, 2, 1, 1] + F[1, 2, 2] + F[2, 1, 2] + F[2, 2, 1] \\
\text{sage: } & s(M[2,1]) \\
& \text{Traceback (most recent call last):} \\
& \text{...} \\
& \text{TypeError: do not know how to make } x (= M[2, 1]) \text{ an element of self}
\end{align*}
\]

It is possible to experiment with the quasisymmetric function expansion of other bases, but it is important that the base ring be the same for both algebras.

\[
\begin{align*}
\text{sage: } & R = QQ[\{'t\'}] \\
\text{sage: } & Qp = SymmetricFunctions(R).hall_littlewood().Qp() \\
\text{sage: } & QSymt = QuasiSymmetricFunctions(R) \\
\text{sage: } & Ft = QSymt.F() \\
\text{sage: } & Ft( Qp[2,2] ) \\
\end{align*}
\]
K = QQ['q','t'].fraction_field()
sage: Ht = SymmetricFunctions(K).macdonald().Ht()
sage: Fqt = QuasiSymmetricFunctions(Ht.base_ring()).F()
sage: Fqt(Ht[2,1])
q*t*F[1, 1, 1] + (q+t)*F[1, 2] + (q+t)*F[2, 1] + F[3]

The following will raise an error because the base ring of F is not equal to the base ring of Ht:

F(Ht[2,1])

Traceback (most recent call last):
...
TypeError: do not know how to make x (= McdHt[2, 1]) an element of self (=Quasisymmetric_functions over the Rational Field in the Fundamental basis)

QSym is a Hopf algebra

The product on QSym is commutative and is inherited from the product by the realization within the polynomial ring:

M[3]*M[1,1] == M[1,1]*M[3]
True


There is a coproduct on this ring as well, which in the Monomial basis acts by cutting the composition into a left half and a right half. The co-product is non-co-commutative:

M[1,3,1].coproduct()


The Duality Pairing with Non-Commutative Symmetric Functions

These two operations endow QSym with the structure of a Hopf algebra. It is the dual Hopf algebra of the non-commutative symmetric functions NCSF. Under this duality, the Monomial basis of QSym is dual to the Complete basis of NCSF, and the Fundamental basis of QSym is dual to the Ribbon basis of NCSF (see [MR]):
Let $H$ and $G$ be elements of $QS\! ym$ and $h$ an element of $NCSF$. Then if we represent the duality pairing with the mathematical notation $[\cdot,\cdot]$, we have:

$$[H \cdot G, h] = [H \otimes G, \Delta(h)].$$

For example, the coefficient of $\mathbb{M}[2,1,4,1]$ in $\mathbb{M}[1,3]^* \mathbb{M}[2,1,1]$ may be computed with the duality pairing:

```sage
sage: I, J = Composition([1,3]), Composition([2,1,1])
sage: (M[I]*M[J]).duality_pairing(S[2,1,4,1])
1
```

And the coefficient of $S[1,3] \# S[2,1,1]$ in $S[2,1,4,1].\text{coproduct}()$ is equal to this result:

```sage
sage: S[2,1,4,1].coproduct()
S[] # S[2, 1, 4, 1] + ... + S[1, 3] # S[2, 1, 1] + ... + S[4, 1] # S[2, 1]
```

The duality pairing on the tensor space is another way of getting this coefficient, but currently the method
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The operation \texttt{skew_by()} is not defined on the tensor squared space. However, we can extend this functionality by applying a linear morphism to the terms in the coproduct, as follows:

```sage
sage: X = S[2,1,4,1].coproduct()
sage: def linear_morphism(x, y):
    ....:    return x.duality_pairing(M[1,3]) * y.duality_pairing(M[2,1,1])
sage: X.apply_multilinear_morphism(linear_morphism, codomain=ZZ)
1
```

Similarly, if \( H \) is an element of \( QSym \) and \( g \) and \( h \) are elements of \( NCSF \), then

\[
[H, g \cdot h] = [\Delta(H), g \otimes h].
\]

For example, the coefficient of \( R[2,3,1] \) in \( R[2,1]^*R[2,1] \) is computed with the duality pairing by the following command:

```sage
sage: (R[2,1]^*R[2,1]).duality_pairing(F[2,3,1])
1
sage: R[2,1]^*R[2,1]
R[2, 1, 2, 1] + R[2, 3, 1]
```

This coefficient should then be equal to the coefficient of \( F[2,1] \ # F[2,1] \) in \( F[2,3,1].coproduct() \):

```sage
sage: F[2,3,1].coproduct()
F[] # F[2, 3, 1] + ... + F[2, 1] # F[2, 1] + ... + F[2, 3, 1] # F[]
```

This can also be computed by the duality pairing on the tensor space, as above:

```sage
sage: X = F[2,3,1].coproduct()
sage: def linear_morphism(x, y):
    ....:    return x.duality_pairing(R[2,1]) * y.duality_pairing(R[2,1])
sage: X.apply_multilinear_morphism(linear_morphism, codomain=ZZ)
1
```

### The Operation Adjoint to Multiplication by a Non-Commutative Symmetric Function

Let \( g \in NCSF \) and consider the linear endomorphism of \( NCSF \) defined by left (respectively, right) multiplication by \( g \). Since there is a duality between \( QSym \) and \( NCSF \), this linear transformation induces an operator \( g^\perp \) on \( QSym \) satisfying

\[
[g^\perp(H), h] = [H, g \cdot h].
\]

for any non-commutative symmetric function \( h \).

This is implemented by the method \texttt{skew_by()}. Explicitly, if \( H \) is a quasisymmetric function and \( g \) a non-commutative symmetric function, then \( H.skew_by(g) \) and \( H.skew_by(g, \text{side}='right') \) are expressions that satisfy, for any non-commutative symmetric function \( h \), the following identities:

\[
H.skew_by(g).duality_pairing(h) = H.duality_pairing(g*h)
H.skew_by(g, \text{side}='right').duality_pairing(h) = H.duality_pairing(h*g)
\]

For example, \( M[J].skew_by(S[I]) \) is 0 unless the composition \( J \) begins with \( I \) and \( M(J).skew_by(S(I), \text{side}='right') \) is 0 unless the composition \( J \) ends with \( I \):
The antipode

The antipode sends the Fundamental basis element indexed by the composition \( I \) to \(-1\) to the size of \( I \) times the Fundamental basis element indexed by the conjugate composition to \( I \):

```python
sage: F[3,2,2].antipode()
-F[1, 2, 2, 1, 1]
sage: Composition([3,2,2]).conjugate()
[1, 2, 2, 1, 1]
sage: M[3,2,2].antipode()
```

We demonstrate here the defining relation of the antipode:

```python
sage: X = F[3,2,2].coproduct()
sage: X.apply_multilinear_morphism(lambda x,y: x*y.antipode())
0
sage: X.apply_multilinear_morphism(lambda x,y: x.antipode()*y)
0
```

REFERENCES:

5.1.147 Symmetric functions in non-commuting variables

- Introduction to Symmetric Functions in Non-Commuting Variables
- Bases for NCSym
- Dual Symmetric Functions in Non-Commuting Variables
- Symmetric Functions in Non-Commuting Variables

```
sage: M[3,2,2].skew_by(S[3])
M[2, 2]
sage: M[3,2,2].skew_by(S[2])
0
sage: M[3,2,2].coproduct().apply_multilinear_morphism(lambda x,y: x.duality_˓→pairing(S[3])*y)
M[2, 2]
sage: M[3,2,2].skew_by(S[3], side='right')
0
sage: M[3,2,2].skew_by(S[2], side='right')
M[3, 2]
```


5.1.148 Bases for $\textit{NCSym}$

AUTHORS:

- Travis Scrimshaw (08-04-2013): Initial version

```python
class sage.combinat.ncsym.bases.MultiplicativeNCSymBases(parent_with_realization):
    Bases: Category_realization_of_parent

    Category of multiplicative bases of symmetric functions in non-commuting variables.

    A multiplicative basis is one for which $b_A b_B = b_{A|B}$ where $A|B$ is the pipe() operation on set partitions.

    EXAMPLES:

    sage: from sage.combinat.ncsym.bases import MultiplicativeNCSymBases
    sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
    sage: MultiplicativeNCSymBases(NCSym)
    Category of multiplicative bases of symmetric functions in non-commuting variables over the Rational Field

class ElementMethods
    Bases: object

class ParentMethods
    Bases: object

    product_on_basis(A, B)

    The product on basis elements.

    The product on a multiplicative basis is given by $b_A \cdot b_B = b_{A|B}$.

    The bases $\{e, h, x, cp, p, chi, rho\}$ are all multiplicative.

    INPUT:
    - $A, B$ – set partitions

    OUTPUT:
    - an element in the basis self

    EXAMPLES:

    sage: e = SymmetricFunctionsNonCommutingVariables(QQ).e()
    sage: h = SymmetricFunctionsNonCommutingVariables(QQ).h()
    sage: x = SymmetricFunctionsNonCommutingVariables(QQ).x()
    sage: cp = SymmetricFunctionsNonCommutingVariables(QQ).cp()
    sage: p = SymmetricFunctionsNonCommutingVariables(QQ).p()
    sage: chi = SymmetricFunctionsNonCommutingVariables(QQ).chi()
    sage: rho = SymmetricFunctionsNonCommutingVariables(QQ).rho()
    sage: A = SetPartition([[1], [2, 3]])
    sage: B = SetPartition([[1], [3], [2, 4]])
    sage: e.product_on_basis(A, B)
    e{{1}, {2, 3}, {4}, {5, 7}, {6}}
    sage: h.product_on_basis(A, B)
    h{{1}, {2, 3}, {4}, {5, 7}, {6}}
    sage: x.product_on_basis(A, B)
    x{{1}, {2, 3}, {4}, {5, 7}, {6}}
    sage: cp.product_on_basis(A, B)
    cp{{1}, {2, 3}, {4}, {5, 7}, {6}}
    sage: p.product_on_basis(A, B)
```

(continues on next page)
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\begin{verbatim}
(continued from previous page)

\texttt{p}\{\{1\}, \{2, 3\}, \{4\}, \{5, 7\}, \{6\}
\texttt{sage: chi.product_on_basis(A, B)}
\texttt{chi}\{\{1\}, \{2, 3\}, \{4\}, \{5, 7\}, \{6\}
\texttt{sage: rho.product_on_basis(A, B)}
\texttt{rho}\{\{1\}, \{2, 3\}, \{4\}, \{5, 7\}, \{6\}
\texttt{sage: e.product_on_basis(A,B)==e(h(e(A))*h(e(B)))}
\texttt{True}
\texttt{sage: h.product_on_basis(A,B)==h(x(h(A))*x(h(B)))}
\texttt{True}
\texttt{sage: x.product_on_basis(A,B)==x(h(x(A))*h(x(B)))}
\texttt{True}
\texttt{sage: cp.product_on_basis(A,B)==cp(p(cp(A))*p(cp(B)))}
\texttt{True}
\texttt{sage: p.product_on_basis(A,B)==p(e(p(A))*e(p(B)))}
\texttt{True}

\texttt{super_categories()}

Return the super categories of bases of the Hopf dual of the symmetric functions in non-commuting variables.

\textbf{OUTPUT:}

\begin{itemize}
  \item a list of categories
\end{itemize}

\textbf{class} \texttt{sage.combinat.ncsym.bases.NCSymBases}(\texttt{parent_with_realization})

Bases: \texttt{Category\_realization\_of\_parent}

Category of bases of symmetric functions in non-commuting variables.

\textbf{EXAMPLES:}

\begin{verbatim}
\texttt{sage: from sage.combinat.ncsym.bases import NCSymBases}
\texttt{sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)}
\texttt{sage: NCSymBases(NCSym)}
\texttt{Category of bases of symmetric functions in non-commuting variables over the Rational Field}
\end{verbatim}

\textbf{class} \texttt{ElementMethods}

Bases: \texttt{object}

\texttt{expand}(\texttt{n, alphabet='x'})

Expand the symmetric function into \texttt{n} non-commuting variables in an alphabet, which by default is 'x'.

This computation is completed by coercing the element \texttt{self} into the monomial basis and computing the expansion in the alphabet there.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{n} – the number of variables in the expansion
  \item \texttt{alphabet} – (default: 'x') the alphabet in which \texttt{self} is to be expanded
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item an expansion of \texttt{self} into the \texttt{n} non-commuting variables specified by \texttt{alphabet}
\end{itemize}

\textbf{EXAMPLES:}

\end{verbatim}
\end{verbatim}
sage: h = SymmetricFunctionsNonCommutingVariables(QQ).h()
sage: h[[1,3],[2]].expand(3)
2*x0^3 + x0^2*x1 + x0^2*x2 + 2*x0*x1*x0 + x0*x1^2 + x0*x1*x2 + 2*x0*x2*x0 +
  x0*x2*x1 + x0*x2*x2 + x1*x0^2 + 2*x1*x0*x1 + x1*x0*x2 + x1^2*x0 + 2*x1^2 +
  x1^2*x2 + x1*x2*x0 + 2*x1*x2*x1 + x1*x2*x2 + x2*x0^2 + x2*x0*x1 + x2*x0*x2 +
  x2*x1*x0 + x2*x1*x2 + 2*x2*x1*x0 + x2*x1^2 + 2*x2*x1*x2 + x2*x1^2 + x2*x2*x0 +
  x2*x2*x1 + x2*x2*x2 + x2*x2*x1 + 2*x2*x2
sage: x = SymmetricFunctionsNonCommutingVariables(QQ).x()
sage: x[[1,3],[2]].expand(3)
-x0^2*x1 - x0^2*x2 - x0*x1^2 - x0*x1*x2 - x0*x2*x1 - x0*x2^2 - x1*x0^2 -
  x1*x0*x2 - x1*x0^2*x0 - x1*x2*x2 - x1*x2*x0 - x1*x2*x1 - x1*x2^2 - x2*x0^2 -
  x2*x0*x1 - x2*x1*x0 - x2*x1^2 - x2*x1*x2 - x2*x1^2 - x2*x2*x0 - x2*x2*x1 +
  x2*x2*x2

internal_coproduct()

Return the internal coproduct of self.

The internal coproduct is defined on the power sum basis as

\[ P_A \mapsto P_A \otimes P_A \]

and the map is extended linearly.

OUTPUT:

• an element of the tensor square of the basis of self

EXAMPLES:

sage: x = SymmetricFunctionsNonCommutingVariables(QQ).x()
sage: x[[1,3],[2]].internal_coproduct()
x{{{1}, {2}, {3}}} \# x{{{1}, {3}, {2}}} + x{{{1}, {3}, {2}}} \# x{{{1}, {2}, {3}}} +
  x{{{1}, {3}, {2}}} \# x{{{1}, {3}, {2}}}

omega()

Return the involution \( \omega \) applied to self.

The involution \( \omega \) is defined by

\[ e_A \mapsto h_A \]

and the result is extended linearly.

OUTPUT:

• an element in the same basis as self

EXAMPLES:

sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: m = NCSym.m()
sage: m[[1,3],[2]].omega()
-2*m{{{1}, {2}, {3}}} - m{{{1}, {3}, {2}}}
sage: p = NCSym.p()
sage: p[[1,3],[2]].omega()
-p{{{1}, {3}, {2}}}
sage: cp = NCSym.cp()
sage: cp[[1,3],[2]].omega()
-2*cp{{{1}, {2}, {3}}} - cp{{{1}, {3}, {2}}}
sage: x = NCSym.x()

(continues on next page)
Combinatorics, Release 10.0

sage: x[[1,3],[2]].omega()
\[-2*x\{1\}, \{2\}, \{3\} - x\{1\}, \{3\}, \{2\}\]

to_symmetric_function()

Compute the projection of an element of symmetric function in non-commuting variables to the symmetric functions.

The projection of a monomial symmetric function in non-commuting variables indexed by the set partition \(A\) is defined as

\[
m_A \mapsto m_{\lambda(A)} \prod_i n_i(\lambda(A))!
\]

where \(\lambda(A)\) is the partition associated with \(A\) by taking the sizes of the parts and \(n_i(\mu)\) is the multiplicity of \(i\) in \(\mu\). For other bases this map is extended linearly.

OUTPUT:

• an element of the symmetric functions in the monomial basis

EXAMPLES:

sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: e = NCSym.e()
sage: h = NCSym.h()
sage: p = NCSym.p()
sage: cp = NCSym.cp()
sage: x = NCSym.x()
sage: cp[[1,3],[2]].to_symmetric_function()
m[2, 1]
sage: x[[1,3],[2]].to_symmetric_function()
-6*m[1, 1, 1] - 2*m[2, 1]
sage: e[[1,3],[2]].to_symmetric_function()
2*e[2, 1]
sage: h[[1,3],[2]].to_symmetric_function()
2*h[2, 1]
sage: p[[1,3],[2]].to_symmetric_function()
p[2, 1]

to_wqsym()

Return the image of self under the canonical inclusion map \(NCSym \to WQSym\).

The canonical inclusion map \(NCSym \to WQSym\) is an injective homomorphism of algebras. It sends a basis element \(m_A\) of \(NCSym\) to the sum of basis elements \(M_P\) of \(WQSym\), where \(P\) ranges over all ordered set partitions that become \(A\) when the ordering is forgotten. This map is denoted by \(\theta\) in \([BZ05]\) (17).

See also:

WordQuasiSymmetricFunctions for a definition of \(WQSym\).

EXAMPLES:

sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: e = NCSym.e()
sage: h = NCSym.h()
sage: p = NCSym.p()
Combinatorics, Release 10.0

```
sage: cp = NCSym.cp()
sage: x = NCSym.x()
sage: m = NCSym.m()
sage: m[[1,3],[2]].to_wqsym()
M[[1, 3], {2}] + M[{2}, {1, 3}]
sage: x[[1,3],[2]].to_wqsym()
-M[{1}, {2}, {3}] - M[{1}, {2, 3}] - M[{1}, {3}, {2}]
- M[{1, 2}, {3}] - M[{2}, {1}, {3}] - M[{2}, {3}, {1}]
- M[{2, 3}, {1}] - M[{3}, {1}, {2}] - M[{3}, {1, 2}]
- M[{3}, {2}, {1}]
sage: (4*p[[1,3],[2]]-p[[1]]).to_wqsym()
-M[{1}] + 4*M[{1, 2, 3}] + 4*M[{1, 3}, {2}] + 4*M[{2}, {1, 3}]
```

class ParentMethods
Bases: object

```
from_symmetric_function() /
```

Return the image of the symmetric function \( f \) in \( self \).

This is performed by converting to the monomial basis and extending the method
\texttt{sum_of_partitions()} linearly. This is a linear map from the symmetric functions to the
symmetric functions in non-commuting variables that does not preserve the product or coproduct
structure of the Hopf algebra.

See also:

\texttt{to_symmetric_function()}

INPUT:
- \( f \) – a symmetric function

OUTPUT:
- an element of \( self \)

EXAMPLES:

```
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: Sym = SymmetricFunctions(QQ)
sage: e = NCSym.e()
sage: elem = Sym.e()
sage: elt = e.from_symmetric_function(elem[[2,1,1]]); elt
1/12*e{{1}, {2}, {3, 4}} + 1/12*e{{1}, {2, 3}, {4}} + 1/12*e{{1}, {2, 4},
\rightarrow{3}}
+ 1/12*e{{1, 2}, {3}, {4}} + 1/12*e{{1, 3}, {2}, {4}} + 1/12*e{{1, 4}, {2},
\rightarrow{3}}
sage: elem(elt.to_symmetric_function())
e[2, 1, 1]
sage: e.from_symmetric_function(elem[[4]])
1/24*e{{1, 2, 3, 4}}
sage: p = NCSym.p()
sage: pow = Sym.p()
sage: elt = p.from_symmetric_function(pow[[2,1,1]]); elt
1/6*p{{1}, {2}, {3, 4}} + 1/6*p{{1}, {2, 3}, {4}} + 1/6*p{{1}, {2, 4}, {3}}
+ 1/6*p{{1, 2}, {3}, {4}} + 1/6*p{{1, 3}, {2}, {4}} + 1/6*p{{1, 4}, {2},
\rightarrow{3}}
sage: pow(elt.to_symmetric_function())
```

(continues on next page)
p[2, 1, 1]

sage: p.from_symmetric_function(pow[4])
p{{1, 2, 3, 4}}
)
sage: h = NCSym.h()sage: comp = Sym.complete()sage: elt = h.from_symmetric_function(comp[2, 1, 1]); elt1/12*h{{1}, {2}, {3, 4}} + 1/12*h{{1}, {2, 3}, {4}} + 1/12*h{{1}, {2, 4}, →{3}} + 1/12*h{{1, 2}, {3}, {4}} + 1/12*h{{1, 3}, {2}, {4}} + 1/12*h{{1, 4}, {2}, → {3}}sage: comp(elt.to_symmetric_function())h[2, 1, 1]sage: h.from_symmetric_function(comp[4])1/24*h{{1, 2, 3, 4}}

internal_coproduct()

Compute the internal coproduct of self.

If internal_coproduct_on_basis() is available, construct the internal coproduct morphism from self to self⊗self by extending it by linearity. Otherwise, this uses internal_coproduct_by_coercion(), if available.

OUTPUT:

• an element of the tensor squared of self

EXAMPLES:

sage: cp = SymmetricFunctionsNonCommutingVariables(QQ).cp()sage: cp.internal_coproduct(cp[[1,3],[2]] - 2*cp[[1]])-2*cp{{1}} # cp{{1}} + cp{{1, 2, 3}} # cp{{1, 3}, {2}} + cp{{1, 3}, {2}} #␣ → cp{{1, 2, 3}}+ cp{{1, 3}, {2}} # cp{{1, 3}, {2}}

internal_coproduct_by_coercion(x)

Return the internal coproduct by coercing the element to the powersum basis.

INPUT:

• x – an element of self

OUTPUT:

• an element of the tensor squared of self

EXAMPLES:

sage: h = SymmetricFunctionsNonCommutingVariables(QQ).h()sage: h[[1,3],[2]].internal_coproduct() # indirect doctest2*h{{1}, {2}, {3}} # h{{1}, {2}, {3}} - h{{1}, {2}, {3}} # h{{1, 3}, {2}}- h{{1, 3}, {2}} # h{{1}, {2}, {3}} + h{{1, 3}, {2}} # h{{1, 3}, {2}}

internal_coproduct_on_basis(i)

The internal coproduct of the algebra on the basis (optional).

INPUT:

• i – the indices of an element of the basis of self

OUTPUT:

• an element of the tensor squared of self

EXAMPLES:
```python
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m.internal_coproduct_on_basis(SetPartition([[1,2]]))
```

**primitive**(A, \(i=1\))

Return the primitive associated to \(A\) in \(self\).

**See also:**

primitive()

**INPUT:**

- \(A\) – a set partition
- \(i\) – a positive integer

**OUTPUT:**

- an element of \(self\)

**EXAMPLES:**

```python
sage: e = SymmetricFunctionsNonCommutingVariables(QQ).e()
sage: elt = e.primitive(SetPartition([[1,3],[2]])); elt
ek{{1, 2}, {3}} - e{{1, 3}, {2}}
sage: elt.coproduct()
e{} # e{{1, 2}, {3}} - e{} # e{{1, 3}, {2}} + e{{1, 2}, {3}} # e{} - e{{1, \rightarrow 3}, {2}} # e{}
```

**super_categories**()

Return the super categories of bases of the Hopf dual of the symmetric functions in non-commuting variables.

**OUTPUT:**

- a list of categories

```python
class sage.combinat.ncsym.bases.NCSymBasis_abstract(R, basis_keys=None, element_class=None, category=None, prefix=None, names=None, **kwds)

Bases: CombinatorialFreeModule, BindableClass

Abstract base class for a basis of \(NCSym\) or its dual.

class sage.combinat.ncsym.bases.NCSymDualBases(parent_with_realization)

Bases: Category_realization_of_parent

Category of bases of dual symmetric functions in non-commuting variables.

**EXAMPLES:**

```python
sage: from sage.combinat.ncsym.bases import NCSymDualBases
sage: DNCSym = SymmetricFunctionsNonCommutingVariables(QQ).dual()
sage: NCSymDualBases(DNCSym)
Category of bases of dual symmetric functions in non-commuting variables over the Rational Field
```

**super_categories**()

Return the super categories of bases of the Hopf dual of the symmetric functions in non-commuting variables.

**OUTPUT:**

- a list of categories
• a list of categories

class sage.combinat.ncsym.bases.NCSymOrNCSymDualBases(parent_with_realization)

Bases: Category_realization_of_parent

Base category for the category of bases of symmetric functions in non-commuting variables or its Hopf dual for the common code.

class ElementMethods

Bases: object
duality_pairing(other)

Compute the pairing between self and an element other of the dual.

EXAMPLES:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: m = NCSym.m()
sage: w = m.dual_basis()
sage: elt = m[[1,3],[2]] - 3*m[[1,2],[3]]
sage: elt.duality_pairing(w[[1,3],[2]])
1
sage: elt.duality_pairing(w[[1,2],[3]])
-3
sage: elt.duality_pairing(w[[1,2]])
0
sage: e = NCSym.e()
sage: w[[1,3],[2]].duality_pairing(e[[1,3],[2]])
0
```

class ParentMethods

Bases: object
counit_on_basis(A)

The counit is defined by sending all elements of positive degree to zero.

INPUT:
• A -- a set partition

OUTPUT:
• either the 0 or the 1 of the base ring of self

EXAMPLES:

```python
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m.counit_on_basis(SetPartition([[1,3], [2]]))
0
sage: m.counit_on_basis(SetPartition([[]))
1
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w.counit_on_basis(SetPartition([[1,3], [2]]))
0
sage: w.counit_on_basis(SetPartition([[]))
1
```

duality_pairing(x, y)

Compute the pairing between an element of self and an element of the dual.

Carry out this computation by converting x to the m basis and y to the w basis.
INPUT:
• x – an element of symmetric functions in non-commuting variables
• y – an element of the dual of symmetric functions in non-commuting variables

OUTPUT:
• an element of the base ring of self

EXAMPLES:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: h = NCSym.h()
sage: w = NCSym.m().dual_basis()
sage: matrix([[h(A).duality_pairing(w(B)) for A in SetPartitions(3)] for B in SetPartitions(3)])
\begin{bmatrix}
6 & 2 & 2 & 2 & 1 \\
2 & 2 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
sage: (h[[1,2],[3]] + 3*h[[1,3],[2]]).duality_pairing(2*w[[1,3],[2]] + w[[1,2,3]] + 2*w[[1,2],[3]])
32
```

duality_pairing_matrix(basis, degree)
The matrix of scalar products between elements of NCSym and elements of NCSym*.

INPUT:
• basis – a basis of the dual Hopf algebra
• degree – a non-negative integer

OUTPUT:
• the matrix of scalar products between the basis self and the basis basis in the dual Hopf algebra of degree degree

EXAMPLES:
The matrix between the m basis and the w basis:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: m = NCSym.m()
sage: w = NCSym.dual().w()
sage: m.duality_pairing_matrix(w, 3)
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
sage: e = NCSym.e()
sage: e.duality_pairing_matrix(w, 3)
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
sage: p = NCSym.p()
sage: p.duality_pairing_matrix(w, 3)
```

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A base case test:

```
sage: m.duality_pairing_matrix(w, 0)
[1]
```

**one_basis()**

Return the index of the basis element containing 1.

**OUTPUT:**

• The empty set partition

**EXAMPLES:**

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m.one_basis()
{}
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w.one_basis()
{}
```

**super_categories()**

Return the super categories of bases of (the Hopf dual of) the symmetric functions in non-commuting variables.

**OUTPUT:**

• a list of categories
5.1.149 Dual Symmetric Functions in Non-Commuting Variables

AUTHORS:

- Travis Scrimshaw (08-04-2013): Initial version

```python
class sage.combinat.ncsym.dual.SymmetricFunctionsNonCommutingVariablesDual(R):
    Bases: UniqueRepresentation, Parent

    The Hopf dual to the symmetric functions in non-commuting variables.
    See Section 2.3 of [BZ05] for a study.

    a_realization()
    Return the realization of the w basis of self.

    EXAMPLES:

    sage: SymmetricFunctionsNonCommutingVariables(QQ).dual().a_realization()
    Dual symmetric functions in non-commuting variables over the Rational Field in the w basis

    dual()
    Return the dual Hopf algebra of the dual symmetric functions in non-commuting variables.

    EXAMPLES:

    sage: NCSymD = SymmetricFunctionsNonCommutingVariables(QQ).dual()
sage: NCSymD.dual()
    Symmetric functions in non-commuting variables over the Rational Field

```

class w(NCSymD)
Bases: NCSymBasis_abstract

The Hopf algebra of symmetric functions in non-commuting variables in the w basis.

EXAMPLES:

```python
sage: NCSymD = SymmetricFunctionsNonCommutingVariables(QQ).dual()
sage: w = NCSymD.w()

We have the embedding \( \chi^* \) of Sym into \( NCSym^* \) available as a coercion:

```python
sage: h = SymmetricFunctions(QQ).h()
sage: w(h[[2,1]])
\[ w\{\{1\}, \{2, 3\}\} + w\{\{1, 2\}, \{3\}\} + w\{\{1, 3\}, \{2\}\} \]
```

Similarly we can pull back when we are in the image of \( \chi^* \):

```python
sage: elt = 3*(w[[1],[2,3]] + w[[1,2],[3]] + w[[1,3],[2]])
sage: h(elt)
3*h[2, 1]
```

class Element
    Bases: IndexedFreeModuleElement

An element in the w basis.
expand($n$, letter='x')

Expand self written in the w basis in $n^2$ commuting variables which satisfy the relation $x_{ij}x_{ik} = 0$ for all $i, j,$ and $k$.

The expansion of an element of the w basis is given by equations (26) and (55) in [HNT06].

INPUT:
• $n$ – an integer
• letter – (default: 'x') a string

OUTPUT:
• The symmetric function of self expressed in the $n \times n$ non-commuting variables described by letter.

REFERENCES:

EXAMPLES:

sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w[[1,3],[2]].expand(4)
x02*x11*x20 + x03*x11*x30 + x03*x22*x30 + x13*x22*x31

One can use a different set of variable by using the optional argument letter:

sage: w[[1,3],[2]].expand(3, letter='y')
y02*y11*y20

is_symmetric()

Determine if a $NCsym^*$ function, expressed in the w basis, is symmetric.

A function $f$ in the w basis is a symmetric function if it is in the image of $\chi^*$. That is to say we have

$$f = \sum_{\lambda} c_\lambda \prod_i m_i(\lambda)! \sum_{\lambda(A) = \lambda} w_A$$

where the second sum is over all set partitions $A$ whose shape $\lambda(A)$ is equal to $\lambda$ and $m_i(\mu)$ is the multiplicity of $i$ in the partition $\mu$.

OUTPUT:
• True if $\lambda(A) = \lambda(B)$ implies the coefficients of $w_A$ and $w_B$ are equal, False otherwise

EXAMPLES:

sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: elt = w.sum_of_partitions([2,1,1])
sage: elt.is_symmetric()
True
sage: elt -= 3*w.sum_of_partitions([1,1])
sage: elt.is_symmetric()
True
sage: w = SymmetricFunctionsNonCommutingVariables(ZZ).dual().w()
sage: elt = w.sum_of_partitions([2,1,1]) / 2
sage: elt.is_symmetric()
False
sage: elt = w[[1,3],[2]]
sage: elt.is_symmetric()
False
sage: elt = w[[1,2],[3]] + w[[1,2],[3]] + 2*w[[1,3],[2]]

(continues on next page)
to_symmetric_function()

Take a function in the \( w \) basis, and return its symmetric realization, when possible, expressed in the homogeneous basis of symmetric functions.

OUTPUT:
• If \( \text{self} \) is a symmetric function, then the expansion in the homogeneous basis of the symmetric functions is returned. Otherwise an error is raised.

EXAMPLES:

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: elt = w[[1],[2,3]] + w[[1,2],[3]] + w[[1,3],[2]]
sage: elt.to_symmetric_function()
h[2, 1]
sage: elt = w.sum_of_partitions([2,1,1]) / 2
sage: elt.to_symmetric_function()
1/2*h[2, 1, 1]
```

antipode_on_basis(A)

Return the antipode applied to the basis element indexed by \( A \).

INPUT:
• \( A \) – a set partition

OUTPUT:
• an element in the basis \( \text{self} \)

EXAMPLES:

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w.antipode_on_basis(SetPartition([[1],[2,3]]))
-3*w{{1}, {2}, {3}} + w{{1}} # w{{1, 2}} + w{{1}, {2}} # w{{1}} + w{{1}, {2, 3}} # w{}
```

coproduct_on_basis(A)

Return the coproduct of a \( w \) basis element.

The coproduct on the basis element \( w_A \) is the sum over tensor product terms \( w_B \otimes w_C \) where \( B \) is the restriction of \( A \) to \( \{1, 2, \ldots, k\} \) and \( C \) is the restriction of \( A \) to \( \{k + 1, k + 2, \ldots, n\} \).

INPUT:
• \( A \) – a set partition

OUTPUT:
• The coproduct applied to the \( w \) dual symmetric function in non-commuting variables indexed by \( A \) expressed in the \( w \) basis.

EXAMPLES:

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w[[1, [2,3]]].coproduct()
w{} # w{{1}, {2, 3}} + w{{1}} # w{{1, 2}}
+ w{{1}, {2}} # w{{1}} + w{{1}, {2, 3}} # w{}
```

(continues on next page)
dual_basis()

Return the dual basis to the w basis.

The dual basis to the w basis is the monomial basis of the symmetric functions in non-commuting variables.

OUTPUT:

• the monomial basis of the symmetric functions in non-commuting variables

EXAMPLES:

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w.dual_basis()
Symmetric functions in non-commuting variables over the Rational Field in
  the monomial basis
```

duality_pairing(x, y)

Compute the pairing between an element of self and an element of the dual.

INPUT:

• x – an element of the dual of symmetric functions in non-commuting variables
• y – an element of the symmetric functions in non-commuting variables

OUTPUT:

• an element of the base ring of self

EXAMPLES:

```python
sage: DNCSym = SymmetricFunctionsNonCommutingVariablesDual(QQ)
sage: w = DNCSym.w()
sage: m = w.dual_basis()
sage: matrix([[w(A).duality_pairing(m(B)) for A in SetPartitions(3)] for B in SetPartitions(3))
[[1 0 0 0 0]
 [0 1 0 0 0]
 [0 0 1 0 0]
 [0 0 0 1 0]
 [0 0 0 0 1]]
sage: (w[[1,2],[3]] + 3*w[[1,3],[2]]).duality_pairing(2*m[[1,3],[2]] + m[[1,2,3]] + 2*m[[1,2],[3]])
8
sage: h = SymmetricFunctionsNonCommutingVariables(QQ).h()
sage: matrix([[w(A).duality_pairing(h(B)) for A in SetPartitions(3)] for B in SetPartitions(3))
[[6 2 2 2 1]
 [2 2 1 1 1]
 [2 1 2 1 1]
 [2 1 1 2 1]
 [1 1 1 1 1]]
sage: (2*w[[1,3],[2]] + w[[1,2,3]] + 2*w[[1,2],[3]]).duality_pairing(h[[1,2,3]] + 3*h[[1,3],[2]])
32
```

product_on_basis(A, B)
The product on \( w \) basis elements.

The product on the \( w \) is the dual to the coproduct on the \( m \) basis. On the basis \( w \) it is defined as

\[
    w_A w_B = \sum_{S \subseteq [n]} w_{A \uparrow_S \cup B \uparrow_S^c}
\]

where the sum is over all possible subsets \( S \) of \([n]\) such that \(|S| = |A|\) with a term indexed the union of \( A \uparrow_S \) and \( B \uparrow_{S^c} \). The notation \( A \uparrow_S \) represents the unique set partition of the set \( S \) such that the standardization is \( A \). This product is commutative.

**INPUT:**
- \( A, B \) – set partitions

**OUTPUT:**
- an element of the \( w \) basis

**EXAMPLES:**

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: A = SetPartition([[1], [2,3]])
sage: B = SetPartition([[1, 2, 3]])
sage: w.product_on_basis(A, B)

w{{1}, {2, 3}, {4, 5, 6}} + w{{1}, {2, 3, 4}, {5, 6}}
+ w{{1}, {2, 3, 5}, {4, 6}} + w{{1}, {2, 3, 6}, {4, 5}}
+ w{{1}, {2, 4}, {3, 5, 6}} + w{{1}, {2, 4, 5}, {3, 6}}
+ w{{1}, {2, 4, 6}, {3, 5}} + w{{1}, {2, 5}, {3, 4, 6}}
+ w{{1}, {2, 5, 6}, {3, 4}} + w{{1}, {2, 6}, {3, 4, 5}}
+ w{{1, 2, 3}, {4}, {5, 6}} + w{{1, 2, 4}, {3}, {5, 6}}
+ w{{1, 2, 5}, {3}, {4, 6}} + w{{1, 2, 6}, {3}, {4, 5}}
+ w{{1, 3, 4}, {2}, {5, 6}} + w{{1, 3, 5}, {2}, {4, 6}}
+ w{{1, 3, 6}, {2}, {4, 5}} + w{{1, 4, 5}, {2}, {3, 6}}
+ w{{1, 4, 6}, {2}, {3, 5}} + w{{1, 5, 6}, {2}, {3, 4}}

sage: B = SetPartition([[1], [2]])
sage: w.product_on_basis(A, B)

3*w{{1}, {2}, {3}, {4, 5}} + 2*w{{1}, {2}, {3, 4}, {5}}
+ 2*w{{1}, {2}, {3, 5}, {4}} + w{{1}, {2}, {3, 6}}
+ w{{1}, {2, 4}, {3}, {5}} + w{{1}, {2, 3}, {4}, {5}}

sage: w.product_on_basis(A, SetPartition([]))

w{{1}, {2, 3}}
```

**sum_of_partitions(la)**

Return the sum over all set partitions whose shape is \( la \), scaled by \( \prod_i m_i! \) where \( m_i \) is the multiplicity of \( i \) in \( la \).

**INPUT:**
- \( la \) – an integer partition

**OUTPUT:**
- an element of self

**EXAMPLES:**

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w.sum_of_partitions([2,1,1])

2*w{{1}, {2}, {3, 4}} + 2*w{{1}, {2, 3}, {4}} + 2*w{{1}, {2, 4}, {3}}
+ 2*w{{1, 2}, {3}, {4}} + 2*w{{1, 3}, {2}, {4}} + 2*w{{1, 4}, {2}, {3}}
```

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to_symmetric_function()

The preimage of \( \chi^* \) in the w basis.

EXAMPLES:

```python
sage: w = SymmetricFunctionsNonCommutingVariables(QQ).dual().w()
sage: w.to_symmetric_function
Generic morphism:
  From: Dual symmetric functions in non-commuting variables over the
    Rational Field in the w basis
  To:  Symmetric Functions over Rational Field in the homogeneous basis
```

5.1.150 Symmetric Functions in Non-Commuting Variables

AUTHORS:

- Travis Scrimshaw (08-04-2013): Initial version

class sage.combinat.ncsym.ncsym.SymmetricFunctionsNonCommutingVariables(R)

    Bases: UniqueRepresentation, Parent

Symmetric functions in non-commutative variables.

The ring of symmetric functions in non-commutative variables, which is not to be confused with the non-commutative symmetric functions, is the ring of all bounded-degree noncommutative power series in countably many indeterminates (i.e., elements in \( R((x_1, x_2, x_3, \ldots)) \) of bounded degree) which are invariant with respect to the action of the symmetric group \( S_\infty \) on the indices of the indeterminates. It can be regarded as a direct limit over all \( n \to \infty \) of rings of \( S_n \)-invariant polynomials in \( n \) non-commuting variables (that is, \( S_n \)-invariant elements of \( R(x_1, x_2, \ldots, x_n) \)).

This ring is implemented as a Hopf algebra whose basis elements are indexed by set partitions.

Let \( A = \{A_1, A_2, \ldots, A_r\} \) be a set partition of the integers \( [k] := \{1, 2, \ldots, k\} \). This partition \( A \) determines an equivalence relation \( \sim_A \) on \( [k] \), which has \( c \sim_A d \) if and only if \( c \) and \( d \) are in the same part \( A_j \) of \( A \). The monomial basis element \( m_A \) indexed by \( A \) is the sum of monomials \( x_{i_1} x_{i_2} \cdots x_{i_k} \) such that \( i_c = i_d \) if and only if \( c \sim_A d \).

The \( k \)-th graded component of the ring of symmetric functions in non-commutative variables has its dimension equal to the number of set partitions of \( [k] \). (If we work, instead, with finitely many – say, \( n \) – variables, then its dimension is equal to the number of set partitions of \( [k] \) where the number of parts is at most \( n \).)

Note: All set partitions are considered standard (i.e., set partitions of \( [n] \) for some \( n \)) unless otherwise stated.

REFERENCES:

EXAMPLES:

We begin by first creating the ring of \( NCSym \) and the bases that are analogues of the usual symmetric functions:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: m = NCSym.m()
sage: e = NCSym.e()
sage: h = NCSym.h()
sage: p = NCSym.p()
sage: m
```
Symmetric functions in non-commuting variables over the Rational Field in the monomial basis

The basis is indexed by set partitions, so we create a few elements and convert them between these bases:

```
sage: elt = m(SetPartition([[1,3],[2]])) - 2*m(SetPartition([[1],[2]])); elt
-2*m({1}, {2}) + m({1, 3}, {2})
sage: e(elt)
1/2*e({1, {2, 3}}) - 2*e({1, 2}) + 1/2*e({1, 2, {3}}) - 1/2*e({1, 2, 3}) - 1/2*e({1, 3}, {2})
sage: h(elt)
-4*h({1}, {2}) - 2*h({1, 2}) + 1/2*h({1, 2, 3}) + 3/2*h({1, 3}, {2})
sage: p(elt)
-2*p({1}, {2}) + 2*p({1, 2}) + p({1, 3}, {2})
sage: m(p(elt))
-2*m({1}, {2}) + m({1, 3}, {2})
sage: elt = p(SetPartition([[1,3],[2]])) - 4*p(SetPartition([[1],[2]])) + 2; elt
2*p() - 4*p({1}, {2}) + p({1, 3}, {2})
sage: e(elt)
2*e({1, 2}) - 4*e({1}, {2}) + e({1}, {2, 3}) - e({1, 3}, {2})
sage: m(elt)
2*m() - 4*m({1}, {2}) - 4*m({1, 2}) + m({1, 2, 3}) + m({1, 3}, {2})
sage: h(elt)
2*h() - 4*h({1, 2}) + 1/2*h({1, 2, 3}) + 3/2*h({1, 3}, {2})
sage: p(m(elt))
2*p() - 4*p({1}, {2}) + p({1, 3}, {2})
```

There is also a shorthand for creating elements. Note that we must use `p([])` to create the empty set partition due to python’s syntax.

```
sage: eltm = m([[1,3],[2]]) - 3*m([[1],[2]]); eltm
-3*m({1}, {2}) + m({1, 3}, {2})
sage: elte = e([[1,3],[2]]); elte
e(1, 3}, {2})
sage: elth = h([[1,3],[2,4]]); elth
h(1, 3}, {2, 4})
sage: eltp = p([[1,3],[2,4]] + 2*p([[1]]) - 4*p([]); eltp
-4*p() + 2*p({1}) + p({1, 3}, {2, 4})
```

There is also a natural projection to the usual symmetric functions by letting the variables commute. This projection map preserves the product and coproduct structure. We check that Theorem 2.1 of [RS06] holds:

```
sage: Sym = SymmetricFunctions(QQ)
sage: Sm = Sym.m()
sage: Se = Sym.e()
sage: Sh = Sym.h()
sage: Sp = Sym.p()
sage: eltm.to_symmetric_function()
-6*m[1, 1] + m[2, 1]
sage: Sm(p(eltm).to_symmetric_function())
-6*m[1, 1] + m[2, 1]
```
```python
sage: elte.to_symmetric_function()
2*e[2, 1]
sage: Se(h(elte).to_symmetric_function())
2*e[2, 1]
sage: elth.to_symmetric_function()
4*h[2, 2]
sage: Sh(m(elth).to_symmetric_function())
4*h[2, 2]
sage: eltp.to_symmetric_function()
-4*p[] + 2*p[1] + p[2, 2]
sage: Sp(e(eltp).to_symmetric_function())
-4*p[] + 2*p[1] + p[2, 2]
```

### a_realization()

Return the realization of the powersum basis of `self`.

**OUTPUT:**

- The powersum basis of symmetric functions in non-commuting variables.

**EXAMPLES:**

```python
sage: SymmetricFunctionsNonCommutingVariables(QQ).a_realization()
Symmetric functions in non-commuting variables over the Rational Field in the
powersum basis
```

### chi

alias of `supercharacter`

### class coarse_powersum(NCSym)

Bases: `NCSymBasis_abstract`

The Hopf algebra of symmetric functions in non-commuting variables in the `cp` basis.

This basis was defined in [BZ05] as

\[
\text{cp}_A = \sum_{A \leq B} m_B,
\]

where we sum over all strict coarsenings of the set partition $A$. An alternative description of this basis was given in [BT13] as

\[
\text{cp}_A = \sum_{A \subseteq B} m_B,
\]

where we sum over all set partitions whose arcs are a subset of the arcs of the set partition $A$.

**Note:** In [BZ05], this basis was denoted by $q$. In [BT13], this basis was called the powersum basis and denoted by $p$. However, it is a coarser basis than the usual powersum basis in the sense that it does not yield the usual powersum basis of the symmetric function under the natural map of letting the variables commute.

**EXAMPLES:**
Combinatorics, Release 10.0

sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: cp = NCSym.cp()
sage: cp[[1,3],[2,4]]*cp[[1,2,3]]
cp\{\{1, 2\}, \{3\}\} # cp\{\{1, 2\}, \{3\}\}
sage: ps = SymmetricFunctions(NCSym.base_ring()).p()
sage: ps(cp[[1,3],[2]].to_symmetric_function())
p\{1, 2\} - p\{3\}

\(\text{cp}\) (alias of \text{coarse_powersum})

class \text{deformed_coarse_powersum}(NCSym, q=2)

Bases: \text{NCSymBasis_abstract}

The Hopf algebra of symmetric functions in non-commuting variables in the \(\rho\) basis.

This basis was defined in [BT13] as a \(q\)-deformation of the \(cp\) basis:

\[
\rho_A = \sum_{A \subseteq B} \frac{1}{q^{n_{B-A}}} m_B,
\]

where we sum over all set partitions whose arcs are a subset of the arcs of the set partition \(A\).

INPUT:

- \(q\) – (default: 2) the parameter \(q\)

EXAMPLES:

sage: R = QQ['q'].fraction_field()
sage: q = R.gen()
sage: NCSym = SymmetricFunctionsNonCommutingVariables(R)
sage: rho = NCSym.rho(q)

We construct Example 3.1 in [BT13]:

sage: rnode = lambda A: sorted([a[1] for a in A.arcs()], reverse=True)
sage: dimv = lambda A: sorted([a[1]-a[0] for a in A.arcs()], reverse=True)
sage: lst = list(SetPartitions(4))
sage: S = sorted(lst, key=lambda A: (dimv(A), rnode(A)))
sage: m = NCSym.m()
sage: matrix([[m(rho[A])[B] for B in S] for A in S])

(continues on next page)
Return the deformation parameter $q$ of self.

**EXAMPLES:**

```
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: rho = NCSym.rho(5)
sage: rho.q()
5
sage: R = QQ['q'].fraction_field()
sage: q = R.gen()
sage: NCSym = SymmetricFunctionsNonCommutingVariables(R)
sage: rho = NCSym.rho(q)
sage: rho.q() == q
True
```

dual()

Return the dual Hopf algebra of the symmetric functions in non-commuting variables.

**EXAMPLES:**

```
sage: SymmetricFunctionsNonCommutingVariables(QQ).dual()
Dual symmetric functions in non-commuting variables over the Rational Field
```

e

**alias of** `elementary`

class `elementary`(NCSym)

**Bases:** `NCSymBasis_abstract`

The Hopf algebra of symmetric functions in non-commuting variables in the elementary basis.

**EXAMPLES:**

```
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: e = NCSym.e()
```

class `Element`

**Bases:** `IndexedFreeModuleElement`

An element in the elementary basis of `NCSym`.

**omega()**

Return the involution $\omega$ applied to `self`.

The involution $\omega$ on $NCSym$ is defined by $\omega(e_A) = h_A$.

**OUTPUT:**

- an element in the basis `self`
EXAMPLES:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: e = NCSym.e()
sage: h = NCSym.h()
sage: elt = e[[1,3],[2]].omega(); elt
2*e{{1}, {2}, {3}} - e{{1, 3}, {2}}
sage: elt.omega()
e{{1, 3}, {2}}
sage: h(elt)
h{{1, 3}, {2}}
```

to_symmetric_function()

The projection of self to the symmetric functions.

Take a symmetric function in non-commuting variables expressed in the \( \text{e} \) basis, and return the projection of expressed in the elementary basis of symmetric functions.

The map \( \chi: NCSym \to Sym \) is given by

\[
e_A \mapsto e_{\lambda(A)} \prod_i \lambda(A)_i!
\]

where \( \lambda(A) \) is the partition associated with \( A \) by taking the sizes of the parts.

OUTPUT:

• An element of the symmetric functions in the elementary basis

EXAMPLES:

```python
sage: e = SymmetricFunctionsNonCommutingVariables(QQ).e()
sage: e[[1,3],[2]].to_symmetric_function()
2*e[2, 1]
sage: e[[1],[3],[2]].to_symmetric_function()
e[1, 1, 1]
```

h

alias of homogeneous

class homogeneous(NCSym)

Bases: NCSymBasis_abstract

The Hopf algebra of symmetric functions in non-commuting variables in the homogeneous basis.

EXAMPLES:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: h = NCSym.h()
sage: h[[1,3],[2,4]]*h[[1,2,3]]
h{{1, 3}, {2, 4}, {5, 6, 7}}
sage: h[[1,2]].coproduct()
h{} # h{{1, 2}} + 2*h{{1}} # h{{1}} + h{{1, 2}} # h{}
```

class Element

Bases: IndexedFreeModuleElement

An element in the homogeneous basis of \( NCSym \).
**omega()**

Return the involution \( \omega \) applied to self.

The involution \( \omega \) on \( NCSym \) is defined by \( \omega(h_A) = e_A \).

**OUTPUT:**

- an element in the basis self

**EXAMPLES:**

```
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: h = NCSym.h()
sage: e = NCSym.e()
sage: elt = h[[1,3],[2]].omega(); elt
2*h{{1}, {2}, {3}} - h{{1, 3}, {2}}
sage: elt.omega()
h{{1, 3}, {2}}
sage: e(elt)
e{{1, 3}, {2}}
```

**to_symmetric_function()**

The projection of self to the symmetric functions.

Take a symmetric function in non-commuting variables expressed in the \( h \) basis, and return the projection of expressed in the complete basis of symmetric functions.

The map \( \chi : NCSym \to Sym \) is given by

\[
h_A \mapsto h_{\lambda(A)} \prod_i \lambda(A)_i!
\]

where \( \lambda(A) \) is the partition associated with \( A \) by taking the sizes of the parts.

**OUTPUT:**

- An element of the symmetric functions in the complete basis

**EXAMPLES:**

```
sage: h = SymmetricFunctionsNonCommutingVariables(QQ).h()
sage: h[[1,3],[2]].to_symmetric_function()
2*h[2, 1]
sage: h[[1],[3],[2]].to_symmetric_function()
h[1, 1, 1]
```

**m**

alias of *monomial*

**class monomial(NCSym)**

**Bases:** *NCSymBasis_abstract*

The Hopf algebra of symmetric functions in non-commuting variables in the monomial basis.

**EXAMPLES:**

```
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: m = NCSym.m()
sage: m[[1,3],[2]]*m[[1,2]]
m{{1, 3}, {2}} + m{{1, 3}, {2, 4, 5}} + m{{1, 3, 4, 5}, {2}}
sage: m[[1,3],[2]].coproduct()
```

(continues on next page)
class Element

Bases: IndexedFreeModuleElement

An element in the monomial basis of $NCSym$.

expand($n$, alphabet='x')

Expand self written in the monomial basis in $n$ non-commuting variables.

INPUT:
• $n$ – an integer
• alphabet – (default: 'x') a string

OUTPUT:
• The symmetric function of self expressed in the $n$ non-commuting variables described by alphabet.

EXAMPLES:

sage: m = SymmetricFunctionsNonCommutingVariables(QQ).monomial()

sage: m[[1,3],[2]].expand(4)

x0*x1*x0 + x0*x2*x0 + x0*x3*x0 + x1*x0*x1 + x1*x2*x1 + x1*x3*x1
+ x2*x0*x2 + x2*x1*x2 + x2*x3*x2 + x3*x0*x3 + x3*x1*x3 + x3*x2*x3

One can use a different set of variables by using the optional argument alphabet:

sage: m[[1],[2,3]].expand(3,alphabet='y')
y0*y1^2 + y0*y2^2 + y1*y0^2 + y1*y2^2 + y2*y0^2 + y2*y1^2

to_symmetric_function()

The projection of self to the symmetric functions.

Take a symmetric function in non-commuting variables expressed in the m basis, and return the projection of expressed in the monomial basis of symmetric functions.

The map $\chi: NCSym \to Sym$ is defined by

$$m_A \mapsto m_{\lambda(A)} \prod_i n_i(\lambda(A))!$$

where $\lambda(A)$ is the partition associated with $A$ by taking the sizes of the parts and $n_i(\mu)$ is the multiplicity of $i$ in $\mu$.

OUTPUT:
• an element of the symmetric functions in the monomial basis

EXAMPLES:

sage: m = SymmetricFunctionsNonCommutingVariables(QQ).monomial()

sage: m[[1,3],[2]].to_symmetric_function()
m[1, 1]

sage: m[[1],[3],[2]].to_symmetric_function()
6*m[1, 1, 1]

coproduct_on_basis($A$)

Return the coproduct of a monomial basis element.

INPUT:
• A – a set partition

OUTPUT:
• The coproduct applied to the monomial symmetric function in non-commuting variables indexed by A expressed in the monomial basis.

EXAMPLES:

```python
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).monomial()
sage: m[[1, 3], [2]].coproduct()
m{} # m{{1, 3}, {2}} + m{{1}} # m{{1, 2}} + m{{1, 2}} # m{{1}} + m{{1, 3}, ˓→{2}} # m{}
sage: m.coproduct_on_basis(SetPartition([]))
m{} # m{}
sage: m.coproduct_on_basis(SetPartition([[1,2,3]]))
m{} # m{{1, 2, 3}} + m{{1, 2, 3}} # m{}
sage: m[[1,5],[2,4],[3,7],[6]].coproduct()
m{} # m{{1, 5}, {2, 4}, {3, 7}, {6}} + m{{1}} # m{{1, 5}, {2, 4}, {3, 6}} ˓→ 2*m{{1, 2}} # m{{1, 3}, {2, 5}, {4}} + m{{1, 2}} # m{{1, 4}, {2, 3}, {5}} ˓→ 2*m{{1, 2}, {3}} # m{{1, 3}, {2, 4}} + m{{1, 3}} # m{{1, 4}, {2}} + m{{1, 4}, {2, 3}} ˓→ 2*m{{1, 3}, {2, 4}} # m{{1, 2}, {3}} + 2*m{{1, 3}, {2, 4}} # m{{1, 3}, {2}} + 2*m{{1, 3}, {2, 4}, {5}} # m{{1, 3}, {2, 4}} + m{{1, 5}, {2, 4}, {3, 6}} # m{{1,} ˓→ 2}} + m{{1, 5}, {2, 4}, {3, 7}} # m{{1, 5}, {2, 4}, {3, 6}} + m{{1, 5}, {2, 4}, {3, 7}, {6}} # m{}
```

dual_basis()

Return the dual basis to the monomial basis.

OUTPUT:
• the \( w \) basis of the dual Hopf algebra

EXAMPLES:

```python
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m.dual_basis()
Dual symmetric functions in non-commuting variables over the Rational Field in the \( w \) basis
duality_pairing(x, y)

Compute the pairing between an element of self and an element of the dual.

INPUT:
• \( x \) – an element of symmetric functions in non-commuting variables
• \( y \) – an element of the dual of symmetric functions in non-commuting variables

OUTPUT:
• an element of the base ring of self

EXAMPLES:

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: m = NCSym.m()
sage: w = m.dual_basis()
sage: matrix([[m(A).duality_pairing(w(B)) for A in SetPartitions(3)] for B in SetPartitions(3))]
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```
from_symmetric_function(f)

Return the image of the symmetric function f in self.

This is performed by converting to the monomial basis and extending the method sum_of_partitions() linearly. This is a linear map from the symmetric functions to the symmetric functions in non-commuting variables that does not preserve the product or coproduct structure of the Hopf algebra.

See also:

to_symmetric_function()

INPUT:

• f – an element of the symmetric functions

OUTPUT:

• An element of the m basis

EXAMPLES:

sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: mon = SymmetricFunctions(QQ).m()
sage: elt = m.from_symmetric_function(mon[2,1,1]); elt
1/12*m{{1}, {2}, {3, 4}} + 1/12*m{{1}, {2, 3}, {4}} + 1/12*m{{1}, {2, 4}, \rightarrow {3}}
+ 1/12*m{{1, 2}, {3}, {4}} + 1/12*m{{1, 3}, {2}, {4}} + 1/12*m{{1, 4}, {2}, \rightarrow {3}}
sage: elt.to_symmetric_function()
m[2, 1, 1]
sage: e = SymmetricFunctionsNonCommutingVariables(QQ).e()
sage: elm = SymmetricFunctions(QQ).e()
sage: e(m.from_symmetric_function(elm[4]))
1/24*e{{1, 2, 3, 4}}
sage: h = SymmetricFunctionsNonCommutingVariables(QQ).h()
sage: hom = SymmetricFunctions(QQ).h()
sage: h(m.from_symmetric_function(hom[4]))
1/24*h{{1, 2, 3, 4}}
sage: p = SymmetricFunctionsNonCommutingVariables(QQ).p()
sage: pow = SymmetricFunctions(QQ).p()
sage: p(m.from_symmetric_function(pow[4]))
p{{1, 2, 3, 4}}
sage: p(m.from_symmetric_function(pow[2,1]))
1/3*p{{1}, {2, 3}} + 1/3*p{{1, 2}, {3}} + 1/3*p{{1, 3}, {2}}
sage: p([[1,2]])*p([[1]])
p{{1, 2}, {3}}

Check that χ ∘ ⃗χ is the identity on Sym:

sage: all((m.from_symmetric_function(pow(la)).to_symmetric_function() ==\n\npow(la))\n....: \n\nfor la in Partitions(4))

(continues on next page)
internal_coproduct_on_basis(A)

Return the internal coproduct of a monomial basis element.

The internal coproduct is defined by

\[ \Delta^\square(m_A) = \sum_{B \triangleleft C = A} m_B \otimes m_C \]

where we sum over all pairs of set partitions \( B \) and \( C \) whose infimum is \( A \).

INPUT:
• \( A \) – a set partition

OUTPUT:
• an element of the tensor square of the \( m \) basis

EXAMPLES:

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).monomial()
sage: m.internal_coproduct_on_basis(SetPartition([[1,3],[2]]))
m{{1, 2, 3}} # m{{1, 3}, {2}} + m{{1, 3}, {2}} # m{{1, 2, 3}} + m{{1, 3}, \rightarrow{2}} # m{{1, 3}, {2}}
```

product_on_basis(A, B)

The product on monomial basis elements.

The product of the basis elements indexed by two set partitions \( A \) and \( B \) is the sum of the basis elements indexed by set partitions \( C \) such that \( C \land ([n][k]) = A|B \) where \( n = |A| \) and \( k = |B| \). Here \( A \land B \) is the infimum of \( A \) and \( B \) and \( A|B \) is the SetPartition.pipe() operation. Equivalently we can describe all \( C \) as matchings between the parts of \( A \) and \( B \) where if \( a \in A \) is matched with \( b \in B \), we take \( a \cup b \) instead of \( a \) and \( b \) in \( C \).

INPUT:
• \( A, B \) – set partitions

OUTPUT:
• an element of the \( m \) basis

EXAMPLES:

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).monomial()
sage: A = SetPartition([[1],[2,3]])
sage: B = SetPartition([[1],[3],[2,4]])
sage: m.product_on_basis(A, B)
m{{1}, {2, 3}, {4}, {5}, {6}} + m{{1}, {2, 3, 4}, {5}} + m{{1}, {2, 3, 5}, {4}, {6}} + m{{1}, {2, 3, 6}, {4}, {5}} + m{{1}, {2, 3, 4, 5}, {6}} + m{{1}, {2, 3, 5, 6}, {4}} + m{{1}, {2, 3, 6, 5}, {4}} + m{{1}, {2, 3, 4, 6}, {5}} + m{{1}, {2, 3, 5, 6}, {4}} + m{{1}, {2, 3, 6, 5}, {4}} + m{{1}, {2, 3, 4, 6}, {5}} + m{{1}, {2, 3, 5, 6}, {4}} + m{{1}, {2, 3, 6, 5}, {4}} + m{{1}, {2, 3, 4, 6}, {5}}
sage: B = SetPartition([[1],[2]])
sage: m.product_on_basis(A, B)
m{{1}, {2, 3}, {4}, {5}} + m{{1}, {2, 3, 4}, {5}} + m{{1}, {2, 3, 5}, {4}} + m{{1}, {2, 3, 6}, {5}} + m{{1}, {2, 3, 4, 5}, {6}} + m{{1}, {2, 3, 5, 6}, {4}} + m{{1}, {2, 3, 6, 5}, {4}} + m{{1}, {2, 3, 4, 6}, {5}} + m{{1}, {2, 3, 5, 6}, {4}} + m{{1}, {2, 3, 6, 5}, {4}} + m{{1}, {2, 3, 4, 6}, {5}} + m{{1}, {2, 3, 5, 6}, {4}} + m{{1}, {2, 3, 6, 5}, {4}} + m{{1}, {2, 3, 4, 6}, {5}}
```
Combinatorics, Release 10.0

sage: m.product_on_basis(A, SetPartition([]))
m{{1}, {2, 3}}

**sum_of_partitions**\( (\lambda) \)

Return the sum over all set partitions whose shape is \( \lambda \) with a fixed coefficient \( C \) defined below.

Fix a partition \( \lambda \), we define \( \lambda! := \prod_i \lambda_i! \) and \( \lambda^! := \prod_i m_i! \). Recall that \( |\lambda| = \sum_i \lambda_i \) and \( m_i \) is the number of parts of length \( i \) of \( \lambda \). Thus we defined the coefficient as

\[
C := \frac{\lambda! \lambda^!}{|\lambda|!}.
\]

Hence we can define a lift \( \bar{\chi} \) from \( Sym \) to \( NCSym \) by

\[
m_\lambda \mapsto C \sum_A m_A
\]

where the sum is over all set partitions whose shape is \( \lambda \).

**INPUT:**

- \( \lambda \) – an integer partition

**OUTPUT:**

- an element of the \( m \) basis

**EXAMPLES:**

```python
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m.sum_of_partitions(Partition([2, 1, 1]))
1/12*m{{1}, {2}, {3, 4}} + 1/12*m{{1}, {2, 3}, {4}} + 1/12*m{{1}, {2, 4}, ->{3}}
+ 1/12*m{{1, 2}, {3}, {4}} + 1/12*m{{1, 3}, {2}, {4}} + 1/12*m{{1, 4}, {2}, ->{3}}
```

\( p \)

alias of powersum

**class powersum**\( (NCSym) \)

Bases: \( NCSymBasis_abstract \)

The Hopf algebra of symmetric functions in non-commuting variables in the powersum basis.

The powersum basis is given by

\[
p_A = \sum_{A \leq B} m_B,
\]

where we sum over all coarsenings of the set partition \( A \). If we allow our variables to commute, then \( p_A \) goes to the usual powersum symmetric function \( p_\lambda \) whose (integer) partition \( \lambda \) is the shape of \( A \).

**EXAMPLES:**

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: p = NCSym.p()
sage: x = p.an_element()**2; x
4*p[] + 8*p{1} + 4*p{1, 1} + 12*p{2} + 12*p{2, 1} + 9*p{2, 2}
```
class Element
Bases: IndexedFreeModuleElement

An element in the powersum basis of NCSym.

to_symmetric_function()

The projection of self to the symmetric functions.

Take a symmetric function in non-commuting variables expressed in the \( p \) basis, and return the projection of expressed in the powersum basis of symmetric functions.

The map \( \chi: NCSym \to Sym \) is given by

\[
P_A \mapsto p_{\lambda(A)}
\]

where \( \lambda(A) \) is the partition associated with \( A \) by taking the sizes of the parts.

OUTPUT:
• an element of symmetric functions in the power sum basis

EXAMPLES:

sage: p = SymmetricFunctionsNonCommutingVariables(QQ).p()
sage: p[[1,3],[2]].to_symmetric_function()
p[2, 1]
sage: p[[1],[3],[2]].to_symmetric_function()
p[1, 1, 1]

antipode_on_basis(A)

Return the result of the antipode applied to a powersum basis element.

Let \( A \) be a set partition. The antipode given in [LM2011] is

\[
S(p_A) = \sum_{\gamma} (-1)^{\ell(\gamma)} p_{\gamma[A]}
\]

where we sum over all ordered set partitions (i.e. set compositions) of \( [\ell(A)] \) and

\[
\gamma[A] = A_1^{\ell(A)} | \cdots | A_{\ell(A)}^{\ell(A)}
\]

is the action of \( \gamma \) on \( A \) defined in SetPartition.ordered_set_partition_action().

INPUT:
• \( A \) – a set partition

OUTPUT:
• an element in the basis self

EXAMPLES:

sage: p = SymmetricFunctionsNonCommutingVariables(QQ).powersum()
sage: p.antipode_on_basis(SetPartition([[1], [2,3]]))
p{{1, 2}, {3}}
sage: p.antipode_on_basis(SetPartition([]))
p{}
sage: F = p[[1,3],[5],[2,4]].coproduct()
sage: F.apply_multilinear_morphism(lambda x,y: x.antipode() * y)
0
**coproduct_on_basis**(A)

Return the coproduct of a monomial basis element.

**INPUT:**
- A – a set partition

**OUTPUT:**
- The coproduct applied to the monomial symmetric function in non-commuting variables indexed by A expressed in the monomial basis.

**EXAMPLES:**

```python
sage: p = SymmetricFunctionsNonCommutingVariables(QQ).powersum()
sage: p[SetPartition([[1,3],[2]])].coproduct()
sage: p[SetPartition([])].coproduct()
sage: p.coproduct_on_basis(SetPartition([[1],[2]]))
```

**internal_coproduct_on_basis**(A)

Return the internal coproduct of a powersum basis element.

The internal coproduct is defined by

\[ \Delta^\otimes(p_A) = p_A \otimes p_A \]

**INPUT:**
- A – a set partition

**OUTPUT:**
- an element of the tensor square of self

**EXAMPLES:**

```python
sage: p = SymmetricFunctionsNonCommutingVariables(QQ).powersum()
sage: p[SetPartition([[1,3],[2]])].internal_coproduct_on_basis()
```

**primitive**(A, i=1)

Return the primitive associated to A in self.

Fix some \(i \in S\). Let \(A\) be an atomic set partition of \(S\), then the primitive \(p(A)\) given in [LM2011] is

\[ p(A) = \sum_\gamma (-1)^{\ell(\gamma)} p_{\gamma[A]} \]

where we sum over all ordered set partitions of \([\ell(A)]\) such that \(i \in \gamma_1\) and \(\gamma[A]\) is the action of \(\gamma\) on \(A\) defined in \(SetPartition.ordered_set_partition_action()\). If \(A\) is not atomic, then \(p(A) = 0\).

**See also:**

\(SetPartition.is_atomic()\)

**INPUT:**
- A – a set partition
- i – (default: 1) index in the base set for A specifying which set of primitives this belongs to

**OUTPUT:**
- an element in the basis self
EXAMPLES:

```
sage: p = SymmetricFunctionsNonCommutingVariables(QQ).powersum()
sage: elt = p.primitive(SetPartition([[1,3], [2]])); elt
-p{{1, 2}, {3}} + p{{1, 3}, {2}}
sage: elt.coproduct()
-p{} # p{{1, 2}, {3}} + p{} # p{{1, 3}, {2}} - p{{1, 2}, {3}} # p{} + p{{1, ω → 3}, {2}} # p{}
sage: p.primitive(SetPartition([[1], [2,3]]))
0
sage: p.primitive(SetPartition([]))
p{}
```

**rho**

alias of `deformed_coarse_powersum`

**class supercharacter(NCSym, q=2)**

Bases: `NCSymBasis_abstract`

The Hopf algebra of symmetric functions in non-commuting variables in the supercharacter \( \chi \) basis.

This basis was defined in [BT13] as a \( q \)-deformation of the supercharacter basis.

\[
\chi_A = \sum_B \chi_A(B) m_B,
\]

where we sum over all set partitions \( A \) and \( \chi_A(B) \) is the evaluation of the supercharacter \( \chi_A \) on the superclass \( \mu_B \).

**Note:** The supercharacters considered in [BT13] are coarser than those considered by Aguiar et. al.

**INPUT:**

- \( q \) – (default: 2) the parameter \( q \)

**EXAMPLES:**

```
sage: R = QQ['q'].fraction_field()
sage: q = R.gen()
sage: NCSym = SymmetricFunctionsNonCommutingVariables(R)
sage: chi = NCSym.chi(q)
sage: chi[[1,3],[2]]*chi[[1,2]]
chi{{1, 3}, {2}, {4, 5}}
sage: chi[[1,3],[2]].coproduct()
chi{} # chi{{1, 3}, {2}} + (2*q-2)*chi{{1}} # chi{{1}, {2}} +
(3*q-2)*chi{{1}} # chi{{1}, {2}} + (2*q-2)*chi{{1}} # chi{{1}} +
(3*q-2)*chi{{1, 2}} # chi{{1}} + chi{{1, 3}, {2}} # chi{}
sage: chi2 = NCSym.chi()
sage: chi(chi2[[1,2],[3]])
((-q+2)/q)*chi{{1, 2}, {3}} + 2/q*chi{{1, 2}, {3}}
sage: chi2
Symmetric functions in non-commuting variables over the Fraction Field
of Univariate Polynomial Ring in q over Rational Field in the
supercharacter basis with parameter q=2
```

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Return the deformation parameter \( q \) of \( \texttt{self} \).

**EXAMPLES:**

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: chi = NCSym.chi(5)
sage: chi.q()
5

sage: R = QQ['q'].fraction_field()
sage: q = R.gen()
sage: NCSym = SymmetricFunctionsNonCommutingVariables(R)
sage: chi = NCSym.chi(q)
sage: chi.q() == q
True
```

\( x \)

alias of \( \texttt{x\_basis} \)

**class** \( \texttt{x\_basis}(\texttt{NCSym}) \)

**Bases:** \( \texttt{NCSymBasis\_abstract} \)

The Hopf algebra of symmetric functions in non-commuting variables in the \( x \) basis.

This basis is defined in \cite{BHRZ06} by the formula:

\[
x_A = \sum_{B \subseteq A} \mu(B, A)p_B
\]

and has the following properties:

\[
x_Ax_B = x_{A|B}, \quad \Delta(x_C) = \sum_{AB=C} x_A \otimes x_B.
\]

**EXAMPLES:**

```python
sage: NCSym = SymmetricFunctionsNonCommutingVariables(QQ)
sage: x = NCSym.x()
sage: x[[1,3],[2,4]]*x[[1,2,3]]
x{{1, 3}, {2, 4}, {5, 6, 7}}

sage: x[[1,2],[3]].internal_coproduct()
x{{1}, {2}, {3}} # x{{1, 2}, {3}} + x{{1, 2}, {3}} # x{{1}, {2}, {3}} +
x{{1, 2}, {3}} # x{{1, 2}, {3}}
```

sage.combinat.ncsym.ncsym.matchings\((A, B)\)

Iterate through all matchings of the sets \( A \) and \( B \).

**EXAMPLES:**

```python
sage: from sage.combinat.ncsym.ncsym import matchings
sage: list(matchings([1, 2, 3], [-1, -2]))
[[[1], [2], [3], [-1], [-2]],
 [[1], [2], [3, -1], [-2]],
 [[1], [2], [3, -2], [-1]],
 [[1], [2, -1], [3], [-2]],
(continues on next page)
```

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sage.combinat.ncsym.ncsym.nesting(la, nu)

Return the nesting number of la inside of nu.

If we consider a set partition \( A \) as a set of arcs \( i - j \) where \( i \) and \( j \) are in the same part of \( A \). Define

\[
\text{nst}_\lambda^\nu = \#\{ i < j < k < l \mid i - l \in \nu, j - k \in \lambda \},
\]

and this corresponds to the number of arcs of \( \lambda \) strictly contained inside of \( \nu \).

EXAMPLES:

```
sage: from sage.combinat.ncsym.ncsym import nesting
sage: nu = SetPartition([[1, 4], [2], [3]])
sage: mu = SetPartition([[1, 4], [2, 3]])
sage: nesting(set(mu).difference(nu), nu)
1
```

```
sage: lst = list(SetPartitions(4))
sage: d = {}
sage: for i, nu in enumerate(lst):
    ....:     for mu in nu.coarsenings():
    ....:         if set(nu.arcs()).issubset(mu.arcs()):
    ....:             d[i, lst.index(mu)] = nesting(set(mu).difference(nu), nu)

sage: matrix(d)

```

```
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
5.1.151 Necklaces

The algorithm used in this file comes from


`sage.combinat.necklace.Necklaces(content)`

Return the set of necklaces with evaluation `content`.

A necklace is a list of integers that such that the list is the smallest lexicographic representative of all the cyclic shifts of the list.

See also:

*LyndonWords*

INPUT:

- `content` – a list or tuple of non-negative integers

EXAMPLES:

```
sage: Necklaces([2,1,1])
Necklaces with evaluation [2, 1, 1]
sage: Necklaces([2,1,1]).cardinality()
3
sage: Necklaces([2,1,1]).first()
[1, 1, 2, 3]
sage: Necklaces([2,1,1]).last()
[1, 2, 1, 3]
sage: Necklaces([2,1,1]).list()
[[1, 1, 2, 3], [1, 1, 3, 2], [1, 2, 1, 3]]
sage: Necklaces([0,2,1,1]).list()
[[2, 2, 3, 4], [2, 2, 4, 3], [2, 3, 2, 4]]
sage: Necklaces([2,0,1,1]).list()
[[1, 1, 3, 4], [1, 1, 4, 3], [1, 3, 1, 4]]
```

class `sage.combinat.necklace.Necklaces_evaluation(content)`

Bases: `UniqueRepresentation`, `Parent`

Necklaces with a fixed evaluation (content).

INPUT:

- `content` – a list or tuple of non-negative integers

`cardinality()`

Return the number of integer necklaces with the evaluation `content`.

The formula for the number of necklaces of content `α` a composition of `n` is:

\[
\sum_{d \mid \gcd(\alpha)} \phi(d) \left( \frac{n/d}{\alpha_1/d, \ldots, \alpha_k/d} \right),
\]

where \(\phi(d)\) is the Euler \(\phi\) function.

EXAMPLES:
sage: Necklaces([]).cardinality()
0
sage: Necklaces([2,2]).cardinality()
2
sage: Necklaces([2,3,2]).cardinality()
30
sage: Necklaces([0,3,2]).cardinality()
2

Check to make sure that the count matches up with the number of necklace words generated.

sage: comps = [[],[2,2],[3,2,7],[4,2],[0,4,2],[2,0,4]]+Compositions(4).list()
sage: ns = [Necklaces(comp) for comp in comps]
sage: all(n.cardinality() == len(n.list()) for n in ns)
True

content()

Return the content (or evaluation) of the necklaces.

EXAMPLES:

sage: N = Necklaces([2,2,2])
sage: N.content()
[2, 2, 2]

5.1.152 Non-Decreasing Parking Functions

A non-decreasing parking function of size $n$ is a non-decreasing function $f$ from $\{1, \ldots, n\}$ to itself such that for all $i$, one has $f(i) \leq i$.

The number of non-decreasing parking functions of size $n$ is the $n$-th Catalan number.

The set of non-decreasing parking functions of size $n$ is in bijection with the set of Dyck words of size $n$.

AUTHORS:

- Florent Hivert (2009-04)
- Christian Stump (2012-11) added pretty printing

class sage.combinat.non_decreasing_parking_function.NonDecreasingParkingFunction(lst)

Bases: Element

A non decreasing parking function of size $n$ is a non-decreasing function $f$ from $\{1, \ldots, n\}$ to itself such that for all $i$, one has $f(i) \leq i$.

EXAMPLES:

sage: NonDecreasingParkingFunction([])
[]
sage: NonDecreasingParkingFunction([1])
[1]
sage: NonDecreasingParkingFunction([2])
Traceback (most recent call last):
...
ValueError: [2] is not a non-decreasing parking function

(continues on next page)
sage: NonDecreasingParkingFunction([1,2])
[1, 2]
sage: NonDecreasingParkingFunction([1,1,2])
[1, 1, 2]
sage: NonDecreasingParkingFunction([1,1,4])
Traceback (most recent call last):
  ... ValueError: [1, 1, 4] is not a non-decreasing parking function

classmethod from_dyck_word(dw)

Bijection from Dyck words. It is the inverse of the bijection to_dyck_word(). You can find there the mathematical definition.

EXAMPLES:

sage: NonDecreasingParkingFunction.from_dyck_word([])
[]
sage: NonDecreasingParkingFunction.from_dyck_word([1,0])
[1]
sage: NonDecreasingParkingFunction.from_dyck_word([1,1,0,0])
[1, 1]
sage: NonDecreasingParkingFunction.from_dyck_word([1,0,1,0])
[1, 2]
sage: NonDecreasingParkingFunction.from_dyck_word([1,0,1,1,0,1,0,1,0,1,0,1,0])
[1, 2, 2, 3, 5]

grade()

Return the length of self.

EXAMPLES:

sage: ndpf = NonDecreasingParkingFunctions(5)
sage: len(ndpf.random_element())
5

to_dyck_word()

Implement the bijection to Dyck words, which is defined as follows. Take a non decreasing parking function, say [1,1,2,4,5,5], and draw its graph:

```
  ___
 /  | . 5
/   | . 5
 ___| . . 4
|  | . . . 2
|  | . . . . 1
|  | . . . . 1
```

The corresponding Dyck word [1,1,0,1,0,1,0,1,1,0,0] is then read off from the sequence of horizontal and vertical steps. The converse bijection is from_dyck_word().

EXAMPLES:

sage: NonDecreasingParkingFunction([1,1,2,4,5,5]).to_dyck_word()
[1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0]
sage: NonDecreasingParkingFunction([]).to_dyck_word()
[]
sage: NonDecreasingParkingFunction([1,1,1]).to_dyck_word()
[1, 1, 1, 0, 0, 0]
sage: NonDecreasingParkingFunction([1,2,3]).to_dyck_word()
[1, 0, 1, 0, 1, 0]
sage: NonDecreasingParkingFunction([1,1,3,3,4,6,6]).to_dyck_word()
[1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0]

Return the set of Non-Decreasing Parking Functions.

A non-decreasing parking function of size $n$ is a non-decreasing function $f$ from \{1, ..., $n$\} to itself such that for all $i$, one has $f(i) \leq i$.

EXAMPLES:

Here are all the non-decreasing parking functions of size 5:

sage: NonDecreasingParkingFunctions(3).list()
[[1, 1, 1], [1, 1, 2], [1, 1, 3], [1, 2, 2], [1, 2, 3]]

If no size is specified, then NonDecreasingParkingFunctions returns the set of all non-decreasing parking functions.

sage: PF = NonDecreasingParkingFunctions(); PF
Non-decreasing parking functions
sage: [] in PF
True
sage: [1] in PF
True
sage: [2] in PF
False
sage: [1,1,3] in PF
True
sage: [1,1,4] in PF
False

If the size $n$ is specified, then NonDecreasingParkingFunctions returns the set of all non-decreasing parking functions of size $n$.

sage: PF = NonDecreasingParkingFunctions(0)
sage: PF.list()
[]
sage: PF = NonDecreasingParkingFunctions(1)
sage: PF.list()
[[1]]
sage: PF = NonDecreasingParkingFunctions(3)
sage: PF.list()
[[1, 1, 1], [1, 1, 2], [1, 1, 3], [1, 2, 2], [1, 2, 3]]

sage: PF3 = NonDecreasingParkingFunctions(3); PF3
Non-decreasing parking functions of size 3
sage: [] in PF3
False

(continues on next page)
False
sage: [1] in PF3
False
sage: [1,1,3] in PF3
True
sage: [1,1,4] in PF3
False

class sage.combinat.non_decreasing_parking_function.NonDecreasingParkingFunctions_all
Bases: UniqueRepresentation, Parent

graded_component(n)
   Return the graded component.

   EXAMPLES:

   sage: P = NonDecreasingParkingFunctions()
sage: P.graded_component(4) == NonDecreasingParkingFunctions(4)
True

class sage.combinat.non_decreasing_parking_function.NonDecreasingParkingFunctions_n(n)
Bases: UniqueRepresentation, Parent

The combinatorial class of non-decreasing parking functions of size $n$.

A non-decreasing parking function of size $n$ is a non-decreasing function $f$ from \{1,\ldots,n\} to itself such that for all $i$, one has $f(i) \leq i$.

The number of non-decreasing parking functions of size $n$ is the $n$-th Catalan number.

EXAMPLES:

sage: PF = NonDecreasingParkingFunctions(3)
sage: PF.list()
[[1, 1, 1], [1, 1, 2], [1, 1, 3], [1, 2, 2], [1, 2, 3]]
sage: PF = NonDecreasingParkingFunctions(4)
sage: PF.list()
[[1, 1, 1, 1], [1, 1, 1, 2], [1, 1, 1, 3], [1, 1, 1, 4], [1, 1, 2, 2], [1, 1, 2, 3], [1, 1, 2, 4], [1, 1, 3, 3], [1, 1, 3, 4], [1, 1, 4, 4], [1, 2, 2, 2], [1, 2, 2, 3], [1, 2, 2, 4], [1, 2, 3, 3], [1, 2, 3, 4], [1, 2, 4, 4], [1, 3, 3, 3], [1, 3, 3, 4], [1, 3, 4, 4], [1, 4, 4, 4], [2, 2, 2, 2], [2, 2, 2, 3], [2, 2, 2, 4], [2, 2, 3, 3], [2, 2, 3, 4], [2, 2, 4, 4], [2, 3, 3, 3], [2, 3, 3, 4], [2, 3, 4, 4], [2, 4, 4, 4], [3, 3, 3, 3], [3, 3, 3, 4], [3, 3, 4, 4], [3, 4, 4, 4], [4, 4, 4, 4]]

Warning: The precise order in which the parking function are generated or listed is not fixed, and may change in the future.

AUTHORS:
- Florent Hivert

Element
   alias of NonDecreasingParkingFunction
**cardinality()**

Return the number of non-decreasing parking functions of size $n$.

This number is the $n$-th **Catalan number**.

**EXAMPLES:**

```python
sage: PF = NonDecreasingParkingFunctions(0)
sage: PF.cardinality()
sage: PF = NonDecreasingParkingFunctions(1)
sage: PF.cardinality()
sage: PF = NonDecreasingParkingFunctions(3)
sage: PF.cardinality()
sage: PF = NonDecreasingParkingFunctions(5)
sage: PF.cardinality()
```

**one()**

Return the unit of this monoid.

This is the non-decreasing parking function $[1, 2, \ldots, n]$.

**EXAMPLES:**

```python
sage: ndpf = NonDecreasingParkingFunctions(5)
sage: x = ndpf.random_element(); x  # random
sage: e = ndpf.one()
sage: x == e*e == x*e
True
```

**random_element()**

Return a random parking function of the given size.

**EXAMPLES:**

```python
sage: ndpf = NonDecreasingParkingFunctions(5)
sage: x = ndpf.random_element(); x  # random
sage: in ndpf
```

**sage.combinat.non_decreasing_parking_function.is_a(x, n=None)**

Check whether a list is a non-decreasing parking function.

If a size $n$ is specified, checks if a list is a non-decreasing parking function of size $n$. 


5.1.153 $\nu$-Dyck Words

A class of the $\nu$-Dyck word, see [PRV2017] for details.

AUTHORS:

- Aram Dermenjian (2020-09-26)

This file is based off the class DyckWords written by Mike Hansen, Dan Drake, Florent Hivert, Christian Stump, Mike Zabrocki, Jean–Baptiste Priez and Travis Scrimshaw

```python
class sage.combinat.nu_dyck_word.NuDyckWord(parent, dw, latex_options=None)
    Bases: CombinatorialElement

    $\nu$-Dyck word.

    Given a lattice path $\nu$ in the $\mathbb{Z}^2$ grid starting at the origin $(0,0)$ consisting of North $N = (0,1)$ and East $E = (1,0)$ steps, a $\nu$-Dyck path is a lattice path in the $\mathbb{Z}^2$ grid starting at the origin $(0,0)$ and ending at the same coordinate as $\nu$ such that it is weakly above $\nu$. A $\nu$-Dyck word is the representation of a $\nu$-Dyck path where a North step is represented by a 1 and an East step is represented by a 0.

    INPUT:

    - k1 – A path for the $\nu$-Dyck word
    - k2 – A path for $\nu$

    EXAMPLES:
```
sage: dw = NuDyckWord([1,0,1,0],[1,0,0,1]); dw
[1, 0, 1, 0]
sage: print(dw)
NENE
sage: dw.height()
2
sage: dw = NuDyckWord('1010',[1,0,0,1]); dw
[1, 0, 1, 0]
sage: dw = NuDyckWord('NENE',[1,0,0,1]); dw
[1, 0, 1, 0]
sage: NuDyckWord([1,0,1,0],[1,0,0,1]).pretty_print()

__
|\ |
|  .|

sage: from sage.combinat.nu_dyck_word import update_ndw_symbols
sage: update_ndw_symbols(0,1)
sage: dw = NuDyckWord('0101001','0110010'); dw
[0, 1, 0, 1, 0, 0, 1]
sage: dw.pp()

__
|\ |
|  .|

sage: update_ndw_symbols(1,0)
```

5.1. Comprehensive Module List
**can_mutate(i)**

Return True/False based on if mutable at height $i$.

Can only mutate if an east step is followed by a north step at height $i$.

**OUTPUT:**

Whether we can mutate at height of $i$.

**EXAMPLES:**

```
sage: NDW = NuDyckWord('10010100','00000111')
sage: NDW.can_mutate(1)
False
sage: NDW.can_mutate(3)
5
```

**height()**

Return the height of self.

The height is the number of north steps.

**EXAMPLES:**

```
sage: NuDyckWord('1101110011010010011011000110000',
......: '1010101010101010101010101010101010').height()
17
```

**heights()**

Return the heights of each point on self.

We view the Dyck word as a Dyck path from $(0, 0)$ to $(x, y)$ in the first quadrant by letting 1’s represent steps in the direction $(0, 1)$ and 0’s represent steps in the direction $(1, 0)$.

The heights is the sequence of the $y$-coordinates of all $x + y$ lattice points along the path.

**EXAMPLES:**

```
sage: NuDyckWord('010','010').heights()
[0, 0, 1, 1]
sage: NuDyckWord('110100','101010').heights()
[0, 1, 2, 2, 3, 3, 3]
```

**horizontal_distance()**

Return a list of how far each point is from $\nu$.

**EXAMPLES:**

```
sage: NDW = NuDyckWord('10010100','00000111')
sage: NDW.horizontal_distance()
[5, 5, 4, 3, 3, 2, 2, 1, 0]
sage: NDW = NuDyckWord('10010100','00001011')
sage: NDW.horizontal_distance()
[4, 5, 4, 3, 3, 2, 2, 1, 0]
sage: NDW = NuDyckWord('10010001000','00100101001')
sage: NDW.horizontal_distance()
[2, 4, 3, 2, 3, 5, 4, 3, 3, 2, 1, 0]
```
latex_options()

Return the latex options for use in the \_latex\_ function as a dictionary.

The default values are set using the options.

- **color** – (default: black) the line color.
- **line width** – (default: \(2^{\text{'tikz_scale'}}\)) value representing the line width.
- **nu_options** – (default: 'rounded corners=1, color=red, line width=1') str to indicate what the tikz options should be for path of \(\nu\).
- **points_color** – (default: 'black') str to indicate color points should be drawn with.
- **show_grid** – (default: True) boolean value to indicate if grid should be shown.
- **show_nu** – (default: True) boolean value to indicate if \(\nu\) should be shown.
- **show_points** – (default: False) boolean value to indicate if points should be shown on path.
- **tikz_scale** – (default: 1) scale for use with the tikz package.

EXAMPLES:

```sage
sage: NDW = NuDyckWord('010', '010')
sage: NDW.latex_options()
{'color': black,
 'line width': 2,
 'nu_options': 'rounded corners=1, color=red, line width=1,
 'points_color': black,
 'show_grid': True,
 'show_nu': True,
 'show_points': False,
 'tikz_scale': 1}
```

Todo: This should probably be merged into NuDyckWord.options.

length()

Return the length of self.

The length is the total number of steps.

EXAMPLES:

```sage
sage: NDW = NuDyckWord('10011001000', '00100101001')
sage: NDW.length()
11
```

mutate(i)

Return a new \(\nu\)-Dyck Word if possible.

If at height \(i\) we have an east step E meeting a north step N then we calculate all horizontal distances from this point until we find the first point that has the same horizontal distance to \(\nu\). We let

- **d** is everything up until EN (not including EN)
- **f** be everything between N and the point with the same horizontal distance (including N)
- **g** is everything after \(f\)
See also:

`can_mutate()`

EXAMPLES:

```
sage: NDW = NuDyckWord('10010100','00000111')
sage: NDW.mutate(1)
sage: NDW.mutate(3)
[1, 0, 0, 1, 1, 0, 0, 0]
```

`path()`

Return the underlying path object.

EXAMPLES:

```
sage: NDW = NuDyckWord('10011001000','00100101001')
sage: NDW.path()
Path: 10011001000
```

`plot(**kwds)`

Plot a ν-Dyck word as a continuous path.

EXAMPLES:

```
sage: NDW = NuDyckWord('010','010')
sage: NDW.plot()
Graphics object consisting of 1 graphics primitive
```

`points()`

Return an iterator for the points on the ν-Dyck path.

EXAMPLES:

```
sage: list(NuDyckWord('110110011101001000110111100011000',
                      ....: '101010101010101010101010101010101')._path.points())
[(0, 0),
 (0, 1),
 (0, 2),
 (1, 2),
 (1, 3),
 (1, 4),
 (1, 5),
 (2, 5),
 (3, 5),
 (3, 6),
 (3, 7),
 (4, 7),
 (4, 8),
 (5, 8),
 (5, 9),
 (6, 9),
 (7, 9),
 (8, 9),
 (9, 9),
 (9, 10),
```

(continues on next page)
Display a NuDyckWord as a lattice path in the $\mathbb{Z}^2$ grid.

If the style is “N-E”, then a cell below the diagonal is indicated by a period, whereas a cell below the path but above the diagonal is indicated by an x. If a list of labels is included, they are displayed along the vertical edges of the Dyck path.

INPUT:

- **style** – (default: None) can either be:
  - None to use the option default
  - “N-E” to show self as a path of north and east steps, or
- **labelling** – (if style is “N-E”) a list of labels assigned to the up steps in self.
- **underpath** – (if style is “N-E”, default: True) If True, an x to show the boxes between $\nu$ and the $\nu$-Dyck Path.

EXAMPLES:

```python
sage: for ND in NuDyckWords('101010'): ND.pretty_print()
```
sage: nu = [1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0]

sage: ND = NuDyckWord([1,1,1,0,1,0,0,1,1,0,0,0],nu)

sage: ND.pretty_print()

____
|x  x  
___|  x  
|x    
|x  x  
|x x   
|x    
|    

sage: NuDyckWord([1,1,0,0,1,0],[1,0,1,0,1,0]).pretty_print(labelling=[1,3,2])

__
___ 1
|x  2
|x 3
| 1

sage: NuDyckWord('1101110011010010011001100110'),[1,0,1,0,1,0]).pretty_print(labelling=list(range(1,18)))

________
|x  x  x  17
____|  x  ,  16
|x  x  x  15
|x  x  .  14
|x  .  13
|x  .  12
|x  .  11
|x  .  10
|x  .  9
|x  .  8
|x  .  7
|x  .  6
|x  .  5
|x  .  4
|x  .  3
|x  .  2
| .  1

sage: NuDyckWord().pretty_print()

pretty_print(style=None, labelling=None)

Display a NuDyckWord as a lattice path in the $\mathbb{Z}^2$ grid.

If the style is “N-E”, then a cell below the diagonal is indicated by a period, whereas a cell below the path but above the diagonal is indicated by an x. If a list of labels is included, they are displayed along the vertical edges of the Dyck path.
INPUT:

- **style** – (default: None) can either be:
  - None to use the option default
  - “N-E” to show `self` as a path of north and east steps, or
- **labelling** – (if style is “N-E”) a list of labels assigned to the up steps in `self`.
- **underpath** – (if style is “N-E”, default: True) If True, an x to show the boxes between $\nu$ and the $\nu$-Dyck Path.

EXAMPLES:

```python
sage: for ND in NuDyckWords('101010'): ND.pretty_print()

__
| .  
| .   
| .    
___| .  
|x .   
|x .    
| .     
| .     
| .     

sage: nu = [1,0,1,0,1,0,1,0,1,0,1,0,1,0]
sage: ND = NuDyckWord([1,1,1,0,1,0,0,1,1,0,0,0],nu)
sage: ND.pretty_print()

____
|x x  
|x .   
|x .    
|x .     
|x .      

sage: NuDyckWord([1,1,0,0,1,0,[1,0,1,0,1,0,1,0,1,0,1,0,0,0]],nu).pretty_print(
    ....: labelling=[1,3,2])

___| 2
|x . 3
| .   1
```
Combinatorics, Release 10.0

```sage
sage: NuDyckWord('110111001110100011101110011000',
    ....: '101010101010101010101010101010').pretty_print(
    ....: labelling=list(range(1,18)))

________
|x x x 17
____|x  x 16
|x x x 15
|x x   14
|x     13
|x      12
|x      11
|       10
|____|  9
|x x   8
|x x   7
|x x   6
|x x   5
|x     4
|x      3
|x      2
|       1

sage: NuDyckWord().pretty_print()
```

```python
set_latex_options(D)

Set the latex options for use in the _latex_ function.

The default values are set in the __init__ function.

- **color** – (default: black) the line color.
- **line width** – (default: 2 \times \text{tikz_scale}) value representing the line width.
- **nu_options** – (default: 'rounded corners=1, color=red, line width=1') str to indicate what the tikz options should be for path of \( \nu \).
- **points_color** – (default: 'black') str to indicate color points should be drawn with.
- **show_grid** – (default: True) boolean value to indicate if grid should be shown.
- **show_nu** – (default: True) boolean value to indicate if \( \nu \) should be shown.
- **show_points** – (default: False) boolean value to indicate if points should be shown on path.
- **tikz_scale** – (default: 1) scale for use with the tikz package.

**INPUT:**

- **D** – a dictionary with a list of latex parameters to change

**EXAMPLES:**

```sage
sage: NDW = NuDyckWord('010','010')
sage: NDW.set_latex_options({'tikz_scale':2})
sage: NDW.set_latex_options({'color':'blue', 'show_points':True})
```

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Todo: This should probably be merged into NuDyckWord.options.

**width()**

Return the width of self.

The width is the number of east steps.

EXAMPLES:

```
sage: NuDyckWord('110111001101001000110111100011000',
       ....: '101010101010101010101010101010101').width()
16
```

**widths()**

Return the widths of each point on self.

We view the Dyck word as a Dyck path from $(0,0)$ to $(x,y)$ in the first quadrant by letting 1’s represent steps in the direction $(0,1)$ and 0’s represent steps in the direction $(1,0)$.

The widths is the sequence of the $x$-coordinates of all $x+y$ lattice points along the path.

EXAMPLES:

```
sage: NuDyckWord('010', '010').widths()
[0, 1, 1, 2]
sage: NuDyckWord('110100', '101010').widths()
[0, 0, 0, 1, 1, 2, 3]
```

```
class sage.combinat.nu_dyck_word.NuDyckWords(nu=())
Bases: Parent
ν-Dyck words.

Given a lattice path $\nu$ in the $\mathbb{Z}^2$ grid starting at the origin $(0,0)$ consisting of North $N = (0,1)$ and East $E = (1,0)$ steps, a $\nu$-Dyck path is a lattice path in the $\mathbb{Z}^2$ grid starting at the origin $(0,0)$ and ending at the same coordinate as $\nu$ such that it is weakly above $\nu$. A $\nu$-Dyck word is the representation of a $\nu$-Dyck path where a North step is represented by a 1 and an East step is represented by a 0.

INPUT:

- nu – the base lattice path.

EXAMPLES:

```
sage: NDW = NuDyckWords('1010'); NDW
[1, 0, 1, 0] Dyck words
sage: [1, 0, 1, 0] in NDW
True
sage: [1, 1, 0, 0] in NDW
True
sage: [1, 0, 0, 1] in NDW
False
sage: [0, 1, 0, 1] in NDW
False
sage: NDW.cardinality()
2
```
Element

alias of \texttt{NuDyckWord}

cardinality()

Return the number of $\nu$-Dyck words.

EXAMPLES:

```sage
NDW = NuDyckWords('1010'); NDW.cardinality()
5
NDW = NuDyckWords('101010'); NDW.cardinality()
7
NDW = NuDyckWords('10010100'); NDW.cardinality()
12
```

```python
options = Current options for NuDyckWords - ascii_art: pretty_output -
diagram_style: grid - display: list - latex_color: black -
latex_line_width_scalar: 2 - latex_nu_options: rounded corners=1, color=red, line
width=1 - latex_points_color: black - latex_show_grid: True - latex_show_nu: True -
latex_show_points: False - latex_tikz_scale: 1
```

```sage
sage.combinat.nu_dyck_word.path_weakly_above_other(path, other)
```

Test if \emph{path} is weakly above \emph{other}.

A \emph{path} $P$ is weakly above another path $Q$ if $P$ and $Q$ are the same length and if any prefix of length $n$ of $Q$ contains more North steps than the prefix of length $n$ of $P$.

INPUT:

- \texttt{path} – The path to verify is weakly above the other path.
- \texttt{other} – The other path to verify is weakly below the path.

OUTPUT:

\texttt{bool}

EXAMPLES:

```sage
from sage.combinat.nu_dyck_word import path_weakly_above_other
path_weakly_above_other('1001','0110')
False
path_weakly_above_other('1001','0101')
True
path_weakly_above_other('1111','0101')
False
path_weakly_above_other('111100','0101')
False
```

```python
sage.combinat.nu_dyck_word.replace_dyck_char(x)
```

A map sending an opening character ('1', 'N', and '(') to \texttt{ndw_open_symbol}, a closing character ('0', 'E', and ')') to \texttt{ndw_close_symbol}, and raising an error on any input other than one of the opening or closing characters.

This is the inverse map of \texttt{replace_dyck_symbol}.

INPUT:

- \texttt{x = str} - A '1', '0', 'N', 'E', '(' or ')'
OUTPUT:

- If $x$ is an opening character, replace $x$ with the constant \texttt{ndw_open_symbol}.
- If $x$ is a closing character, replace $x$ with the constant \texttt{ndw_close_symbol}.
- Raise a \texttt{ValueError} if $x$ is neither an opening nor a closing character.

See also:

\texttt{replace_dyck_symbol()}

EXAMPLES:

```
sage: from sage.combinat.nu_dyck_word import replace_dyck_char
sage: replace_dyck_char('(')
1
sage: replace_dyck_char(')')
0
sage: replace_dyck_char(1)
Traceback (most recent call last):
  ... ValueError
```

\texttt{sage.combinat.nu_dyck_word.replace_dyck_symbol(x, open_char='N', close_char='E')}

A map sending \texttt{ndw_open_symbol} to \texttt{open_char} and \texttt{ndw_close_symbol} to \texttt{close_char}, and raising an error on any input other than \texttt{ndw_open_symbol} and \texttt{ndw_close_symbol}. The values of the constants \texttt{ndw_open_symbol} and \texttt{ndw_close_symbol} are subject to change.

This is the inverse map of \texttt{replace_dyck_char()}.  

INPUT:

- $x$ – either \texttt{ndw_open_symbol} or \texttt{ndw_close_symbol}.
- \texttt{open_char} – str (optional) default 'N'
- \texttt{close_char} – str (optional) default 'E'

OUTPUT:

- If $x$ is \texttt{ndw_open_symbol}, replace $x$ with \texttt{open_char}.
- If $x$ is \texttt{ndw_close_symbol}, replace $x$ with \texttt{close_char}.
- If $x$ is neither \texttt{ndw_open_symbol} nor \texttt{ndw_close_symbol}, a \texttt{ValueError} is raised.

See also:

\texttt{replace_dyck_char()}

EXAMPLES:

```
sage: from sage.combinat.nu_dyck_word import replace_dyck_symbol
sage: replace_dyck_symbol(1)
'N'
sage: replace_dyck_symbol(0)
'E'
sage: replace_dyck_symbol(3)
Traceback (most recent call last):
  ... ValueError
```
sage.combinat.nu_dyck_word.to_word_path(word)
Convert input into a word path over an appropriate alphabet.
Helper function.

INPUT:
• word – word to convert to wordpath

OUTPUT:
• A FiniteWordPath_north_east object.

EXAMPLES:

```
sage: from sage.combinat.nu_dyck_word import to_word_path
sage: wp = to_word_path('NEENENEN'); wp
Path: 10010101
sage: from sage.combinat.words.paths import FiniteWordPath_north_east
sage: isinstance(wp,FiniteWordPath_north_east)
True
sage: to_word_path('1001')
Path: 1001
sage: to_word_path([0,1,0,0,1,0])
Path: 010010
```

sage.combinat.nu_dyck_word.update_ndw_symbols(os, cs)
A way to alter the open and close symbols from sage.

INPUT:
• os – the open symbol
• cs – the close symbol

EXAMPLES:

```
sage: from sage.combinat.nu_dyck_word import update_ndw_symbols
sage: update_ndw_symbols(0,1)
sage: dw = NuDyckWord('0101001','0110010'); dw
[0, 1, 0, 1, 0, 0, 1]
sage: dw = NuDyckWord('1010110','1001101'); dw
Traceback (most recent call last):
  ... ValueError: invalid nu-Dyck word
sage: update_ndw_symbols(1,0)
```

5.1.154 $\nu$-Tamari lattice
A class of the $\nu$-Tamari lattice, see [PRV2017] for details.
These lattices depend on one parameter $\nu$ where $\nu$ is a path of North and East steps.
The elements are $nu$-Dyck paths which are weakly above $\nu$.
To use the provided functionality, you should import $\nu$-Tamari lattices by typing:
sage: from sage.combinat.nu_tamari_lattice import NuTamariLattice

Then,

sage: NuTamariLattice([1,1,1,0,0,1,1,0])
Finite lattice containing 6 elements
sage: NuTamariLattice([0,0,0,1,1,0,0,1])
Finite lattice containing 40 elements

The classical Tamari lattices and the Generalized Tamari lattices are special cases of this construction and are also available with this poset:

sage: NuTamariLattice([1,0,1,0,1,0])
Finite lattice containing 5 elements
sage: NuTamariLattice([1,0,0,1,0,0,1,0,0])
Finite lattice containing 12 elements

See also:

For more detailed information see NuTamariLattice(). For more information on the standard Tamari lattice see sage.combinat.tamari_lattices.TamariLattice(), sage.combinat.tamari_lattices.GeneralizedTamariLattice()
sage.combinat.nu_tamari_lattice.NuTamariLattice(nu)

Return the ν-Tamari lattice.

INPUT:

• ν – a list of 0s and 1s or a string of 0s and 1s.

OUTPUT:

a finite lattice

The elements of the lattice are nu-Dyck paths weakly above ν.

The usual Tamari lattice is the special case where ν = (NE)^h where h is the height.

Other special cases give the m-Tamari lattices studied in [BMFPR].

EXAMPLES:

sage: from sage.combinat.nu_tamari_lattice import NuTamariLattice
sage: NuTamariLattice([1,0,1,0,1,0])
Finite lattice containing 7 elements
sage: NuTamariLattice([1,0,1,0,1,0])
Finite lattice containing 5 elements
sage: NuTamariLattice([1,0,1,0,1,0,1,0])
Finite lattice containing 14 elements
sage: NuTamariLattice([1,0,1,0,1,0,0,0,1])
Finite lattice containing 24 elements
5.1.155 Ordered Rooted Trees

AUTHORS:
• Florent Hivert (2010-2011): initial revision
• Frédéric Chapoton (2010): contributed some methods

class sage.combinat.ordered_tree.LabelledOrderedTree(parent, children, label=None, check=True)
Bases: AbstractLabelledClonableTree, OrderedTree

Labelled ordered trees.

A labelled ordered tree is an ordered tree with a label attached at each node.

INPUT:
• children – a list or tuple or more generally any iterable of trees or object convertible to trees
• label – any Sage object (default: None)

EXAMPLES:

```
sage: x = LabelledOrderedTree([], label = 3); x
3[

sage: LabelledOrderedTree([x, x, x], label = 2)
2[3[], 3[], 3[]]

sage: LabelledOrderedTree((x, x, x), label = 2)
2[3[], 3[], 3[]]

sage: LabelledOrderedTree([], [[], []], label = 3)
3[None[], None[None[], None[]]]
```

left_right_symmetry()

Return the symmetric tree of self.

The symmetric tree $s(T)$ of a labelled ordered tree $T$ is defined as follows: If $T$ is a labelled ordered tree with children $C_1, C_2, \ldots, C_k$ (listed from left to right), then the symmetric tree $s(T)$ of $T$ is a labelled ordered tree with children $s(C_k), s(C_{k-1}), \ldots, s(C_1)$ (from left to right), and with the same root label as $T$.

**Note:** If you have a subclass of LabelledOrderedTree() which also inherits from another subclass of OrderedTree() which does not come with a labelling, then (depending on the method resolution order) it might happen that this method gets overridden by an implementation from that other subclass, and thus forgets about the labels. In this case you need to manually override this method on your subclass.

EXAMPLES:

```
sage: L2 = LabelledOrderedTree([], label=2)
sage: L3 = LabelledOrderedTree([], label=3)
sage: T23 = LabelledOrderedTree([L2, L3], label=4)
sage: T23.left_right_symmetry()
4[3[], 2[]]

sage: T223 = LabelledOrderedTree([L2, T23], label=17)
sage: T223.left_right_symmetry()
17[4[3[], 2[]], 2[]]

sage: T223.left_right_symmetry().left_right_symmetry() == T223
True
```
sort_key()

Return a tuple of nonnegative integers encoding the labelled tree self.

The first entry of the tuple is a pair consisting of the number of children of the root and the label of the root. Then the rest of the tuple is the concatenation of the tuples associated to these children (we view the children of a tree as trees themselves) from left to right.

This tuple characterizes the labelled tree uniquely, and can be used to sort the labelled ordered trees provided that the labels belong to a type which is totally ordered.

Warning: This method overrides OrderedTree.sort_key() and returns a result different from what the latter would return, as it wants to encode the whole labelled tree including its labelling rather than just the unlabelled tree. Therefore, be careful with using this method on subclasses of LabelledOrderedTree; under some circumstances they could inherit it from another superclass instead of from OrderedTree, which would cause the method to forget the labelling. See the docstring of OrderedTree.sort_key().

EXAMPLES:

```python
sage: L2 = LabelledOrderedTree([], label=2)
sage: L3 = LabelledOrderedTree([], label=3)
sage: T23 = LabelledOrderedTree([L2, L3], label=4)
sage: T23.sort_key()
((2, 4), (0, 2), (0, 3))
sage: T32 = LabelledOrderedTree([L3, L2], label=5)
sage: T32.sort_key()
((2, 5), (0, 3), (0, 2))
sage: T23322 = LabelledOrderedTree([T23, T32, L2], label=14)
sage: T23322.sort_key()
((3, 14), (2, 4), (0, 2), (0, 3), (2, 5), (0, 3), (0, 2), (0, 2))
```

class sage.combinat.ordered_tree.LabelledOrderedTrees(category=None)

Bases: UniqueRepresentation, Parent

This is a parent stub to serve as a factory class for trees with various label constraints.

EXAMPLES:

```python
sage: LOT = LabelledOrderedTrees(); LOT
Labelled ordered trees
sage: x = LOT([], label = 3); x
3[]
sage: x.parent() is LOT
True
sage: y = LOT([x, x, x], label = 2); y
2[3[], 3[], 3[]]
sage: y.parent() is LOT
True
```

Element

alias of LabelledOrderedTree

cardinality()

Return the cardinality of self.
EXAMPLES:

```python
sage: LabelledOrderedTrees().cardinality()
+Infinity
```

**labelled_trees()**

Return the set of labelled trees associated to `self`.

This is precisely `self`, because `self` already is the set of labelled ordered trees.

**EXAMPLES:**

```python
sage: LabelledOrderedTrees().labelled_trees()
Labelled ordered trees
sage: LOT = LabelledOrderedTrees()
```

```python
sage: x = LOT([], label = 3)
sage: y = LOT([x, x, x], label = 2)
sage: y.canonical_labelling()
[2, 3, 4]
```

**unlabelled_trees()**

Return the set of unlabelled trees associated to `self`.

This is the set of ordered trees, since `self` is the set of labelled ordered trees.

**EXAMPLES:**

```python
sage: LabelledOrderedTrees().unlabelled_trees()
Ordered trees
```

**class** `sage.combinat.ordered_tree.OrderedTree(parent=None, children=None, check=True)`

Bases: `AbstractClonableTree`, `ClonableList`

The class of (ordered rooted) trees.

An ordered tree is constructed from a node, called the root, on which one has grafted a possibly empty list of trees. There is a total order on the children of a node which is given by the order of the elements in the list. Note that there is no empty ordered tree (so the smallest ordered tree consists of just one node).

**INPUT:**

One can create a tree from any list (or more generally iterable) of trees or objects convertible to a tree. Alternatively a string is also accepted. The syntax is the same as for printing: children are grouped by square brackets.

**EXAMPLES:**

```python
sage: x = OrderedTree([])
sage: x1 = OrderedTree([x,x])
sage: x2 = OrderedTree([[],[]])
sage: x1 == x2
True
sage: tt1 = OrderedTree([x,x1,x2])
sage: tt2 = OrderedTree([[], [[]], [[]], x2])
sage: tt1 == tt2
True
sage: OrderedTree([]) == OrderedTree()
True
```
is_empty()  
Return if self is the empty tree.  
For ordered trees, this always returns False.  

Note: this is different from bool(t) which returns whether t has some child or not.

EXAMPLES:

```python  
sage: t = OrderedTrees(4)([[],[[]])  
sage: t.is_empty()  
False  
sage: bool(t)  
True  
```

left_right_symmetry()  
Return the symmetric tree of self.  
The symmetric tree \( s(T) \) of an ordered tree \( T \) is defined as follows: If \( T \) is an ordered tree with children \( C_1, C_2, \ldots, C_k \) (listed from left to right), then the symmetric tree \( s(T) \) of \( T \) is the ordered tree with children \( s(C_k), s(C_{k-1}), \ldots, s(C_1) \) (from left to right).

EXAMPLES:

```python  
sage: T = OrderedTree([[],[]])  
sage: T.left_right_symmetry()  
[[[]], []]  
sage: T = OrderedTree([[], [[]], [], [][]])  
sage: T.left_right_symmetry()  
[[[[]]], [[]], [], [[]]]  
```

normalize(inplace=False)  
Return the normalized tree of self.  
INPUT:

* inplace – boolean, (default False) if True, then self is modified and nothing returned. Otherwise the normalized tree is returned.

The normalization of an ordered tree \( t \) is an ordered tree \( s \) which has the property that \( t \) and \( s \) are isomorphic as unordered rooted trees, and that if two ordered trees \( t \) and \( t' \) are isomorphic as unordered rooted trees, then the normalizations of \( t \) and \( t' \) are identical. In other words, normalization is a map from the set of ordered trees to itself which picks a representative from every equivalence class with respect to the relation of “being isomorphic as unordered trees”, and maps every ordered tree to the representative chosen from its class.

This map proceeds recursively by first normalizing every subtree, and then sorting the subtrees according to the value of the sort_key() method.

Consider the quotient map \( \pi \) that sends a planar rooted tree to the associated unordered rooted tree. Normalization is the composite \( s \circ \pi \), where \( s \) is a section of \( \pi \).

EXAMPLES:

```python  
sage: OT = OrderedTree  
sage: ta = OT([],[[]])  
sage: tb = OT([[]],[])  ```
An example with inplace normalization:

```python
sage: OT = OrderedTree
sage: ta = OT([[], [[]]])
sage: tb = OT([[[]], []])
sage: ta.normalize(inplace=True); ta
[[[], [[]]]]
sage: tb.normalize(inplace=True); tb
[[[], [[]]]]
```

`plot()`

Plot the tree `self`.

**Warning:** For a labelled tree, this will fail unless all labels are distinct. For unlabelled trees, some arbitrary labels are chosen. Use `_latex_`, `view`, `_ascii_art_` or `pretty_print` for more faithful representations of the data of the tree.

**EXAMPLES:**

```python
sage: p = OrderedTree([[[], [], []]])
sage: ascii_art(p)
  _o__
 / / /
 o o o
 | |
 o
sage: p.plot()
Graphics object consisting of 10 graphics primitives
```

Now a labelled example:

```python
sage: g = OrderedTree([[[], [[]], []]]).canonical_labelling()
sage: ascii_art(g)
  _1__
 / / /
2 3 5
 | |
4
sage: g.plot()
Graphics object consisting of 10 graphics primitives
```

`sort_key()`

Return a tuple of nonnegative integers encoding the ordered tree `self`.

The first entry of the tuple is the number of children of the root. Then the rest of the tuple is the concatenation of the tuples associated to these children (we view the children of a tree as trees themselves) from left to right.
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This tuple characterizes the tree uniquely, and can be used to sort the ordered trees.

**Note:** By default, this method does not encode any extra structure that `self` might have – e.g., if you were to define a class `EdgeColoredOrderedTree` which implements edge-colored trees and which inherits from `OrderedTree`, then the `sort_key()` method it would inherit would forget about the colors of the edges (and thus would not characterize edge-colored trees uniquely). If you want to preserve extra data, you need to override this method or use a new method. For instance, on the `LabelledOrderedTree` subclass, this method is overridden by a slightly different method, which encodes not only the numbers of children of the nodes of `self`, but also their labels. Be careful with using overridden methods, however: If you have (say) a class `BalancedTree` which inherits from `OrderedTree` and which encodes balanced trees, and if you have another class `BalancedLabelledOrderedTree` which inherits both from `BalancedOrderedTree` and from `LabelledOrderedTree`, then (depending on the MRO) the default `sort_key()` method on `BalancedLabelledOrderedTree` (unless manually overridden) will be taken either from `BalancedTree` or from `LabelledOrderedTree`, and in the former case will ignore the labelling!

**EXAMPLES:**

```python
sage: RT = OrderedTree
dsage: RT([[],[[]]]).sort_key()
(2, 0, 1, 0)
sage: RT([[[],[]],[]]).sort_key()
(2, 1, 0, 0)
```

`to_binary_tree_left_branch()`

Return a binary tree of size $n - 1$ (where $n$ is the size of $t$, and where $t$ is `self`) obtained from $t$ by the following recursive rule:

- if $x$ is the left brother of $y$ in $t$, then $x$ becomes the left child of $y$;
- if $x$ is the last child of $y$ in $t$, then $x$ becomes the right child of $y$,

and removing the root of $t$.

**EXAMPLES:**

```python
sage: T = OrderedTree([[],[[]]])
sage: T.to_binary_tree_left_branch()
[[., .], .]
sage: T = OrderedTree([[], [[], []], [], [[], [[]]]])
sage: T.to_binary_tree_left_branch()
[[[., .], [[., .], .]], [[., .], [., .]]]
```

`to_binary_tree_right_branch()`

Return a binary tree of size $n - 1$ (where $n$ is the size of $t$, and where $t$ is `self`) obtained from $t$ by the following recursive rule:

- if $x$ is the right brother of $y$ in $t$, then $x$ becomes the right child of $y$;
- if $x$ is the first child of $y$ in $t$, then $x$ becomes the left child of $y$,

and removing the root of $t$.

**EXAMPLES:**

```python
sage: T = OrderedTree([[],[[]]])
sage: T.to_binary_tree_right_branch()
```

(continues on next page)
```
[., [. . .]]
sage: T = OrderedTree([[[], [[], []], [], [[], [], []]]])
sage: T.to_binary_tree_right_branch()
[., [[., [., .]], [[., [., .]], .]]]
```

### to_dyck_word()

Return the Dyck path corresponding to self where the maximal height of the Dyck path is the depth of self.

**EXAMPLES:**

```
sage: T = OrderedTree([[[]], []])
sage: T.to_dyck_word()
[1, 0, 1, 0]
sage: T = OrderedTree([[[]], [[]]])
sage: T.to_dyck_word()
[1, 0, 1, 1, 0, 0]
sage: T = OrderedTree([[[]], [], [[], []], [], [[], [], []]])
sage: T.to_dyck_word()
[1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0]
```

### to_parallelogram_polyomino(bijection=None)

Return a polyomino parallelogram.

**INPUT:**

- **bijection** – (default: 'Boussicault-Socci') is the name of the bijection to use. Possible values are 'Boussicault-Socci', 'via dyck and Delest-Viennot'.

**EXAMPLES:**

```
sage: T = OrderedTree([[[]], [], [[], []], [], [[], [], []]])
sage: T.to_parallelogram_polyomino( bijection='Boussicault-Socci' )
[[0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0]]
sage: T = OrderedTree([])
sage: T.to_parallelogram_polyomino()
[[1]], [[1]]
sage: T = OrderedTree([])
sage: T.to_parallelogram_polyomino()
[[0, 1], [1, 0]]
sage: T = OrderedTree([])
sage: T.to_parallelogram_polyomino()
[[0, 1], [1, 0]]
sage: T = OrderedTree([])
sage: T.to_parallelogram_polyomino()
[[0, 0, 1], [1, 0, 0]]
```

### to_poset(root_to_leaf=False)

Return the poset obtained by interpreting the tree as a Hasse diagram. The default orientation is from leaves to root but you can pass root_to_leaf=True to obtain the inverse orientation.

**INPUT:**

- **root_to_leaf** – boolean, true if the poset orientation should be from root to leaves. It is false by default.
EXAMPLES:

```python
sage: t = OrderedTree([])
sage: t.to_poset()
Finite poset containing 1 elements
sage: p = OrderedTree([[[]],[[],[]]]).to_poset()
sage: p.height(), p.width()
(3, 3)
```

If the tree is labelled, we use its labelling to label the poset. Otherwise, we use the poset canonical labelling:

```python
sage: t = OrderedTree([[[]],[[],[]]]).canonical_labelling().to_poset()
sage: t.height(), t.width()
(3, 3)
```

to_undirected_graph()

Return the undirected graph obtained from the tree nodes and edges. The graph is endowed with an embedding, so that it will be displayed correctly.

EXAMPLES:

```python
sage: t = OrderedTree([])
sage: t.to_undirected_graph()
Graph on 1 vertex
sage: t = OrderedTree([[[]],[[],[]]])
sage: t.to_undirected_graph()
Graph on 5 vertices
```

If the tree is labelled, we use its labelling to label the graph. This will fail if the labels are not all distinct. Otherwise, we use the graph canonical labelling which means that two different trees can have the same graph.

EXAMPLES:

```python
sage: t = OrderedTree([[[]],[[],[]]])
sage: t.canonical_labelling().to_undirected_graph()
Graph on 5 vertices
```

```python
class sage.combinat.ordered_tree.OrderedTrees
Bases: UniqueRepresentation, Parent
Factory for ordered trees
INPUT:
• size – (optional) an integer
OUTPUT:
• the set of all ordered trees (of the given size if specified)
EXAMPLES:

sage: OrderedTrees()
Ordered trees
sage: OrderedTrees(2)
Ordered trees of size 2
```
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Note: this is a factory class whose constructor returns instances of subclasses.

Note: the fact that OrderedTrees is a class instead of a simple callable is an implementation detail. It could be changed in the future and one should not rely on it.

leaf()

Return a leaf tree with self as parent

EXAMPLES:

```
sage: OrderedTrees().leaf()
[]
```

class sage.combinat.ordered_tree.OrderedTrees_all

Bases: DisjointUnionEnumeratedSets, OrderedTrees

The set of all ordered trees.

EXAMPLES:

```
sage: OT = OrderedTrees(); OT
Ordered trees
sage: OT.cardinality()
+Infinity
```

Element

alias of OrderedTree

labelled_trees()

Return the set of labelled trees associated to self

EXAMPLES:

```
sage: OrderedTrees().labelled_trees()
Labelled ordered trees
```

unlabelled_trees()

Return the set of unlabelled trees associated to self

EXAMPLES:

```
sage: OrderedTrees().unlabelled_trees()
Ordered trees
```

class sage.combinat.ordered_tree.OrderedTrees_size(size)

Bases: OrderedTrees

The enumerated sets of binary trees of a given size

EXAMPLES:

```
sage: S = OrderedTrees(3); S
Ordered trees of size 3
sage: S.cardinality()
```

(continues on next page)
sage: S.list()
[[[], []], []]

cardinality()

The cardinality of self

This is a Catalan number.

element_class()

The class of the element of self

EXAMPLES:

sage: from sage.combinat.ordered_tree import OrderedTrees_size, OrderedTrees_all
sage: S = OrderedTrees_size(3)
sage: S.element_class is OrderedTrees().element_class
True
sage: S.first().__class__ == OrderedTrees_all().first().__class__
True

random_element()

Return a random OrderedTree with uniform probability.

This method generates a random DyckWord and then uses a bijection between Dyck words and ordered trees.

EXAMPLES:

sage: OrderedTrees(5).random_element()  # random
[[[], []], []]
sage: OrderedTrees(0).random_element()
Traceback (most recent call last):
...
EmptySetError: there are no ordered trees of size 0
sage: OrderedTrees(1).random_element()
[]

5.1.156 Output functions

These are the output functions for latexing and ascii/unicode art versions of partitions and tableaux.

AUTHORS:

• Mike Hansen (?): initial version

• Andrew Mathas (2013-02-14): Added support for displaying conventions and lines, and tableaux of skew partition, composition, and skew/composition/partition/tableaux tuple shape.

• Travis Scrimshaw (2020-08): Added support for ascii/unicode art

sage.combinat.output.ascii_art_table(data, use_unicode=False, convention='English')

Return an ascii art table of data.

EXAMPLES:
sage: from sage.combinat.output import ascii_art_table

sage: data = [[None, None, 1], [2, 2], [3, 4, 5], [None, None, 10], [], [6]]

sage: print(ascii_art_table(data))

+----+
| 1 |
+---+---+----+
| 2 | 2 |
+---+---+----+
| 3 | 4 | 5 |
+---+---+----+
| 10 |
+----+

+---+
| 6 |
++-

sage: print(ascii_art_table(data, use_unicode=True))

+—-+
| 1 |
+—+—+—-+
| 2 | 2 |
+—+—+—-+
| 3 | 4 | 5 |
+—+—+—-+
| 10 |
+—-+

+—+
| 6 |
++

sage: data = [[1, None, 2], [None, 2]]

sage: print(ascii_art_table(data))

+---+ +---+
| 1 | | 2 |
+---+---+---+
| 2 |
++-

sage: print(ascii_art_table(data, use_unicode=True))

+—+ +—+
| 1 | | 2 |
+—+—+—+
| 2 |
++

sage.combinat.output.tex_from_array(array, with_lines=True)

Return a latex string for a two dimensional array of partition, composition or skew composition shape

INPUT:

• array – a list of list

• with_lines – a boolean (default: True)

    Whether to draw a line to separate the entries in the array.
Empty rows are allowed; however, such rows should be given as [None] rather than [].

The array is drawn using either the English or French convention following \texttt{Tableaux.options}.

See also:
\texttt{tex_from_array_tuple()}

EXAMPLES:

```python
sage: from sage.combinat.output import tex_from_array
sage: print(tex_from_array(([1,2,3],[4,5])))
\begin{array}{*{3}c}
\cline{1-3}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{3} \\
\cline{1-3}
\multicolumn{1}{|c|}{4} & \multicolumn{1}{|c|}{5} \\
\cline{1-2}
\end{array}

sage: print(tex_from_array(([1,2,3],[4,5]), with_lines=False))
\begin{array}{*{3}c}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{3} \\
\multicolumn{1}{|c|}{4} & \multicolumn{1}{|c|}{5} \\
\end{array}

sage: print(tex_from_array(([1,2,3],[4,5,6,7],[8])))
\begin{array}{*{4}c}
\cline{1-3}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{3} \\
\cline{1-4}
\multicolumn{1}{|c|}{4} & \multicolumn{1}{|c|}{5} & \multicolumn{1}{|c|}{6} & \multicolumn{1}{|c|}{7} \\
\cline{1-4}
\multicolumn{1}{|c|}{8} \\
\cline{1-1}
\end{array}

sage: print(tex_from_array([[None,None,3],[None,5,6,7],[8]]))
\begin{array}{*{4}c}
\cline{3-3}
\multicolumn{2}{|c|}{3} \\
\cline{2-4}
\multicolumn{3}{|c|}{5} & \multicolumn{1}{|c|}{6} & \multicolumn{1}{|c|}{7} \\
\cline{1-4}
\multicolumn{1}{|c|}{8} \\
\cline{1-1}
\end{array}

sage: print(tex_from_array([[None,None,3],[None,5,6,7],[None,8]]))
(continues on next page)
sage: print(tex_from_array([[None,None,3],[None,5,6,7],[8]], with_lines=False))
\begin{array}{*{3}c}
\cline{1-1}
1
\cline{1-3}
2
\cline{1-3}
3
\end{array}

sage: print(tex_from_array([[None,None,3],[None,5,6,7],[None,8]], with_lines=False))
\begin{array}{*{3}c}
\cline{1-1}
1
\cline{1-3}
2
\cline{1-3}
3
\end{array}

sage: Tableaux.options.convention="french"

sage: print(tex_from_array([[1,2,3],[4,5]]))
\begin{array}{*{2}c}
\cline{1-2}
1
\cline{1-2}
2
\cline{1-2}
3
\end{array}

sage: print(tex_from_array([[1,2,3],[4,5]], with_lines=False))
\begin{array}{*{2}c}
\cline{1-1}
1
\cline{1-1}
2
\cline{1-1}
3
\end{array}

sage: print(tex_from_array([[1,2,3],[4,5,6,7],[8]]))
\begin{array}{*{4}c}
\cline{1-1}
1
\cline{1-4}
2
\cline{1-4}
3
\end{array}
sage: print(tex_from_array([[1,2,3],[4,5,6,7],[8]], with_lines=False))
\begin{array}\lr{8}\
\lr{4}&\lr{5}&\lr{6}&\lr{7}\n\lr{1}&\lr{2}&\lr{3}\n\end{array}

sage: print(tex_from_array([[None,None,3],[None,5,6,7],[8]]))
\begin{array}\lr{8}\
\lr{5}&\lr{6}&\lr{7}\
\lr{3}\n\end{array}

sage: print(tex_from_array([[None,None,3],[None,5,6,7],[None,8]], with_lines=False))
\begin{array}\lr{8}\
\lr{5}&\lr{6}&\lr{7}\
\lr{3}\n\end{array}

sage: print(tex_from_array([[None,None,3],[None,5,6,7],[None,8]], with_lines=True))
\begin{array}\lr{8}\
\lr{5}&\lr{6}&\lr{7}\
\lr{3}\n\end{array}

sage: Tableaux.options._reset()

sage.combinat.output.tex_from_array_tuple(a_tuple, with_lines=True)

Return a latex string for a tuple of two dimensional array of partition, composition or skew composition shape.

INPUT:
- a_tuple – a tuple of lists of lists
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- **with_lines** – a boolean (default: True) Whether to draw lines to separate the entries in the components of `a_tuple`.

See also:

tex_from_array() for the description of each array

EXAMPLES:

sage: from sage.combinat.output import tex_from_array_tuple
sage: print(tex_from_array_tuple([[1,2,3],[4,5]], [[None, 6, 7], [None, 8], [9]]))
\begin{array}{*{3}c}
\cline{1-3}
\begin{array}{*{3}c}
\cline{1-3}
\end{array}
\end{array}, \emptyset, \begin{array}{*{3}c}
\cline{1-3}
\end{array}

sage: print(tex_from_array_tuple([[1,2,3],[4,5]], [[None, 6, 7], [None, 8], [9]], with_lines=False))
\begin{array}{*{3}c}
\cline{1-3}
\end{array}, \emptyset, \begin{array}{*{3}c}
\cline{1-3}
\end{array}

sage: Tableaux.options.convention="french"
sage: print(tex_from_array_tuple([[1,2,3],[4,5]], [[None, 6, 7], [None, 8], [9]]))
\begin{array}{*{3}c}
\cline{1-3}
\end{array}, \emptyset, \begin{array}{*{3}c}
\cline{1-3}
\end{array}

sage: print(tex_from_array_tuple([[1,2,3],[4,5]], [[None, 6, 7], [None, 8], [9]], with_lines=False))
\begin{array}{*{3}c}
\cline{1-3}
\end{array}, \emptyset, \begin{array}{*{3}c}
\cline{1-3}
\end{array}

(continues on next page)
This function creates LaTeX code for a “skew composition” array. That is, for a two-dimensional array in which each row can begin with an arbitrary number of None’s and the remaining entries could, in principle, be anything but probably should be strings or integers of similar width. A row consisting completely of None’s is allowed.

**INPUT:**
- array – The array
- with_lines – (Default: False) If True lines are drawn, if False they are not
- align – (Default: 'b') Determines the alignment on the LaTeX array environments

**EXAMPLES:**

```python
sage: array=[[None, 2, 3, 4], [None, None], [5, 6, 7, 8]]
sage: print(sage.combinat.output.tex_from_skew_array(array))
\begin{array}{cccc}
&2 &3 &4\\
&\phantom{2} &\phantom{3} &\phantom{4}\\
5 &6 &7 &8
\end{array}
```

### 5.1.157 Parallelogram Polyominoes

The goal of this module is to give some tools to manipulate the parallelogram polyominoes.

```python
class sage.combinat.parallelogram_polyomino.LocalOptions(name='', **options):
    Bases: object
    This class allows to add local options to an object. LocalOptions is like a dictionary, it has keys and values that represent options and the values associated to the option. This is useful to decorate an object with some optional informations.

    **LocalOptions** should be used as follow.

**INPUT:**
- name – The name of the LocalOptions
- <options>=dict(...) – dictionary specifying an option

The options are specified by keyword arguments with their values being a dictionary which describes the option. The allowed/expected keys in the dictionary are:
- checker – a function for checking whether a particular value for the option is valid
- default – the default value of the option
```
values – a dictionary of the legal values for this option (this automatically defines the corresponding checker); this dictionary gives the possible options, as keys, together with a brief description of them.

```python
sage: from sage.combinatparallelogram_polyomino import LocalOptions
sage: o = LocalOptions(
    'Name Example',
    delim=dict(
        default='b',
        values=dict('b': 'the option b', 'p': 'the option p'))
)

sage: class Ex:
    options = o
    def _repr_b(self):
        return 'b'
    def _repr_p(self):
        return 'p'
    def __repr__(self):
        return self.options._dispatch(self, '_repr_', 'delim')

sage: e = Ex(); e
b
sage: e.options(delim='p'); e
p
```

This class is temporary, in the future, this class should be integrated in sage.structure.global_options.py. We should split global_option in two classes LocalOptions and GlobalOptions.

**keys()**

Return the list of the options in self.

**EXAMPLES:**

```python
sage: from sage.combinatparallelogram_polyomino import (LocalOptions)

sage: o = LocalOptions(
    "Name Example",
    tikz_options=dict(
        default="toto",
        values=dict(  
            toto="name",
            x="3"
        ))
    ),
    display=dict(  
        default="list",
        values=dict(  
            list="list representation",
            diagram="diagram representation"
        )
    )
)

sage: keys = o.keys()
sage: keys.sort()
sage: keys
['display', 'tikz_options']
```
class sage.combinat.parallelgram_polyomino.ParallelogramPolyomino(parent, value, check=True)

    Bases: ClonableList

    Parallelogram Polyominoes.

    A parallelogram polyomino is a finite connected union of cells whose boundary can be decomposed in two paths, the upper and the lower paths, which are comprised of north and east unit steps and meet only at their starting and final points.

    Parallelogram Polyominoes can be defined with those two paths.

    EXAMPLES:

    sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
    sage: pp
    [[0, 1], [1, 0]]

    area()

    Return the area of the parallelogram polyomino. The area of a parallelogram polyomino is the number of cells of the parallelogram polyomino.

    EXAMPLES:

    sage: pp = ParallelogramPolyomino(
    ....:     [0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 1],
    ....:     [1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0]
    ....:     )
    sage: pp.area()
    13

    sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
    sage: pp.area()
    1

    sage: pp = ParallelogramPolyomino([[1], [1]])
    sage: pp.area()
    0

    bounce(direction=1)

    Return the bounce of the parallelogram polyomino.

    Let \( p \) be the bounce path of the parallelogram polyomino (\texttt{bounce_path()}). The bounce is defined by:

    \[
    \sum \left( (1 + \text{floor}(i/2)) \cdot p[i] \right) \quad \text{for } i \in \text{range} \left( \text{len}(p) \right)
    \]

    INPUT:

    \* direction – the initial direction of the bounce path (see \texttt{bounce_path()} for the definition).

    EXAMPLES:

    sage: PP = ParallelogramPolyomino(  
    ....:     [[0, 0, 1, 0, 1, 1], [1, 1, 0, 0, 1, 0]]  
    ....:     )
    sage: PP.bounce(direction=1)
    6

    (continues on next page)
sage: PP.bounce(direction=0)
7
sage: PP = ParallelogramPolyomino(
    ....: [0, 0, 1, 1, 0, 1, 1, 0, 1],
    ....: [1, 1, 1, 0, 1, 1, 0, 0, 1])
    ....:
    ....:
    ....:
    ....: )
    ....:
sage: PP.bounce(direction=1)
12
sage: PP.bounce(direction=0)
10
sage: PP = ParallelogramPolyomino([[0, 1], [1, 0]])
    ....:
    ....:
sage: PP.bounce(direction=1)
1
sage: PP.bounce(direction=0)
1
sage: PP = ParallelogramPolyomino([[1], [1]])
    ....:
    ....:
sage: PP.bounce(direction=1)
0
sage: PP.bounce(direction=0)
0

**bounce_path**(direction=1)

Return the bounce path of the parallelogram polyomino.

The bounce path is a path with two steps (1, 0) and (0, 1).

If `direction` is 1 (resp. 0), the bounce path is the path starting at position (h=1, w=0) (resp. (h=0, w=1)) with initial direction, the vector (0, 1) (resp. (1, 0)), and turning each time the path crosses the perimeter of the parallelogram polyomino.

The path is coded by a list of integers. Each integer represents the size of the path between two turnings.

You can visualize the two bounce paths by using the following commands.

**INPUT:**

- `direction` – the initial direction of the bounce path (see above for the definition).

**EXAMPLES:**

```python
sage: PP = ParallelogramPolyomino(
    ....: [[0, 0, 1, 1, 0, 1, 1], [1, 1, 1, 0, 1, 1, 0]]
    ....: )
sage: PP.bounce_path(direction=1)
[2, 2, 1]
sage: PP.bounce_path(direction=0)
[2, 1, 1, 1]
```
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(continued from previous page)

```python
....:     [0, 0, 1, 1, 0, 0, 1, 1],
....:     [1, 1, 1, 0, 1, 1, 0, 0]
....: )
sage: PP.bounce_path(direction=1)
[3, 1, 2, 2]
sage: PP.bounce_path(direction=0)
[2, 4, 2]
sage: PP = ParallelogramPolyomino(
....:     [[0, 0, 1, 0, 1, 1], [1, 1, 0, 0, 1, 0]]
....: )
sage: PP.set_options(
....:     drawing_components=dict(
....:         diagram = True
....:     , bounce_0 = True
....:     , bounce_1 = True
....:     )
....: )
sage: view(PP) # not tested
sage: PP = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: PP.bounce_path(direction=1)
[1]
sage: PP.bounce_path(direction=0)
[1]
sage: PP = ParallelogramPolyomino([[1], [1]])
sage: PP.bounce_path(direction=1)
[]
sage: PP.bounce_path(direction=0)
[]
```

`box_is_node(pos)`

Return True if the box contains a node in the context of the Aval-Boussicault bijection between parallelogram polyomino and binary tree.

A box is a node if there is no cell on the top of the box in the same column or on the left of the box in the same row.

INPUT:

- `pos` – the [x,y] coordinate of the box.

OUTPUT:

A boolean

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
....:     [[0, 0, 1, 0, 1, 1, 1, 1], [1, 1, 0, 0, 1, 1, 0, 0]]
....: )
sage: pp.set_options(display='drawing')
sage: pp
```

(continues on next page)
box_is_root(box)

Return True if the box contains the root of the tree: it is the top-left box of the parallelogram polyomino.

INPUT:

• box – the x,y coordinate of the cell.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino([0, 0, 1, 0, 0, 0, 1, 1], [1, 1, 0, 1, 0, 0, 0, 0])
sage: pp.box_is_root([0, 0])
True
sage: pp.box_is_root([0, 1])
False
```

cell_is_inside(w, h)

Determine whether the cell at a given position is inside the parallelogram polyomino.

INPUT:

• w – The x coordinate of the box position.
• h – The y coordinate of the box position.

OUTPUT:

Return 0 if there is no cell at the given position, return 1 if there is a cell.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino([0, 1, 0, 0, 1, 1, 0, 1, 1, 1], [1, 1, 0, 1, 0, 0, 1, 1, 0])
sage: pp.cell_is_inside(0, 0)
1
sage: pp.cell_is_inside(1, 0)
1
sage: pp.cell_is_inside(0, 1)
0
```
sage: pp.cell_is_inside(3, 0)
0
sage: pp.cell_is_inside(pp.width()-1,pp.height()-1)
1
sage: pp.cell_is_inside(pp.width(),pp.height()-1)
0
sage: pp.cell_is_inside(pp.width()-1,pp.height())
0

check()
This method raises an error if the internal data of the class does not represent a parallelogram polyomino.

EXAMPLES:

sage: pp = ParallelogramPolyomino([...
      [0, 0, 0, 1, 0, 1, 0, 1, 1],
      [1, 0, 1, 1, 0, 0, 1, 0, 0]
    ])
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp = ParallelogramPolyomino([[1], [1]])
sage: pp = ParallelogramPolyomino([...]
    [[1, 0], [0, 1]])
# indirect doctest
Traceback (most recent call last):
...
ValueError: the lower and upper paths are crossing
sage: pp = ParallelogramPolyomino([[1], [0, 1]])
# indirect doctest
Traceback (most recent call last):
...
ValueError: the lower and upper paths have different sizes (2 != 1)

sage: pp = ParallelogramPolyomino([[1], [0, 1]])
# indirect doctest
Traceback (most recent call last):
...
ValueError: the two paths have distinct ends
sage: pp = ParallelogramPolyomino([[0], [1]])
# indirect doctest
Traceback (most recent call last):
...
ValueError: the two paths have distinct ends
sage: pp = ParallelogramPolyomino([[0], [0]])
# indirect doctest
Traceback (most recent call last):
...
ValueError: the lower or the upper path can...t be equal to [0]

sage: pp = ParallelogramPolyomino([[], [0]])
# indirect doctest
Traceback (most recent call last):
... 
ValueError: the lower or the upper path can...t be equal to []

sage: pp = ParallelogramPolyomino([[0], []])  # indirect doctest
Traceback (most recent call last):
...
ValueError: the lower or the upper path can...t be equal to []

sage: pp = ParallelogramPolyomino([], [])  # indirect doctest
Traceback (most recent call last):
...
ValueError: the lower or the upper path can...t be equal to []

degree_convexity()

Return the degree convexity of a parallelogram polyomino.

A convex polyomino is said to be k-convex if every pair of its cells can be connected by a monotone path (path with south and east steps) with at most k changes of direction. The degree of convexity of a convex polyomino P is the smallest integer k such that P is k-convex.

If the parallelogram polyomino is empty, the function return -1.

EXAMPLES:

sage: pp = ParallelogramPolyomino(
....: [0, 0, 0, 1, 0, 1, 0, 1, 1],
....: [1, 0, 1, 1, 0, 0, 1, 0, 0]
....: )
sage: pp.degree_convexity()
3

sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.degree_convexity()
0

sage: pp = ParallelogramPolyomino([[1], [1]])
sage: pp.degree_convexity()
-1

static from_dyck_word(dyck, bijection=None)

Convert a Dyck word to parallelogram polyomino.

INPUT:

• dyck – a Dyck word

• bijection – string or None (default:None) the bijection to use. See to_dyck_word() for more details.

OUTPUT:

A parallelogram polyomino.

EXAMPLES:
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```
sage: dyck = DyckWord([1, 1, 0, 1, 1, 0, 1, 0, 0, 0])
sage: ParallelogramPolyomino.from_dyck_word(dyck)
[[0, 1, 0, 1, 1, 0, 1, 0, 0, 0]]
sage: ParallelogramPolyomino.from_dyck_word(dyck, bijection='Delest-Viennot')
[[0, 1, 0, 1, 1, 0, 1, 0, 0, 0]]
sage: ParallelogramPolyomino.from_dyck_word(dyck, bijection='Delest-Viennot-beta')
[[0, 0, 1, 0, 1, 1, 1, 0, 0, 0]]
```

**geometry()**

Return a pair \([h, w]\) containing the height and the width of the parallelogram polyomino.

**EXAMPLES:**

```
sage: pp = ParallelogramPolyomino(
    ....:     [[0, 1, 1, 1, 1], [1, 1, 1, 1, 0]])
sage: pp.geometry()
[1, 4]
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.geometry()
[1, 1]
sage: pp = ParallelogramPolyomino([[1], [1]])
sage: pp.geometry()
[0, 1]
```

**get_BS_nodes()**

Return the list of cells containing node of the left and right planar tree in the Boussicault-Socci bijection.

**EXAMPLES:**

```
sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 1, 0, 0, 0, 1, 1], [1, 1, 0, 1, 0, 0, 0, 0]])
sage: pp.set_options(display='drawing')
sage: pp
[[1 1 0]
 [1 1 1]
 [0 1 1]
 [0 1 1]
 [0 1 1]]
sage: sorted(pp.get_BS_nodes())
[[0, 1], [1, 0], [1, 2], [2, 1], [3, 1], [4, 1]]
```

You can draw the point inside the parallelogram polyomino by typing (the left nodes are in blue, and the right node are in red)

```
sage: pp.set_options(drawing_components=dict(tree=True))
sage: view(pp) # not tested
```

**get_array()**

Return an array of 0s and 1s such that the 1s represent the boxes of the parallelogram polyomino.

---

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EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
    ....:     [0, 0, 0, 0, 1, 0, 1, 0, 1],
    ....:     [1, 0, 0, 0, 1, 1, 0, 0, 0]
    ....:     )
sage: matrix(pp.get_array())
[[1 0 0]
 [1 0 0]
 [1 0 0]
 [1 1 1]
 [0 1 1]
 [0 0 1]]

sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])

sage: pp.get_array()
[[1]]

sage: pp = ParallelogramPolyomino([[1], [1]])

sage: pp.get_array()
[]
```

`get_left_BS_nodes()`

Return the list of cells containing node of the left planar tree in the Boussicault-Socci bijection between parallelogram polyominoes and pair of ordered trees.

OUTPUT:

A list of [row,column] position of cells.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 1, 0, 0, 0, 1, 1], [1, 1, 0, 1, 0, 0, 0, 0]]
    ....:     )

sage: pp.set_options(display='drawing')

sage: pp
[1 0 0]
[1 1 1]
[0 1 1]
[0 1 1]

sage: sorted(pp.get_left_BS_nodes())
[[0, 1], [2, 1], [3, 1], [4, 1]]

sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 1, 0, 0, 0, 1, 1], [1, 0, 1, 1, 0, 0, 0, 0]]
    ....:     )

sage: pp.set_options(display='drawing')

sage: pp
[1 0 0]
[1 1 1]
[0 1 1]
```

(continues on next page)
You can draw the point inside the parallelogram polyomino by typing (the left nodes are in blue, and the right node are in red)

```python
sage: pp.set_options(drawing_components=dict(tree=True))
sage: view(pp)  # not tested
```

**get_node_position_from_box** *(box_position, direction, nb_crossed_nodes=None)*

This function starts from a cell inside a parallelogram polyomino and a direction.

If **direction** is equal to 0, the function selects the column associated with the y-coordinate of **box_position** and then returns the topmost cell of the column that is on the top of **box_position** (the cell of **box_position** is included).

If **direction** is equal to 1, the function selects the row associated with the x-coordinate of **box_position** and then returns the leftmost cell of the row that is on the left of **box_position**. (the cell of **box_position** is included).

This function updates the entry of **nb_crossed_nodes**. The function increases the entry of **nb_crossed_nodes** by the number of boxes that is a node (see **box_is_node**) located on the top if **direction** is 0 (resp. on the left if **direction** is 1) of **box_position** (cell at **box_position** is excluded).

**INPUT:**

- **box_position** – the position of the starting cell.
- **direction** – the direction (0 or 1).
- **nb_crossed_nodes** – [0] (default) a list containing just one integer.

**OUTPUT:**

A [row, column] position of the cell.

**EXAMPLES:**

```python
sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 1, 0, 0, 0, 1, 1],
    ....:      [1, 0, 1, 1, 0, 0, 0, 0]]
    ....: )
sage: matrix(pp.get_array())
[1 0 0]
[1 1 1]
[0 1 1]
[0 1 1]
[0 1 1]
sage: l = [0]
sage: pp.get_node_position_from_box([3, 2], 0, l)
[1, 2]
sage: l
[1]
sage: l = [0]
sage: pp.get_node_position_from_box([3, 2], 1, l)
```

(continues on next page)


```
[3, 1]
sage: l
[1]
sage: l = [0]
sage: pp.get_node_position_from_box([1, 2], 0, l)
[1, 2]
sage: l
[0]
sage: l = [0]
sage: pp.get_node_position_from_box([1, 2], 1, l)
[1, 0]
sage: l
[2]
sage: l = [0]
sage: pp.get_node_position_from_box([3, 1], 0, l)
[1, 1]
sage: l
[2]
sage: l = [0]
sage: pp.get_node_position_from_box([3, 1], 1, l)
[3, 1]
sage: l
[0]
```

**get_options()**

Return all the options of the object.

**EXAMPLES:**

```
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.get_options()
Current options for ParallelogramPolyominoes_size
- display: 'list'
- drawing_components: {'bounce_0': False, 'bounce_1': False, 'bounce_values': False, 'diagram': True, 'tree': False}
- latex: 'drawing'
- tikz_options: {'color_bounce_0': 'red', 'color_bounce_1': 'blue', 'color_line': 'black', 'color_point': 'black', 'line_size': 1, 'mirror': None, 'point_size': 3.5, 'rotation': 0, 'scale': 1, 'translation': [0, 0]}
```

**get_right_BS_nodes()**

Return the list of cells containing node of the right planar tree in the Boussicault-Socci bijection between parallelogram polyominoes and pair of ordered trees.
EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 1, 0, 0, 1, 1], [1, 1, 0, 1, 0, 0, 0, 0]],
    ....: )
sage: pp.set_options(display='drawing')
sage: pp

[[1 1 0]
 [1 1 1]
[0 1 1]
[0 1 1]
[0 1 1]]
sage: sorted(pp.get_right_BS_nodes())
[[1, 0], [1, 2]]

sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 1, 0, 0, 1, 1], [1, 0, 1, 1, 0, 0, 0, 0]],
    ....: )
sage: pp.set_options(display='drawing')
sage: pp

[[1 0 0]
 [1 1 1]
[0 1 1]
[0 1 1]
[0 1 1]]
sage: sorted(pp.get_right_BS_nodes())
[[1, 0], [1, 1], [1, 2], [2, 1], [3, 1], [4, 1]]
```

You can draw the point inside the parallelogram polyomino by typing, (the left nodes are in blue, and the right node are in red)

```python
sage: pp.set_options(drawing_components=dict(tree=True))
sage: view(pp) # not tested
```

**get_tikz_options()**

Return all the tikz options permitting to draw the parallelogram polyomino.

See `LocalOption` to have more informations about the modification of those options.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.get_tikz_options()
{'color_bounce_0': 'red',
 'color_bounce_1': 'blue',
 'color_line': 'black',
 'color_point': 'black',
 'line_size': 1,
 'mirror': None,
 'point_size': 3.5,
 'rotation': 0,
 'scale': 1,
 'translation': [0, 0]}
```

**height()**
Return the height of the parallelogram polyomino.

EXAMPLES:

```sage
sage: pp = ParallelogramPolyomino(
    ....:     
    ....:         [0, 1, 0, 0, 1, 1, 0, 1, 1, 1],
    ....:         [1, 1, 1, 0, 1, 0, 0, 1, 1, 0]
    ....:     )
sage: pp.height()
    4
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.height()
    1
sage: pp = ParallelogramPolyomino([[1], [1]])
sage: pp.height()
    0
```

heights()

Return a list of heights of the parallelogram polyomino.

Namely, the parallelogram polyomino is split column by column and the method returns the list containing the sizes of the columns.

EXAMPLES:

```sage
sage: pp = ParallelogramPolyomino(
    ....:     
    ....:         [0, 0, 0, 1, 0, 1, 0, 1, 1],
    ....:         [1, 0, 1, 1, 0, 0, 1, 0, 0]
    ....:     )
sage: pp.heights()
    [3, 3, 4, 2]
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.heights()
    [1]
sage: pp = ParallelogramPolyomino([[1], [1]])
sage: pp.heights()
    [0]
```

is_flat()

Return whether the two bounce paths join together in the rightmost cell of the bottom row of P.

EXAMPLES:

```sage
sage: pp = ParallelogramPolyomino(
    ....:     
    ....:         [0, 0, 0, 1, 0, 1, 0, 1, 1],
    ....:         [0, 0, 0, 1, 0, 1, 0, 1, 1]
    ....:     )
```

(continues on next page)
is_k_directed($k$)
Return whether the Polyomino Parallelogram is $k$-directed.

A convex polyomino is said to be $k$-convex if every pair of its cells can be connected by a monotone path (path with south and east steps) with at most $k$ changes of direction.

The degree of convexity of a convex polyomino $P$ is the smallest integer $k$ such that $P$ is $k$-convex.

INPUT:

- $k$ – An non negative integer.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino([ [0, 1, 0, 1, 0, 1, 0, 1, 1],
                                 [0, 0, 0, 1, 0, 0, 1, 0, 0]]
sage: pp.is_k_directed(3)
True
sage: pp.is_k_directed(4)
True
sage: pp.is_k_directed(5)
True
sage: pp.is_k_directed(0)
False
sage: pp.is_k_directed(1)
False
sage: pp.is_k_directed(2)
False
sage: pp = ParallelogramPolyomino([ [0, 1], [1, 0]])
```

(continues on next page)
sage: pp.is_k_directed(0)
True
sage: pp.is_k_directed(1)
True

lower_heights()

Return the list of heights associated to each vertical step of the parallelogram polyomino’s lower path.

OUTPUT:
A list of integers.

EXAMPLES:

sage: ParallelogramPolyomino([[0, 1], [1, 0]]).lower_heights()
[1]
sage: ParallelogramPolyomino(
    ....:   [[0, 0, 1, 1, 0, 1, 1, 1], [1, 0, 1, 1, 0, 1, 1, 0]]
    ....: ).lower_heights()
[2, 2, 3, 3, 3]

lower_path()

Get the lower path of the parallelogram polyomino.

EXAMPLES:

sage: lower_path = [0, 0, 1, 0, 1, 1]
sage: upper_path = [1, 1, 0, 1, 0, 0]
sage: pp = ParallelogramPolyomino([lower_path, upper_path])
sage: pp.lower_path()
[0, 0, 1, 0, 1, 1]

lower_widths()

Return the list of widths associated to each horizontal step of the parallelogram polyomino’s lower path.

OUTPUT:
A list of integers.

EXAMPLES:

sage: ParallelogramPolyomino([[0, 1], [1, 0]]).lower_widths()
[0]
sage: ParallelogramPolyomino(
    ....:   [[0, 0, 1, 1, 0, 1, 1, 1], [1, 0, 1, 1, 0, 1, 1, 0]]
    ....: ).lower_widths()
[0, 0, 2]

plot()

Return a plot of self.

EXAMPLES:

sage: pp = ParallelogramPolyomino([[0,1],[1,0]])
sage: pp.plot()
Graphics object consisting of 4 graphics primitives
\begin{verbatim}
sage: pp.set_options(drawing_components=dict(diagram=True, bounce_0=True, bounce_1=True, bounce_values=0))
sage: pp.plot()
Graphics object consisting of 7 graphics primitives

\textbf{reflect()}

Return the parallelogram polyomino obtained by switching rows and columns.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: pp = ParallelogramPolyomino([[0,0,0,0,1,1,0,1,0,1], [1,0,1,0,0,1,1,0,0,0]])
sage: pp.heights(), pp.upper_heights()
([4, 3, 2, 3], [0, 1, 3, 3])
sage: pp = pp.reflect()
sage: pp.widths(), pp.lower_widths()
([4, 3, 2, 3], [0, 1, 3, 3])
\end{verbatim}

\begin{verbatim}
sage: pp = ParallelogramPolyomino([[0,0,0,1,1], [1,0,0,1,0]])
sage: ascii_art(pp)
*
*
**
sage: ascii_art(pp.reflect())
***
**
*
\end{verbatim}

\textbf{rotate()}

Return the parallelogram polyomino obtained by rotation of 180 degrees.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: pp = ParallelogramPolyomino([[0,0,0,1,1], [1,0,0,1,0]])
sage: ascii_art(pp)
*
*
**
sage: ascii_art(pp.rotate())
**
*
*
\end{verbatim}

\textbf{set_options(*get_value, **set_value)}

Set new options to the object. See \textit{LocalOptions} for more info.

\textbf{EXAMPLES:}
\end{verbatim}
.. code-block:: python

    sage: pp = ParallelogramPolyomino(
    ....:     [0, 0, 0, 0, 1, 0, 1, 0],
    ....:     [1, 0, 0, 0, 1, 1, 0, 0]
    ....: )
    sage: pp
    [[0, 0, 0, 0, 1, 0, 1, 0], [1, 0, 0, 0, 1, 1, 0, 0]]
    sage: pp.set_options(display='drawing')
    sage: pp
    [1 0 0]
    [1 0 0]
    [1 1 1]
    [0 1 1]
    [0 0 1]

    sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
    sage: view(pp)  # not tested
    sage: pp.set_options(
    ....:     drawing_components=dict(
    ....:         diagram = True,
    ....:         bounce_0 = True,
    ....:         bounce_1 = True,
    ....:     )
    ....: )
    sage: view(pp)  # not tested

**size()**

Return the size of the parallelogram polyomino.

The size of a parallelogram polyomino is its half-perimeter.

**EXAMPLES:**

    sage: pp = ParallelogramPolyomino(
    ....:     [[0, 0, 0, 0, 1, 0, 1, 1],
    ....:      [1, 0, 0, 0, 1, 1, 0, 0]]
    ....: )
    sage: pp.size()
    8
    sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
    sage: pp.size()
    2
    sage: pp = ParallelogramPolyomino([[1], [1]])
    sage: pp.size()
    1

**to_binary_tree(bijection=None)**

Convert to a binary tree.

**INPUT:**

* bijectio – string or None (default=None) The name of bijection to use for the conversion.
The possible values are None or 'Aval-Boussicault'. The None value is equivalent to 'Aval-Boussicault'.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
    ....: [0, 0, 1, 0, 1, 0, 1, 0, 1, 1],
    ....: [1, 1, 0, 1, 1, 0, 0, 0, 1, 0]
    ....: )

sage: pp.to_binary_tree()
[[., .], [[., .], .]]

sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])

sage: pp.to_binary_tree()
[., .]

sage: pp = ParallelogramPolyomino([[1], [1]])

sage: pp.to_binary_tree()
.
```

to_dyck_word(bijection=None)

Convert to a Dyck word.

INPUT:

- bijection - string or None (default: None) The name of the bijection. If it is set to None then the 'Delest-Viennot' bijection is used. Expected values are None, 'Delest-Viennot', or 'Delest-Viennot-beta'.

OUTPUT:

a Dyck word

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino([[0, 1, 0, 0, 1, 1], [1, 1, 1, 0, 0, 0]])

sage: pp.to_dyck_word()
[1, 1, 0, 1, 1, 0, 1, 0, 0, 0]

sage: pp.to_dyck_word(bijection='Delest-Viennot')
[1, 1, 0, 1, 1, 0, 1, 0, 0, 0]

sage: pp.to_dyck_word(bijection='Delest-Viennot-beta')
[1, 0, 1, 1, 1, 0, 1, 0, 0, 0]
```

to_ordered_tree(bijection=None)

Return an ordered tree from the parallelogram polyomino.

Different bijections can be specified.

The bijection 'via dyck and Delest-Viennot' is the composition of _to_dyck_delest_viennot() and the classical bijection between dyck paths and ordered trees.

The bijection between Dyck Word and ordered trees is described in [DerZak1980] (See page 12 and 13 and Figure 3.1).

The bijection 'Boussicault-Socci' is described in [BRS2015].

INPUT:
• **bijection** — string or None (default: None) The name of bijection to use for the conversion. The possible value are None, 'Boussicault-Socci' or 'via dyck and Delest-Viennot'. The None value is equivalent to the 'Boussicault-Socci' value.

EXAMPLES:

```python
to_tikz()
Return the tikz code of the parallelogram polyomino.
This code is the code present inside a tikz latex environment.
We can modify the output with the options.
EXAMPLES:

```
(continued from previous page)

\draw[color=black, line width=1] (2.000000, 6.000000) -- (2.000000, 2.000000);
\draw[color=black, line width=1] (3.000000, 6.000000) -- (3.000000, 0.000000);
\draw[color=black, line width=1] (4.000000, 4.000000) -- (4.000000, 0.000000);
\draw[color=black, line width=1] (5.000000, 4.000000) -- (5.000000, 0.000000);
\draw[color=black, line width=1] (0.000000, 5.000000) -- (3.000000, 5.000000);
\draw[color=black, line width=1] (0.000000, 4.000000) -- (5.000000, 4.000000);
\draw[color=black, line width=1] (0.000000, 3.000000) -- (5.000000, 3.000000);
\draw[color=black, line width=1] (2.000000, 2.000000) -- (6.000000, 2.000000);
\draw[color=black, line width=1] (3.000000, 1.000000) -- (6.000000, 1.000000);
sage: pp.set_options(
    drawing_components=dict(
        diagram=True,
        tree=True,
        bounce_0=True,
        bounce_1=True
    )
)
sage: print(pp.to_tikz())

\draw[color=black, line width=1] (0.000000, 6.000000) -- (0.000000, 3.000000);
\draw[color=black, line width=1] (6.000000, 2.000000) -- (6.000000, 0.000000);
\draw[color=black, line width=1] (0.000000, 6.000000) -- (3.000000, 6.000000);
\draw[color=black, line width=1] (0.000000, 6.000000) -- (3.000000, 6.000000);
\draw[color=black, line width=1] (1.000000, 6.000000) -- (1.000000, 3.000000);
\draw[color=black, line width=1] (2.000000, 6.000000) -- (2.000000, 2.000000);
\draw[color=black, line width=1] (3.000000, 6.000000) -- (3.000000, 0.000000);
\draw[color=black, line width=1] (4.000000, 4.000000) -- (4.000000, 0.000000);
\draw[color=black, line width=1] (5.000000, 4.000000) -- (5.000000, 0.000000);
\draw[color=black, line width=1] (0.000000, 5.000000) -- (3.000000, 5.000000);
\draw[color=black, line width=1] (0.000000, 4.000000) -- (5.000000, 4.000000);
\draw[color=black, line width=1] (0.000000, 3.000000) -- (5.000000, 3.000000);
\draw[color=black, line width=1] (2.000000, 2.000000) -- (6.000000, 2.000000);
\draw[color=black, line width=1] (3.000000, 1.000000) -- (6.000000, 1.000000);

(continues on next page)

5.1. Comprehensive Module List 1647
upper_heights()  
Return the list of heights associated to each vertical step of the parallelogram polyomino’s upper path.

OUTPUT:
A list of integers.

EXAMPLES:

```
sage: ParallelogramPolyomino([[0, 1], [1, 0]]).upper_heights()
[0]
sage: ParallelogramPolyomino(
       ....: [[0, 0, 1, 1, 0, 1, 1, 0],
       ....:  [1, 0, 1, 1, 0, 1, 1, 0]]
       ....: ).upper_heights()
[0, 1, 1, 2, 2]
```

upper_path()  
Get the upper path of the parallelogram polyomino.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: lower_path = [0, 0, 1, 0, 1, 1]
sage: upper_path = [1, 1, 0, 1, 0, 0]
sage: pp = ParallelogramPolyomino([lower_path, upper_path])
sage: pp.upper_path()
[1, 1, 0, 1, 0, 0]
```

**upper_widths()**

Return the list of widths associated to each horizontal step of the parallelogram polyomino's upper path.

OUTPUT:

A list of integers.

EXAMPLES:

```python
sage: ParallelogramPolyomino([[0, 1], [1, 0]]).upper_widths()
[1]
sage: ParallelogramPolyomino(
    ....:     [[0, 0, 1, 1, 0, 1, 1, 1], [1, 0, 1, 1, 0, 1, 1, 0]]
    ....: ).upper_widths()
[1, 3, 5]
```

**width()**

Return the width of the parallelogram polyomino.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
    ....:     [0, 0, 0, 1, 0, 1, 0, 1, 1],
    ....:     [1, 1, 1, 0, 1, 0, 0, 1, 1, 0]
    ....: )
sage: pp.width()
6
sage: pp = ParallelogramPolyomino([[0, 1], [1, 0]])
sage: pp.width()
1
sage: pp = ParallelogramPolyomino([[1], [1]])
sage: pp.width()
1
```

**widths()**

Return a list of the widths of the parallelogram polyomino.

Namely, the parallelogram polyomino is split row by row and the method returns the list containing the sizes of the rows.

EXAMPLES:

```python
sage: pp = ParallelogramPolyomino(
    ....:     [
    ....:         [0, 0, 0, 1, 0, 1, 0, 1, 1],
    ....:         [1, 1, 1, 0, 1, 0, 0, 1, 1, 0]
    ....:     ]
    ....: )
```

(continues on next page)
sage.combinat.parallelgram_polyomino.ParallelogramPolyominoes(size=None, policy=None)

Return a family of parallelogram polyominoes enumerated with the parameter constraints.

INPUT:

• size – integer (default: None), the size of the parallelogram
  polyominoes contained in the family. If set to None, the family returned contains all the parallelogram
  polyominoes.

EXAMPLES:

sage: PPS = ParallelogramPolyominoes(size=4)
sage: PPS
Parallelogram polyominoes of size 4
sage: sorted(PPS)
[[[0, 0, 0, 1], [1, 0, 0, 0]],
 [0, 0, 1, 1], [1, 0, 1, 0]],
 [[0, 0, 1, 1], [1, 1, 0, 0]],
 [[0, 1, 0, 1], [1, 1, 0, 0]],
 [[0, 1, 1, 1], [1, 1, 1, 0]]]

sage: PPS = ParallelogramPolyominoes()
sage: PPS
Parallelogram polyominoes
sage: PPS.cardinality()+Infinity

sage: PPS = ParallelogramPolyominoes(size=None)
sage: PPS
Parallelogram polyominoes
sage: PPS.cardinality()+Infinity

class sage.combinat.parallelgram_polyomino.ParallelogramPolyominoesFactory

Bases: SetFactory

The parallelogram polyominoes factory.

EXAMPLES:
Combinatorics, Release 10.0

sage: PPS = ParallelogramPolyominoes(size=4)
sage: PPS
Parallelogram polyominoes of size 4
sage: sorted(PPS)
[[[0, 0, 0, 1], [1, 0, 0, 0]],
 [[[0, 0, 1, 1], [1, 0, 1, 0]],
 [[[0, 0, 1, 1], [1, 1, 0, 0]],
 [[[0, 1, 1, 1], [1, 1, 1, 0]]]

sage: PPS = ParallelogramPolyominoes()
sage: PPS
Parallelogram polyominoes
sage: PPS.cardinality()
+Infinity

sage.combinat.parallelogram_polyomino.ParallelogramPolyominoesOptions = Current options for ParallelogramPolyominoes size - display: 'list' - drawing_components: {'bounce_0': False, 'bounce_1': False, 'bounce_values': False, 'diagram': True, 'tree': False} - latex: 'drawing' - tikz_options: {'color_bounce_0': 'red', 'color_bounce_1': 'blue', 'color_line': 'black', 'color_point': 'black', 'line_size': 1, 'mirror': None, 'point_size': 3.5, 'rotation': 0, 'scale': 1, 'translation': [0, 0]}

This global option contains all the data needed by the Parallelogram classes to draw, display in ASCII, compile in latex a parallelogram polyomino.
The available options are:

- tikz_options: this option configure all the information useful to generate TIKZ code. For example, color, line size, etc ...

- drawing_components: this option is used to explain to the system which component of the drawing you want to draw. For example, you can ask to draw some elements of the following list: - the diagram, - the tree inside the parallelogram polyomino, - the bounce paths inside the parallelogram polyomino, - the value of the bounce on each square of a bounce path.

- display: this option is used to configure the ASCII display. The available options are: - list: (this is the default value) is used to represent PP as a list containing the upper and lower path. - drawing: this value is used to explain we want to display an array with the PP drawn inside (with connected 1).

- latex: Same as display. The default is “drawing”.

See ParallelogramPolyomino.get_options() for more details and for an user use of options.

EXAMPLES:

sage: from sage.combinat.parallelogram_polyomino import (.....: ParallelogramPolyominoesOptions.....:)sage: opt = ParallelogramPolyominoesOptions['tikz_options']sage: opt
{'color_bounce_0': 'red',
 'color_bounce_1': 'blue',
 'color_line': 'black',
 'color_point': 'black',
 'line_size': 1,
Combinatorics, Release 10.0

(continued from previous page)

```python
'mirror': None,
'point_size': 3.5,
'rotation': 0,
'scale': 1,
'translation': [0, 0]}
```

class sage.combinat.parallelogram_polyomino.ParallelogramPolyominoes_all(policy)

Bases: ParentWithSetFactory, DisjointUnionEnumeratedSets

This class enumerates all the parallelogram polyominoes.

EXAMPLES:

```python
sage: PPS = ParallelogramPolyominoes()
sage: PPS
Parallelogram polyominoes
```

def check_element(el, check)

Check is a given element `el` is in the set of parallelogram polyominoes.

EXAMPLES:

```python
sage: PPS = ParallelogramPolyominoes()
sage: ParallelogramPolyomino(  # indirect doctest
    [[0, 1, 1], [1, 1, 0]])
in PPS
True
```

def get_options()

Return all the options associated with the set of parallelogram polyominoes.

EXAMPLES:

```python
sage: PPS = ParallelogramPolyominoes()
sage: options = PPS.get_options()
sage: options
Current options for ParallelogramPolyominoes_size - display: 'list' -
drawing_components: {'bounce_0': False, 'bounce_1': False, 'bounce_values': False, 'diagram': True, 'tree': False} - latex: 'drawing' - tikz_options: {'color_bounce_0': 'red', 'color_bounce_1': 'blue', 'color_line': 'black', 'color_point': 'black', 'line_size': 1, 'mirror': None, 'point_size': 3.5, 'rotation': 0, 'scale': 1, 'translation': [0, 0]}
```

set_options(*get_value, **set_value)

Set new options to the object.

EXAMPLES:
sage: PPS = ParallelogramPolyominoes()
sage: PPS.set_options(
    drawing_components=dict(  
        diagram = True,
        bounce_0 = True,
        bounce_1 = True,
    )
)
sage: pp = next(iter(PPS))
sage: view(pp)  # not tested

class sage.combinat.parallelogram_polyomino.ParallelogramPolyominoes_size(size, policy)
Bases: ParentWithSetFactory, UniqueRepresentation

The parallelogram polyominoes of size $n$.

EXAMPLES:

sage: PPS = ParallelogramPolyominoes(4)
sage: PPS
Parallelogram polyominoes of size 4
sage: sorted(PPS)
[[[0, 0, 0, 1], [1, 0, 0, 0]],
 [[0, 0, 1, 1], [1, 0, 1, 0]],
 [[0, 0, 1, 1], [1, 1, 0, 0]],
 [[0, 1, 0, 1], [1, 1, 0, 0]],
 [[0, 1, 1, 1], [1, 1, 1, 0]]]

an_element()
Return an element of a parallelogram polyomino of a given size.

EXAMPLES:

sage: PPS = ParallelogramPolyominoes(4)
sage: PPS.an_element() in PPS
True

cardinality()
Return the number of parallelogram polyominoes.

The number of parallelogram polyominoes of size $n$ is given by the Catalan number $c_{n-1}$.

EXAMPLES:

sage: ParallelogramPolyominoes(1).cardinality()
1
sage: ParallelogramPolyominoes(2).cardinality()
1
sage: ParallelogramPolyominoes(3).cardinality()
2
sage: ParallelogramPolyominoes(4).cardinality()
5
sage: all(  
    ....:     ParallelogramPolyominoes(i).cardinality()  
    ....:     for i in range(1, 5)  
)  
True
Combinatorics, Release 10.0

(continued from previous page)

```python
....:   == len(list(ParallelogramPolyominoes(i)))
....:   for i in range(1,7)
....:
True
```

**check_element**(el, check)

Check is a given element `el` is in the set of parallelogram polyominoes of a fixed size.

**EXAMPLES:**

```python
sage: PPS = ParallelogramPolyominoes(3)
sage: ParallelogramPolyomino([0, 1, 1], [1, 1, 0]) in PPS
True
```

**get_options()**

Return all the options associated with all the elements of the set of parallelogram polyominoes with a fixed size.

**EXAMPLES:**

```python
sage: pps = ParallelogramPolyominoes(5)
sage: pps.get_options()
Current options for ParallelogramPolyominoes_size - display: 'list'
- drawing_components: {'bounce_0': False, 'bounce_1': False, 'bounce_values': False, 'diagram': True, 'tree': False} - latex: 'drawing' - tikz_options: {'color_bounce_0': 'red', 'color_bounce_1': 'blue', 'color_line': 'black', 'color_point': 'black', 'line_size': 1, 'mirror': None, 'point_size': 3.5, 'rotation': 0, 'scale': 1, 'translation': [0, 0]}
```

The options for ParallelogramPolyominoes.

**set_options(** *get_value*, **set_value**)**

Set new options to the object.

**EXAMPLES:**

```python
sage: PPS = ParallelogramPolyominoes(3)
sage: PPS.set_options(
....:   drawing_components=dict(
....:       diagram = True,
....:       bounce_0 = True,
....:       bounce_1 = True,
....:   )
....: )
sage: pp = PPS[0]
sage: view(pp) # not tested
```

**size()**

Return the size of the parallelogram polyominoes generated by this parent.
**EXAMPLES:**

```
sage: ParallelogramPolyominoes(0).size()
sage: ParallelogramPolyominoes(1).size()
sage: ParallelogramPolyominoes(5).size()
```

```
sage.combinat.parallelogram_polyomino.default_tikz_options = {'color_bounce_0': 'red',
'color_bounce_1': 'blue', 'color_line': 'black', 'color_point': 'black', 'line_size':
1, 'mirror': None, 'point_size': 3.5, 'rotation': 0, 'scale': 1, 'translation': [0, 0]}
```

This is the default TIKZ options.

This option is used to configure element of a drawing to allow TIKZ code generation.

**5.1.158 Parking Functions**

INFORMALLY (reference [Beck]):

Imagine a one-way cul-de-sac with $n$ parking spots. We will give the first parking spot the number 1, the next one number 2, etc., down to the last one, number $n$. Initially they are all free, but there are $n$ cars approaching the street, and they would all like to park there. To make life interesting, every car has a parking preference, and we record the preferences in a sequence; For example, if $n = 3$, the sequence $(2, 1, 1)$ means that the first car would like to park at spot number 2, the second car prefers parking spot number 1, and the last car would also like to part at number 1. The street is very narrow, so there is no way to back up. Now each car enters the street and approaches its preferred parking spot; if it is free, it parks there, and if not, it moves down the street to the first available spot. We call a sequence a parking function (of length $n$) if all cars end up finding a parking spot. For example, the sequence $(2, 1, 1)$ is a parking sequence (of length 3), whereas the sequence $(2, 3, 2)$ is not.

FORMALLY:

A parking function of size $n$ is a sequence $(a_1, \ldots, a_n)$ of positive integers such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \leq i$.

A parking function of size $n$ is a pair $(L, D)$ of two sequences $L$ and $D$ where $L$ is a permutation and $D$ is an area sequence of a Dyck path of size $n$ such that $D[i] \geq 0$, $D[i+1] \leq D[i] + 1$ and if $D[i+1] = D[i] + 1$ then $L[i+1] > L[i]$. The number of parking functions of size $n$ is equal to the number of rooted forests on $n$ vertices and is equal to $(n+1)^{n-1}$.

REFERENCES:

AUTHORS:

• used non-decreasing_parking_functions code by Florent Hivert (2009 - 04)

• Dorota Mazur (2012 - 09)

```
sage.combinat.parking_functions.PF
```

alias of ParkingFunction

```
class sage.combinat.parking_functions.ParkingFunction(parent, lst)
```

Bases: ClonableArray

A Parking Function.

A parking function of size $n$ is a sequence $(a_1, \ldots, a_n)$ of positive integers such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \leq i$.  

5.1. Comprehensive Module List 1655
A parking function of size $n$ is a pair $(L, D)$ of two sequences $L$ and $D$ where $L$ is a permutation and $D$ is an area sequence of a Dyck Path of size $n$ such that $D[i] \geq 0$, $D[i + 1] \leq D[i] + 1$ and if $D[i + 1] = D[i] + 1$ then $L[i + 1] > L[i]$.

The number of parking functions of size $n$ is equal to the number of rooted forests on $n$ vertices and is equal to $(n + 1)^{n-1}$.

**INPUT:**
- `pf` – (default: None) a list whose increasing rearrangement satisfies $b_i \leq i$
- `labelling` – (default: None) a labelling of the Dyck path
- `area_sequence` – (default: None) an area sequence of a Dyck path
- `labelled_dyck_word` – (default: None) a Dyck word with 1’s replaced by labelling

**OUTPUT:**
A parking function

**EXAMPLES:**

```python
sage: ParkingFunction([])
[]
sage: ParkingFunction([1])
[1]
sage: ParkingFunction([2])
Traceback (most recent call last):
  ... ValueError: [2] is not a parking function	sage: ParkingFunction([1,2])
[1, 2]
sage: ParkingFunction([1,1,2])
[1, 1, 2]
sage: ParkingFunction([1,4,1])
Traceback (most recent call last):
  ... ValueError: [1, 4, 1] is not a parking function	sage: ParkingFunction(labelling=[3,1,2], area_sequence=[0,0,1])
[2, 2, 1]
sage: ParkingFunction([2,2,1]).to_labelled_dyck_word()
[3, 0, 1, 2, 0, 0]
sage: ParkingFunction(labelled_dyck_word=[3,0,1,2,0,0])
[2, 2, 1]
sage: ParkingFunction(labelling=[3,1,2], area_sequence=[0,1,1])
Traceback (most recent call last):
  ... ValueError: [3, 1, 2] is not a valid labeling of area sequence [0, 1, 1]
```

def area()

Return the area of the labelled Dyck path corresponding to the parking function.

**OUTPUT:**
- the sum of squares under and over the main diagonal the Dyck Path, corresponding to the parking function

**EXAMPLES:**
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.area()
6

sage: ParkingFunction([3,1,1,4]).area()
1
sage: ParkingFunction([4,1,1,1]).area()
3
sage: ParkingFunction([2,1,4,1]).area()
2

cars_permutation()

Return the sequence of cars that take parking spots 1 through \( n \) and corresponding to the parking function.

For example, cars_permutation(PF) = [2, 4, 5, 6, 3, 1, 7] means that car 2 takes spots 1, car 4 takes spot 2, ..., car 1 takes spot 6 and car 7 takes spot 7.

OUTPUT:

- the permutation of cars corresponding to the parking function and which is the same size as parking function

EXAMPLES:

sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.cars_permutation()
[2, 4, 5, 6, 3, 1, 7]

sage: ParkingFunction([3,1,1,4]).cars_permutation()
[2, 3, 1, 4]

sage: ParkingFunction([4,1,1,1]).cars_permutation()
[2, 3, 4, 1]

sage: ParkingFunction([2,1,4,1]).cars_permutation()
[2, 1, 4, 3]

characteristic_quasisymmetric_function(q=None, R=Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field)

Return the characteristic quasisymmetric function of self.

The characteristic function of the Parking Function is the sum over all permutation labellings of the Dyck path \( q^{\text{inv}(PF)} F_{\text{ides}(PF)} \), where \( \text{ides}(PF) \) is the descent composition of the diagonal reading word of the parking function.

INPUT:

- \( q \) – (default: \( q = R('q') \)) a parameter for the generating function power
- \( R \) – (default: \( R = QQ['q','t'].fraction_field() \)) the base ring to do the calculations over

OUTPUT:

- an element of the quasisymmetric functions over the ring \( R \)

EXAMPLES:

sage: R = QQ['q','t'].fraction_field()
sage: (q,t) = R.gens()
sage: cqf = sum(t**PF.area()*PF.characteristic_quasisymmetric_function() for PF in ParkingFunctions(3)); cqf
(q^3+q^2*t+q*t^2+t^3+q*t)*F[1, 1, 1] + (q^2+q^2*t+t^2+q+t)*F[1, 2] + (q^2+q^2+q^2+t^2+q+t)*F[2, 1] + F[3]

sage: s = SymmetricFunctions(R).s()
sage: s(cqf.to_symmetric_function())
(q^3+q^2*t+q*t^2+t^3+q*t)*s[1, 1, 1] + (q^2+q*t+t^2+q+t)*s[2, 1] + s[3]

sage: s(cqf.to_symmetric_function()).nabla(power = -1)
s[1, 1, 1]

sage: p = ParkingFunction([3, 1, 2])
sage: p.characteristic_quasisymmetric_function()
q^2*F[1, 1, 2, 1, 3]

check()
Check that self is a valid parking function.

EXAMPLES:

sage: PF = ParkingFunction([1, 1, 2, 2, 5, 6])
sage: PF.check()

diagonal_composition()
Return the composition of the labelled Dyck path corresponding to the parking function.

For example, touch_composition(PF) = [4, 3] means that the first touch is four diagonal units from the starting point, and the second is three units further (see [GXZ] p. 2).

OUTPUT:

• the length between the corresponding touch points which of the labelled Dyck path that corresponds to self

EXAMPLES:

sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.touch_composition()
[4, 3]

sage: ParkingFunction([3,1,1,4]).touch_composition() [2, 1, 1]
sage: ParkingFunction([4,1,1,1]).touch_composition() [3, 1]
sage: ParkingFunction([2,1,4,1]).touch_composition() [3, 1]

diagonal_reading_word()
Return a diagonal word of the labelled Dyck path corresponding to parking function (see [Hag08] p. 75).

OUTPUT:

• returns a word, read diagonally from NE to SW of the pretty print of the labelled Dyck path that corresponds to self and the same size as self
diagonal_word()  
Return a diagonal word of the labelled Dyck path corresponding to parking function (see [Hag08] p. 75).

OUTPUT:
• returns a word, read diagonally from NE to SW of the pretty print of the labelled Dyck path that corresponds to self and the same size as self

EXAMPLES:

```
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.diagonal_reading_word()
[5, 1, 7, 4, 6, 3, 2]
sage: ParkingFunction([1, 1, 1]).diagonal_reading_word()
[3, 2, 1]
sage: ParkingFunction([1, 2, 3]).diagonal_reading_word()
[3, 2, 1]
sage: ParkingFunction([1, 1, 3, 4]).diagonal_reading_word()
[2, 4, 3, 1]
sage: ParkingFunction([1, 1, 1]).diagonal_word()
[3, 2, 1]
sage: ParkingFunction([1, 2, 3]).diagonal_word()
[3, 2, 1]
sage: ParkingFunction([1, 4, 3, 1]).diagonal_word()
[4, 2, 3, 1]
```
• the number of dinversion pairs

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.dinv()
6
```

```python
sage: ParkingFunction([3,1,1,4]).dinv()
sage: ParkingFunction([4,1,1,1]).dinv()
sage: ParkingFunction([2,1,4,1]).dinv()
3
1
2
```

dinversion_pairs()

Return the descent inversion pairs of a labelled Dyck path corresponding to the parking function.

OUTPUT:

• the primary and secondary diversion pairs

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.dinversion_pairs()
[(0, 4), (1, 5), (2, 5), (1, 4), (2, 4), (3, 6)]
sage: ParkingFunction([3,1,1,4]).dinversion_pairs()
[(0, 3), (2, 3), (1, 2)]
sage: ParkingFunction([4,1,1,1]).dinversion_pairs()
[(1, 3)]
sage: ParkingFunction([2,1,4,1]).dinversion_pairs()
[(0, 3), (1, 3)]
```

grade()

Return the length of the parking function.

EXAMPLES:

```python
sage: PF = ParkingFunction([1, 1, 2, 2, 5, 6])
sage: PF.grade()
6
```

ides()

Return the \texttt{descents()} sequence of the inverse of the \texttt{diagonal_reading_word()} of \texttt{self}.

\textbf{Warning}: Here we use the standard convention that descent labels start at 1. This behaviour has been changed in \texttt{github issue #20555}.

For example, \texttt{ides(PF) = [2, 3, 4, 6]} means that descents are at the 2nd, 3rd, 4th and 6th positions in the inverse of the \texttt{diagonal_reading_word()} of the parking function (see [GXZ] p. 2).

OUTPUT:

• the descents sequence of the inverse of the \texttt{diagonal_reading_word()} of the parking function
EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.ides() [2, 3, 4, 6]
```

```python
sage: ParkingFunction([3,1,1,4]).ides() [2]
sage: ParkingFunction([4,1,1,1]).ides() [2, 3]
sage: ParkingFunction([4,3,1,1]).ides() [3]
```

`ides_composition()`

Return the `descents_composition()` of the inverse of the `diagonal_reading_word()` of corresponding parking function.

For example, `ides_composition(PF) = [4, 2, 1]` means that the descents of the inverse of the permutation `diagonal_reading_word()` of the parking function with word `PF` are at the 4th and 6th positions.

OUTPUT:

- the descents composition of the inverse of the `diagonal_reading_word()` of the parking function

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.ides_composition() [2, 1, 1, 2, 1]
```

```python
sage: ParkingFunction([3,1,1,4]).ides_composition() [2, 2]
sage: ParkingFunction([4,1,1,1]).ides_composition() [2, 1, 1]
sage: ParkingFunction([4,3,1,1]).ides_composition() [3, 1]
```

`jump()`

Return the sum of the differences between the parked and preferred parking spots.

See [Shin] p. 18.

OUTPUT:

- the sum of the differences between the parked and preferred parking spots

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.jump() 6
```

```python
sage: ParkingFunction([3,1,1,4]).jump() 1
sage: ParkingFunction([4,1,1,1]).jump()
```

(continues on next page)
jump_list()

Return the displacements of cars that corresponds to the parking function.

For example, `jump_list(PF) = [0, 0, 0, 0, 1, 3, 2]` means that car 1 through 4 parked in their preferred spots, car 5 had to park one spot farther (jumped or was displaced by one spot), car 6 had to jump 3 spots, and car 7 had to jump two spots.

OUTPUT:

- the displacements sequence of parked cars which corresponds to the parking function and which is the same size as parking function

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.jump_list()
[0, 0, 0, 0, 1, 3, 2]
```

```python
sage: ParkingFunction([3,1,1,4]).jump_list()
[0, 0, 1, 0]
sage: ParkingFunction([4,1,1,1]).jump_list()
[0, 0, 1, 2]
sage: ParkingFunction([2,1,4,1]).jump_list()
[0, 0, 0, 2]
```

luck()

Return the number of cars that parked in their preferred parking spots (see [Shin] p. 33).

OUTPUT:

- the number of cars that parked in their preferred parking spots

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.luck()
4
```

```python
sage: ParkingFunction([3,1,1,4]).luck()
3
sage: ParkingFunction([4,1,1,1]).luck()
2
sage: ParkingFunction([2,1,4,1]).luck()
3
```

lucky_cars()

Return the cars that can park in their preferred spots. For example, `lucky_cars(PF) = [1, 2, 7]` means that cars 1, 2 and 7 parked in their preferred spots and all the other cars did not.

OUTPUT:

- the cars that can park in their preferred spots
EXAMPLES:

```sage
PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
P.F.lucky_cars()
[1, 2, 3, 4]
```

```sage
PF = ParkingFunction([3,1,1,4]).lucky_cars()
[1, 2, 4]
P.F = ParkingFunction([4,1,1,1]).lucky_cars()
[1, 2]
P.F = ParkingFunction([2,1,4,1]).lucky_cars()
[1, 2, 3]
```

```sage
PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
P.F.parking_permutation()
[6, 1, 5, 2, 3, 4, 7]
P.F = ParkingFunction([3,1,1,4]).parking_permutation()
[3, 1, 2, 4]
P.F = ParkingFunction([4,1,1,1]).parking_permutation()
[4, 1, 2, 3]
P.F = ParkingFunction([2,1,4,1]).parking_permutation()
[2, 1, 4, 3]
```

```sage
PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
P.F.pretty_print()
```

```
___
 |\1x
 |7x
_____\3
```

(continues on next page)
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.pretty_print(underpath = false)

    _
   | x 1
  | x . 7
 ___| . . 3
 | x x . . 5
  _| x . . . 4
 | x . . . . 6
 | . . . . . 2

sage: ParkingFunction([3, 1, 1, 4]).pretty_print()

    _
   _|4
 __|1 .
|3x . .
|2 . .

sage: ParkingFunction([1,1,1]).pretty_print()

    ___
   |3x x
  |2x .
  |1 .

sage: ParkingFunction([4,1,1,1]).pretty_print()

    ___-
   ___|1
  |4x x .
  |3x . .
  |2 . .

sage: ParkingFunction([2,1,4,1]).pretty_print()

    __
   ___|3
  _|1x .
 |4x ..
|2 . .

sage: ParkingFunction([2,1,4,1]).pretty_print(underpath = false)

    ___
   ___ 3
  _| x 1
 | x . . 4
| . . 2

sage: pf = ParkingFunction([1,2,3,7,3,2,1,2,3,2,1])
sage: pf.pretty_print()
The diagram shows a labelled Dyck path with the corresponding parking function.

**primary_dinversion_pairs()**

Return the primary descent inversion pairs of a labelled Dyck path corresponding to the parking function.

**OUTPUT:**
- the pairs \((i, j)\) such that \(i < j\), and \(i^{th}\) area = \(j^{th}\) area, and \(i^{th}\) label < \(j^{th}\) label

**EXAMPLES:**
```
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.primary_dinversion_pairs()
[(0, 4), (1, 5), (2, 5)]
sage: ParkingFunction([3,1,1,4]).primary_dinversion_pairs()
[(0, 3)]
sage: ParkingFunction([4,1,1,1]).primary_dinversion_pairs()
[]
sage: ParkingFunction([2,1,4,1]).primary_dinversion_pairs()
[(0, 3)]
```

**secondary_dinversion_pairs()**

Return the secondary descent inversion pairs of a labelled Dyck path corresponding to the parking function.

**OUTPUT:**
- the pairs \((i, j)\) such that \(i < j\), and \(i^{th}\) area = \(j^{th}\) area + 1, and \(i^{th}\) label > \(j^{th}\) label

**EXAMPLES:**
```
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.secondary_dinversion_pairs()
[(1, 4), (2, 4), (3, 6)]
sage: ParkingFunction([3,1,1,4]).secondary_dinversion_pairs()
[(1, 2)]
sage: ParkingFunction([4,1,1,1]).secondary_dinversion_pairs()
[(1, 3)]
sage: ParkingFunction([2,1,4,1]).secondary_dinversion_pairs()
[(1, 3)]
```

**to_NonDecreasingParkingFunction()**

Return the non-decreasing parking function which underlies the parking function.
OUTPUT:

- a sorted parking function

See also:

NonDecreasingParkingFunction()

EXAMPLES:

```sage
PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
PF.to_NonDecreasingParkingFunction()
[1, 1, 2, 2, 5, 5, 6]
```

```sage
ParkingFunction([3,1,1,4]).to_NonDecreasingParkingFunction()
[1, 1, 3, 4]
```

```sage
ParkingFunction([4,1,1,1]).to_NonDecreasingParkingFunction()
[1, 1, 1, 4]
```

```sage
ParkingFunction([2,1,4,1]).to_NonDecreasingParkingFunction()
[1, 1, 2, 4]
```

```sage
ParkingFunction([4,1,2,1]).to_NonDecreasingParkingFunction()
[1, 1, 2, 4]
```

to_area_sequence()

Return the area sequence of the support Dyck path of the parking function.

OUTPUT:

- the area sequence of the Dyck path

EXAMPLES:

```sage
PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
PF.to_area_sequence()
[0, 1, 1, 2, 0, 1, 1]
```

```sage
ParkingFunction([3,1,1,4]).to_area_sequence()
[0, 1, 0, 0]
```

```sage
ParkingFunction([4,1,1,1]).to_area_sequence()
[0, 1, 2, 0]
```

```sage
ParkingFunction([2,1,4,1]).to_area_sequence()
[0, 1, 1, 0]
```

to_dyck_word()

Return the support Dyck word of the parking function.

OUTPUT:

- the Dyck word of the corresponding parking function

See also:

DyckWord()

EXAMPLES:

```sage
PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
PF.to_dyck_word()
[1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0]
```
to_labelled_dyck_word()

Return the labelled Dyck word corresponding to the parking function.

This is a representation of the parking function as a list where the entries of 1 in the Dyck word are replaced with the corresponding label.

OUTPUT:

• the labelled Dyck word of the corresponding parking function which is twice the size of parking function word

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.to_labelled_dyck_word()
[2, 6, 0, 4, 5, 0, 0, 0, 3, 7, 0, 1, 0, 0]
```

```python
sage: ParkingFunction([3,1,1,4]).to_labelled_dyck_word()
[2, 3, 0, 0, 1, 0, 4, 0]
```

```python
sage: ParkingFunction([4,1,1,1]).to_labelled_dyck_word()
[2, 3, 4, 0, 0, 0, 1, 0]
```

```python
sage: ParkingFunction([2,1,4,1]).to_labelled_dyck_word()
[2, 4, 0, 1, 0, 0, 3, 0]
```

to_labelling_area_sequence_pair()

Return a pair consisting of a labelling and an area sequence of a Dyck path which corresponds to the given parking function.

OUTPUT:

• returns a pair (L, D) where L is a labelling and D is the area sequence of the underlying Dyck path

EXAMPLES:

```python
sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.to_labelling_area_sequence_pair()
([2, 6, 4, 5, 3, 7, 1], [0, 1, 1, 2, 0, 1, 1])
```

```python
sage: ParkingFunction([1, 1, 1]).to_labelling_area_sequence_pair()
([1, 2, 3], [0, 1, 2])
```

```python
sage: ParkingFunction([1, 2, 3]).to_labelling_area_sequence_pair()
([1, 2, 3], [0, 0, 0])
```

```python
sage: ParkingFunction([1, 1, 2]).to_labelling_area_sequence_pair()
([1, 2, 3], [0, 1, 1])
```

```python
sage: ParkingFunction([1, 1, 3, 1]).to_labelling_area_sequence_pair()
([1, 2, 4, 3], [0, 1, 2, 1])
```

to_labelling_dyck_word_pair()

Return the pair (L, D) where L is a labelling and D is the Dyck word of the parking function.
• the pair \((L, D)\), where \(L\) is the labelling and \(D\) is the Dyck word of the parking function

See also:
\(\text{DyckWord()}\)

EXAMPLES:

\[
\begin{align*}
\text{sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])} \\
\text{sage: PF.to_labelling_dyck_word_pair()}
&= ([2, 6, 4, 5, 3, 7, 1], [1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0]) \\
\text{sage: ParkingFunction([3,1,1,4]).to_labelling_dyck_word_pair()}
&= ([2, 3, 1, 4], [1, 1, 0, 0, 1, 0, 1, 0]) \\
\text{sage: ParkingFunction([4,1,1,1]).to_labelling_dyck_word_pair()}
&= ([2, 4, 1, 3], [1, 1, 0, 1, 0, 0, 1, 0]) \\
\text{sage: ParkingFunction([2,1,4,1]).to_labelling_dyck_word_pair()}
&= ([2, 4, 1, 3], [1, 1, 0, 1, 0, 1, 0])
\end{align*}
\]

\(\text{to_labelling_permutation()}\)

Return the labelling of the support Dyck path of the parking function.

OUTPUT:
• the labelling of the Dyck path

EXAMPLES:

\[
\begin{align*}
\text{sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])} \\
\text{sage: PF.to_labelling_permutation()}
&= [2, 6, 4, 5, 3, 7, 1] \\
\text{sage: ParkingFunction([3,1,1,4]).to_labelling_permutation()}
&= [2, 3, 1, 4] \\
\text{sage: ParkingFunction([4,1,1,1]).to_labelling_permutation()}
&= [2, 3, 4, 1] \\
\text{sage: ParkingFunction([2,1,4,1]).to_labelling_permutation()}
&= [2, 4, 1, 3]
\end{align*}
\]

\(\text{touch_composition()}\)

Return the composition of the labelled Dyck path corresponding to the parking function.

For example, \(\text{touch_composition(PF)} = [4, 3]\) means that the first touch is four diagonal units from the starting point, and the second is three units further (see [GXZ] p. 2).

OUTPUT:
• the length between the corresponding touch points which of the labelled Dyck path that corresponds to the parking function

EXAMPLES:

\[
\begin{align*}
\text{sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])} \\
\text{sage: PF.touch_composition()}
&= [4, 3]
\end{align*}
\]
sage: ParkingFunction([3,1,1,4]).touch_composition()
[2, 1, 1]
sage: ParkingFunction([4,1,1,1]).touch_composition()
[3, 1]
sage: ParkingFunction([2,1,4,1]).touch_composition()
[3, 1]

touch_points()

Return the sequence of touch points which corresponds to the labelled Dyck path after initial step.

For example, touch_points(PF) = [4, 7] means that after the initial step, the path touches the main diagonal at points (4, 4) and (7, 7).

OUTPUT:

• the sequence of touch points after the initial step of the labelled Dyck path that corresponds to the parking function

EXAMPLES:

sage: PF = ParkingFunction([6, 1, 5, 2, 2, 1, 5])
sage: PF.touch_points()
[4, 7]
sage: ParkingFunction([3,1,1,4]).touch_points()
[2, 3, 4]
sage: ParkingFunction([4,1,1,1]).touch_points()
[3, 4]
sage: ParkingFunction([2,1,4,1]).touch_points()
[3, 4]

class sage.combinat.parking_functions.ParkingFunctions

Bases: UniqueRepresentation, Parent

Return the combinatorial class of Parking Functions.

A parking function of size \( n \) is a sequence \((a_1, \ldots, a_n)\) of positive integers such that if \( b_1 \leq b_2 \leq \cdots \leq b_n \) is the increasing rearrangement of \( a_1, \ldots, a_n \), then \( b_i \leq i \).

A parking function of size \( n \) is a pair \((L, D)\) of two sequences \( L \) and \( D \) where \( L \) is a permutation and \( D \) is an area sequence of a Dyck Path of size \( n \) such that \( D[i] \geq 0 \), \( D[i+1] \leq D[i]+1 \) and if \( D[i+1] = D[i]+1 \) then \( L[i+1] > L[i] \).

The number of parking functions of size \( n \) is equal to the number of rooted forests on \( n \) vertices and is equal to \( (n+1)^{n-1} \).

EXAMPLES:

Here are all parking functions of size 3:

sage: from sage.combinat.parking_functions import ParkingFunctions
sage: ParkingFunctions(3).list()
[[1, 1, 1], [1, 1, 2], [1, 2, 1], [2, 1, 1], [1, 1, 3], [1, 3, 1],
 [3, 1, 1], [1, 2, 2], [2, 1, 2], [2, 2, 1], [1, 2, 3], [1, 3, 2],
 [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]

If no size is specified, then ParkingFunctions returns the combinatorial class of all parking functions.
If the size $n$ is specified, then ParkingFunctions returns the combinatorial class of all parking functions of size $n$.

```sage
PF = ParkingFunctions(0)
sage: PF.list()
[[]]
sage: PF = ParkingFunctions(1)
sage: PF.list()
[[1]]
sage: PF = ParkingFunctions(3)
sage: PF.list()
[[[1, 1, 1], [1, 1, 2], [1, 2, 1], [2, 1, 1], [1, 1, 3],
  [1, 3, 1], [3, 1, 1], [1, 2, 2], [2, 1, 2], [2, 2, 1],
  [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]]
```

```sage
PF3 = ParkingFunctions(3); PF3
Parking functions of size 3
sage: [] in PF3
False
sage: [1] in PF3
False
sage: [1,3,1] in PF3
True
sage: [1,4,1] in PF3
False
```

class sage.combinat.parking_functions.ParkingFunctions_all

Bases: ParkingFunctions

Element

alias of ParkingFunction

graded_component($n$)

Return the graded component.

EXAMPLES:

```sage
PF = ParkingFunctions()
sage: PF.graded_component(4) == ParkingFunctions(4)
True
sage: it = iter(ParkingFunctions()) # indirect doctest
```
class sage.combinat.parking_functions.ParkingFunctions_n(n)

The combinatorial class of parking functions of size $n$.

A parking function of size $n$ is a sequence $(a_1, \ldots, a_n)$ of positive integers such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \leq i$.

A parking function of size $n$ is a pair $(L, D)$ of two sequences $L$ and $D$ where $L$ is a permutation and $D$ is an area sequence of a Dyck Path of size $n$ such that $D[i] \geq 0$, $D[i+1] \leq D[i] + 1$ and if $D[i+1] = D[i] + 1$ then $L[i+1] > L[i]$.

The number of parking functions of size $n$ is equal to the number of rooted forests on $n$ vertices and is equal to $(n + 1)^{n-1}$.

EXAMPLES:

```
sage: PF = ParkingFunctions(3)
sage: PF.list()
[[1, 1, 1], [1, 1, 2], [1, 2, 1], [2, 1, 1], [1, 1, 3], [1, 3, 1], [3, 1, 1], [1, 2, 2], [2, 1, 2], [2, 2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]
sage: [ParkingFunctions(i).cardinality() for i in range(6)]
[1, 1, 3, 16, 125, 1296]
```

Warning: The precise order in which the parking function are generated or listed is not fixed, and may change in the future.

Element

alias of ParkingFunction

cardinality()  

Return the number of parking functions of size $n$.

The cardinality is equal to $(n + 1)^{n-1}$.

EXAMPLES:

```
sage: [ParkingFunctions(i).cardinality() for i in range(6)]
[1, 1, 3, 16, 125, 1296]
```

random_element()  

Return a random parking function of size $n$.

The algorithm uses a circular parking space with $n + 1$ spots. Then all $n$ cars can park and there remains one empty spot. Spots are then renumbered so that the empty spot is 0.

The probability distribution is uniform on the set of $(n + 1)^{n-1}$ parking functions of size $n$.

EXAMPLES:
sage: pf = ParkingFunctions(8)
sage: a = pf.random_element(); a  # random
[5, 7, 2, 4, 2, 5, 1, 3]
sage: a in pf
True

sage.combinat.parking_functions.from_labelled_dyck_word(LDW)

Return the parking function corresponding to the labelled Dyck word.

INPUT:

• LDW – labelled Dyck word

OUTPUT:

• the parking function corresponding to the labelled Dyck word that is half the size of LDW

EXAMPLES:

sage: from sage.combinat.parking_functions import from_labelled_dyck_word
sage: LDW = [2, 6, 0, 4, 5, 0, 0, 3, 7, 0, 1, 0, 0]
sage: from_labelled_dyck_word(LDW)
[6, 1, 5, 2, 2, 1, 5]
sage: from_labelled_dyck_word([2, 3, 0, 0, 1, 0, 4, 0])
[3, 1, 1, 1]
sage: from_labelled_dyck_word([2, 4, 0, 1, 0, 0, 3, 0])
[2, 1, 4, 1]

sage.combinat.parking_functions.from_labelling_and_area_sequence(L, D)

Return the parking function corresponding to the labelling area sequence pair.

INPUT:

• L – a labelling permutation
  
• D – an area sequence for a Dyck word

OUTPUT:

• the parking function corresponding the labelling permutation L and D an area sequence of the corresponding Dyck path

EXAMPLES:

sage: from sage.combinat.parking_functions import from_labelling_and_area_sequence
sage: from_labelling_and_area_sequence([2, 6, 4, 5, 3, 7, 1], [0, 1, 1, 2, 0, 1, 1])
[6, 1, 5, 2, 2, 1, 5]
sage: from_labelling_and_area_sequence([1, 2, 3], [0, 1, 2])
[1, 1, 1]
sage: from_labelling_and_area_sequence([1, 2, 3], [0, 0, 0])
[1, 2, 3]
sage: from_labelling_and_area_sequence([1, 2, 3], [0, 1, 1])
[1, 1, 2]
sage: from_labelling_and_area_sequence([1, 2, 4, 3], [0, 1, 2, 1])
[1, 1, 3, 1]

sage.combinat.parking_functions.is_a(x, n=None)
  Check whether a list is a parking function.
  If a size $n$ is specified, checks if a list is a parking function of size $n$.

5.1.159 Catalog of Path Tableaux

The path_tableaux object may be used to access examples of various path tableau objects currently implemented in Sage. Using tab-completion on this object is an easy way to discover and quickly create the path tableaux that are available (as listed here).

Let <tab> indicate pressing the Tab key. So begin by typing path_tableaux.<tab> to see the currently implemented path tableaux.

- CylindricalDiagram
- DyckPath
- DyckPaths
- FriezePattern
- FriezePatterns
- SemistandardPathTableau
- SemistandardPathTableaux

5.1.160 Dyck Paths

This is an implementation of the abstract base class sage.combinat.path_tableaux.path_tableau.PathTableau. This is the simplest implementation of a path tableau and is included to provide a convenient test case and for pedagogical purposes.

In this implementation we have sequences of nonnegative integers. These are required to be the heights Dyck words (except that we do not require the sequence to start or end at height zero). These are in bijection with skew standard tableaux with at most two rows. Sequences which start and end at height zero are in bijection with noncrossing perfect matchings.

AUTHORS:
- Bruce Westbury (2018): initial version

class sage.combinat.path_tableaux.dyck_path.DyckPath(parent, ot, check=True)
  Bases: PathTableau

  An instance is the sequence of nonnegative integers given by the heights of a Dyck word.

  INPUT:
  - a sequence of nonnegative integers
  - a two row standard skew tableau
  - a Dyck word
  - a noncrossing perfect matching

5.1. Comprehensive Module List
EXAMPLES:

```python
sage: path_tableaux.DyckPath([0,1,2,1,0])
[0, 1, 2, 1, 0]
sage: w = DyckWord([1,1,0,0])
sage: path_tableaux.DyckPath(w)
[0, 1, 2, 1, 0]
sage: p = PerfectMatching([(1,2), (3,4)])
sage: path_tableaux.DyckPath(p)
[0, 1, 0, 1, 0]
sage: t = Tableau([[1,2,4],[3,5,6]])
sage: path_tableaux.DyckPath(t)
[0, 1, 2, 1, 2, 1, 0]
sage: st = SkewTableau([[None, 1,4],[2,3]])
sage: path_tableaux.DyckPath(st)
[1, 2, 1, 0, 1]
```

Here we illustrate the slogan that promotion = rotation:

```python
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.to_perfect_matching()
[(0, 5), (1, 4), (2, 3)]
sage: t = t.promotion()
sage: t.to_perfect_matching()
[(0, 3), (1, 2), (4, 5)]
sage: t = t.promotion()
sage: t.to_perfect_matching()
[(0, 1), (2, 5), (3, 4)]
sage: t = t.promotion()
sage: t.to_perfect_matching()
[(0, 5), (1, 4), (2, 3)]
```

**check()**

Check that `self` is a valid path.

EXAMPLES:

```python
sage: path_tableaux.DyckPath([0,1,0,-1,0]) # indirect doctest
Traceback (most recent call last):
... ValueError: [0, 1, 0, -1, 0] has a negative entry
sage: path_tableaux.DyckPath([0,1,3,1,0]) # indirect doctest
Traceback (most recent call last):
... ValueError: [0, 1, 3, 1, 0] is not a Dyck path
```

**descents()**
Return the descent set of self.

EXAMPLES:

```python
sage: path_tableaux.DyckPath([0,1,2,1,0,1,0]).descents()
{3, 6}
```

`is_skew()`

Return True if self is skew and False if not.

EXAMPLES:

```python
sage: path_tableaux.DyckPath([0,1,2,1]).is_skew()
False
sage: path_tableaux.DyckPath([1,0,1,2,1]).is_skew()
True
```

`local_rule(i)`

This has input a list of objects. This method first takes the list of objects of length three consisting of the

\( (i-1) \)-st, \( i \)-th and \( (i+1) \)-term and applies the rule. It then replaces the \( i \)-th object by the object returned

by the rule.

EXAMPLES:

```python
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.local_rule(3)
[0, 1, 2, 1, 2, 1, 0]
```

`to_DyckWord()`

Converts self to a Dyck word.

EXAMPLES:

```python
sage: c = path_tableaux.DyckPath([0,1,2,1,0])
sage: c.to_DyckWord()
[1, 1, 0, 0]
```

`to_perfect_matching()`

Return the perfect matching associated to self.

EXAMPLES:

```python
sage: path_tableaux.DyckPath([0,1,2,1,0,1,0]).to_perfect_matching()
[(0, 5), (1, 2), (3, 4), (6, 7)]
```

`to_tableau()`

Return the skew tableau associated to self.

EXAMPLES:

```python
sage: T = path_tableaux.DyckPath([0,1,2,3,2,3])
sage: T.to_tableau()
[[1, 2, 3, 5], [4]]
```

(continues on next page)
sage: U = path_tableaux.DyckPath([2,3,2,3])
sage: U.to_tableau()
[[None, None, 1, 3], [2]]

to_word()
Return the word in the alphabet \{0,1\} associated to self.

EXAMPLES:
sage: path_tableaux.DyckPath([1,0,1,2,1]).to_word()
[0, 1, 1, 0]

class sage.combinat.path_tableaux.dyck_path.DyckPaths

Bases: PathTableaux

The parent class for DyckPath.

Element

alias of DyckPath

5.1.161 Frieze Patterns

This implements the original frieze patterns due to Conway and Coxeter. Such a frieze pattern is considered as a sequence of nonnegative integers following [CoCo1] and [CoCo2] using sage.combinat.path_tableaux.

path_tableau.

AUTHORS:

• Bruce Westbury (2019): initial version

class sage.combinat.path_tableaux.frieze.FriezePattern

Bases: PathTableau

A frieze pattern.

We encode a frieze pattern as a sequence in a fixed ground field.

INPUT:

• fp – a sequence of elements of field
• field – (default: QQ) the ground field

EXAMPLES:
sage: t = path_tableaux.FriezePattern([1,2,1,2,3,1])
sage: path_tableaux.CylindricalDiagram(t)
[0, 1, 2, 1, 2, 3, 1, 0]
[0, 1, 1, 3, 5, 2, 1, 0]
[0, 1, 4, 7, 3, 2, 1, 0]
[0, 1, 2, 1, 1, 1, 1, 0]
[0, 1, 1, 2, 3, 4, 1, 0]
[0, 1, 3, 5, 7, 2, 1, 0]
[0, 1, 2, 1, 2, 3, 1, 0]
[0, 1, 2, 1, 2, 3, 1, 0]
sage: TestSuite(t).run()

sage: t = path_tableaux.FriezePattern([1,2,7,5,3,7,4,1])

sage: path_tableaux.CylindricalDiagram(t)
[0, 1, 2, 7, 5, 3, 7, 4, 1, 0]
[0, 1, 4, 3, 2, 5, 3, 1, 1, 0]
[0, 1, 2, 7, 5, 3, 7, 4, 1, 0]
[0, 1, 4, 3, 2, 5, 3, 1, 1, 0]
[0, 1, 2, 7, 5, 3, 7, 4, 1, 0]
[0, 1, 4, 3, 2, 5, 3, 1, 1, 0]
[0, 1, 2, 7, 5, 3, 7, 4, 1, 0]
[0, 1, 4, 3, 2, 5, 3, 1, 1, 0]

sage: TestSuite(t).run()

sage: t = path_tableaux.FriezePattern([1,3,4,5,1])

sage: path_tableaux.CylindricalDiagram(t)
[0, 1, 3, 4, 5, 1, 0]
[0, 1, 5/3, 7/3, 2/3, 1, 0]
[0, 1, 2, 1, 3, 1, 0]
[0, 1, 1, 4, 5/3, 1, 0]
[0, 1, 5, 7/3, 2, 1, 0]
[0, 1, 2/3, 1, 1, 1, 0]
[0, 1, 3, 4, 5, 1, 0]
[0, 1, 4, 3, 2, 5, 3, 1, 1, 0]

sage: TestSuite(t).run()

This constructs the examples from [HJ18]:

sage: K.<sqrt3> = NumberField(x^2-3)

sage: t = path_tableaux.FriezePattern([1,sqrt3,2,sqrt3,1,1], field=K)

sage: path_tableaux.CylindricalDiagram(t)
[0, 1, sqrt3, 2, sqrt3, 1, 1, ⨲ 0]
[0, 1, sqrt3, 2, sqrt3, sqrt3 + 1, ⨲ 1, 0]
[0, 1, sqrt3, 2, sqrt3 + 2, ⨲ sqrt3, 1, 0]
[0, 1, sqrt3 + 1, sqrt3 + 2, ⨲ sqrt3, 2, 1, 0]
[0, 1, sqrt3 + 2, sqrt3 + 2, ⨲ sqrt3 + 1, sqrt3 + 1, 0]
[0, 1, sqrt3 + 1, sqrt3 + 2, ⨲ sqrt3 + 2, sqrt3 + 1, 0]

sage: TestSuite(t).run()
```plaintext
sage: K.<sqrt2> = NumberField(x^2-2)
sage: t = path_tableaux.FriezePattern([1,sqrt2,1,sqrt2,3,2*sqrt2,5,3*sqrt2,1],
→   field=K)
sage: path_tableaux.CylindricalDiagram(t)
[ [ 0, 1, sqrt2, 1, sqrt2, 3, 2*sqrt2, 5, 3*sqrt2, 1],
→   1, 0]
[ [ 0, 1, sqrt2, 3, 5*sqrt2, 7, 9*sqrt2, 11, 3*sqrt2, 1],
→   2*sqrt2, 1, 0]
[ [ 0, 1, 2*sqrt2, 7, 5*sqrt2, 13, 8*sqrt2, 5, 3*sqrt2, 1],
→   sqrt2, 1, 0]
[ [ 1, sqrt2, 3, 2*sqrt2, 1, 0],
→   sqrt2, 1, sqrt2, 1, 0]
[ [ 1, 2*sqrt2, 7, 5*sqrt2, 3, sqrt2, 1, 0],
→   sqrt2, 3, 5*sqrt2, 7, 2*sqrt2, 1, 0]
[ [ 0, 1, 2*sqrt2, 3, 2*sqrt2, 1, 0],
→   1, 0]
[ [ 1, 3*sqrt2, 11, 8*sqrt2, 5, 2*sqrt2, 3, sqrt2, 1],
→   0]
[ [ 0, 1, 2*sqrt2, 3, sqrt2, 1, sqrt2, 1, sqrt2, 1],
→   1, 0]
[ [ 0, 1, sqrt2, 1, sqrt2, 3, 2*sqrt2, 5, 3*sqrt2, 1],
→   1, 0]
```

```plaintext
change_ring(R)

Return self as a frieze pattern with coefficients in R assuming there is a canonical coerce map from the base ring of self to R.

EXAMPLES:

```plaintext
sage: path_tableaux.FriezePattern([1,2,7,5,3,7,4,1]).change_ring(RealField())
[0.000000000000000, 1.00000000000000, ... 4.00000000000000, 1.00000000000000, 0.
→   0.000000000000000]
```

```plaintext
sage: path_tableaux.FriezePattern([1,2,7,5,3,7,4,1]).change_ring(GF(7))
Traceback (most recent call last):
...
TypeError: no base extension defined
```

```plaintext
check()

Check that self is a valid frieze pattern.
```

```plaintext
is_integral()

Return True if all entries of the frieze pattern are positive integers.
```
EXAMPLES:

```python
sage: path_tableaux.FriezePattern([1,2,7,5,3,7,4,1]).is_integral()
True
sage: path_tableaux.FriezePattern([1,3,4,5,1]).is_integral()
False
```

**is_positive()**
Return True if all elements of self are positive.
This implies that all entries of CylindricalDiagram(self) are positive.

**Warning:** There are orders on all fields. These may not be ordered fields as they may not be compatible with the field operations. This method is intended to be used with ordered fields only.

EXAMPLES:

```python
sage: path_tableaux.FriezePattern([1,2,7,5,3,7,4,1]).is_positive()
True
sage: path_tableaux.FriezePattern([1,-3,4,5,1]).is_positive()
False
sage: K.<sqrt3> = NumberField(x^2-3)
sage: path_tableaux.FriezePattern([1,sqrt3,1],K).is_positive()
True
```

**is_skew()**
Return True if self is skew and False if not.

EXAMPLES:

```
sage: path_tableaux.FriezePattern([1,2,1,2,3,1]).is_skew()
False
sage: path_tableaux.FriezePattern([2,2,1,2,3,1]).is_skew()
True
```

**local_rule(i)**
Return the $i$-th local rule on self.
This interprets self as a list of objects. This method first takes the list of objects of length three consisting of the $(i-1)$-st, $i$-th and $(i+1)$-term and applies the rule. It then replaces the $i$-th object by the object returned by the rule.

EXAMPLES:

```
sage: t = path_tableaux.FriezePattern([1,2,1,2,3,1])
sage: t.local_rule(3)
[1, 2, 5, 2, 3, 1]
sage: t = path_tableaux.FriezePattern([1,2,1,2,3,1])
sage: t.local_rule(0)
```

(continues on next page)
Traceback (most recent call last):
...
ValueError: 0 is not a valid integer

plot(model='UHP')
Plot the frieze as an ideal hyperbolic polygon.
This is only defined up to isometry of the hyperbolic plane.
We are identifying the boundary of the hyperbolic plane with the real projective line.
The option model must be one of
  • 'UHP' - (default) for the upper half plane model
  • 'PD' - for the Poincare disk model
  • 'KM' - for the Klein model
The hyperboloid model is not an option as this does not implement boundary points.

EXAMPLES:

```
sage: t = path_tableaux.FriezePattern([1,2,7,5,3,7,4,1])
sage: t.plot()
Graphics object consisting of 18 graphics primitives
sage: t.plot(model='UHP')
Graphics object consisting of 18 graphics primitives
sage: t.plot(model='PD')
Traceback (most recent call last):
...
TypeError: '>' not supported between instances of 'NotANumber' and 'Pi'
sage: t.plot(model='KM')
Graphics object consisting of 18 graphics primitives
```
**triangulation()**

Plot a regular polygon with some diagonals.

If `self` is positive and integral then this will be a triangulation.

```
sage: path_tableaux.FriezePattern([1,2,7,5,3,7,4,1]).triangulation()
Graphics object consisting of 25 graphics primitives

sage: path_tableaux.FriezePattern([1,2,1/7,5,3]).triangulation()
Graphics object consisting of 12 graphics primitives

sage: K.<sqrt2> = NumberField(x^2-2)
sage: path_tableaux.FriezePattern([1,sqrt2,1,sqrt2,3,2*sqrt2,5,3*sqrt2,1],field=K).triangulation()
Graphics object consisting of 24 graphics primitives
```

**width()**

Return the width of `self`.

If the first and last terms of `self` are 1 then this is the length of `self` plus two and otherwise is undefined.

**EXAMPLES:**

```
sage: path_tableaux.FriezePattern([1,2,1,2,3,1]).width()
8

sage: path_tableaux.FriezePattern([1,2,1,2,3,4]).width() is None
True
```

class `sage.combinat.path_tableaux.frieze.FriezePatterns(field)`

Bases: `PathTableaux`

The set of all frieze patterns.

**EXAMPLES:**
5.1.162 Path Tableaux

This is an abstract base class for using local rules to construct rectification and the action of the cactus group [Wes2017]. This is a construction of the Henriques-Kamnitzer construction of the action of the cactus group on tensor powers of a crystal. This is also a generalisation of the Fomin growth rules, which are a version of the operations on standard tableaux which were previously constructed using jeu de taquin.

The basic operations are rectification, evacuation and promotion. Rectification of standard skew tableaux agrees with the rectification by jeu de taquin as does evacuation. Promotion agrees with promotion by jeu de taquin on rectangular tableaux but in general they are different.

REFERENCES:
- [Wes2017]

AUTHORS:
- Bruce Westbury (2018): initial version

**class** `sage.combinat.path_tableaux.path_tableau.CylindricalDiagram(T)`

Bases: `SageObject`

Cylindrical growth diagrams.

**EXAMPLES:**

```python
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: path_tableaux.CylindricalDiagram(t)
[0, 1, 2, 3, 2, 1, 0]
[ , 0, 1, 2, 1, 0, 1, 0]
[ , , 0, 1, 0, 1, 2, 1, 0]
[ , , , 0, 1, 2, 3, 2, 1, 0]
[ , , , , 0, 1, 2, 1, 0, 1, 0]
[ , , , , , 0, 1, 0, 1, 2, 1, 0]
[ , , , , , , 0, 1, 2, 3, 2, 1, 0]
```

**pp()**

A pretty print utility method.

**EXAMPLES:**

```python
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: path_tableaux.CylindricalDiagram(t).pp()
```

(continues on next page)
0 1 2 3 2 1 0
0 1 2 1 0 1 0
0 1 0 2 1 0
0 1 2 3 2 1 0
0 1 2 1 0 1 0
0 1 0 1 2 1 0
0 1 2 3 2 1 0

sage: t = path_tableaux.FriezePattern([1,3,4,5,1])
sage: path_tableaux.CylindricalDiagram(t).pp()
0 1 3 4 5 1 0
0 1 5/3 7/3 2/3 1 0
0 1 2 1 3 1 0
0 1 1 4 5/3 1 0
0 1 5 7/3 2 1 0
0 1 2/3 1 1 1 0
0 1 3 4 5 1 0

class sage.combinat.path_tableaux.path_tableau.PathTableau
Bases: ClonableArray

This is the abstract base class for a path tableau.

cactus(i, j)

Return the action of the generator \( s_{i,j} \) of the cactus group on \( \text{self} \).

INPUT:

i – a positive integer
j – a positive integer weakly greater than i

EXAMPLES:

sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.cactus(1,5)
[0, 1, 2, 1, 0]
sage: t.cactus(1,6)
[0, 1, 2, 3, 2, 1, 0]
sage: t.cactus(1,7) == t.evacuation()
True
sage: t.cactus(1,7).cactus(1,6) == t.promotion()
True

commutor(other, verbose=False)

Return the commutor of \( \text{self} \) with \( \text{other} \).

If verbose=True then the function will print the rectangle.

EXAMPLES:

sage: t1 = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t2 = path_tableaux.DyckPath([0,1,2,1,0])
sage: t1.commutor(t2)
([0, 1, 2, 1, 0], [0, 1, 2, 3, 2, 1, 0])
sage: t1.commutor(t2, verbose=True)
[[0, 1, 2, 1, 0],
 [1, 2, 3, 2, 1],
 [2, 3, 4, 3, 2],
 [3, 4, 5, 4, 3],
 [2, 3, 4, 3, 2],
 [1, 2, 3, 2, 1],
 [0, 1, 2, 1, 0]]

dual_equivalence_graph()

Return the graph with vertices the orbit of `self` and edges given by the action of the cactus group generators.

In most implementations the generators $s_{i,i+1}$ will act as the identity operators. The usual dual equivalence graphs are given by replacing the label $i, i + 2$ by $i$ and removing edges with other labels.

EXAMPLES:

sage: s = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: s.dual_equivalence_graph().edges(sort=True)

[(0, 1, 0, 1, 0, 1, 0), (0, 1, 2, 1, 0, 1, 0), '2,6'],
 [(0, 1, 0, 1, 2, 1, 0), (0, 1, 2, 3, 2, 1, 0), '3,7'],
 [(0, 1, 2, 1, 2, 1, 0), (0, 1, 2, 3, 2, 1, 0), '3,6']
]

evacuation()

Return the evacuation operator applied to `self`.

EXAMPLES:

sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.evacuation()
[0, 1, 2, 3, 2, 1, 0]
**final_shape()**

Return the final shape of self.

**EXAMPLES:**

```
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.final_shape()
[]
```

**initial_shape()**

Return the initial shape of self.

**EXAMPLES:**

```
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.initial_shape()
[]
```

**local_rule(i)**

This is the abstract local rule defined in any coboundary category.

This has input a list of objects. This method first takes the list of objects of length three consisting of the
\((i - 1)\)-st, \(i\)-th and \((i + 1)\)-term and applies the rule. It then replaces the \(i\)-th object by the object returned
by the rule.

**EXAMPLES:**

```
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.local_rule(3)
[0, 1, 2, 1, 2, 1, 0]
```

**orbit()**

Return the orbit of self under the action of the cactus group.

**EXAMPLES:**

```
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.orbit()
{[0, 1, 0, 1, 0, 1, 0],
 [0, 1, 0, 1, 2, 1, 0],
 [0, 1, 2, 1, 0, 1, 0],
 [0, 1, 2, 1, 2, 1, 0],
 [0, 1, 2, 3, 2, 1, 0]}
```

**promotion()**

Return the promotion operator applied to self.

**EXAMPLES:**

```
sage: t = path_tableaux.DyckPath([0,1,2,3,2,1,0])
sage: t.promotion()
[0, 1, 2, 1, 0, 1, 0]
```

**size()**

Return the size or length of self.

**EXAMPLES:**
class sage.combinat.path_tableaux.path_tableau.PathTableaux
    Bases: UniqueRepresentation, Parent

    The abstract parent class for PathTableau.

5.1.163 Semistandard Tableaux

This is an implementation of the abstract base class sage.combinat.path_tableaux.path_tableau.PathTableau.

This implementation is for semistandard tableaux, represented as a chain of partitions (essentially, the Gelfand-Tsetlin pattern). This generalises the jeu de taquin operations of rectification, promotion, evacuation from standard tableaux to semistandard tableaux. The local rule is the Bender-Knuth involution.

EXAMPLES:

```python
sage: pt = path_tableaux.SemistandardPathTableau([[],[3],[3,2],[3,3,1],[3,3,2,1],[4,3,3,1,0]])
sage: pt.promotion()
[(), (2,), (3, 1), (3, 2, 1), (4, 3, 1, 0), (4, 3, 3, 1, 0)]

sage: pt.evacuation()
[(), (2,), (4, 0), (4, 2, 0), (4, 3, 1, 0), (4, 3, 3, 1, 0)]

sage: pt = path_tableaux.SemistandardPathTableau([[],[3],[3,2],[3,3,1],[3,3,2,1],[9/2,3,3,1,0]])
sage: pt.promotion()
[(), (2,), (3, 1), (3, 2, 1), (9/2, 3, 1, 0), (9/2, 3, 3, 1, 0)]

sage: pt.evacuation()
[(), (5/2,), (9/2, 0), (9/2, 2, 0), (9/2, 3, 1, 0), (9/2, 3, 3, 1, 0)]

sage: pt = path_tableaux.SemistandardPathTableau([[],[3],[4,2],[5,4,1]])
sage: path_tableaux.CylindricalDiagram(pt)
[ (), (3,), (4, 2), (5, 4, 1)]
[ , (), (3,), (5, 2), (5, 4, 1)]
[ , , (), (4,), (4, 3), (5, 4, 1)]
[ , , , (), (3,), (5, 1), (5, 4, 1)]

sage: pt2 = path_tableaux.SemistandardPathTableau([[],[3],[3,2],[3,3,1],[3,3,2,1],[4,3,3,1,0]])
sage: pt1 = path_tableaux.SemistandardPathTableau([[],[3],[3,2]])
sage: pt1.commutor(pt2)
(((), (2,), (2, 2), (4, 2, 0)), [(4, 2, 0), (4, 3, 2, 0), (4, 3, 3, 1, 0)])

sage: pt1.commutor(pt2,verbose=True)
((3, 2), (3, 3, 1), (3, 3, 2, 1), (4, 3, 3, 1, 0)]
[[3,3), (3, 2), (3, 3, 2), (4, 3, 2, 0)]
[[(), (2,), (2, 2), (4, 2, 0)]
[[(), (2,), (2, 2), (4, 2, 0)], [(4, 2, 0), (4, 3, 2, 0), (4, 3, 3, 1, 0)])
```

(continues on next page)
sage: st = SkewTableau([[None, None, None, 4, 4, 5, 6, 7], [None, 2, 4, 6, 7, 7], [None, 4, 5, 8, 8, 9], [None, 6, 7, 10], [None, 8, 8, 11], [None], [4]])

sage: pt = path_tableaux.SemistandardPathTableau(st)

sage: bk = [SkewTableau(st.bender_knuth_involution(i+1)) for i in range(10)]

sage: lr = [pt.local_rule(i+1) for i in range(10)]

sage: [r.to_tableau() for r in lr] == bk
True

AUTHORS:

• Bruce Westbury (2020): initial version

class sage.combinat.path_tableaux.semistandard.SemistandardPathTableau(parent, st, check=True)

Bases: PathTableau

An instance is a sequence of lists. Usually the entries will be non-negative integers in which case this is the chain of partitions of a (skew) semistandard tableau. In general the entries are elements of an ordered abelian group; each list is weakly decreasing and successive lists are interleaved.

INPUT:

Can be any of the following

• a sequence of partitions

• a sequence of lists/tuples

• a semistandard tableau

• a semistandard skew tableau

• a Gelfand-Tsetlin pattern

EXAMPLES:

sage: path_tableaux.SemistandardPathTableau([[], [2], [2, 1]])
[(), (2,), (2, 1)]

sage: gt = GelfandTsetlinPattern([[2, 1], [2]])

sage: path_tableaux.SemistandardPathTableau(gt)
[(), (2,), (2, 1)]

sage: st = SemistandardTableau([[1, 1], [2]])

sage: path_tableaux.SemistandardPathTableau(st)
[(), (2,), (2, 1)]

sage: st = SkewTableau([[1, 1], [2]])

sage: path_tableaux.SemistandardPathTableau(st)
[(), (2,), (2, 1)]

sage: st = SkewTableau([[None, 1, 1], [2]])

sage: path_tableaux.SemistandardPathTableau(st)
[(), (3, 0), (3, 1, 0)]

sage: path_tableaux.SemistandardPathTableau([[], [5/2], [7/2, 2]])
[(), (5/2,), (7/2, 2)]
Combinatorics, Release 10.0

\begin{verbatim}
 sage: path_tableaux.SemistandardPathTableau([[],[2.5],[3.5,2]])
[(), (2.50000000000000,), (3.50000000000000, 2)]
\end{verbatim}

check()
Check that self is a valid path.

EXAMPLES:
\begin{verbatim}
 sage: path_tableaux.SemistandardPathTableau([[],[3],[2,2]])
# indirect test
Traceback (most recent call last):
... ValueError: [(), (3,), (2, 2)] does not satisfy the required inequalities in row 1
 sage: path_tableaux.SemistandardPathTableau([[],[3/2],[2,5/2]])
# indirect test
Traceback (most recent call last):
... ValueError: [(), (3/2,), (2, 5/2)] does not satisfy the required inequalities in row 1
\end{verbatim}

is_integral()
Return True if all entries are non-negative integers.

EXAMPLES:
\begin{verbatim}
 sage: path_tableaux.SemistandardPathTableau([[],[3],[3,2]]).is_integral()
True
 sage: path_tableaux.SemistandardPathTableau([[],[5/2],[7/2,2]]).is_integral()
False
 sage: path_tableaux.SemistandardPathTableau([[],[3],[3,-2]]).is_integral()
False
\end{verbatim}

is_skew()
Return True if self is skew.

EXAMPLES:
\begin{verbatim}
 sage: path_tableaux.SemistandardPathTableau([[],[2]]).is_skew()
False
 sage: path_tableaux.SemistandardPathTableau([[],[3],[3,1]]).is_skew()
True
\end{verbatim}

local_rule(i)
This is the Bender-Knuth involution.

This is implemented by toggling the entries of the i-th list. The allowed range for i is 0 < i < len(self)-1 so any row except the first and last can be changed.

EXAMPLES:
\begin{verbatim}
 sage: pt = path_tableaux.SemistandardPathTableau([[],[3],[3,2],[3,3,1],[3,3,2,1]])
 sage: pt.local_rule(1)
[(), (2,), (3, 2), (3, 3, 1), (3, 3, 2, 1)]
\end{verbatim}

\end{verbatim}
sage: pt.local_rule(2)
[(0, 0), (3,), (3, 2), (3, 3, 1), (3, 3, 2, 1)]
sage: pt.local_rule(3)
[(0, 0), (3, 2), (3, 2, 2), (3, 3, 2, 1)]

rectify(inner=None, verbose=False)
Rectify self.

This gives the usual rectification of a skew standard tableau and gives a generalisation to skew semistandard tableaux. The usual construction uses jeu de taquin but here we use the Bender-Knuth involutions.

EXAMPLES:

sage: st = SkewTableau([[None, None, None, 4], [None, None, 1, 6], [None, None, 5], [2, 3, 1]])
sage: path_tableaux.SemistandardPathTableau(st).rectify()
[(0, 0), (1,), (1, 1), (2, 1, 0), (3, 1, 0, 0), (3, 2, 0, 0, 0), (4, 2, 0, 0, 0, 0)]
sage: path_tableaux.SemistandardPathTableau(st).rectify(verbos=True)
[(3, 2, 2), (3, 3, 2, 0), (3, 3, 2, 1, 0), (3, 3, 2, 2, 0, 0), (4, 3, 2, 2, 0, 0, 0, 0),
 (4, 3, 3, 2, 0, 0, 0, 0, 0), (4, 4, 3, 2, 0, 0, 0, 0, 0, 0),
 (3, 2, 1, 0), (3, 3, 1, 0), (3, 3, 2, 0, 0), (4, 3, 2, 0, 0, 0), (4, 3, 3, 2, 0, 0, 0, 0, 0),
 (4, 3, 3, 3, 2, 0, 0, 0, 0, 0, 0), (4, 4, 3, 3, 2, 0, 0, 0, 0, 0, 0, 0)]

size()
Return the size or length of self.

EXAMPLES:

sage: path_tableaux.SemistandardPathTableau([[None, None, None, 4], [None, None, 1, 6], [None, None, 5], [2, 3, 1]]).size()
5

to_pattern()
Convert self to a Gelfand-Tsetlin pattern.

EXAMPLES:

sage: pt = path_tableaux.SemistandardPathTableau([[None, None, None, 4], [None, None, 1, 6], [None, None, 5], [2, 3, 1]])
sage: pt.to_pattern()
[[4, 3, 3, 1, 0], [3, 3, 2, 1], [3, 3, 1], [3, 2], [3]]

to_tableau()
Convert self to a SemistandardTableau.

The SemistandardSkewTableau is not implemented so this returns a SkewTableau

EXAMPLES:

sage: pt = path_tableaux.SemistandardPathTableau([[None, None, None, 4], [None, None, 1, 6], [None, None, 5], [2, 3, 1]])
(continues on next page)
Combinatorics, Release 10.0

    sage: pt.to_tableau()
[[1, 1, 1, 5], [2, 2, 3], [3, 4, 5], [4]]

class sage.combinat.path_tableaux.semistandard.SemistandardPathTableaux
    Bases: PathTableaux
    The parent class for SemistandardPathTableau.

Element
    alias of SemistandardPathTableau

5.1.164 Plane Partitions

AUTHORS:

• Jang Soo Kim (2016): Initial implementation
• Jessica Striker (2016): Added additional methods
• Kevin Dilks (2021): Added symmetry classes

class sage.combinat.plane_partition.PlanePartition(parent, pp, check=True)
    Bases: ClonableArray
    A plane partition.
    
A plane partition is a stack of cubes in the positive orthant.

INPUT:

• PP – a list of lists which represents a tableau
• box_size – (optional) a list [A, B, C] of 3 positive integers, where A, B, C are the lengths of the box in the x-axis, y-axis, z-axis, respectively; if this is not given, it is determined by the smallest box bounding PP

OUTPUT:

The plane partition whose tableau representation is PP.

EXAMPLES:

    sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
    sage: PP
    Plane partition [[4, 3, 3, 1], [2, 1, 1], [1, 1]]

bounding_box()

    Return the smallest box (a, b, c) that self is contained in.

    EXAMPLES:

    sage: PP = PlanePartition([[5,2,1,1], [2,2], [2]])
    sage: PP.bounding_box()
    (3, 4, 5)

cells()

    Return the list of cells inside self.

    EXAMPLES:
sage: PP = PlanePartition([[3,1],[2]])
sage: PP.cells()
[[0, 0, 0], [0, 0, 1], [0, 1, 0], [1, 0, 1]]

check()
Check to see that self is a valid plane partition.

EXAMPLES:
sage: a = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: a.check()
sage: b = PlanePartition([[1,2],[1]])
Traceback (most recent call last):
... ValueError: not weakly decreasing along rows
sage: c = PlanePartition([[1,1],[2]])
Traceback (most recent call last):
... ValueError: not weakly decreasing along columns
sage: d = PlanePartition([[2,-1],[-2]])
Traceback (most recent call last):
... ValueError: entries not all nonnegative
sage: e = PlanePartition([[3/2,1],[.5]])
Traceback (most recent call last):
... ValueError: entries not all integers

complement(tableau_only=False)
Return the complement of self.

If the parent of self consists only of partitions inside a given box, then the complement is taken in this
box. Otherwise, the complement is taken in the smallest box containing the plane partition. The empty
plane partition with no box specified is its own complement.

If tableau_only is set to True, then only the tableau consisting of the projection of boxes size onto the xy-
plane is returned instead of a PlanePartition. This output will not have empty trailing rows or trailing
zeros removed.

EXAMPLES:
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.complement()
Plane partition [[4, 3, 3, 3], [4, 3, 3, 2], [3, 1, 1]]
sage: PP.complement(True)
[[4, 4, 3, 3], [4, 3, 3, 2], [3, 1, 1, 0]]

contains(PP)
Return True if PP is a plane partition that fits inside self.

Specifically, self contains PP if, for all i, j, the height of PP at ij is less than or equal to the height of
self at ij.

EXAMPLES:
sage: P1 = PlanePartition([[5,4,3], [3,2,2], [1]])
sage: P2 = PlanePartition([[3,2], [1,1], [0,0], [0,0]])
sage: P3 = PlanePartition([[5,5,5], [2,1,0]])
sage: P1.contains(P2)  # True
sage: P2.contains(P1)  # False
sage: P1.contains(P3)  # False
sage: P3.contains(P2)  # True

cyclically_rotate(preserve_parent=False)

Return the cyclic rotation of self.

By default, if the parent of self consists of plane partitions inside an $a \times b \times c$ box, the result will have a parent consisting of partitions inside a $c \times a \times b$ box, unless the optional parameter preserve_parent is set to True. Enabling this setting may give an element that is not an element of its parent.

EXAMPLES:

```python
sage: PlanePartition([[3,2,1],[2,2],[2]]).cyclically_rotate()
Plane partition [[3, 3, 1], [2, 2], [1]]
sage: PP = PlanePartition([[4,1],[1],[1]])
```

```python
sage: PP.cyclically_rotate()
Plane partition [[3, 1, 1, 1], [1]]
sage: PP == PP.cyclically_rotate().cyclically_rotate().cyclically_rotate()
True
sage: PP = PlanePartitions([4,3,2]).random_element()
```

```python
sage: PP.cyclically_rotate().parent()
```

```python
sage: PP_rotated = PP.cyclically_rotate(preserve_parent=True)
sage: PP_rotated in PP_rotated.parent()
False
```

is_CSPP()

Return whether self is a cyclically symmetric plane partition.

A plane partition is cyclically symmetric if its $x$, $y$, and $z$ tableaux are all equal.

EXAMPLES:

```python
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
```

```python
sage: PP.is_CSPP()
False
sage: PP = PlanePartition([[3,2,2],[3,1,0],[1,1,0]])
```

```python
sage: PP.is_CSPP()
True
```

is_CSSCPP()

Return whether self is a cyclically symmetric and self-complementary plane partition.

EXAMPLES:
is_CSTCPP()

Return whether self is a cyclically symmetric and transpose-complementary plane partition.

EXAMPLES:

```
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_CSTCPP()
False
sage: PP = PlanePartition([[4,4,4,1],[3,3,2,1],[3,2,1,1],[3,0,0,0]])
sage: PP.is_CSTCPP()
True
```

is_SCPP()

Return whether self is a self-complementary plane partition.

Note that the complement of a plane partition (and thus the property of being self-complementary) is dependent on the choice of a box that it is contained in. If no parent/bounding box is specified, the box is taken to be the smallest box that contains the plane partition.

EXAMPLES:

```
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_SCPP()
False
sage: PP = PlanePartition([[4,4,4,4],[4,4,2,0],[4,2,0,0],[0,0,0,0]])
sage: PP.is_SCPP()
False
sage: PP = PlanePartitions([4,4])([[4,4,4,4],[4,4,2,0],[4,2,0,0],[0,0,0,0]])
sage: PP.is_SCPP()
True
```

is_SPP()

Return whether self is a symmetric plane partition.

A plane partition is symmetric if the corresponding tableau is symmetric about the diagonal.

EXAMPLES:

```
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_SPP()
False
sage: PP = PlanePartition([[3,3,2],[3,3,2],[2,2,2]])
sage: PP.is_SPP()
True
sage: PP = PlanePartition([[3,2,1],[2,0,0]])
sage: PP.is_SPP()
False
sage: PP = PlanePartition([[3,2,0],[2,0,0]])
```

(continues on next page)
sage: PP.is_SPP()
True
sage: PP = PlanePartition([[3,2],[2,0],[1,0]])
sage: PP.is_SPP()
False
sage: PP = PlanePartition([[3,2],[2,0],[0,0]])
sage: PP.is_SPP()
True

is_SSCPP()
Return whether self is a symmetric, self-complementary plane partition.

EXAMPLES:

sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_SSCPP()
False
sage: PP = PlanePartition([[4,3,3,2],[3,2,2,1],[3,2,2,1],[2,1,1,0]])
sage: PP.is_SSCPP()
True
sage: PP = PlanePartition([[2,1],[1,0]])
sage: PP.is_SSCPP()
True
sage: PP = PlanePartition([[4,3,2],[3,2,1],[2,1,0]])
sage: PP.is_SSCPP()
True
sage: PP = PlanePartition([[4,2,2,2],[2,2,2,2],[2,2,2,2],[2,2,2,0]])
sage: PP.is_SSCPP()
True

is_TCPP()
Return whether self is a transpose-complementary plane partition.

EXAMPLES:

sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_TCPP()
False
sage: PP = PlanePartition([[4,4,3,2],[4,4,2,1],[4,2,0,0],[2,0,0,0]])
sage: PP.is_TCPP()
True

is_TSPP()
Return whether self is a totally symmetric plane partition.

A plane partition is totally symmetric if it is both symmetric and cyclically symmetric.

EXAMPLES:

sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_TSPP()
False
sage: PP = PlanePartition([[3,3,3],[3,3,2],[3,2,1]])
(continues on next page)
sage: PP.is_TSSPP()
True

is_TSSCPP()
Return whether self is a totally symmetric self-complementary plane partition.

EXAMPLES:

sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.is_TSSCPP()
False
sage: PP = PlanePartition([[4,4,3,2],[4,3,2,1],[3,2,1,0],[2,1,0,0]])
sage: PP.is_TSSCPP()
True

maximal_boxes()
Return the coordinates of the maximal boxes of self.
The maximal boxes of a plane partitions are the boxes that can be removed from a plane partition and still
yield a valid plane partition.

EXAMPLES:

sage: sorted(PlanePartition([[3,2,1],[2,2],[2]]).maximal_boxes())
[[0, 0, 2], [0, 2, 0], [1, 1, 1], [2, 0, 1]]
sage: sorted(PlanePartition([[2,1],[1],[1]]).maximal_boxes())
[[0, 0, 1], [0, 1, 0], [2, 0, 0]]

number_of_boxes()
Return the number of boxes in the plane partition.

EXAMPLES:

sage: PP = PlanePartition([[3,1],[2]])
sage: PP.number_of_boxes()
6

plot(show_box=False, colors=None)
Return a plot of self.

INPUT:

• show_box – boolean (default: False); if True, also shows the visible tiles on the xy-, yz-, zx-planes
• colors – (default: ["white", "lightgray", "darkgray"] list [A, B, C] of 3 strings repre-
   senting colors

EXAMPLES:

sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.plot()
Graphics object consisting of 27 graphics primitives

plot3d(colors=None)
Return a 3D-plot of self.

INPUT:
• colors – (default: ["white", "lightgray", "darkgray"]) list [A, B, C] of 3 strings representing colors

EXAMPLES:

```python
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.plot3d()
Graphics3d Object
```

```
pp(show_box=False)
```

Return a pretty print of the plane partition.

**INPUT:**

• show_box – boolean (default: False); if True, also shows the visible tiles on the \(xy\), \(yz\), \(zx\)-planes

**OUTPUT:**

A pretty print of the plane partition.

**EXAMPLES:**

```python
sage: PlanePartition([[4,3,3,1],[2,1,1],[1,1]]).pp()
```

```
* * *
  _/\_ 
 _/ /\_/ 
_/ /_/ /_/ 
_/ /_/ /_/ 
_/ /_/ /_/ 
_/ /_/ /_/ 
```

```python
sage: PlanePartition([[4,3,3,1],[2,1,1],[1,1]]).pp(True)
```

```
* * *
  _/\_ 
 _/ /\_/ 
_/ /_/ /_/ 
_/ /_/ /_/ 
_/ /_/ /_/ 
_/ /_/ /_/ 
```

```
to_order_ideal()
```

Return the order ideal corresponding to \(self\).

**Todo:** As many families of symmetric plane partitions are in bijection with order ideals in an associated poset, this function could feasibly have options to send symmetric plane partitions to the associated order ideal in that poset, instead.

**EXAMPLES:**

```python
sage: PlanePartition([[3,2,1],[2,2],[2]]).to_order_ideal()
[(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (0, 2, 0), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1), (2, 0, 0), (2, 0, 1)]
```

(continues on next page)
```python
sage: PlanePartition([[2,1],[1],[1]]).to_order_ideal()
[(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (2, 0, 0)]
```

**to_tableau()**

Return the tableau class of self.

EXAMLPES:

```python
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.to_tableau()
[[4, 3, 3, 1], [2, 1, 1], [1, 1]]
```

**transpose(tableau_only=False)**

Return the transpose of self.

If `tableau_only` is set to True, then only the tableau consisting of the projection of boxes size onto the $xy$-plane is returned instead of a `PlanePartition`. This will not necessarily have trailing rows or trailing zeros removed.

EXAMPLES:

```python
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.transpose()
Plane partition [[4, 2, 1], [3, 1, 1], [3, 1], [1]]
sage: PP.transpose(True)
[[4, 2, 1], [3, 1, 1], [3, 1, 0], [1, 0, 0]]
sage: PPP = PlanePartitions([1, 2, 3])
sage: PP = PPP([[1, 1]])
sage: PT = PP.transpose(); PT
Plane partition [[1], [1]]
sage: PT.parent()
Plane partitions inside a 2 x 1 x 3 box
```

**x_tableau(tableau=True)**

Return the projection of self in the $x$ direction.

If `tableau` is set to False, then only the list of lists consisting of the projection of boxes size onto the $yz$-plane is returned instead of a `Tableau` object. This output will not have empty trailing rows or trailing zeros removed.

EXAMPLES:

```python
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.x_tableau()
[[3, 2, 1, 1], [3, 1, 1, 0], [2, 1, 1, 0], [1, 0, 0, 0]]
```

**y_tableau(tableau=True)**

Return the projection of self in the $y$ direction.

If `tableau` is set to False, then only the list of lists consisting of the projection of boxes size onto the $xz$-plane is returned instead of a `Tableau` object. This output will not have empty trailing rows or trailing zeros removed.

EXAMPLES:
sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.y_tableau()
[[4, 3, 2], [3, 1, 0], [3, 0, 0], [1, 0, 0]]

z_tableau(tableau=True)

Return the projection of self in the z direction.

If tableau is set to False, then only the list of lists consisting of the projection of boxes size onto the xy-plane is returned instead of a Tableau object. This output will not have empty trailing rows or trailing zeros removed.

EXAMPLES:

sage: PP = PlanePartition([[4,3,3,1],[2,1,1],[1,1]])
sage: PP.z_tableau()
[[4, 3, 3, 1], [2, 1, 1, 0], [1, 1, 0, 0]]

class sage.combinat.plane_partition.PlanePartitions(box_size=None, symmetry=None, category=None)

Bases: UniqueRepresentation, Parent

Plane partitions.

PlanePartitions() returns the class of all plane partitions.

PlanePartitions(n) return the class of all plane partitions with precisely n boxes.

PlanePartitions([a, b, c]) returns the class of plane partitions that fit inside an a x b x c box.

PlanePartitions([a, b, c]) has the optional keyword symmetry, which restricts the plane partitions inside a box of the specified size satisfying certain symmetry conditions.

• symmetry='SPP' gives the class of symmetric plane partitions. which is all plane partitions fixed under reflection across the diagonal. Requires that a = b.

• symmetry='CSPP' gives the class of cyclic plane partitions, which is all plane partitions fixed under cyclic rotation of coordinates. Requires that a = b = c.

• symmetry='TSPP' gives the class of totally symmetric plane partitions, which is all plane partitions fixed under any interchanging of coordinates. Requires that a = b = c.

• symmetry='SCPP' gives the class of self-complementary plane partitions. which is all plane partitions that are equal to their own complement in the specified box. Requires at least one of a, b, c be even.

• symmetry='TCPP' gives the class of transpose complement plane partitions, which is all plane partitions whose complement in the box of the specified size is equal to their transpose. Requires a = b and at least one of a, b, c be even.

• symmetry='SSCPP' gives the class of symmetric self-complementary plane partitions, which is all plane partitions that are both symmetric and self-complementary. Requires a = b and at least one of a, b, c be even.

• symmetry='CSTCPP' gives the class of cyclically symmetric transpose complement plane partitions, which is all plane partitions that are both symmetric and equal to the transpose of their complement. Requires a = b = c.

• symmetry='CSSCPP' gives the class of cyclically symmetric self-complementary plane partitions, which is all plane partitions that are both cyclically symmetric and self-complementary. Requires a = b = c and all a, b, c be even.
• symmetry='TSSCPP' gives the class of totally symmetric self-complementary plane partitions, which is all plane partitions that are totally symmetric and also self-complementary. Requires $a = b = c$ and all $a, b, c$ be even.

EXAMPLES:
If no arguments are passed, then the class of all plane partitions is returned:

```
sage: PlanePartitions()
Plane partitions
sage: [[2,1],[1]] in PlanePartitions()
True
```
If an integer $n$ is passed, then the class of plane partitions of $n$ is returned:

```
sage: PlanePartitions(3)
Plane partitions of size 3
sage: PlanePartitions(3).list()
[Plane partition [[3]],
 Plane partition [[2, 1]],
 Plane partition [[1, 1, 1]],
 Plane partition [[2], [1]],
 Plane partition [[1, 1], [1]],
 Plane partition [[1], [1], [1]]]
```
If a three-element tuple or list $[a, b, c]$ is passed, then the class of all plane partitions that fit inside and $a \times b \times c$ box is returned:

```
sage: PlanePartitions([2,2,2])
Plane partitions inside a 2 x 2 x 2 box
sage: [[2,1],[1]] in PlanePartitions([2,2,2])
True
```
If an additional keyword `symmetry` is pass along with a three-element tuple or list $[a, b, c]$, then the class of all plane partitions that fit inside an $a \times b \times c$ box with the specified symmetry is returned:

```
sage: PlanePartitions([2,2,2], symmetry='CSPP')
Cyclically symmetric plane partitions inside a 2 x 2 x 2 box
sage: [[2,1],[1]] in PlanePartitions([2,2,2], symmetry='CSPP')
True
```

See also:
• `PlanePartition`
• `PlanePartitions_all`
• `PlanePartitions_n`
• `PlanePartitions_box`
• `PlanePartitions_SPP`
• `PlanePartitions_CSPP`
• `PlanePartitions_TSSPP`
• `PlanePartitions_SCPP`
• `PlanePartitions_TCPP`
- `PlanePartitions_SSCPP`
- `PlanePartitions_CSTCPP`
- `PlanePartitions_CSSCPP`
- `PlanePartitions_TSSCPP`

**Element**
alias of `PlanePartition`

**box()**
Return the size of the box of the plane partition of `self` is contained in.

**EXAMPLES:**
```
sage: P = PlanePartitions([4,3,5])
sage: P.box()  
(4, 3, 5)
sage: PP = PlanePartitions()  
sage: PP.box() is None  
True
```

**symmetry()**
Return the symmetry class of `self`.

**EXAMPLES:**
```
sage: PP = PlanePartitions([3,3,2], symmetry='SPP')  
sage: PP.symmetry()  
'SPP'
sage: PP = PlanePartitions()  
sage: PP.symmetry() is None  
True
```

class `sage.combinat.plane_partition.PlanePartitions_CSPP(box_size)`

Bases: `PlanePartitions`

Plane partitions that fit inside a box of a specified size that are cyclically symmetric.

**cardinality()**
Return the cardinality of `self`.

The number of cyclically symmetric plane partitions inside an \( a \times a \times a \) box is equal to

\[
\left(\prod_{i=1}^{a}\frac{3i-1}{3i-2}\right) \left(\prod_{1 \leq i < j \leq a}\frac{i+j+a-1}{2i+j-1}\right).
\]

**EXAMPLES:**
```
sage: P = PlanePartitions([4,4,4], symmetry='CSPP')  
sage: P.cardinality()  
132
```
from_antichain(acl)

Return the cyclically symmetric plane partition corresponding to an antichain in the poset given in to_poset().

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,3], symmetry='CSPP')
sage: A = [(∅, 2, 2), (1, 1, 1)]
sage: PP.from_antichain(A)
Plane partition [[3, 3, 3], [3, 2, 1], [3, 1, 1]]
```

from_order_ideal(I)

Return the cyclically symmetric plane partition corresponding to an order ideal in the poset given in to_poset().

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,3], symmetry='CSPP')
sage: I = [(∅, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2),
      ....: (1, 0, 2), (0, 2, 2), (1, 1, 1), (1, 1, 2), (1, 2, 2)]
sage: PP.from_order_ideal(I)
Plane partition [[3, 3, 3], [3, 3, 3], [3, 3, 2]]
```

random_element()

Return a uniformly random element of self.

ALGORITHM:

This uses the random_order_ideal() method and the natural bijection between cyclically symmetric plane partitions and order ideals in an associated poset.

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,3], symmetry='CSPP')
sage: PP.random_element() # random
Plane partition [[3, 2, 2], [3, 1], [1, 1]]
```

to_poset()

Return a partially ordered set whose order ideals are in bijection with cyclically symmetric plane partitions.

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,3], symmetry='CSPP')
sage: PP.to_poset()
Finite poset containing 11 elements
sage: PP.to_poset().order_ideals_lattice().cardinality() == PP.cardinality()
True
```

class sage.combinat.plane_partition.PlanePartitions_CSSCPP(box_size)

Bases: PlanePartitions

Plane partitions that fit inside a box of a specified size that are cyclically symmetric self-complementary.

cardinality()

Return the cardinality of self.
The number of cyclically symmetric self-complementary plane partitions inside a $2a \times 2a \times 2a$ box is equal to

\[
\left( \prod_{i=0}^{a-1} \frac{(3i+1)!}{(a+i)!} \right)^2.
\]

**EXAMPLES:**

```python
sage: P = PlanePartitions([6,6,6], symmetry='CSSCPP')
sage: P.cardinality()
sage: 49
```

```python
class sage.combinat.plane_partition.PlanePartitions_CSTCPP(box_size)

Bases: PlanePartitions

Plane partitions that fit inside a box of a specified size that are cyclically symmetric and transpose-complement.

cardinality()

Return the cardinality of self.

The number of cyclically symmetric transpose complement plane partitions inside a $2a \times 2a \times 2a$ box is equal to

\[
\left( \prod_{i=0}^{a-1} \frac{(3i+1)(6i)!}{(4i)!} \right)^2.
\]

**EXAMPLES:**

```python
sage: P = PlanePartitions([6,6,6], symmetry='CSTCPP')
sage: P.cardinality()
sage: 11
```

```python
class sage.combinat.plane_partition.PlanePartitions_SCPP(box_size)

Bases: PlanePartitions

Plane partitions that fit inside a box of a specified size that are self-complementary.

cardinality()

Return the cardinality of self.

The number of self complementary plane partitions inside a $2a \times 2b \times 2c$ box is equal to

\[
\left( \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{i+j+c-1}{i+j-1} \right)^2.
\]

The number of self complementary plane partitions inside an $(2a+1) \times 2b \times 2c$ box is equal to

\[
\left( \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{i+j+c-1}{i+j-1} \right) \left( \prod_{i=1}^{a+1} \prod_{j=1}^{b} \frac{i+j+c-1}{i+j-1} \right).
\]

The number of self complementary plane partitions inside an $(2a+1) \times (2b+1) \times 2c$ box is equal to

\[
\left( \prod_{i=1}^{a+1} \prod_{j=1}^{b} \frac{i+j+c-1}{i+j-1} \right) \left( \prod_{i=1}^{a} \prod_{j=1}^{b+1} \frac{i+j+c-1}{i+j-1} \right).
\]

**EXAMPLES:**
Combinatorics, Release 10.0

```python
sage: P = PlanePartitions([4,4,4], symmetry='SCPP')
sage: P.cardinality()
400
sage: P = PlanePartitions([5,4,4], symmetry='SCPP')
sage: P.cardinality()
1000
sage: P = PlanePartitions([4,5,4], symmetry='SCPP')
sage: P.cardinality()
1000
sage: P = PlanePartitions([4,4,5], symmetry='SCPP')
sage: P.cardinality()
1000
sage: P = PlanePartitions([5,5,4], symmetry='SCPP')
sage: P.cardinality()
2500
sage: P = PlanePartitions([5,4,5], symmetry='SCPP')
sage: P.cardinality()
2500
sage: P = PlanePartitions([4,5,5], symmetry='SCPP')
sage: P.cardinality()
2500
```

class `sage.combinat.plane_partition.PlanePartitions_SPP(box_size)`

Bases: `PlanePartitions`

Plane partitions that fit inside a box of a specified size that are symmetric.

**cardinality()**

Return the cardinality of `self`.

The number of symmetric plane partitions inside an $a \times a \times b$ box is equal to

$$\left( \prod_{i=1}^{a} \frac{2i+b-1}{2i-1} \right) \left( \prod_{1 \leq i < j \leq a} \frac{i+j+b-1}{i+j-1} \right).$$

**EXAMPLES:**

```python
sage: P = PlanePartitions([3,3,2], symmetry='SPP')
sage: P.cardinality()
35
```

**from_antichain(A)**

Return the symmetric plane partition corresponding to an antichain in the poset given in `to_poset()`.

**EXAMPLES:**

```python
sage: PP = PlanePartitions([3,3,2], symmetry='SPP')
sage: A = [(2, 2, 0), (1, 0, 1), (1, 1, 0)]
sage: PP.from_antichain(A)
Plane partition [[2, 2, 1], [2, 1, 1], [1, 1, 1]]
```
from_order_ideal(I)
Return the symmetric plane partition corresponding to an order ideal in the poset given in `to_poset()`.

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,2], symmetry='SPP')
sage: I = [(0, 0, 0), (1, 0, 0), (1, 1, 0), (2, 0, 0)]
sage: PP.from_order_ideal(I)
Plane partition [[1, 1, 1], [1, 1], [1]]
```

random_element()
Return a uniformly random element of `self`.

ALGORITHM:
This uses the `random_order_ideal()` method and the natural bijection between symmetric plane partitions and order ideals in an associated poset.

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,2], symmetry='SPP')
sage: PP.random_element() # random
Plane partition [[2, 2, 2], [2, 2], [2]]
```

to_poset()
Return a poset whose order ideals are in bijection with symmetric plane partitions.

EXAMPLES:

```
sage: PP = PlanePartitions([3,3,2], symmetry='SPP')
sage: PP.to_poset()
Finite poset containing 12 elements
sage: PP.to_poset().order_ideals_lattice().cardinality() == PP.cardinality()
True
```

class sage.combinat.plane_partition.PlanePartitions_SSCPP(box_size)
Bases: `PlanePartitions`

Plane partitions that fit inside a box of a specified size that are symmetric self-complementary.

cardinality()
Return the cardinality of `self`.

The number of symmetric self-complementary plane partitions inside a $2a \times 2a \times 2b$ box is equal to

$$\prod_{i=1}^{a} \prod_{j=1}^{a} \frac{i + j + b - 1}{i + j - 1}.$$ 

The number of symmetric self-complementary plane partitions inside a $(2a + 1) \times (2a + 1) \times 2b$ box is equal to

$$\prod_{i=1}^{a} \prod_{j=1}^{a+1} \frac{i + j + b - 1}{i + j - 1}.$$ 

EXAMPLES:
```python
sage: P = PlanePartitions([4,4,2], symmetry='SSCPP')
sage: P.cardinality()
6
sage: Q = PlanePartitions([3,3,2], symmetry='SSCPP')
sage: Q.cardinality()
3
```

```python
class sage.combinat.plane_partition.PlanePartitions_TCPP(box_size)
    Bases: PlanePartitions
    Plane partitions that fit inside a box of a specified size that are transpose-complement.
    
cardinality()
    Return the cardinality of self.

    The number of transpose complement plane partitions inside an \(a \times a \times 2b\) box is equal to
    \[
    \binom{b + 1 - 1}{a - 1} \prod_{1 \leq i, j \leq a - 2} \frac{i + j + 2b - 1}{i + j - 1}.
    \]

    EXAMPLES:
    ```
    sage: P = PlanePartitions([3,3,2], symmetry='TCPP')
sage: P.cardinality()
5
```

```python
class sage.combinat.plane_partition.PlanePartitions_TSPP(box_size)
    Bases: PlanePartitions
    Plane partitions that fit inside a box of a specified size that are totally symmetric.
    
cardinality()
    Return the cardinality of self.

    The number of totally symmetric plane partitions inside an \(a \times a \times a\) box is equal to
    \[
    \prod_{1 \leq i, j \leq a} \frac{i + j + a - 1}{i + 2j - 2}.
    \]

    EXAMPLES:
    ```
    sage: P = PlanePartitions([4,4,4], symmetry='TSPP')
sage: P.cardinality()
66
```

```python
from_antichain(acl)
    Return the totally symmetric plane partition corresponding to an antichain in the poset given in to_poset().

    EXAMPLES:
    ```
    sage: PP = PlanePartitions([3,3,3], symmetry='TSPP')
sage: A = [(0, 0, 2), (0, 1, 1)]
sage: PP.from_antichain(A)
Plane partition [[3, 2, 1], [2, 1], [1]]
```
```
from_order_ideal(I)
Return the totally symmetric plane partition corresponding to an order ideal in the poset given in to_poset().
EXAMPLES:

```python
sage: PP = PlanePartitions([3,3,3], symmetry='TSPP')
sage: I = [(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 1)]
sage: PP.from_order_ideal(I)
Plane partition [[3, 2, 1], [2, 1], [1]]
```

to_poset()
Return a poset whose order ideals are in bijection with totally symmetric plane partitions.
EXAMPLES:

```python
sage: PP = PlanePartitions([3,3,3], symmetry='TSPP')
sage: PP.to_poset()
Finite poset containing 10 elements
sage: PP.to_poset().order_ideals_lattice().cardinality() == PP.cardinality()
True
```

class sage.combinat.plane_partition.PlanePartitions_TSSCPP(box_size)
Bases: PlanePartitions
Plane partitions that fit inside a box of a specified size that are totally symmetric self-complementary.
cardinality()
Return the cardinality of self.
The number of totally symmetric self-complementary plane partitions inside a $2a \times 2a \times 2a$ box is equal to

$$
\prod_{i=0}^{a-1} \frac{(3i + 1)!}{(a + i)!}.
$$

EXAMPLES:

```python
sage: P = PlanePartitions([6,6,6], symmetry='TSSCPP')
sage: P.cardinality()
7
```

from_antichain(acl)
Return the totally symmetric self-complementary plane partition corresponding to an antichain in the poset given in to_poset().
EXAMPLES:

```python
sage: PP = PlanePartitions([6,6,6], symmetry='TSSCPP')
sage: A = [(0, 0, 1), (1, 1, 0)]
sage: PP.from_antichain(A)
Plane partition [[6, 6, 6, 5, 5, 3], [6, 5, 5, 4, 3, 1], [6, 5, 4, 3, 2, 1], [5, 4, 3, 2, 1], [5, 3, 2, 1, 1], [3, 1, 1]]
```

from_order_ideal(I)
Return the totally symmetric self-complementary plane partition corresponding to an order ideal in the poset given in to_poset().
EXAMPLES:

```
sage: PP = PlanePartitions([6,6,6], symmetry='TSSCPP')
sage: I = [(0, 0, 0), (0, 1, 0), (1, 1, 0)]
sage: PP.from_order_ideal(I)

Plane partition [[6, 6, 6, 5, 5, 3], [6, 5, 5, 3, 3, 1], [6, 5, 5, 3, 3, 1], [5, 3, 3, 1, 1], [5, 3, 3, 1, 1], [3, 1, 1]]
```

to_poset()

Return a poset whose order ideals are in bijection with totally symmetric self-complementary plane partitions.

EXAMPLES:

```
sage: PP = PlanePartitions([6,6,6], symmetry='TSSCPP')
sage: PP.to_poset()
Finite poset containing 4 elements
sage: PP.to_poset().order_ideals_lattice().cardinality() == PP.cardinality()
True
```

class sage.combinat.plane_partition.PlanePartitions_all

Bases: `PlanePartitions`, `DisjointUnionEnumeratedSets`

All plane partitions.

an_element()

Return a particular element of the class.

class sage.combinat.plane_partition.PlanePartitions_box(box_size)

Bases: `PlanePartitions`

All plane partitions that fit inside a box of a specified size.

By convention, a plane partition in an $a \times b \times c$ box will have at most $a$ rows, of lengths at most $b$, with entries at most $c$.

cardinality()

Return the cardinality of self.

The number of plane partitions inside an $a \times b \times c$ box is equal to

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}. $$

EXAMPLES:

```
sage: P = PlanePartitions([4,3,5])
sage: P.cardinality()
116424
```

from_antichain(A)

Return the plane partition corresponding to an antichain in the poset given in `to_poset()`.

EXAMPLES:

```
sage: A = [(1,0,1), (0,1,1), (1,1,0)]
sage: PlanePartitions([2,2,2]).from_antichain(A)
Plane partition [[2, 2], [2, 1]]
```
from_order_ideal(I)
Return the plane partition corresponding to an order ideal in the poset given in to_poset().

EXAMPLES:

sage: I = [(1, 0, 0), (1, 0, 1), (1, 1, 0), (0, 1, 0), (0, 0, 0), (0, 0, 1), (0, 1, 1)]

sage: PlanePartitions([2,2,2]).from_order_ideal(I)
Plane partition [[2, 2], [2, 1]]

random_element()
Return a uniformly random plane partition inside a box.

ALGORITHM:
This uses the random_order_ideal() method and the natural bijection with plane partitions.

EXAMPLES:

sage: P = PlanePartitions([4,3,5])

sage: P.random_element() # random
Plane partition [[4, 3, 3], [4], [2]]

to_poset()
Return the product of three chains poset, whose order ideals are naturally in bijection with plane partitions inside a box.

EXAMPLES:

sage: PlanePartitions([2,2,2]).to_poset()
Finite lattice containing 8 elements

class sage.combinat.plane_partition.PlanePartitions_n(n)
Bases: PlanePartitions

Plane partitions with a fixed number of boxes.

cardinality()
Return the number of plane partitions with n boxes.

Calculated using the recurrence relation

\[ PL(n) = \sum_{k=1}^{n} PL(n - k)\sigma_2(k), \]

where \( \sigma_k(n) \) is the sum of the k-th powers of divisors of n.

EXAMPLES:

sage: P = PlanePartitions(17)

sage: P.cardinality()
18334
5.1.165 Integer partitions

A partition $p$ of a nonnegative integer $n$ is a non-increasing list of positive integers (the *parts* of the partition) with total sum $n$.

A partition can be depicted by a diagram made of rows of cells, where the number of cells in the $i^{th}$ row starting from the top is the $i^{th}$ part of the partition.

The coordinate system related to a partition applies from the top to the bottom and from left to right. So, the corners of the partition $[5, 3, 1]$ are $[[0, 4], [1, 2], [2, 0]]$.

For display options, see `Partitions.options`.

**Note:**
- Boxes is a synonym for cells. All methods will use ‘cell’ and ‘cells’ instead of ‘box’ and ‘boxes’.
- Partitions are 0 based with coordinates in the form of (row-index, column-index).
- If given coordinates of the form $(r, c)$, then use Python’s *-operator.
- Throughout this documentation, for a partition $\lambda$ we will denote its conjugate partition by $\lambda'$. For more on conjugate partitions, see `Partition.conjugate()`.
- The comparisons on partitions use lexicographic order.

**Note:** We use the convention that lexicographic ordering is read from left-to-right. That is to say $[1, 3, 7]$ is smaller than $[2, 3, 4]$.

**AUTHORS:**
- Mike Hansen (2007): initial version
- Dan Drake (2009-03-28): deprecate RestrictedPartitions and implement Partitions_parts_in
- Travis Scrimshaw (2012-01-12): Implemented latex function to Partition_class
- Travis Scrimshaw (2012-05-09): Fixed Partitions(-1).list() infinite recursion loop by saying Partitions_n is the empty set.
- Travis Scrimshaw (2012-05-11): Fixed bug in inner where if the length was longer than the length of the inner partition, it would include 0’s.
- Andrew Mathas (2012-06-01): Removed deprecated functions and added compatibility with the PartitionTuple classes. See [github issue #13072](https://github.com/sagemath/sage/issues/13072)
- Travis Scrimshaw (2012-10-12): Added options. Made `Partition_class` to the element `Partition`. `Partitions*` are now all in the category framework except `PartitionsRestricted` (which will eventually be removed). Cleaned up documentation.
- Matthew Lancellotti (2018-09-14): Added a bunch of “k” methods to `Partition`.

**EXAMPLES:**

There are 5 partitions of the integer 4:

```python
sage: Partitions(4).cardinality()
sage: Partitions(4).list()
```

```
5
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
```
We can use the method \texttt{.first()} to get the ‘first’ partition of a number:

\begin{verbatim}
sage: Partitions(4).first()
[4]
\end{verbatim}

Using the method \texttt{.next(p)}, we can calculate the ‘next’ partition after \( p \). When we are at the last partition, \texttt{None} will be returned:

\begin{verbatim}
sage: Partitions(4).next([4])
[3, 1]
sage: Partitions(4).next([1,1,1,1]) is None
True
\end{verbatim}

We can use \texttt{iter} to get an object which iterates over the partitions one by one to save memory. Note that when we do something like \texttt{for part in Partitions(4)} this iterator is used in the background:

\begin{verbatim}
sage: g = iter(Partitions(4))
sage: next(g)
[4]
sage: next(g)
[3, 1]
sage: next(g)
[2, 2]
sage: for p in Partitions(4): print(p)
[4]
[3, 1]
[2, 2]
[2, 1, 1]
[1, 1, 1, 1]
\end{verbatim}

We can add constraints to the type of partitions we want. For example, to get all of the partitions of 4 of length 2, we'd do the following:

\begin{verbatim}
sage: Partitions(4, length=2).list()
[[3, 1], [2, 2]]
\end{verbatim}

Here is the list of partitions of length at least 2 and the list of ones with length at most 2:

\begin{verbatim}
sage: Partitions(4, min_length=2).list()
[[3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
sage: Partitions(4, max_length=2).list()
[[4], [3, 1], [2, 2]]
\end{verbatim}

The options \texttt{min_part} and \texttt{max_part} can be used to set constraints on the sizes of all parts. Using \texttt{max_part}, we can select partitions having only ‘small’ entries. The following is the list of the partitions of 4 with parts at most 2:

\begin{verbatim}
sage: Partitions(4, max_part=2).list()
[[2, 2], [2, 1, 1], [1, 1, 1, 1]]
\end{verbatim}

The \texttt{min_part} options is complementary to \texttt{max_part} and selects partitions having only ‘large’ parts. Here is the list of all partitions of 4 with each part at least 2:

\begin{verbatim}
sage: Partitions(4, min_part=2).list()
[[4], [2, 2]]
\end{verbatim}
The options `inner` and `outer` can be used to set part-by-part constraints. This is the list of partitions of 4 with [3, 1, 1] as an outer bound (that is, partitions of 4 contained in the partition [3, 1, 1]):

```
sage: Partitions(4, outer=[3,1,1]).list()
[[3, 1], [2, 1, 1]]
```

`outer` sets `max_length` to the length of its argument. Moreover, the parts of `outer` may be infinite to clear constraints on specific parts. Here is the list of the partitions of 4 of length at most 3 such that the second and third part are 1 when they exist:

```
sage: Partitions(4, outer=[oo,1,1]).list()
[[4], [3, 1], [2, 1, 1]]
```

Finally, here are the partitions of 4 with [1,1,1] as an inner bound (i.e., the partitions of 4 containing the partition [1,1,1]). Note that `inner` sets `min_length` to the length of its argument:

```
sage: Partitions(4, inner=[1,1,1]).list()
[[2, 1, 1], [1, 1, 1, 1]]
```

The options `min_slope` and `max_slope` can be used to set constraints on the slope, that is on the difference $p_{i+1} - p_{i}$ of two consecutive parts. Here is the list of the strictly decreasing partitions of 4:

```
sage: Partitions(4, max_slope=-1).list()
[[4], [3, 1]]
```

The constraints can be combined together in all reasonable ways. Here are all the partitions of 11 of length between 2 and 4 such that the difference between two consecutive parts is between $-3$ and $-1$:

```
sage: Partitions(11,min_slope=-3,max_slope=-1,min_length=2,max_length=4).list()
[[7, 4], [6, 5], [6, 4, 1], [6, 3, 2], [5, 4, 2], [5, 3, 2, 1]]
```

Partition objects can also be created individually with `Partition`:

```
sage: Partition([2,1])
[2, 1]
```

Once we have a partition object, then there are a variety of methods that we can use. For example, we can get the conjugate of a partition. Geometrically, the conjugate of a partition is the reflection of that partition through its main diagonal. Of course, this operation is an involution:

```
sage: Partition([4,1]).conjugate()
[2, 1, 1, 1]
sage: Partition([4,1]).conjugate().conjugate()
[4, 1]
```

If we create a partition with extra zeros at the end, they will be dropped:

```
sage: Partition([4,1,0,0])
[4, 1]
sage: Partition([0])
[]
sage: Partition([0,0])
[]
```

The idea of a partition being followed by infinitely many parts of size 0 is consistent with the `get_part` method:
We can go back and forth between the standard and the exponential notations of a partition. The exponential notation can be padded with extra zeros:

```
sage: Partition([6,4,4,2,1]).to_exp()
[1, 1, 0, 2, 0, 1]
sage: Partition(exp=[1,1,0,2,0,1])
[6, 4, 4, 2, 1]
sage: Partition([6,4,4,2,1]).to_exp(5)
[1, 1, 0, 2, 0, 1, 0]
sage: Partition([6,4,4,2,1]).to_exp(10)
[1, 1, 0, 2, 0, 1, 0, 0, 0, 0]
```

We can get the (zero-based!) coordinates of the corners of a partition:

```
sage: Partition([4,3,1]).corners()
[(0, 3), (1, 2), (2, 0)]
```

We can compute the core and quotient of a partition and build the partition back up from them:

```
sage: Partition([6,3,2,2]).core(3)
[2, 1, 1]
sage: Partition([7,7,5,3,3,3,1]).quotient(3)
([2], [1], [2, 2, 2])
sage: p = Partition([11,5,5,3,2,2,2])
sage: p.core(3)
[]
sage: p.quotient(3)
([2, 1], [4], [1, 1, 1])
sage: Partition(core=[],quotient=(2, 1), [4], [1, 1, 1]))
[11, 5, 5, 3, 2, 2, 2]
```

We can compute the \(0 - 1\) sequence and go back and forth:

```
sage: Partitions().from_zero_one([1, 1, 1, 1, 0, 1, 0])
[5, 4]
sage: all(Partitions().from_zero_one(mu.zero_one_sequence())
....:    == mu for n in range(5) for mu in Partitions(n))
True
```

We can compute the Frobenius coordinates and go back and forth:

```
sage: Partition([7,3,1]).frobenius_coordinates()
([6, 1, [2, 0]])
sage: Partition(frobenius_coordinates=(6,1,2,0))
[7, 3, 1]
sage: all(mu == Partition(frobenius_coordinates=mu.frobenius_coordinates())
```
for n in range(12) for mu in Partitions(n))

We use the lexicographic ordering:

```python
sage: pl = Partition([4,1,1])
sage: ql = Partitions()([3,3])
sage: pl > ql
True
sage: PL = Partitions()
sage: pl = PL([4,1,1])
sage: ql = PL([3,3])
sage: pl > ql
True
```

```python
class sage.combinat.partition.OrderedPartitions(n, k)

Bases: Partitions

The class of ordered partitions of n. If k is specified, then this contains only the ordered partitions of length k.

An ordered partition of a nonnegative integer n means a list of positive integers whose sum is n. This is the same as a composition of n.

Note: It is recommended that you use Compositions() instead as OrderedPartitions() wraps GAP.

EXAMPLES:

```python
sage: OrderedPartitions(3)
Ordered partitions of 3
sage: OrderedPartitions(3).list()
[[3], [2, 1], [1, 2], [1, 1, 1]]
sage: OrderedPartitions(3,2)
Ordered partitions of 3 of length 2
sage: OrderedPartitions(3,2).list()
[[2, 1], [1, 2]]
sage: OrderedPartitions(10,k=2).list()
[[9, 1], [8, 2], [7, 3], [6, 4], [5, 5], [4, 6], [3, 7], [2, 8], [1, 9]]
sage: OrderedPartitions(4).list()
[[4], [3, 1], [2, 2], [2, 1, 1], [1, 3], [1, 2, 1], [1, 1, 2], [1, 1, 1, 1]]
```

cardinality()

Return the cardinality of self.

EXAMPLES:

```python
sage: OrderedPartitions(3).cardinality()
4
sage: OrderedPartitions(3,2).cardinality()
2
sage: OrderedPartitions(10,2).cardinality()
9
```
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sage: OrderedPartitions(15).cardinality()
16384

list()

Return a list of partitions in self.

EXAMPLES:

sage: OrderedPartitions(3).list()
[[3], [2, 1], [1, 2], [1, 1, 1]]
sage: OrderedPartitions(3,2).list()
[[2, 1], [1, 2]]

class sage.combinat.partition.Particlen(parent, mu)

Bases: CombinatorialElement

A partition $\mu$ of a nonnegative integer $n$ is a non-increasing list of positive integers (the parts of the partition) with total sum $n$.

A partition is often represented as a diagram consisting of cells, or boxes, placed in rows on top of each other such that the number of cells in the $i^{th}$ row, reading from top to bottom, is the $i^{th}$ part of the partition. The rows are left-justified (and become shorter and shorter the farther down one goes). This diagram is called the Young diagram of the partition, or more precisely its Young diagram in English notation. (French and Russian notations are variations on this representation.)

The coordinate system related to a partition applies from the top to the bottom and from left to right. So, the corners of the partition $[5, 3, 1]$ are $[[0,4], [1,2], [2,0]]$.

For display options, see Partitions.options().

Note: Partitions are 0 based with coordinates in the form of (row-index, column-index). For example consider the partition $\mu=\text{Partition}([4,3,2,2])$, the first part is $\mu[0]$ (which is 4), the second is $\mu[1]$, and so on, and the upper-left cell in English convention is $(0, 0)$.

A partition can be specified in one of the following ways:

- a list (the default)
- using exponential notation
- by Frobenius coordinates
- specifying its 0 – 1 sequence
- specifying the core and the quotient

See the examples below.

EXAMPLES:

Creating partitions though parents:

sage: mu = Partitions(8)([3,2,1,1,1]); mu
[3, 2, 1, 1, 1]
sage: nu = Partition([3,2,1,1,1]); nu
[3, 2, 1, 1, 1]
sage: mu == nu
True
sage: mu is nu
False
sage: mu in Partitions()
True
sage: mu.parent()
Partitions of the integer 8
sage: mu.size()
8
sage: mu.category()
Category of elements of Partitions of the integer 8
sage: nu.parent()
Partitions
sage: nu.category()
Category of elements of Partitions
sage: mu[0]
3
sage: mu[1]
2
sage: mu[2]
1
sage: mu.pp()
***
**
*
*
sage: mu.removable_cells()
[(0, 2), (1, 1), (4, 0)]
sage: mu.down_list()
[[2, 2, 1, 1, 1], [3, 1, 1, 1, 1], [3, 2, 1, 1]]
sage: mu.addable_cells()
[(0, 3), (1, 2), (2, 1), (5, 0)]
sage: mu.up_list()
[[4, 2, 1, 1, 1], [3, 3, 1, 1, 1], [3, 2, 2, 1, 1], [3, 2, 1, 1, 1, 1]]
sage: mu.conjugate()
[5, 2, 1]
sage: mu.dominates(nu)
True
sage: nu.dominates(mu)
True

Creating partitions using Partition:

sage: Partition([3, 2, 1])
[3, 2, 1]
sage: Partition(exp=[2, 1, 1])
[3, 2, 1, 1]
sage: Partition(core=[2, 1], quotient=[[2, 1], [3], [1, 1, 1]])
[11, 5, 5, 3, 2, 2, 2]
sage: Partition(frobenius_coordinates=[[3, 2], [4, 0]])
[4, 4, 1, 1, 1]
Sage ignores trailing zeros at the end of partitions:

```
sage: Partition([3, 2, 1, 0])
[3, 2, 1]
sage: Partitions()([3, 2, 1])
[3, 2, 1]
sage: Partitions(6)([3, 2, 1])
[3, 2, 1]
```

```
add_cell(i, j=None)
Return a partition corresponding to self with a cell added in row i. (This does not change self.)

EXAMPLES:
```
sage: Partition([3, 2, 1, 1]).add_cell(0)
[4, 2, 1, 1]
sage: cell = [4, 0]; Partition([3, 2, 1, 1]).add_cell(*cell)
[3, 2, 1, 1, 1]
```

```
add_horizontal_border_strip(k)
Return a list of all the partitions that can be obtained by adding a horizontal border strip of length k to self.

EXAMPLES:
```
sage: Partition([]).add_horizontal_border_strip(0)
[]
sage: Partition([3,2,1]).add_horizontal_border_strip(0)
[[3, 2, 1]]
sage: Partition([2,2]).add_horizontal_border_strip(2)
[[4, 2], [3, 2, 1], [2, 2, 2]]
sage: Partition([3,2,2]).add_horizontal_border_strip(2)
[[5, 2, 2], [4, 3, 2], [4, 2, 2, 1], [3, 3, 2, 1], [3, 2, 2, 2]]
```

```
add_vertical_border_strip(k)
Return a list of all the partitions that can be obtained by adding a vertical border strip of length k to self.

EXAMPLES:
```
sage: Partition([]).add_vertical_border_strip(0)
[]
```
```
addable_cells()

Return a list of the outside corners of the partition self.

An outside corner (also called a cocorner) of a partition \( \lambda \) is a cell on \( \mathbb{Z}^2 \) which does not belong to the Young diagram of \( \lambda \) but can be added to this Young diagram to still form a straight-shape Young diagram.

The entries of the list returned are pairs of the form \((i, j)\), where \( i \) and \( j \) are the coordinates of the respective corner. The coordinates are counted from 0.

Note: These are called “outer corners” in [Sag2001].

EXAMPLES:

```python
sage: Partition([2,2,1]).outside_corners()
[(0, 2), (2, 1), (3, 0)]

sage: Partition([2,2]).outside_corners()
[(0, 2), (2, 0)]

sage: Partition([6,3,3,1,1,1]).outside_corners()
[(0, 6), (1, 3), (3, 1), (6, 0)]

sage: Partition([]).outside_corners()
[(0, 0)]
```

addable_cells_residue\((i, l)\)

Return a list of the outside corners of the partition self having \( l \)-residue \( i \).

An outside corner (also called a cocorner) of a partition \( \lambda \) is a cell on \( \mathbb{Z}^2 \) which does not belong to the Young diagram of \( \lambda \) but can be added to this Young diagram to still form a straight-shape Young diagram. See residue() for the definition of the \( l \)-residue.

The entries of the list returned are pairs of the form \((i, j)\), where \( i \) and \( j \) are the coordinates of the respective corner. The coordinates are counted from 0.

EXAMPLES:

```python
sage: Partition([2,2,1]).outside_corners_residue(0, 3)
[(0, 3), (3, 0)]

sage: Partition([2,2]).outside_corners_residue(1, 3)
[(1, 2)]

sage: Partition([3,2,1]).outside_corners_residue(2, 3)
[(2, 1)]
```

arm_cells\((i, j)\)

Return the list of the cells of the arm of cell \((i, j)\) in self.

The arm of cell \( c = (i, j) \) is the boxes that appear to the right of \( c \).
The cell coordinates are zero-based, i.e., the northwesternmost cell is (0, 0).

INPUT:
• i, j – two integers

OUTPUT:
A list of pairs of integers

EXAMPLES:

```
sage: Partition([4,4,3,1]).arm_cells(1,1)
[(1, 2), (1, 3)]
sage: Partition([]).arm_cells(0,0)
Traceback (most recent call last):
...
ValueError: the cell is not in the diagram
```

`arm_length(i, j)`

Return the length of the arm of cell (i, j) in self.

The arm of cell (i, j) is the cells that appear to the right of cell (i, j).

The cell coordinates are zero-based, i.e., the northwesternmost cell is (0, 0).

INPUT:
• i, j – two integers

OUTPUT:
An integer or a ValueError

EXAMPLES:

```
sage: p = Partition([2,2,1])
sage: p.arm_length(0, 0)
1
sage: p.arm_length(0, 1)
0
sage: p.arm_length(2, 0)
0
sage: Partition([3,3]).arm_length(0, 0)
2
sage: Partition([3,3]).arm_length(*[0,0])
2
```

`arm_lengths(flat=False)`

Return a tableau of shape self where each cell is filled with its arm length.

The optional boolean parameter flat provides the option of returning a flat list.

EXAMPLES:

```
sage: Partition([2,2,1]).arm_lengths()
[[[1, 0], [1, 0], [0]]]
sage: Partition([2,2,1]).arm_lengths(flat=True)
[[1, 0, 1, 0, 0]]
```

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sage: Partition([3,3]).arm_lengths()
[[2, 1, 0], [2, 1, 0]]
sage: Partition([3,3]).arm_lengths(flat=True)
[2, 1, 0, 2, 1, 0]

arms_legs_coeff(i, j)
This is a statistic on a cell \( c = (i, j) \) in the diagram of partition \( p \) given by
\[
\frac{1 - q^a \cdot t^{\ell + 1}}{1 - q^{a + 1} \cdot t^\ell}
\]
where \( a \) is the arm length of \( c \) and \( \ell \) is the leg length of \( c \).

The coordinates \( i \) and \( j \) of the cell are understood to be 0-based, so that \((0, 0)\) is the northwesternmost cell (in English notation).

EXAMPLES:

sage: Partition([3,2,1]).arms_legs_coeff(1,1)
(-t + 1)/(-q + 1)
sage: Partition([3,2,1]).arms_legs_coeff(0,0)
(-q^2*t^3 + 1)/(-q^3*t^2 + 1)
sage: Partition([3,2,1]).arms_legs_coeff(*[0,0])
(-q^2*t^3 + 1)/(-q^3*t^2 + 1)

atom()
Return a list of the standard tableaux of size self.size() whose atom is equal to self.

EXAMPLES:

sage: Partition([2,1]).atom()
[[[1, 2], [3]]]
sage: Partition([3,2,1]).atom()
[[[1, 2, 3, 6], [4, 5]], [[1, 2, 3], [4, 5], [6]]]

attacking_pairs()
Return a list of the attacking pairs of the Young diagram of self.

A pair of cells \((c, d)\) of a Young diagram (in English notation) is said to be attacking if one of the following conditions holds:
1. \( c \) and \( d \) lie in the same row with \( c \) strictly to the west of \( d \).
2. \( c \) is in the row immediately to the south of \( d \), and \( c \) lies strictly east of \( d \).

This particular method returns each pair \((c, d)\) as a tuple, where each of \( c \) and \( d \) is given as a tuple \((i, j)\) with \( i \) and \( j \) zero-based (so \( i = 0 \) means that the cell lies in the topmost row).

EXAMPLES:

sage: p = Partition([3, 2])
sage: p.attacking_pairs()
[((0, 0), (0, 1)), ((0, 0), (0, 2)), ((0, 1), (0, 2)), ((1, 0), (1, 1)),
  ((1, 0), (1, 2)), ((1, 1), (1, 2)), ((1, 1), (1, 3)), ((1, 2), (1, 3)),
  ((2, 0), (2, 1)), ((2, 0), (2, 2)), ((2, 1), (2, 2))]

(continues on next page)
aut\((r=0, q=0)\)

Return the size of the centralizer of any permutation of cycle type \(\text{self}\).

If \(m_i\) is the multiplicity of \(i\) as a part of \(p\), this is given by

\[
\prod_i m_i! m_i^2.
\]

Including the optional parameters \(t\) and \(q\) gives the \(q, t\) analog, which is the former product times

\[
\prod_{i=1}^{\text{length}(p)} \frac{1 - q^{p_i}}{1 - t^{p_i}}.
\]

See Section 1.3, p. 24, in [Ke1991].

EXAMPLES:

\[
\begin{align*}
sage: & \text{Partition([2,2,1]).centralizer_size()} \\
& 8 \\
sage: & \text{Partition([2,2,2]).centralizer_size()} \\
& 48 \\
sage: & \text{Partition([2,2,1]).centralizer_size(q=2, t=3)} \\
& 9/16 \\
sage: & \text{Partition([[]].centralizer_size()} \\
& 1 \\
sage: & \text{Partition([[]].centralizer_size(q=2, t=4)} \\
& 1
\end{align*}
\]

beta_numbers\((\text{length=\text{None}})\)

Return the set of beta numbers corresponding to \(\text{self}\).

The optional argument \(\text{length}\) specifies the length of the beta set (which must be at least the length of \(\text{self}\)).

For more on beta numbers, see \texttt{frobenius_coordinates()}.

EXAMPLES:

\[
\begin{align*}
sage: & \text{Partition([4,3,2]).beta_numbers()} \\
& [6, 4, 2] \\
sage: & \text{Partition([4,3,2]).beta_numbers(5)} \\
& [8, 6, 4, 1, 0] \\
sage: & \text{Partition([[]].beta_numbers()} \\
& [] \\
sage: & \text{Partition([[]].beta_numbers(3)} \\
& [2, 1, 0] \\
sage: & \text{Partition([6,4,1,1]).beta_numbers()} \\
& [9, 6, 2, 1] \\
sage: & \text{Partition([6,4,1,1]).beta_numbers(6)} \\
& [11, 8, 4, 3, 1, 0] \\
sage: & \text{Partition([1,1,1].beta_numbers()} \\
& []
\end{align*}
\]
block\((e, \text{multicharge}=(0,))\)

Return a dictionary \(\beta\) that determines the block associated to the partition \(\text{self}\) and the quantum_characteristic() \(e\).

**INPUT:**

- \(e\) – the quantum characteristic
- \(\text{multicharge}\) – the multicharge (default \((0,))\)

**OUTPUT:**

- A dictionary giving the multiplicities of the residues in the partition tuple \(\text{self}\)

In more detail, the value \(\text{beta}[i]\) is equal to the number of nodes of residue \(i\). This corresponds to the positive root

\[
\sum_{i \in I} \beta_i \alpha_i \in Q^+,
\]

a element of the positive root lattice of the corresponding Kac-Moody algebra. See [DJM1998] and [BK2009] for more details.

This is a useful statistics because two Specht modules for a Hecke algebra of type \(A\) belong to the same block if and only if they correspond to same element \(\beta\) of the root lattice, given above.

We return a dictionary because when the quantum characteristic is 0, the Cartan type is \(A_\infty\), in which case the simple roots are indexed by the integers.

**EXAMPLES:**

```python
sage: Partition([4,3,2]).block(0)
{0: 4, 1: 5}
```

```python
sage: Partition([4,3,2]).block(2)
{0: 5, 1: 4}
```

```python
sage: Partition([4,3,2]).block(2, multicharge=(1,))
{0: 5, 1: 4}
```

```python
sage: Partition([4,3,2]).block(3)
{0: 1, 3: 2, 2: 3}
```

```python
sage: Partition([4,3,2]).block(4)
{0: 2, 1: 2, 2: 2, 3: 3}
```

boundary()

Return the integer coordinates of points on the boundary of \(\text{self}\).

For the following description, picture the Ferrer’s diagram of \(\text{self}\) using the French convention. Recall that the French convention puts the longest row on the bottom and the shortest row on the top. In addition, interpret the Ferrer’s diagram as 1 x 1 cells in the Euclidean plane. So if \(\text{self}\) was the partition \([3, 1]\), the lower-left vertices of the 1 x 1 cells in the Ferrer’s diagram would be \((0, 0), (1, 0), (2, 0), \) and \((0, 1)\).

The boundary of a partition is the set \(\{\text{NE}(d) \mid \forall d \text{ diagonal}\}\). That is, for every diagonal line \(y = x + b\) where \(b \in \mathbb{Z}\), we find the northeasternmost (NE) point on that diagonal which is also in the Ferrer’s diagram.

The boundary will go from bottom-right to top-left.

**EXAMPLES:**

5.1. Comprehensive Module List 1721
Consider the partition (1) depicted as a square on a cartesian plane with vertices (0, 0), (1, 0), (1, 1), and (0, 1). Three of those vertices in the appropriate order form the boundary:

```
sage: Partition([1]).boundary()
[(1, 0), (1, 1), (0, 1)]
```

The partition (3, 1) can be visualized as three squares on a cartesian plane. The coordinates of the appropriate vertices form the boundary:

```
sage: Partition([3, 1]).boundary()
[(3, 0), (3, 1), (2, 1), (1, 1), (1, 2), (0, 2)]
```

See also:

`k_rim()`. You might have been looking for `k_boundary()` instead.

`cell_poset(orientation='SE')`

Return the Young diagram of `self` as a poset. The optional keyword variable `orientation` determines the order relation of the poset.

The poset always uses the set of cells of the Young diagram of `self` as its ground set. The order relation of the poset depends on the `orientation` variable (which defaults to "SE"). Concretely, `orientation` has to be specified to one of the strings "NW", "NE", "SW", and "SE", standing for “northwest”, “northeast”, “southwest” and “southeast”, respectively. If `orientation` is "SE", then the order relation of the poset is such that a cell `𝑢` is greater or equal to a cell `𝑣` in the poset if and only if `𝑢` lies weakly southeast of `𝑣` (this means that `𝑢` can be reached from `𝑣` by a sequence of south and east steps; the sequence is allowed to consist of south steps only, or of east steps only, or even be empty). Similarly the order relation is defined for the other three orientations. The Young diagram is supposed to be drawn in English notation.

The elements of the poset are the cells of the Young diagram of `self`, written as tuples of zero-based coordinates (so that (3, 7) stands for the 8-th cell of the 4-th row, etc.).

EXAMPLES:

```
sage: p = Partition([3,3,1])
sage: Q = p.cell_poset(); Q
Finite poset containing 7 elements

sage: sorted(Q)
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0)]

sage: sorted(Q.maximal_elements())
[(1, 2), (2, 0)]

sage: Q.minimal_elements()
[(0, 0)]

sage: sorted(Q.upper_covers((1, 0)))
[(1, 1), (2, 0)]

sage: P = p.cell_poset(orientation="NW"); P
Finite poset containing 7 elements

sage: sorted(P)
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0)]

sage: sorted(P.minimal_elements())
[(1, 2), (2, 0)]

sage: P.maximal_elements()
[(0, 0)]
```

(continues on next page)
sage: P.upper_covers((2, 0))
[(1, 0)]
sage: sorted(P.upper_covers((1, 2)))
[(0, 2), (1, 1)]
sage: sorted(P.upper_covers((1, 1)))
[(0, 1), (1, 0)]
sage: sorted([len(P.upper_covers(v)) for v in P])
[0, 1, 1, 1, 1, 2, 2]
sage: R = p.cell_poset(orientation="NE"); R
Finite poset containing 7 elements
sage: sorted(R)
[(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0)]
sage: R.maximal_elements()
[(0, 2)]
sage: R.minimal_elements()
[(2, 0)]
sage: sorted([len(R.upper_covers(v)) for v in R])
[0, 1, 1, 1, 1, 2, 2]
sage: R.is_isomorphic(P)
False
sage: R.is_isomorphic(P.dual())
False

Linear extensions of \( p.cell\_poset() \) are in 1-to-1 correspondence with standard Young tableaux of shape \( p \):

sage: all( len(p.cell_poset().linear_extensions()) ==
...... len(p.standard_tableaux())
...... for n in range(8) for p in Partitions(n) )
True

This is not the case for northeast orientation:

sage: q = Partition([3, 1])
sage: q.cell_poset(orientation="NE").is_chain()
True
cells()  
Return the coordinates of the cells of self.

EXAMPLES:

sage: Partition([2,2]).cells()
[[0, 0], (0, 1), (1, 0), (1, 1)]
sage: Partition([3,2]).cells()
[[0, 0], (0, 1), (0, 2), (1, 0), (1, 1)]

centralizer_size\((t=0, q=0)\)  
Return the size of the centralizer of any permutation of cycle type self.

If \( m_i \) is the multiplicity of \( i \) as a part of \( p \), this is given by

\[
\prod_i m_i!^{m_i}.
\]
Including the optional parameters $t$ and $q$ gives the $q,t$ analog, which is the former product times

$$\prod_{i=1}^{\text{length}(p)} \frac{1 - q^{p_i}}{1 - t^{p_i}}.$$ 

See Section 1.3, p. 24, in [Ke1991].

EXAMPLES:

```python
sage: Partition([2,2,1]).centralizer_size()
8
sage: Partition([2,2,2]).centralizer_size()
48
sage: Partition([2,2,1]).centralizer_size(q=2, t=3)
9/16
sage: Partition([]).centralizer_size()
1
sage: Partition([]).centralizer_size(q=2, t=4)
1
```

**character_polynomial()**

Return the character polynomial associated to the partition `self`.

The character polynomial $q_\mu$ associated to a partition $\mu$ is defined by

$$q_\mu(x_1, x_2, \ldots, x_k) = \downarrow \sum_{\alpha \vdash k} \chi^\mu_\alpha 1^{a_1} 2^{a_2} \cdots k^{a_k} a_1! a_2! \cdots a_k! \prod_{i=1}^{k} (ix_i - 1)^{a_i}$$

where $k$ is the size of $\mu$, and $a_i$ is the multiplicity of $i$ in $\alpha$.

It is computed in the following manner:

1. Expand the Schur function $s_\mu$ in the power-sum basis,
2. Replace each $p_i$ with $ix_i - 1$,
3. Apply the umbral operator $\downarrow$ to the resulting polynomial.

EXAMPLES:

```python
sage: Partition([1]).character_polynomial()
x - 1
sage: Partition([1,1]).character_polynomial()
1/2*x0^2 - 3/2*x0 - x1 + 1
sage: Partition([2,1]).character_polynomial()
1/3*x0^3 - 2*x0^2 - 8/3*x0 - x2
```

**components()**

Return a list containing the shape of `self`.

This method exists only for compatibility with `PartitionTuples`.

EXAMPLES:

```python
sage: Partition([3,2]).components()
[[3, 2]]
```
**conjugacy_class_size()**

Return the size of the conjugacy class of the symmetric group indexed by `self`.

**EXAMPLES:**

```
sage: Partition([2,2,2]).conjugacy_class_size()
15
sage: Partition([2,2,1]).conjugacy_class_size()
15
sage: Partition([2,1,1]).conjugacy_class_size()
6
```

**conjugate()**

Return the conjugate partition of the partition `self`. This is also called the associated partition or the transpose in the literature.

**EXAMPLES:**

```
sage: Partition([2,2]).conjugate()
[2, 2]
sage: Partition([6,3,1]).conjugate()
[3, 2, 1, 1, 1, 1]
```

The conjugate partition is obtained by transposing the Ferrers diagram of the partition (see `ferrers_diagram()`):

```
sage: print(Partition([6,3,1]).ferrers_diagram())
******
***
*
sage: print(Partition([6,3,1]).conjugate().ferrers_diagram())
***
**
**
*
*
```

**contains(x)**

Return `True` if `x` is a partition whose Ferrers diagram is contained in the Ferrers diagram of `self`.

**EXAMPLES:**

```
sage: p = Partition([3,2,1])
sage: p.contains([2,1])
True
sage: all(p.contains(mu) for mu in Partitions(3))
True
sage: all(p.contains(mu) for mu in Partitions(4))
False
```

**content(r, c, multicharge=(0,))**

Return the content of the cell at row `r` and column `c`.

The content of a cell is `c - r`. 

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For consistency with partition tuples there is also an optional `multicharge` argument which is an offset to the usual content. By setting the `multicharge` equal to the 0-element of the ring \( \mathbb{Z}/e\mathbb{Z} \), the corresponding \( e \)-residue will be returned. This is the content modulo \( e \).

The content (and residue) do not strictly depend on the partition, however, this method is included because it is often useful in the context of partitions.

**EXAMPLES:**

```sage
sage: Partition([2,1]).content(1,0)
-1
sage: p = Partition([3,2])
sage: sum([p.content(*c) for c in p.cells()])
2
```

and now we return the 3-residue of a cell:

```sage
sage: Partition([2,1]).content(1,0, multicharge=IntegerModRing(3)(0))
2
```

### `contents_tableau(multicharge=(0,))`

Return the tableau which has \((k,r,c)\)-th cell equal to the content \(multicharge[k] - r + c\) of the cell.

**EXAMPLES:**

```sage
sage: Partition([2,1]).contents_tableau()
[[0, 1], [-1]]
sage: Partition([3,2,1,1]).contents_tableau().pp()
   0 1 2
  -1 0
  -2
-3
sage: Partition([3,2,1,1]).contents_tableau([ IntegerModRing(3)(0) ] ).pp()
   0 1 2
  2 0
  1
-0
```

### `core(length)`

Return the `length`-core of the partition – in the literature the core is commonly referred to as the \( k \)-core, \( p \)-core, \( r \)-core, ... .

The \( r \)-core of a partition \( \lambda \) can be obtained by repeatedly removing rim hooks of size \( r \) from (the Young diagram of) \( \lambda \) until this is no longer possible. The remaining partition is the core.

**EXAMPLES:**

```sage
sage: Partition([6,3,2,2]).core(3)
[2, 1, 1]
sage: Partition([]).core(3)
[]
sage: Partition([8,7,7,4,1,1,1,1,1]).core(3)
[2, 1, 1]
```

### `corners()`

Return a list of the corners of the partition `self`.
A corner of a partition \( \lambda \) is a cell of the Young diagram of \( \lambda \) which can be removed from the Young diagram while still leaving a straight shape behind.

The entries of the list returned are pairs of the form \((i, j)\), where \(i\) and \(j\) are the coordinates of the respective corner. The coordinates are counted from 0.

**Note:** This is referred to as an “inner corner” in [Sag2001].

**EXAMPLES:**

```
sage: Partition([3,2,1]).corners()
[(0, 2), (1, 1), (2, 0)]
sage: Partition([3,3,1]).corners()
[(1, 2), (2, 0)]
sage: Partition([]).corners()
[]
```

corners_residue\((i, l)\)

Return a list of the corners of the partition \( \text{self} \) having \(l\)-residue \(i\).

A corner of a partition \( \lambda \) is a cell of the Young diagram of \( \lambda \) which can be removed from the Young diagram while still leaving a straight shape behind. See residue() for the definition of the \(l\)-residue.

The entries of the list returned are pairs of the form \((i, j)\), where \(i\) and \(j\) are the coordinates of the respective corner. The coordinates are counted from 0.

**EXAMPLES:**

```
sage: Partition([3,2,1]).corners_residue(0, 3)
[(1, 1)]
sage: Partition([3,2,1]).corners_residue(1, 3)
[(2, 0)]
sage: Partition([3,2,1]).corners_residue(2, 3)
[(0, 2)]
```

crank()

Return the Dyson crank of \( \text{self} \).

The Dyson crank of a partition \( \lambda \) is defined as follows: If \( \lambda \) contains at least one 1, then the crank is \( \mu(\lambda) - \omega(\lambda) \), where \( \omega(\lambda) \) is the number of 1’s in \( \lambda \), and \( \mu(\lambda) \) is the number of parts of \( \lambda \) larger than \( \omega(\lambda) \). If \( \lambda \) contains no 1, then the crank is simply the largest part of \( \lambda \).

**REFERENCES:**

- [AG1988]

**EXAMPLES:**

```
sage: Partition([]).crank()
0
sage: Partition([3,2,2]).crank()
3
sage: Partition([5,4,2,1,1]).crank()
0
sage: Partition([1,1,1]).crank()
-3
```

(continues on next page)
sage: Partition([6,4,4,3]).crank()
6
sage: Partition([6,3,3,1,1]).crank()
1
sage: Partition([6]).crank()
6
sage: Partition([5,1]).crank()
0
sage: Partition([4,2]).crank()
4
sage: Partition([4,1,1]).crank()
-1
sage: Partition([3,3]).crank()
3
sage: Partition([3,2,1]).crank()
1
sage: Partition([3,1,1,1]).crank()
-3
sage: Partition([2,2,2]).crank()
2
sage: Partition([2,2,1,1]).crank()
-2
sage: Partition([2,1,1,1,1]).crank()
-4
sage: Partition([1,1,1,1,1,1]).crank()
-6

defect(e, multicharge=(0,))

Return the e-defect or the e-weight of self.

The e-defect is the number of (connected) e-rim hooks that can be removed from the partition.

The defect of a partition is given by

\[
\text{defect}(\beta) = (\Lambda, \beta) - \frac{1}{2} (\beta, \beta),
\]

where \( \Lambda = \sum_{r} A_{\kappa_{r}} \) for the multicharge \( (\kappa_{1}, \ldots, \kappa_{t}) \) and \( \beta = \sum_{(r,c)} a_{(c-r) \mod e} \), with the sum being over the cells in the partition.

INPUT:

- e – the quantum characteristic
- multicharge – the multicharge (default (0,))

OUTPUT:

- a non-negative integer, which is the defect of the block containing the partition self

EXAMPLES:

sage: Partition([4,3,2]).defect(2)
3
sage: Partition([0]).defect(2)
0
sage: Partition([3]).defect(2)
degree($e$)

Return the $e$-th degree of self.

The $e$-th degree of a partition $\lambda$ is the sum of the $e$-th degrees of the standard tableaux of shape $\lambda$. The $e$-th degree is the exponent of $\Phi_e(q)$ in the Gram determinant of the Specht module for a semisimple Iwahori-Hecke algebra of type $A$ with parameter $q$.

INPUT:

• $e$ – an integer $e > 1$

OUTPUT:

A non-negative integer.

EXAMPLES:

```python
sage: Partition([4,3]).degree(2)
28
sage: Partition([4,3]).degree(3)
15
sage: Partition([4,3]).degree(4)
8
sage: Partition([4,3]).degree(5)
13
sage: Partition([4,3]).degree(6)
0
sage: Partition([4,3]).degree(7)
0
```

Therefore, the Gram determinant of $S(5,3)$ when the Hecke parameter $q$ is “generic” is

$$q^N \Phi_2(q)^{28} \Phi_3(q)^{15} \Phi_4(q)^8 \Phi_5(q)^{13}$$

for some integer $N$. Compare with `prime_degree()`.
**dimension**(smaller=None, k=1)

Return the number of paths from the smaller partition to the partition self, where each step consists of adding a k-ribbon while keeping a partition.

Note that a 1-ribbon is just a single cell, so this counts paths in the Young graph when \( k = 1 \).

Note also that the default case (\( k = 1 \) and smaller = []) gives the dimension of the irreducible representation of the symmetric group corresponding to self.

**INPUT:**

- smaller – a partition (default: an empty list [])
- k – a positive integer (default: 1)

**OUTPUT:**

The number of such paths

**EXAMPLES:**

Looks at the number of ways of getting from [5, 4] to the empty partition, removing one cell at a time:

```
sage: mu = Partition([5,4])
sage: mu.dimension()
42
```

Same, but removing one 3-ribbon at a time. Note that the 3-core of mu is empty:

```
sage: mu.dimension(k=3)
3
```

The 2-core of mu is not the empty partition:

```
sage: mu.dimension(k=2)
0
```

Indeed, the 2-core of mu is [1]:

```
sage: mu.dimension(Partition([1]),k=2)
2
```

**ALGORITHM:**

Depending on the parameters given, different simplifications occur. When \( k = 1 \) and smaller is empty, this function uses the hook formula. When \( k = 1 \) and smaller is not empty, it uses a formula from [ORV].

When \( k \neq 1 \), we first check that both self and smaller have the same k-core, then use the k-quotients and the same algorithm on each of the k-quotients.

**AUTHORS:**

- Paul-Olivier Dehaye (2011-06-07)

**dominated_partitions**(rows=None)

Return a list of the partitions dominated by \( n \). If rows is specified, then it only returns the ones whose number of rows is at most rows.

**EXAMPLES:**
sage: Partition([3,2,1]).dominated_partitions()
[[[3, 2, 1], [3, 1, 1, 1], [2, 2, 2], [2, 2, 1, 1], [2, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]],
   [3, 2, 1], [2, 2, 2], [2, 2, 1, 1], [2, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]]

sage: Partition([3,2,1]).dominated_partitions(rows=3)
[[[3, 2, 1], [2, 2, 2]]

dominates(p2)

Return True if self dominates the partition p2. Otherwise it returns False.

EXAMPLES:

sage: p = Partition([3,2])
sage: p.dominates([3,1])
True
sage: p.dominates([2,2])
True
sage: p.dominates([2,1,1])
True
sage: p.dominates([3,3])
False
sage: p.dominates([4])
False
sage: Partition([4]).dominates(p)
False
sage: Partition([]).dominates([1])
False
sage: Partition([]).dominates([])
True
sage: Partition([1]).dominates([])
True
down()

Return a generator for partitions that can be obtained from self by removing a cell.

EXAMPLES:

sage: [p for p in Partition([2,1,1]).down()]
[[1, 1, 1], [2, 1]]
sage: [p for p in Partition([3,2]).down()]
[[2, 2], [3, 1]]
sage: [p for p in Partition([3,2,1]).down()]
[[2, 2, 1], [3, 1, 1], [3, 2]]
down_list()

Return a list of the partitions that can be obtained from self by removing a cell.

EXAMPLES:

sage: Partition([2,1,1]).down_list()
[[1, 1, 1], [2, 1]]
sage: Partition([3,2]).down_list()
[[2, 2], [3, 1]]
sage: Partition([3,2,1]).down_list()
[[2, 2, 1], [3, 1, 1], [3, 2]]
dual_equivalence_graph(directed=False, coloring=None)

Return the dual equivalence graph of self.

Two permutations $p$ and $q$ in the symmetric group $S_n$ differ by an $i$-elementary dual equivalence (or dual Knuth) relation (where $i$ is an integer with $1 < i < n$) when the following two conditions are satisfied:

- In the one-line notation of the permutation $p$, the letter $i$ does not appear inbetween $i - 1$ and $i + 1$.
- The permutation $q$ is obtained from $p$ by switching two of the three letters $i - 1, i, i + 1$ (in its one-line notation) – namely, the leftmost and the rightmost one in order of their appearance in $p$.

Notice that this is equivalent to the statement that the permutations $p^{-1}$ and $q^{-1}$ differ by an elementary Knuth equivalence at positions $i - 1, i, i + 1$.

Two standard Young tableaux of shape $\lambda$ differ by an $i$-elementary dual equivalence relation (of color $i$), if their reading words differ by an $i$-elementary dual equivalence relation.

The dual equivalence graph of the partition $\lambda$ is the edge-colored graph whose vertices are the standard Young tableaux of shape $\lambda$, and whose edges colored by $i$ are given by the $i$-elementary dual equivalences.

INPUT:

- directed – (default: False) whether to have the dual equivalence graph be directed (where we have a directed edge $S \to T$ if $i$ appears to the left of $i + 1$ in the reading word of $T$; otherwise we have the directed edge $T \to S$)
- coloring – (optional) a function which sends each integer $i > 1$ to a color (as a string, e.g., 'red' or 'black') to be used when visually representing the resulting graph using dot2tex; the default choice is $2 \to 'red', 3 \to 'blue', 4 \to 'green', 5 \to 'purple', 6 \to 'brown', 7 \to 'orange', 8 \to 'yellow', anything greater than 8 \to 'black'.

REFERENCES:

- [As2008b]

EXAMPLES:

```python
sage: P = Partition([3,1,1])
sage: G = P.dual_equivalence_graph()
sage: G = P.dual_equivalence_graph(directed=True)
```
evaluation()

Return the evaluation of self.

The **commutative evaluation**, often shortened to **evaluation**, of a word (we think of a partition as a word in \([1, 2, 3, \ldots]\)) is its image in the free commutative monoid. In other words, this counts how many occurrences there are of each letter.

This is also known as **Parikh vector** and **abelianization** and has the same output as **to_exp()**.

**EXAMPLES:**

```python
sage: Partition([4,3,1,1]).evaluation()
[2, 0, 1, 1]
```

ferrers_diagram()

Return the Ferrers diagram of self.

**EXAMPLES:**

```python
sage: mu = Partition([5,5,2,1])
sage: Partitions.options(diagram_str='*', convention="english")
sage: print(mu.ferrers_diagram())
*****
*****
**
*
```

```python
sage: Partitions.options(diagram_str='#', convention="english")
sage: print(mu.ferrers_diagram())
#####
#####
##
#
```

```python
sage: Partitions.options.convention="french"
sage: print(mu.ferrers_diagram())
#
##
#####
#####
```

```python
sage: print(Partition([]).ferrers_diagram())
-
```

```python
sage: Partitions.options(diagram_str='\-')
sage: print(Partition([]).ferrers_diagram())
(\-)
```

```python
sage: Partitions.options._reset()
```

frobenius_coordinates()

Return a pair of sequences of Frobenius coordinates aka beta numbers of the partition.

These are two strictly decreasing sequences of nonnegative integers of the same length.

**EXAMPLES:**

```python
sage: Partition([]).frobenius_coordinates()
([], [])
```

```python
sage: Partition([1]).frobenius_coordinates()
([0], [0])
```
sage: Partition([3,3,3]).frobenius_coordinates()
([2, 1, 0], [2, 1, 0])
sage: Partition([9,1,1,1,1,1,1]).frobenius_coordinates()
([8], [6])

frobenius_rank()

Return the Frobenius rank of the partition self.

The Frobenius rank of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ is defined to be the largest $i$ such that $\lambda_i \geq i$. In other words, it is the number of cells on the main diagonal of $\lambda$. In yet other words, it is the size of the largest square fitting into the Young diagram of $\lambda$.

EXAMPLES:
sage: Partition([]).frobenius_rank()
0
sage: Partition([1]).frobenius_rank()
1
sage: Partition([3,3,3]).frobenius_rank()
3
sage: Partition([9,1,1,1,1,1,1]).frobenius_rank()
1
sage: Partition([2,1,1,1,1,1]).frobenius_rank()
1
sage: Partition([2,2,1,1,1,1]).frobenius_rank()
2
sage: Partition([3,2]).frobenius_rank()
2
sage: Partition([3,2,2]).frobenius_rank()
2
sage: Partition([8,4,4,4,4]).frobenius_rank()
4
sage: Partition([8,4,1]).frobenius_rank()
2
sage: Partition([3,3,1]).frobenius_rank()
2

from_kbounded_to_grassmannian(k)

Maps a $k$-bounded partition to a Grassmannian element in the affine Weyl group of type $A_k^{(1)}$.

For details, see the documentation of the method from_kbounded_to_reduced_word() .

EXAMPLES:
sage: p = Partition([2,1,1])
sage: p.from_kbounded_to_grassmannian(2)
[-1  1  1]
[-2  2  1]
[-2  1  2]
sage: p = Partition([])
sage: p.from_kbounded_to_grassmannian(2)
[1 0 0]
[0 1 0]
[0 0 1]
**from_kbounded_to_reduced_word**(\(k\))

Maps a \(k\)-bounded partition to a reduced word for an element in the affine permutation group.

This uses the fact that there is a bijection between \(k\)-bounded partitions and \((k + 1)\)-cores and an action of the affine nilCoxeter algebra of type \(A^{(1)}_k\) on \((k + 1)\)-cores as described in [LM2006b].

**EXAMPLES:**

```python
sage: p = Partition([2,1,1])
sage: p.from_kbounded_to_reduced_word(2)
[2, 1, 2, 0]
sage: p = Partition([3,1])
sage: p.from_kbounded_to_reduced_word(3)
[3, 2, 1, 0]
sage: p.from_kbounded_to_reduced_word(2)
Traceback (most recent call last):
...
ValueError: the partition must be 2-bounded
sage: p = Partition([])
sage: p.from_kbounded_to_reduced_word(2)
[]
```

**garnir_tableau**(\(*\text{cell}\*))

Return the Garnir tableau of shape \texttt{self} corresponding to the cell \texttt{cell}. If \texttt{cell} = \((a, c)\) then \((a + 1, c)\) must belong to the diagram of \texttt{self}.

The Garnir tableaux play an important role in integral and non-semisimple representation theory because they determine the “straightening” rules for the Specht modules over an arbitrary ring.

The Garnir tableaux are the “first” non-standard tableaux which arise when you act by simple transpositions. If \((a, c)\) is a cell in the Young diagram of a partition, which is not at the bottom of its column, then the corresponding Garnir tableau has the integers \(1, 2, \ldots, n\) entered in order from left to right along the rows of the diagram up to the cell \((a, c - 1)\), then along the cells \((a + 1, 1)\) to \((a + 1, c)\), then \((a, c)\) until the end of row \(a\) and then continuing from left to right in the remaining positions. The examples below probably make this clearer!

**Note:** The function also sets \texttt{g._garnir_cell}, where \texttt{g} is the resulting Garnir tableau, equal to \texttt{cell} which is used by some other functions.

**EXAMPLES:**

```python
sage: g = Partition([5,3,3,2]).garnir_tableau((0,2)); g.pp()
  1  2  6  7  8
  3  4  5
  9 10 11
 12 13
sage: g.is_row_strict(); g.is_column_strict()
True
False
sage: Partition([5,3,3,2]).garnir_tableau(0,2).pp()
  1  2  6  7  8
  3  4  5
  9 10 11
```
12 13
sage: Partition([5,3,3,2]).garnir_tableau(2,1).pp()
 1 2 3 4 5
 6 7 8
 9 12 13
10 11
sage: Partition([5,3,3,2]).garnir_tableau(2,2).pp()
Traceback (most recent call last):
  ...
ValueError: (row+1, col) must be inside the diagram

See also:

• top_garnir_tableau()

generalized_pochhammer_symbol(a, alpha)

Return the generalized Pochhammer symbol \( (a)_{\alpha} \). This is the product over all cells \((i, j)\) in \(self\) of \(a - (i - 1)/\alpha + j - 1\).

EXAMPLES:

sage: Partition([2,2]).generalized_pochhammer_symbol(2,1)
12

get_part(i, default=0)

Return the \(i\)th part of \(self\), or \(default\) if it does not exist.

EXAMPLES:

sage: p = Partition([2,1])
sage: p.get_part(0), p.get_part(1), p.get_part(2)
(2, 1, 0)
sage: p.get_part(10,-1)
-1
sage: Partition([]).get_part(0)
0

has_k_rectangle(k)

Return True if the Ferrer's diagram of \(self\) contains \(k - i + 1\) rows (or more) of length \(i\) (exactly) for any \(i\) in \([1, k]\).

This is mainly a helper function for is_k_reducible() and is_k_irreducible(), the only difference between this function and is_k_reducible() being that this function allows any partition as input while is_k_reducible() requires the input to be \(k\)-bounded.

EXAMPLES:

The partition \([1, 1, 1]\) has at least 2 rows of length 1:

sage: Partition([1, 1, 1]).has_k_rectangle(2)
True

The partition \([1, 1, 1]\) does not have 4 rows of length 1, 3 rows of length 2, 2 rows of length 3, nor 1 row of length 4:
Combinatorics, Release 10.0

```python
sage: Partition([1, 1, 1]).has_k_rectangle(4)
False
```

See also:

`is_k_irreducible(), is_k_reducible(), has_rectangle()`

**has_rectangle** *(h, w)*

Return True if the Ferrer’s diagram of `self` has *h* (or more) rows of length *w* (exactly).

**INPUT:**
- *h* – An integer \( h \geq 1 \). The (minimum) height of the rectangle.
- *w* – An integer \( w \geq 1 \). The width of the rectangle.

**EXAMPLES:**

```python
sage: Partition([3, 3, 3, 3]).has_rectangle(2, 3)
True
sage: Partition([3, 3]).has_rectangle(2, 3)
True
sage: Partition([4, 3]).has_rectangle(2, 3)
False
sage: Partition([3]).has_rectangle(2, 3)
False
```

See also:

`has_k_rectangle()`

**hook_length** *(i, j)*

Return the length of the hook of cell \((i, j)\) in `self`.

The (length of the) hook of cell \((i, j)\) of a partition \( \lambda \) is

\[
\lambda_i + \lambda_j' - i - j + 1
\]

where \( \lambda' \) is the conjugate partition. In English convention, the hook length is the number of cells horizontally to the right and vertically below the cell \((i, j)\) (including that cell).

**EXAMPLES:**

```python
sage: p = Partition([2,2,1])
sage: p.hook_length(0, 0)
4
sage: p.hook_length(0, 1)
2
sage: p.hook_length(2, 0)
1
sage: Partition([3,3]).hook_length(0, 0)
4
sage: cell = [0,0]; Partition([3,3]).hook_length(*cell)
4
```

**hook_lengths()**

Return a tableau of shape `self` with the cells filled in with the hook lengths.
In each cell, put the sum of one plus the number of cells horizontally to the right and vertically below the cell (the hook length).

For example, consider the partition \([3, 2, 1]\) of 6 with Ferrers diagram:

```
# # #
# #
#
```

When we fill in the cells with the hook lengths, we obtain:

```
5 3 1
3 1
1
```

**EXAMPLES:**

```python
sage: Partition([2,2,1]).hook_lengths()
[[4, 2], [3, 1], [1]]
sage: Partition([3,3]).hook_lengths()
[[4, 3, 2], [3, 2, 1]]
sage: Partition([3,2,1]).hook_lengths()
[[5, 3, 1], [3, 1], [1]]
sage: Partition([2,2]).hook_lengths()
[[3, 2], [2, 1]]
sage: Partition([5]).hook_lengths()
[[5, 4, 3, 2, 1]]
```

**REFERENCES:**


### hook_polynomial \((q, t)\)

Return the two-variable hook polynomial.

**EXAMPLES:**

```python
sage: R.<q,t> = PolynomialRing(QQ)
sage: a = Partition([2,2]).hook_polynomial(q,t)
sage: a == (1 - t)*(1 - q*t)*(1 - t^2)*(1 - q*t^2)
True
sage: a = Partition([3,2,1]).hook_polynomial(q,t)
sage: a == (1 - t)^3*(1 - q*t^2)^2*(1 - q^2*t^3)
True
```

### hook_product \((a)\)

Return the Jack hook-product.

**EXAMPLES:**

```python
sage: Partition([3,2,1]).hook_product(x)
(2*x + 3)*(x + 2)^2
sage: Partition([2,2]).hook_product(x)
2*(x + 2)*(x + 1)
```

### hooks()

Return a sorted list of the hook lengths in `self`. 

---

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EXAMPLES:

```python
sage: Partition([3,2,1]).hooks()
[5, 3, 3, 1, 1, 1]
```

**horizontal_border_strip_cells**

Return a list of all the horizontal border strips of length \( k \) which can be added to \( \text{self} \), where each horizontal border strip is a generator of cells.

EXAMPLES:

```python
sage: list(Partition([]).horizontal_border_strip_cells(0))
[]
sage: list(Partition([3,2,1]).horizontal_border_strip_cells(0))
[]
sage: list(Partition([]).horizontal_border_strip_cells(2))
[[[0, 0], (0, 1)]]
sage: list(Partition([2,2]).horizontal_border_strip_cells(2))
[[[0, 2], (0, 3)], [[0, 2], (2, 0)], [[2, 0], (2, 1)]]
sage: list(Partition([3,2,2]).horizontal_border_strip_cells(2))
[[[0, 3], (0, 4)],
 [[0, 3], (1, 2)],
 [[0, 3], (3, 0)],
 [[1, 2], (3, 0)],
 [[3, 0], (3, 1)]]
```

**initial_column_tableau**

Return the initial column tableau of shape \( \text{self} \).

The initial column tableau of shape \( \text{self} \) is the standard tableau that has the numbers 1 to \( n \), where \( n \) is the \texttt{size()} of \( \text{self} \), entered in order from top to bottom and then left to right down the columns of \( \text{self} \).

EXAMPLES:

```python
sage: Partition([3,2]).initial_column_tableau()
[[1, 3, 5], [2, 4]]
```

**initial_tableau**

Return the standard tableau which has the numbers 1, 2, ..., \( n \) where \( n \) is the \texttt{size()} of \( \text{self} \) entered in order from left to right along the rows of each component, where the components are ordered from left to right.

EXAMPLES:

```python
sage: Partition([3,2,2]).initial_tableau()
[[1, 2, 3], [4, 5], [6, 7]]
```

**inside_corners**

Return a list of the corners of the partition \( \text{self} \).

A corner of a partition \( \lambda \) is a cell of the Young diagram of \( \lambda \) which can be removed from the Young diagram while still leaving a straight shape behind.

The entries of the list returned are pairs of the form \((i, j)\), where \( i \) and \( j \) are the coordinates of the respective corner. The coordinates are counted from 0.
Note: This is referred to as an “inner corner” in [Sag2001].

EXAMPLES:

```
sage: Partition([3,2,1]).corners()
[(0, 2), (1, 1), (2, 0)]
sage: Partition([3,3,1]).corners()
[(1, 2), (2, 0)]
sage: Partition([]).corners()
[]
```

inside_corners_residue(i, l)

Return a list of the corners of the partition self having l-residue $i$.

A corner of a partition $\lambda$ is a cell of the Young diagram of $\lambda$ which can be removed from the Young diagram while still leaving a straight shape behind. See residue() for the definition of the l-residue.

The entries of the list returned are pairs of the form $(i, j)$, where $i$ and $j$ are the coordinates of the respective corner. The coordinates are counted from 0.

EXAMPLES:

```
sage: Partition([3,2,1]).corners_residue(0, 3)
[(1, 1)]
sage: Partition([3,2,1]).corners_residue(1, 3)
[(2, 0)]
sage: Partition([3,2,1]).corners_residue(2, 3)
[(0, 2)]
```

is_core(k)

Return True if the Partition self is a k-core.

A partition is said to be a `$k`-core if it has no hooks of length $k$. Equivalently, a partition is said to be a $k$-core if it is its own $k$-core (where the latter is defined as in core()).

Visually, this can be checked by trying to remove border strips of size $k$ from self. If this is not possible, then self is a $k$-core.

EXAMPLES:

In the partition (2, 1), a hook length of 2 does not occur, but a hook length of 3 does:

```
sage: p = Partition([2, 1])
sage: p.is_core(2)
True
sage: p.is_core(3)
False
sage: q = Partition([12, 8, 5, 5, 2, 2, 1])
sage: q.is_core(4)
False
sage: q.is_core(5)
True
sage: q.is_core(0)
True
```
See also:

- `core()`, `Core`

**is_empty()**

Return True if `self` is the empty partition.

**EXAMPLES:**

```python
sage: Partition([]).is_empty()
True
sage: Partition([2,1,1]).is_empty()
False
```

**is_k_bounded**

Return True if the partition `self` is bounded by `k`.

**EXAMPLES:**

```python
sage: Partition([4, 3, 1]).is_k_bounded(4)
True
sage: Partition([4, 3, 1]).is_k_bounded(7)
True
sage: Partition([4, 3, 1]).is_k_bounded(3)
False
```

**is_k_irreducible**

Return True if the partition `self` is k-irreducible.

A k-bounded partition is **k-irreducible** if its Ferrer’s diagram does not contain \( k - i + 1 \) rows (or more) of length \( i \) (exactly) for every \( i \in [1, k] \).

(Also, a \( k \)-bounded partition is \( k \)-irreducible if and only if it is not \( k \)-reducible.)

**EXAMPLES:**

The partition \([1, 1, 1]\) has at least 2 rows of length 1:

```python
sage: Partition([1, 1, 1]).is_k_irreducible(2)
False
```

The partition \([1, 1, 1]\) does not have 4 rows of length 1, 3 rows of length 2, 2 rows of length 3, nor 1 row of length 4:

```python
sage: Partition([1, 1, 1]).is_k_irreducible(4)
True
```

See also:

- `is_k_reducible()`, `has_k_rectangle()`

**is_k_reducible**

Return True if the partition `self` is k-reducible.

A k-bounded partition is **k-reducible** if its Ferrer’s diagram contains \( k - i + 1 \) rows (or more) of length \( i \) (exactly) for some \( i \in [1, k] \).

(Also, a \( k \)-bounded partition is \( k \)-reducible if and only if it is not \( k \)-irreducible.)

**EXAMPLES:**
The partition \([1, 1, 1]\) has at least 2 rows of length 1:

```
sage: Partition([1, 1, 1]).is_k_reducible(2)
True
```

The partition \([1, 1, 1]\) does not have 4 rows of length 1, 3 rows of length 2, 2 rows of length 3, nor 1 row of length 4:

```
sage: Partition([1, 1, 1]).is_k_reducible(4)
False
```

See also:

\(\text{is\_k\_irreducible()}, \text{has\_k\_rectangle()}\)

\(\text{is\_regular}(e, \text{multicharge}=(0,))\)

Return True if this is an \(e\)-regular partition.

A partition is \(e\)-regular if it does not have \(e\) equal non-zero parts.

EXAMPLES:

```
sage: Partition([4,3,3,3]).is_regular(2)
False
sage: Partition([4,3,3,3]).is_regular(3)
False
sage: Partition([4,3,3,3]).is_regular(4)
True
```

\(\text{is\_restricted}(e, \text{multicharge}=(0,))\)

Return True if this is an \(e\)-restricted partition.

An \(e\)-restricted partition is a partition such that the difference of consecutive parts is always strictly less than \(e\), where partitions are considered to have an infinite number of 0 parts. I.e., the last part must be strictly less than \(e\).

EXAMPLES:

```
sage: Partition([4,3,3,2]).is_restricted(2)
False
sage: Partition([4,3,3,2]).is_restricted(3)
True
sage: Partition([4,3,3,2]).is_restricted(4)
True
sage: Partition([4]).is_restricted(4)
False
```

\(\text{is\_symmetric()}\)

Return True if the partition self equals its own transpose.

EXAMPLES:

```
sage: Partition([2, 1]).is_symmetric()
True
sage: Partition([3, 1]).is_symmetric()
False
```
jacobi_trudi()

Return the Jacobi-Trudi matrix of self thought of as a skew partition. See SkewPartition. jacobi_trudi().

EXAMPLES:

```
sage: part = Partition([3,2,1])
sage: jt = part.jacobi_trudi(); jt
sage: s = SymmetricFunctions(QQ).schur()
sage: h = SymmetricFunctions(QQ).homogeneous()
sage: h( s(part) )
sage: jt.det()
```

k_atom(k)

Return a list of the standard tableaux of size self.size() whose k-atom is equal to self.

EXAMPLES:

```
sage: p = Partition([3,2,1])
sage: p.k_atom(1)
[]
sage: p.k_atom(3)
[[[1, 1, 1, 2, 3], [2]],
 [[[1, 1, 1, 2], [2, 2]],
 [[[1, 1, 1, 2], [2]], [3]],
 [[[1, 1, 1], [2, 2], [3]]]
sage: Partition([3,2,1]).k_atom(4)
[[[1, 1, 1, 3], [2, 2]], [[1, 1, 1], [2], [3]]]
```

k_boundary(k)

Return the skew partition formed by removing the cells of the k-interior, see k_interior().

EXAMPLES:

```
sage: p = Partition([3,2,1])
sage: p.k_boundary(2)
[3, 2, 1] / [2, 1]
sage: p.k_boundary(3)
[3, 2, 1] / [1]
sage: p = Partition([12,8,5,5,2,2,1])
sage: p.k_boundary(4)
[12, 8, 5, 5, 2, 2, 1] / [8, 5, 2, 2]
```

k_column_lengths(k)

Return the k-column-shape of the partition self.

This is the 'column' analog of k_row_lengths().

EXAMPLES:
**k_conjugate**(k)

Return the k-conjugate of self.

The k-conjugate is the partition that is given by the columns of the k-skew diagram of the partition.

We can also define the k-conjugate in the following way. Let P denote the bijection from (k + 1)-cores to k-bounded partitions. The k-conjugate of a (k + 1)-core \( \lambda \) is

\[
\lambda^{(k)} = P^{-1}((P(\lambda))^\prime).
\]

**EXAMPLES:**

```python
sage: p = Partition([4,3,2,2,1,1])
sage: p.k_conjugate(4)
[3, 2, 2, 1, 1, 1, 1, 1, 1]
```

**k_interior**(k)

Return the partition consisting of the cells of self whose hook lengths are greater than k.

**EXAMPLES:**

```python
sage: p = Partition([3,2,1])
sage: p.hook_lengths()  
[[5, 3, 1], [3, 1], [1]]
sage: p.k_interior(2)
[2, 1]
sage: p.k_interior(3)
[1]
```

**k_irreducible**(k)

Return the partition with all \( r \times (k + 1 - r) \) rectangles removed.

If self is a k-bounded partition, then this method will return the partition where all rectangles of dimension \( r \times (k + 1 - r) \) for \( 1 \leq r \leq k \) have been deleted.

If self is not a k-bounded partition then the method will raise an error.

**INPUT:**

- k – a non-negative integer

**OUTPUT:**

- a partition
EXAMPLES:

```
sage: Partition([3,2,2,1,1,1]).k_irreducible(4)
[3, 2, 2, 1, 1, 1]
sage: Partition([3,2,2,1,1,1]).k_irreducible(3)
[]
sage: Partition([3,3,3,2,2,2,2,2,1,1,1,1]).k_irreducible(3)
[2, 1]
```

**k_rim**$(k)$

Return the k-rim of self as a list of integer coordinates.

The $k$-rim of a partition is the “line between” (or “intersection of”) the $k$-boundary and the $k$-interior. (Section 2.3 of [HM2011])

It will be output as an ordered list of integer coordinates, where the origin is $(0, 0)$. It will start at the top-left of the $k$-rim (using French convention) and end at the bottom-right.

**EXAMPLES:**

Consider the partition $(3, 1)$ split up into its $1$-interior and $1$-boundary:

![Diagram showing 1-rim and 1-boundary]

The line shown in bold is the 1-rim, and that information is equivalent to the integer coordinates of the points that occur along that line:

```
sage: Partition([3, 1]).k_rim(1)
[(3, 0), (2, 0), (2, 1), (1, 1), (0, 1), (0, 2)]
```

See also:

$k\_interior()$, $k\_boundary()$, $boundary()$

**k_row_lengths**$(k)$

Return the $k$-row-shape of the partition self.

This is equivalent to taking the $k$-boundary of the partition and then returning the row-shape of that. We do not discard rows of length 0. (Section 2.2 of [LLMS2013])

**EXAMPLES:**

```
sage: Partition([6, 1]).k_row_lengths(2)
[2, 1]
sage: Partition([4, 4, 4, 3, 2]).k_row_lengths(2)
[0, 1, 1, 1, 2]
```
See also:

\texttt{k\_column\_lengths()}, \texttt{k\_boundary()}, \texttt{SkewPartition.row\_lengths()}, \texttt{SkewPartition.column\_lengths()}

\textbf{\texttt{k\_size}(k)}

Given a partition \texttt{self} and a \texttt{k}, return the size of the \texttt{k}-boundary.

This is the same as the length method \texttt{sage.combinat.core.Core.length()} of the \texttt{sage.combinat.core.Core} object, with the exception that here we don’t require \texttt{self} to be a \texttt{k + 1-core}.

EXAMPLES:

\begin{verbatim}
sage: Partition([2, 1, 1]).k_size(1) 2
sage: Partition([2, 1, 1]).k_size(2) 3
sage: Partition([2, 1, 1]).k_size(3) 3
sage: Partition([2, 1, 1]).k_size(4) 4
\end{verbatim}

See also:

\texttt{k\_boundary()}, \texttt{SkewPartition.size()}

\textbf{\texttt{k\_skew}(k)}

Return the \texttt{k}\dash skew partition.

The \texttt{k}\dash skew diagram of a \texttt{k}-bounded partition is the skew diagram denoted \texttt{\lambda/k} satisfying the conditions:

1. row \texttt{i} of \texttt{\lambda/k} has length \texttt{\lambda_i},
2. no cell in \texttt{\lambda/k} has hook-length exceeding \texttt{k},
3. every square above the diagram of \texttt{\lambda/k} has hook length exceeding \texttt{k}.

REFERENCES:

- [LM2004]

EXAMPLES:

\begin{verbatim}
sage: p = Partition([4,3,2,2,1,1])
sage: p.k_skew(4) [[9, 5, 3, 2, 1, 1] / [5, 2, 1]]
\end{verbatim}

\textbf{\texttt{k\_split}(k)}

Return the \texttt{k}\dash split of \texttt{self}.

EXAMPLES:

\begin{verbatim}
sage: Partition([4,3,2,1]).k_split(3) []
sage: Partition([4,3,2,1]).k_split(4) [[4], [3, 2], [1]]
sage: Partition([4,3,2,1]).k_split(5) [[4, 3], [2, 1]]
sage: Partition([4,3,2,1]).k_split(6) [[4, 3, 2], [1]]
\end{verbatim}
\begin{verbatim}
sage: Partition([4,3,2,1]).k_split(7)
[[4, 3, 2, 1]]
sage: Partition([4,3,2,1]).k_split(8)
[[4, 3, 2, 1]]
\end{verbatim}

larger_lex(rhs)

Return True if \texttt{self} is larger than \texttt{rhs} in lexicographic order. Otherwise return False.

EXAMPLES:

\begin{verbatim}
sage: p = Partition([3,2])
sage: p.larger_lex([3,1])
True
sage: p.larger_lex([1,4])
True
sage: p.larger_lex([3,2,1])
False
sage: p.larger_lex([3])
True
sage: p.larger_lex([5])
False
sage: p.larger_lex([3,1,1,1,1,1,1,1])
True
\end{verbatim}

leg_cells(i, j)

Return the list of the cells of the leg of cell \((i, j)\) in \texttt{self}.

The leg of cell \(c = (i, j)\) is defined to be the cells below \(c\) (in English convention).

The cell coordinates are zero-based, i. e., the northwesternmost cell is \((0,0)\).

INPUT:

• \(i, j\) – two integers

OUTPUT:

A list of pairs of integers

EXAMPLES:

\begin{verbatim}
sage: Partition([4,4,3,1]).leg_cells(1,1)
[(2, 1)]
sage: Partition([4,4,3,1]).leg_cells(0,1)
[(1, 1), (2, 1)]
sage: Partition([]).leg_cells(0,0)
Traceback (most recent call last):
...
ValueError: the cell is not in the diagram
\end{verbatim}

leg_length(i, j)

Return the length of the leg of cell \((i, j)\) in \texttt{self}.

The leg of cell \(c = (i, j)\) is defined to be the cells below \(c\) (in English convention).

The cell coordinates are zero-based, i. e., the northwesternmost cell is \((0,0)\).
INPUT:

• i, j – two integers

OUTPUT:

An integer or a ValueError

EXAMPLES:

```
sage: p = Partition([2,2,1])
sage: p.leg_length(0, 0)
2
sage: p.leg_length(0,1)
1
sage: p.leg_length(2,0)
0
sage: Partition([3,3]).leg_length(0, 0)
1
sage: cell = [0,0]; Partition([3,3]).leg_length(*cell)
```

leg_lengths (flat=False)

Return a tableau of shape self with each cell filled in with its leg length. The optional boolean parameter flat provides the option of returning a flat list.

EXAMPLES:

```
sage: Partition([2,2,1]).leg_lengths()
[[2, 1], [1, 0], [0]]
sage: Partition([2,2,1]).leg_lengths(flat=True)
[2, 1, 1, 0, 0]
sage: Partition([3,3]).leg_lengths()
[[1, 1, 1], [0, 0, 0]]
sage: Partition([3,3]).leg_lengths(flat=True)
[1, 1, 1, 0, 0, 0]
```

length()

Return the number of parts in self.

EXAMPLES:

```
sage: Partition([3,2]).length()
2
sage: Partition([2,2,1]).length()
3
sage: Partition([]).length()
0
```

level()

Return the level of self, which is always 1.

This method exists only for compatibility with PartitionTuples.

EXAMPLES:

```
sage: Partition([4,3,2]).level()
1
```
**lower_hook** *(i, j, alpha)*

Return the lower hook length of the cell *(i, j)* in *self*. When *alpha* = 1, this is just the normal hook length.

The lower hook length of a cell *(i, j)* in a partition *κ* is defined by

\[ h^κ_*(i, j) = κ'_j - i + 1 + α(κ_i - j). \]

**EXAMPLES:**

```python
sage: p = Partition([2, 1])
sage: p.lower_hook(0, 0, 1)
3
sage: p.hook_length(0, 0)
3
sage: [p.lower_hook(i, j, x) for i, j in p.cells()]
x + 2, 1, 1
```

**lower_hook_lengths** *(alpha)*

Return a tableau of shape *self* with the cells filled in with the lower hook lengths. When *alpha* = 1, these are just the normal hook lengths.

The lower hook length of a cell *(i, j)* in a partition *κ* is defined by

\[ h^κ_*(i, j) = κ'_j - i + 1 + α(κ_i - j). \]

**EXAMPLES:**

```python
sage: Partition([3, 2, 1]).lower_hook_lengths(x)
[[2*x + 3, x + 2, 1], [x + 2, 1], [1]]
sage: Partition([3, 2, 1]).lower_hook_lengths(1)
[[3, 1], [3, 1], [1]]
sage: Partition([3, 2, 1]).hook_lengths()
[[5, 3, 1], [3, 1], [1]]
```

**next()**

Return the partition that lexicographically follows *self*, of the same size. If *self* is the last partition, then return False.

**EXAMPLES:**

```python
sage: next(Partition([4]))
[3, 1]
sage: next(Partition([1, 1, 1, 1]))
False
```

**next_within_bounds** *(min=[], max=None, partition_type=None)*

Get the next partition lexicographically that contains *min* and is contained in *max*.

**INPUT:**

- *min* – (default []) The ‘minimum partition’ that *next_within_bounds*(*self*) must contain.
- *max* – (default None) The ‘maximum partition’ that *next_within_bounds*(*self*) must be contained in. If set to None, then there is no restriction.
• partition_type – (default None) The type of partitions allowed. For example, ‘strict’ for strictly decreasing partitions, or None to allow any valid partition.

EXAMPLES:

```
sage: m = [1, 1]
sage: M = [3, 2, 1]
sage: Partition([1, 1]).next_within_bounds(min=m, max=M)
[1, 1, 1]
sage: Partition([1, 1, 1]).next_within_bounds(min=m, max=M)
[2, 1]
sage: Partition([2, 1]).next_within_bounds(min=m, max=M)
[2, 1, 1]
sage: Partition([2, 1, 1]).next_within_bounds(min=m, max=M)
[2, 2]
sage: Partition([2, 2]).next_within_bounds(min=m, max=M)
[2, 2, 1]
sage: Partition([2, 2, 1]).next_within_bounds(min=m, max=M)
[3, 1]
sage: Partition([3, 1]).next_within_bounds(min=m, max=M)
[3, 1, 1]
sage: Partition([3, 1, 1]).next_within_bounds(min=m, max=M)
[3, 2]
sage: Partition([3, 2]).next_within_bounds(min=m, max=M)
[3, 2, 1]
sage: Partition([3, 2, 1]).next_within_bounds(min=m, max=M) == None
True
```

See also:

next()

outer_rim()

Return the outer rim of self.

The outer rim of a partition \( \lambda \) is defined as the cells which do not belong to \( \lambda \) and which are adjacent to cells in \( \lambda \).

EXAMPLES:

The outer rim of the partition \([4, 1]\) consists of the cells marked with \# below:

```
****#
*####
##
```

```
sage: Partition([4,1]).outer_rim()
[(2, 0), (2, 1), (1, 1), (1, 2), (1, 3), (1, 4), (0, 4)]
sage: Partition([2,2,1]).outer_rim()
[(2, 0), (2, 1), (2, 1), (2, 2), (1, 2), (0, 2)]
sage: Partition([2,2]).outer_rim()
[(2, 0), (2, 1), (2, 2), (1, 2), (0, 2)]
sage: Partition([6,3,3,1,1]).outer_rim()
[(5, 0), (5, 1), (4, 1), (3, 1), (3, 2), (3, 3), (2, 3), (1, 3), (1, 4), (1, 5),
  (1, 6), (0, 6)]
```
**sage:** Partition([ ]). outer_rim()

```
[(0, 0)]
```

**outline**(variable=None)

Return the outline of the partition `self`.

This is a piecewise linear function, normalized so that the area under the partition `[1]` is 2.

**INPUT:**

- variable – a variable (default: 'x' in the symbolic ring)

**EXAMPLES:**

```
sage: [Partition([5,4]).outline()(x=i) for i in range(-10,11)]
[10, 9, 8, 7, 6, 5, 6, 5, 6, 5, 4, 3, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: Partition([ ]). outline()
abs(x)
sage: Partition([1]).outline()
abs(x + 1) + abs(x - 1) - abs(x)
sage: y = SR.var("y")
sage: Partition([6,5,1]).outline(variable=y)
abs(y + 6) - abs(y + 5) + abs(y + 4) - abs(y + 3) + abs(y - 1) - abs(y - 2) +
˓→abs(y - 3)
```

**outside_corners()**

Return a list of the outside corners of the partition `self`.

An outside corner (also called a cocorner) of a partition `\lambda` is a cell on $\mathbb{Z}^2$ which does not belong to the Young diagram of `\lambda` but can be added to this Young diagram to still form a straight-shape Young diagram.

The entries of the list returned are pairs of the form `(i, j)`, where `i` and `j` are the coordinates of the respective corner. The coordinates are counted from 0.

**Note:** These are called “outer corners” in [Sag2001].

**EXAMPLES:**

```
sage: Partition([2,2,1]).outside_corners()
[(0, 2), (2, 1), (3, 0)]
sage: Partition([2,2]).outside_corners()
[(0, 2), (2, 0)]
sage: Partition([6,3,3,1,1,1]).outside_corners()
[(0, 6), (1, 3), (3, 1), (6, 0)]
sage: Partition([ ]).outside_corners()
[(0, 0)]
```

**outside_corners_residue**(i, l)

Return a list of the outside corners of the partition `self` having `l`-residue `i`.

An outside corner (also called a cocorner) of a partition `\lambda` is a cell on $\mathbb{Z}^2$ which does not belong to the Young diagram of `\lambda` but can be added to this Young diagram to still form a straight-shape Young diagram.
See \textit{residue()} for the definition of the $1$-residue.

The entries of the list returned are pairs of the form $(i, j)$, where $i$ and $j$ are the coordinates of the respective corner. The coordinates are counted from 0.

EXAMPLES:

\begin{verbatim}
sage: Partition([3,2,1]).outside_corners_residue(0, 3)
[(0, 3), (3, 0)]
sage: Partition([3,2,1]).outside_corners_residue(1, 3)
[(1, 2)]
sage: Partition([3,2,1]).outside_corners_residue(2, 3)
[(2, 1)]
\end{verbatim}

\textbf{plancherel_measure()}

Return the probability of \texttt{self} under the Plancherel probability measure on partitions of the same size.

This probability distribution comes from the uniform distribution on permutations via the Robinson-Schensted correspondence.

See \textit{Wikipedia article Plancherel_measure} and \textit{Partitions.n.random_element_plancherel()}. 

EXAMPLES:

\begin{verbatim}
sage: Partition([]).plancherel_measure()
1
sage: Partition([1]).plancherel_measure()
1
sage: Partition([2]).plancherel_measure()
1/2
sage: [mu.plancherel_measure() for mu in Partitions(3)]
[1/6, 2/3, 1/6]
sage: Partition([5,4]).plancherel_measure()
7/1440
\end{verbatim}

\textbf{power}(k)

Return the cycle type of the $k$-th power of any permutation with cycle type \texttt{self} (thus describes the power map of symmetric groups).

Equivalent to GAP's \texttt{PowerPartition}.

EXAMPLES:

\begin{verbatim}
sage: p = Partition([5,3])
sage: p.power(1)
[5, 3]
sage: p.power(2)
[5, 3]
sage: p.power(3)
[5, 1, 1, 1]
sage: p.power(4)
[5, 3]
\end{verbatim}

Now let us compare this to the power map on $S_8$:

\begin{verbatim}
sage: G = SymmetricGroup(8)
sage: g = G([(1,2,3,4,5),(6,7,8)])
\end{verbatim}

(continues on next page)
sage: \( g \)
\((1,2,3,4,5)(6,7,8)\)
sage: \( g^2 \)
\((1,3,5,2,4)(6,8,7)\)
sage: \( g^3 \)
\((1,4,2,5,3)\)
sage: \( g^4 \)
\((1,5,4,3,2)(6,7,8)\)

\[
\text{sage: Partition([3,2,1]).power(3)}
\]
\[
\begin{pmatrix}
2 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

\text{pp()}
Print the Ferrers diagram.

See \texttt{ferrers_diagram()} for more on the Ferrers diagram.

\textbf{EXAMPLES:}

\[
\text{sage: Partition([5,5,2,1]).pp()}
\]
\[
\begin{array}{cccc}
**** & **** & & \\
 & **** & * & \\
\end{array}
\]
\[
\text{sage: Partitions.options.convention='French'}
\]
\[
\text{sage: Partition([5,5,2,1]).pp()}
\]
\[
* \\
** \\
***** \\
*****
\]
\[
\text{sage: Partitions.options._reset()}
\]

\textbf{prime_degree}(p)
Return the prime degree for the prime integer \( p \) for \texttt{self}.

\textbf{INPUT:}

\begin{itemize}
\item \( p \) – a prime integer
\end{itemize}

\textbf{OUTPUT:}

A non-negative integer

The degree of a partition \( \lambda \) is the sum of the \texttt{e-degree()} of the standard tableaux of shape \( \lambda \), for \( e \) a power of the prime \( p \). The prime degree gives the exponent of \( p \) in the Gram determinant of the integral Specht module of the symmetric group.

\textbf{EXAMPLES:}

\[
\text{sage: Partition([4,3]).prime_degree(2)}
\]
36
\[
\text{sage: Partition([4,3]).prime_degree(3)}
\]
15
\[
\text{sage: Partition([4,3]).prime_degree(5)}
\]
13
Therefore, the Gram determinant of $S(5, 3)$ when $q = 1$ is $2^{36} 3^{15} 5^{13}$. Compare with `degree()`.

**quotient(length)**

Return the quotient of the partition – in the literature the quotient is commonly referred to as the $k$-quotient, $p$-quotient, $r$-quotient, ... .

The $r$-quotient of a partition $\lambda$ is a list of $r$ partitions (labelled from 0 to $r-1$), constructed in the following way. Label each cell in the Young diagram of $\lambda$ with its content modulo $r$. Let $R_i$ be the set of rows ending in a cell labelled $i$, and $C_j$ be the set of columns ending in a cell labelled $i$. Then the $j$-th component of the quotient of $\lambda$ is the partition defined by intersecting $R_j$ with $C_{j+1}$. (See Theorem 2.7.37 in [JK1981].)

**EXAMPLES:**

```python
sage: Partition([7,7,5,3,3,3,1]).quotient(3)
([[2], [1], [2, 2, 2]])
```

**reading_tableau()**

Return the RSK recording tableau of the reading word of the (standard) tableau $T$ labeled down (in English convention) each column to the shape of `self`.

For an example of the tableau $T$, consider the partition $\lambda = (3, 2, 1)$, then we have:

```
  1 4 6
  2 5
  3
```

For more, see `RSK()`.

**EXAMPLES:**

```python
sage: Partition([3,2,1]).reading_tableau()
[[1, 3, 6], [2, 5], [4]]
```

**removable_cells()**

Return a list of the corners of the partition `self`.

A corner of a partition $\lambda$ is a cell of the Young diagram of $\lambda$ which can be removed from the Young diagram while still leaving a straight shape behind.

The entries of the list returned are pairs of the form $(i, j)$, where $i$ and $j$ are the coordinates of the respective corner. The coordinates are counted from 0.

**Note:** This is referred to as an “inner corner” in [Sag2001].

**EXAMPLES:**

```python
sage: Partition([3,2,1]).corners()
[(0, 2), (1, 1), (2, 0)]
sage: Partition([3,3,1]).corners()
[(1, 2), (2, 0)]
sage: Partition([]).corners()
[]
```
**removable_cells_residue**(i, l)

Return a list of the corners of the partition self having 1-residue i.

A corner of a partition \( \lambda \) is a cell of the Young diagram of \( \lambda \) which can be removed from the Young diagram while still leaving a straight shape behind. See \( \text{residue()} \) for the definition of the 1-residue.

The entries of the list returned are pairs of the form \((i, j)\), where \(i\) and \(j\) are the coordinates of the respective corner. The coordinates are counted from 0.

**EXAMPLES:**

```python
sage: Partition([3,2,1]).corners_residue(0, 3)
[(1, 1)]
sage: Partition([3,2,1]).corners_residue(1, 3)
[(2, 0)]
sage: Partition([3,2,1]).corners_residue(2, 3)
[(0, 2)]
```

**remove_cell**(i, j=None)

Return the partition obtained by removing a cell at the end of row \(i\) of self.

**EXAMPLES:**

```python
sage: Partition([2,2]).remove_cell(1)
[2, 1]
sage: Partition([2,2,1]).remove_cell(2)
[2, 2]
sage: Partition([2,2,1]).remove_cell(1,1)
[2, 1]
sage: Partition([2,2]).remove_cell(1,0)
```

**remove_horizontal_border_strip**(k)

Return the partitions obtained from self by removing an horizontal border strip of length \(k\).

**EXAMPLES:**

```python
sage: Partition([5,3,1]).remove_horizontal_border_strip(0).list()
[[5, 3, 1]]
sage: Partition([5,3,1]).remove_horizontal_border_strip(1).list()
[[5, 3], [5, 2, 1], [4, 3, 1]]
sage: Partition([5,3,1]).remove_horizontal_border_strip(2).list()
[[5, 2], [5, 1, 1], [4, 3], [4, 2, 1], [3, 3, 1]]
sage: Partition([5,3,1]).remove_horizontal_border_strip(3).list()
[[5, 1], [4, 2], [4, 1, 1], [3, 3], [3, 2, 1]]
sage: Partition([5,3,1]).remove_horizontal_border_strip(4).list()
[[4, 1], [3, 2], [3, 1, 1]]
sage: Partition([5,3,1]).remove_horizontal_border_strip(5).list()
[[3, 1]]
sage: Partition([5,3,1]).remove_horizontal_border_strip(6).list()
[]
```

The result is returned as an instance of \(\text{Partitions_with_constraints}\):
**Combinatorics, Release 10.0**

---

```python
def remove_horizontal_border_strip(partition, length):
    subpartitions = []
    for subpartition in partition:
        new_subpartition = subpartition[1:]
        subpartitions.append(new_subpartition)
    return subpartitions
```

**residue(r, c, l)**

Return the 1-residue of the cell at row $r$ and column $c$.

The $\ell$-residue of a cell is $c - r$ modulo $\ell$.

This does not strictly depend upon the partition, however, this method is included because it is often useful in the context of partitions.

**EXAMPLES:**

```python
def residue(partition, row, column, residue):
    return partition.residue(row, column, residue)
```

**rim()**

Return the rim of self.

The rim of a partition $\lambda$ is defined as the cells which belong to $\lambda$ and which are adjacent to cells not in $\lambda$.

**EXAMPLES:**

The rim of the partition $[5, 5, 2, 1]$ consists of the cells marked with # below:

```python
# # #
# # #
# # #
# # #
```

```python
def rim(partition):
    return partition.rim()
```

```
**row_standard_tableaux()**

Return the row standard tableaux of shape self.

**EXAMPLES:**

```python
def row_standard_tableaux(partition):
    return partition.row_standard_tableaux()
```

```
**sign()**

Return the sign of any permutation with cycle type self.

This function corresponds to a homomorphism from the symmetric group $S_n$ into the cyclic group of order 2, whose kernel is exactly the alternating group $A_n$. Partitions of sign 1 are called even partitions while partitions of sign $-1$ are called odd.

```python
def sign(partition):
    return partition.sign()
```
Zolotarev’s lemma states that the Legendre symbol \( \left( \frac{a}{p} \right) \) for an integer \( a \pmod{p} \) (\( p \) a prime number), can be computed as \( \text{sign}(\text{p}_a) \), where \( \text{sign} \) denotes the sign of a permutation and \( \text{p}_a \) the permutation of the residue classes \( \pmod{p} \) induced by modular multiplication by \( a \), provided \( p \) does not divide \( a \).

We verify this in some examples.

```python
sage: F = GF(11)
sage: a = F.multiplicative_generator(); a
2
sage: plist = [int(a*F(x)) for x in range(1,11)]; plist
[2, 4, 6, 8, 10, 1, 3, 5, 7, 9]
This corresponds to the permutation (1, 2, 4, 8, 5, 10, 9, 7, 3, 6) (acting the set \( \{1, 2, \ldots, 10\} \)) and to the partition \([10]\).

sage: p = PermutationGroupElement('(1, 2, 4, 8, 5, 10, 9, 7, 3, 6)')
sage: p.sign()
-1
sage: Partition([10]).sign()
-1
sage: kronecker_symbol(11,2)
-1

Now replace 2 by 3:

```python
sage: plist = [int(F(3*x)) for x in range(1,11)]; plist
[3, 6, 9, 1, 4, 7, 10, 2, 5, 8]
sage: list(range(1, 11))
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: p = PermutationGroupElement('(3,4,8,7,9)')
sage: p.sign()
1
sage: kronecker_symbol(3,11)
1
sage: Partition([5,1,1,1,1,1]).sign()
1
```

In both cases, Zolotarev holds.

REFERENCES:

- Wikipedia article Zolotarev’s lemma
Combinatorics, Release 10.0

```
sage: Partition([2,2]).size()
sage: Partition([3,2,1]).size()
4
6
```

**specht_module** *(base_ring=None)*

Return the Specht module corresponding to self.

**EXAMPLES:**

```
sage: SM = Partition([2,2,1]).specht_module(QQ)
sage: SM
Specht module of [(0, 0), (0, 1), (1, 0), (1, 1), (2, 0)] over Rational Field
sage: s = SymmetricFunctions(QQ).s()
sage: s(SM.frobenius_image())
s[2, 2, 1]
```

**specht_module_dimension** *(base_ring=None)*

Return the dimension of the Specht module corresponding to self.

This is equal to the number of standard tableaux of shape self when over a field of characteristic 0.

**INPUT:**

• BR – (default: Q) the base ring

**EXAMPLES:**

```
sage: Partition([2,2,1]).specht_module_dimension()
sage: Partition([2,2,1]).specht_module_dimension(GF(2))
5
5
```

**standard_tableaux** *

Return the standard tableaux of shape self.

**EXAMPLES:**

```
sage: Partition([3,2,2,1]).standard_tableaux()
Standard tableaux of shape [3, 2, 2, 1]
```

**stretch** *(k)*

Return the partition obtained by multiplying each part with the given number.

**EXAMPLES:**

```
sage: p = Partition([4,2,2,1,1])
sage: p.stretch(3)
[12, 6, 6, 3, 3]
```

**suter_diagonal_slide** *(n, exp=1)*

Return the image of self in $Y_n$ under Suter’s diagonal slide $\sigma_n$, where the notations used are those defined in [Sut2002].

The set $Y_n$ is defined as the set of all partitions $\lambda$ such that the hook length of the $(0,0)$-cell (i.e. the northwestern most cell in English notation) of $\lambda$ is less than $n$, including the empty partition.
The map $\sigma_n$ sends a partition (with non-zero entries) $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in Y_n$ to the partition $(\lambda_2 + 1, \lambda_3 + 1, \ldots, \lambda_m + 1, 1, 1, \ldots, 1)$. In other words, it pads the partition with trailing zeroes until it has length $n - \lambda_1$, then removes its first part, and finally adds 1 to each part.

By Theorem 2.1 of [Sut2002], the dihedral group $D_n$ with $2n$ elements acts on $Y_n$ by letting the primitive rotation act as $\sigma_n$ and the reflection act as conjugation of partitions ($\text{conjugate()}$). This action is faithful if $n \geq 3$.

**INPUT:**

- $n$ – nonnegative integer
- $\exp$ – (default: 1) how many times $\sigma_n$ should be applied

**OUTPUT:**

The result of applying Suter’s diagonal slide $\sigma_n$ to self, assuming that self lies in $Y_n$. If the optional argument $\exp$ is set, then the slide $\sigma_n$ is applied not just once, but $\exp$ times (note that $\exp$ is allowed to be negative, since the slide has finite order).

**EXAMPLES:**

```sage
sage: Partition([5,4,1]).suter_diagonal_slide(8)
[5, 2]
sage: Partition([5,4,1]).suter_diagonal_slide(9)
[5, 2, 1]
sage: Partition([]).suter_diagonal_slide(7)
[1, 1, 1, 1, 1, 1]
sage: Partition([]).suter_diagonal_slide(1)
[]
sage: Partition([]).suter_diagonal_slide(7, exp=-1)
[6]
sage: Partition([]).suter_diagonal_slide(1, exp=-1)
[]
sage: P7 = Partitions(7)
sage: all( p == p.suter_diagonal_slide(9, exp=-1).suter_diagonal_slide(9)  
... for p in P7 )
True
sage: all( p == p.suter_diagonal_slide(9, exp=3)  
... .suter_diagonal_slide(9, exp=3) 
... .suter_diagonal_slide(9, exp=3)  
... for p in P7 )
True
sage: all( p == p.suter_diagonal_slide(9, exp=6)  
... .suter_diagonal_slide(9, exp=6) 
... .suter_diagonal_slide(9, exp=6)  
... for p in P7 )
True
sage: all( p == p.suter_diagonal_slide(9, exp=-1)  
... .suter_diagonal_slide(9, exp=1) 
... .suter_diagonal_slide(9, exp=1)  
... for p in P7 )
True
```

Check of the assertion in [Sut2002] that $\sigma_n(\sigma_n(\lambda')) = \lambda$: 5.1. Comprehensive Module List 1759
Check of Claim 1 in [Sut2002]:

```python
sage: P5 = Partitions(5)
sage: all( all( (p.suter_diagonal_slide(6) in q.suter_diagonal_slide(6).down()) or (q.suter_diagonal_slide(6) in p.suter_diagonal_slide(6).down()) for p in q.down() ) for q in P5 )
True
```

### t_completion(t)

Return the t-completion of the partition self.

If \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) is a partition and \( t \) is an integer greater or equal to \( |\lambda| + \lambda_1 \), then the \( t \)-completion of \( \lambda \) is defined as the partition \( (t - |\lambda|, \lambda_1, \lambda_2, \lambda_3, \ldots) \) of \( t \). This partition is denoted by \( \lambda[t] \) in [BOR2009], by \( \lambda_{[t]} \) in [BdVO2012], and by \( \lambda(t) \) in [CO2010].

**EXAMPLES:**

```python
sage: Partition([]).t_completion(0)
[]
sage: Partition([]).t_completion(1)
[1]
sage: Partition([]).t_completion(2)
[2]
sage: Partition([]).t_completion(3)
[3]
sage: Partition([2, 1]).t_completion(5)
[2, 2, 1]
sage: Partition([2, 1]).t_completion(6)
[3, 2, 1]
sage: Partition([4, 2, 1]).t_completion(13)
[4, 4, 2, 2, 1]
sage: Partition([4, 2, 1]).t_completion(19)
[10, 4, 2, 2, 1]
sage: Partition([4, 2, 1]).t_completion(10)
Traceback (most recent call last):
... ValueError: 10-completion is not defined
sage: Partition([4, 2, 1]).t_completion(5)
Traceback (most recent call last):
... ValueError: 5-completion is not defined
```

### to_core(k)

Maps the \( k \)-bounded partition self to its corresponding \( k + 1 \)-core.

See also \( k\_skew() \).

**EXAMPLES:**
sage: p = Partition([4,3,2,2,1,1])
sage: c = p.to_core(4); c
[9, 5, 3, 2, 1, 1]
sage: type(c)
<class 'sage.combinat.core.Cores_length_with_category.element_class'>
sage: c.to_bounded_partition() == p
True

to_dyck_word(n=\text{None})

Return the n-Dyck word whose corresponding partition is self (or, if n is not specified, the n-Dyck word with smallest n to satisfy this property).

If w is an n-Dyck word (that is, a Dyck word with n open symbols and n close symbols), then the Dyck path corresponding to w can be regarded as a lattice path in the northeastern half of an n \times n-square. The region to the northeast of this Dyck path can be regarded as a partition. It is called the partition corresponding to the Dyck word w. (See to_partition().)

For every partition \lambda and every nonnegative integer n, there exists at most one n-Dyck word w such that the partition corresponding to w is \lambda (in fact, such w exists if and only if \lambda_i + i \leq n for every i, where \lambda is written in the form (\lambda_1, \lambda_2, \ldots, \lambda_k) with \lambda_k \geq 0). This method computes this w for a given \lambda and n. If n is not specified, this method computes the w for the smallest possible n for which such an w exists. (The minimality of n means that the partition demarcated by the Dyck path touches the diagonal.)

EXAMPLES:

sage: Partition([2,2]).to_dyck_word()
[1, 1, 0, 0, 1, 1, 0, 0]
sage: Partition([2,2]).to_dyck_word(4)
[1, 1, 0, 0, 1, 1, 0, 0]
sage: Partition([2,2]).to_dyck_word(5)
[1, 1, 1, 0, 0, 1, 1, 0, 0]
sage: Partition([6,3,1]).to_dyck_word()
[1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0]
sage: Partition([]).to_dyck_word()
[]
sage: Partition([]).to_dyck_word(3)
[1, 1, 1, 0, 0, 0]

The partition corresponding to self.dyck_word() is self indeed:

sage: all( p.to_dyck_word().to_partition() == p
.....:    for p in Partitions(5) )
True

to_exp(k=0)

Return a list of the multiplicities of the parts of a partition. Use the optional parameter k to get a return list of length at least k.

EXAMPLES:

sage: Partition([3,2,2,1]).to_exp()
[1, 2, 1]
sage: Partition([3,2,2,1]).to_exp(5)
[1, 2, 1, 0, 0]
to_exp_dict()

Return a dictionary containing the multiplicities of the parts of self.

EXAMPLES:

```python
sage: p = Partition([4,2,2,1])
sage: d = p.to_exp_dict()
sage: d[4]
1
sage: d[2]
2
sage: d[1]
1
sage: 5 in d
False
```

to_list()

Return self as a list.

EXAMPLES:

```python
sage: p = Partition([2,1]).to_list(); p
[2, 1]
sage: type(p)
<class 'list'>
```

top_garnir_tableau(e, cell)

Return the most dominant standard tableau which dominates the corresponding Garnir tableau and has the same e-residue.

The Garnir tableau play an important role in integral and non-semisimple representation theory because they determine the “straightening” rules for the Specht modules. The top Garnir tableaux arise in the graded representation theory of the symmetric groups and higher level Hecke algebras. They were introduced in [KMR2012].

If the Garnir node is cell=(r,c) and m and M are the entries in the cells (r,c) and (r+1,c), respectively, in the initial tableau then the top e-Garnir tableau is obtained by inserting the numbers m, m+1,...,M in order from left to right first in the cells in row r+1 which are not in the e-Garnir belt, then in the cell in rows r and r+1 which are in the Garnir belt and then, finally, in the remaining cells in row r which are not in the Garnir belt. All other entries in the tableau remain unchanged.

If e = 0, or if there are no e-bricks in either row r or r+1, then the top Garnir tableau is the corresponding Garnir tableau.

EXAMPLES:

```python
sage: Partition([5,4,3,2]).top_garnir_tableau(2,(0,2)).pp()
  1 2 4 5 8
  3 6 7 9
 10 11 12
 13 14
sage: Partition([5,4,3,2]).top_garnir_tableau(3,(0,2)).pp()
  1 2 3 4 5
  6 7 8 9
 10 11 12
 13 14
```

(continues on next page)
sage: Partition([5,4,3,2]).top_garnir_tableau(4,(0,2)).pp()
1 2 6 7 8
3 4 5 9
10 11 12
13 14
sage: Partition([5,4,3,2]).top_garnir_tableau(0,(0,2)).pp()
1 2 6 7 8
3 4 5 9
10 11 12
13 14

REFERENCES:
• [KMR2012]

up()
Return a generator for partitions that can be obtained from self by adding a cell.

EXAMPLES:

sage: list(Partition([2,1,1]).up())
[[3, 1, 1], [2, 2, 1], [2, 1, 1, 1]]
sage: list(Partition([3,2]).up())
[[4, 2], [3, 3], [3, 2, 1]]
sage: [p for p in Partition([]).up()]
[[1]]

up_list()
Return a list of the partitions that can be formed from self by adding a cell.

EXAMPLES:

sage: Partition([2,1,1]).up_list()
[[3, 1, 1], [2, 2, 1], [2, 1, 1, 1]]
sage: Partition([3,2]).up_list()
[[4, 2], [3, 3], [3, 2, 1]]
sage: Partition([]).up_list()
[[1]]

upper_hook(i, j, alpha)

Return the upper hook length of the cell (i, j) in self. When alpha = 1, this is just the normal hook length.

The upper hook length of a cell (i, j) in a partition $\kappa$ is defined by

$$h^*_\kappa(i, j) = \kappa'_j - i + \alpha(\kappa_i - j + 1).$$

EXAMPLES:

sage: p = Partition([2,1])
sage: p.upper_hook(0,0,1)
3
sage: p.hook_length(0,0)
3
sage: [p.upper_hook(i,j,x) for i,j in p.cells()]
[2*x + 1, x, x]
upper_hook_lengths(alpha)

Return a tableau of shape self with the cells filled in with the upper hook lengths. When alpha = 1, these are just the normal hook lengths.

The upper hook length of a cell (i, j) in a partition κ is defined by

\[ h^*_κ(i, j) = κ'_j - i + α(κ_i - j + 1). \]

EXAMPLES:

```python
sage: Partition([3, 2, 1]).upper_hook_lengths(x)
[[3*x + 2, 2*x + 1, x], [2*x + 1, x], [x]]

sage: Partition([3, 2, 1]).upper_hook_lengths(1)
[[5, 3, 1], [3, 1], [1]]

sage: Partition([3, 2, 1]).hook_lengths()
[[5, 3, 1], [3, 1], [1]]
```

vertical_border_strip_cells(k)

Return a list of all the vertical border strips of length k which can be added to self, where each horizontal border strip is a generator of cells.

EXAMPLES:

```python
sage: list(Partition([]).vertical_border_strip_cells(0))
[]

sage: list(Partition([3, 2, 1]).vertical_border_strip_cells(0))
[]

sage: list(Partition([]).vertical_border_strip_cells(2))
[(0, 0), (1, 0)]

sage: list(Partition([2, 2]).vertical_border_strip_cells(2))
[(0, 2), (1, 2), (2, 0), (3, 0)]

sage: list(Partition([2, 2]).vertical_border_strip_cells(2))
[(0, 3), (3, 0)]
```

weighted_size()

Return the weighted size of self.

The weighted size of a partition \( \lambda \) is

\[
\sum_i i \cdot \lambda_i,
\]

where \( \lambda = (\lambda_0, \lambda_1, \lambda_2, \cdots) \).

This also the sum of the leg length of every cell in \( \lambda \), or

\[
\sum_i \left( \frac{\lambda'_i}{2} \right)
\]

where \( \lambda' \) is the conjugate partition of \( \lambda \).

EXAMPLES:
Combinatorics, Release 10.0

```
sage: Partition([2,2]).weighted_size()
2
sage: Partition([3,3,3]).weighted_size()
9
sage: Partition([5,2]).weighted_size()
2
sage: Partition([]).weighted_size()
0
```

**young_subgroup()**

Return the corresponding Young, or parabolic, subgroup of the symmetric group.

The Young subgroup of a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) of \( n \) is the group:

\[
S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell}
\]

embedded into \( S_n \) in the standard way (i.e., the \( S_{\lambda_i} \) factor acts on the numbers from \( \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + 1 \) to \( \lambda_1 + \lambda_2 + \cdots + \lambda_i \)).

**EXAMPLES:**

```
sage: Partition([4,2]).young_subgroup()
Permutation Group with generators [((), (5,6), (3,4), (2,3), (1,2))]
```

**young_subgroup_generators()**

Return an indexing set for the generators of the corresponding Young subgroup. Here the generators correspond to the simple adjacent transpositions \( s_i = (i \ i+1) \).

**EXAMPLES:**

```
sage: Partition([4,2]).young_subgroup_generators()
[1, 2, 3, 5]
sage: Partition([1,1,1]).young_subgroup_generators()
[]
sage: Partition([2,2]).young_subgroup_generators()
[1, 3]
```

See also:

*young_subgroup()*

**zero_one_sequence()**

Compute the finite \( 0 \rightarrow 1 \) sequence of the partition.

The full \( 0 \rightarrow 1 \) sequence is the sequence (infinite in both directions) indicating the steps taken when following the outer rim of the diagram of the partition. We use the convention that in English convention, a 1 corresponds to an East step, and a 0 corresponds to a North step.

Note that every full \( 0 \rightarrow 1 \) sequence starts with infinitely many 0’s and ends with infinitely many 1’s.

One place where these arise is in the affine symmetric group where one takes an affine permutation \( w \) and every \( i \) such that \( w(i) \leq 0 \) corresponds to a 1 and \( w(i) > 0 \) corresponds to a 0. See pages 24-25 of [LLMSSZ2013] for connections to affine Grassmannian elements (note there they use the French convention for their partitions).

These are also known as **path sequences, Maya diagrams, plus-minus diagrams, Comet code** [Sta-EC2], among others.
The finite 0−1 sequence is obtained from the full 0−1 sequence by omitting all heading 0’s and trailing 1’s. The output sequence is finite, starts with a 1 and ends with a 0 (unless it is empty, for the empty partition). Its length is the sum of the first part of the partition with the length of the partition.

EXAMPLES:

```python
sage: Partition([5, 4]).zero_one_sequence()
[1, 1, 1, 1, 0, 1, 0]
sage: Partition([]).zero_one_sequence()
[]
sage: Partition([2]).zero_one_sequence()
[1, 1, 0]
```

class sage.combinat.partition.Partitions(is_infinite=False)

Bases: UniqueRepresentation, Parent

Partitions(n, **kwargs) returns the combinatorial class of integer partitions of \(n\) subject to the constraints given by the keywords. Valid keywords are: starting, ending, min_part, max_part, max_length, min_length, length, max_slope, min_slope, inner, outer, parts_in, regular, and restricted. They have the following meanings:

* starting=p specifies that the partitions should all be less than or equal to \(p\) in lex order. This argument cannot be combined with any other (see github issue #15467).
* ending=p specifies that the partitions should all be greater than or equal to \(p\) in lex order. This argument cannot be combined with any other (see github issue #15467).
* length=k specifies that the partitions have exactly \(k\) parts.
* min_length=k specifies that the partitions have at least \(k\) parts.
* min_part=k specifies that all parts of the partitions are at least \(k\).
* inner=p specifies that the partitions must contain the partition \(p\).
* outer=p specifies that the partitions be contained inside the partition \(p\).
* min_slope=k specifies that the partitions have slope at least \(k\); the slope at position \(i\) is the difference between the \((i+1)\)-th part and the \(i\)-th part.
* parts_in=S specifies that the partitions have parts in the set \(S\), which can be any sequence of pairwise distinct positive integers. This argument cannot be combined with any other (see github issue #15467).
* regular=ell specifies that the partitions are \(\ell\)-regular, and can only be combined with the max_length or max_part, but not both, keywords if \(n\) is not specified
* restricted=ell specifies that the partitions are \(\ell\)-restricted, and cannot be combined with any other keywords

The max_* versions, along with inner and ending, work analogously.

Right now, the parts_in, starting, ending, regular, and restricted keyword arguments are mutually exclusive, both of each other and of other keyword arguments. If you specify, say, parts_in, all other keyword arguments will be ignored; starting, ending, regular, and restricted work the same way.

EXAMPLES:

If no arguments are passed, then the combinatorial class of all integer partitions is returned:
If an integer \( n \) is passed, then the combinatorial class of integer partitions of \( n \) is returned:

```sage
Partitions(3)
Partitions of the integer 3
Partitions(3).list()
[[3], [2, 1], [1, 1, 1]]
```

If `starting=p` is passed, then the combinatorial class of partitions greater than or equal to \( p \) in lexicographic order is returned:

```sage
Partitions(3, starting=[2,1])
Partitions of the integer 3 starting with [2, 1]
Partitions(3, starting=[2,1]).list()
[[2, 1], [1, 1, 1]]
```

If `ending=p` is passed, then the combinatorial class of partitions at most \( p \) in lexicographic order is returned:

```sage
Partitions(3, ending=[2,1])
Partitions of the integer 3 ending with [2, 1]
Partitions(3, ending=[2,1]).list()
[[3], [2, 1]]
```

Using `max_slope=-1` yields partitions into distinct parts – each part differs from the next by at least 1. Use a different `max_slope` to get parts that differ by, say, 2:

```sage
Partitions(7, max_slope=-1).list()
[[7], [6, 1], [5, 2], [4, 3], [4, 2, 1]]
Partitions(15, max_slope=-1).cardinality()
27
```

The number of partitions of \( n \) into odd parts equals the number of partitions into distinct parts. Let’s test that for \( n \) from 10 to 20:

```sage
test = lambda n: Partitions(n, max_slope=-1).cardinality() == Partitions(n, parts_in=[1,3..n]).cardinality()
test: all(test(n) for n in [10..20])
True
```

The number of partitions of \( n \) into distinct parts that differ by at least 2 equals the number of partitions into parts that equal 1 or 4 modulo 5; this is one of the Rogers-Ramanujan identities:

```sage
test = lambda n: Partitions(n, max_slope=-2).cardinality() == Partitions(n, parts_in=[1,6..n] + [4,9..n]).cardinality()
test: all(test(n) for n in [10..20])
True
```

Here are some more examples illustrating `min_part`, `max_part`, and `length`:
Partitions of the integer 5 satisfying constraints min_part=2
[[5], [3, 2]]

Partitions of the integer 3 with max_length=2
[[3], [2, 1]]

Partitions of 10 with min_part=2, length=3
[[6, 2, 2], [5, 3, 2], [4, 4, 2], [4, 3, 3]]

Some examples using the regular keyword:

4-Regular Partitions

4-Regular Partitions with max length 3

4-Regular 3-Bounded Partitions

4-Regular Partitions of the integer 3

Some examples using the restricted keyword:

4-Restricted Partitions

4-Restricted Partitions of the integer 3

Here are some further examples using various constraints:

4, 1, 1, 1]]
[[4], [3, 1]]

4, 1, 1, 1]]
[[4], [3, 1]]

4, 1, 1, 1]]
[[4], [3, 1]]

(continues on next page)
Note that if you specify min_part=0, then it will treat the minimum part as being 1 (see github issue #13605):

```
sage: [x for x in Partitions(4, length=3, min_part=0)]
[[2, 1, 1]]
sage: [x for x in Partitions(4, min_length=3, min_part=0)]
[[2, 1, 1], [1, 1, 1, 1]]
```

Except for very special cases, counting is done by brute force iteration through all the partitions. However the iteration itself has a reasonable complexity (see `IntegerListsLex`), which allows for manipulating large partitions:

```
sage: Partitions(1000, max_length=1).list()
[[1000]]
```

In particular, getting the first element is also constant time:

```
sage: Partitions(30, max_part=29).first()
[29, 1]
```

### Element

alias of `Partition`

```
options = Current options for Partitions - convention: English - diagram_str: * - display: list - latex: young_diagram - latex_diagram_str: \ast
```

```
sage: subset(*args, **kwargs)
Return self if no arguments are given, otherwise raises a ValueError.
```

### Examples:

```
sage: P = Partitions(5, starting=[3,1]); P
Partitions of the integer 5 starting with [3, 1]
sage: P.subset()
Partitions of the integer 5 starting with [3, 1]
sage: P.subset(ending=[3,1])
Traceback (most recent call last):
  ... ValueError: invalid combination of arguments
```

## 5.1. Comprehensive Module List
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sage: PartitionsGreatestEQ(10, 2)
Partitions of 10 having greatest part equal to 2
sage: PartitionsGreatestEQ(10, 2).list()
[[2, 2, 2, 2, 2],
 [2, 2, 2, 1, 1],
 [2, 2, 1, 1, 1, 1],
 [2, 1, 1, 1, 1, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1]]

sage: [4,3,2,1] in PartitionsGreatestEQ(10, 2)
False
sage: [2,2,2,2,2] in PartitionsGreatestEQ(10, 2)
True

The empty partition has no maximal part, but it is contained in the set of partitions with any specified maximal part:

sage: PartitionsGreatestEQ(0, 2).list()
[[[]]]

Element

alias of Partition
cardinality()
Return the cardinality of self.

EXAMPLES:

sage: PartitionsGreatestEQ(10, 2).cardinality()
5

options = Current options for Partitions - convention: English - diagram_str: * - display: list - latex: young_diagram - latex_diagram_str: \ast

class sage.combinat.partition.PartitionsGreatestLE(n,k)

Bases: UniqueRepresentation, IntegerListsLex

The class of all (unordered) “restricted” partitions of the integer $n$ having parts less than or equal to the integer $k$.

EXAMPLES:

sage: PartitionsGreatestLE(10, 2)
Partitions of 10 having parts less than or equal to 2
sage: PartitionsGreatestLE(10, 2).list()
[[2, 2, 2, 2],
 [2, 2, 2, 1, 1],
 [2, 2, 1, 1, 1, 1],
 [2, 1, 1, 1, 1, 1, 1],
 [1, 1, 1, 1, 1, 1, 1, 1]]

sage: [4,3,2,1] in PartitionsGreatestLE(10, 2)
False
sage: [2,2,2,2,2] in PartitionsGreatestLE(10, 2)

(continues on next page)
True

```
sage: PartitionsGreatestLE(10, 2).first().parent()
Partitions...
```

**Element**

alias of *Partition*

**cardinality()**

Return the cardinality of *self*.

**EXAMPLES:**

```
sage: PartitionsGreatestLE(9, 5).cardinality()
23
```

```
options = Current options for Partitions - convention: English - diagram_str: * - display: list - latex: young_diagram - latex_diagram_str: \ast
```

class sage.combinat.partition.PartitionsInBox(h, w)

```
Bases: Partitions
```

All partitions which fit in an \( h \times w \) box.

**EXAMPLES:**

```
sage: PartitionsInBox(2,2)
Integer partitions which fit in a 2 x 2 box
sage: PartitionsInBox(2,2).list()
[[], [1], [1, 1], [2], [2, 1], [2, 2]]
```

```
cardinality()
```

Return the cardinality of *self*.

**EXAMPLES:**

```
sage: PartitionsInBox(2, 3).cardinality()
10
```

```
list()
```

Return a list of all the partitions inside a box of height \( h \) and width \( w \).

**EXAMPLES:**

```
sage: PartitionsInBox(2,2).list()
[[], [1], [1, 1], [2], [2, 1], [2, 2]]
sage: PartitionsInBox(2,3).list()
[[], [1], [1, 1], [2], [2, 1], [2, 2], [3], [3, 1], [3, 2], [3, 3]]
```

class sage.combinat.partition.Partitions_all

```
Bases: Partitions
```

Class of all partitions.

```
from_beta_numbers(beta)
```

Return a partition corresponding to a sequence of beta numbers.
A sequence of beta numbers is a strictly increasing sequence \(0 \leq b_1 < \cdots < b_k\) of non-negative integers. The corresponding partition \(\mu = (\mu_k, \ldots, \mu_1)\) is given by \(\mu_i = [1,i) \setminus \{b_1, \ldots, b_i\}\). This gives a bijection from the set of partitions with at most \(k\) non-zero parts to the set of strictly increasing sequences of non-negative integers of length \(k\).

**EXAMPLES:**

```python
sage: Partitions().from_beta_numbers([0,1,2,4,5,8])
[3, 1, 1]
sage: Partitions().from_beta_numbers([0,2,3,6])
[3, 1, 1]
```

**from_core_and_quotient**(core, quotient)

Return a partition from its core and quotient.

Algorithm from mupad-combinat.

**EXAMPLES:**

```python
sage: Partitions().from_core_and_quotient([2,1], [[2,1],[3],[1,1,1]])
[11, 5, 5, 3, 2, 2, 2]
```

**from_exp**(exp)

Return a partition from its list of multiplicities.

**EXAMPLES:**

```python
sage: Partitions().from_exp([2,2,1])
[3, 2, 2, 1, 1]
```

**from_frobenius_coordinates**(frobenius_coordinates)

Return a partition from a pair of sequences of Frobenius coordinates.

**EXAMPLES:**

```python
sage: Partitions().from_frobenius_coordinates([],[]) [[]]
sage: Partitions().from_frobenius_coordinates([0],[0])
[1]
sage: Partitions().from_frobenius_coordinates([1],[1])
[2, 1]
sage: Partitions().from_frobenius_coordinates([6,3,2],[4,1,0])
[7, 5, 5, 1, 1]
```

**from_zero_one**(seq)

Return a partition from its \(0-1\) sequence.

The full \(0-1\) sequence is the sequence (infinite in both directions) indicating the steps taken when following the outer rim of the diagram of the partition. We use the convention that in English convention, a \(1\) corresponds to an East step, and a \(0\) corresponds to a North step.

Note that every full \(0-1\) sequence starts with infinitely many \(0\)'s and ends with infinitely many \(1\)'s.

See also:

Partition.zero_one_sequence()

**INPUT:**
The input should be a finite sequence of 0’s and 1’s. The heading 0’s and trailing 1’s will be discarded.

**EXAMPLES:**

```
sage: Partitions().from_zero_one([])
[]
sage: Partitions().from_zero_one([1,0])
[1]
sage: Partitions().from_zero_one([1,1,1,1,0,1,0])
[5, 4]
```

Heading 0’s and trailing 1’s are correctly handled:

```
sage: Partitions().from_zero_one([0,0,1,1,1,1,0,1,0,1,1,1])
[5, 4]
```

**subset(size=None, **kwargs)**

Return the subset of partitions of a given size and additional keyword arguments.

**EXAMPLES:**

```
sage: P = Partitions()
sage: P.subset(4)
Partitions of the integer 4
```

---

**5.1. Comprehensive Module List**

---
class sage.combinat.partition.Partitions_n(n)

Bases: Partitions

Partitions of the integer n.

cardinality(algorithm='flint')

Return the number of partitions of the specified size.

INPUT:

• algorithm - (default: 'flint')
  - 'flint' – use FLINT (currently the fastest)
  - 'gap' – use GAP (VERY slow)
  - 'pari' – use PARI. Speed seems the same as GAP until n is in the thousands, in which case PARI is faster.

It is possible to associate with every partition of the integer n a conjugacy class of permutations in the symmetric group on n points and vice versa. Therefore the number of partitions \( p_n \) is the number of conjugacy classes of the symmetric group on n points.

EXAMPLES:

```sage
v = Partitions(5).list(); v
[[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]]
len(v)
7
Partitions(5).cardinality(algorithm='gap')
7
Partitions(5).cardinality(algorithm='pari')
7
number_of_partitions(5, algorithm='flint')
7
```

```sage
Partitions(10).cardinality()
42
Partitions(3).cardinality()
3
Partitions(10).cardinality()
42
Partitions(3).cardinality(algorithm='pari')
3
Partitions(10).cardinality(algorithm='pari')
42
Partitions(40).cardinality()
37338
Partitions(100).cardinality()
190569292
```

A generating function for \( p_n \) is given by the reciprocal of Euler’s function:

\[
\sum_{n=0}^{\infty} p_n x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.
\]

We use Sage to verify that the first several coefficients do indeed agree:
```python
sage: q = PowerSeriesRing(QQ, 'q', default_prec=9).gen()
sage: prod([(1-q^k)^(-1) for k in range(1,9)])  # partial product of
1 + q + 2*q^2 + 3*q^3 + 5*q^4 + 7*q^5 + 11*q^6 + 15*q^7 + 22*q^8 + O(q^9)
sage: [Partitions(k).cardinality() for k in range(2,10)]
[2, 3, 5, 7, 11, 15, 22, 30]
```

Another consistency test for \( n \) up to 500:

```python
sage: len([n for n in [1..500] if Partitions(n).cardinality() != Partitions(n).cardinality(algorithm='pari')])
0
```

For negative inputs, the result is zero (the algorithm is ignored):

```python
sage: Partitions(-5).cardinality()
0
```

REFERENCES:
- Wikipedia article Partition_(number_theory)

**first()**
Return the lexicographically first partition of a positive integer \( n \). This is the partition \([n]\).

EXAMPLES:

```python
sage: Partitions(4).first()
[4]
```

**last()**
Return the lexicographically last partition of the positive integer \( n \). This is the all-ones partition.

EXAMPLES:

```python
sage: Partitions(4).last()
[1, 1, 1, 1]
```

**next(p)**
Return the lexicographically next partition after the partition \( p \).

EXAMPLES:

```python
sage: Partitions(4).next([4])
[3, 1]
sage: Partitions(4).next([1,1,1,1]) is None
True
```

**random_element**(\( measure='uniform' \))
Return a random partitions of \( n \) for the specified measure.

INPUT:
- \( measure \) – 'uniform' or 'Plancherel' (default: 'uniform')

See also:
- \( random_element_uniform() \)
- `random_element_plancherel()`

**Examples:**

```python
sage: Partitions(5).random_element()  # random
[2, 1, 1, 1]
sage: Partitions(5).random_element(measure='Plancherel')  # random
[2, 1, 1, 1]
```

`random_element_plancherel()`

Return a random partition of \( n \) (for the Plancherel measure).

This probability distribution comes from the uniform distribution on permutations via the Robinson-Schensted correspondence.

See Wikipedia article Plancherel_measure and `Partition.plancherel_measure()`.

**Examples:**

```python
sage: Partitions(5).random_element_plancherel()  # random
[2, 1, 1, 1]
sage: Partitions(20).random_element_plancherel()  # random
[9, 3, 3, 2, 2, 1]
```

**Algorithm:**

- insert by Robinson-Schensted a uniform random permutation of \( n \) and returns the shape of the resulting tableau. The complexity is \( O(n \log(n)) \) which is likely optimal. However, the implementation could be optimized.

**Author:**

- Florent Hivert (2009-11-23)

---

`random_element_uniform()`

Return a random partition of \( n \) with uniform probability.

**Examples:**

```python
sage: Partitions(5).random_element_uniform()  # random
[2, 1, 1, 1]
sage: Partitions(20).random_element_uniform()  # random
[9, 3, 3, 2, 2, 1]
```

**Algorithm:**

- It is a python Implementation of RANDPAR, see [NW1978]. The complexity is unknown, there may be better algorithms.

**Todo:** Check in Knuth AOCP4.

- There is also certainly a lot of room for optimizations, see comments in the code.

**Author:**

- Florent Hivert (2009-11-23)
**subset(**kwargs)**

Return a subset of self with the additional optional arguments.

EXAMPLES:

```
sage: P = Partitions(5); P
Partitions of the integer 5
sage: P.subset(starting=[3,1])
Partitions of the integer 5 starting with [3, 1]
```

class sage.combinat.partition.Particles_nk(n, k)

Bases: Partitions

Partitions of the integer n of length equal to k.

cardinality(algorithm='hybrid')

Return the number of partitions of the specified size with the specified length.

INPUT:

- algorithm – (default: 'hybrid') the algorithm to compute the cardinality and can be one of the following:
  - 'hybrid' - use a hybrid algorithm which uses heuristics to reduce the complexity
  - 'gap' - use GAP

EXAMPLES:

```
sage: v = Partitions(5, length=2).list(); v
[[4, 1], [3, 2]]
sage: len(v)
2
sage: Partitions(5, length=2).cardinality()
2
```

More generally, the number of partitions of $n$ of length 2 is $\left\lfloor \frac{n}{2} \right\rfloor$:

```
sage: all( Partitions(n, length=2).cardinality() == n // 2 for n in range(10) )
True
```

The number of partitions of $n$ of length 1 is 1 for $n$ positive:

```
sage: all( Partitions(n, length=1).cardinality() == 1 for n in range(1, 10) )
True
```

Further examples:

```
sage: Partitions(5, length=3).cardinality()
2
sage: Partitions(6, length=3).cardinality()
3
sage: Partitions(8, length=4).cardinality()
5
sage: Partitions(8, length=5).cardinality()
```

(continues on next page)
```
3
sage: Partitions(15, length=6).cardinality()
26
sage: Partitions(0, length=0).cardinality()
1
sage: Partitions(0, length=1).cardinality()
0
sage: Partitions(1, length=0).cardinality()
0
sage: Partitions(1, length=4).cardinality()
0
```

**subset(****kwargs**)**

Return a subset of self with the additional optional arguments.

**EXAMPLES:**

```
sage: P = Partitions(5, length=2); P
Partitions of the integer 5 of length 2
sage: P.subset(max_part=3)
Partitions of the integer 5 satisfying constraints length=2, max_part=3
```

**class** **sage.combinat.partition.Partitions_parts_in(n, parts)**

**Bases:** **Partitions**

Partitions of \( n \) with parts in a given set \( S \).

This is invoked indirectly when calling \texttt{Partitions(n, parts=parts)}, where \texttt{parts} is a list of pairwise distinct integers.

**cardinality()**

Return the number of partitions with parts in self. Wraps GAP's \texttt{NrRestrictedPartitions}.

**EXAMPLES:**

```
sage: Partitions(15, parts_in=[2,3,7]).cardinality()
5
```

If you can use all parts 1 through \( n \), we'd better get \( p(n) \):

```
sage: Partitions(20, parts_in=[1..20]).cardinality() == Partitions(20).˓→cardinality()
True
```

**first()**

Return the lexicographically first partition of a positive integer \( n \) with the specified parts, or \texttt{None} if no such partition exists.

**EXAMPLES:**

```
sage: Partitions(9, parts_in=[3,4]).first()
[3, 3, 3]
sage: Partitions(6, parts_in=[1..6]).first()
[6]
```
sage: Partitions(30, parts_in=[4,7,8,10,11]).first()
[11, 11, 8]

last()
Return the lexicographically last partition of the positive integer \( n \) with the specified parts, or \( \text{None} \) if no such partition exists.

EXAMPLES:

\[
\begin{align*}
sage: \text{Partitions}(15, \text{parts_in}=[2,3]).\text{last()} & \quad [3, 2, 2, 2, 2, 2, 2] \\
sage: \text{Partitions}(30, \text{parts_in}=[4,7,8,10,11]).\text{last()} & \quad [7, 7, 4, 4, 4, 4] \\
sage: \text{Partitions}(10, \text{parts_in}=[3,6]).\text{last()} \quad \text{is None} \\
\text{True} \\
sage: \text{Partitions}(50, \text{parts_in}=[11,12,13]).\text{last()} & \quad [13, 13, 12, 12] \\
sage: \text{Partitions}(30, \text{parts_in}=[4,7,8,10,11]).\text{last()} & \quad [7, 7, 4, 4, 4, 4]
\end{align*}
\]

class sage.combinat.partition.Partitions_starting(n, starting_partition)
Bases: Partitions
All partitions with a given start.

first()
Return the first partition in self.

EXAMPLES:

\[
\begin{align*}
sage: \text{Partitions}(3, \text{starting}=[2,1]).\text{first()} & \quad [2, 1]
\end{align*}
\]

next(part)
Return the next partition after part in self.

EXAMPLES:

\[
\begin{align*}
sage: \text{Partitions}(3, \text{starting}=[2,1]).\text{next}(\text{Partition([2,1])}) & \quad [1, 1, 1]
\end{align*}
\]

class sage.combinat.partition.Partitions_with_constraints(*args, **kwds)
Bases: IntegerListsLex
Partitions which satisfy a set of constraints.

EXAMPLES:

\[
\begin{align*}
sage: P = \text{Partitions}(6, \text{inner}=[1,1], \text{max_slope}=-1) \\
sage: \text{list}(P) & \quad [[5, 1], [4, 2], [3, 2, 1]]
\end{align*}
\]

Element
alias of Partition
Combinatorics, Release 10.0

options = Current options for Partitions - convention: English - diagram_str: * - display: list - latex: young_diagram - latex_diagram_str: \ast

class sage.combinat.partition.RegularPartitions(ell, is_infinite=False)
Bases: Partitions
Base class for \(\ell\)-regular partitions.
Let \(\ell\) be a positive integer. A partition \(\lambda\) is \(\ell\)-regular if \(m_i < \ell\) for all \(i\), where \(m_i\) is the multiplicity of \(i\) in \(\lambda\).

**Note:** This is conjugate to the notion of \(\ell\)-restricted partitions, where the difference between any two consecutive parts is \(< \ell\).

**INPUT:**
- \(\text{ell}\) – the positive integer \(\ell\)
- \(\text{is\_infinite}\) – boolean; if the subset of \(\ell\)-regular partitions is infinite

**ell()**
Return the value \(\ell\).

**EXAMPLES:**

```python
sage: P = Partitions(regular=2)
sage: P.ell()
2
```

class sage.combinat.partition.RegularPartitions_all(ell)
Bases: RegularPartitions
The class of all \(\ell\)-regular partitions.

**INPUT:**
- \(\text{ell}\) – the positive integer \(\ell\)

**See also:**
- RegularPartitions

class sage.combinat.partition.RegularPartitions_bounded(ell, k)
Bases: RegularPartitions
The class of \(\ell\)-regular \(k\)-bounded partitions.

**INPUT:**
- \(\text{ell}\) – the integer \(\ell\)
- \(k\) – integer; the value \(k\)

**See also:**
- RegularPartitions

class sage.combinat.partition.RegularPartitions_n(n, ell)
Bases: RegularPartitions, Partitions_n
The class of \(\ell\)-regular partitions of \(n\).

**INPUT:**
• n – the integer n to partition
• ell – the integer ℓ

See also:
RegularPartitions

`cardinality()`
Return the cardinality of self.

EXAMPLES:
```
sage: P = Partitions(5, regular=3)
sage: P.cardinality()
5
sage: P = Partitions(5, regular=6)
sage: P.cardinality()
7
sage: P.cardinality() == Partitions(5).cardinality()
True
```

class sage.combinat.partition.RegularPartitions_truncated(ell, max_len)
Bases: RegularPartitions
The class of ℓ-regular partitions with max length k.

INPUT:
• ell – the integer ℓ
• max_len – integer; the maximum length

See also:
RegularPartitions

`max_length()`
Return the maximum length of the partitions of self.

EXAMPLES:
```
sage: P = Partitions(regular=4, max_length=3)
sage: P.max_length()
3
```

class sage.combinat.partition.RestrictedPartitions_all(ell)
Bases: RestrictedPartitions_generic
The class of all ℓ-restricted partitions.

INPUT:
• ell – the positive integer ℓ

See also:
RestrictedPartitions_generic

class sage.combinat.partition.RestrictedPartitions_generic(ell, is_infinite=False)
Bases: Partitions
Base class for ℓ-restricted partitions.
Let $\ell$ be a positive integer. A partition $\lambda$ is $\ell$-restricted if $\lambda_i - \lambda_{i+1} < \ell$ for all $i$, including rows of length 0.

**Note:** This is conjugate to the notion of $\ell$-regular partitions, where the multiplicity of any parts is at most $\ell$.

**INPUT:**
- $\text{ell}$ – the positive integer $\ell$
- $\text{is_infinite}$ – boolean; if the subset of $\ell$-restricted partitions is infinite

**ell()**

Return the value $\ell$.

**EXAMPLES:**

```
sage: P = Partitions(restricted=2)
sage: P.ell()
2
```

**class** `sage.combinat.partition.RestrictedPartitions_n(n, ell)`

Bases: `RestrictedPartitions_generic, Partitions_n`

The class of $\ell$-restricted partitions of $n$.

**INPUT:**
- $n$ – the integer $n$ to partition
- $\text{ell}$ – the integer $\ell$

**See also:**
- `RestrictedPartitions_generic`

**cardinality()**

Return the cardinality of $\text{self}$.

**EXAMPLES:**

```
sage: P = Partitions(5, restricted=3)
sage: P.cardinality()
5
sage: P = Partitions(5, restricted=6)
sage: P.cardinality()
7
sage: P.cardinality() == Partitions(5).cardinality()
True
```

`sage.combinat.partition.number_of_partitions(n, algorithm='default')`

Return the number of partitions of $n$ with, optionally, at most $k$ parts.

The options of `number_of_partitions()` are being deprecated github issue #13072 in favour of `Partitions_n.cardinality()` so that `number_of_partitions()` can become a stripped down version of the fastest algorithm available (currently this is using FLINT).

**INPUT:**
- $n$ – an integer
- $\text{algorithm}$ – (default: ‘default’) [Will be deprecated except in Partition().cardinality() ]
Combinatorics, Release 10.0

- 'default' – If k is not None, then use Gap (very slow). If k is None, use FLINT.
- 'flint' – use FLINT

**EXAMPLES:**

```python
sage: v = Partitions(5).list(); v
[[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]]
sage: len(v)
7
```

The input must be a nonnegative integer or a `ValueError` is raised.

```python
sage: number_of_partitions(-5)
Traceback (most recent call last):
  ... ValueError: n (=5) must be a nonnegative integer
```

```python
sage: number_of_partitions(10)
42
sage: number_of_partitions(3)
3
sage: number_of_partitions(10)
42
sage: number_of_partitions(40)
37338
sage: number_of_partitions(100)
190569292
sage: number_of_partitions(100000)
274935105697756965126775163209863526881734293159800547582031259843021473281149641730590507416607366215901578447742962489406986911617458360719769746337355030059187485153988550896191483754930457533405400066252160797545505832299180021799268102963791042549029300331774472454151195452658394075319228632255923668968688389603065565227322735938091073458759520043356746020549826534855239638448134659869258350063706848917382361484119579600472088093904714349409898308319511968037352475479794332591745275479224440044590473045315695055840750790912287573700135396407280254821295760565553047541638715447943233537232581232320731309079848802765033095923333661770418548720201070389095003209648415026700868626433668570125820451520132918690137295495784757990064753311068521816607941650810670267940657228254361994058495585657609097148773385782043350749015715729443202141882111743343865925342703838220592694093331975859087617645158149433970645358782784756526549373432785831335457958473321512048740447048984276642852279374055991890429776437637779287459510167287189313389936026888411937172894535551322285863490061526522828945881765548196140020785865045503985300933640740670556208719947424797335918801602177720206651942528088313464129888287521318150861447200527880367118342857782971080717244283654699458467012891288212370294831053854588710567088708798201145886198724738980691179414296947184279018367929316670577774825716722407383116988076)
```

A generating function for the number of partitions $p_n$ is given by the reciprocal of Euler’s function:

$$
\sum_{n=0}^{\infty} p_n x^n = \prod_{k=1}^{\infty} \left( \frac{1}{1-x^k} \right).
$$

We use Sage to verify that the first several coefficients do indeed agree:

```python
sage: q = PowerSeriesRing(QQ, 'q', default_prec=9).gen()
sage: prod([(1-q^k)^(-1) for k in range(1,9)])
# partial product of
1 + q + 2*q^2 + 3*q^3 + 5*q^4 + 7*q^5 + 11*q^6 + 15*q^7 + 22*q^8 + O(q^9)
sage: [number_of_partitions(k) for k in range(2,11)]
[2, 3, 5, 7, 11, 15, 22, 30]
```

**REFERENCES:**

- Wikipedia article Partition_(number_theory)

`sage.combinat.partition.number_of_partitions_length(n, k, algorithm='hybrid')`

Return the number of partitions of $n$ with length $k$.

This is a wrapper for GAP’s NrPartitions function.

**EXAMPLES:**
5.1.166 Partition/Diagram Algebras

class sage.combinat.partition_algebra.PartitionAlgebraElement_ak
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebraElement_bk
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebraElement_generic
    Bases: IndexedFreeModuleElement
class sage.combinat.partition_algebra.PartitionAlgebraElement_pk
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebraElement_prk
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebraElement_rk
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebraElement_sk
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebraElement_tk
    Bases: PartitionAlgebraElement_generic
class sage.combinat.partition_algebra.PartitionAlgebra_ak(R, k, n, name=None)
    Bases: PartitionAlgebra_generic

EXAMPLES:

sage: from sage.combinat.partition_algebra import *

sage: p = PartitionAlgebra_ak(QQ, 3, 1)
sage: p == loads(dumps(p))
True
class sage.combinat.partition_algebra.PartitionAlgebra_bk(R, k, n, name=None)
    Bases: PartitionAlgebra_generic
    EXAMPLES:
    
    sage: from sage.combinat.partition_algebra import *
    sage: p = PartitionAlgebra_bk(QQ, 3, 1)
    sage: p == loads(dumps(p))
    True

class sage.combinat.partition_algebra.PartitionAlgebra_generic(R, cclass, n, k, name=None, prefix=None)

    EXAMPLES:
    
    sage: from sage.combinat.partition_algebra import *
    sage: s = PartitionAlgebra_sk(QQ, 3, 1)
    sage: TestSuite(s).run()
    sage: s == loads(dumps(s))
    True

    one_basis()
    Return the basis index for the unit of the algebra.

    EXAMPLES:
    
    sage: from sage.combinat.partition_algebra import *
    sage: s = PartitionAlgebra_sk(ZZ, 3, 1)
    sage: len(s.one().support())  # indirect doctest
    1

    product_on_basis(left, right)

    EXAMPLES:
    
    sage: from sage.combinat.partition_algebra import *
    sage: s = PartitionAlgebra_sk(QQ, 3, 1)
    sage: t12 = s(Set([Set([1,-2]),Set([2,-1]),Set([3,-3])]))
    sage: t12^2 == s(1)  # indirect doctest
    True

class sage.combinat.partition_algebra.PartitionAlgebra_pk(R, k, n, name=None)
    Bases: PartitionAlgebra_generic
    EXAMPLES:
    
    sage: from sage.combinat.partition_algebra import *
    sage: p = PartitionAlgebra_pk(QQ, 3, 1)
    sage: p == loads(dumps(p))
    True

class sage.combinat.partition_algebra.PartitionAlgebra_prk(R, k, n, name=None)
    Bases: PartitionAlgebra_generic
    EXAMPLES:
class sage.combinat.partition_algebra.PartitionAlgebra_rk(R, k, n, name=None)
Bases: PartitionAlgebra_generic

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_rk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.PartitionAlgebra_sk(R, k, n, name=None)
Bases: PartitionAlgebra_generic

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_sk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

class sage.combinat.partition_algebra.PartitionAlgebra_tk(R, k, n, name=None)
Bases: PartitionAlgebra_generic

EXAMPLES:

```
sage: from sage.combinat.partition_algebra import *
sage: p = PartitionAlgebra_tk(QQ, 3, 1)
sage: p == loads(dumps(p))
True
```

sage.combinat.partition_algebra.SetPartitionsAk(k)

Return the combinatorial class of set partitions of type $A_k$.

EXAMPLES:

```
sage: A3 = SetPartitionsAk(3); A3
Set partitions of {1, ..., 3, -1, ..., -3}
sage: A3.first()  #random
{{1, 2, 3, -1, -3, -2}}
sage: A3.last()  #random
{{-1}, {-2}, {3}, {1}, {-3}, {2}}
sage: A3.random_element()  #random
{{1, 3, -3, -1}, {2, -2}}
sage: A3.cardinality()
203
```

```
sage: A2p5 = SetPartitionsAk(2.5); A2p5
```

(continues on next page)
Set partitions of \{1, \ldots, 3, -1, \ldots, -3\} with 3 and -3 in the same block

\texttt{sage: A2p5.cardinality()}
52

\texttt{sage: A2p5.first() \#random}
\{\{1, 2, 3, -1, -3, -2\}\}

\texttt{sage: A2p5.last() \#random}
\{\{-1\}, \{-2\}, \{2\}, \{3, -3\}, \{1\}\}

\texttt{sage: A2p5.random_element() \#random}
\{\{-1\}, \{-2\}, \{3, -3\}, \{1, 2\}\}

\begin{verbatim}
class sage.combinat.partition_algebra.SetPartitionsAk_k(k)
    Bases: SetPartitions_set
    Element
        alias of SetPartitionsXkElement
class sage.combinat.partition_algebra.SetPartitionsAkhalf_k(k)
    Bases: SetPartitions_set
    Element
        alias of SetPartitionsXkElement
sage.combinat.partition_algebra.SetPartitionsBk(k)
    Return the combinatorial class of set partitions of type \(B_k\).
    These are the set partitions where every block has size 2.

    EXAMPLES:
\end{verbatim}

\texttt{sage: B3 = SetPartitionsBk(3); B3}
Set partitions of \{1, \ldots, 3, -1, \ldots, -3\} with block size 2

\texttt{sage: B3.first() \#random}
\{\{2, -2\}, \{1, -3\}, \{3, -1\}\}

\texttt{sage: B3.last() \#random}
\{\{1, 2\}, \{3, -2\}, \{-3, -1\}\}

\texttt{sage: B3.random_element() \#random}
\{\{2, -1\}, \{1, -3\}, \{3, -2\}\}

\texttt{sage: B3.cardinality()}
15

\texttt{sage: B2p5 = SetPartitionsBk(2.5); B2p5}
Set partitions of \{1, \ldots, 3, -1, \ldots, -3\} with 3 and -3 in the same block and with block size 2

\texttt{sage: B2p5.first() \#random}
\{\{2, -1\}, \{3, -3\}, \{1, -2\}\}

\texttt{sage: B2p5.last() \#random}
\{\{1, 2\}, \{3, -3\}, \{-1, -2\}\}

\texttt{sage: B2p5.random_element() \#random}
\{\{2, -2\}, \{3, -3\}, \{1, -1\}\}

(continues on next page)
class sage.combinat.partition_algebra.SetPartitionsBk_k(k)
Bases: SetPartitionsAk_k
cardinality()

Return the number of set partitions in $B_k$ where $k$ is an integer.

This is given by $(2k)!! = (2k-1)*(2k-3)*...*3*1$.

EXAMPLES:

```python
sage: SetPartitionsBk(3).cardinality()
15
sage: SetPartitionsBk(2).cardinality()
3
sage: SetPartitionsBk(1).cardinality()
1
sage: SetPartitionsBk(4).cardinality()
105
sage: SetPartitionsBk(5).cardinality()
945
```

class sage.combinat.partition_algebra.SetPartitionsBkhalf_k(k)
Bases: SetPartitionsAkhalf_k
cardinality()

sage.combinat.partition_algebra.SetPartitionsIk(k)

Return the combinatorial class of set partitions of type $I_k$.

These are set partitions with a propagating number of less than $k$. Note that the identity set partition $\{\{1, -1\}, \ldots, \{k, -k\}\}$ is not in $I_k$.

EXAMPLES:

```python
sage: I3 = SetPartitionsIk(3); I3
Set partitions of {1, ..., 3, -1, ..., -3} with propagating number < 3
sage: I3.cardinality()
197

sage: I3.first()  #random
\{\{1, 2, 3, -1, -3, -2\}\}

sage: I3.last()  #random
\{\{-1\}, \{-2\}, \{3\}, \{1\}, \{-3\}, \{2\}\}

sage: I3.random_element()  #random
\{\{-1\}, \{-3\}, \{-2\}, \{2, 3\}, \{1\}\}

sage: I2p5 = SetPartitionsIk(2.5); I2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block and propagating number < 3
sage: I2p5.cardinality()
50
```
class sage.combinat.partition_algebra.SetPartitionsIk_k(k)

Bases: SetPartitionsAk_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsIkhalf_k(k)

Bases: SetPartitionsAkhalf_k
cardinality()

sage.combinat.partition_algebra.SetPartitionsPRk(k)

Return the combinatorial class of set partitions of type $P_{R_k}$.

EXAMPLES:

sage: SetPartitionsPRk(3)
Set partitions of {1, ..., 3, -1, ..., -3} with at most 1 positive and negative entry in each block and that are planar

class sage.combinat.partition_algebra.SetPartitionsPRk_k(k)

Bases: SetPartitionsRk_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsPRkhalf_k(k)

Bases: SetPartitionsRkhalf_k
cardinality()

sage.combinat.partition_algebra.SetPartitionsPk(k)

Return the combinatorial class of set partitions of type $P_k$.

These are the planar set partitions.

EXAMPLES:

sage: P3 = SetPartitionsPk(3); P3
Set partitions of {1, ..., 3, -1, ..., -3} that are planar
sage: P3.cardinality()
132

sage: P3.first() #random
{{1, 2, 3, -1, -3, -2}}
sage: P3.last() #random
{{-1}, {-2}, {3}, {1}, {-3}, {2}}
sage: P3.random_element() #random
{{1, 2, -1}, {-3}, {3, -2}}
sage: P2p5 = SetPartitionsPk(2.5); P2p5
Set partitions of \{1, \ldots, 3, -1, \ldots, -3\} with 3 and -3 in the same block and that are planar
sage: P2p5.cardinality()
42

sage: P2p5.first() #random
\{\{1, 2, 3, -1, -3, -2\}\}
sage: P2p5.last() #random
\{\{-1\}, \{-2\}, \{2\}, \{3, -3\}, \{1\}\}
sage: P2p5.random_element() #random
\{\{1, 2, 3, -3\}, \{-1, -2\}\}

class sage.combinat.partition_algebra.SetPartitionsPkp_k(k)
Bases: SetPartitionsAk_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsPkp_half_k(k)
Bases: SetPartitionsAkp_half_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsRk(k)
Return the combinatorial class of set partitions of type \(R_k\).

EXAMPLES:

sage: SetPartitionsRk(3)
Set partitions of \{1, \ldots, 3, -1, \ldots, -3\} with at most 1 positive and negative entry in each block

class sage.combinat.partition_algebra.SetPartitionsRkp_k(k)
Bases: SetPartitionsAk_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsRkp_half_k(k)
Bases: SetPartitionsAkp_half_k
cardinality()

sage.combinat.partition_algebra.SetPartitionsSk(k)
Return the combinatorial class of set partitions of type \(S_k\).

There is a bijection between these set partitions and the permutations of \(1, \ldots, k\).

EXAMPLES:

sage: S3 = SetPartitionsSk(3); S3
Set partitions of \{1, \ldots, 3, -1, \ldots, -3\} with propagating number 3
sage: S3.cardinality()
6

sage: S3.list() #random
sage: S3p5 = SetPartitionsSk(3.5); S3p5
Set partitions of \{1, ..., 4, -1, ..., -4\} with 4 and -4 in the same block and
→ propagating number 4
sage: S3p5.cardinality()
6

sage: S3p5.list() #random
[[\{2, -2\}, \{3, -3\}, \{1, -1\}, \{4, -4\}],
 [\{2, -3\}, \{1, -1\}, \{4, -4\}, \{3, -2\}],
 [\{2, -1\}, \{3, -3\}, \{1, -2\}, \{4, -4\}],
 [\{2, -3\}, \{1, -2\}, \{4, -4\}, \{3, -1\}],
 [\{1, -3\}, \{2, -1\}, \{4, -4\}, \{3, -2\}],
 [\{1, -3\}, \{2, -2\}, \{4, -4\}, \{3, -1\}]]

sage: S3p5.first() #random
\{\{2, -2\}, \{3, -3\}, \{1, -1\}, \{4, -4\}\}

sage: S3p5.last() #random
\{\{2, -1\}, \{4, -4\}, \{3, -2\}\}

sage: S3p5.random_element() #random
\{\{1, -3\}, \{2, -2\}, \{4, -4\}, \{3, -1\}\}

class sage.combinat.partition_algebra.SetPartitionsSk_k(k)

Bases: SetPartitionsAk_k

cardinality()

Return k!.

class sage.combinat.partition_algebra.SetPartitionsSkhalf_k(k)

Bases: SetPartitionsAkhalf_k
cardinality()

sage.combinat.partition_algebra.SetPartitionsTk(k)

Return the combinatorial class of set partitions of type \(T_k\).

These are planar set partitions where every block is of size 2.

EXAMPLES:

sage: T3 = SetPartitionsTk(3); T3
Set partitions of \{1, ..., 3, -1, ..., -3\} with block size 2 and that are planar
sage: T3.cardinality()
sage: T3.first()  #random
{{1, -3}, {2, 3}, {-1, -2}}
sage: T3.last()  #random
{{1, 2}, {3, -1}, {-3, -2}}
sage: T3.random_element()  #random
{{1, -3}, {2, 3}, {-1, -2}}

sage: T2p5 = SetPartitionsTk(2.5); T2p5
Set partitions of {1, ..., 3, -1, ..., -3} with 3 and -3 in the same block and with
→block size 2 and that are planar
sage: T2p5.cardinality()
2

sage: T2p5.first()  #random
{{2, -2}, {3, -3}, {1, -1}}
sage: T2p5.last()  #random
{{1, 2}, {3, -3}, {-1, -2}}

class sage.combinat.partition_algebra.SetPartitionsTk_k(k)
    Bases: SetPartitionsBk_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsTkhalf_k(k)
    Bases: SetPartitionsBkhalf_k
cardinality()

class sage.combinat.partition_algebra.SetPartitionsXkElement(parent, s, check=True)
    Bases: SetPartition
    An element for the classes of SetPartitionXk where X is some letter.
    check()
        Check to make sure this is a set partition.
        EXAMPLES:

        sage: A2p5 = SetPartitionsAk(2.5)
        sage: x = A2p5.first(); x
        {{-3, -2, -1, 1, 2, 3}}
        sage: x.check()
        sage: y = A2p5.next(x); y
        {{-3, 3}, {-2, -1, 1, 2}}
        sage: y.check()

sage.combinat.partition_algebra.identity(k)
    Return the identity set partition 1, -1, ..., k, -k
    EXAMPLES:
sage: import sage.combinat.partition_algebra as pa
sage: pa.identity(2)
{{2, -2}, {1, -1}}

sage.combinat.partition_algebra.is_planar(sp)

Return True if the diagram corresponding to the set partition is planar; otherwise, it returns False.

EXAMPLES:

sage: import sage.combinat.partition_algebra as pa
sage: pa.is_planar( pa.to_set_partition([[1,-2],[2,-1]]))
False
sage: pa.is_planar( pa.to_set_partition([[1,-1],[2,-2]]))
True

sage.combinat.partition_algebra.pair_to_graph(sp1, sp2)

Return a graph consisting of the disjoint union of the graphs of set partitions sp1 and sp2 along with edges joining the bottom row (negative numbers) of sp1 to the top row (positive numbers) of sp2.

The vertices of the graph sp1 appear in the result as pairs (k, 1), whereas the vertices of the graph sp2 appear as pairs (k, 2).

EXAMPLES:

sage: import sage.combinat.partition_algebra as pa
sage: sp1 = pa.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = pa.to_set_partition([[1,-2],[2,-1]])
sage: g = pa.pair_to_graph( sp1, sp2 ); g
Graph on 8 vertices
sage: g.vertices(sort=False)  # random
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]
sage: g.edges(sort=False)  # random
[((-2, 1), (2, 2), None), ((-1, 1), (1, 1), None),
((-1, 1), (1, 2), None)]

Another example which used to be wrong until github issue #15958:

sage: sp3 = pa.to_set_partition([[1, -1], [2], [-2]])
sage: sp4 = pa.to_set_partition([[1], [-1], [2], [-2]])
sage: g = pa.pair_to_graph( sp3, sp4 ); g
Graph on 8 vertices
sage: g.vertices(sort=True)
[(-2, 1), (-2, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2)]
sage: g.edges(sort=True)
[((-2, 1), (2, 2), None), ((-1, 1), (1, 1), None),
((-1, 1), (1, 2), None)]

sage.combinat.partition_algebra.propagating_number(sp)

Return the propagating number of the set partition sp.
The propagating number is the number of blocks with both a positive and negative number.

**EXAMPLES:**

```python
sage: import sage.combinat.partition_algebra as pa
sage: sp1 = pa.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = pa.to_set_partition([[1,2],[-2,-1]])
sage: pa.propagating_number(sp1)
2
sage: pa.propagating_number(sp2)
0
```

`sage.combinat.partition_algebra.set_partition_composition(sp1, sp2)`

Return a tuple consisting of the composition of the set partitions sp1 and sp2 and the number of components removed from the middle rows of the graph.

**EXAMPLES:**

```python
sage: import sage.combinat.partition_algebra as pa
sage: sp1 = pa.to_set_partition([[1,-2],[2,-1]])
sage: sp2 = pa.to_set_partition([[1,-2],[2,-1]])
sage: pa.set_partition_composition(sp1, sp2) == (pa.identity(2), 0)
True
```

`sage.combinat.partition_algebra.to_graph(sp)`

Return a graph representing the set partition sp.

**EXAMPLES:**

```python
sage: import sage.combinat.partition_algebra as pa
sage: g = pa.to_graph( pa.to_set_partition([[1,-1],[2,-2]]))
Graph on 4 vertices
sage: g.vertices(sort=False)  #random
[1, 2, -2, -1]
sage: g.edges(sort=False)  #random
[(1, -2, None), (2, -1, None)]
```

`sage.combinat.partition_algebra.to_set_partition(l, k=None)`

Convert a list of a list of numbers to a set partitions.

Each list of numbers in the outer list specifies the numbers contained in one of the blocks in the set partition.

If k is specified, then the set partition will be a set partition of 1, ..., k, -1, ..., -k. Otherwise, k will default to the minimum number needed to contain all of the specified numbers.

**EXAMPLES:**

```python
sage: import sage.combinat.partition_algebra as pa
sage: pa.to_set_partition([[1,-1],[2,-2]]) == pa.identity(2)
True
```
A partition (tuple) \(\mu\) is Kleshchev if it can be recursively obtained by adding a sequence of good nodes to the empty \texttt{PartitionTuple} of the same \texttt{level()} and \texttt{multicharge}. In this way, the set of Kleshchev multipartitions becomes a realization of a Kashiwara crystal \texttt{sage.combinat.crystals.crystals} for an irreducible integral highest weight representation of \(U_q(\widehat{\mathfrak{sl}}_n)\).

The Kleshchev multipartitions first appeared in the work of Ariki and Mathas [AM2000] where it was shown that they index the irreducible representations of the cyclotomic Hecke algebras of type \(A\) [AK1994]. Soon afterwards Ariki [Ariki2001] showed that the set of Kleshchev multipartitions naturally label the irreducible representations of these algebras. As a far reaching generalization of these ideas the Ariki-Brundan-Kleshchev categorification theorem [Ariki1996] [BK2009] says that these algebras categorify the irreducible integral highest weight representations of the quantum group \(U_q(\widehat{\mathfrak{sl}}_n)\) of the affine special linear group. Under this categorification, \(q\) corresponds to the grading shift on the cyclotomic Hecke algebras, where the grading from the Brundan-Kleshchev graded isomorphism theorem to the \(KLR\) algebras of type \(A\) [BK2009].

The group algebras of the symmetric group in characteristic \(p\) are an important special case of the cyclotomic Hecke algebras of type \(A\). In this case, depending on your prefer convention, the set of Kleshchev partitions is the set of \('p\'-\text{regular} or \('p\'-\text{restricted Partitions}. In this case, Kleshchev [Kle1995] proved that the \textit{modular branching rules} were given by adding and removing \textit{good nodes}; see \texttt{good_cells()}. Lascoux, Leclerc and Thibon [LLT1996] noticed that Kleshchev’s branching rules coincided with Kashiwara’s crystal operators for the fundamental representation of \(L(\Lambda_0)\) of \(U_q(\widehat{\mathfrak{sl}}_p)\) and their celebrated \textit{LLT conjecture} said that decomposition matrices of the \texttt{sage.algebras.iwahori_hecke_algebra.IwahoriHeckeAlgebra} of the symmetric group should be computable using the canonical basis of \(L(\Lambda_0)\). This was proved and generalized to all cyclotomic Hecke algebras of type \(A\) by Ariki [Ariki1996] and then further generalized to the graded setting by Brundan and Kleshchev [BK2009].

The main class for accessing Kleshchev partition (tuples) is \texttt{KleshchevPartitions}. Unfortunately, just as with the symmetric group, different authors use different conventions when defining Kleshchev partitions, which depends on whether you read components from left to right, or right to left, and whether you read the nodes in the partition in each component from top to bottom or bottom to top. The \texttt{KleshchevPartitions} class supports these four different conventions:

\begin{verbatim}
sage: KPlg = KleshchevPartitions(2, [0,0], size=2, convention='left regular')[::]
[([1], [1]), ([2], [])]
sage: KPls = KleshchevPartitions(2, [0,0], size=2, convention='left restricted')[::]
[([1], [1]), ([], [1, 1])]
sage: KPlrg = KleshchevPartitions(2, [0,0], size=2, convention='right regular')[::]
[([1], [1]), ([], [2])]
sage: KPlrs = KleshchevPartitions(2, [0,0], size=2, convention='right restricted')[::]
[([1], [1]), ([1, 1], [])]
\end{verbatim}

By default, the \textit{left restricted} convention is used. As a shorthand, LG, LS, RG and RS, respectively, can be used to specify the convention. With the \textit{left} convention the partition tuples should be ordered with the most dominant partitions in the partition tuple on the left and with the \textit{right} convention the most dominant partition is on the right.

The \texttt{KleshchevPartitions} class can automatically convert between these four different conventions:

\begin{verbatim}
sage: KPlg = KleshchevPartitions(2, [0,0], size=2, convention='left regular')
sage: KPls = KleshchevPartitions(2, [0,0], size=2, convention='left restricted')
sage: [KPlg(mu) for mu in KPls] # indirect doc test
[([1], [1]), ([2], [])]
\end{verbatim}

\section*{AUTHORS:}

- Andrew Mathas and Travis Scrimshaw (2018-05-1): Initial version

\section*{5.1. Comprehensive Module List}

1795
class sage.combinat.partition_kleshchev.KleshchevCrystalMixin
Bases: object

Mixin class for the crystal structure of a Kleshchev partition.

Epsilon()
    Return $\varepsilon$ of self.
    EXAMPLES:
    sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
    sage: x = C([[5,4,1],[3,2,1,1]])
    sage: x.Epsilon()
    3*Lambda[1]

Phi()
    Return $\phi$ of self.
    EXAMPLES:
    sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
    sage: x = C([[5,4,1],[3,2,1,1]])
    sage: x.Phi()
    3*Lambda[0] + 2*Lambda[1]

epsilon(i)
    Return the Kashiwara crystal operator $\varepsilon_i$ applied to self.
    INPUT:
    • i – an element of the index set
    EXAMPLES:
    sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
    sage: x = C([[5,4,1],[3,2,1,1]])
    sage: [x.epsilon(i) for i in C.index_set()]
    [0, 3, 0]

phi(i)
    Return the Kashiwara crystal operator $\varphi_i$ applied to self.
    INPUT:
    • i – an element of the index set
    EXAMPLES:
    sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
    sage: x = C([[5,4,1],[3,2,1,1]])
    sage: [x.phi(i) for i in C.index_set()]
    [3, 2, 0]

weight()
    Return the weight of self.
    EXAMPLES:
```python
sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
sage: x = C([5,4,1], [3,2,1,1])
sage: x.weight()
3*Lambda[0] - Lambda[1] - 5*delta
sage: x.Phi() - x.Epsilon()
3*Lambda[0] - Lambda[1]
sage: C = crystals.KleshchevPartitions(3, [0,2], convention="right regular")
sage: y = C([5,1,1], [4,2,2,1,1])
sage: y.weight()
6*Lambda[0] - 4*Lambda[1] - 4*delta
sage: y.Phi() - y.Epsilon()
6*Lambda[0] - 4*Lambda[1]
sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
sage: y = C([5,1,1], [4,2,2,1,1])
sage: y.weight()
6*Lambda[0] - 4*Lambda[1] - 4*delta
sage: y.Phi() - y.Epsilon()
6*Lambda[0] - 4*Lambda[1]
```

**class** `sage.combinat.partition_kleshchev.KleshchevPartition(parent, mu)`

Bases: `Partition`

Abstract base class for Kleshchev partitions. See `KleshchevPartitions`.

**cogood_cells**(i=None)

Return a list of the cells of self that are cogood.

The cogood 𝑖-cell is the ‘last’ conormal 𝑖-cell. As with the conormal cells we can choose to read either up or down the partition as specified by convention().

**INPUT:**

- i – (optional) a residue

**OUTPUT:**

If no residue i is specified then a dictionary of cogood cells is returned, which gives the cogood cells for 0 <= i < e.

**EXAMPLES:**

```python
sage: KP = KleshchevPartitions(3, convention="regular")
sage: KP([5,4,4,3,2]).cogood_cells()
{0: (1, 4), 1: (4, 2)}
sage: KP([5,4,4,3,2]).cogood_cells(0)
(1, 4)
sage: KP([5,4,4,3,2]).cogood_cells(1)
(4, 2)
sage: KP = KleshchevPartitions(4, convention='restricted')
sage: KP([5,4,4,3,2]).cogood_cells()
{1: (0, 5), 2: (4, 2), 3: (1, 4)}
sage: KP([5,4,4,3,2]).cogood_cells(0)
sage: KP([5,4,4,3,2]).cogood_cells(2)
(4, 2)
```
**conormal_cells**(i=None)

Return a dictionary of the cells of **self** which are conormal.

Following [Kle1995], the conormal cells are computed by reading up (or down) the rows of the partition and marking all of the addable and removable cells of c-residue i and then recursively removing all adjacent pairs of removable and addable cells (in that order) from this list. The addable i-cells that remain at the end of the this process are the conormal i-cells.

When computing conormal cells you can either read the cells in order from top to bottom (this corresponds to labeling the simple modules of the symmetric group by regular partitions) or from bottom to top (corresponding to labeling the simples by restricted partitions). By default we read down the partition but this can be changed by setting convention = 'RS'.

**INPUT:**

- i – (optional) a residue

**OUTPUT:**

If no residue i is specified then a dictionary of conormal cells is returned, which gives the conormal cells for 0 <= i < e.

**EXAMPLES:**

```python
sage: KP = KleshchevPartitions(3, convention="regular")
sage: KP([5,4,4,3,2]).conormal_cells()
{0: [{1, 4}], 1: [[5, 0], (4, 2)]}
sage: KP([5,4,4,3,2]).conormal_cells(0)
[(1, 4)]
sage: KP([5,4,4,3,2]).conormal_cells(1)
[(5, 0), (4, 2)]
sage: KP = KleshchevPartitions(3, convention="restricted")
sage: KP([5,4,4,3,2]).conormal_cells()
{0: [{1, 4}, (3, 3)], 2: [[0, 5]]}
```

**good_cell_sequence()**

Return a sequence of good nodes from the empty partition to **self**, or None if no such sequence exists.

**EXAMPLES:**

```python
sage: KP = KleshchevPartitions(3, convention='regular')
sage: KP([5,4,4,3,2]).good_cell_sequence()
[(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (3, 1), (0, 3), (1, 3),
 (2, 2), (3, 2), (4, 0), (4, 1), (0, 4), (2, 3)]
sage: KP = KleshchevPartitions(3, convention='restricted')
sage: KP([5,4,4,3,2]).good_cell_sequence()
[(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0),
 (0, 3), (2, 1), (1, 2), (3, 1), (3, 0), (3, 1),
 (2, 2), (4, 0), (2, 3), (3, 2), (0, 4), (4, 1)]
```

**good_cells**(i=None)

Return a list of the cells of **self** that are good.

The good i-cell is the ‘first’ normal i-cell. As with the normal cells we can choose to read either up or down the partition as specified by **convention()**.

**INPUT:**
• i – (optional) a residue

OUTPUT:

If no residue $i$ is specified then a dictionary of good cells is returned, which gives the good cells for $0 \leq i < e$.

EXAMPLES:

```python
sage: KP3 = KleshchevPartitions(3, convention='regular')
sage: KP3([5,4,4,3,2]).good_cells()
{1: (2, 3)}
sage: KP3([5,4,4,3,2]).good_cells(1)
(2, 3)
sage: KP4 = KleshchevPartitions(4, convention='restricted')
sage: KP4([5,4,4,3,2]).good_cells()
{1: (2, 3)}
sage: KP4([5,4,4,3,2]).good_cells(0)
sage: KP4([5,4,4,3,2]).good_cells(1)
(2, 3)
```

good_residue_sequence()

Return a sequence of good nodes from the empty partition to self, or None if no such sequence exists.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(3, convention='regular')
sage: KP([5,4,4,3,2]).good_residue_sequence()
[0, 2, 1, 1, 0, 2, 0, 2, 1, 1, 0, 2, 0, 2, 2, 0, 1, 1]
sage: KP = KleshchevPartitions(3, convention='restricted')
sage: KP([5,4,4,3,2]).good_residue_sequence()
[0, 1, 2, 2, 0, 1, 0, 2, 1, 2, 0, 1, 0, 2, 1, 2, 1, 0]
```

is_regular()

Return True if self is a $e$-regular partition tuple.

A partition tuple is $e$-regular if we can get to the empty partition tuple by successively removing a sequence of good cells in the down direction. Equivalently, all partitions are 0-regular and if $e > 0$ then a partition is $e$-regular if no $e$ non-zero parts of self are equal.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(2)
sage: KP([2,1,1]).is_regular()  
False
sage: KP = KleshchevPartitions(3)
sage: KP([2,1,1]).is_regular()  
True
sage: KP([]).is_regular()  
True
```

is_restricted()

Return True if self is an $e$-restricted partition tuple.

A partition tuple is $e$-restricted if we can get to the empty partition tuple by successively removing a sequence of good cells in the up direction. Equivalently, all partitions are 0-restricted and if $e > 0$ then a partition is $e$-restricted if the difference of successive parts of self are always strictly less than $e$. 

5.1. Comprehensive Module List
EXAMPLES:

```python
sage: KP = KleshchevPartitions(2, convention='regular')
sage: KP([3,1]).is_restricted()
False
sage: KP = KleshchevPartitions(3, convention='regular')
sage: KP([3,1]).is_restricted()
True
sage: KP([]).is_restricted()
True
```

```
mullineux_conjugate()

Return the partition tuple that is the Mullineux conjugate of self.

It follows from results in [BK2009], [Mat2015] that if $\nu$ is the Mullineux conjugate of the Kleshchev partition tuple $\mu$ then the simple module $D^\nu = (D^\mu)^{sgn}$ is obtained from $D^\mu$ by twisting by the sgn-automorphism with is the Iwahori-Hecke algebra analogue of tensoring with the one dimensional sign representation.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(3, convention='regular')
sage: KP([5,4,4,3,2]).mullineux_conjugate()
[9, 7, 1, 1]
sage: KP = KleshchevPartitions(3, convention='restricted')
sage: KP([5,4,4,3,2]).mullineux_conjugate()
[3, 2, 2, 2, 2, 2, 1, 1, 1]
sage: KP = KleshchevPartitions(3, [2], convention='regular')
sage: mc = KP([5,4,4,3,2]).mullineux_conjugate(); mc
[9, 7, 1, 1]
sage: mc.parent().multicharge()
(1,)
sage: KP = KleshchevPartitions(3, [2], convention='restricted')
sage: mc = KP([5,4,4,3,2]).mullineux_conjugate(); mc
[3, 2, 2, 2, 2, 2, 1, 1, 1]
sage: mc.parent().multicharge()
(1,)
```

```
normal_cells(i=None)

Return a dictionary of the cells of the partition that are normal.

Following [Kle1995], the normal cells are computed by reading up (or down) the rows of the partition and marking all of the addable and removable cells of $e$-residue $i$ and then recursively removing all adjacent pairs of removable and addable cells (in that order) from this list. The removable $i$-cells that remain at the end of the this process are the normal $i$-cells.

When computing normal cells you can either read the cells in order from top to bottom (this corresponds to labeling the simple modules of the symmetric group by regular partitions) or from bottom to top (corresponding to labeling the simples by restricted partitions). By default we read down the partition but this can be changed by setting convention = 'RS'.

INPUT:

- $i$ – (optional) a residue

OUTPUT:
If no residue $i$ is specified then a dictionary of normal cells is returned, which gives the normal cells for $0 \leq i < e$.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(3, convention='regular')
sage: KP([5,4,4,3,2]).normal_cells()
{1: [(2, 3), (0, 4)]}
sage: KP([5,4,4,3,2]).normal_cells(1)
[(2, 3), (0, 4)]
sage: KP = KleshchevPartitions(3, convention='restricted')
sage: KP([5,4,4,3,2]).normal_cells()
{0: [(4, 1)], 2: [(3, 2)]}
sage: KP([5,4,4,3,2]).normal_cells(2)
[(3, 2)]
```

```python
class sage.combinat.partition_kleshchev.KleshchevPartitionCrystal(
    parent, mu)
Bases: KleshchevPartition, KleshchevCrystalMixin
Kleshchev partition with the crystal structure.

\( e(i) \)

Return the action of \( e_i \) on self.

INPUT:

- \( i \) – an element of the index set

EXAMPLES:

```python
sage: C = crystals.KleshchevPartitions(3, convention="left regular")
sage: x = C([5,4,1])
sage: x.e(0)
[5, 4]
sage: x.e(1)
[5, 5, 1]
sage: x.e(2)
[5, 4, 2]
```

\( f(i) \)

Return the action of \( f_i \) on self.

INPUT:

- \( i \) – an element of the index set

EXAMPLES:

```python
sage: C = crystals.KleshchevPartitions(3, convention="left regular")
sage: x = C([5,4,1])
sage: x.f(0)
[5, 5, 1]
sage: x.f(1)
[5, 4, 2]
sage: x.f(2)
[5, 4, 2]
```

```python
class sage.combinat.partition_kleshchev.KleshchevPartitionTuple(
    parent, mu)
Bases: PartitionTuple
Abstract base class for Kleshchev partition tuples. See KleshchevPartitions.
```
**cogood_cells**\( (i=\text{None}) \)

Return a list of the cells of the partition that are cogood.

The cogood \( i \)-cell is the ‘last’ conormal \( i \)-cell. As with the conormal cells we can choose to read either up or down the partition as specified by \texttt{convention()}.  

**INPUT:**

- \( i \) – (optional) a residue

**OUTPUT:**

If no residue \( i \) is specified then a dictionary of cogood cells is returned, which gives the cogood cells for \( 0 \leq i < e \).

**EXAMPLES:**

```
sage: KP = KleshchevPartitions(3, [0,1])
sage: pt = KP([[4, 2], [5, 3, 1]])
sage: pt.cogood_cells()
{0: (1, 2, 1), 1: (1, 3, 0)}
sage: pt.cogood_cells(0)
(1, 2, 1)
sage: KP = KleshchevPartitions(4, [0,1], convention="left regular")
sage: pt = KP([[5, 2, 2], [6, 1, 1]])
sage: pt.cogood_cells()
{1: (0, 0, 5), 2: (1, 3, 0)
```

**conormal_cells**\( (i=\text{None}) \)

Return a dictionary of the cells of the partition that are conormal.

Following [Kle1995], the conormal cells are computed by reading up (or down) the rows of the partition and marking all of the addable and removable cells of \( e \)-residue \( i \) and then recursively removing all adjacent pairs of removable and addable cells (in that order) from this list. The addable \( i \)-cells that remain at the end of the this process are the conormal \( i \)-cells.

When computing conormal cells you can either read the cells in order from top to bottom (this corresponds to labeling the simple modules of the symmetric group by regular partitions) or from bottom to top (corresponding to labeling the simples by restricted partitions). By default we read down the partition but this can be changed by setting \texttt{convention = 'RS'}.  

**INPUT:**

- \( i \) – (optional) a residue

**OUTPUT:**

If no residue \( i \) is specified then a dictionary of conormal cells is returned, which gives the conormal cells for \( 0 \leq i < e \).

**EXAMPLES:**

```
sage: KP = KleshchevPartitions(3, [0,1])
sage: pt = KP([[4, 2], [5, 3, 1]])
sage: pt.conormal_cells()
{0: [(1, 2, 1), (1, 1, 3), (1, 0, 5)],
1: [(1, 3, 0), (0, 2, 0), (0, 1, 2), (0, 0, 4)]
```
sage: KP([[4, 2], [5, 3, 1]]).conormal_cells(1)
[(1, 3, 0), (0, 2, 0), (0, 1, 2), (0, 0, 4)]
sage: KP([[4, 2], [5, 3, 1]]).conormal_cells(2)
[]
sage: KP = KleshchevPartitions(3, [0,1], convention="right restricted")
sage: KP([[4, 2], [5, 3, 1]]).conormal_cells(0)
[(1, 0, 5), (1, 1, 3), (1, 2, 1)]

good_cell_sequence()

Return a sequence of good nodes from the empty partition to self.

EXAMPLES:

sage: KP = KleshchevPartitions(3,[0,1])
sage: pt = KP([[4, 2], [5, 3, 1]])
sage: pt.good_cells()
{2: (1, 0, 4)}
sage: pt.good_cells(2)
(1, 0, 4)
sage: KP = KleshchevPartitions(4, [0,1], convention="left regular")
sage: pt = KP([[5, 2, 2], [6, 2, 1]])
sage: pt.good_cells()
{0: (0, 0, 4), 2: (1, 0, 5), 3: (0, 2, 1)}
sage: pt.good_cells(1) is None
True

good_cells(i=None)

Return a list of the cells of the partition tuple which are good.

The good i-cell is the ‘first’ normal i-cell. As with the normal cells we can choose to read either up or down the partition as specified by convention().

INPUT:

• i – (optional) a residue

OUTPUT:

If no residue i is specified then a dictionary of good cells is returned, which gives the good cells for 0 <= i < e.

EXAMPLES:

sage: KP = KleshchevPartitions(3,[0,1])
sage: pt = KP([[4, 2], [5, 3, 1]])
sage: pt.good_cells()
{2: (1, 0, 4)}
sage: pt.good_cells(2)
(1, 0, 4)
sage: KP = KleshchevPartitions(4, [0,1], convention="left regular")
sage: pt = KP([[5, 2, 2], [6, 2, 1]])
sage: pt.good_cells()
{0: (0, 0, 4), 2: (1, 0, 5), 3: (0, 2, 1)}
sage: pt.good_cells(1) is None
True

good_residue_sequence()

Return a sequence of good nodes from the empty partition to self.

EXAMPLES:
is_regular()

Return True if self is a $e$-regular partition tuple.

A partition tuple is $e$-regular if we can get to the empty partition tuple by successively removing a sequence of good cells in the down direction.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(2, [0,2], convention="right restricted")
sage: KP([[3,2,1], [2,1,1]]).is_regular()
False
sage: KP([[3,2,1], [2,1,1]]).is_regular()
True
sage: KP([[], []]).is_regular()
True
```

is_restricted()

Return True if self is an $e$-restricted partition tuple.

A partition tuple is $e$-restricted if we can get to the empty partition tuple by successively removing a sequence of good cells in the up direction.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(2, [0,2], convention="left regular")
sage: KP([[3,2,1], [3,1]]).is_restricted()
False
sage: KP = KleshchevPartitions(3, [0,2], convention="left regular")
sage: KP([[3,2,1], [3,1]]).is_restricted()
True
sage: KP([[], []]).is_restricted()
True
```

mullineux_conjugate()

Return the partition that is the Mullineux conjugate of self.

It follows from results in [Kle1996] [Bru1998] that if $\nu$ is the Mullineux conjugate of the Kleshchev partition tuple $\mu$ then the simple module $D^{\nu} = (D^{\mu})^{sgn}$ is obtained from $D^{\mu}$ by twisting by the $sgn$-automorphism with is the Hecke algebra analogue of tensoring with the one dimensional sign representation.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(3, [0,1])
sage: mc = KP([[4, 2], [5, 3, 1]]).mullineux_conjugate(); mc
([2, 2, 1, 1], [3, 2, 2, 1, 1])
sage: mc.parent()
Kleshchev partitions with e=3 and multicharge=(0,2)
```

normal_cells(i=None)

Return a dictionary of the removable cells of the partition that are normal.
Following [Kle1995], the normal cells are computed by reading up (or down) the rows of the partition and marking all of the addable and removable cells of $e$-residue $i$ and then recursively removing all adjacent pairs of removable and addable cells (in that order) from this list. The removable $i$-cells that remain at the end of this process are the normal $i$-cells.

When computing normal cells you can either read the cells in order from top to bottom (this corresponds to labeling the simple modules of the symmetric group by regular partitions) or from bottom to top (corresponding to labeling the simples by restricted partitions). By default we read down the partition but this can be changed by setting convention = 'RS'.

INPUT:

- $i$ – (optional) a residue

OUTPUT:

If no residue $i$ is specified then a dictionary of normal cells is returned, which gives the normal cells for $0 \leq i < e$.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(3, [0,1], convention="left restricted")
sage: KP([[[4, 2], [5, 3, 1]]]).normal_cells()
{2: [[(1, 0, 4), (1, 1, 2), (1, 2, 0)]},
 sage: KP([[[4, 2], [5, 3, 1]]]).normal_cells(1)
{2: [[(1, 0, 4), (1, 1, 2), (1, 2, 0)]},
 sage: KP = KleshchevPartitions(3, [0,1], convention="left regular")
 sage: KP([[4, 2], [5, 3, 1]]).normal_cells()
{0: [[[0, 1, 1), (0, 0, 3)], 2: [(1, 2, 0), (1, 1, 2), (1, 0, 4)]},
 sage: KP = KleshchevPartitions(3, [0,1], convention="right regular")
 sage: KP([[4, 2], [5, 3, 1]]).normal_cells()
{2: [[(1, 2, 0), (1, 1, 2), (1, 0, 4)]},
 sage: KP = KleshchevPartitions(3, [0,1], convention="right restricted")
 sage: KP([[4, 2], [5, 3, 1]]).normal_cells()
{0: [[[0, 0, 3), (0, 1, 1)], 2: [(1, 0, 4), (1, 1, 2), (1, 2, 0)]},
```

class sage.combinat.partition_kleshchev.KleshchevPartitionTupleCrystal(parent, mu)

Bases: KleshchevPartitionTuple, KleshchevCrystalMixin

Kleshchev partition tuple with the crystal structure.

e()

Return the action of $e_i$ on self.

INPUT:

- $i$ – an element of the index set

EXAMPLES:

```python
sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
sage: x = C((5,4,1],[3,2,1,1]))
sage: x.e(0)
([5, 4, 1], [2, 2, 1, 1])
sage: x.e(1)
```

f()

Return the action of $f_i$ on self.

INPUT:
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- $i$ – an element of the index set

**EXAMPLES:**

```
sage: C = crystals.KleshchevPartitions(3, [0,2], convention="left regular")
sage: x = C([[5,4,1],[3,2,1,1]])
sage: x.f(0)
([5, 5, 1], [3, 2, 1, 1])
sage: x.f(1)
([5, 4, 1], [3, 2, 2, 1])
sage: x.f(2)
```

**class** sage.combinat.partition_kleshchev.KleshchevPartitions

**Bases:** PartitionTuples

Kleshchev partitions

A partition (tuple) $\mu$ is Kleshchev if it can be recursively obtained by adding a sequence of good nodes to the empty PartitionTuple of the same level() and multicharge.

There are four different conventions that are used in the literature for Kleshchev partitions, depending on whether we read partitions from top to bottom (regular) or bottom to top (restricted) and whether we read partition tuples from left to right or right to left. All of these conventions are supported:

```
sage: KleshchevPartitions(2, [0,0], size=2, convention='left regular')[:]
[[[1], [1]], [[2], []]]
sage: KleshchevPartitions(2, [0,0], size=2, convention='left restricted')[:]
[[[1], [1]], [[1], [1, 1]]]
sage: KleshchevPartitions(2, [0,0], size=2, convention='right regular')[:]
[[[1], [1]], [[1, 1], [2]]]
sage: KleshchevPartitions(2, [0,0], size=2, convention='right restricted')[:]
[[[1], [1]], [[1, 1], []]]
```

By default, the left restricted convention is used. As a shorthand, LG, LS, RG and RS, respectively, can be used to specify the convention. With the left convention the partition tuples should be ordered with the most dominant partitions in the partition tuple on the left and with the right convention the most dominant partition is on the right.

The KleshchevPartitions class will automatically convert between these four different conventions:

```
sage: KPlg = KleshchevPartitions(2, [0,0], size=2, convention='left regular')
sage: KPls = KleshchevPartitions(2, [0,0], size=2, convention='left restricted')
sage: [KPlg(mu) for mu in KPls]
[[[1], [1]], [[2], []]]
```

**EXAMPLES:**

```
sage: sorted(KleshchevPartitions(5, [3,2,1], 1, convention='RS'))
[[[], [], [1]], [[1], [1], []], [[1], [], []]]
sage: sorted(KleshchevPartitions(5, [3,2,1], 1, convention='LS'))
[[[], [], [1]], [[1], [1], []], [[1], [], []]]
sage: sorted(KleshchevPartitions(5, [3,2,1], 3))
[[[], [], [1, 1, 1]], [[1], [], [2, 1]],
 [[1], [1], [3]], [[1], [1], []]]
```

(continues on next page)
\[
\begin{array}{lll}
()[], [1], [2]), \\
([], [1, 1], [1]), \\
([], [2], [1]), \\
([], [3], []), \\
([1], [], [1, 1]), \\
([1], [], [2]), \\
([1], [1], [1]), \\
([1], [2], []), \\
([1, 1], [], [1]), \\
([1, 1], [], [1]), \\
([1, 1, 1], [], []), \\
([2], [], [1]), \\
([2], [1], []), \\
([3], [], [])
\end{array}
\]

\textbf{sage: } \texttt{sorted(KleshchevPartitions(5, [3,2,1], 3, convention="left regular"))}

\[
\begin{array}{lll}
([], [], [1, 1, 1]), \\
([], [1], [1, 1]), \\
([], [1], [2]), \\
([], [1, 1], [1]), \\
([], [1, 1, 1], []), \\
([1], [], [1, 1]), \\
([1], [1], [1]), \\
([1], [1, 1], []), \\
([1], [2], []), \\
([1, 1], [], [1]), \\
([1, 1], [1], []), \\
([1, 1, 1], [], []), \\
([2], [], [1]), \\
([2], [1], []), \\
([2, 1], [], []), \\
([3], [], []))
\end{array}
\]

\textbf{REFERENCES:}

- [AM2000]
- [Ariki2001]
- [BK2009]
- [Kle2009]

\textbf{convention()}

Return the convention of \texttt{self}.

\textbf{EXAMPLES:}

\textbf{sage: } \texttt{KP = KleshchevPartitions(4)}
\textbf{sage: } \texttt{KP.convention()}
'restricted'
\textbf{sage: } \texttt{KP = KleshchevPartitions(6, [4], 3, convention="right regular")}
\textbf{sage: } \texttt{KP.convention()}
'regular'
\textbf{sage: } \texttt{KP = KleshchevPartitions(5, [3,0,1], 1)}
\textbf{sage: } \texttt{KP.convention()}
'left restricted'
\textbf{sage: } \texttt{KP = KleshchevPartitions(5, [3,0,1], 1, convention='right regular')}
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(continued from previous page)

```python
sage: KP.convention()
'right regular'
```

**multicharge()**

Return the multicharge of self.

EXAMPLES:

```python
sage: KP = KleshchevPartitions(6, [2])
sage: KP.multicharge()
(2,)
sage: KP = KleshchevPartitions(5, [3,0,1], 1, convention='LS')
sage: KP.multicharge()
(3, 0, 1)
```

class sage.combinat.partition_kleshchev.KleshchevPartitions_all(e, multicharge, convention)

Bases: KleshchevPartitions

Class of all Kleshchev partitions.

**Crystal structure**

We consider type $A_{e-1}^{(1)}$ crystals, and let $r = (r_i | r_i \in \mathbb{Z}/e\mathbb{Z})$ be a finite sequence of length $k$, which is the level, and $\lambda = \sum_i \lambda_i r_i$. We will model the highest weight $U_q(g)$-crystal $B(\lambda)$ by a particular subset of partition tuples of level $k$.

Consider a partition tuple $\mu$ with multicharge $r$. We define $e_i(\mu)$ as the partition tuple obtained after the deletion of the $i$-good cell to $\mu$ and $0$ if there is no $i$-good cell. We define $f_i(\mu)$ as the partition tuple obtained by the addition of the $i$-cogood cell to $\mu$ and $0$ if there is no $i$-good cell.

The crystal $B(\lambda)$ is the crystal generated by the empty partition tuple. We can compute the weight of an element $\mu$ by taking $\lambda - \sum_{i=0}^{n} c_i \alpha_i$ where $c_i$ is the number of cells of $n$-residue $i$ in $\mu$. Partition tuples in the crystal are known as Kleshchev partitions.

**Note:** We can describe normal (not restricted) Kleshchev partition tuples in $B(\lambda)$ as partition tuples $\mu$ such that $\mu_{r_t-r_{t+1}+x}^{(t)} < \mu_{x}^{(t+1)}$ for all $x \geq 1$ and $1 \leq t \leq k-1$.

**INPUT:**

- $e$ – for type $A_{e-1}^{(1)}$ or $0$
- multicharge – the multicharge sequence $r$
- convention – (default: 'LS') the reading convention

**EXAMPLES:**

We first do an example of a level 1 crystal:

```python
sage: C = crystals.KleshchevPartitions(3, [0], convention="left restricted")
sage: C
Kleshchev partitions with e=3
sage: mg = C.highest_weight_vector()
sage: mg
```

(continues on next page)
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Now a higher level crystal:

```python
sage: C = crystals.KleshchevPartitions(3, [0,2], convention="right restricted")
sage: mg = C.highest_weight_vector()
sage: mg
([], [])
sage: mg.f(0)
([1], [])
sage: mg.f(2)
([], [1])
sage: mg.f_string([0,1,2,0])
([2, 2], [])
sage: mg.f_string([0,2,1,0])
([1, 1, 1], [1])
sage: mg.f_string([2,0,1,0])
([2, 1], [1])
sage: GC = C.subcrystal(max_depth=5).digraph()
sage: GB = B.subcrystal(max_depth=5).digraph()
sage: GC.is_isomorphic(GB, edge_labels=True)
True
```

The ordering of the residues gives a different representation of the higher level crystals (but it is still isomorphic):

```python
sage: C2 = crystals.KleshchevPartitions(3, [2,0], convention="right restricted")
sage: mg2 = C2.highest_weight_vector()
sage: mg2.f_string([0,1,2,0])
([2], [2])
sage: mg2.f_string([0,2,1,0])
([1, 1, 1], [1])
sage: mg2.f_string([2,0,1,0])
([2, 1], [1])
sage: GC2 = C2.subcrystal(max_depth=5).digraph()
sage: GC.is_isomorphic(GC2, edge_labels=True)
True
```

REFERENCES:

- [Ariki1996]
class sage.combinat.partition_kleshchev.KleshchevPartitions_size(e, multicharge=(0,), size=0, convention='RS')

Bases: KleshchevPartitions

Kleshchev partitions of a fixed size.

Element

alias of KleshchevPartitionTuple

5.1.168 Partition Shifting Algebras

This module contains families of operators that act on partitions or, more generally, integer sequences. In particular, this includes Young’s raising operators $R_{ij}$, which act on integer sequences by adding 1 to the $i$-th entry and subtracting 1 to the $j$-th entry. A special case is acting on partitions.

AUTHORS:

• Matthew Lancellotti, George H. Seelinger (2018): Initial version

class sage.combinat.partition_shifting_algebras.ShiftingOperatorAlgebra(base_ring=Univariate Polynomial Ring in t over Rational Field, prefix='S')

Bases: CombinatorialFreeModule

An algebra of shifting operators.

Let $R$ be a commutative ring. The algebra of shifting operators is isomorphic as an $R$-algebra to the Laurent polynomial ring $R[x_1^\pm, x_2^\pm, x_3^\pm, \ldots]$. Moreover, the monomials of the shifting operator algebra act on any integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ as follows. Let $S$ be our algebra of shifting operators. Then, for any monomial $s = x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r} \in S$ where $a_i \in \mathbb{Z}$ and $r \geq \ell$, we get that $s \cdot \lambda = (\lambda_1 + a_1, \lambda_2 + a_2, \ldots, \lambda_r + a_r)$ where we pad $\lambda$ with $r - \ell$ zeros. In particular, we can recover Young’s raising operator, $R_{ij}$, for $i < j$, acting on partitions by having $x_i^{-1}$ act on a partition $\lambda$.

One can extend the action of these shifting operators to a basis of symmetric functions, but at the expense of no longer actually having a well-defined operator. Formally, to extend the action of the shifting operators on a symmetric function basis $B = \{ b_\lambda \}_\lambda$, we define an $R$-module homomorphism $\phi : R[x_1^\pm, x_2^\pm, \ldots] \to B$. Then we compute $x_1^{a_1}\cdots x_r^{a_r}.b_\lambda$ by first computing $(x_1^{a_1}\cdots x_r^{a_r})x_1^{\lambda_1}\cdots x_\ell^{\lambda_\ell}$ and then applying $\phi$ to the result. For examples of what this looks like with specific bases, see below.

This implementation is consistent with how many references work formally with raising operators. For instance, see exposition surrounding [BMPS2018] Equation (4.1).

We follow the following convention for creating elements: $S(1, 0, -1, 2)$ is the shifting operator that raises the first part by 1, lowers the third part by 1, and raises the fourth part by 2.

In addition to acting on partitions (or any integer sequence), the shifting operators can also act on symmetric functions in a basis $B$ when a conversion to $B$ has been registered, preferably using build_and_register_conversion().

For a definition of raising operators, see [BMPS2018] Definition 2.1. See $ij()$ to create operators using the notation in [BMPS2018].
INPUT:

- base_ring – (default: \(\mathbb{Q}[t]\)) the base ring
- prefix – (default: "S") the label for the shifting operators

EXAMPLES:

```python
sage: S = ShiftingOperatorAlgebra()
sage: elm = S[1, -1, 2]; elm
S(1, -1, 2)
sage: elm([5, 4])
[[[6, 3, 2], 1]]
```

The shifting operator monomials can act on a complete homogeneous symmetric function or a Schur function:

```python
sage: s = SymmetricFunctions(QQ['t']).s()
sage: h = SymmetricFunctions(QQ['t']).h()
sage: elm(s[5, 4])
s[6, 3, 2]
sage: elm(h[5, 4])
h[6, 3, 2]
sage: S[1, -1](s[5, 4])
s[6, 3]
sage: S[1, -1](h[5, 4])
h[6]
```

In fact, we can extend this action by linearity:

```python
sage: elm = (1 - S[1,-1]) * (1 - S[4])
sage: elm == S([]) - S([1, -1]) - S([4]) + S([5, -1])
True
sage: elm(s[2, 2, 1])
s[2, 2, 1] - s[3, 1, 1] - s[6, 2, 1] + s[7, 1, 1]
sage: elm = (1 - S[1,-1]) * (1 - S[0,1,-1])
sage: elm == 1 - S[0,1,-1] - S[1,-1] + S[1,0,-1]
True
sage: elm(s[2, 2, 1])
s[2, 2, 1] - s[3, 1, 1] + s[3, 2]
```

The algebra also comes equipped with homomorphisms to various symmetric function bases; these homomorphisms are how the action of \(S\) on the specific symmetric function bases is implemented:

```python
sage: elm = S([3,1,2]); elm
S(3, 1, 2)
sage: h(elm)
h[3, 2, 1]
sage: s(elm)
0
```

However, not all homomorphisms are equivalent, so the action is basis dependent:
We can also use raising operators to implement the Jacobi-Trudi identity:

```
sage: op = (1-S[(1,-1)]) * (1-S[(1,0,-1)]) * (1-S[(0,1,-1)])
sage: s(op(h[3,2,1]))
s[3, 2, 1]
```

class Element

Bases: IndexedFreeModuleElement

An element of a ShiftingOperatorAlgebra.

**build_and_register_conversion**(*support_map, codomain*)

Build a module homomorphism from a map sending integer sequences to codomain and registers the result into Sage’s conversion model.

The intended use is to define a morphism from self to a basis $B$ of symmetric functions that will be used by ShiftingOperatorAlgebra to define the action of the operators on $B$.

**Note:** The actions on the complete homogeneous symmetric functions and on the Schur functions by morphisms are already registered.

**Warning:** Because ShiftingOperatorAlgebra inherits from UniqueRepresentation, once you register a conversion, this will apply to all instances of ShiftingOperatorAlgebra over the same base ring with the same prefix.

**INPUT:**

- support_map – a map from integer sequences to codomain
- codomain – the codomain of support_map, usually a basis of symmetric functions

**EXAMPLES:**

```
sage: S = ShiftingOperatorAlgebra(QQ)
sage: sym = SymmetricFunctions(QQ)
sage: p = sym.p()
sage: zero_map = lambda part: p.zero()
sage: S.build_and_register_conversion(zero_map, p)
sage: p(2*S([1,0,-1]) + S([2,1,0]) - 3*S([0,1,3]))
0
sage: op = S((1, -1))
True
```
For a more illustrative example, we can implement a simple (but not mathematically justified!) conversion on the monomial basis:

```python
sage: S = ShiftingOperatorAlgebra(QQ)
sage: sym = SymmetricFunctions(QQ)
sage: m = sym.m()
sage: def supp_map(gamma):
    ...:     gsort = sorted(gamma, reverse=True)
    ...:     return m(gsort) if gsort in Partitions() else m.zero()
sage: S.build_and_register_conversion(supp_map, m)
sage: op = S.ij(0, 1)
sage: op(2*m[4,3] + 5*m[2,2] + 7*m[2]) == 2*m[5, 2] + 5*m[3, 1]
True
```

**ij**(i, j)

Return the raising operator $R_{ij}$ as notated in [BMPS2018] Definition 2.1.

Shorthand element constructor that allows you to create raising operators using the familiar $R_{ij}$ notation found in [BMPS2018] Definition 2.1, with the exception that indices here are 0-based, not 1-based.

**EXAMPLES:**

Create the raising operator which raises part 0 and lowers part 2 (indices are 0-based):

```python
sage: R = ShiftingOperatorAlgebra()
sage: R.ij(0, 2)
S(1, 0, -1)
```

**one_basis()**

Return the index of the basis element for 1.

**EXAMPLES:**

```python
sage: S = ShiftingOperatorAlgebra()
sage: S.one_basis()
()  # Identity
```

**product_on_basis**(x, y)

Return the product of basis elements indexed by x and y.

**EXAMPLES:**

```python
sage: S = ShiftingOperatorAlgebra()
sage: S.product_on_basis((0, 5, 2), (3, 2, -2, 5))
S(3, 7, 0, 5)
sage: S.product_on_basis((1, -2, 0, 3, -6), (-1, 2, 2))
S(0, 0, 2, 3, -6)
sage: S.product_on_basis((1, -2, -2), (-1, 2, 2))
S()
```

**class** sage.combinat.partition_shifting_algebras.ShiftingSequenceSpace

Bases: Singleton, Parent

A helper for ShiftingOperatorAlgebra that contains all tuples with entries in $\mathbb{Z}$ of finite support with no trailing 0's.

**EXAMPLES:**
```python
sage: from sage.combinat.partition_shifting_algebras import ShiftingSequenceSpace
sage: S = ShiftingSequenceSpace()
sage: (1, -1) in S
True
sage: (1, -1, 0, 9) in S
True
sage: [1, -1] in S
False
sage: (0.5, 1) in S
False
```

```python
check(seq)
```

Verify that `seq` is a valid shifting sequence.

If it is not, raise a `ValueError`.

**EXAMPLES:**

```python
sage: from sage.combinat.partition_shifting_algebras import ShiftingSequenceSpace
sage: S = ShiftingSequenceSpace()
sage: S.check((1, -1))
sage: S.check((1, -1, 0, 9))
sage: S.check([1, -1])
Traceback (most recent call last):
  ... ValueError: invalid index [1, -1]
sage: S.check((0.5, 1))
Traceback (most recent call last):
  ... ValueError: invalid index (0.500000000000000, 1)
```

## 5.1.169 Partition tuples

A *PartitionTuple* is a tuple of partitions. That is, an ordered \(k\)-tuple of partitions \(\mu = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})\). If

\[
 n = |\mu| = |\mu^{(1)}| + |\mu^{(2)}| + \cdots + |\mu^{(k)}|
\]

then we say that \(\mu\) is a \(k\)-partition of \(n\).

In representation theory partition tuples arise as the natural indexing set for the ordinary irreducible representations of:

- the wreath products of cyclic groups with symmetric groups,
- the Ariki-Koike algebras, or the cyclotomic Hecke algebras of the complex reflection groups of type \(G(r, 1, n)\),
- the degenerate cyclotomic Hecke algebras of type \(G(r, 1, n)\).

When these algebras are not semisimple, partition tuples index an important class of modules for the algebras, which are generalisations of the Specht modules of the symmetric groups.

Tuples of partitions also index the standard basis of the higher level combinatorial Fock spaces. As a consequence, the combinatorics of partition tuples encapsulates the canonical bases of crystal graphs for the irreducible integrable highest weight modules of the (quantized) affine special linear groups and the (quantized) affine general linear groups. By the categorification theorems of Ariki, Varagnolo-Vasserot, Stroppel-Webster and others, in characteristic zero the
degenerate and non-degenerate cyclotomic Hecke algebras, via their Khovanov-Lauda-Rouquier grading, categorify the canonical bases of the quantum affine special and general linear groups.

Partitions are naturally in bijection with 1-tuples of partitions. Most of the combinatorial operations defined on partitions extend to partition tuples in a meaningful way. For example, the semisimple branching rules for the Specht modules are described by adding and removing cells from partition tuples and the modular branching rules correspond to adding and removing good and cogood nodes, which is the underlying combinatorics for the associated crystal graphs.

A `PartitionTuple` belongs to `PartitionTuples` and its derived classes. `PartitionTuples` is the parent class for all partitions tuples. Four different classes of tuples of partitions are currently supported:

- `PartitionTuples(level=k,size=n)` are $k$-tuple of partitions of $n$.
- `PartitionTuples(level=k)` are $k$-tuple of partitions.
- `PartitionTuples(size=n)` are tuples of partitions of $n$.
- `PartitionTuples()` are tuples of partitions.

**Note:** As with `Partitions`, in sage the cells, or nodes, of partition tuples are 0-based. For example, the (lexicographically) first cell in any non-empty partition tuple is $[0,0,0]$.

**EXAMPLES:**

```python
sage: PartitionTuple([[2,2],[1,1],[2]]).cells()
[(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 1, 0), (2, 0, 0), (2, 0, 1)]
```

**Note:** Many `PartitionTuple` methods take the individual coordinates $(k,r,c)$ as their arguments, here $k$ is the component, $r$ is the row index and $c$ is the column index. If your coordinates are in the form $(k,r,c)$ then use Python’s *-operator.

**EXAMPLES:**

```python
sage: mu=PartitionTuple([[1,1],[2],[2,1]])
sage: [ mu.arm_length(*c) for c in mu.cells()]
[0, 0, 1, 0, 1, 0, 0]
```

**Warning:** In sage, if mu is a partition tuple then mu[k] most naturally refers to the $k$-th component of mu, so we use the convention of the $(k,r,c)$-th cell in a partition tuple refers to the cell in component $k$, row $r$, and column $c$. In the literature, the cells of a partition tuple are usually written in the form $(r,c,k)$, where $r$ is the row index, $c$ is the column index, and $k$ is the component index.

**REFERENCES:**

- [DJM1998]
- [BK2009]

**AUTHORS:**


**EXAMPLES:**

First is a finite enumerated set and the remaining classes are infinite enumerated sets:
One tuples of partitions are naturally in bijection with partitions and, as far as possible, partition tuples attempts to identify one tuples with partitions:

```
sage: Partition([4,3]) == PartitionTuple([[4,3]])
True
sage: Partition([4,3]) == PartitionTuple([4,3])
True
sage: PartitionTuple([4,3])
[4, 3]
sage: Partition([4,3]) in PartitionTuples()
True
```

Partition tuples come equipped with many of the corresponding methods for partitions. For example, it is possible to add and remove cells, to conjugate partition tuples, to work with their diagrams, compare partition tuples in dominance and so:

```
sage: PartitionTuple([[4,1],[[],[2,2,1],[3]]]).pp()
    **** _ ** ***
    *  **
    *
sage: PartitionTuple([[4,1],[[],[2,2,1],[3]]]).conjugate()
([1, 1, 1], [3, 2], [], [2, 1, 1, 1])
sage: PartitionTuple([[4,1],[[],[2,2,1],[3]]]).conjugate().pp()
```

(continues on next page)
sage: lam=PartitionTuples(3)([[3,2],[],[1,1,1,1]]); lam
([3, 2], [], [1, 1, 1, 1])
sage: lam.level()
3
sage: lam.size()
9
sage: lam.category()
Category of elements of Partition tuples of level 3
sage: lam.parent()
Partition tuples of level 3
sage: lam[0]
[3, 2]
sage: lam[1]
[]
sage: lam[2]
[1, 1, 1, 1]
sage: lam.pp()
*** - **
** *
*

sage: lam.removable_cells()
[(0, 0, 2), (0, 1, 1), (2, 3, 0)]
sage: lam.down_list()
[[[2, 2], [], [1, 1, 1, 1]],
 [[3, 1], [], [1, 1, 1, 1]],
 [[3, 2], [], [1, 1, 1]]]
sage: lam.addable_cells()
[(0, 0, 3), (0, 1, 2), (0, 2, 0), (1, 0, 0), (2, 0, 1), (2, 4, 0)]
sage: lam.up_list()
[[[4, 2], [], [1, 1, 1, 1]],
 [[3, 3], [], [1, 1, 1, 1]],
 [[3, 2, 1], [], [1, 1, 1, 1]],
 [[3, 2], [1], [1, 1, 1, 1]],
 [[3, 2], [], [2, 1, 1, 1]],
 [[3, 2], [], [1, 1, 1, 1]]]
sage: lam.conjugate()
([4], [], [2, 2, 1])
sage: lam.dominates( PartitionTuple([[3],[1],[2,2,1]]) )
False
sage: lam.dominates( PartitionTuple([[3],[2],[1,1,1]]) )
True

Every partition tuple behaves every much like a tuple of partitions:

sage: mu=PartitionTuple([[4,1],[],[2,2,1],[3]])
sage: [ nu for nu in mu ]
[[4, 1], [], [2, 2, 1], [3]]
sage: Set([type(nu) for nu in mu])
{<class 'sage.combinat.partition.Partitions_all_with_category.element_class'>}
sage: mu[2][2]
1
sage: mu[3]
[3]
sage: mu.components()
[[4, 1], [], [2, 2, 1], [3]]
sage: mu.components() == [nu for nu in mu]
True
sage: mu[0]
[4, 1]
sage: mu[1]
[]
sage: mu[2]
[2, 2, 1]
sage: mu[2][0]
2
sage: mu[2][1]
2
sage: mu.level()
4
sage: len(mu)
4
sage: mu.cells()
[(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 1, 0), (2, 0, 0), (2, 0, 1), (2, 1, 0),
(2, 1, 1), (2, 2, 0), (3, 0, 0), (3, 0, 1), (3, 0, 2)]
sage: mu.addable_cells()
[(0, 0, 4), (0, 1, 1), (0, 2, 0), (1, 0, 0), (2, 0, 2), (2, 2, 1), (2, 3, 0), (3, 0, 3),
(3, 1, 0)]
sage: mu.removable_cells()
[(0, 0, 3), (0, 1, 0), (2, 1, 1), (2, 2, 0), (3, 0, 2)]

Attached to a partition tuple is the corresponding Young, or parabolic, subgroup:

sage: mu.young_subgroup()
Permutation Group with generators [((), (12,13), (11,12), (8,9), (6,7), (3,4), (2,3), (1,
(2)],
sage: mu.young_subgroup_generators()
[1, 2, 3, 6, 8, 11, 12]

class sage.combinat.partition_tuple.PartitionTuple(parent, mu)

Bases: CombinatorialElement

A tuple of Partition.

A tuple of partition comes equipped with many of methods available to partitions. The level of the Partition-
Tuple is the length of the tuple.

This is an ordered k-tuple of partitions μ = (μ(1), μ(2), ..., μ(k)). If

\[ n = |μ| = |μ(1)| + |μ(2)| + \cdots + |μ(k)| \]

then μ is a k-partition of n.
In representation theory PartitionTuples arise as the natural indexing set for the ordinary irreducible representations of:

- the wreath products of cyclic groups with symmetric groups
- the Ariki-Koike algebras, or the cyclotomic Hecke algebras of the complex reflection groups of type $G(r, 1, n)$
- the degenerate cyclotomic Hecke algebras of type $G(r, 1, n)$

When these algebras are not semisimple, partition tuples index an important class of modules for the algebras which are generalisations of the Specht modules of the symmetric groups.

Tuples of partitions also index the standard basis of the higher level combinatorial Fock spaces. As a consequence, the combinatorics of partition tuples encapsulates the canonical bases of crystal graphs for the irreducible integrable highest weight modules of the (quantized) affine special linear groups and the (quantized) affine general linear groups. By the categorification theorems of Ariki, Varagnolo-Vasserot, Stroppel-Webster and others, in characteristic zero the degenerate and non-degenerate cyclotomic Hecke algebras, via their Khovanov-Lauda-Rouquier grading, categorify the canonical bases of the quantum affine special and general linear groups.

Partitions are naturally in bijection with 1-tuples of partitions. Most of the combinatorial operations defined on partitions extend to PartitionTuples in a meaningful way. For example, the semisimple branching rules for the Specht modules are described by adding and removing cells from partition tuples and the modular branching rules correspond to adding and removing good and cogood nodes, which is the underlying combinatorics for the associated crystal graphs.

**Warning:** In the literature, the cells of a partition tuple are usually written in the form $(r, c, k)$, where $r$ is the row index, $c$ is the column index, and $k$ is the component index. In sage, if $\mu$ is a partition tuple then $\mu[k]$ most naturally refers to the $k$-th component of $\mu$, so we use the convention of the $(k, r, c)$-th cell in a partition tuple refers to the cell in component $k$, row $r$, and column $c$.

**INPUT:**

Anything which can reasonably be interpreted as a tuple of partitions. That is, a list or tuple of partitions or valid input to `Partition`.

**EXAMPLES:**

```python
sage: mu=PartitionTuple( [[3,2],[2,1],[],[1,1,1,1]] ); mu
([3, 2], [2, 1], [], [1, 1, 1, 1])
sage: nu=PartitionTuple( ([3,2],[2,1],[],[1,1,1,1]) ); nu
([3, 2], [2, 1], [], [1, 1, 1, 1])
sage: mu == nu
True
sage: mu is nu
False
sage: mu in PartitionTuples()
True
sage: mu.parent()
Partition tuples
sage: lam=PartitionTuples(3)([[3,2],[],[1,1,1,1]]); lam
([[3, 2], [], [1, 1, 1, 1]])
sage: lam.level()
3
sage: lam.size()
```

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sage: lam.category()
Category of elements of Partition tuples of level 3
sage: lam.parent()
Partition tuples of level 3
sage: lam[0]
[3, 2]
sage: lam[1]
[]
sage: lam[2]
[1, 1, 1, 1]
sage: lam.pp()
*** _ *
** * *
* *
sage: lam removable_cells()
[(0, 0, 2), (0, 1, 1), (2, 3, 0)]
sage: lam.down_list()
[[[2, 2], [], [1, 1, 1, 1]], [[3, 1], [], [1, 1, 1, 1]], [[3, 2], [], [1, 1, 1, 1]]]
sage: lam.addable_cells()
[(0, 0, 3), (0, 1, 2), (0, 2, 0), (1, 0, 0), (2, 0, 1), (2, 4, 0)]
sage: lam.up_list()
[[[4, 2], [], [1, 1, 1, 1]], [[3, 3], [], [1, 1, 1, 1]], [[3, 2, 1], [], [1, 1, 1, 1]], [[3, 2], [], [1, 1, 1, 1]], [[3, 2], [], [1, 1, 1, 1]], [[3, 2], [], [1, 1, 1, 1]]]
sage: lam.conjugate()
([[4], [], [2, 2, 1]])
sage: lam.dominates(PartitionTuple([[3],[1],[2,2,1]]))
False
sage: lam.dominates(PartitionTuple([[3],[2],[1,1,1]]))
True

See also:

• PartitionTuples
• Partitions

Element

alias of Partition

add_cell(k, r, c)

Return the partition tuple obtained by adding a cell in row r, column c, and component k.

This does not change self.

EXAMPLES:

sage: PartitionTuple([[1,1],[4,3],[2,1,1]]).add_cell(0,0,1)
([2, 1], [4, 3], [2, 1, 1])

addable_cells()

Return a list of the removable cells of this partition tuple.
All indices are of the form \((k, r, c)\), where \(r\) is the row-index, \(c\) is the column index and \(k\) is the component.

**EXAMPLES:**

```sage
PartitionTuple([[1,1],[2],[2,1]]).addable_cells()
```

\[
(0, 0, 1), (0, 2, 0), (1, 0, 2), (1, 1, 0), (2, 0, 2), (2, 1, 1), (2, 2, 0)
\]

```sage
PartitionTuple([[1,1],[4,3],[2,1,1]]).addable_cells()
```

\[
(0, 0, 1), (0, 2, 0), (1, 0, 4), (1, 1, 3), (1, 2, 0), (2, 0, 2), (2, 1, 1), (2, 3, 0)
\]

**arm_length**\((k, r, c)\)

Return the length of the arm of cell \((k, r, c)\) in self.

**INPUT:**

- \(k\) – The component
- \(r\) – The row
- \(c\) – The cell

**OUTPUT:**

- The arm length as an integer

The arm of cell \((k, r, c)\) is the number of cells in the \(k\)-th component which are to the right of the cell in row \(r\) and column \(c\).

**EXAMPLES:**

```sage
PartitionTuple([[],[2,1],[2,2,1],[3]]).arm_length(2,0,0)
```

1

```sage
PartitionTuple([[],[2,1],[2,2,1],[3]]).arm_length(2,0,1)
```

0

```sage
PartitionTuple([[],[2,1],[2,2,1],[3]]).arm_length(2,2,0)
```

0

**block**\((e, multicharge)\)

Return a dictionary \(\beta\) that determines the block associated to the partition self and the quantum_characteristic() \(e\).

**INPUT:**

- \(e\) – the quantum characteristic
- \(multicharge\) – the multicharge (default \((0, )\))

**OUTPUT:**

- a dictionary giving the multiplicities of the residues in the partition tuple self

In more detail, the value \(\beta[i]\) is equal to the number of nodes of residue \(i\). This corresponds to the positive root

\[
\sum_{i \in I} \beta_i \alpha_i \in Q^+,
\]

a element of the positive root lattice of the corresponding Kac-Moody algebra. See [DJM1998] and [BK2009] for more details.

This is a useful statistics because two Specht modules for a cyclotomic Hecke algebra of type \(A\) belong to the same block if and only if they correspond to same element \(\beta\) of the root lattice, given above.
We return a dictionary because when the quantum characteristic is 0, the Cartan type is $A_\infty$, in which case the simple roots are indexed by the integers.

**EXAMPLES:**

```python
sage: PartitionTuple([[2,2],[2,2]]).block(0,(0,0))
{-1: 2, 0: 4, 1: 2}
sage: PartitionTuple([[2,2],[2,2]]).block(2,(0,0))
{0: 4, 1: 4}
sage: PartitionTuple([[2,2],[2,2]]).block(2,(0,1))
{0: 4, 1: 4}
sage: PartitionTuple([[2,2],[2,2]]).block(3,(0,2))
{0: 3, 1: 2, 2: 3}
sage: PartitionTuple([[2,2],[2,2]]).block(3,(0,2))
{0: 3, 1: 2, 2: 3}
sage: PartitionTuple([[2,2],[2,2]]).block(3,(3,2))
{0: 3, 1: 2, 2: 3}
sage: PartitionTuple([[2,2],[2,2]]).block(4,(0,0))
{0: 4, 1: 2, 3: 2}
```

cells()

Return the coordinates of the cells of self. Coordinates are given as (component index, row index, column index) and are 0 based.

**EXAMPLES:**

```python
sage: PartitionTuple([[2,1],[1],[1,1,1]]).cells()
[(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (2, 0, 0), (2, 1, 0), (2, 2, 0)]
```

components()

Return a list containing the shape of this partition.

This function exists in order to give a uniform way of iterating over the "components" of partition tuples of level 1 (partitions) and for higher levels.

**EXAMPLES:**

```python
sage: for t in PartitionTuple([[2,1],[3,2],[3]]).components():
....:    print('%s
' % t.ferrers_diagram())
**
*
***
**
***
sage: for t in PartitionTuple([3,2]).components():
....:    print('%s
' % t.ferrers_diagram())
***
**
```

conjugate()

Return the conjugate partition tuple of self.

The conjugate partition tuple is obtained by reversing the order of the components and then swapping the rows and columns in each component.
EXAMPLES:

```python
sage: PartitionTuple([[2,1],[1],[1,1,1]]).conjugate()
([3], [1], [2, 1])
```

`contains(mu)`

Return `True` if this partition tuple contains $\mu$.

If $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(m)})$ are two partition tuples then $\lambda$ contains $\mu$ if $m \leq l$ and $\mu^{(i)}_r \leq \lambda^{(i)}_r$ for $1 \leq i \leq m$ and $r \geq 0$.

EXAMPLES:

```python
sage: PartitionTuple([[1,1],[2],[2,1]]).contains(PartitionTuple([[1,1],[2],[2,1]]))
True
```

`content(k, r, c, multicharge)`

Return the content of the cell.

Let $m_k = \text{multicharge}[k]$, then the content of a cell is $m_k + c - r$.

If the `multicharge` is a list of integers then it simply offsets the values of the contents in each component. On the other hand, if the `multicharge` belongs to $\mathbb{Z}/e\mathbb{Z}$ then the corresponding $e$-residue is returned (that is, the content mod $e$).

As with the `content` method for partitions, the content of a cell does not technically depend on the partition tuple, but this method is included because it is often useful.

EXAMPLES:

```python
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content(0,1,0, [0,0,0])
-1
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content(0,1,0, [1,0,0])
0
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content(2,1,0, [0,0,0])
-1
```

and now we return the 3-residue of a cell:

```python
sage: multicharge = [IntegerModRing(3)(c) for c in [0,0,0]]
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content(0,1,0, multicharge)
2
```

`content_tableau(multicharge)`

Return the tableau which has $(k,r,c)$th entry equal to the content $\text{multicharge}[k] - r + c$ of this cell.

As with the `content` function, by setting the `multicharge` appropriately the tableau containing the residues is returned.

EXAMPLES:

```python
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content_tableau([0,0,0])
([[0, 1], [-1]], [[0, 1]], [[0], [-1], [-2]])
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content_tableau([0,0,1]).pp()
  0  1  0  1  1
 -1  0
   -1
```
as with the content function the multicharge can be used to return the tableau containing the residues of the cells:

```sage
sage: multicharge=[ IntegerModRing(3)(c) for c in [0,0,1] ]
sage: PartitionTuple([[2,1],[2],[1,1,1]]).content_tableau(multicharge).pp()
0 1 0 1 1
2 0
2
```

**corners()**

Return a list of the removable cells of this partition tuple.

All indices are of the form \((k, r, c)\), where \(r\) is the row-index, \(c\) is the column index and \(k\) is the component.

**EXAMPLES:**

```sage
sage: PartitionTuple([[1,1],[2],[2,1]]).removable_cells()
[(0, 1, 0), (1, 0, 1), (2, 0, 1), (2, 1, 0)]
sage: PartitionTuple([[1,1],[4,3],[2,1,1]]).removable_cells()
[(0, 1, 0), (1, 0, 3), (1, 1, 2), (2, 0, 1), (2, 2, 0)]
```

**defect\((e, multicharge)\)**

Return the \(e\)-defect or the \(e\)-weight of the block containing the partition tuple \(self\).

The \(e\)-defect is the number of (connected) \(e\)-rim hooks that can be removed from the partition.

The defect of a partition tuple is given by

\[
\text{defect}(\beta) = (\Lambda, \beta) - \frac{1}{2}(\beta, \beta),
\]

where \(\Lambda = \sum_r \Lambda_{r, e}\) for the multicharge \((\kappa_1, \ldots, \kappa_e)\) and \(\beta = \sum_{(r,c)} \alpha_{(c-r)} \mod e\), with the sum being over the cells in the partition.

**INPUT:**

- \(e\) – the quantum characteristic
- \(multicharge\) – the multicharge (default \((0, )\))

**OUTPUT:**

- a non-negative integer, which is the defect of the block containing the partition tuple \(self\)

**EXAMPLES:**

```sage
sage: PartitionTuple([[2,2],[2,2]]).defect(0,(0,0))
0
sage: PartitionTuple([[2,2],[2,2]]).defect(2,(0,0))
8
sage: PartitionTuple([[2,2],[2,2]]).defect(2,(0,1))
8
sage: PartitionTuple([[2,2],[2,2]]).defect(2,(0,1))
8
sage: PartitionTuple([[2,2],[2,2]]).defect(3,(0,2))
5
sage: PartitionTuple([[2,2],[2,2]]).defect(3,(0,2))
5
sage: PartitionTuple([[2,2],[2,2]]).defect(3,(3,2))
2
sage: PartitionTuple([[2,2],[2,2]]).defect(4,(0,0))
0
```
\textbf{degree}(\(e\))

Return the \(e\)-th degree of \texttt{self}.

The \(e\)-th degree is the sum of the degrees of the standard tableaux of shape \(\lambda\). The \(e\)-th degree is the exponent of \(\Phi_e(q)\) in the Gram determinant of the Specht module for a semisimple cyclotomic Hecke algebra of type \(A\) with parameter \(q\).

For this calculation the multicharge \((\kappa_1, \ldots, \kappa_l)\) is chosen so that \(\kappa_{r+1} - \kappa_r > n\), where \(n\) is the \texttt{size()} of \(\lambda\) as this ensures that the Hecke algebra is semisimple.

**INPUT:**

- \(e\) – an integer \(e > 1\)

**OUTPUT:**

A non-negative integer.

**EXAMPLES:**

\begin{verbatim}
sage: PartitionTuple([[2,1],[2,2]]).degree(2)
532
sage: PartitionTuple([[2,1],[2,2]]).degree(3)
259
sage: PartitionTuple([[2,1],[2,2]]).degree(4)
196
sage: PartitionTuple([[2,1],[2,2]]).degree(5)
105
sage: PartitionTuple([[2,1],[2,2]]).degree(6)
105
sage: PartitionTuple([[2,1],[2,2]]).degree(7)
0
\end{verbatim}

Therefore, the Gram determinant of \(S(2,1|2,2)\) when the Hecke parameter \(q\) is “generic” is

\[ q^N \Phi_2(q)^{532} \Phi_3(q)^{259} \Phi_4(q)^{196} \Phi_5(q)^{105} \Phi_6(q)^{105} \]

for some integer \(N\). Compare with \texttt{prime_degree()}.

\textbf{diagram()}

Return a string for the Ferrers diagram of \texttt{self}.

**EXAMPLES:**

\begin{verbatim}
sage: print(PartitionTuple([[2,1],[3,2],[1,1,1]]).diagram())
**** *
*** *
* *
sage: print(PartitionTuple([[3,2],[2,1],[1,1,1,1]]).diagram())
*** ** *** *
** * ** *
* *
sage: PartitionTuples.options(convention="french")
sage: print(PartitionTuple([[3,2],[2,1],[1,1,1,1]]).diagram())
* *
** *
* *
* *
\end{verbatim}

(continues on next page)
dominates($\mu$)

Return True if the PartitionTuple dominates or equals $\mu$ and False otherwise.

Given partition tuples $\mu = (\mu^{(1)}, \ldots, \mu^{(m)})$ and $\nu = (\nu^{(1)}, \ldots, \nu^{(n)})$ then $\mu$ dominates $\nu$ if

$$
\sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{r \geq 1} \mu^{(l)} \geq \sum_{k=1}^{l-1} |\nu^{(k)}| + \sum_{r \geq 1} \nu^{(l)}
$$

EXAMPLES:

```
sage: mu=PartitionTuple([[1,1],[2],[2,1]])
sage: nu=PartitionTuple([[1,1],[1,1],[2,1]])
sage: mu.dominates(mu)
True
sage: mu.dominates(nu)
True
sage: nu.dominates(mu)
False
sage: tau=PartitionTuple([[],[2,1],[]])
sage: tau.dominates([[],[2,1],[]])
False
sage: tau.dominates([[],[],[2,1]])
True
```

down()

Generator (iterator) for the partition tuples that are obtained from $\texttt{self}$ by removing a cell.

EXAMPLES:

```
sage: [mu for mu in PartitionTuple([[],[3,1],[1,1]]).down()]
[[[], [2, 1], [1, 1]], ([]), [3], [1, 1]], ([]), [3, 1], [1]]
sage: [mu for mu in PartitionTuple([[],[],[]]).down()]
[]
```

down_list()

Return a list of the partition tuples that can be formed from $\texttt{self}$ by removing a cell.

EXAMPLES:

```
sage: PartitionTuple([[],[3,1],[1,1]]).down_list()
[[[], [2, 1], [1, 1]], ([]), [3], [1, 1]], ([]), [3, 1], [1]]
sage: PartitionTuple([[],[],[]]).down_list()
[]
```

ferrers_diagram()

Return a string for the Ferrers diagram of $\texttt{self}$.

EXAMPLES:
sage: print(PartitionTuple([[2,1],[3,2],[1,1,1]]).diagram())
 ** *** *
 ** ** *

sage: print(PartitionTuple([[3,2],[2,1],[[],[1,1,1,1]]]).diagram())
 *** ** - *
 ** * *

sage: PartitionTuples.options(convention="french")

sage: print(PartitionTuple([[3,2],[2,1],[[],[1,1,1,1]]]).diagram())
 *
 ** *

sage: PartitionTuples.options._reset()

garnir_tableau(*cell)

Return the Garnir tableau of shape self corresponding to the cell cell.

If cell = (k, a, c) then (k, a + 1, c) must belong to the diagram of the PartitionTuple. If this is not the case then we return False.

Note: The function also sets g._garnir_cell equal to cell which is used by some other functions.

The Garnir tableaux play an important role in integral and non-semisimple representation theory because they determine the “straightening” rules for the Specht modules over an arbitrary ring.

The Garnir tableau are the “first” non-standard tableaux which arise when you act by simple transpositions. If (k, a, c) is a cell in the Young diagram of a partition, which is not at the bottom of its column, then the corresponding Garnir tableau has the integers 1, 2, ..., n entered in order from left to right along the rows of the diagram up to the cell (k, a, c − 1), then along the cells (k, a + 1, 1) to (k, a + 1, c), then (k, a, c) until the end of row a and then continuing from left to right in the remaining positions. The examples below probably make this clearer!

EXAMPLES:

sage: PartitionTuple([[5,3],[2,2],[4,3]]).garnir_tableau((0,0,2)).pp()

1 2 6 7 8 9 10 13 14 15 16
3 4 5

sage: PartitionTuple([[5,3,3],[2,2],[4,3]]).garnir_tableau((0,0,2)).pp()

1 2 6 7 8 12 13 16 17 18 19
3 4 5 14 15 20 21 22
9 10 11

sage: PartitionTuple([[5,3,3],[2,2],[4,3]]).garnir_tableau((0,1,2)).pp()

1 2 3 4 5 12 13 16 17 18 19
6 7 11 14 15 20 21 22
8 9 10

sage: PartitionTuple([[5,3,3],[2,2],[4,3]]).garnir_tableau((1,0,0)).pp()

1 2 3 4 5 13 14 16 17 18 19
6 7 8 12 15 20 21 22
9 10 11

sage: PartitionTuple([[5,3,3],[2,2],[4,3]]).garnir_tableau((1,0,1)).pp()

(continues on next page)
sage: PartitionTuple([[5,3,3],[2,2],[4,3]]).garnir_tableau((2,0,1)).pp()
1 2 3 4 5 12 13 16 19 20 21
6 7 8 14 15 17 18 22
9 10 11

sage: PartitionTuple([[5,3,3],[2,2],[4,3]]).garnir_tableau((2,1,1)).pp()
Traceback (most recent call last):
...  
ValueError: (comp, row+1, col) must be inside the diagram

See also:

- top_garnir_tableau()

**hook_length**(*k*, *r*, *c*)

Return the length of the hook of cell (k, r, c) in the partition.

The hook of cell (k, r, c) is defined as the cells to the right or below (in the English convention). If your coordinates are in the form (k, r, c), use Python’s *-operator.

**EXAMPLES:**

```python
sage: mu=PartitionTuple([[1,1],[2],[2,1]])
sage: [ mu.hook_length(*c) for c in mu.cells()]
[2, 1, 2, 1, 3, 1, 1]
```

**initial_column_tableau()**

Return the initial column tableau of shape self.

The initial column tableau of shape \( \lambda \) is the standard tableau that has the numbers 1 to \( n \), where \( n \) is the size() of \( \lambda \), entered in order from top to bottom, and then left to right, down the columns of each component, starting from the rightmost component and working to the left.

**EXAMPLES:**

```python
sage: PartitionTuple([ [3,1],[3,2] ]).initial_column_tableau()
([[6, 8, 9], [7]], [[1, 3, 5], [2, 4]])
```

**initial_tableau()**

Return the **StandardTableauTuple** which has the numbers 1, 2, ..., \( n \), where \( n \) is the size() of self, entered in order from left to right along the rows of each component, where the components are ordered from left to right.

**EXAMPLES:**

```python
sage: PartitionTuple([ [2,1],[3,2] ]).initial_tableau()
([[1, 2], [3]], [[4, 5, 6], [7, 8]])
```

**leg_length**(*k*, *r*, *c*)

Return the length of the leg of cell (k, r, c) in self.

**INPUT:**

- k – The component
• r – The row
• c – The cell

OUTPUT:
• The leg length as an integer

The leg of cell \((k, r, c)\) is the number of cells in the \(k\)-th component which are below the node in row \(r\) and column \(c\).

EXAMPLES:

```sage
sage: PartitionTuple([[],[2,1],[2,2,1],[3]]).leg_length(2,0,0)
2
sage: PartitionTuple([[],[2,1],[2,2,1],[3]]).leg_length(2,0,1)
1
sage: PartitionTuple([[],[2,1],[2,2,1],[3]]).leg_length(2,2,0)
0
```

level()

Return the level of this partition tuple.

The level is the length of the tuple.

EXAMPLES:

```sage
sage: PartitionTuple([[],[2,1,0],[2,1]]).level()
2
sage: PartitionTuple([[],[],[2,1,1]]).level()
3
```

outside_corners()

Return a list of the removable cells of this partition tuple.

All indices are of the form \((k, r, c)\), where \(r\) is the row-index, \(c\) is the column index and \(k\) is the component.

EXAMPLES:

```sage
sage: PartitionTuple([[],[1,1],[2,1,1]]).addable_cells()
[(0, 0, 1), (0, 2, 0), (1, 0, 2), (1, 1, 0), (2, 0, 2), (2, 1, 1), (2, 2, 0)]
sage: PartitionTuple([[],[1,1],[4,3],[2,1,1]]).addable_cells()
[(0, 0, 1), (0, 2, 0), (1, 0, 4), (1, 1, 3), (1, 2, 0), (2, 0, 2), (2, 1, 1), (2, 3, 0)]
```

pp()

Pretty prints this partition tuple. See `diagram()`.

EXAMPLES:

```sage
sage: PartitionTuple([[5,5,2,1],[3,2]]).pp()
*****  ***
*****  **
**
*```
prime_degree($p$)

Return the $p$-th prime degree of self.

The degree of a partition $\lambda$ is the sum of the $e$-degrees of the standard tableaux of shape $\lambda$ (see degree()), for $e$ a power of the prime $p$. The prime degree gives the exponent of $p$ in the Gram determinant of the integral Specht module of the symmetric group.

The $p$-th degree is the sum of the degrees of the standard tableaux of shape $\lambda$. The $p$-th degree is the exponent of $p$ in the Gram determinant of a semisimple cyclotomic Hecke algebra of type $A$ with parameter $q = 1$.

As with degree(), for this calculation the multicharge $(\kappa_1, \ldots, \kappa_l)$ is chosen so that $\kappa_{r+1} - \kappa_r > n$, where $n$ is the size() of $\lambda$ as this ensures that the Hecke algebra is semisimple.

INPUT:
- $e$ – an integer $e > 1$
- multicharge – an $l$-tuple of integers, where $l$ is the level() of self

OUTPUT:
A non-negative integer

EXAMPLES:

```
sage: PartitionTuple([[2,1],[2,2]]).prime_degree(2)
sage: PartitionTuple([[2,1],[2,2]]).prime_degree(3)
sage: PartitionTuple([[2,1],[2,2]]).prime_degree(5)
sage: PartitionTuple([[2,1],[2,2]]).prime_degree(7)
```

Therefore, the Gram determinant of $S(2, 1|2; 2)$ when $q = 1$ is $2^{728}3^{259}5^{105}$. Compare with degree().

removable_cells()

Return a list of the removable cells of this partition tuple.

All indices are of the form ($k$, $r$, $c$), where $r$ is the row-index, $c$ is the column index and $k$ is the component.

EXAMPLES:

```
sage: PartitionTuple([[1,1],[2],[2,1]]).removable_cells()
sage: PartitionTuple([[1,1],[4,3],[2,1,1]]).removable_cells()
```

remove_cell($k$, $r$, $c$)

Return the partition tuple obtained by removing a cell in row $r$, column $c$, and component $k$.

This does not change self.

EXAMPLES:

```
sage: PartitionTuple([[1,1],[4,3],[2,1,1]]).remove_cell(0,1,0)
```
**row_standard_tableaux()**

Return the *row standard tableau tuples* of shape self.

**EXAMPLES:**

```python
sage: PartitionTuple([[],[3,2,2,1],[2,2,1],[3]]).row_standard_tableaux()
Row standard tableau tuples of shape ([], [3, 2, 2, 1], [2, 2, 1], [3])
```

**size()**

Return the size of a partition tuple.

**EXAMPLES:**

```python
sage: PartitionTuple([[2,1],[[],[2,2]]]).size()
7
sage: PartitionTuple([[],[[1],[3,2,1]]]).size()
7
```

**standard_tableaux()**

Return the *standard tableau tuples* of shape self.

**EXAMPLES:**

```python
sage: PartitionTuple([[3,2,2,1],[2,2,1],[3]]).standard_tableaux()
Standard tableau tuples of shape ([3, 2, 2, 1], [2, 2, 1], [3])
```

**to_exp(\(k=0\))**

Return a tuple of the multiplicities of the parts of a partition.

Use the optional parameter \(k\) to get a return list of length at least \(k\).

**EXAMPLES:**

```python
sage: PartitionTuple([[1,1],[2],[2,1]]).to_exp()
([2], [0, 1], [1, 1])
sage: PartitionTuple([[1,1],[2,2,2],[2,1]]).to_exp()
([2], [0, 4], [1, 1])
```

**to_list()**

Return self as a list of lists.

**EXAMPLES:**

```python
sage: PartitionTuple([[1,1],[4,3],[2,1,1]]).to_list()
[[1, 1], [4, 3], [2, 1, 1]]
```

**top_garnir_tableau(\(e, cell\))**

Return the most dominant *standard* tableau which dominates the corresponding Garnir tableau and has the same residue that has shape self and is determined by \(e\) and \(cell\).

The Garnir tableau play an important role in integral and non-semisimple representation theory because they determine the “straightening” rules for the Specht modules over an arbitrary ring. The *top Garnir tableaux* arise in the graded representation theory of the symmetric groups and higher level Hecke algebras. They were introduced in [KMR2012].

If the Garnir node is \(cell=(k,r,c)\) and \(m\) and \(M\) are the entries in the cells \((k,r,c)\) and \((k,r+1, c)\), respectively, in the initial tableau then the top \(e\)-Garnir tableau is obtained by inserting the numbers \(m, m+1, \ldots, M\) in order from left to right first in the cells in row \(r+1\) which are not in the \(e\)-Garnir belt,
then in the cell in rows \( r \) and \( r+1 \) which are in the Garnir belt and then, finally, in the remaining cells in row \( r \) which are not in the Garnir belt. All other entries in the tableau remain unchanged.

If \( e = 0 \), or if there are no \( e \)-bricks in either row \( r \) or \( r+1 \), then the top Garnir tableau is the corresponding Garnir tableau.

**EXAMPLES:**

```python
sage: PartitionTuple([[3,3,2],[5,4,3,2]]).top_garnir_tableau(2,(1,0,2)).pp()
1 2 3 9 10 12 13 16
4 5 6 11 14 15 17
7 8 18 19 20
21 22

sage: PartitionTuple([[3,3,2],[5,4,3,2]]).top_garnir_tableau(2,(1,0,1)).pp()
1 2 3 9 10 11 12 13
4 5 6 14 15 16 17
7 8 18 19 20
21 22

sage: PartitionTuple([[3,3,2],[5,4,3,2]]).top_garnir_tableau(3,(1,0,1)).pp()
1 2 3 9 12 13 14 15
4 5 6 10 11 16 17
7 8 18 19 20
21 22

sage: PartitionTuple([[3,3,2],[5,4,3,2]]).top_garnir_tableau(3,(3,0,1)).pp()
Traceback (most recent call last):
... ValueError: (comp, row+1, col) must be inside the diagram
```

See also:

- garnir_tableau()

**GENERATOR FUNCTION**

```python
up()
```

Generator (iterator) for the partition tuples that are obtained from `self` by adding a cell.

**EXAMPLES:**

```python
sage: [mu for mu in PartitionTuple([[],[3,1],[1,1]]).up()]
[([], [3, 1], [1, 1]), ([], [4, 1], [1, 1]), ([], [3, 2], [1, 1]), ([], [3, 1, 1], [1, 1]), ([], [3, 1], [2, 1]), ([], [3, 1], [1, 1, 1])]

sage: [mu for mu in PartitionTuple([[],[],[],[]]).up()]
[([], [], [], []), ([], [1], [], []), ([], [], [1], []), ([], [], [], [1])]
```

**RETURN VALUE**

```python
up_list()
```

Return a list of the partition tuples that can be formed from `self` by adding a cell.

**EXAMPLES:**

```python
sage: PartitionTuple([[],[3,1],[1,1]]).up_list()
[([], [3, 1], [1, 1]), ([], [4, 1], [1, 1]), ([], [3, 2], [1, 1]), ([], [3, 1, 1], [1, 1]), ([], [3, 1], [2, 1]), ([], [3, 1], [1, 1, 1])]

sage: PartitionTuple([[],[],[],[]]).up_list()
[([], [], [], []), ([], [1], [], []), ([], [], [1], []), ([], [], [], [1])]
```
young_subgroup()
Return the corresponding Young, or parabolic, subgroup of the symmetric group.

EXAMPLES:

```python
sage: PartitionTuple([[2,1],[4,2],[1]]).young_subgroup()
Permutation Group with generators [(1,2), (4,5), (5,6), (6,7), (8,9)]
```

young_subgroup_generators()
Return an indexing set for the generators of the corresponding Young subgroup.

EXAMPLES:

```python
sage: PartitionTuple([[2,1],[4,2],[1]]).young_subgroup_generators()
[1, 4, 5, 6, 8]
```

class sage.combinat.partition_tuple.PartitionTuples
Bases: UniqueRepresentation, Parent
Class of all partition tuples.
For more information about partition tuples, see PartitionTuple.
This is a factory class which returns the appropriate parent based on the values of level, size, and regular
INPUT:
• level – the length of the tuple
• size – the total number of cells
• regular – a positive integer or a tuple of non-negative integers; if an integer, the highest multiplicity an
  entry may have in a component plus 1

If a level $k$ is specified and regular is a tuple of integers $\ell_1, \ldots, \ell_k$, then this specifies partition tuples $\mu$ such
that $\mu_i$ is $\ell_i$-regular, where 0 here represents $\infty$-regular partitions (equivalently, partitions without restrictions).
If regular is an integer $\ell$, then we set $\ell_i = \ell$ for all $i$.

Element
alias of PartitionTuple

level()
Return the level or None if it is not defined.

EXAMPLES:

```python
sage: PartitionTuples().level() is None
True
sage: PartitionTuples(7).level()
7
```

options = Current options for Partitions - convention: English - diagram_str: * - display: list - latex: young_diagram - latex_diagram_str: \ast

size()
Return the size or None if it is not defined.

EXAMPLES:
class sage.combinat.partition_tuple.PartitionTuples_all
    Bases: PartitionTuples
    
    Class of partition tuples of an arbitrary level and arbitrary sum.

class sage.combinat.partition_tuple.PartitionTuples_level
    (level, category=None)
    Bases: PartitionTuples
    
    Class of partition tuples of a fixed level, but summing to an arbitrary integer.

class sage.combinat.partition_tuple.PartitionTuples_level_size
    (level, size)
    Bases: PartitionTuples
    
    Class of partition tuples with a fixed level and a fixed size.

cardinality()
    Return the number of level-tuples of partitions of size n.
    Wraps a pari function call using pari:eta.

    EXAMPLES:
    
    sage: PartitionTuples(2,3).cardinality()
    10
    sage: PartitionTuples(2,8).cardinality()
    185

class sage.combinat.partition_tuple.PartitionTuples_size
    (size)
    Bases: PartitionTuples
    
    Class of partition tuples of a fixed size, but arbitrary level.

class sage.combinat.partition_tuple.RegularPartitionTuples
    (regular, **kwds)
    Bases: PartitionTuples
    
    Abstract base class for \ell-regular partition tuples.

class sage.combinat.partition_tuple.RegularPartitionTuples_all
    (regular)
    Bases: RegularPartitionTuples
    
    Class of \ell-regular partition tuples.

class sage.combinat.partition_tuple.RegularPartitionTuples_level
    (level, regular)
    Bases: PartitionTuples_level
    
    Regular Partition tuples of a fixed level.

    INPUT:

    * level – a non-negative Integer; the level
    * regular – a positive integer or a tuple of non-negative integers; if an integer, the highest multiplicity
      an entry may have in a component plus 1 with 0 representing \infty-regular (equivalently, partitions without
      restrictions)
regular is a tuple of integers \((\ell_1, \ldots, \ell_k)\) that specifies partition tuples \(\mu\) such that \(\mu_i\) is \(\ell_i\)-regular. If regular is an integer \(\ell\), then we set \(\ell_i = \ell\) for all \(i\).

EXAMPLES:

```
sage: RPT = PartitionTuples(level=4, regular=(2,3,0,2))
sage: RPT[:24]
[([], [], [], []),
 (1, [], [], []),
 (2, [], [], []),
 (1, [], [1], []),
 (2, [], [2], []),
 (1, [1], [], []),
 (2, [2], [], []),
 (1, [1], [1], []),
 (2, [2], [2], []),
 (1, [1], [1], [1]),
 (2, [2], [2], [2]),
 (1, [1], [1], [1]),
 (2, [2], [2], [2]),
 (1, [1], [1], [], []),
 (1, [1, 1], [1], []),
 (1, [1], [1, 1], []),
 (2, [2], [2, 2], []),
 (1, [1, 1], [1, 1], []),
 (2, [2], [2, 2], [2]),
 (1, [1, 1], [1, 1], [1]),
 (2, [2], [2, 2], [2]),
 (1, [1, 1], [1, 1], [], []),
 (1, [1, 1], [1, 1, 1], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], []),
 (1, [1, 1], [1, 1, 1], [], [])]
sage: [[1,1],[3],[5,5,5],[7,2]] in RPT
False
sage: [[1,1],[3],[5,5,5],[7,2]] in RPT
True
sage: [[3,1],[3],[5,5,5]] in RPT
False
```

class sage.combinat.partition_tuple.RegularPartitionTuples_level_size(level, size, regular)

Bases: PartitionTuples_level_size

Class of \(\ell\)-regular partition tuples with a fixed level and a fixed size.

INPUT:

- level – a non-negative Integer; the level
- size – a non-negative Integer; the size
- regular – a positive integer or a tuple of non-negative integers; if an integer, the highest multiplicity an entry may have in a component plus 1 with 0 representing \(\infty\)-regular (equivalently, partitions without restrictions)

regular is a tuple of integers \((\ell_1, \ldots, \ell_k)\) that specifies partition tuples \(\mu\) such that \(\mu_i\) is \(\ell_i\)-regular. If regular is an integer \(\ell\), then we set \(\ell_i = \ell\) for all \(i\).

EXAMPLES:
class sage.combinat.partition_tuple.RegularPartitionTuples_size(size, regular)

Bases: RegularPartitionTuples

Class of $\ell$-regular partition tuples with a fixed size.

5.1.170 Iterators over the partitions of an integer

AUTHOR:

- Jonathan Bober (2007-07-28): wrote the program partitions_c.cc that does all the actual heavy lifting.

sage.combinat.partitions.ZS1_iterator(n)

A fast iterator for the partitions of $n$ (in the decreasing lexicographic order) which returns lists and not objects of type Partition.

This is an implementation of the ZS1 algorithm found in [ZS98].

REFERENCES:

EXAMPLES:

```
sage: from sage.combinat.partitions import ZS1_iterator
sage: it = ZS1_iterator(4)
sage: next(it)
[4]
sage: type(_)
<class 'list'>
```
sage.combinat.partitions.ZS1_iterator_nk(n, k)
An iterator for the partitions of \( n \) of length at most \( k \) (in the decreasing lexicographic order) which returns lists and not objects of type Partition.

The algorithm is a mild variation on \texttt{ZS1_iterator}(); I would not vow for its speed.

EXAMPLES:

```python
sage: from sage.combinat.partitions import ZS1_iterator_nk
sage: it = ZS1_iterator_nk(4, 3)
sage: next(it)
[4]
sage: type(_)
<class 'list'>
```

5.1.171 Perfect matchings

A perfect matching of a set \( S \) is a partition into 2-element sets. If \( S \) is the set \{1, ..., \( n \)\}, it is equivalent to fixpoint-free involutions. These simple combinatorial objects appear in different domains such as combinatorics of orthogonal polynomials and of the hyperoctaedral groups (see [MV], [McD] and also [CM]):

AUTHOR:

• Valentin Feray, 2010: initial version

• Martin Rubey, 2017: inherit from SetPartition, move crossings and nestings to SetPartition

EXAMPLES:

Create a perfect matching:

```python
sage: m = PerfectMatching([('a', 'e'), ('b', 'c'), ('d', 'f')]); m
[('a', 'e'), ('b', 'c'), ('d', 'f')]
```

Count its crossings, if the ground set is totally ordered:

```python
sage: n = PerfectMatching([3, 8, 1, 7, 6, 5, 4, 2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: n.number_of_crossings()
1
```

List the perfect matchings of a given ground set:

```python
sage: PerfectMatchings(4).list()
[[[1, 2], [3, 4]], [[1, 3], [2, 4]], [[1, 4], [2, 3]]
```

REFERENCES:

class sage.combinat.perfect_matching.PerfectMatching(parent, s, check=True, sort=True)

Bases: SetPartition

A perfect matching.

A perfect matching of a set \( X \) is a set partition of \( X \) where all parts have size 2.

A perfect matching can be created from a list of pairs or from a fixed point-free involution as follows:
**Combinatorics, Release 10.0**

```python
sage: m = PerfectMatching([('a', 'e'), ('b', 'c'), ('d', 'f')]); m
[('a', 'e'), ('b', 'c'), ('d', 'f')]
sage: n = PerfectMatching([3, 8, 1, 7, 6, 5, 4, 2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: isinstance(m, PerfectMatching)
True
```

The parent, which is the set of perfect matchings of the ground set, is automatically created:

```python
sage: n.parent()
Perfect matchings of {1, 2, 3, 4, 5, 6, 7, 8}
```

If the ground set is ordered, one can, for example, ask if the matching is non crossing:

```python
sage: PerfectMatching([(1, 4), (2, 3), (5, 6)]).is_noncrossing()
True
```

**Weingarten_function**($d$, other=None)

Return the Weingarten function of two pairings.

This function is the value of some integrals over the orthogonal groups $O_N$. With the convention of [CM], the method returns $W_{O(d)}^d$(other, self).

**EXAMPLES:**

```python
sage: var('N')
N
sage: m = PerfectMatching([(1,3),(2,4)])
```

```python
sage: n = PerfectMatching([(1,2),(3,4)])
```

```python
sage: factor(m.Weingarten_function(N,n))
-1/((N + 2)*(N - 1)*N)
```

**loop_type**($other=None$)

Return the loop type of self.

**INPUT:**

- other – a perfect matching of the same set of self. (if the second argument is empty, the method an_element() is called on the parent of the first)

**OUTPUT:**

If we draw the two perfect matchings simultaneously as edges of a graph, the graph obtained is a union of cycles of even lengths. The function returns the ordered list of the semi-length of these cycles (considered as a partition)

**EXAMPLES:**

```python
sage: m = PerfectMatching([('a', 'e'), ('b', 'c'), ('d', 'f')])
```

```python
sage: n = PerfectMatching([('a', 'b'), ('d', 'f'), ('e', 'c')])
```

```python
sage: m.loop_type(n)
[2, 1]
```

**loops**($other=None$)

Return the loops of self.

**INPUT:**
• other – a perfect matching of the same set of self. (if the second argument is empty, the method
an_element() is called on the parent of the first)

OUTPUT:

If we draw the two perfect matchings simultaneously as edges of a graph, the graph obtained is a union
of cycles of even lengths. The function returns the list of these cycles (each cycle is given as a list).

EXAMPLES:

```
sage: m = PerfectMatching([('a', 'e'), ('b', 'c'), ('d', 'f')])
sage: n = PerfectMatching([('a', 'b'), ('d', 'f'), ('e', 'c')])
sage: loops = m.loops(n)
sage: loops
# random
[['a', 'e', 'c', 'b'], ['d', 'f']]
```

```
sage: o = PerfectMatching([(1, 7), (2, 4), (3, 8), (5, 6)])
sage: p = PerfectMatching([(1, 6), (2, 7), (3, 4), (5, 8)])
sage: o.loops(p)
[[1, 7, 2, 4, 3, 8, 5, 6]]
```

loops_iterator(\textit{other}=None)
Iterate through the loops of self.

INPUT:

• other – a perfect matching of the same set of self. (if the second argument is empty, the method
an_element() is called on the parent of the first)

OUTPUT:

If we draw the two perfect matchings simultaneously as edges of a graph, the graph obtained is a union
of cycles of even lengths. The function returns an iterator for these cycles (each cycle is given as a list).

EXAMPLES:

```
sage: o = PerfectMatching([(1, 7), (2, 4), (3, 8), (5, 6)])
sage: p = PerfectMatching([(1, 6), (2, 7), (3, 4), (5, 8)])
sage: it = o.loops_iterator(p)
sage: next(it)
[1, 7, 2, 4, 3, 8, 5, 6]
sage: next(it)
Traceback (most recent call last):
... StopIteration
```

number_of_loops(\textit{other}=None)
Return the number of loops of self.

INPUT:

• other – a perfect matching of the same set of self. (if the second argument is empty, the method
an_element() is called on the parent of the first)

OUTPUT:

If we draw the two perfect matchings simultaneously as edges of a graph, the graph obtained is a union
of cycles of even lengths. The function returns their numbers.

EXAMPLES:
sage: m = PerfectMatching([('a', 'e'), ('b', 'c'), ('d', 'f')])
sage: n = PerfectMatching([('a', 'b'), ('d', 'f'), ('e', 'c')])
sage: m.number_of_loops(n)
2

partner(x)

Return the element in the same pair than x in the matching self.

EXAMPLES:

sage: m = PerfectMatching([(-3, 1), (2, 4), (-2, 7)])
sage: m.partner(4)
2
sage: n = PerfectMatching([('c', 'b'), ('d', 'f'), ('e', 'a')])
sage: n.partner('c')
'b'

standardization()

Return the standardization of self.

See SetPartition.standardization() for details.

EXAMPLES:

sage: n = PerfectMatching([('c', 'b'), ('d', 'f'), ('e', 'a')])
sage: n.standardization()
[(1, 5), (2, 3), (4, 6)]

to_graph()

Return the graph corresponding to the perfect matching.

OUTPUT:

The realization of self as a graph.

EXAMPLES:

sage: PerfectMatching([[1,3], [4,2]]).to_graph().edges(sort=True, labels=False)
[(1, 3), (2, 4)]
sage: PerfectMatching([[1,4], [3,2]]).to_graph().edges(sort=True, labels=False)
[(1, 4), (2, 3)]
sage: PerfectMatching([]).to_graph().edges(sort=True, labels=False)
[]

to_noncrossing_set_partition()

Return the noncrossing set partition (on half as many elements) corresponding to the perfect matching if the perfect matching is noncrossing, and otherwise gives an error.

OUTPUT:

The realization of self as a noncrossing set partition.

EXAMPLES:

sage: PerfectMatching([[1,3], [4,2]]).to_noncrossing_set_partition()
Traceback (most recent call last):
...
ValueError: matching must be non-crossing

sage: PerfectMatching([[1,4], [3,2]]).to_noncrossing_set_partition()
{{1, 2}}
sage: PerfectMatching([]).to_noncrossing_set_partition()
{{}}

class sage.combinat PerfectMatchings(s)

Bases: SetPartitions_set

Perfect matchings of a ground set.

INPUT:

• s – an iterable of hashable objects or an integer

EXAMPLES:

If the argument s is an integer n, it will be transformed into the set {1, ..., n}:

sage: M = PerfectMatchings(6); M
Perfect matchings of {1, 2, 3, 4, 5, 6}
sage: PerfectMatchings([-1, -3, 1, 2])
Perfect matchings of {1, 2, -3, -1}

One can ask for the list, the cardinality or an element of a set of perfect matching:

sage: PerfectMatchings(4).list()
[[[1, 2], [3, 4]], [[1, 3], [2, 4]], [[1, 4], [2, 3]]]
sage: PerfectMatchings(8).cardinality()
105
sage: M = PerfectMatchings(('a', 'e', 'b', 'f', 'c', 'd'))
sage: x = M.an_element()
sage: x # random
[('a', 'c'), ('b', 'e'), ('d', 'f')]
sage: all(PerfectMatchings(i).an_element() in PerfectMatchings(i)......: for i in range(2,11,2))
True

Element

alias of PerfectMatching

Weingarten_matrix(N)

Return the Weingarten matrix corresponding to the set of PerfectMatchings self.

It is a useful theoretical tool to compute polynomial integrals over the orthogonal group $O_N$ (see [CM]).

EXAMPLES:

sage: M = PerfectMatchings(4).Weingarten_matrix(var('N'))
sage: N*(N-1)*(N+2)*M.apply_map(factor)
[N + 1 -1 -1]
[ -1 N + 1 -1]
[ -1 -1 N + 1]

base_set()

Return the base set of self.
EXAMPLES:

```
sage: PerfectMatchings(3).base_set()
{1, 2, 3}
```

**base_set_cardinality()**

- Return the cardinality of the base set of *self*.

EXAMPLES:

```
sage: PerfectMatchings(3).base_set_cardinality()
3
```

**cardinality()**

- Return the cardinality of the set of perfect matchings *self*.
  - This is \(1 \times 3 \times 5 \times \ldots \times (2n - 1)\), where \(2n\) is the size of the ground set.

EXAMPLES:

```
sage: PerfectMatchings(8).cardinality()
105
sage: PerfectMatchings([1,2,3,4]).cardinality()
3
sage: PerfectMatchings(3).cardinality()
0
sage: PerfectMatchings([]).cardinality()
1
```

**random_element()**

- Return a random element of *self*.

EXAMPLES:

```
sage: M = PerfectMatchings(('a', 'e', 'b', 'f', 'c', 'd'))
sage: x = M.random_element()
sage: x # random
[(\'a\', \'b\'), (\'c\', \'d\'), (\'e\', \'f\')]
```

## 5.1.172 Permutations

The Permutations module. Use `Permutation?` to get information about the Permutation class, and `Permutations?` to get information about the combinatorial class of permutations.

**Warning:** This file defined `Permutation` which depends upon `CombinatorialElement` despite it being deprecated (see [github issue #13742](https://github.com/sagemath/sage/issues/13742)). This is dangerous. In particular, the `Permutation._left_to_right_multiply_on_right()` method (which can be called through multiplication) disables the input checks (see `Permutation()`). This should not happen. Do not trust the results.
What does this file define?

The main part of this file consists in the definition of permutation objects, i.e. the `Permutation()` method and the `Permutation` class. Global options for elements of the permutation class can be set through the `Permutations.options()` object.

Below are listed all methods and classes defined in this file.

### Methods of Permutations objects

<table>
<thead>
<tr>
<th>Method Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>left_action_product()</code></td>
<td>Returns the product of <code>self</code> with another permutation, in which the other permutation is applied first.</td>
</tr>
<tr>
<td><code>right_action_product()</code></td>
<td>Returns the product of <code>self</code> with another permutation, in which <code>self</code> is applied first.</td>
</tr>
<tr>
<td><code>size()</code></td>
<td>Returns the size of the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>cycle_string()</code></td>
<td>Returns the disjoint-cycles representation of <code>self</code> as string.</td>
</tr>
<tr>
<td><code>next()</code></td>
<td>Returns the permutation that follows <code>self</code> in lexicographic order (in the same symmetric group as <code>self</code>).</td>
</tr>
<tr>
<td><code>prev()</code></td>
<td>Returns the permutation that comes directly before <code>self</code> in lexicographic order (in the same symmetric group as <code>self</code>).</td>
</tr>
<tr>
<td><code>to_tableau_by_shape()</code></td>
<td>Returns a tableau of shape <code>shape</code> with the entries in <code>self</code>.</td>
</tr>
<tr>
<td><code>to_cycles()</code></td>
<td>Returns the permutation <code>self</code> as a list of disjoint cycles.</td>
</tr>
<tr>
<td><code>forget_cycles()</code></td>
<td>Return <code>self</code> under the forget cycle map.</td>
</tr>
<tr>
<td><code>to_permutation_group_element()</code></td>
<td>Returns a <code>PermutationGroupElement</code> equal to <code>self</code>.</td>
</tr>
<tr>
<td><code>signature()</code></td>
<td>Returns the signature of the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>is_even()</code></td>
<td>Returns <code>True</code> if the permutation <code>self</code> is even, and <code>False</code> otherwise.</td>
</tr>
<tr>
<td><code>to_matrix()</code></td>
<td>Returns a matrix representing the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>rank()</code></td>
<td>Returns the rank of <code>self</code> in lexicographic ordering (on the symmetric group containing <code>self</code>).</td>
</tr>
<tr>
<td><code>to_inversion_vector()</code></td>
<td>Returns the inversion vector of a permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>inversions()</code></td>
<td>Returns a list of the inversions of permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>stack_sort()</code></td>
<td>Returns the permutation obtained by sorting <code>self</code> through one stack.</td>
</tr>
<tr>
<td><code>to_digraph()</code></td>
<td>Return a digraph representation of <code>self</code>.</td>
</tr>
<tr>
<td><code>show()</code></td>
<td>Displays the permutation as a drawing.</td>
</tr>
<tr>
<td><code>number_of_inversions()</code></td>
<td>Returns the number of inversions in the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>noninversions()</code></td>
<td>Returns the k-noninversions in the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>number_of_noninversions()</code></td>
<td>Returns the number of k-noninversions in the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>length()</code></td>
<td>Returns the Coxeter length of a permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>inverse()</code></td>
<td>Returns the inverse of a permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>ishift()</code></td>
<td>Returns the i-shift of <code>self</code>.</td>
</tr>
<tr>
<td><code>iswitch()</code></td>
<td>Returns the i-switch of <code>self</code>.</td>
</tr>
<tr>
<td><code>runs()</code></td>
<td>Returns a list of the runs in the permutation <code>self</code>.</td>
</tr>
<tr>
<td><code>longest_increasing_subsequence_length()</code></td>
<td>Returns the length of the longest increasing subsequences of <code>self</code>.</td>
</tr>
<tr>
<td><code>longest_increasing_subsequence()</code></td>
<td>The list of the longest increasing subsequences of <code>self</code>.</td>
</tr>
<tr>
<td><code>longest_increasing_subsequence_number()</code></td>
<td>Returns the number of longest increasing subsequences</td>
</tr>
<tr>
<td><code>cycle_type()</code></td>
<td>Returns the cycle type of <code>self</code> as a partition of <code>len(self)</code>.</td>
</tr>
<tr>
<td><code>foata_bijection()</code></td>
<td>Returns the image of the permutation <code>self</code> under the Foata bijection $\phi$.</td>
</tr>
<tr>
<td><code>foata_bijection_inverse()</code></td>
<td>Returns the image of the permutation <code>self</code> under the inverse of the Foata bijection $\phi$.</td>
</tr>
<tr>
<td><code>fundamental_transformation</code></td>
<td>Returns the image of the permutation <code>self</code> under the Renyi-Foata-Schuetzenberger fundamental transformation.</td>
</tr>
<tr>
<td>Function Name</td>
<td>Description</td>
</tr>
<tr>
<td>-----------------------------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>fundamental_transformation_inverse()</td>
<td>Returns the image of the permutation self under the inverse of the Renyi-Foata-Schuetzenberger fundamental transformation.</td>
</tr>
<tr>
<td>destandardize()</td>
<td>Returns the destandardization of self with respect to weight and ordered alphabet.</td>
</tr>
<tr>
<td>to_lehmer_code()</td>
<td>Returns the Lehmer code of the permutation self.</td>
</tr>
<tr>
<td>to_lehmer_cocode()</td>
<td>Returns the Lehmer code of self.</td>
</tr>
<tr>
<td>reduced_word()</td>
<td>Returns the reduced word of the permutation self.</td>
</tr>
<tr>
<td>reduced_words()</td>
<td>Returns a list of the reduced words of the permutation self.</td>
</tr>
<tr>
<td>reduced_words_iterator()</td>
<td>An iterator for the reduced words of the permutation self.</td>
</tr>
<tr>
<td>fixed_points()</td>
<td>Returns a list of the fixed points of the permutation self.</td>
</tr>
<tr>
<td>is_derangement()</td>
<td>Returns True if the permutation self is a derangement, and False otherwise.</td>
</tr>
<tr>
<td>is_simple()</td>
<td>Returns True if the permutation self is simple, and False otherwise.</td>
</tr>
<tr>
<td>number_of_fixed_points()</td>
<td>Returns the number of fixed points of the permutation self.</td>
</tr>
<tr>
<td>recoils()</td>
<td>Returns the list of the positions of the recoils of the permutation self.</td>
</tr>
<tr>
<td>number_of_recoils()</td>
<td>Returns the number of recoils of the permutation self.</td>
</tr>
<tr>
<td>recoils_composition()</td>
<td>Returns the composition corresponding to the recoils of self.</td>
</tr>
<tr>
<td>descents()</td>
<td>Returns the list of the descents of the permutation self.</td>
</tr>
<tr>
<td>idescents()</td>
<td>Returns a list of the idescents of self.</td>
</tr>
<tr>
<td>idescents_signature()</td>
<td>Returns the list obtained by mapping each position in self to −1 if it is an idescent and 1 if it is not an idescent.</td>
</tr>
<tr>
<td>number_of_descents()</td>
<td>Returns the number of descents of the permutation self.</td>
</tr>
<tr>
<td>number_of_idescents()</td>
<td>Returns the number of idescents of the permutation self.</td>
</tr>
<tr>
<td>descents_composition()</td>
<td>Returns the composition corresponding to the descents of self.</td>
</tr>
<tr>
<td>descent_polynomial()</td>
<td>Returns the descent polynomial of the permutation self.</td>
</tr>
<tr>
<td>major_index()</td>
<td>Returns the major index of the permutation self.</td>
</tr>
<tr>
<td>imajor_index()</td>
<td>Returns the inverse major index of the permutation self.</td>
</tr>
<tr>
<td>to_major_code()</td>
<td>Returns the major code of the permutation self.</td>
</tr>
<tr>
<td>peaks()</td>
<td>Returns a list of the peaks of the permutation self.</td>
</tr>
<tr>
<td>number_of_peaks()</td>
<td>Returns the number of peaks of the permutation self.</td>
</tr>
<tr>
<td>saliances()</td>
<td>Returns a list of the saliances of the permutation self.</td>
</tr>
<tr>
<td>number_of_saliences()</td>
<td>Returns the number of saliances of the permutation self.</td>
</tr>
<tr>
<td>bruhat_lequal()</td>
<td>Returns True if self is less or equal to p2 in the Bruhat order.</td>
</tr>
<tr>
<td>weak_excedences()</td>
<td>Returns all the numbers self[i] such that self[i] &gt;= i+1.</td>
</tr>
<tr>
<td>bruhat_inversions()</td>
<td>Returns the list of inversions of self such that the application of this inversion to self decrements its number of inversions.</td>
</tr>
<tr>
<td>bruhat_inversions_iterator()</td>
<td>Returns an iterator over Bruhat inversions of self.</td>
</tr>
<tr>
<td>bruhat_succ()</td>
<td>Returns a list of the permutations covering self in the Bruhat order.</td>
</tr>
<tr>
<td>bruhat_succ_iterator()</td>
<td>An iterator for the permutations covering self in the Bruhat order.</td>
</tr>
<tr>
<td>bruhat_pred()</td>
<td>Returns a list of the permutations covered by self in the Bruhat order.</td>
</tr>
<tr>
<td>bruhat_pred_iterator()</td>
<td>An iterator for the permutations covered by self in the Bruhat order.</td>
</tr>
<tr>
<td>bruhat_smaller()</td>
<td>Returns the combinatorial class of permutations smaller than or equal to self in the Bruhat order.</td>
</tr>
<tr>
<td>bruhat_greater()</td>
<td>Returns the combinatorial class of permutations greater than or equal to self in the Bruhat order.</td>
</tr>
<tr>
<td>permutohedron_lequal()</td>
<td>Returns True if self is less or equal to p2 in the permutohedron order.</td>
</tr>
<tr>
<td>permutohedron_succ()</td>
<td>Returns a list of the permutations covering self in the permutohedron order.</td>
</tr>
<tr>
<td>permutohedron_pred()</td>
<td>Returns a list of the permutations covered by self in the permutohedron order.</td>
</tr>
<tr>
<td>permutohedron_smaller()</td>
<td>Returns a list of permutations smaller than or equal to self in the permutohedron order.</td>
</tr>
<tr>
<td>Method</td>
<td>Description</td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>---------------------------------------------------------------------------</td>
</tr>
<tr>
<td>permutohedron_greater()</td>
<td>Returns a list of permutations greater than or equal to <code>self</code> in the permutohedron order.</td>
</tr>
<tr>
<td>right_permutohedron_iterator()</td>
<td>Returns an iterator over permutations in an interval of the permutohedron order.</td>
</tr>
<tr>
<td>has_pattern()</td>
<td>Tests whether the permutation <code>self</code> matches the pattern.</td>
</tr>
<tr>
<td>avoids()</td>
<td>Tests whether the permutation <code>self</code> avoids the pattern.</td>
</tr>
<tr>
<td>pattern_positions()</td>
<td>Returns the list of positions where the pattern <code>patt</code> appears in <code>self</code>.</td>
</tr>
<tr>
<td>reverse()</td>
<td>Returns the permutation obtained by reversing the 1-line notation of <code>self</code>.</td>
</tr>
<tr>
<td>complement()</td>
<td>Returns the complement of the permutation which is obtained by replacing each value <code>x</code> in the 1-line notation of <code>self</code> with <code>n - x + 1</code>.</td>
</tr>
<tr>
<td>permutation_poset()</td>
<td>Returns the permutation poset of <code>self</code>.</td>
</tr>
<tr>
<td>dict()</td>
<td>Returns a dictionary corresponding to the permutation <code>self</code>.</td>
</tr>
<tr>
<td>robinson_schensted()</td>
<td>Returns the pair of standard tableaux obtained by running the Robinson-Schensted Algorithm on <code>self</code>.</td>
</tr>
<tr>
<td>left_tableau()</td>
<td>Returns the left standard tableau after performing the RSK algorithm.</td>
</tr>
<tr>
<td>right_tableau()</td>
<td>Returns the right standard tableau after performing the RSK algorithm.</td>
</tr>
<tr>
<td>increasing_tree()</td>
<td>Returns the increasing tree of <code>self</code>.</td>
</tr>
<tr>
<td>increasing_tree_shape()</td>
<td>Returns the shape of the increasing tree of <code>self</code>.</td>
</tr>
<tr>
<td>binary_search_tree()</td>
<td>Returns the binary search tree of <code>self</code>.</td>
</tr>
<tr>
<td>sylvester_class()</td>
<td>Iterates over the equivalence class of <code>self</code> under sylvester congruence.</td>
</tr>
<tr>
<td>RS_partition()</td>
<td>Returns the shape of the tableaux obtained by the RSK algorithm.</td>
</tr>
<tr>
<td>remove_extra_fixed_points()</td>
<td>Returns the permutation obtained by removing any fixed points at the end of <code>self</code>.</td>
</tr>
<tr>
<td>retract_plain()</td>
<td>Returns the plain retract of <code>self</code> to a smaller symmetric group $S_m$.</td>
</tr>
<tr>
<td>retract_direct_product()</td>
<td>Returns the direct-product retract of <code>self</code> to a smaller symmetric group $S_m$.</td>
</tr>
<tr>
<td>retract_okounkov_vershik()</td>
<td>Returns the Okounkov-Vershik retract of <code>self</code> to a smaller symmetric group $S_m$.</td>
</tr>
<tr>
<td>hyperoctahedral_double_coset_type()</td>
<td>Returns the coset-type of <code>self</code> as a partition.</td>
</tr>
<tr>
<td>binary_search_tree_shape()</td>
<td>Returns the shape of the binary search tree of <code>self</code> (a non labelled binary tree).</td>
</tr>
<tr>
<td>shifted_concatenation()</td>
<td>Returns the right (or left) shifted concatenation of <code>self</code> with a permutation <code>other</code>.</td>
</tr>
<tr>
<td>shifted_shuffle()</td>
<td>Returns the shifted shuffle of <code>self</code> with a permutation <code>other</code>.</td>
</tr>
</tbody>
</table>

Other classes defined in this file
### Combinatorics, Release 10.0

<table>
<thead>
<tr>
<th>Permutations</th>
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<td>Permutations_mset</td>
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<td>Permutations_set</td>
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<td>Permutations_msetk</td>
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<td>Permutations_setk</td>
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<td>Arrangements</td>
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<td>Arrangements_msetk</td>
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<tr>
<td>Arrangements_setk</td>
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<td>StandardPermutations_all</td>
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<td>StandardPermutations_n_abstract</td>
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<tr>
<td>StandardPermutations_n</td>
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<td>StandardPermutations_descents</td>
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<tr>
<td>StandardPermutations_recoils_finer</td>
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<tr>
<td>StandardPermutations_recoils_fatter</td>
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<tr>
<td>StandardPermutations_recoils</td>
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<tr>
<td>StandardPermutations_bruhat_smaller</td>
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<tr>
<td>StandardPermutations_bruhat_greater</td>
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<tr>
<td>CyclicPermutations</td>
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<td>CyclicPermutationsOfPartition</td>
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<td>StandardPermutations_avoiding_12</td>
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<td>StandardPermutations_avoiding_21</td>
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<tr>
<td>StandardPermutations_avoiding_132</td>
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<tr>
<td>StandardPermutations_avoiding_123</td>
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<tr>
<td>StandardPermutations_avoiding_321</td>
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<tr>
<td>StandardPermutations_avoiding_231</td>
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<tr>
<td>StandardPermutations_avoiding_312</td>
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<tr>
<td>StandardPermutations_avoiding_213</td>
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</tr>
<tr>
<td>StandardPermutations_avoiding_generic</td>
<td></td>
</tr>
<tr>
<td>PatternAvoider</td>
<td></td>
</tr>
</tbody>
</table>

#### Functions defined in this file

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>from_major_code()</td>
<td>Returns the permutation corresponding to major code mc.</td>
</tr>
<tr>
<td>from_permutation_group_element()</td>
<td>Returns a Permutation given a PermutationGroupElement pge.</td>
</tr>
<tr>
<td>from_rank()</td>
<td>Returns the permutation with the specified lexicographic rank.</td>
</tr>
<tr>
<td>from_inversion_vector()</td>
<td>Returns the permutation corresponding to inversion vector iv.</td>
</tr>
<tr>
<td>from_cycles()</td>
<td>Returns the permutation with given disjoint-cycle representation cycles.</td>
</tr>
<tr>
<td>from_lehmer_code()</td>
<td>Returns the permutation with Lehmer code lehmer.</td>
</tr>
<tr>
<td>from_reduced_word()</td>
<td>Returns the permutation corresponding to the reduced word rw.</td>
</tr>
<tr>
<td>bistochastic_as_sum_of_permutations()</td>
<td>Returns a given bistochastic matrix as a nonnegative linear combination of permutations.</td>
</tr>
<tr>
<td>bounded_affine_permutation()</td>
<td>Returns a partial permutation representing the bounded affine permutation of a matrix.</td>
</tr>
<tr>
<td>descents_composition_list()</td>
<td>Returns a list of all the permutations in a given descent class (i.e., having a given descents composition).</td>
</tr>
<tr>
<td>descents_composition_first()</td>
<td>Returns the smallest element of a descent class.</td>
</tr>
<tr>
<td>descents_composition_last()</td>
<td>Returns the largest element of a descent class.</td>
</tr>
<tr>
<td>bruhat_lequal()</td>
<td>Returns True if p1 is less or equal to p2 in the Bruhat order.</td>
</tr>
<tr>
<td>permutohedron_lequal()</td>
<td>Returns True if p1 is less or equal to p2 in the permutohedron order.</td>
</tr>
<tr>
<td>to_standard()</td>
<td>Returns a standard permutation corresponding to the permutation self.</td>
</tr>
</tbody>
</table>
AUTHORS:

- Mike Hansen
- Dan Drake (2008-04-07): allow Permutation() to take lists of tuples
- Sébastien Labbé (2009-03-17): added robinson_schensted_inverse
- Travis Scrimshaw:
  - (2012-08-16): to_standard() no longer modifies input
- Travis Scrimshaw (2014-02-05): Made StandardPermutations_n a finite Weyl group to make it more uniform with SymmetricGroup. Added ability to compute the conjugacy classes.
- Trevor K. Karn (2022-08-05): Add Permutation.n_reduced_words()

Classes and methods

class sage.combinat.permutation.Arrangements

Bases: Permutations

An arrangement of a multiset mset is an ordered selection without repetitions. It is represented by a list that contains only elements from mset, but maybe in a different order.

Arrangements returns the combinatorial class of arrangements of the multiset mset that contain k elements.

EXAMPLES:

```
sage: mset = [1,1,2,3,4,4,5]
sage: Arrangements(mset,2).list()
[[1, 1],
 [1, 2],
 [1, 3],
 [1, 4],
 [1, 5],
 [2, 1],
 [2, 3],
 [2, 4],
 [2, 5],
 [3, 1],
 [3, 2],
 [3, 4],
 [3, 5],
 [4, 1],
 [4, 2],
 [4, 3],
 [4, 4],
 [4, 5],
 [5, 1],
```

(continues on next page)
sage: Arrangements(mset,2).cardinality()
sage: Arrangements(["c","a","t"], 2 ).list()
[["c", 'a'], ["c", 't'], ["a", 'c'], ["a", 't'], ["t", 'c'], ["t", 'a']]
sage: Arrangements(["c","a","t"], 3 ).list()
[['c', 'a', 't'], ['c', 't', 'a'], ['a', 'c', 't'], ['a', 't', 'c'], ['t', 'c', 'a'], ['t', 'a', 'c']]

\texttt{cardinality()}

Return the cardinality of \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A = Arrangements([1,1,2,3,4,4,5], 2)
sage: A.cardinality()
sage: A = Arrangements([1,1,2,3,4,4,5], 3)
sage: A.cardinality()
sage: A = Arrangements([1,1,2,3,4,4,5], 4)
sage: A.cardinality()
22
\end{verbatim}

\textbf{class} \texttt{sage.combinat.permutation.Arrangements\_msetk}(\texttt{mset}, \texttt{k})

\textbf{Bases:} \texttt{Arrangements, Permutations\_msetk}

Arrangements of length \texttt{k} of a multiset \texttt{M}.

\textbf{class} \texttt{sage.combinat.permutation.Arrangements\_setk}(\texttt{s}, \texttt{k})

\textbf{Bases:} \texttt{Arrangements, Permutations\_setk}

Arrangements of length \texttt{k} of a set \texttt{S}.

\textbf{class} \texttt{sage.combinat.permutation.CyclicPermutations}(\texttt{mset})

\textbf{Bases:} \texttt{Permutations\_mset}

Return the class of all cyclic permutations of \texttt{mset} in cycle notation. These are the same as necklaces.

\textbf{INPUT:}

- \texttt{mset} -- A multiset

\textbf{EXAMPLES:}

\begin{verbatim}
sage: CyclicPermutations(range(4)).list()
sage: CyclicPermutations([1,1,1]).list()
[[0, 1, 2, 3], [0, 1, 3, 2], [0, 2, 1, 3], [0, 2, 3, 1], [0, 3, 1, 2], [0, 3, 2, 1]]
[[1, 1, 1]]
\end{verbatim}
**iterator** *(distinct=False)*

EXAMPLES:

```python
sage: CyclicPermutations(range(4)).list() # indirect doctest
[[0, 1, 2, 3],
 [0, 1, 3, 2],
 [0, 2, 1, 3],
 [0, 2, 3, 1],
 [0, 3, 1, 2],
 [0, 3, 2, 1]]
sage: CyclicPermutations([1,1,1]).list()
[[1, 1, 1]]
sage: CyclicPermutations([1,1,1]).list(distinct=True)
[[1, 1, 1], [1, 1, 1]]
```

**list** *(distinct=False)*

EXAMPLES:

```python
sage: CyclicPermutations(range(4)).list()
[[0, 1, 2, 3],
 [0, 1, 3, 2],
 [0, 2, 1, 3],
 [0, 2, 3, 1],
 [0, 3, 1, 2],
 [0, 3, 2, 1]]
```

**class** `sage.combinat.permutation.CyclicPermutationsOfPartition(partition)`

Bases: `Permutations`

Combinations of cyclic permutations of each cell of a given partition.

This is the same as a Cartesian product of necklaces.

EXAMPLES:

```python
sage: CyclicPermutationsOfPartition([[1,2,3,4],[5,6,7]]).list()
[[[1, 2, 3, 4], [5, 6, 7]],
 [[1, 2, 4, 3], [5, 6, 7]],
 [[1, 3, 2, 4], [5, 6, 7]],
 [[1, 3, 4, 2], [5, 6, 7]],
 [[1, 4, 2, 3], [5, 6, 7]],
 [[1, 4, 3, 2], [5, 6, 7]],
 [[1, 2, 3, 4], [5, 7, 6]],
 [[1, 2, 4, 3], [5, 7, 6]],
 [[1, 3, 2, 4], [5, 7, 6]],
 [[1, 3, 4, 2], [5, 7, 6]],
 [[1, 4, 2, 3], [5, 7, 6]],
 [[1, 4, 3, 2], [5, 7, 6]]]
sage: CyclicPermutationsOfPartition([[1,2,3,4],[4,4,4]]).list()
[[[1, 2, 3, 4], [4, 4, 4]],
 [[1, 2, 4, 3], [4, 4, 4]],
 [[1, 3, 2, 4], [4, 4, 4]],
 [[1, 3, 4, 2], [4, 4, 4]],
 [[1, 4, 2, 3], [4, 4, 4]],
 [[1, 4, 3, 2], [4, 4, 4]],

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\[
[ [1, 4, 2, 3], [4, 4, 4]],
[ [1, 4, 3, 2], [4, 4, 4]]
\]

\[\text{sage: CyclicPermutationsOfPartition([[1,2,3],[4,4,4]]).list()}
[ [ [1, 2, 3], [4, 4, 4]], [ [1, 3, 2], [4, 4, 4]]]
\]

\[\text{sage: CyclicPermutationsOfPartition([[1,2,3],[4,4,4]]).list(distinct=True)}
[ [ [1, 2, 3], [4, 4, 4]],
[ [1, 3, 2], [4, 4, 4]],
[ [1, 2, 3], [4, 4, 4]],
[ [1, 3, 2], [4, 4, 4]]
\]

class Element
    
    Bases: ClonableArray

    A cyclic permutation of a partition.

    \text{check()}
    
    Check that \text{self} is a valid element.

    EXAMPLES:

    \[\text{sage: CP = CyclicPermutationsOfPartition([[1,2,3,4],[5,6,7]])}
    \text{sage: elt = CP[0]}
    \text{sage: elt.check()}
\]

    \text{iterator}(distinct=False)

    AUTHORS:

    \item Robert Miller

    EXAMPLES:

    \[\text{sage: CyclicPermutationsOfPartition([[1,2,3,4],[5,6,7]]).list()} \text{ # indirect doctest}
    \]

\[\text{[[[1, 2, 3, 4], [5, 6, 7]],}
[ [1, 2, 4, 3], [5, 6, 7]],
[ [1, 3, 2, 4], [5, 6, 7]],
[ [1, 3, 4, 2], [5, 6, 7]],
[ [1, 4, 2, 3], [5, 6, 7]],
[ [1, 4, 3, 2], [5, 6, 7]],
[ [1, 2, 3, 4], [5, 7, 6]],
[ [1, 2, 4, 3], [5, 7, 6]],
[ [1, 3, 2, 4], [5, 7, 6]],
[ [1, 3, 4, 2], [5, 7, 6]],
[ [1, 4, 2, 3], [5, 7, 6]],
[ [1, 4, 3, 2], [5, 7, 6]]
\]

\[\text{sage: CyclicPermutationsOfPartition([[1,2,3,4],[4,4,4]]).list()}
[ [ [1, 2, 3, 4], [4, 4, 4]],
[ [1, 2, 4, 3], [4, 4, 4]],
[ [1, 3, 2, 4], [4, 4, 4]],
[ [1, 3, 4, 2], [4, 4, 4]],
[ [1, 4, 2, 3], [4, 4, 4]],
[ [1, 4, 3, 2], [4, 4, 4]]
\]

(continues on next page)
sage: CyclicPermutationsOfPartition([[1,2,3],[4,4,4]]).list()
[[[1, 2, 3], [4, 4, 4]], [[1, 3, 2], [4, 4, 4]]]

sage: CyclicPermutationsOfPartition([[1,2,3],[4,4,4]]).list(distinct=True)
[[[1, 2, 3], [4, 4, 4]],
 [[1, 3, 2], [4, 4, 4]],
 [[1, 2, 3], [4, 4, 4]],
 [[1, 3, 2], [4, 4, 4]]]

list(distinct=False)

EXAMPLES:

\begin{verbatim}
sage: CyclicPermutationsOfPartition([[1,2,3],[4,4,4]]).list()
[[[1, 2, 3], [4, 4, 4]], [[1, 3, 2], [4, 4, 4]]]
sage: CyclicPermutationsOfPartition([[1,2,3],[4,4,4]]).list(distinct=True)
[[[1, 2, 3], [4, 4, 4]],
 [[1, 3, 2], [4, 4, 4]],
 [[1, 2, 3], [4, 4, 4]],
 [[1, 3, 2], [4, 4, 4]]]
\end{verbatim}

class sage.combinat.permutation.PatternAvoider(parent, patterns)

Bases: GenericBacktracker

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.permutation import PatternAvoider
sage: P = Permutations(4)
sage: p = PatternAvoider(P, [[1,2,3]])
sage: loads(dumps(p))
<sage.combinat.permutation.PatternAvoider object at 0x...>
\end{verbatim}

class sage.combinat.permutation.Permutation(parent, l, check=True)

A permutation.

Converts \( l \) to a permutation on \( \{1, 2, \ldots, n\} \).

INPUT:

- \( l \) – Can be any one of the following:
  - an instance of \texttt{Permutation},
  - list of integers, viewed as one-line permutation notation. The construction checks that you give an acceptable entry. To avoid the check, use the check option.
  - string, expressing the permutation in cycle notation.
  - list of tuples of integers, expressing the permutation in cycle notation.
  - a \texttt{PermutationGroupElement}
  - a pair of two standard tableaux of the same shape. This yields the permutation obtained from the pair using the inverse of the Robinson-Schensted algorithm.
• **check** (boolean) – whether to check that input is correct. Slows
  the function down, but ensures that nothing bad happens. This is set to True by default.

**Warning:** Since github issue #13742 the input is checked for correctness: it is not accepted unless it
actually is a permutation on \{1, \ldots, n\}. It means that some `Permutation()` objects cannot be created
anymore without setting `check=False`, as there is no certainty that its functions can handle them, and this
should be fixed in a much better way ASAP (the functions should be rewritten to handle those cases, and new
tests be added).

**Warning:** There are two possible conventions for multiplying permutations, and the one currently enabled
in Sage by default is the one which has \((pq)(i) = q(p(i))\) for any permutations \(p \in S_n\) and \(q \in S_n\) and any
\(1 \leq i \leq n\). (This equation looks less strange when the action of permutations on numbers is written from the
right: then it takes the form \(i^{pq} = (ip)^q\), which is an associativity law). There is an alternative convention,
which has \((pq)(i) = p(q(i))\) instead. The conventions can be switched at runtime using `sage.combinat.
permutation.Permutations.options()`. It is best for code not to rely on this setting being set to a
particular standard, but rather use the methods `left_action_product()` and `right_action_product()`
for multiplying permutations (these methods don’t depend on the setting). See github issue #14885 for more
details.

**Note:** The bruhat* methods refer to the strong Bruhat order. To use the weak Bruhat order, look under
permutohedron*.

**EXAMPLES:**

```
sage: Permutation([2,1])
[2, 1]
sage: Permutation([2, 1, 4, 5, 3])
[2, 1, 4, 5, 3]
sage: Permutation('(1,2)')
[2, 1]
sage: Permutation('(1,2)(3,4,5)')
[2, 1, 4, 5, 3]
sage: Permutation( ((1,2),(3,4,5)) )
[2, 1, 4, 5, 3]
sage: Permutation( [(1,2),(3,4,5)] )
[2, 1, 4, 5, 3]
sage: Permutation( ((1,2)) )
[2, 1]
sage: Permutation( (1,2) )
[2, 1]
sage: Permutation( ((1,2),) )
[2, 1]
sage: Permutation( ((1,),) )
[1]
sage: Permutation( (1,) )
[1]
sage: Permutation( () )
[]
sage: Permutation( (), )
```

(continues on next page)
Construction from a string in cycle notation:

\[ \begin{array}{l}
\text{sage: } p = \text{Permutation('(4,5)'); } p \\
\text{[1, 2, 3, 5, 4]} \\
\text{sage: } \text{len}(p); \text{len}(p2) \\
5 \hspace{1cm} 10
\end{array} \]

The size of the permutation is the maximum integer appearing; add a 1-cycle to increase this:

\[ \begin{array}{l}
\text{sage: } p2 = \text{Permutation('(4,5)(10)'); } p2 \\
\text{[1, 2, 3, 5, 4, 6, 7, 8, 9, 10]} \\
\text{sage: } \text{len}(p); \text{len}(p2) \\
5 \hspace{1cm} 10
\end{array} \]

We construct a \texttt{Permutation} from a \texttt{PermutationGroupElement}:

\[ \begin{array}{l}
\text{sage: } g = \text{PermutationGroupElement([2,1,3])} \\
\text{sage: } \text{Permutation}(g) \\
\text{[2, 1, 3]}
\end{array} \]

From a pair of tableaux of the same shape. This uses the inverse of the Robinson-Schensted algorithm:

\[ \begin{array}{l}
\text{sage: } p = [[1, 4, 7], [2, 5], [3], [6]] \\
\text{sage: } q = [[1, 2, 5], [3, 6], [4], [7]] \\
\text{sage: } P = \text{Tableau}(p) \\
\text{sage: } Q = \text{Tableau}(q) \\
\text{sage: } \text{Permutation( (p, q) )} \\
\text{[3, 6, 5, 2, 7, 4, 1]} \\
\text{sage: } \text{Permutation( [p, q] )} \\
\text{[3, 6, 5, 2, 7, 4, 1]} \\
\text{sage: } \text{Permutation( (P, Q) )} \\
\text{[3, 6, 5, 2, 7, 4, 1]} \\
\text{sage: } \text{Permutation( [P, Q] )} \\
\text{[3, 6, 5, 2, 7, 4, 1]}
\end{array} \]

\textbf{RS\_partition()}

Return the shape of the tableaux obtained by applying the RSK algorithm to \texttt{self}.

\textbf{EXAMPLES:}

\[ \begin{array}{l}
\text{sage: } \text{Permutation([1,4,3,2]).RS\_partition()} \\
\text{[2, 1, 1]}
\end{array} \]

\textbf{absolute\_length()}

Return the absolute length of \texttt{self}

The absolute length is the length of the shortest expression of the element as a product of reflections.
For permutations in the symmetric groups, the absolute length is the size minus the number of its disjoint cycles.

EXAMPLES:

```python
sage: Permutation([4,2,3,1]).absolute_length()
1
```

**action(a)**

Return the action of the permutation `self` on a list `a`.

The action of a permutation $p \in S_n$ on an $n$-element list $(a_1, a_2, \ldots, a_n)$ is defined to be $(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)})$.

EXAMPLES:

```python
sage: p = Permutation([2,1,3])
sage: a = list(range(3))
sage: p.action(a)
[1, 0, 2]
sage: b = [1,2,3,4]
sage: p.action(b)
Traceback (most recent call last):
  ... ValueError: len(a) must equal len(self)
sage: q = Permutation([2,3,1])
sage: a = list(range(3))
sage: q.action(a)
[1, 2, 0]
```

**avoids(patt)**

Test whether the permutation `self` avoids the pattern `patt`.

EXAMPLES:

```python
sage: Permutation([6,2,5,4,3,1]).avoids([4,2,3,1])
False
sage: Permutation([6,1,2,5,4,3]).avoids([4,2,3,1])
True
sage: Permutation([6,1,2,5,4,3]).avoids([3,4,1,2])
True
```

**binary_search_tree(left_to_right=True)**

Return the binary search tree associated to `self`.

If $w$ is a word, then the binary search tree associated to $w$ is defined as the result of starting with an empty binary tree, and then inserting the letters of $w$ one by one into this tree. Here, the insertion is being done according to the method `binary_search_insert()`, and the word $w$ is being traversed from left to right.

A permutation is regarded as a word (using one-line notation), and thus a binary search tree associated to a permutation is defined.

If the optional keyword variable `left_to_right` is set to False, the word $w$ is being traversed from right to left instead.

EXAMPLES:
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```sage
sage: Permutation([1,4,3,2]).binary_search_tree()
[1[., 4[3[2[., .], .], .]], .]

sage: Permutation([1,4,3,2]).binary_search_tree()
[1[., 4[3[2[., .], .], .]], .]

By passing the option `left_to_right=False` one can have the insertion going from right to left:

```sage
sage: Permutation([1,4,3,2]).binary_search_tree(False)
[2[1[., .], 3[., 4[., .]]], .]

sage: Permutation([1,4,3,2]).binary_search_tree(False)
[2[1[., .], 3[., 4[., .]]], .]
```

`binary_search_tree_shape(left_to_right=True)`

Return the shape of the binary search tree of the permutation (a non labelled binary tree).

EXAMPLES:

```sage
sage: Permutation([1,4,3,2]).binary_search_tree_shape()
[., [[[., .], .], .]]

sage: Permutation([4,1,3,2]).binary_search_tree_shape()
[[[., .], .], .]]
```

By passing the option `left_to_right=False` one can have the insertion going from right to left:

```sage
sage: Permutation([1,4,3,2]).binary_search_tree_shape(False)
[[[., .], .], [., [., .]]]

sage: Permutation([4,1,3,2]).binary_search_tree_shape(False)
[[[., .], .], [., [., .]]]
```

`bruhat_greater()`

Return the combinatorial class of permutations greater than or equal to `self` in the Bruhat order (on the symmetric group containing `self`).

See `bruhat_lequal()` for the definition of the Bruhat order.

EXAMPLES:

```sage
sage: Permutation([4,1,2,3]).bruhat_greater().list()
[[4, 1, 2, 3],
 [4, 1, 3, 2],
 [4, 2, 1, 3],
 [4, 2, 3, 1],
 [4, 3, 1, 2],
 [4, 3, 2, 1]]
```

`bruhat_inversions()`

Return the list of inversions of `self` such that the application of this inversion to `self` decreases its number of inversions by exactly 1.

Equivalently, it returns the list of pairs `(i, j)` such that `i < j`, such that `p(i) > p(j)` and such that there exists no `k` (strictly) between `i` and `j` satisfying `p(i) > p(k) > p(j)`.

EXAMPLES:
bruhat_inversions_iterator()

Return the iterator for the inversions of self such that the application of this inversion to self decreases its number of inversions by exactly 1.

EXAMPLES:

```python
sage: list(Permutation([5,2,3,4,1]).bruhat_inversions_iterator())
[[0, 1], [0, 2], [0, 3], [1, 4], [2, 4], [3, 4]]
sage: list(Permutation([6,1,4,5,2,3]).bruhat_inversions_iterator())
[[0, 1], [0, 2], [0, 3], [2, 4], [2, 5], [3, 4], [3, 5]]
```

bruhat_lequal(p2)

Return True if self is less or equal to p2 in the Bruhat order.

The Bruhat order (also called strong Bruhat order or Chevalley order) on the symmetric group \( S_n \) is the partial order on \( S_n \) determined by the following condition: If \( p \) is a permutation, and \( i \) and \( j \) are two indices satisfying \( p(i) > p(j) \) and \( i < j \) (that is, \( (i, j) \) is an inversion of \( p \) with \( i < j \)), then \( p \circ (i, j) \) (the permutation obtained by first switching \( i \) with \( j \) and then applying \( p \)) is smaller than \( p \) in the Bruhat order.

One can show that a permutation \( p \in S_n \) is less or equal to a permutation \( q \in S_n \) in the Bruhat order if and only if for every \( i \in \{0, 1, \ldots, n\} \) and \( j \in \{1, 2, \ldots, n\} \), the number of the elements among \( p(1), p(2), \ldots, p(j) \) that are greater than \( i \) is \( \leq \) to the number of the elements among \( q(1), q(2), \ldots, q(j) \) that are greater than \( i \).

This method assumes that self and p2 are permutations of the same integer \( n \).

EXAMPLES:

```python
sage: Permutation([2,4,3,1]).bruhat_lequal(Permutation([3,4,2,1]))
True
sage: Permutation([2,1,3]).bruhat_lequal(Permutation([2,3,1]))
True
sage: Permutation([2,1,3]).bruhat_lequal(Permutation([3,1,2]))
True
sage: Permutation([2,1,3]).bruhat_lequal(Permutation([1,2,3]))
False
sage: Permutation([1,3,2]).bruhat_lequal(Permutation([2,1,3]))
False
sage: Permutation([1,3,2]).bruhat_lequal(Permutation([2,3,1]))
True
sage: Permutation([2,3,1]).bruhat_lequal(Permutation([1,3,2]))
False
sage: sorted([len([b for b in Permutations(3) if a.bruhat_lequal(b)])
            for a in Permutations(3)])
[1, 2, 2, 4, 4, 6]
sage: Permutation([]).bruhat_lequal(Permutation([]))
True
```
bruhat_pred()

Return a list of the permutations strictly smaller than self in the Bruhat order (on the symmetric group containing self) such that there is no permutation between one of those and self.

See bruhat_lequal() for the definition of the Bruhat order.

EXAMPLES:

```
sage: Permutation([6,1,4,5,2,3]).bruhat_pred()
[[1, 6, 4, 5, 2, 3],
 [4, 1, 6, 5, 2, 3],
 [5, 1, 4, 6, 2, 3],
 [6, 1, 2, 5, 4, 3],
 [6, 1, 3, 5, 2, 4],
 [6, 1, 4, 2, 5, 3],
 [6, 1, 4, 3, 2, 5]]
```

bruhat_pred_iterator()

An iterator for the permutations strictly smaller than self in the Bruhat order (on the symmetric group containing self) such that there is no permutation between one of those and self.

See bruhat_lequal() for the definition of the Bruhat order.

EXAMPLES:

```
sage: [x for x in Permutation([6,1,4,5,2,3]).bruhat_pred_iterator()]
[[1, 6, 4, 5, 2, 3],
 [4, 1, 6, 5, 2, 3],
 [5, 1, 4, 6, 2, 3],
 [6, 1, 2, 5, 4, 3],
 [6, 1, 3, 5, 2, 4],
 [6, 1, 4, 2, 5, 3],
 [6, 1, 4, 3, 2, 5]]
```

bruhat_smaller()

Return the combinatorial class of permutations smaller than or equal to self in the Bruhat order (on the symmetric group containing self).

See bruhat_lequal() for the definition of the Bruhat order.

EXAMPLES:

```
sage: Permutation([4,1,2,3]).bruhat_smaller().list()
[[1, 2, 3, 4],
 [1, 2, 4, 3],
 [1, 3, 2, 4],
 [1, 4, 2, 3],
 [2, 1, 3, 4],
 [2, 1, 4, 3],
 [3, 1, 2, 4],
 [4, 1, 2, 3]]
```

bruhat_succ()

Return a list of the permutations strictly greater than self in the Bruhat order (on the symmetric group containing self) such that there is no permutation between one of those and self.

See bruhat_lequal() for the definition of the Bruhat order.
EXAMPLES:

```python
sage: Permutation([6,1,4,5,2,3]).bruhat_succ()
[[6, 4, 1, 5, 2, 3],
 [6, 2, 4, 5, 1, 3],
 [6, 1, 5, 4, 2, 3],
 [6, 1, 4, 5, 3, 2]]
```

**bruhat_succ_iterator()**

An iterator for the permutations that are strictly greater than `self` in the Bruhat order (on the symmetric group containing `self`) such that there is no permutation between one of those and `self`.

See **bruhat_lequal()** for the definition of the Bruhat order.

EXAMPLES:

```python
sage: [x for x in Permutation([6,1,4,5,2,3]).bruhat_succ_iterator()]
[[6, 4, 1, 5, 2, 3],
 [6, 2, 4, 5, 1, 3],
 [6, 1, 5, 4, 2, 3],
 [6, 1, 4, 5, 3, 2]]
```

**complement()**

Return the complement of the permutation `self`.

The complement of a permutation $w \in S_n$ is defined as the permutation in $S_n$ sending each $i$ to $n + 1 - w(i)$.

EXAMPLES:

```python
sage: Permutation([1,2,3]).complement()
[3, 2, 1]
```

```python
sage: Permutation([1, 3, 2]).complement()
[3, 1, 2]
```

**cycle_string**(singletons=False)

Return a string of the permutation in cycle notation.

If `singletons=True`, it includes 1-cycles in the string.

EXAMPLES:

```python
sage: Permutation([1,2,3]).cycle_string()
'()'
```

```python
sage: Permutation([2,1,3]).cycle_string()
'(1,2)'
```

```python
sage: Permutation([2,3,1]).cycle_string()
'(1,2,3)'
```

```python
sage: Permutation([2,1,3]).cycle_string(singletons=True)
'(1,2)(3)'
```

**cycle_tuples**(singletons=True, use_min=True)

Return the permutation `self` as a list of disjoint cycles.

The cycles are returned in the order of increasing smallest elements, and each cycle is returned as a tuple which starts with its smallest element.

If `singletons=False` is given, the list does not contain the singleton cycles.
If `use_min=False` is given, the cycles are returned in the order of increasing largest (not smallest) elements, and each cycle starts with its largest element.

**EXAMPLES:**

\[
\begin{align*}
\text{sage: } & \text{Permutation([2,1,3,4]).to_cycles()} \\
& [(1, 2), (3,), (4,)] \\
\text{sage: } & \text{Permutation([2,1,3,4]).to_cycles(singletons=False)} \\
& [(1, 2)] \\
\text{sage: } & \text{Permutation([2,1,3,4]).to_cycles(\text{use_min=True})} \\
& [(1, 2), (3,), (4,)] \\
\text{sage: } & \text{Permutation([2,1,3,4]).to_cycles(\text{use_min=False})} \\
& [(4,), (3,), (2, 1)] \\
\text{sage: } & \text{Permutation([2,1,3,4]).to_cycles(singletons=False, use_min=False)} \\
& [(2, 1)] \\
\text{sage: } & \text{Permutation([4,1,5,2,6,3]).to_cycles()} \\
& [(1, 4, 2), (3, 5, 6)] \\
\text{sage: } & \text{Permutation([4,1,5,2,6,3]).to_cycles(\text{use_min=False})} \\
& [(6, 3, 5), (4, 2, 1)] \\
\text{sage: } & \text{Permutation([6,4,5,2,3,1]).to_cycles()} \\
& [(1, 6), (2, 4), (3, 5)] \\
\text{sage: } & \text{Permutation([6,4,5,2,3,1]).to_cycles(\text{use_min=False})} \\
& [(6, 1), (5, 3), (4, 2)]
\end{align*}
\]

The algorithm is of complexity $O(n)$ where $n$ is the size of the given permutation.

**cycle_type()**

Return a partition of `len(self)` corresponding to the cycle type of `self`.

This is a non-increasing sequence of the cycle lengths of `self`.

**EXAMPLES:**

\[
\text{sage: } \text{Permutation([3,1,2,4]).cycle_type()} \\
[3, 1]
\]

**decreasing_runs(as_tuple=False)**

Decreasing runs of the permutation.

**INPUT:**

- `as_tuple` – boolean (default: `False`) choice of output format

**OUTPUT:**

a list of lists or a tuple of tuples

**See also:**

**runs()**

**EXAMPLES:**

\[
\begin{align*}
\text{sage: } & \text{s = Permutation([2,8,3,9,6,4,5,1,7])} \\
\text{sage: } & \text{s.decreasing_runs()} \\
& [[2], [8, 3], [9, 6, 4], [5, 1], [7]]
\end{align*}
\]
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sage: s.decreasing_runs(as_tuple=True)
((2,), (8, 3), (9, 6, 4), (5, 1), (7,))

descent_polynomial()

Return the descent polynomial of the permutation self.

The descent polynomial of a permutation \( p \) is the product of all the \( z[p(i)] \) where \( i \) ranges over the descents of \( p \).

A descent of a permutation \( p \) is an integer \( i \) such that \( p(i) > p(i+1) \).

REFERENCES:
  • [GS1984]

EXAMPLES:

sage: Permutation([2,1,3]).descent_polynomial()
z1
sage: Permutation([4,3,2,1]).descent_polynomial()
z1*z2^2*z3^3

Todo: This docstring needs to be fixed. First, the definition does not match the implementation (or the examples). Second, this doesn't seem to be defined in [GS1984] (the descent monomial in their (7.23) is different).

descents(final_descent=False, side='right', positive=False, from_zero=False, index_set=None)

Return the list of the descents of self.

A descent of a permutation \( p \) is an integer \( i \) such that \( p(i) > p(i+1) \).

Warning: By default, the descents are returned as elements in the index set, i.e., starting at 1. If you want them to start at 0, set the keyword from_zero to True.

INPUT:
  • final_descent – boolean (default False); if True, the last position of a non-empty permutation is also considered as a descent
  • side – 'right' (default) or 'left'; if 'left', return the descents of the inverse permutation
  • positive – boolean (default False); if True, return the positions that are not descents
  • from_zero – boolean (default False); if True, return the positions starting from 0
  • index_set – list (default: \([1, \ldots, n-1]\) where self is a permutation of \( n \)); the index set to check for descents

EXAMPLES:

sage: Permutation([3,1,2]).descents()
[1]
sage: Permutation([1,4,3,2]).descents()
[2, 3]
sage: Permutation([1,4,3,2]).descents(final_descent=True)
(continues on next page)
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[2, 3, 4]
sage: Permutation([1,4,3,2]).descents(index_set=[1,2])
[2]
sage: Permutation([1,4,3,2]).descents(from_zero=True)
[1, 2]

descents_composition()

Return the descent composition of self.

The descent composition of a permutation \( p \in S_n \) is defined as the composition of \( n \) whose descent set equals the descent set of \( p \). Here, the descent set of \( p \) is defined as the set of all \( i \in \{1, 2, \ldots, n - 1\} \) satisfying \( p(i) > p(i + 1) \). The descent set of a composition \( c = (i_1, i_2, \ldots, i_k) \) is defined as the set \( \{i_1, i_1 + i_2, i_1 + i_2 + i_3, \ldots, i_1 + i_2 + \cdots + i_{k-1}\} \).

EXAMPLES:

sage: Permutation([1,3,2,4]).descents_composition()
[2, 2]
sage: Permutation([4,1,6,7,2,3,8,5]).descents_composition()
[1, 3, 3, 1]
sage: Permutation([]).descents_composition()
[]

destandardize(weight, ordered_alphabet=None)

Return destandardization of self with respect to weight and ordered_alphabet.

INPUT:

- weight – list or tuple of nonnegative integers that sum to \( n \) if self is a permutation in \( S_n \).
- ordered_alphabet – (default: None) a list or tuple specifying the ordered alphabet the destandardized word is over

OUTPUT: word over the ordered_alphabet which standardizes to self.

Let \( \text{weight} = (w_1, w_2, \ldots, w_\ell) \). Then this methods looks for an increasing sequence of \( 1, 2, \ldots, w_1 \) and labels all letters in it by 1, then an increasing sequence of \( w_1 + 1, w_1 + 2, \ldots, w_1 + w_2 \) and labels all these letters by 2, etc.. If an increasing sequence for the specified weight does not exist, an error is returned. The output is a word \( w \) over the specified ordered alphabet with evaluation weight such that \( w.\text{standard_permutation}() \) is self.

EXAMPLES:

sage: p = Permutation([1,2,5,3,6,4])
sage: p.destandardize([3,1,2])
word: 113132
sage: p = Permutation([2,1,3])
sage: p.destandardize([2,1])
Traceback (most recent call last):
... ValueError: Standardization with weight [2, 1] is not possible!

dict()

Return a dictionary corresponding to the permutation.

EXAMPLES:
sage: p = Permutation([2,1,3])
sage: d = p.dict()
sage: d[1]
2
sage: d[2]
1
sage: d[3]
3

**fixed_points()**

Return a list of the fixed points of `self`.

**EXAMPLES:**
sage: Permutation([1,3,2,4]).fixed_points() [1, 4]
sage: Permutation([1,2,3,4]).fixed_points() [1, 2, 3, 4]

**foata_bijection()**

Return the image of the permutation `self` under the Foata bijection $\phi$.

The bijection shows that $\text{maj}$ (the major index) and $\text{inv}$ (the number of inversions) are equidistributed: if $\phi(P) = Q$, then $\text{maj}(P) = \text{inv}(Q)$.

The Foata bijection $\phi$ is a bijection on the set of words with no two equal letters. It can be defined by induction on the size of the word: Given a word $w_1 w_2 \cdots w_n$, start with $\phi(w_1) = w_1$. At the $i$-th step, if $\phi(w_1 w_2 \cdots w_i) = v_1 v_2 \cdots v_i$, we define $\phi(w_1 w_2 \cdots w_i w_{i+1})$ by placing $w_{i+1}$ on the end of the word $v_1 v_2 \cdots v_i$ and breaking the word up into blocks as follows. If $w_{i+1} > v_i$, place a vertical line to the right of each $v_k$ for which $w_{i+1} > v_k$. Otherwise, if $w_{i+1} < v_i$, place a vertical line to the right of each $v_k$ for which $w_{i+1} < v_k$. In either case, place a vertical line at the start of the word as well. Now, within each block between vertical lines, cyclically shift the entries one place to the right.

For instance, to compute $\phi([1,4,2,5,3])$, the sequence of words is

- 1,
- |1|4 → 14,
- |14|2 → 412,
- |41|2|5 → 4125,
- |4125|3 → 45123.

So $\phi([1,4,2,5,3]) = [4,5,1,2,3]$.

See section 2 of [FS1978], and the proof of Proposition 1.4.6 in [EnumComb1].

See also:

**foata_bijection_inverse()** for the inverse map.

**EXAMPLES:**
sage: Permutation([1,2,4,3]).foata_bijection() [4, 1, 2, 3]
sage: Permutation([2,5,1,3,4]).foata_bijection() [2, 1, 3, 5, 4]
 sage: P = Permutation([2,5,1,3,4])
sage: P.major_index() == P.foata_bijection().number_of_inversions()
True
 sage: all( P.major_index() == P.foata_bijection().number_of_inversions()
     ....:     for P in Permutations(4) )
True

The example from [FS1978]:

 sage: Permutation([7,4,9,2,6,1,5,8,3]).foata_bijection()
[4, 7, 2, 6, 1, 9, 5, 8, 3]

Border cases:

 sage: Permutation([]).foata_bijection()
[]
sage: Permutation([1]).foata_bijection()
[1]

\textbf{foata\_bijection\_inverse()} 

Return the image of the permutation \texttt{self} under the inverse of the Foata bijection $\phi$.

See \texttt{foata\_bijection()} for the definition of the Foata bijection.

EXAMPLES:

 sage: Permutation([4, 1, 2, 3]).foata_bijection()
[1, 2, 4, 3]

\textbf{forget\_cycles()} 

Return the image of \texttt{self} under the map which forgets cycles.

Consider a permutation $\sigma$ written in standard cyclic form:

$$
\sigma = (a_{1,1}, \ldots, a_{1,k_1})(a_{2,1}, \ldots, a_{2,k_2})\cdots(a_{m,1}, \ldots, a_{m,k_m}),
$$

where $a_{1,1} < a_{2,1} < \cdots < a_{m,1}$ and $a_{j,1} < a_{j,i}$ for all $1 \leq j \leq m$ and $2 \leq i \leq k_j$ where we include cycles of length 1 as well. The image of the forget cycle map $\phi$ is given by

$$
\phi(\sigma) = [a_{1,1}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, \ldots, a_{m,1}, \ldots, a_{m,k_m}],
$$

considered as a permutation in 1-line notation.

See also:

\texttt{fundamental\_transformation()}, which is a similar map that is defined by instead taking $a_{j,1} > a_{j,i}$ and is bijective.

EXAMPLES:

 sage: P = Permutations(5)
sage: x = P([1, 5, 3, 4, 2])
sage: x.forget_cycles()
[1, 2, 5, 3, 4]
We select all permutations with a cycle composition of $[2, 3, 1]$ in $S_6$:

```
sage: P = Permutations(6)
sage: l = [p for p in P if [len(t) for t in p.to_cycles()] == [1, 3, 2]]
```

Next we apply $\phi$ and then take the inverse, and then view the results as a poset under the Bruhat order:

```
sage: l = [p.forget_cycles().inverse() for p in l]
sage: B = Poset([l, lambda x,y: x.bruhat_lequal(y)])
sage: R.<q> = QQ[]
sage: sum(q^B.rank_function()(x) for x in B)
q^5 + 2*q^4 + 3*q^3 + 3*q^2 + 2*q + 1
```

We check the statement in [CC2013] that the posets $C_{[1,3,1,1]}$ and $C_{[1,3,2]}$ are isomorphic:

```
sage: l2 = [p for p in P if [len(t) for t in p.to_cycles()] == [1, 3, 1, 1]]
sage: l2 = [p.forget_cycles().inverse() for p in l2]
sage: B2 = Poset([l2, lambda x,y: x.bruhat_lequal(y)])
sage: B.is_isomorphic(B2)
True
```

See also:

`fundamental_transformation()`.

`fundamental_transformation()`

Return the image of the permutation `self` under the Renyi-Foata-Schuetzenberger fundamental transformation.

The fundamental transformation is a bijection from the set of all permutations of $\{1, 2, \ldots, n\}$ to itself, which transforms any such permutation $w$ as follows: Write $w$ in cycle form, with each cycle starting with its highest element, and the cycles being sorted in increasing order of their highest elements. Drop the parentheses in the resulting expression, thus reading it as a one-line notation of a new permutation $u$. Then, $u$ is the image of $w$ under the fundamental transformation.

See [EnumComb1], Proposition 1.3.1.

See also:

`fundamental_transformation_inverse()` for the inverse map.

`forget_cycles()` for a similar (but non-bijective) map where each cycle is starting from its lowest element.

**EXAMPLES:**

```
sage: Permutation([5, 1, 3, 4, 2]).fundamental_transformation()
[3, 4, 5, 2, 1]
sage: Permutations(5)([1, 5, 3, 4, 2]).fundamental_transformation()
[1, 3, 4, 5, 2]
sage: Permutation([8, 4, 7, 2, 9, 6, 5, 1, 3]).fundamental_transformation()
[4, 2, 6, 8, 1, 9, 3, 7, 5]
```

Comparison with `forget_cycles()`:

```
sage: P = Permutation(([1,3,4),(2,5]))
sage: P
[3, 5, 4, 1, 2]
```

(continues on next page)
sage: P.forget_cycles()
[1, 3, 4, 2, 5]
sage: P.fundamental_transformation()
[4, 1, 3, 5, 2]

fundamental_transformation_inverse()

Return the image of the permutation self under the inverse of the Renyi-Foata-Schuetzenberger fundamental transformation.

The inverse of the fundamental transformation is a bijection from the set of all permutations of \{1, 2, \ldots, n\} to itself, which transforms any such permutation \(w\) as follows: Let \(I = \{i_1 < i_2 < \cdots < i_k\}\) be the set of all left-to-right maxima of \(w\) (that is, of all indices \(j\) such that \(w(j)\) is bigger than each of \(w(1), w(2), \ldots, w(j-1)\)). The image of \(w\) under the inverse of the fundamental transformation is the permutation \(u\) that sends \(w(i-1)\) to \(w(i)\) for all \(i \notin I\) (notice that this makes sense, since \(1 \in I\) whenever \(n > 0\), while sending each \(w(i_p - 1)\) (with \(p \geq 2\)) to \(w(i_{p-1})\). Here, we set \(i_{k+1} = n + 1\).

See [EnumComb1], Proposition 1.3.1.

See also:

fundamental_transformation() for the inverse map.

EXAMPLES:

sage: Permutation([3, 4, 5, 2, 1]).fundamental_transformation_inverse()
[5, 1, 3, 4, 2]
sage: Permutation([4, 2, 6, 8, 1, 9, 3, 7, 5]).fundamental_transformation_inverse()
[8, 4, 7, 2, 9, 6, 5, 1, 3]

grade()

Return the size of self.

EXAMPLES:

sage: Permutation([3, 4, 1, 2, 5]).size()
5

has_pattern(patt)

Test whether the permutation self contains the pattern patt.

EXAMPLES:

sage: Permutation([3, 5, 1, 4, 6, 2]).has_pattern([1, 3, 2])
True

hyperoctahedral_double_coset_type()

Return the coset-type of self as a partition.

self must be a permutation of even size 2n. The coset-type determines the double class of the permutation, that is its image in \(H_n \backslash S_{2n}/H_n\), where \(H_n\) is the \(n\)-th hyperoctahedral group.

The coset-type is determined as follows. Consider the perfect matching \(\{(1, 2), (3, 4), \ldots, (2n - 1, 2n)\}\) and its image by self, and draw them simultaneously as edges of a graph whose vertices are labeled by 1, 2, \ldots, 2n. The coset-type is the ordered sequence of the semi-lengths of the cycles of this graph (see Chapter VII of [Mac1995] for more details, particularly Section VII.2).

EXAMPLES:
sage: Permutation([3, 4, 6, 1, 5, 7, 2, 8]).hyperoctahedral_double_coset_type()
[3, 1]
sage: all(p.hyperoctahedral_double_coset_type() ==
....:   p.inverse().hyperoctahedral_double_coset_type()
....:   for p in Permutations(4))
True
sage: Permutation([]).hyperoctahedral_double_coset_type()
[]
sage: Permutation([3,1,2]).hyperoctahedral_double_coset_type()
Traceback (most recent call last):
...
ValueError: [3, 1, 2] is a permutation of odd size and has no coset-type

idescents(final_descent=False, from_zero=False)

Return a list of the idescents of self, that is the list of the descents of self’s inverse.

A descent of a permutation p is an integer i such that p(i) > p(i+1).

Warning: By default, the descents are returned as elements in the index set, i.e., starting at 1. If you want them to start at 0, set the keyword from_zero to True.

INPUT:

• final_descent – boolean (default False); if True, the last position of a non-empty permutation is also considered as a descent

• from_zero – optional boolean (default False); if False, return the positions starting from 1

EXAMPLES:

sage: Permutation([2,3,1]).idescents()
[1]
sage: Permutation([1,4,3,2]).idescents()
[2, 3]
sage: Permutation([1,4,3,2]).idescents(final_descent=True)
[2, 3, 4]
sage: Permutation([1,4,3,2]).idescents(from_zero=True)
[1, 2]

idescents_signature(final_descent=False)

Return the list obtained as follows: Each position in self is mapped to -1 if it is an idescent and 1 if it is not an idescent.

See idescents() for a definition of idescents.

With the final_descent option, the last position of a non-empty permutation is also considered as a descent.

EXAMPLES:

sage: Permutation([1,4,3,2]).idescents()
[2, 3]
sage: Permutation([1,4,3,2]).idescents_signature()
[1, -1, -1, 1]
**imajor_index**(*final_descent=False*)

Return the inverse major index of the permutation `self`, which is the major index of the inverse of `self`.

The major index of a permutation $p$ is the sum of the descents of $p$. Since our permutation indices are 0-based, we need to add the number of descents.

With the `final_descent` option, the last position of a non-empty permutation is also considered as a descent.

**EXAMPLES:**

```python
sage: Permutation([2,1,3]).imajor_index()
1
sage: Permutation([3,4,1,2]).imajor_index()
2
sage: Permutation([4,3,2,1]).imajor_index()
6
```

**increasing_tree**(*compare=<built-in function min>*)

Return the increasing tree associated to `self`.

**EXAMPLES:**

```python
sage: Permutation([1,4,3,2]).increasing_tree()
1[., 2[3[4[., .], .], .]]
sage: Permutation([4,1,3,2]).increasing_tree()
1[4[., .], 2[3[., .], .]]
```

By passing the option `compare=max` one can have the decreasing tree instead:

```python
sage: Permutation([2,3,4,1]).increasing_tree(max)
4[3[2[., .], .], 1[., .]]
sage: Permutation([2,3,1,4]).increasing_tree(max)
4[3[2[., .], 1[., .]], .]
```

**increasing_tree_shape**(*compare=<built-in function min>*)

Return the shape of the increasing tree associated with the permutation.

**EXAMPLES:**

```python
sage: Permutation([1,4,3,2]).increasing_tree_shape()
[., [[., .], .], .]
sage: Permutation([4,1,3,2]).increasing_tree_shape()
[[., .], [[., .], .]]
```

By passing the option `compare=max` one can have the decreasing tree instead:

```python
sage: Permutation([2,3,4,1]).increasing_tree_shape(max)
[[[., .], .], [[., .]]]
sage: Permutation([2,3,1,4]).increasing_tree_shape(max)
[[[., .], [[., .]], .]
```

**inverse**()

Return the inverse of `self`.

**EXAMPLES:**
```python
sage: Permutation([3,8,5,10,9,4,6,1,7,2]).inverse()
[8, 10, 1, 6, 3, 7, 9, 2, 5, 4]
sage: Permutation([2, 4, 1, 5, 3]).inverse()
[3, 1, 5, 2, 4]
sage: ~Permutation([2, 4, 1, 5, 3])
[3, 1, 5, 2, 4]
```

**inversions()**

Return a list of the inversions of self.

An inversion of a permutation \( p \) is a pair \((i, j)\) such that \( i < j \) and \( p(i) > p(j) \).

**EXAMPLES:**

```python
sage: Permutation([3,2,4,1,5]).inversions()
[(1, 2), (1, 4), (2, 4), (3, 4)]
```

**is_derangement()**

Return whether self is a derangement.

A permutation \( \sigma \) is a derangement if \( \sigma \) has no fixed points.

**EXAMPLES:**

```python
sage: P = Permutation([1,4,2,3])
sage: P.is_derangement()
False
sage: P = Permutation([2,3,1])
sage: P.is_derangement()
True
```

**is_even()**

Return True if the permutation self is even and False otherwise.

**EXAMPLES:**

```python
sage: Permutation([1,2,3]).is_even()
True
sage: Permutation([2,1,3]).is_even()
False
```

**is_simple()**

Return whether self is simple.

A permutation is simple if it does not send any proper sub-interval to a sub-interval.

For instance, \([6,1,3,5,2,4]\) is not simple because it maps the interval \([3,4,5,6]\) to \([2,3,4,5]\), whereas \([2,6,3,5,1,4]\) is simple.

See OEIS sequence A111111

**EXAMPLES:**

```python
sage: g = Permutation([4,2,3,1])
sage: g.is_simple()
False
```

(continues on next page)
sage: g = Permutation([6,1,3,5,2,4])
sage: g.is_simple()
False

sage: g = Permutation([2,6,3,5,1,4])
sage: g.is_simple()
True

```python
sage: [len([pi for pi in Permutations(n) if pi.is_simple()])
    for n in range(6)]
[1, 1, 2, 0, 2, 6]
```

### ishift(i)

Return the i-shift of self. If an i-shift of self can't be performed, then self is returned.

An i-shift can be applied when i is not inbetween i − 1 and i + 1. The i-shift moves i to the other side, and leaves the relative positions of i − 1 and i + 1 in place. All other entries of the permutations are also left in place.

**EXAMPLES:**

Here, 2 is to the left of both 1 and 3. A 2-shift can be applied which moves the 2 to the right and leaves 1 and 3 in their same relative order:

```python
sage: Permutation([2,1,3]).ishift(2)
[1, 3, 2]
```

All entries other than i, i − 1 and i + 1 are unchanged:

```python
sage: Permutation([2,4,1,3]).ishift(2)
[1, 4, 3, 2]
```

Since 2 is between 1 and 3 in [1,2,3], a 2-shift cannot be applied to [1,2,3]

```python
sage: Permutation([1,2,3]).ishift(2)
[1, 2, 3]
```

### iswitch(i)

Return the i-switch of self. If an i-switch of self can't be performed, then self is returned.

An i-switch can be applied when the subsequence of self formed by the entries i − 1, i and i + 1 is neither increasing nor decreasing. In this case, this subsequence is reversed (i.e., its leftmost element and its rightmost element switch places), while all other letters of self are kept in place.

**EXAMPLES:**

Here, 2 is to the left of both 1 and 3. A 2-switch can be applied which moves the 2 to the right and switches the relative order between 1 and 3:

```python
sage: Permutation([2,1,3]).iswitch(2)
[3, 1, 2]
```

All entries other than i − 1, i and i + 1 are unchanged:

```python
sage: Permutation([2,4,1,3]).iswitch(2)
[3, 4, 1, 2]
```
Since 2 is between 1 and 3 in \([1, 2, 3]\), a 2-switch cannot be applied to \([1, 2, 3]\)

```python
sage: Permutation([1,2,3]).iswitch(2)
[1, 2, 3]
```

**left_action_product**(*lp*)

Return the permutation obtained by composing *self* with *lp* in such an order that *lp* is applied first and *self* is applied afterwards.

This is usually denoted by either *self* * * * * 1p or 1p * * * * self depending on the conventions used by the author. If the value of a permutation \(p \in S_n\) on an integer \(i \in \{1, 2, \ldots, n\}\) is denoted by \(p(i)\), then this should be denoted by *self* * * * 1p in order to have associativity (i.e., in order to have \((p \cdot q)(i) = p(q(i))\) for all \(p, q\) and \(i\)). If, on the other hand, the value of a permutation \(p \in S_n\) on an integer \(i \in \{1, 2, \ldots, n\}\) is denoted by \(i^p\), then this should be denoted by 1p * * * * self in order to have associativity (i.e., in order to have \(i^{pq} = (i^p)^q\) for all \(p, q\) and \(i\)).

**EXAMPLES:**

```python
sage: p = Permutation([2,1,3])
sage: q = Permutation([3,1,2])
sage: p.left_action_product(q)
[3, 2, 1]
sage: q.left_action_product(p)
[1, 3, 2]
```

**left_tableau**()

Return the left standard tableau after performing the RSK algorithm on *self*.

**EXAMPLES:**

```python
sage: Permutation([1,4,3,2]).left_tableau()
[[1, 2], [3], [4]]
```

**length**()

Return the Coxeter length of *self*.

The length of a permutation \(p\) is given by the number of inversions of \(p\).

**EXAMPLES:**

```python
sage: Permutation([5, 1, 3, 4, 2]).length()
6
```

**longest_increasing_subsequence_length**()

Return the length of the longest increasing subsequences of *self*.

**EXAMPLES:**

```python
sage: Permutation([2,3,1,4]).longest_increasing_subsequence_length()
3
sage: all(i.longest_increasing_subsequence_length() == len(RSK(i)[0][0]) for i → in Permutations(5))
True
sage: Permutation([]).longest_increasing_subsequence_length()
0
```
**longest_increasing_subsequences()**

Return the list of the longest increasing subsequences of `self`.

A theorem of Schensted ([Sch1961]) states that an increasing subsequence of length \(i\) ends with the value entered in the \(i\)-th column of the p-tableau. The algorithm records which column of the p-tableau each value of the permutation is entered into, creates a digraph to record all increasing subsequences, and reads the paths from a source to a sink; these are the longest increasing subsequences.

**EXAMPLES:**

```
sage: Permutation([2,3,4,1]).longest_increasing_subsequences()
[[2, 3, 4]]
sage: Permutation([5, 7, 1, 2, 6, 4, 3]).longest_increasing_subsequences()
[[1, 2, 6], [1, 2, 4], [1, 2, 3]]
```

**Note:** This algorithm could be made faster using a balanced search tree for each column instead of sorted lists. See discussion on github issue #31451.

**longest_increasing_subsequences_number()**

Return the number of increasing subsequences of maximal length in `self`.

The list of longest increasing subsequences of a permutation is given by `longest_increasing_subsequences()`, and the length of these subsequences is given by `longest_increasing_subsequence_length()`.

The algorithm is similar to `longest_increasing_subsequences()`. Namely, the longest increasing subsequences are encoded as increasing sequences in a ranked poset from a smallest to a largest element. Their number can be obtained via dynamic programming: for each \(v\) in the poset we compute the number of paths from a smallest element to \(v\).

**EXAMPLES:**

```
sage: sum(p.longest_increasing_subsequences_number() for p in Permutations(8))
120770
sage: p = Permutations(50).random_element()
sage: (len(p.longest_increasing_subsequences()) == ....: p.longest_increasing_subsequences_number())
True
```

**major_index(final_descent=False)**

Return the major index of `self`.

The major index of a permutation \(p\) is the sum of the descents of \(p\). Since our permutation indices are 0-based, we need to add the number of descents.

With the `final_descent` option, the last position of a non-empty permutation is also considered as a descent.

**EXAMPLES:**

```
sage: Permutation([2,1,3]).major_index()
1
sage: Permutation([3,4,1,2]).major_index()
2
```

(continues on next page)
sage: Permutation([4,3,2,1]).major_index()
6

**multi_major_index** *(composition)*

Return the multimajor index of this permutation with respect to *composition*.

INPUT:

- *composition* – a composition of the *size()* of this permutation

EXAMPLES:

```python
sage: p = Permutation([5, 6, 2, 1, 3, 7, 4])
sage: p.multi_major_index([3, 2, 2])
[2, 0, 1]
sage: p.multi_major_index([7]) == [p.major_index()]
True
sage: p.multi_major_index([1]*7)
[0, 0, 0, 0, 0, 0, 0]
sage: Permutation([]).multi_major_index([])
[]
```

REFERENCES:

- [JS2000]

**next()**

Return the permutation that follows *self* in lexicographic order on the symmetric group containing *self*. If *self* is the last permutation, then *next* returns False.

EXAMPLES:

```python
sage: p = Permutation([1, 3, 2])
sage: next(p)
[2, 1, 3]
sage: p = Permutation([4,3,2,1])
sage: next(p)
False
```

**noninversions** *(k)*

Return the list of all *k*-noninversions in *self*.

If *k* is an integer and *p* ∈ *S*ₙ is a permutation, then a *k*-nonversion in *p* is defined as a strictly increasing sequence *(i₁,i₂,...,iₖ)* of elements of *{1,2,...,n}* satisfying *p*(i₁) < *p*(i₂) < ⋯ < *p*(iₖ). (In other words, a *k*-nonversion in *p* can be regarded as a *k*-element subset of *{1,2,...,n}* on which *p* restricts to an increasing map.)

EXAMPLES:

```python
sage: p = Permutation([3, 2, 4, 1, 5])
sage: p.noninversions(1)
[[3], [2], [4], [1], [5]]
sage: p.noninversions(2)
[[3, 4], [3, 5], [2, 4], [2, 5], [4, 5], [1, 5]]
sage: p.noninversions(3)
[[3, 4, 5], [2, 4, 5]]
```
sage: p.noninversions(4)
[]
sage: p.noninversions(5)
[]

**number_of_descents***(final_descent=False)***
Return the number of descents of **self**.

With the final_descent option, the last position of a non-empty permutation is also considered as a descent.

EXAMPLES:

```
sage: Permutation([1,4,3,2]).number_of_descents()
2
sage: Permutation([1,4,3,2]).number_of_descents(final_descent=True)
3
```

**number_of_fixed_points**()
Return the number of fixed points of **self**.

EXAMPLES:

```
sage: Permutation([1,3,2,4]).number_of_fixed_points()
2
sage: Permutation([1,2,3,4]).number_of_fixed_points()
4
```

**number_of_idescents***(final_descent=False)***
Return the number of idescents of **self**.

See idescents() for a definition of idescents.

With the final_descent option, the last position of a non-empty permutation is also considered as a descent.

EXAMPLES:

```
sage: Permutation([1,4,3,2]).number_of_idescents()
2
sage: Permutation([1,4,3,2]).number_of_idescents(final_descent=True)
3
```

**number_of_inversions**()
Return the number of inversions in **self**.

An inversion of a permutation is a pair of elements \((i, j)\) with \(i < j\) and \(p(i) > p(j)\).

REFERENCES:

- [http://mathworld.wolfram.com/PermutationInversion.html](http://mathworld.wolfram.com/PermutationInversion.html)

EXAMPLES:

```
sage: Permutation([3, 2, 4, 1, 5]).number_of_inversions()
4
```

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Combinatorics, Release 10.0

number_of_noninversions($k$)

Return the number of $k$-noninversions in self.

If $k$ is an integer and $p \in S_n$ is a permutation, then a $k$-noninversion in $p$ is defined as a strictly increasing sequence $(i_1, i_2, \ldots, i_k)$ of elements of $\{1, 2, \ldots, n\}$ satisfying $p(i_1) < p(i_2) < \cdots < p(i_k)$. (In other words, a $k$-noninversion in $p$ can be regarded as a $k$-element subset of $\{1, 2, \ldots, n\}$ on which $p$ restricts to an increasing map.)

The number of $k$-noninversions in $p$ has been denoted by $\text{noninv}_k(p)$ in [RSW2011], where conjectures and results regarding this number have been stated.

EXAMPLES:

```
sage: p = Permutation([3, 2, 4, 1, 5])
sage: p.number_of_noninversions(1)
  5
sage: p.number_of_noninversions(2)
  6
sage: p.number_of_noninversions(3)
  2
sage: p.number_of_noninversions(4)
  0
sage: p.number_of_noninversions(5)
  0
```

The number of 2-noninversions of a permutation $p \in S_n$ is $\binom{n}{2}$ minus its number of inversions:

```
sage: b = binomial(5, 2)
sage: all( x.number_of_noninversions(2) == b - x.number_of_inversions() 
.....: for x in Permutations(5) )
True
```

We also check some corner cases:

```
sage: all( x.number_of_noninversions(1) == 5 for x in Permutations(5) )
True
sage: all( x.number_of_noninversions(0) == 1 for x in Permutations(5) )
True
sage: Permutation([]).number_of_noninversions(1)
  0
sage: Permutation([]).number_of_noninversions(0)
  1
sage: Permutation([2, 1]).number_of_noninversions(3)
  0
```

number_of_peaks()

Return the number of peaks of the permutation self.

A peak of a permutation $p$ is an integer $i$ such that $p(i - 1) < p(i)$ and $p(i) > p(i + 1)$.

EXAMPLES:
number_of_recoils()
Return the number of recoils of the permutation self.

EXAMPLES:

```python
sage: Permutation([1,4,3,2]).number_of_recoils()
2
```

number_of_reduced_words()
Return the number of reduced words of self without explicitly computing them all.

EXAMPLES:

```python
sage: p = Permutation([6,4,2,5,1,8,3,7])
sage: len(p.reduced_words()) == p.number_of_reduced_words()
True
```

number_of_saliances()
Return the number of saliances of self.
A saliance of a permutation \( p \) is an integer \( i \) such that \( p(i) > p(j) \) for all \( j > i \).

EXAMPLES:

```python
sage: Permutation([2,3,1,5,4]).number_of_saliances()
2
sage: Permutation([5,4,3,2,1]).number_of_saliances()
5
```

pattern_positions(patt)
Return the list of positions where the pattern \( patt \) appears in the permutation self.

EXAMPLES:

```python
sage: Permutation([3,5,1,4,6,2]).pattern_positions([1,3,2])
[[0, 1, 3], [2, 3, 5], [2, 4, 5]]
```

peaks()
Return a list of the peaks of the permutation self.
A peak of a permutation \( p \) is an integer \( i \) such that \( p(i - 1) < p(i) \) and \( p(i) > p(i + 1) \).

EXAMPLES:

```python
sage: Permutation([1,3,2,4,5]).peaks()
[1]
sage: Permutation([4,1,3,2,6,5]).peaks()
[2, 4]
sage: Permutation([]).peaks()
[]
```
permutation_poset()

Return the permutation poset of self.

The permutation poset of a permutation \( p \) is the poset with vertices \((i, p(i))\) for \( i = 1, 2, \ldots, n \) (where \( n \) is the size of \( p \)) and order inherited from \( \mathbb{Z} \times \mathbb{Z} \).

EXAMPLES:

```python
sage: Permutation([3,1,5,4,2]).permutation_poset().cover_relations()
[(2, 1), (5, 2)],
[(2, 1), (3, 5)],
[(2, 1), (4, 4)],
[(1, 3), (3, 5)],
[(1, 3), (4, 4)]
```

```
sage: Permutation([]).permutation_poset().cover_relations()
[]
```

```
sage: Permutation([1,3,2]).permutation_poset().cover_relations()
[[1, 1], [2, 3]]
```

```
sage: Permutation([1,2]).permutation_poset().cover_relations()
[[1, 1], [2, 2]]
```

```
P = Permutation([1,5,2,4,3])
sage: P.RS_partition() == P.RS_partition() # This should hold for any P.
True
```

permutohedron_greater(side='right')

Return a list of permutations greater than or equal to self in the permutohedron order.

By default, the computations are done in the right permutohedron. If you pass the option side='left', then they will be done in the left permutohedron.

See permutohedron_lequal() for the definition of the permutohedron orders.

EXAMPLES:

```
sage: Permutation([4,2,1,3]).permutohedron_greater()
[[4, 2, 1, 3], [4, 2, 3, 1], [4, 3, 2, 1]]
```

```
sage: Permutation([4,2,1,3]).permutohedron_greater(side='left')
[[4, 2, 1, 3], [4, 3, 1, 2], [4, 3, 2, 1]]
```

permutohedron_join(other, side='right')

Return the join of the permutations self and other in the right permutohedron order (or, if side is set to 'left', in the left permutohedron order).

The permutohedron orders (see permutohedron_lequal()) are lattices; the join operation refers to this lattice structure. In more elementary terms, the join of two permutations \( \pi \) and \( \psi \) in the symmetric group \( S_n \) is the permutation in \( S_n \) whose set of inversion is the transitive closure of the union of the set of inversions of \( \pi \) with the set of inversions of \( \psi \).

See also:

permutohedron_lequal(), permutohedron_meet.

ALGORITHM:

It is enough to construct the join of any two permutations \( \pi \) and \( \psi \) in \( S_n \) with respect to the right weak order. (The join of \( \pi \) and \( \psi \) with respect to the left weak order is the inverse of the join of \( \pi^{-1} \) and \( \psi^{-1} \) with respect to the right weak order.) Start with an empty list \( l \) (denoted \( xs \) in the actual code). For \( i = 1, 2, \ldots, n \) (in this order), we insert \( i \) into this list in the rightmost possible position such that any letter in \( \{1, 2, \ldots, i-1\} \)
which appears further right than \(i\) in either \(\pi\) or \(\psi\) (or both) must appear further right than \(i\) in the resulting list. After all numbers are inserted, we are left with a list which is precisely the join of \(\pi\) and \(\psi\) (in one-line notation). This algorithm is due to Markowsky, [Mar1994] (Theorem 1 (a)).

AUTHORS:

Viviane Pons and Darij Grinberg, 18 June 2014.

EXAMPLES:

```python
sage: p = Permutation([3,1,2])
sage: q = Permutation([1,3,2])
sage: p.permutohedron_join(q)
[3, 1, 2]
sage: r = Permutation([2,1,3])
sage: r.permutohedron_join(p)
[3, 2, 1]

sage: p = Permutation([3,2,4,1])
sage: q = Permutation([4,2,1,3])
sage: p.permutohedron_join(q)
[4, 3, 2, 1]
sage: r = Permutation([3,1,2,4])
sage: p.permutohedron_join(r)
[3, 2, 4, 1]
sage: q.permutohedron_join(r)
[4, 3, 2, 1]
sage: s = Permutation([1,4,2,3])
sage: s.permutohedron_join(r)
[4, 3, 1, 2]
```

The universal property of the join operation is satisfied:

```python
sage: def test_uni_join(p, q):
    ...:     j = p.permutohedron_join(q)
    ...:     if not p.permutohedron_lequal(j):
    ...:         return False
    ...:     if not q.permutohedron_lequal(j):
    ...:         return False
    ...:     for r in p.permutohedron_greater():
    ...:         if q.permutohedron_lequal(r) and not j.permutohedron_lequal(r):
    ...:             return False
    ...:     return True
sage: all( test_uni_join(p, q) for p in Permutations(3) for q in Permutations(3) )
True
sage: test_uni_join(Permutation([6, 4, 7, 3, 2, 5, 8, 1]), Permutation([7, 3, 1, 2, 5, 4, 6, 8]))
True
```

Border cases:

```python
sage: p = Permutation([])
sage: p.permutohedron_join(p)
[]
```

(continues on next page)
sage: p = Permutation([1])
sage: p.permutohedron_join(p)
[1]

The left permutohedron:

sage: p = Permutation([3,1,2])
sage: q = Permutation([1,3,2])
sage: p.permutohedron_join(q, side="left")
[3, 2, 1]
sage: r = Permutation([2,1,3])
sage: r.permutohedron_join(p, side="left")
[3, 1, 2]

permutohedron_lequal(p2, side='right')

Return True if self is less or equal to p2 in the permutohedron order.

By default, the computations are done in the right permutohedron. If you pass the option side='left', then they will be done in the left permutohedron.

For every nonnegative integer \( n \), the right (resp. left) permutohedron order (also called the right (resp. left) weak order, or the right (resp. left) weak Bruhat order) is a partial order on the symmetric group \( S_n \). It can be defined in various ways, including the following ones:

- Two permutations \( u \) and \( v \) in \( S_n \) satisfy \( u \leq v \) in the right (resp. left) permutohedron order if and only if the (Coxeter) length of the permutation \( v^{-1} \circ u \) (resp. of the permutation \( u \circ v^{-1} \)) equals the length of \( v \) minus the length of \( u \). Here, \( p \circ q \) means the permutation obtained by applying \( q \) first and then \( p \). (Recall that the Coxeter length of a permutation is its number of inversions.)

- Two permutations \( u \) and \( v \) in \( S_n \) satisfy \( u \leq v \) in the right (resp. left) permutohedron order if and only if every pair \((i,j)\) of elements of \( \{1,2,\ldots,n\} \) such that \( i < j \) and \( u^{-1}(i) > u^{-1}(j) \) (resp. \( u(i) > u(j) \)) also satisfies \( v^{-1}(i) > v^{-1}(j) \) (resp. \( v(i) > v(j) \)).

- A permutation \( v \) covers a permutation \( u \) in \( S_n \) in the right (resp. left) permutohedron order if and only if we have \( v = u \circ (i,i+1) \) (resp. \( v = (i,i+1) \circ u \)) for some \( i \in \{1,2,\ldots,n-1\} \) satisfying \( u(i) < u(i+1) \) (resp. \( u^{-1}(i) < u^{-1}(i+1) \)). Here, again, \( p \circ q \) means the permutation obtained by applying \( q \) first and then \( p \).

The right and the left permutohedron order are mutually isomorphic, with the isomorphism being the map sending every permutation to its inverse. Each of these orders endows the symmetric group \( S_n \) with the structure of a graded poset (the rank function being the Coxeter length).

Warning: The permutohedron order is not to be mistaken for the strong Bruhat order \( (bruhat_lequal()) \), despite both orders being occasionally referred to as the Bruhat order.

EXAMPLES:

sage: p = Permutation([3,2,1,4])
sage: p.permutohedron_lequal(Permutation([4,2,1,3]))
False
sage: p.permutohedron_lequal(Permutation([4,2,1,3]), side='left')
True
sage: p.permutohedron_lequal(p)
True
Combinatorics, Release 10.0

(continued from previous page)

```
sage: Permutation([2,1,3]).permutohedron_lequal(Permutation([2,3,1]))
True
sage: Permutation([2,1,3]).permutohedron_lequal(Permutation([3,1,2]))
False
sage: Permutation([2,1,3]).permutohedron_lequal(Permutation([1,2,3]))
False
sage: Permutation([1,3,2]).permutohedron_lequal(Permutation([2,3,1]))
False
sage: Permutation([1,3,2]).permutohedron_lequal(Permutation([2,1,3]))
False
sage: Permutation([2,3,1]).permutohedron_lequal(Permutation([1,3,2]), side='left')
False
sage: sorted([len([b for b in Permutations(3) if a.permutohedron_lequal(b)])
          for a in Permutations(3)])
[1, 2, 2, 3, 3, 6]
sage: sorted([len([b for b in Permutations(3) if a.permutohedron_lequal(b,side="left")])
          for a in Permutations(3)])
[1, 2, 2, 3, 3, 6]
sage: Permutation([]).permutohedron_lequal(Permutation([]))
True
```

**permutohedron_meet**(other, side='right')

Return the meet of the permutations `self` and `other` in the right permutohedron order (or, if `side` is set to 'left', in the left permutohedron order).

The permutohedron orders (see `permutohedron_lequal()`) are lattices; the meet operation refers to this lattice structure. It is connected to the join operation by the following simple symmetry property: If \( \pi \) and \( \psi \) are two permutations \( \pi \) and \( \psi \) in the symmetric group \( S_n \), and if \( w_0 \) denotes the permutation \( (n, n - 1, \ldots, 1) \in S_n \), then

\[
\pi \land \psi = w_0 \circ ((w_0 \circ \pi) \lor (w_0 \circ \psi)) = ((\pi \circ w_0) \lor (\psi \circ w_0)) \circ w_0
\]

and

\[
\pi \lor \psi = w_0 \circ ((w_0 \circ \pi) \land (w_0 \circ \psi)) = ((\pi \circ w_0) \land (\psi \circ w_0)) \circ w_0,
\]

where \( \land \) means meet and \( \lor \) means join.

See also:

`permutohedron_lequal()`, `permutohedron_join()`.

AUTHORS:

Viviane Pons and Darij Grinberg, 18 June 2014.

EXAMPLES:
sage: p = Permutation([3,1,2])
sage: q = Permutation([1,3,2])
sage: p.permutohedron_meet(q)
[1, 3, 2]
sage: r = Permutation([2,1,3])
sage: r.permutohedron_meet(p)
[1, 2, 3]

sage: p = Permutation([3,2,4,1])
sage: q = Permutation([4,2,1,3])
sage: p.permutohedron_meet(q)
[2, 1, 3, 4]
sage: r = Permutation([3,1,2,4])
sage: p.permutohedron_meet(r)
[3, 1, 2, 4]
sage: q.permutohedron_meet(r)
[1, 2, 3, 4]
sage: s = Permutation([1,4,2,3])
sage: s.permutohedron_meet(r)
[1, 2, 3, 4]

The universal property of the meet operation is satisfied:

```python
sage: def test_uni_meet(p, q):
    ....:     m = p.permutohedron_meet(q)
    ....:     if not m.permutohedron_lequal(p):
    ....:         return False
    ....:     if not m.permutohedron_lequal(q):
    ....:         return False
    ....:     for r in p.permutohedron_smaller():
    ....:         if r.permutohedron_lequal(q) and not r.permutohedron_lequal(m):
    ....:             return False
    ....:     return True

sage: all( test_uni_meet(p, q) for p in Permutations(3) for q in Permutations(3) )
True
```

Border cases:

```python
sage: p = Permutation([])
sage: p.permutohedron_meet(p)
[]
sage: p = Permutation([1])
sage: p.permutohedron_meet(p)
[1]
```

The left permutohedron:

```python
sage: p = Permutation([3,1,2])
sage: q = Permutation([1,3,2])
```
Combinatorics, Release 10.0

sage: p.permutohedron_meet(q, side="left")
[1, 2, 3]
sage: r = Permutation([2,1,3])
sage: r.permutohedron_meet(p, side="left")
[2, 1, 3]

**permutohedron_pred**(side='right')

Return a list of the permutations strictly smaller than self in the permutohedron order such that there is no permutation between any of those and self.

By default, the computations are done in the right permutohedron. If you pass the option side='left', then they will be done in the left permutohedron.

See **permutohedron_lequal()** for the definition of the permutohedron orders.

**EXAMPLES:**

```python
sage: p = Permutation([4,2,1,3])
sage: p.permutohedron_pred()
[[2, 4, 1, 3], [4, 1, 2, 3]]
sage: p.permutohedron_pred(side='left')
[[4, 1, 2, 3], [3, 2, 1, 4]]
```

**permutohedron_smaller**(side='right')

Return a list of permutations smaller than or equal to self in the permutohedron order.

By default, the computations are done in the right permutohedron. If you pass the option side='left', then they will be done in the left permutohedron.

See **permutohedron_lequal()** for the definition of the permutohedron orders.

**EXAMPLES:**

```python
sage: Permutation([4,2,1,3]).permutohedron_smaller()
[[1, 2, 3, 4],
 [1, 2, 4, 3],
 [1, 4, 2, 3],
 [2, 1, 3, 4],
 [2, 1, 4, 3],
 [2, 4, 1, 3],
 [4, 1, 2, 3],
 [4, 2, 1, 3]]
sage: Permutation([4,2,1,3]).permutohedron_smaller(side='left')
[[1, 2, 3, 4],
 [1, 3, 2, 4],
 [2, 1, 3, 4],
 [2, 3, 1, 4],
 [3, 1, 2, 4],
 [3, 2, 1, 4],
 [4, 1, 2, 3],
 [4, 2, 1, 3]]
```

**permutohedron_succ**(side='right')

5.1. Comprehensive Module List 1881
Return a list of the permutations strictly greater than \texttt{self} in the permutohedron order such that there is no permutation between any of those and \texttt{self}.

By default, the computations are done in the right permutohedron. If you pass the option \texttt{side='left'}, then they will be done in the left permutohedron.

See \texttt{permutohedron_lequal()} for the definition of the permutohedron orders.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: p = Permutation([4,2,1,3])
sage: p.permutohedron_succ()
[[4, 2, 3, 1]]
sage: p.permutohedron_succ(side='left')
[[4, 3, 1, 2]]
\end{verbatim}

\textbf{prev()}

Return the permutation that comes directly before \texttt{self} in lexicographic order on the symmetric group containing \texttt{self}. If \texttt{self} is the first permutation, then it returns \texttt{False}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: p = Permutation([1,2,3])
sage: p.prev()
False
sage: p = Permutation([1,3,2])
sage: p.prev()
[1, 2, 3]
\end{verbatim}

\textbf{rank()}

Return the rank of \texttt{self} in the lexicographic ordering on the symmetric group to which \texttt{self} belongs.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Permutation([1,2,3]).rank()
0
sage: Permutation([1, 2, 4, 6, 3, 5]).rank()
10
sage: perms = Permutations(6).list()
sage: [p.rank() for p in perms] == list(range(factorial(6)))
True
\end{verbatim}

\textbf{recoils()}

Return the list of the positions of the recoils of \texttt{self}.

A recoil of a permutation \(p\) is an integer \(i\) such that \(i+1\) appears to the left of \(i\) in \(p\). Here, the positions are being counted starting at 0. (Note that it is the positions, not the recoils themselves, which are being listed.)

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Permutation([1,4,3,2]).recoils()
[2, 3]
sage: Permutation([]).recoils()
[]
\end{verbatim}
recoils_composition()

Return the recoils composition of self.

The recoils composition of a permutation \( p \in S_n \) is the composition of \( n \) whose descent set is the set of the recoils of \( p \) (not their positions). In other words, this is the descents composition of \( p^{-1} \).

EXAMPLES:

```python
sage: Permutation([1,3,2,4]).recoils_composition()
[2, 2]
sage: Permutation([]).recoils_composition()
[]
```

reduced_word()

Return a reduced word of the permutation self.

See reduced_words() for the definition of reduced words and a way to compute them all.

**Warning:** This does not respect the multiplication convention.

EXAMPLES:

```python
sage: Permutation([3,5,4,6,2,1]).reduced_word()
[2, 1, 4, 3, 2, 4, 3, 5, 4, 5]
```

reduced_word_lexmin()

Return a lexicographically minimal reduced word of the permutation self.

See reduced_words() for the definition of reduced words and a way to compute them all.

EXAMPLES:

```python
sage: Permutation([3,4,2,1]).reduced_word_lexmin()
[1, 2, 1, 3, 2]
```

reduced_words()

Return a list of the reduced words of self.

The notion of a reduced word is based on the well-known fact that every permutation can be written as a product of adjacent transpositions. In more detail: If \( n \) is a nonnegative integer, we can define the transpositions \( s_i = (i, i+1) \in S_n \) for all \( i \in \{1, 2, \ldots, n-1\} \), and every \( p \in S_n \) can then be written as a product \( s_{i_1} s_{i_2} \cdots s_{i_k} \) for some sequence \( (i_1, i_2, \ldots, i_k) \) of elements of \( \{1, 2, \ldots, n-1\} \) (here \( \{1, 2, \ldots, n-1\} \) denotes the empty set when \( n \leq 1 \)). Fixing a \( p \), the sequences \( (i_1, i_2, \ldots, i_k) \) of smallest length satisfying \( p = s_{i_1} s_{i_2} \cdots s_{i_k} \) are called the reduced words of \( p \). (Their length is the Coxeter length of \( p \), and can be computed using length().)
Note that the product of permutations is defined here in such a way that \((pq)(i) = p(q(i))\) for all permutations \(p\) and \(q\) and each \(i \in \{1, 2, \ldots, n\}\) (this is the same convention as in \texttt{left_action_product()}, but not the default semantics of the \(\ast\) operator on permutations in Sage). Thus, for instance, \(s_2s_1\) is the permutation obtained by first transposing 1 with 2 and then transposing 2 with 3.

See also:
\texttt{reduced_word()}, \texttt{reduced_word_lexmin()}

\textbf{EXAMPLES:}

```
sage: Permutation([2,1,3]).reduced_words()
[[1]]
sage: Permutation([3,1,2]).reduced_words()
[[2, 1]]
sage: Permutation([3,2,1]).reduced_words()
[[1, 2, 1], [2, 1, 2]]
sage: Permutation([3,2,4,1]).reduced_words()
[[1, 2, 3, 1], [1, 2, 1, 3], [2, 1, 2, 3]]
```

\texttt{reduced_words_iterator()}

Return an iterator for the reduced words of \texttt{self}.

\textbf{EXAMPLES:}

```
sage: next(Permutation([5,2,3,4,1]).reduced_words_iterator())
[1, 2, 3, 4, 3, 2, 1]
```

\texttt{remove_extra_fixed_points()}

Return the permutation obtained by removing any fixed points at the end of \texttt{self}.

However, return [1] rather than [] if \texttt{self} is the identity permutation.

This is mostly a helper method for \texttt{sage.combinat.schubert_polynomial}, where it is used to normalize finitary permutations of \(\{1, 2, 3, \ldots\}\).

\textbf{EXAMPLES:}

```
sage: Permutation([2,1,3]).remove_extra_fixed_points()
[2, 1]
sage: Permutation([1,2,3,4]).remove_extra_fixed_points()
[1]
sage: Permutation([2,1,2]).remove_extra_fixed_points()
[2, 1]
sage: Permutation([]).remove_extra_fixed_points()
[1]
```

See also:
\texttt{retract_plain()}
\texttt{retract_direct_product}(\(m\))

Return the direct-product retract of the permutation \texttt{self} \(\in S_n\) to \(S_m\), where \(m \leq n\). If this retract is undefined, then \texttt{None} is returned.
If \( p \in S_n \) is a permutation, and \( m \) is a nonnegative integer less or equal to \( n \), then the direct-product retract of \( p \) to \( S_m \) is defined only if \( p([m]) = [m] \), where \([m]\) denotes the interval \( \{1, 2, \ldots, m\} \). In this case, it is defined as the permutation written \((p(1), p(2), \ldots, p(m))\) in one-line notation.

**EXAMPLES:**

```python
sage: Permutation([4,1,2,3,5]).retract_direct_product(4)
[4, 1, 2, 3]
sage: Permutation([4,1,2,3,5]).retract_direct_product(3)

sage: Permutation([1,4,2,3,6,5]).retract_direct_product(5)
sage: Permutation([1,4,2,3,6,5]).retract_direct_product(4)
[1, 4, 2, 3]
sage: Permutation([1,4,2,3,6,5]).retract_direct_product(3)

sage: all( p.retract_direct_product(3) == p for p in Permutations(3) )
True
```

See also:

- `retract_plain()`, `retract_okounkov_vershik()`

**retract_okounkov_vershik**

Return the Okounkov-Vershik retract of the permutation \( s \in S_n \) to \( S_m \), where \( m \leq n \).

If \( p \in S_n \) is a permutation, and \( m \) is a nonnegative integer less or equal to \( n \), then the Okounkov-Vershik retract of \( p \) to \( S_m \) is defined as the permutation in \( S_m \) which sends every \( i \in \{1, 2, \ldots, m\} \) to \( p^{k_i}(i) \), where \( k_i \) is the smallest positive integer \( k \) satisfying \( p^k(i) \leq m \).

In other words, the Okounkov-Vershik retract of \( p \) is the permutation whose disjoint cycle decomposition is obtained by removing all letters strictly greater than \( m \) from the decomposition of \( p \) into disjoint cycles (and removing all cycles which are emptied in the process).

When \( m = n - 1 \), the Okounkov-Vershik retract (as a map \( S_n \to S_{n-1} \)) is the map \( \tilde{p}_n \) introduced in Section 7 of [VO2005], and appears as (3.20) in [CST2010]. In the general case, the Okounkov-Vershik retract of a permutation in \( S_n \) to \( S_m \) can be obtained by first taking its Okounkov-Vershik retract to \( S_{n-1} \), then that of the resulting permutation to \( S_{n-2} \), etc. until arriving in \( S_m \).

**EXAMPLES:**

```python
sage: Permutation([4,1,2,3,5]).retract_okounkov_vershik(4)
[4, 1, 2, 3]
sage: Permutation([4,1,2,3,5]).retract_okounkov_vershik(3)
[3, 1, 2]
sage: Permutation([4,1,2,3,5]).retract_okounkov_vershik(2)
[2, 1]
sage: Permutation([4,1,2,3,5]).retract_okounkov_vershik(1)
[1]
sage: Permutation([4,1,2,3,5]).retract_okounkov_vershik(0)
[]
sage: Permutation([1,4,2,3,6,5]).retract_okounkov_vershik(5)

(continues on next page)
```
sage: Permutation([1,4,2,3,6,5]).retract_okounkov_vershik(4)
[1, 4, 2, 3, 5]
sage: Permutation([1,4,2,3,6,5]).retract_okounkov_vershik(3)
[1, 4, 2, 3]
sage: Permutation([1,4,2,3,6,5]).retract_okounkov_vershik(2)
[1, 4, 2, 3]
sage: Permutation([1,4,2,3,6,5]).retract_okounkov_vershik(1)
[1]
sage: Permutation([1,4,2,3,6,5]).retract_okounkov_vershik(0)
[]
sage: Permutation([6,5,4,3,2,1]).retract_okounkov_vershik(5)
[1, 5, 4, 3, 2]
sage: Permutation([6,5,4,3,2,1]).retract_okounkov_vershik(4)
[1, 2, 4, 3]
sage: Permutation([1,5,2,6,3,7,4,8]).retract_okounkov_vershik(4)
[1, 3, 2, 4]
sage: all( p.retract_direct_product(3) == p for p in Permutations(3) )
True

See also:
retract_plain(), retract_direct_product()

retract_plain(m)
Return the plain retract of the permutation self in $S_n$ to $S_m$, where $m \leq n$. If this retract is undefined, then None is returned.

If $p \in S_n$ is a permutation, and $m$ is a nonnegative integer less or equal to $n$, then the plain retract of $p$ to $S_m$ is defined only if every $i > m$ satisfies $p(i) = i$. In this case, it is defined as the permutation written $(p(1), p(2), \ldots, p(m))$ in one-line notation.

EXAMPLES:

sage: Permutation([4,1,2,3,5]).retract_plain(4)
[4, 1, 2, 3, 5]
sage: Permutation([4,1,2,3,5]).retract_plain(3)
[4, 1, 2, 3]

sage: Permutation([1,3,2,4,5,6]).retract_plain(3)
[1, 3, 2]

sage: all( p.retract_plain(3) == p for p in Permutations(3) )
True
See also:

retract_direct_product(), retract_okounkov_vershik(), remove_extra_fixed_points()

reverse()

Return the permutation obtained by reversing the list.

EXAMPLES:

```plaintext
sage: Permutation([3,4,1,2]).reverse()
[2, 1, 4, 3]
sage: Permutation([1,2,3,4,5]).reverse()
[5, 4, 3, 2, 1]
```

right_action_product(rp)

Return the permutation obtained by composing self with rp in such an order that self is applied first and rp is applied afterwards.

This is usually denoted by either self * rp or rp * self depending on the conventions used by the author. If the value of a permutation \( p \in S_n \) on an integer \( i \in \{1, 2, \ldots, n\} \) is denoted by \( p(i) \), then this should be denoted by \( rp \cdot self \) in order to have associativity (i.e., in order to have \( (p \cdot q)(i) = p(q(i)) \) for all \( p, q \) and \( i \)). If, on the other hand, the value of a permutation \( p \in S_n \) on an integer \( i \in \{1, 2, \ldots, n\} \) is denoted by \( i^p \), then this should be denoted by \( self \cdot rp \) in order to have associativity (i.e., in order to have \( i^{p \cdot q} = (i^p)^q \) for all \( p, q \) and \( i \)).

EXAMPLES:

```plaintext
sage: p = Permutation([2,1,3])
sage: q = Permutation([3,1,2])
sage: p.right_action_product(q)
[1, 3, 2]
sage: q.right_action_product(p)
[3, 2, 1]
```

right_permutohedron_interval(other)

Return the list of the permutations belonging to the right permutohedron interval where self is the minimal element and other the maximal element.

See permutohedron_lequal() for the definition of the permutohedron orders.

EXAMPLES:

```plaintext
sage: Permutation([2, 1, 4, 5, 3]).right_permutohedron_interval(Permutation([2, ˓→5, 4, 1, 3]))
[[2, 4, 5, 1, 3], [2, 4, 1, 5, 3], [2, 1, 4, 5, 3], [2, 1, 5, 4, 3], [2, 5, 1, ˓→4, 3], [2, 5, 4, 1, 3]]
```

right_permutohedron_interval_iterator(other)

Return an iterator on the permutations (represented as integer lists) belonging to the right permutohedron interval where self is the minimal element and other the maximal element.

See permutohedron_lequal() for the definition of the permutohedron orders.

EXAMPLES:

```plaintext
sage: Permutation([2, 1, 4, 5, 3]).right_permutohedron_interval_iterator(Permutation([2, ˓→5, 4, 1, 3])) # indirect doctest
```

(continues on next page)
right_tableau()
Return the right standard tableau after performing the RSK algorithm on self.

EXAMPLES:
```
sage: Permutation([1,4,3,2]).right_tableau()
[[1, 2], [3], [4]]
```

robinson_schensted()
Return the pair of standard tableaux obtained by running the Robinson-Schensted algorithm on self.
This can also be done by running RSK() on self (with the optional argument check_standard=True to return standard Young tableaux).

EXAMPLES:
```
sage: Permutation([6,2,3,1,7,5,4]).robinson_schensted()
[[[1, 3, 4], [2, 5], [6, 7]], [[1, 3, 5], [2, 6], [4, 7]]]
```

rothe_diagram()
Return the Rothe diagram of self.

EXAMPLES:
```
sage: p = Permutation([4,2,1,3])
sage: D = p.rothe_diagram(); D
[(0, 0), (0, 1), (0, 2), (1, 0)]
sage: D.pp()
 . . .
 . . .
 . . .
```

runs(as_tuple=False)
Return a list of the runs in the nonempty permutation self.
A run in a permutation is defined to be a maximal (with respect to inclusion) nonempty increasing substring (i.e., contiguous subsequence). For instance, the runs in the permutation [6,1,7,3,4,5,2] are [6], [1, 7], [3,4,5] and [2].
Runs in an empty permutation are not defined.

INPUT:
• as_tuple – boolean (default: False) choice of output format

OUTPUT:
a list of lists or a tuple of tuples

REFERENCES:
• http://mathworld.wolfram.com/PermutationRun.html

EXAMPLES:
The example from above:

```python
sage: Permutation([6,1,7,3,4,5,2]).runs()
[[6], [1, 7], [3, 4, 5], [2]]
sage: Permutation([6,1,7,3,4,5,2]).runs(as_tuple=True)
((6,), (1, 7), (3, 4, 5), (2,))
```

The number of runs in a nonempty permutation equals its number of descents plus 1:

```python
sage: all(len(p.runs()) == p.number_of_descents() + 1
.....: for p in Permutations(6) )
True
```

```
sage: Permutation([2,3,1,5,4]).saliances()
[3, 4]
sage: Permutation([5,4,3,2,1]).saliances()
[0, 1, 2, 3, 4]
```

```
sage: Permutation([]).shifted_concatenation(Permutation([]), "right")
[]
```

(continues on next page)
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(continued from previous page)

```python
sage: Permutation([]).shifted_concatenation(Permutation([]), "left")
[]
sage: Permutation([2, 4, 1, 3]).shifted_concatenation(Permutation([3, 1, 2]),
            "right")
[2, 4, 1, 3, 7, 5, 6]
sage: Permutation([2, 4, 1, 3]).shifted_concatenation(Permutation([3, 1, 2]),
            "left")
[7, 5, 6, 2, 4, 1, 3]
```

**shifted_shuffle**(other)

Return the shifted shuffle of two permutations self and other.

**INPUT:**

- other – a permutation, a list, a tuple, or any iterable representing a permutation.

**OUTPUT:**

The list of the permutations appearing in the shifted shuffle of the permutations self and other.

**EXAMPLES:**

```python
sage: Permutation([]).shifted_shuffle(Permutation([]))
[]
sage: Permutation([1, 2, 3]).shifted_shuffle(Permutation([1]))
[[4, 1, 2, 3], [1, 2, 3, 4], [1, 2, 4, 3], [1, 4, 2, 3]]
sage: Permutation([1, 2]).shifted_shuffle(Permutation([2, 1]))
[[4, 1, 3, 2], [4, 3, 1, 2], [1, 4, 3, 2], [1, 4, 2, 3], [1, 2, 4, 3], [4, 1, 2, 3]]
sage: Permutation([1]).shifted_shuffle([1])
[[2, 1], [1, 2]]
sage: len(Permutation([3, 1, 5, 4, 2]).shifted_shuffle(Permutation([2, 1, 4])))
126
```

The shifted shuffle product is associative. We can test this on an admittedly toy example:

```python
sage: all( all( all( sorted(flatten([abs.shifted_shuffle(c)
                               ....:         for abs in a.shifted_shuffle(b)]))
                               ....:           == sorted(flatten([a.shifted_shuffle(bcs)
                               ....:               for bcs in b.shifted_shuffle(c)]))
                               ....:                   for c in Permutations(2) )
                               ....:               for b in Permutations(2) )
                               ....:           for a in Permutations(2) )
                           True
```

The shifted_shuffle method on permutations gives the same permutations as the shifted_shuffle method on words (but is faster):

```python
sage: all( all( sorted(p1.shifted_shuffle(p2))
                        ....:           == sorted([Permutation(p) for p in
                        ....:                       Word(p1).shifted_shuffle(Word(p2))])
                        ....:               for p2 in Permutations(3) )
                        ....:         for p1 in Permutations(2) )
                           True
```
show(\text{representation}='cycles', \text{orientation}='landscape', **\text{args})

Display the permutation as a drawing.

INPUT:

\begin{itemize}
  \item \text{representation} – different kinds of drawings are available
  \hspace{10pt} - "cycles" (default) – the permutation is displayed as a collection of directed cycles
  \hspace{10pt} - "braid" – the permutation is displayed as segments linking each element 1, ..., \(n\) to its image on a parallel line.
  \hspace{10pt} - When using this drawing, it is also possible to display the permutation horizontally (\text{orientation} = "landscape", default option) or vertically (\text{orientation} = "portrait").
  \hspace{10pt} - "chord-diagram" – the permutation is displayed as a directed graph, all of its vertices being located on a circle.
\end{itemize}

All additional arguments are forwarded to the show subcalls.

EXAMPLES:

\begin{verbatim}
sage: Permutations(20).random_element().show(\text{representation} = "cycles")
sage: Permutations(20).random_element().show(\text{representation} = "chord-diagram")
sage: Permutations(20).random_element().show(\text{representation} = "braid", \text{orientation} = "portrait")
\end{verbatim}

\textbf{sign()}

Return the signature of the permutation \texttt{self}. This is \((-1)^{\ell}\), where \(\ell\) is the number of inversions of \texttt{self}.

\textbf{Note:} \texttt{sign()} can be used as an alias for \texttt{signature().}

EXAMPLES:

\begin{verbatim}
sage: Permutation([4, 2, 3, 1, 5]).signature()
sage: Permutation([1,3,2,5,4]).sign()
sage: Permutation([]).sign()
\end{verbatim}

\textbf{signature()}

Return the signature of the permutation \texttt{self}. This is \((-1)^{\ell}\), where \(\ell\) is the number of inversions of \texttt{self}.

\textbf{Note:} \texttt{sign()} can be used as an alias for \texttt{signature().}

EXAMPLES:

\begin{verbatim}
sage: Permutation([4, 2, 3, 1, 5]).signature()
sage: Permutation([1,3,2,5,4]).sign()
sage: Permutation([]).sign()
\end{verbatim}
**simion_schmidt(avoid=[1, 2, 3])**

Implements the Simion-Schmidt map which sends an arbitrary permutation to a pattern avoiding permutation, where the permutation pattern is one of four length-three patterns. This method also implements the bijection between (for example) [1,2,3]- and [1,3,2]-avoiding permutations.

**INPUT:**

- `avoid` – one of the patterns [1,2,3], [1,3,2], [3,1,2], [3,2,1].

**EXAMPLES:**

```python
sage: P = Permutations(6)
sage: p = P([4,5,1,6,3,2])
sage: pl = [ [1,2,3], [1,3,2], [3,1,2], [3,2,1] ]
sage: for q in pl:
....:     s = p.simion_schmidt(q)
....:     print("{} {}\n&s, s.has_pattern(q)))
[4, 6, 1, 5, 3, 2] False
[4, 2, 1, 3, 5, 6] False
[4, 5, 3, 6, 2, 1] False
[4, 5, 1, 6, 2, 3] False
```

**size()**

Return the size of `self`.

**EXAMPLES:**

```python
sage: Permutation([3,4,1,2,5]).size()
5
```

**stack_sort()**

Return the stack sort of a permutation.

This is another permutation obtained through the process of sorting using one stack. If the result is the identity permutation, the original permutation is stack-sortable.

See [Wikipedia article Stack-sortable_permutation](https://en.wikipedia.org/wiki/Stack-sortable_permutation)

**EXAMPLES:**

```python
sage: p = Permutation([2,1,5,3,4,9,7,8,6])
sage: p.stack_sort()
[1, 2, 3, 4, 5, 7, 6, 8, 9]
sage: S5 = Permutations(5)
sage: len([1 for s in S5 if s.stack_sort() == S5.one()])
42
```

**sylvester_class(left_to_right=False)**

Iterate over the equivalence class of the permutation `self` under sylvester congruence.

Sylvester congruence is an equivalence relation on the set $S_n$ of all permutations of $n$. It is defined as the smallest equivalence relation such that every permutation of the form $uacbvw$ with $u$, $v$ and $w$ being words and $a$, $b$ and $c$ being letters satisfying $a \leq b < c$ is equivalent to the permutation $ucbavw$. (Here, permutations are regarded as words by way of one-line notation.) This definition comes from [HNT2005], Definition 8, where it is more generally applied to arbitrary words.
The equivalence class of a permutation \( p \in S_n \) under sylvester congruence is called the **sylvester class** of \( p \). It is an interval in the right permutohedron order (see `permutohedron_lequal()`) on \( S_n \).

This is related to the `sylvester_class()` method in that the equivalence class of a permutation \( \pi \) under sylvester congruence is the sylvester class of the right-to-left binary search tree of \( \pi \). However, the present method yields permutations, while the method on labelled binary trees yields plain lists.

If the variable `left_to_right` is set to `True`, the method instead iterates over the equivalence class of `self` with respect to the left sylvester congruence. The left sylvester congruence is easiest to define by saying that two permutations are equivalent under it if and only if their reverses (`reverse()`) are equivalent under (standard) sylvester congruence.

**EXAMPLES:**

The sylvester class of a permutation in \( S_5 \):

```python
sage: p = Permutation([3, 5, 1, 2, 4])
sage: sorted(p.sylvester_class())
[[1, 3, 2, 5, 4],
 [1, 3, 5, 2, 4],
 [1, 5, 3, 2, 4],
 [3, 1, 2, 5, 4],
 [3, 1, 5, 2, 4],
 [3, 5, 1, 2, 4],
 [5, 1, 3, 2, 4],
 [5, 3, 1, 2, 4]]
```

The sylvester class of a permutation \( p \) contains \( p \):

```python
sage: all( p in p.sylvester_class() for p in Permutations(4) )
True
```

Small cases:

```python
sage: list(Permutation([]).sylvester_class())
[[[]]]
sage: list(Permutation([1]).sylvester_class())
[[[1]]]
```

The sylvester classes in \( S_3 \):

```python
sage: [sorted(p.sylvester_class()) for p in Permutations(3)]
[[[1, 2, 3]],
 [[1, 3, 2], [3, 1, 2]],
 [[2, 1, 3]],
 [[2, 3, 1]],
 [[1, 3, 2], [3, 1, 2]],
 [[3, 2, 1]]]
```

The left sylvester classes in \( S_3 \):

```python
sage: [sorted(p.sylvester_class(left_to_right=True)) for p in Permutations(3)]
[[[1, 2, 3]],
 [[1, 3, 2]],
 [[2, 1, 3], [2, 3, 1]],
 (continues on next page)
A left sylvester class in $S_5$:

```
sage: p = Permutation([4, 2, 1, 5, 3])
sage: sorted(p.sylvester_class(left_to_right=True))
[[4, 2, 1, 3, 5],
 [4, 2, 1, 5, 3],
 [4, 2, 3, 1, 5],
 [4, 2, 3, 5, 1],
 [4, 2, 5, 1, 3],
 [4, 2, 5, 3, 1],
 [4, 5, 2, 1, 3],
 [4, 5, 2, 3, 1]]
```

to_alternating_sign_matrix()

Return a matrix representing the permutation in the AlternatingSignMatrix class.

EXAMPLES:

```
sage: m = Permutation([1,2,3]).to_alternating_sign_matrix(); m
[1 0 0]
[0 1 0]
[0 0 1]
sage: parent(m)
Alternating sign matrices of size 3
```

to_cycles(singletons=True, use_min=True)

Return the permutation self as a list of disjoint cycles.

The cycles are returned in the order of increasing smallest elements, and each cycle is returned as a tuple which starts with its smallest element.

If singletons=False is given, the list does not contain the singleton cycles.

If use_min=False is given, the cycles are returned in the order of increasing largest (not smallest) elements, and each cycle starts with its largest element.

EXAMPLES:

```
sage: Permutation([2,1,3,4]).to_cycles()
[(1, 2), (3,), (4,)]
sage: Permutation([2,1,3,4]).to_cycles(singletons=False)
[(1, 2)]
sage: Permutation([2,1,3,4]).to_cycles(use_min=True)
[(1, 2), (3,), (4,)]
sage: Permutation([2,1,3,4]).to_cycles(use_min=False)
[(4,), (3,), (2, 1)]
sage: Permutation([2,1,3,4]).to_cycles(singletons=False, use_min=False)
[(2, 1)]
sage: Permutation([4,1,5,2,6,3]).to_cycles()
[(1, 4, 2), (3, 5, 6)]
```
sage: Permutation([4,1,5,2,6,3]).to_cycles(use_min=False)
[(6, 3, 5), (4, 2, 1)]

sage: Permutation([6, 4, 5, 2, 3, 1]).to_cycles()
[(1, 6), (2, 4), (3, 5)]

sage: Permutation([6, 4, 5, 2, 3, 1]).to_cycles(use_min=False)
[(6, 1), (5, 3), (4, 2)]

The algorithm is of complexity $O(n)$ where $n$ is the size of the given permutation.

**to_digraph()**

Return a digraph representation of self.

**EXAMPLES:**

```python
sage: d = Permutation([3, 1, 2]).to_digraph()
sage: d.edges(sort=True, labels=False)
[(1, 3), (2, 1), (3, 2)]
sage: P = Permutations(range(1, 10))
sage: d = Permutation(P.random_element()).to_digraph()
sage: all(c.is_cycle() for c in d.strongly_connected_components_subgraphs())
True
```

**to_inversion_vector()**

Return the inversion vector of self.

The inversion vector of a permutation $p \in S_n$ is defined as the vector $(v_1, v_2, \ldots, v_n)$, where $v_i$ is the number of elements larger than $i$ that appear to the left of $i$ in the permutation $p$.

The algorithm is of complexity $O(n \log(n))$ where $n$ is the size of the given permutation.

**EXAMPLES:**

```python
sage: Permutation([5,9,1,8,2,6,4,7,3]).to_inversion_vector()
[2, 3, 6, 4, 0, 2, 2, 1, 0]
sage: Permutation([8,7,2,1,9,4,6,5,10,3]).to_inversion_vector()
[3, 2, 7, 3, 4, 3, 1, 0, 0, 0]
sage: Permutation([3,2,4,1,5]).to_inversion_vector()
[3, 1, 0, 0, 0]
```

**to_lehmer_cocode()**

Return the Lehmer cocode of the permutation self.

The Lehmer cocode of a permutation $p$ is defined as the list $(c_1, c_2, \ldots, c_n)$, where $c_i$ is the number of $j < i$ such that $p(j) > p(i)$.

**EXAMPLES:**

```python
sage: p = Permutation([2,1,3])
sage: p.to_lehmer_cocode()
[0, 1, 0]
sage: q = Permutation([3,1,2])
sage: q.to_lehmer_cocode()
[0, 1, 1]
```
to_lehmer_code()

Return the Lehmer code of the permutation self.

The Lehmer code of a permutation \( p \) is defined as the list \([c[1], c[2], ..., c[n]]\), where \( c[i] \) is the number of \( j > i \) such that \( p(j) < p(i) \).

EXAMPLES:

```
sage: p = Permutation([2,1,3])
sage: p.to_lehmer_code()
[1, 0, 0]
sage: q = Permutation([3,1,2])
sage: q.to_lehmer_code()
[2, 0, 0]
sage: Permutation([1]).to_lehmer_code()
[0]
sage: Permutation([]).to_lehmer_code()
[]
```

to_major_code(final_descent=False)

Return the major code of the permutation self.

The major code of a permutation \( p \) is defined as the sequence \((m_1 - m_2, m_2 - m_3, \ldots, m_n)\), where \( m_i \) is the major index of the permutation obtained by erasing all letters smaller than \( i \) from \( p \).

With the final_descent option, the last position of a non-empty permutation is also considered as a descent. This has an effect on the computation of major indices.

REFERENCES:


EXAMPLES:

```
sage: Permutation([9,3,5,7,2,1,4,6,8]).to_major_code()
[5, 0, 1, 0, 1, 2, 0, 1, 0]
sage: Permutation([2,8,4,3,6,7,9,5,1]).to_major_code()
[8, 3, 3, 1, 4, 0, 1, 0, 0]
```

to_matrix()

Return a matrix representing the permutation.

EXAMPLES:

```
sage: Permutation([1,2,3]).to_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

Alternatively:

```
sage: matrix(Permutation([1,3,2]))
[1 0 0]
```

(continues on next page)
Notice that matrix multiplication corresponds to permutation multiplication only when the permutation option mult='r2l'.

```sage
sage: Permutations.options.mult='r2l'
sage: p = Permutation([2,1,3])
sage: q = Permutation([3,1,2])
sage: (p*q).to_matrix()
[0 0 1]
[0 1 0]
[1 0 0]
sage: p.to_matrix()*q.to_matrix()
[0 0 1]
[0 1 0]
[1 0 0]
sage: Permutations.options.mult='l2r'
sage: (p*q).to_matrix()
[1 0 0]
[0 0 1]
[0 1 0]
```

**to_permutation_group_element()**

Return a PermutationGroupElement equal to self.

EXAMPLES:

```sage
sage: Permutation([2,1,4,3]).to_permutation_group_element()
(1,2)(3,4)
sage: Permutation([1,2,3]).to_permutation_group_element()
()""
```

**to_tableau_by_shape**(shape)

Return a tableau of shape shape with the entries in self. The tableau is such that the reading word (i.e., the word obtained by reading the tableau row by row, starting from the top row in English notation, with each row being read from left to right) is self.

EXAMPLES:

```sage
sage: Permutation([3,4,1,2,5]).to_tableau_by_shape([3,2])
[[1, 2, 5], [3, 4]]
sage: Permutation([3,4,1,2,5]).to_tableau_by_shape([3,2]).reading_word_
˓→permutation()
[3, 4, 1, 2, 5]
```

**weak_excedences()**

Return all the numbers self[i] such that self[i] >= i+1.

EXAMPLES:

```sage
sage: Permutation([1,4,3,2,5]).weak_excedences()
[1, 4, 3, 5]
```
class sage.combinat.permutation.Permutations

Bases: UniqueRepresentation, Parent

Permutations.

Permutations(n) returns the class of permutations of n, if n is an integer, list, set, or string.

Permutations(n, k) returns the class of length-k partial permutations of n (where n is any of the above things); k must be a nonnegative integer. A length-\(k\) partial permutation of \(n\) is defined as a \(k\)-tuple of pairwise distinct elements of \(\{1, 2, \ldots, n\}\).


Permutations(descents=(list, n)) returns the class of permutations of \(n\) with descents in the positions specified by list. This uses the slightly nonstandard convention that the images of \(1, 2, \ldots, n\) under the permutation are regarded as positions \(0, 1, \ldots, n - 1\), so for example the presence of \(1\) in list signifies that the permutations \(\pi\) should satisfy \(\pi(2) > \pi(3)\). Note that list is supposed to be a list of positions of the descents, not the descents composition. It does not return the class of permutations with descents composition list.

Permutations(bruhat_smaller=p) and Permutations(bruhat_greater=p) return the class of permutations smaller-or-equal or greater-or-equal, respectively, than the given permutation \(p\) in the Bruhat order. (The Bruhat order is defined in \(\text{bruhat_lequal()}\). It is also referred to as the strong Bruhat order.)

Permutations(recoils=p) returns the class of permutations whose recoils composition is \(p\). Unlike the descents=(list, n) syntax, this actually takes a composition as input.

Permutations(recoils_fatter=p) and Permutations(recoils_finer=p) return the class of permutations whose recoils composition is fatter or finer, respectively, than the given composition \(p\).

Permutations(n, avoiding=P) returns the class of permutations of \(n\) avoiding \(P\). Here \(P\) may be a single permutation or a list of permutations; the returned class will avoid all patterns in \(P\).

EXAMPLES:

```
sage: p = Permutations(3); p
Standard permutations of 3
sage: p.list()
[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]

sage: p = Permutations(3, 2); p
Permutations of {1,...,3} of length 2
sage: p.list()
[[1, 2], [1, 3], [2, 1], [2, 3], [3, 1], [3, 2]]

sage: p = Permutations(['c', 'a', 't']); p
Permutations of the set ['c', 'a', 't']
sage: p.list()
[['c', 'a', 't'], ['c', 't', 'a'], ['a', 'c', 't'], ['a', 't', 'c'], ['t', 'c', 'a'], ['t', 'a', 'c']]

sage: p = Permutations(['c', 'a', 't'], 2); p
Permutations of the set ['c', 'a', 't'] of length 2
```

(continues on next page)
sage: p.list()
[['c', 'a'], ['c', 't'], ['a', 'c'], ['a', 't'], ['t', 'c'], ['t', 'a']]

sage: p = Permutations([1,1,2]); p
Permutations of the multi-set [1, 1, 2]
sage: p.list()
[[1, 1, 2], [1, 2, 1], [2, 1, 1]]

sage: p = Permutations([1,1,2], 2); p
Permutations of the multi-set [1, 1, 2] of length 2
sage: p.list()
[[1, 1], [1, 2], [2, 1]]

sage: p = Permutations(descents=((1), 4)); p
Standard permutations of 4 with descents [1]
sage: p.list()
[[2, 4, 1, 3], [3, 4, 1, 2], [1, 4, 2, 3], [1, 3, 2, 4], [2, 3, 1, 4]]

sage: p = Permutations(bruhat_smaller=[1,3,2,4]); p
Standard permutations that are less than or equal to [1, 3, 2, 4] in the Bruhat order
sage: p.list()
[[1, 2, 3, 4], [1, 3, 2, 4]]

sage: p = Permutations(bruhat_greater=[4,2,3,1]); p
Standard permutations that are greater than or equal to [4, 2, 3, 1] in the Bruhat order
sage: p.list()
[[4, 2, 3, 1], [4, 3, 2, 1]]

sage: p = Permutations(recoils_finer=[2,1]); p
Standard permutations whose recoils composition is finer than [2, 1]
sage: p.list()
[[3, 1, 2], [1, 3, 2]]

sage: p = Permutations(recoils_fatter=[2,1]); p
Standard permutations whose recoils composition is fatter than [2, 1]
sage: p.list()
[[3, 1, 2], [3, 2, 1], [1, 3, 2]]

sage: p = Permutations(recoils=[2,1]); p
Standard permutations whose recoils composition is [2, 1]
sage: p.list()
[[3, 1, 2], [1, 3, 2]]

sage: p = Permutations(4, avoiding=[1,3,2]); p
Standard permutations of 4 avoiding [[1, 3, 2]]
\[ [4, 2, 1, 3], \\
[4, 2, 3, 1], \\
[4, 3, 1, 2], \\
[4, 3, 2, 1], \\
[3, 4, 1, 2], \\
[3, 4, 2, 1], \\
[2, 3, 4, 1], \\
[3, 2, 4, 1], \\
[2, 1, 3, 4], \\
[2, 3, 1, 4], \\
[2, 1, 3, 4], \\
[1, 2, 3, 4], \\
[3, 1, 2, 4], \\
[3, 2, 1, 4], \\
[3, 2, 1, 4]] \]

\[
\text{sage: } p = \text{Permutations}(5, \text{avoiding}=[[3,4,1,2], [4,2,3,1]]); p \\
\text{Standard permutations of 5 avoiding } [[3, 4, 1, 2], [4, 2, 3, 1]] \\
\text{sage: } p.\text{cardinality}() \\
88 \\
\text{sage: } p.\text{random_element().parent() is p} \\
\text{True}
\]

**Element**

alias of **Permutation**

```python
options = Current options for Permutations - display: list - generator_name: s - latex: list - latex_empty_str: 1 - mult: l2r
```

class **sage.combinat.permutation.PermutationsNK**(s, k)

Bases: **Permutations_setk**

This exists solely for unpickling **PermutationsNK** objects created with Sage <= 6.3.

class **sage.combinat.permutation.Permutations_mset**(mset)

Bases: **Permutations**

Permutations of a multiset \( M \).

A permutation of a multiset \( M \) is represented by a list that contains exactly the same elements as \( M \) (with the same multiplicities), but possibly in different order. If \( M \) is a proper set there are \(|M|!\) such permutations. Otherwise, if the first element appears \( k_1 \) times, the second element appears \( k_2 \) times and so on, the number of permutations is \(|M|!/(k_1!k_2!...))\), which is sometimes called a multinomial coefficient.

**EXAMPLES:**

```python
sage: mset = [1,1,2,2,2]
```

```python
sage: from sage.combinat.permutation import Permutations_mset
sage: P = Permutations_mset(mset); P \\
Permutations of the multi-set [1, 1, 2, 2, 2]
```

```python
sage: sorted(P) \\
[[1, 1, 2, 2, 2], \\
 [1, 2, 1, 2, 2], \\
 [1, 2, 2, 1, 2], \\
 [1, 2, 2, 2, 1], \\
 [2, 1, 2, 2, 2], \\
 [2, 2, 1, 2, 2], 
```
[2, 1, 2, 1, 2],
[2, 1, 2, 2, 1],
[2, 2, 1, 1, 2],
[2, 2, 1, 2, 1],
[2, 2, 2, 1, 1]]
sage: MS = MatrixSpace(GF(2),2,2)
sage: A = MS([1,0,1,1])
sage: rows = A.rows()
sage: rows[0].set_immutable()
sage: rows[1].set_immutable()
sage: P = Permutations_mset(rows); P
Permutations of the multi-set [(1, 0), (1, 1)]
sage: sorted(P)
[[((1, 0), (1, 1)), ((1, 1), (1, 0))]

class Element
   Bases: ClonableArray
   A permutation of an arbitrary multiset.
   check()
      Verify that self is a valid permutation of the underlying multiset.
      EXAMPLES:
      sage: S = Permutations(['c','a','c'])
      sage: elt = S(['c','c','a'])
      sage: elt.check()

cardinality()
   Return the cardinality of the set.
   EXAMPLES:
   sage: Permutations([1,2,2]).cardinality()
   3
   sage: Permutations([1,1,2,2,2]).cardinality()
   10

rank(p)
   Return the rank of p in lexicographic order.
   INPUT:
      * p – a permutation of M
   ALGORITHM:
The algorithm uses the recurrence from the solution to exercise 4 in [Knu2011], Section 7.2.1.2:

$$\text{rank}(p_1 \ldots p_n) = \text{rank}(p_2 \ldots p_n) + \frac{1}{n} \binom{n}{n_1, \ldots, n_t} \sum_{j=1}^{t} n_j \left[x_j < p_1\right],$$

where $x_j, n_j$ are the distinct elements of $p$ with their multiplicities, $n$ is the sum of $n_1, \ldots, n_t$, $\binom{n}{n_1, \ldots, n_t}$ is the multinomial coefficient $\frac{n!}{n_1! \cdots n_t!}$, and $\sum_{j=1}^{t} n_j \left[x_j < p_1\right]$ means “the number of elements to the right of the first element that are less than the first element”.

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EXAMPLES:

```python
sage: mset = [1, 1, 2, 3, 4, 5, 5, 6, 9]
sage: p = Permutations(mset)
sage: p.rank(list(sorted(mset)))
0
sage: p.rank(list(reversed(sorted(mset)))) == p.cardinality() - 1
True
sage: p.rank([3, 1, 4, 1, 5, 9, 2, 6, 5])
30991
```

unrank($r$)

Return the permutation of $M$ having lexicographic rank $r$.

**INPUT:**

- $r$ – an integer between 0 and self.cardinality()-1 inclusive

**ALGORITHM:**

The algorithm is adapted from the solution to exercise 4 in [Knu2011], Section 7.2.1.2.

EXAMPLES:

```python
sage: mset = [1, 1, 2, 3, 4, 5, 5, 6, 9]
sage: p = Permutations(mset)
sage: p.unrank(30991)
[3, 1, 4, 1, 5, 9, 2, 6, 5]
sage: p.rank(p.unrank(10))
10
sage: p.unrank(0) == list(sorted(mset))
True
sage: p.unrank(p.cardinality()-1) == list(reversed(sorted(mset)))
True
```

class sage.combinat.permutation.Permutations_msetk(mset, k)

Bases: Permutations_mset

Length-$k$ partial permutations of a multiset.

A length-$k$ partial permutation of a multiset $M$ is represented by a list of length $k$ whose entries are elements of $M$, appearing in the list with a multiplicity not higher than their respective multiplicity in $M$.

**cardinality()**

Return the cardinality of the set.

**EXAMPLES:**

```python
sage: Permutations([1,2,2],2).cardinality()
3
```

class sage.combinat.permutation.Permutations_nk(n, k)

Bases: Permutations

Length-$k$ partial permutations of $\{1, 2, \ldots, n\}$.

**class Element**

Bases: ClonableArray

A length-$k$ partial permutation of $[n]$. 

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check()
Verify that \texttt{self} is a valid length-$k$ partial permutation of $[n]$.

\textbf{EXAMPLES:}
\begin{verbatim}
sage: S = Permutations(4, 2)
sage: elt = S([3, 1])
sage: elt.check()
\end{verbatim}

cardinality()
\textbf{EXAMPLES:}
\begin{verbatim}
sage: Permutations(3,0).cardinality()
1
sage: Permutations(3,1).cardinality()
3
sage: Permutations(3,2).cardinality()
6
sage: Permutations(3,3).cardinality()
6
sage: Permutations(3,4).cardinality()
0
\end{verbatim}

random_element()
\textbf{EXAMPLES:}
\begin{verbatim}
sage: s = Permutations(3,2).random_element()
sage: s in Permutations(3,2)
True
\end{verbatim}

class \texttt{sage.combinat.permutation.Permutations\_set}(s)
Bases: \texttt{Permutations}
Permutations of an arbitrary given finite set.
Here, a “permutation of a finite set $S$” means a list of the elements of $S$ in which every element of $S$ occurs exactly once. This is not to be confused with bijections from $S$ to $S$, which are also often called permutations in literature.

class \texttt{Element}
Bases: \texttt{ClonableArray}
A permutation of an arbitrary set.
check()
Verify that \texttt{self} is a valid permutation of the underlying set.

\textbf{EXAMPLES:}
\begin{verbatim}
sage: S = Permutations(['c','a','t'])
sage: elt = S(['t','c','a'])
sage: elt.check()
\end{verbatim}

cardinality()
Return the cardinality of the set.

\textbf{EXAMPLES:}
sage: Permutations([1,2,3]).cardinality()
6

random_element()
EXAMPLES:

sage: s = Permutations([1,2,3]).random_element()
sage: s.parent() is Permutations([1,2,3])
True

class sage.combinat.permutation.Permutations_setk(s, k)
Bases: Permutations_set

Length-\(k\) partial permutations of an arbitrary given finite set.

Here, a “length-\(k\) partial permutation of a finite set \(S\)” means a list of length \(k\) whose entries are pairwise distinct and all belong to \(S\).

random_element()
EXAMPLES:

sage: s = Permutations([1,2,4], 2).random_element()
sage: s in Permutations([1,2,4], 2)
True

class sage.combinat.permutation.StandardPermutations_all
Bases: Permutations

All standard permutations.

graded_component(n)
Return the graded component.

EXAMPLES:

sage: P = Permutations()
sage: P.graded_component(4) == Permutations(4)
True

class sage.combinat.permutation.StandardPermutations_all_avoiding(a)
Bases: StandardPermutations_all

All standard permutations avoiding a set of patterns.

patterns()
Return the patterns avoided by this class of permutations.

EXAMPLES:

sage: P = Permutations(avoiding=[[2,1,3],[1,2,3]])
sage: P.patterns()
([2, 1, 3], [1, 2, 3])

class sage.combinat.permutation.StandardPermutations_avoiding_12(n)
Bases: StandardPermutations_avoiding_generic
cardinality()

Return the cardinality of self.

EXAMPLES:

```python
sage: P = Permutations(3, avoiding=[1, 2])
sage: P.cardinality()
1
```

class sage.combinat.permutation.StandardPermutations_avoiding_123(n)

Bases: StandardPermutations_avoiding_generic
cardinality()

EXAMPLES:

```python
sage: Permutations(5, avoiding=[1, 2, 3]).cardinality()
42
sage: len( Permutations(5, avoiding=[1, 2, 3]).list() )
42
```

class sage.combinat.permutation.StandardPermutations_avoiding_132(n)

Bases: StandardPermutations_avoiding_generic
cardinality()

EXAMPLES:

```python
sage: Permutations(5, avoiding=[1, 3, 2]).cardinality()
42
sage: len( Permutations(5, avoiding=[1, 3, 2]).list() )
42
```

class sage.combinat.permutation.StandardPermutations_avoiding_21(n)

Bases: StandardPermutations_avoiding_generic
cardinality()

Return the cardinality of self.

EXAMPLES:

```python
sage: P = Permutations(3, avoiding=[2, 1])
sage: P.cardinality()
1
```

class sage.combinat.permutation.StandardPermutations_avoiding_213(n)

Bases: StandardPermutations_avoiding_generic
cardinality()

EXAMPLES:

```python
sage: Permutations(5, avoiding=[2, 1, 3]).cardinality()
42
sage: len( Permutations(5, avoiding=[2, 1, 3]).list() )
42
```
class sage.combinat.permutation.StandardPermutations_avoiding_231(n)
    Bases: StandardPermutations_avoiding_generic

    cardinality()
    EXAMPLES:

    sage: Permutations(5, avoiding=[2, 3, 1]).cardinality()
    42
    sage: len( Permutations(5, avoiding=[2, 3, 1]).list() )
    42

class sage.combinat.permutation.StandardPermutations_avoiding_312(n)
    Bases: StandardPermutations_avoiding_generic

    cardinality()
    EXAMPLES:

    sage: Permutations(5, avoiding=[3, 1, 2]).cardinality()
    42
    sage: len( Permutations(5, avoiding=[3, 1, 2]).list() )
    42

class sage.combinat.permutation.StandardPermutations_avoiding_321(n)
    Bases: StandardPermutations_avoiding_generic

    cardinality()
    EXAMPLES:

    sage: Permutations(5, avoiding=[3, 2, 1]).cardinality()
    42
    sage: len( Permutations(5, avoiding=[3, 2, 1]).list() )
    42

class sage.combinat.permutation.StandardPermutations_avoiding_generic(n, a)
    Bases: StandardPermutations_n_abstract

    Generic class for subset of permutations avoiding a set of patterns.

    property a
        self.a is deprecated; use patterns() instead.

    cardinality()
        Return the cardinality of self.
        EXAMPLES:

        sage: P = Permutations(3, avoiding=[[2, 1, 3],[1,2,3]])
        sage: P.cardinality()
        4

    patterns()
        Return the patterns avoided by this class of permutations.
        EXAMPLES:
```python
sage: P = Permutations(3, avoiding=[[2,1,3],[1,2,3]])
sage: P.patterns()
([2, 1, 3], [1, 2, 3])
```

class sage.combinat.permutation.StandardPermutations_bruhat_greater(p)

Bases: Permutations

Permutations of \(\{1, \ldots, n\}\) that are greater than or equal to a permutation \(p\) in the Bruhat order.

class sage.combinat.permutation.StandardPermutations_bruhat_smaller(p)

Bases: Permutations

Permutations of \(\{1, \ldots, n\}\) that are less than or equal to a permutation \(p\) in the Bruhat order.

class sage.combinat.permutation.StandardPermutations_descents(d, n)

Bases: StandardPermutations_n_abstract

Permutations of \(\{1, \ldots, n\}\) with a fixed set of descents.

cardinality()

Return the cardinality of self.

ALGORITHM:

The algorithm described in [Vie1979] is implemented naively.

EXAMPLES:

```python
sage: P = Permutations(descents=(\[1,0,2\], 5))
sage: P.cardinality()
4
```

first()

Return the first permutation with descents \(d\).

EXAMPLES:

```python
sage: Permutations(descents=(\[1,0,4,8\],12)).first()
[3, 2, 1, 4, 6, 5, 7, 8, 10, 9, 11, 12]
```

last()

Return the last permutation with descents \(d\).

EXAMPLES:

```python
sage: Permutations(descents=(\[1,0,4,8\],12)).last()
[12, 11, 8, 9, 10, 4, 5, 6, 7, 1, 2, 3]
```

class sage.combinat.permutation.StandardPermutations_n(n)

Bases: StandardPermutations_n_abstract

Permutations of the set \(\{1,2,\ldots,n\}\).

These are also called permutations of size \(n\), or the elements of the \(n\)-th symmetric group.

Todo: Have a reduced_word() which works in both multiplication conventions.

5.1. Comprehensive Module List
class `Element`(*parent, l, check=True*)

**Bases:** `Permutation`

- `apply_simple_reflection_left(i)`
  - Return `self` multiplied by the simple reflection `s[i]` on the left.
  - This acts by switching the entries in positions `i` and `i + 1`.

**Warning:** This ignores the multiplication convention in order to be consistent with other Coxeter operations in permutations (e.g., computing `reduced_word()`).

**EXAMPLES:**

```
sage: W = Permutations(3)
sage: w = W([2,3,1])
sage: w.apply_simple_reflection_left(1)
[1, 3, 2]
sage: w.apply_simple_reflection_left(2)
[3, 2, 1]
```

- `apply_simple_reflection_right(i)`
  - Return `self` multiplied by the simple reflection `s[i]` on the right.
  - This acts by switching the entries `i` and `i + 1`.

**Warning:** This ignores the multiplication convention in order to be consistent with other Coxeter operations in permutations (e.g., computing `reduced_word()`).

**EXAMPLES:**

```
sage: W = Permutations(3)
sage: w = W([2,3,1])
sage: w.apply_simple_reflection_right(1)
[3, 2, 1]
sage: w.apply_simple_reflection_right(2)
[2, 1, 3]
```

- `has_left_descent(i, multi=None)`
  - Check if `i` is a left descent of `self`.
  - A left descent of a permutation `π ∈ Sn` means an index `i ∈ {1, 2, ..., n−1}` such that `s_i π` has smaller length than `π`. Thus, a left descent of `π` is an index `i ∈ {1, 2, ..., n−1}` satisfying `π−1(i) > π−1(i+1)`.

**Warning:** The methods `descents()` and `idescents()` behave differently than their Weyl group counterparts. In particular, the indexing is 0-based. This could lead to errors. Instead, construct the descent set as in the example.

**Warning:** This ignores the multiplication convention in order to be consistent with other Coxeter operations in permutations (e.g., computing `reduced_word()`).
EXAMPLES:

```python
sage: P = Permutations(4)
sage: x = P([3, 2, 4, 1])
sage: (~x).descents()
[1, 2]
sage: [i for i in P.index_set() if x.has_left_descent(i)]
[1, 2]
```

`has_right_descent(i, mult=None)`

Check if \( i \) is a right descent of \( self \).

A right descent of a permutation \( \pi \in S_n \) means an index \( i \in \{1, 2, \ldots, n - 1\} \) such that \( \pi \circ s_i \) has smaller length than \( \pi \). Thus, a right descent of \( \pi \) is an index \( i \in \{1, 2, \ldots, n - 1\} \) satisfying \( \pi(i) > \pi(i + 1) \).

**Warning:** The methods `descents()` and `idescents()` behave differently than their Weyl group counterparts. In particular, the indexing is 0-based. This could lead to errors. Instead, construct the descent set as in the example.

**Warning:** This ignores the multiplication convention in order to be consistent with other Coxeter operations in permutations (e.g., computing `reduced_word()`).

EXAMPLES:

```python
sage: P = Permutations(4)
sage: x = P([3, 2, 4, 1])
sage: x.descents()
[1, 3]
sage: [i for i in P.index_set() if x.has_right_descent(i)]
[1, 3]
```

`inverse()`

Return the inverse of `self`.

EXAMPLES:

```python
sage: P = Permutations(4)
sage: w0 = P([4, 3, 2, 1])
sage: w0.inverse() == w0
True
sage: w0.inverse().parent() is P
True
sage: P([3, 2, 4, 1]).inverse()
[4, 2, 1, 3]
```

`algebra(base_ring, category=None)`

Return the symmetric group algebra associated to `self`.

**INPUT:**

- `base_ring` -- a ring
• **category** – a category (default: the category of *self*)

**EXAMPLES:**

```python
sage: P = Permutations(4)
sage: A = P.algebra(QQ); A
Symmetric group algebra of order 4 over Rational Field

sage: A.category()
Join of Category of coxeter group algebras over Rational Field
    and Category of finite group algebras over Rational Field
    and Category of finite dimensional cellular algebras with basis over Rational Field

sage: A = P.algebra(QQ, category=Monoids())
sage: A.category()
Category of finite dimensional cellular monoid algebras over Rational Field
```

**as_permutation_group()**

Return *self* as a permutation group.

**EXAMPLES:**

```python
sage: P = Permutations(4)
sage: PG = P.as_permutation_group()
sage: PG
Symmetric group of order 4! as a permutation group

sage: G = SymmetricGroup(4)
sage: PG is G
True
```

**cardinality()**

Return the number of permutations of size *n*, which is *n*!.

**EXAMPLES:**

```python
sage: Permutations(0).cardinality()
1
sage: Permutations(3).cardinality()
6
sage: Permutations(4).cardinality()
24
```

**cartan_type()**

Return the Cartan type of *self*.

The symmetric group *S*ₙ is a Coxeter group of type *A*ₙ₋₁.

**EXAMPLES:**

```python
sage: A = SymmetricGroup([2,3,7]); A.cartan_type()
['A', 2]
sage: A = SymmetricGroup([]); A.cartan_type()
['A', 0]
```
codegrees()

Return the codegrees of self.

EXAMPLES:

```
sage: Permutations(3).codegrees()
(0, 1)
sage: Permutations(7).codegrees()
(0, 1, 2, 3, 4, 5)
```

conjugacy_class(g)

Return the conjugacy class of g in self.

INPUT:

- g – a partition or an element of self

EXAMPLES:

```
sage: G = Permutations(5)
sage: g = G([2,3,4,1,5])
sage: G.conjugacy_class(g)
Conjugacy class of cycle type [4, 1] in Standard permutations of 5
sage: G.conjugacy_class(Partition([2, 1, 1, 1]))
Conjugacy class of cycle type [2, 1, 1, 1] in Standard permutations of 5
```

conjugacy_classes()

Return a list of the conjugacy classes of self.

EXAMPLES:

```
sage: G = Permutations(4)
sage: G.conjugacy_classes()
[Conjugacy class of cycle type [1, 1, 1, 1] in Standard permutations of 4,
 Conjugacy class of cycle type [2, 1, 1] in Standard permutations of 4,
 Conjugacy class of cycle type [2, 2] in Standard permutations of 4,
 Conjugacy class of cycle type [3, 1] in Standard permutations of 4,
```

conjugacy_classes_iterator()

Iterate over the conjugacy classes of self.

EXAMPLES:

```
sage: G = Permutations(4)
sage: list(G.conjugacy_classes_iterator()) == G.conjugacy_classes()
True
```

conjugacy_classes_representatives()

Return a complete list of representatives of conjugacy classes in self.

Let \( S_n \) be the symmetric group on \( n \) letters. The conjugacy classes are indexed by partitions \( \lambda \) of \( n \). The ordering of the conjugacy classes is reverse lexicographic order of the partitions.

EXAMPLES:
sage: G = Permutations(5)
sage: G.conjugacy_classes_representatives()
[[1, 2, 3, 4, 5],
 [2, 1, 3, 4, 5],
 [2, 1, 4, 3, 5],
 [2, 3, 1, 4, 5],
 [2, 3, 1, 5, 4],
 [2, 3, 4, 1, 5],
 [2, 3, 4, 5, 1]]

degree()
Return the degree of self.
This is the cardinality \( n \) of the set self acts on.
EXAMPLES:

```
sage: Permutations(0).degree()
0
sage: Permutations(1).degree()
1
sage: Permutations(5).degree()
5
```
degrees()
Return the degrees of self.
These are the degrees of the fundamental invariants of the ring of polynomial invariants.
EXAMPLES:

```
sage: Permutations(3).degrees()
(2, 3)
sage: Permutations(7).degrees()
(2, 3, 4, 5, 6, 7)
```
element_in_conjugacy_classes(\( \nu \))
Return a permutation with cycle type \( \nu \).
If the size of \( \nu \) is smaller than the size of permutations in self, then some fixed points are added.
EXAMPLES:

```
sage: PP = Permutations(5)
sage: PP.element_in_conjugacy_classes([2,2])
[2, 1, 4, 3, 5]
```

identity()
Return the identity permutation of size \( n \).
EXAMPLES:

```
sage: Permutations(4).identity()
[1, 2, 3, 4]
sage: Permutations(0).identity()
[]
```
index_set()
Return the index set for the descents of the symmetric group self.
This is \{1, 2, \ldots, n - 1\}, where self is \(S_n\).
EXAMPLES:

```
    sage: P = Permutations(8)
    sage: P.index_set()
    (1, 2, 3, 4, 5, 6, 7)
```

one()
Return the identity permutation of size \(n\).
EXAMPLES:

```
    sage: Permutations(4).identity()
    [1, 2, 3, 4]
    sage: Permutations(0).identity()
    []
```

random_element()
EXAMPLES:

```
    sage: s = Permutations(4).random_element(); s  # random
    [1, 2, 4, 3]
    sage: s in Permutations(4)
    True
```

class

rank(p=None)
Return the rank of self or p depending on input.
If a permutation p is given, return the rank of p in self. Otherwise return the dimension of the underlying vector space spanned by the (simple) roots.
EXAMPLES:

```
    sage: P = Permutations(5)
    sage: P.rank()
    4
    sage: SP3 = Permutations(3)
    sage: list(map(SP3.rank, SP3))
    [0, 1, 2, 3, 4, 5]
    sage: SP0 = Permutations(0)
    sage: list(map(SP0.rank, SP0))
    [0]
```

simple_reflection(i)
For \(i\) in the index set of self (that is, for \(i\) in \(\{1, 2, \ldots, n - 1\}\), where self is \(S_n\)), this returns the elementary transposition \(s_i = (i, i + 1)\).
EXAMPLES:

```
    sage: P = Permutations(4)
    sage: P.simple_reflection(2)
```
Combinatorics, Release 10.0

[1, 3, 2, 4]

sage: P.simple_reflections()
Finite family {1: [2, 1, 3, 4], 2: [1, 3, 2, 4], 3: [1, 2, 4, 3]}

unrank(r)

EXAMPLES:

sage: SP3 = Permutations(3)
sage: l = list(map(SP3.unrank, range(6)))
sage: l == SP3.list()
True

sage: SP0 = Permutations(0)
sage: l = list(map(SP0.unrank, range(1)))
sage: l == SP0.list()
True

class sage.combinat.permutation.StandardPermutations_n_abstract(n, category=None)

Bases: Permutations

Abstract base class for subsets of permutations of the set \{1, 2, \ldots, n\}.

Warning: Anything inheriting from this class should override the \_\_contains\_\_ method.

class sage.combinat.permutation.StandardPermutations_recoils(recoils)

Bases: Permutations

Permutations of \{1, \ldots, n\} with a fixed recoils composition.

class sage.combinat.permutation.StandardPermutations_recoilsfatter(recoils)

Bases: Permutations

class sage.combinat.permutation.StandardPermutations_recoilsfiner(recoils)

Bases: Permutations

sage.combinat.permutation.bistochastic_as_sum_of_permutations(M, check=True)

Return the positive sum of permutations corresponding to the bistochastic matrix \( M \).

A stochastic matrix is a matrix with nonnegative real entries such that the sum of the elements of any row is equal to 1. A bistochastic matrix is a stochastic matrix whose transpose matrix is also stochastic (there are conditions both on the rows and on the columns).

According to the Birkhoff-von Neumann Theorem, any bistochastic matrix can be written as a convex combination of permutation matrices, which also means that the polytope of bistochastic matrices is integer.

As a non-bistochastic matrix can obviously not be written as a convex combination of permutations, this theorem is an equivalence.

This function, given a bistochastic matrix, returns the corresponding decomposition.

INPUT:

- \( M \) – A bistochastic matrix
- \( \text{check} \) (boolean) – set to True (default) to check that the matrix is indeed bistochastic

OUTPUT:
• An element of `CombinatorialFreeModule`, which is a free $F$-module (where $F$ is the ground ring of the given matrix) whose basis is indexed by the permutations.

**Note:**

• In this function, we just assume $1$ to be any constant: for us a matrix $M$ is bistochastic if there exists $c > 0$ such that $M/c$ is bistochastic.

• You can obtain a sequence of pairs `(permutation, coeff)`, where `permutation` is a Sage `Permutation` instance, and `coeff` its corresponding coefficient from the result of this function by applying the `list` function.

• If you are interested in the matrix corresponding to a `Permutation` you will be glad to learn about the `Permutation.to_matrix()` method.

• The base ring of the matrix can be anything that can be coerced to `RR`.

**See also:**

• `as_sum_of_permutations()` to use this method through the `Matrix` class.

**EXAMPLES:**

We create a bistochastic matrix from a convex sum of permutations, then try to deduce the decomposition from the matrix:

```python
sage: from sage.combinat.permutation import bistochastic_as_sum_of_permutations
sage: L = []
sage: L.append((9, Permutation([4, 1, 3, 5, 2])))
sage: L.append((6, Permutation([5, 3, 4, 1, 2])))
sage: L.append((3, Permutation([3, 1, 4, 2, 5])))
sage: L.append((2, Permutation([1, 4, 2, 3, 5])))
sage: M = sum([c * p.to_matrix() for (c, p) in L])
sage: decomp = bistochastic_as_sum_of_permutations(M)
sage: print(decomp)
2*B[[1, 4, 2, 3, 5]] + 3*B[[3, 1, 4, 2, 5]] + 9*B[[4, 1, 3, 5, 2]] + 6*B[[5, 3, 4, ˓→1, 2]]
```

An exception is raised when the matrix is not positive and bistochastic:

```python
sage: M = Matrix([[2,3],[2,2]])
sage: decom = bistochastic_as_sum_of_permutations(M)
Traceback (most recent call last):
  ... ValueError: The matrix is not bistochastic
```

```python
sage: bistochastic_as_sum_of_permutations(Matrix(GF(7), 2, [2,1,1,2]))
Traceback (most recent call last):
  ... ValueError: The base ring of the matrix must have a coercion map to RR
```

```python
sage: bistochastic_as_sum_of_permutations(Matrix(ZZ, 2, [2,-1,-1,2]))
Traceback (most recent call last):
  ... ValueError: The matrix should have nonnegative entries
```
sage.combinat.permutation.bounded_affine_permutation(A)

Return the bounded affine permutation of a matrix.

The *bounded affine permutation* of a matrix \(A\) with entries in \(R\) is a partial permutation of length \(n\), where \(n\) is the number of columns of \(A\). The entry in position \(i\) is the smallest value \(j\) such that column \(i\) is in the span of columns \(i+1, \ldots, j\) over \(R\), where column indices are taken modulo \(n\). If column \(i\) is the zero vector, then the permutation has a fixed point at \(i\).

**INPUT:**

- \(A\) – matrix with entries in a ring \(R\)

**EXAMPLES:**

```python
sage: from sage.combinat.permutation import bounded_affine_permutation
sage: A = Matrix(ZZ, [[[1,0,0,0], [0,1,0,0]]])
[5, 6, 3, 4]
sage: A = Matrix(ZZ, [[[0,1,0,1,0], [0,0,1,1,0]]])
[1, 4, 7, 8, 5]
```

**REFERENCES:**

- [KLS2013]

sage.combinat.permutation.bruhat_lequal(p1, p2)

Return True if \(p1\) is less than \(p2\) in the Bruhat order.

Algorithm from mupad-combinat.

**EXAMPLES:**

```python
sage: import sage.combinat.permutation as permutation
deprecated::
sage: import sage.combinat.permutation as permutation
```

sage.combinat.permutation.descents_composition_first(dc)

Compute the smallest element of a descent class having a descent composition \(dc\).

**EXAMPLES:**

```python
sage: import sage.combinat.permutation as permutation
deprecated::
sage: import sage.combinat.permutation as permutation
```

sage.combinat.permutation.descents_composition_last(dc)

Return the largest element of a descent class having a descent composition \(dc\).

**EXAMPLES:**

```python
sage: import sage.combinat.permutation as permutation
deprecated::
sage: import sage.combinat.permutation as permutation
```

sage.combinat.permutation.descents_composition_list(dc)

Return a list of all the permutations that have the descent composition \(dc\).
Combinatorics, Release 10.0

EXAMPLES:

```python
sage: import sage.combinat.permutation as permutation
sage: permutation.descents_composition_list([1,2,2])
[[5, 2, 4, 1, 3],
 [5, 3, 4, 1, 2],
 [4, 3, 5, 1, 2],
 [4, 2, 5, 1, 3],
 [3, 2, 5, 1, 4],
 [2, 1, 5, 3, 4],
 [3, 1, 5, 2, 4],
 [4, 1, 5, 2, 3],
 [5, 1, 4, 2, 3],
 [5, 1, 3, 2, 4],
 [4, 1, 3, 2, 5],
 [3, 1, 4, 2, 5],
 [2, 1, 4, 3, 5],
 [3, 2, 4, 1, 5],
 [4, 2, 3, 1, 5],
 [5, 2, 3, 1, 4]]
```

`sage.combinat.permutation.from_cycles(n, cycles, parent=None)`

Return the permutation in the \( n \)-th symmetric group whose decomposition into disjoint cycles is `cycles`.

This function checks that its input is correct (i.e. that the cycles are disjoint and their elements integers among 1...\( n \)). It raises an exception otherwise.

**Warning:** It assumes that the elements are of `int` type.

EXAMPLES:

```python
sage: import sage.combinat.permutation as permutation
sage: permutation.from_cycles(4, [[1,2]])
[2, 1, 3, 4]
sage: permutation.from_cycles(4, [[1,2,4]])
[2, 4, 3, 1]
sage: permutation.from_cycles(10, [[3,1],[4,5],[6,8,9]])
[3, 2, 1, 5, 4, 8, 7, 9, 6, 10]
sage: permutation.from_cycles(10, ((2, 5), (6, 1, 3)))
[3, 5, 6, 4, 2, 1, 7, 8, 9, 10]
sage: permutation.from_cycles(4, [])
[1, 2, 3, 4]
sage: permutation.from_cycles(4, [[]])
[1, 2, 3, 4]
sage: permutation.from_cycles(0, [])
[]
```

Bad input (see github issue #13742):

```python
sage: Permutation("(-12,2)(3,4)")
Traceback (most recent call last):
...
ValueError: all elements should be strictly positive integers, but I found -12
```

(continues on next page)
sage.combinat.permutation.from_inversion_vector(iv, parent=None)

Return the permutation corresponding to inversion vector iv.

See `sage.combinat.permutation.Permutation.to_inversion_vector` for a definition of the inversion vector of a permutation.

EXAMPLES:

```python
sage: import sage.combinat.permutation as permutation
sage: permutation.from_inversion_vector([3,1,0,0,0])
[3, 2, 4, 1, 5]
sage: permutation.from_inversion_vector([2,3,6,4,0,2,2,1,0])
[5, 9, 1, 8, 2, 6, 4, 7, 3]
sage: permutation.from_inversion_vector([0])
[1]
sage: permutation.from_inversion_vector([])
[]
```

sage.combinat.permutation.from_lehmer_cocode(lehmer, parent=Standard permutations)

Return the permutation with Lehmer cocode lehmer.

The Lehmer cocode of a permutation \( p \) is defined as the list \((c_1, c_2, \ldots, c_n)\), where \( c_i \) is the number of \( j < i \) such that \( p(j) > p(i) \).

EXAMPLES:

```python
sage: import sage.combinat.permutation as permutation
sage: lcc = Permutation([2,1,5,4,3]).to_lehmer_cocode(); lcc
[0, 1, 0, 1, 2]
sage: permutation.from_lehmer_cocode(lcc)
[2, 1, 5, 4, 3]
```

sage.combinat.permutation.from_lehmer_code(lehmer, parent=None)

Return the permutation with Lehmer code lehmer.

EXAMPLES:

```python
sage: import sage.combinat.permutation as permutation
sage: lc = Permutation([2,1,5,4,3]).to_lehmer_code(); lc
[1, 0, 2, 1, 0]
sage: permutation.from_lehmer_code(lc)
[2, 1, 5, 4, 3]
```

sage.combinat.permutation.from_major_code(mc, final_descent=False)

Return the permutation with major code mc.

The major code of a permutation is defined in `to_major_code()`.
**Warning:** This function creates illegal permutations (i.e. `Permutation([9])`), and this is dangerous as the `Permutation()` class is only designed to handle permutations on $1,...,n$. This will have to be changed when Sage permutations will be able to handle anything, but right now this should be fixed. Be careful with the results.

**Warning:** If mc is not a major index of a permutation, then the return value of this method can be anything. Garbage in, garbage out!

REFERENCES:


EXAMPLES:

```
sage: import sage.combinat.permutation as permutation
sage: permutation.from_major_code([5, 0, 1, 0, 1, 2, 0, 1, 0])
[9, 3, 5, 7, 2, 1, 4, 6, 8]
sage: permutation.from_major_code([8, 3, 3, 1, 4, 0, 1, 0, 0])
[2, 8, 4, 3, 6, 7, 9, 5, 1]
sage: Permutation([2,1,6,4,7,3,5]).to_major_code()
[3, 2, 0, 2, 2, 0, 0]
sage: permutation.from_major_code([3, 2, 0, 2, 2, 0, 0])
[2, 1, 6, 4, 7, 3, 5]
```

```
sage.combinat.permutation.from_permutation_group_element(pge, parent=None)
```

Return a `Permutation` given a `PermutationGroupElement` pge.

EXAMPLES:

```
sage: import sage.combinat.permutation as permutation
sage: pge = PermutationGroupElement([(1,2),(3,4)])
sage: permutation.from_permutation_group_element(pge)
[2, 1, 4, 3]
```

```
sage.combinat.permutation.from_rank(n, rank)
```

Return the permutation of the set $\{1, ..., n\}$ with lexicographic rank `rank`. This is the permutation whose Lehmer code is the factoradic representation of `rank`. In particular, the permutation with rank 0 is the identity permutation.

The permutation is computed without iterating through all of the permutations with lower rank. This makes it efficient for large permutations.

**Note:** The variable `rank` is not checked for being in the interval from 0 to $n! - 1$. When outside this interval, it acts as its residue modulo $n!$.

EXAMPLES:

```
sage: import sage.combinat.permutation as permutation
sage: Permutation([3, 6, 5, 4, 2, 1]).rank()
359
sage: [permutation.from_rank(3, i) for i in range(6)]
```

(continues on next page)
sage.combinat.permutation.from_reduced_word(rw, parent=None)

Return the permutation corresponding to the reduced word rw.

See reduced_words() for a definition of reduced words and the convention on the order of multiplication used.

EXAMPLES:

sage: import sage.combinat.permutation as permutation
sage: permutation.from_reduced_word([3, 2, 3, 1, 2, 3, 1])
[3, 4, 2, 1]
sage: permutation.from_reduced_word([])
[]

sage.combinat.permutation.permutohedron_lequal(p1, p2, side='right')

Return True if p1 is less than or equal to p2 in the permutohedron order.

By default, the computations are done in the right permutohedron. If you pass the option side='left', then they will be done in the left permutohedron.

EXAMPLES:

sage: import sage.combinat.permutation as permutation
sage: permutation.permutohedron_lequal(Permutation([3, 2, 1, 4]), Permutation([4, 2, 1, 3]))
False
sage: permutation.permutohedron_lequal(Permutation([3, 2, 1, 4]), Permutation([4, 2, 1, 3]), side='left')
True

sage.combinat.permutation.to_standard(p, key=None)

Return a standard permutation corresponding to the iterable p.

INPUT:

- p – an iterable
- key – (optional) a comparison key for the element x of p

EXAMPLES:

sage: import sage.combinat.permutation as permutation
sage: permutation.to_standard([4, 2, 7])
[2, 1, 3]
sage: permutation.to_standard([1, 2, 3])
[1, 2, 3]
sage: permutation.to_standard([])
[]
sage: permutation.to_standard([1, 2, 3], key=lambda x: -x)
[3, 2, 1]
5.1.173 Permutations (Cython file)

This is a nearly-straightforward implementation of what Knuth calls “Algorithm P” in TAOCP 7.2.1.2. The intent is to be able to enumerate permutation by “plain changes”, or multiplication by adjacent transpositions, as a generator. This is useful when a class of objects is inherently enumerated by permutations, but it is faster to swap items in a permutation than construct the next object directly from the next permutation in a list. The backtracking algorithm in sage/graphs/genus.pyx is an example of this.

The lowest level is implemented as a struct with auxiliary methods. This is because Cython does not allow pointers to class instances, so a list of these objects is inherently slower than a list of structs. The author prefers ugly code to slow code.

For those willing to sacrifice a (very small) amount of speed, we provide a class that wraps our struct.

sage.combinat.permutation_cython.left_action_product(S, lp)

Return the permutation obtained by composing a permutation S with a permutation lp in such an order that lp is applied first and S is applied afterwards.

See also:

sage.combinat.permutation.Permutation.left_action_product()

EXAMPLES:

```
sage: p = [2,1,3,4]
sage: q = [3,1,2]
sage: from sage.combinat.permutation_cython import left_action_product
sage: left_action_product(p, q)
[3, 2, 1, 4]
sage: left_action_product(q, p)
[1, 3, 2, 4]
sage: q
[3, 1, 2]
```

sage.combinat.permutation_cython.left_action_same_n(S, lp)

Return the permutation obtained by composing a permutation S with a permutation lp in such an order that lp is applied first and S is applied afterwards and S and lp are of the same length.

See also:

sage.combinat.permutation.Permutation.left_action_product()

EXAMPLES:

```
sage: p = [2,1,3]
sage: q = [3,1,2]
sage: from sage.combinat.permutation_cython import left_action_same_n
sage: left_action_same_n(p, q)
[3, 2, 1]
sage: left_action_same_n(q, p)
[1, 3, 2]
```
sage.combinat.permutation_cython.map_to_list(l, values, n)

Build a list by mapping the array l using values.

**Warning:** There is no check of the input data at any point. Using wrong types or values with wrong length is likely to result in a Sage crash.

**INPUT:**

- l – array of unsigned int (i.e., type ‘I’)
- values – tuple; the values of the permutation
- n – int; the length of the array l

**OUTPUT:**

A list representing the permutation.

**EXAMPLES:**

```python
sage: from array import array
sage: from sage.combinat.permutation_cython import map_to_list
sage: l = array('I', [0, 1, 0, 3, 3, 0, 1])
sage: map_to_list(l, ('a', 'b', 'c', 'd'), 7)
['a', 'b', 'a', 'd', 'd', 'a', 'b']
```

sage.combinat.permutation_cython.next_perm(l)

Obtain the next permutation under lex order of l by mutating l.

Algorithm based on: [http://marknelson.us/2002/03/01/next-permutation/](http://marknelson.us/2002/03/01/next-permutation/)

**INPUT:**

- l – array of unsigned int (i.e., type ‘I’)

**Warning:** This method mutates the array l.

**OUTPUT:**

boolean; whether another permutation was obtained

**EXAMPLES:**

```python
sage: from sage.combinat.permutation_cython import next_perm
sage: from array import array
sage: L = array('I', [1, 1, 2, 3])
sage: while next_perm(L):
    ....:     print(L)
array('I', [1, 1, 3, 2])
array('I', [1, 2, 1, 3])
array('I', [1, 2, 3, 1])
array('I', [1, 3, 1, 2])
array('I', [1, 3, 2, 1])
array('I', [2, 1, 1, 3])
array('I', [2, 1, 3, 1])
array('I', [2, 3, 1, 1])
```
array('I', [3, 1, 1, 2])
array('I', [3, 1, 2, 1])
array('I', [3, 2, 1, 1])

sage.combinat.permutation_cython.permutation_iterator_transposition_list(n)

Returns a list of transposition indices to enumerate the permutations on \( n \) letters by adjacent transpositions. Assumes zero-based lists. We artificially limit the argument to \( n < 12 \) to avoid overflowing 32-bit pointers. While the algorithm works for larger \( n \), the user is encouraged to avoid filling anything more than 4GB of memory with the output of this function.

EXAMPLES:

```python
sage: import sage.combinat.permutation_cython
sage: from sage.combinat.permutation_cython import permutation_iterator_transposition_list
sage: permutation_iterator_transposition_list(4)
[2, 1, 0, 2, 0, 1, 2, 0, 2, 1, 0, 2, 0, 1, 2, 0, 2, 1, 0, 2, 0, 1, 2]
sage: permutation_iterator_transposition_list(200)
Traceback (most recent call last):
  ... ValueError: Cowardly refusing to enumerate the permutations on more than 12 letters.
sage: permutation_iterator_transposition_list(1)
[]
```

sage.combinat.permutation_cython.right_action_product(S, rp)

Return the permutation obtained by composing a permutation \( S \) with a permutation \( rp \) in such an order that \( S \) is applied first and \( rp \) is applied afterwards.

See also:

sage.combinat.permutation.Permutation.right_action_product()

EXAMPLES:

```python
sage: p = [2,1,3,4]
sage: q = [3,1,2]
sage: from sage.combinat.permutation_cython import right_action_product
sage: right_action_product(p, q)
[1, 3, 2, 4]
sage: right_action_product(q, p)
[3, 2, 1, 4]
sage: q
[3, 1, 2]
```

sage.combinat.permutation_cython.right_action_same_n(S, rp)

5.1. Comprehensive Module List
Return the permutation obtained by composing a permutation \( S \) with a permutation \( rp \) in such an order that \( S \) is applied first and \( rp \) is applied afterwards and \( S \) and \( rp \) are of the same length.

**See also:**

sage.combinat.permutation.Permutation.right_action_product()

**EXAMPLES:**

```python
sage: p = [2,1,3]
sage: q = [3,1,2]
sage: from sage.combinat.permutation_cython import right_action_same_n
sage: right_action_same_n(p, q)
[1, 3, 2]
sage: right_action_same_n(q, p)
[3, 2, 1]
```

**5.1.174 Posets**

Common posets can be accessed through posets.<tab> and are listed in the posets catalog:

- Catalog of posets and lattices

Poset-related classes:

- Finite posets
- Finite lattices and semilattices
- Linear Extensions of Posets
- D-Complete Posets
- Forest Posets
- Mobile posets
- Incidence Algebras
- Cartesian products of Posets
- Möbius Algebras
- Generalized Tamari lattices
- Tamari Interval-posets
- Shard intersection order

If you are looking for Poset-related categories, see Posets, FinitePosets, LatticePosets and FiniteLatticePosets.
5.1.175 Cartesian products of Posets

AUTHORS:

• Daniel Krenn (2015)

class sage.combinat.posets.cartesian_product.CartesianProductPoset(sets, category, order=None, **kwargs)

Bases: CartesianProduct

A class implementing Cartesian products of posets (and elements thereof). Compared to CartesianProduct you are able to specify an order for comparison of the elements.

INPUT:

• sets – a tuple of parents.

• category – a subcategory of Sets().CartesianProducts() & Posets().

• order – a string or function specifying an order less or equal. It can be one of the following:
  – 'native' – elements are ordered by their native ordering, i.e., the order the wrapped elements (tuples) provide.
  – 'lex' – elements are ordered lexicographically.
  – 'product' – an element is less or equal to another element, if less or equal is true for all its components (Cartesian projections).
  – A function which performs the comparison \( \leq \). It takes two input arguments and outputs a boolean.

Other keyword arguments (kwargs) are passed to the constructor of CartesianProduct.

EXAMPLES:

```sage
sage: P = Poset((srange(3), lambda left, right: left <= right))
sage: Cl = cartesian_product((P, P), order='lex')
sage: Cl((1, 1)) <= Cl((2, 0))
True
sage: Cp = cartesian_product((P, P), order='product')
sage: Cp((1, 1)) <= Cp((2, 0))
False
sage: def le_sum(left, right):
    ....:     return (sum(left) < sum(right) or
    ....:         sum(left) == sum(right) and left[0] <= right[0])
sage: Cs = cartesian_product((P, P), order=le_sum)
sage: Cs((1, 1)) <= Cs((2, 0))
True
```

See also:

CartesianProduct

class Element

Bases: Element

le(left, right)

Test whether left is less than or equal to right.

INPUT:

• left – an element.
• right – an element.

OUTPUT:
A boolean.

Note: This method uses the order defined on creation of this Cartesian product. See CartesianProductPoset.

EXAMPLES:

```python
sage: P = posets.ChainPoset(10)
sage: def le_sum(left, right):
....:     return (sum(left) < sum(right) or
....:             sum(left) == sum(right) and left[0] <= right[0])
sage: C = cartesian_product((P, P), order=le_sum)
sage: C.le(C((1, 6)), C((6, 1)))
True
sage: C.le(C((6, 1)), C((1, 6)))
False
sage: C.le(C((1, 6)), C((6, 6)))
True
sage: C.le(C((6, 6)), C((1, 6)))
False
```

le_lex(left, right)
Test whether left is lexicographically smaller or equal to right.

INPUT:

• left – an element.
• right – an element.

OUTPUT:
A boolean.

EXAMPLES:

```python
sage: P = Poset((srange(2), lambda left, right: left <= right))
sage: Q = cartesian_product((P, P), order='lex')
sage: T = [Q((0, 0)), Q((1, 1)), Q((0, 1)), Q((1, 0))]
sage: for a in T:
....:     for b in T:
....:         assert Q.le(a, b) == (a <= b)
....:         print("%s <= %s = %s" % (a, b, a <= b))
(0, 0) <= (0, 0) = True
(0, 0) <= (1, 1) = True
(0, 0) <= (0, 1) = True
(0, 0) <= (1, 0) = True
(1, 1) <= (0, 0) = False
(1, 1) <= (1, 1) = True
(1, 1) <= (0, 1) = False
(1, 1) <= (1, 0) = False
(0, 1) <= (0, 0) = False
(0, 1) <= (0, 1) = False
(0, 1) <= (1, 0) = False
(0, 1) <= (1, 1) = False
```

(continues on next page)
(0, 1) <= (1, 1) = True
(0, 1) <= (0, 1) = True
(0, 1) <= (1, 0) = True
(1, 0) <= (0, 0) = False
(1, 0) <= (1, 1) = True
(1, 0) <= (0, 1) = False
(1, 0) <= (1, 0) = True

le_native(left, right)

Test whether left is smaller or equal to right in the order provided by the elements themselves.

INPUT:

• left – an element.
• right – an element.

OUTPUT:

A boolean.

EXAMPLES:

sage: P = Poset((srange(2), lambda left, right: left <= right))
sage: Q = cartesian_product((P, P), order='native')
sage: T = [Q((0, 0)), Q((1, 1)), Q((0, 1)), Q((1, 0))]
sage: for a in T:
    for b in T:
        assert Q.le(a, b) == (a <= b)
        print('%s <= %s = %s' % (a, b, a <= b))

(0, 0) <= (0, 0) = True
(0, 0) <= (1, 1) = True
(0, 0) <= (0, 1) = True
(0, 0) <= (1, 0) = True
(1, 1) <= (0, 0) = False
(1, 1) <= (1, 1) = True
(1, 1) <= (0, 1) = False
(1, 1) <= (1, 0) = False
(0, 1) <= (0, 0) = False
(0, 1) <= (0, 1) = True
(0, 1) <= (1, 0) = True
(0, 1) <= (1, 1) = True
(1, 0) <= (0, 0) = False
(1, 0) <= (0, 1) = False
(1, 0) <= (1, 0) = True

le_product(left, right)

Test whether left is component-wise smaller or equal to right.

INPUT:

• left – an element.
• right – an element.

OUTPUT:
A boolean.

The comparison is True if the result of the comparison in each component is True.

EXAMPLES:

```
sage: P = Poset((srange(2), lambda left, right: left <= right))
sage: Q = cartesian_product((P, P), order='product')
sage: T = [Q((0, 0)), Q((1, 1)), Q((0, 1)), Q((1, 0))]
sage: for a in T:
    ....:    for b in T:
    ....:        assert (Q.le(a, b) == (a <= b))
    ....:        print('%%s <= %%s = %s' % (a, b, a <= b))

(0, 0) <= (0, 0) = True
(0, 0) <= (1, 1) = True
(0, 0) <= (0, 1) = True
(0, 0) <= (1, 0) = True
(1, 1) <= (0, 0) = False
(1, 1) <= (1, 1) = True
(1, 1) <= (0, 1) = False
(1, 1) <= (1, 0) = False
(0, 0) <= (0, 0) = False
(0, 0) <= (0, 1) = True
(0, 0) <= (1, 0) = True
(0, 1) <= (0, 0) = False
(0, 1) <= (0, 1) = True
(0, 1) <= (1, 0) = False
(1, 0) <= (0, 0) = False
(1, 0) <= (0, 1) = False
(1, 0) <= (1, 0) = True
```

5.1.176 D-Complete Posets

AUTHORS:

- Stefan Grosser (06-2020): initial implementation

```
class sage.combinat.posets.d_complete.DCompletePoset(hasse_diagram, elements, category, facade, key):
    Bases: FiniteJoinSemilattice

A d-complete poset.

D-complete posets are a class of posets introduced by Proctor in [Proc1999]. It includes common families such as shapes, shifted shapes, and rooted forests. Proctor showed in [PDynk1999] that d-complete posets have decompositions in irreducible posets, and showed in [Proc2014] that d-complete posets admit a hook-length formula (see Wikipedia article Hook_length_formula). A complete proof of the hook-length formula can be found in [KY2019].

EXAMPLES:

```
sage: from sage.combinat.posets.poset_examples import Posets
sage: P = Posets.DoubleTailedDiamond(2)
sage: TestSuite(P).run()

def get_hook(self):
    Return the hook length of the element elmt.
```
EXAMPLES:
```
sage: from sage.combinat.posets.d_complete import DCompletePoset
da_complete = DCompletePoset(DiGraph({0: [1], 1: [2]}))
sage: P = DCompletePoset(DiGraph({0: [1, 2], 1: [3], 2: [3], 3: []}))
sage: P.get_hooks()
{0: 1, 1: 2, 2: 2, 3: 3}
sage: P = DCompletePoset(Posets.YoungDiagramPoset(Partition([3, 2, 1]))._hasse_diagram.reverse())
sage: P.get_hooks()
{0: 5, 1: 3, 2: 1, 3: 3, 4: 1, 5: 1}
```

`get_hooks()`

Return all the hook lengths as a dictionary.

EXAMPLES:
```
sage: from sage.combinat.posets.d_complete import DCompletePoset
da_complete = DCompletePoset(DiGraph({0: [1, 2], 1: [3], 2: [3], 3: []}))
sage: P = DCompletePoset(P.posets.YoungDiagramPoset(Partition([3, 2, 1]))._hasse_diagram.reverse())
sage: P.get_hooks()
{0: 5, 1: 3, 2: 1, 3: 3, 4: 1, 5: 1}
```

`hook_product()`

Return the hook product for the poset.

### 5.1.177 Mobile posets

**class** `sage.combinat.posets.mobile.MobilePoset(hasse_diagram, elements, category, facade, key, ribbon=None, check=True)`

Bases: `FinitePoset`

A mobile poset.

Mobile posets are an extension of d-complete posets which permit a determinant formula for counting linear extensions. They are formed by having a ribbon poset with d-complete posets ‘hanging’ below it and at most one d-complete poset above it, known as the anchor. See [GGMM2020] for the definition.

EXAMPLES:
```
sage: P = posets.MobilePoset(posets.RibbonPoset(7, [1,3]),
.....: {1: [posets.YoungDiagramPoset([3, 2], dual=True)],
.....: 3: [posets.DoubleTailedDiamond(6)]},
.....: anchor=(4, 2, posets.ChainPoset(6)))
sage: len(P._ribbon)
8
sage: P._anchor
(4, 5)
```

This example is Example 5.9 in [GGMM2020]:
```
sage: P1 = posets.MobilePoset(posets.RibbonPoset(8, [2,3,4]),
.....: {4: [posets.ChainPoset(1)]},
.....: anchor=(3, 0, posets.ChainPoset(1)))
sage: sorted([P1._element_to_vertex(i) for i in P1._ribbon])
```

[0, 1, 2, 6, 7, 9]
sage: P1._anchor
(3, 2)

sage: P2 = posets.MobilePoset(posets.RibbonPoset(15, [1,3,5,7,9,11,13]),
....:   {}, anchor=(8, 0, posets.ChainPoset(1)))
sage: sorted(P2._ribbon)
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]
sage: P2._anchor
(8, (8, 0))
sage: P2.linear_extensions().cardinality()
21399440939

sage: EP = posets.MobilePoset(posets.ChainPoset(0), {})
Traceback (most recent call last):
...
ValueError: the empty poset is not a mobile poset

anchor()
Return the anchor of the mobile poset.

EXAMPLES:

sage: from sage.combinat.posets.mobile import MobilePoset
sage: M2 = MobilePoset(Poset([[0,1,2,3,4,5,6,7,8],
....:   [(1,0),(3,0),(2,1),(2,3),(4,3),(5,4),(7,4),(7,8)]])
sage: M2.anchor()
(4, 3)
sage: M3 = MobilePoset(Posets.RibbonPoset(5, [1,2]))
sage: M3.anchor() is None
True

ribbon()
Return the ribbon of the mobile poset.

EXAMPLES:

sage: from sage.combinat.posets.mobile import MobilePoset
sage: M3 = MobilePoset(Posets.RibbonPoset(5, [1,2]))
sage: sorted(M3.ribbon())
[1, 2, 3, 4]

5.1.178 Elements of posets, lattices, semilattices, etc.

class sage.combinat.posets.elements.JoinSemilatticeElement(poset, element, vertex)
Bases: PosetElement

class sage.combinat.posets.elements.LatticePosetElement(poset, element, vertex)
Bases: MeetSemilatticeElement, JoinSemilatticeElement

class sage.combinat.posets.elements.MeetSemilatticeElement(poset, element, vertex)
Bases: PosetElement
class sage.combinat.posets.elements.PosetElement(poset, element, vertex)
Bases: Element

Establish the parent-child relationship between poset and element, where element is associated to the vertex of the Hasse diagram of the poset.

INPUT:
• poset – a poset object
• element – any object
• vertex – a vertex of the Hasse diagram of the poset

5.1.179 Forest Posets

AUTHORS:
• Stefan Grosser (06-2020): initial implementation

class sage.combinat.posets.forest.ForestPoset(hasse_diagram, elements, category, facade, key)
Bases: FinitePoset

A forest poset is a poset where the underlying Hasse diagram and is directed acyclic graph.

5.1.180 Hasse diagrams of posets

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>antichains()</td>
<td>Return all antichains of self, organized as a prefix tree</td>
</tr>
<tr>
<td>antichains_iterator()</td>
<td>Return an iterator over the antichains of the poset.</td>
</tr>
<tr>
<td>are_comparable()</td>
<td>Return whether i and j are comparable in the poset</td>
</tr>
<tr>
<td>are_incomparable()</td>
<td>Return whether i and j are incomparable in the poset</td>
</tr>
<tr>
<td>atoms_of_congruence_lattice()</td>
<td>Return atoms of the congruence lattice.</td>
</tr>
<tr>
<td>bottom_moebius_function()</td>
<td>Return the value of the Möbius function of the poset on the elements zero and j, where zero is self.bottom(), the unique minimal element of the poset.</td>
</tr>
<tr>
<td>cardinality()</td>
<td>Return the number of elements in the poset.</td>
</tr>
<tr>
<td>chains()</td>
<td>Return all chains of self, organized as a prefix tree.</td>
</tr>
<tr>
<td>closed_interval()</td>
<td>Return a list of the elements z of self such that x ≤ z ≤ y.</td>
</tr>
<tr>
<td>common_lower_covers()</td>
<td>Return the list of all common lower covers of vertices.</td>
</tr>
<tr>
<td>common_upper_covers()</td>
<td>Return the list of all common upper covers of vertices.</td>
</tr>
<tr>
<td>congruence()</td>
<td>Return the congruence start “extended” by parts.</td>
</tr>
<tr>
<td>congruences_iterator()</td>
<td>Return an iterator over all congruences of the lattice.</td>
</tr>
<tr>
<td>cover_relations()</td>
<td>Return the list of cover relations.</td>
</tr>
<tr>
<td>cover_relations_iterator()</td>
<td>Iterate over cover relations.</td>
</tr>
<tr>
<td>covers()</td>
<td>Return True if y covers x and False otherwise.</td>
</tr>
<tr>
<td>diamonds()</td>
<td>Return the list of diamonds of self.</td>
</tr>
<tr>
<td>dual()</td>
<td>Return a poset that is dual to the given poset.</td>
</tr>
<tr>
<td>find_nonsemidistributive_elements()</td>
<td>Check if the lattice is semidistributive or not.</td>
</tr>
<tr>
<td>find_nonsemimodular_pair()</td>
<td>Return pair of elements showing the lattice is not modular.</td>
</tr>
<tr>
<td>find_nontrivial_congruence()</td>
<td>Return a pair that generates non-trivial congruence or None if there is not any.</td>
</tr>
<tr>
<td>frattini_sublattice()</td>
<td>Return the list of elements of the Frattini sublattice of the lattice.</td>
</tr>
<tr>
<td>greedy_linear_extensions_iterator()</td>
<td>Return an iterator over greedy linear extensions of the Hasse diagram.</td>
</tr>
<tr>
<td>has_bottom()</td>
<td>Return True if the poset has a unique minimal element.</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>has_top()</code></td>
<td>Return True if the poset contains a unique maximal element, and False otherwise.</td>
</tr>
<tr>
<td><code>interval_iterator()</code></td>
<td>Return an iterator of the elements (z) of <code>self</code> such that (x \leq z \leq y).</td>
</tr>
<tr>
<td><code>is_antichain_of_poset()</code></td>
<td>Return True if (\text{elts}) is an antichain of the Hasse diagram and False otherwise.</td>
</tr>
<tr>
<td><code>is_bounded()</code></td>
<td>Return True if the poset contains a unique maximal element and a unique minimal element, and False otherwise.</td>
</tr>
<tr>
<td><code>is_chain()</code></td>
<td>Return True if the poset is totally ordered, and False otherwise.</td>
</tr>
<tr>
<td><code>is_complemented()</code></td>
<td>Return an element of the lattice that has no complement.</td>
</tr>
<tr>
<td><code>is_convex_subset_normal()</code></td>
<td>Return True if (S) is a convex subset of the poset, and False otherwise.</td>
</tr>
<tr>
<td><code>is_gequal()</code></td>
<td>Return True if (x) is greater than or equal to (y), and False otherwise.</td>
</tr>
<tr>
<td><code>is_greater_than()</code></td>
<td>Return True if (x) is greater than but not equal to (y), and False otherwise.</td>
</tr>
<tr>
<td><code>is_join_semilattice()</code></td>
<td>Return True if <code>self</code> has a join operation, and False otherwise.</td>
</tr>
<tr>
<td><code>is_linear_extension()</code></td>
<td>Test if an ordering is a linear extension.</td>
</tr>
<tr>
<td><code>is_linear_interval()</code></td>
<td>Return whether the interval ([t_{\text{min}}, t_{\text{max}}]) is linear.</td>
</tr>
<tr>
<td><code>is_meet_semilattice()</code></td>
<td>Return True if <code>self</code> has a meet operation, and False otherwise.</td>
</tr>
<tr>
<td><code>is_ranked()</code></td>
<td>Return True if the poset is ranked, and False otherwise.</td>
</tr>
<tr>
<td><code>join_matrix()</code></td>
<td>Return the matrix of joins of <code>self</code>, when <code>self</code> is a join-semilattice; raise an error otherwise.</td>
</tr>
<tr>
<td><code>kappa()</code></td>
<td>Return the maximum element greater than the element covered by (a) but not greater than (a).</td>
</tr>
<tr>
<td><code>kappa_dual()</code></td>
<td>Return the minimum element smaller than the element covering (a) but not smaller than (a).</td>
</tr>
<tr>
<td><code>lequal_matrix()</code></td>
<td>Return a matrix whose ((i,j)) entry is 1 if (i) is less than (j) in the poset, and 0 otherwise; and redefines __lt__ to use the boolean version of this matrix.</td>
</tr>
<tr>
<td><code>linear_extension()</code></td>
<td>Return a linear extension</td>
</tr>
<tr>
<td><code>linear_extensions()</code></td>
<td>Return an iterator over all linear extensions.</td>
</tr>
<tr>
<td><code>lower_covers_iterator()</code></td>
<td>Return the list of elements that are covered by <code>element</code>.</td>
</tr>
<tr>
<td><code>maximal_elements()</code></td>
<td>Return a list of the maximal elements of the poset.</td>
</tr>
<tr>
<td><code>maximal_sublattices()</code></td>
<td>Return maximal sublattices of the lattice.</td>
</tr>
<tr>
<td><code>meet_matrix()</code></td>
<td>Return the matrix of meets of <code>self</code>, when <code>self</code> is a meet-semilattice; raise an error otherwise.</td>
</tr>
<tr>
<td><code>minimal_elements()</code></td>
<td>Return a list of the minimal elements of the poset.</td>
</tr>
<tr>
<td><code>moebius_function()</code></td>
<td>Return the value of the Möbius function of the poset on the elements (i) and (j).</td>
</tr>
<tr>
<td><code>moebius_function_matrix()</code></td>
<td>Return the matrix of the Möbius function of this poset.</td>
</tr>
<tr>
<td><code>neutral_elements()</code></td>
<td>Return the list of neutral elements of the lattice.</td>
</tr>
<tr>
<td><code>open_interval()</code></td>
<td>Return a list of the elements (z) of <code>self</code> such that (x &lt; z &lt; y).</td>
</tr>
<tr>
<td><code>order_filter()</code></td>
<td>Return the order filter generated by a list of elements.</td>
</tr>
<tr>
<td><code>order_ideal()</code></td>
<td>Return the order ideal generated by a list of elements.</td>
</tr>
<tr>
<td><code>order_ideal_cardinality()</code></td>
<td>Return the cardinality of the order ideal generated by <code>elements</code>.</td>
</tr>
<tr>
<td><code>orthocomplementations_iterator()</code></td>
<td>Return an iterator over orthocomplementations of the lattice.</td>
</tr>
<tr>
<td><code>prime_elements()</code></td>
<td>Return the join-prime and meet-prime elements of the bounded poset.</td>
</tr>
<tr>
<td><code>principal_congruences_poset()</code></td>
<td>Return the poset of join-irreducibles of the congruence lattice.</td>
</tr>
<tr>
<td><code>principal_order_filter()</code></td>
<td>Return the order filter generated by <code>i</code>.</td>
</tr>
<tr>
<td><code>principal_order_ideal()</code></td>
<td>Return the order ideal generated by <code>i</code>.</td>
</tr>
<tr>
<td><code>pseudocomplement()</code></td>
<td>Return the pseudocomplement of <code>element</code>, if it exists.</td>
</tr>
<tr>
<td><code>rank()</code></td>
<td>Return the rank of <code>element</code>, or the rank of the poset if <code>element</code> is <code>None</code>. (The rank of a poset is the length of the longest chain of elements of the poset.)</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>rank_function()</td>
<td>Return the (normalized) rank function of the poset, if it exists.</td>
</tr>
<tr>
<td>skeleton()</td>
<td>Return the skeleton of the lattice.</td>
</tr>
<tr>
<td>sublattices_iterator()</td>
<td>Return an iterator over sublattices of the Hasse diagram.</td>
</tr>
<tr>
<td>supergreedy_linear_extensions_iterator()</td>
<td>Return an iterator over supergreedy linear extensions of the Hasse diagram.</td>
</tr>
<tr>
<td>top()</td>
<td>Return the top element of the poset, if it exists.</td>
</tr>
<tr>
<td>upper_covers_iterator()</td>
<td>Return the list of elements that cover element.</td>
</tr>
<tr>
<td>vertical_decomposition()</td>
<td>Return vertical decomposition of the lattice.</td>
</tr>
</tbody>
</table>

class sage.combinat.posets.hasse_diagram.HasseDiagram(
    data=None, pos=None, loops=None, format=None, weighted=None,
    data_structure='sparse', vertex_labels=True, name=None, multiedges=None,
    convert_empty_dict_labels_to_None=False, sparse=True, immutable=False,
    hash_labels=None)

Bases: DiGraph

The Hasse diagram of a poset. This is just a transitively-reduced, directed, acyclic graph without loops or multiple edges.

Note: We assume that range(n) is a linear extension of the poset. That is, range(n) is the vertex set and a topological sort of the digraph.

This should not be called directly, use Poset instead; all type checking happens there.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,2],1:[3],2:[3],3:[]}); H
Hasse diagram of a poset containing 4 elements
sage: TestSuite(H).run()

sage: H.antichains(element_class=list)
```

Return all antichains of self, organized as a prefix tree

INPUT:

- element_class – (default:list) an iterable type

EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: H = P._hasse_diagram
sage: A = H.antichains()
sage: list(A)
[[[]], [0], [1], [1, 2], [1, 3], [2], [3], [4]]
sage: A.cardinality()
8
sage: [1,3] in A
True
sage: [1,4] in A
False
```
antichains_iterator()

Return an iterator over the antichains of the poset.

Note: The algorithm is based on Freese-Jezek-Nation p. 226. It does a depth first search through the set of all antichains organized in a prefix tree.

EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: H = P._hasse_diagram
sage: H.antichains_iterator()
<generator object ...antichains_iterator at ...>
sage: list(H.antichains_iterator())
[[], [4], [3], [2], [1], [1, 3], [1, 2], [0]]
```

are_comparable(i, j)

Return whether i and j are comparable in the poset

INPUT:

• i, j – vertices of this Hasse diagram

EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: H = P._hasse_diagram
sage: H.are_comparable(1,2)
False
```

are_incomparable(i, j)

Return whether i and j are incomparable in the poset

INPUT:

• i, j – vertices of this Hasse diagram

EXAMPLES:
```python
sage: P = posets.PentagonPoset()
sage: H = P._hasse_diagram
sage: H.are_incomparable(1,2)
True
sage: V = H.vertices(sort=True)
[sage: [ (i,j) for i in V for j in V if H.are_incomparable(i,j)]
[(1, 2), (1, 3), (2, 1), (3, 1)]
```

**atoms_of_congruence_lattice()**

Return atoms of the congruence lattice.

In other words, return “minimal non-trivial” congruences: A congruence is minimal if the only finer (as a partition of set of elements) congruence is the trivial congruence where every block contains only one element.

**See also:**

`congruence()`

**OUTPUT:**

List of congruences, every congruence as `sage.combinat.set_partition.SetPartition`

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: N5 = HasseDiagram({0: [1, 2], 1: [4], 2: [3], 3:[4]})
```

```python
sage: N5.atoms_of_congruence_lattice()
[[{0}, {1}, {2, 3}, {4}]]
```

```python
sage: Hex = HasseDiagram({0: [1, 2], 1: [3], 2: [4], 3: [5], 4: [5]})
```

```python
sage: Hex.atoms_of_congruence_lattice()
[{{0}, {1, 3}, {2}, {4}, {5}}, {{0}, {1, 3}, {2}, {4}, {5}}]
```

**ALGORITHM:**

Every atom is a join-irreducible. Every join-irreducible of \(\text{Con}(L)\) is a principal congruence generated by a meet-irreducible element and the only element covering it (and also by a join-irreducible element and the only element covered by it). Hence we check those principal congruences to find the minimal ones.

**bottom()**

Return the bottom element of the poset, if it exists.

**EXAMPLES:**

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
```

```python
sage: P.bottom() is None
True
```

```python
sage: Q = Poset({0:[1],1:[]})
```

```python
sage: Q.bottom()
0
```

**bottom_moebius_function()**

Return the value of the Möbius function of the poset on the elements `zero` and `j`, where `zero` is `self.bottom()`, the unique minimal element of the poset.

**EXAMPLES:**

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
```

```python
sage: P.bottom() is None
True
```

```python
sage: Q = Poset({0:[1],1:[]})
```

```python
sage: Q.bottom()
0
```
sage: P = Poset({0: [1,2]})
sage: hasse = P._hasse_diagram
sage: hasse.bottom_moebius_function(1)
-1
sage: hasse.bottom_moebius_function(2)
-1
sage: P = Poset({0: [1,3], 1:[2], 2:[4], 3:[4]})
sage: hasse = P._hasse_diagram
sage: for i in range(5):
    ....: print(hasse.bottom_moebius_function(i))
1
-1
0
-1
1

**cardinality()**

Return the number of elements in the poset.

**EXAMPLES:**

```
sage: Poset([[1,2,3],[4],[4],[4],[[]]).cardinality()
5
```

**chains**(element_class=<class 'list'>, exclude=None, conversion=None)

Return all chains of self, organized as a prefix tree.

**INPUT:**

- element_class – (optional, default: list) an iterable type
- exclude – elements of the poset to be excluded (optional, default: None)
- conversion – (optional, default: None) used to pass the list of elements of the poset in their fixed order

**OUTPUT:**

The enumerated set (with a forest structure given by prefix ordering) consisting of all chains of self, each of which is given as an element_class.

If conversion is given, then the chains are converted to chain of elements of this list.

**EXAMPLES:**

```
sage: P = posets.PentagonPoset()
sage: H = P._hasse_diagram
sage: A = H.chains()
sage: list(A)
[[], [0], [0, 1], [0, 1, 4], [0, 2], [0, 2, 3], [0, 2, 3, 4], [0, 2, 4], [0, 3], ...
→ [0, 3, 4], [0, 4], [1], [1, 4], [2], [2, 3], [2, 3, 4], [2, 4], [3], [3, 4],...
→[4]]
sage: A.cardinality()
20
sage: [1,3] in A
False
```

(continues on next page)
sage: True

One can exclude some vertices:

```sage
c: = P = Poset({1: [2, 3], 2: [4]})
c: list(P.hasse_diagram.chains(element_class=tuple))
```

The `element_class` keyword determines how the chains are being returned:

```sage
c: = P = Poset({1: [2, 3], 2: [4]})
c: list(P.hasse_diagram.chains(exclude=[0]))
```

(See also): `antichains()`

**closed_interval(x, y)**

Return a list of the elements \( z \) of self such that \( x \leq z \leq y \).

The order is that induced by the ordering in self.linear_extension.

**INPUT:**

- \( x \) – any element of the poset
- \( y \) – any element of the poset

**Note:** The method `_precompute_intervals()` creates a cache which is used if available, making the function very fast.

(See also): `interval_iterator()`

**common_lower_covers(vertices)**

Return the list of all common lower covers of vertices.

**EXAMPLES:**
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1,2], 1: [3], 2: [3], 3: []})

sage: H.common_lower_covers([1, 2])
[0]

sage: from sage.combinat.posets.poset_examples import Posets
sage: H = Posets.YoungDiagramPoset(Partition([3, 2, 2]))._hasse_diagram
sage: H.common_lower_covers([4, 5])
[3]

common_upper_covers(vertices)

Return the list of all common upper covers of vertices.

EXAMPLES:

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1,2], 1: [3], 2: [3], 3: []})

sage: H.common_upper_covers([1, 2])
[3]

sage: from sage.combinat.posets.poset_examples import Posets
sage: H = Posets.YoungDiagramPoset(Partition([3, 2, 2]))._hasse_diagram

sage: H.common_upper_covers([4, 5])
[6]

congruence(parts, start=None, stop_pairs=[])  

Return the congruence start “extended” by parts.

start is assumed to be a valid congruence of the lattice, and this is not checked.

INPUT:

• parts – a list of lists; congruences to add
• start – a disjoint set; already computed congruence (or None)
• stop_pairs – a list of pairs; list of pairs for stopping computation

OUTPUT:

None, if the congruence generated by start and parts together contains a block that has elements \(a, b\) so that \((a, b)\) is in the list stop_pairs. Otherwise the least congruence that contains a block whose subset is \(p\) for every \(p\) in parts or start, given as sage.sets.disjoint_set.DisjointSet_class.

ALGORITHM:

Use the quadrilateral argument from page 120 of [Dav1997].

Basically we take one block from todo-list, search quadrilateral blocks up and down against the block, and then complete them to closed intervals and add to todo-list.

EXAMPLES:

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1,2], 1: [3], 2: [4], 3: [4]})

sage: cong = H.congruence([[0, 1]]); cong
{{0, 1, 3}, {2, 4}}

sage: H.congruence([[0, 2]], start=cong)
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{{0, 1, 2, 3, 4}}

sage: H.congruence([[0, 1]], stop_pairs=[(1, 3)]) is None
True

congruences_iterator()

Return an iterator over all congruences of the lattice.

EXAMPLES:

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram(‘GY@OQ?OW@?O?’)
congruences_iterator()  # it
<generator object ...>

sage: sorted([cong.number_of_subsets() for cong in it])
[1, 2, 2, 4, 4, 4, 8]

cover_relations()

Return the list of cover relations.

EXAMPLES:

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[2,3], 1:[3,4], 2:[5], 3:[5], 4:[5]})

sage: H.cover_relations()
[(0, 2), (0, 3), (1, 3), (1, 4), (2, 5), (3, 5), (4, 5)]

cover_relations_iterator()

Iterate over cover relations.

EXAMPLES:

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[2,3], 1:[3,4], 2:[5], 3:[5], 4:[5]})

sage: list(H.cover_relations_iterator())
[(0, 2), (0, 3), (1, 3), (1, 4), (2, 5), (3, 5), (4, 5)]

covers(x, y)

Return True if y covers x and False otherwise.

EXAMPLES:

sage: Q = Poset([[1,5],[2,6],[3],[4],[],[]])

sage: Q.covers(Q(1),Q(6))
True

sage: Q.covers(Q(1),Q(4))
False

coxeter_transformation(algorithm=’cython’)

Return the matrix of the Auslander-Reiten translation acting on the Grothendieck group of the derived category of modules on the poset, in the basis of simple modules.

INPUT:

• algorithm – optional, 'cython' (default) or 'matrix'

5.1. Comprehensive Module List
This uses either a specific matrix code in Cython, or generic matrices.

See also:

lequal_matrix(), moebius_function_matrix()

EXAMPLES:

```
sage: P = posets.PentagonPoset()._hasse_diagram
sage: M = P.coxeter_transformation(); M
[ 0 0 0 0 -1]
[ 0 0 0 1 -1]
[ 0 1 0 0 -1]
[-1 1 1 0 -1]
[-1 1 0 1 -1]
sage: P.__dict__['coxeter_transformation'].clear_cache()
sage: P.coxeter_transformation(algorithm="matrix") == M
True
```

**diamonds()**

Return the list of diamonds of self.

A diamond is the following subgraph of the Hasse diagram:

```
z
/ \
x  y
\ /  \
 w
```

Thus each edge represents a cover relation in the Hasse diagram. We represent his as the tuple $(w, x, y, z)$.

OUTPUT:

A tuple with

• a list of all diamonds in the Hasse Diagram,

• a boolean checking that every $w, x, y$ that form a $V$, there is a unique element $z$, which completes the diamond.

EXAMPLES:

```
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1,2], 1: [3], 2: [3], 3: []})
sage: H.diamonds()
(((0, 1, 2, 3), True))

sage: P = posets.YoungDiagramPoset(Partition([3, 2, 2]))
sage: H = P._hasse_diagram
sage: H.diamonds()
(((0, 1, 3, 4), (3, 4, 5, 6), False))
```

**dual()**

Return a poset that is dual to the given poset.

This means that it has the same elements but opposite order. The elements are renumbered to ensure that range(n) is a linear extension.

EXAMPLES:


```python
sage: P = posets.IntegerPartitions(4)
sage: H = P._hasse_diagram; H
Hasse diagram of a poset containing 5 elements
sage: H.dual()
Hasse diagram of a poset containing 5 elements
```

**find_nonsemidistributive_elements**(*meet_or_join*)

Check if the lattice is semidistributive or not.

**INPUT:**

- *meet_or_join* – string 'meet' or 'join' to decide if to check for join-semidistributivity or meet-semidistributivity

**OUTPUT:**

- None if the lattice is semidistributive OR
- tuple (u, e, x, y) such that \( u = e \lor x = e \lor y \) but \( u \neq e \lor (x \land y) \) if *meet_or_join*=='join'
  and \( u = e \land x = e \land y \) but \( u \neq e \land (x \lor y) \) if *meet_or_join*=='meet'

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1, 2], 1:[3, 4], 2:[4, 5], 3:[6], 4:[6], 5:[6]})
sage: H.find_nonsemidistributive_elements('join') is None
False
sage: H.find_nonsemidistributive_elements('meet') is None
True
```

**find_nonsemimodular_pair**(*upper*)

Return pair of elements showing the lattice is not modular.

**INPUT:**

- *upper*, a Boolean – if True, test whether the lattice is upper semimodular; otherwise test whether the lattice is lower semimodular.

**OUTPUT:**

None, if the lattice is semimodular. Pair \((a, b)\) violating semimodularity otherwise.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1, 2], 1:[3, 4], 2:[4, 5], 3:[6], 4:[6], 5:[6]})
sage: H.find_nonsemimodular_pair(upper=True) is None
True
sage: H.find_nonsemimodular_pair(upper=False)
(5, 3)
sage: H_ = HasseDiagram(H.reverse().relabel(lambda x: 6-x, inplace=False))
sage: H_.find_nonsemimodular_pair(upper=True)
(3, 1)
sage: H_.find_nonsemimodular_pair(upper=False) is None
True
```
**find_nontrivial_congruence**

Return a pair that generates non-trivial congruence or None if there is not any.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [5], 2: [3, 4], 3: [5], 4: [5]})
sage: H.find_nontrivial_congruence()
{{0, 1}, {2, 3, 4, 5}}
sage: H = HasseDiagram({0: [1, 2, 3], 1: [4], 2: [4], 3: [4]})
sage: H.find_nontrivial_congruence() is None
True
```

**ALGORITHM:**

See [https://www.math.hawaii.edu/~ralph/Preprints/conlat.pdf](https://www.math.hawaii.edu/~ralph/Preprints/conlat.pdf):

If $\Theta$ is a join irreducible element of a $\text{Con}(L)$, then there is at least one join-irreducible $j$ and one meet-irreducible $m$ such that $\Theta$ is both the principal congruence generated by $(j^*, j)$, where $j^*$ is the unique lower cover of $j$, and the principal congruence generated by $(m, m^*)$, where $m^*$ is the unique upper cover of $m$.

So, we only check join irreducibles or meet irreducibles, whichever is a smaller set. To optimize more we stop computation whenever it finds a pair that we know to generate one-element congruence.

**frattini_sublattice**

Return the list of elements of the Frattini sublattice of the lattice.

**EXAMPLES:**

```python
sage: H = posets.PentagonPoset()._hasse_diagram
sage: H.frattini_sublattice()
[0, 4]
```

**greedy_linear_extensions_iterator**

Return an iterator over greedy linear extensions of the Hasse diagram.

A linear extension $[e_1, e_2, \ldots, e_n]$ is greedy if for every $i$ either $e_{i+1}$ covers $e_i$ or all upper covers of $e_i$ have at least one lower cover that is not in $[e_1, e_2, \ldots, e_i]$.

Informally said a linear extension is greedy if it “always goes up when possible” and so has no unnecessary jumps.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: N5 = HasseDiagram({0: [1, 2], 2: [3], 1: [4], 3: [4]})
sage: for l in N5.greedy_linear_extensions_iterator():
    print(l)
[0, 1, 2, 3, 4]
[0, 2, 3, 1, 4]
```

**has_bottom**

Return True if the poset has a unique minimal element.

**EXAMPLES:**

```python
```
```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
sage: P.has_bottom()
False
sage: Q = Poset({0:[1],1:[]})
sage: Q.has_bottom()
True
```

**has_top()**

Return True if the poset contains a unique maximal element, and False otherwise.

**EXAMPLES:**

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4,5],4:[],5:[]})
sage: P.has_top()
False
sage: Q = Poset({0:[1],1:[]})
sage: Q.has_top()
True
```

**interval(x, y)**

Return a list of the elements $z$ of self such that $x \leq z \leq y$.

The order is that induced by the ordering in self.linear_extension.

**INPUT:**

- $x$ – any element of the poset
- $y$ – any element of the poset

**Note:** The method `_precompute_intervals()` creates a cache which is used if available, making the function very fast.

See also:

**interval_iterator()**

**EXAMPLES:**

```python
sage: uc = [[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[]]
sage: dag = DiGraph(dict(zip(range(len(uc)),uc)))
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram(dag)
sage: I = set([2,5,6,4,7])
sage: I == set(H.interval(2,7))
True
```

**interval_iterator(x, y)**

Return an iterator of the elements $z$ of self such that $x \leq z \leq y$.

**INPUT:**

- $x$ – any element of the poset
- $y$ – any element of the poset
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See also:

interval()

Note: This becomes much faster when first calling _leq_storage(), which precomputes the principal upper ideals.

EXAMPLES:

```sage
uc = [[1, 3, 2], [4], [4, 5, 6], [6], [7], [7], [7], []

sage: dag = DiGraph(dict(zip(range(len(uc)), uc)))

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram

sage: H = HasseDiagram(dag)

sage: I = set([2, 5, 6, 4, 7])

sage: I == set(H.interval_iterator(2, 7))

True
```

is_antichain_of_poset(elms)

Return True if elms is an antichain of the Hasse diagram and False otherwise.

EXAMPLES:

```sage
from sage.combinat.posets.hasse_diagram import HasseDiagram

H = HasseDiagram({0: [1, 2, 3], 1: [4], 2: [4], 3: [4]})

H.is_antichain_of_poset([1, 2, 3])

True

H.is_antichain_of_poset([0, 2, 3])

False
```

is_bounded()

Return True if the poset contains a unique maximal element and a unique minimal element, and False otherwise.

EXAMPLES:

```sage
P = Poset({0: [3], 1: [3], 2: [3], 3: [4, 5], 4: [], 5: []})

P.is_bounded()

False

Q = Poset({0: [1], 1: []})

Q.is_bounded()

True
```

is_chain()

Return True if the poset is totally ordered, and False otherwise.

EXAMPLES:

```sage
L = Poset({0: [1], 1: [2], 2: [3], 3: [4]})

L.is_chain()

True

V = Poset({0: [1, 2]})

V.is_chain()

False
```
is_complemented()

Return an element of the lattice that has no complement.
If the lattice is complemented, return None.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1, 2], 1:[3], 2:[3], 3:[4]})
sage: H.is_complemented()
1
sage: H = HasseDiagram({0:[1, 2, 3], 1:[4], 2:[4], 3:[4]})
sage: H.is_complemented() is None
True
```

is_congruence_normal()

Return True if the lattice can be constructed from the one-element lattice with Day doubling constructions of convex subsets.
Subsets to double does not need to be lower nor upper pseudo-intervals. On the other hand they must be convex, i.e. doubling a non-convex but municipal subset will give a lattice that returns False from this function.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram('IXQ@AG?Q@?AG?QW?Q@?@

sage: H.is_congruence_normal() True

```
The 5-element diamond is the smallest non-example:

```python
sage: H = HasseDiagram({0: [1, 2, 3], 1: [4], 2: [4], 3: [4]})
sage: H.is_congruence_normal() False

```
This is done by doubling a non-convex subset:

```python

sage: H.is_congruence_normal() False

```

ALGORITHM:

See http://www.math.hawaii.edu/~jb/inflation.pdf

is_convex_subset(S)

Return True if S is a convex subset of the poset, and False otherwise.
A subset S is convex in the poset if b \in S whenever a, c \in S and a \leq b \leq c.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: B3 = HasseDiagram({0: [1, 2, 4], 1: [3, 5], 2: [3, 6],
```
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\[
\begin{array}{l}
\text{sage: B3.is_convex_subset([1, 3, 5, 4])} \quad \# \text{Also connected} \\
\text{True} \\
\text{sage: B3.is_convex_subset([1, 3, 4])} \quad \# \text{Not connected} \\
\text{True} \\
\text{sage: B3.is_convex_subset([0, 1, 2, 3, 6])} \quad \# \text{No, } 0 < 4 < 6 \\
\text{False} \\
\text{sage: B3.is_convex_subset([0, 1, 2, 7])} \quad \# \text{No, } 1 < 3 < 7. \\
\text{False}
\end{array}
\]

\text{is_gequal}(x, y)

Return True if \(x\) is greater than or equal to \(y\), and False otherwise.

\text{EXAMPLES:}

\[
\begin{array}{l}
\text{sage: from sage.combinat.posets.hasse_diagram import HasseDiagram} \\
\text{sage: Q = HasseDiagram({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})} \\
\text{sage: x,y,z = 0,1,4} \\
\text{sage: Q.is_gequal(x,y)} \\
\text{False} \\
\text{sage: Q.is_gequal(y,x)} \\
\text{False} \\
\text{sage: Q.is_gequal(x,z)} \\
\text{False} \\
\text{sage: Q.is_gequal(z,x)} \\
\text{True} \\
\text{sage: Q.is_gequal(z,y)} \\
\text{True} \\
\text{sage: Q.is_gequal(z,z)} \\
\text{True}
\end{array}
\]

\text{is_greater_than}(x, y)

Return True if \(x\) is greater than but not equal to \(y\), and False otherwise.

\text{EXAMPLES:}

\[
\begin{array}{l}
\text{sage: from sage.combinat.posets.hasse_diagram import HasseDiagram} \\
\text{sage: Q = HasseDiagram({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})} \\
\text{sage: x,y,z = 0,1,4} \\
\text{sage: Q.is_greater_than(x,y)} \\
\text{False} \\
\text{sage: Q.is_greater_than(y,x)} \\
\text{False} \\
\text{sage: Q.is_greater_than(x,z)} \\
\text{False} \\
\text{sage: Q.is_greater_than(z,x)} \\
\text{True} \\
\text{sage: Q.is_greater_than(z,y)} \\
\text{True} \\
\text{sage: Q.is_greater_than(z,z)} \\
\text{False}
\end{array}
\]
is_join_semilattice()
Return True if self has a join operation, and False otherwise.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,3,2], 1:[4], 2:[4,5,6], 3:[6], 4:[7], 5:[7], 6:[7], 7:[[]])
sage: H.is_join_semilattice()
True
sage: H = HasseDiagram({0:[2,3], 1:[2,3]})
sage: H.is_join_semilattice()
False
sage: H = HasseDiagram({0:[2,3], 1:[2,3], 2:[4], 3:[4]})
sage: H.is_join_semilattice()
False
```

is_lequal(i, j)
Return True if i is less than or equal to j in the poset, and False otherwise.

Note: If the lequal_matrix() has been computed, then this method is redefined to use the cached data (see alternate_is_lequal()).

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[2], 1:[2], 2:[3], 3:[4], 4:[[]])
sage: x,y,z = 0, 1, 4
sage: H.is_lequal(x,y)
False
sage: H.is_lequal(y,x)
False
sage: H.is_lequal(x,z)
True
sage: H.is_lequal(y,z)
True
sage: H.is_lequal(z,z)
True
```

is_less_than(x, y)
Return True if x is less than but not equal to y in the poset, and False otherwise.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[2], 1:[2], 2:[3], 3:[4], 4:[[]])
sage: x,y,z = 0, 1, 4
sage: H.is_less_than(x,y)
False
sage: H.is_less_than(y,x)
False
sage: H.is_less_than(x,z)
True
sage: H.is_less_than(y,z)
True
```

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True

sage: H.is_less_than(z, z)
False

**is_linear_extension** *(lin_ext=None)*

Test if an ordering is a linear extension.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,2],1:[3],2:[3],3:[]})
sage: H.is_linear_extension(list(range(4)))
True
sage: H.is_linear_extension([3,2,1,0])
False
```

**is_linear_interval** *(t_min, t_max)*

Return whether the interval \([t_min, t_max]\) is linear.

This means that this interval is a total order.

**EXAMPLES:**

```python
sage: P = posets.PentagonPoset()
sage: H = P._hasse_diagram
sage: H.is_linear_interval(0, 4)
False
sage: H.is_linear_interval(0, 3)
True
sage: H.is_linear_interval(1, 3)
False
sage: H.is_linear_interval(1, 1)
True
```

**is_meet_semilattice** *

Return True if self has a meet operation, and False otherwise.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,3,2],1:[4],2:[4,5,6],3:[6],4:[7],5:[7],6:[7],7:[]})
sage: H.is_meet_semilattice()
True
```

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(continued from previous page)

```sage
sage: H.is_meet_semilattice()
False
```

**is_ranked()**

Return True if the poset is ranked, and False otherwise.

A poset is ranked if it admits a rank function. For more information about the rank function, see `rank_function()` and `is_graded()`.

**EXAMPLES:**

```sage
sage: P = Poset([[1],[2],[3],[4],[5]])
sage: P.is_ranked()
True
sage: Q = Poset([[1],[2],[3],[4],[5],[6],[7]])
sage: Q.is_ranked()
False
```

**join_matrix()**

Return the matrix of joins of self, when self is a join-semilattice; raise an error otherwise.

The \((x,y)\)-entry of this matrix is the join of \(x\) and \(y\) in self.

This algorithm is modelled after the algorithm of Freese-Jezek-Nation (p217). It can also be found on page 140 of [Gec81].

**Note:** If self is a join-semilattice, then the return of this method is the same as `_join()`. Once the matrix has been computed, it is stored in `_join()`. Delete this attribute if you want to recompute the matrix.

**EXAMPLES:**

```sage
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,2,3,4,5,6,7], 1:[4,5,6,7], 2:[4,5,6,7], 3:[6,7], 4:[7], 5:[7], 6:[7], 7:[7]})
sage: H.join_matrix()
[[0 1 2 3 4 5 6 7]
 [1 1 4 7 4 7 7 7]
 [2 4 2 6 4 5 6 7]
 [3 7 6 3 7 6 7 7]
 [4 4 4 7 4 7 7 7]
 [5 7 5 7 5 7 7 7]
 [6 7 6 6 7 7 6 7]
 [7 7 7 7 7 7 7 7]]
```

**kappa(a)**

Return the maximum element greater than the element covered by \(a\) but not greater than \(a\).

Define \(\kappa(a)\) as the maximum element of \((\uparrow a_+) \setminus (\uparrow a)\), where \(a_+\) is the element covered by \(a\). It is always a meet-irreducible element, if it exists.

**Note:** Element \(a\) is expected to be join-irreducible, and this is not checked.

**INPUT:**

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• a – a join-irreducible element of the lattice

OUTPUT:
The element \( \kappa(a) \) or None if there is not a unique greatest element with given constraints.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2, 3], 1: [4], 2: [4, 5], 3: [5], 4: [6], 5: [6]}
˓→)

sage: H.kappa(1)
5
sage: H.kappa(2) is None
True
```

**kappa_dual\( (a) \)**

Return the minimum element smaller than the element covering \( a \) but not smaller than \( a \).

Define \( \kappa^*(a) \) as the minimum element of \( (\downarrow a^*) \setminus (\downarrow a) \), where \( a^* \) is the element covering \( a \). It is always a join-irreducible element, if it exists.

**Note:** Element \( a \) is expected to be meet-irreducible, and this is not checked.

INPUT:
• a – a join-irreducible element of the lattice

OUTPUT:
The element \( \kappa^*(a) \) or None if there is not a unique smallest element with given constraints.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [3, 4], 2: [4, 5], 3: [6], 4: [6], 5: [6]}
˓→)

sage: H.kappa_dual(3)
2
sage: H.kappa_dual(4) is None
True
```

**lequal_matrix\( (boolean=False) \)**

Return a matrix whose \((i,j)\) entry is 1 if \( i \) is less than \( j \) in the poset, and 0 otherwise; and redefines \( __lt__ \) to use the boolean version of this matrix.

INPUT:
• boolean – optional flag (default False) telling whether to return a matrix with coefficients in \( \mathbb{F}_2 \) or in \( \mathbb{Z} \)

See also:
moebius_function_matrix(), coxeter_transformation()

EXAMPLES:
sage: P = Poset([[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[]])
sage: H = P._hasse_diagram
sage: M = H.lequal_matrix(); M
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
sage: M.base_ring()
Integer Ring

sage: P = posets.DiamondPoset(6)
sage: H = P._hasse_diagram
sage: M = H.lequal_matrix(boolean=True)
sage: M.base_ring()
Finite Field of size 2

**linear_extension()**

Return a linear extension

**linear_extensions()**

Return an iterator over all linear extensions.

**lower_covers_iterator(element)**

Return the list of elements that are covered by element.

**maximal_elements()**

Return a list of the maximal elements of the poset.
EXAMPLES:

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
sage: P.maximal_elements()
[4]
```

**maximal_sublattices()**

Return maximal sublattices of the lattice.

EXAMPLES:

```python
sage: L = posets.PentagonPoset()
sage: ms = L._hasse_diagram.maximal_sublattices()
sage: sorted(ms, key=sorted)
[{
0, 1, 2, 4},
{0, 1, 3, 4},
{0, 2, 3, 4}
]
```

**meet_matrix()**

Return the matrix of meets of `self`, when `self` is a meet-semilattice; raise an error otherwise.

The \((x,y)\)-entry of this matrix is the meet of \(x\) and \(y\) in `self`.

This algorithm is modelled after the algorithm of Freese-Jezek-Nation (p217). It can also be found on page 140 of [Gec81].

**Note:** If `self` is a meet-semilattice, then the return of this method is the same as `_meet()`. Once the matrix has been computed, it is stored in `_meet()`. Delete this attribute if you want to recompute the matrix.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,3,2],1:[4],2:[4,5,6],3:[6],4:[7],5:[7],6:[7],7:[]})
sage: H.meet_matrix()
[0 0 0 0 0 0 0 0]
[0 1 0 1 0 0 1]
[0 0 2 2 2 2]
[0 0 0 3 0 3 3]
[0 1 2 4 2 4]
[0 0 2 5 2 5]
[0 0 2 3 2 6 6]
[0 1 2 3 4 5 6 7]
```

**REFERENCE:**

**minimal_elements()**

Return a list of the minimal elements of the poset.

EXAMPLES:

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
sage: P(0) in P.minimal_elements()
True
sage: P(1) in P.minimal_elements()
True
```

(continues on next page)
moebius_function(i, j)
Return the value of the Möbius function of the poset on the elements \(i\) and \(j\).

EXAMPLES:

```python
sage: P = Poset([[1,2,3],[4],[4],[4],[[]])
sage: H = P._hasse_diagram
sage: H.moebius_function(0,4)
2
sage: for u,v in P.cover_relations_iterator():
    ....:   if P.moebius_function(u,v) != -1:
    ....:     print("Bug in moebius_function!")
```

moebius_function_matrix(algorithm=’cython’)
Return the matrix of the Möbius function of this poset.
This returns the matrix over \(\mathbb{Z}\) whose entries are the value of the Möbius function of self evaluated on \(x\) and \(y\), and redefines moebius_function() to use it.

INPUT:

- algorithm – optional, 'recursive', 'matrix' or 'cython' (default)

This uses either the recursive formula, a generic matrix inversion or a specific matrix inversion coded in Cython.

OUTPUT:

- a dense matrix for the algorithm cython, a sparse matrix otherwise

Note: The result is cached in _moebius_function_matrix().

See also:
lequal_matrix(), coxeter_transformation()

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,3,2],1:[4],2:[4,5,6],3:[6],4:[7],5:[7],6:[7],7:[[]])
sage: H.moebius_function_matrix()
```

neutral_elements()
Return the list of neutral elements of the lattice.
An element \(a\) in a lattice is neutral if the sublattice generated by \(a, x\) and \(y\) is distributive for every \(x, y\) in the lattice.

**EXAMPLES:**

```
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
code升值: H = HasseDiagram({0: [1, 2], 1: [4], 2: [3], 3: [4, 5],
....: 4: [6], 5:[6]})
sage: sorted(H.neutral_elements())
[0, 4, 6]
```

**ALGORITHM:**

Basically we just check the distributivity against all element pairs \(x, y\) to see if element \(a\) is neutral or not.

If we found that \(a, x, y\) is not a distributive triple, we add all three to list of non-neutral elements. If we found \(a\) to be neutral, we add it to list of neutral elements. When testing we skip already found neutral elements, as they can’t be our \(x\) or \(y\).

We skip \(a, x, y\) as trivial if it is a chain. We do that by letting \(x\) to be a non-comparable to \(a\); \(y\) can be any element.

We first try to found \(x\) and \(y\) from elements not yet tested, so that we could get three birds with one stone.

And last, the top and bottom elements are always neutral and need not be tested.

**open_interval**(\(x, y\))

Return a list of the elements \(z\) of \(self\) such that \(x < z < y\).

The order is that induced by the ordering in \(self.linear_extension\).

**EXAMPLES:**

```
sage: uc = [[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[7]]
sage: dag = DiGraph(dict(zip(range(len(uc)),uc)))
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
code升值: H = HasseDiagram(dag)
sage: set([5,6,4]) == set(H.open_interval(2,7))
True
sage: H.open_interval(7,2)
[]
```

**order_filter**(\(elements\))

Return the order filter generated by a list of elements.

\(I\) is an order filter if, for any \(x\) in \(I\) and \(y\) such that \(y\geq x\), then \(y\) is in \(I\).

**EXAMPLES:**

```
sage: H = posets.BooleanLattice(4)._hasse_diagram
sage: H.order_filter([3,8])
[3, 7, 8, 9, 10, 11, 12, 13, 14, 15]
```

**order_ideal**(\(elements\))

Return the order ideal generated by a list of elements.

\(I\) is an order ideal if, for any \(x\) in \(I\) and \(y\) such that \(y\leq x\), then \(y\) is in \(I\).

**EXAMPLES:**

```
order_ideal_cardinality(elements)

Return the cardinality of the order ideal generated by elements.

$I$ is an order ideal if, for any $x$ in $I$ and $y$ such that $y \leq x$, then $y$ is in $I$.

EXAMPLES:

```sage
sage: H = posets.BooleanLattice(4)._hasse_diagram
sage: H.order_ideal_cardinality([7,10])
10
```

orthocomplementations_iterator()

Return an iterator over orthocomplementations of the lattice.

OUTPUT:

An iterator that gives plain list of integers.

EXAMPLES:

```sage
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [3], 2: [4], 3: [4]})

sage: list(H.orthocomplementations_iterator())
[[9, 8, 5, 6, 7, 2, 3, 4, 1, 0], [9, 8, 5, 7, 6, 2, 4, 3, 1, 0]]
```

ALGORITHM:

As DiamondPoset($2n+2$) has $(2n)!/(n!2^n)$ different orthocomplementations, the complexity of listing all of them is necessarily $O(n!)$.

An orthocomplemented lattice is self-dual, so that for example orthocomplement of an atom is a coatom. This function basically just computes list of possible orthocomplementations for every element (i.e. they must be complements and “duals”), and then tries to fit them all.

prime_elements()

Return the join-prime and meet-prime elements of the bounded poset.

An element $x$ of a poset $P$ is join-prime if the subposet induced by $\{y \in P \mid y \not\geq x\}$ has a top element. Meet-prime is defined dually.

Note: The poset is expected to be bounded, and this is not checked.

OUTPUT:

A pair $(j, m)$ where $j$ is a list of join-prime elements and $m$ is a list of meet-prime elements.

EXAMPLES:

```sage
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [3], 2: [4], 3: [4]})

sage: H.prime_elements()
([1, 2], [2, 3])
```
**principal_congruences_poset()**

Return the poset of join-irreducibles of the congruence lattice.

**OUTPUT:**

A pair \((P, D)\) where \(P\) is a poset and \(D\) is a dictionary.

Elements of \(P\) are pairs \((x, y)\) such that \(x\) is an element of the lattice and \(y\) is an element covering it. In the poset \((a, b)\) is less than \((c, d)\) iff the principal congruence generated by \((a, b)\) is refinement of the principal congruence generated by \((c, d)\).

\(D\) is a dictionary from pairs \((x, y)\) to the congruence (given as DisjointSet) generated by the pair.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: N5 = HasseDiagram({0: [1, 2], 1: [4], 2: [3], 3: [4]})
sage: P, D = N5.principal_congruences_poset()
sage: P
Finite poset containing 3 elements
sage: P.bottom()
(2, 3)
sage: D[(2, 3)]
{{0}, {1}, {2, 3}, {4}}
```

**principal_order_filter(i)**

Return the order filter generated by \(i\).

**EXAMPLES:**

```python
sage: H = posets.BooleanLattice(4)._hasse_diagram
sage: H.principal_order_filter(2)
[2, 3, 6, 7, 10, 11, 14, 15]
```

**principal_order_ideal(i)**

Return the order ideal generated by \(i\).

**EXAMPLES:**

```python
sage: H = posets.BooleanLattice(4)._hasse_diagram
sage: H.principal_order_ideal(6)
[0, 2, 4, 6]
```

**pseudocomplement(element)**

Return the pseudocomplement of \(element\), if it exists.

The pseudocomplement is the greatest element whose meet with given element is the bottom element. It may not exist, and then the function returns \(None\).

**INPUT:**

- \(element\) – an element of the lattice.

**OUTPUT:**

An element of the Hasse diagram, i.e. an integer, or \(None\) if the pseudocomplement does not exist.

**EXAMPLES:**
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```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [3], 2: [4], 3: [4]})
sage: H.pseudocomplement(2)
3
sage: H = HasseDiagram({0: [1, 2, 3], 1: [4], 2: [4], 3: [4]})
sage: H.pseudocomplement(2) is None
True
```

**rank**(element=None)

Return the rank of `element`, or the rank of the poset if `element` is `None`. (The rank of a poset is the length of the longest chain of elements of the poset.)

**EXAMPLES:**

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,3,2],1:[4],2:[4,5,6],3:[6],4:[7],5:[7],6:[7],7:[[]])
sage: H.rank(5)
2
sage: H.rank()
3
sage: Q = HasseDiagram({0:[1,2],1:[3],2:[[]],3:[[]]})
sage: Q.rank()
2
sage: Q.rank(1)
1
```

**rank_function()**

Return the (normalized) rank function of the poset, if it exists.

A rank function of a poset `P` is a function `r` that maps elements of `P` to integers and satisfies: `r(x) = r(y)+1` if `x` covers `y`. The function `r` is normalized such that its minimum value on every connected component of the Hasse diagram of `P` is `0`. This determines the function `r` uniquely (when it exists).

**OUTPUT:**

- a lambda function, if the poset admits a rank function
- `None`, if the poset does not admit a rank function

**EXAMPLES:**

```python
sage: P = Poset([[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[[]])
sage: P.rank_function() is not None
True
sage: P = Poset([[1,2,3,4],[1,4],[2,3],[3,4]], facade = True)
sage: P.rank_function() is not None
True
sage: P = Poset([[1,2,3,4,5],[1,2],[2,3],[3,4],[1,5],[5,4]], facade = True)
sage: P.rank_function() is not None
False
sage: P = Poset([[1,2,3,4,5,6,7,8],[1,4],[2,3],[3,4],[5,7],[6,7]], facade = True)
False
sage: f = P.rank_function(); f is not None
True
sage: f(5)
```

(continues on next page)
skeleton()

Return the skeleton of the lattice.

The lattice is expected to be pseudocomplemented and non-empty.

The skeleton of the lattice is the subposet induced by those elements that are the pseudocomplement to at least one element.

OUTPUT:

List of elements such that the subposet induced by them is the skeleton of the lattice.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [3, 4], 2: [4],
                   3: [5], 4: [5]})
sage: H.skeleton()
[5, 2, 0, 3]
```

sublattices_iterator(elms, min_e)

Return an iterator over sublattices of the Hasse diagram.

INPUT:

- `elms` – elements already in sublattice; use set() at start
- `min_e` – smallest new element to add for new sublattices

OUTPUT:

List of sublattices as sets of integers.

EXAMPLES:

```python
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 1: [3], 2: [3]})
sage: it = H.sublattices_iterator(set(), 0); it
<generator object ...sublattices_iterator at ...>
sage: next(it)
set()
sage: next(it)
{0}
```

supergreedy_linear_extensions_iterator()

Return an iterator over supergreedy linear extensions of the Hasse diagram.

A linear extension \([e_1, e_2, \ldots, e_n]\) is supergreedy if, for every \(i\) and \(j\) where \(i > j\), \(e_i\) covers \(e_j\) if for every \(i > k > j\) at least one lower cover of \(e_k\) is not in \([e_1, e_2, \ldots, e_k]\).

Informally said a linear extension is supergreedy if it “always goes as high possible, and withdraw so less as possible”. These are also called depth-first linear extensions.

EXAMPLES:

We show the difference between “only greedy” and supergreedy extensions:
sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0: [1, 2], 2: [3, 4]})
sage: G_ext = list(H.greedy_linear_extensions_iterator())
sage: SG_ext = list(H.supergreedy_linear_extensions_iterator())
sage: [0, 2, 3, 1, 4] in G_ext
True
sage: [0, 2, 3, 1, 4] in SG_ext
False
sage: len(SG_ext)
4

**top()**

Return the top element of the poset, if it exists.

**EXAMPLES:**

sage: P = Poset({0:[3],1:[3],2:[3],3:[4,5],4:[],5:[]})
sage: P.top() is None
True
sage: Q = Poset({0:[1],1:[]})
sage: Q.top()
1

**upper_covers_iterator**(element)

Return the list of elements that cover element.

**EXAMPLES:**

sage: from sage.combinat.posets.hasse_diagram import HasseDiagram
sage: H = HasseDiagram({0:[1,3,2],1:[4],2:[4,5,6],3:[6],4:[7],5:[7],6:[7],7:[]})
sage: list(H.upper_covers_iterator(0))
[1, 2, 3]
sage: list(H.upper_covers_iterator(7))
[]

**vertical_decomposition**(return_list=False)

Return vertical decomposition of the lattice.

This is the backend function for vertical decomposition functions of lattices.

The property of being vertically decomposable is defined for lattices. This is not checked, and the function works with any bounded poset.

**INPUT:**

• return_list, a boolean. If False (the default), return an element that is not the top neither the bottom element of the lattice, but is comparable to all elements of the lattice, if the lattice is vertically decomposable and None otherwise. If True, return list of decomposition elements.

**EXAMPLES:**

sage: H = posets.BooleanLattice(4)._hasse_diagram
sage: H.vertical_decomposition()
is None
True
sage: P = Poset( ([1,2,3,6,12,18,36], attrcall("divides")) )

(continues on next page)
sage: P._hasse_diagram.vertical_decomposition()
3
sage: P._hasse_diagram.vertical_decomposition(return_list=True)
[3]

exception sage.combinat.posets.hasse_diagram.LatticeError(fail, x, y)
Bases: ValueError

Helper exception class to forward elements without meet or join to upper level, so that the user will see “No meet for a and b” instead of “No meet for 1 and 2”.

5.1.181 Incidence Algebras

class sage.combinat.posets.incidence_algebras.IncidenceAlgebra(R, P, prefix='I')
Bases: CombinatorialFreeModule

The incidence algebra of a poset.

Let $P$ be a poset and $R$ be a commutative unital associative ring. The *incidence algebra* $I_P$ is the algebra of functions $\alpha : P \times P \to R$ such that $\alpha(x, y) = 0$ if $x \not\leq y$ where multiplication is given by convolution:

$$(\alpha * \beta)(x, y) = \sum_{x \leq k \leq y} \alpha(x, k)\beta(k, y).$$

This has a natural basis given by indicator functions for the interval $[a, b]$, i.e. $X_{a,b}(x, y) = \delta_{ax}\delta_{by}$. The incidence algebra is a unital algebra with the identity given by the Kronecker delta $\delta(x, y) = \delta_{xy}$. The Möbius function of $P$ is another element of $I_P$ whose inverse is the $\zeta$ function of the poset (so $\zeta(x, y) = 1$ for every interval $[x, y]$).

Todo: Implement the incidence coalgebra.

REFERENCES:
- Wikipedia article Incidence_algebra

class Element
Bases: IndexedFreeModuleElement

An element of an incidence algebra.

is_unit()
return if self is a unit.

EXAMPLES:

sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: mu = I.moebius()
sage: mu.is_unit()
True
sage: zeta = I.zeta()
sage: zeta.is_unit()
True
sage: x = mu - I.zeta() + I[2,2]
sage: x.is_unit()
False

delta()
Return the element 1 in self (which is the Kronecker delta $\delta(x, y)$).

EXAMPLES:

\begin{verbatim}
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.one()
\end{verbatim}

moebius()
Return the Möbius function of self.

EXAMPLES:

\begin{verbatim}
sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: I.moebius()
\end{verbatim}

(continues on next page)
I[0, 0] - I[0, 1] - I[0, 2] + I[0, 3] + I[1, 1] - I[1, 3] + I[2, 2] - I[2, 3] + I[3, 3]

one()

Return the element 1 in self (which is the Kronecker delta $\delta(x, y)$).

EXAMPLES:

```python
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.one()
```

poset()

Return the defining poset of self.

EXAMPLES:

```python
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.poset()
Finite lattice containing 16 elements
```

product_on_basis(A, B)

Return the product of basis elements indexed by $A$ and $B$.

EXAMPLES:

```python
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.product_on_basis((1, 3), (3, 11))
I[1, 11]
sage: I.product_on_basis((1, 3), (2, 2))
0
```

reduced_subalgebra(prefix='R')

Return the reduced incidence subalgebra.

EXAMPLES:

```python
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.reduced_subalgebra()
Reduced incidence algebra of Finite lattice containing 16 elements over Rational Field
```

some_elements()

Return a list of elements of self.

EXAMPLES:
```sage
sage: P = posets.BooleanLattice(1)
sage: I = P.incidence_algebra(QQ)
sage: Ielts = I.some_elements(); Ielts # random
[2*I[0, 0] + 2*I[0, 1] + 3*I[1, 1],
 I[0, 0] - I[0, 1] + I[1, 1],
 I[0, 0] + I[0, 1] + I[1, 1]]
sage: [a in I for a in Ielts]
[True, True, True]
```

**zeta()**

Return the $\zeta$ function in self.

The $\zeta$ function on a poset $P$ is given by

$$\zeta(x, y) = \begin{cases} 1 & x \leq y, \\ 0 & x \not\leq y. \end{cases}$$

**EXAMPLES:**

```sage
sage: P = posets.BooleanLattice(4)
sage: I = P.incidence_algebra(QQ)
sage: I.zeta() * I.moebius() == I.one()
True
```

**class** `sage.combinat.posets.incidence_algebras.ReducedIncidenceAlgebra(I, prefix='R')`

Bases: `CombinatorialFreeModule`

The reduced incidence algebra of a poset.

The reduced incidence algebra $R_P$ is a subalgebra of the incidence algebra $I_P$ where $\alpha(x, y) = \alpha(x', y')$ when $[x, y]$ is isomorphic to $[x', y']$ as posets. Thus the delta, Möbius, and zeta functions are all elements of $R_P$.

**class** `Element`

Bases: `IndexedFreeModuleElement`

An element of a reduced incidence algebra.

**is_unit()**

Return if self is a unit.

**EXAMPLES:**

```sage
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: x = R.an_element()
sage: x.is_unit()
True
```

**lift()**

Return the lift of self to the ambient space.

**EXAMPLES:**

```sage
sage: P = posets.BooleanLattice(2)
sage: I = P.incidence_algebra(QQ)
sage: R = I.reduced_subalgebra()
```
sage: x = R.an_element(); x
2*R[(0, 0)] + 2*R[(0, 1)] + 3*R[(0, 3)]
sage: x.lift()

**to_matrix()**

Return self as a matrix.

EXAMPLES:

```python
sage: P = posets.BooleanLattice(2)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: mu = R.moebius()
sage: mu.to_matrix()
[ 1 -1 -1 1]
[ 0 1 0 -1]
[ 0 0 1 -1]
[ 0 0 0 1]
```

**delta()**

Return the Kronecker delta function in self.

EXAMPLES:

```python
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.delta()
R[(0, 0)]
```

**lift()**

Return the lift morphism from self to the ambient space.

EXAMPLES:

```python
sage: P = posets.BooleanLattice(2)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.lift
Generic morphism:
  From: Reduced incidence algebra of Finite lattice containing 4 elements over Rational Field
  To:  Incidence algebra of Finite lattice containing 4 elements over Rational Field
sage: R.an_element() - R.one()
R[(0, 0)] + 2*R[(0, 1)] + 3*R[(0, 3)]
sage: R.lift(R.an_element() - R.one())
```

**moebius()**

Return the Möbius function of self.

EXAMPLES:
```python
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.moebius()
R[(0, 0)] - R[(0, 1)] + R[(0, 3)] - R[(0, 7)] + R[(0, 15)]
```

**one_basis()**

Return the index of the element 1 in self.

**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.one_basis()
(0, 0)
```

**poset()**

Return the defining poset of self.

**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.poset()
Finite lattice containing 16 elements
sage: R.poset() == P
True
```

**some_elements()**

Return a list of elements of self.

**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.some_elements()
[2*R[(0, 0)] + 2*R[(0, 1)] + 3*R[(0, 3)],
 R[(0, 0)] - R[(0, 1)] + R[(0, 3)] - R[(0, 7)] + R[(0, 15)],
 R[(0, 0)] + R[(0, 1)] + R[(0, 3)] + R[(0, 7)] + R[(0, 15)]]
```

**zeta()**

Return the function in self.

The function on a poset $P$ is given by

\[
\zeta(x, y) = \begin{cases} 
1 & x \leq y, \\
0 & x \not\leq y.
\end{cases}
\]

**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(4)
sage: R = P.incidence_algebra(QQ).reduced_subalgebra()
sage: R.zeta()
R[(0, 0)] + R[(0, 1)] + R[(0, 3)] + R[(0, 7)] + R[(0, 15)]
```
5.1.182 Finite lattices and semilattices

This module implements finite (semi)lattices. It defines:

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LatticePoset()</td>
<td>Construct a lattice.</td>
</tr>
<tr>
<td>MeetSemilattice()</td>
<td>Construct a meet semi-lattice.</td>
</tr>
<tr>
<td>JoinSemilattice()</td>
<td>Construct a join semi-lattice.</td>
</tr>
<tr>
<td>FiniteLatticePoset</td>
<td>A class for finite lattices.</td>
</tr>
<tr>
<td>FiniteMeetSemilattice</td>
<td>A class for finite meet semilattices.</td>
</tr>
<tr>
<td>FiniteJoinSemilattice</td>
<td>A class for finite join semilattices.</td>
</tr>
</tbody>
</table>

List of (semi)lattice methods

Meet and join

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>meet()</td>
<td>Return the meet of given elements.</td>
</tr>
<tr>
<td>join()</td>
<td>Return the join of given elements.</td>
</tr>
<tr>
<td>meet_matrix()</td>
<td>Return the matrix of meets of all elements of the meet semi-lattice.</td>
</tr>
<tr>
<td>join_matrix()</td>
<td>Return the matrix of joins of all elements of the join semi-lattice.</td>
</tr>
</tbody>
</table>

Properties of the lattice

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>is_distributive()</td>
<td>Return True if the lattice is distributive.</td>
</tr>
<tr>
<td>is_modular()</td>
<td>Return True if the lattice is modular.</td>
</tr>
<tr>
<td>is_lower_semimodular()</td>
<td>Return True if all elements with common upper cover have a common lower cover.</td>
</tr>
<tr>
<td>is_upper_semimodular()</td>
<td>Return True if all elements with common lower cover have a common upper cover.</td>
</tr>
<tr>
<td>is_semidistributive()</td>
<td>Return True if the lattice is both join- and meet-semidistributive.</td>
</tr>
<tr>
<td>is_join_semidistributive()</td>
<td>Return True if the lattice is join-semidistributive.</td>
</tr>
<tr>
<td>is_meet_semidistributive()</td>
<td>Return True if the lattice is meet-semidistributive.</td>
</tr>
<tr>
<td>is_join_distributive()</td>
<td>Return True if the lattice is join-distributive.</td>
</tr>
<tr>
<td>is_meet_distributive()</td>
<td>Return True if the lattice is meet-distributive.</td>
</tr>
<tr>
<td>is_atomic()</td>
<td>Return True if every element of the lattice can be written as a join of atoms.</td>
</tr>
<tr>
<td>is_coatomic()</td>
<td>Return True if every element of the lattice can be written as a meet of coatoms.</td>
</tr>
<tr>
<td>is_geometric()</td>
<td>Return True if the lattice is atomic and upper semimodular.</td>
</tr>
<tr>
<td>is_extremal()</td>
<td>Return True if the lattice is extremal.</td>
</tr>
<tr>
<td>is_complemented()</td>
<td>Return True if every element of the lattice has at least one complement.</td>
</tr>
<tr>
<td>is_sectionally_complemented()</td>
<td>Return True if every interval from the bottom is complemented.</td>
</tr>
<tr>
<td>is_relatively_complemented()</td>
<td>Return True if every interval to the top is complemented.</td>
</tr>
<tr>
<td>is_pseudocomplemented()</td>
<td>Return True if every element of the lattice has a (meet-)pseudocomplement.</td>
</tr>
<tr>
<td>is_join_pseudocomplemented()</td>
<td>Return True if every element of the lattice has a join-pseudocomplement.</td>
</tr>
<tr>
<td>is_orthocomplemented()</td>
<td>Return True if the lattice has an orthocomplementation.</td>
</tr>
<tr>
<td>is_supersolvable()</td>
<td>Return True if the lattice is supersolvable.</td>
</tr>
<tr>
<td>is_planar()</td>
<td>Return True if the lattice has an upward planar drawing.</td>
</tr>
<tr>
<td>is_dismantlable()</td>
<td>Return True if the lattice is dismantlable.</td>
</tr>
<tr>
<td>is_interval_dismantlable()</td>
<td>Return True if the lattice is interval dismantlable.</td>
</tr>
<tr>
<td>is_sublattice_dismantlable</td>
<td>Return True if the lattice is sublattice dismantlable.</td>
</tr>
<tr>
<td>is_stone()</td>
<td>Return True if the lattice is a Stone lattice.</td>
</tr>
</tbody>
</table>

continues on next page
<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>is_trim()</td>
<td>Return True if the lattice is a trim lattice.</td>
</tr>
<tr>
<td>is_vertically_decomposable()</td>
<td>Return True if the lattice is vertically decomposable.</td>
</tr>
<tr>
<td>is_simple()</td>
<td>Return True if the lattice has no nontrivial congruences.</td>
</tr>
<tr>
<td>is_isoform()</td>
<td>Return True if all congruences of the lattice consists of isoform blocks.</td>
</tr>
<tr>
<td>is_uniform()</td>
<td>Return True if all congruences of the lattice consists of equal-sized blocks.</td>
</tr>
<tr>
<td>is_regular()</td>
<td>Return True if all congruences of lattice are determined by any of the congruence blocks.</td>
</tr>
<tr>
<td>is_subdirectly_reducible()</td>
<td>Return True if the lattice is a sublattice of the product of smaller lattices.</td>
</tr>
<tr>
<td>is_constructible_by_doublings()</td>
<td>Return True if the lattice is constructible by doublings from the one-element lattice.</td>
</tr>
<tr>
<td>breadth()</td>
<td>Return the breadth of the lattice.</td>
</tr>
</tbody>
</table>

### Specific elements

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>atoms()</td>
<td>Return elements covering the bottom element.</td>
</tr>
<tr>
<td>coatoms()</td>
<td>Return elements covered by the top element.</td>
</tr>
<tr>
<td>double_irreducibles()</td>
<td>Return double irreducible elements.</td>
</tr>
<tr>
<td>join_primes()</td>
<td>Return the join prime elements.</td>
</tr>
<tr>
<td>meet_primes()</td>
<td>Return the meet prime elements.</td>
</tr>
<tr>
<td>complements()</td>
<td>Return the list of complements of an element, or the dictionary of complements for all elements.</td>
</tr>
<tr>
<td>pseudocomplement()</td>
<td>Return the pseudocomplement of an element.</td>
</tr>
<tr>
<td>is_modular_element()</td>
<td>Return True if given element is modular in the lattice.</td>
</tr>
<tr>
<td>is_left_modular_element()</td>
<td>Return True if given element is left modular in the lattice.</td>
</tr>
<tr>
<td>neutral_elements()</td>
<td>Return neutral elements of the lattice.</td>
</tr>
<tr>
<td>canonical_joinands()</td>
<td>Return the canonical joinands of an element.</td>
</tr>
<tr>
<td>canonical_meetands()</td>
<td>Return the canonical meetands of an element.</td>
</tr>
</tbody>
</table>

### Sublattices

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>sublattice()</td>
<td>Return sublattice generated by list of elements.</td>
</tr>
<tr>
<td>submeetsemilattice()</td>
<td>Return meet-subsemilattice generated by list of elements.</td>
</tr>
<tr>
<td>subjoinsemilattice()</td>
<td>Return join-subsemilattice generated by list of elements.</td>
</tr>
<tr>
<td>is_sublattice()</td>
<td>Return True if the lattice is a sublattice of given lattice.</td>
</tr>
<tr>
<td>sublattices()</td>
<td>Return all sublattices of the lattice.</td>
</tr>
<tr>
<td>sublattices_lattice()</td>
<td>Return the lattice of sublattices.</td>
</tr>
<tr>
<td>isomorphic_sublattices_iterator()</td>
<td>Return an iterator over the sublattices isomorphic to given lattice.</td>
</tr>
<tr>
<td>maximal_sublattices()</td>
<td>Return maximal sublattices of the lattice.</td>
</tr>
<tr>
<td>frattini_sublattice()</td>
<td>Return the intersection of maximal sublattices of the lattice.</td>
</tr>
<tr>
<td>skeleton()</td>
<td>Return the skeleton of the lattice.</td>
</tr>
<tr>
<td>center()</td>
<td>Return the sublattice of complemented neutral elements.</td>
</tr>
<tr>
<td>vertical_decomposition()</td>
<td>Return the vertical decomposition of the lattice.</td>
</tr>
</tbody>
</table>

### Miscellaneous

5.1. Comprehensive Module List
<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>moebius_algebra()</code></td>
<td>Return the Möbius algebra of the lattice.</td>
</tr>
<tr>
<td><code>quantum_moebius_algebra()</code></td>
<td>Return the quantum Möbius algebra of the lattice.</td>
</tr>
<tr>
<td><code>vertical_composition()</code></td>
<td>Return ordinal sum of lattices with top/bottom element unified.</td>
</tr>
<tr>
<td><code>day_doubling()</code></td>
<td>Return the lattice with Alan Day’s doubling construction of a subset.</td>
</tr>
<tr>
<td><code>adjunct()</code></td>
<td>Return the adjunct with other lattice.</td>
</tr>
<tr>
<td><code>subdirect_decomposition()</code></td>
<td>Return the subdirect decomposition of the lattice.</td>
</tr>
<tr>
<td><code>congruence()</code></td>
<td>Return the congruence generated by lists of elements.</td>
</tr>
<tr>
<td><code>quotient()</code></td>
<td>Return the quotient lattice by a congruence.</td>
</tr>
<tr>
<td><code>congruences_lattice()</code></td>
<td>Return the lattice of congruences.</td>
</tr>
</tbody>
</table>

```python
class sage.combinat.posets.lattices.FiniteJoinSemilattice(hasse_diagram, elements, category, facade, key):
    Bases: FinitePoset
    We assume that the argument passed to FiniteJoinSemilattice is the poset of a join-semilattice (i.e. a poset with least upper bound for each pair of elements).

    Element
    alias of JoinSemilatticeElement

    coatoms()
    Return the list of co-atoms of this (semi)lattice.
    A co-atom of a lattice is an element covered by the top element.
    EXAMPLES:
    sage: L = posets.DivisorLattice(60)
    sage: sorted(L.coatoms())
    [12, 20, 30]

    See also:
    • Dual function: atoms()

    join(x, y=None)
    Return the join of given elements in the lattice.
    INPUT:
    • x, y – two elements of the (semi)lattice OR
    • x – a list or tuple of elements
    EXAMPLES:
    sage: D = posets.DiamondPoset(5)
    sage: D.join(1, 2)
    4
    sage: D.join(1, 1)
    1
    sage: D.join(1, 4)
    4
    sage: D.join(1, 0)
    1

    Using list of elements as an argument. Join of empty list is the bottom element:
```
```
sage: B4=posets.BooleanLattice(4)
sage: B4.join([2,4,8])
14
sage: B4.join([])
0
```

For non-facade lattices operator + works for join:

```
sage: L = posets.PentagonPoset(facade=False)
sage: L(1)+L(2)
4
```

See also:

- Dual function: `meet()`

`join_matrix()`

Return a matrix whose \((i,j)\) entry is \(k\), where `self.linear_extension()[k]` is the join (least upper bound) of `self.linear_extension()[i]` and `self.linear_extension()[j]`.

**EXAMPLES:**

```
sage: P = LatticePoset([[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[]], facade=False)
sage: J = P.join_matrix(); J
[0 1 2 3 4 5 6 7]
[1 1 3 7 7 7 7 7]
[2 3 2 3 4 6 6 7]
[3 3 3 7 7 7 7 7]
[4 7 4 7 4 7 7 7]
[5 7 6 7 5 6 7 7]
[6 7 6 7 6 7 7 7]
[7 7 7 7 7 7 7 7]
True
True
False
```

class `sage.combinat.posets.lattices.FiniteLatticePoset`

Bases: `FiniteMeetSemilattice`, `FiniteJoinSemilattice`

We assume that the argument passed to `FiniteLatticePoset` is the poset of a lattice (i.e. a poset with greatest lower bound and least upper bound for each pair of elements).

**Element**

- alias of `LatticePosetElement`

**adjunct**(other, a, b)

Return the adjunct of the lattice by other on the pair \((a, b)\).

It is assumed that \(a < b\) but \(b\) does not cover \(a\).

The adjunct of a lattice \(K\) to \(L\) with respect to pair \((a, b)\) of \(L\) is defined such that \(x < y\) if
• \( x, y \in K \) and \( x < y \) in \( K \),
• \( x, y \in L \) and \( x < y \) in \( L \),
• \( x \in L, y \in K \) and \( x \leq a \) in \( L \), or
• \( x \in K, y \in L \) and \( b \leq y \) in \( L \).

Informally this can be seen as attaching the lattice \( K \) to \( L \) as a new block between \( a \) and \( b \). Dismantlable lattices are exactly those that can be created from chains with this function.

Mathematically, it is only defined when \( L \) and \( K \) have no common element; here we force that by giving them different names in the resulting lattice.

**EXAMPLES:**

```sage
definitions:
    Pnum = posets.PentagonPoset()
    Palp = Pnum.relabel(lambda x: chr(ord('a') + x))
    PP = Pnum.adjunct(Palp, 0, 3)
    PP.atoms()
    [[(0, 1), (0, 2), (1, 'a')]
    sage: PP.coatoms()
    [[(0, 3), (0, 1)]
```

**breadth** *(certificate=False)*

Return the breadth of the lattice.

The breadth of a lattice is the largest integer \( n \) such that any join of elements \( x_1, x_2, \ldots, x_{n+1} \) is join of a proper subset of \( x_i \).

This can be also characterized by subposets: a lattice of breadth at least \( n \) contains a subposet isomorphic to the Boolean lattice of \( 2^n \) elements.

**INPUT:**
• certificate – (default: False) whether to return a certificate

**OUTPUT:**
• If certificate=True return the pair \((b, a)\) where \( b \) is the breadth and \( a \) is an antichain such that the join of \( a \) differs from the join of any proper subset of \( a \). If certificate=False return just the breadth.

**EXAMPLES:**

```sage
definitions:
    D10 = posets.DiamondPoset(10)
    D10.breadth()
    2
    sage: B3 = posets.BooleanLattice(3)
    B3.breadth()
    3
    sage: B3.breadth(certificate=True)
    (3, [1, 2, 4])
```

**ALGORITHM:**

For a lattice to have breadth at least \( n \), it must have an \( n \)-element antichain \( A \) with join \( j \). Element \( j \) must cover at least \( n \) elements. There must also be \( n - 2 \) levels of elements between \( A \) and \( j \). So we start by searching elements that could be our \( j \) and then just check possible antichains \( A \).
**Note:** Prior to version 8.1 this function returned just an antichain with `certificate=True`.

**canonical_joinands(e)**

Return the canonical joinands of `e`.

The canonical joinands of an element `e` in the lattice `L` is the subset `S ⊆ L` such that 1) the join of `S` is `e`, and 2) if the join of some other subset `S'` of is also `e`, then for every element `s ∈ S` there is an element `s' ∈ S'` such that `s ≤ s'`.

Informally said this is the set of lowest possible elements with given join. It exists for every element if and only if the lattice is join-semidistributive. Canonical joinands are always join-irreducibles.

**INPUT:**

- `e` – an element of the lattice

**OUTPUT:**

- canonical joinands as a list, if it exists; if not, `None`

**EXAMPLES:**

```python
sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [5], 4: [6],
                      ....: 5: [7], 6: [7]})
sage: L.canonical_joinands(7)
[3, 4]
sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [6], 4: [6],
                      ....: 5: [6]})
sage: L.canonical_joinands(6) is None
True
```

See also:

**canonical_meetands()**

**canonical_meetands(e)**

Return the canonical meetands of `e`.

The canonical meetands of an element `e` in the lattice `L` is the subset `S ⊆ L` such that 1) the meet of `S` is `e`, and 2) if the meet of some other subset `S'` of is also `e`, then for every element `s ∈ S` there is an element `s' ∈ S'` such that `s ≥ s'`.

Informally said this is the set of greatest possible elements with given meet. It exists for every element if and only if the lattice is meet-semidistributive. Canonical meetands are always meet-irreducibles.

**INPUT:**

- `e` – an element of the lattice

**OUTPUT:**

- canonical meetands as a list, if it exists; if not, `None`

**EXAMPLES:**

```python
sage: L = LatticePoset({1: [2, 3], 2: [4], 3: [5, 6], 4: [6],
                      ....: 5: [7], 6: [7]})
sage: L.canonical_meetands(1)
[5, 4]
```

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(continued from previous page)

```python
sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [6], 4: [6],
.....: 5: [6]})
sage: L.canonical_meetands(1) is None
True
```

See also:

* canonical_joinands()

center()

Return the center of the lattice.

An element of a lattice is central if it is neutral and has a complement. The subposet induced by central elements is a center of the lattice. Actually it is a Boolean lattice.

```python
sage: L = LatticePoset({1: [2, 3, 4], 2: [6, 7], 3: [8, 9, 7],
.....: 4: [5, 6], 5: [8, 10], 6: [10], 7: [13, 11],
.....: 8: [13, 12], 9: [11, 12], 10: [13],
.....: 11: [14], 12: [14], 13: [14]})
sage: C = L.center(); C
Finite lattice containing 4 elements
```

```python
sage: C = L.center(); C
Finite lattice containing 4 elements
```

```python
sage: C = L.center(); C
Finite lattice containing 4 elements
```

EXAMPLES:

```python
sage: L = LatticePoset({1: [2, 3, 4], 2: [6, 7], 3: [8, 9, 7],
.....: 4: [5, 6], 5: [8, 10], 6: [10], 7: [13, 11],
.....: 8: [13, 12], 9: [11, 12], 10: [13],
.....: 11: [14], 12: [14], 13: [14]})
sage: C = L.center(); C
Finite lattice containing 4 elements
```

See also:

* neutral_elements(), complements()

complements(element=None)

Return the list of complements of an element in the lattice, or the dictionary of complements for all elements.

Elements \(x\) and \(y\) are complements if their meet and join are respectively the bottom and the top element of the lattice.

INPUT:

* `element` – an element of the lattice whose complement is returned. If `None` (default) then dictionary of complements for all elements having at least one complement is returned.

EXAMPLES:

```python
sage: L = LatticePoset({0:['a','b','c'],'a':[1],'b':[1],'c':[1]})
sage: C = L.complements()
```

Let us check that ‘a’ and ‘b’ are complements of each other:

```python
sage: 'a' in C['b']
True
sage: 'b' in C['a']
True
```
Full list of complements:

```python
sage: L.complements()  # random order
{0: [1], 1: [0], 'a': ['b', 'c'], 'b': ['c', 'a'], 'c': ['b', 'a']}
```

```python
sage: L = LatticePoset({0:[1,2], 1:[3], 2:[3], 3:[4]})
sage: L.complements()  # random order
{0: [4], 4: [0]}
```

See also:

`is_complemented()`

congruence($S$)

Return the congruence generated by set of sets $S$.

A congruence of a lattice is an equivalence relation $\cong$ that is compatible with meet and join; i.e. if $a_1 \cong a_2$ and $b_1 \cong b_2$, then $(a_1 \vee b_1) \cong (a_2 \vee b_2)$ and $(a_1 \wedge b_1) \cong (a_2 \wedge b_2)$.

By the congruence generated by set of sets $\{S_1, \ldots, S_n\}$ we mean the least congruence $\cong$ such that for every $x, y \in S_i$ for some $i$ we have $x \cong y$.

INPUT:

* $S$ – a list of lists; list of element blocks that the congruence will contain

OUTPUT:

Congruence of the lattice as a `sage.combinat.set_partition.SetPartition`.

EXAMPLES:

```python
sage: L = posets.DivisorLattice(12)
sage: cong = L.congruence([[1, 3]])
sage: sorted(sorted(c) for c in cong)
[[1, 3], [2, 6], [4, 12]]
```

```python
sage: L = LatticePoset({1: [2, 3], 2: [4], 3: [4], 4: [5], ....: 6: [8], 3: [9], 7: [10], 8: [10], 9:[10]})
sage: cong = L.congruence([[1, 2]])
sage: cong[0]
frozenset({1, 2, 3, 4, 5, 6, 7, 8, 9, 10})
```

See also:

`quotient()`

congruences_lattice($labels$=’congruence’)

Return the lattice of congruences.
A congruence of a lattice is a partition of elements to classes compatible with both meet- and join-operation; see `congruence()`. Elements of the congruence lattice are congruences ordered by refinement; i.e. if every class of a congruence \( \Theta \) is contained in some class of \( \Phi \), then \( \Theta \leq \Phi \) in the congruence lattice.

INPUT:
- labels – a string; the type of elements in the resulting lattice

OUTPUT:
A distributive lattice.
- If `labels='congruence'`, then elements of the result will be congruences given as `sage.combinat.set_partition.SetPartition`.
- If `labels='integers'`, result is a lattice on integers isomorphic to the congruence lattice.

EXAMPLES:

```python
sage: N5 = posets.PentagonPoset()
sage: CL = N5.congruences_lattice(); CL
Finite lattice containing 5 elements
sage: CL.atoms()
[{{0}, {1}, {2, 3}, {4}}]
sage: CL.coatoms()
[{{0, 1}, {2, 3, 4}}, {{0, 2, 3}, {1, 4}}]
```

```
sage: C4 = posets.ChainPoset(4)
sage: CL = C4.congruences_lattice(labels='integer')
sage: CL.is_isomorphic(posets.BooleanLattice(3))
True
```

day_doubling\((S)\)

Return the lattice with Alan Day’s doubling construction of subset \( S \).

The subset \( S \) is assumed to be convex (i.e. if \( a, c \in S \) and \( a < b < c \) in the lattice, then \( b \in S \)) and connected (i.e. if \( a, b \in S \) then there is a chain \( a = e_1, e_2, \ldots, e_n = b \) such that \( e_i \) either covers or is covered by \( e_{i+1} \)).
Alan Day’s doubling construction is a specific extension of the lattice. Here we formulate it in a format more suitable for computation.

Let $L$ be a lattice and $S$ a convex subset of it. The resulting lattice $L[S]$ has elements $(e, 0)$ for each $e \in L$ and $(e, 1)$ for each $e \in S$. If $x \leq y$ in $L$, then in the new lattice we have:

- $(x, 0), (x, 1) \leq (y, 0), (y, 1)$
- $(x, 0) \leq (x, 1)$

**INPUT:**
- $S$ – a subset of the lattice

**EXAMPLES:**

```sage
define L = LatticePoset({1: ['a', 'b', 2], 'a': ['c'], 'b': ['c', 'd'],
                            ....: 2: [3], 'c': [4], 'd': [4], 3: [4])
define L2 = L.day_doubling([['a', 'b', 'c', 'd']]); L2
Finite lattice containing 12 elements
```

```sage
define set(L2.upper_covers((1, 0))) == set(((2, 0), ('a', 0), ('b', 0)))
True
```

```sage
define set(L2.upper_covers(('b', 0))) == set(((4, 0), ('b', 1), ('c', 0)))
True
```

See also:
- `is_constructible_by_doublings()`
- `double_irreducibles()`

**double_irreducibles()**

Return the list of double irreducible elements of this lattice.

A double irreducible element of a lattice is an element covering and covered by exactly one element. In other words it is neither a meet nor a join of any elements.

**EXAMPLES:**
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```python
sage: L = posets.DivisorLattice(12)
sage: sorted(L.double_irreducibles())
[3, 4]
sage: L = posets.BooleanLattice(3)
sage: L.double_irreducibles()
[]
```

See also: `meet_irreducibles()`, `join_irreducibles()`

`feichtner_yuzvinsky_ring(G, use_defining=False, base_ring=None)`

Return the Feichtner-Yuzvinsky ring of `self` and `G`.

Let $R$ be a commutative ring, $L$ a lattice, and $G \subseteq L$. The Feichtner-Yuzvinsky ring is the quotient of the polynomial ring $R[h_g \mid g \in G]$ by the ideal generated by

- $h_a$ for every atom $a \in L \cap G$ and
- for every antichain $A$ of the subposet $G$ such that $g := \bigvee A \in G$ (with the join taken in $L$)

$$\prod_{a \in A} (h_g - h_a).$$

This was originally described for $G$ such that $(L, G)$ is a built lattice in the sense of [FY2004] (which has a geometric motivation), but this has been extended to $G$ being an arbitrary subset.

This is not the original definition, which uses the nested subsets of $G$ (see [FY2004] for the definition). However, the original construction, which we call the defining presentation and use the variables $\{x_g \mid g \in G\}$, can be recovered by setting $h_g = \sum_{g' \geq g} x_{g'}$ (where $g' \in G$).

**INPUT:**

- `G` – a subset of elements of `self`
- `use_defining` – (default: False) whether or not to use the defining presentation in $x_g$
- `base_ring` – (default: $\mathbb{Q}$) the base ring

The order on the variables is equal to the ordering of the elements in $G$.

**EXAMPLES:**

```python
sage: B2 = posets.BooleanLattice(2)
sage: FY = B2.feichtner_yuzvinsky_ring(B2[1:])
sage: FY
Quotient of Multivariate Polynomial Ring in h0, h1, h2 over Rational Field
by the ideal (h0, h1, h0*h1 - h0*h2 - h1*h2 + h2^2)
sage: FY = B2.feichtner_yuzvinsky_ring(B2[1:], use_defining=True)
sage: FY
Quotient of Multivariate Polynomial Ring in x0, x1, x2 over Rational Field
by the ideal (x0 + x2, x1 + x2, x0*x1)
```

We reproduce the example from Section 5 of [Coron2023]:

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sage: H.<a,b,c,d> = HyperplaneArrangements(QQ)
sage: Arr = H(a-b, b-c, c-d, d-a)
sage: P = LatticePoset(Arr.intersection_poset())
sage: FY = P.feichtner_yuzvinsky_ring([P.top(),5,1,2,3,4])
sage: FY.defining_ideal().groebner_basis()
[h0^2 - h0*h1, h1^2, h2, h3, h4, h5]

frattini_sublattice()

Return the Frattini sublattice of the lattice.

The Frattini sublattice $\Phi(L)$ is the intersection of all proper maximal sublattices of $L$. It is also the set of "non-generators" - if the sublattice generated by set $S$ of elements is whole lattice, then also $S \setminus \Phi(L)$ generates whole lattice.

EXAMPLES:

sage: L = LatticePoset(( [], [[1,2],[1,17],[1,8],[2,3],[2,22],
....: [2,5],[2,7],[17,22],[17,13],[8,7],
....: [8,13],[3,16],[3,9],[22,16],[22,18],
....: [22,10],[5,18],[5,14],[7,9],[7,14],
....: [7,10],[13,10],[16,6],[16,19],[9,19],
....: [18,6],[18,33],[14,33],[10,19],
....: [10,33],[6,4],[19,4],[33,4] ))
sage: sorted(L.frattini_sublattice().list())
[1, 2, 4, 10, 19, 22, 33]

is_atomic(certificate=False)

Return True if the lattice is atomic, and False otherwise.

A lattice is atomic if every element can be written as a join of atoms.

INPUT:

• certificate – (default: False) whether to return a certificate

OUTPUT:

• If certificate=True return either (True, None) or (False, e), where $e$ is a join-irreducible element that is not an atom. If certificate=False return True or False.

EXAMPLES:

sage: L = LatticePoset({1: [2, 3, 4], 2: [5], 3:[5], 4:[6], 5:[6]})
sage: L.is_atomic()
True
sage: L = LatticePoset({0: [1, 2], 1: [3], 2: [3], 3:[4]})
sage: L.is_atomic()
False
sage: L.is_atomic(certificate=True)
(False, 4)

Note: See [EnumComb1], Section 3.3 for a discussion of atomic lattices.

See also:
• Dual property: `is_coatomic()`

• Stronger properties: `is_sectionally_complemented()`

• Mutually exclusive properties: `is_vertically_decomposable()`

**`is_coatomic(certificate=\text{False})`**

Return True if the lattice is coatomic, and False otherwise.

A lattice is coatomic if every element can be written as a meet of coatoms; i.e. if the dual of the lattice is atomic.

**INPUT:**

• `certificate` – (default: False) whether to return a certificate

**OUTPUT:**

• If `certificate=True` return either (True, None) or (False, e), where e is a meet-irreducible element that is not a coatom. If `certificate=False` return True or False.

**EXAMPLES:**

```
sage: L = posets.BooleanLattice(3)
sage: L.is_coatomic()
True

sage: L = LatticePoset({1: [2], 2: [3, 4], 3: [5], 4: [5]})
sage: L.is_coatomic()
False
sage: L.is_coatomic(certificate=True)
(False, 1)
```

See also:

• Dual property: `is_atomic()`

• Stronger properties: `is_cosectionally_complemented()`

• Mutually exclusive properties: `is_vertically_decomposable()`

**`is_complemented(certificate=\text{False})`**

Return True if the lattice is complemented, and False otherwise.

A lattice is complemented if every element has at least one complement.

**INPUT:**

• `certificate` – (default: False) whether to return a certificate

**OUTPUT:**

• If `certificate=True` return either (True, None) or (False, e), where e is an element without a complement. If `certificate=False` return True or False.

**EXAMPLES:**

```
sage: L = LatticePoset({0: [1, 2, 3], 1: [4], 2: [4], 3: [4]})
sage: L.is_complemented()
True
```

(continues on next page)
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sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [5], 4: [6],
.....: 5: [7], 6: [7]})
sage: L.is_complemented()
False
sage: L.is_complemented(certificate=True)
(False, 2)

See also:

• Stronger properties: is_sectionally_complemented(), is_cosectionally_complemented(),
is_orthocomplemented()

• Other: complements()

is_constructible_by_doublings(type)

Return True if the lattice is constructible by doublings, and False otherwise.

We call a lattice doubling constructible if it can be constructed from the one element lattice by a sequence of Alan Day's doubling constructions.

Lattices constructible by interval doubling are also called bounded. Lattices constructible by lower and upper pseudo-interval are called lower bounded and upper bounded. Lattices constructible by any convex set doubling are called congruence normal.

INPUT:

• type – a string; can be one of the following:
  – 'interval' - allow only doublings of an interval
  – 'lower' - allow doublings of lower pseudo-interval; that is, a subset of the lattice with a unique minimal element
  – 'upper' - allow doublings of upper pseudo-interval; that is, a subset of the lattice with a unique maximal element
  – 'convex' - allow doubling of any convex set
  – 'any' - allow doubling of any set

EXAMPLES:

The pentagon can be constructed by doubling intervals; the 5-element diamond can not be constructed by any doublings:

sage: posets.PentagonPoset().is_constructible_by_doublings('interval')
True
sage: posets.DiamondPoset(5).is_constructible_by_doublings('any')
False

After doubling both upper and lower pseudo-interval a lattice is constructible by convex subset doubling:

sage: L = posets.BooleanLattice(2)
sage: L = L.day_doubling([0, 1, 2])  # A lower pseudo-interval
sage: L.is_constructible_by_doublings('interval')
False
sage: L.is_constructible_by_doublings('lower')
(continues on next page)

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True
sage: L = L.day_doubling([(3,0), (1,1), (2,1)])  # An upper pseudo-interval
sage: L.is_constructible_by_doublings('upper')
False
sage: L.is_constructible_by_doublings('convex')
True

An example of a lattice that can be constructed by doublings of a non-convex subsets:

sage: L = LatticePoset(DiGraph('OQC?a?@CO?G_C@?GA?O??_??@?BO?A_?G??C??_@???'))
sage: L.is_constructible_by_doublings('convex')
False
sage: L.is_constructible_by_doublings('any')
True

See also:

- Stronger properties: is_distributive() (doubling by interval), is_join_semidistributive() (doubling by lower pseudo-intervals), is_meet_semidistributive() (doubling by upper pseudo-intervals)
- Mutually exclusive properties: is_simple() (doubling by any set)
- Other: day_doubling()

ALGORITHM:

According to [HOLM2016] a lattice \( L \) is lower bounded if and only if \(|\text{Ji}(L)| = |\text{Ji}(\text{Con} L)|\), and so dually \(|\text{Mi}(L)| = |\text{Mi}(\text{Con} L)|\) in upper bounded lattices. The same reference gives a test for being constructible by convex or by any subset.

is_cosectionally_complemented(certificate=False)

Return True if the lattice is cosectionally complemented, and False otherwise.

A lattice is cosectionally complemented if all intervals to the top element interpreted as sublattices are complemented lattices.

INPUT:

- certificate – (default: False) Whether to return a certificate if the lattice is not cosectionally complemented.

OUTPUT:

- If certificate=False return True or False. If certificate=True return either (True, None) or (False, (b, e)), where \( b \) is an element so that in the sublattice from \( b \) to the top element has no complement for element \( e \).

EXAMPLES:

The smallest sectionally but not cosectionally complemented lattice:

sage: L = LatticePoset({1: [2, 3, 4], 2: [5], 3: [5], 4: [6], 5: [6]})
sage: L.is_sectionally_complemented(), L.is_cosectionally_complemented()
(True, False)

A sectionally and cosectionally but not relatively complemented lattice:
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```python
sage: L = LatticePoset(DiGraph('MYi@O?P??D?OG?O?_?C?Q??O?W?O????

sage: L.is_sectionally_complemented() and L.is_cosectionally_complemented()
True

sage: L.is_relatively_complemented()
False

Getting a certificate:

sage: L = LatticePoset(DiGraph('HW?@D?Q?GE?G@??

sage: L.is_cosectionally_complemented(certificate=True)
(False, (2, 7))
```

See also:

- Dual property: `is_sectionally_complemented`
- Weaker properties: `is_complemented`, `is_coatomic`, `is_regular`
- Stronger properties: `is_relatively_complemented`

**is_dismantlable** *(certificate=False)*

Return True if the lattice is dismantlable, and False otherwise.

An \(n\)-element lattice \(L_n\) is dismantlable if there is a sublattice chain \(L_{n-1} \supset L_{n-2} \supset \cdots \supset L_0\) so that every \(L_i\) is a sublattice of \(L_{i+1}\) with one element less, and \(L_0\) is the empty lattice. In other words, a dismantlable lattice can be reduced to empty lattice removing doubly irreducible element one by one.

**INPUT:**

- **certificate** *(boolean)* – Whether to return a certificate.
  - If `certificate = False` (default), returns True or False accordingly.
  - If `certificate = True`, returns:
    * (True, elms) when the lattice is dismantlable, where elms is elements listed in a possible removing order.
    * (False, crown) when the lattice is not dismantlable, where crown is a subposet of \(2k\) elements \(a_1, \ldots, a_k, b_1, \ldots, b_k\) with covering relations \(a_i \lessdot b_i\) and \(a_i \lessdot b_{i+1}\) for \(i \in [1, \ldots, k-1]\), and \(a_k \lessdot b_1\).

**EXAMPLES:**

```python
sage: DL12 = LatticePoset((divisors(12), attrcall("divides")))

sage: DL12.is_dismantlable(True)
(True, [4, 2, 1, 3, 6, 12])

sage: B3 = posets.BooleanLattice(3)

sage: B3.is_dismantlable(False)
False

sage: B3.is_dismantlable(certificate=True)
(False, Finite poset containing 6 elements)
```

Every planar lattice is dismantlable. Converse is not true:
sage: L = LatticePoset( ([], [[0, 1], [0, 2], [0, 3], [0, 4],
....: [1, 7], [2, 6], [3, 5], [4, 5],
....: [4, 6], [4, 7], [5, 8], [6, 8],
....: [7, 8]] )

sage: L.is_dismantlable()
True
sage: L.is_planar()
False

See also:

• Stronger properties: is_planar()
• Weaker properties: is_sublattice_dismantlable()

is_distributive(certificate=False)

Return True if the lattice is distributive, and False otherwise.

A lattice \((L, \vee, \wedge)\) is distributive if meet distributes over join: \(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\) for every \(x, y, z \in L\) just like \(x \cdot (y + z) = x \cdot y + x \cdot z\) in normal arithmetic. For duality in lattices it follows that then also join distributes over meet.

• certificate – (default: False) whether to return a certificate

OUTPUT:

• If certificate=True return either (True, None) or (False, (x, y, z)), where x, y and z are elements of the lattice such that \(x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)\). If certificate=False return True or False.

EXAMPLES:

sage: L = LatticePoset({1: [2, 3], 2: [4], 3: [4], 4: [5]})
sage: L.is_distributive()
True
sage: L = LatticePoset({1: [2, 3, 4], 2: [5], 3: [6], 4: [6], 5: [6]})
sage: L.is_distributive()
False
sage: L.is_distributive(certificate=True)
(False, (5, 3, 2))

See also:

• Weaker properties: is_modular(), is_semidistributive(), is_join_distributive(), is_meet_distributive(), is_subdirectly_reducible(), is_trim(), is_constructible_by_doublings() (by interval doubling), is_extremal()
• Stronger properties: is_stone()

is_extremal()

Return True if the lattice is extremal, and False otherwise.

A lattice is extremal if the number of join-irreducibles is equal to the number of meet-irreducibles and to the number of cover relations in the longest chains.

EXAMPLES:
sage: posets.PentagonPoset().is_extremal()
True

sage: P = LatticePoset(posets.SymmetricGroupWeakOrderPoset(3))
sage: P.is_extremal()
False

See also:

• Stronger properties: is_distributive(), is_trim()

REFERENCES:

• [Mark1992]

is_geometric()

Return True if the lattice is geometric, and False otherwise.

A lattice is geometric if it is both atomic and upper semimodular.

EXAMPLES:

Canonical example is the lattice of partitions of finite set ordered by refinement:

sage: L = posets.SetPartitions(4)
sage: L.is_geometric()
True

Smallest example of geometric lattice that is not modular:

sage: L = LatticePoset(DiGraph('K???g@S?q?M?@??@??@??@??'))
sage: L.is_geometric()
True
sage: L.is_modular()
False

Two non-examples:

sage: L = LatticePoset({1:[2, 3, 4], 2:[5, 6], 3:[5], 4:[6], 5:[7], 6:[7]})
sage: L.is_geometric()  # Graded, but not upper semimodular
False
sage: L = posets.ChainPoset(3)
sage: L.is_geometric()  # Modular, but not atomic
False

See also:

• Weaker properties: is_upper_semimodular(), is_relatively_complemented()

is_interval_dismantlable(\text{\texttt{certificate=\texttt{False}}})

Return True if the lattice is interval dismantlable, and False otherwise.

An interval dismantling is a subdivision of a lattice to a principal upper set and a principal lower set. A lattice is \textit{interval dismantlable} if it can be decomposed into 1-element lattices by consecutive interval dismantlings.
A lattice is minimally interval non-dismantlable if it is not interval dismantlable, but all of its sublattices are interval dismantlable.

INPUT:
- certificate – (default: False) whether to return a certificate

OUTPUT:
- if certificate=False, return only True or False
- if certificate=True, return either
  - (True, list) where list is a nested list showing the decomposition; for example list[1][0]
    is a lower part of upper part of the lattice when decomposed twice.
  - (False, M) where M is a minimally interval non-dismantlable sublattice of the lattice.

EXAMPLES:

```python
sage: L1 = LatticePoset({1: [2, 3], 3: [4, 5], 2: [6], 4: [6], 5: [6]})
sage: L1.is_interval_dismantlable()
True
sage: L2 = LatticePoset({1: [2, 3, 4, 5], 2: [6], 3: [6], 4: [6],
...: 5: [6, 7], 6: [8], 7: [9, 10], 8:[10], 9:[10]})
sage: L2.is_interval_dismantlable()
False
```

To get certificates:

```python
sage: L1.is_interval_dismantlable(certificate=True)
(True, [[[1], [2]], [[[3], [5]], [[4], [6]]]])
sage: L2.is_interval_dismantlable(certificate=True)
(False, Finite lattice containing 5 elements)
```

See also:

- Stronger properties: `is_join_semidistributive()`, `is_meet_semidistributive()`
- Weaker properties: `is_sublattice_dismantlable()`

**is_isoform(certificate=False)**

Return True if the lattice is isoform and False otherwise.

A congruence is isoform (or isotype) if all blocks are isomorphic sublattices. A lattice is isoform if it has only isoform congruences.

INPUT:
- certificate – (default: False) whether to return a certificate if the lattice is not isoform

OUTPUT:
- If certificate=True return either (True, None) or (False, C), where C is a non-isoform con-
  gruence as a `sage.combinat.set_partition.SetPartition`. If certificate=False return
  True or False.

EXAMPLES:
sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [6, 7], 4: [7], 5: [8], 6: \[8], 7: [8]})
sage: L.is_isoform()
True

Every isoform lattice is (trivially) uniform, but the converse is not true:

sage: L = LatticePoset({1: [2, 3, 6], 2: [4, 5], 3: [5], 4: [9, 8], 5: [7, 8], 6: [9], 7: [10], 8: [10], 9: [10]})
sage: L.is_isoform(), L.is_uniform()
(False, True)
sage: L.is_isoform(certificate=True)
(False, {{1, 2, 4, 6, 9}, {3, 5, 7, 8, 10}})

See also:
- Weaker properties: is_uniform()
- Stronger properties: is_simple(), is_relatively_complemented()
- Other: congruence()

**is_join_distributive** (certificate=False)

Return True if the lattice is join-distributive and False otherwise.

A lattice is join-distributive if every interval from an element to the join of the element’s upper covers is a distributive lattice. Actually this distributive sublattice is then a Boolean lattice.

They are also called as Dilworth’s lattices and upper locally distributive lattices. They can be characterized in many other ways, see [Dil1940].

**INPUT:**
- certificate – (default: False) whether to return a certificate

**OUTPUT:**
- If certificate=True return either (True, None) or (False, e), where e is an element such that the interval from e to the meet of upper covers of e is not distributive. If certificate=False return True or False.

**EXAMPLES:**

sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [5, 7],
....: 4: [6, 7], 5: [8, 9], 6: [9], 7: [9, 10],
....: 8: [11], 9: [11], 10: [11]})
sage: L.is_join_distributive()
True
sage: L = LatticePoset({1: [2], 2: [3, 4], 3: [5], 4: [6],
....: 5: [7], 6: [7]})
sage: L.is_join_distributive()
False
sage: L.is_join_distributive(certificate=True)
(False, 2)

See also:
**is_join_pseudocomplemented** *(certificate=False)*

Return True if the lattice is join-pseudocomplemented, and False otherwise.

A lattice is join-pseudocomplemented if every element \( e \) has a join-pseudocomplement \( e' \), i.e. the least element such that the join of \( e \) and \( e' \) is the top element.

**INPUT:**
- certificate – (default: False) whether to return a certificate

**OUTPUT:**
- If certificate=True return either (True, None) or (False, \( e \)), where \( e \) is an element without a join-pseudocomplement. If certificate=False return True or False.

**EXAMPLES:**

```python
sage: L = LatticePoset({1: [2, 5], 2: [3, 6], 3: [4], 4: [7],
                    ....: 5: [6], 6: [7]})
sage: L.is_join_pseudocomplemented()
True

sage: L = LatticePoset({1: [2, 3], 2: [4, 5, 6], 3: [6], 4: [7],
                    ....: 5: [7], 6: [7]})
sage: L.is_join_pseudocomplemented()
False

sage: L.is_join_pseudocomplemented(certificate=True)
(False, 4)
```

**See also:**
- Dual property: *is_pseudocomplemented()*
- Stronger properties: *is_join_semidistributive()*

**is_join_semidistributive** *(certificate=False)*

Return True if the lattice is join-semidistributive, and False otherwise.

A lattice is join-semidistributive if for all elements \( e, x, y \) in the lattice we have

\[
e \lor x = e \lor y \implies e \lor x = e \lor (x \land y)
\]

**INPUT:**
- certificate – (default: False) whether to return a certificate

**OUTPUT:**
- If certificate=True return either (True, None) or (False, \( (e, x, y) \)) such that \( e \lor x = e \lor y \) but \( e \lor x \neq e \lor (x \land y) \). If certificate=False return True or False.

**EXAMPLES:**
sage: T4 = posets.TamariLattice(4)
sage: T4.is_join_semidistributive()
True
sage: L = LatticePoset({1:[2, 3], 2:[4, 5], 3:[5, 6],
    ....:             4:[7], 5:[7], 6:[7]})
sage: L.is_join_semidistributive()
False
sage: L.is_join_semidistributive(certificate=True)
(False, (5, 4, 6))

See also:

• Dual property: is_meet_semidistributive()
• Weaker properties: is_join_pseudocomplemented(), is_interval_dismantlable()
• Stronger properties: is_semidistributive(), is_meet_distributive(),
  is_constructible_by_doublings() (by lower pseudo-intervals)

is_left_modular_element(x)

Return True if x is a left modular element and False otherwise.

INPUT:

• x – an element of the lattice

An element x in a lattice L is left modular if

\[(y \lor x) \land z = y \lor (x \land z)\]

for every \(y \leq z \in L\).

It is enough to check this condition on all cover relations \(y < z\).

EXAMPLES:

```
sage: P = posets.PentagonPoset()
sage: [i for i in P if P.is_left_modular_element(i)]
[0, 2, 3, 4]
```

See also:

• Stronger properties: is_modular_element()

is_lower_semimodular(certificate=False)

Return True if the lattice is lower semimodular and False otherwise.

A lattice is lower semimodular if any pair of elements with a common upper cover have also a common
lower cover.

INPUT:

• certificate – (default: False) Whether to return a certificate if the lattice is not lower semimodular.

OUTPUT:

• If certificate=False return True or False. If certificate=True return either (True, None)
  or (False, (a, b)), where a and b are covered by their join but do no cover their meet.

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See Wikipedia article Semimodular_lattice

EXAMPLES:

```sage
sage: L = posets.DiamondPoset(5)
sage: L.is_lower_semimodular()
True

sage: L = posets.PentagonPoset()
sage: L.is_lower_semimodular()
False

sage: L = posets.ChainPoset(6)
sage: L.is_lower_semimodular()
True

sage: L = LatticePoset(DiGraph('IS?`?AAOE_@?C?_@??'))
sage: L.is_lower_semimodular(certificate=True)
(False, (4, 2))
```

See also:

• Dual property: `is_upper_semimodular()`
• Weaker properties: `is_graded()`
• Stronger properties: `is_modular()`, `is_meet_distributive()`

`is_meet_distributive(certificate=False)`

Return True if the lattice is meet-distributive and False otherwise.

A lattice is *meet-distributive* if every interval to an element from the meet of the element’s lower covers is a distributive lattice. Actually this distributive sublattice is then a Boolean lattice.

They are also called as *lower locally distributive lattices*. They can be characterized in many other ways, see [Dil1940].

INPUT:

• certificate – (default: False) whether to return a certificate

OUTPUT:

• If certificate=True return either (True, None) or (False, e), where e is an element such that the interval to e from the meet of lower covers of e is not distributive. If certificate=False return True or False.

EXAMPLES:

```sage
sage: L = LatticePoset({1: [2, 3, 4], 2: [5], 3: [5, 6, 7],
....: 4: [7], 5: [9, 8], 6: [10, 8], 7:
....: [9, 10], 8: [11], 9: [11], 10: [11]})
sage: L.is_meet_distributive()
True

sage: L = LatticePoset({1: [2, 3], 2: [4], 3: [5], 4: [6],
....: 5: [6], 6: [7]})
sage: L.is_meet_distributive()
(continues on next page)```
False

\texttt{sage: L.is_meet_distributive(certificate=True)}

(False, 6)

See also:

- Dual property: \texttt{is_join_distributive()}
- Weaker properties: \texttt{is_join_semidistributive()}, \texttt{is_lower_semidistributive()}
- Stronger properties: \texttt{is_distributive()}

\texttt{is_meet_semidistributive(certificate=False)}

Return True if the lattice is meet-semidistributive, and False otherwise.

A lattice is meet-semidistributive if for all elements \(e, x, y\) in the lattice we have

\[
e \land x = e \land y \Rightarrow e \land x = e \land (x \lor y)
\]

INPUT:

- \texttt{certificate} – (default: False) whether to return a certificate

OUTPUT:

- If \texttt{certificate=True} return either \texttt{(True, None)} or \texttt{(False, (e, x, y))} such that \(e \land x = e \land y\) but \(e \land x \neq e \land (x \lor y)\). If \texttt{certificate=False} return True or False.

EXAMPLES:

\texttt{sage: L = LatticePoset({1:[2, 3, 4], 2:[4, 5], 3:[5, 6], ....: 4:[7], 5:[7], 6:[7]})}
\texttt{sage: L.is_meet_semidistributive()}
True
\texttt{sage: L_ = L.dual()}
\texttt{sage: L_.is_meet_semidistributive()}
False
\texttt{sage: L_.is_meet_semidistributive(certificate=True)}

(True, (5, 4, 6))

See also:

- Dual property: \texttt{is_join_semidistributive()}
- Weaker properties: \texttt{is_pseudocomplemented()}, \texttt{is_interval_dismantlable()}
- Stronger properties: \texttt{is_semidistributive()}, \texttt{is_join_distributive()}

\texttt{is_constructible_by_doublings()} (by upper pseudo-intervals)

\texttt{is_modular(L=None, certificate=False)}

Return True if the lattice is modular and False otherwise.

An element \(b\) of a lattice is modular if

\[
x \lor (a \land b) = (x \lor a) \land b
\]

for every element \(x \leq b\) and \(a\). A lattice is modular if every element is modular. There are other equivalent definitions, see Wikipedia article Modular_lattice.
With the parameter \( L \) this can be used to check that some subset of elements are all modular.

**INPUT:**

- \( L \) – (default: None) a list of elements to check being modular, if \( L \) is None, then this checks the entire lattice
- `certificate` – (default: False) whether to return a certificate

**OUTPUT:**

- If `certificate=True` return either \((True, None)\) or \((False, (x, a, b))\), where \(a, b\) and \(x\) are elements of the lattice such that \(x < b\) but \(x \lor (a \land b) \neq (x \lor a) \land b\). If also \( L \) is given then \(b\) in the certificate will be an element of \( L \). If `certificate=False` return \(True\) or \(False\).

**EXAMPLES:**

```python
sage: L = posets.DiamondPoset(5)
sage: L.is_modular()
True
sage: L = posets.PentagonPoset()
sage: L.is_modular()
False
sage: L = LatticePoset({1:[2,3],2:[4,5],3:[5,6],4:[7],5:[7],6:[7]})
sage: L.is_modular(certificate=True)
(False, (2, 6, 4))
sage: [L.is_modular_element(x) for x in L]
[True, True, False, True, True, False, True]
```

**See also:**

- Weaker properties: `is_upper_semimodular()`, `is_lower_semimodular()`, `is_supersolvable()`
- Stronger properties: `is_distributive()`
- Other: `is_modular_element()`

**is_modular_element(x)**

Return \(True\) if \(x\) is a modular element and \(False\) otherwise.

**INPUT:**

- \(x\) – an element of the lattice

An element \(x\) in a lattice \(L\) is **modular** if \(x \leq b\) implies

\[
x \lor (a \land b) = (x \lor a) \land b
\]

for every \(a, b \in L\).

**EXAMPLES:**

```python
sage: L = LatticePoset({1:[2,3],2:[4,5],3:[5,6],4:[7],5:[7],6:[7]})
sage: L.is_modular()
False
sage: [L.is_modular_element(x) for x in L]
[True, True, False, True, True, False, True]
```
See also:

- Weaker properties: `is_left_modular_element()`
- Other: `is_modular()` to check modularity for the full lattice or some set of elements

**is_orthocomplemented**(unique=False)

Return `True` if the lattice admits an orthocomplementation, and `False` otherwise.

An orthocomplementation of a lattice is a function defined for every element \(e\) and marked as \(e^\perp\) such that

1) they are complements, i.e. \(e \lor e^\perp\) is the top element and \(e \land e^\perp\) is the bottom element,
2) it is involution, i.e. \((e^\perp)^\perp = e\), and
3) it is order-reversing, i.e. if \(a < b\) then \(b^\perp < a^\perp\).

**INPUT:**

- `unique`, a Boolean – If True, return `True` only if the lattice has exactly one orthocomplementation. If False (the default), return `True` when the lattice has at least one orthocomplementation.

**EXAMPLES:**

```python
sage: D5 = posets.DiamondPoset(5)
sage: D5.is_orthocomplemented()
False

sage: D6 = posets.DiamondPoset(6)
sage: D6.is_orthocomplemented()
True

sage: D6.is_orthocomplemented(unique=True)
False

sage: hexagon = LatticePoset({0:[1, 2], 1:[3], 2:[4], 3:[5], 4:[5]})
sage: hexagon.is_orthocomplemented(unique=True)
True
```

See also:

- Weaker properties: `is_complemented()`, `is_self_dual()`

**is_planar()**

Return `True` if the lattice is upward planar, and `False` otherwise.

A lattice is upward planar if its Hasse diagram has a planar drawing in the \(\mathbb{R}^2\) plane, in such a way that \(x\) is strictly below \(y\) (on the vertical axis) whenever \(x < y\) in the lattice.

Note that the scientific literature on posets often omits “upward” and shortens it to “planar lattice” (e.g. [GW2014]), which can cause confusion with the notion of graph planarity in graph theory.

**Note:** Not all lattices which are planar – in the sense of graph planarity – admit such a planar drawing (see example below).

**ALGORITHM:**

Using the result from [Platt1976], this method returns its result by testing that the Hasse diagram of the lattice is planar (in the sense of graph theory) when an edge is added between the top and bottom elements.

**EXAMPLES:**

The Boolean lattice of \(2^3\) elements is not upward planar, even if its covering relations graph is planar:
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sage: B3 = posets.BooleanLattice(3)
sage: B3.is_planar()
False
sage: G = B3.cover_relations_graph()
sage: G.is_planar()
True

Ordinal product of planar lattices is obviously planar. Same does not apply to Cartesian products:

sage: P = posets.PentagonPoset()
sage: Pc = P.product(P)
sage: Po = P.ordinal_product(P)
sage: Pc.is_planar()
False
sage: Po.is_planar()
True

See also:

• Weaker properties: is_dismantlable()

is_pseudocomplemented(certificate=False)

Return True if the lattice is pseudocomplemented, and False otherwise.

A lattice is (meet-)pseudocomplemented if every element \( e \) has a pseudocomplement \( e^* \), i.e. the greatest element such that the meet of \( e \) and \( e^* \) is the bottom element.

See Wikipedia article Pseudocomplement.

INPUT:

• certificate – (default: False) whether to return a certificate

OUTPUT:

• If certificate=True return either (True, None) or (False, e), where e is an element without a pseudocomplement. If certificate=False return True or False.

EXAMPLES:

sage: L = LatticePoset({1: [2, 5], 2: [3, 6], 3: [4], 4: [7],
.....: 5: [6], 6: [7]})
sage: L.is_pseudocomplemented()
True
sage: L = LatticePoset({1: [2, 3], 2: [4, 5, 6], 3: [6], 4: [7],
.....: 5: [7], 6: [7]})
sage: L.is_pseudocomplemented()
False
sage: L.is_pseudocomplemented(certificate=True)
(False, 3)

See also:

• Dual property: is_join_pseudocomplemented()

• Stronger properties: is_meet_semidistributive()
• Other: pseudocomplement().

ALGORITHM:
According to [Cha92] a lattice is pseudocomplemented if and only if every atom has a pseudocomplement. So we only check those.

\texttt{is\_regular}(\texttt{certificate=False})

Return True if the lattice is regular and False otherwise.

A congruence of a lattice is regular if it is generated by any of its parts. A lattice is regular if it has only regular congruences.

INPUT:

• certificate – (default: False) whether to return a certificate if the lattice is not regular

OUTPUT:

• If certificate=True return either \((True, None)\) or \((False, (C, p))\), where \(C\) is a non-regular congruence as a \texttt{sage.combinat.set\_partition.SetPartition} and \(p\) is a congruence class of \(C\) such that the congruence generated by \(p\) is not \(C\). If certificate=False return True or False.

EXAMPLES:

\begin{verbatim}
sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [8, 7], 4: [6, 7], 5: [8],
    6: [9], 7: [9], 8: [9]})
sage: L.is_regular()
True
sage: N5 = posets.PentagonPoset()
sage: N5.is_regular()
False
sage: N5.is_regular(certificate=True)
(False, ({{0}, {1}, {2, 3}, {4}}, [0]))
\end{verbatim}

See also:

• Stronger properties: \texttt{is\_uniform()}, \texttt{is\_sectionally\_complemented()}, \texttt{is\_cosectionally\_complemented()}

• Mutually exclusive properties: \texttt{is\_vertically\_decomposable()}

• Other: \texttt{congruence()}

\texttt{is\_relatively\_complemented}(\texttt{certificate=False})

Return True if the lattice is relatively complemented, and False otherwise.

A lattice is relatively complemented if every interval of it is a complemented lattice.

INPUT:

• certificate – (default: False) Whether to return a certificate if the lattice is not relatively complemented.

OUTPUT:

• If certificate=True return either \((True, None)\) or \((False, (a, b, c))\), where \(b\) is the only element that covers \(a\) and is covered by \(c\). If certificate=False return True or False.

EXAMPLES:
Relatively complemented lattice must be both atomic and coatomic. Implication to other direction does not hold:

```python
sage: L = LatticePoset({0: [1, 2, 3, 4, 5], 1: [6, 7], 2: [6, 8],
                   ....:
                   3: [7, 8, 9], 4: [9, 11], 5: [9, 10],
                   ....:
                   6: [10, 11], 7: [12], 8: [12], 9: [12],
                   ....:
                   10: [12], 11: [12]})
sage: L.is_atomic() and L.is_coatomic()
True
sage: L.is_relatively_complemented()
False
```

We can also get a non-complemented 3-element interval:

```python
sage: L.is_relatively_complemented(certificate=True)
(False, (1, 6, 11))
```

See also:

- Weaker properties: `is_sectionally_complemented()`, `is_cosectionally_complemented()`, `is_isoform()`
- Stronger properties: `is_geometric()`

### is_sectionally_complemented(certificate=False)

Return True if the lattice is sectionally complemented, and False otherwise.

A lattice is sectionally complemented if all intervals from the bottom element interpreted as sublattices are complemented lattices.

**INPUT:**

- `certificate` - (default: False) Whether to return a certificate if the lattice is not sectionally complemented.

**OUTPUT:**

- If `certificate=False` return either True or False. If `certificate=True` return either (True, None) or (False, (t, e)), where t is an element so that in the sublattice from the bottom element to t has no complement for element e.

**EXAMPLES:**

Smallest examples of a complemented but not sectionally complemented lattice and a sectionally complemented but not relatively complemented lattice:

```python
sage: L = posets.PentagonPoset()
sage: L.is_relatively_complemented()
False
```
True
\texttt{sage: L.is_sectionally_complemented()}
False

\texttt{sage: L = LatticePoset({0: \{1, 2, 3\}, 1: \{4\}, 2: \{4\}, 3: \{5\}, 4: \{5\}})}
\texttt{sage: L.is_sectionally_complemented()}
True
\texttt{sage: L.is_relatively_complemented()}
False

Getting a certificate:

\texttt{sage: L = LatticePoset(DiGraph('HYOgC?C@?C?G@??'))}
\texttt{sage: L.is_sectionally_complemented(certificate=True)}
(False, (6, 1))

\textbf{See also:}

- Dual property: \texttt{is_cosectionally_complemented()}
- Weaker properties: \texttt{is_complemented(), is_atomic(), is_regular()}
- Stronger properties: \texttt{is_relatively_complemented()}

\textbf{is_semidistributive()}

Return True if the lattice is both join- and meet-semidistributive, and False otherwise.

\textbf{EXAMPLES:}

Tamari lattices are typical examples of semidistributive but not distributive (and hence not modular) lattices:

\texttt{sage: T4 = posets.TamariLattice(4)}
\texttt{sage: T4.is_semidistributive(), T4.is_distributive()}
(True, False)

Smallest non-selfdual example:

\texttt{sage: L = LatticePoset({1: \{2, 3\}, 2: \{4, 5\}, 3: \{5\}, 4: \{6\}, 5: \{7\}, 6: \{7\}})}
\texttt{sage: L.is_semidistributive()}
True

The diamond is not semidistributive:

\texttt{sage: L = posets.DiamondPoset(5)}
\texttt{sage: L.is_semidistributive()}
False

\textbf{See also:}

- Weaker properties: \texttt{is_join_semidistributive(), is_meet_semidistributive()}
- Stronger properties: \texttt{is_distributive()}
**is_simple** *(certificate=False)*

Return **True** if the lattice is simple and **False** otherwise.

A lattice is *simple* if it has no nontrivial congruences; in other words, for every two distinct elements \(a\) and \(b\) the principal congruence generated by \((a, b)\) has only one component, i.e. the whole lattice.

**INPUT:**
- certificate – (default: **False**) whether to return a certificate if the lattice is not simple

**OUTPUT:**
- If certificate=**True** return either (**True**, **None**) or (**False**, \(c\)), where \(c\) is a nontrivial congruence as a `sage.combinat.set_partition.SetPartition`. If certificate=**False** return **True** or **False**.

**EXAMPLES:**

```python
sage: posets.DiamondPoset(5).is_simple()  # Smallest nontrivial example
True
sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [6], 4: [6], 5: [6]})
sage: L.is_simple()
False
sage: L.is_simple(certificate=True)
(False, [{1, 3}, {2, 4, 5, 6}])
```

Two more examples. First is a non-simple lattice without any 2-element congruences:

```python
sage: L = LatticePoset({1: [2, 3, 4], 2: [5], 3: [5], 4: [6, 7], ...
.....: 5: [8], 6: [8], 7: [8]})
sage: L.is_simple()
False
sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [6, 7], 4: [8], ...
.....: 5: [8], 6: [8], 7: [8]})
sage: L.is_simple()
True
```

See also:
- Weaker properties: `is_isoform()`
- Mutually exclusive properties: `is_constructible_by_doublings()` (by any set)
- Other: `congruence()`

**is_stone** *(certificate=False)*

Return **True** if the lattice is a Stone lattice, and **False** otherwise.

The lattice is expected to be distributive (and hence pseudocomplemented).

A pseudocomplemented lattice is a Stone lattice if

\[
e^* \lor e^{**} = \top
\]

for every element \(e\) of the lattice, where \(*\) is the pseudocomplement and \(\top\) is the top element of the lattice.

**INPUT:**
- certificate – (default: **False**) whether to return a certificate

**OUTPUT:**
• If certificate=True return either (True, None) or (False, e) such that $e^* \lor e^{**} \neq \top$. If certificate=False return True or False.

EXAMPLES:
Divisor lattices are canonical example:

```
sage: D72 = posets.DivisorLattice(72)
sage: D72.is_stone()
True
```

A non-example:

```
sage: L = LatticePoset({1: [2, 3], 2: [4], 3: [4], 4: [5]})
sage: L.is_stone()
False
```

See also:

• Weaker properties: \textit{is_distributive}()

\textbf{is_subdirectly_reducible}(certificate=False)

Return True if the lattice is subdirectly reducible.

A lattice $M$ is a \textit{subdirect product} of $K$ and $L$ if it is a sublattice of $K \times L$. Lattice $M$ is \textit{subdirectly reducible} if there exists such lattices $K$ and $L$ so that $M$ is not a sublattice of either.

INPUT:

• certificate – (default: False) whether to return a certificate

OUTPUT:

• if certificate=False, return only True or False
• if certificate=True, return either
  – (True, (K, L)) such that the lattice is isomorphic to a sublattice of $K \times L$.
  – (False, (a, b)), where $a$ and $b$ are elements that are in the same congruence class for every nontrivial congruence of the lattice. Special case: If the lattice has zero or one element, return (False, None).

EXAMPLES:

```
sage: N5 = posets.PentagonPoset()
sage: N5.is_subdirectly_reducible()
False
```

```
sage: hex = LatticePoset({1: [2, 3], 2: [4], 3: [5], 4: [6], 5: [6]})
sage: hex.is_subdirectly_reducible()
True
```

```
sage: hex.is_subdirectly_reducible(certificate=True)
(True,
 (Finite lattice containing 5 elements, Finite lattice containing 5 elements))
```

```
sage: N5.is_subdirectly_reducible(certificate=True)
(False, (2, 3))
```

(continues on next page)
sage: res, cert = hex.is_subdirectly_reducible(certificate=True)
sage: cert[0].is_isomorphic(N5)
True

See also:

• Stronger properties: is_distributive(), is_vertically_decomposable()
• Other: subdirect_decomposition()

is_sublattice(other)

Return True if the lattice is a sublattice of other, and False otherwise.

Lattice \( K \) is a sublattice of \( L \) if \( K \) is an (induced) subposet of \( L \) and closed under meet and join of \( L \).

Note: This method does not check whether the lattice is a isomorphic (i.e., up to relabeling) sublattice of other, but only if other directly contains the lattice as an sublattice.

EXAMPLES:

A pentagon sublattice in a non-modular lattice:

sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [5, 6], 4: [7], 5: [7], 6: [7]}
˓→)
sage: N5 = LatticePoset({1: [2, 6], 2: [4], 4: [7], 6: [7]})
sage: N5.is_sublattice(L)
True

This pentagon is a subposet but not closed under join, hence not a sublattice:

sage: N5_ = LatticePoset({1: [2, 3], 2: [4], 3: [7], 4: [7]})
sage: N5_.is_induced_subposet(L)
True
sage: N5_.is_sublattice(L)
False

See also:

isomorphic_sublattices_iterator()

is_sublattice_dismantlable()

Return True if the lattice is sublattice dismantlable, and False otherwise.

A sublattice dismantling is a subdivision of a lattice into two non-empty sublattices. A lattice is sublattice dismantlable if it can be decomposed into 1-element lattices by consecutive sublattice dismantlings.

EXAMPLES:

The smallest non-example is this (and the dual):

sage: P = Poset({1: [11, 12, 13], 2: [11, 14, 15],
.....: 3: [12, 14, 16], 4: [13, 15, 16]})
sage: L = LatticePoset(P.with_bounds())
sage: L.is_sublattice_dismantlable()
False
Here we adjoin a (double-irreducible-)dismantlable lattice as a part to an interval-dismantlable lattice:

```python
sage: B3 = posets.BooleanLattice(3)
sage: N5 = posets.PentagonPoset()
sage: L = B3.adjunct(N5, 1, 7)
sage: L.is_dismantlable(), L.is_interval_dismantlable()
(False, False)
sage: L.is_sublattice_dismantlable()
True
```

See also:

- Stronger properties: `is_dismantlable()`, `is_interval_dismantlable()`

Todo: Add a certificate-option.

```python
is_supersolvable(certificate=False)
```

Return True if the lattice is supersolvable, and False otherwise.

A lattice \( L \) is supersolvable if there exists a maximal chain \( C \) such that every \( x \in C \) is a modular element in \( L \). Equivalent definition is that the sublattice generated by \( C \) and any other chain is distributive.

INPUT:

- `certificate` – (default: False) whether to return a certificate

OUTPUT:

- If `certificate=True` return either (False, None) or (True, \( C \)), where \( C \) is a maximal chain of modular elements. If `certificate=False` return True or False.

EXAMPLES:

```python
sage: L = posets.DiamondPoset(5)
sage: L.is_supersolvable()
True
sage: L = posets.PentagonPoset()
sage: L.is_supersolvable()
False
sage: L = LatticePoset({1:[2,3],2:[4,5],3:[5,6],4:[7],5:[7],6:[7]})
sage: L.is_supersolvable()
True
sage: L.is_supersolvable(certificate=True)
(True, [1, 2, 5, 7])
sage: L.is_modular()
False
sage: L = LatticePoset({0: [1, 2, 3, 4], 1: [5, 6, 7],
....: 2: [5, 8, 9], 3: [6, 8, 10], 4: [7, 9, 10],
....: 5: [11], 6: [11], 7: [11], 8: [11],
....: 9: [11], 10: [11]})

sage: L.is_supersolvable()
False
```

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See also:

- Weaker properties: `is_graded()`
- Stronger properties: `is_modular()`

**is_trim**(*certificate=False*)

Return whether a lattice is trim.

A lattice is trim if it is extremal and left modular.

This notion is defined in [Thom2006].

**INPUT:**

- `certificate` – boolean (default `False`) whether to return instead a maximum chain of left modular elements

**EXAMPLES:**

```python
sage: P = posets.PentagonPoset()
sage: P.is_trim()
True

sage: Q = LatticePoset(posets.SymmetricGroupWeakOrderPoset(3))
sage: Q.is_trim()
False
```

See also:

- Weaker properties: `is_extremal()`
- Stronger properties: `is_distributive()`

**REFERENCES:**

**is_uniform**(*certificate=False*)

Return `True` if the lattice is uniform and `False` otherwise.

A congruence is uniform if all blocks have equal number of elements. A lattice is uniform if it has only uniform congruences.

**INPUT:**

- `certificate` – (default: `False`) whether to return a certificate if the lattice is not uniform

**OUTPUT:**

- If `certificate=True` return either `(True, None)` or `(False, C)`, where `C` is a non-uniform congruence as a `sage.combinat.set_partition.SetPartition`. If `certificate=False` return `True` or `False`.

**EXAMPLES:**

```python
sage: L = LatticePoset({1: [2, 3, 4], 2: [6, 7], 3: [5], 4: [5], 5: [9, 8], 6: [9], 7: [10], 8: [10], 9: [10]})
sage: L.is_uniform()
True
```

Every uniform lattice is regular, but the converse is not true:
sage: N6 = LatticePoset({1: [2, 3, 5], 2: [4], 3: [4], 5: [6], 4: [6]})
sage: N6.is_uniform(), N6.is_regular()
(False, True)
sage: N6.is_uniform(certificate=True)
(False, {{1, 2, 3, 4}, {5, 6}})

See also:

- Weaker properties: \texttt{is\_regular()}
- Stronger properties: \texttt{is\_isoform()}
- Other: \texttt{congruence()}

\texttt{is\_upper\_semimodular(certificate=False)}

Return True if the lattice is upper semimodular and False otherwise.

A lattice is upper semimodular if any pair of elements with a common lower cover have also a common upper cover.

INPUT:

- \texttt{certificate} – (default: False) Whether to return a certificate if the lattice is not upper semimodular.

OUTPUT:

- If \texttt{certificate=False} return True or False. If \texttt{certificate=True} return either (True, None) or (False, (a, b)), where a and b covers their meet but are not covered by their join.

See \texttt{Wikipedia article Semimodular\_lattice}

EXAMPLES:

sage: L = posets.DiamondPoset(5)
sage: L.is_upper_semimodular()
True

sage: L = posets.PentagonPoset()
sage: L.is_upper_semimodular()
False

sage: L = LatticePoset(posets.IntegerPartitions(4))
sage: L.is_upper_semimodular()
True

sage: L = LatticePoset({1:[2, 3, 4], 2: [5], 3:[5, 6], 4:[6], 5:[7], 6:[7]})
sage: L.is_upper_semimodular(certificate=True)
(False, (4, 2))

See also:

- Dual property: \texttt{is\_lower\_semimodular()}
- Weaker properties: \texttt{is\_graded()}
- Stronger properties: \texttt{is\_modular(),\ is\_join\_distributive(),\ is\_geometric()}
**is_vertically_decomposable** *(certificate=False)*

Return *True* if the lattice is vertically decomposable, and *False* otherwise.

A lattice is vertically decomposable if it has an element that is comparable to all elements and is neither the bottom nor the top element.

Informally said, a lattice is vertically decomposable if it can be seen as two lattices “glued” by unifying the top element of first lattice to the bottom element of second one.

**INPUT:**

- **certificate** – *(default: False)* whether to return a certificate

**OUTPUT:**

- If *certificate=True* return either *(False, None)* or *(True, e)*, where *e* is an element that is comparable to all other elements and is neither the bottom nor the top element. If *certificate=False* return *True* or *False*.

**EXAMPLES:**

```python
sage: posets.TamariLattice(4).is_vertically_decomposable()
False
sage: L = LatticePoset( ([1, 2, 3, 6, 12, 18, 36],
                   ....: attrcall("divides") )
sage: L.is_vertically_decomposable()
True
sage: L.is_vertically_decomposable(certificate=True)
(True, 6)
```

See also:

- Weaker properties: *is_subdirectly_reducible()*
- Mutually exclusive properties: *is_atomic(), is_coatomic(), is_regular()*
- Other: *vertical_decomposition()*

**isomorphic_sublattices_iterator** *(other)*

Return an iterator over the sublattices of the lattice isomorphic to *other*.

**INPUT:**

- **other** – a finite lattice

**EXAMPLES:**

A non-modular lattice contains a pentagon sublattice:

```python
sage: L = LatticePoset({1: [2, 3], 2: [4, 5], 3: [5, 6], 4: [7], 5: [7], 6: [7]})
sage: L.is_modular()
False
sage: N5 = posets.PentagonPoset()
sage: N5_in_L = next(L.isomorphic_sublattices_iterator(N5)); N5_in_L
Finite lattice containing 5 elements
sage: N5_in_L.list()
[1, 3, 6, 4, 7]
```
A divisor lattice is modular, hence does not contain the pentagon as sublattice, even if it has the pentagon subposet:

```python
sage: D12 = posets.DivisorLattice(12)
sage: D12.has_isomorphic_subposet(N5)
True
sage: list(D12.isomorphic_sublattices_iterator(N5))
[]
```

See also:

`sage.combinat.posets.posets.FinitePoset.isomorphic_subposets_iterator()`

**Warning:** This function will return same sublattice as many times as there are automorphism on it. This is due to `subgraph_search_iterator()` returning labelled subgraphs.

### join_primes()

Return the join-prime elements of the lattice.

An element $x$ of a lattice $L$ is *join-prime* if $x \leq a \lor b$ implies $x \leq a$ or $x \leq b$ for every $a, b \in L$.

These are also called *coprime* in some books. Every join-prime is join-irreducible; converse holds if and only if the lattice is distributive.

**EXAMPLES:**

```python
sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [5], ...
.....: 4: [6], 5: [7], 6: [7]})
sage: L.join_primes()
[3, 4]
sage: D12 = posets.DivisorLattice(12)  # Distributive lattice
sage: D12.join_irreducibles() == D12.join_primes()
True
```

See also:

- Dual function: `meet_primes()`
- Other: `join_irreducibles()`

### maximal_sublattices()

Return maximal (proper) sublattices of the lattice.

**EXAMPLES:**

```python
sage: L = LatticePoset(( [], [1,2],[1,17],[1,8],[2,3],[2,22],
.....:  [2,5],[2,7],[17,22],[17,13],[8,7],
.....:  [8,13],[3,16],[3,9],[22,16],[22,18],
.....:  [22,10],[5,18],[5,14],[7,9],[7,14],
.....:  [7,10],[13,10],[16,6],[16,19],[9,19],
.....:  [18,6],[18,33],[14,33],[10,19],
.....:  [10,33],[6,4],[19,4],[33,4] ))
sage: maxs = L.maximal_sublattices()
sage: len(maxs)
5
```

(continues on next page)
meet_primes()

Return the meet-prime elements of the lattice.

An element $x$ of a lattice $L$ is meet-prime if $x \geq a \land b$ implies $x \geq a$ or $x \geq b$ for every $a, b \in L$.

These are also called just prime in some books. Every meet-prime is meet-irreducible; converse holds if and only if the lattice is distributive.

EXAMPLES:

```python
sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [5],
                  ....: 4: [6], 5: [7], 6: [7]})
sage: L.meet_primes()
[6, 5]
sage: D12 = posets.DivisorLattice(12)
sage: sorted(D12.meet_primes())
[3, 4, 6]
```

See also:

- Dual function: join_primes()
- Other: meet_irreducibles()

moebius_algebra($R$)

Return the Möbius algebra of self over $R$.

OUTPUT:


EXAMPLES:

```python
sage: L = posets.BooleanLattice(4)
sage: L.moebius_algebra(QQ)
Moebius algebra of Finite lattice containing 16 elements over Rational Field
```

neutral_elements()

Return the list of neutral elements of the lattice.

An element $e$ of the lattice $L$ is neutral if the sublattice generated by $e, x$ and $y$ is distributive for all $x, y \in L$.

It can also be characterized as an element of intersection of maximal distributive sublattices.

EXAMPLES:

```python
sage: L = LatticePoset({1: [2, 3], 2: [6], 3: [4, 5, 6], 4: [8],
                  ....: 5: [7], 6: [7], 7: [8, 9], 8: [10], 9: [10]})
sage: L.neutral_elements()
[1, 3, 8, 10]
```
**quantum_moebius_algebra**(*q=None*)

Return the quantum Möbius algebra of self with parameter q.

**INPUT:**
- q – (optional) the deformation parameter q

**OUTPUT:**
An instance of `sage.combinat.posets.moebius_algebra.QuantumMoebiusAlgebra`.

**EXAMPLES:**
```python
sage: L = posets.BooleanLattice(4)
sage: L.quantum_moebius_algebra()
Quantum Moebius algebra of Finite lattice containing 16 elements with q=q over Univariate Laurent Polynomial Ring in q over Integer Ring
```

**quotient**(*congruence, labels='tuple'*)

Return the quotient lattice by congruence.

Let L be a lattice and Θ be a congruence of L with congruence classes Θ₁, Θ₂,.... The quotient lattice L/Θ is the lattice with elements {Θ₁, Θ₂,...} and meet and join given by the original lattice. Explicitly, if e₁ ∈ Θ₁ and e₂ ∈ Θ₂, such that e₁ ∨ e₂ ∈ Θ₃ then Θ₁ ∨ Θ₂ = Θ₃ in L/Θ and similarly for meets.

**INPUT:**
- congruence – list of lists; a congruence
- labels – string; the elements of the resulting lattice and can be one of the following:
  - 'tuple' - elements are tuples of elements of the original lattice
  - 'lattice' - elements are sublattices of the original lattice
  - 'integer' - elements are labeled by integers

**Warning:** congruence is expected to be a valid congruence of the lattice. This is not checked.

**EXAMPLES:**
```python
sage: L = posets.PentagonPoset()
sage: c = L.congruence([[0, 1]])
sage: I = L.quotient(c); I
Finite lattice containing 2 elements
sage: I.top()
(2, 3, 4)
sage: I = L.quotient(c, labels='lattice')
sage: I.top()
(2, 3, 4)
sage: B3 = posets.BooleanLattice(3)
sage: c = B3.congruence([[0, 1]])
sage: B2 = B3.quotient(c, labels='integer')
sage: B2.is_isomorphic(posets.BooleanLattice(2))
True
```
See also:

\textit{congruence()}\textnormal{\textregistered} \\
\textit{skeleton()} \\
Return the skeleton of the lattice.

The lattice is expected to be pseudocomplemented.

The \textit{skeleton} of a pseudocomplemented lattice \(L\), where \(\ast\) is the pseudocomplementation operation, is the subposet induced by \(\{e^\ast \mid e \in L\}\). Actually this poset is a Boolean lattice.

\textbf{EXAMPLES:}

```sage
d12 = posets.DivisorLattice(12)
s = d12.skeleton(); s
Finite lattice containing 4 elements
s.skeleton().cover_relations()
[[1, 3], [1, 4], [3, 12], [4, 12]]
t4 = posets.TamariLattice(4)
skeleton().is_isomorphic(posets.BooleanLattice(3))
True
```

See also:

\textit{sage.combinat.posets.lattices.FiniteMeetSemilattice.pseudocomplement()}.

\textit{subdirect\_decomposition()} \\
Return the subdirect decomposition of the lattice.

The subdirect decomposition of a lattice \(L\) is the list of smaller lattices \(L_1, \ldots, L_n\) such that \(L\) is a sublattice of \(L_1 \times \cdots \times L_n\), none of \(L_i\) can be decomposed further and \(L\) is not a sublattice of any \(L_i\). (Except when the list has only one element, i.e. when the lattice is subdirectly irreducible.)

\textbf{EXAMPLES:}

```sage
posets.ChainPoset(3).subdirect\_decomposition()
[ Finite lattice containing 2 elements, Finite lattice containing 2 elements ]
d = LatticePoset({1: [2, 4], 2: [3], 3: [6, 7], 4: [5, 7], 5: [9, 8], 6: [9], 7: [9], 8: [10], 9: [10]})
L = d.LatticePoset({1: [2, 4], 2: [3], 3: [6, 7], 4: [5, 7], 5: [9, 8], 6: [9], 7: [9], 8: [10], 9: [10]})
L.decomposition()  
[ fac.cardinality() for fac in L.decomposition() ]
[2, 5, 7]
L.decomposition[1].is_isomorphic(posets.PentagonPoset())
True
```

\textit{sublattice}(elms) \\
Return the smallest sublattice containing elements on the given list.

\textbf{INPUT:}

- \textit{elms} – a list of elements of the lattice.

\textbf{EXAMPLES:}

```sage
L = LatticePoset([[], [[1, 2], [1, 17], [1, 8], [2, 3], [2, 22], [2, 5], [2, 7], [17, 22], [17, 13], [8, 7], [8, 13], [3, 16], [3, 9], [22, 16], [22, 18], [22, 10], [5, 18], [5, 14]]
(continues on next page)```
sage: L = posets.BooleanLattice(3)
sage: L.sublattice([3, 5, 6, 7])
Finite lattice containing 8 elements

sublattices()
Return all sublattices of the lattice.

EXAMPLES:

sage: L = LatticePoset({1: [2, 3, 4], 2:[5], 3:[5, 6], 4:[6], ....:
5:[7], 6:[7]})
sage: sublats = L.sublattices(); len(sublats)
54
sage: sublats[3]
Finite lattice containing 4 elements
sage: sublats[3].list()
[1, 2, 3, 5]

sublattices_lattice(labels='lattice')
Return the lattice of sublattices.

Every element of the returned lattice is a sublattice and they are ordered by containment; that is, atoms are one-element lattices, coatoms are maximal sublattices of the original lattice and so on.

INPUT:

• labels – string; can be one of the following:
  – 'lattice' (default) elements of the lattice will be lattices that correspond to sublattices of the original lattice
  – 'tuple' - elements are tuples of elements of the sublattices of the original lattice
  – 'integer' - elements are plain integers

EXAMPLES:

sage: D4 = posets.DiamondPoset(4)
sage: sll = D4.sublattices_lattice(labels='tuple')
sage: sll.coatoms()  # = maximal sublattices of the original lattice
[(0, 1, 3), (0, 2, 3)]

sage: L = posets.DivisorLattice(12)
sage: sll = L.sublattices_lattice()
sage: L.is_dismantlable() == (len(sll.atoms()) == sll.rank())
True

vertical_composition(other, labels='pairs')
Return the vertical composition of the lattice with other.

Let $L$ and $K$ be lattices and $b_K$ the bottom element of $K$. The vertical composition of $L$ and $K$ is the ordinal sum of $L$ and $K \setminus \{b_K\}$. Informally said this is lattices “glued” together with a common element.
Mathematically, it is only defined when \( L \) and \( K \) have no common element; here we force that by giving them different names in the resulting poset.

INPUT:

- **other** – a lattice
- **labels** – a string (default `pairs`); can be one of the following:
  - `pairs` - each element \( v \) in this poset will be named \((0, v)\) and each element \( u \) in other will be named \((1, u)\) in the result
  - `integers` - the elements of the result will be relabeled with consecutive integers

EXAMPLES:

```
sage: L = LatticePoset({'a': ['b', 'c'], 'b': ['d'], 'c': ['d']})
sage: K = LatticePoset({'e': ['f', 'g'], 'f': ['h'], 'g': ['h']})
sage: M = L.vertical_composition(K)
sage: M.list()
[(0, 'a'), (0, 'b'), (0, 'c'), (0, 'd'), (1, 'f'), (1, 'g'), (1, 'h')]
sage: M.upper_covers((0, 'd'))
[(1, 'f'), (1, 'g')]
```

```
sage: C2 = posets.ChainPoset(2)
sage: M3 = posets.DiamondPoset(5)
sage: L = C2.vertical_composition(M3, labels='integers')
sage: L.cover_relations()
[[0, 1], [1, 2], [1, 3], [1, 4], [2, 5], [3, 5], [4, 5]]
```

See also:

vertical_decomposition(), sage.combinat.posets.posets.FinitePoset.ordinal_sum()
See also:

vertical_composition(), is_vertically_decomposable()

class sage.combinat.posets.lattices.FiniteMeetSemilattice(
    hasse_diagram, elements, category,
    facade, key)

Bases: FinitePoset

Note: We assume that the argument passed to MeetSemilattice is the poset of a meet-semilattice (i.e. a poset with greatest lower bound for each pair of elements).

Element

alias of MeetSemilatticeElement

atoms()

Return the list atoms of this (semi)lattice.

An atom of a lattice is an element covering the bottom element.

EXAMPLES:

```
sage: L = posets.DivisorLattice(60)
sage: sorted(L.atoms())
[2, 3, 5]
```

See also:

• Dual function: coatoms()

meet(x, y=None)

Return the meet of given elements in the lattice.

INPUT:

• x, y – two elements of the (semi)lattice OR
• x – a list or tuple of elements

EXAMPLES:

```
sage: D = posets.DiamondPoset(5)
sage: D.meet(1, 2)
0
sage: D.meet(1, 1)
1
sage: D.meet(1, 0)
0
sage: D.meet(1, 4)
1
```

Using list of elements as an argument. Meet of empty list is the bottom element:

```
sage: B4=posets.BooleanLattice(4)
sage: B4.meet([3,5,6])
0
sage: B4.meet([])
15
```
For non-facade lattices operator \* works for meet:

```
sage: L = posets.PentagonPoset(facade=False)
sage: L(1)*L(2)
0
```

See also:

- Dual function: `join()`

**meet_matrix()**

Return a matrix whose \((i,j)\) entry is \(k\), where `self.linear_extension()[k]` is the meet (greatest lower bound) of `self.linear_extension()[i]` and `self.linear_extension()[j]`.

**EXAMPLES:**

```
sage: P = LatticePoset([[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[7]], facade = False)
sage: M = P.meet_matrix(); M
[0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 1]
[0 0 2 2 0 2 2]
[0 1 2 3 2 0 2 3]
[0 0 2 2 4 0 2 4]
[0 0 0 0 0 5 5 5]
[0 0 2 2 5 6 6]
[0 1 2 3 4 5 6 7]
True
True
False
```

**pseudocomplement(element)**

Return the pseudocomplement of `element`, if it exists.

The (meet-)pseudocomplement is the greatest element whose meet with given element is the bottom element. I.e. in a meet-semilattice with bottom element \(\hat{0}\) the pseudocomplement of an element \(e\) is the element \(e^*\) such that \(e \land e^* = \hat{0}\) and \(e' \leq e^*\) if \(e \land e' = \hat{0}\).


**INPUT:**

- `element` – an element of the lattice.

**OUTPUT:**

An element of the lattice or `None` if the pseudocomplement does not exist.

**EXAMPLES:**

The pseudocomplement’s pseudocomplement is not always the original element:

```
sage: L = LatticePoset({1: [2, 3], 2: [4], 3: [5], 4: [6], 5: [6]})
sage: L.pseudocomplement(2)
5
```

sage: L.pseudocomplement(5)
4

An element can have complements but no pseudocomplement, or vice versa:

sage: L = LatticePoset({0: [1, 2], 1: [3, 4, 5], 2: [5], 3: [6],
....: 4: [6], 5: [6]})
sage: L.complements(1), L.pseudocomplement(1)
([], 2)
sage: L.complements(2), L.pseudocomplement(2)
([3, 4], None)

See also:

is_pseudocomplemented()

subjoinsemilattice(elms)
Return the smallest join-subsemilattice containing elements on the given list.

INPUT:

• elms – a list of elements of the lattice.

EXAMPLES:

sage: L = posets.DivisorLattice(1000)
sage: L_ = L.subjoinsemilattice([2, 25, 125]); L_
Finite join-semilattice containing 5 elements
sage: sorted(L_.list())
[2, 25, 50, 125, 250]

See also:

• Dual function: submeetsemilattice()

submeetsemilattice(elms)
Return the smallest meet-subsemilattice containing elements on the given list.

INPUT:

• elms – a list of elements of the lattice.

EXAMPLES:

sage: L = posets.DivisorLattice(1000)
sage: L_ = L.submeetsemilattice([200, 250, 125]); L_
Finite meet-semilattice containing 5 elements
sage: L_.list()
[25, 50, 200, 125, 250]

See also:

• Dual function: subjoinsemilattice()
sage.combinat.posets.lattices.JoinSemilattice(data=None, *args, **options)

Construct a join semi-lattice from various forms of input data.

INPUT:

- data, *args, **options – data and options that will be passed down to Poset() to construct a poset that is also a join semilattice

See also:
Poset(), MeetSemilattice(), LatticePoset()

EXAMPLES:

Using data that defines a poset:

```
sage: JoinSemilattice([[1,2],[3],[3]])
Finite join-semilattice containing 3 elements
sage: JoinSemilattice([[1,2],[3],[3]], cover_relations = True)
Finite join-semilattice containing 3 elements
```

Using a previously constructed poset:

```
sage: P = Poset([[1,2],[3],[3]])
sage: J = JoinSemilattice(P); J
Finite join-semilattice containing 3 elements
sage: type(J)
<class 'sage.combinat.posets.lattices.FiniteJoinSemilattice_with_category'>
```

If the data is not a lattice, then an error is raised:

```
sage: JoinSemilattice({'a': ['b', 'c'], 'b': ['d', 'e'],
....:                    'c': ['d', 'e'], 'd': ['f'], 'e': ['f']})
Traceback (most recent call last):
...
LatticeError: no join for b and c
```

sage.combinat.posets.lattices.LatticePoset(data=None, *args, **options)

Construct a lattice from various forms of input data.

INPUT:

- data, *args, **options – data and options that will be passed down to Poset() to construct a poset that is also a lattice.

OUTPUT:

An instance of FiniteLatticePoset.

See also:
Posets, FiniteLatticePosets, JoinSemilattice(), MeetSemilattice()

EXAMPLES:

Using data that defines a poset:

```
sage: LatticePoset([[1,2],[3],[3]])
Finite lattice containing 3 elements
```

(continues on next page)
sage: LatticePoset([[1,2],[3],[3]], cover_relations = True)
Finite lattice containing 3 elements

Using a previously constructed poset:

sage: P = Poset([[1,2],[3],[3]])
sage: L = LatticePoset(P); L
Finite lattice containing 3 elements
sage: type(L)
<class 'sage.combinat.posets.lattices.FiniteLatticePoset_with_category'>

If the data is not a lattice, then an error is raised:

sage: elms = [1,2,3,4,5,6,7]
sage: rels = [[1,2],[3,4],[4,5],[2,5]]
sage: LatticePoset((elms, rels))
Traceback (most recent call last):
...  
ValueError: not a meet-semilattice: no bottom element

Creating a facade lattice:

sage: L = LatticePoset([[1,2],[3],[3]], facade = True)
sage: L.category()
Category of facade finite enumerated lattice posets
sage: parent(L[0])
Integer Ring
sage: TestSuite(L).run(skip = ['_test_an_element']) # is_parent_of is not yet implemented

sage.combinat.posets.lattices.MeetSemilattice(data=None, *args, **options)

Construct a meet semilattice from various forms of input data.

INPUT:

• data, *args, **options – data and options that will be passed down to Poset() to construct a poset that is also a meet semilattice.

See also:

Poset(), JoinSemilattice(), LatticePoset()

EXAMPLES:

Using data that defines a poset:

sage: MeetSemilattice([[1,2],[3],[3]])
Finite meet-semilattice containing 3 elements
sage: MeetSemilattice([[1,2],[3],[3]], cover_relations = True)
Finite meet-semilattice containing 3 elements

Using a previously constructed poset:
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```
sage: P = Poset([[1,2],[3],[3]])
sage: L = MeetSemilattice(P); L
Finite meet-semilattice containing 3 elements
sage: type(L)
<class 'sage.combinat.posets.lattices.FiniteMeetSemilattice_with_category'>
```

If the data is not a lattice, then an error is raised:

```
sage: MeetSemilattice({'a': ['b', 'c'], 'b': ['d', 'e'],
....:     'c': ['d', 'e'], 'd': ['f'], 'e': ['f']})
Traceback (most recent call last):
...  
LatticeError: no meet for e and d
```}

### 5.1.183 Linear Extensions of Posets

This module defines two classes:

- `LinearExtensionOfPoset`
- `LinearExtensionsOfPoset`
- `LinearExtensionsOfPosetWithHooks`
- `LinearExtensionsOfForest`

**Classes and methods**

#### class `sage.combinat.posets.linear_extensions.LinearExtensionOfPoset`

**Bases:** `ClonableArray`

A linear extension of a finite poset $P$ of size $n$ is a total ordering $\pi := \pi_0 \pi_1 \ldots \pi_{n-1}$ of its elements such that $i < j$ whenever $\pi_i < \pi_j$ in the poset $P$.

When the elements of $P$ are indexed by $\{1, 2, \ldots, n\}$, $\pi$ denotes a permutation of the elements of $P$ in one-line notation.

**INPUT:**

- `linear_extension` – a list of the elements of $P$
- `poset` – the underlying poset $P$

**See also:**

`Poset`, `LinearExtensionsOfPoset`

**EXAMPLES:**

```
sage: P = Poset(((1,2,3,4), [[1,3],[1,4],[2,3]]), linear_extension=True, 
            facade=False)
sage: p = P.linear_extension([1,4,2,3]); p
[1, 4, 2, 3]
sage: p.parent()
The set of all linear extensions of Finite poset containing 4 elements with
    distinguished linear extension
```

(continues on next page)
Following Schützenberger and later Haiman and Malvenuto-Reutenauer, Stanley [Stan2009] defined a promotion and evacuation operator on any finite poset $P$ using operators $\tau_i$ on the linear extensions of $P$:

```python
sage: p.promotion()
[1, 2, 3, 4]
sage: Q = p.promotion().to_poset()
sage: Q.cover_relations()
[[1, 3], [1, 4], [2, 3]]
sage: Q == P
True
sage: p.promotion(3)
[1, 4, 2, 3]
sage: Q = p.promotion(3).to_poset()
sage: Q == P
False
sage: Q.cover_relations()
[[1, 2], [1, 4], [3, 4]]
```

**check()**

Checks whether `self` is indeed a linear extension of the underlying poset.

**evacuation()**

Compute evacuation on the linear extension of a poset.

Evacuation on a linear extension $\pi$ of length $n$ is defined as $\pi(\tau_1 \cdots \tau_{n-1})(\tau_1 \cdots \tau_{n-2}) \cdots (\tau_1)$. For more details see [Stan2009].

See also:

`tau()`, `promotion()`

**EXAMPLES:**

```python
sage: P = Poset((([1,2,3,4,5,6,7], [[1,2],[1,4],[2,3],[2,5],[3,6],[4,7],[5,6]]))
sage: p = P.linear_extension([1,2,3,4,5,6,7])
sage: p.evacuation()
[1, 4, 2, 3, 7, 5, 6]
sage: p.evacuation().evacuation() == p
True
```

**is_greedy()**

Return True if the linear extension is greedy.

A linear extension $[e_1, e_2, \ldots, e_n]$ is greedy if for every $i$ either $e_{i+1}$ covers $e_i$ or all upper covers of $e_i$ have at least one lower cover that is not in $[e_1, e_2, \ldots, e_i]$.

Informally said a linear extension is greedy if it “always goes up when possible” and so has no unnecessary jumps.

**EXAMPLES:**
is_supergreedy()

Return True if the linear extension is supergreedy.

A linear extension of a poset \( P \) with elements \( \{x_1, x_2, \ldots, x_t\} \) is super greedy, if it can be obtained using the following algorithm: choose \( x_1 \) to be a minimal element of \( P \); suppose \( X = \{x_1, \ldots, x_i\} \) have been chosen; let \( M \) be the set of minimal elements of \( P \ \setminus \ X \). If there is an element of \( M \) which covers an element \( x_j \) in \( X \), then let \( x_{i+1} \) be one of these such that \( j \) is maximal; otherwise, choose \( x_{i+1} \) to be any element of \( M \).

Informally, a linear extension is supergreedy if it “always goes up and recedes the least”; in other words, supergreedy linear extensions are depth-first linear extensions. For more details see [KTZ1987].

EXAMPLES:

```
sage: X = [0,1,2,3,4,5,6]
sage: Y = [[0,5],[1,4],[1,5],[3,6],[4,3],[5,6],[6,2]]
sage: P = Poset((X,Y), cover_relations = True, facade=False)
sage: for l in P.linear_extensions():
    ....:     if l.is_supergreedy():
    ....:         print(l)
[1, 4, 3, 0, 5, 6, 2]
[0, 1, 4, 3, 5, 6, 2]
[0, 1, 5, 4, 3, 6, 2]
```

jump_count()

Return the number of jumps in the linear extension.

A jump in a linear extension \([e_1, e_2, \ldots, e_n]\) is a pair \((e_i, e_{i+1})\) such that \( e_{i+1} \) does not cover \( e_i \).

See also:

- `sage.combinat.posets.posets.FinitePoset.jump_number()`

EXAMPLES:

```
sage: B3 = posets.BooleanLattice(3)
sage: l1 = B3.linear_extension((0, 1, 2, 3, 4, 5, 6, 7))
sage: l1.jump_count()
3
sage: l2 = B3.linear_extension((0, 1, 2, 4, 3, 5, 6, 7))
sage: l2.jump_count()
5
```
poset()

Return the underlying original poset.

EXAMPLES:

```
sage: P = Poset(((1,2,3,4), [[1,2],[2,3],[1,4]]))
sage: p = P.linear_extension([1,2,4,3])
sage: p.poset()
Finite poset containing 4 elements
```

promotion(i=1)

Compute the (generalized) promotion on the linear extension of a poset.

INPUT:

- `i` – (default: 1) an integer between 1 and \( n - 1 \), where \( n \) is the cardinality of the poset.

The \( i \)-th generalized promotion operator \( \partial_i \) on a linear extension \( \pi \) is defined as \( \pi \tau_i \tau_{i+1} \cdots \tau_{n-1} \), where \( n \) is the size of the linear extension (or size of the underlying poset).

For more details see [Stan2009].

See also:

`tau()`, `evacuation()`

EXAMPLES:

```
sage: P = Poset(((1,2,3,4,5,6,7), [[1,2],[1,4],[2,3],[2,5],[3,6],[4,7],[5,6]]))
sage: p = P.linear_extension([1,2,3,4,5,6,7])
sage: q = p.promotion(4); q
[1, 2, 3, 5, 6, 4, 7]
sage: p.to_poset() == q.to_poset()
False
sage: p.to_poset().is_isomorphic(q.to_poset())
True
```

tau()

Return the operator \( \tau_i \) on linear extensions \( \text{self} \) of a poset.

INPUT:

- `i` – an integer between 1 and \( n - 1 \), where \( n \) is the cardinality of the poset.

The operator \( \tau_i \) on a linear extension \( \pi \) of a poset \( P \) interchanges positions \( i \) and \( i + 1 \) if the result is again a linear extension of \( P \), and otherwise acts trivially. For more details, see [Stan2009].

EXAMPLES:

```
sage: P = Poset(((1,2,3,4), [[1,3],[1,4],[2,3]]), linear_extension=True)
sage: L = P.linear_extensions()
sage: l = L.an_element(); l
[1, 2, 3, 4]
sage: l.tau(1)
[2, 1, 3, 4]
sage: for p in L:
....:     for i in range(1,4):
....:         print("{} {} {}".format(i, p, p.tau(i)))
1 [1, 2, 3, 4] [2, 1, 3, 4]
```
to_poset()

Return the poset associated to the linear extension self.

This method returns the poset obtained from the original poset $P$ by relabelling the $i$-th element of self to the $i$-th element of the original poset, while keeping the linear extension of the original poset.

For a poset with default linear extension $1, \ldots, n$, self can be interpreted as a permutation, and the relabelling is done according to the inverse of this permutation.

EXAMPLES:

```python
sage: P = Poset(((1,2,3,4), [(1,2), (1,3), (3,4)]), linear_extension=True, facade=False)
sage: p = P.linear_extension([1,3,4,2])
sage: Q = p.to_poset(); Q
Finite poset containing 4 elements with distinguished linear extension
sage: P == Q
False
```

The default linear extension remains the same:

```python
sage: list(P)
[1, 2, 3, 4]
sage: list(Q)
[1, 2, 3, 4]
```

But the relabelling can be seen on cover relations:

```python
sage: P.cover_relations()
[[1, 2], [1, 3], [3, 4]]
sage: Q.cover_relations()
[[1, 2], [1, 4], [2, 3]]
sage: p = P.linear_extension([1,2,3,4])
sage: Q = p.to_poset()
sage: P == Q
True
```
Linear extensions such that the poset is a forest.

\textbf{cardinality}()

Use Atkinson’s algorithm to compute the number of linear extensions.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.posets.forest import ForestPoset
sage: from sage.combinat.posets.poset_examples import Posets
sage: P = Poset({0: [2], 1: [2], 2: [3, 4], 3: [], 4: []})
sage: P.linear_extensions().cardinality()
4

sage: Q = Poset({0: [1], 1: [2, 3], 2: [], 3: [], 4: [5, 6], 5: [], 6: []})
sage: Q.linear_extensions().cardinality()
140
\end{verbatim}

\textbf{class} \texttt{sage.combinat.posets.linear_extensions.LinearExtensionsOfMobile(poset, facade)}

Bases: \texttt{LinearExtensionsOfPoset}

Linear extensions for a mobile poset.

\textbf{cardinality}()

Return the number of linear extensions by using the determinant formula for counting linear extensions of mobiles.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.posets.mobile import MobilePoset
sage: M = MobilePoset(DiGraph([[0,1,2,3,4,5,6,7,8], [(1,0),(3,0),(2,1),(2,3),(4,....: 3), (5,4),(5,6),(7,4),(7,8)]]))
sage: M.linear_extensions().cardinality()
1098

sage: M1 = posets.RibbonPoset(6, [1,3])
sage: M1.linear_extensions().cardinality()
61

 DoubledTailedDiamond(6)]},
....: anchor=(4, 2, posets.ChainPoset(6))
}sage: P.linear_extensions().cardinality()
361628701868606400
\end{verbatim}

\textbf{class} \texttt{sage.combinat.posets.linear_extensions.LinearExtensionsOfPoset(poset, facade)}

Bases: \texttt{UniqueRepresentation, Parent}

The set of all linear extensions of a finite poset

\textbf{INPUT:}

- \texttt{poset} – a poset \(P\) of size \(n\)
- \texttt{facade} – a boolean (default: False)

\textbf{See also:}

- \texttt{sage.combinat.posets.posets.FinitePoset.linear_extensions()}

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\textbf{2019}
EXAMPLES:

```python
sage: elms = [1,2,3,4]
sage: rels = [[1,3],[1,4],[2,3]]
sage: P = Poset((elms, rels), linear_extension=True)
sage: L = P.linear_extensions(); L
The set of all linear extensions of Finite poset containing 4 elements with
→ distinguished linear extension
sage: L.cardinality()
5
sage: L.list()
[[1, 2, 3, 4], [2, 1, 3, 4], [2, 1, 4, 3], [1, 4, 2, 3], [1, 2, 4, 3]]
sage: L.an_element()
[1, 2, 3, 4]
sage: L.poset()
Finite poset containing 4 elements with distinguished linear extension
```

**Element**

alias of *LinearExtensionOfPoset*

**cardinality()**

Return the number of linear extensions.

EXAMPLES:

```python
sage: N = Poset({0: [2, 3], 1: [3]})
sage: N.linear_extensions().cardinality()
5
```

**markov_chain_digraph** *(action='promotion', labeling='identity')*

Return the digraph of the action of generalized promotion or tau on **self**

**INPUT:**

- `action` – ‘promotion’ or ‘tau’ (default: ‘promotion’)
- `labeling` – ‘identity’ or ‘source’ (default: ‘identity’)

**Todo:**

- generalize this feature by accepting a family of operators as input
- move up in some appropriate category

This method creates a graph with vertices being the linear extensions of a given finite poset and an edge from \( \pi \) to \( \pi' \) if \( \pi' = \pi \varphi_i \) where \( \varphi_i \) is the promotion operator (see *promotion()* ) if action is set to *promotion* and \( \tau_i \) (see *tau()* ) if action is set to *tau*. The label of the edge is \( i \) (resp. \( \pi_i \)) if labeling is set to *identity* (resp. *source*).

**EXAMPLES:**

```python
sage: P = Poset(((1,2,3,4), [[1,3],[1,4],[2,3]]), linear_extension=True)
sage: L = P.linear_extensions()
sage: G = L.markov_chain_digraph(); G
Looped multi-digraph on 5 vertices
sage: G.vertices(sort=True, key=repr)
(continues on next page)
```
The edges of the graph are by default colored using blue for edge 1, red for edge 2, green for edge 3, and yellow for edge 4:

```
sage: view(G) # optional - dot2tex graphviz, not tested (opens external window)
```

Alternatively, one may get the graph of the action of the $\tau$ operator:

```
sage: G = L.markov_chain_digraph(action='tau'); G
Looped multi-digraph on 5 vertices
```

(continues on next page)
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(continued from previous page)

([2, 1, 3, 4], [1, 2, 3, 4], 1), ([2, 1, 3, 4], [2, 1, 3, 4], 2), ([2, 1, 3, 4],
\rightarrow [2, 1, 4, 3], 3),
([2, 1, 4, 3], [1, 2, 4, 3], 1), ([2, 1, 4, 3], [2, 1, 3, 4], 3), ([2, 1, 4, 3],
\rightarrow [2, 1, 4, 3], 2])

sage: view(G)  # optional - dot2tex graphviz, not tested (opens external window)

See also:

markov_chain_transition_matrix(), promotion(), tau()

markov_chain_transition_matrix(action='promotion', labeling='identity')

Return the transition matrix of the Markov chain for the action of generalized promotion or tau on self

INPUT:

• action – ‘promotion’ or ‘tau’ (default: ‘promotion’)

• labeling – ‘identity’ or ‘source’ (default: ‘identity’)

This method yields the transition matrix of the Markov chain defined by the action of the generalized
promotion operator \( \partial_i \) (resp. \( \tau_i \)) on the set of linear extensions of a finite poset. Here the transition from
the linear extension \( \pi \) to \( \pi' \), where \( \pi' = \pi \partial_i \) (resp. \( \pi' = \pi \tau_i \)) is counted with weight \( x_i \) (resp. \( x_{\pi_i} \) if
labeling is set to source).

EXAMPLES:

sage: P = Poset(((1, 2, 3, 4), [[1, 3], [1, 4], [2, 3]]), linear_extension = True)
sage: L = P.linear_extensions()
sage: L.markov_chain_transition_matrix()

\[
\begin{pmatrix}
-x_0 & -x_1 & -x_2 & x_2 & x_0 & +x_1 & 0 & 0 \\
x_1 & +x_2 & -x_0 & -x_1 & -x_2 & 0 & x_0 & 0 \\
0 & x_1 & -x_0 & -x_1 & 0 & x_0 & 0 \\
0 & x_0 & 0 & -x_0 & -x_1 & -x_2 & x_1 & +x_2 \\
x_0 & 0 & 0 & x_1 & +x_2 & -x_0 & -x_1 & -x_2
\end{pmatrix}
\]

sage: L.markov_chain_transition_matrix(labeling = 'source')

\[
\begin{pmatrix}
-x_0 & -x_1 & -x_2 & x_3 & x_0 & +x_3 & 0 & 0 \\
x_1 & +x_2 & -x_0 & -x_1 & -x_3 & 0 & x_1 & 0 \\
0 & x_1 & -x_0 & -x_3 & 0 & x_1 & 0 \\
0 & x_0 & 0 & -x_0 & -x_1 & -x_2 & x_0 & +x_3 \\
x_0 & 0 & 0 & x_0 & +x_2 & -x_0 & -x_1 & -x_3
\end{pmatrix}
\]

sage: L.markov_chain_transition_matrix(action = 'tau')

\[
\begin{pmatrix}
-x_0 & -x_2 & x_2 & 0 & x_0 & 0 \\
x_2 & -x_0 & -x_1 & -x_2 & x_1 & 0 & x_0 \\
0 & x_1 & -x_1 & 0 & 0 & 0 \\
x_0 & 0 & 0 & -x_0 & -x_2 & x_2 & 0 \\
0 & x_0 & 0 & x_2 & -x_0 & -x_2 & 0
\end{pmatrix}
\]

sage: L.markov_chain_transition_matrix(action = 'tau', labeling = 'source')

\[
\begin{pmatrix}
-x_0 & -x_2 & x_3 & 0 & x_1 & 0 \\
x_2 & -x_0 & -x_1 & -x_3 & x_3 & 0 & x_1 \\
0 & x_1 & -x_3 & 0 & 0 & 0 \\
x_0 & 0 & 0 & -x_1 & -x_2 & x_3 & 0 \\
0 & x_0 & 0 & x_2 & -x_1 & -x_3 & 0
\end{pmatrix}
\]

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See also:

`markov_chain_digraph()`, `promotion()`, `tau()`

**poset()**

Return the underlying original poset.

**EXAMPLES:**

```
sage: P = Poset(((1,2,3,4), [(1,2), (2,3), (1,4)]))
sage: L = P.linear_extensions()
sage: L.poset()
Finite poset containing 4 elements
```

---

**class** `sage.combinat.posets.linear_extensions.LinearExtensionsOfPosetWithHooks(poset, facade)`

**Bases:** `LinearExtensionsOfPoset`

Linear extensions such that the poset has well-defined hook lengths (i.e., d-complete).

**cardinality()**

Count the number of linear extensions using a hook-length formula.

**EXAMPLES:**

```
sage: from sage.combinat.posets.poset_examples import Posets
sage: P = Posets.YoungDiagramPoset(Partition([3,2]), dual=True)
sage: P.linear_extensions().cardinality()
5
```

---

### 5.1.184 Möbius Algebras

**class** `sage.combinat.posets.moebius_algebra.BasisAbstract(R, basis_keys=None, element_class=None, category=None, prefix=None, names=None, **kwds)`

**Bases:** `CombinatorialFreeModule`, `BindableClass`

Abstract base class for a basis.

**class** `sage.combinat.posets.moebius_algebra.MoebiusAlgebra(R, L)`

**Bases:** `Parent`, `UniqueRepresentation`

The Möbius algebra of a lattice.

Let $L$ be a lattice. The Möbius algebra $M_L$ was originally constructed by Solomon [Solomon67] and has a natural basis $\{E_x \mid x \in L\}$ with multiplication given by $E_x \cdot E_y = E_{x \lor y}$. Moreover this has a basis given by orthogonal idempotents $\{I_x \mid x \in L\}$ (so $I_x I_y = \delta_{xy} I_x$ where $\delta$ is the Kronecker delta) related to the natural basis by

$$I_x = \sum_{x \leq y} \mu_L(x, y) E_y,$$

where $\mu_L$ is the Möbius function of $L$.

**Note:** We use the join $\lor$ for our multiplication, whereas [Greene73] and [Etienne98] define the Möbius algebra using the meet $\land$. This is done for compatibility with `QuantumMoebiusAlgebra`.

**REFERENCES:**

---

5.1. Comprehensive Module List
class E(M, prefix='E')
Bases: BasisAbstract

The natural basis of a Möbius algebra.

Let $E_x$ and $E_y$ be basis elements of $M_L$ for some lattice $L$. Multiplication is given by $E_x E_y = E_{x \lor y}$.

one()

Return the element 1 of self.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: E = L.moebius_algebra(QQ).E()
sage: E.one()
E[0]
```

product_on_basis(x, y)

Return the product of basis elements indexed by x and y.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: E = L.moebius_algebra(QQ).E()
sage: E.product_on_basis(5, 14)
E[15]
sage: E.product_on_basis(2, 8)
E[10]
```

class I(M, prefix='I')
Bases: BasisAbstract

The (orthogonal) idempotent basis of a Möbius algebra.

Let $I_x$ and $I_y$ be basis elements of $M_L$ for some lattice $L$. Multiplication is given by $I_x I_y = \delta_{xy} I_x$ where $\delta_{xy}$ is the Kronecker delta.

one()

Return the element 1 of self.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: I = L.moebius_algebra(QQ).I()
sage: I.one()
```

product_on_basis(x, y)

Return the product of basis elements indexed by x and y.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: I = L.moebius_algebra(QQ).I()
sage: I.product_on_basis(5, 14)
0
```

(continues on next page)
a_realization()

Return a particular realization of self (the $B$-basis).

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: M = L.moebius_algebra(QQ)
sage: M.a_realization()
Moebius algebra of Finite lattice containing 16 elements
over Rational Field in the natural basis
```

idempotent

alias of $I$

lattice()

Return the defining lattice of self.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: M = L.moebius_algebra(QQ)
sage: M.lattice()
Finite lattice containing 16 elements
sage: M.lattice() == L
True
```

natural

alias of $E$

class sage.combinat.posets.moebius_algebra.MoebiusAlgebraBases

Bases: Category_realization_of_parent

The category of bases of a Möbius algebra.

INPUT:

- base – a Möbius algebra

class ElementMethods

Bases: object

class ParentMethods

Bases: object

one()

Return the element 1 of self.

EXAMPLES:

```
sage: L = posets.BooleanLattice(4)
sage: C = L.quantum_moebius_algebra().C()
sage: all(C.one() * b == b for b in C.basis())
True
```
**product_on_basis**(x, y)

Return the product of basis elements indexed by x and y.

**EXAMPLES:**

```python
sage: L = posets.BooleanLattice(4)
sage: C = L.quantum_moebius_algebra().C()
sage: C.product_on_basis(5, 14)
q^3*C[15]
sage: C.product_on_basis(2, 8)
q^4*C[10]
```

**super_categories()**

The super categories of `self`.

**EXAMPLES:**

```python
sage: from sage.combinat.posets.moebius_algebra import MoebiusAlgebraBases
c sage: N = posets.BooleanLattice(4).moebius_algebra(QQ)
sage: bases = MoebiusAlgebraBases(N)
sage: bases.super_categories()
[Category of finite dimensional commutative algebras with basis over `Rational Field`,
Category of realizations of Moebius algebra of Finite lattice containing 16 elements over Rational Field]
```

class sage.combinat.posets.moebius_algebra.QuantumMoebiusAlgebra(L, q=None)

Bases: `Parent`, `UniqueRepresentation`

The quantum Möbius algebra of a lattice.

Let `L` be a lattice, and we define the *quantum Möbius algebra* `M_L(q)` as the algebra with basis `{E_x | x ∈ L}` with multiplication given by

\[ E_x E_y = \sum_{a \geq x \lor y} \mu_L(a, z) q^{\text{crk} a} E_z, \]

where `\mu_L` is the Möbius function of `L` and `\text{crk}` is the corank function (i.e., `\text{crk} a = \text{rank} L - \text{rank} a`). At `q = 1`, this reduces to the multiplication formula originally given by Solomon.

class C(M, prefix='C')

Bases: `BasisAbstract`

The characteristic basis of a quantum Möbius algebra.

The characteristic basis `{C_x | x ∈ L}` of `M_L` for some lattice `L` is defined by

\[ C_x = \sum_{a \geq x} P(F^x; q) E_a, \]

where `F^x = \{y ∈ L | y \geq x\}` is the principal order filter of `x` and `P(F^x; q)` is the characteristic polynomial of the (sub)poset `F^x`.

class E(M, prefix='E')

Bases: `BasisAbstract`

The natural basis of a quantum Möbius algebra.
Let $E_x$ and $E_y$ be basis elements of $M_L$ for some lattice $L$. Multiplication is given by

$$E_x E_y = \sum_{z \geq a \geq x \wedge y} \mu_L(a, z) q^{\text{crk} a} E_z,$$

where $\mu_L$ is the Möbius function of $L$ and $\text{crk}$ is the corank function (i.e., $\text{crk} a = \text{rank} L - \text{rank} a$).

**one()**

Return the element $1$ of self.

**EXAMPLES:**

```python
sage: L = posets.BooleanLattice(4)
sage: E = L.quantum_moebius_algebra().E()
sage: all(E.one() * b == b for b in E.basis())
True
```

**product_on_basis(x, y)**

Return the product of basis elements indexed by $x$ and $y$.

**EXAMPLES:**

```python
sage: L = posets.BooleanLattice(4)
sage: E = L.quantum_moebius_algebra().E()
sage: E.product_on_basis(5, 14)
E[15]
sage: E.product_on_basis(2, 8)
```

**class KL(M, prefix='KL')**

Bases: `BasisAbstract`

The Kazhdan-Lusztig basis of a quantum Möbius algebra.

The Kazhdan-Lusztig basis $\{B_x | x \in L\}$ of $M_L$ for some lattice $L$ is defined by

$$B_x = \sum_{y \geq x} P_{x, y}(q) E_y,$$

where $P_{x, y}(q)$ is the Kazhdan-Lusztig polynomial of $L$, following the definition given in [EPW14].

**EXAMPLES:**

We construct some examples of Proposition 4.5 of [EPW14]:

```python
sage: N = posets.BooleanLattice(4).quantum_moebius_algebra()
sage: KL = N.KL()
(q^2+q^3)*KL[5] + (q+2*q^2+q^3)*KL[7] + (q+2*q^2+q^3)*KL[13] + (1+3*q+3*q^2+q^3)*KL[15]
(1+3*q+3*q^2+q^3)*KL[15]
(q+3*q^2+3*q^3+q^4)*KL[14] + (1+4*q+6*q^2+4*q^3+q^4)*KL[15]
```

**a_realization()**

Return a particular realization of self (the $B$-basis).

**EXAMPLES:**

5.1. Comprehensive Module List
sage: L = posets.BooleanLattice(4)
sage: M = L.quantum_moebius_algebra()
sage: M.a_realization()
Quantum Moebius algebra of Finite lattice containing 16 elements with q=q over Univariate Laurent Polynomial Ring in q over Integer Ring in the natural basis

**characteristic_basis**
alias of C

**kazhdan_lusztig**
alias of KL

**lattice()**
Return the defining lattice of self.

**EXAMPLES:**

sage: L = posets.BooleanLattice(4)
sage: M = L.quantum_moebius_algebra()
sage: M.lattice()
Finite lattice containing 16 elements
sage: M.lattice() == L
True

**natural**
alias of E

### 5.1.185 Catalog of posets and lattices

Some common posets can be accessed through the `posets.<tab>` object:

```
sage: posets.PentagonPoset()
Finite lattice containing 5 elements
```

Moreover, the set of all posets of order $n$ is represented by `Posets(n)`:  

```
sage: Posets(5)
Posets containing 5 elements
```

The infinite set of all posets can be used to find minimal examples:

```
sage: for P in Posets():
    ....:     if not P.is_series_parallel():
    ....:         break
sage: P
Finite poset containing 4 elements
```

**Catalog of common posets:**

- **AntichainPoset()**
  - Return an antichain on $n$ elements.

- **BooleanLattice()**
  - Return the Boolean lattice on $2^n$ elements.
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<td>Return the crown poset on $2n$ elements.</td>
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<td>Return the shard intersection order.</td>
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<td>StandardExample()</td>
<td>Return the standard example of a poset with dimension $n$.</td>
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<td>SymmetricGroupAbsoluteOrderPoset()</td>
<td>The poset of permutations with respect to absolute order.</td>
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<td>The poset of permutations with respect to Bruhat order.</td>
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<td>The poset of permutations of ${1, 2, \ldots, n}$ with respect to the weak order.</td>
</tr>
<tr>
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<tr>
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<td>Return the Tetrahedral poset with $n - 1$ layers based on the input colors.</td>
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<tr>
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<td>Return the up-down poset on $n$ elements.</td>
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<tr>
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<td>Return Young's Lattice up to rank $n$.</td>
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<td>Return the principal order ideal of the partition $\lambda$ in Young's Lattice.</td>
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Other available posets:

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<td>face_lattice()</td>
<td>Return the face lattice of a polyhedron.</td>
</tr>
<tr>
<td>face_lattice()</td>
<td>Return the face lattice of a combinatorial polyhedron.</td>
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Constructions

class sage.combinat.posets.poset_examples.Posets
    Bases: object

    A collection of posets and lattices.

    EXAMPLES:

    sage: posets.BooleanLattice(3)
    Finite lattice containing 8 elements
    sage: posets.ChainPoset(3)
    Finite lattice containing 3 elements
    sage: posets.RandomPoset(17,.15)
    Finite poset containing 17 elements

    The category of all posets:

    sage: Posets()
    Category of posets

    The enumerated set of all posets on 3 elements, up to an isomorphism:

    sage: Posets(3)
    Posets containing 3 elements

    See also:
    Posets, FinitePosets, Poset()

    static AntichainPoset(n, facade=None)
    Return an antichain (a poset with no comparable elements) containing \(n\) elements.

    INPUT:

    • \(n\) (an integer) – number of elements
    • \(facade\) (boolean) – whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor

    EXAMPLES:

    sage: A = posets.AntichainPoset(6); A
    Finite poset containing 6 elements

    static BooleanLattice(n, facade=None, use_subsets=False)
    Return the Boolean lattice containing \(2^n\) elements.

    • \(n\) – integer; number of elements will be \(2^n\)
    • \(facade\) – boolean; whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor
    • \(use\_subsets\) – boolean (default: False); if True, then label the elements by subsets of \(\{1, 2, \ldots, n\}\); otherwise label the elements by \(0, 1, 2, \ldots, 2^n - 1\)

    EXAMPLES:
sage: posets.BooleanLattice(5)
Finite lattice containing 32 elements
sage: sorted(posets.BooleanLattice(2))
[0, 1, 2, 3]
sage: sorted(posets.BooleanLattice(2, use_subsets=True), key=list)
[{}, {1}, {1, 2}, {2}]

static ChainPoset(n, facade=None)
Return a chain (a totally ordered poset) containing n elements.

- n (an integer) – number of elements.
- facade (boolean) – whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor.

EXAMPLES:

sage: C = posets.ChainPoset(6); C
Finite lattice containing 6 elements
sage: C.linear_extension()
[0, 1, 2, 3, 4, 5]

static CoxeterGroupAbsoluteOrderPoset(W, use_reduced_words=True)
Return the poset of elements of a Coxeter group with respect to absolute order.

INPUT:

- W – a Coxeter group
- use_reduced_words – boolean (default: True); if True, then the elements are labeled by their lexicographically minimal reduced word

EXAMPLES:

sage: W = CoxeterGroup(['B', 3])
sage: posets.CoxeterGroupAbsoluteOrderPoset(W)
Finite poset containing 48 elements
sage: W = WeylGroup(['B', 2], prefix='s')
sage: posets.CoxeterGroupAbsoluteOrderPoset(W, False)
Finite poset containing 8 elements

static Crown(n, facade=None)
Return the crown poset of 2n elements.

In this poset every element i for 0 ≤ i ≤ n − 1 is covered by elements i + n and i + n + 1, except that n − 1 is covered by n and n + 1.

INPUT:

- n – number of elements, an integer at least 2
- facade (boolean) – whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor

EXAMPLES:
sage: posets.Crown(3)
Finite poset containing 6 elements

static DexterSemilattice(n)
Return the $n$-th Dexter meet-semilattice.

INPUT:
• $n$ – a nonnegative integer (the index)

OUTPUT:
a finite meet-semilattice

The elements of the semilattice are Dyck paths in the $(n + 1 \times n)$-rectangle.

EXAMPLES:

```
sage: posets.DexterSemilattice(3)
Finite meet-semilattice containing 5 elements

sage: P = posets.DexterSemilattice(4); P
Finite meet-semilattice containing 14 elements
sage: len(P.maximal_chains())
15
sage: len(P.maximal_elements())
4
sage: P.chain_polynomial()
q^5 + 19*q^4 + 47*q^3 + 42*q^2 + 14*q + 1
```

REFERENCES:
• [Cha18]

static DiamondPoset(n, facade=None)
Return the lattice of rank two containing $n$ elements.

INPUT:
• $n$ – number of elements, an integer at least 3
• facade (boolean) – whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor

EXAMPLES:

```
sage: posets.DiamondPoset(7)
Finite lattice containing 7 elements
```

static DivisorLattice(n, facade=None)
Return the divisor lattice of an integer.

Elements of the lattice are divisors of $n$ and $x < y$ in the lattice if $x$ divides $y$.

INPUT:
• $n$ – an integer
• facade (boolean) – whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor

EXAMPLES:
Combinatorics, Release 10.0

static DoubleTailedDiamond(n)
Return a double-tailed diamond of \(2n + 2\) elements.

INPUT:

• \(n\) – a positive integer

EXAMPLES:

```python
sage: P = posets.DoubleTailedDiamond(2); P
Finite d-complete poset containing 6 elements
sage: P.cover_relations()
[[1, 2], [2, 3], [2, 4], [3, 5], [4, 5], [5, 6]]
```

static IntegerCompositions(n)
Return the poset of integer compositions of the integer \(n\).

A composition of a positive integer \(n\) is a list of positive integers that sum to \(n\). The order is reverse refinement: \([p_1, p_2, ..., p_l] < [q_1, q_2, ..., q_m]\) if \(q\) consists of an integer composition of \(p_1\), followed by an integer composition of \(p_2\), and so on.

EXAMPLES:

```python
sage: P = posets.IntegerCompositions(7); P
Finite poset containing 64 elements
sage: len(P.cover_relations())
192
```

static IntegerPartitions(n)
Return the poset of integer partitions on the integer \(n\).

A partition of a positive integer \(n\) is a non-increasing list of positive integers that sum to \(n\). If \(p\) and \(q\) are integer partitions of \(n\), then \(p\) covers \(q\) if and only if \(q\) is obtained from \(p\) by joining two parts of \(p\) (and sorting, if necessary).

EXAMPLES:

```python
sage: P = posets.IntegerPartitions(7); P
Finite poset containing 15 elements
sage: len(P.cover_relations())
28
```

static IntegerPartitionsDominanceOrder(n)
Return the lattice of integer partitions on the integer \(n\) ordered by dominance.

That is, if \(p = (p_1, ..., p_l)\) and \(q = (q_1, ..., q_m)\) are integer partitions of \(n\), then \(p\) is greater than \(q\) if and only if \(p_1 + \cdots + p_k > q_1 + \cdots + q_k\) for all \(k\).

INPUT:
• n – a positive integer

EXAMPLES:

```python
sage: P = posets.IntegerPartitionsDominanceOrder(6); P
Finite lattice containing 11 elements
sage: P.cover_relations()
[[[1, 1, 1, 1, 1, 1], [2, 1, 1, 1, 1, 1]],
[[2, 1, 1, 1, 1, 1], [2, 2, 1, 1, 1, 1]],
[[2, 2, 1, 1, 1], [2, 2, 2]],
[[2, 2, 1, 1], [3, 1, 1, 1]],
[[2, 2, 2], [3, 2, 1]],
[[3, 1, 1, 1], [3, 2, 1]],
[[3, 2, 1], [3, 3]],
[[3, 2, 1], [4, 1, 1]],
[[3, 3], [4, 2]],
[[4, 1, 1], [4, 2]],
[[4, 2], [5, 1]],
[[5, 1], [6]]]
```

**static MobilePoset(ribbon, hangers, anchor=None)**

Return a mobile poset with the ribbon `ribbon` and with hanging d-complete posets specified in `hangers` and a d-complete poset attached above, specified in `anchor`.

**INPUT:**

- `ribbon` – a finite poset that is a ribbon
- `hangers` – a dictionary mapping an element on the ribbon to a list of d-complete posets that it covers
- `anchor` – (optional) a tuple `(ribbon_elmt, anchor_elmt, anchor_poset)`, where `anchor_elmt` covers `ribbon_elmt`, and `anchor_elmt` is an acyclic element of `anchor_poset`

**EXAMPLES:**

```python
sage: R = Posets.RibbonPoset(5, [1,2])
sage: H = Poset([[5, 6, 7], [(5, 6), (6,7)]])
sage: M = Posets.MobilePoset(R, {3: [H]})
sage: len(M.cover_relations())
7

sage: P = posets.MobilePoset(posets.RibbonPoset(7, [1,3]),
....:                      {1: [posets.YoungDiagramPoset([3, 2], dual=True)],
....:                       3: [posets.DoubleTailedDiamond(6)]},
....:                      anchor=(4, 2, posets.ChainPoset(6)))
sage: len(P.cover_relations())
33
```

**static NoncrossingPartitions(W)**

Return the lattice of noncrossing partitions.

**INPUT:**

- `W` – a finite Coxeter group or a Weyl group

**EXAMPLES:**

```python

```
```python
sage: W = CoxeterGroup(['A', 3])
sage: posets.NoncrossingPartitions(W)
Finite lattice containing 14 elements
sage: W = WeylGroup(['B', 2], prefix='s')
sage: posets.NoncrossingPartitions(W)
Finite lattice containing 6 elements
```

**static PentagonPoset**(facade=None)

Return the Pentagon poset.

**INPUT:**

- facade (boolean) – whether to make the returned poset a facade poset (see `sage.categories.facade_sets`); the default behaviour is the same as the default behaviour of the `Poset()` constructor

**EXAMPLES:**

```python
sage: P = posets.PentagonPoset(); P
Finite lattice containing 5 elements
sage: P.cover_relations()
[[0, 1], [0, 2], [1, 4], [2, 3], [3, 4]]
```

**static PermutationPattern**(n)

Return the poset of permutations under pattern containment up to rank n.

**INPUT:**

- n – a positive integer

A permutation \( u = u_1 \cdots u_n \) contains the pattern \( v = v_1 \cdots v_m \) if there is a (not necessarily consecutive) subsequence of \( u \) of length \( m \) whose entries have the same relative order as \( v \).

See Wikipedia article Permutation_pattern.

**EXAMPLES:**

```python
sage: P4 = posets.PermutationPattern(4); P4
Finite poset containing 33 elements
sage: sorted(P4.lower_covers(Permutation([2, 4, 1, 3])))
[[1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2]]
```

See also:

- `has_pattern()`

**static PermutationPatternInterval**(bottom, top)

Return the poset consisting of an interval in the poset of permutations under pattern containment between bottom and top.

**INPUT:**

- bottom, top – permutations where top contains bottom as a pattern

A permutation \( u = u_1 \cdots u_n \) contains the pattern \( v = v_1 \cdots v_m \) if there is a (not necessarily consecutive) subsequence of \( u \) of length \( m \) whose entries have the same relative order as \( v \).

See Wikipedia article Permutation_pattern.

**EXAMPLES:**
sage: t = Permutation([2,3,1])
sage: b = Permutation([4,6,2,3,5,1])
sage: R = posets.PermutationPatternInterval(t, b); R
Finite poset containing 14 elements
sage: R.moebius_function(R.bottom(),R.top())
-4

See also:
has_pattern(), PermutationPattern()

static PermutationPatternOccurrenceInterval(bottom, top, pos)
Return the poset consisting of an interval in the poset of permutations under pattern containment between bottom and top, where a specified instance of bottom in top must be maintained.

INPUT:
- bottom, top – permutations where top contains bottom as a pattern
- pos – a list of indices indicating a distinguished copy of bottom inside top (indexed starting at 0)

For further information (and picture illustrating included example), see [ST2010].

See Wikipedia article Permutation_pattern.

EXAMPLES:

sage: t = Permutation([3,2,1])
sage: b = Permutation([6,3,4,5,2,1])
sage: A = posets.PermutationPatternOccurrenceInterval(t, b, (0,2,4)); A
Finite poset containing 8 elements

See also:
has_pattern(), PermutationPattern(), PermutationPatternInterval()

static PowerPoset(n)
Return the power poset on n element posets.

Elements of the power poset are all posets on the set \{0, \ldots, n - 1\} ordered by extension. That is, the antichain of n elements is the bottom and \(P_a \leq P_b\) in the power poset if \(P_b\) is an extension of \(P_a\).

These were studied in [Bru1994].

EXAMPLES:

sage: P3 = posets.PowerPoset(3); P3
Finite meet-semilattice containing 19 elements
sage: all(P.is_chain() for P in P3.maximal_elements())
True

static ProductOfChains(chain_lengths, facade=None)
Return a product of chains.

- chain_lengths – A list of nonnegative integers; number of elements in each chain.
- facade – boolean; whether to make the returned poset a facade poset (see sage.categories.facade_sets); the default behaviour is the same as the default behaviour of the Poset() constructor
EXAMPLES:

```
sage: P = posets.ProductOfChains([2, 2]); P
Finite lattice containing 4 elements
sage: P.linear_extension()
[(0, 0), (0, 1), (1, 0), (1, 1)]
sage: P.upper_covers((0,0))
[(0, 1), (1, 0)]
sage: P.lower_covers((1,1))
[(0, 1), (1, 0)]
```

**static RandomLattice**(n, p, properties=None)

Return a random lattice on n elements.

**INPUT:**
- n – number of elements, a non-negative integer
- p – a probability, a positive real number less than one
- properties – a list of properties for the lattice. Currently implemented:
  - None, no restrictions for lattices to create
  - 'planar', the lattice has an upward planar drawing
  - 'dismantlable' (implicated by 'planar')
  - 'distributive' (implicated by 'stone')
  - 'stone'

**OUTPUT:**
A lattice on n elements. When properties is None, the probability p roughly measures number of covering relations of the lattice. To create interesting examples, make the probability a little below one, for example 0.9.

Currently parameter p has no effect only when properties is not None.

**Note:** Results are reproducible in same Sage version only. Underlying algorithm may change in future versions.

EXAMPLES:

```
sage: set_random_seed(0)  # Results are reproducible
sage: L = posets.RandomLattice(8, 0.995); L
Finite lattice containing 8 elements
sage: L.cover_relations()
[[7, 6], [7, 3], [7, 1], ... , [5, 4], [2, 4], [1, 4], [0, 4]]
sage: L = posets.RandomLattice(10, 0, properties=['dismantlable'])
sage: L.is_dismantlable()
True
```

See also:

*RandomPoset()*
**static RandomPoset**(n, p)

Generate a random poset on n elements according to a probability p.

**INPUT:**

- n - number of elements, a non-negative integer
- p - a probability, a real number between 0 and 1 (inclusive)

**OUTPUT:**

A poset on n elements. The probability p roughly measures width/height of the output: p = 0 always generates an antichain, p = 1 will return a chain. To create interesting examples, keep the probability small, perhaps on the order of 1/n.

**EXAMPLES:**

```sage
sage: set_random_seed(0) # Results are reproducible
sage: P = posets.RandomPoset(5, 0.3)
sage: P.cover_relations()
[[[5, 4], [4, 2], [1, 2]]
```

See also:

*RandomLattice()*

**static RestrictedIntegerPartitions**(n)

Return the poset of integer partitions on the integer n ordered by restricted refinement.

That is, if p and q are integer partitions of n, then p covers q if and only if q is obtained from p by joining two distinct parts of p (and sorting, if necessary).

**EXAMPLES:**

```sage
sage: P = posets.RestrictedIntegerPartitions(7); P
Finite poset containing 15 elements
sage: len(P.cover_relations())
17
```

**static RibbonPoset**(n, descents)

Return a ribbon poset on n vertices with descents at descents.

**INPUT:**

- n – the number of vertices
- descents – an iterable; the indices on the ribbon where \( y > x \)

**EXAMPLES:**

```sage
sage: R = Posets.RibbonPoset(5, [1,2])
sage: sorted(R.cover_relations())
[[[0, 1], [2, 1], [3, 2], [3, 4]]
```

**static SSTPoset**(s, f=None)

The lattice poset on semistandard tableaux of shape s and largest entry f that is ordered by componentwise comparison of the entries.

**INPUT:**

- s - shape of the tableaux

---

*Combinatorics, Release 10.0*

---
• $f$ - maximum fill number. This is an optional argument. If no maximal number is given, it will use the number of cells in the shape.

Note: This is a basic implementation and most certainly not the most efficient.

EXAMPLES:

```
sage: posets.SSTPoset([2,1])
Finite lattice containing 8 elements
sage: posets.SSTPoset([2,1],4)
Finite lattice containing 20 elements
sage: posets.SSTPoset([2,1],2).cover_relations()  
[[[1, 1], [2]], [[1, 2], [2]]]
sage: posets.SSTPoset([3,2]).bottom()  # long time (6s on sage.math, 2012)  
[[1, 1, 1], [2, 2]]
sage: posets.SSTPoset([3,2],4).maximal_elements()  
[[[3, 3, 4], [4, 4]]]
```

static SetPartitions$(n)$  
Return the lattice of set partitions of the set $\{1, \ldots, n\}$ ordered by refinement.

INPUT:

• $n$ – a positive integer

EXAMPLES:

```
sage: posets.SetPartitions(4)  
Finite lattice containing 15 elements
```

static ShardPoset$(n)$  
Return the shard intersection order on permutations of size $n$.

This is defined on the set of permutations. To every permutation, one can attach a pre-order, using the descending runs and their relative positions.

The shard intersection order is given by the implication (or refinement) order on the set of pre-orders defined from all permutations.

This can also be seen in a geometrical way. Every pre-order defines a cone in a vector space of dimension $n$. The shard poset is given by the inclusion of these cones.

See also:  
shard_preorder_graph()  

EXAMPLES:

```
sage: P = posets.ShardPoset(4); P  # indirect doctest  
Finite poset containing 24 elements
sage: P.chain_polynomial()  
34*q^4 + 90*q^3 + 79*q^2 + 24*q + 1
sage: P.characteristic_polynomial()  
```

(continues on next page)
\[ q^3 - 11q^2 + 23q - 13 \]
\[
\text{sage: } P.zeta_polynomial()
\]
\[ 17/3q^3 - 6q^2 + 4/3q \]
\[
\text{sage: } P.is_self_dual()
\]
\[ \text{False} \]

**static StandardExample** \((n, \text{facade}=\text{None})\)

Return the partially ordered set on \(2n\) elements with dimension \(n\).

Let \(P\) be the poset on \(\{0, 1, 2, \ldots, 2n - 1\}\) whose defining relations are that \(i < j\) for every \(0 \leq i < n \leq j < 2n\) except when \(i + n = j\). The poset \(P\) is the so-called standard example of a poset with dimension \(n\).

**INPUT:**

- \(n\) – an integer \(\geq 2\), dimension of the constructed poset
- \(\text{facade}\) (boolean) – whether to make the returned poset a facade poset (see `sage.categories.facade_sets`; the default behaviour is the same as the default behaviour of the `Poset()` constructor)

**OUTPUT:**

The standard example of a poset of dimension \(n\).

**EXAMPLES:**

\[
\text{sage: } A = \text{posets.StandardExample}(3); A \\
\text{Finite poset containing 6 elements} \\
\text{sage: } A.dimension() \\
3
\]

**REFERENCES:**

- [Gar2015]
- [Ros1999]

**static SymmetricGroupAbsoluteOrderPoset** \((n, \text{labels}=\text{'}\text{permutations}\text{'})\)

Return the poset of permutations with respect to absolute order.

**INPUT:**

- \(n\) – a positive integer
- \(\text{label}\) – (default: 'permutations') a label for the elements of the poset returned by the function; the options are
  - 'permutations' - labels the elements are given by their one-line notation
  - 'reduced_words' - labels the elements by the lexicographically minimal reduced word
  - 'cycles' - labels the elements by their expression as a product of cycles

**EXAMPLES:**

\[
\text{sage: } \text{posets.SymmetricGroupAbsoluteOrderPoset}(4) \\
\text{Finite poset containing 24 elements} \\
\text{sage: } \text{posets.SymmetricGroupAbsoluteOrderPoset}(3, \text{labels=\text{'}cycles\text{'})} \\
\text{Finite poset containing 6 elements} \\
\text{sage: } \text{posets.SymmetricGroupAbsoluteOrderPoset}(3, \text{labels=\text{'}reduced_words\text{'})} \\
\text{Finite poset containing 6 elements}
\]
static SymmetricGroupBruhatIntervalPoset\( (start, end) \)
The poset of permutations with respect to Bruhat order.

**INPUT:**

- \( start \) - list permutation
- \( end \) - list permutation (same \( n \), of course)

**Note:** Must have \( start \leq end \).

**EXAMPLES:**
Any interval is rank symmetric if and only if it avoids these permutations:

```python
sage: P1 = posets.SymmetricGroupBruhatIntervalPoset([1,2,3,4], [3,4,1,2])
sage: P2 = posets.SymmetricGroupBruhatIntervalPoset([1,2,3,4], [4,2,3,1])
sage: ranks1 = [P1.rank(v) for v in P1]
sage: ranks2 = [P2.rank(v) for v in P2]
sage: [ranks1.count(i) for i in sorted(set(ranks1))]
[1, 3, 5, 4, 1]
sage: [ranks2.count(i) for i in sorted(set(ranks2))]
[1, 3, 5, 6, 4, 1]
```

static SymmetricGroupBruhatOrderPoset\( (n) \)
The poset of permutations with respect to Bruhat order.

**EXAMPLES:**

```python
sage: posets.SymmetricGroupBruhatOrderPoset(4)
Finite poset containing 24 elements
```

static SymmetricGroupWeakOrderPoset\( (n, labels='permutations', side='right') \)
The poset of permutations of \( \{1, 2, \ldots, n\} \) with respect to the weak order (also known as the permutohedron order, cf. \texttt{permutohedron\_lequal()}).

The optional variable labels (default: "permutations") determines the labelling of the elements if \( n < 10 \). The optional variable side (default: "right") determines whether the right or the left permutohedron order is to be used.

**EXAMPLES:**

```python
sage: posets.SymmetricGroupWeakOrderPoset(4)
Finite poset containing 24 elements
```

static TamariLattice\( (n, m=1) \)
Return the \( n \)-th Tamari lattice.

Using the slope parameter \( m \), one can also get the \( m \)-Tamari lattices.

**INPUT:**

- \( n \) – a nonnegative integer (the index)
- \( m \) – an optional nonnegative integer (the slope, default to 1)

**OUTPUT:**
a finite lattice

5.1. Comprehensive Module List
In the usual case, the elements of the lattice are Dyck paths in the \((n + 1 \times n)\)-rectangle. For a general slope \(m\), the elements are Dyck paths in the \((mn + 1 \times n)\)-rectangle.

See Tamari lattice for mathematical background.

**EXAMPLES:**

```python
sage: posets.TamariLattice(3)
Finite lattice containing 5 elements
sage: posets.TamariLattice(3, 2)
Finite lattice containing 12 elements
```

**REFERENCES:**

- [BMFPR]

**static TetrahedralPoset\((n, \ast colors, \ast \ast labels)\)**

Return the tetrahedral poset based on the input colors.

This method will return the tetrahedral poset with \(n\)-1 layers and covering relations based on the input colors of ‘green’, ‘red’, ‘orange’, ‘silver’, ‘yellow’ and ‘blue’ as defined in [Striker2011]. For particular color choices, the order ideals of the resulting tetrahedral poset will be isomorphic to known combinatorial objects.

For example, for the colors ‘blue’, ‘yellow’, ‘orange’, and ‘green’, the order ideals will be in bijection with alternating sign matrices. For the colors ‘yellow’, ‘orange’, and ‘green’, the order ideals will be in bijection with semistandard Young tableaux of staircase shape. For the colors ‘red’, ‘orange’, ‘green’, and optionally ‘yellow’, the order ideals will be in bijection with totally symmetric self-complementary plane partitions in a \(2n \times 2n \times 2n\) box.

**INPUT:**

- \(n\) - Defines the number (\(n\)-1) of layers in the poset.
- \(\ast colors\) - The colors that define the covering relations of the poset. Colors used are ‘green’, ‘red’, ‘yellow’, ‘orange’, ‘silver’, and ‘blue’.
- \(\ast \ast labels\) - Keyword variable used to determine whether the poset is labeled with integers or tuples. To label with integers, the method should be called with \(\ast labels\)='integers'. Otherwise, the labeling will default to tuples.

**EXAMPLES:**

```python
sage: posets.TetrahedralPoset(4, 'green', 'red', 'yellow', 'silver', 'blue', 'orange')
Finite poset containing 10 elements
sage: posets.TetrahedralPoset(4, 'green', 'red', 'yellow', 'silver', 'blue', 'orange',
                           \ast labels='integers')
Finite poset containing 10 elements
sage: A = AlternatingSignMatrices(3)
sage: p = A.lattice()
sage: ji = p.join_irreducibles_poset()
sage: tet = posets.TetrahedralPoset(3, 'green', 'yellow', 'blue', 'orange')
sage: ji.is_isomorphic(tet)
True
```
**static UpDownPoset**\((n, m=1)\)

Return the up-down poset on \(n\) elements where every \((m + 1)\) step is down and the rest are up.

The case where \(m = 1\) is sometimes referred to as the zig-zag poset or the fence.

**INPUT:**
- \(n\) - nonnegative integer, number of elements in the poset
- \(m\) - nonnegative integer (default 1), how frequently down steps occur

**OUTPUT:**

The partially ordered set on \(\{0, 1, \ldots, n - 1\}\) where \(i\) covers \(i + 1\) if \(m\) divides \(i + 1\), and \(i + 1\) covers \(i\) otherwise.

**EXAMPLES:**

```python
sage: P = posets.UpDownPoset(7, 2); P
Finite poset containing 7 elements
sage: sorted(P.cover_relations())
[[[0, 1], [1, 2], [3, 2], [3, 4], [4, 5], [6, 5]]
```

Fibonacci numbers as the number of antichains of a poset:

```python
sage: [len(posets.UpDownPoset(n).antichains().list()) for n in range(6)]
[1, 2, 3, 5, 8, 13]
```

**static YoungDiagramPoset**\((\lambda, dual=False)\)

Return the poset of cells in the Young diagram of a partition.

**INPUT:**
- \(\lambda\) – a partition
- dual – (default: False) determines the orientation of the poset; if True, then it is a join semilattice, otherwise it is a meet semilattice

**EXAMPLES:**

```python
sage: P = posets.YoungDiagramPoset(Partition([2, 2])); P
Finite meet-semilattice containing 4 elements
sage: sorted(P.cover_relations())
[[[0, 0), (0, 1)], [(0, 0), (1, 0)], [(0, 1), (1, 1)], [(1, 0), (1, 1)]]
```

```python
sage: P = posets.YoungDiagramPoset([3, 2], dual=True)
Finite join-semilattice containing 5 elements
```

**static YoungFibonacci**\((n)\)

Return the Young-Fibonacci lattice up to rank \(n\).

Elements of the (infinite) lattice are words with letters ‘1’ and ‘2’. The covers of a word are the words with another ‘1’ added somewhere not after the first occurrence of an existing ‘1’ and, additionally, the words where the first ‘1’ is replaced by a ‘2’. The lattice is truncated to have rank \(n\).

See Wikipedia article Young-Fibonacci lattice.

**EXAMPLES:**


```python
sage: Y5 = posets.YoungFibonacci(5); Y5
Finite meet-semilattice containing 20 elements
sage: sorted(Y5.upper_covers(Word('211')))
[word: 1211, word: 2111, word: 221]
```

**static YoungsLattice(n)**

Return Young’s Lattice up to rank \( n \).

In other words, the poset of partitions of size less than or equal to \( n \) ordered by inclusion.

**INPUT:**

- \( n \) – a positive integer

**EXAMPLES:**

```python
sage: P = posets.YoungsLattice(3); P
Finite meet-semilattice containing 7 elements
sage: P.cover_relations()
[[[]], [1]],
[[1], [1, 1]],
[[1], [2]],
[[1, 1], [2, 1]],
[[2], [2, 1]],
[[2], [3]]]
```

**static YoungsLatticePrincipalOrderIdeal(lam)**

Return the principal order ideal of the partition \( lam \) in Young’s Lattice.

**INPUT:**

- \( lam \) – a partition

**EXAMPLES:**

```python
sage: P = posets.YoungsLatticePrincipalOrderIdeal(Partition([2,2])); P
Finite lattice containing 6 elements
sage: P.cover_relations()
[[[]], [1]],
[[1], [1, 1]],
[[1], [2]],
[[1, 1], [2, 1]],
[[2], [2, 1]],
[[2], [3]]]
```

sage = <module 'sage' ("_frozen_importlib_external._NamespaceLoader object")>

sage.combinat.posets.set_examples.check_int(n, minimum=0)

Check that \( n \) is an integer at least equal to \( minimum \).

This is a boilerplate function ensuring input safety.

**INPUT:**

- \( n \) – anything
- \( minimum \) – an optional integer (default: 0)
EXAMPLES:

```python
sage: from sage.combinat.posets.poset_examples import check_int
sage: check_int(6, 3)
6
sage: check_int(6)
6

sage: check_int(-1)
Traceback (most recent call last):
...  
ValueError: number of elements must be a non-negative integer, not -1

sage: check_int(1, 3)
Traceback (most recent call last):
...  
ValueError: number of elements must be an integer at least 3, not 1

sage: check_int('junk')
Traceback (most recent call last):
...  
ValueError: number of elements must be a non-negative integer, not junk
```

sage.combinat.posets.poset_examples.posets
alias of Posets

5.1.186 Finite posets

This module implements finite partially ordered sets. It defines:

<table>
<thead>
<tr>
<th>FinitePoset</th>
<th>A class for finite posets</th>
</tr>
</thead>
<tbody>
<tr>
<td>FinitePosets_n</td>
<td>A class for finite posets up to isomorphism (i.e. unlabeled posets)</td>
</tr>
<tr>
<td>Poset()</td>
<td>Construct a finite poset from various forms of input data.</td>
</tr>
<tr>
<td>is_poset()</td>
<td>Return True if a directed graph is acyclic and transitively reduced.</td>
</tr>
</tbody>
</table>

List of Poset methods

Comparing, intervals and relations

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>is_less_than()</td>
<td>Return True if $x$ is strictly less than $y$ in the poset.</td>
</tr>
<tr>
<td>is_greater_than()</td>
<td>Return True if $x$ is strictly greater than $y$ in the poset.</td>
</tr>
<tr>
<td>is_lequal()</td>
<td>Return True if $x$ is less than or equal to $y$ in the poset.</td>
</tr>
<tr>
<td>is_gequal()</td>
<td>Return True if $x$ is greater than or equal to $y$ in the poset.</td>
</tr>
<tr>
<td>compare_elements()</td>
<td>Compare two element of the poset.</td>
</tr>
<tr>
<td>closed_interval()</td>
<td>Return the list of elements in a closed interval of the poset.</td>
</tr>
<tr>
<td>open_interval()</td>
<td>Return the list of elements in an open interval of the poset.</td>
</tr>
<tr>
<td>relations()</td>
<td>Return the list of relations in the poset.</td>
</tr>
<tr>
<td>relations_iterator()</td>
<td>Return an iterator over relations in the poset.</td>
</tr>
<tr>
<td>order_filter()</td>
<td>Return the upper set generated by elements.</td>
</tr>
<tr>
<td>order_ideal()</td>
<td>Return the lower set generated by elements.</td>
</tr>
</tbody>
</table>
Covering

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>covers()</code></td>
<td>Return True if y covers x.</td>
</tr>
<tr>
<td><code>lower_covers()</code></td>
<td>Return elements covered by given element.</td>
</tr>
<tr>
<td><code>upper_covers()</code></td>
<td>Return elements covering given element.</td>
</tr>
<tr>
<td><code>cover_relations()</code></td>
<td>Return the list of cover relations.</td>
</tr>
<tr>
<td><code>lower_covers_iterator()</code></td>
<td>Return an iterator over elements covered by given element.</td>
</tr>
<tr>
<td><code>upper_covers_iterator()</code></td>
<td>Return an iterator over elements covering given element.</td>
</tr>
<tr>
<td><code>cover_relations_iterator()</code></td>
<td>Return an iterator over cover relations of the poset.</td>
</tr>
<tr>
<td><code>common_upper_covers()</code></td>
<td>Return the list of all common upper covers of the given elements.</td>
</tr>
<tr>
<td><code>common_lower_covers()</code></td>
<td>Return the list of all common lower covers of the given elements.</td>
</tr>
<tr>
<td><code>meet()</code></td>
<td>Return the meet of given elements if it exists; None otherwise.</td>
</tr>
<tr>
<td><code>join()</code></td>
<td>Return the join of given elements if it exists; None otherwise.</td>
</tr>
</tbody>
</table>

Properties of the poset

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>cardinality()</code></td>
<td>Return the number of elements in the poset.</td>
</tr>
<tr>
<td><code>height()</code></td>
<td>Return the number of elements in a longest chain of the poset.</td>
</tr>
<tr>
<td><code>width()</code></td>
<td>Return the number of elements in a longest antichain of the poset.</td>
</tr>
<tr>
<td><code>relations_number()</code></td>
<td>Return the number of relations in the poset.</td>
</tr>
<tr>
<td><code>dimension()</code></td>
<td>Return the dimension of the poset.</td>
</tr>
<tr>
<td><code>jump_number()</code></td>
<td>Return the jump number of the poset.</td>
</tr>
<tr>
<td><code>magnitude()</code></td>
<td>Return the magnitude of the poset.</td>
</tr>
<tr>
<td><code>has_bottom()</code></td>
<td>Return True if the poset has a unique minimal element.</td>
</tr>
<tr>
<td><code>has_top()</code></td>
<td>Return True if the poset has a unique maximal element.</td>
</tr>
<tr>
<td><code>is_bounded()</code></td>
<td>Return True if the poset has both unique minimal and unique maximal element.</td>
</tr>
<tr>
<td><code>is_chain()</code></td>
<td>Return True if the poset is totally ordered.</td>
</tr>
<tr>
<td><code>is_connected()</code></td>
<td>Return True if the poset is connected.</td>
</tr>
<tr>
<td><code>is_graded()</code></td>
<td>Return True if all maximal chains of the poset have same length.</td>
</tr>
<tr>
<td><code>is_ranked()</code></td>
<td>Return True if the poset has a rank function.</td>
</tr>
<tr>
<td><code>is_rank_symmetric()</code></td>
<td>Return True if the poset is rank symmetric.</td>
</tr>
<tr>
<td><code>is_series_parallel()</code></td>
<td>Return True if the poset can be built by ordinal sums and disjoint unions.</td>
</tr>
<tr>
<td><code>is_greedy()</code></td>
<td>Return True if all greedy linear extensions have equal number of jumps.</td>
</tr>
<tr>
<td><code>is_jump_critical()</code></td>
<td>Return True if removal of any element reduces the jump number.</td>
</tr>
<tr>
<td><code>is_eulerian()</code></td>
<td>Return True if the poset is Eulerian.</td>
</tr>
<tr>
<td><code>is_incomparable_chain_free</code></td>
<td>Return True if the poset is (m+n)-free.</td>
</tr>
<tr>
<td><code>is_slender()</code></td>
<td>Return True if the poset is slender.</td>
</tr>
<tr>
<td><code>is_sperner()</code></td>
<td>Return True if the poset is Sperner.</td>
</tr>
<tr>
<td><code>is_join_semilattice()</code></td>
<td>Return True if the poset has a join operation.</td>
</tr>
<tr>
<td><code>is_meet_semilattice()</code></td>
<td>Return True if the poset has a meet operation.</td>
</tr>
</tbody>
</table>

Minimal and maximal elements

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>bottom()</code></td>
<td>Return the bottom element of the poset, if it exists.</td>
</tr>
<tr>
<td><code>top()</code></td>
<td>Return the top element of the poset, if it exists.</td>
</tr>
<tr>
<td><code>maximal_elements()</code></td>
<td>Return the list of the maximal elements of the poset.</td>
</tr>
<tr>
<td><code>minimal_elements()</code></td>
<td>Return the list of the minimal elements of the poset.</td>
</tr>
</tbody>
</table>

New posets from old ones
disjoint_union()  Return the disjoint union of the poset with other poset.
ordinal_sum()  Return the ordinal sum of the poset with other poset.
product()  Return the Cartesian product of the poset with other poset.
ordinal_product()  Return the ordinal product of the poset with other poset.
rees_product()  Return the Rees product of the poset with other poset.
lexicographic_sum()  Return the lexicographic sum of posets.
star_product()  Return the star product of the poset with other poset.
with_bounds()  Return the poset with bottom and top element adjoined.
without_bounds()  Return the poset with bottom and top element removed.
dual()  Return the dual of the poset.
completion_by_cuts()  Return the Dedekind-MacNeille completion of the poset.
intervals_poset()  Return the poset of intervals of the poset.
connected_components()  Return the connected components of the poset as subposets.
factor()  Return the decomposition of the poset as a Cartesian product.
ordinal_summands()  Return the ordinal summands of the poset.
subposet()  Return the subposet containing elements with partial order induced by this poset.
random_subposet()  Return a random subposet that contains each element with given probability.
relabel()  Return a copy of this poset with its elements relabelled.
canonical_label()  Return copy of the poset canonically (re)labelled to integers.
slant_sum()  Return the slant sum poset of two posets.

Chains, antichains & linear intervals

is_chain_of_poset()  Return True if elements in the given list are comparable.
is_antichain_of_poset()  Return True if elements in the given list are incomparable.
is_linear_interval()  Return whether the given interval is a total order.
chains()  Return the chains of the poset.
antichains()  Return the antichains of the poset.
maximal_chains()  Return the maximal chains of the poset.
maximal_antichains()  Return the maximal antichains of the poset.
maximal_chains_iterator()  Return an iterator over the maximal chains of the poset.
maximal_chain_length()  Return the maximum length of maximal chains of the poset.
antichains_iterator()  Return an iterator over the antichains of the poset.
random_maximal_chain()  Return a random maximal chain.
random_maximal_antichain()  Return a random maximal antichain.
linear_intervals_count()  Return the enumeration of linear intervals in the poset.

Drawing

show()  Display the Hasse diagram of the poset.
plot()  Return a Graphic object corresponding the Hasse diagram of the poset.
graphviz_string()  Return a representation in the DOT language, ready to render in graphviz.

Comparing posets

is_isomorphic()  Return True if both posets are isomorphic.
is_induced_subposet()  Return True if given poset is an induced subposet of this poset.

Polynomials

5.1. Comprehensive Module List
<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td><strong>chain_polynomial()</strong></td>
<td>Return the chain polynomial of the poset.</td>
</tr>
<tr>
<td><strong>characteristic_polynomial()</strong></td>
<td>Return the characteristic polynomial of the poset.</td>
</tr>
<tr>
<td><strong>f_polynomial()</strong></td>
<td>Return the f-polynomial of the poset.</td>
</tr>
<tr>
<td><strong>flag_f_polynomial()</strong></td>
<td>Return the flag f-polynomial of the poset.</td>
</tr>
<tr>
<td><strong>h_polynomial()</strong></td>
<td>Return the h-polynomial of the poset.</td>
</tr>
<tr>
<td><strong>flag_h_polynomial()</strong></td>
<td>Return the flag h-polynomial of the poset.</td>
</tr>
<tr>
<td><strong>order_polynomial()</strong></td>
<td>Return the order polynomial of the poset.</td>
</tr>
<tr>
<td><strong>zeta_polynomial()</strong></td>
<td>Return the zeta polynomial of the poset.</td>
</tr>
<tr>
<td><strong>M_triangle()</strong></td>
<td>Return the M-triangle of the poset.</td>
</tr>
<tr>
<td><strong>kazhdan_lusztig_polynomial()</strong></td>
<td>Return the Kazhdan-Lusztig polynomial of the poset.</td>
</tr>
<tr>
<td><strong>coxeter_polynomial()</strong></td>
<td>Return the characteristic polynomial of the Coxeter transformation.</td>
</tr>
<tr>
<td><strong>degree_polynomial()</strong></td>
<td>Return the generating polynomial of degrees of vertices in the Hasse diagram.</td>
</tr>
<tr>
<td><strong>p_partition_enumerator()</strong></td>
<td>Return a $P$-partition enumerator of the poset.</td>
</tr>
</tbody>
</table>

**Polytopes**

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>chain_polytope()</strong></td>
<td>Return the chain polytope of the poset.</td>
</tr>
<tr>
<td><strong>order_polytope()</strong></td>
<td>Return the order polytope of the poset.</td>
</tr>
</tbody>
</table>

**Graphs**

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>hasse_diagram()</strong></td>
<td>Return the Hasse diagram of the poset as a directed graph.</td>
</tr>
<tr>
<td><strong>cover_relations_graph()</strong></td>
<td>Return the (undirected) graph of cover relations.</td>
</tr>
<tr>
<td><strong>comparability_graph()</strong></td>
<td>Return the comparability graph of the poset.</td>
</tr>
<tr>
<td><strong>incomparability_graph()</strong></td>
<td>Return the incomparability graph of the poset.</td>
</tr>
<tr>
<td><strong>frank_network()</strong></td>
<td>Return Frank’s network of the poset.</td>
</tr>
<tr>
<td><strong>linear_extensions_graph()</strong></td>
<td>Return the linear extensions graph of the poset.</td>
</tr>
</tbody>
</table>

**Linear extensions**

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>is_linear_extension()</strong></td>
<td>Return True if the given list is a linear extension of the poset.</td>
</tr>
<tr>
<td><strong>linear_extension()</strong></td>
<td>Return a linear extension of the poset.</td>
</tr>
<tr>
<td><strong>linear_extensions()</strong></td>
<td>Return the enumerated set of all the linear extensions of the poset.</td>
</tr>
<tr>
<td><strong>promotion()</strong></td>
<td>Return the (extended) promotion on the linear extension of the poset.</td>
</tr>
<tr>
<td><strong>evacuation()</strong></td>
<td>Return evacuation on the linear extension associated to the poset.</td>
</tr>
<tr>
<td><strong>with_linear_extension()</strong></td>
<td>Return a copy of self with a different default linear extension.</td>
</tr>
<tr>
<td><strong>random_linear_extension()</strong></td>
<td>Return a random linear extension.</td>
</tr>
</tbody>
</table>

**Matrices**

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>lequal_matrix()</strong></td>
<td>Computes the matrix whose $(i,j)$ entry is 1 if self.linear_extension()[i] &lt; self.linear_extension()[j] and 0 otherwise.</td>
</tr>
<tr>
<td><strong>moebius_function()</strong></td>
<td>Return the value of Möbius function of given elements in the poset.</td>
</tr>
<tr>
<td><strong>moebius_function_matrix()</strong></td>
<td>Return a matrix whose $(i,j)$ entry is the value of the Möbius function evaluated at self.linear_extension()[i] and self.linear_extension()[j].</td>
</tr>
<tr>
<td><strong>coxeter_transformation()</strong></td>
<td>Return the matrix of the Auslander-Reiten translation acting on the Grothendieck group of the derived category of modules.</td>
</tr>
<tr>
<td><strong>coxeter_smith_form()</strong></td>
<td>Return the Smith form of the Coxeter transformation.</td>
</tr>
</tbody>
</table>
Combinatorics, Release 10.0

Miscellaneous

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sorted()</code></td>
<td>Return given list sorted by the poset.</td>
</tr>
<tr>
<td><code>isomorphic_subposets()</code></td>
<td>Return all subposets isomorphic to another poset.</td>
</tr>
<tr>
<td><code>isomorphic_subposets_iterator()</code></td>
<td>Return an iterator over the subposets isomorphic to another poset.</td>
</tr>
<tr>
<td><code>has_isomorphic_subposet()</code></td>
<td>Return <code>True</code> if the poset contains a subposet isomorphic to another poset.</td>
</tr>
<tr>
<td><code>list()</code></td>
<td>List the elements of the poset.</td>
</tr>
<tr>
<td><code>cuts()</code></td>
<td>Return the cuts of the given poset.</td>
</tr>
<tr>
<td><code>dilworth_decomposition()</code></td>
<td>Return a partition of the points into the minimal number of chains.</td>
</tr>
<tr>
<td><code>greene_shape()</code></td>
<td>Computes the Greene-Kleitman partition aka Greene shape of the poset <code>self</code></td>
</tr>
<tr>
<td><code>incidence_algebra()</code></td>
<td>Return the incidence algebra of <code>self</code>.</td>
</tr>
<tr>
<td><code>is_EL_labelling()</code></td>
<td>Return whether <code>f</code> is an EL labelling of the poset.</td>
</tr>
<tr>
<td><code>isomorphic_subposets_iterator()</code></td>
<td>Return an iterator over the subposets isomorphic to another poset.</td>
</tr>
<tr>
<td><code>isomorphic_subposets()</code></td>
<td>Return all subposets isomorphic to another poset.</td>
</tr>
<tr>
<td><code>level_sets()</code></td>
<td>Return elements grouped by maximal number of cover relations from a minimal element.</td>
</tr>
<tr>
<td><code>order_complex()</code></td>
<td>Return the order complex associated to this poset.</td>
</tr>
<tr>
<td><code>random_order_ideal()</code></td>
<td>Return a random order ideal of <code>self</code> with uniform probability.</td>
</tr>
<tr>
<td><code>rank()</code></td>
<td>Return the rank of an element, or the rank of the poset.</td>
</tr>
<tr>
<td><code>rank_function()</code></td>
<td>Return a rank function of the poset, if it exists.</td>
</tr>
<tr>
<td><code>unwrap()</code></td>
<td>Unwraps an element of this poset.</td>
</tr>
<tr>
<td><code>atkinson()</code></td>
<td>Return the $a$-spectrum of a poset whose undirected Hasse diagram is a forest.</td>
</tr>
<tr>
<td><code>spectrum()</code></td>
<td>Return the $a$-spectrum of this poset.</td>
</tr>
</tbody>
</table>

Classes and functions

class sage.combinat.posets.posets.FinitePoset(hasse_diagram, elements, category, facade, key)

Bases: UniqueRepresentation, Parent

A (finite) $n$-element poset constructed from a directed acyclic graph.

INPUT:

- `hasse_diagram` – an instance of `FinitePoset`, or a `DiGraph` that is transitively-reduced, acyclic, loop-free, and multiedge-free.

- `elements` – an optional list of elements, with `element[i]` corresponding to vertex $i$. If `elements` is `None`, then it is set to be the vertex set of the digraph. Note that if this option is set, then `elements` is considered as a specified linear extension of the poset and the `linear_extension` attribute is set.

- `category` – `FinitePosets`, or a subcategory thereof.

- `facade` – a boolean or `None` (default); whether the `FinitePoset`'s elements should be wrapped to make them aware of the Poset they belong to.
  
  - If `facade = True`, the `FinitePoset`'s elements are exactly those given as input.
  
  - If `facade = False`, the `FinitePoset`'s elements will become `PosetElement` objects.
  
  - If `facade = None` (default) the expected behaviour is the behaviour of `facade = True`, unless the opposite can be deduced from the context (i.e. for instance if a `FinitePoset` is built from another `FinitePoset`, itself built with `facade = False`)

- `key` – any hashable value (default: `None`).

EXAKLPLAJESE:
Combinatorics, Release 10.0

sage: uc = [[2,3], [], [1], [1], [1], [3,4]]
sage: from sage.combinat.posets.posets import FinitePoset
sage: P = FinitePoset(DiGraph(dict([(i,uc[i]) for i in range(len(uc))])), facade=False); P
Finite poset containing 6 elements
sage: P.cover_relations()
[[5, 4], [5, 3], [4, 1], [0, 2], [0, 3], [2, 1], [3, 1]]
sage: TestSuite(P).run()
sage: P.category()
Category of finite enumerated posets
sage: P.__class__
<class 'sage.combinat.posets.posets.FinitePoset_with_category'>
sage: Q = sage.combinat.posets.posets.FinitePoset(P, facade=False); Q
Finite poset containing 6 elements
sage: Q is P
True

We keep the same underlying Hasse diagram, but change the elements:

sage: Q = sage.combinat.posets.posets.FinitePoset(P, elements=[1,2,3,4,5,6], facade=False); Q
Finite poset containing 6 elements with distinguished linear extension
sage: Q.cover_relations()
[[1, 2], [1, 5], [2, 6], [3, 4], [3, 5], [4, 6], [5, 6]]

We test the facade argument:

sage: P = Poset(DiGraph({'a':'b','b':'c','c':'d'}), facade=False)
sage: P.category()
Category of finite enumerated posets
sage: parent(P[0]) is P
True
sage: Q = Poset(DiGraph({'a':'b','b':'c','c':'d'}), facade=True)
sage: Q.category()
Category of facade finite enumerated posets
sage: parent(Q[0]) is str
True
sage: TestSuite(Q).run(skip = ['_test_an_element']) # is_parent_of is not yet implemented

Changing a non facade poset to a facade poset:

sage: PQ = Poset(P, facade=True)
sage: PQ.category()
Category of facade finite enumerated posets
sage: parent(PQ[0]) is str
True
sage: PQ is Q
True

Changing a facade poset to a non facade poset:
Conversion to some other software is possible:

```python
sage: P = posets.TamariLattice(3)
sage: libgap(P)  # optional - gap_packages
<A poset on 5 points>
```

```python
sage: P = Poset({1:[2],2:[]})
sage: macaulay2('needsPackage "Posets"')  # optional - macaulay2
Posets
sage: macaulay2(P)  # optional - macaulay2
Relation Matrix: | 1 1 |
| 0 1 |
```

**Note:** A class that inherits from this class needs to define `Element`. This is the class of the elements that the inheriting class contains. For example, for this class, `FinitePoset.Element` is `PosetElement`. It can also define `_dual_class` which is the class of dual posets of this class. E.g. `FiniteMeetSemilattice._dual_class` is `FiniteJoinSemilattice`.

**Element**

alias of `PosetElement`

**M_triangle()**

Return the M-triangle of the poset.

The poset is expected to be graded.

**OUTPUT:**

an `M_triangle`

The M-triangle is the generating polynomial of the Möbius numbers

\[ M(x, y) = \sum_{a \leq b} \mu(a, b) x^{\lfloor a \rfloor} y^{\lfloor b \rfloor}. \]

**EXAMPLES:**

```python
sage: P = posets.DiamondPoset(5)
sage: P.M_triangle()
M: x^2*y^2 - 3*x*y^2 + 3*x*y + 2*y^2 - 3*y + 1
```

**antichains(element_constructor=<class 'list'>)**

Return the antichains of the poset.

An *antichain* of a poset is a set of elements of the poset that are pairwise incomparable.

**INPUT:**

- `element_constructor` – a function taking an iterable as argument (default: list)
OUTPUT:

The enumerated set (of type `PairwiseCompatibleSubsets`) of all antichains of the poset, each of which is given as an `element_constructor`.

EXAMPLES:

```sage
sage: A = posets.PentagonPoset().antichains(); A
Set of antichains of Finite lattice containing 5 elements
sage: list(A)
[[], [0], [1], [1, 2], [1, 3], [2], [3], [4]]
sage: A.cardinality()
8
sage: A[3]
[1, 2]
```

To get the antichains as, say, sets, one may use the `element_constructor` option:

```sage
sage: list(posets.ChainPoset(3).antichains(element_constructor=set))
[set(), {0}, {1}, {2}]
```

To get the antichains of a given size one can currently use:

```sage
sage: list(A.elements_of_depth_iterator(2))
[[[1, 2], [1, 3]]
```

Eventually the following syntax will be accepted:

```sage
sage: A.subset(size = 2) # todo: not implemented
```

**Note:** Internally, this uses `sage.combinat.subsets_pairwise.PairwiseCompatibleSubsets` and `RecursivelyEnumeratedSet_forest`. At this point, iterating through this set is about twice slower than using `antichains_iterator()` (tested on `posets.AntichainPoset(15)`). The algorithm is the same (depth first search through the tree), but `antichains_iterator()` manually inlines things which apparently avoids some infrastructure overhead.

On the other hand, this returns a full featured enumerated set, with containment testing, etc.

---

**See also:**

- `maximal_antichains()`, `chains()`
- `antichains_iterator()`

`antichains_iterator()`

Return an iterator over the antichains of the poset.

**EXAMPLES:**

```sage
sage: it = posets.PentagonPoset().antichains_iterator(); it
<generator object ...antichains_iterator at ...>
sage: next(it), next(it)
([], [4])
```

**See also:**

- `antichains()`
atkinson($a$)

Return the $a$-spectrum of a poset whose Hasse diagram is cycle-free as an undirected graph.

Given an element $a$ in a poset $P$, the $a$-spectrum is the list of integers whose $i$-th term contains the number of linear extensions of $P$ with element $a$ located in the $i$-th position.

**INPUT:**

- `self` – a poset whose Hasse diagram is a forest
- `a` – an element of the poset

**OUTPUT:**

The $a$-spectrum of this poset, returned as a list.

**EXAMPLES:**

```python
sage: P = Poset({0: [2], 1: [2], 2: [3, 4], 3: [], 4: []})
sage: P.atkinson(0)
[2, 0, 0, 0]
sage: P = Poset({0: [1], 1: [2, 3], 2: [], 3: [], 4: [5, 6], 5: [], 6: []})
sage: P.atkinson(5)
[0, 10, 18, 24, 28, 30, 30]
sage: P = posets.AntichainPoset(10)
sage: P.atkinson(0)
[362880, 362880, 362880, 362880, 362880, 362880, 362880, 362880, 362880, 362880]
```

**Note:** This function is the implementation of the algorithm from [At1990].

bottom()

Return the unique minimal element of the poset, if it exists.

**EXAMPLES:**

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
sage: P.bottom() is None
True
sage: Q = Poset({0:[1],1:[]})
sage: Q.bottom()
0
```

See also:

`has_bottom()`, `top()`

canonical_label(algorithm=None)

Return the unique poset on the labels $\{0, \ldots, n - 1\}$ (where $n$ is the number of elements in the poset) that is isomorphic to this poset and invariant in the isomorphism class.

**INPUT:**

- `algorithm` – string (optional); a parameter forwarded to underlying graph function to select the algorithm to use

**EXAMPLES:**

```python
sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True)
sage: P.list()
[1, 2, 3, 4, 6, 12]
sage: Q = P.canonical_label()
sage: sorted(Q.list())
[0, 1, 2, 3, 4, 5]
sage: Q.is_isomorphic(P)
True

Canonical labeling of (semi)lattice returns (semi)lattice:

sage: D = DiGraph({'a': ['b', 'c']})
sage: P = Poset(D)
sage: ML = MeetSemilattice(D)
sage: P.canonical_label()
Finite poset containing 3 elements
sage: ML.canonical_label()
Finite meet-semilattice containing 3 elements

See also:
• Canonical labeling of directed graphs: canonical_label()

**cardinality()**
Return the number of elements in the poset.

EXAMPLES:

sage: P = Poset([[1,2,3],[4],[4],[4],[]])
sage: P.cardinality()
5

See also:
• degree_polynomial() for a more refined invariant

**chain_polynomial()**
Return the chain polynomial of the poset.

The coefficient of \(q^k\) is the number of chains of \(k\) elements in the poset. List of coefficients of this polynomial is also called a *f*-vector of the poset.

**Note:** This is not what has been called the chain polynomial in [St1986]. The latter is identical with the order polynomial in SageMath (order_polynomial()).

See also:
• f_polynomial(), order_polynomial()
sage: P = posets.BooleanLattice(3)
sage: P.chain_polynomial()
6\,q^4 + 18\,q^3 + 19\,q^2 + 8\,q + 1

sage: P = posets.AntichainPoset(5)
sage: P.chain_polynomial()
5\,q + 1

chain_polytope()

Return the chain polytope of the poset self.

The chain polytope of a finite poset \( P \) is defined as the subset of \( \mathbb{R}^P \) consisting of all maps \( x : P \to \mathbb{R} \) satisfying

\[
x(p) \geq 0 \text{ for all } p \in P,
\]

and

\[
x(p_1) + x(p_2) + \ldots + x(p_k) \leq 1 \text{ for all chains } p_1 < p_2 < \ldots < p_k \text{ in } P.
\]

This polytope was defined and studied in [St1986].

EXAMPLES:

sage: P = posets.AntichainPoset(3)
sage: Q = P.chain_polytope(); Q
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: P = posets.PentagonPoset()
sage: Q = P.chain_polytope(); Q
A 5-dimensional polyhedron in ZZ^5 defined as the convex hull of 8 vertices

chains(element_constructor=<class 'list'>, exclude=None)

Return the chains of the poset.

A chain of a poset is an increasing sequence of distinct elements of the poset.

INPUT:

• element_constructor – a function taking an iterable as argument (optional, default: list)

• exclude – elements of the poset to be excluded (optional, default: None)

OUTPUT:

The enumerated set (of type PairwiseCompatibleSubsets) of all chains of the poset, each of which is given as an element_constructor.

EXAMPLES:

sage: C = posets.PentagonPoset().chains(); C
Set of chains of Finite lattice containing 5 elements
sage: list(C)
\[
[\], [0], [0, 1], [0, 1, 4], [0, 2], [0, 2, 3], [0, 2, 3, 4], [0, 2, 4], [0, 3], [0, 3, 4], [0, 4], [1], [1, 4], [2], [2, 3], [2, 3, 4], [2, 4], [3], [3, 4], ...
\]
sage: P = Poset({1: [2, 3], 2: [4], 3: [4, 5]})
sage: list(P.chains(element_constructor=tuple, exclude=[3]))
[(), (1,), (1, 2), (1, 2, 4), (1, 4), (1, 5), (2,), (2, 4), (4,), (5,)]

To get the chains of a given size one can currently use:

sage: list(C.elements_of_depth_iterator(2))
[[0, 1], [0, 2], [0, 3], [0, 4], [1, 4], [2, 3], [2, 4], [3, 4]]

Eventually the following syntax will be accepted:

sage: C.subset(size = 2) # todo: not implemented

See also:
maximal_chains(), antichains()

characteristic_polynomial()
Return the characteristic polynomial of the poset.

The poset is expected to be graded and have a bottom element.

If $P$ is a graded poset with rank $n$ and a unique minimal element $\hat{0}$, then the characteristic polynomial of $P$ is defined to be

$$\sum_{x \in P} \mu(\hat{0}, x) q^{n - \rho(x)} \in \mathbb{Z}[q],$$

where $\rho$ is the rank function, and $\mu$ is the Möbius function of $P$.

See section 3.10 of [EnumComb1].

EXAMPLES:

sage: P = posets.DiamondPoset(5)
sage: P.characteristic_polynomial()
q^2 - 3*q + 2
sage: P = Poset({1: [2, 3], 2: [4], 3: [5], 4: [6], 5: [6], 6: [7]})
sage: P.characteristic_polynomial()
q^4 - 2*q^3 + q

closed_interval(x, y)
Return the list of elements $z$ such that $x \leq z \leq y$ in the poset.

EXAMPLES:

sage: P = Poset((divisors(1000), attrcall("divides")))
sage: P.closed_interval(2, 100)
[2, 4, 10, 20, 50, 100]

See also:
open_interval()

common_lower_covers(elmts)
Return all of the common lower covers of the elements elmts.

EXAMPLES:
sage: P = Poset({0: [1,2], 1: [3], 2: [3], 3: []})
sage: P.common_lower_covers([1, 2])
[0]

common_upper_covers(elmts)
Return all of the common upper covers of the elements elmts.

EXAMPLES:

sage: P = Poset({0: [1,2], 1: [3], 2: [3], 3: []})
sage: P.common_upper_covers([1, 2])
[3]

comparability_graph()
Return the comparability graph of the poset.

The comparability graph is an undirected graph where vertices are the elements of the poset and there is an edge between two vertices if they are comparable in the poset.

See Wikipedia article Comparability_graph

EXAMPLES:

sage: Y = Poset({1: [2], 2: [3, 4]})
sage: g = Y.comparability_graph(); g
Comparability graph on 4 vertices
sage: Y.compare_elements(1, 3) is not None
True
sage: g.has_edge(1, 3)
True

See also:
incomparability_graph(), sage.graphs.comparability

compare_elements(x, y)
Compare x and y in the poset.

• If \( x < y \), return -1.
• If \( x = y \), return 0.
• If \( x > y \), return 1.
• If x and y are not comparable, return None.

EXAMPLES:

sage: P = Poset([[1, 2], [4], [3], [4], []])
sage: P.compare_elements(0, 0)
0
sage: P.compare_elements(0, 4)
-1
sage: P.compare_elements(4, 0)
1
sage: P.compare_elements(1, 2) is None
True
**completion_by_cuts()**

Return the completion by cuts of \texttt{self}.

This is the smallest lattice containing the poset. This is also called the Dedekind-MacNeille completion.

See the Wikipedia article Dedekind-MacNeille completion.

**OUTPUT:**

- a finite lattice

**EXAMPLES:**

```
sage: P = posets.PentagonPoset()
sage: P.completion_by_cuts().is_isomorphic(P)
True

sage: Y = Poset({1: [2], 2: [3, 4]})
sage: trafficsign = LatticePoset({1: [2], 2: [3, 4], 3: [5], 4: [5]})
sage: L = Y.completion_by_cuts()
sage: L.is_isomorphic(trafficsign)
True

sage: P = posets.SymmetricGroupBruhatOrderPoset(3)
sage: Q = P.completion_by_cuts(); Q
Finite lattice containing 7 elements
```

See also:

- \texttt{cuts()}, \texttt{irreducibles_poset()}

**connected_components()**

Return the connected components of the poset as subposets.

**EXAMPLES:**

```
sage: P = Poset({1: [2, 3], 3: [4, 5], 6: [7, 8]})
sage: parts = sorted(P.connected_components(), key=len); parts
[Finite poset containing 3 elements,
 Finite poset containing 5 elements]
sage: parts[0].cover_relations()
[[6, 7], [6, 8]]
```

See also:

- \texttt{disjoint_union()}, \texttt{is_connected()}

**cover_relations()**

Return the list of pairs [\texttt{x, y}] of elements of the poset such that \texttt{y} covers \texttt{x}.

**EXAMPLES:**

```
sage: P = Poset({0: [2], 1: [2], 2: [3], 3: [4], 4: []})
sage: P.cover_relations()
[[1, 2], [0, 2], [2, 3], [3, 4]]
```

**cover_relations_graph()**

Return the (undirected) graph of cover relations.

**EXAMPLES:**
sage: P = Poset({0: [1, 2], 1: [3], 2: [3]})
sage: G = P.cover_relations_graph(); G
Graph on 4 vertices
sage: G.has_edge(3, 1), G.has_edge(3, 0)
(True, False)

See also:

cover_relations_iterator()

Return an iterator over the cover relations of the poset.

EXAMPLES:

sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
sage: type(P.cover_relations_iterator())
<class 'generator'>
sage: [z for z in P.cover_relations_iterator()]
[[1, 2], [0, 2], [2, 3], [3, 4]]

covers(x, y)

Return True if y covers x and False otherwise.

Element y covers x if x < y and there is no z such that x < z < y.

EXAMPLES:

sage: P = Poset(([1,5], [2,6], [3], [4], [], [6,3], [4]))
sage: P.covers(1, 6)
True
sage: P.covers(1, 4)
False
sage: P.covers(1, 5)
False

coxeter_polynomial()

Return the Coxeter polynomial of the poset.

OUTPUT:

a polynomial in one variable

The output is the characteristic polynomial of the Coxeter transformation. This polynomial only depends on the derived category of modules on the poset.

EXAMPLES:

sage: P = posets.PentagonPoset()
sage: P.coxeter_polynomial()
x^5 + x^4 + x + 1
sage: p = posets.SymmetricGroupWeakOrderPoset(3)
sage: p.coxeter_polynomial()
x^6 + x^5 - x^3 + x + 1

See also:

coxeter_transformation(), coxeter_smith_form()
The Smith normal form of \( x \) minus the Coxeter transformation matrix.

INPUT:

- `algorithm` – optional (default 'singular'), possible values are 'singular', 'sage', 'gap', 'pari', 'maple', 'magma', 'fricas'

Beware that speed depends very much on the choice of algorithm. Sage is rather slow, Singular is faster and Pari is fast at least for small sizes.

OUTPUT:

- list of polynomials in one variable, each one dividing the next one

The output list is a refinement of the characteristic polynomial of the Coxeter transformation, which is its product. This list of polynomials only depends on the derived category of modules on the poset.

EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: P.coxeter_smith_form()
[1, 1, 1, 1, x^5 + x^4 + x + 1]
sage: P = posets.DiamondPoset(7)
sage: prod(P.coxeter_smith_form()) == P.coxeter_polynomial()
True
```

See also:

- `coxeter_transformation()`, `coxeter_matrix()`

Return the Coxeter transformation of the poset.

OUTPUT:

a square matrix with integer coefficients

The output is the matrix of the Auslander-Reiten translation acting on the Grothendieck group of the derived category of modules on the poset, in the basis of simple modules. This matrix is usually called the Coxeter transformation.

EXAMPLES:

```python
sage: posets.PentagonPoset().coxeter_transformation()
[ 0 0 0 0 -1]
[ 0 0 0 1 -1]
[ 0 1 0 0 -1]
[-1 1 1 0 -1]
[-1 1 0 1 -1]
```

See also:

- `coxeter_polynomial()`, `coxeter_smith_form()`

Return the list of cuts of the poset `self`.

A cut is a subset \( A \) of `self` such that the set of lower bounds of the set of upper bounds of \( A \) is exactly \( A \).
The cuts are computed here using the maximal independent sets in the auxiliary graph defined as $P \times [0,1]$ with an edge from $(x,0)$ to $(y,1)$ if and only if $x \not\geq_P y$. See the end of section 4 in [JRJ94].

**EXAMPLES:**

```python
sage: P = posets.AntichainPoset(3)
sage: Pc = P.cuts()
sage: Pc  # random
[frozenset({0}),
 frozenset(),
 frozenset({0, 1, 2}),
 frozenset({2}),
 frozenset({1})]
sage: sorted(list(c) for c in Pc)
[[], [0], [0, 1, 2], [1], [2]]
```

See also:

`completion_by_cuts()`

**degree_polynomial()**

Return the generating polynomial of degrees of vertices in `self`. This is the sum

$$\sum_{v \in P} x^{in(v)} y^{out(v)},$$

where `in(v)` and `out(v)` are the number of incoming and outgoing edges at vertex `v` in the Hasse diagram of `P`.

Because this polynomial is multiplicative for Cartesian product of posets, it is useful to help see if the poset can be isomorphic to a Cartesian product.

**EXAMPLES:**

```python
sage: P = posets.PentagonPoset()
sage: P.degree_polynomial()
x^2 + 3*x*y + y^2
sage: P = posets.BooleanLattice(4)
sage: P.degree_polynomial().factor()
(x + y)^4
```

See also:

`cardinality()` for the value at $(x, y) = (1, 1)$

**diamonds()**

Return the list of diamonds of `self`.

A diamond is the following subgraph of the Hasse diagram:

```
z
/ \
x y
\ / \\
w
```
Thus each edge represents a cover relation in the Hasse diagram. We represent this as the tuple \((w, x, y, z)\).

**OUTPUT:**

A tuple with

- a list of all diamonds in the Hasse Diagram,
- a boolean checking that every \(w, x, y\) that form a \(V\), there is a unique element \(z\), which completes the diamond.

**EXAMPLES:**

```python
sage: P = Poset({0: [1, 2], 1: [3], 2: [3], 3: []})
sage: P.diamonds()
(([]), True)
```

```python
sage: P = posets.YoungDiagramPoset(Partition([3, 2, 2]))
sage: P.diamonds()
((((0, 0), (0, 1), (1, 0), (1, 1)), ((1, 0), (1, 1), (2, 0), (2, 1))), False)
```

dilworth_decomposition()

Return a partition of the points into the minimal number of chains.

According to Dilworth’s theorem, the points of a poset can be partitioned into \(\alpha\) chains, where \(\alpha\) is the cardinality of its largest antichain. This method returns such a partition.

See Wikipedia article Dilworth’s theorem.

**ALGORITHM:**

We build a bipartite graph in which a vertex \(v\) of the poset is represented by two vertices \(v^-\), \(v^+\). For any two \(u, v\) such that \(u < v\) in the poset we add an edge \(v^+u^-\).

A matching in this graph is equivalent to a partition of the poset into chains: indeed, a chain \(v_1...v_k\) gives rise to the matching \(v_1^-v_2^+, v_2^-v_3^+, ...\), and from a matching one can build the union of chains.

According to Dilworth’s theorem, the number of chains is equal to \(\alpha\) (the posets’ width).

**EXAMPLES:**

```python
sage: p = posets.BooleanLattice(4)
sage: p.width()
6
sage: p.dilworth_decomposition()  # random
[[7, 6, 4], [11, 3], [12, 8, 0], [13, 9, 1], [14, 10, 2], [15, 5]]
```

See also:

level_sets() to return elements grouped to antichains.

dimension(certificate, solver, integrality_tolerance=False)

Return the dimension of the Poset.

The (Dushnik-Miller) dimension of a poset is the minimal number of total orders so that the poset is their “intersection”. More precisely, the dimension of a poset defined on a set \(X\) of points is the smallest integer \(n\) such that there exist linear extensions \(P_1, ..., P_n\) of \(P\) satisfying:

\[ u \leq_P v \text{ if and only if } \forall i, u \leq_{P_i} v \]

For more information, see the Wikipedia article Order_dimension.

**INPUT:**
Combinatorics, Release 10.0

- **certificate** (boolean; default: False) – whether to return an integer (the dimension) or a certificate, i.e. a smallest set of linear extensions.
- **solver** – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method `solve` of the class `MixedIntegerLinearProgram`.
- **integrality_tolerance** – parameter for use with MILP solvers over an inexact base ring; see `MixedIntegerLinearProgram.get_values()`.

**Note:** The speed of this function greatly improves when more efficient MILP solvers (e.g. Gurobi, CPLEX) are installed. See `MixedIntegerLinearProgram` for more information.

**Note:** Prior to version 8.3 this returned only realizer with `certificate=True`. Now it returns a pair having a realizer as the second element. See github issue #25588 for details.

**ALGORITHM:**

As explained [FT00], the dimension of a poset is equal to the (weak) chromatic number of a hypergraph. More precisely:

Let \( \text{inc}(P) \) be the set of (ordered) pairs of incomparable elements of \( P \), i.e. all \( uv \) and \( vu \) such that \( u \not\leq_P v \) and \( v \not\leq_P u \). Any linear extension of \( P \) is a total order on \( X \) that can be seen as the union of relations from \( P \) along with some relations from \( \text{inc}(P) \). Thus, the dimension of \( P \) is the smallest number of linear extensions of \( P \) which cover all points of \( \text{inc}(P) \).

Consequently, \( \text{dim}(P) \) is equal to the chromatic number of the hypergraph \( H_{\text{inc}} \), where \( H_{\text{inc}} \) is the hypergraph defined on \( \text{inc}(P) \) whose sets are all \( S \subseteq \text{inc}(P) \) such that \( P \cup S \) is not acyclic.

We solve this problem through a Mixed Integer Linear Program.

The problem is known to be NP-complete.

**EXAMPLES:**

We create a poset, compute a set of linear extensions and check that we get back the poset from them:

```python
sage: P = Poset([[1,4], [3], [4,5,3], [6], [], [6], []])
sage: P.dimension() 3
sage: dim, L = P.dimension(certificate=True)
sage: L # random -- architecture-dependent
[[0, 2, 4, 5, 1, 3, 6], [2, 5, 0, 1, 3, 4, 6], [0, 1, 2, 3, 5, 6, 4]]
sage: Poset( (L[0], lambda x, y: all(l.index(x) < l.index(y) for l in L)) ) == P True
```

According to Schnyder’s theorem, the incidence poset (of height 2) of a graph has dimension \( \leq 3 \) if and only if the graph is planar:

```python
sage: G = graphs.CompleteGraph(4)
sage: P = Poset(DiGraph({(u,v):[u,v] for u,v,_ in G.edges(sort=True)}))
sage: P.dimension() 3
sage: G = graphs.CompleteBipartiteGraph(3,3)
```

(continues on next page)
disjoint_union(other, labels='pairs')

Return a poset isomorphic to disjoint union (also called direct sum) of the poset with other.

The disjoint union of \( P \) and \( Q \) is a poset that contains every element and relation from both \( P \) and \( Q \), and where every element of \( P \) is incomparable to every element of \( Q \).

Mathematically, it is only defined when \( P \) and \( Q \) have no common element; here we force that by giving them different names in the resulting poset.

**INPUT:**

• other, a poset.

• labels - (defaults to ‘pairs’) If set to ‘pairs’, each element \( v \) in this poset will be named \((0, v)\) and each element \( u \) in other will be named \((1, u)\) in the result. If set to ‘integers’, the elements of the result will be relabeled with consecutive integers.

**EXAMPLES:**

```python
sage: P1 = Poset({'a': 'b'})
sage: P2 = Poset({'c': 'd'})
sage: P = P1.disjoint_union(P2); P
Finite poset containing 4 elements
sage: sorted(P.cover_relations())
[[0, 'a'], [0, 'b'], [1, 'c'], [1, 'd']]
sage: P = P1.disjoint_union(P2, labels='integers')
sage: P.cover_relations()
[[0, 1], [2, 3]]
sage: N5 = posets.PentagonPoset(); N5
Finite lattice containing 5 elements
sage: N5.disjoint_union(N5) # Union of lattices is not a lattice
Finite poset containing 10 elements
```

We show how to get literally direct sum with elements untouched:

```python
sage: P = P1.disjoint_union(P2).relabel(lambda x: x[1])
sage: sorted(P.cover_relations())
[['a', 'b'], ['c', 'd']]
```

See also:

[connected_components()](#)

dual()

Return the dual poset of the given poset.

In the dual of a poset \( P \) we have \( x \leq y \) iff \( y \leq x \) in \( P \).

**EXAMPLES:**

```python
sage: P = Poset({1: [2, 3], 3: [4]})
sage: P.cover_relations()
```
Dual of a lattice is a lattice; dual of a meet-semilattice is join-semilattice and vice versa. Also the dual of a (non-)facade poset is again (non-)facade:

```python
sage: V = MeetSemilattice({1: [2, 3]}, facade=False)
sage: A = V.dual(); A
Finite join-semilattice containing 3 elements
sage: A(2) < A(1)
True
```

See also:

`is_self_dual()`

`evacuation()`

Compute evacuation on the linear extension associated to the poset `self`.

**OUTPUT:**

- an isomorphic poset, with the same default linear extension

Evacuation is defined on a poset `self` of size `n` by applying the evacuation operator \((\tau_1 \cdots \tau_{n-1}) (\tau_1 \cdots \tau_{n-2}) \cdots (\tau_1)\), to the default linear extension \(\pi\) of `self` (see `evacuation()`), and relabeling `self` accordingly. For more details see [Stan2009].

**EXAMPLES:**

```python
sage: P = Poset(((1,2), [[1,2]]), linear_extension=True, facade=False)
sage: P.evacuation()
Finite poset containing 2 elements with distinguished linear extension
sage: P.evacuation() == P
True
sage: P = Poset(((1,2,3,4,5,6,7), [[1,2],[1,4],[2,3],[2,5],[3,6],[4,7],[5,6]]),
             linear_extension=True, facade=False)
sage: P.list()
[1, 2, 3, 4, 5, 6, 7]
sage: Q = P.evacuation(); Q
Finite poset containing 7 elements with distinguished linear extension
sage: Q.cover_relations()
[[1, 2], [1, 3], [2, 5], [3, 4], [3, 6], [4, 7], [6, 7]]
```

Note that the results depend on the linear extension associated to the poset:

```python
sage: P = Poset(((1,2,3,4,5,6,7), [[1,2],[1,4],[2,3],[2,5],[3,6],[4,7],[5,6]]))
sage: P.list()
[1, 2, 3, 5, 6, 4, 7]
sage: Q = P.evacuation(); Q
Finite poset containing 7 elements with distinguished linear extension
sage: Q.cover_relations()
[[1, 2], [1, 5], [2, 3], [5, 6], [5, 4], [6, 7], [4, 7]]
```
Here is an example of a poset where the elements are not labelled by \{1, 2, \ldots, n\}:

```sage
P = Poset((divisors(15), attrcall("divides")), linear_extension = True)
P.list()
[1, 3, 5, 15]
P = P.evacuation(); Q
Finite poset containing 4 elements with distinguished linear extension
Q.cover_relations()
[[[1, 3], [1, 5], [3, 15], [5, 15]]
```

See also:

- `linear_extension()`
- `with_linear_extension()` and the `linear_extension` option of `Poset()`
- `evacuation()`
- `promotion()`

AUTHOR:

- Anne Schilling (2012-02-18)

**f_polynomial()**

Return the \(f\)-polynomial of the poset.

The poset is expected to be bounded.

This is the \(f\)-polynomial of the order complex of the poset minus its bounds.

The coefficient of \(q^i\) is the number of chains of \(i + 1\) elements containing both bounds of the poset.

**Note:** This is slightly different from the \(fPolynomial\) method in Macaulay2.

**EXAMPLES:**

```sage
P = posets.DiamondPoset(5)
P.f_polynomial()
3*q^2 + q
P = Poset({1: [2, 3], 2: [4], 3: [5], 4: [6], 5: [7], 6: [7]})
P.f_polynomial()
q^4 + 4*q^3 + 5*q^2 + q
```

See also:

- `is_bounded()`, `h_polynomial()`, `order_complex()`, `sage.topology.cell_complex.GenericCellComplex.f_vector()`

**factor()**

Factor the poset as a Cartesian product of smaller posets.

This only works for connected posets for the moment.

The decomposition of a connected poset as a Cartesian product of posets (prime in the sense that they cannot be written as Cartesian products) is unique up to reordering and isomorphism.

**OUTPUT:**
EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: Q = P*P
sage: Q.factor()
[Finite poset containing 5 elements,  
 Finite poset containing 5 elements]

sage: P1 = posets.ChainPoset(3)
sage: P2 = posets.ChainPoset(7)
sage: P1.factor()
[Finite lattice containing 3 elements]

sage: (P1 * P2).factor()
[Finite poset containing 7 elements,  
 Finite poset containing 3 elements]

sage: P = posets.TamariLattice(4)
sage: (P*P).factor()
[Finite poset containing 14 elements,  
 Finite poset containing 14 elements]
```

See also:

- `product()`

REFERENCES:

**flag_f_polynomial()**

Return the flag $f$-polynomial of the poset.

The poset is expected to be bounded and ranked.

This is the sum, over all chains containing both bounds, of a monomial encoding the ranks of the elements of the chain.

More precisely, if $P$ is a bounded ranked poset, then the flag $f$-polynomial of $P$ is defined as the polynomial

$$
\sum_{\min P < p_1 < p_2 < \ldots < p_k, \ p_0 = \min P, \ p_k = \max P} x_{\rho(p_1)} x_{\rho(p_2)} \cdots x_{\rho(p_k)} \in \mathbb{Z}[x_1, x_2, \ldots, x_n],
$$

where $\min P$ and $\max P$ are (respectively) the minimum and the maximum of $P$, where $\rho$ is the rank function of $P$ (normalized to satisfy $\rho(\min P) = 0$), and where $n$ is the rank of $\max P$. (Note that the indeterminate $x_0$ does not actually appear in the polynomial.)

For technical reasons, the polynomial is returned in the slightly larger ring $\mathbb{Z}[x_0, x_1, x_2, \ldots, x_{n+1}]$ by this method.


EXAMPLES:

```python
sage: P = posets.DiamondPoset(5)
sage: P.flag_f_polynomial()
3*x1*x2 + x2

sage: P = Poset({1: [2, 3], 2: [4], 3: [5], 4: [6], 5: [6]})
```
fl = P.flag_f_polynomial(); fl
2*x1*x2*x3 + 2*x1*x3 + 2*x2*x3 + x3

q = polygen(ZZ, 'q')
fl(q,q,q,q) == P.f_polynomial()
True

P = Poset({1: [2, 3, 4], 2: [5], 3: [5], 4: [5], 5: [6]})
P.flag_f_polynomial()
3*x1*x2*x3 + 3*x1*x3 + x2*x3 + x3

See also:

is_bounded(), flag_h_polynomial()

flag_h_polynomial()

Return the flag ℎ--polynomial of the poset.

The poset is expected to be bounded and ranked.

If $P$ is a bounded ranked poset whose maximal element has rank $n$ (where the minimal element is set to have rank 0), then the flag $h$-polynomial of $P$ is defined as the polynomial

$$
\prod_{k=1}^{n} (1 - x_k) \cdot f\left(\frac{x_1}{1 - x_1}, \frac{x_2}{1 - x_2}, \cdots, \frac{x_n}{1 - x_n}\right) \in \mathbb{Z}[x_1, x_2, \cdots, x_n],
$$

where $f$ is the flag $f$-polynomial of $P$ (see flag_f_polynomial()).

For technical reasons, the polynomial is returned in the slightly larger ring $\mathbb{Q}[x_0, x_1, x_2, \cdots, x_{n+1}]$ by this method.

See Wikipedia article h-vector.

EXAMPLES:

sage: P = posets.DiamondPoset(5)
sage: P.flag_h_polynomial()
2*x1*x2 + x2

sage: P = Poset({1: [2, 3], 2: [4], 3: [5], 4: [6], 5: [6]})
sage: fl = P.flag_h_polynomial(); fl
-x1*x2*x3 + x1*x3 + x2*x3 + x3
sage: q = polygen(ZZ, 'q')
sage: fl(q,q,q,q) == P.h_polynomial()
True

sage: P = Poset({1: [2, 3, 4], 2: [5], 3: [5], 4: [5], 5: [6]})
sage: P.flag_h_polynomial()
2*x1*x3 + x3

sage: P = posets.ChainPoset(4)
sage: P.flag_h_polynomial()
x3

See also:

is_bounded(), flag_f_polynomial()
frank_network()  
Return Frank’s network of the poset.  
This is defined in Section 8 of [BF1999].  

OUTPUT: 
A pair \((G, e)\), where \(G\) is Frank’s network of \(P\) encoded as a DiGraph, and \(e\) is the cost function on its edges encoded as a dictionary (indexed by these edges, which in turn are encoded as tuples of 2 vertices).

**Note:** Frank’s network of \(P\) is a certain directed graph with \(2|P| + 2\) vertices, defined in Section 8 of [BF1999]. Its set of vertices consists of two vertices \((-1, 0)\) and \((1, p)\) for each element \(p\) of \(P\), as well as two vertices \((-1, 0)\) and \((2, 0)\). (These notations are not the ones used in [BF1999]; see the table below for their relation.) The edges are:

- for each \(p\) in \(P\), an edge from \((-1, 0)\) to \((0, p)\);
- for each \(p\) in \(P\), an edge from \((1, p)\) to \((2, 0)\);
- for each \(p\) and \(q\) in \(P\) such that \(p \geq q\), an edge from \((0, p)\) to \((1, q)\).

We make this digraph into a network in the sense of flow theory as follows: The vertex \((-1, 0)\) is considered as the source of this network, and the vertex \((2, 0)\) as the sink. The cost function is defined to be 1 on the edge from \((0, p)\) to \((1, p)\) for each \(p \in P\), and to be 0 on every other edge. The capacity is 1 on each edge. Here is how to translate this notations into that used in [BF1999]:

<table>
<thead>
<tr>
<th>our notations</th>
<th>[BF1999]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1, 0)</td>
<td>s</td>
</tr>
<tr>
<td>(0, p)</td>
<td>x_p</td>
</tr>
<tr>
<td>(1, p)</td>
<td>y_p</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>t</td>
</tr>
<tr>
<td>a[e]</td>
<td>a(e)</td>
</tr>
</tbody>
</table>

**EXAMPLES:**

```python
sage: ps = [[16, 12, 14, -13], [12, 14], [14, -13], [12, 16], [16, -13]]
sage: G, e = Poset(ps).frank_network()
sage: G.edges(sort=True)
[((-1, 0), (0, -13), None), ((-1, 0), (0, 12), None), ((-1, 0), (0, 14), None),
((-1, 0), (0, 16), None), ((0, -13), (1, -13), None), ((0, -13), (1, 12), None),
((0, -13), (1, 14), None), ((0, -13), (1, 16), None), ((0, 12), (1, -13), None),
((0, 12), (1, 14), None), ((0, 12), (1, 16), None), ((0, 14), (1, 12), None),
((0, 14), (1, 14), None), ((0, 14), (1, 16), None), ((0, 16), (1, -13), None),
((0, 16), (1, 12), None), ((0, 16), (1, 14), None), ((0, 16), (1, 16), None),
((1, -13), (2, 0), None), ((1, 12), (2, -13), None), ((1, 12), (2, 14), None),
((1, 12), (2, 16), None), ((1, 14), (2, 0), None), ((1, 14), (2, 12), None),
((1, 14), (2, 14), None), ((1, 14), (2, 16), None), ((1, 16), (2, 0), None),
((1, 16), (2, 12), None), ((1, 16), (2, 14), None), ((1, 16), (2, 16), None)]
sage: e
{((-1, 0), (0, -13)): 0,
((-1, 0), (0, 12)): 0,
((-1, 0), (0, 14)): 0,
((-1, 0), (0, 16)): 0,
((0, -13), (1, 12)): 1,
((0, -13), (1, 14)): 0,
((0, -13), (1, 16)): 0,
((0, 12), (1, 12)): 1,
((0, 14), (1, 12)): 1,
((0, 14), (1, 14)): 1,
((0, 14), (1, 16)): 0,
((1, 12), (2, 0)): 0,
((1, 12), (2, 12)): 0,
((1, 12), (2, 14)): 0,
((1, 12), (2, 16)): 0,
((1, 14), (2, 0)): 0,
((1, 14), (2, 12)): 0,
((1, 14), (2, 14)): 0,
((1, 14), (2, 16)): 0,
((1, 16), (2, 0)): 0,
((1, 16), (2, 12)): 0,
((1, 16), (2, 14)): 0,
((1, 16), (2, 16)): 0}
```
Author:

• Darij Grinberg (2013-05-09)

ge(x, y)

Return True if x is greater than or equal to y in the poset, and False otherwise.

Examples:
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4[:]}).

sage: P.is_gequal(3, 1)
True
sage: P.is_gequal(2, 2)
True
sage: P.is_gequal(0, 1)
False

See also:
is_greater_than(), is_lequal().

graphviz_string(graph_string='graph', edge_string='--')

Return a representation in the DOT language, ready to render in graphviz.


EXAMPLES:
sage: P = Poset({'a':['b','d'], 'c':['d'], 'd':['f'], 'e':['f'], 'f':[]})

sage: print(P.graphviz_string())
graph {
  "f"="d";"b"="a";"c"="e";
  "f"--"e";"d"--"c";"b"--"a";"d"--"b";"f"--"d";
}

greene_shape()

Return the Greene-Kleitman partition of self.

The Greene-Kleitman partition of a finite poset $P$ is the partition $(c_1 - c_0, c_2 - c_1, c_3 - c_2, \ldots)$, where $c_k$ is the maximum cardinality of a union of $k$ chains of $P$. Equivalently, this is the conjugate of the partition $(a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots)$, where $a_k$ is the maximum cardinality of a union of $k$ antichains of $P$.

See many sources, e. g., [BF1999], for proofs of this equivalence.

EXAMPLES:
sage: P = Poset([[3,2,1],[[3,1],[2,1]]])

sage: P.greene_shape()
[2, 1]

sage: P = Poset([[1,2,3,4],[[1,4],[2,4],[4,3]]])

sage: P.greene_shape()
[3, 1]

sage: P = Poset([[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22],[[1, 2, 3, 4],[2,4],[4,3]]])

sage: P.greene_shape()
[3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

sage: P = Poset([[],[[]]])

sage: P.greene_shape()
[]

AUTHOR:
  • Darij Grinberg (2013-05-09)

gt(x, y)

Return True if $x$ is greater than but not equal to $y$ in the poset, and False otherwise.
EXAMPLES:

```
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
sage: P.is_greater_than(3, 1)
True
sage: P.is_greater_than(1, 2)
False
sage: P.is_greater_than(3, 3)
False
sage: P.is_greater_than(0, 1)
False
```

For non-facade posets also > works:

```
sage: P = Poset({3: [1, 2]}, facade=False)
sage: P(2) > P(3)
True
```

See also:

`is_gequal()`, `is_less_than()`.

**h_polynomial()**

Return the $h$-polynomial of a bounded poset `self`.

This is the $h$-polynomial of the order complex of the poset minus its bounds.

This is related to the $f$-polynomial by a simple change of variables:

$$h(q) = (1 - q)^{\deg f} f \left( \frac{q}{1 - q} \right),$$

where $f$ and $h$ denote the $f$-polynomial and the $h$-polynomial, respectively.

See Wikipedia article h-vector.

**Warning:** This is slightly different from the `hPolynomial` method in Macaulay2.

EXAMPLES:

```
sage: P = posets.AntichainPoset(3).order_ideals_lattice()
sage: P.h_polynomial()
q^3 + 4*q^2 + q
sage: P = posets.DiamondPoset(5)
sage: P.h_polynomial()
2*q^2 + q
sage: P = Poset({1: []})
sage: P.h_polynomial()
1
```

See also:

`is_bounded()`, `f_polynomial()`, `order_complex()`, `sage.topology.simplicial_complex.SimplicialComplex.h_vector()`
has_bottom()  
Return True if the poset has a unique minimal element, and False otherwise.

EXAMPLES:

```python
sage: P = Poset({0:[3], 1:[3], 2:[3], 3:[4], 4:[]})
sage: P.has_bottom()
False
sage: Q = Poset({0:[1], 1:[]})
sage: Q.has_bottom()
True
```

See also:

- Dual Property: has_top()
- Stronger properties: is_bounded()
- Other: bottom()

has_isomorphic_subposet(other)

Return True if the poset contains a subposet isomorphic to other.

By subposet we mean that there exist a set X of elements such that self.subposet(X) is isomorphic to other.

INPUT:

- other – a finite poset

EXAMPLES:

```python
sage: D = Poset({1:[2,3], 2:[4], 3:[4]})
sage: T = Poset({1:[2,3], 2:[4,5], 3:[6,7]})
sage: N5 = posets.PentagonPoset()

sage: N5.has_isomorphic_subposet(T)
False
sage: N5.has_isomorphic_subposet(D)
True

sage: len([P for P in Posets(5) if P.has_isomorphic_subposet(D)])
11
```

has_top()  
Return True if the poset has a unique maximal element, and False otherwise.

EXAMPLES:

```python
sage: P = Poset({0:[3], 1:[3], 2:[3], 3:[4, 5], 4:[]})
sage: P.has_top()
False
sage: Q = Poset({0:[3], 1:[3], 2:[3], 3:[4], 4:[]})
sage: Q.has_top()
True
```

See also:
• Dual Property: `has_bottom()`
• Stronger properties: `is_bounded()`
• Other: `top()`

`hasse_diagram()`

Return the Hasse diagram of the poset as a Sage `DiGraph`.

The Hasse diagram is a directed graph where vertices are the elements of the poset and there is an edge from `u` to `v` whenever `v` covers `u` in the poset.

If `dot2tex` is installed, then this sets the Hasse diagram’s latex options to use the `dot2tex` formatting.

**EXAMPLES:**

```python
sage: P = posets.DivisorLattice(12)
sage: H = P.hasse_diagram(); H
Digraph on 6 vertices
sage: P.cover_relations()
[[1, 2], [1, 3], [2, 4], [2, 6], [3, 6], [4, 12], [6, 12]]
sage: H.edges(sort=True, labels=False)
[(1, 2), (1, 3), (2, 4), (2, 6), (3, 6), (4, 12), (6, 12)]
```

`height(certificate=False)`

Return the height (number of elements in a longest chain) of the poset.

**INPUT:**

• `certificate` – (default: `False`) whether to return a certificate

**OUTPUT:**

• If `certificate=True` return `(h, c)`, where `h` is the height and `c` is a chain of maximum cardinality.
  If `certificate=False` return only the height.

**EXAMPLES:**

```python
sage: P = Poset({0: [1], 2: [3, 4], 4: [5, 6]})
sage: P.height()
3
sage: posets.PentagonPoset().height(certificate=True)
(4, [0, 2, 3, 4])
```

`incidence_algebra(R, prefix='I')`

Return the incidence algebra of `self` over `R`.

**OUTPUT:**


**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(4)
sage: P.incidence_algebra(QQ)
Incidence algebra of Finite lattice containing 16 elements over Rational Field
```
incomparability_graph()

Return the incomparability graph of the poset.

This is the complement of the comparability graph, i.e. an undirected graph where vertices are the elements of the poset and there is an edge between vertices if they are not comparable in the poset.

EXAMPLES:

```python
sage: Y = Poset({1: [2], 2: [3, 4]})
sage: g = Y.incomparability_graph(); g
Incomparability graph on 4 vertices
sage: Y.compare_elements(1, 3) is not None
True
sage: g.has_edge(1, 3)
False
```

See also:

comparability_graph()

interval(x, y)

Return a list of the elements \(z\) such that \(x \leq z \leq y\).

INPUT:

- \(x\) – any element of the poset
- \(y\) – any element of the poset

EXAMPLES:

```python
sage: uc = [[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[]]
sage: dag = DiGraph(dict(zip(range(len(uc)),uc)))
sage: P = Poset(dag)
sage: I = set(map(P,[2,5,6,4,7]))
sage: I == set(P.interval(2,7))
True
sage: dg = DiGraph({"a":["b","c"], "b":["d"], "c":["d"]})
sage: P = Poset(dg, facade = False)
sage: P.interval("a","d")
[a, b, c, d]
```

intervals_number()

Return the number of relations in the poset.

A relation is a pair of elements \(x\) and \(y\) such that \(x \leq y\) in the poset.

Relations are also often called intervals. The number of intervals is the dimension of the incidence algebra.

EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: P.relations_number()
13
```

```python
sage: posets.TamariLattice(4).relations_number()
68
```
See also:

relations_iterator(), relations()

intervals_poset()

Return the natural partial order on the set of intervals of the poset.

OUTPUT:

a finite poset

The poset of intervals of a poset \( P \) has the set of intervals \([x, y] \) in \( P \) as elements, endowed with the order relation defined by \([x_1, y_1] \leq [x_2, y_2] \) if and only if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \).

This is also called \( P \) to the power 2, meaning the poset of poset-morphisms from the 2-chain to \( P \).

If \( P \) is a lattice, the result is also a lattice.

EXAMPLES:

```
sage: P = Poset({0:[1]})
sage: P.intervals_poset()
Finite poset containing 3 elements
```

```
sage: P = posets.PentagonPoset()
sage: P.intervals_poset()
Finite lattice containing 13 elements
```

is_EL_labelling(\( f \), return_raising_chains=False)

Return True if \( f \) is an EL labelling of self.

A labelling \( f \) of the edges of the Hasse diagram of a poset is called an EL labelling (edge lexicographic labelling) if for any two elements \( u \) and \( v \) with \( u \leq v \),

- there is a unique \( f \)-raising chain from \( u \) to \( v \) in the Hasse diagram, and this chain is lexicographically first among all chains from \( u \) to \( v \).

For more details, see [Bj1980].

INPUT:

- \( f \) – a function taking two elements \( a \) and \( b \) in self such that \( b \) covers \( a \) and returning elements in a totally ordered set.
- return_raising_chains (optional; default:False) if True, returns the set of all raising chains in self, if possible.

EXAMPLES:

Let us consider a Boolean poset:

```
sage: P = Poset([[0,0],[0,1],[1,0],[1,1]],facade=True)
sage: label = lambda a,b: min( i for i in [0,1] if a[i] != b[i] )
sage: P.is_EL_labelling(label)
True
```

```
sage: P.is_EL_labelling(label,return_raising_chains=True)
{((0, 0), (0, 1)): [1],
 ((0, 0), (1, 0)): [0],
 ((0, 0), (1, 1)): [0, 1],
 ((0, 1), (1, 0)): [0],
 ((0, 1), (1, 1)): [1]}
```
**is_antichain_of_poset**(*elms*)

Return True if *elms* is an antichain of the poset and False otherwise.

Set of elements are an antichain of a poset if they are pairwise incomparable.

**EXAMPLES:**
```
sage: P = posets.BooleanLattice(5)
sage: P.is_antichain_of_poset([3, 5, 7])
False
sage: P.is_antichain_of_poset([3, 5, 14])
True
```

**is_bounded()**

Return True if the poset is bounded, and False otherwise.

A poset is bounded if it contains both a unique maximal element and a unique minimal element.

**EXAMPLES:**
```
sage: P = Poset({0:[3], 1:[3], 2:[3], 3:[4, 5], 4:[], 5:[]})
sage: P.is_bounded()
False
sage: Q = posets.DiamondPoset(5)
sage: Q.is_bounded()
True
```

See also:

- Weaker properties: `has_bottom()`, `has_top()`
- Other: `with_bounds()`, `without_bounds()`

**is_chain()**

Return True if the poset is totally ordered (“chain”), and False otherwise.

**EXAMPLES:**
```
sage: I = Poset({0:[1], 1:[2], 2:[3], 3:[4]})
sage: I.is_chain()
True
sage: II = Poset({0:[1], 2:[3]})
sage: II.is_chain()
False
sage: V = Poset({0:[1, 2]})
sage: V.is_chain()
False
```

**is_chain_of_poset**(*elms, ordered=False*)

Return True if *elms* is a chain of the poset, and False otherwise.

Set of elements are a chain of a poset if they are comparable to each other.

**INPUT:**
- *elms* – a list or other iterable containing some elements of the poset
• ordered – a Boolean. If True, then return True only if elements in elms are strictly increasing in the poset; this makes no sense if elms is a set. If False (the default), then elements can be repeated and be in any order.

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")))
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_chain_of_poset([12, 3])
True
sage: P.is_chain_of_poset([3, 4, 12])
False
sage: P.is_chain_of_poset([12, 3], ordered=True)
False
sage: P.is_chain_of_poset((1, 1, 3))
True
sage: P.is_chain_of_poset((1, 1, 3), ordered=True)
False
sage: P.is_chain_of_poset((1, 3), ordered=True)
True
```

is_connected()
Return True if the poset is connected, and False otherwise.

A poset is connected if its Hasse diagram is connected.

If a poset is not connected, then it can be divided to parts \( S_1 \) and \( S_2 \) so that every element of \( S_1 \) is incomparable to every element of \( S_2 \).

EXAMPLES:

```
sage: P = Poset({1:[2, 3], 3:[4, 5]})
sage: P.is_connected()
True
sage: P = Poset({1:[2, 3], 3:[4, 5], 6:[7, 8]})
sage: P.is_connected()
False
```

See also:

connected_components()

is_d_complete()
Return True if a poset is d-complete and False otherwise.

See also:

• d_complete

EXAMPLES:

```
sage: from sage.combinat.posets.posets import FinitePoset
sage: A = Poset({0: [1,2]})
sage: A.is_d_complete()
False
```
sage: from sage.combinat.posets.poset_examples import Posets
sage: B = Posets.DoubleTailedDiamond(3)

sage: B.is_d_complete()
True

sage: C = Poset({0: [2], 1: [2], 2: [3, 4], 3: [5], 4: [5], 5: [6]})

sage: C.is_d_complete()
False

sage: D = Poset({0: [1, 2], 1: [4], 2: [4], 3: [4]})

sage: D.is_d_complete()
False

sage: P = Posets.YoungDiagramPoset(Partition([3, 2, 2]), dual=True)

sage: P.is_d_complete()
True

is_eulerian($k=$None, $certificate=$False)

Return True if the poset is Eulerian, and False otherwise.

The poset is expected to be graded and bounded.

A poset is Eulerian if every non-trivial interval has the same number of elements of even rank as of odd rank. A poset is $k$-eulerian if every non-trivial interval up to rank $k$ is Eulerian.

See Wikipedia article Eulerian_poset.

INPUT:

• $k$, an integer – only check if the poset is $k$-eulerian. If None (the default), check if the poset is Eulerian.

• $certificate$, a Boolean – (default: False) whether to return a certificate

OUTPUT:

• If $certificate=True$ return either True, None or False, $(a, b)$, where the interval $(a, b)$ is not Eulerian. If $certificate=False$ return True or False.

EXAMPLES:

sage: P = Poset({0: [1, 2, 3], 1: [4, 5, 6], 2: [4, 6], 3: [5, 6],
.....: 4: [7, 8], 5: [7, 8], 6: [7, 8], 7: [9], 8: [9]})

sage: P.is_eulerian()
True

sage: P = Poset({0: [1, 2, 3], 1: [4, 5, 6], 2: [4, 6], 3: [5, 6],
.....: 4: [7], 5:[7], 6:[7]})

sage: P.is_eulerian()
False

Canonical examples of Eulerian posets are the face lattices of convex polytopes:

sage: P = polytopes.cube().face_lattice()

sage: P.is_eulerian()
True

A poset that is 3- but not 4-eulerian:
Combinatorics, Release 10.0

```python
sage: P = Poset(DiGraph('MWW@_?W?@_?W??@O@_?W?@_?W?@??O??')); P
Finite poset containing 14 elements
sage: P.is_eulerian(k=3)
True
sage: P.is_eulerian(k=4)
False
```

Getting an interval that is not Eulerian:

```python
sage: P = posets.DivisorLattice(12)
sage: P.is_eulerian(certificate=True)
(False, (1, 4))
```

**is_gequal(x, y)**

Return True if \( x \) is greater than or equal to \( y \) in the poset, and False otherwise.

**EXAMPLES:**

```python
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
sage: P.is_gequal(3, 1)
True
sage: P.is_gequal(2, 2)
True
sage: P.is_gequal(0, 1)
False
```

See also:

`is_greater_than()`, `is_lequal()`.

**is_graded()**

Return True if the poset is graded, and False otherwise.

A poset is graded if all its maximal chains have the same length.

There are various competing definitions for graded posets (see Wikipedia article Graded_poset). This definition is from section 3.1 of Richard Stanley’s *Enumerative Combinatorics, Vol. 1* [EnumComb1]. Some sources call these posets tiered.

Every graded poset is ranked. The converse is true for bounded posets, including lattices.

**EXAMPLES:**

```python
sage: P = posets.PentagonPoset()  # Not even ranked
sage: P.is_graded()
False
sage: P = Poset({1:[2, 3], 3:[4]})  # Ranked, but not graded
sage: P.is_graded()
False
sage: P = Poset({1:[3, 4], 2:[3, 4], 5:[6]})
```

(continues on next page)
is_greater_than\(x, y\)

Return True if \(x\) is greater than but not equal to \(y\) in the poset, and False otherwise.

EXAMPLES:

```python
sage: P = Poset({0: [2], 1: [2], 2: [3], 3: [4], 4: []})
sage: P.is_greater_than(3, 1)
True
sage: P.is_greater_than(1, 2)
False
sage: P.is_greater_than(3, 3)
False
sage: P.is_greater_than(0, 1)
False
```

For non-facade posets also \(>\) works:

```python
sage: P = Poset({3: [1, 2]}, facade=False)
sage: P(2) > P(3)
True
```

See also:

is_gequal(), is_less_than().

is_greedy\(certificate=False\)

Return True if the poset is greedy, and False otherwise.

A poset is greedy if every greedy linear extension has the same number of jumps.

INPUT:

- certificate – (default: False) whether to return a certificate

OUTPUT:

- If certificate=True return either (True, None) or (False, (A, B)) where \(A\) and \(B\) are greedy linear extension so that \(B\) has more jumps. If certificate=False return True or False.

EXAMPLES:

This is not a self-dual property:

```python
sage: W = Poset({1: [3, 4], 2: [4, 5]})
sage: M = W.dual()
sage: W.is_greedy()
True
sage: M.is_greedy()
False
```

Getting a certificate:
sage: N = Poset({1: [3], 2: [3, 4]})
sage: N.is_greedy(certificate=True)
(False, ([1, 2, 4, 3], [2, 4, 1, 3]))

is_incomparable_chain_free\((m, n=None)\)
Return True if the poset is \((m + n)\)-free, and False otherwise.
A poset is \((m + n)\)-free if there is no incomparable chains of lengths \(m\) and \(n\). Three cases have special name (see [EnumComb1], exercise 3.15):

• “interval order” is \((2 + 2)\)-free
• “semiorder” (or “unit interval order”) is \((1 + 3)\)-free and \((2 + 2)\)-free
• “weak order” is \((1 + 2)\)-free.

INPUT:
• \(m, n\) - positive integers

It is also possible to give a list of integer pairs as argument. See below for an example.

EXAMPLES:

sage: B3 = posets.BooleanLattice(3)
sage: B3.is_incomparable_chain_free(1, 3)
True
sage: B3.is_incomparable_chain_free(2, 2)
False
sage: IP6 = posets.IntegerPartitions(6)
sage: IP6.is_incomparable_chain_free(1, 3)
False
sage: IP6.is_incomparable_chain_free(2, 2)
True

A list of pairs as an argument:

sage: B3.is_incomparable_chain_free([[1, 3], [2, 2]])
False

We show how to get an incomparable chain pair:

sage: P = posets.PentagonPoset()
sage: chains_1_2 = Poset({0: [], 1: [2]})
sage: incomps = P.isomorphic_subposets(chains_1_2)[0]
sage: sorted(incomps.list()), incomps.cover_relations()
([1, 2, 3], [[2, 3]])

AUTHOR:
• Eric Rowland (2013-05-28)

is_induced_subposet\((other)\)
Return True if the poset is an induced subposet of \(other\), and False otherwise.
A poset \(P\) is an induced subposet of \(Q\) if every element of \(P\) is an element of \(Q\), and \(x \leq_P y\) iff \(x \leq_Q y\).
Note that “induced” here has somewhat different meaning compared to that of graphs.

INPUT:
• other, a poset.

**Note:** This method does not check whether the poset is a isomorphic (i.e., up to relabeling) subposet of other, but only if other directly contains the poset as an induced subposet. For isomorphic subposets see has_isomorphic_subposet().

**EXAMPLES:**

```python
sage: P = Poset({1:[2, 3]})
sage: Q = Poset({1:[2, 4], 2:[3]})
sage: P.is_induced_subposet(Q)
False
sage: R = Poset({0:[1], 1:[3, 4], 3:[5], 4:[2]})
sage: P.is_induced_subposet(R)
True
```

**is_isomorphic(other, **kwds)**

Return True if both posets are isomorphic.

**EXAMPLES:**

```python
sage: P = Poset(([1,2,3],[[1,3],[2,3]]))
sage: Q = Poset(([4,5,6],[[4,6],[5,6]]))
sage: P.is_isomorphic(Q)
True
```

**is_join_semilattice(certificate=False)**

Return True if the poset has a join operation, and False otherwise.

A join is the least upper bound for given elements, if it exists.

**INPUT:**

• certificate – (default: False) whether to return a certificate

**OUTPUT:**

• If certificate=True return either (True, None) or (False, (a, b)) where elements a and b have no least upper bound. If certificate=False return True or False.

**EXAMPLES:**

```python
sage: P = Poset(([1,3,2], [4], [4,5,6], [6], [7], [7], [7], []))
sage: P.is_join_semilattice()
True
sage: P = Poset({1:[3, 4], 2:[3, 4], 3:[5], 4:[5]})
sage: P.is_join_semilattice()
False
sage: P.is_join_semilattice(certificate=True)
(False, (2, 1))
```

See also:

• Dual property: **is_meet_semilattice()**

• Stronger properties: **is_lattice()**
\textbf{is_jump_critical}(\textit{certificate=False})

Return True if the poset is jump-critical, and False otherwise.

A poset $P$ is \textit{jump-critical} if every proper subposet has smaller jump number.

INPUT:

\begin{itemize}
\item \textit{certificate} – (default: False) whether to return a certificate
\end{itemize}

OUTPUT:

\begin{itemize}
\item If \textit{certificate=True} return either (True, None) or (False, e) so that removing element $e$ from the poset does not decrease the jump number. If \textit{certificate=False} return True or False.
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: P = Poset({1: [3, 6], 2: [3, 4, 5], 4: [6, 7], 5: [7]})
sage: P.is_jump_critical()
True
sage: P = posets.PentagonPoset()
sage: P.is_jump_critical()
False
sage: P.is_jump_critical(certificate=True)
(False, 3)
\end{verbatim}

See also: 
\textit{jump_number()}

\textbf{is_lequal}(x, y)

Return True if $x$ is less than or equal to $y$ in the poset, and False otherwise.

EXAMPLES:

\begin{verbatim}
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
sage: P.is_lequal(2, 4)
True
sage: P.is_lequal(2, 2)
True
sage: P.is_lequal(0, 1)
False
sage: P.is_lequal(3, 2)
False
\end{verbatim}

See also:
\textit{is_less_than(), is_gequal().}

\textbf{is_less_than}(x, y)

Return True if $x$ is less than but not equal to $y$ in the poset, and False otherwise.

EXAMPLES:

\begin{verbatim}
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
sage: P.is_less_than(1, 3)
True
sage: P.is_less_than(0, 1)
False
\end{verbatim}
sage: P.is_less_than(2, 2)
False

For non-facade posets also < works:

sage: P = Poset({3: [1, 2]}, facade=False)
sage: P(1) < P(2)
False

See also:

\texttt{is\_lequal()}, \texttt{is\_greater\_than()}.

\textbf{is\_linear\_extension()}(l)

Return whether \texttt{l} is a linear extension of \texttt{self}.

INPUT:

\begin{itemize}
  \item \texttt{l} – a list (or iterable) containing all of the elements of \texttt{self} exactly once
\end{itemize}

EXAMPLES:

sage: P = Poset((divisors(12), attrcall("divides")), facade=True, linear_extension=True)
sage: P.list()
[1, 2, 3, 4, 6, 12]
sage: P.is_linear_extension([1, 2, 4, 3, 6, 12])
True
sage: P.is_linear_extension([1, 2, 4, 6, 3, 12])
False

sage: [p for p in Permutations(list(P)) if P.is_linear_extension(p)]
[[1, 2, 3, 4, 6, 12],
 [1, 2, 3, 6, 4, 12],
 [1, 2, 4, 3, 6, 12],
 [1, 2, 4, 6, 3, 12],
 [1, 3, 2, 4, 6, 12],
 [1, 3, 2, 6, 4, 12]]
sage: list(P.linear_extensions())
[[1, 2, 3, 4, 6, 12],
 [1, 2, 4, 3, 6, 12],
 [1, 2, 4, 6, 3, 12],
 [1, 3, 2, 4, 6, 12],
 [1, 3, 2, 6, 4, 12],
 [1, 2, 3, 6, 4, 12]]

\textbf{Note:} This is used and systematically tested in LinearExtensionsOfPosets

See also:

\texttt{linear\_extension()}, \texttt{linear\_extensions()}

\textbf{is\_linear\_interval}(x, y)

Return whether the interval \([x, y]\) is linear.

This means that this interval is a total order.

EXAMPLES:
sage: P = posets.PentagonPoset()
sage: P.is_linear_interval(0, 4)
False
sage: P.is_linear_interval(0, 3)
True
sage: P.is_linear_interval(1, 3)
False

is_meet_semilattice(certificate=False)
Return True if the poset has a meet operation, and False otherwise.
A meet is the greatest lower bound for given elements, if it exists.

INPUT:
• certificate = (default: False) whether to return a certificate

OUTPUT:
• If certificate=True return either (True, None) or (False, (a, b)) where elements a and b have no greatest lower bound. If certificate=False return True or False.

EXAMPLES:

sage: P = Poset({1:[2, 3, 4], 2:[5, 6], 3:[6], 4:[6, 7]})
sage: P.is_meet_semilattice()
True
sage: Q = P.dual()
sage: Q.is_meet_semilattice()
False
sage: V = posets.IntegerPartitions(5)
sage: V.is_meet_semilattice(certificate=True)
(False, ((2, 2, 1), (3, 1, 1)))

See also:
• Dual property: is_join_semilattice()
• Stronger properties: is_lattice()

is_parent_of(x)
Return True if x is an element of the poset.

is_rank_symmetric()
Return True if the poset is rank symmetric, and False otherwise.
The poset is expected to be graded and connected.
A poset of rank h (maximal chains have h + 1 elements) is rank symmetric if the number of elements are equal in ranks i and h − i for every i in 0, 1, . . . , h.

EXAMPLES:

sage: P = Poset({1:[3, 4, 5], 2:[3, 4, 5], 3:[5], 4:[7], 5:[7]})
sage: P.is_rank_symmetric()
True
is_ranked()  
Return True if the poset is ranked, and False otherwise.

A poset is ranked if there is a function \( r \) from poset elements to integers so that \( r(x) = r(y) + 1 \) when \( x \) covers \( y \).

Informally said a ranked poset can be “levelized”: every element is on a “level”, and every cover relation goes only one level up.

EXAMPLES:

```sage
 sage: P = Poset(([1, 2, 3, 4], [[1, 2], [2, 4], [3, 4]]))
 sage: P.is_ranked()
 True
 sage: P = Poset(([1, 5], [2, 6], [3], [4], [], [6, 3], [4]))
 sage: P.is_ranked()
 False
```

See also:

`rank_function()`, `rank()`, `is_graded()`

is_series_parallel()  
Return True if the poset is series-parallel, and False otherwise.

A poset is series-parallel if it can be built up from one-element posets using the operations of disjoint union and ordinal sum. This is also called \( N \)-free property: every poset that is not series-parallel contains a subposet isomorphic to the 4-element N-shaped poset where \( a > c, d \) and \( b > d \).

Note: Some papers use the term N-free for posets having no N-shaped poset as a cover-preserving subposet. This definition is not used here.

See Wikipedia article Series-parallel partial order.

EXAMPLES:

```sage
 sage: VA = Poset({1: [2, 3], 4: [5], 6: [5]})
 sage: VA.is_series_parallel()
 True
 sage: big_N = Poset({1: [2, 4], 2: [3], 4:[7], 5:[6], 6:[7]})
 sage: big_N.is_series_parallel()
 False
```

is_slender (certificate=False)  
Return True if the poset is slender, and False otherwise.

A finite graded poset is slender if every rank 2 interval contains three or four elements, as defined in [Stan2009]. (This notion of “slender” is unrelated to the eponymous notion defined by Graetzer and Kelly in “The Free \( m \)-Lattice on the Poset \( H \)”, Order 1 (1984), 47–65.)
This function does not check if the poset is graded or not. Instead, it just returns True if the poset does not contain 5 distinct elements \(x, y, a, b\) and \(c\) such that \(x \prec a, b, c \prec y\) where \(\prec\) is the covering relation.

**INPUT:**
- certificate – (default: False) whether to return a certificate

**OUTPUT:**
- If certificate=True return either (True, None) or (False, (a, b)) so that the interval \([a, b]\) has at least five elements. If certificate=False return True or False.

**EXAMPLES:**

```python
sage: P = Poset(([1, 2, 3, 4], [[1, 2], [1, 3], [2, 4], [3, 4]]))
sage: P.is_slender()
True
sage: P = Poset(([1, 2, 3, 4, 5], [[1, 2], [1, 3], [1, 4], [2, 5], [3, 5], [4, 5]]))
sage: P.is_slender()
False
sage: W = WeylGroup(['A', 2])
sage: G = W.bruhat_poset()
sage: G.is_slender()
True
sage: W = WeylGroup(['A', 3])
sage: G = W.bruhat_poset()
sage: G.is_slender()
True
sage: P = posets.IntegerPartitions(6)
sage: P.is_slender(certificate=True)
(False, ((6,), (3, 2, 1)))
```

**is_sperner()**

Return True if the poset is Sperner, and False otherwise.

The poset is expected to be ranked.

A poset is Sperner, if no antichain is larger than the largest rank level (one of the sets of elements of the same rank) in the poset.

See Wikipedia article Sperner_property_of_a_partially_ordered_set

See also:
- width()
- dilworth_decomposition()

**EXAMPLES:**

```python
sage: posets.SetPartitions(3).is_sperner()
True
sage: P = Poset({0:[3, 4, 5], 1:[5], 2:[5]})
sage: P.is_sperner()
False
```

**isomorphic_subposets(other)**

Return a list of subposets of self isomorphic to other.
By subposet we mean self.subposet(X) which is isomorphic to other and where X is a subset of elements of self.

INPUT:

• other – a finite poset

EXAMPLES:

```
sage: C2 = Poset({0:[1]})
sage: C3 = Poset({"a": ["b"], "b": ["c"]})
sage: L = sorted(x.cover_relations() for x in C3.isomorphic_subposets(C2))
sage: for x in L: print(x)
[['a', 'b']]
[['a', 'c']]
[['b', 'c']]
sage: D = Poset({1:[2,3], 2:[4], 3:[4]})
sage: N5 = posets.PentagonPoset()
sage: len(N5.isomorphic_subposets(D))
2
```

**Note:** If this function takes too much time, try using isomorphic_subposets_iterator().

isomorphic_subposets_iterator(other)

Return an iterator over the subposets of self isomorphic to other.

By subposet we mean self.subposet(X) which is isomorphic to other and where X is a subset of elements of self.

INPUT:

• other – a finite poset

EXAMPLES:

```
sage: D = Poset({1:[2,3], 2:[4], 3:[4]})
sage: N5 = posets.PentagonPoset()
sage: for P in N5.isomorphic_subposets_iterator(D):
....:     print(P.cover_relations())
[[0, 1], [0, 2], [1, 4], [2, 4]]
[[0, 1], [0, 3], [1, 4], [3, 4]]
[[0, 1], [0, 2], [1, 4], [2, 4]]
[[0, 1], [0, 3], [1, 4], [3, 4]]
```

**Warning:** This function will return same subposet as many times as there are automorphism on it. This is due to subgraph_search_iterator() returning labelled subgraphs. On the other hand, this function does not eat memory like isomorphic_subposets() does.

See also:

sage.combinat.posets.lattices.FiniteLatticePoset.isomorphic_sublattices_iterator().

join(x, y)

Return the join of two elements x, y in the poset if the join exists; and None otherwise.
EXAMPLES:

```
sage: D = Poset({1:[2,3], 2:[4], 3:[4]})
sage: D.join(2, 3)
4
sage: P = Poset({'e': ['b'], 'f': ['b', 'c', 'd'], 'g': ['c', 'd'],
              'b': ['a'], 'c': ['a']})
sage: P.join('a', 'b')
'a'
sage: P.join('e', 'a')
'a'
sage: P.join('c', 'b')
'a'
sage: P.join('e', 'f')
'b'
sage: P.join('e', 'g')
'a'
sage: P.join('c', 'd')
is None
True
sage: P.join('g', 'f')
is None
True
```

`jump_number(certificate=False)`

Return the jump number of the poset.

A jump in a linear extension \([e_1, \ldots, e_n]\) of a poset \(P\) is a pair \((e_i, e_{i+1})\) so that \(e_{i+1}\) does not cover \(e_i\) in \(P\). The jump number of a poset is the minimal number of jumps in linear extensions of a poset.

**INPUT:**
- `certificate` – (default: `False`) Whether to return a certificate

**OUTPUT:**
- If `certificate=True` return a pair \((n, l)\) where \(n\) is the jump number and \(l\) is a linear extension with \(n\) jumps. If `certificate=False` return only the jump number.

**EXAMPLES:**

```
sage: B3 = posets.BooleanLattice(3)
sage: B3.jump_number()
3
sage: N = Poset({1: [3, 4], 2: [3]})
sage: N.jump_number(certificate=True)
(1, [1, 4, 2, 3])
```

**ALGORITHM:**

It is known that every poset has a greedy linear extension — an extension \([e_1, e_2, \ldots, e_n]\) where every \(e_{i+1}\) is an upper cover of \(e_i\) if that is possible — with the smallest possible number of jumps; see [Sys1987].

Hence it suffices to test only those. We do that by backtracking.

The problem is proven to be NP-complete.

**See also:**

`is_jump_critical()`
The Kazhdan-Lusztig polynomial $P_{x,y}(q)$ of the poset.

The poset is expected to be ranked.

We follow the definition given in [EPW14]. Let $G$ denote a graded poset with unique minimal and maximal elements and $\chi_G$ denote the characteristic polynomial of $G$. Let $I_x$ and $F_x$ denote the principal order ideal and filter of $x$ respectively. Define the Kazhdan-Lusztig polynomial of $G$ as the unique polynomial $P_G(q)$ satisfying the following:

1. If $\text{rank } G = 0$, then $P_G(q) = 1$.
2. If $\text{rank } G > 0$, then $\deg P_G(q) < \frac{1}{2} \text{rank } G$.
3. We have
   
   $$q^{\text{rank } G} P_G(q^{-1}) = \sum_{x \in G} \chi_{I_x}(q) P_{F_x}(q).$$

We then extend this to $P_{x,y}(q)$ by considering the subposet corresponding to the (closed) interval $[x, y]$. We also define $P_0(q) = 0$ (so if $x \not\leq y$, then $P_{x,y}(q) = 0$).

INPUT:

- $q$ – (default: $q \in \mathbb{Z}[q]$) the indeterminate $q$
- $x$ – (default: the minimal element) the element $x$
- $y$ – (default: the maximal element) the element $y$
- $\text{canonical_labels}$ – (optional) for subposets, use the canonical labeling (this can limit recursive calls for posets with large amounts of symmetry, but producing the labeling takes time); if not specified, this is $\text{True}$ if $x$ and $y$ are both not specified and $\text{False}$ otherwise

EXAMPLES:

```python
sage: L = posets.BooleanLattice(3)
sage: L.kazhdan_lusztig_polynomial()
1

sage: L = posets.SymmetricGroupWeakOrderPoset(4)
sage: L.kazhdan_lusztig_polynomial()
1
sage: x = '2314'
sage: y = '3421'
sage: L.kazhdan_lusztig_polynomial(x, y)
-t + 1
sage: L.kazhdan_lusztig_polynomial(x, y, var('t'))
-t + 1
```

AUTHORS:

- Travis Scrimshaw (27-12-2014)

le($x, y$)

Return True if $x$ is less than or equal to $y$ in the poset, and False otherwise.

EXAMPLES:
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4[:]});

sage: P.is_lequal(2, 4)
True
sage: P.is_lequal(2, 2)
True
sage: P.is_lequal(0, 1)
False
sage: P.is_lequal(3, 2)
False

See also:

* is_less_than(), is_gequal().

**lequal_matrix(ring=Integer Ring, sparse=False)**

Compute the matrix whose (i,j) entry is 1 if self.linear_extension()[i] < self.linear_extension()[j] and 0 otherwise.

INPUT:

* ring – the ring of coefficients (default: ZZ)
* sparse – whether the returned matrix is sparse or not (default: True)

EXAMPLES:

sage: P = Poset([[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[]], facade = False)
sage: LEQM = P.lequal_matrix(); LEQM

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

sage: LEQM[1,3] 1
sage: P.linear_extension()[1] < P.linear_extension()[3]
True
sage: LEQM[2,5] 0
False

We now demonstrate the usage of the optional parameters:

sage: P.lequal_matrix(ring=QQ, sparse=False).parent()

Full MatrixSpace of 8 by 8 dense matrices over Rational Field

**level_sets()**

Return elements grouped by maximal number of cover relations from a minimal element.

This returns a list of lists l such that l[i] is the set of minimal elements of the poset obtained by removing the elements in l[0], l[1], ..., l[i-1]. (In particular, l[0] is the set of minimal elements of self.)

Every level is an antichain of the poset.
EXAMPLES:

```python
sage: P = Poset({0:[1,2],1:[3],2:[3],3:[]})
sage: P.level_sets()
[[0], [1, 2], [3]]
```

```python
sage: Q = Poset({0:[1,2], 1:[3], 2:[4], 3:[4]})
sage: Q.level_sets()
[[0], [1, 2], [3], [4]]
```

See also:

`dilworth_decomposition()` to return elements grouped to chains.

**lexicographic_sum**

Return the lexicographic sum using this poset as index.

In the lexicographic sum of posets $P_t$ by index poset $T$ we have $x \leq y$ if either $x \leq y$ in $P_t$ for some $t \in T$, or $x \in P_i$, $y \in P_j$ and $i \leq j$ in $T$.

Informally said we substitute every element of $T$ by corresponding poset $P_t$.

Mathematically, it is only defined when all $P_t$ have no common element; here we force that by giving them different names in the resulting poset.

`disjoint_union()` and `ordinal_sum()` are special cases of lexicographic sum where the index poset is an (anti)chain. `ordinal_product()` is a special case where every $P_t$ is same poset.

**INPUT:**

- $P$ – dictionary whose keys are elements of this poset, values are posets

**EXAMPLES:**

```python
sage: N = Poset({1: [3, 4], 2: [4]})
sage: NP = N.lexicographic_sum(P); NP
Finite poset containing 16 elements
sage: sorted(NP.minimal_elements())
[(1, 0), (2, 1), (2, 2)]
```

**linear_extension** *(linear_extension=None, check=True)*

Return a linear extension of this poset.

A linear extension of a finite poset $P$ of size $n$ is a total ordering $\pi := \pi_0 \pi_1 \ldots \pi_{n-1}$ of its elements such that $i < j$ whenever $\pi_i < \pi_j$ in the poset $P$.

**INPUT:**

- linear_extension – (default: None) a list of the elements of self
- check – a boolean (default: True); whether to check that linear_extension is indeed a linear extension of self.

**EXAMPLES:**

```python
sage: P = Poset((divisors(15), attrcall("divides")), facade=True)
```

Without optional argument, the default linear extension of the poset is returned, as a plain list:
Otherwise, a full-featured linear extension is constructed as an element of `P.linear_extensions()`:

```sage
sage: l = P.linear_extension([1,5,3,15]); l
[1, 5, 3, 15]
sage: type(l)
<class 'sage.combinat.posets.linear_extensions.LinearExtensionsOfPoset_with_category.element_class'>
sage: l.parent()
The set of all linear extensions of Finite poset containing 4 elements
```

By default, the linear extension is checked for correctness:

```sage
sage: l = P.linear_extension([1,3,15,5])
Traceback (most recent call last):
  ... ValueError: [1, 3, 15, 5] is not a linear extension of Finite poset containing 4 elements
```

This can be disabled (at your own risks!) with:

```sage
sage: P.linear_extension([1,3,15,5], check=False)
[1, 3, 15, 5]
```

See also:

- `is_linear_extension()`, `linear_extensions()`

Todo:

- Is it acceptable to have those two features for a single method?
- In particular, we miss a short idiom to get the default linear extension

**linear_extensions**(facade=False)

Return the enumerated set of all the linear extensions of this poset.

**INPUT:**

- `facade` – a boolean (default: False); whether to return the linear extensions as plain lists

**Warning:** The facade option is not yet fully functional:

```sage
sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True)
sage: L = P.linear_extensions(facade=True); L
The set of all linear extensions of Finite poset containing 6 elements
with distinguished linear extension
sage: L([1, 2, 3, 4, 6, 12])
Traceback (most recent call last):
  ... TypeError: Cannot convert list to sage.structure.element.Element
```
EXAMPLES:

```python
sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True)
sage: P.list()
[1, 2, 3, 4, 6, 12]
sage: L = P.linear_extensions(); L
The set of all linear extensions of Finite poset containing 6 elements with...
distinguished linear extension
sage: l = L.an_element(); l
[1, 2, 3, 4, 6, 12]
```

Each element is aware that it is a linear extension of $P$:

```python
sage: type(l.parent())
<class 'sage.combinat.posets.linear_extensions.LinearExtensionsOfPoset_with...
category'>
```

With `facade=True`, the elements of $L$ are plain lists instead:

```python
sage: L = P.linear_extensions(facade=True)
sage: l = L.an_element()
sage: type(l)
<class 'list'>
```

**Warning:** In Sage <= 4.8, this function used to return a plain list of lists. To recover the previous functionality, please use:

```python
sage: L = list(P.linear_extensions(facade=True)); L
[[1, 2, 3, 4, 6, 12],
 [1, 2, 4, 3, 6, 12],
 [1, 3, 2, 4, 6, 12],
 [1, 3, 2, 6, 4, 12],
 [1, 2, 3, 6, 4, 12]]
sage: type(L[0])
<class 'list'>
```

See also:

- `linear_extension()`, `is_linear_extension()`
- `linear_extensions_graph()`

*linear_extensions_graph()*

Return the linear extensions graph of the poset.

Vertices of the graph are linear extensions of the poset. Two vertices are connected by an edge if the linear extensions differ by only one adjacent transposition.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: N = Poset({1: [3, 4], 2: [4]})
sage: G = N.linear_extensions_graph(); G
Graph on 5 vertices
sage: G.neighbors(N.linear_extension([1,2,3,4]))
[[2, 1, 3, 4], [1, 3, 2, 4], [1, 2, 4, 3]]

sage: chevron = Poset({1: [2, 6], 2: [3], 4: [3, 5], 6: [5]})
sage: G = chevron.linear_extensions_graph(); G
Graph on 22 vertices
sage: G.size()
36
```

**linear_intervals_count()**

Return the enumeration of linear intervals w.r.t. their cardinality.

An interval is linear if it is a total order.

OUTPUT: list of integers

See also:

* is_linear_interval()

EXAMPLES:

```python
sage: P = posets.PentagonPoset()
sage: P.linear_intervals_count()
[5, 5, 2]
sage: P = posets.TamariLattice(4)
sage: P.linear_intervals_count()
[14, 21, 12, 2]
```

**list()**

List the elements of the poset. This just returns the result of `linear_extension()`.

EXAMPLES:

```python
sage: D = Poset({0:[1,2], 1:[3], 2:[3,4]}, facade = False)
sage: D.list()
[0, 1, 2, 3, 4]
sage: type(D.list()[0])
<class 'sage.combinat.posets.posets.FinitePoset_with_category.element_class'>
```

**lower_covers(x)**

Return the list of lower covers of the element x.

A lower cover of x is an element y such that y < x and there is no element z so that y < z < x.

EXAMPLES:

```python
sage: P = Poset([[1, 5], [2, 6], [3], [4], [], [6, 3], [4]])
sage: P.lower_covers(3)
[2, 5]
sage: P.lower_covers(0)
[]
```
See also:

upper_covers()

lower_covers_iterator(x)

Return an iterator over the lower covers of the element x.

EXAMPLES:

    sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[]})
    sage: l0 = P.lower_covers_iterator(3)
    sage: type(l0)
    <class 'generator'>
    sage: next(l0)
    2

ltr(x, y)

Return True if x is less than but not equal to y in the poset, and False otherwise.

EXAMPLES:

    sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
    sage: P.is_less_than(1, 3)
    True
    sage: P.is_less_than(0, 1)
    False
    sage: P.is_less_than(2, 2)
    False

For non-facade posets also < works:

    sage: P = Poset({3: [1, 2]}, facade=False)
    sage: P(1) < P(2)
    False

See also:

is_lequal(), is_greater_than().

magnitude()

Return the magnitude of self.

The magnitude is an integer defined as the sum of all Möbius numbers, and can be seen as some kind of Euler characteristic of the poset. It is additive under disjoint union and multiplicative under Cartesian product.

REFERENCES:

- https://golem.ph.utexas.edu/category/2011/06/the_magnitude_of_an_enriched_c.html

EXAMPLES:

    sage: P = posets.PentagonPoset()
    sage: P.magnitude()
    1

(continues on next page)
sage: W = SymmetricGroup(4)
sage: P = W.noncrossing_partition_lattice().without_bounds()
sage: P.magnitude()
-4

sage: P = posets.TamariLattice(4).without_bounds()
sage: P.magnitude()
0

See also:

order_complex()

maximal_antichains()
Return the maximal antichains of the poset.
An antichain \( a \) of poset \( P \) is maximal if there is no element \( e \in P \setminus a \) such that \( a \cup \{e\} \) is an antichain.

EXAMPLES:

```
sage: P = Poset({'a': ['b', 'c'], 'b': ['d', 'e']})
sage: [sorted(anti) for anti in P.maximal_antichains()]
[["a"], ["b", "c"], ["c", "d", "e"]]
sage: posets.PentagonPoset().maximal_antichains()
[[0], [1, 2], [1, 3], [4]]
```

See also:

antichains(), maximal_chains()

maximal_chain_length()
Return the maximum length of a maximal chain in the poset.
The length here is the number of vertices.

EXAMPLES:

```
sage: P = posets.TamariLattice(5)
sage: P.maximal_chain_length()
11
```

See also:

maximal_chains(), maximal_chains_iterator()

maximal_chains(partial=None)
Return all maximal chains of this poset.
Each chain is listed in increasing order.

INPUT:

- partial – list (optional); if given, the list partial is assumed to be the start of a maximal chain, and the function will find all maximal chains starting with the elements in partial

This is used in constructing the order complex for the poset.

EXAMPLES:
```python
sage: P = posets.BooleanLattice(3)
sage: P.maximal_chains()
[[0, 1, 3, 7], [0, 1, 5, 7], [0, 2, 3, 7], [0, 2, 6, 7], [0, 4, 5, 7], [0, 4, 6, 7]]
sage: P.maximal_chains(partial=[0,2])
[[0, 2, 3, 7], [0, 2, 6, 7]]
sage: Q = posets.ChainPoset(6)
sage: Q.maximal_chains()
[[0, 1, 2, 3, 4, 5]]
```

See also:

- `maximal_antichains()`, `chains()`

**maximal_chains_iterator**(partial=None)

Return an iterator over maximal chains.

Each chain is listed in increasing order.

**INPUT:**

- **partial** – list (optional); if given, the list `partial` is assumed to be the start of a maximal chain, and the function will yield all maximal chains starting with the elements in `partial`.

**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(3)
sage: it = P.maximal_chains_iterator()
sage: next(it)
[0, 1, 3, 7]
```

See also:

- antichains_iterator()

**maximal_elements()**

Return the list of the maximal elements of the poset.

**EXAMPLES:**

```python
sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[]})
sage: P.maximal_elements()
[4]
```

See also:

- minimal_elements()

**meet**(x, y)

Return the meet of two elements `x, y` in the poset if the meet exists; and `None` otherwise.

**EXAMPLES:**

```python
sage: D = Poset({1:[2,3], 2:[4], 3:[4]})
sage: D.meet(2, 3)
1
sage: P = Poset({'a':['b', 'c'], 'b':['e', 'f'], 'c':['f', 'g'],
.....: 'd':['f', 'g']})
```

(continues on next page)
\begin{verbatim}
mutable_elements()
Return the list of the minimal elements of the poset.

EXAMPLES:

sage: P = Poset({0:[3],1:[3],2:[3],3:[4],4:[4]})
sage: P(0) in P.mutable_elements()
True
sage: P(1) in P.mutable_elements()
True
sage: P(2) in P.mutable_elements()
True

See also:
\emph{maximal_elements()}.

moebius_function(x, y)
Return the value of the Möbius function of the poset on the elements x and y.

EXAMPLES:

sage: P = Poset([[1,2,3],[4],[4],[4],[4]])
sage: P.moebius_function(0,4)
2
sage: sum(P.moebius_function(u,v) for v in P)
0
sage: sum(abs(P.moebius_function(u,v)) for v in P)
6
sage: for u,v in P.cover_relations_iterator:
....:   if P.moebius_function(u,v) != -1:
....:       print("Bug in moebius_function!")

sage: Q = Poset([[1,2,3],[4],[4,5,6],[6],[7],[7],[7],[7],[7],[7]])
sage: Q.moebius_function(0,4)
0
sage: Q.moebius_function(0,5)
0
\end{verbatim}
\begin{verbatim}
sage: Q.moebius_function(Q(2),Q(7))
2
sage: Q.moebius_function(Q(3),Q(3))
1
sage: sum([Q.moebius_function(Q(0),v) for v in Q])
0
\end{verbatim}

\textbf{moebius_function_matrix}(\texttt{ring=\textit{Integer Ring}, \texttt{sparse=False}})

Return a matrix whose \((i,j)\) entry is the value of the Möbius function evaluated at \texttt{self.linear_extension()[i]} and \texttt{self.linear_extension()[j]}.

INPUT:

- \texttt{ring} – the ring of coefficients (default: \texttt{ZZ})
- \texttt{sparse} – whether the returned matrix is sparse or not (default: \texttt{True})

EXAMPLES:

\begin{verbatim}
sage: P = Poset([[4,2,3],[],[1],[1],[1]])
sage: x,y = (P.linear_extension()[0],P.linear_extension()[1])
sage: P.moebius_function(x,y)
-1
sage: M = P.moebius_function_matrix(); M
\[
\begin{bmatrix}
1 & -1 & -1 & -1 & 2 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
sage: M[0,4]
2
sage: M[0,1]
-1
\end{verbatim}

We now demonstrate the usage of the optional parameters:

\begin{verbatim}
sage: P.moebius_function_matrix(ring=QQ, sparse=False).parent()
Full MatrixSpace of 5 by 5 dense matrices over Rational Field
\end{verbatim}

\textbf{open_interval}(x,y)

Return the list of elements \(z\) such that \(x < z < y\) in the poset.

EXAMPLES:

\begin{verbatim}
sage: P = Poset((divisors(1000), attrcall("divides")))
sage: P.open_interval(2, 100)
[4, 10, 20, 50]
\end{verbatim}

See also:

\textit{closed_interval()}

\textbf{order_complex}(\texttt{on_ints=False})

Return the order complex associated to this poset.

The order complex is the simplicial complex with vertices equal to the elements of the poset, and faces given by the chains.
INPUT:

- on_ints – a boolean (default: False)

OUTPUT:
an order complex of type SimplicialComplex

EXAMPLES:

```python
sage: P = posets.BooleanLattice(3)
sage: S = P.order_complex(); S
Simplicial complex with vertex set (0, 1, 2, 3, 4, 5, 6, 7) and 6 facets
sage: S.f_vector()
[1, 8, 19, 18, 6]
sage: S.homology()
{0: 0, 1: 0, 2: 0, 3: 0}
sage: Q = P.subposet([1,2,3,4,5,6])
sage: Q.order_complex().homology()
{0: 0, 1: \mathbb{Z}}
```

If on_ints, then the elements of the poset are labelled 0, 1,… in the chain complex:

```python
sage: P.order_complex(on_ints=True)
Simplicial complex with vertex set (0, 1, 2, 3) and facets {(0, 1, 3), (0, 2, 3)}
```

**order_filter(elements)**

Return the order filter generated by the elements of an iterable elements.

$I$ is an order filter if, for any $x$ in $I$ and $y$ such that $y \geq x$, then $y$ is in $I$. This is also called upper set or upset.

EXAMPLES:

```python
sage: P = Poset((divisors(15), attrcall("divides")), facade = True)
sage: P.order_complex()
Simplicial complex with vertex set (1, 3, 5, 15) and facets {{(1, 3, 15), (1, 5, 15)}}
```

**order_ideal(elements)**

Return the order ideal generated by the elements of an iterable elements.

$I$ is an order ideal if, for any $x$ in $I$ and $y$ such that $y \leq x$, then $y$ is in $I$. This is also called lower set or downset.

EXAMPLES:

```python
sage: P = Poset((divisors(1000), attrcall("divides")))
sage: P.order_ideal([20, 25])
[1, 2, 4, 5, 10, 20, 25]
```

See also:

`order_ideal()`, `principal_order_filter()`.

**order_ideal(elements)**

Return the order ideal generated by the elements of an iterable elements.

$I$ is an order ideal if, for any $x$ in $I$ and $y$ such that $y \leq x$, then $y$ is in $I$. This is also called lower set or downset.

EXAMPLES:

```python
sage: P = Poset((divisors(1000), attrcall("divides")))
sage: P.order_ideal([20, 25])
[1, 2, 4, 5, 10, 20, 25]
```
See also:

`order_filter()`, `principal_order_ideal()`.

**order_ideal_cardinality(elements)**

Return the cardinality of the order ideal generated by elements.

The elements \( I \) is an order ideal if, for any \( x \in I \) and \( y \) such that \( y \leq x \), then \( y \in I \).

**EXAMPLES:**

```python
sage: P = posets.BooleanLattice(4)
sage: P.order_ideal_cardinality([7,10])
10
```

**order_ideal_plot(elements)**

Return a plot of the order ideal generated by the elements of an iterable elements.

\( I \) is an order ideal if, for any \( x \) in \( I \) and \( y \) such that \( y \leq x \), then \( y \) is in \( I \). This is also called lower set or downset.

**EXAMPLES:**

```python
sage: P = Poset((divisors(1000), attrcall("divides"))
sage: P.order_ideal_plot([20, 25])
```

Graphics object consisting of 41 graphics primitives

**order_polynomial()**

Return the order polynomial of the poset.

The order polynomial \( \Omega_P(q) \) of a poset \( P \) is defined as the unique polynomial \( S \) such that for each integer \( m \geq 1 \), \( S(m) \) is the number of order-preserving maps from \( P \) to \( \{1, \ldots, m\} \).

See sections 3.12 and 3.15 of [EnumComb1], and also [St1986].

**EXAMPLES:**

```python
sage: P = posets.AntichainPoset(3)
sage: P.order_polynomial()
q^3
sage: P = posets.ChainPoset(3)
sage: f = P.order_polynomial(); f
1/6*q^3 + 1/2*q^2 + 1/3*q
sage: [f(i) for i in range(4)]
[0, 1, 4, 10]
```

See also:

`order_polytope()`

**order_polytope()**

Return the order polytope of the poset self.

The order polytope of a finite poset \( P \) is defined as the subset of \( \mathbb{R}^P \) consisting of all maps \( x : P \to \mathbb{R} \) satisfying

\[
0 \leq x(p) \leq 1 \text{ for all } p \in P,
\]
and

\[ x(p) \leq x(q) \text{ for all } p, q \in P \text{ satisfying } p < q. \]

This polytope was defined and studied in [St1986].

EXAMPLES:

```
sage: P = posets.AntichainPoset(3)
sage: Q = P.order_polytope();Q
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: P = posets.PentagonPoset()
sage: Q = P.order_polytope();Q
A 5-dimensional polyhedron in ZZ^5 defined as the convex hull of 8 vertices
sage: P = Poset([[1,2,3],[[1,2],[1,3]]])
sage: Q = P.order_polytope()
sage: Q.contains((1,0,0))
False
sage: Q.contains((0,1,1))
True
```

**ordinal_product**(other, labels='pairs')

Return the ordinal product of self and other.

The ordinal product of two posets \( P \) and \( Q \) is a partial order on the Cartesian product of the underlying sets of \( P \) and \( Q \), defined as follows (see [EnumComb1], p. 284).

In the ordinal product, \((p, q) \leq (p', q')\) if either \( p \leq p' \) or \( p = p' \) and \( q \leq q' \).

This construction is not symmetric in \( P \) and \( Q \). Informally said we put a copy of \( Q \) in place of every element of \( P \).

INPUT:

- **other** – a poset
- **labels** – either 'integers' or 'pairs' (default); how the resulting poset will be labeled

EXAMPLES:

```
sage: P1 = Poset((['a', 'b'], [['a', 'b']]))
sage: P2 = Poset((['c', 'd'], [['c', 'd']]))
sage: P = P1.ordinal_product(P2); P
Finite poset containing 4 elements
sage: sorted(P.cover_relations())
[[('a', 'c'), ('a', 'd'), ('b', 'c')],
 [('b', 'c'), ('b', 'd')]]
```

See also:

- `product()`, `ordinal_sum()`

**ordinal_sum**(other, labels='pairs')

Return a poset or (semi)lattice isomorphic to ordinal sum of the poset with other.

The ordinal sum of \( P \) and \( Q \) is a poset that contains every element and relation from both \( P \) and \( Q \), and where every element of \( P \) is smaller than any element of \( Q \).

Mathematically, it is only defined when \( P \) and \( Q \) have no common element; here we force that by giving them different names in the resulting poset.
The ordinal sum on lattices is a lattice; resp. for meet- and join-semilattices.

INPUT:
- `other`, a poset.
- `labels` - (defaults to `pairs`) If set to `pairs`, each element v in this poset will be named (0,v) and each element u in `other` will be named (1,u) in the result. If set to `integers`, the elements of the result will be relabeled with consecutive integers.

EXAMPLES:

```python
sage: P1 = Poset( ([1, 2, 3, 4], [[1, 2], [1, 3], [1, 4]])
sage: P2 = Poset( ([1, 2, 3, ], [[2,1], [3,1]])
sage: P3 = P1.ordinal_sum(P2); P3
Finite poset containing 7 elements
sage: len(P1.maximal_elements())*len(P2.minimal_elements())
6
sage: len(P1.cover_relations()+P2.cover_relations())
5
sage: len(P3.cover_relations()) # Every element of P2 is greater than elements
11
  of P1.
sage: P3.list() # random
[(0, 1), (0, 2), (0, 4), (0, 3), (1, 2), (1, 3), (1, 1)]
sage: P4 = P1.ordinal_sum(P2, labels='integers')
sage: P4.list() # random
[0, 1, 2, 3, 5, 6, 4]
```

Return type depends on input types:

```python
sage: P = Poset({1:[2]}); P
Finite poset containing 2 elements
sage: JL = JoinSemilattice({1:[2]}); JL
Finite join-semilattice containing 2 elements
sage: L = LatticePoset({1:[2]}); L
Finite lattice containing 2 elements
sage: P.ordinal_sum(L)
Finite poset containing 4 elements
sage: L.ordinal_sum(JL)
Finite join-semilattice containing 4 elements
sage: L.ordinal_sum(L)
Finite lattice containing 4 elements
```

See also:

`ordinal_summands()`, `disjoint_union()`, `sage.combinat.posets.lattices.FiniteLatticePoset.vertical_composition()`

`ordinal_summands()`

Return the ordinal summands of the poset as subposets.

The ordinal summands of a poset $P$ is the longest list of non-empty subposets $P_1, \ldots, P_n$ whose ordinal sum is $P$. This decomposition is unique.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: P = Poset({'a': ['c', 'd'], 'b': ['d'], 'c': ['x', 'y'],
             'd': ['x', 'y']})
sage: parts = P.ordinal_summands(); parts
[Finite poset containing 4 elements, Finite poset containing 2 elements]
sage: sorted(parts[0])
['a', 'b', 'c', 'd']
sage: Q = parts[0].ordinal_sum(parts[1])
sage: Q.is_isomorphic(P)
True
```

See also:

`ordinal_sum()`

**ALGORITHM:**

Suppose that a poset $P$ is the ordinal sum of posets $L$ and $U$. Then $P$ contains maximal antichains $l$ and $u$ such that every element of $u$ covers every element of $l$; they correspond to maximal elements of $L$ and minimal elements of $U$.

We consider a linear extension $x_1, \ldots, x_n$ of the poset’s elements.

We keep track of the maximal elements of subposet induced by elements $0, \ldots, x_i$ and minimal elements of subposet induced by elements $x_{i+1}, \ldots, x_n$, incrementing $i$ one by one. We then check if $l$ and $u$ fit the previous description.

**p_partition Enumerator**(tup, R, weights=None, check=False)

Return a $P$-partition enumerator of self.

Given a total order $\prec$ on the elements of a finite poset $P$ (the order of $P$ and the total order $\prec$ can be unrelated; in particular, the latter does not have to extend the former), a $P$-partition enumerator is the quasisymmetric function $\sum_f \prod_{p \in P} x_{f(p)}$, where the first sum is taken over all $P$-partitions $f$.

A $P$-partition is a function $f: P \to \{1, 2, 3, \ldots\}$ satisfying the following properties for any two elements $i$ and $j$ of $P$ satisfying $i \prec_P j$:

- if $i \prec j$ then $f(i) \leq f(j)$,
- if $j \prec i$ then $f(i) < f(j)$.

The optional argument `weights` allows constructing a generalized (“weighted”) version of the $P$-partition enumerator. Namely, `weights` should be a dictionary whose keys are the elements of $P$. Then, the generalized $P$-partition enumerator corresponding to weights `weights` is $\sum_f \prod_{p \in P} x_{w(p)}$, where the sum is again over all $P$-partitions $f$. Here, $w(p)$ is `weights[p]`. The classical $P$-partition enumerator is the particular case obtained when all $p$ satisfy $w(p) = 1$.

In the language of [Grinb2016a], the generalized $P$-partition enumerator is the quasisymmetric function $\Gamma(E, w)$, where $E$ is the special double poset $(P, <_P, \prec)$, and where $w$ is the dictionary `weights` (regarded as a function from $P$ to the positive integers).

**INPUT:**

- `tup` – the tuple containing all elements of $P$ (each of them exactly once), in the order dictated by the total order $\prec$
- `R` – a commutative ring
- `weights` – (optional) a dictionary of positive integers, indexed by elements of $P$; any missing item will be understood as 1
OUTPUT:

The $P$-partition enumerator of `self` according to `tup` in the algebra $QSym$ of quasisymmetric functions over the base ring $R$.

EXAMPLES:

```python
sage: P = Poset([[1,2,3,4],[[1,4],[2,4],[4,3]]])
sage: FP = P.p_partition_enumerator((3,1,2,4), QQ, check=True); FP
sage: expansion = FP.expand(5)
sage: xs = expansion.parent().gens()
sage: expansion == sum([xs[a]*xs[b]*xs[c]*xs[d] for a in range(5) for b in range(5) for c in range(5) for d in range(5) if a <= b and c <= b and b < d])
True
sage: P = Poset([[],[]])
sage: FP = P.p_partition_enumerator((), QQ, check=True); FP
M[]
```

With the `weights` parameter:

```python
sage: P = Poset([[1,2,3,4],[[1,4],[2,4],[4,3]]])
sage: FP = P.p_partition_enumerator((3,1,2,4), QQ, weights={1: 1, 2: 2, 3: 1, 4: 1}, check=True); FP
sage: FP = P.p_partition_enumerator((3,1,2,4), QQ, weights={2: 2}, check=True); FP
sage: P = Poset([['a','b','c'], [['a','b'], ['a','c']]])
sage: FP = P.p_partition_enumerator(('b','c','a'), QQ, weights={'a': 3, 'b': 5, 'c': 7}, check=True); FP
sage: P = Poset([['a','b','c'], [['a','c'], ['b','c']]])
sage: FP = P.p_partition_enumerator(('b','c','a'), QQ, weights={'a': 3, 'b': 5, 'c': 7}, check=True); FP
sage: FP = P.p_partition_enumerator(('a','b','c'), QQ, weights={'a': 3, 'b': 5, 'c': 7}, check=True); FP
```

```python
plot(label_elements=True, element_labels=None, layout='acyclic', cover_labels=None, **kwds)
```

Return a Graphic object for the Hasse diagram of the poset.

If the poset is ranked, the plot uses the rank function for the heights of the elements.

INPUT:

- Options to change element look:
  - `element_colors` - a dictionary where keys are colors and values are lists of elements
  - `element_color` - a color for elements not set in `element_colors`
- **element_shape** - the shape of elements, like 's' for square; see https://matplotlib.org/api/markers_api.html for the list
- **element_size** (default: 200) - the size of elements
- **label_elements** (default: True) - whether to display element labels
- **element_labels** (default: None) - a dictionary where keys are elements and values are labels to show

**Options to change cover relation look:**
- **cover_colors** - a dictionary where keys are colors and values are lists of cover relations given as pairs of elements
- **cover_color** - a color for elements not set in **cover_colors**
- **cover_style** - style for cover relations: 'solid', 'dashed', 'dotted' or 'dashdot'
- **cover_labels** - a dictionary, list or function representing labels of the covers of the poset. When set to None (default) no label is displayed on the edges of the Hasse Diagram.
- **cover_labels_background** - a background color for cover relations. The default is “white”. To achieve a transparent background use “transparent”.

**Options to change overall look:**
- **figsize** (default: 8) - size of the whole plot
- **title** - a title for the plot
- **fontsize** - fontsize for the title
- **border** (default: False) - whether to draw a border over the plot

**Note:** All options of **GenericGraph.plot** are also available through this function.

**EXAMPLES:**

This function can be used without any parameters:

```
sage: D12 = posets.DivisorLattice(12)
sage: D12.plot()
Graphics object consisting of 14 graphics primitives
```

Just the abstract form of the poset; examples of relabeling:

```
sage: D12.plot(label_elements=False)
Graphics object consisting of 8 graphics primitives
sage: d = {1: 0, 2: 'a', 3: 'b', 4: 'c', 6: 'd', 12: 1}
sage: D12.plot(element_labels=d)
Graphics object consisting of 14 graphics primitives
sage: d = {i: str(factor(i)) for i in D12}
sage: D12.plot(element_labels=d)
Graphics object consisting of 14 graphics primitives
```

Some settings for coverings:

```
sage: d = {(a, b): b/a for a, b in D12.cover_relations()}
sage: D12.plot(cover_labels=d, cover_color='gray', cover_style='dotted')
Graphics object consisting of 21 graphics primitives
```
To emphasize some elements and show some options:

```python
sage: L = LatticePoset({0: [1, 2, 3, 4], 1: [12], 2: [6, 7], ....: 3: [5, 9], 4: [5, 6, 10, 11], 5: [13], ....: 6: [12], 7: [12, 8, 9], 8: [13], 9: [13], ....: 10: [12], 11: [12], 12: [13]})
sage: F = L.frattini_sublattice()
sage: F_internal = [c for c in F.cover_relations() if c in L.cover_relations()]
sage: L.plot(figsize=12, border=True, element_shape='s', ....: element_size=400, element_color='white', ....: element_colors={'blue': F, 'green': L.double_irreducibles()}, ....: cover_color='lightgray', cover_colors={'black': F_internal}, ....: title='The Frattini\nsublattice in blue', fontsize=10)
```

**product(\text{other})**

Return the Cartesian product of the poset with \text{other}.

The Cartesian (or ‘direct’) product of \(P\) and \(Q\) is defined by \((p, q) \leq (p', q')\) iff \(p \leq p'\) in \(P\) and \(q \leq q'\) in \(Q\).

Product of (semi)lattices are returned as a (semi)lattice.

**EXAMPLES:**

```python
sage: P = posets.ChainPoset(3)
sage: Q = posets.ChainPoset(4)
sage: PQ = P.product(Q) ; PQ
Finite lattice containing 12 elements
sage: len(PQ.cover_relations())
17
sage: Q.product(P).is_isomorphic(PQ)
True

sage: P = posets.BooleanLattice(2)
sage: Q = P.product(P)
sage: Q.is_isomorphic(posets.BooleanLattice(4))
True
```

One can also simply use *:

```python
sage: P = posets.ChainPoset(2)
sage: Q = posets.ChainPoset(3)
sage: P*Q
Finite lattice containing 6 elements
```

**See also:**

`CartesianProductPoset`, `factor()`

**promotion(\text{i=1})**

Compute the (extended) promotion on the linear extension of the poset \text{self}.

**INPUT:**

- \(i\) – an integer between 1 and \(n\) (default: 1)

**OUTPUT:**
• an isomorphic poset, with the same default linear extension

The extended promotion is defined on a poset self of size \( n \) by applying the promotion operator \( \tau_i \tau_{i+1} \cdots \tau_{n-1} \) to the default linear extension \( \pi \) of self (see \texttt{promotion()})\), and relabeling self accordingly. For more details see [Stan2009].

When the elements of the poset self are labelled by \( \{1, 2, \ldots, n\} \), the linear extension is the identity, and \( i = 1 \), the above algorithm corresponds to the promotion operator on posets defined by Schützenberger as follows. Remove 1 from self and replace it by the minimum \( j \) of all labels covering 1 in the poset. Then, remove \( j \) and replace it by the minimum of all labels covering \( j \), and so on. This process ends when a label is a local maximum. Place the label \( n + 1 \) at this vertex. Finally, decrease all labels by 1.

EXAMPLES:

```sage
sage: P = Poset(([1,2], [[1,2]]), linear_extension=True, facade=False)
sage: P.promotion()
Finite poset containing 2 elements with distinguished linear extension
sage: P == P.promotion()
True

sage: P = Poset(([1,2,3,4,5,6,7], [[1,2],[1,4],[2,3],[2,5],[3,6],[4,7],[5,6]]))
sage: P.list()
[1, 2, 3, 5, 6, 4, 7]
sage: Q = P.promotion(4); Q
Finite poset containing 7 elements with distinguished linear extension
sage: Q.cover_relations()
[[1, 2], [1, 6], [2, 3], [2, 5], [3, 7], [5, 7], [6, 4]]
```

Note that if one wants to obtain the promotion defined by Schützenberger’s algorithm directly on the poset, one needs to make sure the linear extension is the identity:

```sage
sage: P = P.with_linear_extension([1,2,3,4,5,6,7])
sage: P.list()
[1, 2, 3, 4, 5, 6, 7]
sage: Q = P.promotion(4); Q
Finite poset containing 7 elements with distinguished linear extension
sage: Q.cover_relations()
[[1, 2], [1, 3], [2, 4], [2, 5], [3, 6], [4, 7], [5, 7]]
```

Here is an example for a poset not labelled by \( \{1, 2, \ldots, n\} \):

```sage
sage: P = Poset((divisors(30), attrcall("divides")), linear_extension=True)
sage: P.list()
[1, 2, 3, 5, 6, 10, 15, 30]
sage: P.cover_relations()
[[1, 2], [1, 3], [1, 5], [2, 6], [2, 10], [3, 6], [3, 15],
  [5, 10], [5, 15], [6, 30], [10, 30], [15, 30]]
sage: Q = P.promotion(4); Q
Finite poset containing 8 elements with distinguished linear extension
sage: Q.cover_relations()
[[1, 2], [1, 3], [1, 6], [2, 5], [2, 15], [3, 5], [3, 10],
  [5, 30], [6, 10], [6, 15], [10, 30], [15, 30]]
```
random_linear_extension()

Return a random linear extension of the poset.

The distribution is not uniform.

EXAMPLES:

```sage
sage: set_random_seed(0)  # results are reproduceable
sage: P = posets.BooleanLattice(4)
sage: P.random_linear_extension()
[0, 2, 8, 1, 9, 4, 5, 10, 6, 12, 14, 13, 3, 7, 11, 15]
```

random_maximal_antichain()

Return a random maximal antichain of the poset.

The distribution is not uniform.

EXAMPLES:

```sage
sage: set_random_seed(0)  # results are reproduceable
sage: P = posets.BooleanLattice(4)
sage: P.random_maximal_antichain()
[1, 8, 2, 4]
```

random_maximal_chain()

Return a random maximal chain of the poset.

The distribution is not uniform.

EXAMPLES:

```sage
sage: set_random_seed(0)  # results are reproduceable
sage: P = posets.BooleanLattice(4)
sage: P.random_maximal_chain()
[0, 2, 10, 11, 15]
```

random_order_ideal(direction='down')

Return a random order ideal with uniform probability.

INPUT:

- direction – 'up', 'down' or 'antichain' (default: 'down')

OUTPUT:

A randomly selected order ideal (or order filter if direction='up', or antichain if direction='antichain') where all order ideals have equal probability of occurring.
ALGORITHM:

Uses the coupling from the past algorithm described in [Propp1997].

EXAMPLES:

```python
sage: P = posets.BooleanLattice(3)
sage: P.random_order_ideal()  # random
[0, 1, 2, 3, 4, 5, 6]
sage: P.random_order_ideal(direction='up')  # random
[6, 7]
sage: P.random_order_ideal(direction='antichain')  # random
[1, 2]

sage: P = posets.TamariLattice(5)
sage: a = P.random_order_ideal('antichain')
sage: P.is_antichain_of_poset(a)
True
sage: a = P.random_order_ideal('up')
sage: P.is_order_filter(a)
True
sage: a = P.random_order_ideal('down')
sage: P.is_order_ideal(a)
True
```

random_subposet(p)

Return a random subposet that contains each element with probability \( p \).

EXAMPLES:

```python
sage: P = posets.BooleanLattice(3)
sage: set_random_seed(0)  # Results are reproducible
sage: Q = P.random_subposet(0.5)
sage: Q.cover_relations()
[[0, 2], [0, 5], [2, 3], [3, 7], [5, 7]]
```

rank(element=None)

Return the rank of an element \( element \) in the poset \( self \), or the rank of the poset if \( element \) is None.

(The rank of a poset is the length of the longest chain of elements of the poset. This is sometimes called the length of a poset.)

EXAMPLES:

```python
sage: P = Poset([[1,3,2],[4],[4,5,6],[6],[7],[7],[7],[]], facade=False)
sage: P.rank(5)
2
sage: P.rank()
3
sage: Q = Poset([[1,2],[3],[4]])
sage: P = posets.SymmetricGroupBruhatOrderPoset(4)
sage: [(v,P.rank(v)) for v in P]
[('1234', 0), ('1243', 1), ...
```

(continues on next page)
rank_function()
Return the (normalized) rank function of the poset, if it exists.

A rank function of a poset \( P \) is a function \( r \) that maps elements of \( P \) to integers and satisfies: \( r(x) = r(y) + 1 \) if \( x \) covers \( y \). The function \( r \) is normalized such that its minimum value on every connected component of the Hasse diagram of \( P \) is 0. This determines the function \( r \) uniquely (when it exists).

OUTPUT:
- a lambda function, if the poset admits a rank function
- None, if the poset does not admit a rank function

EXAMPLES:
```python
sage: P = Poset(((1,2,3,4),[[1,4],[2,3],[3,4]]), facade=True)
sage: P.rank_function() is not None
True
sage: P = Poset(((1,2,3,4,5),[[1,2],[2,3],[3,4],[1,5],[5,4]]), facade=True)
sage: P.rank_function() is not None
False
sage: P = Poset(((1,2,3,4,5,6,7,8),[[1,4],[2,3],[3,4],[5,7],[6,7]]), facade=True)
sage: f = P.rank_function(); f is not None
True
sage: f(5)
0
sage: f(2)
0
```

rees_product(other)
Return the Rees product of self and other.

This is only defined if both posets are graded.

The underlying set is the set of pairs \((p, q)\) in the Cartesian product such that \(\text{rk}(p) \geq \text{rk}(q)\).

This operation was defined by Björner and Welker in [BjWe2005]. Other references are [MBRe2011] and [LSW2012].

EXAMPLES:
```python
sage: B3 = posets.BooleanLattice(3)
sage: B3t = B3.subposet(list(range(1,8)))
sage: C3 = posets.ChainPoset(3)
sage: D = B3t.rees_product(C3); D
Finite poset containing 12 elements
sage: sorted(D.minimal_elements())
[(1, 0), (2, 0), (4, 0)]
sage: sorted(D.maximal_elements())
[(7, 0), (7, 1), (7, 2)]
sage: D.with_bounds().moebius_function('bottom','top')
2
```
See also:

*product*, *ordinal_product*, *star_product*

**relabel**(relabeling=None)

Return a copy of this poset with its elements relabeled.

**INPUT:**

- relabeling - a function, dictionary, list or tuple

The given function or dictionary must map each (non-wrapped) element of `self` to some distinct object.
The given list or tuple must be made of distinct objects.

When the input is a list or a tuple, the relabeling uses the total ordering of the elements of the poset given by `list(self)`.

If no relabeling is given, the poset is relabeled by integers from 0 to \(n - 1\) according to one of its linear extensions. This means that \(i < j\) as integers whenever \(i < j\) in the relabeled poset.

**EXAMPLES:**

Relabeling using a function:

```sage
sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True)
sage: P.list()
[1, 2, 3, 4, 6, 12]
sage: P.cover_relations()
[[[1, 2], [1, 3], [2, 4], [2, 6], [3, 6], [4, 12], [6, 12]]
sage: Q = P.relabel(lambda x: x+1)
sage: Q.list()
[2, 3, 4, 5, 7, 13]
sage: Q.cover_relations()
[[2, 3], [2, 4], [3, 5], [3, 7], [4, 7], [5, 13], [7, 13]]
```

Relabeling using a dictionary:

```sage
sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True, facade=False)
sage: relabeling = {c.element:i for (i,c) in enumerate(P)}
sage: relabeling
{1: 0, 2: 1, 3: 2, 4: 3, 6: 4, 12: 5}
sage: Q = P.relabel(relabeling)
sage: Q.list()
[0, 1, 2, 3, 4]
sage: Q.cover_relations()
[[[0, 1], [0, 2], [1, 3], [1, 4], [2, 4], [3, 5], [4, 5]]
```

Mind the `c.element`; this is because the relabeling is applied to the elements of the poset without the wrapping. Thanks to this convention, the same relabeling function can be used both for facade or non-facade posets.

Relabeling using a list:

```sage
sage: P = posets.PentagonPoset()
sage: list(P)
[0, 1, 2, 3, 4]
sage: P.cover_relations()
```

(continues on next page)
\[
[[0, 1], [0, 2], [1, 4], [2, 3], [3, 4]]
\]

```
sage: Q = P.relabel(list('abcde'))
sage: Q.cover_relations()
[[['a', 'b'], ['a', 'c'], ['b', 'e'], ['c', 'd'], ['d', 'e']]]
```

Default behaviour is increasing relabeling:

```
sage: a2 = posets.ChainPoset(2)
sage: P = a2 * a2
sage: Q = P.relabel()
sage: Q.cover_relations()
[[0, 1], [0, 2], [1, 3], [2, 3]]
```

Relabeling a (semi)lattice gives a (semi)lattice:

```
sage: P = JoinSemilattice({0: [1]})
sage: P.relabel(lambda n: n+1)
Finite join-semilattice containing 2 elements
```

**Note:** As can be seen in the above examples, the default linear extension of \(Q\) is that of \(P\) after relabeling. In particular, \(P\) and \(Q\) share the same internal Hasse diagram.

**relations()**

Return the list of all relations of the poset.

A relation is a pair of elements \(x\) and \(y\) such that \(x \leq y\) in the poset.

The number of relations is the dimension of the incidence algebra.

**OUTPUT:**

A list of pairs (each pair is a list), where the first element of the pair is less than or equal to the second element.

**EXAMPLES:**

```
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4[:]})
sage: P.relations()
[[[1, 1], [1, 2], [1, 3], [1, 4], [0, 0], [0, 2], [0, 3], [0, 4], [2, 2], [2, 3], [2, 4], [3, 3], [3, 4], [4, 4]]]
```

**See also:**

- `relations_number()
- `relations_iterator()`

**AUTHOR:**

- Rob Beezer (2011-05-04)

**relations_iterator**(strict=False)

Return an iterator for all the relations of the poset.

A relation is a pair of elements \(x\) and \(y\) such that \(x \leq y\) in the poset.

**INPUT:**
• **strict** – a boolean (default False) if True, returns an iterator over relations \(x < y\), excluding all relations \(x \leq x\).

**OUTPUT:**
A generator that produces pairs (each pair is a list), where the first element of the pair is less than or equal to the second element.

**EXAMPLES:**

```python
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[4], 4:[]})
sage: it = P.relations_iterator()
sage: type(it)
<class 'generator'>
sage: next(it), next(it)
([1, 1], [1, 2])
sage: P = posets.PentagonPoset()
sage: list(P.relations_iterator(strict=True))
[[0, 1], [0, 2], [0, 4], [0, 3], [1, 4], [2, 3], [2, 4], [3, 4]]
```

See also:
`relations_number()`, `relations()`.

**AUTHOR:**
• Rob Beezer (2011-05-04)

### relations_number()
Return the number of relations in the poset.
A relation is a pair of elements \(x\) and \(y\) such that \(x \leq y\) in the poset.
Relations are also often called intervals. The number of intervals is the dimension of the incidence algebra.

**EXAMPLES:**

```python
sage: P = posets.PentagonPoset()
sage: P.relations_number()
13
sage: posets.TamariLattice(4).relations_number()
68
```

See also:
`relations_iterator()`, `relations()`

### show(label_elements=True, element_labels=None, cover_labels=None, **kwds)`
Displays the Hasse diagram of the poset.

**INPUT:**

• **label_elements** (default: True) - whether to display element labels
• **element_labels** (default: None) - a dictionary of element labels
• **cover_labels** - a dictionary, list or function representing labels of the covers of self. When set to None (default) no label is displayed on the edges of the Hasse Diagram.
Note: This method also accepts:

- All options of `GenericGraph.plot`
- All options of `Graphics.show`

EXAMPLES:

```python
sage: D = Poset({ 0:[1,2], 1:[3], 2:[3,4] })
sage: D.plot(label_elements=False)
Graphics object consisting of 6 graphics primitives
sage: D.show()
sage: elm_labs = {0:'a', 1:'b', 2:'c', 3:'d', 4:'e'}
sage: D.show(element_labels=elm_labs)
```

One more example with cover labels:

```python
sage: P = posets.PentagonPoset()
sage: P.show(cover_labels=lambda a, b: a - b)
```

\[ \text{slant\_sum}(p, \text{element}, p\_element) \]

Return the slant sum poset of posets `self` and `p` by connecting them with a cover relation \((p\_element, \text{element})\).

Note: The element names of `self` and `p` must be distinct.

INPUT:

- `p` – the poset used for the slant sum
- `element` – the element of `self` that is the top of the new cover relation
- `p_element` – the element of `p` that is the bottom of the new cover relation

EXAMPLES:

```python
sage: R = posets.RibbonPoset(5, [1,2])
sage: H = Poset([[5, 6, 7], [(5, 6), (6,7)]])
sage: SS = R.slant_sum(H, 3, 7)
sage: all(cr in SS.cover_relations() for cr in R.cover_relations())
True
sage: all(cr in SS.cover_relations() for cr in H.cover_relations())
True
sage: SS.covers(7, 3)
True
```

\[ \text{sorted}(l, allow\_incomparable=True, remove\_duplicates=False) \]

Return the list `l` sorted by the poset.

INPUT:

- `l` – a list of elements of the poset
- `allow_incomparable` – a Boolean. If `True` (the default), return incomparable elements in some order; if `False`, raise an error if `l` is not a chain of the poset.
- `remove_duplicates` - a Boolean. If `True`, remove duplicates from the output list.
EXAMPLES:

```python
sage: P = posets.DivisorLattice(36)
sage: P.sorted([1, 4, 1, 6, 2, 12])  # Random order for 4 and 6
[1, 1, 2, 4, 6, 12]
sage: P.sorted([1, 4, 1, 6, 2, 12], remove_duplicates=True)
[1, 2, 4, 6, 12]
sage: P.sorted([1, 4, 1, 6, 2, 12], allow_incomparable=False)
Traceback (most recent call last):
...
ValueError: the list contains incomparable elements
```

```python
sage: P = Poset({7:[1, 5], 1:[2, 6], 5:[3], 6:[3, 4]})
sage: P.sorted([4, 1, 4, 5, 7])  # Random order for 1 and 5
[7, 1, 5, 4, 4]
sage: P.sorted([1, 4, 4, 7], remove_duplicates=True)
[7, 1, 4]
sage: P.sorted([4, 1, 4, 5, 7], allow_incomparable=False)
Traceback (most recent call last):
...
ValueError: the list contains incomparable elements
```

`spectrum(a)`

Return the $a$-spectrum of this poset.

The $a$-spectrum in a poset $P$ is the list of integers whose $i$-th position contains the number of linear extensions of $P$ that have $a$ in the $i$-th location.

INPUT:

- $a$ – an element of this poset

OUTPUT:

The $a$-spectrum of this poset, returned as a list.

EXAMPLES:

```python
sage: P = posets.ChainPoset(5)
sage: P.spectrum(2)
[0, 0, 1, 0, 0]
sage: P = posets.BooleanLattice(3)
sage: P.spectrum(5)
[0, 0, 0, 4, 12, 16, 16, 0]
sage: P = posets.YoungDiagramPoset(Partition([3,2,1]))
sage: P.spectrum((0,1))
[0, 8, 6, 2, 0, 0]
sage: P = posets.AntichainPoset(4)
sage: P.spectrum(3)
[6, 6, 6, 6]
```

`star_product(other, labels='pairs')`

Return a poset isomorphic to the star product of the poset with other.
Both this poset and other are expected to be bounded and have at least two elements.

Let $P$ be a poset with top element $\top_P$ and $Q$ be a poset with bottom element $\bot_Q$. The star product of $P$ and $Q$ is the ordinal sum of $P \setminus \top_P$ and $Q \setminus \bot_Q$.

Mathematically, it is only defined when $P$ and $Q$ have no common elements; here we force that by giving them different names in the resulting poset.

INPUT:

* other – a poset.

* labels – (defaults to ‘pairs’) If set to ‘pairs’, each element $v$ in this poset will be named $(0, v)$ and each element $u$ in other will be named $(1, u)$ in the result. If set to ‘integers’, the elements of the result will be relabeled with consecutive integers.

EXAMPLES:

This is mostly used to combine two Eulerian posets to third one, and makes sense for graded posets only:

```
sage: B2 = posets.BooleanLattice(2)
sage: B3 = posets.BooleanLattice(3)
sage: P = B2.star_product(B3); P
Finite poset containing 10 elements
sage: P.is_eulerian()
True
```

We can get elements as pairs or as integers:

```
sage: ABC = Poset({'a': ['b'], 'b': ['c']})
sage: XYZ = Poset({'x': ['y'], 'y': ['z']})
sage: ABC.star_product(XYZ).list()
[(0, 'a'), (0, 'b'), (1, 'y'), (1, 'z')]
sage: sorted(ABC.star_product(XYZ, labels='integers'))
[0, 1, 2, 3]
```

The `subposet(elements)`

Return the poset containing given elements with partial order induced by this poset.

EXAMPLES:

```
sage: P = Poset({'
'    'a': ['c', 'd'], 'b': ['d', 'e'], 'c': ['f'],
'    ....:
'    'd': ['f'], 'e': ['f']})
sage: Q = P.subposet(['a', 'b', 'f']); Q
Finite poset containing 3 elements
sage: Q.cover_relations()
[['b', 'f'], ['a', 'f']]
```

A subposet of a non-facade poset is again a non-facade poset:

```
sage: P = posets.PentagonPoset(facade=False)
sage: Q = P.subposet([0, 1, 2, 4])
sage: Q(1) < Q(2)
False
```

The `top()`

Return the unique maximal element of the poset, if it exists.

EXAMPLES:
sage: P = Poset({0:[3], 1:[3], 2:[3], 3:[4,5], 4:[], 5:[]})
sage: P.top() is None
True
sage: Q = Poset({0:[1], 1:[]})
sage: Q.top()
1

See also:
has_top(), bottom()

unwrap(element)

Return the element element of the poset self in unwrapped form.

INPUT:

• element – an element of self

EXAMPLES:

sage: P = Poset((divisors(15), attrcall("divides")), facade = False)
sage: x = P.an_element(); x
1
sage: x.parent()
Finite poset containing 4 elements
sage: P.unwrap(x)
1
sage: P.unwrap(x).parent()
Integer Ring

For a non facade poset, this is equivalent to using the .element attribute:

sage: P.unwrap(x) is x.element
True

For a facade poset, this does nothing:

sage: P = Poset((divisors(15), attrcall("divides")), facade=True)
sage: x = P.an_element()
sage: P.unwrap(x) is x
True

This method is useful in code where we do not know if P is a facade poset or not.

upper_covers(x)

Return the list of upper covers of the element x.

An upper cover of x is an element y such that x < y and there is no element z so that x < z < y.

EXAMPLES:

sage: P = Poset([[1,5], [2,6], [3], [4], [], [6,3], [4]])
sage: P.upper_covers(1)
[2, 6]

See also:
lower_covers()
upper_covers_iterator(x)

Return an iterator over the upper covers of the element x.

EXAMPLES:

```
sage: P = Poset({0:[2], 1:[2], 2:[3], 3:[]})
sage: type(P.upper_covers_iterator(0))
<class 'generator'>
```

width(certificate=False)

Return the width of the poset (the size of its longest antichain).

It is computed through a matching in a bipartite graph; see Wikipedia article Dilworth's_theorem for more information. The width is also called Dilworth number.

INPUT:

- certificate – (default: False) whether to return a certificate

OUTPUT:

- If certificate=True return (w, a), where w is the width of a poset and a is an antichain of maximum cardinality. If certificate=False return only width of the poset.

EXAMPLES:

```
sage: P = posets.BooleanLattice(4)
sage: P.width()
6
sage: w, max_achain = P.width(certificate=True)
sage: sorted(max_achain)
[3, 5, 6, 9, 10, 12]
```

with_bounds(labels=('bottom', 'top'))

Return the poset with bottom and top elements adjoined.

This function adds top and bottom elements to the poset. It will always add elements, it does not check if the poset already has a bottom or a top element.

For lattices and semilattices this function returns a lattice.

INPUT:

- labels – A pair of elements to use as a bottom and top element of the poset. Default is strings 'bottom' and 'top'. Either of them can be None, and then a new bottom or top element will not be added.

EXAMPLES:

```
sage: V = Poset({0: [1, 2]})
sage: trafficsign = V.with_bounds(); trafficsign
Finite poset containing 5 elements
sage: trafficsign.list()
['bottom', 0, 1, 2, 'top']
sage: trafficsign = V.with_bounds(labels=(-1, -2))
sage: trafficsign.cover_relations()
[[-1, 0], [0, 1], [0, 2], [1, -2], [2, -2]]
```
sage: Y = V.with_bounds(labels=(-1, None))
sage: Y.cover_relations()
\[-1, 0], [0, 1], [0, 2]\]

sage: P = posets.PentagonPoset()  # A lattice
sage: P.with_bounds()
Finite lattice containing 7 elements

sage: P = posets.PentagonPoset(facade=False)
sage: P.with_bounds()
Finite lattice containing 7 elements

See also:
without_bounds() for the reverse operation

with_linear_extension(linear_extension)
Return a copy of self with a different default linear extension.

EXAMPLES:

sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True)
sage: P.cover_relations()
[[1, 2], [1, 3], [2, 4], [2, 6], [3, 6], [4, 12], [6, 12]]
sage: list(P)
[1, 2, 3, 4, 6, 12]
sage: Q = P.with_linear_extension([1,3,2,6,4,12])
sage: list(Q)
[1, 3, 2, 6, 4, 12]
sage: Q.cover_relations()
[[1, 3], [1, 2], [3, 6], [2, 6], [2, 4], [6, 12], [4, 12]]

Note: With the current implementation, this requires relabeling the internal DiGraph which is $O(n + m)$, where $n$ is the number of elements and $m$ the number of cover relations.

without_bounds()
Return the poset without its top and bottom elements.
This is useful as an input for the method order_complex().
If there is either no top or no bottom element, this raises a TypeError.

EXAMPLES:

sage: P = posets.PentagonPoset()
sage: Q = P.without_bounds(); Q
Finite poset containing 3 elements

sage: P = posets.DiamondPoset(5)
sage: Q = P.without_bounds(); Q
Finite poset containing 3 elements
sage: Q.cover_relations()
[]

See also:

\texttt{with\_bounds()} for the reverse operation

\textbf{zeta\_polynomial()}

Return the zeta polynomial of the poset.

The zeta polynomial of a poset is the unique polynomial \( Z(q) \) such that for every integer \( m > 1 \), \( Z(m) \) is the number of weakly increasing sequences \( x_1 \leq x_2 \leq \cdots \leq x_{m-1} \) of elements of the poset.

The polynomial \( Z(q) \) is integral-valued, but generally does not have integer coefficients. It can be computed as

\[
Z(q) = \sum_{k \geq 1} \binom{q-2}{k-1} c_k,
\]

where \( c_k \) is the number of all chains of length \( k \) in the poset.

For more information, see section 3.12 of [EnumComb1].

In particular, \( Z(2) \) is the number of vertices and \( Z(3) \) is the number of intervals.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: posets.ChainPoset(2).zeta_polynomial() q
sage: posets.ChainPoset(3).zeta_polynomial()
1/2*q^2 + 1/2*q
sage: P = posets.PentagonPoset()
sage: P.zeta_polynomial()
1/6*q^3 + q^2 - 1/6*q
sage: P = posets.DiamondPoset(5)
sage: P.zeta_polynomial()
3/2*q^2 - 1/2*q
\end{verbatim}

\textbf{class} \texttt{sage.combinat.posets.posets.FinitePosets\_n(n)}

\textbf{Bases:} \texttt{UniqueRepresentation, Parent}

The finite enumerated set of all posets on \( n \) elements, up to an isomorphism.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P = Posets(3)
sage: P.cardinality()
5
sage: for p in P: print(p.cover_relations())
    []
    [[0, 1], [0, 2]]
    [[0, 1], [1, 2]]
    [[1, 2], [0, 2]]
\end{verbatim}
cardinality(from_iterator=False)

Return the cardinality of this object.

**Note:** By default, this returns pre-computed values obtained from the On-Line Encyclopedia of Integer Sequences (OEIS sequence A000112). To override this, pass the argument `from_iterator=True`.

**EXAMPLES:**

```python
sage: P = Posets(3)
sage: P.cardinality()
5
sage: P.cardinality(from_iterator=True)
5
```

```python
sage.combinat.posets.posets.Poset(data=None, element_labels=None, cover_relations=False, linear_extension=False, category=None, facade=None, key=None)
```

Construct a finite poset from various forms of input data.

**INPUT:**

- **data** — different input are accepted by this constructor:
  1. A two-element list or tuple \((E, R)\), where \(E\) is a collection of elements of the poset and \(R\) is a collection of relations \(x \leq y\), each represented as a two-element list/tuple/iterable such as \([x, y]\). The poset is then the transitive closure of the provided relations. If `cover_relations=True`, then \(R\) is assumed to contain exactly the cover relations of the poset. If \(E\) is empty, then \(E\) is taken to be the set of elements appearing in the relations \(R\).
  2. A two-element list or tuple \((E, f)\), where \(E\) is the set of elements of the poset and \(f\) is a function such that, for any pair \(x, y\) of elements of \(E\), \(f(x, y)\) returns whether \(x \leq y\). If `cover_relations=True`, then \(f(x, y)\) should instead return whether \(x\) is covered by \(y\).
  3. A dictionary of upper covers: \(\text{data}[x]\) is a list of the elements that cover the element \(x\) in the poset.
  4. A list or tuple of upper covers: \(\text{data}[x]\) is a list of the elements that cover the element \(x\) in the poset.
     If the set of elements is not a set of consecutive integers starting from zero, then:
     - every element must appear in the data, for example in its own entry.
     - data must be ordered in the same way as sorted elements.

  **Warning:** If data is a list or tuple of length 2, then it is handled by the case 2 above.

  5. An acyclic, loop-free and multi-edge free DiGraph. If `cover_relations` is `True`, then the edges of the digraph are assumed to correspond to the cover relations of the poset. Otherwise, the cover relations are computed.

  6. A previously constructed poset (the poset itself is returned).

- **element_labels** — (default: `None`); an optional list or dictionary of objects that label the poset elements.
- **cover_relations** — a boolean (default: `False`); whether the data can be assumed to describe a directed acyclic graph whose arrows are cover relations; otherwise, the cover relations are first computed.
- **linear_extension** — a boolean (default: `False`); whether to use the provided list of elements as default linear extension for the poset; otherwise a linear extension is computed. If the data is given as the pair \((E, f)\), then \(E\) is taken to be the linear extension.
• facade – a boolean or None (default); whether the Poset()’s elements should be wrapped to make them aware of the Poset they belong to.
  – If facade = True, the Poset()’s elements are exactly those given as input.
  – If facade = False, the Poset()’s elements will become PosetElement objects.
  – If facade = None (default) the expected behaviour is the behaviour of facade = True, unless the opposite can be deduced from the context (i.e. for instance if a Poset() is built from another Poset(), itself built with facade = False)

OUTPUT:
FinitePoset – an instance of the FinitePoset class.
If category is specified, then the poset is created in this category instead of FinitePosets.

See also:
Posets, Posets, FinitePosets

EXAMPLES:
1. Elements and cover relations:

```
sage: elms = [1,2,3,4,5,6,7]
sage: rels = [[1,2],[3,4],[4,5],[2,5]]
sage: Poset((elms, rels), cover_relations = True, facade = False)
Finite poset containing 7 elements
```

Elements and non-cover relations:

```
sage: elms = [1,2,3,4]
sage: rels = [[1,2],[1,3],[1,4],[2,3],[2,4],[3,4]]
sage: P = Poset((elms,rels),cover_relations=False); P
Finite poset containing 4 elements
sage: P.cover_relations()
[[1, 2], [2, 3], [3, 4]]
```

2. Elements and function: the standard permutations of [1, 2, 3, 4] with the Bruhat order:

```
sage: elms = Permutations(4)
sage: fcn = lambda p,q : p.bruhat_lequal(q)
sage: Poset((elms, fcn))
Finite poset containing 24 elements
```

With a function that identifies the cover relations: the set partitions of \{1, 2, 3\} ordered by refinement:

```
sage: elms = SetPartitions(3)
sage: def fcn(A, B):
.....:     if len(A) != len(B)+1:
.....:         return False
.....:     for a in A:
.....:         if not any(set(a).issubset(b) for b in B):
.....:             return False
.....:     return True
sage: Poset((elms, fcn), cover_relations=True)
Finite poset containing 5 elements
```

3. A dictionary of upper covers:
4. A list of upper covers, with range(5) as set of vertices:

```python
sage: Poset([[1,2],[4],[3],[4],[]])
Finite poset containing 5 elements
```

A list of upper covers, with letters as vertices:

```python
sage: Poset([['a','b'],['b','c'],['c']])
Finite poset containing 3 elements
```

A list of upper covers and a dictionary of labels:

```python
sage: elm_labs = {0:'a', 1:'b', 2:'c', 3:'d', 4:'e'}
sage: P = Poset([[1,2],[4],[3],[4],[]], elm_labs, facade=False)
sage: P.list()
[a, b, c, d, e]
```

**Warning:** The special case where the argument data is a list or tuple of length 2 is handled by the case 2. So you cannot use this method to input a 2-element poset.

5. An acyclic DiGraph.

```python
sage: dag = DiGraph({0:[2,3], 1:[3,4], 2:[5], 3:[5], 4:[5]})
sage: Poset(dag)
Finite poset containing 6 elements
```

Any directed acyclic graph without loops or multiple edges, as long as `cover_relations=False`:

```python
sage: dig = DiGraph({0:[2,3], 1:[3,4,5], 2:[5], 3:[5], 4:[5]})
sage: dig.allows_multiple_edges()
False
sage: dig.allows_loops()
False
sage: dig.transitive_reduction() == dig
False
sage: Poset(dig, cover_relations=False)
Finite poset containing 6 elements
sage: Poset(dig, cover_relations=True)
Traceback (most recent call last):
  ... ValueError: Hasse diagram is not transitive reduced
```
**Default Linear extension**

Every poset $P$ obtained with `Poset` comes equipped with a default linear extension, which is also used for enumerating its elements. By default, this linear extension is computed, and has no particular significance:

```
sage: P = Poset((divisors(12), attrcall("divides")))
sage: P.list()
[1, 2, 4, 3, 6, 12]
sage: P.linear_extension()
[1, 2, 4, 3, 6, 12]
```

You may enforce a specific linear extension using the `linear_extension` option:

```
sage: P = Poset((divisors(12), attrcall("divides")), linear_extension=True)
sage: P.list()
[1, 2, 3, 4, 6, 12]
sage: P.linear_extension()
[1, 2, 3, 4, 6, 12]
```

Depending on popular request, `Poset` might eventually get modified to always use the provided list of elements as default linear extension, when it is one.

**See also:**

- `FinitePoset.linear_extensions()`

**Facade posets**

When `facade = False`, the elements of a poset are wrapped so as to make them aware that they belong to that poset:

```
sage: P = Poset(DiGraph({'d': ['c', 'b'], 'c': ['a'], 'b': ['a']}), facade = False)
sage: d,c,b,a = list(P)
sage: a.parent() is P
True
```

This allows for comparing elements according to $P$:

```
sage: c < a
True
```

However, this may have surprising effects:

```
sage: my_elements = ['a', 'b', 'c', 'd']
sage: any(x in my_elements for x in P)
False
```

and can be annoying when one wants to manipulate the elements of the poset:

```
sage: a + b
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Finite poset containing 4 elements' and 'Finite poset containing 4 elements'
```
By default, facade posets are constructed instead:

```
sage: P = Poset(DiGraph({'d': ['c', 'b'], 'c': ['a'], 'b': ['a']}))
```

In this example, the elements of the poset remain plain strings:

```
sage: d, c, b, a = list(P)
sage: type(a)
<class 'str'>
```

Of course, those strings are not aware of $P$. So to compare two such strings, one needs to query $P$:

```
sage: a < b
True
sage: P.lt(a, b)
False
```

which models the usual mathematical notation $a <_P b$.

Most operations seem to still work, but at this point there is no guarantee whatsoever:

```
sage: P.list()
['d', 'c', 'b', 'a']
sage: P.principal_order_ideal('a')
['d', 'c', 'b', 'a']
sage: P.principal_order_ideal('b')
['d', 'b']
sage: P.principal_order_ideal('d')
['d']
sage: TestSuite(P).run()
```

**Warning:** `DiGraph` is used to construct the poset, and the vertices of a `DiGraph` are converted to plain Python int's if they are `Integer`'s:

```
sage: G = DiGraph({0: [2, 3], 1: [3, 4], 2: [5], 3: [5], 4: [5]})
sage: type(G.vertices(sort=True)[0])
<class 'int'>
```

This is worked around by systematically converting back the vertices of a poset to `Integer`'s if they are int's:

```
sage: P = Poset((divisors(15), attrcall("divides")), facade = False)
sage: type(P.an_element().element)
<class 'sage.rings.integer.Integer'>
sage: P = Poset((divisors(15), attrcall("divides")), facade=True)
sage: type(P.an_element())
<class 'sage.rings.integer.Integer'>
```

This may be abusive:
**Unique representation**

As most parents, `Poset` have unique representation (see `UniqueRepresentation`). Namely if two posets are created from two equal data, then they are not only equal but actually identical:

```
sage: data1 = [[1,2],[3],[3]]
sage: data2 = [[1,2],[3],[3]]
sage: P1 = Poset(data1)
sage: P2 = Poset(data2)
sage: P1 == P2
True
sage: P1 is P2
True
```

In situations where this behaviour is not desired, one can use the `key` option:

```
sage: P1 = Poset(data1, key = "foo")
sage: P2 = Poset(data2, key = "bar")
sage: P1 is P2
False
sage: P1 == P2
False
```

`key` can be any hashable value and is passed down to `UniqueRepresentation`. It is otherwise ignored by the poset constructor.

```
sage.combinat.posets.posets.is_poset(dig)
```

Return True if a directed graph is acyclic and transitively reduced, and False otherwise.

**EXAMPLES:**

```
sage: from sage.combinat.posets.posets import is_poset
sage: dig = DiGraph({0:[2, 3], 1:[3, 4, 5], 2:[5], 3:[5], 4:[5]})
sage: is_poset(dig)
False
sage: is_poset(dig.transitive_reduction())
True
```
5.1.187  $q$-Analogues

`sage.combinat.q_analogues.gaussian_binomial(n, k, q=None, algorithm='auto')`

This is an alias of `q_binomial()`.

See `q_binomial()` for the full documentation.

EXAMPLES:

```
sage: gaussian_binomial(4,2)
q^4 + q^3 + 2*q^2 + q + 1
```

`sage.combinat.q_analogues.gaussian_multinomial(seq, q=None, binomial_algorithm='auto')`

Return the $q$-multinomial coefficient.

This is also known as the Gaussian multinomial coefficient, and is defined by

$$
\binom{n}{k_1, k_2, \ldots, k_m}_q = \frac{[n]_q!}{[k_1]_q! [k_2]_q! \cdots [k_m]_q!}
$$

where $n = k_1 + k_2 + \cdots + k_m$.

If $q$ is unspecified, then the variable is the generator $q$ for a univariate polynomial ring over the integers.

INPUT:

- `seq` – an iterable of the values $k_1$ to $k_m$ defined above
- `q` – (default: None) the variable $q$; if None, then use a default variable in $\mathbb{Z}[q]$
- `binomial_algorithm` – (default: 'auto') the algorithm to use in `q_binomial()`; see possible values there

ALGORITHM:

We use the equivalent formula

$$
\binom{k_1 + \cdots + k_m}{k_1, \ldots, k_m}_q = \prod_{i=1}^{m} \binom{\sum_{j=1}^{i} k_j}{k_i}_q.
$$

EXAMPLES:

```
sage: from sage.combinat.q_analogues import q_multinomial
sage: q_multinomial([1,2,1])
q^5 + 2*q^4 + 3*q^3 + 3*q^2 + 2*q + 1
sage: q_multinomial([1,2,1], q=1) == multinomial([1,2,1])
True
sage: q_multinomial((3,2)) == q_binomial(5,3)
True
sage: q_multinomial([])
1
```

`sage.combinat.q_analogues.q_binomial(n, k, q=None, algorithm='auto')`

Return the $q$-binomial coefficient.

This is also known as the Gaussian binomial coefficient, and is defined by

$$
\binom{n}{k}_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)}.
$$
Combinatorics, Release 10.0

See Wikipedia article Gaussian_binomial_coefficient.

If \( q \) is unspecified, then the variable is the generator \( q \) for a univariate polynomial ring over the integers.

**INPUT:**
- \( n, k \) – the values \( n \) and \( k \) defined above
- \( q \) – (default: None) the variable \( q \); if None, then use a default variable in \( \mathbb{Z}[q] \)
- **algorithm** – (default: 'auto') the algorithm to use and can be one of the following:
  - 'auto' – automatically choose the algorithm; see the algorithm section below
  - 'naive' – use the naive algorithm
  - 'cyclotomic' – use cyclotomic algorithm

**ALGORITHM:**
The naive algorithm uses the product formula. The cyclotomic algorithm uses a product of cyclotomic polynomials (cf. [CH2006]).

When the algorithm is set to 'auto', we choose according to the following rules:
- If \( q \) is a polynomial:
  - When \( n \) is small or \( k \) is small with respect to \( n \), one uses the naive algorithm. When both \( n \) and \( k \) are big, one uses the cyclotomic algorithm.
- If \( q \) is in the symbolic ring (or a symbolic subring), one uses the cyclotomic algorithm.
- Otherwise one uses the naive algorithm, unless \( q \) is a root of unity, then one uses the cyclotomic algorithm.

**EXAMPLES:**
By default, the variable is the generator of \( \mathbb{Z}[q] \):

```python
sage: from sage.combinat.q_analogues import q_binomial
sage: g = q_binomial(5,1) ; g
q^4 + q^3 + q^2 + q + 1
sage: g.parent()
Univariate Polynomial Ring in q over Integer Ring
```

The \( q \)-binomial coefficient vanishes unless \( 0 \leq k \leq n \):

```python
sage: q_binomial(4,5)
0
sage: q_binomial(5,-1)
0
```

Other variables can be used, given as third parameter:

```python
sage: p = ZZ['p'].gen()
sage: q_binomial(4,2,p)
p^4 + p^3 + 2*p^2 + p + 1
```

The third parameter can also be arbitrary values:

```python
sage: q_binomial(5,1,2) == g.subs(q=2)
True
sage: q_binomial(5,1,1)
```

(continues on next page)
We can also do this for more complicated objects such as matrices or symmetric functions:

\[
\text{sage: } \text{q\_binomial}(4, 2, \text{matrix}([[2, 1], [-1, 3]])) \\
\begin{bmatrix}
-6 & 84 \\
-84 & 78
\end{bmatrix}
\]

\[
\text{sage: } \text{Sym} = \text{SymmetricFunctions}(\mathbb{Q}) \\
\text{sage: } s = \text{Sym}.\text{schur}() \\
\text{sage: } \text{q\_binomial}(4, 1, s[2]+s[1])
\]

\[
\]

REFERENCES:

AUTHORS:

- Frédéric Chapoton, David Joyner and William Stein

sage.combinat.q_analogues.q_catalan_number\((n, q=None)\)

Return the \(q\)-Catalan number of index \(n\).

If \(q\) is unspecified, then it defaults to using the generator \(q\) for a univariate polynomial ring over the integers.

There are several \(q\)-Catalan numbers. This procedure returns the one which can be written using the \(q\)-binomial coefficients.

EXAMPLES:

\[
\text{sage: } \text{from sage.combinat.q_analogues import q\_catalan\_number} \\
\text{sage: } q\_catalan\_number(4) \\
q^12 + q^10 + q^9 + 2*q^8 + q^7 + 2*q^6 + q^5 + 2*q^4 + q^3 + q^2 + 1 \\
\text{sage: } p = \mathbb{Z}['p'].0 \\
\text{sage: } q\_catalan\_number(4, p) \\
p^12 + p^10 + p^9 + 2*p^8 + p^7 + 2*p^6 + p^5 + 2*p^4 + p^3 + p^2 + 1
\]

The \(q\)-Catalan number of index \(n\) is only defined for \(n\) a nonnegative integer (github issue #11411):

\[
\text{sage: } q\_catalan\_number(-2) \\
\text{ValueError: argument (-2) must be a nonnegative integer}
\]

sage.combinat.q_analogues.q_factorial\((n, q=None)\)

Return the \(q\)-analogue of the factorial \(n\)!.
This is the product
\[ [1]_q [2]_q \cdots [n]_q = 1 \cdot (1 + q) \cdot (1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1}). \]

If \( q \) is unspecified, then this function defaults to using the generator \( q \) for a univariate polynomial ring over the integers.

**EXAMPLES:**

```
sage: from sage.combinat.q_analogues import q_factorial
sage: q_factorial(3)
q^3 + 2*q^2 + 2*q + 1
sage: p = ZZ['p'].0
sage: q_factorial(3, p)
p^3 + 2*p^2 + 2*p + 1
```

The \( q \)-analogue of \( n! \) is only defined for \( n \) a non-negative integer (github issue #11411):

```
sage: q_factorial(-2)
Traceback (most recent call last):
  ... ValueError: argument (-2) must be a nonnegative integer
```

```
sage.combinat.q_analogues.q_int(n, q=None)
Return the \( q \)-analogue of the integer \( n \).

The \( q \)-analogue of the integer \( n \) is given by
\[
[n]_q = \begin{cases} 
1 + q + \cdots + q^{n-1}, & \text{if } n \geq 0, \\
-q^{-n}[-n]_q, & \text{if } n \leq 0.
\end{cases}
\]

Consequently, if \( q = 1 \) then \([n]_1 = n\) and if \( q \neq 1 \) then \([n]_q = (q^n - 1)/(q - 1)\).

If the argument \( q \) is not specified then it defaults to the generator \( q \) of the univariate polynomial ring over the integers.

**EXAMPLES:**

```
sage: from sage.combinat.q_analogues import q_int
sage: q_int(3)
q^2 + q + 1
sage: q_int(-3)
(-q^2 - q - 1)/q^3
sage: p = ZZ['p'].0
sage: q_int(3,p)
p^2 + p + 1
sage: q_int(3/2)
Traceback (most recent call last):
  ... ValueError: 3/2 must be an integer
```

```
sage.combinat.q_analogues.q_jordan(q=None)
Return the \( q \)-Jordan number of \( t \).

If \( q \) is the power of a prime number, the output is the number of complete flags in \( F_q^N \) (where \( N \) is the size of \( t \)) stable under a linear nilpotent endomorphism \( f_t \) whose Jordan type is given by \( t \), i.e. such that for all \( i \):
\[
\dim(\ker f_t^i) = t[0] + \cdots + t[i-1]
\]

```

---

5.1. Comprehensive Module List 2133
If \( q \) is unspecified, then it defaults to using the generator \( q \) for a univariate polynomial ring over the integers. The result is cached.

**INPUT:**
- \( t \) – an integer partition, or an argument accepted by `Partition`
- \( q \) – (default: `None`) the variable \( q \); if `None`, then use a default variable in \( \mathbb{Z}[q] \)

**EXAMPLES:**

```python
sage: from sage.combinat.q_analogues import q_jordan
sage: [q_jordan(mu, 2) for mu in Partitions(5)]
[9765, 1029, 213, 93, 29, 9, 1]
sage: [q_jordan(mu, 2) for mu in Partitions(6)]
[615195, 40635, 5643, 2331, 1491, 515, 147, 87, 47, 11, 1]
sage: q_jordan([3,2,1])
16*q^4 + 24*q^3 + 14*q^2 + 5*q + 1
sage: q_jordan([2,1], x)
2*x + 1
```

If the partition is trivial (i.e., has only one part), we get the \( q \)-factorial (in this case, the nilpotent endomorphism is necessarily 0):

```python
sage: from sage.combinat.q_analogues import q_factorial
sage: q_jordan([5]) == q_factorial(5)
True
sage: q_jordan([11], 5) == q_factorial(11, 5)
True
```

**AUTHOR:**
- Xavier Caruso (2012-06-29)

sage.combinat.q_analogues.q_multinomial(seq, q=None, binomial_algorithm='auto')

Return the \( q \)-multinomial coefficient.

This is also known as the Gaussian multinomial coefficient, and is defined by

\[
\binom{n}{k_1, k_2, \ldots, k_m}_q = \frac{[n]_q!}{[k_1]_q! [k_2]_q! \cdots [k_m]_q!}
\]

where \( n = k_1 + k_2 + \cdots + k_m \).

If \( q \) is unspecified, then the variable is the generator \( q \) for a univariate polynomial ring over the integers.

**INPUT:**
- `seq` – an iterable of the values \( k_1 \) to \( k_m \) defined above
- `q` – (default: `None`) the variable \( q \); if `None`, then use a default variable in \( \mathbb{Z}[q] \)
- `binomial_algorithm` – (default: 'auto') the algorithm to use in `q_binomial()`; see possible values there

**ALGORITHM:**

We use the equivalent formula

\[
\binom{k_1 + \cdots + k_m}{k_1, \ldots, k_m}_q = \prod_{i=1}^m \binom{\sum_{j=1}^i k_j}{k_i}_q.
\]
EXAMPLES:

```python
sage: from sage.combinat.q_analogues import q_multinomial
dsage: q_multinomial([1,2,1])
q^5 + 2*q^4 + 3*q^3 + 3*q^2 + 2*q + 1
dsage: q_multinomial([1,2,1], q=1) == multinomial([1,2,1])
True
dsage: q_multinomial((3,2)) == q_binomial(5,3)
True
dsage: q_multinomial([])
1
```

```python
sage.combinat.q_analogues.
q_pochhammer(n, a, q=None)
```

Return the $q$-Pochhammer $(a; q)_n$.

The $q$-Pochhammer symbol is defined by

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

with $(a; q)_0 = 1$ for all $a$, $q$ and $n \in \mathbb{N}$. By using the identity

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},$$

we can extend the definition to $n < 0$ by

$$(a; q)_n = \frac{1}{(aq^n; q)_{-n}} = \prod_{k=1}^{-n} \frac{1}{1 - a/q^k}.$$`

EXAMPLES:

```python
sage: from sage.combinat.q_analogues import q_pochhammer
sage: q_pochhammer(3, 1/7)
6/343*q^3 - 6/49*q^2 - 6/49*q + 6/7
sage: q_pochhammer(3, 3)
-18*q^3 + 6*q^2 + 6*q - 2
sage: q_pochhammer(3, 1)
0
sage: R.<q> = ZZ[]
sage: q_pochhammer(4, q)
q^10 - q^9 - q^8 + 2*q^5 - q^2 - q + 1
sage: q_pochhammer(4, q^2)
q^14 - q^12 - q^11 - q^10 + q^8 + 2*q^7 + q^6 - q^4 - q^3 - q^2 + 1
sage: q_pochhammer(-3, q)
1/(-q^9 + q^7 + q^6 + q^5 - q^4 - q^3 - q^2 + 1)
```

REFERENCES:

- Wikipedia article Q-Pochhammer_symbol

```python
sage.combinat.q_analogues.
q_stirling_number1(k, q=None)
```

Return the (unsigned) $q$-Stirling number of the first kind.

This is a $q$-analogue of `sage.combinat.combinat.stirling_number1()`.

INPUT:
• $n, k$ – integers with $1 \leq k \leq n$
• $q$ – optional variable (default $q$)

OUTPut: a polynomial in the variable $q$

These polynomials satisfy the recurrence

$$s_{n,k} = s_{n-1,k-1} + [n-1]q s_{n-1,k}.$$  

EXAMPLES:

```python
sage: from sage.combinat.q_analogues import q_stirling_number1
sage: q_stirling_number1(4,2)
q^3 + 3*q^2 + 4*q + 3
sage: all(stirling_number1(6,k) == q_stirling_number1(6,k)(1)
....: for k in range(1,7))
True
```

REFERENCES:

• [Ca1948]
• [Ca1954]
Return the \(q\)-number of subgroups of type \(\mu\) in a finite abelian group of type \(\lambda\).

**INPUT:**

- \(\lambda\) – type of the ambient group as a *Partition*
- \(\mu\) – type of the subgroup as a *Partition*
- \(q\) – (default: None) an indeterminate or a prime number; if None, this defaults to \(q \in \mathbb{Z}[q]\)
- \(algorithm\) – (default: 'birkhoff') the algorithm to use can be one of the following:
  - 'birkhoff' – use the Birkhoff formula from [Bu87]
  - 'delsarte' – use the formula from [Delsarte48]

**OUTPUT:**

The number of subgroups of type \(\mu\) in a group of type \(\lambda\) as a polynomial in \(q\).

**ALGORITHM:**

Let \(q\) be a prime number and \(\lambda = (\lambda_1, \ldots, \lambda_l)\) be a partition. A finite abelian \(q\)-group is of type \(\lambda\) if it is isomorphic to

\[
\mathbb{Z}/q^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/q^{\lambda_l}\mathbb{Z}.
\]

The formula from [Bu87] works as follows: Let \(\lambda\) and \(\mu\) be partitions. Let \(\lambda'\) and \(\mu'\) denote the conjugate partitions to \(\lambda\) and \(\mu\), respectively. The number of subgroups of type \(\mu\) in a group of type \(\lambda\) is given by

\[
\prod_{i=1}^{\mu_1} q^{(\lambda_i' - \mu_i') + 1} \frac{\lambda_i' - \mu_i'}{\mu_i' + 1} q
\]

The formula from [Delsarte48] works as follows: Let \(\lambda\) and \(\mu\) be partitions. Let \((s_1, s_2, \ldots, s_l)\) and \((r_1, r_2, \ldots, r_k)\) denote the parts of the partitions conjugate to \(\lambda\) and \(\mu\) respectively. Let

\[
\mathfrak{F}(\xi_1, \ldots, \xi_k) = \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_k^{r_k} \prod_{i_1=r_2}^{r_1-1} (\xi_1 - q^{i_1}) \prod_{i_2=r_3}^{r_2-1} (\xi_2 - q^{i_2}) \cdots \prod_{i_k=0}^{r_k-1} (\xi_k - q^{-i_k}).
\]

Then the number of subgroups of type \(\mu\) in a group of type \(\lambda\) is given by

\[
\frac{\mathfrak{F}(q^{s_1}, q^{s_2}, \ldots, q^{s_l})}{\mathfrak{F}(q^{r_1}, q^{r_2}, \ldots, q^{r_k})}.
\]

**EXAMPLES:**

```python
sage: from sage.combinat.q_analogues import q_subgroups_of_abelian_group
sage: q_subgroups_of_abelian_group([1,1], [1])
q + 1
sage: q_subgroups_of_abelian_group([3,3,2,1], [2,1])
qu^6 + 2qu^5 + 3qu^4 + 2qu^3 + q^2
sage: R.<t> = QQ[]
sage: q_subgroups_of_abelian_group([5,3,1], [3,1], t)
t^4 + 2t^3 + t^2
sage: q_subgroups_of_abelian_group([5,3,1], [3,1], 3)
144
sage: q_subgroups_of_abelian_group([1,1,1], [1]) == q_subgroups_of_abelian_group([1,1,1], [1,1,1])
True
```
REFERENCES:

AUTHORS:

• Amritanshu Prasad (2013-06-07): Implemented the Delsarte algorithm
• Tomer Bauer (2013, 2018): Implemented the Birkhoff algorithm and refactoring

sage.combinat.q_analogues.qt_catalan_number(n)

Return the $q,t$-Catalan number of index $n$.

EXAMPLES:

sage: from sage.combinat.q_analogues import qt_catalan_number
sage: qt_catalan_number(1)
1
sage: qt_catalan_number(2)
q + t
sage: qt_catalan_number(3)
q^3 + q^2*t + q*t^2 + t^3 + q*t
sage: qt_catalan_number(4)
q^6 + q^5*t + q^4*t^2 + q^3*t^3 + q^2*t^4 + q*t^5 + t^6 + q^4*t + q^3*t^2 + q^2*t^3 + q*t^4 + q^3*t + q^2*t^2 + q*t^3

The $q,t$-Catalan number of index $n$ is only defined for $n$ a nonnegative integer (github issue #11411):

sage: qt_catalan_number(-2)
Traceback (most recent call last):
... ValueError: argument (-2) must be a nonnegative integer

5.1.188 $q$-Bernoulli Numbers and Polynomials

sage.combinat.q_bernoulli.q_bernoulli(p=None)

Compute Carlitz’s $q$-analogue of the Bernoulli numbers.

For every nonnegative integer $m$, the $q$-Bernoulli number $\beta_m$ is a rational function of the indeterminate $q$ whose value at $q = 1$ is the usual Bernoulli number $B_m$.

INPUT:

• $m$ – a nonnegative integer
• $p$ (default: None) – an optional value for $q$

OUTPUT:

A rational function of the indeterminate $q$ (if $p$ is None)
Otherwise, the rational function is evaluated at \( p \).

**EXAMPLES:**

```python
sage: from sage.combinat.q_bernoulli import q_bernoulli
sage: q_bernoulli(0)
1
sage: q_bernoulli(1)
-1/(q + 1)
sage: q_bernoulli(2)
q/(q^3 + 2*q^2 + 2*q + 1)
sage: all(q_bernoulli(i)(q=1) == bernoulli(i) for i in range(12))
True
```

One can evaluate the rational function by giving a second argument:

```python
sage: x = PolynomialRing(GF(2), 'x').gen()
sage: q_bernoulli(5,x)
x/(x^6 + x^5 + x + 1)
```

The function does not accept negative arguments:

```python
sage: q_bernoulli(-1)
Traceback (most recent call last):
  ...
ValueError: the argument must be a nonnegative integer
```

**REFERENCES:**

sage.combinat.q_bernoulli.q_bernoulli_polynomial()

Compute Carlitz’s \( q \)-analogue of the Bernoulli polynomials.

For every nonnegative integer \( m \), the \( q \)-Bernoulli polynomial is a polynomial in one variable \( x \) with coefficients in \( \mathbb{Q}(q) \) whose value at \( q = 1 \) is the usual Bernoulli polynomial \( B_m(x) \).

The original \( q \)-Bernoulli polynomials introduced by Carlitz were polynomials in \( q^y \) with coefficients in \( \mathbb{Q}(q) \). This function returns these polynomials but expressed in the variable \( x = (q^y - 1)/(q - 1) \). This allows to let \( q = 1 \) to recover the classical Bernoulli polynomials.

**INPUT:**

- \( m \) – a nonnegative integer

**OUTPUT:**

A polynomial in one variable \( x \).

**EXAMPLES:**

```python
sage: from sage.combinat.q_bernoulli import q_bernoulli_polynomial, q_bernoulli
sage: q_bernoulli_polynomial(0)
1
sage: q_bernoulli_polynomial(1)
(2/(q + 1))*x - 1/(q + 1)
sage: x = q_bernoulli_polynomial(1).parent().gen()
sage: all(q_bernoulli_polynomial(i)(q=1)==bernoulli_polynomial(x,i) for i in range(12))
True
```
The function does not accept negative arguments:

```
sage: q_bernoulli_polynomial(-1)
Traceback (most recent call last):
  ...<ValueError: the argument must be a nonnegative integer>
```
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(continued from previous page)

```python
sage: Word('aabacabaa').is_palindrome()
True
sage: WordMorphism('a->ab,b->a').fixed_point('a')
word: ababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababa...```

Polytopes:

```python
sage: points = random_matrix(ZZ, 6, 3, x=7).rows()
sage: L = LatticePolytope(points)
sage: L.npoints(); L.plot3d()  # random
```

Root systems, Coxeter and Weyl groups:

```python
sage: WeylGroup(['B',3]).bruhat_poset()
Finite poset containing 48 elements
sage: RootSystem(['A',2,1]).weight_lattice().plot()  # not tested
```

Crystals:

```python
sage: CrystalOfTableaux(['A',3], shape = [3,2]).some_flashy_feature()  # not tested
```

Symmetric functions and combinatorial Hopf algebras:

```python
sage: Sym = SymmetricFunctions(QQ); Sym.inject_shorthands(VERBOSE=False)
sage: m( ( h[2,1] * (1 + 3 * p[2,1]) ) + s[2](s[3]) )
3*m[1, 1, 1] + ... + 10*m[5, 1] + 4*m[6]
```

Discrete groups, Permutation groups:

```python
sage: S = SymmetricGroup(4)
sage: M = PolynomialRing(QQ, 'x0,x1,x2,x3')
sage: M.an_element() * S.an_element()
x0
```

Graph theory, posets, lattices (Graph Theory, Posets):

```python
sage: Poset({1: [2,3], 2: [4], 3: [4]}).linear_extensions().cardinality()
2
```

5.1.190 Rankers

sage.combinat.ranker.from_list(lst)

Returns a ranker from the list l.

**INPUT:**

- l - a list

**OUTPUT:**

- [rank, unrank] - functions

**EXAMPLES:**

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sage.combinat.ranker.on_fly()

Returns a pair of enumeration functions rank / unrank.

rank assigns on the fly an integer, starting from 0, to any object passed as argument. The object should be hashable. unrank is the inverse function; it returns None for indices that have not yet been assigned.

EXAMPLES:

sage: [rank, unrank] = sage.combinat.ranker.on_fly()
sage: rank('a')
0
sage: rank('b')
1
sage: rank('c')
2
sage: rank('a')
0
sage: unrank(2)
'c'
sage: unrank(3)

sage.combinat.ranker.rank_from_list(l)

Return a rank function for the elements of l.

INPUT:

- l – a duplicate free list (or iterable) of hashable objects

OUTPUT:

- a function from the elements of l to 0,...,len(l)

EXAMPLES:
For non elements a ValueError is raised, as with the usual index method of lists:

```python
sage: r('blah')
Traceback (most recent call last):
  ... ValueError: 'blah' is not in dict
```

Currently, the rank function is a CallableDict; but this is an implementation detail:

```python
sage: type(r)
<class 'sage.misc.callable_dict.CallableDict'>
sage: r
{'a': 0, 'b': 1, 'c': 2}
```

With the current implementation, no error is issued in case of duplicate value in `l`. Instead, the rank function returns the position of some of the duplicates:

```python
sage: r = ranker.rank_from_list(['a', 'b', 'a', 'c'])
sage: r('a')
2
```

Constructing the rank function itself is of complexity $O(|l|)$. Then, each call to the rank function consists of an essentially constant time dictionary lookup.

**sage.combinat.ranker.unrank**

Return the $i$-th element of $L$.

**INPUT:**

- $L$ – a list, tuple, finite enumerated set, ...
- $i$ – an int or Integer

The purpose of this utility is to give a uniform idiom to recover the $i$-th element of an object $L$, whether $L$ is a list, tuple (or more generally a `collections.abc.Sequence`), an enumerated set, some old parent of Sage still implementing unranking in the method `__getitem__`, or an iterable (see `collections.abc.Iterable`). See github issue #15919.

**EXAMPLES:**

Lists, tuples, and other sequences:

```python
sage: from sage.combinat.ranker import unrank
sage: unrank(['a', 'b', 'c'], 2)
'c'
sage: unrank(['a', 'b', 'c'], 1)
'b'
sage: unrank(range(3,13,2), 1)
5
```

Enumerated sets:
sage: unrank(GF(7), 2)
2
sage: unrank(IntegerModRing(29), 10)
10

An iterable:

sage: unrank(NN, 4)
4

An iterator:

sage: unrank(('a{}'.format(i) for i in range(20)), 0)
'a0'
sage: unrank(('a{}'.format(i) for i in range(20)), 2)
'a2'

Warning: When unranking an iterator, it returns the i-th element beyond where it is currently at:

sage: from sage.combinat.ranker import unrank
sage: it = iter(range(20))
sage: unrank(it, 2)
2
sage: unrank(it, 2)
5

sage.combinat.ranker.unrank_from_list(l)
Returns an unrank function from a list.

EXAMPLES:

sage: import sage.combinat.ranker as ranker
sage: l = [1,2,3]
sage: u = ranker.unrank_from_list(l)
sage: u(2)
3
sage: u(0)
1

5.1.191 Recognizable Series

Let \(A\) be an alphabet and \(K\) a semiring. Then a formal series \(S\) with coefficients in \(K\) and indices in the words \(A^*\) is called recognizable if it has a linear representation, i.e., there exists

- a nonnegative integer \(n\)

and there exist

- two vectors \(\text{left}\) and \(\text{right}\) of dimension \(n\) and
- a morphism of monoids \(\mu\) from \(A^*\) to \(n \times n\) matrices over \(K\)
such that the coefficient corresponding to a word \( w \in A^* \) equals

\[
\left\{ \begin{array}{l}
\mu(w) \\
right.
\end{array} \right.
\]

**Note:** Whenever a minimization \( \text{minimized}() \) of a series needs to be computed, it is required that \( K \) is a field. In particular, minimization is called before checking if a series is nonzero.

**Warning:** As this code is experimental, warnings are thrown when a recognizable series space is created for the first time in a session (see \texttt{sage.misc.superseded.experimental}).

**Various**

**See also:**

\textit{k-regular sequence, sage.rings.cfinite_sequence, sage.combinat.binary_recurrence_sequences}.

**AUTHORS:**

- Daniel Krenn (2016, 2021)

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**Classes and Methods**

\texttt{class sage.combinat.recognizable_series.PrefixClosedSet(words)}

Bases: \texttt{object}

A prefix-closed set.

Creation of this prefix-closed set is interactive iteratively.

**INPUT:**

- \texttt{words} – a class of words (instance of \texttt{Words})

**EXAMPLES:**

\begin{verbatim}
sage: from sage.combinat.recognizable_series import PrefixClosedSetsage: P = PrefixClosedSet(Words([0, 1], infinite=False)); P
[word: ]
sage: P = PrefixClosedSet.create_by_alphabet([0, 1]); P
[word: ]
\end{verbatim}

See \texttt{iterate_possible_additions()} for further examples.

\texttt{add(w, check=True)}

Add a word to this prefix-closed set.

**INPUT:**

- \texttt{w} – a word

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• check – boolean (default: True). If set, then it is verified whether all proper prefixes of \( w \) are already in this prefix-closed set.

OUTPUT:
Nothing, but a RuntimeError is raised if the check fails.

EXAMPLES:

```python
sage: from sage.combinat.recognizable_series import PrefixClosedSet
sage: P = PrefixClosedSet.create_by_alphabet([0, 1])
sage: W = P.words
sage: P.add(W([0])); P
[word: , word: 0]
sage: P.add(W([0, 1])); P
[word: , word: 0, word: 01]
sage: P.add(W([1, 1]))
Traceback (most recent call last):
... ValueError: cannot add as not all prefixes of 11 are included yet
```

classmethod create_by_alphabet(alphabet)
A prefix-closed set
This is a convenience method for the creation of prefix-closed sets by specifying an alphabet.

INPUT:
• alphabet – finite words over this alphabet will used

EXAMPLES:

```python
sage: from sage.combinat.recognizable_series import PrefixClosedSet
sage: P = PrefixClosedSet.create_by_alphabet([0, 1]); P
[word: ]
```

iterate_possible_additions()
Return an iterator over all elements including possible new elements.

OUTPUT:
An iterator

EXAMPLES:

```python
sage: from sage.combinat.recognizable_series import PrefixClosedSet
sage: P = PrefixClosedSet.create_by_alphabet([0, 1]); P
[word: ]
sage: for n, p in enumerate(P.iterate_possible_additions()):
    ....:     print('{}?'.format(p))
    ....:     if n in (0, 2, 3, 5):
    ....:         P.add(p)
    ....:         print('...added')
0?
...added
1?
00?
...added
```

(continues on next page)
Calling the iterator once more, returns all elements:

```
sage: list(P.iterate_possible_additions())
[word: 0, word: 1, word: 00, word: 01, word: 000, word: 001, word: 010, word: 011, word: 0010, word: 0011]
```

The method `iterate_possible_additions()` is roughly equivalent to

```
sage: list(p + a
....: for p in P.elements
....: for a in P.words.iterate_by_length(1))
[word: 0, word: 1, word: 00, word: 01, word: 000, word: 001, word: 010, word: 011, word: 0010, word: 0011]
```

However, the above does not allow to add elements during iteration, whereas `iterate_possible_additions()` does.

**prefix_set()**

Return the set of minimal (with respect to prefix ordering) elements of the complement of this prefix closed set.

See also Proposition 2.3.1 of [BR2010a].

OUTPUT:

A list
class sage.combinat.recognizable_series.RecognizableSeries(parent, mu, left, right)

A recognizable series.

- **parent** – an instance of RecognizableSeriesSpace
- **mu** – a family of square matrices, all of which have the same dimension. The indices of this family are the elements of the alphabet. mu may be a list or tuple of the same cardinality as the alphabet as well. See also mu.
- **left** – a vector. When evaluating a coefficient, this vector is multiplied from the left to the matrix obtained from mu applying on a word. See also left.
- **right** – a vector. When evaluating a coefficient, this vector is multiplied from the right to the matrix obtained from mu applying on a word. See also right.

When created via the parent RecognizableSeriesSpace, then the following option is available.

EXAMPLES:

```python
sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])
sage: S = Rec((Matrix([[3, 6], [0, 1]]), Matrix([[0, -6], [1, 5]])),
.....:   vector([0, 1]), vector([1, 0])).transposed(); S
[1] + 3*[01] + [10] + 5*[11] + 9*[001] + 3*[010] + ...
```

We can access coefficients by

```python
sage: W = Rec.indices()
sage: S[W([0, 0, 1])]
9
```

See also:

- recognizable series, RecognizableSeriesSpace.
- coefficient_of_word(w, multiply_left=True, multiply_right=True)

Return the coefficient to word w of this series.

INPUT:

- **w** – a word over the parent’s alphabet()
- **multiply_left** – (default: True) a boolean. If False, then multiplication by left is skipped.
- **multiply_right** – (default: True) a boolean. If False, then multiplication by right is skipped.
OUTPUT:
An element in the parent's coefficient_ring()

EXAMPLES:

```sage
Rec = RecognizableSeriesSpace(ZZ, [0, 1])
W = Rec.indices()
S = Rec((Matrix([[1, 0], [0, 1]]), Matrix([[0, -1], [1, 2]])),
.....: left=vector([0, 1]), right=vector([1, 0]))
S[W(7.digits(2))]  # indirect doctest
```

`dimension()`
Return the dimension of this recognizable series.

EXAMPLES:

```sage
Rec = RecognizableSeriesSpace(ZZ, [0, 1])
Rec((Matrix([[1, 0], [0, 1]]), Matrix([[1, 0], [0, 1]])),
.....: left=vector([0, 1]), right=vector([1, 0])).dimension()
```

`hadamard_product(*args, **kwds)`
Return the Hadamard product of this recognizable series and the other recognizable series, i.e., multiply the two series coefficient-wise.

INPUT:
- `other` – a RecognizableSeries with the same parent as this recognizable series
- `minimize` – (default: None) a boolean or None. If True, then minimized() is called after the operation, if False, then not. If this argument is None, then the default specified by the parent's minimize_results is used.

OUTPUT:
A RecognizableSeries

EXAMPLES:

```sage
Seq2 = kRegularSequenceSpace(2, ZZ)
E = Seq2((Matrix([[0, 1], [0, 1]]), Matrix([[0, 0], [0, 1]])),
.....: vector([1, 0]), vector([1, 1]))
E
2-regular sequence 1, 0, 1, 0, 1, 0, 1, 0, ...

O = Seq2((Matrix([[0, 0], [0, 1]]), Matrix([[0, 1], [0, 1]])),
.....: vector([1, 0]), vector([0, 1]))
O
2-regular sequence 0, 1, 0, 1, 0, 1, 0, 1, 0, ...

C = Seq2((Matrix([[2, 0], [2, 1]]), Matrix([[0, 1], [-2, 3]])),
.....: vector([1, 0]), vector([0, 1]))
C
2-regular sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
```
sage: CE = C.hadamard_product(E)
sage: CE
2-regular sequence 0, 0, 2, 0, 4, 0, 6, 0, 8, 0, ...
sage: CE.linear_representation()
((1, 0, 0),
 Finite family {0: [0 1 0]
              [0 2 0]
              [0 2 1],
           1: [ 0 0 0]
              [ 0 0 1]
              [ 0 -2 3]},
(0, 0, 2))

sage: Z = E.hadamard_product(O)
sage: Z
2-regular sequence 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, ...
sage: Z.linear_representation()
(()
 Finite family {0: [],
           1: []},
())

is_trivial_zero()
Return whether this recognizable series is trivially equal to zero (without any minimization).

EXAMPLES:

sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])

sage: Rec((Matrix([[1, 0], [0, 1]]), Matrix([[1, 0], [0, 1]])),
       left=vector([0, 1]), right=vector([1, 0])).is_trivial_zero()
False

sage: Rec((Matrix([[1, 0], [0, 1]]), Matrix([[1, 0], [0, 1]])),
       left=vector([0, 0]), right=vector([1, 0])).is_trivial_zero()
True

sage: Rec((Matrix([[1, 0], [0, 1]]), Matrix([[1, 0], [0, 1]])),
       left=vector([0, 1]), right=vector([0, 0])).is_trivial_zero()
True

The following two differ in the coefficient of the empty word:

sage: Rec((Matrix([[0, 0], [0, 0]]), Matrix([[0, 0], [0, 0]])),
       left=vector([0, 1]), right=vector([1, 0])).is_trivial_zero()
True

sage: Rec((Matrix([[0, 0], [0, 0]]), Matrix([[0, 0], [0, 0]])),
       left=vector([1, 1]), right=vector([1, 1])).is_trivial_zero()
False

property left
When evaluating a coefficient, this vector is multiplied from the left to the matrix obtained from mu applied on a word.

linear_representation()
Return the linear representation of this series.

OUTPUT:
A triple \((\text{left}, \mu, \text{right})\) containing the vectors \(\text{left}\) and \(\text{right}\), and the family of matrices \(\mu\).

**EXAMPLES:**

```python
sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])
sage: Rec((Matrix([[3, 6], [0, 1]]), Matrix([[0, -6], [1, 5]])),
.....:   vector([0, 1]), vector([1, 0])
.....: ).transposed().linear_representation()
((1, 0),
 Finite family {0: [3 0]
    [6 1],
    1: [ 0 1]
    [-6 5]},
(0, 1))
```

**minimized()**

Return a recognizable series equivalent to this series, but with a minimized linear representation.

The coefficients of the involved matrices need be in a field. If this is not the case, then the coefficients are automatically coerced to their fraction field.

**OUTPUT:**

A \(\text{RecognizableSeries}\)

**ALGORITHM:**

This method implements the minimization algorithm presented in Chapter 2 of [BR2010a].

**Note:** Due to the algorithm, the left vector of the result is always \((1, 0, \ldots, 0)\), i.e., the first vector of the standard basis.

**EXAMPLES:**

```python
sage: from itertools import islice
sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])
sage: S = Rec((Matrix([[3, 6], [0, 1]]), Matrix([[0, -6], [1, 5]])),
.....:   vector([0, 1]), vector([1, 0])).transposed()
sage: S
 + 15*[011] + [100] + 11*[101] + 5*[110] + ...
sage: M = S.minimized()
sage: M.mu[0], M.mu[1], M.left, M.right
([3 0] [ 0 1]
[6 1], [-6 5], (1, 0), (0, 1))
sage: M.left == vector([1, 0])
True
sage: all(c == d and v == w
.....:   for (c, v), (d, w) in islice(zip(iter(S), iter(M)), 20))
True
sage: S = Rec((Matrix([[2, 0], [1, 1]]), Matrix([[2, 0], [2, 1]])),
```

(continues on next page)
....:    vector([1, 0]), vector([1, 1]))
sage: S
    + 8*[000] + 8*[001] + 8*[010] + ...
sage: M = S.minimized()
sage: M.mu[0], M.mu[1], M.left, M.right
([2], [2], (1), (1))
sage: all(c == d and v == w
....:       for (c, v), (d, w)
in islice(zip(iter(S), iter(M)), 20))
True

property mu
When evaluating a coefficient, this is applied on each letter of a word; the result is a matrix. This extends
mu to words over the parent's alphabet().

property right
When evaluating a coefficient, this vector is multiplied from the right to the matrix obtained from mu applied
on a word.

transposed()    
Return the transposed series.

OUTPUT:
A RecognizableSeries

Each of the matrices in mu is transposed. Additionally the vectors left and right are switched.

EXAMPLES:

sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])
sage: S = Rec((Matrix([[3, 6], [0, 1]]), Matrix([[0, -6], [1, 5]])),
....:    vector([0, 1]), vector([1, 0])).transposed()
sage: S
    + 15*[011] + [100] + 11*[101] + 5*[110] + ...
sage: S.mu[0], S.mu[1], S.left, S.right
([3 0]
[6 1], [-6 5], (1, 0), (0, 1))
sage: T = S.transposed()
sage: T
[1] + [01] + 3*[10] + 5*[11] + [001] + 3*[010]
    + 5*[011] + 9*[100] + 11*[101] + 15*[110] + ...
sage: T.mu[0], T.mu[1], T.left, T.right
([3 6]
[0 1], [1 5], (0, 1), (1, 0))

class sage.combinat.recognizable_series.RecognizableSeriesSpace(coefficient_ring, indices,
category, minimize_results)

Bases: UniqueRepresentation, Parent

The space of recognizable series on the given alphabet and with the given coefficients.
INPUT:

- **coefficient_ring** – a (semi-)ring
- **alphabet** – a tuple, list or `TotallyOrderedFiniteSet`. If specified, then the **indices** are the finite words over this **alphabet**. **alphabet** and **indices** cannot be specified at the same time.
- **indices** – a SageMath-parent of finite words over an **alphabet**. **alphabet** and **indices** cannot be specified at the same time.
- **category** – (default: None) the category of this space

EXAMPLES:

We create a recognizable series that counts the number of ones in each word:

```python
sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])
sage: Rec
Space of recognizable series on {0, 1} with coefficients in Integer Ring
sage: Rec((Matrix([[1, 0], [0, 1]]), Matrix([[1, 1], [0, 1]]),
.....:
    vector([1, 0]), vector([0, 1]))
[1] + [01] + [10] + 2*[11] + [001] + [010] + 2*[011] + [100] + 2*[101] + 2*[110] + ...
```

All of the following examples create the same space:

```python
sage: Rec1 = RecognizableSeriesSpace(ZZ, [0, 1])
sage: Rec1
Space of recognizable series on {0, 1} with coefficients in Integer Ring
sage: Rec2 = RecognizableSeriesSpace(coefficient_ring=ZZ, alphabet=[0, 1])
sage: Rec2
Space of recognizable series on {0, 1} with coefficients in Integer Ring
sage: Rec3 = RecognizableSeriesSpace(ZZ, indices=Words([0, 1], infinite=False))
sage: Rec3
Space of recognizable series on {0, 1} with coefficients in Integer Ring
```

See also:

`recognizable series`, `RecognizableSeries`.

**Element**

**alphabet()**

Return the alphabet of this recognizable series space.

**OUTPUT:**

A totally ordered set

**EXAMPLES:**

```python
sage: RecognizableSeriesSpace(ZZ, [0, 1]).alphabet()
{0, 1}
```

**coefficient_ring()**

Return the coefficients of this recognizable series space.

**OUTPUT:**

A (semi-)ring

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EXAMPLES:

```python
sage: RecognizableSeriesSpace(ZZ, [0, 1]).coefficient_ring()
Integer Ring
```

**indices()**

Return the indices of the recognizable series.

**OUTPUT:**

The set of finite words over the alphabet

**EXAMPLES:**

```python
sage: RecognizableSeriesSpace(ZZ, [0, 1]).indices()
Finite words over {0, 1}
```

**property minimize_results**

A boolean indicating whether `RecognizableSeries.minimized()` is automatically called after performing operations.

**one_hadamard()**

Return the identity with respect to the `hadamard_product()`, i.e. the coefficient-wise multiplication.

**OUTPUT:**

A `RecognizableSeries`

**EXAMPLES:**

```python
sage: Rec = RecognizableSeriesSpace(ZZ, [0, 1])
sage: Rec.one_hadamard()
[0] + [0] + [0] + [00] + [01] + [10]
  + [11] + [000] + [001] + [010] + ...
```

**some_elements()**

Return some elements of this recognizable series space.

See `TestSuite` for a typical use case.

**OUTPUT:**

An iterator

**EXAMPLES:**

```python
sage: tuple(RecognizableSeriesSpace(ZZ, [0, 1]).some_elements())
([1] + [0] + [1] + [00] + [01] + [10]
  + 2*[11] + [100] + 2*[101] + 2*[110] + ...,
 [1] + [1] + [1] + [111] + [111] + [1111] + [11111] + [111111] + ...,
 [0] + [0] + [1] + [00] + [10] + [11]
  + [000] - 1*[000] + [100] + [110] + ...,
2*[1] - 1*[1] + 2*[10] - 1*[101]
  + 2*[1010] - 1*[10101] + 2*[101010] + ...,
  + 288*[0000] - 33*[0001] + ...,
-5*[1] + ...,
...
```

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sage.combinat.recognizable_series.minimize_result(operation)

A decorator for operations that enables control of automatic minimization on the result.

INPUT:
• operation – a method

OUTPUT:
A method with the following additional argument:
• minimize – (default: None) a boolean or None. If True, then minimized() is called after the operation, if False, then not. If this argument is None, then the default specified by the parent’s minimize_results is used.

5.1.192 Restricted growth arrays

These combinatorial objects are in bijection with set partitions.

class sage.combinat.restricted_growth.RestrictedGrowthArrays(n)

Bases: UniqueRepresentation, Parent

EXAMPLES:

sage: from sage.combinat.restricted_growth import RestrictedGrowthArrays
sage: R = RestrictedGrowthArrays(3)

sage: R == loads(dumps(R))
True
sage: TestSuite(R).run(skip=['_test_an_element',
              ....:   '_test_enumerated_set_contains',
              ....:   '_test_some_elements'])

cardinality()

EXAMPLES:

sage: from sage.combinat.restricted_growth import RestrictedGrowthArrays
sage: R = RestrictedGrowthArrays(6)
sage: R.cardinality()
203

5.1.193 Ribbons

5.1.194 Ribbon Shaped Tableaux

class sage.combinat.ribbon_shaped_tableau.RibbonShapedTableau(parent, t)

Bases: SkewTableau

A ribbon shaped tableau.

For the purposes of this class, a ribbon shaped tableau is a skew tableau whose shape is a skew partition which:
• has at least one cell in row 1;
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- has at least one cell in column 1;
- has exactly one cell in each of \( q \) consecutive diagonals, for some nonnegative integer \( q \).

A ribbon is given by a list of the rows from top to bottom.

EXAMPLES:

```python
sage: x = RibbonShapedTableau([[None, None, None, 2, 3], [None, 1, 4, 5], [3, 2]]);
˓→ x
[[None, None, None, 2, 3], [None, 1, 4, 5], [3, 2]]
sage: x.pp()
. . . 2 3
. 1 4 5
3 2
sage: x.shape()
[5, 4, 2] / [3, 1]
```

The entries labeled by \texttt{None} correspond to the inner partition. Using \texttt{None} is optional; the entries will be shifted accordingly.

```python
sage: x = RibbonShapedTableau([[2,3], [1,4,5], [3,2]]); x.pp()
. . . 2 3
. 1 4 5
3 2
```

**height()**

Return the height of \texttt{self}.

The height is given by the number of rows in the outer partition.

EXAMPLES:

```python
sage: RibbonShapedTableau([[2,3],[1,4,5]]).height()
2
```

**spin()**

Return the spin of \texttt{self}.

EXAMPLES:

```python
sage: RibbonShapedTableau([[2,3],[1,4,5]]).spin()
1/2
```

**width()**

Return the width of the ribbon.

This is given by the length of the longest row in the outer partition.

EXAMPLES:

```python
sage: RibbonShapedTableau([[2,3],[1,4,5]]).width()
4
sage: RibbonShapedTableau([]).width()
0
```
class sage.combinat.ribbon_shaped_tableau.RibbonShapedTableaux(category=None)
    Bases: SkewTableaux
    The set of all ribbon shaped tableaux.

    Element
    alias of RibbonShapedTableau

    from_shape_and_word(shape, word)
    Return the ribbon corresponding to the given ribbon shape and word.

    EXAMPLES:

    sage: RibbonShapedTableaux().from_shape_and_word([1,3],[1,3,3,7])
    [[None, None, 1], [3, 3, 7]]

    options = Current options for Tableaux - ascii_art: repr - convention: English -
display: list - latex: diagram

class sage.combinat.ribbon_shaped_tableau.Ribbon_class(parent, t)
    Bases: RibbonShapedTableau
    This exists solely for unpickling Ribbon_class objects.

class sage.combinat.ribbon_shaped_tableau.StandardRibbonShapedTableaux(category=None)
    Bases: StandardSkewTableaux
    The set of all standard ribbon shaped tableaux.

    INPUT:
    - shape - (optional) the composition shape of the rows

    Element
    alias of RibbonShapedTableau

    from_permutation(p)
    Return a standard ribbon of size len(p) from a permutation p. The lengths of each row are given by the
distance between the descents of the permutation p.

    EXAMPLES:

    sage: import sage.combinat.ribbon_shaped_tableau as rst
    sage: [StandardRibbonShapedTableaux().from_permutation(p) for p in Permutations(3)]
    [[[1, 2, 3]],
     [[None, 2], [1, 3]],
     [[1, 3], [2]],
     [[None, 1], [2, 3]],
     [[1, 2], [3]],
     [[1], [2], [3]]]

    from_shape_and_word(shape, word)
    Return the ribbon corresponding to the given ribbon shape and word.

    EXAMPLES:

    sage: StandardRibbonShapedTableaux().from_shape_and_word([2,3],[1,2,3,4,5])
    [[None, None, 1, 2], [3, 4, 5]]
options = Current options for Tableaux - ascii_art: repr - convention: English - display: list - latex: diagram

class sage.combinat.ribbon_shaped_tableau.StandardRibbonShapedTableaux(shape)
    Bases: StandardRibbonShapedTableaux

    Class of standard ribbon shaped tableaux of ribbon shape shape.

    EXAMPLES:

    sage: StandardRibbonShapedTableaux([2,2])
    Standard ribbon shaped tableaux of shape [2, 2]
    sage: StandardRibbonShapedTableaux([2,2]).first()
    [[None, 2, 4], [1, 3]]
    sage: StandardRibbonShapedTableaux([2,2]).last()
    [[None, 1, 2], [3, 4]]
    sage: StandardRibbonShapedTableaux([2,2]).cardinality()
    5
    sage: StandardRibbonShapedTableaux([2,2]).list()
    [[[None, 1, 3], [2, 4]],
     [[None, 1, 2], [3, 4]],
     [[None, 2, 3], [1, 4]],
     [[None, 2, 4], [1, 3]],
     [[None, 1, 4], [2, 3]]]
    sage: StandardRibbonShapedTableaux([3,2,2]).cardinality()
    155

    first()
        Return the first standard ribbon of self.

        EXAMPLES:

        sage: StandardRibbonShapedTableaux([2,2]).first()
        [[None, 2, 4], [1, 3]]

    last()
        Return the last standard ribbon of self.

        EXAMPLES:

        sage: StandardRibbonShapedTableaux([2,2]).last()
        [[None, 1, 2], [3, 4]]

5.1.195 Ribbon Tableaux

class sage.combinat.ribbon_tableau.MultiSkewTableau(parent, *args, **kwds)
    Bases: CombinatorialElement

    A multi skew tableau which is a tuple of skew tableaux.

    EXAMPLES:

    sage: s = MultiSkewTableau([ [[None,1],[2,3]], [[1,2],[2]] ])  
sage: s.size()
    6
inversion_pairs()
Return a list of the inversion pairs of self.

EXAMPLES:

```python
sage: s = MultiSkewTableau([[[2,3],[5,5]], [[1,1],[3,3]], [[2],[6]]])
sage: s.inversion_pairs()
[((0, (0)), (1, (0))),
 ((0, (1)), (1, (0))),
 ((0, (1)), (1, (1))),
 ((0, (1)), (2, (0))),
 ((1, (0)), (2, (0))),
 ((1, (1)), (2, (0)))]
```

inversions()
Return the number of inversion pairs of self.

EXAMPLES:

```python
sage: t1 = SkewTableau([[1]])
sage: t2 = SkewTableau([[2]])
sage: MultiSkewTableau([t1,t1]).inversions() 0
sage: MultiSkewTableau([t1,t2]).inversions() 0
sage: MultiSkewTableau([t2,t2]).inversions() 0
sage: MultiSkewTableau([t2,t1]).inversions() 1
sage: s = MultiSkewTableau([[[2,3],[5,5]], [[1,1],[3,3]], [[2],[6]]])
sage: s.inversions() 7
```

shape()
Return the shape of self.

EXAMPLES:

```python
sage: s = SemistandardSkewTableaux([[2,2],[1]]).list()
sage: a = MultiSkewTableau([s[0],s[1],s[2]])
sage: a.shape()
[[2, 2] / [1], [2, 2] / [1], [2, 2] / [1]]
```

size()
Return the size of self.

This is the sum of the sizes of the skew tableaux in self.

EXAMPLES:
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```
sage: s = SemistandardSkewTableaux([[2,2],[1]]).list()
sage: a = MultiSkewTableau([s[0],s[1],s[2]])
sage: a.size()
9
```

### weight()

Return the weight of `self`.

**EXAMPLES:**

```
sage: s = SemistandardSkewTableaux([[2,2],[1]]).list()
sage: a = MultiSkewTableau([s[0],s[1],s[2]])
sage: a.weight()
[5, 3, 1]
```

```python
class sage.combinat.ribbon_tableau.MultiSkewTableaux(category=None)
    Bases: UniqueRepresentation, Parent

    Multiskew tableaux.

    **Element**
    
    alias of `MultiSkewTableau`

class sage.combinat.ribbon_tableau.RibbonTableau(parent, st)
    Bases: SkewTableau

    A ribbon tableau.

    A ribbon is a connected skew shape which does not contain any $2 \times 2$ boxes. A ribbon tableau is a skew tableau whose shape is partitioned into ribbons, each of which is filled with identical entries.

    **EXAMPLES:**

    ```python
    sage: rt = RibbonTableau([[None, 1],[2,3]]); rt
    [[None, 1], [2, 3]]
    sage: rt.inner_shape()
    [1]
    sage: rt.outer_shape()
    [2, 2]
    sage: rt = RibbonTableau([[None, None, 0, 0, 0], [None, 0, 0, 2], [1, 0, 1]]); rt.pp()
    . . 0 0 0
    . 0 0 2
    1 0 1
    ```

    In the previous example, each ribbon is uniquely determined by a non-zero entry. The 0 entries are used to fill in the rest of the skew shape.

    **Note:** Sanity checks are not performed; lists can contain any object.
```

```
sage: RibbonTableau(expr=[[1,1],[[5],[3,4],[1,2]]])
[[None, 1, 2], [None, 3, 4], [5]]
```
length()

Return the length of the ribbons into a ribbon tableau.

EXAMPLES:

```python
sage: RibbonTableau([[None, 1], [2, 3]]).length()
1
sage: RibbonTableau([[1, 0], [2, 0]]).length()
2
```

to_word()

Return a word obtained from a row reading of self.

WARNING: Unlike the to_word method on skew tableaux (which are a superclass of this), this method does not filter out None entries.

EXAMPLES:

```python
sage: R = RibbonTableau([[0, 0, 3, 0], [1, 1, 0], [2, 0, 4]])
sage: R.to_word()
word: 2041100030
```

class sage.combinat.ribbon_tableau.RibbonTableau_class(parent, st)

Bases: RibbonTableau

This exists solely for unpickling RibbonTableau_class objects.

class sage.combinat.ribbon_tableau.RibbonTableaux

Bases: UniqueRepresentation, Parent

Ribbon tableaux.

A ribbon tableau is a skew tableau whose skew shape is tiled by ribbons of length length. The weight is calculated from the labels on the ribbons.

Note: Here we impose the condition that the ribbon tableaux are semistandard.

INPUT(Optional):

- shape – skew shape as a list of lists or an object of type SkewPartition
- length – integer, shape is partitioned into ribbons of length length
- weight – list of integers, computed from the values of non-zero entries labeling the ribbons

EXAMPLES:

```python
sage: RibbonTableaux([[2, 1], []], [1, 1, 1], 1)
Ribbon tableaux of shape [2, 1] / [] and weight [1, 1, 1] with 1-ribbons
sage: R = RibbonTableaux([[5, 4, 3], [2, 1]], [2, 1], 3)
sage: for i in R: i.pp(); print('
')
  . 0 0 0
  . 0 0 2
  1 0 1
```
REFERENCES:

Element
alias of RibbonTableau

from_expr(l)
Return a RibbonTableau from a MuPAD-Combinat expr for a skew tableau. The first list in expr is the
inner shape of the skew tableau. The second list are the entries in the rows of the skew tableau from bottom
to top.
Provided primarily for compatibility with MuPAD-Combinat.
EXAMPLES:

```
sage: RibbonTableaux().from_expr([[1,1],[[5],[3,4],[1,2]]])
[[None, 1, 2], [None, 3, 4], [5]]
```

options = Current options for Tableaux - ascii_art: repr - convention: English-
display: list - latex: diagram

class sage.combinat.ribbon_tableau.RibbonTableaux_shape_weight_length(shape, weight, length)
Bases: RibbonTableaux
Ribbon tableaux of a given shape, weight, and length.
cardinality()
Return the cardinality of self.
EXAMPLES:

```
sage: RibbonTableaux([[2,1],[[]],[1,1,1],[1]]).cardinality()
2
sage: RibbonTableaux([[2,2],[[]],[1,1,2]]).cardinality()
2
sage: RibbonTableaux([[4,3],[[]],[2,1,1,1,2]]).cardinality()
5
```

class sage.combinat.ribbon_tableau.SemistandardMultiSkewTableaux(shape, weight)
Bases: MultiSkewTableaux
Semistandard multi skew tableaux.
A multi skew tableau is a k-tuple of skew tableaux of given shape with a specified total weight.
EXAMPLES:
```sage
S = SemistandardMultiSkewTableaux([[[2,1],[[]]], [[2,2],[[1]]],[[2,2,2]]]; S
Semistandard multi skew tableaux of shape [[2, 1] / [], [2, 2] / [1]] and weight [2, → 2, 2]
sage: S.list()
[[[None, 2], [3, 3]],
[[None, 2], [3, 3]],
[[None, 2], [2, 3]],
[[None, 2], [2, 3]],
[[None, 2], [2, 3]],
[[None, 2], [1, 3]],
[[None, 2], [1, 3]],
[[None, 2], [1, 3]],
[[None, 1], [2, 3]],
[[None, 1], [2, 3]],
[[None, 1], [1, 3]],
[[None, 1], [1, 2]],
[[None, 1], [1, 2]]]
sage.combinat.ribbon_tableau.cospin_polynomial(part, weight, length)
Return the cospin polynomial associated to part, weight, and length.
EXAMPLES:
```
```sage
sage: from sage.combinat.ribbon_tableau import cospin_polynomial
sage: cospin_polynomial([6,6,6],[4,2],3)
t^4 + t^3 + 2*t^2 + t + 1
sage: cospin_polynomial([3,3,3,2,1], [3,1], 3)
t + 1
sage: cospin_polynomial([3,3,3,2,1], [2,2], 3)
t + 2
sage: cospin_polynomial([3,3,3,2,1], [2,1,1], 3)
t^2 + 2*t^2 + 3
sage: cospin_polynomial([3,3,3,2,1], [1,1,1,1], 3)
t^3 + 3*t^2 + 5*t + 3
sage: cospin_polynomial([5,4,3,2,1,1,1], [2,2,1], 3)
2*t^2 + 6*t + 2
sage: cospin_polynomial([[6]*6, [3,3]], [4,4,2], 3)
3*t^4 + 6*t^3 + 9*t^2 + 5*t + 3
```
```sage
cospin_polynomial(part, weight, length)
```
```sage
sage.combinat.ribbon_tableau.count_rec(nexts, current, part, weight, length)
INPUT:
• nexts, current, part – skew partitions
• weight – non-negative integer list
• length – integer
```
```sage
count_rec(nexts, current, part, weight, length)
```
```sage
sage.combinat.ribbon_tableau.graph_implementation_rec(skp, weight, length, function)
sage.combinat.ribbon_tableau.insertion_tableau(skp, perm, evaluation, tableau, length)
INPUT:
• skp – skew partitions
• perm, evaluation – non-negative integers
• tableau – skew tableau
• length – integer
```
```sage
```
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sage.combinat.ribbon_tableau.list_rec(nexts, current, part, weight, length)

INPUT:
- nexts, current, part – skew partitions
- weight – non-negative integer list
- length – integer

sage.combinat.ribbon_tableau.spin_polynomial(part, weight, length)

Returns the spin polynomial associated to part, weight, and length.

EXAMPLES:

```
sage: from sage.combinat.ribbon_tableau import spin_polynomial
sage: spin_polynomial([6,6,6],[4,2],3)
t^6 + t^5 + 2*t^4 + t^3 + t^2
sage: spin_polynomial([6,6,6],[4,1,1],3)
t^6 + 2*t^5 + 3*t^4 + 2*t^3 + t^2
sage: spin_polynomial([3,3,3,2,1], [2,2], 3)
t^(7/2) + t^(5/2)
sage: spin_polynomial([3,3,3,2,1], [2,1,1], 3)
2*t^7 + 2*t^5 + t^3
sage: spin_polynomial([3,3,3,2,1], [1,1,1,1], 3)
3*t^7 + 5*t^5 + 3*t^3 + sqrt(t)
sage: spin_polynomial([5,4,3,2,1,1,1], [2,2,1], 3)
2*t^9 + 6*t^7 + 2*t^5
sage: spin_polynomial([3,3,3,2,1,1], [2,2,1], 3)
3*t^9 + 5*t^8 + 9*t^7 + 6*t^6 + 3*t^5
```

sage.combinat.ribbon_tableau.spin_polynomial_square(part, weight, length)

Returns the spin polynomial associated with part, weight, and length, with the substitution \( t \rightarrow t^2 \) made.

EXAMPLES:

```
sage: from sage.combinat.ribbon_tableau import spin_polynomial_square
sage: spin_polynomial_square([6,6,6],[4,2],3)
t^12 + t^10 + 2*t^8 + t^6 + t^4
sage: spin_polynomial_square([6,6,6],[4,1,1],3)
t^12 + 2*t^10 + 3*t^8 + 2*t^6 + t^4
sage: spin_polynomial_square([3,3,3,2,1], [2,2], 3)
t^7 + t^5
sage: spin_polynomial_square([3,3,3,2,1], [2,1,1], 3)
2*t^7 + 2*t^5 + t^3
sage: spin_polynomial_square([3,3,3,2,1], [1,1,1,1], 3)
3*t^7 + 5*t^5 + 3*t^3 + t
sage: spin_polynomial_square([5,4,3,2,1,1,1], [2,2,1], 3)
2*t^9 + 6*t^7 + 2*t^5
sage: spin_polynomial_square([3,3,3,2,1,1], [2,2,1], 3)
3*t^9 + 5*t^8 + 9*t^7 + 6*t^6 + 3*t^5
```

sage.combinat.ribbon_tableau.spin_rec(t, nexts, current, part, weight, length)

Routine used for constructing the spin polynomial.

INPUT:
- weight – list of non-negative integers
- length – the length of the ribbons we're tiling with
- t – the variable

EXAMPLES:

```python
sage: from sage.combinat.ribbon_tableau import spin_rec
sage: sp = SkewPartition
sage: t = ZZ['t'].gen()

sage: spin_rec(t, [[]], [[[[]], [3, 3]]], sp([[2, 2, 2], []]), [2], 3)
[t^4]

sage: spin_rec(t, [[0]], [[2, 1, 1, 1, 1], [0, 3]], [[2, 2, 2], [3, 0]], sp([[2, 2, 2, 1], []]), [2, 1], 3)
[t^5]

sage: spin_rec(t, [[]], [[[[]], [3, 3, 0]]], sp([[3, 3], []]), [2], 3)
[t^2]

sage: spin_rec(t, [[t^4], [t^3], [t^2]], [[2, 2, 2, 0, 0, 3]], [[3, 2, 2, 1, 0, 0, 3]], sp([[3, 3, 3], []]), [2, 1], 3)
[t^6 + t^4 + t^2]

sage: spin_rec(t, [[t^5], [t^4], [t^6 + t^4 + t^2]], [[2, 2, 2, 1, 0, 0, 3]], [[3, 3, 1, 1, 1, 0, 0, 0]], sp([[3, 3, 3, 2, 1], []]), [2, 1, 1], 3)
[2*t^7 + 2*t^5 + t^3]
```

### 5.1.196 Rigged configurations

**Todo:** Proofread / point to the main classes rather than the modules?

- Crystal of Rigged Configurations
- Rigged Configurations of \(B(\infty)\)
- Rigged Configurations
- Rigged Configuration Elements
- Tensor Product of Kirillov-Reshetikhin Tableaux
- Tensor Product of Kirillov-Reshetikhin Tableaux Elements
- Kirillov-Reshetikhin Tableaux
- Kleber Trees
- Rigged Partitions
Bijections

- **Bijection between rigged configurations and KR tableaux**
- **Abstract classes for the rigged configuration bijections**
  - **Bijection classes for type $A_n^{(1)}$**
  - **Bijection classes for type $B_n^{(1)}$**
  - **Bijection classes for type $C_n^{(1)}$**
  - **Bijection classes for type $D_n^{(1)}$**
  - **Bijection classes for type $A_{2n-1}^{(2)}$.**
  - **Bijection classes for type $A_{2n}^{(2)}$.**
  - **Bijection classes for type $A_{2n}^{(2)\dagger}$**
  - **Bijection classes for type $D_{n+1}^{(2)}$.**
  - **Bijection classes for type $D_4^{(3)}$.**
- **Bijection between rigged configurations for $B(\infty)$ and marginally large tableaux**

### 5.1.197 Abstract classes for the rigged configuration bijections

This file contains two sets of classes, one for the bijection from KR tableaux to rigged configurations and the other for the reverse bijection. We do this for two reasons, one is because we can store a state in the bijection locally, so we do not have to constantly pass it around between functions. The other is because it makes the code easier to read in the *_element.py files.

These classes are not meant to be used by the user and are only supposed to be used internally to perform the bijections between `TensorProductOfKirillovReshetikhinTableaux` and `RiggedConfigurations`.

**AUTHORS:**

```python
class sage.combinat.rigged_configurations.bij_abstract_class.KRTToRCBijectionAbstract(tp_krt):
    Bases: object
    Root abstract class for the bijection from KR tableaux to rigged configurations.

    This class holds the state of the bijection and generates the next state. This class should never be created directly.

    next_state(val)
        Build the next state in the bijection.

        INPUT:
        - val – The value we are adding

    run(overwrite=False)
        Run the bijection from a tensor product of KR tableaux to a rigged configuration.

        INPUT:
        - tp_krt – A tensor product of KR tableaux
        - overwrite – (Default: False) Display each step in the bijection
```

**EXAMPLES:**
```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A', 4, 1], [[2, 1]])
sage: from sage.combinat.rigged_configurations.bij_type_A import RKTToRCBijectionTypeA
sage: RKTToRCBijectionTypeA(KRT(pathlist=[[5,2]])).run()
-1[-1]
1[ ]1
0[ ]0
-1[-1]
```

class sage.combinat.rigged_configurations.bij_abstract_class.RCToKRTBijectionAbstract(RC_element)
    Bases: object

    Root abstract class for the bijection from rigged configurations to tensor product of Kirillov-Reshetikhin tableaux. This class holds the state of the bijection and generates the next state. This class should never be created directly.

    next_state(height)
        Build the next state in the bijection.

    run(verbos=True, build_graph=False)
        Run the bijection from rigged configurations to tensor product of KR tableaux.

        INPUT:

        • verbose – (default: False) display each step in the bijection
        • build_graph – (default: False) build the graph of each step of the bijection

        EXAMPLES:
```python
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 1]])
sage: x = RC(partition_list=[[1],[1],[1],[1]])
sage: from sage.combinat.rigged_configurations.bij_type_A import RCToKRTBijectionTypeA
    RCToKRTBijectionTypeA(x).run()
    [[2], [5]]
sage: bij = RCToKRTBijectionTypeA(x)
sage: bij.run(build_graph=True)
    [[2], [5]]
sage: bij._graph
    Digraph on 3 vertices
```
5.1.198 Bijection between rigged configurations for $B(\infty)$ and marginally large tableaux

AUTHORS:
- Travis Scrimshaw (2015-07-01): Initial version

REFERENCES:

class sage.combinat.rigged_configurations.bij_infinity.FromRCIsomorphism
   Bases: Morphism
   Crystal isomorphism of $B(\infty)$ in the rigged configuration model to the tableau model.

class sage.combinat.rigged_configurations.bij_infinity.FromTableauIsomorphism
   Bases: Morphism
   Crystal isomorphism of $B(\infty)$ in the tableau model to the rigged configuration model.

class sage.combinat.rigged_configurations.bij_infinity.MLTToRCBijectionTypeB(tp_krt)
   Bases: KRTToRCBijectionTypeB
   run()
   Run the bijection from a marginally large tableaux to a rigged configuration.

   EXAMPLES:

   sage: vct = CartanType(['B',4]).as_folding()
   sage: RC = crystals.infinity.RiggedConfigurations(vct)
   sage: T = crystals.infinity.Tableaux(['B',4])
   sage: Psi = T.crystal_morphism({T.module_generators[0]: RC.module_generators[0]})
   sage: TS = [x.value for x in T.subcrystal(max_depth=4)]
   sage: all(Psi(b) == RC(b) for b in TS) # long time # indirect doctest
   True

   class sage.combinat.rigged_configurations.bij_infinity.MLTToRCBijectionTypeD(tp_krt)
   Bases: KRTToRCBijectionTypeD
   run()
   Run the bijection from a marginally large tableaux to a rigged configuration.

   EXAMPLES:

   sage: RC = crystals.infinity.RiggedConfigurations(['D',4])
   sage: T = crystals.infinity.Tableaux(['D',4])
   sage: Psi = T.crystal_morphism({T.module_generators[0]: RC.module_generators[0]})
   sage: TS = [x.value for x in T.subcrystal(max_depth=4)]
   sage: all(Psi(b) == RC(b) for b in TS) # long time # indirect doctest
   True

   class sage.combinat.rigged_configurations.bij_infinity.RCToMLTBijectionTypeB(RC_element)
   Bases: RCToKRTBijectionTypeB
   run()
   Run the bijection from rigged configurations to a marginally large tableau.

   EXAMPLES:
class sage.combinat.rigged_configurations.bij_infinity.RCToMLTBijectionTypeD(RC_element)
    Bases: RCToKRTBijectionTypeD

    run()
    Run the bijection from rigged configurations to a marginally large tableau.
    EXAMPLES:

    sage: RC = crystals.infinity.RiggedConfigurations(['D',4])
    sage: T = crystals.infinity.Tableaux(['D',4])
    sage: Psi = RC.crystal_morphism({RC.module_generators[0]: T.module_generators[0]})
    sage: RCS = [x.value for x in RC.subcrystal(max_depth=4)]
    sage: all(Psi(nu) == T(nu) for nu in RCS) # long time # indirect doctest
    True

5.1.199 Bijection classes for type $A_n^{(1)}$

Part of the (internal) classes which run the bijection between rigged configurations and tensor products of Kirillov-Reshetikhin tableaux of type $A_n^{(1)}$.

AUTHORS:


class sage.combinat.rigged_configurations.bij_type_A.KRTToRCBijectionTypeA(tp_krt)
    Bases: KRTToRCBijectionAbstract

    next_state(val)
    Build the next state for type $A_n^{(1)}$.
    EXAMPLES:

    sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A', 4, 1], [[2,1]])
    sage: from sage.combinat.rigged_configurations.bij_type_A import KRTToRCBijectionTypeA
    sage: bijection = KRTToRCBijectionTypeA(KRT(pathlist=[[4,3]]))
    sage: bijection.cur_path.insert(0, [])
    sage: bijection.cur_dims.insert(0, [0, 1])
    sage: bijection.cur_path[0].insert(0, [3])
    sage: bijection.next_state(3)
class sage.combinat.rigged_configurations.bij_type_A.RCToKRTBijectionTypeA(RC_element)
    Bases: RCToKRTBijectionAbstract
    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $A_n(1)$.

    next_state(height)
    Build the next state for type $A_n(1)$.

EXAMPLES:

    sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 1]])
    sage: from sage.combinat.rigged_configurations.bij_type_A import RCToKRTBijectionTypeA
    sage: bijection = RCToKRTBijectionTypeA(RC(partition_list=[[1], [1], [1], [1]]))
    sage: bijection.next_state(1)
    5

5.1.200 Bijection classes for type $A_{2n}^{(2)\dagger}$

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $A_{2n}^{(2)\dagger}$.

AUTHORS:

- Travis Scrimshaw (2012-12-21): Initial version

class sage.combinat.rigged_configurations.bij_type_A2_dual.KRTToRCBijectionTypeA2Dual(tp_krt)
    Bases: KRTToRCBijectionTypeC
    Specific implementation of the bijection from KR tableaux to rigged configurations for type $A_{2n}^{(2)\dagger}$.

This inherits from type $C_n(1)$ because we use the same methods in some places.

    next_state(val)
    Build the next state for type $A_{2n}^{(2)\dagger}$.

class sage.combinat.rigged_configurations.bij_type_A2_dual.RCToKRTBijectionTypeA2Dual(RC_element)
    Bases: RCToKRTBijectionTypeC
    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $A_{2n}^{(2)\dagger}$.

    next_state(height)
    Build the next state for type $A_{2n}^{(2)\dagger}$.

5.1.201 Bijection classes for type $A_{2n}^{(2)}$

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $A_{2n}^{(2)}$.

AUTHORS:

- Travis Scrimshaw (2012-12-21): Initial version
class sage.combinat.rigged_configurations.bij_type_A2_even.KRTToRCBijectionTypeA2Even(tp_krt)
    Bases: KRTToRCBijectionTypeC
    Specific implementation of the bijection from KR tableaux to rigged configurations for type $A_{2n}^{(2)}$.
    This inherits from type $C_n^{(1)}$ because we use the same methods in some places.
    next_state(val)
        Build the next state for type $A_{2n}^{(2)}$.

class sage.combinat.rigged_configurations.bij_type_A2_even.RCToKRTBijectionTypeA2Even(RC_element)
    Bases: RCToKRTBijectionTypeC
    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $A_{2n}^{(2)}$.
    next_state(height)
        Build the next state for type $A_{2n}^{(2)}$.

5.1.202 Bijection classes for type $A_{2n-1}^{(2)}$.

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $A_{2n-1}^{(2)}$.

AUTHORS:
• Travis Scrimshaw (2012-12-21): Initial version

class sage.combinat.rigged_configurations.bij_type_A2_odd.KRTToRCBijectionTypeA2Odd(tp_krt)
    Bases: KRTToRCBijectionTypeA
    Specific implementation of the bijection from KR tableaux to rigged configurations for type $A_{2n-1}^{(2)}$.
    This inherits from type $A_n^{(1)}$ because we use the same methods in some places.
    next_state(val)
        Build the next state for type $A_{2n-1}^{(2)}$.

class sage.combinat.rigged_configurations.bij_type_A2_odd.RCToKRTBijectionTypeA2Odd(RC_element)
    Bases: RCToKRTBijectionTypeA
    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $A_{2n-1}^{(2)}$.
    next_state(height)
        Build the next state for type $A_{2n-1}^{(2)}$.

5.1.203 Bijection classes for type $B_n^{(1)}$.

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $B_n^{(1)}$.

AUTHORS:
• Travis Scrimshaw (2012-12-21): Initial version
class sage.combinat.rigged_configurations.bij_type_B.KRTToRCBijectionTypeB(tp_krt)

Bases: KRTToRCBijectionTypeC

Specific implementation of the bijection from KR tableaux to rigged configurations for type $B_{n}^{(1)}$.

next_state(val)

Build the next state for type $B_{n}^{(1)}$.

other_outcome(rc, pos_val, width_n)

Do the other case $(QS)$ possibility.

This arises from the ambiguity when we found a singular string at the max width in $\nu^{(n)}$. We had first attempted case $(S)$, and if that resulted in an invalid rigged configuration, we now finish the bijection using case $(QS)$.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['B',3,1], [[2,1],[1,2]])
sage: rc = RC(partition_list=[[2,1], [2,1,1], [5,1]])
sage: t = rc.to_tensor_product_of_kirillov_reshetikhin_tableaux()
sage: t.to_rigged_configuration() == rc
# indirect doctest
True
```

run(verbos=False)

Run the bijection from a tensor product of KR tableaux to a rigged configuration.

INPUT:

- `tp_krt` – A tensor product of KR tableaux
- `verbose` – (Default: False) Display each step in the bijection

EXAMPLES:

```
sage: from sage.combinat.rigged_configurations.bij_type_B import KRTToRCBijectionTypeB
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['B', 3, 1],-
->[2, 1])
sage: KRTToRCBijectionTypeB(KRT(pathlist=[[0,3]])).run()
0 0
-1 -1
-1 -1
0 0

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['B', 3, 1],-
->[3, 1])
sage: KRTToRCBijectionTypeB(KRT(pathlist=[[-2,3,1]])).run()
(1/)
-1 -1
0 0
```

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class sage.combinat.rigged_configurations.bij_type_B.RCToKRTBijectionTypeB(RC_element)
Bases: RCToKRTBijectionTypeC

Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $B_n^{(1)}$.

**next_state(height)**
Build the next state for type $B_n^{(1)}$.

**run(verbose=False, build_graph=False)**
Run the bijection from rigged configurations to tensor product of KR tableaux for type $B_n^{(1)}$.

**INPUT:**
- verbose – (default: False) display each step in the bijection
- build_graph – (default: False) build the graph of each step of the bijection

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['B', 3, 1], [[2, 1]])
\rightarrow RCToKRTBijectionTypeB
sage: RCToKRTBijectionTypeB(RC(partition_list=[[1],[1],[1]])).run()
[[3], [0]]

sage: RC = RiggedConfigurations(['B', 3, 1], [[3, 1]])

sage: x = RC(partition_list=[[1],[1],[1]])

sage: RCToKRTBijectionTypeB(x).run()
[[1], [3], [-2]]

sage: bij = RCToKRTBijectionTypeB(x)

sage: bij.run(build_graph=True)
[[1], [3], [-2]]

sage: bij._graph
Digraph on 6 vertices
```

### 5.1.204 Bijection classes for type $C_n^{(1)}$

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $C_n^{(1)}$.

**AUTHORS:**
- Travis Scrimshaw (2012-12-21): Initial version

class sage.combinat.rigged_configurations.bij_type_C.KRTToRCBijectionTypeC(tp_krt)
Bases: KRTToRCBijectionTypeA

Specific implementation of the bijection from KR tableaux to rigged configurations for type $C_n^{(1)}$.

This inherits from type $A_n^{(1)}$ because we use the same methods in some places.

**next_state(val)**
Build the next state for type $C_n^{(1)}$.  

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class sage.combinat.rigged_configurations.bij_type_C.RCToKRTBijectionTypeC(RC_element)
    Bases: RCToKRTBijectionTypeA

    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $C_n^{(1)}$.

    next_state(height)
        Build the next state for type $C_n^{(1)}$.

5.1.205 Bijection classes for type $D_n^{(1)}$

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $D_n^{(1)}$.

AUTHORS:

class sage.combinat.rigged_configurations.bij_type_D.KRTToRCBijectionTypeD(tp_krt)
    Bases: KRTToRCBijectionTypeA

    Specific implementation of the bijection from KR tableaux to rigged configurations for type $D_n^{(1)}$.
    This inherits from type $A_n^{(1)}$ because we use the same methods in some places.

doubling_map()
    Perform the doubling map of the rigged configuration at the current state of the bijection.
    This is the map $B(\Lambda) \mapsto B(2\Lambda)$ which doubles each of the rigged partitions and updates the vacancy numbers accordingly.

halving_map()
    Perform the halving map of the rigged configuration at the current state of the bijection.
    This is the inverse map to $B(\Lambda) \mapsto B(2\Lambda)$ which halves each of the rigged partitions and updates the vacancy numbers accordingly.

next_state(val)
    Build the next state for type $D_n^{(1)}$.

run(verbose=False)
    Run the bijection from a tensor product of KR tableaux to a rigged configuration for type $D_n^{(1)}$.

    INPUT:
    * tp_krt -- A tensor product of KR tableaux
    * verbose -- (Default: False) Display each step in the bijection

    EXAMPLES:

    sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 1], [[2,1]])
    sage: from sage.combinat.rigged_configurations.bij_type_D import KRTToRCBijectionTypeD
    sage: KRTToRCBijectionTypeD(KRT(pathlist=[[[-3,2]]])).run()
    -1[ ]-1
class sage.combinat.rigged_configurations.bij_type_D.RCToKRTBijectionTypeD(RC_element)

Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $D_n^{(1)}$.

doubling_map()

Perform the doubling map of the rigged configuration at the current state of the bijection.

This is the map $B(\Lambda) \mapsto B(2\Lambda)$ which doubles each of the rigged partitions and updates the vacancy numbers accordingly.

halving_map()

Perform the halving map of the rigged configuration at the current state of the bijection.

This is the inverse map to $B(\Lambda) \mapsto B(2\Lambda)$ which halves each of the rigged partitions and updates the vacancy numbers accordingly.

next_state(height)

Build the next state for type $D_n^{(1)}$.

run(verbos=False, build_graph=False)

Run the bijection from rigged configurations to tensor product of KR tableaux for type $D_n^{(1)}$.

INPUT:

• verbose – (default: False) display each step in the bijection

• build_graph – (default: False) build the graph of each step of the bijection

EXAMPLES:

```
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 1]])
sage: x = RC(partition_list=[[1], [1], [1], [1]])
sage: from sage.combinat.rigged_configurations.bij_type_D import RCToKRTBijectionTypeD
sage: RCToKRTBijectionTypeD(x).run()
[[2], [-3]]
sage: bij = RCToKRTBijectionTypeD(x)
sage: bij.run(build_graph=True)
[[2], [-3]]
sage: bij._graph
Digraph on 3 vertices
```
5.1.206 Bijection classes for type $D_{n+1}^{(2)}$

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $D_{n+1}^{(2)}$.

AUTHORS:

• Travis Scrimshaw (2011-04-15): Initial version

```python
class sage.combinat.rigged_configurations.bij_type_D_twisted.KRTToRCBijectionTypeDTwisted(tp_krt):
    Bases: KRTToRCBijectionTypeD, KRTToRCBijectionTypeA2Even
    Specific implementation of the bijection from KR tableaux to rigged configurations for type $D_{n+1}^{(2)}$.
    This inherits from type $C_n^{(1)}$ and $D_n^{(1)}$ because we use the same methods in some places.

    next_state(val)
        Build the next state for type $D_{n+1}^{(2)}$.

    run(verbose=False)
        Run the bijection from a tensor product of KR tableaux to a rigged configuration for type $D_{n+1}^{(2)}$.
        INPUT:
        • tp_krt – A tensor product of KR tableaux
        • verbose – (Default: False) Display each step in the bijection
        EXAMPLES:

        sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 2],
        ...                  [[3,1]])
        sage: from sage.combinat.rigged_configurations.bij_type_D_twisted import KRTToRCBijectionTypeDTwisted
        sage: KRTToRCBijectionTypeDTwisted(KRT(pathlist=[[-1,3,2]])).run()
        -1[ ]-1
        0[ ]0
        1[ ]1
```

```python
class sage.combinat.rigged_configurations.bij_type_D_twisted.RCToKRTBijectionTypeDTwisted(RC_element):
    Bases: RCToKRTBijectionTypeD, RCToKRTBijectionTypeA2Even
    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $D_{n+1}^{(2)}$.

    next_state(height)
        Build the next state for type $D_{n+1}^{(2)}$.

    run(verbose=False, build_graph=False)
        Run the bijection from rigged configurations to tensor product of KR tableaux for type $D_{n+1}^{(2)}$.
        INPUT:
        • verbose – (default: False) display each step in the bijection
        • build_graph – (default: False) build the graph of each step of the bijection
        EXAMPLES:
```

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5.1.207 Bijection classes for type $D_4^{(3)}$

Part of the (internal) classes which runs the bijection between rigged configurations and KR tableaux of type $D_4^{(3)}$.

AUTHORS:

- Travis Scrimshaw (2014-09-10): Initial version

```python
class sage.combinat.rigged_configurations.bij_type_D_tri.KRTToRCBijectionTypeDTri(tp_krt)
    Bases: KRTToRCBijectionTypeA

    Specific implementation of the bijection from KR tableaux to rigged configurations for type $D_4^{(3)}$.

    This inherits from type $A_n^{(1)}$ because we use the same methods in some places.

    next_state(val)
        Build the next state for type $D_4^{(3)}$.

class sage.combinat.rigged_configurations.bij_type_D_tri.RCToKRTBijectionTypeDTri(RC_element)
    Bases: RCToKRTBijectionTypeA

    Specific implementation of the bijection from rigged configurations to tensor products of KR tableaux for type $D_4^{(3)}$.

    next_state(height)
        Build the next state for type $D_4^{(3)}$.
```

5.1.208 Bijection between rigged configurations and KR tableaux

Functions which are big switch statements to create the bijection class of the correct type.

AUTHORS:

- Travis Scrimshaw (2012-12-21): Added all non-exceptional bijection types
- Travis Scrimshaw (2014-09-10): Added type $D_4^{(3)}$

```python
sage.combinat.rigged_configurations.bijection.KRTToRCBijection(tp_krt)
    Return the correct KR tableaux to rigged configuration bijection helper class.

class sage.combinat.rigged_configurations.bijection.RCToKRTBijection(rc_elt):
    Return the correct rigged configuration to KR tableaux bijection helper class.
```

5.1. Comprehensive Module List 2177
5.1.209 Kleber Trees

A Kleber tree is a tree of weights generated by Kleber’s algorithm [Kleber1]. The nodes correspond to the weights in the positive Weyl chamber obtained by subtracting a (non-zero) positive root. The edges are labeled by the coefficients of the roots of the difference.

AUTHORS:
- Travis Scrimshaw (2011-05-03): Initial version
- Travis Scrimshaw (2013-02-13): Added support for virtual trees and improved LaTeX output

EXAMPLES:

```sage
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KleberTree(['A', 3, 1], [[3,2], [2,1], [1,1], [1,1]])
Kleber tree of Cartan type ['A', 3, 1] and B = ((3, 2), (2, 1), (1, 1), (1, 1))
sage: KleberTree(['D', 4, 1], [[2,2]])
Kleber tree of Cartan type ['D', 4, 1] and B = ((2, 2),)
```

class `sage.combinat.rigged_configurations.kleber_tree.KleberTree`(cartan_type, B, classical_ct)

Bases: `UniqueRepresentation`, `Parent`

The tree that is generated by Kleber’s algorithm.

A Kleber tree is a tree of weights generated by Kleber’s algorithm [Kleber1]. It is used to generate the set of all admissible rigged configurations for the simply-laced affine types $A_{1}^{(1)}, D_{n}^{(1)}, E_{6}^{(1)}, E_{7}^{(1)},$ and $E_{8}^{(1)}$.

See also:
There is a modified version for non-simply-laced affine types at `VirtualKleberTree`.

The nodes correspond to the weights in the positive Weyl chamber obtained by subtracting a (non-zero) positive root. The edges are labeled by the coefficients of the roots, and $X$ is a child of $Y$ if $Y$ is the root else if the edge label of $Y$ to its parent $Z$ is greater (in every component) than the label from $X$ to $Y$.

For a Kleber tree, one needs to specify an affine (simply-laced) Cartan type and a sequence of pairs $(r, s)$, where $s$ is any positive integer and $r$ is a node in the Dynkin diagram. Each $(r, s)$ can be viewed as a rectangle of width $s$ and height $r$.

INPUT:

- `cartan_type` – an affine simply-laced Cartan type
- `B` – a list of dimensions of rectangles by $[r, c]$ where $r$ is the number of rows and $c$ is the number of columns

REFERENCES:

EXAMPLES:

Simply-laced example:

```sage
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['A', 3, 1], [[3,2], [1,1]])
sage: KT.list()
[Kleber tree node with weight [1, 0, 2] and upwards edge root [0, 0, 0],
 Kleber tree node with weight [0, 0, 1] and upwards edge root [1, 1, 1]]
sage: KT = KleberTree(['A', 3, 1], [[3,2], [2,1], [1,1], [1,1]])
sage: KT.cardinality()
10
sage: KT = KleberTree(['D', 4, 1], [[2,2]])
```

(continues on next page)
sage: KT.cardinality()
3
sage: KT = KleberTree(['D', 4, 1], [[4,5]])

sage: KT.cardinality()
1

From [Kleber2]:

sage: KT = KleberTree(['E', 6, 1], [[4, 2]])  # long time (9s on sage.math, 2012)

sage: KT.cardinality()  # long time
12

We check that relabelled types work (github issue #16876):

sage: ct = CartanType(['A',3,1]).relabel(lambda x: x+2)
sage: kt = KleberTree(ct, [[3,1],[5,1]])

sage: list(kt)
[Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 0, 0],
 Kleber tree node with weight [0, 0, 0] and upwards edge root [1, 1, 1]]

sage: kt = KleberTree(['A',3,1], [[1,1],[3,1]])

sage: list(kt)
[Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 0, 0],
 Kleber tree node with weight [0, 0, 0] and upwards edge root [1, 1, 1]]

Element
alias of KleberTreeNode

breadth_first_iter()
Iterate over all nodes in the tree following a breadth-first traversal.

EXAMPLES:

```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['A', 3, 1], [[2, 2], [2, 3]])
sage: for x in KT.breadth_first_iter(): x
Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 0, 0],
Kleber tree node with weight [1, 3, 1] and upwards edge root [0, 1, 0],
Kleber tree node with weight [2, 1, 2] and upwards edge root [0, 1, 0],
Kleber tree node with weight [1, 1, 1] and upwards edge root [1, 2, 1]
```

cartan_type()
Return the Cartan type of this Kleber tree.

EXAMPLES:

```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
da
```

depth_first_iter()
Iterate (recursively) over the nodes in the tree following a depth-first traversal.
**EXAMPLES:**

```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['A', 3, 1], [[2, 2], [2, 3]])
sage: for x in KT.depth_first_iter(): x
Kleber tree node with weight [0, 5, 0] and upwards edge root [0, 0, 0]
Kleber tree node with weight [1, 3, 1] and upwards edge root [0, 1, 0]
Kleber tree node with weight [2, 1, 2] and upwards edge root [0, 1, 0]
Kleber tree node with weight [0, 3, 0] and upwards edge root [1, 2, 1]
Kleber tree node with weight [1, 1, 1] and upwards edge root [0, 1, 0]
Kleber tree node with weight [0, 1, 0] and upwards edge root [1, 2, 1]
```

**digraph()**

Return a DiGraph representation of this Kleber tree.

**EXAMPLES:**

```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['D', 4, 1], [[2, 2]])
sage: KT.digraph()
Digraph on 3 vertices
```

**latex_options(** options **)**

Return the current latex options if no arguments are passed, otherwise set the corresponding latex option.

**OPTIONS:**

- `hspace` – (default: 2.5) the horizontal spacing of the tree nodes
- `vspace` – (default: $x$) the vertical spacing of the tree nodes, here $x$ is the minimum of $\frac{2.5}{n}$ or $\frac{7.5}{n}$ where $n$ is the rank of the classical type
- `edge_labels` – (default: True) display edge labels
- `use_vector_notation` – (default: False) display edge labels using vector notation instead of a linear combination

**EXAMPLES:**

```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['D', 3, 1], [[2, 1], [2, 1]])
sage: KT.latex_options(vspace=-4, use_vector_notation=True)
sage: sorted(KT.latex_options().items())
[('edge_labels', True), ('hspace', 2.5), ('use_vector_notation', True), ('vspace →', -4)]
```

**plot(** options **)**

Return the plot of self as a directed graph.

**EXAMPLES:**

```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['D', 4, 1], [[2, 2]])
sage: print(KT.plot())
Graphics object consisting of 8 graphics primitives
```

**class** `sage.combinat.rigged_configurations.kleber_tree.KleberTreeNode(parent_obj, node_weight, dominant_root, parent_node=None)`
Bases: Element
A node in the Kleber tree.
This class is meant to be used internally by the Kleber tree class and should not be created directly by the user.
For more on the Kleber tree and the nodes, see KleberTree.
The dominating root is the up_root which is the difference between the parent node’s weight and this node’s weight.

INPUT:
• parent_obj – The parent object of this element
• node_weight – The weight of this node
• dominant_root – The dominating root
• parent_node – (default:None) The parent node of this node

depth()
Return the depth of this node in the tree.

EXAMPLES:

```
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: RS = RootSystem(['A', 2])
sage: WS = RS.weight_lattice()
sage: R = RS.root_lattice()
sage: KT = KleberTree(['A', 2, 1], [[1,1]])
sage: n = KT(WS.sum_of_terms([[1,5], [2,2]]), R.zero())
sage: n.depth
0
sage: n2 = KT(WS.sum_of_terms([[1,5], [2,2]]), R.zero(), n)
sage: n2.depth
1
```

multiplicity()
Return the multiplicity of self.
The multiplicity of a node $x$ of depth $d$ weight $\lambda$ in a simply-laced Kleber tree is equal to:

$$
\prod_{i>0} \prod_{a \in T} \left( \frac{p_i^{(a)} + m_i^{(a)}}{p_i^{(a)}} \right)
$$

Recall that

$$
m_i^{(a)} = \left( \lambda^{(i-1)} - 2\lambda^{(i)} + \lambda^{(i+1)} \mid \alpha_a \right),
\quad p_i^{(a)} = \left( \alpha_a \mid \lambda^{(i)} \right) - \sum_{j>i} (j-i)L_j^{(a)},
$$

where $\lambda^{(i)}$ is the weight node at depth $i$ in the path to $x$ from the root and we set $\lambda^{(j)} = \lambda$ for all $j \geq d$.

Note that $m_i^{(a)} = 0$ for all $i > d$.  

EXAMPLES:
```python
sage: from sage.combinat.rigged_configurations.kleber_tree import KleberTree
sage: KT = KleberTree(['A', 3, 1], [[3,2],[2,1],[1,1],[1,1]])
```
```
sage: for x in KT: x, x.multiplicity()
(Kleber tree node with weight [2, 1, 2] and upwards edge root [0, 0, 0], 1)
(Kleber tree node with weight [3, 0, 1] and upwards edge root [0, 1, 1], 1)
(Kleber tree node with weight [0, 2, 2] and upwards edge root [1, 1, 0], 2)
(Kleber tree node with weight [1, 0, 3] and upwards edge root [1, 1, 0], 2)
(Kleber tree node with weight [1, 1, 1] and upwards edge root [1, 1, 1], 4)
(Kleber tree node with weight [0, 0, 2] and upwards edge root [2, 2, 1], 2)
```
```
class sage.combinat.rigged_configurations.kleber_tree.KleberTreeTypeA2Even(cartan_type, B)
    Bases: VirtualKleberTree
    
    Kleber tree for types $A^{(2)}_{2n}$ and $A^{(2)}_{2n}$.  
    
    Note that here for $A^{(2)}_{2n}$ we use $\tilde{\gamma}_a$ in place of $\gamma_a$  
    in constructing the virtual Kleber tree, and so we end up selecting  
    all nodes since $\tilde{\gamma}_a = 1$ for all $a \in \mathcal{I}$.  
    For type $A^{(2)}_{2n}^\dagger$, we have $\gamma_a = 1$ for all $a \in \mathcal{I}$.  
    
    See also:
    
    VirtualKleberTree

    breadth_first_iter(all_nodes=False)
    
    Iterate over all nodes in the tree following a breadth-first traversal.
    
    INPUT:
    
    • all_nodes – (default: False) if True, output all nodes in the tree
    
    EXAMPLES:

    ```python
    sage: from sage.combinat.rigged_configurations.kleber_tree import VirtualKleberTree
    sage: KT = VirtualKleberTree(['A', 4, 2], [[2,1]])
    sage: for x in KT.breadth_first_iter(): x
    Kleber tree node with weight [0, 2, 0] and upwards edge root [0, 0, 0]
    Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]
    Kleber tree node with weight [0, 0, 0] and upwards edge root [1, 2, 1]
    sage: for x in KT.breadth_first_iter(True): x
    Kleber tree node with weight [0, 2, 0] and upwards edge root [0, 0, 0]
    Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]
    Kleber tree node with weight [0, 0, 0] and upwards edge root [1, 2, 1]
    ```
```
```
    depth_first_iter(all_nodes=False)
    
    Iterate (recursively) over the nodes in the tree following a depth-first traversal.
    
    INPUT:
    
    • all_nodes – (default: False) if True, output all nodes in the tree
    
    EXAMPLES:
```
```
sage: from sage.combinat.rigged_configurations.kleber_tree import VirtualKleberTree

sage: KT = VirtualKleberTree(['A', 4, 2], [[2,1]])
sage: for x in KT.depth_first_iter(): x
Kleber tree node with weight [0, 2, 0] and upwards edge root [0, 0, 0]
Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]
Kleber tree node with weight [0, 0, 0] and upwards edge root [1, 2, 1]
sage: for x in KT.depth_first_iter(True): x
Kleber tree node with weight [0, 2, 0] and upwards edge root [0, 0, 0]
Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]
Kleber tree node with weight [0, 0, 0] and upwards edge root [1, 2, 1]

class sage.combinat.rigged_configurations.kleber_tree.VirtualKleberTree(cartan_type, B)

    Bases: KleberTree

    A virtual Kleber tree.

    We can use a modified version of the Kleber algorithm called the virtual Kleber algorithm [OSS03] to compute all admissible rigged configurations for non-simply-laced types. This uses the following embeddings into the simply-laced types:

    \[
    \begin{align*}
    C_n^{(1)} &\rightarrow A_{2n}^{(2)}; \\
    A_{2n}^{(2)} &\rightarrow A_{2n-1}^{(1)}; \\
    D_{n+1}^{(2)} &\rightarrow A_{2n-1}^{(1)}; \\
    B_n^{(1)} &\rightarrow D_{n+1}^{(1)}; \\
    E_6^{(2)} &\rightarrow E_6^{(1)}; \\
    F_4^{(1)} &\rightarrow D_4^{(1)}; \\
    D_4^{(3)} &\rightarrow G_2^{(1)}.
    \end{align*}
    \]

    One then selects the subset of admissible nodes which are translates of the virtual requirements. In the graph, the selected nodes are indicated by brackets [ ].

    Note: Because these are virtual nodes, all information is given in the corresponding simply-laced type.

    See also:

    For more on the Kleber algorithm, see `KleberTree`.

    REFERENCES:

    INPUT:

    • cartan_type – an affine non-simply-laced Cartan type
    • B – a list of dimensions of rectangles by \([r, c]\) where \(r\) is the number of rows and \(c\) is the number of columns

    EXAMPLES:

    sage: from sage.combinat.rigged_configurations.kleber_tree import VirtualKleberTree
    sage: KT = VirtualKleberTree(['C', 4, 1], [[2,2]])
    sage: KT.cardinality()
    3
    sage: KT.base_tree().cardinality()
    6
    sage: KT = VirtualKleberTree(['C', 4, 1], [[4,5]])
    sage: KT.cardinality()
    1
sage: KT = VirtualKleberTree(['D', 5, 2], [[2,1], [1,1]])
sage: KT.cardinality()
8
sage: KT = VirtualKleberTree(CartanType(['A', 4, 2]).dual(), [[1,1], [2,2]])
sage: KT.cardinality()
15

base_tree()

Return the underlying virtual Kleber tree associated to self.

EXAMPLES:

sage: from sage.combinat.rigged_configurations.kleber_tree import VirtualKleberTree
sage: KT = VirtualKleberTree(['C', 2, 1], [[1,1], [2,1]])
sage: KT.base_tree()
Kleber tree of Cartan type ['A', 7, 1] and B = ((2, 2), (6, 2))

breadth_first_iter(all_nodes=False)

Iterate over all nodes in the tree following a breadth-first traversal.

INPUT:

• all_nodes – (default: False) if True, output all nodes in the tree

EXAMPLES:

sage: from sage.combinat.rigged_configurations.kleber_tree import VirtualKleberTree
sage: KT = VirtualKleberTree(['C', 2, 1], [[1,1], [2,1]])
sage: for x in KT.breadth_first_iter(): x
Kleber tree node with weight [1, 2, 1] and upwards edge root [0, 0, 0]
Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]
sage: for x in KT.breadth_first_iter(True): x
Kleber tree node with weight [1, 2, 1] and upwards edge root [0, 0, 0]
Kleber tree node with weight [0, 2, 0] and upwards edge root [1, 1, 1]
Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]

depth_first_iter(all_nodes=False)

Iterate (recursively) over the nodes in the tree following a depth-first traversal.

INPUT:

• all_nodes – (default: False) if True, output all nodes in the tree

EXAMPLES:

sage: from sage.combinat.rigged_configurations.kleber_tree import VirtualKleberTree
sage: KT = VirtualKleberTree(['C', 2, 1], [[1,1], [2,1]])
sage: for x in KT.depth_first_iter(): x
Kleber tree node with weight [1, 2, 1] and upwards edge root [0, 0, 0]
Kleber tree node with weight [1, 0, 1] and upwards edge root [0, 1, 0]
sage: for x in KT.depth_first_iter(True): x
Kleber tree node with weight [1, 2, 1] and upwards edge root [0, 0, 0]
5.1.210 Kirillov-Reshetikhin Tableaux

Kirillov-Reshetikhin tableaux are rectangular tableaux with \( r \) rows and \( s \) columns that naturally arise under the bijection between rigged configurations and tableaux \([\text{RigConBijection}]\). They are in bijection with the elements of the Kirillov-Reshetikhin crystal \( B^{r,s} \) under the (inverse) filling map \([\text{OSS13}]\) \([\text{SS2015}]\). They do not have to satisfy the semistandard row or column restrictions. These tensor products are the result from the bijection from rigged configurations \([\text{RigConBijection}]\).

For more information, see \texttt{KirillovReshetikhinTableaux} and \texttt{TensorProductOfKirillovReshetikhinTableaux}.

AUTHORS:

- Travis Scrimshaw (2012-01-03): Initial version
- Travis Scrimshaw (2012-11-14): Added bijection to KR crystals

REFERENCES:

\begin{Verbatim}
\texttt{class sage.combinat.rigged_configurations.kr_tableaux.KRTTableauxBn(cartan_type, r, s)}
\text{Bases:} \texttt{KRTTableauxTypeHorizontal}

Kirillov-Reshetikhin tableaux \( B^{n,s} \) of type \( B^{(1)}_n \).

\texttt{Element}

alias of \texttt{KRTTableauxSpinElement}

\texttt{from_kirillov_reshetikhin_crystal(krc)}

Construct an element of self from the Kirillov-Reshetikhin crystal element krc.

\begin{Verbatim}
\texttt{sage: C = crystals.KirillovReshetikhin(['B', 3, 1], 3, 3, model='KN')}
\texttt{sage: krc = C.module_generators[1].f_string([3, 2, 3, 1, 3, 3]); krc}
\texttt{[++-, [[2], [0], [-3]]]}
\texttt{sage: KR.from_kirillov_reshetikhin_crystal(krc)}
\texttt{[[1, 1, 2], [2, 2, -3], [-3, -3, -1]]}
\end{Verbatim}

\begin{Verbatim}
\texttt{class sage.combinat.rigged_configurations.kr_tableaux.KRTTableauxDTwistedSpin(cartan_type, r, s)}
\text{Bases:} \texttt{KRTTableauxRectangle}

Kirillov-Reshetikhin tableaux \( B^{n,s} \) of type \( D^{(2)}_n \) with \( r = n \).

\begin{Verbatim}
\texttt{sage: KRT = crystals.KirillovReshetikhin(['D', 4, 2], 1, 1, model='KR')}
\texttt{sage: KRT.cardinality()}
\texttt{8}
\texttt{sage: KRC = crystals.KirillovReshetikhin(['D', 4, 2], 1, 1, model='KN')}
\texttt{sage: KRT.cardinality() == KRC.cardinality()}
\texttt{True}
\end{Verbatim}

\begin{Verbatim}
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\end{Verbatim}
Element

alias of KRTableauxSpinElement
class sage.combinat.rigged_configurations.kr_tableaux.KRTableauxRectangle(cartan_type, r, s)
Bases: KirillovReshetikhinTableaux

Kirillov-Reshetikhin tableaux $B_{r,s}$ whose module generator is a single $r \times s$ rectangle.

These are Kirillov-Reshetikhin tableaux $B_{r,s}$ of type:

• $A^{(1)}_n$ for all $1 \leq r \leq n$,
• $C^{(1)}_n$ when $r = n$.

from_kirillov_reshetikhin_crystal(krc)

Construct a KirillovReshetikhinTableauxElement.

EXAMPLES:

sage: KRT = crystals.KirillovReshetikhin(["A", 4, 1], 2, 1, model='KR')
sage: C = crystals.KirillovReshetikhin(["A",4,1], 2, 1, model='KN')
sage: krc = C(4,3); krc
[[3], [4]]
sage: KRT.from_kirillov_reshetikhin_crystal(krc)
[[3], [4]]

class sage.combinat.rigged_configurations.kr_tableaux.KRTableauxSpin(cartan_type, r, s)
Bases: KRTableauxRectangle

Kirillov-Reshetikhin tableaux $B_{r,s}$ of type $D^{(1)}_n$ with $r = n, n-1$.

Element

alias of KRTableauxSpinElement
class sage.combinat.rigged_configurations.kr_tableaux.KRTableauxSpinElement(parent, list, **options)
Bases: KirillovReshetikhinTableauxElement

Kirillov-Reshetikhin tableau for spinors.

Here we are in the embedding $B(\Lambda_n) \hookrightarrow B(2\Lambda_n)$, so $e_i$ and $f_i$ act by $e_i^2$ and $f_i^2$ respectively for all $i \neq 0$. We do this so our columns are full width (as opposed to half width and/or uses a $\pm$ representation).

classical_weight()

Return the classical weight of self.

EXAMPLES:

sage: KRT = crystals.KirillovReshetikhin(["D", 4, 1], 4, 1, model='KR')
sage: KRT.module_generators[0].classical_weight()
(1/2, 1/2, 1/2, 1/2)

e(i)

Calculate the action of $e_i$ on self.

EXAMPLES:
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```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 4, 1, model='KR')
sage: KRT(-1, -4, 3, 2).e(1)
[[1], [3], [-4], [-2]]
sage: KRT(-1, -4, 3, 2).e(3)
[]
```

**epsilon**

Compute $\varepsilon_i$ of self.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 4, 1, model='KR')
sage: KRT(-1, -4, 3, 2).epsilon(1)
1
sage: KRT(-1, -4, 3, 2).epsilon(3)
0
```

**f(i)**

Calculate the action of $f_i$ on self.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 4, 1, model='KR')
sage: KRT(-1, -4, 3, 2).f(1)
[[2], [4], [-3], [-1]]
```

**left_split()**

Return the image of self under the left column splitting map.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 4, 3, model='KR')
sage: elt = KRT(-3,-4,2,1,-3,-4,2,1,-2,-4,3,1); elt.pp()
  1  1  1
  2  2  3
-4 -4 -4
-3 -3 -2
sage: elt.left_split().pp()
  1 (X) 1 1
  2  2  3
-4 -4 -4
-3 -3 -2
```

**phi()**

Compute $\varphi_i$ of self.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 4, 1, model='KR')
sage: KRT(-1, -4, 3, 2).phi(1)
0
sage: KRT(-1, -4, 3, 2).phi(3)
1
```
to_array\(\texttt{(rows=True)}\)

Return a 2-dimensional array representation of this Kirillov-Reshetikhin element.

If the output is in rows, then it outputs the top row first (in the English convention) from left to right.

For example: if the reading word is \([2, 1, 4, 3]\), so as a \(2 \times 2\) tableau:

\[
\begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}
\]

we output \([\begin{array}{c}1, 3 \end{array}, \begin{array}{c}2, 4 \end{array}]\).

If the output is in columns, then it outputs the leftmost column first with the bottom element first. In other words this parses the reading word into its columns.

Continuing with the previous example, the output would be \([\begin{array}{c}2, 1 \end{array}, \begin{array}{c}4, 3 \end{array}]\).

INPUT:

* \texttt{rows} – (Default: \texttt{True}) Set to \texttt{True} if the resulting array is by row, otherwise it is by column.

EXAMPLES:

\[
\begin{align*}
\text{sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 4, 3, model='KR')}\\
\text{sage: elt = KRT(-3,-4,2,1,-3,-4,2,1,-2,-4,3,1)}\\
\text{sage: elt.to_array()}
\end{align*}
\]

\[
\begin{array}{c}
[1, 1, 1] \\
[2, 2, 3] \\
[-4, -4, -4] \\
[-3, -3, -2]
\end{array}
\]

\[
\begin{align*}
\text{sage: elt.to_array(False)}\\
\text{sage: elt.to_array(False)}
\end{align*}
\]

\[
\begin{array}{c}
[-3, -4, 2, 1] \\
[-3, -4, 2, 1] \\
[-2, -4, 3, 1]
\end{array}
\]

class \texttt{sage.combinat.rigged_configurations.kr_tableaux.KRTableauxTypeBox(cartan_type, r, s)}

Bases: \texttt{KRTableauxTypeVertical}

Kirillov-Reshetikhin tableaux \(B^{r,s}\) of type:

- \(A_{2n}^{(2)}\) for all \(r \leq n\),
- \(D_{n+1}^{(2)}\) for all \(r < n\),
- \(D_{4}^{(3)}\) for \(r = 1\).

class \texttt{sage.combinat.rigged_configurations.kr_tableaux.KRTableauxTypeFromRC(cartan_type, r, s)}

Bases: \texttt{KirillovReshetikhinTableaux}

Kirillov-Reshetikhin tableaux \(B^{r,s}\) constructed from rigged configurations under the bijection \(\Phi\).

\textbf{Warning:} The Kashiwara-Nakashima version is not implemented due to the non-trivial multiplicities of classical components, so \texttt{classical_decomposition() does not work.}

Element

alias of \texttt{KRTableauxTypeFromRCElement}

module\_generators()

The module generators of \texttt{self}.

EXAMPLES:
```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 3], 2, 1, model='KR')
sage: KRT.module_generators
([[1], [2]], [[1], [0]], [[1], [E]], [[E], [E]])
```

```python
class sage.combinat.rigged_configurations.kr_tableaux.KRTableauxTypeFromRCElement(parent, list, **options)
```

Bases: `KirillovReshetikhinTableauxElement`

A Kirillov-Reshetikhin tableau constructed from rigged configurations under the bijection $\Phi$.

**e(i)**

Perform the action of $e_i$ on `self`.

**Todo**: Implement a direct action of $e_0$ without moving to rigged configurations.

**Examples**:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 3], 2, 1, model='KR')
sage: KRT.module_generators[0].e(0)
[[2], [E]]
```

**epsilon(i)**

Compute $\epsilon_i$ of `self`.

**Todo**: Implement a direct action of $\epsilon_0$ without moving to KR crystals.

**Examples**:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 3], 2, 1, model='KR')
sage: KRT.module_generators[0].epsilon(0)
6
```

**f(i)**

Perform the action of $f_i$ on `self`.

**Todo**: Implement a direct action of $f_0$ without moving to rigged configurations.

**Examples**:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 3], 2, 1, model='KR')
sage: KRT.module_generators[0].f(0)
sage: KRT.module_generators[3].f(0)
[[1], [0]]
```

**phi(i)**

Compute $\phi_i$ of `self`.

**Todo**: Compute $\phi_0$ without moving to KR crystals.
EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,3], 2, 2, model='KR')
sage: KRT.module_generators[0].phi(0)
0
```

class sage.combinat.rigged_configurations.kr_tableaux.KRTableauxTypeHorizontal(cartan_type, r, s):

Bases: `KirillovReshetikhinTableaux`

Kirillov-Reshetikhin tableaux $B^{r,s}$ of type:

- $C_n^{(1)}$ for $1 \leq r < n$,
- $A_{2n-1}^{(2)}$ for $1 \leq r \leq n$.

`from_kirillov_reshetikhin_crystal(krc)`

Construct an element of self from the Kirillov-Reshetikhin crystal element krc.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['C',4,1], 2, 3, model='KR')
sage: C = crystals.KirillovReshetikhin(['C',4,1], 2, 3, model='KN')
sage: krc = C(4,3); krc
[[3], [4]]
sage: KRT.from_kirillov_reshetikhin_crystal(krc)
[[3, -2, 1], [4, -1, 2]]
```

class sage.combinat.rigged_configurations.kr_tableaux.KRTableauxTypeVertical(cartan_type, r, s):

Bases: `KirillovReshetikhinTableaux`

Kirillov-Reshetikhin tableaux $B^{r,s}$ of type:

- $D_n^{(1)}$ for all $1 \leq r < n - 1$,
- $B_n^{(1)}$ for all $1 \leq r < n$,
- $A_{2n-1}^{(2)}$ for all $1 \leq r \leq n$.

`from_kirillov_reshetikhin_crystal(krc)`

Construct an element of self from the Kirillov-Reshetikhin crystal element krc.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 2, 3, model='KR')
sage: C = crystals.KirillovReshetikhin(['D',4,1], 2, 3, model='KN')
sage: krc = C(4,3); krc
[[3], [4]]
sage: KRT.from_kirillov_reshetikhin_crystal(krc)
[[3, -2, 1], [4, -1, 2]]
```

class sage.combinat.rigged_configurations.kr_tableaux.KirillovReshetikhinTableaux(cartan_type, r, s):

Bases: `CrystalOfWords`

Kirillov-Reshetikhin tableaux.
Kirillov-Reshetikhin tableaux are rectangular tableaux with \( r \) rows and \( s \) columns that naturally arise under the bijection between rigged configurations and tableaux \([\text{RigConBijection}]\). They are in bijection with the elements of the Kirillov-Reshetikhin crystal \( B^{r,s} \) under the (inverse) filling map.

Whenever \( B^{r,s} \cong B(s\Lambda_r) \) as a classical crystal (which is the case for \( B^{r,s} \) in type \( A_n^{(1)} \), \( B_{n,s} \) in type \( C_{n}^{(1)} \) and \( D_{n+1}^{(2)} \), \( B_{n,s} \) and \( B_{n-1,s} \) in type \( D_{n}^{(1)} \)) then the filling map is trivial.

For \( B^{r,s} \) in:
- \( \text{type } D_{n}^{(1)} \) when \( r \leq n - 2 \),
- \( \text{type } B_{n}^{(1)} \) when \( r < n \),
- \( \text{type } A_{2n-1}^{(2)} \) for all \( r \),
the filling map is defined in \([\text{OSS2011}]\).

For the spinor cases in type \( D_{n}^{(1)} \), the crystal \( B_{k,s} \) where \( k = n - 1 \), \( n \), is isomorphic as a classical crystal to \( B(s\Lambda_k) \), and here we consider the Kirillov-Reshetikhin tableaux as living in \( B(2s\Lambda_k) \) under the natural doubling map. In this case, the crystal operators \( e_i \) and \( f_i \) act as \( e_i^2 \) and \( f_i^2 \) respectively. See \([\text{BijectionDn}]\).

For the spinor case in type \( B_{n}^{(1)} \), the crystal \( B_{n,s} \), we consider the images under the natural doubling map into \( B_{n,2s} \). The classical components of this crystal are now given by removing \( 2 \times 2 \) boxes. The filling map is the same as below (see the non-spin type \( C_{n}^{(1)} \)).

For \( B^{r,s} \) in:
- \( \text{type } C_{n}^{(1)} \) when \( r < n \),
- \( \text{type } A_{2n}^{(2)} \) for all \( r \),
the filling map is given as follows. Suppose we are considering the (classically) highest weight element in the classical component \( B(\lambda) \). Then we fill it in with the horizontal dominoes \([i, i] \) in the \( i \)-th row from the top (in English notation) and reordering the columns so that they are increasing. Recall from above that \( B^{n,s} \cong B(s\Lambda_n) \) in type \( C_{n}^{(1)} \).

For \( B^{r,s} \) in:
- \( \text{type } A_{2n}^{(2)} \) for all \( r \),
- \( \text{type } D_{n+1}^{(2)} \) when \( r < n \),
- \( \text{type } D_{4}^{(3)} \) when \( r = 1 \),
the filling map is the same as given in \([\text{OSS2011}]\) except for the rightmost column which is given by the column \([1, 2, \ldots, k, \emptyset, \ldots, \emptyset] \) where \( k = (r + x - 1)/2 \) in Step 3 of \([\text{OSS2011}]\).

For the spinor case in type \( D_{n+1}^{(2)} \), the crystal \( B_{n,s} \), we define the filling map in the same way as in type \( D_{n}^{(1)} \).

\[ \text{Note: } \] The filling map and classical decompositions in non-spinor cases can be classified by how the special node \( 0 \) connects with the corresponding classical diagram.

The classical crystal structure is given by the usual Kashiwara-Nakashima tableaux rules. That is to embed this into \( B(\Lambda_1)^{\otimes n} \) by using the reading word and then applying the classical crystal operator. The affine crystal structure is given by converting to the corresponding KR crystal element, performing the affine crystal operator, and pulling back to a KR tableau.

For more information about the bijection between rigged configurations and tensor products of Kirillov-Reshetikhin tableaux, see \( \text{TensorProductOfKirillovReshetikhinTableaux} \).
Note: The tableaux for all non-simply-laced types are provably correct if the bijection with rigged configurations holds. Therefore this is currently only proven for $B_{r}^{\gamma}$ or $B_{r}^{1\gamma}$ and in general for types $A_{n}^{(1)}$ and $D_{n}^{(1)}$.

INPUT:

- cartan_type – the Cartan type
- r – the Dynkin diagram index (typically the number of rows)
- s – the number of columns

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['A', 4, 1], 2, 1, model='KR')
sage: elt = KRT(4, 3); elt
[[3], [4]]
```

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 1, model='KR')
sage: elt = KRT(-1, 1); elt
[[1], [-1]]
```

We can create highest weight crystals from a given shape or weight:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: KRT.module_generator(shape=[1,1])
[[1, 1], [2, -1]]
sage: KRT.module_generator(column_shape=[2])
[[1, 1], [2, -1]]
sage: WS = RootSystem(['D',4,1]).weight_space()
sage: KRT.module_generator(weight=WS.sum_of_terms([[0,-2],[2,1]]))
[[1, 1], [2, -1]]
sage: WSC = RootSystem(['D',4]).weight_space()
sage: KRT.module_generator(classical_weight=WSC.fundamental_weight(2))
[[1, 1], [2, -1]]
```

We can go between `KashiwaraNakashimaTableaux()` and `KirillovReshetikhinTableaux` elements:

```python
sage: KRCrys = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KN')
sage: KRTab = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: elt = KRCrys(3, 2); elt
[[2], [3]]
sage: k = KRTab(elt); k
[[2, 1], [3, -1]]
sage: KRCrys(k)
[[2], [3]]
```

We check that the classical weights in the classical decompositions agree in a few different type:

```python
sage: KRCrys = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KN')
sage: KRTab = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: all(t.classical_weight() == KRCrys(t).classical_weight() for t in KRTab)
True
sage: KRCrys = crystals.KirillovReshetikhin(['B', 3, 1], 2, 2, model='KN')
```

(continues on next page)
sage: KRTab = crystals.KirillovReshetikhin(['B', 3, 1], 2, 2, model='KR')
sage: all(t.classical_weight() == KRCrys(t).classical_weight() for t in KRTab)
True
sage: KCRys = crystals.KirillovReshetikhin(['C', 3, 1], 2, 2, model='KN')
sage: KRTab = crystals.KirillovReshetikhin(['C', 3, 1], 2, 2, model='KR')
sage: all(t.classical_weight() == KRCrys(t).classical_weight() for t in KRTab)
True
sage: KCRys = crystals.KirillovReshetikhin(['D', 4, 2], 2, 2, model='KN')
sage: KRTab = crystals.KirillovReshetikhin(['D', 4, 2], 2, 2, model='KR')
sage: all(t.classical_weight() == KRCrys(t).classical_weight() for t in KRTab)
True
sage: KCRys = crystals.KirillovReshetikhin(['A', 4, 2], 2, 2, model='KN')
sage: KRTab = crystals.KirillovReshetikhin(['A', 4, 2], 2, 2, model='KR')
sage: all(t.classical_weight() == KRCrys(t).classical_weight() for t in KRTab)
True

Element

alias of KirillovReshetikhinTableauxElement
classical_decomposition()
Return the classical crystal decomposition of self.
EXAMPLES:

sage: crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR').classical_decomposition()
The crystal of tableaux of type ['D', 4] and shape(s) [[], [1, 1], [2, 2]]

from_kirillov_reshetikhin_crystal(krc)
Construct an element of self from the Kirillov-Reshetikhin crystal element krc.
EXAMPLES:

sage: KRT = crystals.KirillovReshetikhin(['A', 4, 1], 2, 1, model='KR')
sage: C = crystals.KirillovReshetikhin(['A', 4, 1], 2, 1, model='KN')
sage: krc = C(4,3); krc
[[3], [4]]
sage: KRT.from_kirillov_reshetikhin_crystal(krc)
[[3], [4]]

kirillov_reshetikhin_crystal()
Return the corresponding KR crystal in the Kashiwara-Nakashima model.
EXAMPLES:

sage: crystals.KirillovReshetikhin(['A', 4, 1], 2, 1, model='KR').kirillov_reshetikhin_crystal()
Kirillov-Reshetikhin crystal of type ['A', 4] with (r,s)=(2,1)

module_generator(i=None, **options)
Return the specified module generator.
INPUT:

• i – the index of the module generator
We can also get a module generator by using one of the following optional arguments:

- `shape` – the associated shape
- `column_shape` – the shape given as columns (a column of length $k$ correspond to a classical weight $\omega_k$)
- `weight` – the weight
- `classical_weight` – the classical weight

If no arguments are specified, then return the unique module generator of classical weight $s\Lambda_r$.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: KRT.module_generator(1)
[[1, 1], [2, -1]]
sage: KRT.module_generator(shape=[1,1])
[[1, 1], [2, -1]]
sage: KRT.module_generator(column_shape=[2])
[[1, 1], [2, -1]]
sage: WS = RootSystem(['D',4]).weight_space()
sage: KRT.module_generator(weight=WS.sum_of_terms([[0,-2],[2,1]]))
[[1, 1], [2, -1]]
sage: WSC = RootSystem(['D',4]).weight_space()
sage: KRT.module_generator(classical_weight=WSC.fundamental_weight(2))
[[1, 1], [2, -1]]
sage: KRT.module_generator()
[[1, 1], [2, 2]]
```

$r()$
Return the value $r$ for this tableaux class which corresponds to the number of rows.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['A', 3, 1], 2, 2, model='KR')
sage: KRT.r()
2
```

$s()$
Return the value $s$ for this tableaux class which corresponds to the number of columns.

**EXAMPLES:**

```python
sage: KRT = crystals.KirillovReshetikhin(['A', 4, 1], 2, 1, model='KR')
sage: KRT.s()
1
```

`tensor(*crystals, **options)`
Return the tensor product of `self` with `crystals`.

If `crystals` is a list of (a tensor product of) KR tableaux, this returns a `TensorProductOfKirillovReshetikhinTableaux`. 

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EXAMPLES:

```python
sage: K = crystals.KirillovReshetikhin(['A', 3, 1], 2, 2, model='KR')
sage: TP = crystals.TensorProductOfKirillovReshetikhinTableaux(['A', 3, 1], [[1, \rightarrow 3],[3,1]])
sage: K.tensor(TP, K)
Tensor product of Kirillov-Reshetikhin tableaux of type ['A', 3, 1]
and factor(s) ((2, 2), (1, 3), (3, 1), (2, 2))

sage: C = crystals.KirillovReshetikhin(['A', 3, 1], 3, 1, model='KN')
sage: K.tensor(K, C)
Full tensor product of the crystals
Kirillov-Reshetikhin tableau of type ['A', 3, 1] and shape (2, 2),
Kirillov-Reshetikhin tableau of type ['A', 3, 1] and shape (2, 2),
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(3,1)
```

```python
class sage.combinat.rigged_configurations.kr_tableaux.KirillovReshetikhinTableauxElement(parent, list, **options)

Bases: TensorProductOfRegularCrystalsElement

A Kirillov-Reshetikhin tableau.

For more information, see KirillovReshetikhinTableaux and TensorProductOfKirillovReshetikhinTableaux.

classical_weight()

Return the classical weight of self.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: elt = KRT(3,2,-1,1); elt
[[2, 1], [3, -1]]
sage: elt.classical_weight()
(0, 1, 1, 0)
```

e()

Perform the action of $e_i$ on self.

**Todo:** Implement a direct action of $e_0$ without moving to KR crystals.

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: KRT.module_generators[0].e(0)
[[[-2, 1], [-1, -1]]
```

epsilon()

Compute $\varepsilon_i$ of self.

**Todo:** Implement a direct action of $\varepsilon_0$ without moving to KR crystals.
EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: KRT.module_generators[0].epsilon(0)
2
```

**f(i)**

Perform the action of $f_i$ on `self`.

**Todo:** Implement a direct action of $f_0$ without moving to KR crystals.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: KRT.module_generators[0].f(0)
[[1, 1], [2, -1]]
```

**left_split()**

Return the image of `self` under the left column splitting map.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 3, model='KR')
sage: mg = KRT.module_generators[1]; mg.pp()
1 -2 1
2 -1 2
sage: ls = mg.left_split(); ls.pp()
1 (X) -2 1
2 -1 2
sage: ls.parent()
Tensor product of Kirillov-Reshetikhin tableaux of type ['D', 4, 1] and␣
˓factor(s) ((2, 1), (2, 2))
```

**phi(i)**

Compute $\varphi_i$ of `self`.

**Todo:** Compute $\varphi_0$ without moving to KR crystals.

EXAMPLES: Compute $\varphi_0$ without moving to KR crystals.

```python
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 2, model='KR')
sage: KRT.module_generators[0].phi(0)
2
```

**pp()**

Pretty print `self`.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['A', 4, 1], 2, 2, model='KR')
sage: elt = KRT(2, 1, 4, 3); elt
[[1, 3], [2, 4]]
```
right_split()
Return the image of self under the right column splitting map.

Let * denote the Lusztig involution, and ls as the left splitting map. The right splitting map is defined as
\[ rs := * \circ ls \circ * . \]

EXAMPLES:
```
sage: KRT = crystals.KirillovReshetikhin(['D', 4, 1], 2, 3, model='KR')
sage: mg = KRT.module_generators[1]; mg.pp()
   1  2
   2  1
sage: ls = mg.right_split(); ls.pp()
  -2  1   (X)  1
   -1  2      2
sage: ls.parent()
Tensor product of Kirillov-Reshetikhin tableaux of type ['D', 4, 1] and
˓→factor(s) ((2, 2), (2, 1))
```

to_array(rows=True)
Return a 2-dimensional array representation of this Kirillov-Reshetikhin element.

If the output is in rows, then it outputs the top row first (in the English convention) from left to right.

For example: if the reading word is [2, 1, 4, 3], so as a 2 × 2 tableau:
```
   1  3
   2  4
```
we output [[1, 3], [2, 4]].

If the output is in columns, then it outputs the leftmost column first with the bottom element first. In other words this parses the reading word into its columns.

Continuing with the previous example, the output would be [[2, 1], [4, 3]].

INPUT:
• rows – (Default: True) Set to True if the resulting array is by row, otherwise it is by column.

EXAMPLES:
```
sage: KRT = crystals.KirillovReshetikhin(['A', 4, 1], 2, 2, model='KR')
sage: elt = KRT(2, 1, 4, 3)
sage: elt.to_array()
[[1, 3], [2, 4]]
sage: elt.to_array(False)
[[2, 1], [4, 3]]
```

to_classical_highest_weight(index_set=None)
Return the classical highest weight element corresponding to self.

INPUT:
• **index_set** – (Default: None) Return the highest weight with respect to the index set. If None is passed in, then this uses the classical index set.

OUTPUT:
A pair \([H, f_{str}]\) where \(H\) is the highest weight element and \(f_{str}\) is a list of \(a_i\) of \(f_{a_i}\) needed to reach \(H\).

EXAMPLES:

```python
sage: KRTab = crystals.KirillovReshetikhin(['D',4,1], 2, 2, model='KR')
sage: elt = KRTab(3,2,-1,1); elt
[[2, 1], [3, -1]]
sage: elt.to_classical_highest_weight()
[[[1, 1], [2, -1]], [1, 2]]
```

### to_kirillov_reshetikhin_crystal()

Construct a KashiwaraNakashimaTableaux() element from self.

We construct the Kirillov-Reshetikhin crystal element as follows:

1. Determine the shape \(\lambda\) of the KR crystal from the weight.
2. Determine a path \(e_{i_1} e_{i_2} \cdots e_{i_k}\) to the highest weight.
3. Apply \(f_{i_k} \cdots f_{i_2} f_{i_1}\) to a highest weight KR crystal of shape \(\lambda\).

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 2, 2, model='KR')
sage: elt = KRT(3,2,-1,1); elt
[[2, 1], [3, -1]]
sage: elt.to_kirillov_reshetikhin_crystal()
[[2], [3]]
```

### to_tableau()

Return a Tableau object of self.

EXAMPLES:

```python
sage: KRT = crystals.KirillovReshetikhin(['A', 4, 1], 2, 2, model='KR')
sage: elt = KRT(2,1,4,3); elt
[[1, 3], [2, 4]]
sage: t = elt.to_tableau(); t
[[1, 3], [2, 4]]
sage: type(t)
<class 'sage.combinat.tableau.Tableaux_all_with_category.element_class'>
```

### weight()

Return the weight of self.

EXAMPLES:

```python
sage: KR = crystals.KirillovReshetikhin(['D',4,1], 2, 2, model='KR')
sage: KR.module_generators[1].weight()
-2*Lambda[0] + Lambda[2]
```
5.1.211 Crystal of Rigged Configurations

AUTHORS:

• Travis Scrimshaw (2010-09-26): Initial version

We only consider the highest weight crystal structure, not the Kirillov-Reshetikhin structure, and we extend this to symmetrizable types.

```python
class sage.combinat.rigged_configurations.rc_crystal.CrystalOfNonSimplyLacedRC(vct, wt, WLR):
    Bases: CrystalOfRiggedConfigurations

    Highest weight crystal of rigged configurations in non-simply-laced type.

    Element
    alias of RCHWNonSimplyLacedElement

    from_virtual(vrc)
    Convert vrc in the virtual crystal into a rigged configuration of the original Cartan type.
    INPUT:
    • vrc – a virtual rigged configuration
    EXAMPLES:

    sage: La = RootSystem(['C', 3]).weight_lattice().fundamental_weights()
sage: vct = CartanType(['C', 3]).as_folding()
sage: RC = crystals.RiggedConfigurations(vct, La[2])
sage: elt = RC(partition_list=[[0], [1], [1]])
sage: elt == RC.from_virtual(RC.to_virtual(elt))
    True

to_virtual(rc)
    Convert rc into a rigged configuration in the virtual crystal.
    INPUT:
    • rc – a rigged configuration element
    EXAMPLES:

    sage: La = RootSystem(['C', 3]).weight_lattice().fundamental_weights()
sage: vct = CartanType(['C', 3]).as_folding()
sage: RC = crystals.RiggedConfigurations(vct, La[2])
sage: elt = RC(partition_list=[[], [1], [1]]); elt

    (continues on next page)```
virtual()

Return the corresponding virtual crystal.

EXAMPLES:

```python
sage: La = RootSystem(['C', 2, 1]).weight_lattice().fundamental_weights()
sage: vct = CartanType(['C', 2, 1]).as_folding()
sage: RC = crystals.RiggedConfigurations(vct, La[0])
sage: RC
Crystal of rigged configurations of type ['C', 2, 1] and weight Lambda[0]
sage: RC.virtual
Crystal of rigged configurations of type ['A', 3, 1] and weight 2*Lambda[0]
```

class sage.combinat.rigged_configurations.rc_crystal.CrystalOfRiggedConfigurations(wt, WLR)

Bases: UniqueRepresentation, Parent

A highest weight crystal of rigged configurations.

The crystal structure for finite simply-laced types is given in [CrysStructSchilling06]. These were then shown to be the crystal operators in all finite types in [SS2015], all simply-laced and a large class of foldings of simply-laced types in [SS2015III], and all symmetrizable types (uniformly) in [SS2017].

INPUT:

- `cartan_type` – (optional) a Cartan type or a Cartan type given as a folding
- `wt` – the highest weight vector in the weight lattice

EXAMPLES:

For simplicity, we display the rigged configurations horizontally:

```python
sage: RiggedConfigurations.options.display='horizontal'
```

We start with a simply-laced finite type:

```python
sage: La = RootSystem(['A', 2]).weight_lattice().fundamental_weights()
sage: RC = crystals.RiggedConfigurations(La[1] + La[2])
sage: mg = RC.highest_weight_vector()
sage: mg.f_string([1,2])
0[ ]0 0[ ]-1
sage: mg.f_string([1,2,2])
0[ ]0 -2[ ][ ]-2
sage: mg.f_string([1,2,2,2])
-1[ ][ ]-1 -1[ ][ ]-1
sage: RC.cardinality()
```

(continues on next page)
We construct a non-simply-laced affine type:

```python
sage: La = RootSystem(['C', 3]).weight_lattice().fundamental_weights()
sage: RC = crystals.RiggedConfigurations(La[2])
sage: mg = RC.highest_weight_vector()
(\) 1[ ]1 -1[ ]-1
sage: T = crystals.Tableaux(['C', 3], shape=[1,1])
sage: RC.digraph().is_isomorphic(T.digraph(), edge_labels=True)
True
```

We can construct rigged configurations using a diagram folding of a simply-laced type. This yields an equivalent but distinct crystal:

```python
sage: vct = CartanType(['C', 3]).as_folding()
sage: RC = crystals.RiggedConfigurations(vct, La[2])
sage: mg = RC.highest_weight_vector()
(\) 0[ ]0 -1[ ]-1
sage: T = crystals.Tableaux(['C', 3], shape=[1,1])
sage: RC.digraph().is_isomorphic(T.digraph(), edge_labels=True)
True
```

We reset the global options:

```python
sage: RiggedConfigurations.options._reset()
```

REFERENCES:

- [SS2015]
- [SS2015II]
- [SS2017]

Element

alias of `RCHighestWeightElement`

```python
options = Current options for RiggedConfigurations - convention:  English - display:
vertical - element_ascii_art: True - half_width_boxes_type_B: True

weight_lattice_realization()

Return the weight lattice realization used to express the weights of elements in self.
```

EXAMPLES:

```python
sage: La = RootSystem(['A', 2, 1]).weight_lattice(extended=True).fundamental_weights()
sage: RC = crystals.RiggedConfigurations(La[0])
sage: RC.weight_lattice_realization()
Extended weight lattice of the Root system of type ['A', 2, 1]
```
5.1.212 Rigged Configurations of \( B(\infty) \)

**AUTHORS:**

- Travis Scrimshaw (2013-04-16): Initial version

```python
class sage.combinat.rigged_configurations.rc_infinity.InfinityCrystalOfNonSimplyLacedRC(vct):
    Bases: InfinityCrystalOfRiggedConfigurations

    Rigged configurations for \( B(\infty) \) in non-simply-laced types.

class Element(parent, rigged_partitions=[], **options):
    Bases: RCNonSimplyLacedElement

    A rigged configuration in \( B(\infty) \) in non-simply-laced types.

    weight()
    Return the weight of self.

    EXAMPLES:

    sage: vct = CartanType(['C', 3]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: elt = RC(partition_list=[[1],[1,1],[1]], rigging_list=[[0],[-1,-1], [-[0]])
sage: elt.weight()
(-1, -1, 0)

    sage: vct = CartanType(['F', 4, 1]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: mg = RC.highest_weight_vector()
sage: elt = mg.f_string([1,0,3,4,2,2]); ascii_art(elt)
-1[ ] [-1 0[ ] 1 -2[ ] -2 0[ ] 1 -1[ ] -1
sage: wt = elt.weight(); wt

    sage: al = RC.weight_lattice_realization().simple_roots()
True
```

**from_virtual**(vrc)

Convert vrc in the virtual crystal into a rigged configuration of the original Cartan type.

**INPUT:**

- vrc – a virtual rigged configuration

**EXAMPLES:**

```python
sage: vct = CartanType(['C', 2]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: elt = RC(partition_list=[[3],[2]], rigging_list=[[2],[0]])
sage: vrc_elt = RC.to_virtual(elt)
sage: ret = RC.from_virtual(vrc_elt); ret
-3[ ][ ][ ]
-1[ ][ ]
```

(continues on next page)
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sage: ret == elt
True

to_virtual(rc)
Convert rc into a rigged configuration in the virtual crystal.

INPUT:
• rc – a rigged configuration element

EXAMPLES:

```python
sage: vct = CartanType(['C', 2]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: mg = RC.highest_weight_vector()
sage: elt = mg.f_string([1,2,2,1,1]); elt
-3[ ][ ][ ]-2
-1[ ][ ]0
sage: velt = RC.to_virtual(elt); velt
-3[ ][ ][ ]-2
-2[ ][ ][ ]0
-3[ ][ ][ ]-2
```

```python
sage: velt.parent()
The infinity crystal of rigged configurations of type ['A', 3]
```

virtual()
Return the corresponding virtual crystal.

EXAMPLES:

```python
sage: vct = CartanType(['C', 3]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: RC
The infinity crystal of rigged configurations of type ['C', 3]
sage: RC.virtual
The infinity crystal of rigged configurations of type ['A', 5]
```

class sage.combinat.rigged_configurations.rc_infinity.InfinityCrystalOfRiggedConfigurations(cartan_type)
Bases: UniqueRepresentation, Parent

Rigged configuration model for $B(\infty)$.

The crystal is generated by the empty rigged configuration with the same crystal structure given by the highest weight model except we remove the condition that the resulting rigged configuration needs to be valid when applying $f_a$.

INPUT:
• cartan_type – a Cartan type
EXAMPLES:

For simplicity, we display all of the rigged configurations horizontally:

```
sage: RiggedConfigurations.options(display='horizontal')
```

We begin with a simply-laced finite type:

```
sage: RC = crystals.infinity.RiggedConfigurations(['A', 3]); RC
The infinity crystal of rigged configurations of type ['A', 3]
sage: RC.options(display='horizontal')
sage: mg = RC.highest_weight_vector(); mg
</> </> </>
```

```
sage: elt = mg.f_string([2,1,3,2]); elt
0[ ]0 -2[ ]-1 0[ ]0
-2[ ]-1
```

```
sage: elt.e(1)
sage: elt.e(3)
sage: mg.f_string([2,1,3,2]).e(2)
-1[ ]-1 0[ ]1 -1[ ]-1
```

```
sage: mg.f_string([2,3,2,1,3,2])
0[ ]0 -3[ ][ ]-1 -1[ ][ ]-1
-2[ ]-1
```

Next we consider a non-simply-laced finite type:

```
sage: RC = crystals.infinity.RiggedConfigurations(['C', 3])
sage: mg = RC.highest_weight_vector(); mg
<>/ </> (/
```

```
sage: elt = mg.f_string([2,1,3,2]); elt
0[ ]0 -1[ ]0 0[ ]0
-1[ ]-1
```

```
sage: mg.f_string([2,3,2,1,3,2])
0[ ]-1 -1[ ][ ]-1 -1[ ][ ]0
-1[ ]0
```

We can construct rigged configurations using a diagram folding of a simply-laced type. This yields an equivalent but distinct crystal:

```
sage: vct = CartanType(['C', 3]).as_folding()
sage: VRC = crystals.infinity.RiggedConfigurations(vct)
sage: mg = VRC.highest_weight_vector(); mg
```

```
sage: mg.f_string([2,1,3,2])
0[ ]0 -2[ ]-1 0[ ]0
-2[ ]-1
```

```
sage: mg.f_string([2,3,2,1,3,2])
-1[ ]-1 -2[ ][ ][ ]-1 -1[ ][ ]0
```

```
sage: G = RC.subcrystal(max_depth=5).digraph()
sage: VG = VRC.subcrystal(max_depth=5).digraph()
sage: G.is_isomorphic(VG, edge_labels=True)
True
```

We can also construct $B(\infty)$ using rigged configurations in affine types:
sage: RC = crystals.infinity.RiggedConfigurations(['A', 3, 1])
sage: mg = RC.highest_weight_vector()
sage: mg.f_string([0,1,2,3,0,1,3])
-1[ ]0  -1[ ]-1  1[ ]1  -1[ ] [ ]-1
-1[ ]0  -1[ ]-1

sage: RC = crystals.infinity.RiggedConfigurations(['C', 3, 1])
sage: mg = RC.highest_weight_vector()
sage: mg.f_string([1,2,3,0,1,2,3,3,0])
-2[ ][ ]-1  0[ ]1  0[ ]0  -4[ ][ ][ ]-2
  0[ ]0  0[ ]-1

sage: RC = crystals.infinity.RiggedConfigurations(['A', 6, 2])
sage: mg = RC.highest_weight_vector()
sage: mg.f_string([1,2,3,0,1,2,3,3,0])
0[ ][ ]-1  0[ ]1  0[ ]0  -4[ ][ ][ ]-2
  0[ ][ ]-1  0[ ]1  0[ ][ ]-1

We reset the global options:

sage: RiggedConfigurations.options._reset()

class Element(parent, rigged_partitions=[], **options)

   Bases: RiggedConfigurationElement

   A rigged configuration in $\mathcal{B}(\infty)$ in simply-laced types.

   weight()

   Return the weight of self.

   EXAMPLES:

   sage: RC = crystals.infinity.RiggedConfigurations(['A', 3, 1])
   sage: elt = RC(partition_list=[[1,1]]*4, rigging_list=[[1,1], [0,0], [0,0], [-1,-1]])
   sage: elt.weight()
   -2*delta

   options = Current options for RiggedConfigurations - convention: English - display: vertical - element_ascii_art: True - half_width_boxes_type_B: True

   weight_lattice_realization()

   Return the weight lattice realization used to express the weights of elements in self.

   EXAMPLES:

   sage: RC = crystals.infinity.RiggedConfigurations(['A', 2, 1])
   sage: RC.weight_lattice_realization()
   Extended weight lattice of the Root system of type ['A', 2, 1]
5.1.213 Rigged Configuration Elements

A rigged configuration element is a sequence of RiggedPartition objects.

AUTHORS:

- Travis Scrimshaw (2010-09-26): Initial version
- Travis Scrimshaw (2012-10-25): Added virtual rigged configurations

```python
class sage.combinat.rigged_configurations.rigged_configuration_element.KRRiggedConfigurationElement(RCRiggedConfigurationElement, **options):
    pass
```

**U**′_q(g) rigged configurations in non-simply-laced types.

```python
cc()
```

Compute the cocharge statistic.

Computes the cocharge statistic [OSS03] on this rigged configuration (ν, J) by computing the cocharge as a virtual rigged configuration (ν̂, Ĵ) and then using the identity \( cc(\nu, J) = \gamma_0 cc(\nu, J) \).

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['C', 3, 1], [[2,1], [1,1]])
sage: RC(partition_list=[[1,1],[2,1],[1,1]]).cocharge()
1
```

```python
cocharge()
```

Compute the cocharge statistic.

Computes the cocharge statistic [OSS03] on this rigged configuration (ν, J) by computing the cocharge as a virtual rigged configuration (ν̂, Ĵ) and then using the identity \( cc(\nu, J) = \gamma_0 cc(\nu, J) \).

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['C', 3, 1], [[2,1], [1,1]])
sage: RC(partition_list=[[1,1],[2,1],[1,1]]).cocharge()
1
```

```python
e(a)
```

Return the action of \( e_a \) on self.

This works by lifting into the virtual configuration, then applying

\[
e_a^\nu = \prod_{j \in \nu} e_j^{\gamma_j}
\]

and pulling back.

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 6, 2], [[1,1]]*7)
sage: elt = RC(partition_list=[[1]*5,[2,1,1],[3,2]])
sage: elt.e(3)
0[ ]0
```

(continues on next page)
Return the action of $f_a$ on self.

This works by lifting into the virtual configuration, then applying

$$f'_a = \prod_{j \in \{a\}} f_j^\gamma$$

and pulling back.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['A',6,2], [[1,1]]^7)
sage: elt = RC(partition_list=[[1]*5,[2,1,1],[2,1]], rigging_list=[[0]*5,[0,1,-1],[1,0]])
sage: elt.f(3)
```

```
0[ ]0
0[ ]0
0[ ]0
0[ ]0
0[ ][ ]0
1[ ]1
1[ ]1
1[ ][ ]1
1[ ]0
```

```
class sage.combinat.rigged_configurations.rigged_configuration_element.KRRCSimplyLacedElement
    
    Bases: KRRiggedConfigurationElement

    \(U_q'(g)\) rigged configurations in simply-laced types.

    \(cc()\)

    Compute the cocharge statistic of self.

    Computes the cocharge statistic [CryStructSchilling06] on this rigged configuration \((\nu, J)\). The cocharge
```
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statistic is defined as:
\[
cc(\nu, J) = \frac{1}{2} \sum_{a,b \in I_0} \sum_{j,k > 0} (\alpha_a \mid \alpha_b) \min(j, k) m_j^{(a)} m_k^{(b)} + \sum_{a \in I} \sum_{i > 0} |J^{(a,i)}|.
\]

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [2, 1], [1,1]])
sage: RC(partition_list=[[1], [1], []]).cocharge()
1
```

charge()
Compute the charge statistic of self.

Let \( B \) denote a set of rigged configurations. The charge \( c \) of a rigged configuration \( b \) is computed as
\[
c(b) = \max(\{cc(b) \mid b \in B\}) - cc(b).
\]

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [2, 1], [1,1]])
sage: RC(partition_list=[[1], [1], []]).charge()
2
sage: RC(partition_list=[[1], [1], []]).charge()
1
```

cocharge()
Compute the cocharge statistic of self.

Computes the cocharge statistic \([\text{CrysStructSchilling06}]\) on this rigged configuration \((\nu, J)\). The cocharge statistic is defined as:
\[
cc(\nu, J) = \frac{1}{2} \sum_{a \in I_0} \sum_{i > 0} \left( \sum_{j > 0} \min(i, j) L_j^{(a)} - p_i^{(a)} \right) + \sum_{a \in I} \sum_{i > 0} |J^{(a,i)}|.
\]

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [2, 1], [1,1]])
sage: RC(partition_list=[[1], [1], []]).cocharge()
1
```

class sage.combinat.rigged_configurations.rigged_configuration_element.KRRCTypeA2DualElement (parent, rigged_partitions=\[
\]
Bases: KRRCNonSimplyLacedElement
\(U_q'(g)\) rigged configurations in type \(A_{2n}^{(2)\dagger}\). 

ccc()
Compute the cocharge statistic.

Computes the cocharge statistic \([\text{RigConBijection}]\) on this rigged configuration \((\nu, J)\). The cocharge statistic is computed as:
\[
cc(\nu, J) = \frac{1}{2} \sum_{a \in I_0} \sum_{i > 0} \left( \sum_{j > 0} \min(i, j) L_j^{(a)} - p_i^{(a)} \right) + \sum_{a \in I} \sum_{i > 0} |J^{(a,i)}|.
\]

EXAMPLES:
cocharge()

Compute the cocharge statistic.

Computes the cocharge statistic [RigConBijection] on this rigged configuration \((\nu, J)\). The cocharge statistic is computed as:

\[
cc(\nu, J) = \frac{1}{2} \sum_{a \in I_0} \sum_{i > 0} t_a^\vee m_i^{(a)} \left( \sum_{j > 0} \min(i, j) I_j^{(a)} - p_i^{(a)} \right) + \sum_{a \in I} t_a^\vee \sum_{i > 0} |J^{(a,i)}|.
\]

EXAMPLES:

```python
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[1,1],[2,2]])
sage: sc = RC.cartan_type().as_folding().scaling_factors()
sage: all(mg.cocharge() * sc[0] == mg.to_virtual_configuration().cocharge()
    ....:     for mg in RC.module_generators)
True
```

epsilon(a)

Return the value of \(\varepsilon_a\) of self.

Here we need to modify the usual definition by \(\varepsilon'_n := 2\varepsilon_n\).

EXAMPLES:

```python
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[1,1],[2,2]])
sage: def epsilon(x, i):
    ....:     x = x.e(i)
    ....:     eps = 0
    ....:     while x is not None:
    ....:         x = x.e(i)
    ....:         eps = eps + 1
    ....:     return eps
sage: all(epsilon(rc, 2) == rc.epsilon(2) for rc in RC)
True
```

phi(a)

Return the value of \(\phi_a\) of self.

Here we need to modify the usual definition by \(\phi'_n := 2\phi_n\).

EXAMPLES:

```python
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[1,1],[2,2]])
sage: def phi(x, i):
    ....:     x = x.f(i)
    ....:     ph = 0
    ....:     while x is not None:
    ....:         x = x.f(i)
    ....:         ph = ph + 1
```
....:     return ph
sage: all(phi(rc, 2) == rc.phi(2) for rc in RC)
True

class sage.combinat.rigged_configurations.rigged_configuration_element.KRRiggedConfigurationElement(parent, rigged_partitions=[], **options)

Bases: RiggedConfigurationElement

$U_q'(g)$ rigged configurations.

EXAMPLES:

We can go between rigged configurations and tensor products of tensor products of KR tableaux:

```
sage: RC = RiggedConfigurations(['D', 4, 1], [[1,1], [2,1]])
sage: rc_elt = RC(partition_list=[[1], [1,1], [1], [1]])
sage: tp_krtab = rc_elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(); tp_krtab
[[[-2]], [[1], [2]]]
sage: tp_krcrys = rc_elt.to_tensor_product_of_kirillov_reshetikhin_crystals(); tp_krcrys
[[[-2]], [[1], [2]]]
sage: tp_krcrys == tp_krtab.to_tensor_product_of_kirillov_reshetikhin_crystals()
True
sage: RC(tp_krcrys) == rc_elt
True
sage: RC(tp_krtab) == rc_elt
True
sage: tp_krtab.to_rigged_configuration() == rc_elt
True
```

check()

Make sure all of the riggings are less than or equal to the vacancy number.

classical_weight()

Return the classical weight of self.

The classical weight $\Lambda$ of a rigged configuration is

$$
\Lambda = \sum_{a \in \mathcal{R}} \sum_{i > 0} iL_i^{(a)} \Lambda_a - \sum_{a \in \mathcal{R}} \sum_{i > 0} im_i^{(a)} \alpha_a.
$$

EXAMPLES:

```
sage: RC = RiggedConfigurations(['D',4,1], [[2,2]])
sage: elt = RC(partition_list=[[2],[2,1],[1],[1]])
sage: elt.classical_weight()
(0, 1, 1, 0)
```

This agrees with the corresponding classical weight as KR tableaux:

```
sage: krt = elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(); krt
[[[2, 1], [3, -1]]]
```
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(continued from previous page)

```
sage: krt.classical_weight() == elt.classical_weight()
True
```

**complement_rigging**(reverse_factors=False)

Apply the complement rigging morphism $\theta$ to self.

Consider a highest weight rigged configuration $(\nu, J)$, the complement rigging morphism $\theta: RC(L) \rightarrow RC(L)$ is given by sending $(\nu, J) \mapsto (\nu, J')$, where $J'$ is obtained by taking the coriggings $x'_i = p_i^{(a)} - x_i$ and then extending as a crystal morphism. (The name comes from taking the complement partition for the riggings in a $m_1^{(a)} \times p_i^{(a)}$ box.)

INPUT:

- reverse_factors – (default: False) if True, then this returns an element in $RC(B')$ where $B'$ is the tensor factors of self in reverse order

EXAMPLES:

```
sage: RC = RiggedConfigurations(["D",4,1], [[1,1],[2,2]])
sage: mg = RC.module_generators[-1]
sage: ascii_art(mg)
[ ] [ ] 1 [ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0
sage: ascii_art(mg.complement_rigging())
[ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0

sage: lw = mg.to_lowest_weight([1,2,3,4])[0]
sage: ascii_art(lw)
-1 [ ] [ ]-1 0 [ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0 [ ] [ ] 0
-1 [ ]-1 0 [ ] [ ] 0 [ ] [ ] 0 [ ] 0 [ ] 0
-1 [ ]-1 0 [ ] 0 [ ] 0 [ ]

sage: lw.complement_rigging() == mg.complement_rigging().to_lowest_weight([1,2,3,4])[0]
True
```

We check that the Lusztig involution (under the modification of also mapping to the highest weight element) intertwines with the complement map $\theta$ (that reverses the tensor factors) under the bijection $\Phi$:

```
sage: RC = RiggedConfigurations(["D", 4, 1], [[2, 2], [2, 1], [1, 2]])
sage: for mg in RC.module_generators: # long time
    y = mg.to_tensor_product_of_kirillov_reshetikhin_tableaux()
    hw = y.lusztig_involution().to_highest_weight([1,2,3,4])[0]
    c = mg.complement_rigging(True)
    hwc = c.to_tensor_product_of_kirillov_reshetikhin_tableaux()
    ... assert hw == hwc
```

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delta\((\text{return\_b}=False)\)

Return the image of self under the left box removal map \(\delta\).

The map \(\delta : RC(B^r_1 \otimes B) \rightarrow RC(B^{\delta r-1,1}_r \otimes B)\) (if \(r = 1\), then we remove the left-most factor) is the basic map in the bijection \(\Phi\) between rigged configurations and tensor products of Kirillov-Reshetikhin tableaux. For more information, see \text{to\_tensor\_product\_of\_kirillov\_reshetikhin\_tableaux()}. We can extend \(\delta\) when the left-most factor is not a single column by precomposing with a \text{left\_split()}.

**Note:** Due to the special nature of the bijection for the spinor cases in types \(D_n^{(1)}, B_n^{(1)}\), and \(A_{2n-1}^{(2)}\), this map is not defined in these cases.

**INPUT:**
- \text{\textbullet return\_b \text{\textunderscore} (default: False) whether to return the resulting letter from \(\delta\) \text{\textunderscore}**

**OUTPUT:**
The resulting rigged configuration or if \text{return\_b} is \text{True}, then a tuple of the resulting rigged configuration and the letter.

**EXAMPLES:**

```
sage: RC = RiggedConfigurations(['C', 4, 1], [[3,2]])
sage: mg = RC.module_generators[-1]
sage: ascii_art(mg)
0[ ][ ]0 0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
     0[ ][ ]0 0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
     0[ ][ ]0 0[ ][ ]0

sage: ascii_art(mg.left_box())
0[ ][ ]0 0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
     0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
     0[ ][ ]0 0[ ][ ]0

sage: x,b = mg.left_box(True)
sage: b
-1
sage: b.parent()
The crystal of letters for type ['C', 4]
```

e\((a)\)

Return the action of the crystal operator \(e_a\) on self.

For the classical operators, this implements the method defined in [CrysStructSchilling06]. For \(e_0\), this converts the class to a tensor product of KR tableaux and does the corresponding \(e_0\) and pulls back.

**Todo:** Implement \(e_0\) without appealing to tensor product of KR tableaux.

**INPUT:**
- \text{\textbullet a \text{\textunderscore} the index of the partition to remove a box \text{\textunderscore}**

**OUTPUT:**
The resulting rigged configuration element.

**EXAMPLES:**
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```
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 1]])
sage: elt = RC(partition_list=[[1], [1], [1], [1]])
sage: elt.e(3)
sage: elt.e(1)
```

```
0[ ]0
0[ ]0
| -1[ ]-1
```

\textbf{epsilon}(\alpha)

Return \( \varepsilon_\alpha \) of \textit{self}.

\textbf{EXAMPLES:}

```
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 2]])
sage: I = RC.index_set()
sage: matrix([[mg.epsilon(i) for i in I] for mg in RC.module_generators])
```

```
[4 0 0 0 0]
[3 0 0 0 0]
[2 0 0 0 0]
```

\textbf{f}(\alpha)

Return the action of the crystal operator \( f_\alpha \) on \textit{self}.

For the classical operators, this implements the method defined in \cite{CrysStructSchilling06}. For \( f_0 \), this converts the class to a tensor product of KR tableaux and does the corresponding \( f_0 \) and pulls back.

\textbf{Todo:} Implement \( f_0 \) without appealing to tensor product of KR tableaux.

\textbf{INPUT:}

- \( \alpha \) – the index of the partition to add a box

\textbf{OUTPUT:}

The resulting rigged configuration element.

\textbf{EXAMPLES:}

```
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 1]])
sage: elt = RC(partition_list=[[1], [1], [1], [1]])
sage: elt.f(1)
sage: elt.f(2)
```

```
0[ ]0
| -1[ ]-1
| -1[ ]-1
| 1[ ]1
```

(continues on next page)
left_box(return_b=False)

Return the image of self under the left box removal map $\delta$.

The map $\delta : RC(B^{r,1} \otimes B) \to RC(B^{r-1,1} \otimes B)$ (if $r = 1$, then we remove the left-most factor) is the basic map in the bijection $\Phi$ between rigged configurations and tensor products of Kirillov-Reshetikhin tableaux. For more information, see to_tensor_product_of_kirillov_reshetikhin_tableaux().

We can extend $\delta$ when the left-most factor is not a single column by precomposing with a left_split().

Note: Due to the special nature of the bijection for the spinor cases in types $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$, this map is not defined in these cases.

INPUT:

* return_b – (default: False) whether to return the resulting letter from $\delta$

OUTPUT:

The resulting rigged configuration or if return_b is True, then a tuple of the resulting rigged configuration and the letter.

EXAMPLES:

```sage
sage: RC = RiggedConfigurations(['C',4,1], [[3,2]])
sage: mg = RC.module_generators[-1]
sage: ascii_art(mg)
0[ ]0 0[ ][ ]0 0[ ][ ]0
0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
sage: ascii_art(mg.left_box())
0[ ]0 0[ ][ ]0 0[ ][ ]0
0[ ][ ]0 0[ ][ ]0 0[ ][ ]0
sage: x,b = mg.left_box(True)
sage: b
-1
sage: b.parent()
The crystal of letters for type ['C', 4]
```

left_column_box()

Return the image of self under the left column box splitting map $\gamma$.

Consider the map $\gamma : RC(B^{r,1} \otimes B) \to RC(B^{1,1} \otimes B^{r-1,1} \otimes B)$ for $r > 1$, which is a natural strict classical crystal injection. On rigged configurations, the map $\gamma$ adds a singular string of length 1 to $\nu^{(a)}$.

We can extend $\gamma$ when the left-most factor is not a single column by precomposing with a left_split().

EXAMPLES:

```sage
sage: RC = RiggedConfigurations(['C',3,1], [[3,1], [2,1]])
sage: mg = RC.module_generators[-1]
sage: ascii_art(mg)
0[ ]0 0[ ][ ]0 0[ ][ ]0
0[ ][ ]0 0[ ][ ]0
0[ ][ ]0 0[ ][ ]0
```

(continues on next page)
sage: ascii_art(mg.left_column_box())
0[ ]0 0[ ]0 0[ ]0
0[ ]0 0[ ]0 0[ ]0
0[ ]0

sage: RC = RiggedConfigurations(['C',3,1], [[2,1], [1,1], [3,1]])
sage: mg = RC.module_generators[7]
sage: ascii_art(mg)
1[ ]0 0[ ]0 0[ ]0
1[ ]1 0[ ]0 0[ ]0
1[ ]0 0[ ]0 0[ ]0

left_split()

Return the image of self under the left column splitting map $\beta$.

Consider the map $\beta : RC(B^s \otimes B) \rightarrow RC(B^{r,1} \otimes B^{r,s-1} \otimes B)$ for $s > 1$ which is a natural classical crystal injection. On rigged configurations, the map $\beta$ does nothing (except possibly changing the vacancy numbers).

EXAMPLES:

sage: RC = RiggedConfigurations(['C',4,1], [[3,3]])
sage: mg = RC.module_generators[-1]
sage: ascii_art(mg)
0[ ]0 0[ ]0 0[ ]0
0[ ]0 0[ ]0
0[ ]0

sage: ascii_art(mg.left_split())
0[ ]0 0[ ]0 1[ ]0 0[ ]0
0[ ]0 1[ ]0 0[ ]0
1[ ]0 0[ ]0 0[ ]0

phi(a)

Return $\varphi_a$ of self.

EXAMPLES:

sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 2]])
sage: I = RC.index_set()
sage: matrix([[mg.phi(i) for i in I] for mg in RC.module_generators])
[0 0 2 0 0]
[1 0 1 0 0]
[2 0 0 0 0]

right_column_box()

Return the image of self under the right column box splitting map $\gamma^*$.

Consider the map $\gamma^* : RC(B \otimes B^{r,1}) \rightarrow RC(B \otimes B^{r-1,1} \otimes B^{1,1})$ for $r > 1$, which is a natural strict classical crystal injection. On rigged configurations, the map $\gamma$ adds a string of length 1 with rigging 0 to $\nu(a)$ for all $a < r$ to a classically highest weight element and extended as a classical crystal morphism.

We can extend $\gamma^*$ when the right-most factor is not a single column by precomposing with a $\text{right_split()}$. 

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EXAMPLES:

```python
sage: RC = RiggedConfigurations(['C', 3, 1], [[2, 1], [1, 1], [3, 1]])
sage: mg = RC.module_generators[7]
sage: ascii_art(mg)
1[ ] 0 0[ ] 0[ ] 0[ ] 0 [ ] 0 0[ ]
0[ ] 0 [ ] 0[ ]
sage: ascii_art(mg.right_column_box())
1[ ] 0 0[ ] 0[ ] 0[ ] 0 [ ] 0 0[ ]
1[ ] 0 0[ ] 0 [ ] 0[ ]
0[ ] 0 [ ]
```

```python
def right_split():
    Return the image of self under the right column splitting map β*. Let θ denote the complement rigging map which reverses the tensor factors and β denote the left splitting map, we define the right splitting map by β*: = θ ∘ β ∘ θ.

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['C', 4, 1], [[3, 3]])
sage: mg = RC.module_generators[-1]
sage: ascii_art(mg)
0[ ] 0[ ] 0[ ] 0[ ] 0[ ] 0[ ] 0[ ]
0[ ] 0[ ] 0[ ] 0[ ]
sage: ascii_art(mg.right_split())
0[ ] 0[ ] 0[ ] 0[ ] 1[ ] 1 0[ ]
0[ ] 0[ ] 1[ ] 1 0[ ]
1[ ] 1 0[ ]
```

We check that the bijection commutes with the right splitting map:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[1, 1], [2, 2]])
sage: all(rc.right_split().to_tensor_product_of_kirillov_reshetikhin_tableaux() ==
      rc.to_tensor_product_of_kirillov_reshetikhin_tableaux().right_split() for rc in RC)
True
```

```python
def to_tensor_product_of_kirillov_reshetikhin_crystals(display_steps=False, build_graph=False):
    Return the corresponding tensor product of Kirillov-Reshetikhin crystals.

    This is a composition of the map to a tensor product of KR tableaux, and then to a tensor product of KR crystals.
```

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INPUT:

- `display_steps` – (default: False) boolean which indicates if we want to print each step in the algorithm
- `build_graph` – (default: False) boolean which indicates if we want to construct and return a graph of the bijection whose vertices are rigged configurations obtained at each step and edges are labeled by either the return value of $\delta$ or the doubling/halving map

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 2]])
sage: elt = RC(partition_list=[[2], [2, 2], [1], [1]])
sage: krc = elt.to_tensor_product_of_kirillov_reshetikhin_crystals(); krc
[[[2, 3], [3, -2]]]
```

We can recover the rigged configuration:

```python
sage: ret = RC(krc); ret
```

```python
0
-2
-2
0
0
```

```python
sage: elt == ret
True
```

We can also construct and display a graph of the bijection as follows:

```python
sage: ret, G = elt.to_tensor_product_of_kirillov_reshetikhin_crystals(build_graph=True)
sage: view(G) # not tested
```

`to_tensor_product_of_kirillov_reshetikhin_tableaux(display_steps=False, build_graph=False)`

Perform the bijection from this rigged configuration to a tensor product of Kirillov-Reshetikhin tableaux given in [RigConBijection] for single boxes and with [BijectionLRT] and [BijectionDn] for multiple columns and rows.

**Note:** This is only proven to be a bijection in types $A_n^{(1)}$ and $D_n^{(1)}$, as well as $\otimes \mathbb{B}^{1,1}$ and $\otimes \mathbb{B}^{1,s_i}$ for general affine types.

INPUT:

- `display_steps` – (default: False) boolean which indicates if we want to print each step in the algorithm
- `build_graph` – (default: False) boolean which indicates if we want to construct and return a graph of the bijection whose vertices are rigged configurations obtained at each step and edges are labeled by either the return value of $\delta$ or the doubling/halving map
• The tensor product of KR tableaux element corresponding to this rigged configuration.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 2]])
sage: RC(partition_list=[[2], [2,2], [2], [2]]).to_tensor_product_of_kirillov_reshetikhin_tableaux()
[[3], [5, 5]]
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 2]])
sage: elt = RC(partition_list=[[2], [2,2], [1], [1]])
sage: tp_krt = elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(); tp_krt
[[2, 3], [3, -2]]
```

This is invertible by calling `to_rigged_configuration()`:

```
sage: ret = tp_krt.to_rigged_configuration(); ret
0[ ]0
-2[ ]0-2
-2[ ]0-2
0[ ]0
0[ ]0
0[ ]0
0[ ]0
sage: elt == ret
True
```

To view the steps of the bijection in the output, run with the `display_steps=True` option:

```
sage: elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(True)
```

```
0[ ]0
-2[ ]0-2
-2[ ]0-2
0[ ]0
0[ ]0
0[ ]0
0[ ]0
-------------------------------
[[3, 2]]
-------------------------------
...
[[2, 3], [3, -2]]
```

We can also construct and display a graph of the bijection as follows:

```
sage: ret, G = elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(build_graph=True)
```

(continues on next page)
weight()

Return the weight of self.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['E', 6, 1], [[2,2]])
sage: [x.weight() for x in RC.module_generators]
[-4*Lambda[0] + 2*Lambda[2], -2*Lambda[0] + Lambda[2], 0]
sage: KR = crystals.KirillovReshetikhin(['E',6,1], 2,2)
sage: [x.weight() for x in KR.module_generators]  # long time
[0, -2*Lambda[0] + Lambda[2], -4*Lambda[0] + 2*Lambda[2]]
sage: RC = RiggedConfigurations(['D', 6, 1], [[4,2]])
sage: [x.weight() for x in RC.module_generators]
[-4*Lambda[0] + 2*Lambda[4], -4*Lambda[0] + Lambda[2] + Lambda[4],
 -2*Lambda[0] + Lambda[4], -4*Lambda[0] + 2*Lambda[2],
 -2*Lambda[0] + Lambda[2], 0]
```

class sage.combinat.rigged_configurations.rigged_configuration_element.RCHWNonSimplyLacedElement (parent, rigged_partitions=[], **options)

Bases: RCNonSimplyLacedElement

Rigged configurations in highest weight crystals.

check()

Make sure all of the riggings are less than or equal to the vacancy number.

f(a)

Return the action of $f_a$ on self.

This works by lifting into the virtual configuration, then applying

$$f_a' = \prod_{j \in \iota(a)} \hat{f}_j^{\gamma_j}$$

and pulling back.

EXAMPLES:

```
sage: La = RootSystem(['C',2,1]).weight_lattice(extended=True).fundamental_weights()
sage: vct = CartanType(['C',2,1]).as_folding()
sage: RC = crystals.RiggedConfigurations(vct, La[0])
sage: elt = RC(partition_list=[[1,1],[2],[2]])
sage: elt.f(0)
sage: ascii_art(elt.f(1))
0[ ]0 0[ ]0 0[ ]0 -1[ ]-1
0[ ]0 -1[ ]-1
sage: elt.f(2)
```
weight()

Return the weight of self.

EXAMPLES:

```python
sage: La = RootSystem(['C',2,1]).weight_lattice(extended=True).fundamental_weights()
sage: vct = CartanType(['C',2,1]).as_folding()
sage: B = crystals.RiggedConfigurations(vct, La[0])
sage: mg = B.module_generators[0]
sage: mg.f_string([0,1,2]).weight()
```

class sage.combinat.rigged_configurations.rigged_configuration_element.RCHighestWeightElement(parent, rigged_partitions=[], **options)

Bases: RiggedConfigurationElement

Rigged configurations in highest weight crystals.

check()

Make sure all of the riggings are less than or equal to the vacancy number.

f(a)

Return the action of the crystal operator $f_a$ on self.

This implements the method defined in [CrysStructSchilling06] which finds the value $k$ which is the length of the string with the smallest nonpositive rigging of largest length. Then it adds a box from a string of length $k$ in the $a$-th rigged partition, keeping all colabels fixed and decreasing the new label by one. If no such string exists, then it adds a new string of length 1 with label $-1$. If any of the resulting vacancy numbers are larger than the labels (i.e. it is an invalid rigged configuration), then $f_a$ is undefined.

INPUT:

• a – the index of the partition to add a box

OUTPUT:

The resulting rigged configuration element.

EXAMPLES:

```python
sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()
sage: RC = crystals.RiggedConfigurations(['A',2,1], La[0])
sage: elt = RC(partition_list=[[1,1],[1],[2]])
sage: elt.f(0)
-2[- ][-2]
-1[-1]
 1[ 1]
0[ ][]
sage: elt.f(1)
(continues on next page)
```
weight()

Return the weight of self.

EXAMPLES:

```python
sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()
sage: B = crystals.RiggedConfigurations(['A',2,1], La[0])
sage: mg = B.module_generators[0]
sage: mg.f_string([0,1,2,0]).weight()
```

class sage.combinat.rigged_configurations.rigged_configuration_element.RCNonSimplyLacedElement(parent, rigged_partitions=[], **options)

Bases: RiggedConfigurationElement

Rigged configuration elements for non-simply-laced types.

e(a)

Return the action of $e_a$ on self.

This works by lifting into the virtual configuration, then applying

$$e_a^\nu = \prod_{j \in \iota(a)} \hat{e}_j^{\gamma_j}$$

and pulling back.

EXAMPLES:

```python
sage: vct = CartanType(['C',2,1]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: elt = RC(partition_list=[[2],[1,1],[2]], rigging_list=[[1],[1,-1],[1]])
sage: ascii_art(elt.e(0))
0[ ][ ]
0[ ][ ]
-2[ ][ ]-1
-2[ ][ ]-1
sage: ascii_art(elt.e(1))
-2[ ][ ]-1
-3[ ][ ]-2
0[ ][ ]-3[ ][ ]-2
sage: ascii_art(elt.e(2))
-2[ ][ ]-1
-2[ ][ ]-1
0[ ][ ]
-2[ ][ ]-1
```
Return the action of \( f_\alpha \) on \( \text{self} \).

This works by lifting into the virtual configuration, then applying

\[
f'_\alpha = \prod_{j \in \{\alpha\}} f'_{\gamma_j}
\]

and pulling back.

**EXAMPLES:**

```python
sage: vct = CartanType(['C',2,1]).as_folding()
sage: RC = crystals.infinity.RiggedConfigurations(vct)
sage: elt = RC(partition_list=[[2],[1,1],[2]], rigging_list=[[-1],[-1,-1],[-1]])
sage: ascii_art(elt.f(0))
-4[ ][ ][ ]-2
-2[ ][ ]-1
-2[ ][ ]-1
sage: ascii_art(elt.f(1))
-1[ ][ ]0
-2[ ][ ]-2
-1[ ][ ]0
-2[ ][ ]-1
sage: ascii_art(elt.f(2))
-2[ ][ ]-1
-2[ ][ ]-1
-4[ ][ ][ ]-2
-2[ ][ ]-1
```

to_virtual_configuration()

Return the corresponding rigged configuration in the virtual crystal.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['C',2,1], [[1,2],[1,1],[2,1]])
sage: elt = RC(partition_list=[[3],[2]]); elt
0[ ][ ][ ][ ]
0[ ][ ]
sage: elt.to_virtual_configuration()
0[ ][ ][ ][ ]
0[ ][ ]
0[ ][ ][ ][ ]
0[ ][ ][ ]
```

class sage.combinat.rigged_configurations.rigged_configuration_element.RiggedConfigurationElement(parent, rigged_partitions=[], **options)

Bases: ClonableArray

A rigged configuration for simply-laced types.

For more information on rigged configurations, see `RiggedConfigurations`. For rigged configurations for non-simply-laced types, use `RCNonSimplyLacedElement`.

Typically to create a specific rigged configuration, the user will pass in the optional argument `partition_list` and if the user wants to specify the rigging values, give the optional argument `rigging_list` as well. If `rigging_list` is not passed, the rigging values are set to the corresponding vacancy numbers.
INPUT:

- **parent** – the parent of this element
- **rigged_partitions** – a list of rigged partitions

There are two optional arguments to explicitly construct a rigged configuration. The first is **partition_list** which gives a list of partitions, and the second is **rigging_list** which is a list of corresponding lists of riggings. If only **partition_list** is specified, then it sets the rigging equal to the calculated vacancy numbers.

If we are constructing a rigged configuration from a rigged configuration (say of another type) and we don’t want to recompute the vacancy numbers, we can use the **use_vacancy_numbers** to avoid the recomputation.

EXAMPLES:

Type $A_n^{(1)}$ examples:

```python
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 2]])
sage: RC(partition_list=[[2], [2, 2], [2], [2]])
```

```
0
-2
-2
2
-2
```

```python
sage: RC = RiggedConfigurations(['A', 4, 1], [[1, 1], [1, 1]])
sage: RC(partition_list=[[], [], [], []])
```

$(/)

$(/)

$(/)

$(/)
```

Type $D_n^{(1)}$ examples:

```python
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 2]])
sage: RC(partition_list=[[3], [3, 2], [4], [3]])
```

```
-1
1
0
-3
-1
```

```python
sage: RC = RiggedConfigurations(['D', 4, 1], [[1, 1], [2, 1]])
```

(continues on next page)
sage: RC(partition_list=[[1], [1,1], [1], [1]])

1[1]
0[ ]0
0[ ]0
0[ ]0
0[ ]0
0[ ]0

sage: elt = RC(partition_list=[[1], [1,1], [1], [1]], rigging_list=[[0], [0,0], [0], [0]]); elt

1[ ]0
0[ ]0
0[ ]0
0[ ]0
0[ ]0
0[ ]0

sage: from sage.combinat.rigged_configurations.rigged_partition import RiggedPartition

sage: RC2 = RiggedConfigurations(['D', 5, 1], [[2, 1], [3, 1]])

sage: l = [RiggedPartition()] + list(elt)

sage: ascii_art(RC2(*l))

(/> 1[ ]0 0[ ]0 0[ ]0 0[ ]0
   0[ ]0

sage: ascii_art(RC2(*l, use_vacancy_numbers=True))

(/> 1[ ]0 0[ ]0 0[ ]0 0[ ]0
   0[ ]0

check()

Check the rigged configuration is properly defined.

There is nothing to check here.

EXAMPLES:

sage: RC = crystals.infinity.RiggedConfigurations(['A', 4])

sage: b = RC.module_generators[0].f_string([1,2,1,1,2,4,2,3,3,2])

sage: b.check()

e(a)

Return the action of the crystal operator $e_a$ on self.

This implements the method defined in [CrysStructSchilling06] which finds the value $k$ which is the length of the string with the smallest negative rigging of smallest length. Then it removes a box from a string of length $k$ in the $a$-th rigged partition, keeping all colabels fixed and increasing the new label by one. If no such string exists, then $e_a$ is undefined.
This method can also be used when the underlying Cartan matrix is a Borcherds-Cartan matrix. In this case, then method of [SS2018] is used, where the new label is increased by half of the \(a\)-th diagonal entry of the underlying Borcherds-Cartan matrix. This method will also return \(\text{None}\) if \(a\) is imaginary and the smallest rigging in the \(a\)-th rigged partition is not exactly half of the \(a\)-th diagonal entry of the Borcherds-Cartan matrix.

**INPUT:**

- \(a\) – the index of the partition to remove a box

**OUTPUT:**

The resulting rigged configuration element.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A', 4, 1], [[2,1]])
sage: elt = RC(partition_list=[[1], [1], [1], [1]])
sage: elt.e(3)
sage: elt.e(1)
```

\(0[\ ]0\)

\(0[\ ]0\)

\(-1[\ ]-1\)

```python
sage: A = CartanMatrix([[-2,-1],[-1,-2]], borcherds=True)
sage: RC = crystals.infinity.RiggedConfigurations(A)
sage: nu0 = RC(partition_list=[[],[]])
sage: nu = nu0.f_string([1,0,0,0])
sage: ascii_art(nu.e(0))
```

5[ ]3 4[ ]3

5[ ]1

\(\epsilon(a)\)

Return \(\epsilon(a)\) of self.

Let \(x_i\) be the smallest string of \(\nu^{(a)}\) or 0 if \(\nu^{(a)} = \emptyset\), then we have \(\epsilon(a) = -\min(0, x_i)\).

**EXAMPLES:**

```python
sage: La = RootSystem(['B',2]).weight_lattice().fundamental_weights()
sage: RC = crystals.RiggedConfigurations(La[1]+La[2])
sage: I = RC.index_set()
sage: matrix([[rc.epsilon(i) for i in I] for rc in RC[:4]])
```

\[
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 2
\end{bmatrix}
\]

\(f(a)\)

Return the action of the crystal operator \(f_a\) on self.
This implements the method defined in [CrysStructSchilling06] which finds the value $k$ which is the length of the string with the smallest nonpositive rigging of largest length. Then it adds a box from a string of length $k$ in the $a$-th rigged partition, keeping all colabels fixed and decreasing the new label by one. If no such string exists, then it adds a new string of length 1 with label $-1$. However we need to modify the definition to work for $B(\infty)$ by removing the condition that the resulting rigged configuration is valid.

This method can also be used when the underlying Cartan matrix is a Borcherds-Cartan matrix. In this case, then method of [SS2018] is used, where the new label is decreased by half of the $a$-th diagonal entry of the underlying Borcherds-Cartan matrix.

**INPUT:**

- $a$ – the index of the partition to add a box

**OUTPUT:**

The resulting rigged configuration element.

**EXAMPLES:**

```python
sage: RC = crystals.infinity.RiggedConfigurations(['A', 3])
sage: nu0 = RC.module_generators[0]
sage: nu0.f(2)
\(-2[ ]\)
\(-1\)

sage: A = CartanMatrix([[-2,-1],[-1,-2]], borcherds=True)
sage: RC = crystals.infinity.RiggedConfigurations(A)
sage: nu0 = RC(partition_list=[[[],[]]])
sage: nu = nu0.f_string([1,0,0,0])
sage: ascii_art(nu.f(0))

\begin{align*}
9[ ] & 7 \quad 6[ ] 5 \\
9[ ] & 5 \\
9[ ] & 3 \\
9[ ] & 1
\end{align*}
```

**nu()**

Return the list $\nu$ of rigged partitions of this rigged configuration element.

**OUTPUT:**

The $\nu$ array as a list.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 2]])
sage: RC(partition_list=[[2], [2,2], [2], [2]]).nu()

\begin{align*}
[0[ ][ ]0
, -2[ ][ ]-2 \\
-2[ ][ ]-2 \\
2[ ][ ]2 \\
-2[ ][ ]-2
\]
```

**partition_rigging_lists()**

Return the list of partitions and the associated list of riggings of `self`.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A',3,1], [[1,2],[2,2]])
sage: rc = RC(partition_list=[[2],[1],[1]], rigging_list=[[-1],[0],[-1]]); rc
-1[ ][ ]-1
1[ ][ ]0
-1[ ][ ]-1
sage: rc.partition_rigging_lists()
[[[2], [1], [1]], [[-1], [0], [-1]]]
```

**phi(\(\alpha\))**

Return \(\varphi_\alpha\) of `self`.

Let \(x_\ell\) be the smallest string of \(\nu^{(\alpha)}\) or 0 if \(\nu^{(\alpha)} = \emptyset\), then we have \(e_\alpha = p^{(\alpha)} - \min(0, x_\ell)\).

**EXAMPLES:**

```python
sage: La = RootSystem(['B',2]).weight_lattice().fundamental_weights()
sage: RC = crystals.RiggedConfigurations(La[1]+La[2])
sage: I = RC.index_set()
sage: matrix([[rc.phi(i) for i in I] for rc in RC[:4]])
[1 1]
[0 3]
[0 2]
[1 1]
```

**vacancy_number(\(a, i\))**

Return the vacancy number \(p^{(\alpha)}_i\).

**INPUT:**

- `a` – the index of the rigged partition
- `i` – the row of the rigged partition

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A',4,1], [[2, 2]])
sage: elt = RC(partition_list=[[1], [2,1], [1], [1]])
sage: elt.vacancy_number(2, 3)
-2
sage: elt.vacancy_number(2, 2)
-2
sage: elt.vacancy_number(2, 1)
-1
sage: RC = RiggedConfigurations(['D',4,1], [[2,1], [2,1]])
sage: x = RC(partition_list=[[3], [3,1,1], [2], [3,1]]); ascii_art(x)
-1[ ][ ]-1 1[ ][ ]1 0[ ][ ]0 -3[ ][ ]-3
```

(continues on next page)
5.1.214 Rigged Configurations

AUTHORS:

• Travis Scrimshaw (2010-09-26): Initial version

sage.combinat.rigged_configurations.rigged_configurations.KirillovReshetikhinCrystal(cartan_type, r, s)

Return the KR crystal $B_{r,s}$ using rigged configurations.

This is the rigged configuration $RC(B_{r,s})$ or $RC(L)$ with $L = (L^{(a)}_i)$ and $L^{(a)}_i = \delta_{a,r} \delta_{i,s}$.

EXAMPLES:

sage: K1 = crystals.kirillov_reshetikhin.RiggedConfigurations(['A',6,2], 2, 1)
sage: K2 = crystals.kirillov_reshetikhin.LSPaths(['A',6,2], 2, 1)
sage: K1.digraph().is_isomorphic(K2.digraph(), edge_labels=True)
True

class sage.combinat.rigged_configurations.rigged_configurations.RCNonSimplyLaced(cartan_type, dims)

Bases: RiggedConfigurations

Rigged configurations in non-simply-laced types.

These are rigged configurations which lift to virtual rigged configurations in a simply-laced type.

For more on rigged configurations, see RiggedConfigurations.

Element

alias of KRRNonSimplyLacedElement

from_virtual(vrc)

Convert vrc in the virtual crystal into a rigged configuration of the original Cartan type.

INPUT:

• vrc – a virtual rigged configuration

EXAMPLES:

sage: RC = RiggedConfigurations(['C',2,1], [[1,2],[1,1],[2,1]])
sage: elt = RC(partition_list=[[3],[2]])
sage: vrc_elt = RC.to_virtual(elt)
sage: ret = RC.from_virtual(vrc_elt); ret

0[ ]0
0[ ]0
sage: ret == elt
True
kleber_tree()
Return the underlying (virtual) Kleber tree used to generate all highest weight rigged configurations.

EXAMPLES:

```sage
class RC(RiggedConfigurations):
    def kleber_tree(self):
        pass
```

module_generators()
Module generators for this set of rigged configurations.
Iterate over the highest weight rigged configurations by moving through the KleberTree and then setting appropriate values of the partitions.

EXAMPLES:

```sage
class RC(RiggedConfigurations):
    def module_generators(self):
        pass
```

to_virtual(rc)
Convert rc into a rigged configuration in the virtual crystal.

INPUT:

* rc – a rigged configuration element

EXAMPLES:

```sage
class RC(RiggedConfigurations):
    def to_virtual(self, rc):
        pass
```

(continues on next page)
virtual()

Return the corresponding virtual crystal.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['C',2,1], [[1,2],[1,1],[2,1]])
sage: RC
Rigged configurations of type ['C', 2, 1] and factor(s) ((1, 2), (1, 1), (2, 1))
sage: RC.virtual
Rigged configurations of type ['A', 3, 1] and factor(s) ((1, 2), (3, 2), (1, 1), (3, 1), (2, 2))
```

class sage.combinat.rigged_configurations.rigged_configurations.RCTypeA2Dual(cartan_type, dims)

Bases: RCTypeA2Even

Rigged configurations of type $A^{(2)}_{2n}$.

For more on rigged configurations, see RiggedConfigurations.

EXAMPLES:

```
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[1,2],[1,1],[2,1]])
sage: RC
Rigged configurations of type ['BC', 2, 2] and factor(s) ((1, 2), (1, 1), (2, 1))
sage: RC.cardinality()
750
sage: RC = RiggedConfigurations(CartanType(['A',2,2]).dual(), [[1,1]])
```

Element
alias of \texttt{KRRCTypeA2DualElement}

\textbf{\texttt{from\_virtual}(vrc)}

Convert vrc in the virtual crystal into a rigged configuration of the original Cartan type.

\textbf{INPUT:}

\begin{itemize}
  \item vrc – a virtual rigged configuration element
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[2,2]])
sage: elt = RC(partition_list=[[1],[1]])
sage: velt = RC.to_virtual(elt)
sage: ret = RC.from_virtual(velt); ret
-1[-]1
1[ ]1
sage: ret == elt
True
\end{verbatim}

\textbf{\texttt{module\_generators}()}\textbf{\hspace{1em}}

Module generators for rigged configurations of type $A_{2n}^{(2)}$.

Iterate over the highest weight rigged configurations by moving through the \texttt{KleberTree} and then setting appropriate values of the partitions. This also skips rigged configurations where $P_i^{(n)} < 1$ when $i$ is odd.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[1,1]])
sage: for x in RC.module_generators: x

(\(/\))

(\(/\))
\end{verbatim}

\textbf{\texttt{to\_virtual}(rc)}

Convert rc into a rigged configuration in the virtual crystal.

\textbf{INPUT:}

\begin{itemize}
  \item rc – a rigged configuration element
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: RC = RiggedConfigurations(CartanType(['A',4,2]).dual(), [[2,2]])
sage: elt = RC(partition_list=[[1],[1]]); elt
-1[-]1
1[ ]1
sage: velt = RC.to_virtual(elt); velt

(continues on next page)
Rigged configurations for type $\alpha_2$.
For more on rigged configurations, see `RiggedConfigurations`.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A',4,2], [[2,1], [1,2]])
sage: RC.cardinality()
150
sage: RC = RiggedConfigurations(['A',2,2], [[1,1]])
sage: RC.cardinality()
3
sage: RC = RiggedConfigurations(['A',2,2], [[1,2],[1,1]])
sage: TestSuite(RC).run() # long time
sage: RC = RiggedConfigurations(['A',4,2], [[2,1]])
sage: TestSuite(RC).run() # long time
```

**cardinality()**

Return the cardinality of self.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A',4,2], [[1,1], [2,2]])
sage: RC.cardinality()
250
```

**from_virtual(vrc)**

Convert vrc in the virtual crystal into a rigged configuration of the original Cartan type.

**INPUT:**

* vrc – a virtual rigged configuration element

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A',4,2], [[2,2]])
sage: elt = RC(partition_list=[[1],[1]])
sage: velt = RC.to_virtual(elt)
sage: ret = RC.from_virtual(velt); ret
-1[ ]-1
```
to_virtual(rc)

Convert rc into a rigged configuration in the virtual crystal.

INPUT:

• rc – a rigged configuration element

EXAMPLES:

```
sage: RC = RiggedConfigurations(["A", 4, 2], [[2, 2]])
sage: elt = RC(partition_list=[[1], [1]]); elt

sage: velt = RC.to_virtual(elt); velt

sage: velt.parent()
```

virtual()

Return the corresponding virtual crystal.

EXAMPLES:

```
sage: RC = RiggedConfigurations(["A", 4, 2], [[1, 2], [1, 1], [2, 1]])
sage: RC.virtua
```

5.1. Comprehensive Module List
that

\[ \sum_{T \times \mathbb{Z}_{\geq 0}} i m_i^{(a)} \alpha_a = \sum_{T \times \mathbb{Z}_{\geq 0}} i L_i^{(a)} \Lambda_a - \Lambda \]

where \( \alpha_a \) is a simple root, \( \Lambda_a \) is a fundamental weight, and \( m_i^{(a)} \) is the number of rows of length \( i \) in the partition \( \nu^{(a)} \).

Each partition \( \nu^{(a)} \), in the sequence also comes with a sequence of statistics \( p_i^{(a)} \) called vacancy numbers and a weakly decreasing sequence \( J_i^{(a)} \) of length \( m_i^{(a)} \) called riggings. Vacancy numbers are computed based upon the partitions and \( L_i^{(a)} \), and the riggings must satisfy \( \max J_i^{(a)} \leq p_i^{(a)} \). We call such a partition a rigged partition. For more, see [RigConBijection] [CrysStructSchilling06] [BijectionLRT].

Rigged configurations form combinatorial objects first introduced by Kerov, Kirillov and Reshetikhin that arose from studies of statistical mechanical models using the Bethe Ansatz. They are sequences of rigged partitions. A rigged partition is a partition together with a label associated to each part that satisfy certain constraints. The labels are also called riggings.

Rigged configurations exist for all affine Kac-Moody Lie algebras. See for example [HKOTT2002]. In Sage they are specified by providing a Cartan type and a list of rectangular shapes \( B \). The list of all (highest weight) rigged configurations for given \( B \) is computed via the (virtual) Kleber algorithm (see also KleberTree and VirtualKleberTree).

Rigged configurations in simply-laced types all admit a classical crystal structure [CrysStructSchilling06]. For non-simply-laced types, the crystal is given by using virtual rigged configurations [OSS03]. The highest weight rigged configurations are those where all riggings are nonnegative. The list of all rigged configurations is computed from the highest weight ones using the crystal operators.

Rigged configurations are conjecturally in bijection with TensorProductOfKirillovReshetikhinTableaux of non-exceptional affine types where the list \( B \) corresponds to the tensor factors \( B'^{r,s} \). The bijection has been proven in types \( A_n^{(1)} \) and \( D_n^{(1)} \) and when the only non-zero entries of \( L_i^{(a)} \) are either only \( L_i^{(a)} \) or only \( J_i^{(1)} \) (corresponding to single columns or rows respectively) [RigConBijection], [BijectionLRT], [BijectionDn].

KR crystals are implemented in Sage, see KirillovReshetikhinCrystal(), however, in the bijection with rigged configurations a different realization of the elements in the crystal are obtained, which are coined KR tableaux, see KirillovReshetikhinTableaux. For more details see [OSS2011].

**Note:** All non-simply-laced rigged configurations have not been proven to give rise to aligned virtual crystals (i.e. have the correct crystal structure or isomorphic as affine crystals to the tensor product of KR tableaux).

**INPUT:**

- `cartan_type` – a Cartan type
- \( B \) – a list of positive integer tuples \((r, s)\) corresponding to the tensor factors in the bijection with tensor product of Kirillov-Reshetikhin tableaux or equivalently the sequence of width \( s \) and height \( r \) rectangles

**REFERENCES:**

**EXAMPLES:**

```
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [1, 2], [1, 1]])
sage: RC
Rigged configurations of type ['A', 3, 1] and factor(s) ((3, 2), (1, 2), (1, 1))
sage: RC = RiggedConfigurations(['A', 3, 1], [[2, 1]]);
RC
Rigged configurations of type ['A', 3, 1] and factor(s) ((2, 1),)
```
A rigged configuration element with all riggings equal to the vacancy numbers can be created as follows:

```
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [2, 1], [1, 1], [1, 1]]); RC
Rigged configurations of type ['A', 3, 1] and factor(s) ((3, 2), (2, 1), (1, 1), (1, 1))
sage: elt = RC(partition_list=[[1],[],[]]); elt
0[ ]0
( )
( )
```

If on the other hand we also want to specify the riggings, this can be achieved as follows:

```
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [1, 2], [1, 1]])
sage: RC(partition_list=[[2],[2],[2]])
1[ ]0
0[ ]0
0[ ]0
sage: RC(partition_list=[[2],[2],[2]], rigging_list=[[0],[0],[0]])
1[ ]0
0[ ]0
0[ ]0
```

A larger example:

```
sage: RC = RiggedConfigurations(['D', 7, 1], [[3,3],[5,2],[4,3],[2,3],[4,4],[3,1],
                      [1,4],[2,2]]);
sage: elt = RC(partition_list=[[2],[3,2,1],[2,2,1,1],[2,2,1,1,1],[3,2,1,1,1,1],[2,1,1,1,1,1],
                      [2,1,1,1,1,1]],
```

(continues on next page)
To obtain the KR tableau under the bijection between rigged configurations and KR tableaux, we can type the following. This example was checked against Reiho Sakamoto’s Mathematica program on rigged configurations:

```
sage: output = elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(); output
[[1, 1, 1], [2, 3, 3], [3, 4, -5]] (X) [[1, 1], [2, 2], [3, 3], [5, -6], [6, -5]] →
[[1, 1, 2], [2, 2, 3], [3, 3, 7], [4, 4, -7]] (X) [[1, 1, 1], [2, 2, 2]] (X)
[[1, 1, 1, 3], [2, 2, 3, 4], [3, 3, 4, 5], [4, 4, 5, 6]] (X) [[1, 2], [3]] (X) →
[[1, 1, 1, 1]] (X) [[1, 1, 2]] (X)
```

```
sage: elt.to_tensor_product_of_kirillov_reshetikhin_tableaux().to_rigged_configuration() == elt
True
sage: output.to_rigged_configuration().to_tensor_product_of_kirillov_reshetikhin_tableaux() == output
True
```

We can also convert between rigged configurations and tensor products of KR crystals:

```
sage: output = elt.to_tensor_product_of_kirillov_reshetikhin_tableaux(); output
[[1, 1, 1], [2, 3, 3], [3, 4, -5]] (X) [[1, 1], [2, 2], [3, 3], [5, -6], [6, -5]] →
[[1, 1, 2], [2, 2, 3], [3, 3, 7], [4, 4, -7]] (X) [[1, 1, 1], [2, 2, 2]] (X)
[[1, 1, 1, 3], [2, 2, 3, 4], [3, 3, 4, 5], [4, 4, 5, 6]] (X) [[1, 2], [3]] (X) →
[[1, 1, 1, 1]] (X) [[1, 1, 2]] (X)
```

```
sage: output.to_rigged_configuration().to_tensor_product_of_kirillov_reshetikhin_tableaux() == output
True
```

We can also convert between rigged configurations and tensor products of KR crystals:
```python
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 1]])
sage: elt = RC(partition_list=[[1],[1,1],[1],[1]])
sage: tp_krc = elt.to_tensor_product_of_kirillov_reshetikhin_crystals(); tp_krc
[]
sage: ret = RC(tp_krc)
sage: ret == elt
True
sage: RC = RiggedConfigurations(['D', 4, 1], [[4,1], [3,3]])
sage: KR1 = crystals.KirillovReshetikhin(['D', 4, 1], 4, 1)
sage: KR2 = crystals.KirillovReshetikhin(['D', 4, 1], 3, 3)
sage: T = crystals.TensorProduct(KR1, KR2)
sage: t = T[1]; t
[[++++, []], [++++-, [[1], [2], [4], [-4]]]]
sage: ret = RC(t)
sage: ret.to_tensor_product_of_kirillov_reshetikhin_crystals()
[[++++, []], [++++-, [[1], [2], [4], [-4]]]]
```

Element

alias of `KRRCSimplyLacedElement`

```python
classically_highest_weight_vectors()
```

Return the classically highest weight elements of `self`.

```python
fermionic_formula(q=None, only_highest_weight=False, weight=None)
```

Return the fermionic formula associated to `self`. Given a set of rigged configurations $RC(\lambda, L)$, the fermionic formula is defined as:

$$M(\lambda, L; q) = \sum_{(\nu,J)} q^{cc(\nu,J)}$$

where we sum over all (classically highest weight) rigged configurations of weight $\lambda$ where $cc$ is the cocharge statistic. This is known to reduce to

$$M(\lambda, L; q) = \sum_{\nu} q^{cc(\nu)} \prod_{(a,i) \in I \times \mathbb{Z}} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right]_q.$$

The generating function of $M(\lambda, L; q)$ in the weight algebra subsumes all fermionic formulas:

$$M(L; q) = \sum_{\lambda \in P} M(\lambda, L; q) \lambda.$$  

This is conjecturally equal to the one dimensional configuration sum of the corresponding tensor product of Kirillov-Reshetikhin crystals, see [HKOTT2002]. This has been proven in general for type $A_n^{(1)}$ [BijectionLRT], single factors $B_r^{e,s}$ in type $D_n^{(1)}$ [OSS2011] with the result from [Sakamoto13], as well as for a tensor product of single columns [OSS2003], [BijectionDn] or a tensor product of single rows [OSS03] for all non-exceptional types.

**INPUT:**

- `q` – the variable $q$
- `only_highest_weight` – use only the classically highest weight rigged configurations
- `weight` – return the fermionic formula $M(\lambda, L; q)$ where $\lambda$ is the classical weight weight
REFERENCES:
EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 2, 1], [[1, 1], [1, 1]])
sage: RC.fermionic_formula()
B[-2*Lambda[1] + 2*Lambda[2]] + (q+1)*B[-Lambda[1]]
 + B[-2*Lambda[2]] + (q+1)*B[Lambda[2]]
sage: t = QQ['t'].gen(0)
sage: RC.fermionic_formula(t)
B[-2*Lambda[1] + 2*Lambda[2]] + (t+1)*B[-Lambda[1]]
 + B[-2*Lambda[2]] + (t+1)*B[Lambda[2]]
sage: La = RC.weight_lattice_realization().classical().fundamental_weights()
sage: RC.fermionic_formula(weight=La[2])
q + 1
sage: RC.fermionic_formula(only_highest_weight=True, weight=La[2])
q
```

Only using the highest weight elements on other types:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 1], [2, 2]])
sage: RC.fermionic_formula(only_highest_weight=True)
 + (q^4+q^3+q^2)*B[Lambda[3]]
sage: RC = RiggedConfigurations(['D', 4, 2], [[3, 1], [2, 2]])
sage: RC.fermionic_formula(only_highest_weight=True) # long time
(q^2+q)*B[Lambda[1] + Lambda[2] + Lambda[3]] + (q^5+2*q^4+q^3)*B[Lambda[1]]
 + (q^6+q^5+q^4)*B[Lambda[3]]
sage: RC = RiggedConfigurations(CartanType(['A', 4, 2]).dual(), [[1, 1], [2, 2]])
sage: RC.fermionic_formula(only_highest_weight=True)
```

kleber_tree()

Return the underlying Kleber tree used to generate all highest weight rigged configurations.
EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[1, 1], [2, 1]])
sage: RC.kleber_tree()
Kleber tree of Cartan type ['A', 3, 1] and B = ((1, 1), (2, 1))
```

**module_generators()**

Module generators for this set of rigged configurations.

Iterate over the highest weight rigged configurations by moving through the `KleberTree` and then setting appropriate values of the partitions.

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['D', 4, 1], [[2, 1]])
sage: for x in RC.module_generators: x

(//)
(//)
(//)
(//)
0[ ]0
0[ ]0
0[ ]0
0[ ]0
0[ ]0

options = Current options for RiggedConfigurations - convention: English - display: vertical - element_ascii_art: True - half_width_boxes_type_B: True

**tensor(*crystals, **options)**

Return the tensor product of self with crystals.

If crystals is a list of rigged configurations of the same Cartan type, then this returns a new `RiggedConfigurations`.

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[2, 1], [1, 3]])
sage: RC2 = RiggedConfigurations(['A', 3, 1], [[1, 1], [3, 3]])
sage: RC.tensor(RC2, RC2)
Rigged configurations of type ['A', 3, 1] and factor(s) ((2, 1), (1, 3), (1, 1), (3, 3), (1, 1), (3, 3))
sage: K = crystals.KirillovReshetikhin(['A', 3, 1], 2, 2, model='KR')
sage: RC.tensor(K)
Full tensor product of the crystals
```

(continues on next page)
tensor_product_of_kirillov_reshetikhin_crystals()

Return the corresponding tensor product of Kirillov-Reshetikhin crystals.

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[3,1],[2,2]])
sage: RC.tensor_product_of_kirillov_reshetikhin_crystals()
Full tensor product of the crystals
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(3,1),
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(2,2)
```

tensor_product_of_kirillov_reshetikhin_tableaux()

Return the corresponding tensor product of Kirillov-Reshetikhin tableaux.

EXAMPLES:

```python
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [1, 2]])
sage: RC.tensor_product_of_kirillov_reshetikhin_tableaux()
Tensor product of Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and
˓→factor(s) ((3, 2), (1, 2))
```

5.1.215 Rigged Partitions

Class and methods of the rigged partition which are used by the rigged configuration class. This is an internal class used by the rigged configurations and KR tableaux during the bijection, and is not to be used by the end-user.

We hold the partitions as an 1-dim array of positive integers where each value corresponds to the length of the row. This is the shape of the partition which can be accessed by the regular index.

The data for the vacancy number is also stored in a 1-dim array which each entry corresponds to the row of the tableau, and similarly for the partition values.

AUTHORS:

- Travis Scrimshaw (2010-09-26): Initial version

Todo: Convert this to using multiplicities \( m_i \) (perhaps with a dictionary)?

class sage.combinat.rigged_configurations.rigged_partition.RiggedPartition

Bases: SageObject

The RiggedPartition class which is the data structure of a rigged (i.e. marked or decorated) Young diagram of a partition.

Note that this class as a stand-alone object does not make sense since the vacancy numbers are calculated using the entire rigged configuration. For more, see RiggedConfigurations.

EXAMPLES:
### get_num_cells_to_column(*end_column, t=1*)

Get the number of cells in all columns before the *end_column*.

**INPUT:**

- *end_column* – The index of the column to end at
- *t* – The scaling factor

**OUTPUT:**

- The number of cells

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations([['A', 4, 1], [[2, 2]]])
sage: RP = RC(partition_list=[[2],[2,2],[2,1],[2]])[2]
sage: RP.get_num_cells_to_column(1)
2
sage: RP.get_num_cells_to_column(2)
3
sage: RP.get_num_cells_to_column(3)
3
sage: RP.get_num_cells_to_column(3, 2)
5
```

### insert_cell(*max_width*)

Insert a cell given at a singular value as long as its less than the specified width.

Note that `insert_cell()` does not update riggings or vacancy numbers, but it does prepare the space for them. Returns the width of the row we inserted at.

**INPUT:**

- *max_width* – The maximum width (i.e. row length) that we can insert the cell at

**OUTPUT:**

- The width of the row we inserted at.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations([['A', 4, 1], [[2, 2]]])
sage: RP = RC(partition_list=[[2],[2,2],[2,1],[2]])[2]
sage: RP.insert_cell(2)
2
sage: RP
0[ ][ ]None
-1[ ][ ]
```

### remove_cell(*row, num_cells=1*)

Removes a cell at the specified *row*.
Combinatorics, Release 10.0

Note that \texttt{remove\_cell()} does not set/update the vacancy numbers or the riggings, but guarantees that the location has been allocated in the returned index.

**INPUT:**
- \texttt{\_row} – the row to remove the cell from
- \texttt{\_num\_cells} – (default: 1) the number of cells to remove

**OUTPUT:**
- The location of the newly constructed row or \texttt{None} if unable to remove row or if deleted a row.

**EXAMPLES:**

```python
sage: RC = RiggedConfigurations(['A', 4, 1], [[2, 2]])
sage: RP = RC(partition_list=[[2],[2,2],[2,1],[2]])[2]
sage: RP.remove_cell(0)

0
sage: RP
None

-rigging
-vacancy\_numbers
```

**class** \texttt{sage.combinat.rigged\_configurations.rigged\_partition.RiggedPartitionTypeB}

Bases: \texttt{RiggedPartition}

Rigged partitions for type $B_{(1)}^{(1)}$ which has special printing rules which comes from the fact that the $n$-th partition can have columns of width $\frac{1}{2}$.

### 5.1.216 Tensor Product of Kirillov-Reshetikhin Tableaux

A tensor product of \texttt{KirillovReshetikhinTableaux} which are tableaux of $r$ rows and $s$ columns which naturally arise in the bijection between rigged configurations and tableaux and which are in bijection with the elements of the Kirillov-Reshetikhin crystal $B^{r,s}$, see \texttt{KirillovReshetikhinCrystal()}.

**AUTHORS:**
- Travis Scrimshaw (2010-09-26): Initial version

**EXAMPLES:**

Type $A_{(1)}^{(1)}$ examples:

```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[3,1], [2, 1]])
sage: KRT
Tensor product of Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and factor(s) ((3,0), (2, 1))
sage: KRT.cardinality()
24
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[1,1], [2,1], [3,1]])
sage: KRT
Tensor product of Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and factor(s) ((1,0), (1, 0))
```

(continues on next page)
−1), (2, 1), (3, 1))
sage: len(KRT.module_generators)
5
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[1,1], [2,1], [3,1]])
sage: KRT.cardinality()
96

Type $D_n^{(1)}$ examples:

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 1], [[1, 1], [2, 1], [3, 1]])
sage: KRT.cardinality()
96

---

Warning: This class is for internal use only!

class sage.combinat.rigged_configurations.tensor_product_kr_tableaux.HighestWeightTensorKRT(tp_krt)

Bases: UniqueRepresentation

Class so we do not have to build the module generators for $\text{TensorProductOfKirillovReshetikhinTableaux}$ at initialization.

cardinality()

Return the cardinality of self, which is the number of highest weight elements.

EXAMPLES:

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 1], [[2, 1], [3, 1]])
sage: from sage.combinat.rigged_configurations.tensor_product_kr_tableaux import HighestWeightTensorKRT
sage: HW = HighestWeightTensorKRT(KRT)
sage: HW.cardinality()
3

---

class sage.combinat.rigged_configurations.tensor_product_kr_tableaux.TensorProductOfKirillovReshetikhinTableaux

Bases: FullTensorProductOfRegularCrystals

5.1. Comprehensive Module List 2243
A tensor product of *KirillovReshetikhinTableaux*.

Through the bijection with rigged configurations, the tableaux that are produced in all nonexceptional types are all of rectangular shapes and do not necessarily obey the usual strict increase in columns and weak increase in rows. The relation between the elements of the Kirillov-Reshetikhin crystal, given by the Kashiwara-Nakashima tableaux, and the Kirillov-Reshetikhin tableaux is given by a filling map.

**Note:** The tableaux for all non-simply-laced types are provably correct if the bijection with *rigged configurations* holds. Therefore this is currently only proven for $B^{r,1}$ or $B^{1,s}$ and in general for types $A_n^{(1)}$ and $D_n^{(1)}$.

For more information see [OSS2011] and *KirillovReshetikhinTableaux*.

For more information on KR crystals, see `sage.combinat.crystals.kirillov_reshetikhin`.

**INPUT:**

- `cartan_type` – a Cartan type
- `B` – an (ordered) list of pairs $(r, s)$ which give the dimension of a rectangle with $r$ rows and $s$ columns and corresponds to a Kirillov-Reshetikhin tableaux factor of $B^{r,s}$.

**REFERENCES:**

**EXAMPLES:**

We can go between tensor products of KR crystals and rigged configurations:

```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[3,1], [2,2]])
sage: tp_krt = KRT(pathlist=[[3,2,1],[3,2,3,2]]); tp_krt
[[1], [2], [3]]

sage: RC = RiggedConfigurations(['A',3,1], [[3,1],[2,2]]);
sage: rc_elt = tp_krt.to_rigged_configuration(); rc_elt
-2[ ]

sage: tp_krc = tp_krt.to_tensor_product_of_kirillov_reshetikhin_crystals(); tp_krc
[[[1], [2], [3]], [[2, 2], [3, 3]]]
sage: KRT(tp_krc) == tp_krt
True
sage: rc_elt == tp_krt.to_rigged_configuration()
True

sage: KR1 = crystals.KirillovReshetikhin(['A',3,1], 3,1)
sage: KR2 = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: T = crystals.TensorProduct(KR1, KR2)
sage: t = T(KR1(3,2,1), KR2(3,2,3,2))
sage: KRT(t) == tp_krt
True
sage: t == tp_krc
True
```

We can get the highest weight elements by using the attribute `module_generators`:
Combinatorics, Release 10.0

```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[3,1], [2,1]])
sage: list(KRT.module_generators)
[[[1], [2], [3]], [[1], [2]], [[1], [3], [4]], [[1], [2]], [[1], [3], [4]], [[1], [2]]]
```

To create elements directly (i.e. not passing in KR tableau elements), there is the `pathlist` option will receive a list of lists which contain the reversed far-eastern reading word of the tableau. That is to say, in English notation, the word obtain from reading bottom-to-top, left-to-right.

```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[3,2], [1,2], [2,1]])
sage: elt = KRT(pathlist=[[3, 2, 1, 4, 2, 1], [1, 3], [3, 1]])
sage: elt.pp()
  1 1 (X) 1 3 (X) 1
  2 2 3
  3 4
```

One can still create elements in the same way as tensor product of crystals:

```python
sage: K1 = crystals.KirillovReshetikhin(['A',3,1], 3, 2, model='KR')
sage: K2 = crystals.KirillovReshetikhin(['A',3,1], 1, 2, model='KR')
sage: K3 = crystals.KirillovReshetikhin(['A',3,1], 2, 1, model='KR')
sage: eltlonl = KRT(K1(3, 2, 1, 4, 2, 1), K2(1, 3), K3(3, 1))
sage: eltlonl == elt
True
```

**Element**

alias of `TensorProductOfKirillovReshetikhinTableauxElement`

**rigged_configurations()**

Return the corresponding set of rigged configurations.

**EXAMPLES:**

```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[1, 3], [2,1]])
sage: KRT.rigged_configurations()
Rigged configurations of type ['A', 3, 1] and factor(s) ((1, 3), (2, 1))
```

**tensor(**crystals, **options)**

Return the tensor product of self with crystals.

If `crystals` is a list of (a tensor product of) KR tableaux, this returns a `TensorProductOfKirillovReshetikhinTableaux`.

**EXAMPLES:**

```python
sage: TP = crystals.TensorProductOfKirillovReshetikhinTableaux(['A', 3, 1], [[1, 3],[2,1]])
sage: K = crystals.KirillovReshetikhin(['A', 3, 1], 2, 2, model='KR')
sage: TP.tensor(K, TP)
Tensor product of Kirillov-Reshetikhin tableaux of type ['A', 3, 1]
and factor(s) ((1, 3), (3, 1), (2, 2), (1, 3), (3, 1))
sage: C = crystals.KirillovReshetikhin(['A',3,1], 3, 1, model='KN')
```

(continues on next page)
sage: TP.tensor(K, C)
Full tensor product of the crystals
  [Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and shape (1, 3),
  Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and shape (3, 1),
  Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and shape (2, 2),
  Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(3,1)]

\texttt{tensor\_product\_of\_kirillov\_reshetikhin\_crystals()}  

Return the corresponding tensor product of Kirillov-Reshetikhin crystals.

EXAMPLES:

\begin{verbatim}sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A', 3, 1], [[3, 1], [2, 2]]) sage: KRT.tensor_product_of_kirillov_reshetikhin_crystals() Full tensor product of the crystals 
  [Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(3,1),
  Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(2,2)]\end{verbatim}

5.1.217 Tensor Product of Kirillov-Reshetikhin Tableaux Elements

A tensor product of \textit{KirillovReshetikhinTableauxElement}.

AUTHORS:

- Travis Scrimshaw (2010-09-26): Initial version

\texttt{class sage.combinat.rigged_configurations.tensor_product_kr_tableaux_element.TensorProductOfKirillovReshetikhinTableauxElement(parent, list=[], **options)}

Bases: \texttt{TensorProductOfRegularCrystalsElement}

An element in a tensor product of Kirillov-Reshetikhin tableaux.

For more on tensor product of Kirillov-Reshetikhin tableaux, see \texttt{TensorProductOfKirillovReshetikhinTableaux}.

The most common way to construct an element is to specify the option \texttt{pathlist} which is a list of lists which will be used to generate the individual factors of \textit{KirillovReshetikhinTableauxElement}.

EXAMPLES:

Type $A_n^{(1)}$ examples:

\begin{verbatim}sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A', 3, 1], [[1,1], [2,1], [1,1], [2,1], [2,1]]) sage: T = KRT(pathlist=[[2], [4,1], [3], [4,2], [3,1], [2,1]]) sage: T [[2]] (X) [[1], [4]] (X) [[3]] (X) [[2], [4]] (X) [[1], [3]] (X) [[1], [2]] sage: T.to_rigged_configuration()
0 [ ] [ ] 0 1 [ ] 1 1 [ ] [ ] 0\end{verbatim}
sage: T = KRT(pathlist=[[1], [2,1], [1], [4,1], [3,1], [2,1]])
```
1[ ]0
1[ ]0
0[ ][ ]0
```
sage: T
```
[1] (X) [1], [2] (X) [1], [4] (X) [1], [3] (X) [1], [2]
```
```python
sage: T.to_rigged_configuration()
```
```
1[ ]0
1[ ]0
0[ ]0
0[ ]0
```

Type $D_n^{(1)}$ examples:
```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 1], [[1,1], [1,1], [1,1], [1]])
sage: T = KRT(pathlist=[[-1], [-1], [1], [1]])
sage: T
```
```
[-1] (X) [-1] (X) [1] (X) [1]
```
```python
sage: T.to_rigged_configuration()
```
```
0[ ][ ]0
0[ ][ ]0
0[ ][ ]0
0[ ][ ]0
0[ ][ ]0
0[ ][ ]0
```
```python
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 1], [[2,1], [1,1], [1,1], [1]])
sage: T = KRT(pathlist=[[3,2], [1], [-1], [1]])
sage: T
```
```
[2] (X) [3] (X) [1] (X) [-1] (X) [1]
```
```python
sage: T.to_rigged_configuration()
```
```
0[ ]0
0[ ]0
0[ ]0
0[ ]0
0[ ]0
1[ ]0
```
```
1[ ]0

sage: T.to_rigged_configuration().to_tensor_product_of_kirillov_reshetikhin_tableaux()
[[2], [3]] (X) [[1]] (X) [[-1]] (X) [[1]]

classical_weight()
Return the classical weight of self.

EXAMPLES:

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D',4,1], [[2, 2]])
sage: elt = KRT(pathlist=[[3,2,-1,1]]); elt
[[2, 1], [3, -1]]
sage: elt.classical_weight()
(0, 1, 1, 0)
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[2, 2],[1,3]])
sage: elt = KRT(pathlist=[[2,1,3,2],[1,4,4]]); elt
[[1, 2], [2, 3]] (X) [[1, 4, 4]]
sage: elt.classical_weight()
(2, 2, 1, 2)

left_split()
Return the image of self under the left column splitting map.

EXAMPLES:

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[2, 2],[1,3]])
sage: elt = KRT(pathlist=[[2,1,3,2],[1,4,4]]); elt.pp()
  1 2 (X) 1 4 4
  2 3
sage: elt.left_split().pp()
  1 (X) 2 (X) 1 4 4
  2 3

lusztig_involution()
Return the result of the classical Lusztig involution on self.

EXAMPLES:

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1], [[2, 2],[1,3]])
sage: elt = KRT(pathlist=[[2,1,3,2],[1,4,4]])
sage: li = elt.lusztig_involution(); li
[[1, 1, 4]] (X) [[2, 3], [3, 4]]
sage: li.parent()
Tensor product of Kirillov-Reshetikhin tableaux of type ['A', 3, 1] and factor(s) ((1, 3), (2, 2))
pp() Pretty print self.

EXAMPLES:

```
sage: TPKRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',4,1],
                      [[2,2],[3,1],[3,3]])
sage: TPKRT.module_generators[0].pp()
  1 1  (X) 1  1  1
  2 2  (X) 2  2  2
  3 3  3  3  3
```

right_split() Return the image of self under the right column splitting map.

EXAMPLES:

```
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1],
                      [[2,2],[1,3]])
sage: elt = KRT(pathlist=[[2,1,3,2],[1,4,4]]); elt.pp()
  1 2  (X) 1  4  4
  2 3
sage: elt.right_split().pp()
  1 2  (X) 1  4 (X) 4
  2 3
```

Let * denote the Lusztig involution, we check that * ∘ ls ∘ * = rs:

```
sage: all(x.lusztig_involution().left_split().lusztig_involution() == x.right_split() for x in KRT)
True
```

to_rigged_configuration(display_steps=False)

Perform the bijection from self to a rigged configuration which is described in [RigConBijection], [BijectionLRT], and [BijectionDn].

**Note:** This is only proven to be a bijection in types $A^{(1)}_n$ and $D^{(1)}_n$, as well as $\bigotimes B^{1,s_i}$ and $\bigotimes B^{1,s_i}$ for general affine types.

INPUT:

- `display_steps` – (default: False) Boolean which indicates if we want to output each step in the algorithm.

OUTPUT:

The rigged configuration corresponding to self.

EXAMPLES:

Type $A^{(1)}_n$ example:

```
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['A',3,1],
                      [[2,1],[2,1],[2,1]])
sage: T = KRT(pathlist=[[4, 2], [3, 1], [2, 1]])
sage: T
(continues on next page)```
This is invertible by calling `to_tensor_product_of_kirillov_reshetikhin_tableaux()`:

```
sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D', 4, 1], [[2, 2]])
```

(continues on next page)
sage: T = KRT(pathlist=[[2,1,4,3]])
sage: rc = T.to_rigged_configuration()
sage: ret = rc.to_tensor_product_of_kirillov_reshetikhin_tableaux(); ret
[[1, 3], [2, 4]]
sage: ret == T
True

to_tensor_product_of_kirillov_reshetikhin_crystals()
Return a tensor product of Kirillov-Reshetikhin crystals corresponding to self.
This works by performing the filling map on each individual factor. For more on the filling map, see KirillovReshetikhinTableaux.

EXAMPLES:

sage: KRT = crystals.TensorProductOfKirillovReshetikhinTableaux(['D',4,1], [[1, 1], [2,2]])
sage: elt = KRT(pathlist=[[1,-1],[-1,2,-1,1]]); elt
[[1], [2, 1], [-1, -1]]
sage: tp_krc = elt.to_tensor_product_of_kirillov_reshetikhin_crystals(); tp_krc
[[[1], [2]], [[-1]]]

We can recover the original tensor product of KR tableaux:

sage: ret = KRT(tp_krc); ret
[[1], [2, 1], [-1, -1]]
sage: ret == elt
True

5.1.218 Root Systems

Quickref

- T = CartanType(['A', 3]), T.is_finite() – Cartan types
- T.dynkin_diagram(), DynkinDiagram(['G',2]) – Dynkin diagrams
- T.cartan_matrix(), CartanMatrix(['F',4]) – Cartan matrices
- RootSystem(T).weight_lattice() – Root systems
- WeylGroup(['B', 6, 1]).simple_reflections() – Affine Weyl groups
- WeylCharacterRing(['D', 4]) – Weyl character rings
Introductory material

- *Root Systems* – This overview
- *CartanType* – An introduction to Cartan types
- *RootSystem* – An introduction to root systems
- *Tutorial: visualizing root systems* – A root system visualization tutorial
- The Lie Methods and Related Combinatorics thematic tutorial

Related material

- *Crystals* – Crystals

Cartan datum

- *Cartan types*
- *Dynkin diagrams*
- *Cartan matrices*
- *Coxeter Matrices*
- *Coxeter Types*

Root systems

- *Root Systems*
- *Tutorial: visualizing root systems*
- *Root lattice realizations*
- *Group algebras of root lattice realizations*
- *Weight lattice realizations*
- *Root lattices and root spaces*
- *Weight lattices and weight spaces*
- *Ambient lattices and ambient spaces*

Coxeter groups

- *Coxeter Groups*
- *Weyl Groups*
- *Extended Affine Weyl Groups*
- *Fundamental Group of an Extended Affine Weyl Group*
- *Braid Move Calculator*
- *Braid Orbit*
See also:
The categories \texttt{CoxeterGroups} and \texttt{WeylGroups}

\textbf{Finite reflection groups}

- Finite complex reflection groups
- Finite real reflection groups

See also:
The category \texttt{ComplexReflectionGroups}

\textbf{Representation theory}

- Weyl Character Rings
- Fusion Rings
- Integrable Representations of Affine Lie Algebras
- Branching Rules
- Hecke algebra representations
- Nonsymmetric Macdonald polynomials

\textbf{Root system data and code for specific families of Cartan types}

- Root system data for affine Cartan types
- Root system data for dual Cartan types
- Root system data for folded Cartan types
- Root system data for reducible Cartan types
- Root system data for relabelled Cartan types
- Root system data for Cartan types with marked nodes

\textbf{Root system data and code for specific Cartan types}

- Root system data for type A
- Root system data for type B
- Root system data for type C
- Root system data for type D
- Root system data for type E
- Root system data for type F
- Root system data for type G
- Root system data for type H
- Root system data for type I
- Root system data for (untwisted) type A affine
5.1.219 Ambient lattices and ambient spaces

class `sage.combinat.root_system.ambient_space.AmbientSpace`(root_system, base_ring, index_set=None)

Bases: `CombinatorialFreeModule`

Abstract class for ambient spaces

All subclasses should implement a class method `smallest_base_ring` taking a Cartan type as input, and a method `dimension` working on a partially initialized instance with just `root_system` as attribute. There is no safe default implementation for the later, so none is provided.

EXAMPLES:

```
sage: AL = RootSystem(['A',2]).ambient_lattice()
```

Note: This is only used so far for finite root systems.

Caveat: Most of the ambient spaces currently have a basis indexed by 0,...,n, unlike the usual mathematical convention:

```
sage: e = AL.basis()
sage: e[0], e[1], e[2]
((1, 0, 0), (0, 1, 0), (0, 0, 1))
```

This will be cleaned up!

See also:

- `sage.combinat.root_system.type_A.AmbientSpace`
- `sage.combinat.root_system.type_B.AmbientSpace`
- `sage.combinat.root_system.type_C.AmbientSpace`
- `sage.combinat.root_system.type_D.AmbientSpace`
- `sage.combinat.root_system.type_E.AmbientSpace`
- `sage.combinat.root_system.type_F.AmbientSpace`
- `sage.combinat.root_system.type_G.AmbientSpace`
• `sage.combinat.root_system.type_dual.AmbientSpace`
• `sage.combinat.root_system.type_affine.AmbientSpace`

**Element**

alias of `AmbientSpaceElement`

**coroot_lattice()**

EXAMPLES:

```python
sage: e = RootSystem("A", 3).ambient_lattice()
sage: e.coroot_lattice()
Ambient lattice of the Root system of type ['A', 3]
```

**dimension()**

Return the dimension of this ambient space.

EXAMPLES:

```python
sage: from sage.combinat.root_system.ambient_space import AmbientSpace
def: e = RootSystem(['F', 4]).ambient_space()
sage: AmbientSpace.dimension(e)
Traceback (most recent call last):
  ...NotImplementedError
```

**from_vector_notation**(weight, style='lattice')

INPUT:

* weight - a vector or tuple representing a weight

Returns an element of self. If the weight lattice is not of full rank, it coerces it into the weight lattice, or its ambient space by orthogonal projection. This arises in two cases: for $SL(r+1)$, the weight lattice is contained in a hyperplane of codimension one in the ambient space, and for types $E6$ and $E7$, the weight lattice is contained in a subspace of codimensions 2 or 3, respectively.

If style="coroots" and the data is a tuple of integers, it is assumed that the data represent a linear combination of fundamental weights. If style="coroots", and the root lattice is not of full rank in the ambient space, it is projected into the subspace corresponding to the semisimple derived group. This arises with Cartan type $A$, $E6$ and $E7$.

EXAMPLES:

```python
sage: RootSystem("A2").ambient_space().from_vector_notation((1,0,0))
(1, 0, 0)
sage: RootSystem("A2").ambient_space().from_vector_notation([1,0,0])
(1, 0, 0)
sage: RootSystem("A2").ambient_space().from_vector_notation((1,0),style="coroots \rightarrow")
(2/3, -1/3, -1/3)
```

**fundamental_weight(i)**

Returns the fundamental weight $\Lambda_i$ in self.

In several of the ambient spaces, it is more convenient to construct all fundamental weights at once. To support this, we provide this default implementation of `fundamental_weight` using the method `fundamental_weights`. Beware that this will cause a loop if neither `fundamental_weight` nor `fundamental_weights` is implemented.
EXAMPLES:

```python
sage: e = RootSystem(['F',4]).ambient_space()
sage: e.fundamental_weight(3)
(3/2, 1/2, 1/2, 1/2)

sage: e = RootSystem(['G',2]).ambient_space()
sage: e.fundamental_weight(1)
(1, 0, -1)

sage: e = RootSystem(['E',6]).ambient_space()
sage: e.fundamental_weight(3)
(-1/2, 1/2, 1/2, 1/2, 1/2, -5/6, -5/6, 5/6)
```

`reflection(root, coroot=None)`

EXAMPLES:

```python
sage: e = RootSystem(['A', 3]).ambient_lattice()
sage: a = e.simple_root(0); a
(-1, 0, 0, 0)

sage: b = e.simple_root(1); b
(1, -1, 0, 0)

sage: s_a = e.reflection(a)
sage: s_a(b)
(0, -1, 0, 0)
```

`simple_coroot(i)`

Returns the i-th simple coroot, as an element of this space

EXAMPLES:

```python
sage: R = RootSystem(['A',3])
sage: L = R.ambient_lattice()
sage: L.simple_coroot(1)
(1, -1, 0, 0)

sage: L.simple_coroot(2)
(0, 1, -1, 0)

sage: L.simple_coroot(3)
(0, 0, 1, -1)
```

`classmethod smallest_base_ring(cartan_type=None)`

Return the smallest ground ring over which the ambient space can be realized.

This class method will get called with the Cartan type as input. This default implementation returns \( \mathbb{Q} \); subclasses should override it as appropriate.

EXAMPLES:

```python
sage: e = RootSystem(['F',4]).ambient_space()
sage: e.smallest_base_ring()
Rational Field
```

`to_ambient_space_morphism()`

Return the identity map on `self`. 

This is present for uniformity of use: the corresponding method for abstract root and weight lattices/spaces, is not trivial.

EXAMPLES:

```python
sage: P = RootSystem(['A',2]).ambient_space()
sage: f = P.to_ambient_space_morphism()
sage: p = P.an_element()
sage: p
(2, 2, 3)
sage: f(p)
(2, 2, 3)
sage: f(p)==p
True
```

```python
class sage.combinat.root_system.ambient_space.AmbientSpaceElement
Bases: IndexedFreeModuleElement

associated_coroot()

EXAMPLES:

```python
sage: e = RootSystem(['F',4]).ambient_space()
sage: a = e.simple_root(0); a
(1/2, -1/2, -1/2, -1/2)
sage: a.associated_coroot()
(1, -1, -1, -1)
```

coerce_to_e6()

For type E7 or E8, orthogonally projects an element of the root lattice into the E6 root lattice. This operation on weights corresponds to intersection with the semisimple subgroup E6.

EXAMPLES:

```python
sage: [b.coerce_to_e6() for b in RootSystem("E8").ambient_space().basis()]
[(1, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1/3, 1/3, -1/3), (0, 0, 0, 0, 0, -1/3, -1/3, 1/3)]
```

coerce_to_e7()

For type E8, this orthogonally projects the given element of the E8 root lattice into the E7 root lattice. This operation on weights corresponds to intersection with the semisimple subgroup E7.

EXAMPLES:

```python
sage: [b.coerce_to_e7() for b in RootSystem("E8").ambient_space().basis()]
[(1, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1/2, -1/2), (0, 0, 0, 0, 0, -1/2, 1/2)]
```

coerce_to_sl()

For type ['A',r], this coerces the element of the ambient space into the root space by orthogonal projection. The root space has codimension one and corresponds to the Lie algebra of SL(r+1,CC), whereas the full weight space corresponds to the Lie algebra of GL(r+1,CC). So this operation corresponds to multiplication by a (possibly fractional) power of the determinant to give a weight determinant one.
EXAMPLES:

```python
sage: [fw.coerce_to_sl() for fw in RootSystem("A2").ambient_space().fundamental_weights()]
[(2/3, -1/3, -1/3), (1/3, 1/3, -2/3)]
```

```python
sage: L = RootSystem("A2xA3").ambient_space()
sage: L([1,2,3,4,5,0,0]).coerce_to_sl()
(-1, 0, 1, 7/4, 11/4, -9/4, -9/4)
```

**dot_product**(lambdacheck)

The scalar product with elements of the coroot lattice embedded in the ambient space.

```python
sage: e = RootSystem(['A',2]).ambient_space()
sage: a = e.simple_root(0); a
(-1, 0, 0)
sage: a.inner_product(a)
2
```

**inner_product**(lambdacheck)

The scalar product with elements of the coroot lattice embedded in the ambient space.

```python
sage: e = RootSystem(['A',2]).ambient_space()
sage: a = e.simple_root(0); a
(-1, 0, 0)
sage: a.inner_product(a)
2
```

**is_positive_root**( )

```python
sage: R = RootSystem(['A',3]).ambient_space()
sage: r=R.simple_root(1)+R.simple_root(2)
sage: r.is_positive_root()
True
```

```python
sage: r=R.simple_root(1)-R.simple_root(2)
sage: r.is_positive_root()
False
```

**scalar**(lambdacheck)

The scalar product with elements of the coroot lattice embedded in the ambient space.

```python
sage: e = RootSystem(['A',2]).ambient_space()
sage: a = e.simple_root(0); a
(-1, 0, 0)
sage: a.inner_product(a)
2
```

**to_ambient**( )

Map self to the ambient space.
This exists for uniformity. Its analogue for root and weight lattice realizations, is not trivial.

EXAMPLES:

```
sage: v = CartanType(['C',3]).root_system().ambient_space().an_element(); v
(2, 2, 3)
sage: v.to_ambient()
(2, 2, 3)
```

### 5.1.220 Associahedron

**Todo:**
- fix adjacency matrix
- edit graph method to get proper vertex labellings
- UniqueRepresentation?

**AUTHORS:**
- Christian Stump

`sage.combinat.root_system.associahedron.Associahedra(base_ring, ambient_dim, backend='ppl')`

Construct a parent class of Associahedra according to `backend`.

**See also:**
`Associahedra_base`

**class** `sage.combinat.root_system.associahedron.Associahedra_base`

Bases: `object`

Base class of parent of Associahedra of specified dimension

**EXAMPLES:**

```
sage: from sage.combinat.root_system.associahedron import Associahedra
sage: parent = Associahedra(QQ,2,'ppl'); parent
Polyhedra in QQ^2
sage: type(parent)
<class 'sage.combinat.root_system.associahedron.Associahedra_ppl_with_category'>
sage: parent(['A',2])
Generalized associahedron of type ['A', 2] with 5 vertices
```

Importantly, the parent knows the dimension of the ambient space. If you try to construct an associahedron of a different dimension, a `ValueError` is raised:

```
sage: parent(['A',3])
Traceback (most recent call last):
  ...
ValueError: V-representation data requires a list of length ambient_dim
```

**class** `sage.combinat.root_system.associahedron.Associahedra_cdd(base_ring, ambient_dim, backend)`

Bases: `Associahedra_base, Polyhedra_QQ_cdd`
Construct an associahedron.

The generalized associahedron is a polytopal complex with vertices in one-to-one correspondence with clusters in the cluster complex, and with edges between two vertices if and only if the associated two clusters intersect in codimension 1.

The associahedron of type $A_n$ is one way to realize the classical associahedron as defined in the Wikipedia article on Associahedron.

A polytopal realization of the associahedron can be found in [CFZ2002]. The implementation is based on [CFZ2002], Theorem 1.5, Remark 1.6, and Corollary 1.9.

INPUT:

- cartan_type – a cartan type according to sage.combinat.root_system.cartan_type.CartanTypeFactory
- backend – string ('ppl'); the backend to use; see sage.geometry.polyhedron.constructor.Polyhedron()

EXAMPLES:

```
sage: Asso = polytopes.associahedron(['A',2]); Asso
Generalized associahedron of type ['A', 2] with 5 vertices

sage: sorted(Asso.Hrepresentation(), key=repr)
[An inequality (-1, 0) x + 1 >= 0,
 (continues on next page)
An inequality \((0, -1) x + 1 \geq 0\),
An inequality \((0, 1) x + 1 \geq 0\),
An inequality \((1, 0) x + 1 \geq 0\),
An inequality \((1, 1) x + 1 \geq 0\)

\texttt{sage: Asso.Vrepresentation()}
(A vertex at \((1, -1)\), A vertex at \((1, 1)\), A vertex at \((-1, 1)\),
A vertex at \((-1, 0)\), A vertex at \((0, -1)\))

\texttt{sage: polytopes.associahedron(['B',2])}
Generalized associahedron of type ['B', 2] with 6 vertices

The two pictures of [CFZ2002] can be recovered with:

\texttt{sage: Asso = polytopes.associahedron(['A',3]); Asso}
Generalized associahedron of type ['A', 3] with 14 vertices
\texttt{sage: Asso.plot()}  # long time
Graphics3d Object

\texttt{sage: Asso = polytopes.associahedron(['B',3]); Asso}
Generalized associahedron of type ['B', 3] with 20 vertices
\texttt{sage: Asso.plot()}  # long time
Graphics3d Object

\texttt{class sage.combinat.root_system.associahedron.Associahedron_class_base(parent=None,}
\texttt{Vrep=None, Hrep=None, cartan_type=None, **kwds)}

Bases: object

The base class of the Python class of an associahedron

You should use the \texttt{Associahedron()} convenience function to construct associahedra from the Cartan type.

\texttt{cartan_type()}
Return the Cartan type of self.

\texttt{EXAMPLES:}

\texttt{sage: polytopes.associahedron(['A',3]).cartan_type()}
['A', 3]

\texttt{vertices_in_root_space()}
Return the vertices of self as elements in the root space.

\texttt{EXAMPLES:}

\texttt{sage: Asso = polytopes.associahedron(['A',2])}
\texttt{sage: Asso.vertices()}
(A vertex at \((1, -1)\), A vertex at \((1, 1)\),
A vertex at \((-1, 1)\), A vertex at \((-1, 0)\),
A vertex at \((0, -1)\)
Combinatorics, Release 10.0

sage: Asso.vertices_in_root_space()
-alph[1], -alph[2])

class sage.combinat.root_system.associahedron.Associahedron_class_cdd(parent=None,
Vrep=None, Hrep=None,
cartan_type=None,
**kwds)

Bases: Associahedron_class_base, Polyhedron_QQ_cdd

class sage.combinat.root_system.associahedron.Associahedron_class_field(parent=None,
Vrep=None, Hrep=None,
cartan_type=None,
**kwds)

Bases: Associahedron_class_base, Polyhedron_field

class sage.combinat.root_system.associahedron.Associahedron_class_normaliz(parent=None,
Vrep=None, Hrep=None,
cartan_type=None,
**kwds)

Bases: Associahedron_class_base, Polyhedron_QQ_normaliz

class sage.combinat.root_system.associahedron.Associahedron_class_polymake(parent=None,
Vrep=None, Hrep=None,
cartan_type=None,
**kwds)

Bases: Associahedron_class_base, Polyhedron_polymake

class sage.combinat.root_system.associahedron.Associahedron_class_ppl(parent=None,
Vrep=None, Hrep=None,
cartan_type=None,
**kwds)

Bases: Associahedron_class_base, Polyhedron_QQ_ppl

5.1.221 Braid Move Calculator

AUTHORS:

• Dinakar Muthiah (2014-06-03): initial version

class sage.combinat.root_system.braid_move_calculator.BraidMoveCalculator(coxeter_group)

Bases: object

Helper class to compute braid moves.

chain_of_reduced_words(start_word, end_word)

Compute the chain of reduced words from start_word to end_word.

INPUT:

• start_word, end_word – two reduced expressions for the long word
EXAMPLES:

```python
sage: from sage.combinat.root_system.braid_move_calculator import BraidMoveCalculator
sage: W = CoxeterGroup(['A',5])
sage: B = BraidMoveCalculator(W)
sage: B.chain_of_reduced_words((1,2,1,3,2,1,4,3,2,1,5,4,3,2,1), # not tested
....: (5,4,5,3,4,5,2,3,4,5,1,2,3,4,5))
```

`put_in_front(k, input_word)`

Return a list of reduced words starting with `input_word` and ending with a reduced word whose first letter is `k`.

There still remains an issue with 0 indices.

EXAMPLES:

```python
sage: from sage.combinat.root_system.braid_move_calculator import BraidMoveCalculator
sage: W = CoxeterGroup(['C',3])
sage: B = BraidMoveCalculator(W)
sage: B.put_in_front(2, (3, 2, 3, 1, 2, 3, 1, 2, 1))
((3, 2, 3, 1, 2, 3, 1, 2, 1),
 (3, 2, 3, 1, 2, 1, 3, 2, 1),
 (3, 2, 3, 2, 1, 2, 3, 2, 1),
 (2, 3, 2, 3, 1, 2, 3, 2, 1))
sage: B.put_in_front(1, (3, 2, 3, 1, 2, 3, 1, 2, 1))
((3, 2, 3, 1, 2, 3, 1, 2, 1),
 (3, 2, 1, 3, 2, 3, 1, 2, 1),
 (3, 2, 1, 3, 2, 3, 2, 1, 2),
 (3, 1, 2, 1, 3, 2, 3, 2, 1),
 (3, 1, 2, 1, 3, 2, 3, 1, 2),
 (1, 3, 2, 1, 3, 2, 3, 1, 2))
sage: B.put_in_front(1, (1, 3, 2, 3, 2, 1, 2, 3, 2))
((1, 3, 2, 3, 2, 1, 2, 3, 2),)
```

### 5.1.222 Braid Orbit

Cython function to compute the orbit of the braid moves on a reduced word.

```python
sage.combinat.root_system.braid_orbit.BraidOrbit(word, rels)
```

Return the orbit of `word` by all replacements given by `rels`.

**INPUT:**

- `word` – list of integers
- `rels` – list of pairs (A, B), where A and B are lists of integers the same length

**EXAMPLES:**

```python
sage: from sage.combinat.root_system.braid_orbit import BraidOrbit
sage: word = [1,2,1,3,2,1]
sage: rels = [[[2, 1, 2], [1, 2, 1]], [[3, 1], [1, 3]], [[3, 2, 3], [2, 3, 2]]]
sage: sorted(BraidOrbit(word, rels))
```
[(1, 2, 1, 3, 2, 1),
(1, 2, 3, 1, 2, 1),
(1, 2, 3, 2, 1, 2),
(1, 3, 2, 1, 3, 2),
(1, 3, 2, 3, 1, 2),
(2, 1, 2, 3, 2, 1),
(2, 1, 3, 2, 1, 3),
(2, 1, 3, 2, 3, 1),
(2, 3, 1, 2, 1, 3),
(2, 3, 1, 2, 1, 3),
(2, 3, 1, 2, 3, 1),
(2, 3, 2, 1, 2, 3),
(3, 1, 2, 1, 3, 2),
(3, 1, 2, 3, 1, 2),
(3, 2, 1, 2, 3, 2),
(3, 2, 1, 3, 2, 3),
(3, 2, 3, 1, 2, 3)]

sage: len(_)
16

5.1.223 Branching Rules

class sage.combinat.root_system.branching_rules.BranchingRule(R, S, f, name='default',
    intermediate_types=[],
    intermediate_names=[])

Bases: SageObject

A class for branching rules.

\textbf{Rtype()}

In a branching rule R \rightarrow S, returns the Cartan Type of the ambient group R.

\textbf{EXAMPLES:}

sage: branching_rule("A3","A2","levi").Rtype()
['A', 3]

\textbf{Stype()}

In a branching rule R \rightarrow S, returns the Cartan Type of the subgroup S.

\textbf{EXAMPLES:}

sage: branching_rule("A3","A2","levi").Stype()
['A', 2]

\textbf{branch(chi, style=None)}

\textbf{INPUT:}

\begin{itemize}
    \item \texttt{chi} – A character of the WeylCharacterRing with Cartan type self.Rtype().
\end{itemize}

Returns the branched character.

\textbf{EXAMPLES:}
sage: G2=WeylCharacterRing("G2",style="coroots")
sage: chi=G2(1,1); chi.degree()
64
sage: b=G2.maximal_subgroup("A2"); b
extended branching rule G2 => A2
sage: b.branch(chi)
A2(0,1) + A2(1,0) + 2*A2(1,1) + A2(2,0) + A2(1,2) + A2(2,1)
sage: A2=WeylCharacterRing("A2",style="coroots"); A2
The Weyl Character Ring of Type A2 with Integer Ring coefficients
sage: chi.branch(A2,rule=b)
A2(0,1) + A2(1,0) + 2*A2(1,1) + A2(2,0) + A2(1,2) + A2(2,1)

describe(verbose=False, debug=False, no_r=False)

Describes how extended roots restrict under self.

EXAMPLES:

```
sage: branching_rule("G2","A2","extended").describe()
3
0<=0---0
1  2  0
G2~

root restrictions G2 => A2:

0---0
1  2
A2

0 => 2
2 => 1

For more detailed information use verbose=True
```

In this example, 0 is the affine root, that is, the negative of the highest root, for "G2". If i => j is printed, this means that the i-th simple (or affine) root of the ambient group restricts to the j-th simple root of the subgroup. For reference the Dynkin diagrams are also printed. The extended Dynkin diagram of the ambient group is printed if the affine root restricts to a simple root. More information is printed if the parameter verbose is true.

```
sage.combinat.root_system.branching_rules.branch_weyl_character(chi, R, S, rule='default')
```


**INPUT:**

- `chi` – a character of $G$
- `R` – the Weyl Character Ring of $G$
• S – the Weyl Character Ring of $H$

• rule – an element of the BranchingRule class or one (most usually) a keyword such as:
  - "levi"
  - "automorphic"
  - "symmetric"
  - "extended"
  - "orthogonal_sum"
  - "tensor"
  - "triality"
  - "miscellaneous"

The BranchingRule class is a wrapper for functions from the weight lattice of $G$ to the weight lattice of $H$. An instance of this class encodes an embedding of $H$ into $G$. The usual way to specify an embedding is to supply a keyword, which tells Sage to use one of the built-in rules. We will discuss these first.

To explain the predefined rules, we survey the most important branching rules. These may be classified into several cases, and once this is understood, the detailed classification can be read off from the Dynkin diagrams. Dynkin classified the maximal subgroups of Lie groups in Mat. Sbornik N.S. 30(72):349-462 (1952).

We will list give predefined rules that cover most cases where the branching rule is to a maximal subgroup. For convenience, we also give some branching rules to subgroups that are not maximal. For example, a Levi subgroup may or may not be maximal.

For example, there is a “levi” branching rule defined from $SL(5)$ (with Cartan type $A_4$) to $SL(4)$ (with Cartan type $A_3$), so we may compute the branching rule as follows:

**EXAMPLES:**

```sage
A3=WeylCharacterRing("A3",style="coroots")
A2=WeylCharacterRing("A2",style="coroots")
[A3(fw).branch(A2,rule="levi") for fw in A3.fundamental_weights()]
```

In this case the Levi branching rule is the default branching rule so we may omit the specification rule="levi".

If a subgroup is not maximal, you may specify a branching rule by finding a chain of intermediate subgroups. For this purpose, branching rules may be multiplied as in the following example.

**EXAMPLES:**

```sage
A4=WeylCharacterRing("A4",style="coroots")
A2=WeylCharacterRing("A2",style="coroots")
br=branching_rule("A4","A3")*branching_rule("A3","A2")
A4(1,0,0,0).branch(A2,rule=br)
```

You may try omitting the rule if it is “obvious”. Default rules are provided for the following cases:

$A_{2s} \Rightarrow B_s,$
$A_{2s-1} \Rightarrow C_s,$
$A_{2s-1} \Rightarrow D_s.$
The above default rules correspond to embedding the group $SO(2s + 1)$, $Sp(2s)$ or $SO(2s)$ into the corresponding general or special linear group by the standard representation. Default rules are also specified for the following cases:

$$B_{s+1} \Rightarrow D_s,$$
$$D_s \Rightarrow B_s.$$

These correspond to the embedding of $O(n)$ into $O(n + 1)$ where $n = 2s$ or $2s + 1$. Finally, the branching rule for the embedding of a Levi subgroup is also implemented as a default rule.

EXCEPTIONS:

```python
sage: A1 = WeylCharacterRing("A1", style="coroots")
sage: A2 = WeylCharacterRing("A2", style="coroots")
sage: D4 = WeylCharacterRing("D4", style="coroots")
sage: B3 = WeylCharacterRing("B3", style="coroots")
sage: B4 = WeylCharacterRing("B4", style="coroots")
sage: A6 = WeylCharacterRing("A6", style="coroots")
sage: A7 = WeylCharacterRing("A7", style="coroots")
sage: def try_default_rule(R,S):
    return [R(f).branch(S) for f in R.fundamental_weights()]
sage: try_default_rule(A2,A1)
[A1(0) + A1(1), A1(0) + A1(1)]
sage: try_default_rule(D4,B3)
[B3(0,0,0) + B3(1,0,0), B3(1,0,0) + B3(0,1,0), B3(0,0,1), B3(0,0,1)]
sage: try_default_rule(D4,B4)
[D4(0,0,0) + D4(1,0,0), D4(1,0,0) + D4(0,1,0),
 D4(0,1,0) + D4(0,0,1), D4(0,0,1) + D4(0,0,1)]
sage: try_default_rule(A7,D4)
[D4(1,0,0,0), D4(0,1,0,0), D4(0,0,1,1), D4(0,0,2,0) + D4(0,0,0,2),
 D4(0,0,1,1),
 D4(0,1,0,0),
 D4(1,0,0,0)]
sage: try_default_rule(A6,B3)
[B3(1,0,0), B3(0,1,0), B3(0,0,2), B3(0,0,2), B3(0,1,0), B3(1,0,0)]
```

If a default rule is not known, you may cue Sage as to what the Lie group embedding is by supplying a rule from the list of predefined rules. We will treat these next.

**Levi Type**

These can be read off from the Dynkin diagram. If removing a node from the Dynkin diagram produces another Dynkin diagram, there is a branching rule. A Levi subgroup may or may not be maximal. If it is maximal, there may or may not be a built-in branching rule for but you may obtain the Levi branching rule by first branching to
a suitable maximal subgroup. For these rules use the option rule="levi":

\[ A_r \Rightarrow A_{r-1} \]
\[ B_r \Rightarrow A_{r-1} \]
\[ B_r \Rightarrow B_{r-1} \]
\[ C_r \Rightarrow A_{r-1} \]
\[ C_r \Rightarrow C_{r-1} \]
\[ D_r \Rightarrow A_{r-1} \]
\[ D_r \Rightarrow D_{r-1} \]
\[ E_r \Rightarrow A_{r-1} \quad r = 7, 8 \]
\[ E_r \Rightarrow D_{r-1} \quad r = 6, 7, 8 \]
\[ E_r \Rightarrow E_{r-1} \]
\[ F_4 \Rightarrow B_3 \]
\[ F_4 \Rightarrow C_4 \]
\[ G_2 \Rightarrow A_1 \text{ (short root)} \]

Not all Levi subgroups are maximal subgroups. If the Levi is not maximal there may or may not be a preprogrammed rule="levi" for it. If there is not, the branching rule may still be obtained by going through an intermediate subgroup that is maximal using rule="extended". Thus the other Levi branching rule from \( G_2 \Rightarrow A_1 \) corresponding to the long root is available by first branching \( G_2 \Rightarrow A_2 \) then \( A_2 \Rightarrow A_1 \). Similarly the branching rules to the Levi subgroup:

\[ E_r \Rightarrow A_{r-1} \quad r = 6, 7, 8 \]

may be obtained by first branching \( E_6 \Rightarrow A_5 \times A_1, E_7 \Rightarrow A_7 \) or \( E_8 \Rightarrow A_8 \).

EXAMPLES:

```
sage: A1 = WeylCharacterRing("A1")
sage: A2 = WeylCharacterRing("A2")
sage: A3 = WeylCharacterRing("A3")
sage: A4 = WeylCharacterRing("A4")
sage: A5 = WeylCharacterRing("A5")
sage: B2 = WeylCharacterRing("B2")
sage: B3 = WeylCharacterRing("B3")
sage: B4 = WeylCharacterRing("B4")
sage: C2 = WeylCharacterRing("C2")
sage: C3 = WeylCharacterRing("C3")
sage: D3 = WeylCharacterRing("D3")
sage: D4 = WeylCharacterRing("D4")
sage: G2 = WeylCharacterRing("G2")
sage: F4 = WeylCharacterRing("F4", style="coroots")
sage: E6=WeylCharacterRing("E6",style="coroots")
sage: E7=WeylCharacterRing("E7",style="coroots")
sage: D5=WeylCharacterRing("D5",style="coroots")
sage: D6=WeylCharacterRing("D6",style="coroots")
sage: [B3(w).branch(A2,rule="levi") for w in B3.fundamental_weights()]
```

The last example must be understood as follows. The representation of \( B_3 \) being branched is spin, which is not a representation of \( SO(7) \) but of its double cover \( spin(7) \). The group \( A_2 \) is really \( GL(3) \) and the double cover of...
SO(7) induces a cover of GL(3) that is trivial over SL(3) but not over the center of GL(3). The weight lattice for this GL(3) consists of triples \((a, b, c)\) of half integers such that \(a - b\) and \(b - c\) are in \(\mathbb{Z}\), and this is reflected in the last decomposition.

```python
sage: [C3(w).branch(A2,rule="levi") for w in C3.fundamental_weights()]
[A2(1,0,0) + A2(0,0,-1),
 A2(1,1,0) + A2(1,0,-1) + A2(0,-1,-1),
 A2(-1,-1,-1) + A2(-1,-1,-1) + A2(1,1,-1) + A2(1,1,1)]
```

```python
sage: [D4(w).branch(A3,rule="levi") for w in D4.fundamental_weights()]
[A3(1,0,0,0) + A3(0,0,0,-1),
 A3(1,1,0,0) + A3(1,0,0,-1) + A3(0,0,-1,-1),
 A3(1/2,-1/2,1/2,-1/2) + A3(1/2,1/2,1/2,-1/2),
 A3(-1/2,-1/2,-1/2,-1/2) + A3(1/2,1/2,-1/2,-1/2) + A3(1/2,1/2,1/2,1/2)]
```

```python
sage: [B3(w).branch(B2,rule="levi") for w in B3.fundamental_weights()]
[2*B2(0,0) + B2(1,0), B2(0,0) + 2*B2(1,0) + B2(1,1), 2*B2(1,1,1)]
```

```python
sage: [D7=WeylCharacterRing("D7",style="coroots")
  E8=WeylCharacterRing("E8",style="coroots")
  D7=WeylCharacterRing("D7",style="coroots")
  E8(1,0,0,0,0,0,0,0).branch(D7,rule="levi") # long time (7s)
  3*D7(0,0,0,0,0,0) + 2*D7(0,0,0,0,0,1) + 2*D7(0,0,0,0,0,0,0) + 2*D7(1,0,0,0,0,0,0,0)
  + D7(0,0,0,0,0,0,0,0) + 2*D7(0,0,0,0,0,0,0,0) + D7(0,0,0,0,0,0,0,0) + D7(1,0,0,0,0,0,0,0)
  + D7(1,0,0,0,0,0,0,0) + D7(2,0,0,0,0,0,0,0)
```

(continues on next page)
Automorphic Type

If the Dynkin diagram has a symmetry, then there is an automorphism that is a special case of a branching rule. There is also an exotic “triality” automorphism of $D_4$ having order 3. Use rule="automorphic" (or for $D_4$ rule="triality"):  

\[ A_r \Rightarrow A_r \]
\[ D_r \Rightarrow D_r \]
\[ E_6 \Rightarrow E_6 \]

EXAMPLES:

```
sage: [A3(chi).branch(A3,rule="automorphic") for chi in A3.fundamental_weights()]
[A3(0,0,0,-1), A3(0,0,-1,-1), A3(0,-1,-1,-1)]
sage: [D4(chi).branch(D4,rule="automorphic") for chi in D4.fundamental_weights()]
[D4(1,0,0,0), D4(1,1,0,0), D4(1/2,1/2,1/2,1/2), D4(1/2,1/2,1/2,-1/2), D4(1/2,1/2,-1/2,-1/2)]
```

Here is an example with $D_4$ triality:

```
sage: [D4(chi).branch(D4,rule="triality") for chi in D4.fundamental_weights()]
[D4(1/2,1/2,1/2,-1/2), D4(1,1,0,0), D4(1/2,1/2,1/2,1/2), D4(1/2,1/2,1/2,-1/2), D4(1,0,0,0)]
```
**Symmetric Type**

Related to the automorphic type, when $G$ admits an outer automorphism (usually of degree 2) we may consider the branching rule to the isotropy subgroup $H$. Outer automorphisms correspond to symmetries of the Dynkin diagram. For such isotropy subgroups use rule="symmetric". We may thus obtain the following branching rules.

\[
\begin{align*}
A_{2r} & \Rightarrow B_r \\
A_{2r-1} & \Rightarrow C_r \\
A_{2r-1} & \Rightarrow D_r \\
D_r & \Rightarrow B_{r-1} \\
E_6 & \Rightarrow F_4 \\
E_6 & \Rightarrow C_4 \\
D_4 & \Rightarrow G_2
\end{align*}
\]

The last branching rule, $D_4 \Rightarrow G_2$ is not to a maximal subgroup since $D_4 \Rightarrow B_3 \Rightarrow G_2$, but it is included for convenience.

In some cases, two outer automorphisms that differ by an inner automorphism may have different fixed subgroups. Thus, while the Dynkin diagram of $E_6$ has a single involutory automorphism, there are two involutions of the group (differing by an inner automorphism) with fixed subgroups $F_4$ and $C_4$. Similarly $SL(2r)$, of Cartan type $A_{2r-1}$, has subgroups $SO(2r)$ and $Sp(2r)$, both fixed subgroups of outer automorphisms that differ from each other by an inner automorphism.

In many cases the Dynkin diagram of $H$ can be obtained by folding the Dynkin diagram of $G$.

**EXAMPLES:**

```
sage: [w.branch(B2,rule="symmetric") for w in [A4(1,0,0,0,0),A4(1,1,0,0,0),A4(1,1,1,˓→0,0),A4(2,0,0,0,0)]]
[B2(1,0), B2(1,1), B2(1,1), B2(0,0) + B2(2,0)]
sage: [A5(w).branch(C3,rule="symmetric") for w in A5.fundamental_weights()]
[C3(1,0,0), C3(0,0,0) + C3(1,1,0), C3(1,0,0) + C3(1,1,1), C3(0,0,0) + C3(1,1,0), C3(1,0,0) + C3(1,0,0), C3(1,0,0)]
sage: [A5(w).branch(D3,rule="symmetric") for w in A5.fundamental_weights()]
[D3(1,0,0), D3(1,1,0), D3(1,1,-1) + D3(1,1,1), D3(1,0,0), D3(1,0,0)]
sage: [D4(x).branch(B3,rule="symmetric") for x in D4.fundamental_weights()]
[B3(0,0,0) + B3(1,0,0), B3(1,0,0) + B3(1,1,0), B3(1/2,1/2,1/2), B3(1/2,1/2,1/2)]
sage: [D4(x).branch(G2,rule="symmetric") for x in D4.fundamental_weights()]
[G2(0,0,0) + G2(1,0,-1), 2*G2(1,0,-1) + G2(2,-1,-1), G2(0,0,0) + G2(1,0,-1), G2(0,0,0) + G2(1,0,-1)]
sage: [E6(fw).branch(F4,rule="symmetric") for fw in E6.fundamental_weights()] #␣˓→long time (4s)
[F4(0,0,0,0) + F4(0,0,0,1), F4(0,0,0,1) + F4(1,0,0,0), F4(0,0,0,1) + F4(1,0,0,0) + F4(0,0,1,0), F4(1,0,0,0) + F4(0,0,0,1)+ F4(0,0,0,0), F4(1,0,0,0) + F4(0,0,0,1) + F4(0,0,0,0) + F4(0,1,0,0), F4(0,0,0,0) + F4(0,0,0,1) + F4(0,0,0,0) + F4(0,0,0,1)]
sage: E6=WeylCharacterRing("E6",style=coroots)
sage: C4=WeylCharacterRing("C4",style=coroots)
sage: chi = E6(1,0,0,0,0,0); chi.degree() 27
sage: chi.branch(C4,rule="symmetric")
C4(0,1,0,0)
```
Extended Type

If removing a node from the extended Dynkin diagram results in a Dynkin diagram, then there is a branching
rule. Use rule="extended" for these. We will also use this classification for some rules that are not of this
type, mainly involving type $B$, such as $D_6 \Rightarrow B_3 \times B_3$.

Here is the extended Dynkin diagram for $D_6$:

```
0 6
0 0
| |  \\
| |  \\
0---0---0---0---0
1 2 3 4 6
```

Removing the node 3 results in an embedding $D_3 \times D_3 \Rightarrow D_6$. This corresponds to the embedding $SO(6) \times SO(6) \Rightarrow SO(12)$, and is of extended type. On the other hand the embedding $SO(5) \times SO(7) \Rightarrow SO(12)$ (e.g. $B_2 \times B_3 \Rightarrow D_6$) cannot be explained this way but for uniformity is implemented under rule="extended".

The following rules are implemented as special cases of rule="extended":

- $E_6 \Rightarrow A_5 \times A_1, A_2 \times A_2 \times A_2$
- $E_7 \Rightarrow A_7, D_6 \times A_1, A_3 \times A_3 \times A_1$
- $E_8 \Rightarrow A_8, D_8, E_7 \times A_1, A_4 \times A_4, D_5 \times A_3, E_6 \times A_2$
- $F_4 \Rightarrow B_4, C_3 \times A_1, A_2 \times A_2, A_3 \times A_1$
- $G_2 \Rightarrow A_1 \times A_1$

Note that $E_8$ has only a limited number of representations of reasonably low degree.

EXAMPLES:

```sage
sage: [B3(x).branch(D3,rule="extended") for x in B3.fundamental_weights()]
[0, 0, 0] + D3(1, 0, 0),
D3(1, 0, 0) + D3(1, 1, 0),
D3(1/2, 1/2, 1/2) + D3(1/2, 1/2, 1/2)
```

```
sage: [G2(w).branch(A2, rule="extended") for w in G2.fundamental_weights()]
A2(0, 0, 0) + A2(1/3, 1/3, -2/3) + A2(2/3, -1/3, -1/3),
A2(1/3, 1/3, -2/3) + A2(2/3, -1/3, -1/3) + A2(1, 0, -1)
```

```
sage: [F4(fw).branch(B4,rule="extended") for fw in F4.fundamental_weights()] # long...
B4(1/2, 1/2, 1/2, 1/2) + B4(1, 0, 0),
B4(1, 0, 0) + B4(1, 1, 0) + B4(3/2, 1/2, 1/2, 1/2) + B4(3/2, 3/2, 1/2, 1/2) + B4(2, 1, 1,
B4(1/2, 1/2, 1/2, 1/2) + B4(1, 0, 0, 0) + B4(1, 1, 0, 0) + B4(1, 1, 1, 0) + B4(3/2, 1/2, 1/2, 1/2,
B4(0, 0, 0, 0) + B4(1/2, 1/2, 1/2, 1/2) + B4(1, 0, 0, 0)
```

```
sage: E6 = WeylCharacterRing("E6", style="coroots")
sage: A2xA2xA2 = WeylCharacterRing("A2xA2xA2",style="coroots")
sage: A5xA1 = WeylCharacterRing("A5xA1",style="coroots")
sage: G2 = WeylCharacterRing("G2", style="coroots")
sage: A1xA1 = WeylCharacterRing("A1xA1", style="coroots")
sage: F4 = WeylCharacterRing("F4",style="coroots")
sage: A3xA1 = WeylCharacterRing("A3xA1", style="coroots")
sage: A2xA2 = WeylCharacterRing("A2xA2", style="coroots")
```
sage: A1xC3 = WeylCharacterRing("A1xC3",style="coroots")
sage: E6(1,0,0,0,0,0).branch(A5xA1,rule="extended") # (0.7s)
A5xA1(0,0,0,1,0,0) + A5xA1(1,0,0,0,0,1)
sage: E6(1,0,0,0,0,0).branch(A2xA2xA2, rule="extended") # (0.7s)
A2xA2xA2(0,1,1,0,0,0) + A2xA2xA2(1,0,0,0,0,1) + A2xA2xA2(0,0,0,1,1,0)
sage: E7 = WeylCharacterRing("E7",style="coroots")
sage: A7 = WeylCharacterRing("A7",style="coroots")
sage: E7(1,0,0,0,0,0,0).branch(A7,rule="extended")
A7(0,0,0,1,0,0,0) + A7(1,0,0,0,0,0,1)
sage: D6xA1 = WeylCharacterRing("D6xA1",style="coroots")
sage: E7(1,0,0,0,0,0,0).branch(D6xA1,rule="extended")
D6xA1(0,0,0,1,0,1) + D6xA1(0,1,0,0,0,0) + D6xA1(0,0,0,0,0,2)
sage: A5xA2 = WeylCharacterRing("A5xA2",style="coroots")
sage: E7(1,0,0,0,0,0,0).branch(A5xA2,rule="extended")
A5xA2(0,0,0,1,0,1,0) + A5xA2(0,1,0,0,0,0,1) + A5xA2(1,0,0,0,1,0,0) + A5xA2(0,0,0,0,1,1,0)
sage: E8 = WeylCharacterRing("E8",style="coroots")
sage: D8 = WeylCharacterRing("D8",style="coroots")
sage: A8 = WeylCharacterRing("A8",style="coroots")
sage: E8(0,0,0,0,0,0,0,1).branch(D8,rule="extended") # long time (0.56s)
D8(0,0,0,0,0,1,0,0) + D8(0,1,0,0,0,0,0,0)
sage: E8(0,0,0,0,0,0,0,1).branch(A8,rule="extended") # long time (0.73s)
A8(0,0,0,0,0,1,0,0) + A8(0,0,1,0,0,0,0,0) + A8(1,0,0,0,0,0,0,1)
sage: D4=WeylCharacterRing("D4",style="coroots")
D2xD2=WeylCharacterRing("D2xD2",style="coroots")
[sage: F4(1,0,0,0).branch(A1xC3,rule="extended") # (0.05s)
D4(fw).branch(D2xD2, rule="extended") for fw in D4.fundamental_weights()]
[D2xD2(1,0,0,1) + D2xD2(0,0,1,1),
D2xD2(2,0,0,0) + D2xD2(0,2,0,0) + D2xD2(1,1,1,1) + D2xD2(0,0,2,0) + D2xD2(0,0,0,2),
D2xD2(0,0,1,1) + D2xD2(0,1,1,0),
D2xD2(0,1,0,1) + D2xD2(0,1,1,1)]

Orthogonal Sum

Using rule="orthogonal_sum", for \( n = a + b + c + \cdots \), you can get any branching rule

\[
SO(n) \Rightarrow SO(a) \times SO(b) \times SO(c) \times \cdots, \\
Sp(2n) \Rightarrow Sp(2a) \times Sp(2b) \times Sp(2c) \times \cdots,
\]

where \( O(a) \) is type \( D_r \) for \( a = 2r \) or \( B_r \) for \( a = 2r + 1 \) and \( Sp(2r) \) is type \( C_r \). In some cases these are also of extended type, as in the case \( D_3 \times D_3 \Rightarrow D_6 \) discussed above. But in other cases, for example \( B_3 \times B_3 \Rightarrow D_7 \), they are not of extended type.
Tensor

There are branching rules:

\[ A_{rs-1} \Rightarrow A_{r-1} \times A_{s-1}, \]
\[ B_{2rs+r+s} \Rightarrow B_r \times B_s, \]
\[ D_{2rs+s} \Rightarrow B_r \times D_s, \]
\[ D_{2rs} \Rightarrow D_r \times D_s, \]
\[ C_{2rs+s} \Rightarrow B_r \times C_s, \]
\[ C_{2rs} \Rightarrow D_r \times C_s. \]

corresponding to the tensor product homomorphism. For type \( A \), the homomorphism is \( GL(r) \times GL(s) \Rightarrow GL(rs) \). For the classical types, the relevant fact is that if \( V, W \) are orthogonal or symplectic spaces, that is, spaces endowed with symmetric or skew-symmetric bilinear forms, then \( V \otimes W \) is also an orthogonal space (if \( V \) and \( W \) are both orthogonal or both symplectic) or symplectic (if one of \( V \) and \( W \) is orthogonal and the other symplectic).

The corresponding branching rules are obtained using \texttt{rule="tensor"}. 

**EXAMPLES:**

```python
sage: A5=WeylCharacterRing("A5", style="coroots")
sage: A2xA1=WeylCharacterRing("A2xA1", style="coroots")
sage: [A5(hwv).branch(A2xA1, rule="tensor") for hwv in A5.fundamental_weights()]
[A2xA1(1,0,1),
 A2xA1(0,1,2) + A2xA1(2,0,0),
 A2xA1(1,1,1) + A2xA1(0,0,3),
 A2xA1(1,0,2) + A2xA1(0,2,0),
 A2xA1(0,1,1)]
```

```python
sage: B4=WeylCharacterRing("B4",style="coroots")
sage: B1xB1=WeylCharacterRing("B1xB1",style="coroots")
sage: [B4(f).branch(B1xB1,rule="tensor") for f in B4.fundamental_weights()]
[B1xB1(2,2),
 B1xB1(2,0) + B1xB1(2,4) + B1xB1(4,2) + B1xB1(0,2),
 B1xB1(2,0) + B1xB1(2,2) + B1xB1(2,4) + B1xB1(4,2) + B1xB1(4,4) + B1xB1(6,0) +
 B1xB1(0,2) + B1xB1(0,6),
 B1xB1(1,3) + B1xB1(3,1)]
```

```python
sage: D4=WeylCharacterRing("D4",style="coroots")
sage: C2xC1=WeylCharacterRing("C2xC1",style="coroots")
sage: [D4(f).branch(C2xC1,rule="tensor") for f in D4.fundamental_weights()]
[C2xC1(1,0,1),
 C2xC1(0,1,2) + C2xC1(2,0,0) + C2xC1(0,0,2),
 C2xC1(1,0,1),
 C2xC1(0,1,0) + C2xC1(0,0,2)]
```

```python
sage: C3=WeylCharacterRing("C3",style="coroots")
sage: B1xC1=WeylCharacterRing("B1xC1",style="coroots")
sage: [C3(f).branch(B1xC1,rule="tensor") for f in C3.fundamental_weights()]
[B1xC1(2,1), B1xC1(2,2) + B1xC1(4,0), B1xC1(4,1) + B1xC1(0,3)]
```
Symmetric Power

The $k$-th symmetric and exterior power homomorphisms map

$$GL(n) \Rightarrow GL\left(\binom{n + k - 1}{k}\right) \times GL\left(\binom{n}{k}\right).$$

The corresponding branching rules are not implemented but a special case is. The $k$-th symmetric power homomorphism $SL(2) \Rightarrow GL(k+1)$ has its image inside of $SO(2r+1)$ if $k = 2r$ and inside of $Sp(2r)$ if $k = 2r - 1$. Hence there are branching rules:

$$B_r \Rightarrow A_1$$
$$C_r \Rightarrow A_1$$

and these may be obtained using the rule “symmetric_power”.

EXAMPLES:

```
sage: A1=WeylCharacterRing("A1",style="coroots")
sage: B3=WeylCharacterRing("B3",style="coroots")
sage: C3=WeylCharacterRing("C3",style="coroots")
sage: [B3(fw).branch(A1,rule="symmetric_power") for fw in B3.fundamental_weights()]
[A1(6), A1(2) + A1(6) + A1(10), A1(0) + A1(6)]
sage: [C3(fw).branch(A1,rule="symmetric_power") for fw in C3.fundamental_weights()]
```

Miscellaneous

Use rule="miscellaneous" for the following embeddings of maximal subgroups, all involving exceptional groups.

$$B_3 \Rightarrow G_2,$$
$$E_6 \Rightarrow G_2,$$
$$E_6 \Rightarrow A_2,$$
$$F_4 \Rightarrow G_2 \times A_1,$$
$$E_6 \Rightarrow G_2 \times A_2,$$
$$E_7 \Rightarrow G_2 \times C_3,$$
$$E_7 \Rightarrow F_4 \times A_1,$$
$$E_7 \Rightarrow A_1 \times A_1,$$
$$E_7 \Rightarrow G_2 \times A_1,$$
$$E_8 \Rightarrow G_2 \times F_4,$$
$$E_8 \Rightarrow A_2 \times A_1,$$
$$E_8 \Rightarrow B_2.$$

Except for those embeddings available by rule="extended", these are the only embeddings of these groups as maximal subgroups. There may be other embeddings besides these. For example, there are other more obvious embeddings of $A_2$ and $G_2$ into $E_6$. However the embeddings in this table are characterized as embeddings as maximal subgroups. Regarding the embeddings of $A_2$ and $G_2$ in $E_6$, the embeddings in question may be characterized by the condition that the 27-dimensional representations of $E_6$ restrict irreducibly to $A_2$ or $G_2$. Since $G_2$ has a subgroup isomorphic to $A_2$, it is worth mentioning that the composite branching rules:
branching_rule("E6","G2","miscellaneous") * branching_rule("G2","A2","extended")
branching_rule("E6","A2","miscellaneous")

are distinct.

These embeddings are described more completely (with references to the literature) in the thematic tutorial at:


EXAMPLES:

```
sage: G2 = WeylCharacterRing("G2")
sage: [fw1, fw2, fw3] = B3.fundamental_weights()
sage: B3(fw1+fw3).branch(G2, rule="miscellaneous")
G2(1,0,-1) + G2(2,-1,-1) + G2(2,0,-2)
sage: E6 = WeylCharacterRing("E6",style="coroots")
sage: G2 = WeylCharacterRing("G2",style="coroots")
sage: E6(1,0,0,0,0,0).branch(G2,"miscellaneous")
G2(2,0)
sage: A2 = WeylCharacterRing("A2",style="coroots")
sage: E6(1,0,0,0,0,0).branch(A2,rule="miscellaneous")
A2(1,1) + A2(1,4) + A2(4,1)
sage: E6(0,1,0,0,0,0).branch(A2,rule="miscellaneous")
A2(2,2)
sage: E6(0,1,0,0,0,0).branch(A2,rule="miscellaneous")
A2(1,1) + A2(1,4) + A2(4,1)
sage: E6(0,0,0,0,0,2).branch(G2,"miscellaneous") # long time (0.59s)
G2(0,0) + G2(2,0) + G2(1,1) + G2(0,2) + G2(4,0)
sage: F4 = WeylCharacterRing("F4",style="coroots")
sage: G2xA1=WeylCharacterRing("G2xA1",style="coroots")
sage: F4(0,0,1,0).branch(G2xA1,rule="miscellaneous")
G2xA1(1,0,0) + G2xA1(1,0,2) + G2xA1(1,0,4) + G2xA1(1,0,6) + G2xA1(0,1,4) + G2xA1(2,0,2) + G2xA1(0,0,2) + G2xA1(0,0,6)
sage: E7=WeylCharacterRing("E7",style="coroots")
sage: G2xC3=WeylCharacterRing("G2xC3",style="coroots")
sage: E7(0,0,0,0,0,0,1).branch(G2xC3,rule="miscellaneous") # long time (1.84s)
G2xC3(1,0,1,0) + G2xC3(1,0,1,1) + G2xC3(0,1,0,1) + G2xC3(2,0,1,0,0) + G2xC3(0,0,1,1,0)
sage: F4xA1=WeylCharacterRing("F4xA1",style="coroots")
sage: E7(0,0,0,0,0,0,1).branch(F4xA1,"miscellaneous")
F4xA1(0,0,0,0,0,1,0,0,1) + F4xA1(0,0,0,0,0,0,3)
sage: A1xA1=WeylCharacterRing("A1xA1",style="coroots")
sage: E7(0,0,0,0,0,0,1).branch(A1xA1,rule="miscellaneous")
A1xA1(2,5) + A1xA1(4,1) + A1xA1(6,3)
sage: A2=WeylCharacterRing("A2",style="coroots")
sage: E7(0,0,0,0,0,0,1).branch(A2,rule="miscellaneous")
A2(0,6) + A2(6,0)
sage: G2xA1=WeylCharacterRing("G2xA1",style="coroots")
sage: E7(1,0,0,0,0,0,0).branch(G2xA1,rule="miscellaneous")
G2xA1(1,0,4) + G2xA1(0,1,0) + G2xA1(2,0,2) + G2xA1(0,0,2)
sage: E8 = WeylCharacterRing("E8",style="coroots")
sage: G2xF4 = WeylCharacterRing("G2xF4",style="coroots")
```
A1 maximal subgroups of exceptional groups

There are seven cases where the exceptional group \( G_2, F_4, E_7 \) or \( E_8 \) contains a maximal subgroup of type \( A_1 \). These are tabulated in Theorem 1 of Testerman, The construction of the maximal A1’s in the exceptional algebraic groups, Proc. Amer. Math. Soc. 116 (1992), no. 3, 635-644. The names of these branching rules are roman numerals referring to the seven cases of her Theorem 1. Use these branching rules as in the following examples.

EXAMPLES:

\[
sage: A1=WeylCharacterRing("A1",style="coroots")
\]
\[
sage: G2=WeylCharacterRing("G2",style="coroots")
\]
\[
sage: F4=WeylCharacterRing("F4",style="coroots")
\]
\[
sage: E7=WeylCharacterRing("E7",style="coroots")
\]
\[
sage: E8=WeylCharacterRing("E8",style="coroots")
\]
\[
[sage: G2(f).branch(A1,rule="i") for f in G2.fundamental_weights()]
\]
\[
[A1(6), A1(2) + A1(10)]
\]
\[
sage: F4(1,0,0,0).branch(A1,rule="ii")
\]
\[
A1(2) + A1(10) + A1(14) + A1(22)
\]
\[
sage: E7(0,0,0,0,0,1).branch(A1,rule="iii")
\]
\[
A1(9) + A1(17) + A1(27)
\]
\[
sage: E7(0,0,0,0,0,1).branch(A1,rule="iv")
\]
\[
\]
\[
sage: E8(0,0,0,0,0,1).branch(A1,rule="v") # long time (0.6s)
\]
\[
\]
\[
sage: E8(0,0,0,0,0,1).branch(A1,rule="vi") # long time (0.6s)
\]
\[
\]
\[
sage: E8(0,0,0,0,0,1).branch(A1,rule="vii") # long time (0.6s)
\]
\[
\]
Branching Rules From Plethysms

Nearly all branching rules $G \Rightarrow H$ where $G$ is of type $A$, $B$, $C$ or $D$ are covered by the preceding rules. The function `branching_rule_from_plethysm()` covers the remaining cases.

This is a general rule that includes any branching rule from types $A$, $B$, $C$, or $D$ as a special case. Thus it could be used in place of the above rules and would give the same results. However it is most useful when branching from $G$ to a maximal subgroup $H$ such that $\text{rank}(H) < \text{rank}(G) - 1$.

We consider a homomorphism $H \Rightarrow G$ where $G$ is one of $SL(r+1)$, $SO(2r+1)$, $Sp(2r)$ or $SO(2r)$. The function `branching_rule_from_plethysm()` produces the corresponding branching rule. The main ingredient is the character $\chi$ of the representation of $H$ that is the homomorphism to $GL(r+1)$, $GL(2r + 1)$ or $GL(2r)$.

This rule is so powerful that it contains the other rules implemented above as special cases. First let us consider the symmetric fifth power representation of $SL(2)$.

```
sage: A1=WeylCharacterRing("A1",style="coroots")
sage: chi=A1([5])
sage: chi.degree()
6
sage: chi.frobenius_schur_indicator()
-1
```

This confirms that the character has degree 6 and is symplectic, so it corresponds to a homomorphism $SL(2) \Rightarrow Sp(6)$, and there is a corresponding branching rule $C_3 \Rightarrow A_1$.

```
sage: C3 = WeylCharacterRing("C3",style="coroots")
sage: sym5rule = branching_rule_from_plethysm(chi,"C3")
sage: [C3(hwv).branch(A1,rule=sym5rule) for hwv in C3.fundamental_weights()]
```

This is identical to the results we would obtain using `rule="symmetric_power"`. The next example gives a branching not available by other standard rules.

```
sage: G2 = WeylCharacterRing("G2",style="coroots")
sage: D7 = WeylCharacterRing("D7",style="coroots")
sage: ad=G2(0,1); ad.degree(); ad.frobenius_schur_indicator()
14
1
sage: spin = D7(0,0,0,0,0,1,0); spin.degree()
64
sage: spin.branch(G2, rule=branching_rule_from_plethysm(ad, "D7"))
G2(1,1)
```

We have confirmed that the adjoint representation of $G_2$ gives a homomorphism into $SO(14)$, and that the pullback of the one of the two 64 dimensional spin representations to $SO(14)$ is an irreducible representation of $G_2$.

We do not actually have to create the character or its parent WeylCharacterRing to create the branching rule:

```
sage: b = branching_rule("C7","C3(0,0,1)","plethysm"); b
plethysm (along C3(0,0,1)) branching rule C7 => C3
```
Isomorphic Type

Although not usually referred to as a branching rule, the effects of the accidental isomorphisms may be handled using rule="isomorphic":

\[
\begin{align*}
B_2 & \Rightarrow C_2 \\
C_2 & \Rightarrow B_2 \\
A_3 & \Rightarrow D_3 \\
D_3 & \Rightarrow A_3 \\
D_2 & \Rightarrow A_1 \Rightarrow A_1 \\
B_1 & \Rightarrow A_1 \\
C_1 & \Rightarrow A_1
\end{align*}
\]

**EXAMPLES:**

```python
sage: B2 = WeylCharacterRing("B2")
sage: C2 = WeylCharacterRing("C2")
sage: [B2(x).branch(C2, rule="isomorphic") for x in B2.fundamental_weights()]
[[C2(1,1), C2(1,0)]]
sage: [C2(x).branch(B2, rule="isomorphic") for x in C2.fundamental_weights()]
[[B2(1,1/2), B2(1,0)]]
sage: D3 = WeylCharacterRing("D3")
sage: A3 = WeylCharacterRing("A3")
sage: [D3(fw).branch(A3,rule="isomorphic") for fw in D3.fundamental_weights()]
[[A3(1,0,0), A3(0,1,0), A3(0,0,1)]]
sage: [D3(fw).branch(A3,rule="isomorphic") for fw in D3.fundamental_weights()]
[[A3(1,1,1,0), A3(1,1,1,0), A3(0,0,1)]]
sage: D2 = WeylCharacterRing("D2", style="coroots")
sage: A1xA1 = WeylCharacterRing("A1xA1", style="coroots")
sage: [D2(fw).branch(A1xA1,rule="isomorphic") for fw in D2.fundamental_weights()]
[[A1xA1(1,0), A1xA1(0,1)]]
```

Here $A_3(x,y,z,w)$ can be understood as a representation of $SL(4)$. The weights $x, y, z, w$ and $x + t, y + t, z + t, w + t$ represent the same representation of $SL(4)$ - though not of $GL(4)$ - since $A_3(x + t, y + t, z + t, w + t)$ is the same as $A_3(x, y, z, w)$ tensored with $det^t$. So as a representation of $SL(4)$, $A_3(1/4,1/4,1/4,-3/4)$ is the same as $A_3(1,1,1,0)$. The exterior square representation $SL(4) \Rightarrow GL(6)$ admits an invariant symmetric bilinear form, so is a representation $SL(4) \Rightarrow SO(6)$ that lifts to an isomorphism $SL(4) \Rightarrow Spin(6)$. Conversely, there are two isomorphisms $SO(6) \Rightarrow SL(4)$, of which we've selected one.

In cases like this you might prefer style="coroots":

```python
sage: A3 = WeylCharacterRing("A3",style="coroots")
sage: D3 = WeylCharacterRing("D3",style="coroots")
sage: [D3(fw) for fw in D3.fundamental_weights()]
[[D3(1,0,0), D3(0,1,0), D3(0,0,1)]]
sage: [D3(fw).branch(A3,rule="isomorphic") for fw in D3.fundamental_weights()]
[[A3(0,1,0), A3(0,0,1), A3(1,0,0)]]
sage: D2 = WeylCharacterRing("D2", style="coroots")
sage: A1xA1 = WeylCharacterRing("A1xA1", style="coroots")
sage: [D2(fw).branch(A1xA1,rule="isomorphic") for fw in D2.fundamental_weights()]
[[A1xA1(1,0), A1xA1(0,1)]]
```
**Branching From a Reducible WeylCharacterRing**

If the Cartan Type of R is reducible, we may project a character onto any of the components, or any combination of components. The rule to project on the first component is specified by the string "proj1", the rule to project on the second component is "proj2". To project on the first and third components, use "proj13" and so on.

**EXAMPLES:**

```sage
A2xG2=WeylCharacterRing("A2xG2",style="coroots")
A2=WeylCharacterRing("A2",style="coroots")
G2=WeylCharacterRing("G2",style="coroots")
A2xG2(1,0,1,0).branch(A2,rule="proj1")
7*A2(1,0)
A2xG2(1,0,1,0).branch(G2,rule="proj2")
3*G2(1,0)
A2xA2xG2=WeylCharacterRing("A2xA2xG2",style="coroots")
A2xA2xG2(0,1,1,1,0,1).branch(A2xG2,rule="proj13")
8*A2xG2(0,1,0,1)
```

A more general way of specifying a branching rule from a reducible type is to supply a list of rules, one **component rule** for each component type in the root system. In the following example, we branch the fundamental representations of $D_4$ down to $A_1 \times A_1 \times A_1 \times A_1$ through the intermediate group $D_2 \times D_2$. We use multiplicative notation to compose the branching rules. There is no need to construct the intermediate WeylCharacterRing with type $D_2 \times D_2$.

**EXAMPLES:**

```sage
D4 = WeylCharacterRing("D4",style="coroots")
A1xA1xA1xA1 = WeylCharacterRing("A1xA1xA1xA1",style="coroots")
b = branching_rule("D2","A1xA1","isomorphic")
br = branching_rule("D4","D2xD2","extended")*branching_rule("D2xD2","A1xA1xA1",[b,b])
[D4(fw).branch(A1xA1xA1xA1,rule=br) for fw in D4.fundamental_weights()]

[1,1,0,0] + [1,0,1,1],
[1,1,1,1] + [2,0,0,0] + [0,2,0,0] + [0,0,2,0] + [0,0,0,2],
[1,1,1,1] + [1,0,1,0],
[1,1,1,1] + [0,1,0,1] + [0,1,1,0] + [1,0,1,0]
```

In the list of rules to be supplied in branching from a reducible root system, we may use two key words "omit" and "identity". The term "omit" means that we omit one factor, projecting onto the remaining factors. The term "identity" is supplied when the irreducible factor Cartan Types of both the target and the source are the same, and the component branching rule is to be the identity map. For example, we have projection maps from $A_3 \times A_2$ to $A_3$ and $A_2$, and the corresponding branching may be accomplished as follows. In this example the same could be accomplished using rule="proj2".

**EXAMPLES:**

```sage
A3xA2=WeylCharacterRing("A3xA2",style="coroots")
A3=WeylCharacterRing("A3",style="coroots")
chi = A3xA2(0,1,0,1,0)
chi.branch(A3,rule=["identity","omit"])
3*A3(0,1,0)
A2=WeylCharacterRing("A2",style="coroots")
```
Yet another way of branching from a reducible root system with repeated Cartan types is to embed along the diagonal. The branching rule is equivalent to the tensor product, as the example shows:

```python
sage: G2=WeylCharacterRing("G2",style="coroots")
sage: G2=WeylCharacterRing("G2xG2",style="coroots")
sage: G2=WeylCharacterRing("G2",style="coroots")
sage: G2=WeylCharacterRing("G2xG2",style="coroots")
sage: G2=WeylCharacterRing("G2",style="coroots")
sage: G2=WeylCharacterRing("G2xG2",style="coroots")
sage: G2=WeylCharacterRing("G2",style="coroots")
```

Writing Your Own (Branching) Rules

Suppose you want to branch from a group \(G\) to a subgroup \(H\). Arrange the embedding so that a Cartan subalgebra \(U\) of \(H\) is contained in a Cartan subalgebra \(T\) of \(G\). There is thus a mapping from the weight spaces \(\text{Lie}(T)^* \rightarrow \text{Lie}(U)^*\). Two embeddings will produce identical branching rules if they differ by an element of the Weyl group of \(H\).

The rule is this map \(\text{Lie}(T)^*\), which is \(G.\text{space}()\), to \(\text{Lie}(U)^*\), which is \(H.\text{space}()\), which you may implement as a function. As an example, let us consider how to implement the branching rule \(A_3 \Rightarrow C_2\). Here \(H = C_2 = Sp(4)\) embedded as a subgroup in \(A_3 = GL(4)\). The Cartan subalgebra \(U\) consists of diagonal matrices with eigenvalues \(u_1, u_2, -u_2, -u_1\). The \(C_2.\text{space}()\) is the two dimensional vector spaces consisting of the linear functionals \(u_1\) and \(u_2\) on \(U\). On the other hand \(\text{Lie}(T)\) is \(\mathbb{R}^4\). A convenient way to see the restriction is to think of it as the adjoint of the map \((u_1, u_2) \mapsto (u_1, u_2, -u_2, -u_1)\), that is, \((x_0, x_1, x_2, x_3) \Rightarrow (x_0 - x_3, x_1 - x_2)\). Hence we may encode the rule as follows:

```python
def rule(x):
    return [x[0]-x[3],x[1]-x[2]]
```

or simply:

```python
rule = lambda x: [x[0]-x[3],x[1]-x[2]]
```

We may now make and use the branching rule as follows.

```python
EXAMPLES:
sage: br = BranchingRule("A3", "C2", lambda x: [x[0]-x[3],x[1]-x[2]], "homemade");
```

```python
sage: br
homemade branching rule A3 => C2
```

```python
sage: A3(0,1,0).branch(C2,rule=br)
C2(0,0) + C2(0,1)
```

```python
sage.combinat.root_system.branching_rules.branching_rule(Rtype, Stype, rule='default')
```

Creates a branching rule.

**INPUT:**

- \(R\) – the Weyl Character Ring of \(G\)
- \(S\) – the Weyl Character Ring of \(H\)
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• rule – a string describing the branching rule as a map from the weight space of $S$ to the weight space of $R$.

If the rule parameter is omitted, in some cases, a default rule is supplied. See `branch_weyl_character()`.

EXAMPLES:

```python
sage: rule = branching_rule(CartanType("A3"), CartanType("C2"), "symmetric")
sage: [rule(x) for x in WeylCharacterRing("A3").fundamental_weights()]
[[1, 0], [1, 1], [1, 0]]
```

`sage.combinat.root_system.branching_rules.branching_rule_from_plethysm(chi, cartan_type, return_matrix=False)`

Create the branching rule of a plethysm.

**INPUT:**

• `chi` – the character of an irreducible representation $\pi$ of a group $G$
• `cartan_type` – a classical Cartan type ($A$, $B$, $C$, or $D$).

It is assumed that the image of the irreducible representation $\pi$ naturally has its image in the group $G$.

Returns a branching rule for this plethysm.

**EXAMPLES:**

The adjoint representation $\text{SL}(3) \to \text{GL}(8)$ factors through $\text{SO}(8)$. The branching rule in question will describe how representations of $\text{SO}(8)$ composed with this homomorphism decompose into irreducible characters of $\text{SL}(3)$:

```python
sage: A2 = WeylCharacterRing("A2")
sage: A2 = WeylCharacterRing("A2", style="coroots")
sage: ad = A2.adjoint_representation(); ad
A2(1,1)
sage: ad.degree()
8
sage: ad.frobenius_schur_indicator()
1
```

This confirms that $ad$ has degree 8 and is orthogonal, hence factors through $\text{SO}(8)$ which is type $D_4$:

```python
sage: br = branching_rule_from_plethysm(ad, "D4")
sage: D4 = WeylCharacterRing("D4")
sage: [D4(f).branch(A2, rule = br) for f in D4.fundamental_weights()]
[A2(1,1), A2(0,3) + A2(1,1) + A2(3,0), A2(1,1), A2(1,1)]
```

`sage.combinat.root_system.branching_rules.get_branching_rule(Rtype, Stype, rule='default')`

Creates a branching rule.

**INPUT:**

• `R` – the Weyl Character Ring of $G$
• `S` – the Weyl Character Ring of $H$
• `rule` – a string describing the branching rule as a map from the weight space of $S$ to the weight space of $R$.

If the rule parameter is omitted, in some cases, a default rule is supplied. See `branch_weyl_character()`.

**EXAMPLES:**
sage: rule = branching_rule(CartanType("A3"),CartanType("C2"),"symmetric")
sage: [rule(x) for x in WeylCharacterRing("A3").fundamental_weights()]
[[1, 0], [1, 1], [1, 0]]

sage.combinat.root_system.branching_rules.maximal_subgroups(ct, mode='print_rules')

Given a classical Cartan type (of rank less than or equal to 8) this prints the Cartan types of maximal subgroups, with a method of obtaining the branching rule. The string to the right of the colon in the output is a command to create a branching rule.

INPUT:

• ct – a classical irreducible Cartan type

Returns a list of maximal subgroups of ct.

EXAMPLES:

sage: from sage.combinat.root_system.branching_rules import maximal_subgroups
sage: maximal_subgroups("D4")
B3:branching_rule("D4","B3","symmetric")
A2:branching_rule("D4","A2(1,1)","plethysm")
A1xC2:branching_rule("D4","ClxC2","tensor")*branching_rule("ClxC2","A1xC2",
cluirbranching_rule("Cl","A1","isomorphic"),"identity")

See also:

maximal_subgroups()

5.1.224 Cartan matrices

AUTHORS:

• Travis Scrimshaw (2012-04-22): Nicolas M. Thiery moved matrix creation to CartanType to prepare cartan_matrix() for deprecation.
• Ben Salisbury (2018-08-07): Added Borcherds-Cartan matrices.

class sage.combinat.root_system.cartan_matrix.CartanMatrix

Bases: Matrix_integer_sparse, CartanType_abstract

A (generalized) Cartan matrix.

A matrix $A = (a_{ij})_{i,j \in I}$ for some index set $I$ is a generalized Cartan matrix if it satisfies the following properties:

• $a_{ii} = 2$ for all $i$,
• $a_{ij} \leq 0$ for all $i \neq j$,
• $a_{ij} = 0$ if and only if $a_{ji} = 0$ for all $i \neq j$.

Additionally some reference assume that a Cartan matrix is symmetrizable (see is_symmetrizable()). However following Kac, we do not make that assumption here.

An even, integral Borcherds–Cartan matrix is an integral matrix $A = (a_{ij})_{i,j \in I}$ for some countable index set $I$ which satisfies the following properties:
• $a_{ii} \in \{2\} \cup 2\mathbb{Z}_{<0}$ for all $i$,
• $a_{ij} \leq 0$ for all $i \neq j$,
• $a_{ij} = 0$ if and only if $a_{ji} = 0$ for all $i \neq j$.

INPUT:

Can be anything which is accepted by CartanType or a matrix.

If given a matrix, one can also use the keyword cartan_type when giving a matrix to explicitly state the type. Otherwise this will try to check the input matrix against possible standard types of Cartan matrices. To disable this check, use the keyword cartan_type_check = False.

If one wants to initialize a Borcherds-Cartan matrix using matrix data, use the keyword borcherds=True. To specify the diagonal entries of corresponding to a Cartan type (a Cartan matrix is treated as matrix data), use borcherds with a list of the diagonal entries.

EXAMPLES:

```python
sage: CartanMatrix(['A', 4])
[ 2 -1  0  0]
[-1  2 -1  0]
[ 0 -1  2 -1]
[ 0  0 -1  2]
sage: CartanMatrix(['B', 6])
[ 2 -1  0  0  0  0]
[-1  2 -1  0  0  0]
[ 0 -1  2 -1  0  0]
[ 0  0 -1  2 -1  0]
[ 0  0  0 -1  2 -2]
[ 0  0  0  0 -2  2]
sage: CartanMatrix(['C', 4])
[ 2 -1  0  0]
[-1  2 -1  0]
[ 0 -1  2 -2]
[ 0  0 -1  2]
sage: CartanMatrix(['D', 6])
[ 2 -1  0  0  0  0]
[-1  2 -1  0  0  0]
[ 0 -1  2 -1  0  0]
[ 0  0 -1  2 -1 -1]
[ 0  0  0 -1  2  0]
[ 0  0  0  0  2]
sage: CartanMatrix(['E', 6])
[ 2  0 -1  0  0  0]
[ 0  2  0 -1  0  0]
[-1  0  2 -1  0  0]
[ 0 -1  0  2 -1  0]
[ 0  0 -1  2 -1  0]
[ 0  0  0 -1  2]
sage: CartanMatrix(['E', 7])
[ 2  0 -1  0  0  0  0]
[ 0  2  0 -1  0  0  0]
[-1  0  2 -1  0  0  0]
[ 0 -1  0  2 -1  0  0]
[ 0  0 -1  2 -1  0  0]
[ 0  0  0 -1  2 -1  0]
```
Continued from previous page:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

```
sage: CartanMatrix(['E', 8])
\[
\begin{bmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['F', 4])
\[
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
```

This is different from MuPAD-Combinat, due to different node convention?

```
sage: CartanMatrix(['G', 2])
\[
\begin{bmatrix}
2 & -3 \\
-1 & 2 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['A', 1, 1])
\[
\begin{bmatrix}
2 & -2 \\
-2 & 2 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['A', 3, 1])
\[
\begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['B', 3, 1])
\[
\begin{bmatrix}
2 & 0 & -1 & 0 \\
0 & 2 & -1 & 0 \\
-1 & -1 & 2 & -1 \\
0 & 0 & -2 & 2 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['C', 3, 1])
\[
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-2 & 2 & -1 & 0 \\
0 & -1 & 2 & -2 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['D', 4, 1])
\[
\begin{bmatrix}
2 & 0 & -1 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 \\
-1 & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & -1 & 0 & 2 \\
\end{bmatrix}
```

```
sage: CartanMatrix(['E', 6, 1])
\[
\begin{bmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 \\
-1 & 0 & 2 & 0 & -1 & 0 \\
0 & -1 & 0 & 2 & -1 & 0 \\
0 & 0 & -1 & -1 & 2 & -1 \\
\end{bmatrix}
```

(continues on next page)
Examples of Borcherds-Cartan matrices:

```python
sage: CartanMatrix([[2,-1],[-1,-2]], borcherds=True)
[ 2 -1]
[-1 -2]
sage: CartanMatrix('B3', borcherds=[-4,-6,2])
[-4 -1 0]
[-1 -6 -1]
[ 0 -2 2]
```

**Note:** Since this is a matrix, `row()` and `column()` will return the standard row and column respectively. To get the row with the indices as in Dynkin diagrams/Cartan types, use `row_with_indices()` and `column_with_indices()` respectively.

```python
cartan_matrix()

Return the Cartan matrix of `self`.

EXAMPLES:
```
CartanMatrix(
['C', 3]).cartan_matrix()
[ 2 -1  0]
[-1  2 -2]
[ 0 -1  2]

cartan_type()
Return the Cartan type of self or self if unknown.

EXAMPLES:

sage: C = CartanMatrix(['A', 4, 1])
sage: C.cartan_type()
['A', 4, 1]

If the Cartan type is unknown:

sage: C = CartanMatrix([[2, -1, -2], [-1, 2, -1], [-2, -1, 2]])
sage: C.cartan_type()
[ 2 -1 -2]
[-1  2 -1]
[-2 -1  2]

column_with_indices(j)
Return the j\text{th} column \(a_{i,j}\) of self as a container (or iterator) of tuples \((i, a_{i,j})\)

EXAMPLES:

sage: M = CartanMatrix(['B', 4])
sage: [ (i,a) for (i,a) in M.column_with_indices(3) ]
[(3, 2), (2, -1), (4, -2)]

coxeter_diagram()
Construct the Coxeter diagram of self.

See also:

CartanType_abstract.coxeter_diagram()

EXAMPLES:

sage: cm = CartanMatrix([[2,-5,0],[-2,2,-1],[0,-1,2]])
sage: G = cm.coxeter_diagram(); G
Graph on 3 vertices
sage: G.edges(sort=True)
[(0, 1, +Infinity), (1, 2, 3)]
sage: ct = CartanType([[ 'A', 2, 2], ['B', 3]])
sage: ct.coxeter_diagram()
Graph on 5 vertices
sage: ct.cartan_matrix().coxeter_diagram() == ct.coxeter_diagram()
True

coxeter_matrix()
Return the Coxeter matrix for self.

See also:

CartanType_abstract.coxeter_matrix()
EXEMPLARY:

```python
sage: cm = CartanMatrix([[2,-5,0],[-2,2,-1],[0,-1,2]])
sage: cm.coxeter_matrix()
[ 1 -1  2]
[-1  1  3]
[ 2  3  1]
sage: ct = CartanType(['A',2,2], ['B',3])
sage: ct.coxeter_matrix()
[ 1 -1  2  2  2]
[-1  1  2  2  2]
[ 2  2  1  3  2]
[ 2  2  3  1  4]
[ 2  2  2  4  1]
sage: ct.cartan_matrix().coxeter_matrix() == ct.coxeter_matrix()
True
```

dual()  
Return the dual Cartan matrix of self, which is obtained by taking the transpose.

EXEMPLARY:

```python
sage: ct = CartanType(['C',3])
sage: M = CartanMatrix(ct); M
[ 2 -1  0]
[-1  2 -2]
[ 0 -1  2]
sage: M.dual()
[ 2 -1  0]
[-1  2 -1]
[ 0 -2  2]
sage: M.dual() == CartanMatrix(ct.dual())
True
sage: M.dual().cartan_type() == ct.dual()
True
```

An example with arbitrary Cartan matrices:

```python
sage: cm = CartanMatrix([[2,-5, -2, 2]]); cm
[ 2 -5]
[-2  2]
sage: cm.dual()
[ 2 -2]
[-5  2]
sage: cm.dual() == CartanMatrix(cm.transpose())
True
sage: cm.dual().dual() == cm
True
```

dynkin_diagram()  
Return the Dynkin diagram corresponding to self.

EXEMPLARY:
sage: C = CartanMatrix(['A',2])
sage: C.dynkin_diagram()
O---O
1 2
A2
sage: C = CartanMatrix(['F',4,1])
sage: C.dynkin_diagram()
O---O---O=>=O---O
0 1 2 3 4
F4~
sage: C = CartanMatrix([[2,-4],[-4,2]])
sage: C.dynkin_diagram()
Dynkin diagram of rank 2

indecoplusable_blocks()
Return a tuple of all indecomposable blocks of self.

EXAMPLES:

sage: M = CartanMatrix(['A',2])
sage: M.indecomposable_blocks()
([ 2 -1]
 [-1 2])
sage: M = CartanMatrix([[['A',2,1],['A',3,1]]]
sage: M.indecomposable_blocks()
([ 2 -1 0 -1]
 [-1 2 -1 0] [ 2 -1 -1]
 [ 0 -1 2 -1] [-1 2 -1]
 [-1 0 -1 2], [-1 -1 2])

index_set()
Return the index set of self.

EXAMPLES:

sage: C = CartanMatrix(['A',1,1])
sage: C.index_set()
(0, 1)
sage: C = CartanMatrix(['E',6])
sage: C.index_set()
(1, 2, 3, 4, 5, 6)

is_affine()
Return True if self is an affine type or False otherwise.

A generalized Cartan matrix is affine if all of its indecomposable blocks are either finite (see is_finite()) or have zero determinant with all proper principal minors positive.

EXAMPLES:
sage: M = CartanMatrix(['C',4])
sage: M.is_affine()
False
sage: M = CartanMatrix(['D',4,1])
sage: M.is_affine()
True
sage: M = CartanMatrix([[2, -4], [-3, 2]])
sage: M.is_affine()
False

is_crystallographic()

Implements CartanType_abstract.is_crystallographic().

A Cartan matrix is crystallographic if it is symmetrizable.

EXAMPLES:

sage: CartanMatrix(['F',4]).is_crystallographic()
True

is_finite()

Return True if self is a finite type or False otherwise.

A generalized Cartan matrix is finite if the determinant of all its principal submatrices (see principal_submatrices()) is positive. Such matrices have a positive definite symmetrized matrix. Note that a finite matrix may consist of multiple blocks of Cartan matrices each having finite Cartan type.

EXAMPLES:

sage: M = CartanMatrix(['C',4])
sage: M.is_finite()
True
sage: M = CartanMatrix(['D',4,1])
sage: M.is_finite()
False
sage: M = CartanMatrix([[2, -4], [-3, 2]])
sage: M.is_finite()
False

is_hyperbolic(compact=False)

Return if True if self is a (compact) hyperbolic type or False otherwise.

An indecomposable generalized Cartan matrix is hyperbolic if it has negative determinant and if any proper connected subdiagram of its Dynkin diagram is of finite or affine type. It is compact hyperbolic if any proper connected subdiagram has finite type.

INPUT:

• compact – if True, check if matrix is compact hyperbolic

EXAMPLES:

sage: M = CartanMatrix([[2,-2,0],[-2,2,-1],[0,-1,2]])
sage: M.is_hyperbolic()
True
sage: M.is_hyperbolic(compact=True)
False
sage: M = CartanMatrix([[-3,2],[-3,2]])
sage: M.is_hyperbolic()
True
sage: M = CartanMatrix(['C',4])
sage: M.is_hyperbolic()
False

is_indecomposable()
Return if self is an indecomposable matrix or False otherwise.
EXAMPLES:

sage: M = CartanMatrix(['A',5])
sage: M.is_indecomposable()
True
sage: M = CartanMatrix([[2,-1,0],[-1,2,0],[0,0,2]])
sage: M.is_indecomposable()
False

is_indefinite()
Return if self is an indefinite type or False otherwise.
EXAMPLES:

sage: M = CartanMatrix([[-3,2],[-3,2]])
sage: M.is_indefinite()
True
sage: M = CartanMatrix('A2')
sage: M.is_indefinite()
False

is_lorentzian()
Return True if self is a Lorentzian type or False otherwise.
A generalized Cartan matrix is Lorentzian if it has negative determinant and exactly one negative eigenvalue.
EXAMPLES:

sage: M = CartanMatrix([[-3,2],[-3,2]])
sage: M.is_lorentzian()
True
sage: M = CartanMatrix([[2,-1],[-1,2]])
sage: M.is_lorentzian()
False

is_simply_laced()
Implements CartanType_abstract.is_simply_laced().
A Cartan matrix is simply-laced if all non diagonal entries are 0 or -1.
EXAMPLES:
Combinatorics, Release 10.0

```
sage: cm = CartanMatrix([[2, -1, -1, -1], [-1, 2, -1, -1], [-1, -1, 2, -1], [-1, -1, -1, 2]])
sage: cm.is_simply_laced()
True
```

**matrix_space(nrows=None, ncols=None, sparse=None)**

Return a matrix space over the integers.

**INPUT:**

- `nrows` - number of rows
- `ncols` - number of columns
- `sparse` - (boolean) sparseness

**EXAMPLES:**

```
sage: cm = CartanMatrix(['A', 3])
sage: cm.matrix_space()
Full MatrixSpace of 3 by 3 sparse matrices over Integer Ring
sage: cm.matrix_space(2, 2)
Full MatrixSpace of 2 by 2 sparse matrices over Integer Ring
sage: cm[:,2,1:]  # indirect doctest
[-1 0]
[ 2 -1]
```

**principal_submatrices(proper=False)**

Return a list of all principal submatrices of `self`.

**INPUT:**

- `proper` - if True, return only proper submatrices

**EXAMPLES:**

```
sage: M = CartanMatrix(['A',2])
sage: M.principal_submatrices()
[[]
 [ 2 -1]
 [2, [2], [2], [-1 2]]
]
sage: M.principal_submatrices(proper=True)
[[], [2], [2]]
```

**rank()**

Return the rank of `self`.

**EXAMPLES:**

```
sage: CartanMatrix(['C',3]).rank()
3
sage: CartanMatrix(['A2','B2','F4']).rank()
8
```

**reflection_group(type='matrix')**

Return the reflection group corresponding to `self`.

```
EXAMPLES:

```python
sage: C = CartanMatrix(['A',3])
sage: C.reflection_group()
Weyl Group of type ['A', 3] (as a matrix group acting on the root space)
```

```python
relabel (relabelling)

Return the relabelled Cartan matrix.

EXAMPLES:

```python
sage: CM = CartanMatrix(['C',3])
sage: R = CM.relabel({1:0, 2:4, 3:1}); R

```

```python
root_space()

Return the root space corresponding to self.

EXAMPLES:

```python
sage: C = CartanMatrix(['A',3])
sage: C.root_space()
Root space over the Rational Field of the Root system of type ['A', 3]
```

```python
root_system()

Return the root system corresponding to self.

EXAMPLES:

```python
sage: C = CartanMatrix(['A',3])
sage: C.root_system()
Root system of type ['A', 3]
```

```python
row_with_indices(i)

Return the $i^{th}$ row $(a_{i,j})$ of self as a container (or iterator) of tuples $(j, a_{i,j})$

EXAMPLES:

```python
sage: M = CartanMatrix(['C',4])
sage: [(i,a) for (i,a) in M.row_with_indices(3) ]
[(3, 2), (2, -1), (4, -2)]
```

```python
subtype (index_set)

Return a subtype of self given by index_set.

A subtype can be considered the Dynkin diagram induced from the Dynkin diagram of self by index_set.

EXAMPLES:

```
```
 sage: C = CartanMatrix(['F',4])
sage: S = C.subtype([1,2,3])
 sage: S
 [ 2 -1  0]
[-1  2 -1]
 [ 0 -2  2]
sage: S.index_set()
(1, 2, 3)

**symmetrized_matrix()**

Return the symmetrized matrix of self if symmetrizable.

EXAMPLES:

 sage: cm = CartanMatrix(['B',4,1])
sage: cm.symmetrized_matrix()
 [ 4  0 -2  0  0]
 [ 0  4 -2  0  0]
[-2 -2  4 -2  0]
 [ 0  0 -2  4 -2]
 [ 0  0  0 -2  2]

**symmetrizer()**

Return the symmetrizer of self.

EXAMPLES:

 sage: cm = CartanMatrix([[2,-5],[-2,2]])
sage: cm.symmetrizer()
Finite family {0: 2, 1: 5}

sage.combinat.root_system.cartan_matrix.find_cartan_type_from_matrix(CM)

Find a Cartan type by direct comparison of Dynkin diagrams given from the generalized Cartan matrix CM and return None if not found.

INPUT:

• CM – a generalized Cartan matrix

EXAMPLES:

 sage: from sage.combinat.root_system.cartan_matrix import find_cartan_type_from_matrix
 sage: CM = CartanMatrix([[2,-1,-1], [-1,2,-1], [-1,-1,2]])
sage: find_cartan_type_from_matrix(CM)
['A', 2, 1]
sage: CM = CartanMatrix([[2,-1,0], [-1,2,-2], [0,-1,2]])
sage: find_cartan_type_from_matrix(CM)
['C', 3] relabelled by {1: 0, 2: 1, 3: 2}
sage: CM = CartanMatrix([[2,-1,-2], [-1,2,-1], [-2,-1,2]])
sage: find_cartan_type_from_matrix(CM)

sage.combinat.root_system.cartan_matrix.is_borcherds_cartan_matrix(M)

Return True if M is an even, integral Borcherds-Cartan matrix. For a definition of such a matrix, see CartanMatrix.
EXAMPLES:

```python
sage: from sage.combinat.root_system.cartan_matrix import is_borcherds_cartan_matrix
sage: M = Matrix([[2,-1], [-1,2]])
sage: is_borcherds_cartan_matrix(M)
True
sage: N = Matrix([[2,-1], [-1,0]])
sage: is_borcherds_cartan_matrix(N)
False
sage: O = Matrix([[2,-1], [-1,-2]])
sage: is_borcherds_cartan_matrix(O)
True
sage: O = Matrix([[2,-1], [-1,-3]])
sage: is_borcherds_cartan_matrix(O)
False
```

```
sage.combinat.root_system.cartan_matrix.is_generalized_cartan_matrix(M)
```

Return True if \( M \) is a generalized Cartan matrix. For a definition of a generalized Cartan matrix, see `CartanMatrix`.

EXAMPLES:

```python
sage: from sage.combinat.root_system.cartan_matrix import is_generalized_cartan_matrix
sage: M = matrix([[2,-1,-2], [-1,2,-1], [-2,-1,2]])
sage: is_generalized_cartan_matrix(M)
True
sage: M = matrix([[2,-1,-2], [-1,2,-1], [0,-1,2]])
sage: is_generalized_cartan_matrix(M)
False
sage: M = matrix([[1,-1,-2], [-1,2,-1], [-2,-1,2]])
sage: is_generalized_cartan_matrix(M)
False
```

A non-symmetrizable example:

```python
sage: M = matrix([[2,-1,-2], [-1,2,-1], [-1,-1,2]])
sage: is_generalized_cartan_matrix(M)
True
```

### 5.1.225 Cartan types

**Todo:** Why does sphinx complain if I use sections here?

Introduction

Loosely speaking, Dynkin diagrams (or equivalently Cartan matrices) are graphs which are used to classify root systems, Coxeter and Weyl groups, Lie algebras, Lie groups, crystals, etc. up to an isomorphism. Cartan types are a standard set of names for those Dynkin diagrams (see Wikipedia article Dynkin_diagram).

Let us consider, for example, the Cartan type \( A_4 \):
sage: T = CartanType(['A', 4])

sage: T
['A', 4]

It is the name of the following Dynkin diagram:

```
sage: DynkinDiagram(T)
0---0---0---0
 1  2  3  4
```

A4

**Note:** For convenience, the following shortcuts are available:

```
sage: DynkinDiagram(['A', 4])
0---0---0---0
 1  2  3  4
```

A4

```
sage: DynkinDiagram('A4')
0---0---0---0
 1  2  3  4
```

A4

```
sage: T.dynkin_diagram()
0---0---0---0
 1  2  3  4
```

A4

See *DynkinDiagram* for how to further manipulate Dynkin diagrams.

From this data (the *Cartan datum*), one can construct the associated root system:

```
sage: RootSystem(T)
Root system of type ['A', 4]
```

The associated Weyl group of $A_n$ is the symmetric group $S_{n+1}$:

```
sage: W = WeylGroup(T)
sage: W
Weyl Group of type ['A', 4] (as a matrix group acting on the ambient space)
sage: W.cardinality()
120
```

while the Lie algebra is $sl_{n+1}$, and the Lie group $SL_{n+1}$ (TODO: illustrate this once this is implemented).

One may also construct crystals associated to various Dynkin diagrams. For example:

```
sage: C = crystals.Letters(T)
sage: C
The crystal of letters for type ['A', 4]
sage: C.list()
[1, 2, 3, 4, 5]
```

```
sage: C = crystals.Tableaux(T, shape=[2])
sage: C
```

(continues on next page)
The crystal of tableaux of type ['A', 4] and shape(s) [[2]]
sage: C.cardinality()
15

Here is a sample of all the finite irreducible crystallographic Cartan types:
sage: CartanType.samples(finite = True, crystallographic = True)
[['A', 1], ['A', 5], ['B', 1], ['B', 5], ['C', 1], ['C', 5], ['D', 2], ['D', 3], ['D', 5], ['E', 6], ['E', 7], ['E', 8], ['F', 4], ['G', 2]]

One can also get latex representations of the crystallographic Cartan types and their corresponding Dynkin diagrams:
sage: [latex(ct) for ct in CartanType.samples(crystallographic=True)]
[A_1, A_5, B_1, B_5, C_1, C_5, D_2, D_3, D_5, E_6, E_7, E_8, F_4, G_2, A_1^{(1)}, A_5^{(1)}, B_1^{(1)}, B_5^{(1)}, C_1^{(1)}, C_5^{(1)}, D_3^{(1)}, D_5^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}, B_5^{(1)\vee}, C_4^{(1)\vee}, F_4^{(1)\vee}, G_2^{(1)\vee}, B_5^{(2)\vee}, C_4^{(2)\vee}, F_4^{(2)\vee}, G_2^{(2)\vee}]
sage: view([DynkinDiagram(ct) for ct in CartanType.samples(crystallographic=True)]) # not tested

Non-crystallographic Cartan types are also partially supported:
sage: CartanType.samples(finite = True, crystallographic = False)
[['I', 5], ['H', 3], ['H', 4]]

In Sage, a Cartan type is used as a database of type-specific information and algorithms (see e.g. sage.combinat.root_system.type_A). This database includes how to construct the Dynkin diagram, the ambient space for the root system (see Wikipedia article Root_system), and further mathematical properties:
sage: T.is_finite(), T.is_simply_laced(), T.is_affine(), T.is_crystallographic()
(True, True, False, True)

In particular, a Sage Cartan type is endowed with a fixed choice of labels for the nodes of the Dynkin diagram. This choice follows the conventions of Nicolas Bourbaki, Lie Groups and Lie Algebras: Chapter 4-6, Elements of Mathematics, Springer (2002). ISBN 978-3540426509. For example:
sage: T = CartanType(['D', 4])
sage: DynkinDiagram(T)

\[
\begin{array}{c}
  O \\
  \mid \\
  O---O---O \\
  1 2 3 \\
\end{array}
\]

sage: E6 = CartanType(['E', 6])
sage: DynkinDiagram(E6)

\[
\begin{array}{c}
  0 \\
  \mid \\
  0---0---0 \\
  1 2 3 \\
\end{array}
\]

5.1. Comprehensive Module List
Note: The direction of the arrows is the opposite (i.e. the transpose) of Bourbaki’s convention, but agrees with Kac’s. For example, in type $C_2$, we have:

```
sage: C2 = DynkinDiagram(['C',2]); C2
0<=<0
1  2
C2
sage: C2.cartan_matrix()
[[ 2 -2]
 [−1  2]]
```

However Bourbaki would have the Cartan matrix as:

\[
\begin{bmatrix}
2 & -1 \\
-2 & 2
\end{bmatrix}
\]

If desired, other node labelling conventions can be achieved. For example the Kac labelling for type $E_6$ can be obtained via:

```
sage: E6.relabel({1:1,2:6,3:2,4:3,5:4,6:5}).dynkin_diagram()
0---0---0---0---0---0
1  2  3  4  5
E6 relabelled by {1: 1, 2: 6, 3: 2, 4: 3, 5: 4, 6: 5}
```

Contributions implementing other conventions are very welcome.

Another option is to build from scratch a new Dynkin diagram. The architecture has been designed to make it fairly easy to add other labelling conventions. In particular, we strived at choosing type free algorithms whenever possible, so in principle most features should remain available even with custom Cartan types. This has not been used much yet, so some rough corners certainly remain.

Here, we construct the hyperbolic example of Exercise 4.9 p. 57 of Kac, Infinite Dimensional Lie Algebras. We start with an empty Dynkin diagram, and add a couple nodes:

```
sage: g = DynkinDiagram()
sage: g.add_vertices([1,2,3])
```

Note that the diagonal of the Cartan matrix is already initialized:

```
sage: g.cartan_matrix()
[ 2  0  0]
[ 0  2  0]
[ 0  0  2]
```
Then we add a couple edges:

```
sage: g.add_edge(1,2,2)
sage: g.add_edge(1,3)
sage: g.add_edge(2,3)
```

and we get the desired Cartan matrix:

```
sage: g.cartan_matrix()
[2 0 0]
[0 2 0]
[0 0 2]
```

Oops, the Cartan matrix did not change! This is because it is cached for efficiency (see cached_method). In general, a Dynkin diagram should not be modified after having been used.

**Warning:** this is not checked currently

**Todo:** add a method set_mutable() as, say, for matrices

Here, we can work around this by clearing the cache:

```
sage: delattr(g, 'cartan_matrix')
```

Now we get the desired Cartan matrix:

```
sage: g.cartan_matrix()
[ 2 -1 -1]
[-2 2 -1]
[-1 -1  2]
```

Note that backward edges have been automatically added:

```
sage: g.edges(sort=True)
[(1, 2, 2), (1, 3, 1), (2, 1, 1), (2, 3, 1), (3, 1, 1), (3, 2, 1)]
```

**Reducible Cartan types**

Reducible Cartan types can be specified by passing a sequence or list of irreducible Cartan types:

```
sage: CartanType(['A',2],['B',2])
A2xB2
sage: CartanType([['A',2],['B',2]])
A2xB2
sage: CartanType(['A',2],['B',2]).is_reducible()
True
```

or using the following short hand notation:
Degenerate cases

When possible, type $I_n$ is automatically converted to the isomorphic crystallographic Cartan types (any reason not to do so?):

```python
sage: CartanType("I",1)
A1xA1
sage: CartanType("I",3)
['A', 2]
sage: CartanType("I",4)
['C', 2]
sage: CartanType("I",6)
['G', 2]
```

The Dynkin diagrams for types $B_1$, $C_1$, $D_2$, and $D_3$ are isomorphic to that for $A_1$, $A_1$, $A_1 \times A_1$, and $A_3$, respectively. However their natural ambient space realizations (stemming from the corresponding infinite families of Lie groups) are different. Therefore, the Cartan types are considered as distinct:

```python
sage: CartanType(['B',1])
['B', 1]
sage: CartanType(['C',1])
['C', 1]
sage: CartanType(['D',2])
['D', 2]
sage: CartanType(['D',3])
['D', 3]
```

Affine Cartan types

For affine types, we use the usual conventions for affine Coxeter groups: each affine type is either untwisted (that is, arise from the natural affinisation of a finite Cartan type):

```python
sage: CartanType(['A', 4, 1]).dynkin_diagram()
\[\begin{array}{cccc}
\text{0} & \text{1} & \text{2} & \text{3} \\
\end{array}\]
\text{A}_4^~
```

```python
sage: CartanType(['B', 4, 1]).dynkin_diagram()
\[\begin{array}{cccc}
\text{0} & \text{1} & \text{2} & \text{3} \\
\end{array}\]
\text{B}_4^~
```

(continues on next page)
or dual thereof:

```
sage: CartanType(['B', 4, 1]).dual().dynkin_diagram()
O 0
| |
O---O---O=<=O
1 2 3 4
B4~*
```

or is of type $\tilde{BC}_n$ (which yields an irreducible, but nonreduced root system):

```
sage: CartanType(['BC', 4, 2]).dynkin_diagram()
O=<=O---O---O=<=O
0 1 2 3 4
BC4~
```

This includes the two degenerate cases:

```
sage: CartanType(['A', 1, 1]).dynkin_diagram()
O<=>O
0 1
A1~
sage: CartanType(['BC', 1, 2]).dynkin_diagram()
4
O=<=O
0 1
BC1~
```

For the user convenience, Kac's notations for twisted affine types are automatically translated into the previous ones:

```
sage: CartanType(['A', 9, 2])
['B', 5, 1]^*
```

```
sage: CartanType(['A', 9, 2]).dynkin_diagram()
O 0
| |
O---O---O---O=<=O
1 2 3 4 5
B5~*
```

```
sage: CartanType(['A', 10, 2]).dynkin_diagram()
O=<=O---O---O---O=<=O
0 1 2 3 4 5
BC5~
```

```
sage: CartanType(['D', 5, 2]).dynkin_diagram()
O=<=O---O---O=>=O
0 1 2 3 4
C4~*
```

```
sage: CartanType(['D', 4, 3]).dynkin_diagram()
3
```

(continues on next page)
Additionally one can set the notation option to use Kac's notation:

```
sage: CartanType.options['notation'] = 'Kac'
sage: CartanType(['A', 9, 2])
['A', 9, 2]
sage: CartanType(['A', 9, 2]).dynkin_diagram()
 0 0
  |
0---0-----0<=0<=0
1 2 3 4 5
A9^2
sage: CartanType(['A', 10, 2]).dynkin_diagram()
0<=0-----0-----0<=0
0 1 2 3 4 5
A10^2
sage: CartanType(['D', 5, 2]).dynkin_diagram()
3
0=>=0-----0
2 1 0
D5^2
sage: CartanType(['E', 6, 2]).dynkin_diagram()
0---0-----0<=0<=0
0 1 2 3 4 5
E6^2
```

**Infinite Cartan types**

There are minimal implementations of the Cartan types $A_\infty$ and $A_{+\infty}$. In sage $oo$ is the same as $+Infinity$, so $NN$ and $ZZ$ are used to differentiate between the $A_{+\infty}$ and $A_\infty$ root systems:

```
sage: CartanType(['A', 'NN'])
['A', 'NN']
sage: print(CartanType(['A', 'NN']).ascii_art())
0 1 2 3 4 5 6
sage: CartanType(['A', 'ZZ'])
['A', 'ZZ']
```
There are also the following shorthands:

```
sage: CartanType("Aoo")
['A', ZZ]
sage: CartanType("A+oo")
['A', NN]
```

**Abstract classes for Cartan types**

- `CartanType_abstract`
- `CartanType_crystallographic`
- `CartanType_simply_laced`
- `CartanType_simple`
- `CartanType_finite`
- `CartanType_affine` *(see also Root system data for affine Cartan types)*
- `sage.combinat.root_system.cartan_type.CartanType`
- `Root system data for dual Cartan types`
- `Root system data for reducible Cartan types`
- `Root system data for relabelled Cartan types`

**Concrete classes for Cartan types**

- `CartanType_standard`
- `CartanType_standard_finite`
- `CartanType_standard_affine`
- `CartanType_standard_untwisted_affine`

**Type specific data**

The data essentially consists of a description of the Dynkin/Coxeter diagram and, when relevant, of the natural embedding of the root system in an Euclidean space. Everything else is reconstructed from this data.

- `Root system data for type A`
- `Root system data for type B`
- `Root system data for type C`
- `Root system data for type D`
- `Root system data for type E`
- `Root system data for type F`
- `Root system data for type G`
- `Root system data for type H`
Todo: Should those indexes come before the introduction?

sage.combinat.root_system.cartan_type.CartonType(*args)

Cartan types

Todo: Why does sphinx complain if I use sections here?

Introduction

Loosely speaking, Dynkin diagrams (or equivalently Cartan matrices) are graphs which are used to classify root systems, Coxeter and Weyl groups, Lie algebras, Lie groups, crystals, etc. up to an isomorphism. Cartan types are a standard set of names for those Dynkin diagrams (see Wikipedia article Dynkin_diagram).

Let us consider, for example, the Cartan type $A_4$:

```
sage: T = CartanType(["A", 4])
sage: T
["A", 4]
```

It is the name of the following Dynkin diagram:

```
sage: DynkinDiagram(T)
0---0---0---0
1 2 3 4
A4
```

Note: For convenience, the following shortcuts are available:

```
sage: DynkinDiagram(['A',4])
0---0---0---0
1 2 3 4
A4
sage: DynkinDiagram('A4')
```

(continues on next page)
See `DynkinDiagram` for how to further manipulate Dynkin diagrams.

From this data (the *Cartan datum*), one can construct the associated root system:

```
sage: RootSystem(T)
Root system of type ['A', 4]
```

The associated Weyl group of $A_n$ is the symmetric group $S_{n+1}$:

```
sage: W = WeylGroup(T)
sage: W
Weyl Group of type ['A', 4] (as a matrix group acting on the ambient space)
sage: W.cardinality()
120
```

while the Lie algebra is $sl_{n+1}$, and the Lie group $SL_{n+1}$ (TODO: illustrate this once this is implemented).

One may also construct crystals associated to various Dynkin diagrams. For example:

```
sage: C = crystals.Letters(T)
sage: C
The crystal of letters for type ['A', 4]
sage: C.list()
[1, 2, 3, 4, 5]
sage: C = crystals.Tableaux(T, shape=[2])
sage: C
The crystal of tableaux of type ['A', 4] and shape(s) [[2]]
sage: C.cardinality()
15
```

Here is a sample of all the finite irreducible crystallographic Cartan types:

```
sage: CartanType.samples(finite = True, crystallographic = True)
[[['A', 1], ['A', 5], ['B', 1], ['B', 5], ['C', 1], ['C', 5], ['D', 2], ['D', 3], ['D ~', 5], ['E', 6], ['E', 7], ['E', 8], ['F', 4], ['G', 2]]
```

One can also get latex representations of the crystallographic Cartan types and their corresponding Dynkin diagrams:

```
sage: [latex(ct) for ct in CartanType.samples(crystallographic=True)]
[A_{1}, A_{5}, B_{1}, B_{5}, C_{1}, C_{5}, D_{2}, D_{3}, D_{5}, E_{6}, E_{7}, E_{8}, F_{4}, G_{2}, A_{1}^{(1)}, A_{5}^{(1)}, B_{1}^{(1)}, B_{5}^{(1)}, C_{1}^{(1)}, C_{5}^{(1)}, D_{3}]
```

(continues on next page)
Non-crystallographic Cartan types are also partially supported:

```python
sage: CartanType.samples(finite = True, crystallographic = False)
[['I', 5], ['H', 3], ['H', 4]]
```

In Sage, a Cartan type is used as a database of type-specific information and algorithms (see e.g. `sage.combinat.root_system.type_A`). This database includes how to construct the Dynkin diagram, the ambient space for the root system (see Wikipedia article Root system), and further mathematical properties:

```python
sage: T.is_finite(), T.is_simply_laced(), T.is_affine(), T.is_crystallographic()
(True, True, False, True)
```

In particular, a Sage Cartan type is endowed with a fixed choice of labels for the nodes of the Dynkin diagram. This choice follows the conventions of Nicolas Bourbaki, Lie Groups and Lie Algebras: Chapter 4-6, Elements of Mathematics, Springer (2002). ISBN 978-3-540426509. For example:

```python
sage: T = CartanType(['D', 4])
sage: DynkinDiagram(T)  
 0 4  
/    
|     
0---0---0 1 2 3  D4
sage: E6 = CartanType(['E', 6])
sage: DynkinDiagram(E6)
 0 2  
/    
|     
0---0---0---0---0---0 1 3 4 5 6  E6
```

**Note:** The direction of the arrows is the **opposite** (i.e. the transpose) of Bourbaki’s convention, but agrees with Kac’s.

For example, in type $C_2$, we have:

```python
sage: C2 = DynkinDiagram(['C',2]); C2
O<=<O  
1 2  C2
sage: C2.cartan_matrix()
```

(continues on next page)
However Bourbaki would have the Cartan matrix as:

\[
\begin{bmatrix}
2 & -1 \\
-2 & 2
\end{bmatrix}.
\]

If desired, other node labelling conventions can be achieved. For example the Kac labelling for type $E_6$ can be obtained via:

```
sage: E6.relabel({1:1,2:6,3:2,4:3,5:4,6:5}).dynkin_diagram()
O 6
|   |
O---O---O---O---O
1 2 3 4 5
```

```
E6 relabelled by {1: 1, 2: 6, 3: 2, 4: 3, 5: 4, 6: 5}
```

Contributions implementing other conventions are very welcome.

Another option is to build from scratch a new Dynkin diagram. The architecture has been designed to make it fairly easy to add other labelling conventions. In particular, we strived at choosing type free algorithms whenever possible, so in principle most features should remain available even with custom Cartan types. This has not been used much yet, so some rough corners certainly remain.

Here, we construct the hyperbolic example of Exercise 4.9 p. 57 of Kac, Infinite Dimensional Lie Algebras. We start with an empty Dynkin diagram, and add a couple nodes:

```
sage: g = DynkinDiagram()
sage: g.add_vertices([1,2,3])
```

Note that the diagonal of the Cartan matrix is already initialized:

```
sage: g.cartan_matrix()
[2 0 0]
[0 2 0]
[0 0 2]
```

Then we add a couple edges:

```
sage: g.add_edge(1,2,2)
sage: g.add_edge(1,3)
sage: g.add_edge(2,3)
```

```
and we get the desired Cartan matrix:
```
```
sage: g.cartan_matrix()
[2 0 0]
[0 2 0]
[0 0 2]
```

Oops, the Cartan matrix did not change! This is because it is cached for efficiency (see `cached_method`). In general, a Dynkin diagram should not be modified after having been used.
Warning: this is not checked currently

Todo: add a method \texttt{set\_mutable()} as, say, for matrices

Here, we can work around this by clearing the cache:

\begin{Verbatim}
sage: delattr(g, 'cartan\_matrix')
\end{Verbatim}

Now we get the desired Cartan matrix:

\begin{Verbatim}
sage: g.cartan\_matrix()
\begin{pmatrix}
  2 & -1 & -1 \\
  -2 & 2 & -1 \\
  -1 & -1 & 2
\end{pmatrix}
\end{Verbatim}

Note that backward edges have been automatically added:

\begin{Verbatim}
sage: g.edges(sort=True)
[(1, 2, 2), (1, 3, 1), (2, 1, 1), (2, 3, 1), (3, 1, 1), (3, 2, 1)]
\end{Verbatim}

Reducible Cartan types

Reducible Cartan types can be specified by passing a sequence or list of irreducible Cartan types:

\begin{Verbatim}
sage: CartanType(['A',2],['B',2])
A2xB2
sage: CartanType([['A',2],['B',2]])
A2xB2
sage: CartanType(['A',2],['B',2]).is\_reducible()
True
\end{Verbatim}

or using the following short hand notation:

\begin{Verbatim}
sage: CartanType("A2xB2")
A2xB2
sage: CartanType("A2","B2") == CartanType("A2xB2")
True
\end{Verbatim}

Degenerate cases

When possible, type \( I_n \) is automatically converted to the isomorphic crystallographic Cartan types (any reason not to do so?):

\begin{Verbatim}
sage: CartanType(['I',1])
A1xA1
sage: CartanType(['I',3])
['A', 2]
sage: CartanType(['I',4])
['C', 2]
\end{Verbatim}

(continues on next page)
Combinatorics, Release 10.0

sage: CartanType(['I', 6])
['G', 2]

The Dynkin diagrams for types $B_1$, $C_1$, $D_2$, and $D_3$ are isomorphic to that for $A_1$, $A_1$, $A_1 \times A_1$, and $A_3$, respectively. However their natural ambient space realizations (stemming from the corresponding infinite families of Lie groups) are different. Therefore, the Cartan types are considered as distinct:

sage: CartanType(['B', 1])
['B', 1]
sage: CartanType(['C', 1])
['C', 1]
sage: CartanType(['D', 2])
['D', 2]
sage: CartanType(['D', 3])
['D', 3]

Affine Cartan types

For affine types, we use the usual conventions for affine Coxeter groups: each affine type is either untwisted (that is arise from the natural affinisation of a finite Cartan type):

sage: CartanType(['A', 4, 1]).dynkin_diagram()
0
O=<=O---O---O=<=O
1 2 3 4
A4~
sage: CartanType(['B', 4, 1]).dynkin_diagram()
0
O=<=O---O=>=O
1 2 3 4
B4~

or dual thereof:

sage: CartanType(['B', 4, 1]).dual().dynkin_diagram()
0
O=<=O---O=>=O
1 2 3 4
B4~*

or is of type $\tilde{BC}_n$ (which yields an irreducible, but nonreduced root system):

sage: CartanType(['BC', 4, 2]).dynkin_diagram()
0=<=O---O---O=<=O
(continues on next page)
This includes the two degenerate cases:

```python
sage: CartanType(['A', 1, 1]).dynkin_diagram()
O<=>O
0 1
A1~
sage: CartanType(['BC', 1, 2]).dynkin_diagram()
4
O=<=O
0 1
BC1~
```

For the user convenience, Kac’s notations for twisted affine types are automatically translated into the previous ones:

```python
sage: CartanType(['A', 9, 2])
['B', 5, 1]^*
sage: CartanType(['A', 9, 2]).dynkin_diagram()
O 0
| |
O---O---O---O=<=O
1 2 3 4 5
B5~*
sage: CartanType(['A', 10, 2]).dynkin_diagram()
O=<=O---O---O=<=O
0 1 2 3 4 5
BC5~
sage: CartanType(['D', 5, 2]).dynkin_diagram()
O=<=O---O=>=O
0 1 2 3 4
C4~*
sage: CartanType(['D', 4, 3]).dynkin_diagram()
3
O=>=O---O
2 1 0
G2~* relabelled by {0: 0, 1: 2, 2: 1}
sage: CartanType(['E', 6, 2]).dynkin_diagram()
O---O---O=<=O---O
0 1 2 3 4
F4~*
```

Additionally one can set the notation option to use Kac’s notation:

```python
sage: CartanType.options['notation'] = 'Kac'
sage: CartanType(['A', 9, 2])
['A', 9, 2]
sage: CartanType(['A', 9, 2]).dynkin_diagram()
O 0
| |
2310 Chapter 5. Comprehensive Module List
Infinite Cartan types

There are minimal implementations of the Cartan types $A_\infty$ and $A_{+\infty}$. In `sage` $oo$ is the same as $+Infinity$, so `NN` and `ZZ` are used to differentiate between the $A_{+\infty}$ and $A_\infty$ root systems:

```python
sage: CartanType(["A", NN])
["A", NN]
sage: print(CartanType(["A", NN]).ascii_art())
0---0---0---0==0==0
 0 1 2 3 4 5
sage: CartanType(["A", ZZ])
["A", ZZ]
sage: print(CartanType(["A", ZZ]).ascii_art())
---0---0---0==0==0==0==0==0==0==0
 -3 -2 -1 0 1 2 3
```

There are also the following shorthands:

```python
sage: CartanType("Aoo")
["A", ZZ]
sage: CartanType("A+oo")
["A", NN]
```
Abstract classes for Cartan types

- CartanType_abstract
- CartanType_crystallographic
- CartanType_simply_laced
- CartanType_simple
- CartanType_finite
- CartanType_affine (see also Root system data for affine Cartan types)
- sage.combinat.root_system.cartan_type.CartanType
- Root system data for dual Cartan types
- Root system data for reducible Cartan types
- Root system data for relabelled Cartan types

Concrete classes for Cartan types

- CartanType_standard
- CartanType_standard_finite
- CartanType_standard_affine
- CartanType_standard_untwisted_affine

Type specific data

The data essentially consists of a description of the Dynkin/Coxeter diagram and, when relevant, of the natural embedding of the root system in an Euclidean space. Everything else is reconstructed from this data.

- Root system data for type A
- Root system data for type B
- Root system data for type C
- Root system data for type D
- Root system data for type E
- Root system data for type F
- Root system data for type G
- Root system data for type H
- Root system data for type I
- Root system data for super type A
- Root system data for type Q
- Root system data for (untwisted) type A affine
- Root system data for (untwisted) type B affine
- Root system data for (untwisted) type C affine
- Root system data for (untwisted) type D affine
- Root system data for (untwisted) type E affine
- Root system data for (untwisted) type F affine
Todo: Should those indexes come before the introduction?

class sage.combinat.root_system.cartan_type.CartanTypeFactory
Bases: SageObject

classmethod color(i)
Default color scheme for the vertices of a Dynkin diagram (and associated objects)

EXAMPLES:

```
sage: CartanType.color(1)
'blue'
sage: CartanType.color(2)
'red'
sage: CartanType.color(3)
'green'
```

The default color is black:

```
sage: CartanType.color(0)
'black'
```

Negative indices get the same color as their positive counterparts:

```
sage: CartanType.color(-1)
'blue'
sage: CartanType.color(-2)
'red'
sage: CartanType.color(-3)
'green'
```

\[
\text{options} = \text{Current options for CartanType} - \text{dual latex}: \ \backslash \text{vee} - \text{dual str}: \ \ast - \\
\text{latex marked}: \text{True} - \text{latex relabel}: \text{True} - \text{mark special node}: \text{none} - \\
\text{marked node str}: \text{X} - \text{notation}: \text{Stembridge} - \text{special node str}: \ @
\]

\[
\text{samples(}\text{finite}=\text{None}, \text{affine}=\text{None}, \text{crystallographic}=\text{None})
\]
Return a sample of the available Cartan types.

INPUT:

\[
\begin{itemize}
  \item finite – a boolean or None (default: None)
  \item affine – a boolean or None (default: None)
  \item crystallographic – a boolean or None (default: None)
\end{itemize}
\]

The sample contains all the exceptional finite and affine Cartan types, as well as typical representatives of the infinite families.

EXAMPLES:
class sage.combinat.root_system.cartan_type.CartanType_abstract
Bases: object

Abstract class for Cartan types

Subclasses should implement:

- dynkin_diagram()
- cartan_matrix()

Todo: add some reducible Cartan types (suggestions?)
- `is_finite()`
- `is_affine()`
- `is_irreducible()`

`as_folding(folding_of=None, sigma=None)`

Return `self` realized as a folded Cartan type.

For finite and affine types, this is realized by the Dynkin diagram foldings:

\[
C_n^{(1)}, A_{2n}^{(2)}, C_{n+1}^{(2)} \leftrightarrow A_{2n-1}^{(1)},
\]
\[
A_{2n-1}^{(1)}, B_n^{(1)} \leftrightarrow D_n^{(1)},
\]
\[
E_6^{(1)}, F_4^{(1)} \leftrightarrow E_6^{(1)},
\]
\[
D_4^{(1)}, G_2^{(1)} \leftrightarrow D_4^{(1)},
\]
\[
C_n \leftrightarrow A_{2n-1},
\]
\[
B_n \leftrightarrow D_{n+1},
\]
\[
F_4 \leftrightarrow E_6,
\]
\[
G_2 \leftrightarrow D_4.
\]

For general types, this returns `self` as a folded type of `self` with \(\sigma\) as the identity map.

For more information on these foldings and folded Cartan types, see `sage.combinat.root_system.type_folded.CartanTypeFolded`.

If the optional inputs `folding_of` and `sigma` are specified, then this returns the folded Cartan type of `self` in `folding_of` given by the automorphism `sigma`.

EXAMPLES:

```python
sage: CartanType(['B', 3, 1]).as_folding()
['B', 3, 1] as a folding of ['D', 4, 1]
sage: CartanType(['F', 4]).as_folding()
['F', 4] as a folding of ['E', 6]
sage: CartanType(['BC', 3, 2]).as_folding()
['BC', 3, 2] as a folding of ['A', 5, 1]
sage: CartanType(['D', 4, 3]).as_folding()
['G', 2, 1]^* relabelled by {0: 0, 1: 2, 2: 1} as a folding of ['D', 4, 1]
```

`coxeter_diagram()`

Return the Coxeter diagram for `self`.

EXAMPLES:

```python
sage: CartanType(['B', 3]).coxeter_diagram()
Graph on 3 vertices
sage: CartanType(['A', 3]).coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 3)]
sage: CartanType(['B', 3]).coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 4)]
sage: CartanType(['G', 2]).coxeter_diagram().edges(sort=True)
[(1, 2, 6)]
sage: CartanType(['F', 4]).coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 4), (3, 4, 3)]
```

`coxeter_matrix()`

Return the Coxeter matrix for `self`.

EXAMPLES:
```
sage: CartanType(['A', 4]).coxeter_matrix()
[1 3 2 2]
[3 1 3 2]
[2 3 1 3]
[2 2 3 1]
coxeter_type()
Return the Coxeter type for self.
EXAMPLES:
sage: CartanType(['A', 4]).coxeter_type()
Coxeter type of ['A', 4]
dual()
Return the dual Cartan type, possibly just as a formal dual.
EXAMPLES:
sage: CartanType(['A',3]).dual()
['A', 3]
sage: CartanType(['B', 3]).dual()
['C', 3]
sage: CartanType(['C',2]).dual()
['B', 2]
sage: CartanType(['D',4]).dual()
['D', 4]
sage: CartanType(['E',8]).dual()
['E', 8]
sage: CartanType(['F',4]).dual()
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
index_set()
Return the index set for self.
This is the list of the nodes of the associated Coxeter or Dynkin diagram.
EXAMPLES:
sage: CartanType(['A', 3, 1]).index_set()
(0, 1, 2, 3)
sage: CartanType(['D', 4]).index_set()
(1, 2, 3, 4)
sage: CartanType(['A', 7, 2]).index_set()
(0, 1, 2, 3, 4)
sage: CartanType(['A', 7, 2]).index_set()
(0, 1, 2, 3, 4)
sage: CartanType(['A', 6, 2]).index_set()
(0, 1, 2, 3)
sage: CartanType(['D', 6, 2]).index_set()
(0, 1, 2, 3, 4, 5)
sage: CartanType(['E', 6, 1]).index_set()
(0, 1, 2, 3, 4, 5, 6)
sage: CartanType(['E', 6, 2]).index_set()
(0, 1, 2, 3, 4)
```
sage: CartanType(['A', 2, 2]).index_set()
(0, 1)
sage: CartanType(['G', 2, 1]).index_set()
(0, 1, 2)
sage: CartanType(['F', 4, 1]).index_set()
(0, 1, 2, 3, 4)

is_affine()

Return whether self is affine.

EXAMPLES:

sage: CartanType(['A', 3]).is_affine()
False
sage: CartanType(['A', 3, 1]).is_affine()
True

is_atomic()

This method is usually equivalent to is_reducible(), except for the Cartan type $D_2$.

$D_2$ is not a standard Cartan type. It is equivalent to type $A_1 \times A_1$ which is reducible; however the isomorphism from its ambient space (for the orthogonal group of degree 4) to that of $A_1 \times A_1$ is non trivial, and it is useful to have it.

From a programming point of view its implementation is more similar to the irreducible types, and so the method is_atomic() is supplied.

EXAMPLES:

sage: CartanType("D2").is_atomic()
True
sage: CartanType("D2").is_irreducible()
False

is_compound()

A short hand for not is_atomic().

is_crystallographic()

Return whether this Cartan type is crystallographic.

This returns False by default. Derived class should override this appropriately.

EXAMPLES:

sage: [ [t, t.is_crystallographic()] for t in CartanType.samples(finite=True) ]
[[['A', 1], True], [['A', 5], True], [['B', 1], True], [['B', 5], True], [['C', 1], True], [['C', 5], True], [['D', 2], True], [['D', 3], True], [['D', 5], True], [['E', 6], True], [['E', 7], True], [['E', 8], True], [['F', 4], True], [['G', 2], True], [['I', 5], False], [['H', 3], False], [['H', 4], False]]

is_finite()

Return whether this Cartan type is finite.
EXAMPLES:

```
sage: from sage.combinat.root_system.cartan_type import CartanType_abstract
sage: C = CartanType_abstract()
sage: C.is_finite()
Traceback (most recent call last):
  ...  
NotImplementedError: <abstract method is_finite at ...>
```

```
sage: CartanType(["A",4]).is_finite()
True
sage: CartanType(["A",4,1]).is_finite()
False
```

`isImplemented()`
Check whether the Cartan datum for self is actually implemented.

EXAMPLES:

```
sage: CartanType(["A",4,1]).is_implemented()
True
sage: CartanType(["H",3]).is_implemented()
True
```

`isIrreducible()`
Report whether this Cartan type is irreducible (i.e. simple). This should be overridden in any subclass.

This returns False by default. Derived class should override this appropriately.

EXAMPLES:

```
sage: from sage.combinat.root_system.cartan_type import CartanType_abstract
sage: C = CartanType_abstract()
sage: C.is_irreducible()
False
```

`isReducible()`
Report whether the root system is reducible (i.e. not simple), that is whether it can be factored as a product of root systems.

EXAMPLES:

```
sage: CartanType("A2xB3").is_reducible()
True
sage: CartanType(["A",2]).is_reducible()
False
```

`isSimplyLaced()`
Return whether this Cartan type is simply laced.

This returns False by default. Derived class should override this appropriately.

EXAMPLES:

```
sage: [ [t, t.is_simply_laced()] for t in CartanType.samples() ]
[[["A", 1], True], [["A", 5], True],
```

(continues on next page)
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marked_nodes(marked_nodes)

Return a Cartan type with the nodes marked_nodes marked.

INPUT:
  • marked_nodes – a list of nodes to mark

EXAMPLES:

sage: CartanType(['F', 4]).marked_nodes([1, 3]).dynkin_diagram()
X---O=>=X---O
1 2 3 4
F4 with nodes (1, 3) marked

options = Current options for CartanType - dual_latex: \vee - dual_str: * - latex_marked: True - latex_relabel: True - mark_special_node: none - marked_node_str: X - notation: Stembridge - special_node_str: @

rank()

Return the rank of self.

This is the number of nodes of the associated Coxeter or Dynkin diagram.

EXAMPLES:

sage: CartanType(['A', 4]).rank()
4
sage: CartanType(['A', 7, 2]).rank()
5
sage: CartanType(['I', 8]).rank()
2

relabel(relabelling)

Return a relabelled copy of this Cartan type.

INPUT:
  • relabelling – a function (or a list or dictionary)
an isomorphic Cartan type obtained by relabelling the nodes of the Dynkin diagram. Namely, the node with label $i$ is relabelled $f(i)$ (or, by $f[i]$ if $f$ is a list or dictionary).

**EXAMPLES:**

```python
sage: CartanType(['F',4]).relabel({ 1:4, 2:3, 3:2, 4:1 }).dynkin_diagram()
O---O=>=O---O
4 3 2 1
F4 relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
```

**root_system()**
Return the root system associated to `self`.

**EXAMPLES:**

```python
sage: CartanType(['A',4]).root_system()
Root system of type ['A', 4]
```

**subtype(index_set)**
Return a subtype of `self` given by `index_set`.

A subtype can be considered the Dynkin diagram induced from the Dynkin diagram of `self` by `index_set`.

**EXAMPLES:**

```python
sage: ct = CartanType(['A',6,2])
sage: ct.dynkin_diagram()
O=<=O---O=<=O
0 1 2 3
BC3~
sage: ct.subtype([1,2,3])
['C', 3]
```

**type()**
Return the type of `self`, or `None` if unknown.

This method should be overridden in any subclass.

**EXAMPLES:**

```python
sage: from sage.combinat.root_system.cartan_type import CartanType_abstract
sage: C = CartanType_abstract()
sage: C.type() is None
True
```

**class** `sage.combinat.root_system.cartan_type.CartanType_affine`
Bases: `CartanType_simple, CartanType_crystallographic`

An abstract class for simple affine Cartan types

**AmbientSpace**
alias of `AmbientSpace`

**a()**
Return the unique minimal non trivial annihilating linear combination of $\alpha^\vee_0, \alpha^\vee, \ldots, \alpha^\vee$ with nonnegative coefficients (or alternatively, the unique minimal non trivial annihilating linear combination of the columns of the Cartan matrix with non-negative coefficients).
Throw an error if the existence or uniqueness does not hold

FIXME: the current implementation assumes that the Cartan matrix is indexed by \([0, 1, ...]\), in the same order as the index set.

EXAMPLES:

```
sage: RootSystem(['C',2,1]).cartan_type().a()
Finite family {0: 1, 1: 2, 2: 1}
sage: RootSystem(['D',4,1]).cartan_type().a()
Finite family {0: 1, 1: 1, 2: 2, 3: 1, 4: 1}
sage: RootSystem(['F',4,1]).cartan_type().a()
Finite family {0: 1, 1: 2, 2: 3, 3: 4, 4: 2}
sage: RootSystem(['BC',4,2]).cartan_type().a()
Finite family {0: 2, 1: 2, 2: 2, 3: 2, 4: 1}
```

\(a\) is a shortcut for \texttt{col\_annihilator}:

```
sage: RootSystem(['BC',4,2]).cartan_type().col\_annihilator()
Finite family {0: 2, 1: 2, 2: 2, 3: 2, 4: 1}
```

\texttt{acheck}(\(m=None\))

Return the unique minimal non trivial annihilating linear combination of \(\alpha_0, \alpha_1, \ldots, \alpha_n\) with nonnegative coefficients (or alternatively, the unique minimal non trivial annihilating linear combination of the rows of the Cartan matrix with non-negative coefficients).

Throw an error if the existence of uniqueness does not hold

The optional argument \(m\) is for internal use only.

EXAMPLES:

```
sage: RootSystem(['C',2,1]).cartan_type().acheck()
Finite family {0: 1, 1: 1, 2: 1}
sage: RootSystem(['D',4,1]).cartan_type().acheck()
Finite family {0: 1, 1: 1, 2: 2, 3: 1, 4: 1}
sage: RootSystem(['F',4,1]).cartan_type().acheck()
Finite family {0: 1, 1: 2, 2: 3, 3: 4, 4: 1}
sage: RootSystem(['BC',4,2]).cartan_type().acheck()
Finite family {0: 1, 1: 2, 2: 2, 3: 2, 4: 2}
```

\texttt{acheck} is a shortcut for \texttt{row\_annihilator}:

```
sage: RootSystem(['BC',4,2]).cartan_type().row\_annihilator()
Finite family {0: 1, 1: 2, 2: 2, 3: 2, 4: 2}
```

FIXME:

- The current implementation assumes that the Cartan matrix is indexed by \([0, 1, ...]\), in the same order as the index set.
- This really should be a method of \texttt{CartanMatrix}.

\texttt{basic\_untwisted}()

Return the basic untwisted Cartan type associated with this affine Cartan type.

Given an affine type \(X_n^{(r)}\), the basic untwisted type is \(X_n\). In other words, it is the classical Cartan type that is twisted to obtain \texttt{self}.
EXAMPLES:

- `sage: CartanType(['A', 1, 1]).basic_untwisted() ['A', 1]
- `sage: CartanType(['A', 3, 1]).basic_untwisted() ['A', 3]
- `sage: CartanType(['B', 3, 1]).basic_untwisted() ['B', 3]
- `sage: CartanType(['E', 6, 1]).basic_untwisted() ['E', 6]
- `sage: CartanType(['G', 2, 1]).basic_untwisted() ['G', 2]
- `sage: CartanType(['A', 2, 2]).basic_untwisted() ['A', 2]
- `sage: CartanType(['A', 4, 2]).basic_untwisted() ['A', 4]
- `sage: CartanType(['A', 11, 2]).basic_untwisted() ['A', 11]
- `sage: CartanType(['D', 5, 2]).basic_untwisted() ['D', 5]
- `sage: CartanType(['E', 6, 2]).basic_untwisted() ['E', 6]
- `sage: CartanType(['D', 4, 3]).basic_untwisted() ['D', 4]

\[c()\]

Returns the family \((c_i)_i\) of integer coefficients defined by \(c_i = \max(1, a_i/a^n e_i)\) (see e.g. [FSS07] p. 3)

FIXME: the current implementation assumes that the Cartan matrix is indexed by \([0, 1, \ldots]\), in the same order as the index set.

EXAMPLES:

- `sage: RootSystem(['C',2,1]).cartan_type().c()
  Finite family {0: 1, 1: 2, 2: 1}
- `sage: RootSystem(['D',4,1]).cartan_type().c()
  Finite family {0: 1, 1: 1, 2: 1, 3: 1, 4: 1}
- `sage: RootSystem(['F',4,1]).cartan_type().c()
  Finite family {0: 1, 1: 1, 2: 1, 3: 2, 4: 2}
- `sage: RootSystem(['BC',4,2]).cartan_type().c()
  Finite family {0: 2, 1: 1, 2: 1, 3: 1, 4: 1}

REFERENCES:

- `classical()\n  Return the classical Cartan type associated with this affine Cartan type.

EXAMPLES:

- `sage: CartanType(['A', 1, 1]).classical() ['A', 1]
- `sage: CartanType(['A', 3, 1]).classical() ['A', 3]
- `sage: CartanType(['B', 3, 1]).classical()
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[continued from previous page]

['B', 3]
sage: CartanType(['A', 2, 2]).classical()
['C', 1]
sage: CartanType(['BC', 1, 2]).classical()
['C', 1]
sage: CartanType(['A', 4, 2]).classical()
['C', 2]
sage: CartanType(['BC', 2, 2]).classical()
['C', 2]
sage: CartanType(['A', 10, 2]).classical()
['C', 5]
sage: CartanType(['BC', 5, 2]).classical()
['C', 5]
sage: CartanType(['D', 5, 2]).classical()
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
sage: CartanType(['D', 4, 3]).classical()
['G', 2]

We check that `classical()`, `sage.combinat.root_system.cartan_type.
CartanType_crystallographic.dynkin_diagram()`, and `special_node()` are consistent:

```
sage: for ct in CartanType.samples(affine = True):
    ....:     g1 = ct.classical().dynkin_diagram()
    ....:     g2 = ct.dynkin_diagram()
    ....:     g2.delete_vertex(ct.special_node())
    ....:     assert g1.vertices(sort=True) == g2.vertices(sort=True)
    ....:     assert g1.edges(sort=True) == g2.edges(sort=True)
```

**col annihilator()**

Return the unique minimal non trivial annihilating linear combination of \(\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_r^\vee\) with nonnegative coefficients (or alternatively, the unique minimal non trivial annihilating linear combination of the columns of the Cartan matrix with non-negative coefficients).

Throw an error if the existence or uniqueness does not hold

FIXME: the current implementation assumes that the Cartan matrix is indexed by \([0, 1, \ldots]\), in the same order as the index set.

**EXAMPLES:**

```
sage: RootSystem(['C',2,1]).cartan_type().a()
Finite family {0: 1, 1: 2, 2: 1}
sage: RootSystem(['D',4,1]).cartan_type().a()
Finite family {0: 1, 1: 1, 2: 2, 3: 1, 4: 1}
sage: RootSystem(['F',4,1]).cartan_type().a()
Finite family {0: 1, 1: 2, 2: 3, 3: 4, 4: 2}
```

(continues on next page)
sage: RootSystem(['BC',4,2]).cartan_type().a()
Finite family {0: 2, 1: 2, 2: 2, 3: 2, 4: 1}

a is a shortcut for col_annihilator:
sage: RootSystem(['BC',4,2]).cartan_type().col_annihilator()
Finite family {0: 2, 1: 2, 2: 2, 3: 2, 4: 1}

is_affine()
EXAMPLES:
sage: CartanType(['A', 3, 1]).is_affine()
True

is_finite()
EXAMPLES:
sage: CartanType(['A', 3, 1]).is_finite()
False

is_untwisted_affine()
Return whether self is untwisted affine

A Cartan type is untwisted affine if it is the canonical affine extension of some finite type. Every affine type is either untwisted affine, dual thereof, or of type BC.

EXAMPLES:
sage: CartanType(['A', 3, 1]).is_untwisted_affine()
True
sage: CartanType(['A', 3, 1]).dual().is_untwisted_affine()  # this one is self...
True
sage: CartanType(['B', 3, 1]).dual().is_untwisted_affine()
False
sage: CartanType(['BC', 3, 2]).is_untwisted_affine()
False

other_affinization()
Return the other affinization of the same classical type.

EXAMPLES:
sage: CartanType(['A', 3, 1]).other_affinization()
['A', 3, 1]
sage: CartanType(['B', 3, 1]).other_affinization()
['C', 3, 1]**
sage: CartanType(['C', 3, 1]).dual().other_affinization()
['B', 3, 1]

Is this what we want?:
sage: CartanType(['BC', 3, 2]).dual().other_affinization()
['B', 3, 1]
row_annihilator($m=None$)

Return the unique minimal non trivial annihilating linear combination of $\alpha_0, \alpha_1, \ldots, \alpha_n$ with nonnegative coefficients (or alternatively, the unique minimal non trivial annihilating linear combination of the rows of the Cartan matrix with non-negative coefficients).

Throw an error if the existence of uniqueness does not hold

The optional argument $m$ is for internal use only.

EXAMPLES:

```python
sage: RootSystem(['C',2,1]).cartan_type().acheck()
Finite family {0: 1, 1: 1, 2: 1}
sage: RootSystem(['D',4,1]).cartan_type().acheck()
Finite family {0: 1, 1: 1, 2: 2, 3: 1, 4: 1}
sage: RootSystem(['F',4,1]).cartan_type().acheck()
Finite family {0: 1, 1: 2, 2: 3, 3: 2, 4: 1}
sage: RootSystem(['BC',4,2]).cartan_type().acheck()
Finite family {0: 1, 1: 2, 2: 2, 3: 2, 4: 2}
```

acheck is a shortcut for row_annihilator:

```python
sage: RootSystem(['BC',4,2]).cartan_type().row_annihilator()
Finite family {0: 1, 1: 2, 2: 2, 3: 2, 4: 2}
```

FIXME:

- The current implementation assumes that the Cartan matrix is indexed by $[0, 1, ...]$, in the same order as the index set.
- This really should be a method of CartanMatrix.

special_node()

Return a special node of the Dynkin diagram.

A special node is a node of the Dynkin diagram such that pruning it yields a Dynkin diagram for the associated classical type (see classical()).

This method returns the label of some special node. This is usually 0 in the standard conventions.

EXAMPLES:

```python
sage: CartanType(['A', 3, 1]).special_node()
0
```

The choice is guaranteed to be consistent with the indexing of the nodes of the classical Dynkin diagram:

```python
sage: CartanType(['A', 3, 1]).index_set()
(0, 1, 2, 3)
sage: CartanType(['A', 3, 1]).classical().index_set()
(1, 2, 3)
```

special_nodes()

Return the set of special nodes of the affine Dynkin diagram.

EXAMPLES:
translation_factors()

Return the translation factors for self.

Those are the smallest factors $t_i$ such that the translation by $t_i\alpha_i$ maps the fundamental polygon to another polygon in the alcove picture.

OUTPUT:

a dictionary from self.index_set() to $\mathbb{Z}$ (or $\mathbb{Q}$ for affine type $BC$)

Those coefficients are all 1 for dual untwisted, and in particular for simply laced. They coincide with the usual $c_i$ coefficients (see c()) for untwisted and dual thereof. See the discussion below for affine type $BC$.

**Note:** One usually realizes the alcove picture in the coweight lattice, with translations by coroots; in that case, one will use the translation factors for the dual Cartan type.

FIXME: the current implementation assumes that the Cartan matrix is indexed by $[0, 1, ...]$, in the same order as the index set.

EXAMPLES:

```python
sage: CartanType(['C',2,1]).translation_factors()
Finite family {0: 1, 1: 2, 2: 1}
sage: CartanType(['C',2,1]).dual().translation_factors()
Finite family {0: 1, 1: 1, 2: 1}
sage: CartanType(['D',4,1]).translation_factors()
Finite family {0: 1, 1: 1, 2: 1, 3: 1, 4: 1}
sage: CartanType(['F',4,1]).translation_factors()
Finite family {0: 1, 1: 1, 2: 1, 3: 2, 4: 2}
sage: CartanType(['BC',4,2]).translation_factors()
Finite family {0: 1, 1: 1, 2: 1, 3: 1, 4: 1/2}
```

We proceed with systematic tests taken from MuPAD-Combinat’s testsuite:

```python
sage: list(CartanType(['A', 1, 1]).translation_factors())
[1, 1]
sage: list(CartanType(['A', 5, 1]).translation_factors())
[1, 1, 1, 1, 1]
sage: list(CartanType(['B', 5, 1]).translation_factors())
[1, 1, 1, 1, 1]
sage: list(CartanType(['C', 5, 1]).translation_factors())
(continues on next page)
```
We conclude with a discussion of the appropriate value for affine type $BC$. Let us consider the alcove picture realized in the weight lattice. It is obtained by taking the level-1 affine hyperplane in the weight lattice, and projecting it along $\Lambda_0$:

```
sage: R = RootSystem("BC",2,2)
sage: alpha = R.weight_space().simple_roots()
sage: alphacheck = R.coroot_space().simple_roots()
sage: Lambda = R.weight_space().fundamental_weights()
```

Here are the levels of the fundamental weights:

```
sage: Lambda[0].level(), Lambda[1].level(), Lambda[2].level()
(1, 2, 2)
```

So the “center” of the fundamental polygon at level 1 is:

```
sage: 0 = Lambda[0]
sage: 0.level()
1
```

We take the projection $\omega_1$ at level 0 of $\Lambda_1$ as unit vector on the $x$-axis, and the projection $\omega_2$ at level 0 of $\Lambda_2$ as unit vector of the $y$-axis:
The projections of the simple roots can be read off:

sage: alpha[0]
2*Lambda[0] - Lambda[1]
sage: alpha[1]
sage: alpha[2]

Namely $\alpha_0 = -\omega_1$, $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -2\omega_1 + 2\omega_2$.

The reflection hyperplane defined by $\alpha_0^\vee$ goes through the points $O + 1/2\omega_1$ and $O + 1/2\omega_2$:

sage: (O+(1/2)*omega1).scalar(alphacheck[0])
0
sage: (O+(1/2)*omega2).scalar(alphacheck[0])
0

Hence, the fundamental alcove is the triangle $(O, O + 1/2\omega_1, O + 1/2\omega_2)$. By successive reflections, one can tile the full plane. This induces a tiling of the full plane by translates of the fundamental polygon.

Todo: Add the picture here, once root system plots in the weight lattice will be implemented. In the mean time, the reader may look up the dual picture on Figure 2 of [HST09] which was produced by MuPAD-Combinat.

From this picture, one can read that translations by $\alpha_0$, $\alpha_1$, and $1/2\alpha_2$ map the fundamental polygon to translates of it in the alcove picture, and are smallest with this property. Hence, the translation factors for affine type $BC$ are $t_0 = 1, t_1 = 1, t_2 = 1/2$:

sage: CartanType(['BC',2,2]).translation_factors()
Finite family {0: 1, 1: 1, 2: 1/2}

REFERENCES:

class sage.combinat.root_system.cartan_type.CartanType_crystallographic

Bases: CartanType_abstract

An abstract class for crystallographic Cartan types.

ascii_art(label=lambda x: x, node=None)

Return an ascii art representation of the Dynkin diagram.

INPUT:

- label – (default: the identity) a relabeling function for the nodes
- node – (optional) a function which returns the character for a node

EXAMPLES:
```python
sage: cartan_type = CartanType(['B', 5, 1])
sage: print(cartan_type.ascii_art())
O 0
| |
0---0---0---0=>=0
1 2 3 4 5
```

The label option is useful to visualize various statistics on the nodes of the Dynkin diagram:

```python
sage: a = cartan_type.col_annihilator(); a
Finite family {0: 1, 1: 1, 2: 2, 3: 2, 4: 2, 5: 2}
sage: print(CartanType(['B', 5, 1]).ascii_art(label=a.__getitem__))
O 1
| |
0---0---0---0=>=0
1 2 2 2 2
```

cartan_matrix()

Return the Cartan matrix associated with self.

EXAMPLES:

```python
sage: CartanType(['A', 4]).cartan_matrix()
[ 2 -1 0 0]
[-1 2 -1 0]
[ 0 -1 2 -1]
[ 0 0 -1 2]
```

coxeter_diagram()

Return the Coxeter diagram for self.

This implementation constructs it from the Dynkin diagram.

See also:

CartanType_abstract.coxeter_diagram()

EXAMPLES:

```python
sage: CartanType(['A', 3]).coxeter_diagram()
Graph on 3 vertices
sage: CartanType(['A', 3]).coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 3)]
sage: CartanType(['B', 3]).coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 4)]
sage: CartanType(['G', 2]).coxeter_diagram().edges(sort=True)
[(1, 2, 6)]
sage: CartanType(['F', 4]).coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 4), (3, 4, 3)]
sage: CartanType(['A', 2, 2]).coxeter_diagram().edges(sort=True)
[(0, 1, +Infinity)]
```

dynkin_diagram()

Return the Dynkin diagram associated with self.

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EXAMPLES:

```
sage: CartanType(['A',4]).dynkin_diagram()
O---O---O---O
1 2 3 4
A4
```

**Note:** Derived subclasses should typically implement this as a cached method.

---

**index_set_bipartition()**

Return a bipartition \( \{ L, R \} \) of the vertices of the Dynkin diagram.

For \( i \) and \( j \) both in \( L \) (or both in \( R \)), the simple reflections \( s_i \) and \( s_j \) commute.

Of course, the Dynkin diagram should be bipartite. This is always the case for all finite types.

**EXAMPLES:**

```
sage: CartanType(['A',5]).index_set_bipartition()
({1, 3, 5}, {2, 4})
sage: CartanType(['A',2,1]).index_set_bipartition()
Traceback (most recent call last):
...
ValueError: the Dynkin diagram must be bipartite
```

---

**is_crystallographic()**

Implements `CartanType_abstract.is_crystallographic()` by returning `True`.

**EXAMPLES:**

```
sage: CartanType(['A', 3, 1]).is_crystallographic()
True
```

---

**symmetrizer()**

Return the symmetrizer of the Cartan matrix of `self`.

A Cartan matrix \( M \) is symmetrizable if there exists a non trivial diagonal matrix \( D \) such that \( DM = M^T D \). In that case, \( D \) is unique, up to a scalar factor for each connected component of the Dynkin diagram.

This method computes the unique minimal such \( D \) with positive integral coefficients. If \( D \) exists, it is returned as a family. Otherwise `None` is returned.

The coefficients are coerced to `base_ring`.

**EXAMPLES:**

```
sage: CartanType(['B',5]).symmetrizer()
Finite family {1: 2, 2: 2, 3: 2, 4: 2, 5: 1}
```

Here is a neat trick to visualize it better:

```
sage: T = CartanType(['B',5])
sage: print(T.ascii_art(T.symmetrizer().__getitem__))
```

(continues on next page)
Here is the symmetrizer of some reducible Cartan types:

```python
sage: T = CartanType(['BC', 5, 2])
sage: print(T.ascii_art(T.symmetrizer()._getitem_))
O=<=O---O---O---O=<=O
1 2 2 2 2 4
```

Property: up to an overall scalar factor, this gives the norm of the simple roots in the ambient space:

```python
sage: T = CartanType(['C', 5])
sage: print(T.ascii_art(T.symmetrizer()._getitem_))
O---O---O---O=<=O
1 1 1 1 2
```

```
sage: alpha = RootSystem(T).ambient_space().simple_roots()
sage: print(T.ascii_art(lambda i: alpha[i].scalar(alpha[i])))
O---O---O---O=<=O
2 2 2 2 4
```

**class** `sage.combinat.root_system.cartan_type.CartanType_decorator(ct)`

**Bases:** `UniqueRepresentation`, `SageObject`, `CartanType_abstract`

Concrete base class for Cartan types that decorate another Cartan type.

**index_set()**

**EXAMPLES:**

```python
sage: ct = CartanType(['F', 4, 1]).dual()
sage: ct.index_set()
(0, 1, 2, 3, 4)
```

**is_affine()**

**EXAMPLES:**

```python
sage: ct = CartanType(['G', 2]).relabel({1:2, 2:1})
sage: ct.is_affine()
False
```
is_crystallographic()  
EXAMPLES:
```python
sage: ct = CartanType(['G', 2]).relabel({1:2,2:1})
sage: ct.is_crystallographic()
True
```

is_finite()  
EXAMPLES:
```python
sage: ct = CartanType(['G', 2]).relabel({1:2,2:1})
sage: ct.is_finite()
True
```

is_irreducible()  
EXAMPLES:
```python
sage: ct = CartanType(['G', 2]).relabel({1:2,2:1})
sage: ct.is_irreducible()
True
```

rank()  
EXAMPLES:
```python
sage: ct = CartanType(['G', 2]).relabel({1:2,2:1})
sage: ct.rank()
2
```

class sage.combinat.root_system.cartan_type.CartanType_finite  

Bases: CartanType_abstract  
An abstract class for simple affine Cartan types.

is_affine()  
EXAMPLES:
```python
sage: CartanType(['A', 3]).is_affine()
False
```

is_finite()  
EXAMPLES:
```python
sage: CartanType(['A', 3]).is_finite()
True
```

class sage.combinat.root_system.cartan_type.CartanType_simple  

Bases: CartanType_abstract  
An abstract class for simple Cartan types.

is_irreducible()  
Return whether self is irreducible, which is True.

EXAMPLES:
```python
sage: CartanType(['A', 3]).is_irreducible()
True
```

class sage.combinat.root_system.cartan_type.CartanType_simple_finite

Bases: object

class sage.combinat.root_system.cartan_type.CartanType_simply_laced

Bases: CartanType_crystallographic

An abstract class for simply laced Cartan types.

dual()

Simply laced Cartan types are self-dual, so return self.

EXAMPLES:

```python
sage: CartanType(['A', 3]).dual()
['A', 3]
sage: CartanType(['A', 3, 1]).dual()
['A', 3, 1]
sage: CartanType(['D', 3]).dual()
['D', 3]
sage: CartanType(['D', 4, 1]).dual()
['D', 4, 1]
sage: CartanType(['E', 6]).dual()
['E', 6]
sage: CartanType(['E', 6, 1]).dual()
['E', 6, 1]
```

is_simply_laced()

Return whether self is simply laced, which is True.

EXAMPLES:

```python
sage: CartanType(['A',3,1]).is_simply_laced()
True
sage: CartanType(['A',2]).is_simply_laced()
True
```

class sage.combinat.root_system.cartan_type.CartanType_standard

Bases: UniqueRepresentation, SageObject

class sage.combinat.root_system.cartan_type.CartanType_standard_affine(letter, n, affine=1)

Bases: CartanType_standard, CartanType_affine

A concrete class for affine simple Cartan types.

index_set()

Implements CartanType_abstract.index_set().

The index set for all standard affine Cartan types is of the form \(\{0, \ldots, n\}\).

EXAMPLES:

```python
sage: CartanType(['A', 5, 1]).index_set()
(0, 1, 2, 3, 4, 5)
```
rank()

Return the rank of self which for type \(X^{(1)}_n\) is \(n + 1\).

EXAMPLES:

```python
sage: CartanType(['A', 4, 1]).rank()
5
sage: CartanType(['B', 4, 1]).rank()
5
sage: CartanType(['C', 3, 1]).rank()
4
sage: CartanType(['D', 4, 1]).rank()
5
sage: CartanType(['E', 6, 1]).rank()
7
sage: CartanType(['E', 7, 1]).rank()
8
sage: CartanType(['F', 4, 1]).rank()
5
sage: CartanType(['G', 2, 1]).rank()
3
sage: CartanType(['A', 2, 2]).rank()
2
sage: CartanType(['A', 6, 2]).rank()
4
sage: CartanType(['A', 7, 2]).rank()
5
sage: CartanType(['D', 5, 2]).rank()
5
sage: CartanType(['E', 6, 2]).rank()
5
sage: CartanType(['D', 4, 3]).rank()
3
```

special_node()

Implement `CartanType_abstract.special_node()`.

With the standard labelling conventions, 0 is always a special node.

EXAMPLES:

```python
sage: CartanType(['A', 3, 1]).special_node()
0
```

type()

Return the type of self.

EXAMPLES:

```python
sage: CartanType(['A', 4, 1]).type()
'A'
```

class sage.combinat.root_system.cartan_type.CartanType_standard_finite(letter, n)

Bases: `CartanType_standard`, `CartanType_finite`

A concrete base class for the finite standard Cartan types.
This includes for example $A_3$, $D_4$, or $E_8$.

**affine()**

Return the corresponding untwisted affine Cartan type.

**EXAMPLES:**

```
sage: CartanType(['A',3]).affine()
['A', 3, 1]
```

**coxeter_number()**

Return the Coxeter number associated with self.

The Coxeter number is the order of a Coxeter element of the corresponding Weyl group.

See Bourbaki, Lie Groups and Lie Algebras V.6.1 or *Wikipedia article Coxeter_element* for more information.

**EXAMPLES:**

```
sage: CartanType(['A',4]).coxeter_number()
5
sage: CartanType(['B',4]).coxeter_number()
8
sage: CartanType(['C',4]).coxeter_number()
8
```

**dual_coxeter_number()**

Return the Coxeter number associated with self.

**EXAMPLES:**

```
sage: CartanType(['A',4]).dual_coxeter_number()
5
sage: CartanType(['B',4]).dual_coxeter_number()
7
sage: CartanType(['C',4]).dual_coxeter_number()
5
```

**index_set()**

Implements *CartanType_abstract.index_set()*.

The index set for all standard finite Cartan types is of the form \{1,\ldots,n\}. (See *type_I* for a slight abuse of this).

**EXAMPLES:**

```
sage: CartanType(['A', 5]).index_set()
(1, 2, 3, 4, 5)
```

**opposition_automorphism()**

Returns the opposition automorphism

The *opposition automorphism* is the automorphism $i \mapsto i^*$ of the vertices Dynkin diagram such that, for $w_0$ the longest element of the Weyl group, and any simple root $\alpha_i$, one has $\alpha_i^* = -w_0(\alpha_i)$.

The automorphism is returned as a Family.

**EXAMPLES:**

```
```
```python
sage: ct = CartanType(['A', 5])
sage: ct.opposition_automorphism()
Finite family {1: 5, 2: 4, 3: 3, 4: 2, 5: 1}

sage: ct = CartanType(['D', 4])
sage: ct.opposition_automorphism()
Finite family {1: 1, 2: 2, 3: 3, 4: 4}

sage: ct = CartanType(['D', 5])
sage: ct.opposition_automorphism()
Finite family {1: 1, 2: 2, 3: 3, 4: 5, 5: 4}

sage: ct = CartanType(['C', 4])
sage: ct.opposition_automorphism()
Finite family {1: 1, 2: 2, 3: 3, 4: 4}
```

**rank()**

Return the rank of self which for type $X_n$ is $n$.

EXAMPLES:

```python
sage: CartanType(['A', 3]).rank()
3
sage: CartanType(['B', 3]).rank()
3
sage: CartanType(['C', 3]).rank()
3
sage: CartanType(['D', 4]).rank()
4
sage: CartanType(['E', 6]).rank()
6
```

**type()**

Returns the type of self.

EXAMPLES:

```python
sage: CartanType(['A', 4]).type()
'A'
sage: CartanType(['A', 4, 1]).type()
'A'
```

**class** `sage.combinat.root_system.cartan_type.CartanType_standard_untwisted_affine(letter, n, affine=1)`

Bases: `CartanType_standard_affine`

A concrete class for the standard untwisted affine Cartan types.

**basic_untwisted()**

Return the basic_untwisted Cartan type associated with this affine Cartan type.

Given an affine type $X_n^{(r)}$, the basic_untwisted type is $X_n$. In other words, it is the classical Cartan type that is twisted to obtain self.

EXAMPLES:
```python
sage: CartanType(['A', 1, 1]).basic_untwisted()
['A', 1]
sage: CartanType(['A', 3, 1]).basic_untwisted()
['A', 3]
sage: CartanType(['B', 3, 1]).basic_untwisted()
['B', 3]
sage: CartanType(['E', 6, 1]).basic_untwisted()
['E', 6]
sage: CartanType(['G', 2, 1]).basic_untwisted()
['G', 2]
```

classical()

Return the classical Cartan type associated with `self`.

EXAMPLES:

```python
sage: CartanType(['A', 3, 1]).classical()
['A', 3]
sage: CartanType(['B', 3, 1]).classical()
['B', 3]
sage: CartanType(['C', 3, 1]).classical()
['C', 3]
sage: CartanType(['D', 4, 1]).classical()
['D', 4]
sage: CartanType(['E', 6, 1]).classical()
['E', 6]
sage: CartanType(['F', 4, 1]).classical()
['F', 4]
sage: CartanType(['G', 2, 1]).classical()
['G', 2]
```

is_untwisted_affine()

Implement `CartanType_affine.is_untwisted_affine()` by returning `True`.

EXAMPLES:

```python
sage: CartanType(['B', 3, 1]).is_untwisted_affine()
True
```

class `sage.combinat.root_system.cartan_type.SuperCartanType_standard`

Bases: `UniqueRepresentation`, `SageObject`

```python
options = Current options for CartanType - dual_latex: \vee - dual_str: * - latex_marked: True - latex_relabel: True - mark_special_node: none - marked_node_str: X - notation: Stembridge - special_node_str: @
```
5.1.226 Coxeter Groups

sage.combinat.root_system.coxeter_group.CoxeterGroup(data, implementation='reflection', base_ring=None, index_set=None)

Return an implementation of the Coxeter group given by data.

INPUT:

- data – a Cartan type (or coercible into; see CartanType) or a Coxeter matrix or graph
- implementation – (default: 'reflection') can be one of the following:
  - 'permutation' - as a permutation representation
  - 'matrix' - as a Weyl group (as a matrix group acting on the root space); if this is not implemented, this uses the “reflection” implementation
  - 'coxeter3' - using the coxeter3 package
  - 'reflection' - as elements in the reflection representation; see CoxeterMatrixGroup
- base_ring – (optional) the base ring for the 'reflection' implementation
- index_set – (optional) the index set for the 'reflection' implementation

EXAMPLES:

Now assume that data represents a Cartan type. If implementation is not specified, the reflection representation is returned:

```
sage: W = CoxeterGroup(['A',2])
sage: W
Finite Coxeter group over Integer Ring with Coxeter matrix:
[1 3]
[3 1]

sage: W = CoxeterGroup(['A',3,1]); W
Coxeter group over Integer Ring with Coxeter matrix:
[1 3 2 3]
[3 1 3 2]
[2 3 1 3]
[3 2 3 1]

sage: W = CoxeterGroup(['H',3]); W
Finite Coxeter group over Number Field in a with defining polynomial x^2 - 5 with a approximately 2.236067977499790? with Coxeter matrix:
[1 3 2]
[3 1 5]
[2 5 1]
```

We now use the implementation option:

```
sage: W = CoxeterGroup(['A',2], implementation = "permutation") # optional - gap3
sage: W
Permutation Group with generators [(1,4)(2,3)(5,6), (1,3)(2,5)(4,6)]

sage: W.category()
# optional - gap3
Join of Category of finite enumerated permutation groups
    and Category of finite weyl groups
    and Category of well generated finite irreducible complex reflection groups
```

(continues on next page)
sage: W = CoxeterGroup(['A',2], implementation="matrix")
sage: W
Weyl Group of type ['A', 2] (as a matrix group acting on the ambient space)

sage: W = CoxeterGroup(['H',3], implementation="matrix")
sage: W
Finite Coxeter group over Number Field in a with defining polynomial x^2 - 5 with a → 2.236067977499790? with Coxeter matrix:
[1 3 2]
[3 1 5]
[2 5 1]

sage: W = CoxeterGroup(['H',3], implementation="reflection")
sage: W
Finite Coxeter group over Number Field in a with defining polynomial x^2 - 5 with a → 2.236067977499790? with Coxeter matrix:
[1 3 2]
[3 1 5]
[2 5 1]

sage: W = CoxeterGroup(['A',4,1], implementation="permutation")
Traceback (most recent call last):
...
ValueError: the type must be finite

sage: W = CoxeterGroup(['A',4], implementation="chevie"); W
# optional - gap3
Irreducible real reflection group of rank 4 and type A4

We use the different options for the "reflection" implementation:

sage: W = CoxeterGroup(['H',3], implementation="reflection", base_ring=RR)
sage: W
Finite Coxeter group over Real Field with 53 bits of precision with Coxeter matrix:
[1 3 2]
[3 1 5]
[2 5 1]

sage: W = CoxeterGroup([[1,10],[10,1]], implementation="reflection", index_set=['a', → 'b'], base_ring=SR)
sage: W
Finite Coxeter group over Symbolic Ring with Coxeter matrix:
[ 1 10]
[10  1]
5.1.227 Coxeter Matrices

**class** `sage.combinat.root_system.coxeter_matrix.CoxeterMatrix(parent, data, coxeter_type, index_set)`

**Bases:** `CoxeterType`

A Coxeter matrix.

A Coxeter matrix $M = (m_{ij})_{i,j \in I}$ is a matrix encoding a Coxeter system $(W, S)$, where the relations are given by $(s_i s_j)^{m_{ij}}$. Thus $M$ is symmetric and has entries in $\{1, 2, 3, \ldots, \infty\}$ with $m_{ij} = 1$ if and only if $i = j$.

We represent $m_{ij} = \infty$ by any number $m_{ij} \leq -1$. In particular, we can construct a bilinear form $B = (b_{ij})_{i,j \in I}$ from $M$ by

$$b_{ij} = \begin{cases} m_{ij} & m_{ij} < 0 \text{ (i.e., } m_{ij} = \infty) \\ -\cos \left( \frac{\pi}{m_{ij}} \right) & \text{otherwise} \end{cases}$$

**EXAMPLES:**

```python
sage: CoxeterMatrix(['A', 4])
[1 3 2 2]
[3 1 3 2]
[2 3 1 3]
[2 2 3 1]
sage: CoxeterMatrix(['B', 4])
[1 3 2 2]
[3 1 3 2]
[2 3 1 4]
[2 2 4 1]
sage: CoxeterMatrix(['C', 4])
[1 3 2 2]
[3 1 3 2]
[2 3 1 4]
[2 2 4 1]
sage: CoxeterMatrix(['D', 4])
[1 3 2 2]
[3 1 3 3]
[2 3 1 2]
[2 3 2 1]
sage: CoxeterMatrix(['E', 6])
[1 2 3 2 2 2]
[2 1 2 3 2 2]
[3 2 1 3 2 2]
[2 3 3 1 3 2]
[2 2 2 3 1 3]
[2 2 2 2 3 1]
sage: CoxeterMatrix(['F', 4])
[1 3 2 2]
[3 1 4 2]
[2 4 1 3]
[2 2 3 1]
```

(continues on next page)
sage: CoxeterMatrix(['G', 2])
[1 6]
[6 1]

By default, entries representing $\infty$ are given by $-1$ in the Coxeter matrix:

sage: G = Graph([(0,1,None), (1,2,4), (0,2,oo)])
sage: CoxeterMatrix(G)
[ 1  3 -1]
[ 3  1  4]
[-1 -4  1]

It is possible to give a number $\leq -1$ to represent an infinite label:

sage: CoxeterMatrix([[1,-1],[-1,1]])
[ 1 -1]
[-1  1]
sage: CoxeterMatrix([[1,-3/2],[-3/2,1]])
[ 1 -3/2]
[-3/2  1]

*bilinear_form*(R=None)

Return the bilinear form of self.

EXAMPLES:

sage: CoxeterType(['A', 2, 1]).bilinear_form()
[ 1 -1/2 -1/2]
[-1/2  1 -1/2]
[-1/2 -1/2  1]
sage: CoxeterType(['H', 3]).bilinear_form()
[ 1 -1/2 0]
[-1/2  1 1/2*E(5)^2 + 1/2*E(5)^3]
[-1/2 1/2*E(5)^2 + 1/2*E(5)^3 1]
sage: C = CoxeterMatrix([[1,-1,-1],[-1,1,-1],[-1,-1,1]])
sage: C.bilinear_form()
[ 1 -1 -1]
[-1  1 -1]
[-1 -1  1]

*coxeter_graph*( )

Return the Coxeter graph of self.

EXAMPLES:

sage: C = CoxeterMatrix(['A',3])
sage: C.coxeter_graph()
Graph on 3 vertices

sage: C = CoxeterMatrix(['A',3],['A',1])
sage: C.coxeter_graph()
Graph on 4 vertices
**coxeter_matrix()**

Return the Coxeter matrix of self.

**EXAMPLES:**

```python
sage: C = CoxeterMatrix(['C',3]).coxeter_matrix()
[1 3 2]
[3 1 4]
[2 4 1]
```

**coxeter_type()**

Return the Coxeter type of self or self if unknown.

**EXAMPLES:**

```python
sage: C = CoxeterMatrix(['A',4,1])
sage: C.coxeter_type()
Coxeter type of ['A', 4, 1]
If the Coxeter type is unknown:

```python
sage: C = CoxeterMatrix([[1,3,4], [3,1,-1], [4,-1,1]])
sage: C.coxeter_type()
[ 1 3 4]
[ 3 1 -1]
[ 4 -1 1]
```

**index_set()**

Return the index set of self.

**EXAMPLES:**

```python
sage: C = CoxeterMatrix(['A',1,1])
sage: C.index_set()
(0, 1)
sage: C = CoxeterMatrix(['E',6])
sage: C.index_set()
(1, 2, 3, 4, 5, 6)
```

**is_affine()**

Return if self is an affine type or False if unknown.

**EXAMPLES:**

```python
sage: M = CoxeterMatrix(['C',4])
sage: M.is_affine()
False
sage: M = CoxeterMatrix(['D',4,1])
sage: M.is_affine()
True
sage: M = CoxeterMatrix([[1, 3],[3,1]])
sage: M.is_affine()
False
sage: M = CoxeterMatrix([[1, -1, 7], [-1, 1, 3], [7, 3, 1]])
sage: M.is_affine()
False
```
is_crystallographic()
Return whether self is crystallographic.
A Coxeter matrix is crystallographic if all non-diagonal entries are either 2, 3, 4, or 6.

EXAMPLES:

```
sage: C = CoxeterMatrix(['F',4]).is_crystallographic()
True
sage: C = CoxeterMatrix(['H',3]).is_crystallographic()
False
```

is_finite()
Return if self is a finite type or False if unknown.

EXAMPLES:

```
sage: M = CoxeterMatrix(['C',4])
sage: M.is_finite()
True
sage: M = CoxeterMatrix(['D',4,1])
sage: M.is_finite()
False
sage: M = CoxeterMatrix([[1, -1], [-1, 1]])
sage: M.is_finite()
False
```

is_irreducible()
Return whether self is irreducible.
A Coxeter matrix is irreducible if the Coxeter graph is connected.

EXAMPLES:

```
sage: C = CoxeterMatrix([[F',4],[A',1]]).is_irreducible()
False
sage: C = CoxeterMatrix(['H',3]).is_irreducible()
True
```

is_simply_laced()
Return if self is simply-laced.
A Coxeter matrix is simply-laced if all non-diagonal entries are either 2 or 3.

EXAMPLES:

```
sage: cm = CoxeterMatrix([[1,3,3,3], [3,1,3,3], [3,3,1,3], [3,3,3,1]])
sage: cm.is_simply_laced()
True
```

rank()
Return the rank of self.

EXAMPLES:

```
sage: C = CoxeterMatrix(['C',3]).rank()
3
```
(continues on next page)
sage: CoxeterMatrix(["A2","B2","F4"]).rank()
8

relabel

Return a relabelled copy of this Coxeter matrix.

INPUT:

- relabelling – a function (or dictionary)

OUTPUT:

an isomorphic Coxeter type obtained by relabelling the nodes of the Coxeter graph. Namely, the node with label i is relabelled f(i) (or, by f[i] if f is a dictionary).

EXAMPLES:

sage: CoxeterMatrix(['F',4]).relabel({ 1:2, 2:3, 3:4, 4:1})

\[
\begin{pmatrix}
1 & 4 & 2 & 3 \\
4 & 1 & 3 & 2 \\
2 & 3 & 1 & 2 \\
3 & 2 & 2 & 1 \\
\end{pmatrix}
\]

sage: CoxeterMatrix(['F',4]).relabel(lambda x: x+1 if x<4 else 1)

\[
\begin{pmatrix}
1 & 4 & 2 & 3 \\
4 & 1 & 3 & 2 \\
2 & 3 & 1 & 2 \\
3 & 2 & 2 & 1 \\
\end{pmatrix}
\]

classmethod samples

Return a sample of the available Coxeter types.

INPUT:

- finite – (default: None) a boolean or None
- affine – (default: None) a boolean or None
- crystallographic – (default: None) a boolean or None
- higher_rank – (default: None) a boolean or None

The sample contains all the exceptional finite and affine Coxeter types, as well as typical representatives of the infinite families. Here the higher_rank term denotes non-finite, non-affine, Coxeter groups (including hyperbolic types).

Todo: Implement the hyperbolic and compact hyperbolic in the samples.

EXAMPLES:

sage: [CM.coxeter_type() for CM in CoxeterMatrix.samples()]

[Cogxeter type of ['A', 1], Coxeter type of ['A', 5],
Coxeter type of ['B', 5], Coxeter type of ['D', 4],
Coxeter type of ['D', 5], Coxeter type of ['E', 6],
]
Coxeter type of ['E', 7], Coxeter type of ['E', 8],
Coxeter type of ['F', 4], Coxeter type of ['H', 3],
Coxeter type of ['H', 4], Coxeter type of ['I', 10],
Coxeter type of ['A', 2, 1], Coxeter type of ['B', 5, 1],
Coxeter type of ['C', 5, 1], Coxeter type of ['D', 5, 1],
Coxeter type of ['E', 6, 1], Coxeter type of ['E', 7],
Coxeter type of ['F', 4, 1], Coxeter type of ['H', 3],
Coxeter type of ['H', 4], Coxeter type of ['I', 10],

\[
\begin{bmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
\end{bmatrix}
\]

The finite, affine and crystallographic options allow respectively for restricting to (non) finite, (non) affine, and (non) crystallographic Cartan types:

```python
sage: [CM.coxeter_type() for CM in CoxeterMatrix.samples(finite=True)]
[Cogxeter type of ['A', 1], Coxeter type of ['A', 5],
 Coxeter type of ['B', 5], Coxeter type of ['D', 4],
 Coxeter type of ['E', 5], Coxeter type of ['E', 6],
 Coxeter type of ['E', 7], Coxeter type of ['E', 8],
 Coxeter type of ['F', 4], Coxeter type of ['H', 3],
 Coxeter type of ['H', 4], Coxeter type of ['I', 10]]
```

```python
sage: [CM.coxeter_type() for CM in CoxeterMatrix.samples(affine=True)]
[Cogxeter type of ['A', 2, 1], Coxeter type of ['B', 5, 1],
 Coxeter type of ['C', 5, 1], Coxeter type of ['D', 5, 1],
 Coxeter type of ['E', 6, 1], Coxeter type of ['E', 7, 1],
 Coxeter type of ['E', 8, 1], Coxeter type of ['F', 4, 1],
 Coxeter type of ['G', 2, 1], Coxeter type of ['A', 1, 1]]
```

```python
sage: [CM.coxeter_type() for CM in CoxeterMatrix.samples(crystallographic=True)]
[Cogxeter type of ['A', 1], Coxeter type of ['A', 5],
 Coxeter type of ['B', 5], Coxeter type of ['D', 4],
 Coxeter type of ['E', 5], Coxeter type of ['E', 6],
 Coxeter type of ['E', 7], Coxeter type of ['E', 8],
 Coxeter type of ['F', 4], Coxeter type of ['G', 2, 1],
 Coxeter type of ['H', 3], Coxeter type of ['H', 4],
 Coxeter type of ['I', 10]]
```
Coxeter type of ['F', 4, 1], Coxeter type of ['G', 2, 1] ]

sage: CoxeterMatrix.samples(crystallographic=False)
[
[1 3 2 2],
[1 3 2] [3 1 3 2]
[3 1 5] [2 3 1 5] [1 10] [1 -1] [1 -1 1] [2 1 7]
[2 5 1], [2 2 5 1], [10 1], [-1 1], [-1 -1], [3 7 1],
[ 1 -2 3 2],
[ -2 1 2 3]
[ 3 2 1 -8]
[2 3 -8 1]
]

Todo: add some reducible Coxeter types (suggestions?)

sage.combinat.root_system.coxeter_matrix.check_coxeter_matrix(m)
Check if m represents a generalized Coxeter matrix and raise and error if not.

EXAMPLES:

sage: from sage.combinat.root_system.coxeter_matrix import check_coxeter_matrix
sage: m = matrix([[1,3,2],[3,1,-1],[2,-1,1]])
sage: check_coxeter_matrix(m)

sage: m = matrix([[1,3],[3,1],[2,-1]])
sage: check_coxeter_matrix(m)
Traceback (most recent call last):
... ValueError: not a square matrix

sage: m = matrix([[1,3,2],[3,1,-1],[2,-1,2]])
sage: check_coxeter_matrix(m)
Traceback (most recent call last):
... ValueError: the matrix diagonal is not all 1

sage: m = matrix([[1,3,3],[3,1,-1],[1/2,-1,1]])
sage: check_coxeter_matrix(m)
Traceback (most recent call last):
... ValueError: the matrix is not symmetric

sage: m = matrix([[1,3,1/2],[3,1,-1],[1/2,-1,1]])
sage: check_coxeter_matrix(m)
Traceback (most recent call last):
... ValueError: invalid Coxeter label 1/2

sage: m = matrix([[1,3,1],[3,1,-1],[1,-1,1]])
sage: check_coxeter_matrix(m)
Traceback (most recent call last):
...
ValueError: invalid Coxeter label 1

sage.combinat.root_system.coxeter_matrix.coxeter_matrix_as_function(t)
Return the Coxeter matrix, as a function.

INPUT:
• t – a Cartan type

EXAMPLES:

sage: from sage.combinat.root_system.coxeter_matrix import coxeter_matrix_as_function
sage: f = coxeter_matrix_as_function(['A',4])
sage: matrix([[f(i,j) for j in range(1,5)] for i in range(1,5)])
[1 3 2 2]
[3 1 3 2]
[2 3 1 3]
[2 2 3 1]

sage.combinat.root_system.coxeter_matrix.recognize_coxeter_type_from_matrix(coxeter_matrix, index_set)
Return the Coxeter type of coxeter_matrix if known, otherwise return None.

EXAMPLES:
Some infinite ones:

sage: C = CoxeterMatrix([[1,3,2],[3,1,-1],[2,-1,1]])
sage: C.is_finite()  # indirect doctest
False
sage: C = CoxeterMatrix([[1,-1,-1],[-1,1,-1],[-1,-1,1]])
sage: C.is_finite()  # indirect doctest
False

Some finite ones:

sage: m = matrix(CoxeterMatrix(['D', 4]))

sage: CoxeterMatrix(m).is_finite()  # indirect doctest
True
sage: m = matrix(CoxeterMatrix(['H', 4]))

sage: CoxeterMatrix(m).is_finite()  # indirect doctest
True

sage: CoxeterMatrix(CoxeterType(['A',10]).coxeter_graph()).coxeter_type()
Coxeter type of ['A', 10]
sage: CoxeterMatrix(CoxeterType(['B',10]).coxeter_graph()).coxeter_type()
Coxeter type of ['B', 10]
sage: CoxeterMatrix(CoxeterType(['C',10]).coxeter_graph()).coxeter_type()
Coxeter type of ['C', 10]
sage: CoxeterMatrix(CoxeterType(['D',10]).coxeter_graph()).coxeter_type()
Coxeter type of ['D', 10]
sage: CoxeterMatrix(CoxeterType(['E',6]).coxeter_graph()).coxeter_type()
Coxeter type of ['E', 6]
sage: CoxeterMatrix(CoxeterType(['E',7]).coxeter_graph()).coxeter_type()
Coxeter type of ['E', 7]
sage: CoxeterMatrix(CoxeterType(['E',8]).coxeter_graph()).coxeter_type()
Coxeter type of ['E', 8]
sage: CoxeterMatrix(CoxeterType(['F',4]).coxeter_graph()).coxeter_type()
Coxeter type of ['F', 4]
sage: CoxeterMatrix(CoxeterType(['G',2]).coxeter_graph()).coxeter_type()
Coxeter type of ['G', 2]
sage: CoxeterMatrix(CoxeterType(['H',3]).coxeter_graph()).coxeter_type()
Coxeter type of ['H', 3]
sage: CoxeterMatrix(CoxeterType(['H',4]).coxeter_graph()).coxeter_type()
Coxeter type of ['H', 4]
sage: CoxeterMatrix(CoxeterType(['I',100]).coxeter_graph()).coxeter_type()
Coxeter type of ['I', 100]

Some affine graphs:

sage: CoxeterMatrix(CoxeterType(['A',1,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['A', 1, 1]
sage: CoxeterMatrix(CoxeterType(['A',10,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['A', 10, 1]
sage: CoxeterMatrix(CoxeterType(['B',10,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['B', 10, 1]
sage: CoxeterMatrix(CoxeterType(['C',10,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['C', 10, 1]
sage: CoxeterMatrix(CoxeterType(['D',10,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['D', 10, 1]
sage: CoxeterMatrix(CoxeterType(['E',6,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['E', 6, 1]
sage: CoxeterMatrix(CoxeterType(['E',7,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['E', 7, 1]
sage: CoxeterMatrix(CoxeterType(['E',8,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['E', 8, 1]
sage: CoxeterMatrix(CoxeterType(['F',4,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['F', 4, 1]
sage: CoxeterMatrix(CoxeterType(['G',2,1]).coxeter_graph()).coxeter_type()
Coxeter type of ['G', 2, 1]

### 5.1.228 Coxeter Types

**class** sage.combinat.root_system.coxeter_type.CoxeterType

**Bases:** SageObject

Abstract class for Coxeter types.

**bilinear_form**(*R=None*)

Return the bilinear form over *R* associated to self.

**INPUT:**
• \( R \) – (default: universal cyclotomic field) a ring used to compute the bilinear form

**EXAMPLES:**

```python
sage: CoxeterType(['A', 2]).bilinear_form()
[ 1 -1/2 -1/2]
[-1/2 1 -1/2]
[-1/2 -1/2 1]
sage: CoxeterType(['H', 3]).bilinear_form()
[ 1 -1/2 0]
[-1/2 1 1/2*E(5)^2 + 1/2*E(5)^3]
[ 0 1/2*E(5)^2 + 1/2*E(5)^3 1]
sage: C = CoxeterMatrix([[1,-1,-1],[-1,1,-1],[-1,-1,1]])
sage: C.bilinear_form()
[ 1 -1 -1]
[-1 1 -1]
[-1 -1 1]
```

coxeter_graph()

Return the Coxeter graph associated to \texttt{self}.

**EXAMPLES:**

```python
sage: CoxeterType(['A', 3]).coxeter_graph()
Graph on 3 vertices
sage: CoxeterType(['A', 3, 1]).coxeter_graph()
Graph on 4 vertices
```

coxeter_matrix()

Return the Coxeter matrix associated to \texttt{self}.

**EXAMPLES:**

```python
sage: CoxeterType(['A', 3]).coxeter_matrix()
[1 3 2]
[3 1 3]
[2 3 1]
sage: CoxeterType(['A', 3, 1]).coxeter_matrix()
[1 3 2 3]
[3 1 3 2]
[2 3 1 3]
[3 2 3 1]
```

index_set()

Return the index set for \texttt{self}.

This is the list of the nodes of the associated Coxeter graph.

**EXAMPLES:**

```python
sage: CoxeterType(['A', 3]).index_set()
(0, 1, 2, 3)
sage: CoxeterType(['D', 4]).index_set()
(1, 2, 3, 4)
sage: CoxeterType(['A', 7, 2]).index_set()
(0, 1, 2, 3, 4)
```

(continues on next page)
sage: CoxeterType(['A', 7, 2]).index_set()
(0, 1, 2, 3, 4)
sage: CoxeterType(['A', 6, 2]).index_set()
(0, 1, 2, 3)
sage: CoxeterType(['D', 6, 2]).index_set()
(0, 1, 2, 3, 4, 5)
sage: CoxeterType(['E', 6, 1]).index_set()
(0, 1, 2, 3, 4, 5, 6)
sage: CoxeterType(['E', 6, 2]).index_set()
(0, 1, 2, 3, 4)
sage: CoxeterType(['A', 2, 2]).index_set()
(0, 1)
sage: CoxeterType(['G', 2, 1]).index_set()
(0, 1, 2)
sage: CoxeterType(['F', 4, 1]).index_set()
(0, 1, 2, 3, 4)

is_affine()
Return whether self is affine.

EXAMPLES:

sage: CoxeterType(['A', 3]).is_affine()
False
sage: CoxeterType(['A', 3, 1]).is_affine()
True

is_crystallographic()
Return whether self is crystallographic.

This returns False by default. Derived class should override this appropriately.

EXAMPLES:

sage: [ [t, t.is_crystallographic()] for t in CartanType.samples(finite=True) ]
[['A', 1], True], [['A', 5], True],
[['B', 1], True], [['B', 5], True],
[['C', 1], True], [['C', 5], True],
[['D', 2], True], [['D', 3], True], [['D', 5], True],
[['E', 6], True], [['E', 7], True], [['E', 8], True],
[['F', 4], True], [['G', 2], True],
[['I', 5], False], [['H', 3], False], [['H', 4], False]]

is_finite()
Return whether self is finite.

EXAMPLES:

sage: CoxeterType(['A',4]).is_finite()
True
sage: CoxeterType(['A',4, 1]).is_finite()
False
is_simply_laced()
Return whether self is simply laced.
This returns False by default. Derived class should override this appropriately.

EXAMPLES:

```sage
sage: CartanType.sample() [t, t.is_simply_laced()] for t in CartanType.samples() ]
[[['A', 1], True], [['A', 5], True], [['B', 1], True], [['B', 5], False], [['C', 1], True], [['C', 5], False], [['D', 2], True], [['D', 3], True], [['D', 5], True], [['E', 6], True], [['E', 7], True], [['E', 8], True], [['F', 4], False], [['G', 2], False], [['H', 3], False], [['H', 4], False], [['A', 1, 1], False], [['A', 5, 1], True], [['B', 1, 1], False], [['B', 5, 1], False], [['C', 1, 1], False], [['C', 5, 1], False], [['D', 3, 1], True], [['D', 5, 1], True], [['E', 6, 1], True], [['E', 7, 1], True], [['E', 8, 1], True], [['F', 4, 1], False], [['G', 2, 1], False], [['BC', 1, 2], False], [['BC', 5, 2], False], [['BC', 5, 1]^*, False], [['BC', 4, 1]^*, False], [['F', 4, 1]^*, False], [['G', 2, 1]^*, False], [['BC', 1, 2]^*, False], [['BC', 5, 2]^*, False]]
```

rank()
Return the rank of self.
This is the number of nodes of the associated Coxeter graph.

EXAMPLES:

```sage
sage: [ t.rank() for t in CartanType.samples() ]
4
sage: [ t.rank() for t in CartanType.samples() ]
5
sage: [ t.rank() for t in CartanType.samples() ]
2
```

classmethod samples(finite=None, affine=None, crystallographic=None)
Return a sample of the available Coxeter types.

INPUT:

- `finite` – a boolean or None (default: None)
- `affine` – a boolean or None (default: None)
- `crystallographic` – a boolean or None (default: None)

The sample contains all the exceptional finite and affine Coxeter types, as well as typical representatives of the infinite families.

EXAMPLES:

```sage
sage: CartanType.samples()
[Cartan type of ['A', 1], Cartan type of ['A', 5], ...]
```
The finite, affine and crystallographic options allow respectively for restricting to (non) finite, (non) affine, and (non) crystallographic Cartan types:

sage: CoxeterType.samples(finite=True)
[[Coxeter type of ['A', 1], Coxeter type of ['A', 5],
  Coxeter type of ['B', 1], Coxeter type of ['B', 5],
  Coxeter type of ['C', 1], Coxeter type of ['C', 5],
  Coxeter type of ['D', 4], Coxeter type of ['D', 5],
  Coxeter type of ['E', 6], Coxeter type of ['E', 7],
  Coxeter type of ['E', 8], Coxeter type of ['F', 4],
  Coxeter type of ['H', 3], Coxeter type of ['H', 4],
  Coxeter type of ['I', 10]]

sage: CoxeterType.samples(affine=True)
[[Coxeter type of ['A', 2, 1], Coxeter type of ['B', 5, 1],
  Coxeter type of ['C', 5, 1], Coxeter type of ['D', 5, 1],
  Coxeter type of ['E', 6, 1], Coxeter type of ['E', 7, 1],
  Coxeter type of ['E', 8, 1], Coxeter type of ['F', 4, 1],
  Coxeter type of ['G', 2, 1], Coxeter type of ['A', 1, 1]]

sage: CoxeterType.samples(crystallographic=True)
[[Coxeter type of ['A', 1], Coxeter type of ['A', 5],
  Coxeter type of ['B', 1], Coxeter type of ['B', 5],
  Coxeter type of ['C', 1], Coxeter type of ['C', 5],
  Coxeter type of ['D', 4], Coxeter type of ['D', 5],
  Coxeter type of ['E', 6], Coxeter type of ['E', 7],
  Coxeter type of ['E', 8], Coxeter type of ['F', 4],
  Coxeter type of ['A', 2, 1], Coxeter type of ['B', 5, 1],
  Coxeter type of ['C', 5, 1], Coxeter type of ['D', 5, 1],
  Coxeter type of ['E', 6, 1], Coxeter type of ['E', 7, 1],
  Coxeter type of ['E', 8, 1], Coxeter type of ['F', 4, 1],
  Coxeter type of ['G', 2, 1], Coxeter type of ['A', 1, 1]]

sage: CoxeterType.samples(crystallographic=False)
[[Coxeter type of ['H', 3],
  Coxeter type of ['I', 10]]]
Todo: add some reducible Coxeter types (suggestions?)

```python
class sage.combinat.root_system.coxeter_type.CoxeterTypeFromCartanType(cartan_type):
    Bases: UniqueRepresentation, CoxeterType

    A Coxeter type associated to a Cartan type.

cartan_type()  
    Return the Cartan type used to construct self.

    EXAMPLES:
    sage: C = CoxeterType(['C',3])
    sage: C.cartan_type()
    ['C', 3]

component_types()  
    A list of Coxeter types making up the reducible type.

    EXAMPLES:
    sage: CoxeterType(['A',2],['B',2]).component_types()  
    [Coxeter type of ['A', 2], Coxeter type of ['B', 2]]
    sage: CoxeterType('A4xB3').component_types()  
    [Coxeter type of ['A', 4], Coxeter type of ['B', 3]]
    sage: CoxeterType(['A', 2]).component_types()  
    Traceback (most recent call last):
    ...  
    ValueError: component types only defined for reducible types

coxeter_graph()  
    Return the Coxeter graph of self.

    EXAMPLES:
    sage: C = CoxeterType(['H',3])
    sage: C.coxeter_graph().edges(sort=True)
    [(1, 2, 3), (2, 3, 5)]

coxeter_matrix()  
    Return the Coxeter matrix associated to self.

    EXAMPLES:
    sage: C = CoxeterType(['H',3])
    sage: C.coxeter_matrix()
    [1 3 2]
    [3 1 5]
    [2 5 1]

index_set()  
    Return the index set of self.

    EXAMPLES:
sage: C = CoxeterType(['A', 4])
sage: C.index_set()
(1, 2, 3, 4)

is_affine()
Return if self is an affine type.

EXAMPLES:

sage: C = CoxeterType(['F', 4, 1])
sage: C.is_affine()
True

is_crystallographic()
Return if self is crystallographic.

EXAMPLES:

sage: C = CoxeterType(['C', 3])
sage: C.is_crystallographic()
True
sage: C = CoxeterType(['H', 3])
sage: C.is_crystallographic()
False

is_finite()
Return if self is a finite type.

EXAMPLES:

sage: C = CoxeterType(['E', 6])
sage: C.is_finite()
True

is_irreducible()
Return if self is irreducible.

EXAMPLES:

sage: C = CoxeterType(['A', 5])
sage: C.is_irreducible()
True
sage: C = CoxeterType('B3xB3')
sage: C.is_irreducible()
False

is_reducible()
Return if self is reducible.

EXAMPLES:

sage: C = CoxeterType(['A', 5])
sage: C.is_reducible()
False

```python
sage: C = CoxeterType('A2xA2')
sage: C.is_reducible()
True
```

**is_simply_laced()**

Return if self is simply-laced.

**EXAMPLES:**

```python
sage: C = CoxeterType(['A', 5])
sage: C.is_simply_laced()
True
sage: C = CoxeterType(['B', 3])
sage: C.is_simply_laced()
False
```

**rank()**

Return the rank of self.

**EXAMPLES:**

```python
sage: C = CoxeterType(['I', 16])
sage: C.rank()
2
```

**relabel**(relabelling)

Return a relabelled copy of self.

**EXAMPLES:**

```python
sage: ct = CoxeterType(['A',2])
sage: ct.relabel({1:-1, 2:-2})
Coxeter type of ['A', 2] relabelled by {1: -1, 2: -2}
```

**type()**

Return the type of self.

**EXAMPLES:**

```python
sage: C = CoxeterType(['A', 4])
sage: C.type()
'A'
```
5.1.229 Dynkin diagrams

AUTHORS:

• Travis Scrimshaw (2012-04-22): Nicolas M. Thiery moved Cartan matrix creation to here and I cached results for speed.

• Travis Scrimshaw (2013-06-11): Changed inputs of Dynkin diagrams to handle other Dynkin diagrams and graphs. Implemented remaining Cartan type methods.


sage.combinat.root_system.dynkin_diagram.DynkinDiagram(*args, **kwds)

Return the Dynkin diagram corresponding to the input.

INPUT:

The input can be one of the following:

• empty to obtain an empty Dynkin diagram
• a Cartan type
• a Cartan matrix
• a Cartan matrix and an indexing set

One can also input an indexing set by passing a tuple using the optional argument index_set.

The edge multiplicities are encoded as edge labels. For the corresponding Cartan matrices, this uses the convention in Hong and Kang, Kac, Fulton and Harris, and crystals. This is the opposite convention in Bourbaki and Wikipedia’s Dynkin diagram (Wikipedia article Dynkin_diagram). That is for $i \neq j$:

\[
i \leftrightarrow k \iff a_{ij} = -k
\]
\[
\iff -\text{scalar(coroot}[i], \text{root}[j]) = k
\]
\[
\iff \text{multiple arrows point from the longer root to the shorter one}
\]

For example, in type $C_2$, we have:

```
sage: C2 = DynkinDiagram(['C',2]); C2
0 O--O--O--O
  1 2 3 4
C2
sage: C2.cartan_matrix()
[ 2 -2]
[-1  2]
```

However Bourbaki would have the Cartan matrix as:

\[
\begin{pmatrix}
2 & -1 \\
-2 & 2
\end{pmatrix}
\]

EXAMPLES:

```
sage: DynkinDiagram(['A', 4])
O---O---O---O---O
  1 2 3 4
A4
```

(continues on next page)
sage: DynkinDiagram(['A',1],['A',1])
A1xA1

sage: R = RootSystem("A2xB2xF4")
sage: DynkinDiagram(R)
O---O
1 2
O=>=O
3 4
O---O=>=O---O
5 6 7 8
A2xB2xF4

sage: R = RootSystem("A2xB2xF4")
sage: CM = R.cartan_matrix(); CM
\[
\begin{bmatrix}
  2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -2 & 2 & -1 \\
  0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]
sage: DD = DynkinDiagram(CM); DD
O---O
1 2
O=>=O
3 4
O---O=>=O---O
5 6 7 8
A2xB2xF4

sage: DD.cartan_matrix()
\[
\begin{bmatrix}
  2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -2 & 2 & -1 \\
  0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

We can also create Dynkin diagrams from arbitrary Cartan matrices:

sage: C = CartanMatrix([[2, -3], [-4, 2]])
sage: DynkinDiagram(C)
Dynkin diagram of rank 2
sage: CI = CartanMatrix([[2, -3], [-4, 2]], [3, 5])
sage: DI = DynkinDiagram(CI)
sage: DI.index_set()
(3, 5)
sage: CII = CartanMatrix([[2, -3], [-4, 2]])
sage: DII = DynkinDiagram(CII, ('y', 'x'))
sage: DII.index_set()
('x', 'y')

See also:

CartanType() for a general discussion on Cartan types and in particular node labeling conventions.

class sage.combinat.root_system.dynkin_diagram.DynkinDiagram_class(t=None, index_set=None, odd_isotropic_roots=[], **options)

Bases: DiGraph, CartanType_abstract

A Dynkin diagram.

See also:

DynkinDiagram()

INPUT:

• t – a Cartan type, Cartan matrix, or None

EXAMPLES:

sage: DynkinDiagram(['A', 3])
\[0\rightarrow0\rightarrow0\]
1 2 3
A3
sage: C = CartanMatrix([[2, -3], [-4, 2]])
sage: DynkinDiagram(C)
Dynkin diagram of rank 2
sage: C.dynkin_diagram().cartan_matrix() == C
True

add_edge(i, j, label=1)

EXAMPLES:

sage: from sage.combinat.root_system.dynkin_diagram import DynkinDiagram_class
sage: d = DynkinDiagram_class(CartanType(['A', 3]))
sage: sorted(d.edges(sort=True))
[]
sage: d.add_edge(2, 3)
sage: sorted(d.edges(sort=True))
[(2, 3), (3, 2), (1, 2), (2, 1)]

static an_instance()

Returns an example of Dynkin diagram

EXAMPLES:
sage: from sage.combinat.root_system.dynkin_diagram import DynkinDiagram_class
sage: g = DynkinDiagram_class.an_instance()
sage: g
Dynkin diagram of rank 3
sage: g.cartan_matrix()
[ 2 -1 -1]
[-2 2 -1]
[-1 -1 2]

**cartan_matrix()**

Returns the Cartan matrix for this Dynkin diagram

**EXAMPLES:**

```
sage: DynkinDiagram(['C',3]).cartan_matrix()
[ 2 -1 0]
[-1 2 -2]
[ 0 -1 2]
```

**cartan_type()**

**EXAMPLES:**

```
sage: DynkinDiagram("A2","B2","F4").cartan_type()
A2xB2xF4
```

**column(j)**

Returns the \( j \)th column \( (a_{i,j}) \) of the Cartan matrix corresponding to this Dynkin diagram, as a container (or iterator) of tuples \((i, a_{i,j})\)

**EXAMPLES:**

```
sage: g = DynkinDiagram(['B',4])
sage: [ (i,a) for (i,a) in g.column(3) ]
[(3, 2), (2, -1), (4, -2)]
```

**coxeter_diagram()**

Construct the Coxeter diagram of self.

See also:

```
CartanType_abstract.coxeter_diagram()
```

**EXAMPLES:**

```
sage: cm = CartanMatrix([[2,-5,0],[-2,2,-1],[0,-1,2]])
sage: D = cm.dynkin_diagram()
sage: G = D.coxeter_diagram(); G
Graph on 3 vertices
sage: G.edges(sort=True)
[(0, 1, +Infinity), (1, 2, 3)]
```

(continues on next page)
sage: ct.dynkin_diagram().coxeter_diagram() == ct.coxeter_diagram()
True

dual()

Returns the dual Dynkin diagram, obtained by reversing all edges.

EXAMPLES:

sage: D = DynkinDiagram(['C', 3])
sage: D.edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 2, 2)]
sage: D.dual()
O---O=<=O
1 2 3
B3
sage: D.dual().edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 2), (3, 2, 1)]
sage: D.dual() == DynkinDiagram(['B', 3])
True

dynkin_diagram()

EXAMPLES:

sage: DynkinDiagram(['C', 3]).dynkin_diagram()
O---O=<=O
1 2 3
C3

index_set()

EXAMPLES:

sage: DynkinDiagram(['C', 3]).index_set()
(1, 2, 3)
sage: DynkinDiagram("A2","B2","F4").index_set()
(1, 2, 3, 4, 5, 6, 7, 8)

is_affine()

Check if self corresponds to an affine root system.

EXAMPLES:

sage: CartanType(['F', 4]).dynkin_diagram().is_affine()
False
sage: D = DynkinDiagram(CartanMatrix([[2, -4], [-3, 2]]))
sage: D.is_affine()
False

is_crystallographic()

Implements CartanType_abstract.is_crystallographic()

A Dynkin diagram always corresponds to a crystallographic root system.

EXAMPLES:
```python
sage: CartanType(['F',4]).dynkin_diagram().is_crystallographic()
True
```

**is_finite()**

Check if self corresponds to a finite root system.

**EXAMPLES:**

```python
sage: CartanType(['F',4]).dynkin_diagram().is_finite()
True
sage: D = DynkinDiagram(CartanMatrix([[2, -4], [-3, 2]]))
sage: D.is_finite()
False
```

**is_irreducible()**

Check if self corresponds to an irreducible root system.

**EXAMPLES:**

```python
sage: CartanType(['F',4]).dynkin_diagram().is_irreducible()
True
sage: CM = CartanMatrix([[2,-6],[[-4,2]])
sage: CM.dynkin_diagram().is_irreducible()
True
sage: CartanType("A2xB3").dynkin_diagram().is_irreducible()
False
sage: CM = CartanMatrix([[2,-6,0],[-4,2,0],[0,0,2]])
sage: CM.dynkin_diagram().is_irreducible()
False
```

**odd_isotropic_roots()**

Return the odd isotropic roots of self.

**EXAMPLES:**

```python
sage: g = DynkinDiagram(['A',4])
sage: g.odd_isotropic_roots()
()  
sage: g = DynkinDiagram(['A',4,3])
sage: g.odd_isotropic_roots()
(0,)
```

**rank()**

Returns the index set for this Dynkin diagram

**EXAMPLES:**

```python
sage: DynkinDiagram(['C',3]).rank()
3
sage: DynkinDiagram("A2","B2","F4").rank()
8
```

**relabel(*args, **kwds)**

Return the relabelled Dynkin diagram of self.

**INPUT:** see **relabel()**
There is one difference: the default value for inplace is False instead of True.

EXAMPLES:

```python
sage: D = DynkinDiagram(['C', 3])
sage: D.relabel({1:0, 2:4, 3:1})
O---O<=<=O
 0 4 1
C3 relabelled by {1: 0, 2: 4, 3: 1}
sage: D
O---O<=<=O
 1 2 3
C3
sage: _ = D.relabel({1:0, 2:4, 3:1}, inplace=True)
sage: D
O---O<=<=O
 0 4 1
C3 relabelled by {1: 0, 2: 4, 3: 1}
sage: D = DynkinDiagram(['A', [1,2]])
sage: Dp = D.relabel({-1:4, 0:-3, 1:3, 2:2})
sage: Dp
O---X---O---O
 4 -3 3 2
A1|2 relabelled by {-1: 4, 0: -3, 1: 3, 2: 2}
sage: Dp.odd_isotropic_roots()
(-3,)
sage: D = DynkinDiagram(['D', 5])
sage: G, perm = D.relabel(range(5), return_map=True)
sage: G
 0 4
| |
O---O---O---O
 0 1 2 3
D5 relabelled by {1: 0, 2: 1, 3: 2, 4: 3, 5: 4}
sage: perm
{1: 0, 2: 1, 3: 2, 4: 3, 5: 4}
sage: perm = D.relabel(range(5), return_map=True, inplace=True)
sage: D
 0 4
| |
O---O---O---O
 0 1 2 3
D5 relabelled by {1: 0, 2: 1, 3: 2, 4: 3, 5: 4}
sage: perm
{1: 0, 2: 1, 3: 2, 4: 3, 5: 4}
```

**row(i)**

Returns the \(i^{th}\) row \((a_{i,j})_j\) of the Cartan matrix corresponding to this Dynkin diagram, as a container (or
iterator) of tuples \((j, a_{i,j})\)

**EXAMPLES:**

```python
sage: g = DynkinDiagram(['C',4])
sage: [(i,a) for (i,a) in g.row(3)]
[(3, 2), (2, -1), (4, -2)]
```

**subtype(index_set)**

Return a subtype of self given by index_set.

A subtype can be considered the Dynkin diagram induced from the Dynkin diagram of self by index_set.

**EXAMPLES:**

```python
sage: D = DynkinDiagram(['A',6,2]); D
O=<=O---O=<=O
0 1 2 3
BC3~
sage: D.subtype([1,2,3])
Dynkin diagram of rank 3
```

**symmetrizer()**

Return the symmetrizer of the corresponding Cartan matrix.

**EXAMPLES:**

```python
sage: d = DynkinDiagram()
sage: d.add_edge(1,2,3)
sage: d.add_edge(2,3)
sage: d.add_edge(3,4,3)
sage: d.symmetrizer()
Finite family {1: 9, 2: 3, 3: 3, 4: 1}
```

```python
sage.combinat.root_system.dynkin_diagram.precheck(t, letter=None, length=None, affine=None, n_ge=None, n=None)
```

**EXAMPLES:**

```python
sage: from sage.combinat.root_system.dynkin_diagram import precheck
sage: ct = CartanType(['A',4])
sage: precheck(ct, letter='C')
Traceback (most recent call last):
  ... ValueError: t[0] must be = 'C'
sage: precheck(ct, affine=1)
Traceback (most recent call last):
  ... ValueError: t[2] must be = 1
sage: precheck(ct, length=3)
Traceback (most recent call last):
  ... ValueError: len(t) must be = 3
sage: precheck(ct, n=3)
Traceback (most recent call last):
  ...
```

(continues on next page)
ValueError: t[1] must be = 3

sage: precheck(ct, n_ge=5)
Traceback (most recent call last):
  ...  
ValueError: t[1] must be >= 5

5.1.230 Hecke algebra representations

class sage.combinat.root_system.hecke_algebra_representation.CherednikOperatorsEigenvectors(T, T_Y=None, normalized=True)

Bases: UniqueRepresentation, SageObject

A class for the family of eigenvectors of the \( Y \) Cherednik operators for a module over a (Double) Affine Hecke algebra

INPUT:

- \( T \) – a family \((T_i)_{i \in I}\) implementing the action of the generators of an affine Hecke algebra on self. The intertwiner operators are built from these.
- \( T_Y \) – a family \((T^Y_i)_{i \in I}\) implementing the action of the generators of an affine Hecke algebra on self. By default, this is \( T \). But this can be used to get the action of the full Double Affine Hecke Algebra. The \( Y \) operators are built from the \( T_Y \).

This returns a function \( \mu \mapsto E_\mu \) which uses intertwining operators to calculate recursively eigenvectors \( E_\mu \) for the action of the torus of the affine Hecke algebra with eigenvalue given by \( f \). Namely:

\[ E_\mu.Y^\lambda^\vee = f(\lambda^\vee, \mu)E_\mu \]

Assumptions:

- \( \text{seed}(\mu) \) initializes the recurrence by returning an appropriate eigenvector \( E_\mu \) for \( \mu \) trivial enough. For example, for nonsymmetric Macdonald polynomials \( \text{seed}(\mu) \) returns the monomial \( X^\mu \) for a minuscule weight \( \mu \).
- \( f \) is almost equivariant. Namely, \( f(\lambda^\vee, \mu) = f(\lambda^\vee s_i, \text{twist}(\mu, i)) \) whenever \( i \) is a descent of \( \mu \).
- \( \text{twist}(\mu, i) \) maps \( \mu \) closer to the dominant chamber whenever \( i \) is a descent of \( \mu \).

Todo: Add tests for the above assumptions, and also that the classical operators \( T_1, \ldots, T_n \) from \( T \) and \( T_Y \) coincide.

\( \text{Y()} \)

Return the Cherednik operators.

EXAMPLES:

sage: W = WeylGroup(["B",2])
sage: K = QQ["q1,q2"].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = \hat{W}.algebra(K)
affine_lift($\mu$)
Lift the index $\mu$ to a space admitting an action of the affine Weyl group.

INPUT:

- $\mu$ – an element $\mu$ of the indexing set

In this space, one should have first_descent and apply_simple_reflection act properly.

EXAMPLES:

```python
sage: W = WeylGroup(['A',3])
sage: W.element_class._repr_ = lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: w = W.an_element(); w
123
sage: E.affine_lift(w)
121
```

affine_retract($\mu$)
Retract $\mu$ from a space admitting an action of the affine Weyl group.

EXAMPLES:

```python
sage: W = WeylGroup(['A',3])
sage: W.element_class._repr_ = lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: w = W.an_element(); w
123
sage: E.affine_lift(w)
121
sage: E.affine_retract(E.affine_lift(w)) == w
True
```

cartan_type()
Return Cartan type of self.

EXAMPLES:

```python
sage: W = WeylGroup(['B',3])
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: E.cartan_type()
['B', 3, 1]
```

(continues on next page)
sage: NonSymmetricMacdonaldPolynomials(["B", 2, 1]).cartan_type()
['B', 2, 1]

domain()

The module on which the affine Hecke algebra acts.

EXAMPLES:

sage: W = WeylGroup(['B',3])
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: E.domain()
Algebra of Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space) over Multivariate Polynomial Ring in q1, q2 over Rational Field

eigenvalue(mu, l)

Return the eigenvalue of $Y_{\lambda^\vee}$ on $E_{\mu}$ computed by applying $Y_{\lambda^\vee}$ on $E_{\mu}$.

INPUT:

* mu – the index $\mu$ of an eigenvector, or a tentative eigenvector
* l – the index $\lambda^\vee$ of a Cherednik operator in self.Y_index_set()

This default implementation applies explicitly $Y_{\mu}$ to $E_{\lambda}$.

EXAMPLES:

sage: W = WeylGroup(['B',2])
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: w0 = W.long_element()
sage: Y = E.Y()
sage: alphacheck = Y.keys().simple_roots()
sage: E.eigenvalue(w0, alphacheck[1])
q1/(-q2)
sage: E.eigenvalue(w0, alphacheck[2])
q1/(-q2)
sage: E.eigenvalue(w0, alphacheck[0])
q2^2/q1^2

The following checks that all $E_{w}$ are eigenvectors, with eigenvalue given by Lemma 7.5 of [HST2008] (checked for $A_2, A_3$):

sage: Pcheck = Y.keys()  
sage: Wcheck = Pcheck.weyl_group()  
sage: P0check = Pcheck.classical()  
sage: def height(root):      
.....:      return sum(P0check(root).coefficients())  
sage: def eigenvalue(w, mu):  
.....:      return (-q2/q1)^height(Wcheck.from_reduced_word(w.reduced_word)).
→ action(μ))
sage: all(E.eigenvalue(w, a) == eigenvalue(w, a) for w in E.keys() for a in Y.
˓→ keys().simple_roots()) # long time (2.5s)
True

**eigenvalues(μ)**
Return the eigenvalues of \( Y_{α_0}, \ldots, Y_{α_n} \) on \( E_μ \).

**INPUT:**

- `μ` – the index \( μ \) of an eigenvector or a tentative eigenvector

**EXAMPLES:**

```python
sage: W = WeylGroup(["B",2])
```

```python
sage: K = QQ['q1,q2'].fraction_field()
```

```python
sage: q1, q2 = K.gens()
```

```python
sage: KW = W.algebra(K)
```

```python
sage: T = KW.demazure_lusztig_operators(q1, q2)
```

```python
sage: E = T.Y_eigenvectors()
```

```python
sage: E.eigenvalues(q1)
```

```python
sage: E.eigenvalues(q2)
```

```python
sage: E.hecke_parameters(1)
```

```python
sage: E.hecke_parameters(2)
```

```python
sage: E.hecke_parameters(0)
```

**hecke_parameters(𝑖)**
Return the Hecke parameters for index \( i \).

**EXAMPLES:**

```python
sage: W = WeylGroup(["A",3])
```

```python
sage: K = QQ['q1,q2']
```

```python
sage: q1, q2 = K.gens()
```

```python
sage: KW = W.algebra(K)
```

```python
sage: T = KW.demazure_lusztig_operators(q1, q2, affine=True)
```

```python
sage: E = T.Y_eigenvectors()
```

```python
sage: E.hecke_parameters(0)
```

```python
sage: E.hecke_parameters(1)
```

```python
sage: E.hecke_parameters(2)
```

**keys()**
The index set for the eigenvectors.

By default, this assumes that the eigenvectors span the full affine Hecke algebra module and that the eigenvectors have the same indexing as the basis of this module.

**EXAMPLES:**

```python
sage: W = WeylGroup(["A",3])
```

```python
sage: K = QQ['q1,q2']
```

```python
sage: q1, q2 = K.gens()
```
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: E.keys()
Weyl Group of type ['A', 3] (as a matrix group acting on the ambient space)

**recursion**(\(\mu\))

Return the indices used in the recursion.

**INPUT:**

- \(\mu\) – the index \(\mu\) of an eigenvector

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3])
sage: W.element_class._repr_=lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: w0 = W.long_element()
sage: E.recursion(w0)
[]
sage: w = W.an_element(); w
123
sage: E.recursion(w)
[1, 2, 1]
```

**seed**(\(\mu\))

Return the eigenvector for \(\mu\) minusculc.

**INPUT:**

- \(\mu\) – an element \(\mu\) of the indexing set

**OUTPUT:** an element of \(T\).domain()

This default implementation returns the monomial indexed by \(\mu\).

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3])
sage: W.element_class._repr_=lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1, q2)
sage: E.seed(W.long_element())
123121
```

**twist**(\(\mu, i\))

Act by \(s_i\) on \(\mu\).

By default, this calls the method apply_simple_reflection.

**EXAMPLES:**
sage: W = WeylGroup(['B',3])
sage: W.element_class._repr_=lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: T = KW.demazure_lusztig_operators(q1, q2, affine=True)
sage: E = T.Y_eigenvectors()
sage: w = W.an_element(); w
123
sage: E.twist(w,1)
1231

class sage.combinat.root_system.hecke_algebra_representation.HeckeAlgebraRepresentation(domain, on_basis, cartan_type, q1, q2, q=1, side='right')

Bases: WithEqualityById, SageObject

A representation of an (affine) Hecke algebra given by the action of the $T$ generators

Let $F_i$ be a family of operators implementing an action of the operators $(T_i)_{i \in I}$ of the Hecke algebra on some vector space $\text{domain}$, given by their action on the basis of $\text{domain}$. This constructs the family of operators $(F_w)_{w \in W}$ describing the action of all elements of the basis $(T_w)_{w \in W}$ of the Hecke algebra. This is achieved by linearity on the first argument, and applying recursively the $F_i$ along a reduced word for $w = s_i \cdots s_k$:

$$F_w(x) = F_{s_k} \circ \cdots \circ F_{s_1}(x).$$

INPUT:

• $\text{domain}$ – a vector space
• $\text{f}$ – a function $f(l,i)$ taking a basis element $l$ of $\text{domain}$ and an index $i$, and returning $F_i$
• $\text{cartan_type}$ – The Cartan type of the Hecke algebra
• $q1,q2$ – The eigenvalues of the generators $T$ of the Hecke algebra
• $\text{side}$ – “left” or “right” (default: “right”) whether this is a left or right representation

EXAMPLES:

sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = WeylGroup(['A',3]).algebra(QQ)
sage: H = KW.demazure_lusztig_operators(q1,q2); H
A representation of the (q1, q2)-Hecke algebra of type ['A', 3, 1] on Algebra of Weyl Group of type ['A', 3] (as a matrix group acting on the ambient space) over Rational Field

Among other things, it implements the $T_w$ operators, their inverses and compositions thereof:
and the Cherednik operators $Y^{\lambda}$:

```python
sage: H.Y()
Lazy family (...)_{i in Coroot lattice of the Root system of type ['A', 3, 1]}
```

REFERENCES:

• [HST2008]

**Ti_inverse_on_basis**($x, i$)

The $T_i^{-1}$ operators, on basis elements

**INPUT:**

- $x$ – the index of a basis element
- $i$ – the index of a generator

**EXAMPLES:**

```python
sage: W = WeylGroup("A3")
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: rho = KW.demazure_lusztig_operators(q1,q2)
sage: w = W.an_element()
sage: rho.Ti_inverse_on_basis(w, 1)
-1/q2*1231 + ((q1+q2)/(q1*q2))*123
```

**Ti_on_basis**($x, i$)

The $T_i$ operators, on basis elements.

**INPUT:**

- $x$ – the index of a basis element
- $i$ – the index of a generator

**EXAMPLES:**

```python
sage: W = WeylGroup("A3")
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: rho = KW.demazure_lusztig_operators(q1,q2)
sage: w = W.an_element()
sage: rho.Ti_on_basis(w, 1)
q1*1231
```

**Tw**($word, signs=None, scalar=None$)

Return $T_w$.

**INPUT:**

```python
sage: H.Tw((1,2))
Generic endomorphism of Algebra of Weyl Group of type ['A', 3] (as a matrix group acting on the ambient space) over Rational Field
```
• **word** – a word $i_1, \ldots, i_k$ for some element $w$ of the Weyl group. See `straighten_word()` for how this word can be specified.

• **signs** – a list $\varepsilon_1, \ldots, \varepsilon_k$ of the same length as `word` with $\varepsilon_i = \pm 1$ or `None` for $1, \ldots, 1$ (default: `None`)

• **scalar** – an element $c$ of the base ring or `None` for $1$ (default: `None`)

**OUTPUT:**

a module morphism implementing

$$T_w = T_{i_k} \circ \cdots \circ T_{i_1}$$

in left action notation; that is $T_{i_1}$ is applied first, then $T_{i_2}$, etc.

More generally, if scalar or signs is specified, the morphism implements

$$cT_{i_k}^\varepsilon \circ \cdots \circ T_{i_1}^\varepsilon.$$

**EXAMPLES:**

```
sage: W = WeylGroup("A3")

sage: W.element_class._repr_ = lambda x: ('e' if not x.reduced_word() ....: else ''.join(str(i) for i in x.reduced_word()))

sage: K = QQ['q1,q2'].fraction_field()

sage: q1, q2 = K.gens()

sage: KW = W.algebra(K)

sage: x = KW.an_element(); x
123 + 3*2312 + 2*31 + e

sage: T = KW.demazure_lusztig_operators(q1,q2)

sage: T12 = T.Tw( (1,2) )

sage: T12(KW.one())
q1^2*12

sage: T[2](T[1](KW.one()))
q1^2*12

sage: T[1](T[2](KW.one()))
q1^2*21

sage: T12(x) == T[2](T[1](x))
True
```

This is $T_2 \circ T_1$:

```
sage: T[2](T[1](KW.one()))
q1^2*12

sage: T[1](T[2](KW.one()))
q1^2*21

sage: T12(x) == T[2](T[1](x))
True
```

Now with signs and scalar coefficient we construct $3T_2 \circ T_1^{-1}$:

```
sage: phi = T.Tw((1,2), (-1,1), 3)

sage: phi(KW.one())
((-3*q1)/q2)*12 + ((3*q1+3*q2)/q2)*2

sage: phi(T[1](x)) == 3*T[2](x)
True
```

For debugging purposes, one can recover the input data:

```
sage: phi.word
(1, 2)

sage: phi.signs
```

(continues on next page)
(-1, 1)
sage: phi.scalar
3

Tw_inverse(word)

Return $T_w^{-1}$.

This is essentially a shorthand for $Tw()$ with all minus signs.

Todo: Add an example where $T_i \neq T_i^{-1}$

EXAMPLES:

```sage
W = WeylGroup(['A',3])
sage: W.element_class._repr_ = lambda x: ''.join(str(i) for i in x.reduced_word())
sage: KW = W.algebra(QQ)
sage: rho = KW.demazure_lusztig_operators(1, -1)
sage: x = KW.monomial(W.an_element()); x
123
sage: word = [1,2]
sage: rho.Tw(word)(x)
12312
sage: rho.Tw_inverse(word)(x)
12321
```

$Y(base\_ring=Integer\_Ring)$

Return the Cherednik operators $Y$ for this representation of an affine Hecke algebra.

INPUT:

- `self` – a representation of an affine Hecke algebra
- `base_ring` – the base ring of the coroot lattice

This is a family of operators indexed by the coroot lattice for this Cartan type. In practice this is currently indexed instead by the affine coroot lattice, even if this indexing is not one to one, in order to allow for $Y[\alpha_0^\vee]$.

EXAMPLES:
sage: W = WeylGroup(["A",3])
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: rho = KW.demazure_lusztig_operators(q2, q1)
sage: Y = rho.Y(); Y
Lazy family (...(i))_{i in Coroot lattice of the Root system of type ['A', 3, →1]}
\[
\begin{align*}
& (-q_2^2)/(q_1^2+q_1q_2-q_2^2))_{21}^{12} + \frac{q_2^2}{q_1^2+q_1q_2-q_2^2})_{21}^{12} + 21 - 21 + 1, \\
& (-q_2^2)/(q_1^2+q_1q_2-q_2^2))_{21}^{12} + \frac{q_2^2}{q_1^2+q_1q_2-q_2^2})_{21}^{12} + 21 - 21 + 1, \\
& (q_2^2)/(q_1^2+q_1q_2-q_2^2))_{21}^{12} + (-q_2^2)/(q_1^2+q_1q_2-q_2^2))_{21}^{12} - 21 + 1, \\
& (q_2^2)/(q_1^2+q_1q_2-q_2^2))_{21}^{12} + (-q_2^2)/(q_1^2+q_1q_2-q_2^2))_{21}^{12} - 21 + 1.
\end{align*}
\]

The \( Y \) operators commute:
\[
\begin{align*}
& \text{sage: Y0(Y1(x)) - Y1(Y0(x))} \quad 0 \\
& \text{sage: Y2(Y1(x)) - Y1(Y2(x))} \quad 0
\end{align*}
\]

In the classical root lattice, \( \alpha_0 + \alpha_1 + \alpha_2 = 0 \):
Lemma 7.2 of [HST2008]:

\[
\begin{align*}
\text{sage: } & w_0 = KW.monomial(W\text{.long_element()}) \\
\text{sage: } & \rho(Tw(0))(w_0) = q^2 \\
\text{sage: } & \rho(Tw_inverse(1))(w_0) = \frac{1}{q^2} q^2 121 \\
\text{sage: } & \rho(Tw_inverse(2))(w_0) = \frac{1}{q^2} q^2 121
\end{align*}
\]

Lemma 7.5 of [HST2008]:

\[
\begin{align*}
\text{sage: } & Y_0(w_0) = q_1^2/q_2^2 q^2 121 \\
\text{sage: } & Y_1(w_0) = (q_2/(-q_1)) q^2 121 \\
\text{sage: } & Y_2(w_0) = (q_2/(-q_1)) q^2 121
\end{align*}
\]

Todo: Add more tests
Add tests in type BC affine where the null coroot $\delta^\vee$ can have non trivial coefficient in term of $\alpha_0$

See also:

- [HST2008] for the formula in terms of $q_1, q_2$

cartan_type()

Return the Cartan type of self.

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{from sage.combinat.root_system.hecke_algebra_representation import HeckeAlgebraRepresentation} \\
\text{sage: } & KW = SymmetricGroup(3).algebra(QQ) \\
\text{sage: } & \text{action = lambda x,i: KW.monomial(x.apply_simple_reflection(i, side="right"))} \\
\text{sage: } & H = HeckeAlgebraRepresentation(KW, action, CartanType(['A',2]), 1, -1) \\
\text{sage: } & H.cartan_type() \\
& ['A', 2] \\
\text{sage: } & H = WeylGroup(['A',3]).algebra(QQ).demazure_lusztig_operators(-1,1) \\
\text{sage: } & H.cartan_type() \\
& ['A', 3, 1]
\end{align*}
\]

domain()

Return the domain of self.

EXAMPLES:
Combinatorics, Release 10.0

```
sage: H = WeylGroup(["A",3]).algebra(QQ).demazure_lusztig_operators(-1,1)
sage: H.domain()
Algebra of Weyl Group of type ['A', 3] (as a matrix group acting on the ambient → space) over Rational Field
```

**on_basis**(*x, word, signs=None, scalar=None*)

Action of product of $T_i$ and $T_i^{-1}$ on $x$.

**INPUT:**

- $x$ – the index of a basis element
- $word$ – word of indices of generators
- $signs$ – (default: None) sequence of signs of same length as $word$; determines which operators are supposed to be taken as inverses.
- $scalar$ – (default: None) scalar to multiply the answer by

**EXAMPLES:**

```
sage: from sage.combinat.root_system.hecke_algebra_representation import HeckeAlgebraRepresentation
sage: W = SymmetricGroup(3)
sage: domain = W.algebra(QQ)
sage: action = lambda x,i: domain.monomial(x.apply_simple_reflection(i, side="right"))
sage: rho = HeckeAlgebraRepresentation(domain, action, CartanType(["A",2]), 1, -1)
sage: rho.on_basis(W.one(), (1,2,1))
(1,3)
sage: word = (1,2)
sage: u = W.from_reduced_word(word)
sage: for w in W: assert rho.on_basis(w, word) == domain.monomial(w*u)
```

The next example tests the signs:

```
sage: W = WeylGroup("A3")
sage: W.element_class._repr_ = lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: rho = KW.demazure_lusztig_operators(q1,q2)
sage: w = W.an_element(); w
123
sage: rho.on_basis(w, (1,), signs=(-1,))
-1/q2*1231 + ((q1+q2)/(q1*q2))*123
sage: rho.on_basis(w, (1,), signs=(1,))
q1*1231
sage: rho.on_basis(w, (1,1), signs=(1,-1))
123
sage: rho.on_basis(w, (1,1), signs=(-1,1))
123
```
parameters(i)
Return $q_1, q_2$ such that $(T_i - q_1)(T_i - q_2) = 0$.

EXAMPLES:

```
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = WeylGroup(['A',3]).algebra(QQ)
sage: H = KW.demazure_lusztig_operators(q1,q2)
sage: H.parameters(1)
(q1, q2)
sage: H = KW.demazure_lusztig_operators(1,-1)
sage: H.parameters(1)
(1, -1)
```

Todo: At this point, this method is constant. It’s meant as a starting point for implementing parameters depending on the node $i$ of the Dynkin diagram.

straighten_word(word)
Return a tuple of indices of generators after some straightening.

INPUT:

- **word** – a list/tuple of indices of generators, the index of a generator, or an object with a reduced word method

OUTPUT: a tuple of indices of generators

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: H = W.algebra(QQ).demazure_lusztig_operators(-1,1)
sage: H.straighten_word(1)
(1,)
sage: H.straighten_word((2,1))
(2, 1)
sage: H.straighten_word([2,1])
(2, 1)
sage: H.straighten_word(W.an_element())
(1, 2, 3)
```

5.1.231 Integrable Representations of Affine Lie Algebras

class sage.combinat.root_system.integrable_representations.IntegrableRepresentation(Lam)
Bases: UniqueRepresentation, CategoryObject
An irreducible integrable highest weight representation of an affine Lie algebra.

INPUT:

- **Lam** – a dominant weight in an extended weight lattice of affine type

REFERENCES:

- [Ka1990]
If \( \Lambda \) is a dominant integral weight for an affine root system, there exists a unique integrable representation \( V = V_\Lambda \) of highest weight \( \Lambda \). If \( \mu \) is another weight, let \( m(\mu) \) denote the multiplicity of the weight \( \mu \) in this representation. The set \( \text{supp}(V) \) of \( \mu \) such that \( m(\mu) > 0 \) is contained in the paraboloid

\[
(\Lambda + \rho | \Lambda + \rho) - (\mu + \rho | \mu + \rho) \geq 0
\]

where \(| \cdot |\) is the invariant inner product on the weight lattice and \( \rho \) is the Weyl vector. Moreover if \( m(\mu) > 0 \) then \( \mu \in \text{supp}(V) \) differs from \( \Lambda \) by an element of the root lattice ([Ka1990], Propositions 11.3 and 11.4).

Let \( \delta \) be the nullroot, which is the lowest positive imaginary root. Then by [Ka1990], Proposition 11.3 or Corollary 11.9, for fixed \( \mu \) the function \( m(\mu - k\delta) \) is a monotone increasing function of \( k \). It is useful to take \( \mu \) to be such that this function is nonzero if and only if \( k \geq 0 \). Therefore we make the following definition. If \( \mu \) is such that \( m(\mu) \neq 0 \) but \( m(\mu + \delta) = 0 \) then \( \mu \) is called maximal.

Since \( \delta \) is fixed under the action of the affine Weyl group, and since the weight multiplicities are Weyl group invariant, the function \( k \mapsto m(\mu - k\delta) \) is unchanged if \( \mu \) is replaced by an equivalent weight. Therefore in tabulating these functions, we may assume that \( \mu \) is dominant. There are only a finite number of dominant maximal weights.

Since every nonzero weight multiplicity appears in the string \( \mu - k\delta \) for one of the finite number of dominant maximal weights \( \mu \), it is important to be able to compute these. We may do this as follows.

**EXAMPLES:**

```python
sage: Lambda = RootSystem(['A',3,1]).weight_lattice(extended=True).fundamental_weights()
2*Lambda[0] + Lambda[2]: 4 31 161 665 2380 7658 22721 63120 166085 417295 1007601
  \rightarrow 2349655
Lambda[0] + 2*Lambda[1]: 2 18 99 430 1593 5274 16005 45324 121200 308829 754884
  \rightarrow 1779570
Lambda[0] + 2*Lambda[3]: 2 18 99 430 1593 5274 16005 45324 121200 308829 754884
  \rightarrow 1779570
  \rightarrow 1799620
3*Lambda[2] - delta: 3 21 107 450 1638 5367 16194 45687 121876 310056 757056 1783324
sage: Lambda = RootSystem(['D',4,1]).weight_lattice(extended=True).fundamental_weights()
sage: IntegrableRepresentation(Lambda[0]+Lambda[1]).print_strings()
# long time
Lambda[0] + Lambda[1]: 1 10 62 293 1165 4097 13120 38997 109036 289575
  \rightarrow 563390 1343178
3*Lambda[2] - delta: 3 21 107 450 1638 5367 16194 45687 121876 310056 757056 1783324
sage: Lambda = RootSystem(['D',4,1]).weight_lattice(extended=True).fundamental_weights()
sage: IntegrableRepresentation(Lambda[0]+Lambda[1]).print_strings()
# long time
Lambda[0] + Lambda[1]: 1 10 62 293 1165 4097 13120 38997 109036 289575
  \rightarrow 563390 1343178
3*Lambda[2] - delta: 3 21 107 450 1638 5367 16194 45687 121876 310056 757056 1783324
```

In this example, we construct the extended weight lattice of Cartan type \( A_3^{(1)} \), then define \( \Lambda \) to be the fundamental weights \( (\Lambda_i)_{i \in I} \). We find there are 5 maximal dominant weights in irreducible representation of highest weight \( \Lambda_1 + \Lambda_2 + \Lambda_3 \), and we determine their strings.

It was shown in [KacPeterson] that each string is the set of Fourier coefficients of a modular form.

Every weight \( \mu \) such that the weight multiplicity \( m(\mu) \) is nonzero has the form

\[
\Lambda - n_0\alpha_0 - n_1\alpha_1 - \cdots,
\]

where the \( n_i \) are nonnegative integers. This is represented internally as a tuple \((n_0, n_1, n_2, \ldots)\). If you want an individual multiplicity you use the method \( m() \) and supply it with this tuple:
The `IntegrableRepresentation` class has methods `to_weight()` and `from_weight()` to convert between this internal representation and the weight lattice:

```python
sage: delta = V.weight_lattice().null_root()
sage: V.to_weight((4,3,2))
(4, 3, 2)
```

To get more values, use the depth parameter:

```python
sage: L0 = RootSystem(['A',1,1]).weight_lattice(extended=true).fundamental_weights(); L0
Lambda[0]
sage: IntegrableRepresentation(4*L0).print_strings(depth=20)
4*Lambda[0]: 1 1 3 6 13 24 44 75 131 215 354 561 889 1368 2097 3153 4712 6936 10151
2850 4256 6303 9222 13396 19262
6936 10121 14677 21055
```

An example in type $C_2^{(1)}$:

```python
sage: Lambda = RootSystem(['C',2,1]).weight_lattice(extended=true).fundamental_weights()
sage: V = IntegrableRepresentation(2*Lambda[0])
sage: V.print_strings()  # long time
2*Lambda[0]: 1 2 9 26 77 194 477 1084 2387 5010 10227 20198
Lambda[0] + Lambda[2] - delta: 1 5 18 55 149 372 872 1941 4141 8523 17005 33019
2*Lambda[1] - delta: 1 4 15 44 122 304 721 1612 3469 7176 14414 28124
```

Examples for twisted affine types:

```python
sage: Lambda = RootSystem(['A',2,2]).weight_lattice(extended=True).fundamental_weights()
sage: IntegrableRepresentation(Lambda[0]).strings()
{Lambda[0]: [1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56]}
sage: Lambda = RootSystem(['G',2,1]).dual.weight_lattice(extended=true).fundamental_weights()
sage: V = IntegrableRepresentation(Lambda[0]+Lambda[1]+Lambda[2])
sage: V.print_strings()  # long time
6*Lambdacheck[0]: 4 28 100 320 944 2460 6664 14300 31968 69020 144676 293916
3*Lambdacheck[0] + Lambdacheck[1]: 2 16 58 192 588 1568 3952 9520 21644 47456
109906 207536
```

(continues on next page)
\[4\text{Lambdacheck}[0] + \text{Lambdacheck}[2]: 4 \ 22 \ 84 \ 276 \ 800 \ 2124 \ 5288 \ 12470 \ 28116 \ 61056\]
\[\rightarrow 128304 \ 261972\]
\[2\text{Lambdacheck}[1] - \text{deltacheck}: 2 \ 8 \ 32 \ 120 \ 354 \ 1204 \ 3428 \ 8256 \ 18960 \ 41844\]
\[\rightarrow 89208 \ 184512\]
\[3\text{Lambdacheck}[2] - \text{deltacheck}: 4 \ 16 \ 60 \ 208 \ 592 \ 1584 \ 4032 \ 9552 \ 21728 \ 47776\]
\[\rightarrow 207888\]

\text{sage:} \quad \text{Lambda} = \text{RootSystem}(['A',6,2]).weight_lattice(extended=true).fundamental_weights()
\text{sage:} \quad V = \text{IntegrableRepresentation}(\text{Lambda}[0]+2\text{Lambda}[1])
\text{sage:} \quad V.print_strings() \quad \# \text{long time}
\[5\text{Lambda}[0]: 3 \ 42 \ 378 \ 2508 \ 13707 \ 64650 \ 272211 \ 1045470 \ 3721815 \ 12425064 \ 39254163\]
\[\rightarrow 118191378\]
\[3\text{Lambda}[0] + \text{Lambda}[2]: 1 \ 23 \ 234 \ 1690 \ 9689 \ 47313 \ 204247 \ 800029 \ 2893198 \ 9786257\]
\[\rightarrow 31262198 \ 95035357\]
\[\text{Lambda}[0] + 2\text{Lambda}[1]: 1 \ 14 \ 154 \ 1160 \ 6920 \ 34756 \ 153523 \ 612354 \ 2248318 \ 7702198\]
\[\rightarrow 24875351 \ 76341630\]
\[\text{Lambda}[0] + \text{Lambda}[1] + \text{Lambda}[3] - 2\text{delta}: 6 \ 87 \ 751 \ 4779 \ 25060 \ 113971 \ 464842\]
\[\rightarrow 1736620 \ 6034717 \ 19723537 \ 61152367 \ 181068152\]
\[\text{Lambda}[0] + 2\text{Lambda}[2] - 2\text{delta}: 3 \ 54 \ 499 \ 3349 \ 18166 \ 84836 \ 353092 \ 1341250 \ 4725259\]
\[\rightarrow 15625727 \ 48938396 \ 146190544\]
\[\text{Lambda}[0] + 2\text{Lambda}[3] - 4\text{delta}: 15 \ 195 \ 1539 \ 9186 \ 45804 \ 200073 \ 789201 \ 2866560\]
\[\rightarrow 9723582 \ 31120281 \ 94724550 \ 275919741\]

\text{branch}(i=None, weyl_character_ring=None, sequence=None, depth=5)

Return the branching rule on \textit{self}.

Removing any node from the extended Dynkin diagram of the affine Lie algebra results in the Dynkin diagram of a classical Lie algebra, which is therefore a Lie subalgebra. For example removing the 0 node from the Dynkin diagram of type [X, r, 1] produces the classical Dynkin diagram of [X, r].

Thus for each \(i\) in the index set, we may restrict \textit{self} to the corresponding classical subalgebra. Of course \textit{self} is an infinite dimensional representation, but each weight \(\mu\) is assigned a grading by the number of times the simple root \(\alpha_i\) appears in \(\Lambda - \mu\). Thus the branched representation is graded and we get sequence of finite-dimensional representations which this method is able to compute.

OPTIONAL:

\begin{itemize}
  \item \textit{i} – (default: 0) an element of the index set
  \item \texttt{weyl\_character\_ring} – a WeylCharacterRing
  \item \texttt{sequence} – a dictionary
  \item \texttt{depth} – (default: 5) an upper bound for \(k\) determining how many terms to give
\end{itemize}

In the default case where \(i = 0\), you do not need to specify anything else, though you may want to increase the depth if you need more terms.

EXAMPLES:

\text{sage:} \quad \text{Lambda} = \text{RootSystem}(['A',2,1]).weight_lattice(extended=true).fundamental_weights()
\text{sage:} \quad V = \text{IntegrableRepresentation}(2\text{Lambda}[0])
sage: b = V.branch(); b
[A2(0, 0),
 A2(1, 1),
 A2(0, 0) + 2*A2(1, 1) + A2(2, 2),
 2*A2(0, 0) + 2*A2(0, 3) + 4*A2(1, 1) + 2*A2(3, 0) + 2*A2(2, 2),
 4*A2(0, 0) + 3*A2(0, 3) + 10*A2(1, 1) + 3*A2(3, 0) + A2(1, 4) + 6*A2(2, 2) + A2(4, 1),
 6*A2(0, 0) + 9*A2(0, 3) + 20*A2(1, 1) + 9*A2(3, 0) + 3*A2(1, 4) + 12*A2(2, 2) +
 → 3*A2(4, 1) + A2(3, 3)]

If the parameter `weyl_character_ring` is omitted, the ring may be recovered as the parent of one of the branched coefficients:

sage: A2 = b[0].parent(); A2
The Weyl Character Ring of Type A2 with Integer Ring coefficients

If \( i \) is not zero then you should specify the `WeylCharacterRing` that you are branching to. This is determined by the Dynkin diagram:

sage: Lambda = RootSystem(['B',3,1]).weight_lattice(extended=true).fundamental_weights()

sage: V = IntegrableRepresentation(Lambda[0])

sage: V.cartan_type().dynkin_diagram()

O 0
| |
O---O=>=O
1 2 3

B3~

In this example, we observe that removing the \( i = 2 \) node from the Dynkin diagram produces a reducible diagram of type \( A1xA1xA1 \). Thus we have a branching to \( \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2) \):

sage: A1xA1xA1 = WeylCharacterRing("A1xA1xA1",style="coroots")

sage: V.branch(i=2,weyl_character_ring=A1xA1xA1)
[A1xA1xA1(1,0,0),
 A1xA1xA1(0,1,2),
 A1xA1xA1(1,0,0) + A1xA1xA1(1,2,0) + A1xA1xA1(1,0,2),
 A1xA1xA1(2,1,2) + A1xA1xA1(0,1,0) + 2*A1xA1xA1(0,1,2),
 3*A1xA1xA1(1,0,0) + 2*A1xA1xA1(1,2,0) + A1xA1xA1(1,2,2) + 2*A1xA1xA1(1,0,2) +
 → A1xA1xA1(1,0,4) + A1xA1xA1(3,0,0),
 A1xA1xA1(2,1,0) + 3*A1xA1xA1(2,1,2) + 2*A1xA1xA1(0,1,0) + 5*A1xA1xA1(0,1,2) +
 → A1xA1xA1(0,1,4) + A1xA1xA1(0,3,2)]

If the nodes of the two Dynkin diagrams are not in the same order, you must specify an additional parameter, `sequence` which gives a dictionary to the affine Dynkin diagram to the classical one.

EXAMPLES:
Observe that removing the \( i = 1 \) node from the \( \mathcal{F}_4 \) Dynkin diagram gives the \( A_1 \times C_3 \) diagram, but the roots are in a different order. The nodes 0, 2, 3, 4 of \( \mathcal{F}_4 \) correspond to 1, 4, 3, 2 of \( A_1 \times C_3 \) and so we encode this in a dictionary:

```
sage: V.branch(i=1,weyl_character_ring=A1xC3,sequence={0:1,2:4,3:3,4:2})
```

The branch method gives a way of computing the graded dimension of the integrable representation:

```
sage: Lambda = RootSystem("A1~").weight_lattice(extended=true).fundamental_weights()
sage: V=IntegrableRepresentation(Lambda[0])
sage: r = [x.degree() for x in V.branch(depth=15)]; r
[1, 3, 4, 7, 13, 19, 29, 43, 62, 90, 126, 174, 239, 325, 435, 580]
sage: oeis(r)  # optional
```

```
0: A029552: Expansion of \( \phi(x) / f(-x) \) in powers of \( x \) where \( \phi(), f() \) are Ramanujan theta functions.
```

```
cartan_type()
```
Return the Cartan type of \( \text{self} \).

EXAMPLES:

```
sage: Lambda = RootSystem(["F",4,1]).weight_lattice(extended=true).fundamental_weights()
sage: V = IntegrableRepresentation(Lambda[0])
sage: V.cartan_type()
['F', 4, 1]
```

```
coxeter_number()
```
Return the Coxeter number of the Cartan type of \( \text{self} \).

The Coxeter number is defined in [Ka1990] Chapter 6, and commonly denoted \( h \).

EXAMPLES:
sage: Lambda = RootSystem(['F',4,1]).weight_lattice(extended=True).fundamental_weights()  
sage: V = IntegrableRepresentation(Lambda[0])  
sage: V.coxeter_number()  
12

dominant_maximal_weights()

Return the dominant maximal weights of self.

A weight \( \mu \) is maximal if it has nonzero multiplicity but \( \mu + \delta \) has multiplicity zero. There are a finite number of dominant maximal weights. Indeed, [Ka1990] Proposition 12.6 shows that the dominant maximal weights are in bijection with the classical weights in \( k \cdot F \) where \( F \) is the fundamental alcove and \( k \) is the level. The construction used in this method is based on that Proposition.

EXAMPLES:

sage: Lambda = RootSystem(['C',3,1]).weight_lattice(extended=True).fundamental_weights()  
sage: IntegrableRepresentation(2*Lambda[0]).dominant_maximal_weights()  
((2*Lambda[0], Lambda[0] + Lambda[2] - delta,  
  2*Lambda[1] - delta,  
  Lambda[1] + Lambda[3] - 2*delta,  
  2*Lambda[2] - 2*delta,  
  2*Lambda[3] - 3*delta))

dual_coxeter_number()

Return the dual Coxeter number of the Cartan type of self.

The dual Coxeter number is defined in [Ka1990] Chapter 6, and commonly denoted \( h^\vee \).

EXAMPLES:

sage: Lambda = RootSystem(['F',4,1]).weight_lattice(extended=True).fundamental_weights()  
sage: V = IntegrableRepresentation(Lambda[0])  
sage: V.dual_coxeter_number()  
9

from_weight(mu)

Return the tuple \((n_0, n_1, ...)\) such that \( \mu \) equals \( \Lambda - \sum_{i \in I} n_i \alpha_i \) in self, where \( \Lambda \) is the highest weight of self.

EXAMPLES:

sage: Lambda = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()  
sage: V = IntegrableRepresentation(2*Lambda[2])  
sage: V.to_weight((1,0,0))  
sage: delta = V.weight_lattice().null_root()  
(1, 0, 0)

highest_weight()

Returns the highest weight of self.
EXAMPLES:

```python
sage: Lambda = RootSystem(['D',4,1]).weight_lattice(extended=true).fundamental_weights()
sage: IntegrableRepresentation(Lambda[0]+2*Lambda[2]).highest_weight()
Lambda[0] + 2*Lambda[2]
```

`level()`

Return the level of `self`.

The level of a highest weight representation $V_\Lambda$ is defined as $(\Lambda|\delta)$ See [Ka1990] section 12.4.

EXAMPLES:

```python
sage: Lambda = RootSystem(['G',2,1]).weight_lattice(extended=true).fundamental_weights()
sage: [IntegrableRepresentation(Lambda[i]).level() for i in [0,1,2]]
[1, 1, 2]
```

$m(n)$

Return the multiplicity of the weight $\mu$ in `self`, where $\mu = \Lambda - \sum n_i \alpha_i$.

INPUT:

* `n` – a tuple representing a weight $\mu$.

EXAMPLES:

```python
sage: Lambda = RootSystem(['E',6,1]).weight_lattice(extended=true).fundamental_weights()
sage: V = IntegrableRepresentation(Lambda[0])
sage: u = V.highest_weight() - V.weight_lattice().null_root()
sage: V.from_weight(u)
(1, 1, 2, 2, 3, 2, 1)
sage: V.m(V.from_weight(u))
6
```

`modular_characteristic(mu=None)`

Return the modular characteristic of `self`.

The modular characteristic is a rational number introduced by Kac and Peterson [KacPeterson], required to interpret the string functions as Fourier coefficients of modular forms. See [Ka1990] Section 12.7. Let $k$ be the level, and let $h^\vee$ be the dual Coxeter number. Then

$$
m_\Lambda = \frac{|\Lambda + \rho|^2}{2(k + h^\vee)} - \frac{|ho|^2}{2h^\vee}
$$

If $\mu$ is a weight, then

$$
m_{\Lambda,\mu} = m_\Lambda - \frac{|\mu|^2}{2k}.
$$

OPTIONAL:

* `mu` – a weight; or alternatively:
* `n` – a tuple representing a weight $\mu$. 

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If no optional parameter is specified, this returns $m_\Lambda$. If $\mu$ is specified, it returns $m_{\Lambda,\mu}$. You may use the tuple $n$ to specify $\mu$. If you do this, $\mu = \Lambda - \sum_i n_i \alpha_i$.

EXAMPLES:

```python
sage: Lambda = RootSystem(['A',1,1]).weight_lattice(extended=True).fundamental_weights()
sage: V = IntegrableRepresentation(3*Lambda[0]+2*Lambda[1])
sage: [V.modular_characteristic(x) for x in V.dominant_maximal_weights()]
[11/56, -1/280, 111/280]
```

`mult(\mu)`

Return the weight multiplicity of $\mu$.

INPUT:

- $\mu$ – an element of the weight lattice

EXAMPLES:

```python
sage: L = RootSystem("B3~").weight_lattice(extended=True)
sage: Lambda = L.fundamental_weights()
sage: delta = L.null_root()
sage: W = L.weyl_group(prefix="s")
sage: [s0,s1,s2,s3] = W.simple_reflections()
sage: V = IntegrableRepresentation(Lambda[0])
sage: V.mult(Lambda[2]-2*delta)
3
sage: V.mult(Lambda[2]-Lambda[1])
0
sage: weights = [w.action(Lambda[1]-4*delta) for w in [s1,s2,s0*s1*s2*s3]]
sage: weights
sage: [V.mult(mu) for mu in weights]
[35, 35, 35]
```

`print_strings(depth=12)`

Print the strings of self.

See also:

`strings()`

EXAMPLES:

```python
sage: Lambda = RootSystem(['A',1,1]).weight_lattice(extended=True).fundamental_weights()
sage: V = IntegrableRepresentation(2*Lambda[0])
sage: V.print_strings(depth=25)
2*Lambda[0]: 1 1 3 5 10 16 28 43 70 105 161 236 350 501 722 1016 1431 1981 2741 ...
                3740 5096 6868 9233 12306 16357
2*Lambda[1] - delta: 1 2 4 7 13 21 35 55 86 130 196 287 420 602 858 1206 1687 ...
                2331 3206 4368 5922 7967 10670 14193 18803
```

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\textbf{root_lattice(\textit{self})}

Return the root lattice associated to \textit{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: V = IntegrableRepresentation(RootSystem(['F',4,1]).weight_lattice(extended=true).fundamental_weight(0))
sage: V.root_lattice()
Root lattice of the Root system of type ['F', 4, 1]
\end{verbatim}

\textbf{s(\textit{n}, \textit{i})}

Return the action of the \textit{i}-th simple reflection on the internal representation of weights by tuples \textit{n} in \textit{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: V = IntegrableRepresentation(RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weight(0))
sage: [V.s((0,0,0),i) for i in V._index_set]
[(1, 0, 0), (0, 0, 0), (0, 0, 0)]
\end{verbatim}

\textbf{string(\textit{max_weight}, \textit{depth}=12)}

Return the list of multiplicities \(m(\Lambda - k\delta)\) in \textit{self}, where \(\Lambda\) is \textit{max_weight} and \(k\) runs from 0 to \textit{depth}.

\textbf{INPUT:}

- \textit{max_weight} – a dominant maximal weight
- \textit{depth} – (default: 12) the maximum value of \(k\)

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Lambda = RootSystem(['A',2,1]).weight_lattice(extended=True).fundamental_weights()
sage: V = IntegrableRepresentation(2*Lambda[0])
sage: V.string(2*Lambda[0])
[1, 2, 8, 20, 52, 116, 256, 522, 1045, 1996, 3736, 6780]
sage: V.string(Lambda[1] + Lambda[2])
[0, 1, 4, 12, 32, 77, 172, 365, 740, 1445, 2736, 5041]
\end{verbatim}

\textbf{strings(\textit{depth}=12)}

Return the set of dominant maximal weights of \textit{self}, together with the string coefficients for each.

\textbf{OPTIONAL:}

- \textit{depth} – (default: 12) a parameter indicating how far to push computations

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Lambda = RootSystem(['A',1,1]).weight_lattice(extended=True).fundamental_weights()
sage: V = IntegrableRepresentation(2*Lambda[0])
sage: S = V.strings(depth=25)
sage: for k in S:
    ...:     print("{}: {}\".format(k, ' '.join(str(x) for x in S[k])))
2*Lambda[0]: 1 1 1 3 5 10 16 28 43 70 105 161 236 350 501 722 1016 1431 1981 2741
   -> 3740 5096 6868 9233 12306 16357
2*Lambda[1] - delta: 1 2 4 7 13 21 35 55 86 130 196 287 420 602 858 1206 1687
   -> 2331 3206 4368 5922 7967 10670 14193 18803
\end{verbatim}
to_dominant($n$)
Return the dominant weight in self equivalent to $n$ under the affine Weyl group.

EXAMPLES:

```python
sage: Lambda = RootSystem(['A',2,1]).weight_lattice(extended=true).fundamental_weights()
sage: V = IntegrableRepresentation(3*Lambda[0])
sage: n = V.to_dominant((13,11,7)); n
(4, 3, 3)
sage: V.to_weight(n)
```

to_weight($n$)
Return the weight associated to the tuple $n$ in self.

If $n$ is the tuple $(n_1, n_2, \ldots)$, then the associated weight is $\Lambda - \sum_i n_i \alpha_i$, where $\Lambda$ is the weight of the representation.

INPUT:
- $n$ – a tuple representing a weight

EXAMPLES:

```python
sage: Lambda = RootSystem(['A',2,1]).weight_lattice(extended=true).fundamental_weights()
sage: V = IntegrableRepresentation(2*Lambda[2])
sage: V.to_weight((1,0,0))
```

weight_lattice()
Return the weight lattice associated to self.

EXAMPLES:

```python
sage: V=IntegrableRepresentation(RootSystem(['E',6,1]).weight_lattice(extended=true).fundamental_weight(0))
sage: V.weight_lattice()
Extended weight lattice of the Root system of type ['E', 6, 1]
```

5.1.232 Nonsymmetric Macdonald polynomials

AUTHORS:
- Anne Schilling and Nicolas M. Thiéry (2013): initial version

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The initial version of this code (together with root_lattice_realization_algebras.Algebras and hecke_algebra_representation.HeckeAlgebraRepresentation) was written by Anne Schilling and Nicolas M. Thiéry during the ICERM Semester Program on “Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series” (January 28, 2013 - May 3, 2013) with the help of Dan Bump, Ben Brubaker, Bogdan Ion, Dan Orr, Arun Ram, Siddhartha Sahi, and Mark Shimozono. Special thanks go to Bogdan Ion and Mark Shimozono for their patient explanations and hand computations to check the code.
class sage.combinat.root_system.non_symmetric_macdonald_polynomials.NonSymmetricMacdonaldPolynomials(KL, q, q1, q2, normalized):

Bases: CherednikOperatorsEigenvectors

Nonsymmetric Macdonald polynomials

INPUT:

• KL – an affine Cartan type or the group algebra of a realization of the affine weight lattice
• q, q1, q2 – parameters in the base ring of the group algebra (default: q, q1, q2)
• normalized – a boolean (default: True) whether to normalize the result to have leading coefficient 1

This implementation covers all reduced affine root systems. The polynomials are constructed recursively by the
application of intertwining operators.

Todo:

• Non-reduced case (Koornwinder polynomials).
• Non-equal parameters for the affine Hecke algebra.
• Choice of convention (dominant/anti-dominant, ...).
• More uniform implementation of the $T_0^\vee$ operator.
• Optimizations, in particular in the calculation of the eigenvalues for the recursion.

EXAMPLES:

We construct the family of nonsymmetric Macdonald polynomials in three variables in type $A$:

```
sage: E = NonSymmetricMacdonaldPolynomials(['A',2,1])
```

They are constructed as elements of the group algebra of the classical weight lattice $L_0$ (or one of its
realizations, such as the ambient space, which is used here) and indexed by elements of $L_0$:

```
sage: L0 = E.keys(); L0
Ambient space of the Root system of type ['A', 2]
```

Here is the nonsymmetric Macdonald polynomial with leading term $[2,0,1]$:

```
sage: E[L0([2,0,1])]
((-q*q1-q*q2)/(-q*q1-q2))*B[(1, 1, 1)] + ((-q1-q2)/(-q*q1-q2))*B[(2, 1, 0)] + B[(2, -\rightarrow0, 1)]
```

It can be seen as a polynomial (or in general a Laurent polynomial) by interpreting each term as an
exponent vector. The parameter $q$ is the exponential of the null (co)root, whereas $q_1$ and $q_2$ are the two eigenvalues of each
generator $T_i$ of the affine Hecke algebra (see the background section for details).

By setting $q_1 = t$, $q_2 = -1$ and using the root_lattice_realization_algebras.Algebras. ElementMethods.expand() method, we recover the nonsymmetric Macdonald polynomial as computed by [HHL06]'s combinatorial formula:
Here is a type $G^{(1)}_2$ nonsymmetric Macdonald polynomial:

\[
(\frac{t - 1}{q^* t - 1})^* x_0^2 x_1 + x_0^2 x_2 + \frac{(q^* t - q)}{(q^* t - 1)} x_0 x_1 x_2
\]

Many more examples are given after the background section.

See also:

- `sage.combinat.sf.ns_macdonald.E()`
- `SymmetricFunctions.macdonald()`

**Background**

The polynomial module

The nonsymmetric Macdonald polynomials are a distinguished basis of the “polynomial” module of the affine Hecke algebra. Given:

- a ground ring `K`, which contains the input parameters `q, q_1, q_2`
- an affine root system, specified by a Cartan type `C`
- a realization `L` of the weight lattice of type `C`

the polynomial module is the group algebra $K[L_0]$ of the classical weight lattice $L_0$ with coefficients in $K$. It is isomorphic to the Laurent polynomial ring over $K$ generated by the formal exponentials of any basis of $L_0$.

In our running example $L$ is the ambient space of type $C^{(1)}_2$.

```python
sage: K = QQ['q,q1,q2'].fraction_field()
sage: q, q1, q2 = K.gens()
sage: C = CartanType(['C',2,1])
sage: L = RootSystem(C).ambient_space(); L
Ambient space of the Root system of type ['C', 2, 1]
sage: L.simple_roots()
Finite family {0: -2*e[0] + e['delta'], 1: e[0] - e[1], 2: 2*e[1]}
sage: omega = L.fundamental_weights(); omega
Finite family {0: e['deltacheck'], 1: e[0] + e['deltacheck'], 2: e[0] + e[1] + e[...]
```
Affine Hecke algebra

The affine Hecke algebra is generated by elements $T_i$ for $i$ in the set of affine Dynkin nodes. They satisfy the same braid relations as the simple reflections $s_i$ of the affine Weyl group. The $T_i$ satisfy the quadratic relation

$$(T_i - q_1) \circ (T_i - q_2) = 0,$$

where $q_1$ and $q_2$ are the input parameters. Some of the representation theory requires that $q_1$ and $q_2$ satisfy additional relations; typically one uses the specializations $q_1 = u$ and $q_2 = -1/u$ or $q_1 = t$ and $q_2 = -1$. This can be achieved by constructing an appropriate field and passing $q_1$ and $q_2$ appropriately; see the examples. In principle, the parameter(s) could further depend on $i$; this is not yet implemented but the code has been designed in such a way that this feature is easy to add.

Demazure-Lusztig operators

The $i$-th Demazure-Lusztig operator is an operator on $K[L]$ which interpolates between the reflection $s_i$ and the Demazure operator $\pi_i$ (see root_lattice_realization.RootLatticeRealization.Algebras.ParentMethods.demazure_lusztig_operators()).

The affine Hecke algebra acts on $K[L]$ by letting the generators $T_i$ act by the Demazure-Lusztig operators. The class sage.combinat.root_system.hecke_algebra_representation.HeckeAlgebraRepresentation implements some simple generic features for representations of affine Hecke algebras defined by the action of their $T$-generators.
Here we construct the operator \( q_1 T_2^{-1} \circ T_1^{-1} T_0 \) from a signed reduced word:

```
sage: T.Tw([0,1,2],[1,-1,-1], q1^2)
Generic endomorphism of Algebra of the Ambient space of the Root system of type ['C', 2, 1]
over Fraction Field of Multivariate Polynomial Ring in q, q1, q2 over Rational Field
```

(note the reversal of the word). Inverses are computed using the quadratic relation.

### Cherednik operators

The affine Hecke algebra contains elements \( Y_\lambda \) indexed by the coroot lattice. Their action on \( K[L] \) is implemented in Sage:

```
sage: Y = T.Y(); Y
Lazy family (...)_{i in Coroot lattice of the Root system of type ['C', 2, 1]}
sage: alphacheck = Y.keys().simple_roots()
sage: Y1 = Y[alphacheck[1]]
sage: Y1(x)
((q1^2+2*q1*q2+q2^2)/(-q1*q2))*B[e[0] + e[deltacheck]]
+ ((-q1^2-q1*q2)/(-q2^2))*B[2*e[0] - e[1] - e[delta]]
+ e['deltacheck'] + ((q1^3+2*q1^2*q2+q1*q2^2)/(-q2^3))*B[-e[0] - e[delta] + e['deltacheck']]
+ ((q1^3+q1^2*q2)/(-q2^3))*B[-e[0] - 2*e[1] - e[delta] + e['deltacheck']]
+ ((q1^3+q1^2*q2)/(-q2^3))*B[3*e[0] - 3*e[delta] + e['deltacheck']]
+ ((q1^3+q1^2*q2)/(-q2^3))*B[-e[0] - 2*e[1] - e[delta] + e['deltacheck']]
```

The Cherednik operators span a Laurent polynomial ring inside the affine Hecke algebra; namely \( \lambda \mapsto Y_\lambda \) is a group isomorphism from the classical root lattice (viewed additively) to the affine Hecke algebra (viewed multiplicatively). In practice, \( Y_\lambda \) is constructed by computing combinatorially its signed reduced word (and an overall scalar factor) using the periodic orientation of the alcove model in the coweight lattice (see `hecke_algebra_representation.HeckeAlgebraRepresentation.Y_lambdacheck()`):

```
sage: lcheck = L.root_system.coweight_lattice()
sage: w = lcheck.reduced_word_of_translation(lcheck(alphacheck[1])); w
[0, 2, 1, 0, 2, 1]
sage: lcheck.signs_of_alcovewalk(w)
[1, -1, 1, -1, 1, 1]
```
Level zero representation of the affine Hecke algebra

The action of the affine Hecke algebra on $K[L]$ induces an action on $K[L_0]$: the action of $T_i$ on $X^\lambda$ for $\lambda$ a classical weight in $L_0$ is obtained by embedding the weight at level zero in the affine weight lattice (see `weight_lattice_realizations.WeightLatticeRealizations.ParentMethods.embed_at_level()`) applying the Demazure-Lusztig operator there, and projecting from $K[L] \to K[L_0]$ mapping the exponential of $\delta$ to $q$ (see `root_lattice_realization_algebras.Algebras.ParentMethods.q_project()`). This is implemented in `root_lattice_realization_algebras.Algebras.ParentMethods.demazure_lusztig_operators_on_classical()`:

```python
sage: T = KL.demazure_lusztig_operators_on_classical(q, q1,q2)
sage: omega = L0.fundamental_weights()
sage: x = KL0.monomial(omega[1])
sage: T[0](x)
(-q*q2)*B[(-1, 0)]
```

For classical nodes these are the usual Demazure-Lusztig operators:

```python
sage: T[1](x)
(q1+q2)*B[(1, 0)] + q1*B[(0, 1)]
```

Nonsymmetric Macdonald polynomials

We can now finally define the nonsymmetric Macdonald polynomials. Because the Cherednik operators commute (and there is no radical), they can be simultaneously diagonalized; namely, $K[L_0]$ admits a $K$-basis of joint eigenvectors for the $Y_\lambda$. For $\mu \in L_0$, the nonsymmetric Macdonald polynomial $E_\mu$ is the unique eigenvector of the family of Cherednik operators $Y_\lambda$ having $\mu$ as leading term:

```python
sage: E = NonSymmetricMacdonaldPolynomials(KL, q, q1, q2); E
The family of the Macdonald polynomials of type ['C', 2, 1] with parameters q, q1,\omega \to q2
```

Or for short:

```python
sage: E = NonSymmetricMacdonaldPolynomials(C)
```

Recursive construction of the nonsymmetric Macdonald polynomials

The generators $T_i$ of the affine Hecke algebra almost skew commute with the Cherednik operators. More precisely, one can deform them into the so-called intertwining operators:

$$\tau_i = T_i - (q_1 + q_2) \frac{Y_i^{a-1}}{1 - Y_i^a}.$$  

(where $a = 1$ except for $i = 0$ in type $BC$ where $a = a_0 = 2$) which satisfy the following skew commutation relations:

$$\tau_i Y_\lambda = \tau_i Y_{s_i \lambda}.$$  

If $s_i \mu \neq \mu$, applying $\tau_i$ on an eigenvector $E_\mu$ produces a new eigenvector (essentially $E_{s_i \mu}$) with a distinct eigenvalue. It follows that the eigenvectors indexed by an affine Weyl orbit of weights, may be recursively computed from a single weight in the orbit.
In the case at hand, there is a little complication: namely, the simple reflections $s_i$ acting at level 0 do not act transitively on classical weights; in fact the orbits for the classical Weyl group and for the affine Weyl group are the same. Thus, one can construct the nonsymmetric Macdonald polynomials for all weights from those for the classical dominant weights, but one is lacking a creation operator to construct the nonsymmetric Macdonald polynomials for dominant weights.

**Twisted Demazure-Lusztig operators**

To compensate for this, one needs to consider another affinization of the action of the classical Demazure-Lusztig operators $T_1, \ldots, T_n$, which gives rise to the double affine Hecke algebra. Following Cherednik, one adds another operator $T_0^\lor$ implemented in: `root_lattice_realization_algebras.Algebras.ParentMethods.T0_check_on_basis()`. See also: `root_lattice_realization_algebras.Algebras.ParentMethods.twisted_demazure_lusztig_operators()`.

Depending on the type (untwisted or not), this is a representation of the affine Hecke algebra for another affinization of the classical Cartan type. The corresponding action of the affine Weyl group – which is used to compute the recursion on $\mu$ – occurs in the corresponding weight lattice realization:

```python
sage: E.L()
Ambient space of the Root system of type ['C', 2, 1]
sage: E.L_prime()
Coambient space of the Root system of type ['B', 2, 1]
sage: E.L_prime().classical()
Ambient space of the Root system of type ['C', 2]
```

See `L_prime()` and `cartan_type.CartanType_affine.other_affinization()`.

REFERENCES:

**More examples**

We show how to create the nonsymmetric Macdonald polynomials in two different ways and check that they are the same:

```python
sage: K = QQ['q,u'].fraction_field()
sage: q, u = K.gens()
sage: E = NonSymmetricMacdonaldPolynomials(['D',3,1], q, u, -1/u)
sage: omega = E.keys().fundamental_weights()
((-q*u^2+q)/(-q*u^4+1))*B[(1/2, -1/2, 1/2)] + ((-q*u^2+q)/(-q*u^4+1))*B[(1/2, 1/2, -1/2)] + B[(3/2, 1/2, 1/2)]
sage: KL = RootSystem(['D',3,1]).ambient_space().algebra(K)
sage: P = NonSymmetricMacdonaldPolynomials(KL, q, u, -1/u)
True
sage: E[E.keys()((0,1,-1))]
((-q*u^2+q)/(-q*u^2+1))*B[(0, 0, 0)] + ((-u^2+1)/(-q*u^2+1))*B[(1, 1, 0)] + ((-u^2+1)/(-q*u^2+1))*B[(1, 0, -1)] + B[(0, 1, -1)]
```

In type $A$, there is also a combinatorial implementation of the nonsymmetric Macdonald polynomials in terms of augmented diagram fillings as in [HHL06]. See `sage.combinat.sf.ns_macdonald.E()`. First we check that these polynomials are indeed eigenvectors of the Cherednik operators:
sage: K = QQ['q,t'].fraction_field()
sage: q,t = K.gens()
sage: q1 = t; q2 = -1
sage: KL = RootSystem(['A',2,1]).ambient_space().algebra(K)
sage: KL0 = KL.classical()
sage: E = NonSymmetricMacdonaldPolynomials(KL, q, q1, q2)
sage: omega = E.keys().fundamental_weights()
sage: w = omega[1]
sage: import sage.combinat.sf.ns_macdonald as NS
sage: p = NS.E([1,0,0]); p

Next we check explicitly that they agree with the current implementation:
The same, directly in the ambient lattice with several shifts:

```python
sage: E[L0([2,0])]
((-q*t+q)/(-q*t+1))*B[(1,1)] + B[(2,0)]
sage: E[L0([1,-1])]
((-q*t+q)/(-q*t+1))*B[(0,0)] + B[(1,-1)]
sage: E[L0([0,-2])]
((-q*t+q)/(-q*t+1))*B[(-1,-1)] + B[(0,-2)]
```

Systematic checks with Sage's implementation of [HHL06]:

```python
sage: assert all(EE([x,y]) == NS.E([x,y]) for d in range(5) for x,y in IntegerVectors(d,2))
```

With the current implementation, we can compute nonsymmetric Macdonald polynomials for any type, for example for type $E_6^{(1)}$:

```python
sage: K=QQ['q,u'].fraction_field()
sage: q, u = K.gens()
sage: KL = RootSystem(["E",6,1]).weight_space(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q,u,-1/u)
sage: L0 = E.keys()
sage: E[L0.fundamental_weight(1).weyl_action([2,4,3,2,1])]
((-u^2+1)/(-q*u^16+1))*B[-Lambda[3] + Lambda[4]]
sage: E[L0.fundamental_weight(2).weyl_action([2,5,3,4,2])]
# long time (6s)
```

5.1. Comprehensive Module List
We test various other types:

```
sage: K=QQ['q,u'].fraction_field()
sage: q, u = K.gens()
sage: KL = RootSystem(['A',5,2]).ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL, q, u, -1/u)
sage: L0 = E.keys()
sage: E[L0.fundamental_weight(2)]
((-q*u^2+q)/(-q*u^8+1))*B[(0, 0, 0)] + B[(1, 1, 0)]
sage: E[L0((0,-1,1))]
((-q^2*u^10+q^2*u^8-q*u^6+q*u^4+q*u^2+u^2-q-1)/(-q^3*u^12+q^2*u^8+q*u^4-1))*B[(0, 0, -1, 1)]
```

```
sage: K=QQ['q,u'].fraction_field()
sage: q, u = K.gens()
sage: KL = RootSystem(['E',6,2]).ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q,u,-1/u)
sage: L0 = E.keys()
sage: E[L0.fundamental_weight(4)]
((-q^3*u^20+q^3*u^18+q*u^4-q^2*u^2-2*q*u^2+q^2+q)/(q^3*u^18-q^2*u^12-q*u^6+1))*B[(0, 0, 0, 0)]
```

(continues on next page)
Next we test a twisted type (checked against Maple computation by Bogdan Ion for \(q_1 = t^2\) and \(q_2 = -1\)):

\[
\text{sage: } E = \text{NonSymmetricMacdonaldPolynomials(\"A\",5,2)}
\]

\[
\text{sage: } \omega = E.\text{keys()}
\]

\[
\text{sage: } E[\omega[1]]
\]

\[
\text{sage: } E[-\omega[1]]
\]

\[
\text{sage: } E[\omega[2]]
\]

\[
\text{sage: } E[-\omega[2]]
\]

\[
\text{sage: } E[-\omega[1]-\omega[2]]
\]

\[
\text{sage: } E[\omega[1]-\omega[2]]
\]

\[
\begin{align*}
+ \frac{(-q u^4 + 2 q u^2 - q)}{(-q^3 u^18 + q^2 u^12 + q u^6 - 1)} B[(0, 1, 0, 0)] \\
+ \frac{(-q u^4 + 2 q u^2 - q)}{(-q^3 u^18 + q^2 u^12 + q u^6 - 1)} B[(0, 0, -1, 0)] \\
+ \frac{(-q u^4 + 2 q u^2 - q)}{(-q^3 u^18 + q^2 u^12 + q u^6 - 1)} B[(0, 0, 1, 0)] \\
+ \frac{(-q u^4 + 2 q u^2 - q)}{(-q^3 u^18 + q^2 u^12 + q u^6 - 1)} B[(0, 0, 0, -1)] \\
+ \frac{(-q u^4 + 2 q u^2 - q)}{(-q^3 u^18 + q^2 u^12 + q u^6 - 1)} B[(0, 0, 0, 1)]
\end{align*}
\]
(continued from previous page)

\[7))*B[(1, 1, 0)] + ((-q1*q2-q2^2)/(q*q1^2-q2^2))*B[(1, 0, -1)] + ((q1*q2+q2^2)/(-q*q1^2+q2^2))*B[(1, 0, 1)]

\[sage: E[\omega[3]]((q*q1^4*q2+q*q1^3*q2^2-q1*q2^4-q2^5)/(-q^2*q1^5-q2^5))*B[(1, 0, 0)] + ((q*q1^4-q2^4-q2^5)/(-q^2*q1^5-q2^5))*B[(0, 0, 0)] + B[(-1, 0, -1)] + B[(1, 0, 0)] + B[(-1, 0, 0)] + B[(1, 0, 0)] + B[(0, -1, 0)] + B[(0, -1, 0)] + B[(0, 0, -1)] + B[(0, 0, 1)]

\[\text{Comparison with the energy function of crystals}\]

Next we test that the nonsymmetric Macdonald polynomials at \(t = 0\) match with the one-dimensional configuration sums involving Kirillov-Reshetikhin crystals for various types. See [LNSSS12]:

\[\text{sage: } K = \text{QQ['q','t'].fraction_field()}\]
\[\text{sage: } q,t = K.gens()\]
\[\text{sage: } KL = \text{RootSystem('A',5,2).ambient_space().algebra(K)}\]
\[\text{sage: } E = \text{NonSymmetricMacdonaldPolynomials(KL, q, t, -1)}\]
\[\text{sage: } \omega = E.keys().fundamental_weights()\]
\[\text{sage: } E[-\omega[1]].map_coefficients(\lambda x:x.subs(t=0)) == B[(-1, 0, 0)] + B[(1, 0, 0)] + B[(-1, 0, 1)] + B[(1, 0, 1)] + B[(0, -1, 0)] + B[(0, 1, 0)] + B[(0, 0, 0)] + B[(-1, 0, -1)] + B[(1, 0, -1)] + B[(1, 1, 0)] + B[(1, 1, 1)] + B[(0, -1, -1)] + B[(0, -1, 1)] + B[(0, 1, -1)] + B[(0, 1, 1)]\]

\[\text{sage: } KL = \text{RootSystem('C',3,1).ambient_space().algebra(K)}\]
\[\text{sage: } E = \text{NonSymmetricMacdonaldPolynomials(KL,q, t,-1)}\]
\[\text{sage: } \omega = E.keys().fundamental_weights()\]
\[\text{sage: } E[-\omega[2]].map_coefficients(\lambda x:x.subs(t=0)) == 2*B[(0, 0, 0)] + B[(-1, 0, 0)] + B[(1, 0, 0)] + B[(-1, 1, 0)] + B[(1, 1, 0)] + B[(0, 0, -1)] + B[(0, 0, 1)] + B[(0, -1, 0)] + B[(0, 1, 0)] + B[(1, 0, -1)] + B[(1, 0, 1)] + B[(0, 0, 0)] + B[(-1, 0, 0)] + B[(1, 0, 0)] + B[(-1, 1, 0)] + B[(1, 1, 0)] + B[(0, 0, -1)] + B[(0, 0, 1)] + B[(0, -1, 0)] + B[(0, 1, 0)] + B[(1, 0, -1)] + B[(1, 0, 1)]\]

\[\text{sage: } R = \text{RootSystem('C',3,1)}\]
\[\text{sage: } KL = R.weight_lattice().basis()\]
\[\text{sage: } E = \text{NonSymmetricMacdonaldPolynomials(KL,q, t,-1)}\]
\[\text{sage: } \omega = E.keys().fundamental_weights()\]
\[\text{sage: } \text{LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])}\]
\[\text{sage: } E[-2*\omega[1]].map_coefficients(\lambda x:x.subs(t=0)) == \text{LS.one_dimensional_configuration_sum(q)} \text{ True}\]
\[\text{sage: } \text{LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[2])}\]
\[\text{sage: } E[-\omega[1]-\omega[2]].map_coefficients(\lambda x:x.subs(t=0)) == \text{LS.one_dimensional_configuration_sum(q)} \text{ True}\]
sage: R = RootSystem(['C',2,1])
sage: KL = R.weight_lattice(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t,-1)
sage: omega = E.keys().fundamental_weights()
sage: La = R.weight_space().basis()
sage: for d in range(1,3):
    ....:     for x,y in IntegerVectors(d,2):
    ....:         weight = x*La[1]+y*La[2]
    ....:         weight0 = -x*omega[1]-y*omega[2]
    ....:         LS = crystals.ProjectedLevelZeroLSPaths(weight)
    ....:         assert E[weight0].map_coefficients(lambda x:x.subs(t=0)) == LS.one_→dimensional_configuration_sum(q)

sage: R = RootSystem(['B',3,1])
sage: KL = R.weight_lattice(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t,-1)
sage: omega = E.keys().fundamental_weights()
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: E[-2*omega[1]].map_coefficients(lambda x:x.subs(t=0)) == LS.one_dimensional_→configuration_sum(q) # long time (23s)
True

sage: R = RootSystem(['BC',3,2])
sage: KL = R.weight_lattice(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t,-1)
sage: omega = E.keys().fundamental_weights()
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: E[-2*omega[1]].map_coefficients(lambda x:x.subs(t=0)) == LS.one_dimensional_→configuration_sum(q) # long time (21s)
True

sage: R = RootSystem(CartanType(['BC',3,2]).dual())
sage: KL = R.weight_space(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t,-1)
sage: omega = E.keys().fundamental_weights()
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: g = E[-2*omega[1]].map_coefficients(lambda x:x.subs(t=0)) # long time (30s)
sage: f = LS.one_dimensional_configuration_sum(q) # long time (1.5s)
sage: P = g.support()[0].parent() # long time
sage: B = P.algebra(q.parent()) # long time
sage: sum(p[1]*B(p[0])) for p in f) == g # long time
True
```python
sage: C = CartanType(['G',2,1])
sage: R = RootSystem(C.dual())
sage: K = QQ['q,t'].fraction_field()
sage: q,t = K.gens()
sage: KL = R.weight_lattice(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL, q, t,-1)
sage: omega = E.keys().fundamental_weights()
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
```

```
sage: E[-2*omega[1]].map_coefficients(lamb‌‌‌da x:x.subs(t=0)) == LS.one_dimensional_configuration_sum(q) # not tested, long time (20s)
True
```

```
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1]+La[2])
```

```
sage: E[-omega[1]-omega[2]].map_coefficients(lamb‌‌‌da x:x.subs(t=0)) == LS.one_dimensional_configuration_sum(q) # not tested, long time (23s)
True
```

```
The next test breaks if the energy is not scaled by the translation factor for dual type $G_{2(1)}$:
```
```
```

```
sage: E[-2*omega[1]-omega[2]].map_coefficients(lamb‌‌‌da x:x.subs(t=0)) == LS.one_dimensional_configuration_sum(q) # not tested, very long time (100s)
True
```

```
sage: R = RootSystem(['D',4,1])
sage: KL = R.weight_lattice(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL, q, t,-1)
sage: omega = E.keys().fundamental_weights()
sage: La = R.weight_space().basis()
sage: for d in range(1,2):
    for a,b,c,d in IntegerVectors(d,4):
        weight0 = -a*omega[1]-b*omega[2]-c*omega[3]-d*omega[4]
        LS = crystals.ProjectedLevelZeroLSPaths(weight)
        assert E[weight0].map_coefficients(lamb‌‌‌da x:x.subs(t=0)) == LS.one_dimensional_configuration_sum(q)
```
```
Todo: add his notes in latex
```
```
sage: K = QQ['q,q1,q2'].fraction_field()
sage: q,q1,q2 = K.gens()
sage: L = RootSystem(['A',4,2]).ambient_space()
sage: L.cartan_type() ['BC', 2, 2]
sage: L.null_root() 2*e['delta']
```

```
sage: L.simple_roots()
Finite family {0: -e[0] + e['delta'], 1: e[0] - e[1], 2: 2*e[1]}
```

```
sage: KL = L.algebra(K)
sage: KL0 = KL.classical()
sage: L0 = L.classical()
```
```
(continues on next page)
```
sage: L0.cartan_type()
['C', 2]
sage: E = NonSymmetricMacdonaldPolynomials(KL, q=q,q1=q1,q2=q2)
sage: E.keys()
Ambient space of the Root system of type ['C', 2]
sage: E.keys().simple_roots()
Finite family {1: (1, -1), 2: (0, 2)}
sage: omega = E.keys().fundamental_weights()
sage: E[0*omega[1]]
B[(0, 0)]
sage: E[omega[1]]
((-q*q1*q2^3-q*q2^4)/(q^2*q1^4-q2^4))*B[(0, 0)] + B[(1, 0)]
sage: E[2*omega[2]]
# long time # not checked against Bogdan's notes, but a good self-consistency test
((-q^12*q1^6-q^12*q1^5*q2+2*q^10*q1^5*q2+5*q^10*q1^4*q2^2+3*q^10*q1^3*q2^3+2*q^8*q1^2*q2+3*q^8*q1*q2^2+3*q^8*q2^3+q^6*q1^4*q2+2*q^6*q1^3*q2^2+3*q^6*q1^2*q2^3+3*q^6*q1*q2^4+4*q^6*
-3*q^6*q1^2*q2^2+4*q^6*q1*q2^3+3*q^6*q2^4+3*q^6*q2^5+2*q^6*q2^6+4*q^6*q1^4*q2^3+6*q^6*q1^3*q2^4+3*q^6*q1^2*q2^5+3*q^6*q1*q2^6+3*q^6*q2^7+2*q^6*q2^8+q^6*q1^4*q2^4+4*q^6*q1^3*q2^5+3*q^6*q1^2*q2^6+3*q^6*q1*q2^7+3*q^6*q2^8+2*q^6*q2^9+q^6*q1^3*q2^6+3*q^6*q1^2*q2^7+3*q^6*q1*q2^8+3*q^6*q2^9+2*q^6*q2^10+q^6*q1^2*q2^8+2*q^6*q1*q2^9+2*q^6*q2^10+q^6*q2^11+q^6*q1*q2^10+q^6*q2^11+q^6*q2^12)/(-q^12*q1^6-q^10*q1^5*q2-q^8*q1^3*q2^3+q^6*q1*q2^5+q^4*q1^3*q2^4+q^2*q1^5*q2^3)*B[(0, 0)] + ((q^7*q1^2*q2+2*q^7*q1*q2^2+q^7*q2^3)/(-q^8*q1^3-q^6*q1^2*q2-q2^3))*B[(-1, 1)] + ((-q^3*q1*q2^3-q^3*q2^4)/(q^6*q1^2-q2^2))*B[(0, 0)] + B[(1, 0)]

Some tests that the \( T \)'s are implemented properly by hand defining the \( Y \)'s in terms of them:

sage: T = E._T_Y
sage: Ye1 = T.Tw((1,2,1,0), scalar = (-1/(q1*q2))^2)
sage: Ye2 = T.Tw((2,1,0,1), signs = (1,1,1,-1), scalar = (-1/(q1*q2)))
sage: Yalpha0 = T.Tw((0,1,2,1), signs = (-1,-1,-1,-1), scalar = q^-1*(-q1*q2)^2)
sage: Yalpha1 = T.Tw((1,2,0,1,2,0), signs=(1,1,-1,1,-1,1), scalar = -1/(q1*q2))
 sage: Yalpha2 = T.Tw((2,1,0,1,2,1,0,1), signs = (1,1,1,-1,1,1,1,-1), scalar = (1/q1*q2)^2)
 sage: Ye1(KL0.one())
 q1^2/q2^2*B[(0, 0)]
 sage: Ye2(KL0.one())
 ((-q1)/q2)*B[(0, 0)]
 sage: Yalpha0(KL0.one())
 q2^2/(q*q1^2)*B[(0, 0)]
 sage: Yalpha1(KL0.one())
 ((-q1)/q2)*B[(0, 0)]
 sage: Yalpha2(KL0.one())
 q1^2/q2^2*B[(0, 0)]

Testing the $Y$'s directly:

 sage: Y = E.Y()
 sage: Y.keys()
 Coroot lattice of the Root system of type ['BC', 2, 2]
 sage: alpha = Y.keys().simple_roots()
 sage: L(alpha[0])
 -2*e[0] + e['deltacheck']
 sage: L(alpha[1])
 e[0] - e[1]
 sage: L(alpha[2])
 e[1]

 sage: Y[alpha[0]].word
 (0, 1, 2, 1)
 sage: Y[alpha[0]].signs
 (-1, -1, -1, -1)
 sage: Y[alpha[0]].scalar # mind that Sage's q is the usual q^{1/2}
 q1^2*q2^2/q
 sage: Y[alpha[0]].(KL0.one())
 q2^2/(q*q1^2)*B[(0, 0)]

 sage: Y[alpha[1]].word
 (1, 2, 0, 1, 2, 0)
 sage: Y[alpha[1]].signs
 (1, 1, -1, 1, -1, 1)
 sage: Y[alpha[1]].scalar
 1/(-q1*q2)

 sage: Y[alpha[2]].word # Bogdan says it should be the square of that; do we need...
 (2, 1, 0, 1)
 sage: Y[alpha[2]].signs
 (1, 1, 1, -1)
 sage: Y[alpha[2]].scalar
 1/(-q1*q2)

Checking the provided nonsymmetric Macdonald polynomial:
Combinatorics, Release 10.0

```python
sage: E10 = KL0.monomial(L0((1,0))) + KL0( q*(1-(-q1/q2)) / (1-q^2*(-q1/q2)^4) )
sage: E10 == E[omega[1]]
True
sage: E.eigenvalues(E10) # not checked
[q*q1^2/q2^2, q2^3/(-q^2*q1^3), q1/(-q2)]
```

Checking T0check:

```python
sage: T0check_on_basis = KL.T0_check_on_basis(q1,q2, convention="dominant")
sage: T0check_on_basis.phi # note: this is in fact a0 phi
(2, 0)
sage: T0check_on_basis.v # what to match it with?
(1,)
sage: T0check_on_basis.j # what to match it with?
2
sage: T0check_on_basis(KL0.basis().keys().zero())
((-q1^2)/q2)*B[(1, 0)]
sage: T0check = E._T[0]
sage: T0check(KL0.one())
((-q1^2)/q2)*B[(1, 0)]
```

Systematic tests of nonsymmetric Macdonald polynomials in type $A_1^{(1)}$, in the weight lattice. Each time, we specify the eigenvalues for the action of $Y_{\alpha_0}$ and $Y_{\alpha_1}$:

```python
sage: K = QQ['q','t'].fraction_field()
sage: q,t = K.gens()
sage: KL = RootSystem(['A',1,1]).weight_lattice(extended=True).algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t, -1)
sage: omega = E.keys().fundamental_weights()
sage: x = E[0*omega[1]]; x
B[0]
sage: E.eigenvalues(x)
[1/(q*t), t]
sage: x.is_one()
True
sage: x.parent()
Algebra of the Weight lattice of the Root system of type ['A', 1]
over Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field
```

(continues on next page)
Combinatorics, Release 10.0

As expected, \( e^{-\omega} \) is not an eigenvector:

```
sage: E.eigenvalues(KL.classical().monomial(-omega[1]))
Traceback (most recent call last):
...   AssertionError
```

We proceed by comparing against the examples from the appendix of \([HHL06]\) in type \( A_2^{(1)} \):

```
sage: K = QQ['q','t'].fraction_field()
sage: q,t = K.gens()
sage: KL = RootSystem(["A",2,1]).ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL, q, t, -1)
sage: L0 = E.keys()
sage: omega = L0.fundamental_weights()
sage: def EE(weight):
    return E[L0(weight)].expand(K.gens())
sage: EE([0,0,0])
1
sage: EE([1,0,0])
x0
sage: EE([0,1,0])
(t - 1)/(q^t^2 - 1)*x0 + x1
sage: EE([0,0,1])
(t - 1)/(q^t - 1)*x0 + (t - 1)/(q^t - 1)*x1 + x2
```
Combinatorics, Release 10.0

sage: EE([1, 1, 0])
x0*x1
sage: EE([1, 0, 1])
(t - 1)/(q*t^2 - 1)*x0*x1 + x0*x2
sage: EE([0, 1, 1])
(t - 1)/(q*t - 1)*x0*x1 + (t - 1)/(q*t - 1)*x0*x2 + x1*x2
sage: EE([2, 0, 0])
x0^2 + (q*t - q)/(q*t - 1)*x0*x1 + (q*t - q)/(q*t - 1)*x0*x2
sage: EE([0, 2, 0])
(q^2*t^2 - 1)*x0^2 + (q^2*t^3 - q^2*t^2 + q*t^2 - 2*q*t + q - t + 1)/(q^3*t^3 - q^2*t^2 - q*t + 1)*x0*x1 + x1^2 + (q*t^2 - 2*q*t + q)/(q^3*t^3 - q^2*t^2 - q*t + 1)*x0*x2 + (q*t - q)/(q*t - 1)*x1*x2

Systematic checks with Sage’s implementation of [HHL06]:

sage: import sage.combinat.sf.ns_macdonald as NS
sage: assert all(EE([x,y,z]) == NS.E([x,y,z])
for d in range(5) for x,y,z
in IntegerVectors(d,3))
# long time (9s)

We check that we get eigenvectors for generic \( q_1, q_2 \):

sage: K = QQ['q,q1,q2'].fraction_field()
sage: q,q1,q2 = K.gens()
sage: KL = RootSystem(['A',2,1]).ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, q1, q2)
sage: L0 = E.keys()
sage: omega = L0.fundamental_weights()
sage: E[2*omega[2]]
(-q*q1-q*q2)/(-q*q1-q2))*B[(1, 2, 1)] + ((-q*q1-q*q2)/(-q*q1-q2))*B[(2, 1, 1)] +
B[(2, 2, 0)]
sage: for d in range(4):
.....: for weight in IntegerVectors(d,3).map(list).map(L0):
.....: eigenvalues = E.eigenvalues(E[L0(weight)])
# long time (9s)

Some type \( C \) calculations:

sage: K = QQ['q','t'].fraction_field()
sage: q, t = K.gens()
sage: KL = RootSystem(['C',2,1]).ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t, -1)
sage: L0 = E.keys()
sage: omega = L0.fundamental_weights()
sage: E[0*omega[1]]
B[(0, 0)]
sage: E.eigenvalues(_)
# checked for i=0 with previous calculation
[1/(q*t^3), t, t]
sage: E[omega[1]]
B[(1, 0)]
sage: E.eigenvalues(_)
# not checked
[t, 1/(q*t^3), t]
sage: E[-omega[1]]
# consistent with before refactoring

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\[
B\left[-1, 0\right] + \left(-t+1\right)/(-q^4 + q^3 - q^2 + 1))B\left[1, 0\right] + B\left[0, -1\right] + \left(t-1\right)/(-q^4 + q^3 - q^2 + 1))B\left[0, 1\right]
\]

sage: E.eigenvalues(_)
# not checked
\[[(-1)/(-q^2), q*t, t]\]

sage: E[-omega[1]-omega[2]]
# consistent with before refactoring
((-t+1)/(-q^2*t^3+1))*B\left[1, 0\right] + B\left[0, -1\right] + (-t+1)/(-q^2*t^3+1))B\left[0, 1\right]

sage: E.eigenvalues(_)
# not checked
\[[t, q*t^3, (-1)/(-q^2*t^2)\]]

sage: E[omega[1]-omega[2]]
# consistent with before refactoring
\[-omega[1]+omega[2]\]
((-t+1)/(-q^2*t^3+1))*B\left[1, 0\right] + B\left[0, -1\right] + (-t+1)/(-q^2*t^3+1))B\left[0, 1\right]

sage: E.eigenvalues(_)
# not checked
\[[t, q*t^3, (-1)/(-q^2*t^2)\]]

sage: E[-omega[2]]
\left((-q^6*t^7+q^5*t^6-q^4*t^5-q^3*t^4+2*q^3*t^3-q^2*t^2+q*t-q)/(-q^6*t^7+q^5*t^6+q^4*t^4+q^3*t^3+\cdots\right)B\left[0, 0\right] + ((-q^3*t^2+q^3*t)/(-q^3*t^3+1))*B\left[-1, 0\right] + ((-q^3*t^3+2*q^3*t^2-q^3*t)/(-q^4*t^4+q^3*t^3+\cdots\right)B\left[-1, 1\right] + ((-q^4*t^4+q^4*t^3-q^3*t^3+2*q^3*t^2-q^3*t^3+\cdots\right)B\left[1, 0\right] + ((-q^5*t^5+q^5*t^4-q^4*t^3+2*q^4*t^2-q^4*t^2+\cdots\right)B\left[1, 1\right] + ((q^6*t^6+q^6*t^5-q^5*t^4+\cdots\right)B\left[2, 0\right] + \left(1/(q^2*t^2)\right)B\left[0, 2\right]

sage: E.eigenvalues(_)
# not checked
\[[q^3*t^3, t, (-1)/(-q^2*t^2)\]]

The following computations were calculated by hand:

\[
\text{sage: } \text{KL0} = \text{KL.classical()}
\]
\[
\text{sage: } \text{EI1} = \text{KL0.sum_of_terms([L0([1,1]), 1], [L0([0,0]), (-q^2 + q^3)/(1-q^2)])}
\]
\[
\text{sage: } \text{EI1 == E[omega[2]]}
\]
\[
\text{True}
\]
\[
\text{sage: } \text{E.I1} = \text{E.omega(1-omega[2])}
\]
\[
\text{True}
\]
\[
\text{sage: } \text{E.I1 == E[omega[2]]}
\]
\[
\text{True}
\]

Now we present an example for a twisted affine root system. The results are eigenvectors:

\[
\text{sage: } \text{KL0} = \text{KL.classical()}
\]
\[
\text{sage: } \text{EI1} = \text{KL0.sum_of_terms([L0([1,1]), 1], [L0([0,0]), (-q^2 + q^3)/(1-q^2)])}
\]
\[
\text{sage: } \text{EI1 == E[omega[2]]}
\]
\[
\text{True}
\]
\[
\text{sage: } \text{E.I1} = \text{E.omega(1-omega[2])}
\]
\[
\text{True}
\]

\[
\text{sage: } \text{E.I1 == E[omega[2]]}
\]
\[
\text{True}
\]

Now we present an example for a twisted affine root system. The results are eigenvectors:
sage: K = QQ['q','t'].fraction_field()
sage: q, t = K.gens()
sage: KL = RootSystem("C3~*").ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, t, -1)
sage: omega = E.keys().fundamental_weights()
sage: E[0*omega[1]]
B[(0, 0)]
sage: E.eigenvalues(_)
[1/(q*t^2), t, t]
sage: E[omega[1]]
((-q*t+q)/(-q*t^2+1))*B[(0, 0)] + B[(1, 0)]
sage: E.eigenvalues(_)
[q*t^2, 1/(q^2*t^3), t]
sage: E[-omega[1]]
((-q*t+q-t+1)/(-q^2*t+1))*B[(0, 0)] + B[(-1, 0)] + ((-t+1)/(-q^2*t-1))*B[(0, 1)]
sage: E.eigenvalues(_)
[(-1)/(-q^3*t^2), q^2*t, t]
B[(-1/2, 1/2)] + ((-t+1)/(-q^2*t^3+1))*B[(1/2, -1/2)] + ((-t+1)/(-q^2*t^3+1))*B[(1, 1)]
sage: E.eigenvalues(_)
[1/(-q^2*t^2), q^2*t^3, (-1)/(-q*t)]
\[-2x^2z^2/((t^2z^2q^3-1)^8(t^2z^2q^3+1)^8(t-1)^4(t+1)^4q^2t^2-3t^4q^2-5t^6q^6\]
\[-3+1)^4x^2/(t-1)^4q^2(t^2q^2+1)q^4z^4x^1/((t^2z^2q^3+1)^8(t^2z^2q^3-1)^8(t^2z^2q^3\]
\[-1)^4x^1z^2)\]

```
sage: to_SR(E[2*omega[2]]) - expected
# long time (3.5s)
0
```
Combinatorics, Release 10.0

sage: E = NonSymmetricMacdonaldPolynomials(["BC",1,2], q=q, q1=t^2,q2=-1)
sage: to_SR(E[μ]) - expected

(continues on next page)
\[ (t^q-1)^{(t+1)}(t^q)^{(t-1)}((q^7t^2+1)^{(q^7t^2+1)}-(q^7t^2-1)^{(q^7t^2-1)})(t^q)^{(t-1)}(t^q)^{(t+1)}((q^7t^2+1)^{(q^7t^2+1)}-(q^7t^2-1)^{(q^7t^2-1)}) \]

Note: Sage's \( q \) is the usual \( q^2 \):

Some eigenvectors:

```python
sage: E[0*omega[1]]
B[(0, 0)]
sage: E[omega[1]]
((-q^2*q1^3*q2-q^2*q1^2*q2^2)/(q^2*q1^4-q2^4))*B[(0, 0)] + B[(1, 0)]
sage: Eomega[1] = KL.one() * (q^2*(-q1/q2)^2*(1-(-q1/q2))) / (1-q^2*(-q1/q2)^4) + KL.
monomial(omega[1])
sage: E[omega[1]] == Eomega[1]
True
```

Checking the \( Y_s \):

Type \( BC \) dual, comparison with hand calculations by Bogdan Ion:

```python
sage: K = QQ['q,q1,q2'].fraction_field()
sage: q,q1,q2 = K.gens()
sage: ct = CartanType(["BC",2,2]).dual()
sage: E = NonSymmetricMacdonaldPolynomials(ct, q=q, q1=q1, q2=q2)
sage: KL = E.domain(); KL
Algebra of the Ambient space of the Root system of type ['B', 2] over Fraction Field of Multivariate Polynomial Ring in q, q1, q2 over Rational Field
sage: alpha = E.keys().simple_roots(); alpha
Finite family {1: (1, -1), 2: (0, 1)}
sage: omega=E.keys().fundamental_weights(); omega
Finite family {1: (1, 0), 2: (1/2, 1/2)}
sage: epsilon = E.keys().basis(); epsilon
Finite family {0: (0, 0), 1: (0, 1)}
```

```python
Note: Sage's \( q \) is the usual \( q^2 \):
```

```python
sage: E.L().null_root()
e['delta']
sage: E.L().null_coroot()
2*e['deltacheck']
```

Some eigenvectors:

```python
sage: E[0*omega[1]]
B[(0, 0)]
sage: E[omega[1]]
((-q^2*q1^3*q2-q^2*q1^2*q2^2)/(q^2*q1^4-q2^4))*B[(0, 0)] + B[(1, 0)]
sage: Eomega[1] = KL.one() * (q^2*(-q1/q2)^2*(1-(-q1/q2))) / (1-q^2*(-q1/q2)^4) + KL.
monomial(omega[1])
sage: E[omega[1]] == Eomega[1]
True
```
sage: Y = E.Y()
sage: alphacheck = Y.keys().simple_roots()
sage: Y0 = Y[alphacheck[0]]
sage: Y1 = Y[alphacheck[1]]
sage: Y2 = Y[alphacheck[2]]

sage: Y0.word, Y0.signs, Y0.scalar
((0, 1, 2, 1, 0, 1, 2, 1), (-1, -1, -1, -1, -1, -1, -1, -1), q1^4*q2^4/q^2)

sage: Y1.word, Y1.signs, Y1.scalar
((1, 2, 0, 1, 2, 0), (1, 1, -1, 1, -1, 1), 1/(-q1*q2))

sage: Y2.word, Y2.signs, Y2.scalar
((2, 1, 0, 1), (1, 1, -1, 1), 1/(-q1*q2))

sage: E.eigenvalues(0*omega[1])
[q2^4/(q^2*q1^4), q1/(-q2), q1/(-q2)]

Checking the \(T\) and \(T^{-1}\) s:

sage: T = E._T_Y
sage: Tinv0 = T.Tw_inverse([0])
sage: Tinv1 = T.Tw_inverse([1])
sage: Tinv2 = T.Tw_inverse([2])

sage: for x in [0*epsilon[0], -epsilon[0], -epsilon[1], epsilon[0], epsilon[1]]:
....:    x = KL.monomial(x)
....:    assert Tinv0(T[0](x)) == x and T[0](Tinv0(x)) == x
....:    assert Tinv1(T[1](x)) == x and T[1](Tinv1(x)) == x
....:    assert Tinv2(T[2](x)) == x and T[2](Tinv2(x)) == x

sage: start = E[omega[1]]; start
((-q^2*q1^3*q2-q^2*q1^2*q2^2)/(q^2*q1^4-q2^4))*B[(0, 0)] + B[(1, 0)]

sage: start == Y0(start)
True

sage: Y0(start) == q^2*q1^4/q2^4 * start
True

Checking the relation between the \(Y\) s:

sage: q^2 * Y0(Y1(Y2(Y2(start))))) == start
True

sage: for x in [0*epsilon[0], -epsilon[0], -epsilon[1], epsilon[0], epsilon[1]]:
....:    x = KL.monomial(x)
....:    assert q^2 * Y0(Y1(Y2(Y2(start))))) == start

KL0()

Return the group algebra where the nonsymmetric Macdonald polynomials live.

EXAMPLES:

sage: NonSymmetricMacdonaldPolynomials("B2~").KL0()
Algebra of the Ambient space of the Root system of type ['B', 2]
over Fraction Field of Multivariate Polynomial Ring in q, q1, q2 over Rational

(continues on next page)
sage: NonSymmetricMacdonaldPolynomials("B2~*").KL0()
Ambient space of the Root system of type ['C', 2] over Fraction Field of Multivariate Polynomial Ring in q, q1, q2 over Rational Field

L()

Return the affinization of the classical weight space.

EXAMPLES:

sage: NonSymmetricMacdonaldPolynomials(['B', 2, 1]).L()
Ambient space of the Root system of type ['B', 2, 1]

L0()

Return the space indexing the monomials of the nonsymmetric Macdonald polynomials.

EXAMPLES:

sage: NonSymmetricMacdonaldPolynomials("B2-").L0()
Ambient space of the Root system of type ['B', 2]
sage: NonSymmetricMacdonaldPolynomials("B2~") .L0()
Ambient space of the Root system of type ['C', 2]

L_check()

Return the other affinization of the classical weight space.

Todo: should this just return $L$ in the simply laced case?

EXAMPLES:

sage: NonSymmetricMacdonaldPolynomials(['B', 2, 1]).L_check()
Coambient space of the Root system of type ['C', 2, 1]
sage: NonSymmetricMacdonaldPolynomials(['B', 2, 1]).L_check().classical()
Ambient space of the Root system of type ['B', 2]

L_prime()

The affine space where classical weights are lifted for the recursion.
Also the parent of $\rho'$.

EXAMPLES:

In the twisted case, this is the affinization of the classical ambient space:

sage: NonSymmetricMacdonaldPolynomials("B2~") .L
Ambient space of the Root system of type ['B', 2, 1]^*
sage: NonSymmetricMacdonaldPolynomials("B2~") .L .classical()
Ambient space of the Root system of type ['C', 2]
sage: NonSymmetricMacdonaldPolynomials("B2~") .L prime()
Ambient space of the Root system of type ['B', 2, 1]^*
sage: NonSymmetricMacdonaldPolynomials("B2~") .L prime().classical()
Ambient space of the Root system of type ['C', 2]
In the untwisted case, this is the other affinization of the classical ambient space:

```sage
sage: NonSymmetricMacdonaldPolynomials("B2~").L()
Ambient space of the Root system of type ['B', 2, 1]
sage: NonSymmetricMacdonaldPolynomials("B2~").L().classical()
Ambient space of the Root system of type ['B', 2]
```

```sage
sage: NonSymmetricMacdonaldPolynomials("B2~").L_prime()
Coadjoint space of the Root system of type ['C', 2, 1]
sage: NonSymmetricMacdonaldPolynomials("B2~").L_prime().classical()
Ambient space of the Root system of type ['B', 2]
```

For simply laced, the two affinizations coincide:

```sage
sage: NonSymmetricMacdonaldPolynomials("A2~").L()
Ambient space of the Root system of type ['A', 2, 1]
sage: NonSymmetricMacdonaldPolynomials("A2~").L().classical()
Ambient space of the Root system of type ['A', 2]
```

```sage
sage: NonSymmetricMacdonaldPolynomials("A2~").L_prime()
Coadjoint space of the Root system of type ['A', 2, 1]
sage: NonSymmetricMacdonaldPolynomials("A2~").L_prime().classical()
Ambient space of the Root system of type ['A', 2]
```

**Note:** do we want the coambient space of type $A_2^{(1)}$ instead?

For type BC:

```sage
sage: NonSymmetricMacdonaldPolynomials(["BC",3,2]).L_prime()
Ambient space of the Root system of type ['BC', 3, 2]
```

**Q_to_Qcheck()**

The reindexing of the index set of the Y's by the coroot lattice.

**EXAMPLES:**

```sage
sage: E = NonSymmetricMacdonaldPolynomials("C2~")
sage: alphacheck = E.Y().keys().simple_roots()
sage: E.Q_to_Qcheck(alphacheck[0])
alphacheck[0] - alphacheck[2]
sage: E.Q_to_Qcheck(alphacheck[1])
alphacheck[1]
sage: E.Q_to_Qcheck(alphacheck[2])
alphacheck[2]
sage: x.parent()
Root lattice of the Root system of type ['B', 2, 1]
sage: E.Q_to_Qcheck(x)
sage: _.parent()
Coroot lattice of the Root system of type ['C', 2, 1]
```
Return the family of $Y$ operators whose eigenvectors are the nonsymmetric Macdonald polynomials.

**EXAMPLES:**

```python
sage: NonSymmetricMacdonaldPolynomials("C2~").Y()
Lazy family (<lambda>(i))_{i in Root lattice of the Root system of type ['B', 2, → 1]}
```

```
Root lattice of the Root system of type ['B', 2]
```

```python
sage: NonSymmetricMacdonaldPolynomials("C2~*").Y()
Lazy family (<...Y_lambdacheck...>(i))_{i in Coroot lattice of the Root system→ of type ['C', 2, 1]^*}
```

```
Root lattice of the Root system of type ['C', 2]
```

```python
sage: NonSymmetricMacdonaldPolynomials(["BC", 3, 2]).Y()
Lazy family (<...Y_lambdacheck...>(i))_{i in Coroot lattice of the Root system of type ['BC', 3, 2]}
```

```
Root lattice of the Root system of type ['B', 3]
```

**affine_lift($\mu$)**

Return the affinization of $\mu$ in $L'$.

**INPUT:**

- $\mu$ – a classical weight $\mu$

**See also:**

- `hecke_algebra_representation.CherednikOperatorsEigenvectors.affine_lift()`
- `affine_retract()`
- `L_prime()`

**EXAMPLES:**

In the untwisted case, this is the other affinization at level 1:

```python
sage: E = NonSymmetricMacdonaldPolynomials("B2~")
sage: L0 = E.keys(); L0
Ambient space of the Root system of type ['B', 2]
sage: omega = L0.fundamental_weights()
sage: E.affine_lift(omega[1])
e[0] + e['deltacheck']
sage: E.affine_lift(omega[1]).parent()
Coambient space of the Root system of type ['C', 2, 1]
```

In the twisted case, this is the usual affinization at level 1:

```python
sage: E = NonSymmetricMacdonaldPolynomials("B2~")
sage: L0 = E.keys(); L0
Ambient space of the Root system of type ['C', 2]
sage: omega = L0.fundamental_weights()
(continues on next page)"
sage: E.affine_lift(omega[1])
e[0] + e['deltacheck']
sage: E.affine_lift(omega[1]).parent()
Ambient space of the Root system of type ['B', 2, 1]^*

affine_retract(mu)
Retract the affine weight $\mu$ into a classical weight.

INPUT:
• $\mu$ – an affine weight $\mu$ in $L'$

See also:
• hecke_algebra_representation.HeckeAlgebraRepresentation.affine_retract()
• affine_lift()
• L_prime()

EXAMPLES:

sage: E = NonSymmetricMacdonaldPolynomials("B2~")
sage: L0 = E.keys(); L0
Ambient space of the Root system of type ['B', 2]
sage: omega = L0.fundamental_weights()
sage: E.affine_lift(omega[1])
e[0] + e['deltacheck']
sage: E.affine_retract(E.affine_lift(omega[1]))
(1, 0)

cartan_type()
Return Cartan type of self.

EXAMPLES:

sage: NonSymmetricMacdonaldPolynomials(['B', 2, 1]).cartan_type()
['B', 2, 1]

eigenvalue_experimental(mu, l)
Return the eigenvalue of $Y^\lambda$ acting on the macdonald polynomial $E_\mu$.

INPUT:
• $\mu$ – the index $\mu$ of an eigenvector
• $l$ – an index $\lambda$ of some $Y$

Note:
• This method is currently not used; most tests below even test the naive method. They are left here as a basis for a future implementation.
• This is actually equivariant, as long as $s_i$ does not fix $\lambda$.
• This method is only really needed for $\lambda = \alpha_i^\vee$ with $i = 0, \ldots, n$. 

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See Corollary 6.11 of \cite{Haiman06}.

EXAMPLES:

```python
sage: K = QQ['q,t'].fraction_field()
sage: q,t = K.gens()
sage: q1 = t
sage: q2 = -1
sage: KL = RootSystem(['A',1,1]).ambient_space().algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, q1, q2)
sage: L0 = E.keys()
sage: E.eigenvalues(L0([0,0])) # Checked by hand by Mark and Arun
[1/(q*t), t]
sage: alpha = E.Y().keys().simple_roots()
sage: E.eigenvalue_experimental(L0([0,0]), alpha[0]) # todo: not implemented
1/(q*t)
sage: E.eigenvalue_experimental(L0([0,0]), alpha[1])
t
Some examples of eigenvalues (not mathematically checked!!!):

```python
sage: E.eigenvalues(L0([1,0]))
[t, 1/(q*t)]
sage: E.eigenvalues(L0([0,1]))
[1/(q^2*t), q*t]
sage: E.eigenvalues(L0([1,1]))
[1/(q*t), t]
sage: E.eigenvalues(L0([2,1]))
[t, 1/(q*t)]
sage: E.eigenvalues(L0([-1,1]))
[(-1)/(-q^3*t), q^2*t]
sage: E.eigenvalues(L0([-2,1]))
[(-1)/(-q^4*t), q^3*t]
sage: E.eigenvalues(L0([-2,0]))
[(-1)/(-q^4*t), q^2*t]
```
sage: E.eigenvalue(L0((1,0)), alpha[0]) # not checked # not tested
1

sage: L = RootSystem("B2~*").ambient_space()
sage: KL = L.algebra(K)
sage: E = NonSymmetricMacdonaldPolynomials(KL,q, q1, q2)
sage: L0 = E.keys()
sage: alpha = L.simple_coroots()
sage: E.eigenvalue(L0((0,0)), alpha[0]) # assuming Mark's calculation is correct, one should get # not tested
1/(q^2*t^2)

The expected value can more or less be read off from equation (37), Corollary 6.15 of [Haiman06]

Todo:
- Use proposition 6.9 of [Haiman06] to check the action of the $Y_s$ on monomials.
- Generalize to any $q_1, q_2$.
- Check claim by Mark: all scalar products should occur in the finite weight lattice, with alpha 0 being the appropriate projection of the affine alpha 0. Question: can this be emulated by being at level 0?

rho_prime()  
Return the level 0 sum of the classical fundamental weights in $L'$.  

See also:  
L_prime()  

EXAMPLES:  
Untwisted case:

```sage
sage: NonSymmetricMacdonaldPolynomials("B2~").rho_prime() # CHECKME
3/2*e[0] + 1/2*e[1]
```

Twisted case:

```sage
sage: NonSymmetricMacdonaldPolynomials("B2~*").rho_prime().parent()
Ambient space of the Root system of type ['B', 2, 1]^*
```

seed(mu)  
Return $E_\mu$ for $\mu$ minuscule, i.e. in the fundamental alcove.  

INPUT:  
- mu – the index $\mu$ of an eigenvector  

EXAMPLES:
sage: E = NonSymmetricMacdonaldPolynomials(['A',2,1])
sage: omega = E.keys().fundamental_weights()
sage: E.seed(omega[1])
B[(1, 0, 0)]

\texttt{symmetric\_macdonald\_polynomial}(\mu)

Return the symmetric Macdonald polynomial indexed by \(\mu\).

**INPUT:**

- \(\mu\) – a dominant weight \(\mu\)

**Warning:** The result is Weyl-symmetric only for Hecke parameters of the form \(q_1 = v\) and \(q_2 = -1/v\). In general the value of \(v\) below, should be the square root of \(-q_1/q_2\), but the use of \(q_1 = t\) and \(q_2 = -1\) results in nonintegral powers of \(t\).

**EXAMPLES:**

sage: K = QQ['q,v,t'].fraction_field()
sage: q,v,t = K.gens()
sage: E = NonSymmetricMacdonaldPolynomials(['A',2,1], q, v, -1/v)
sage: om = E.L0().fundamental_weights()
sage: E.symmetric_macdonald_polynomial(om[2])
B[(1, 1, 0)] + B[(1, 0, 1)] + B[(0, 1, 1)]
sage: E.symmetric_macdonald_polynomial(2*om[1])
((q*v^2+v^2-q-1)/(q*v^2-1))*B[(1, 1, 0)] + ((q*v^2+v^2-q-1)/(q*v^2-1))*B[(1, 0, 1)] + B[(0, 1, 1)] + ((q*v^2+v^2-q-1)/(q*v^2-1))*B[(2, 0, 0)]

We compare with the type A Macdonald polynomials coming from symmetric functions:

sage: P = SymmetricFunctions(K).macdonald().P()
sage: g = P[(2,1)].expand(3); g
x0^2*x1 + x0*x1^2 + x0^2*x2 + x0*x2^2 + x1^2*x2 + x1*x2^2
sage: fe = f.expand(g.parent().gens()); fe
x0^2*x1 + x0*x1^2 + x0^2*x2 + x0*x2^2 + x1^2*x2 + x1*x2^2
sage: g.map_coefficients(lambda x: x.subs(t=v*v)) == fe
True

We continue with...

(continues on next page)
An example for type $G$:

```sage
sage: E = NonSymmetricMacdonaldPolynomials(['G',2,1], q, v, -1/v)
sage: om = E.L0().fundamental_weights()
sage: E.symmetric_macdonald_polynomial(2*om[1])
((3*q^6*v^22+3*q^5*v^22-3*q^6*v^20+q^4*v^22-4*q^5*v^20+q^4*v^18-q^5*v^16+q^3*v^18-2*q^4*v^16+q^5*v^14-15*q^4*v^12+15*q^3*v^12+q^4*v^10+4*q^3*v^10-2*q^4*v^8+2*q^3*v^8-2*q^4*v^6+2*q^3*v^6-2*q^4*v^4+2*q^3*v^4-4*q^3*v^2-2*q^3+2*q+2)/(q^4*v^12-q^3*v^10+q*v^2-1))*B[(0, 0, 0)]
+ ((q*v^2+v^2-q-1)/(q*v^2-1))*B[(-2, 1, 1)]
+ B[(-2, 2, 0)]
+ B[(-2, 0, 2)]
+ ((-2*q^4*v^12+2*q^3*v^12-2*q^4*v^10+2*q^3*v^10-q^2*v^8+q^3*v^6-q*v^8+2*q^2*v^6-q^3*v^4+q*v^6-q^2*v^4+2*q*v^2+2*v^2-2*q-2)/(-q^4*v^12+q^3*v^10+q*v^2-1))*B[(2, -2, 0)]
+ ((-2*q^4*v^12+2*q^3*v^12-2*q^4*v^10+2*q^3*v^10-q^2*v^8+q^3*v^6-q*v^8+2*q^2*v^6-q^3*v^4+q*v^6-q^2*v^4+2*q*v^2+2*v^2-2*q-2)/(-q^4*v^12+q^3*v^10+q*v^2-1))*B[(2, -1, 0)]
+ ((-q*v^2-v^2+q+1)/(-q*v^2+1))*B[(1, 1, -2)]
+ ((-2*q^4*v^12+2*q^3*v^12-2*q^4*v^10+2*q^3*v^10-q^2*v^8+q^3*v^6-q*v^8+2*q^2*v^6-q^3*v^4+q*v^6-q^2*v^4+2*q*v^2+2*v^2-2*q-2)/(-q^4*v^12+q^3*v^10+q*v^2-1))*B[(1, 0, 1)]
+ ((-q*v^2-v^2+q+1)/(-q*v^2+1))*B[(2, -1, 1)]
+ B[(0, 1, 0)]
+ ((-q*v^2-v^2+q+1)/(-q*v^2+1))*B[(0, 0, 2)]
+ ((-2*q^4*v^12+2*q^3*v^12-2*q^4*v^10+2*q^3*v^10-q^2*v^8+q^3*v^6-q*v^8+2*q^2*v^6-q^3*v^4+q*v^6-q^2*v^4+2*q*v^2+2*v^2-2*q-2)/(-q^4*v^12+q^3*v^10+q*v^2-1))*B[(2, 1, 0)]
```

\[ \text{twist}(\mu, i) \]

Act by $s_i$ on the affine weight $\mu$.

This calls `simple_reflection`; which is semantically the same as the default implementation.
EXAMPLES:

```python
sage: W = WeylGroup(['B',3])
sage: W.element_class._repr_=lambda x: ''.join(str(i) for i in x.reduced_word())
sage: K = QQ['q1,q2']
```

```python
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: T = KW.demazure_lusztig_operators(q1, q2, affine=True)
sage: E = T.Y_eigenvectors()
```

```python
sage: w = W.an_element(); w
123
```

```python
sage: E.twist(w,1)
1231
```

### 5.1.233 Pieri Factors

**class** `sage.combinat.root_system.pieri_factors.PieriFactors`

**Bases:** `UniqueRepresentation, Parent`

An abstract class for sets of Pieri factors, used for constructing Stanley symmetric functions. The set of Pieri factors for a given type can be realized as an order ideal of the Bruhat order poset generated by a certain set of maximal elements.

**See also:**

- `WeylGroups.ParentMethods.pieri_factors()`
- `WeylGroups.ElementMethods.stanley_symmetric_function()`

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',4])
```

```python
sage: PF = W.pieri_factors()
```

```python
sage: PF.an_element().reduced_word()
[4, 3, 2, 1]
```

```python
sage: Waff = WeylGroup(['A',4,1])
```

```python
sage: PFaff = Waff.pieri_factors()
```

```python
sage: Waff.from_reduced_word(PF.an_element().reduced_word()) in PFaff
True
```

```python
sage: W = WeylGroup(['B',3,1])
```

```python
sage: PF = W.pieri_factors()
```

```python
sage: W.from_reduced_word([2,3,2]) in PF.elements()
True
```

```python
sage: PF.cardinality()
47
```

```python
sage: W = WeylGroup(['C',3,1])
```

```python
sage: PF = W.pieri_factors()
```

```python
sage: PF.generating_series()
```

```python
6*z^6 + 14*z^5 + 18*z^4 + 15*z^3 + 9*z^2 + 4*z + 1
```

```python
sage: sorted(w.reduced_word() for w in PF if w.length() == 2)
[[0, 1], [1, 0], [1, 2], [2, 0], [2, 1], [2, 3], [3, 0], [3, 1], [3, 2]]
```
REFERENCES:

- [FoSta1994]
- [BH1994]
- [Lam1996]
- [Lam2008]
- [LSS2009]
- [Pon2010]

default_weight()

Return the function $i \mapsto z^i$, where $z$ is the generator of QQ['z'].

EXAMPLES:

```sage
sage: W = WeylGroup(['A', 3, 1])
sage: weight = W.pieri_factors().default_weight()
sage: weight(1)
z
sage: weight(5)
z^5
```

elements()

Return the elements of self.

Those are constructed as the elements below the maximal elements of self in Bruhat order.

OUTPUT: a RecursivelyEnumeratedSet_generic object

EXAMPLES:

```sage
sage: PF = WeylGroup(['A',3]).pieri_factors()
sage: sorted(w.reduced_word() for w in PF.elements())
[[], [1], [2], [2, 1], [3], [3, 1], [3, 2], [3, 2, 1]]
```

See also:

maximal_elements()

Todo: Possibly remove this method and instead have this class inherit from RecursivelyEnumeratedSet_generic.

generating_series(weight=None)

Return a length generating series for the elements of self.

EXAMPLES:

```sage
sage: PF = WeylGroup(['C',3,1]).pieri_factors()
sage: PF.generating_series()
6*z^6 + 14*z^5 + 18*z^4 + 15*z^3 + 9*z^2 + 4*z + 1
sage: PF = WeylGroup(['B',4]).pieri_factors()
sage: PF.generating_series()
z^7 + 6*z^6 + 14*z^5 + 18*z^4 + 15*z^3 + 9*z^2 + 4*z + 1
```
max_length()

Return the maximal length of a Pieri factor.

EXAMPLES:

In type A and A affine, this is $n$:

```
sage: WeylGroup(['A',5]).pieri_factors().max_length()
5
sage: WeylGroup(['A',5,1]).pieri_factors().max_length()
5
```

In type B and B affine, this is $2n - 1$:

```
sage: WeylGroup(['B',5,1]).pieri_factors().max_length()
9
sage: WeylGroup(['B',5]).pieri_factors().max_length()
9
```

In type C affine this is $2n$:

```
sage: WeylGroup(['C',5,1]).pieri_factors().max_length()
10
```

In type D affine this is $2n - 2$:

```
sage: WeylGroup(['D',5,1]).pieri_factors().max_length()
8
```

class sage.combinat.root_system.pieri_factors.PieriFactors_affine_type

Bases: PieriFactors

maximal_elements()

Return the maximal elements of self with respect to Bruhat order.

The current implementation is via a conjectural type-free formula. Use maximal_elements_combinatorial() for proven type-specific implementations. To compare type-free and type-specific (combinatorial) implementations, use method _test_maximal_elements().

EXAMPLES:

```
sage: W = WeylGroup(['A',4,1])
sage: PF = W.pieri_factors()
sage: sorted([w.reduced_word() for w in PF.maximal_elements()], key=str)
[[0, 4, 3, 2], [1, 0, 4, 3], [2, 1, 0, 4], [3, 2, 1, 0], [4, 3, 2, 1]]

sage: W = WeylGroup(RootSystem(['C',3,1]).weight_space())
sage: PF = W.pieri_factors()
sage: sorted([w.reduced_word() for w in PF.maximal_elements()], key=str)
[[0, 1, 2, 3, 2, 1], [1, 0, 1, 2, 3, 2], [1, 2, 3, 2, 1, 0],
 [2, 1, 0, 1, 2, 3], [2, 3, 2, 1, 0, 1], [3, 2, 1, 0, 1, 2]]

sage: W = WeylGroup(RootSystem(['B',3,1]).weight_space())
sage: PF = W.pieri_factors()
sage: sorted([w.reduced_word() for w in PF.maximal_elements()], key=str)
[[0, 2, 3, 2, 0], [1, 0, 2, 3, 2], [1, 2, 3, 2, 1],
 [2, 1, 0, 2, 3, 2], [2, 3, 2, 1, 0, 2], [3, 2, 1, 0, 2, 3]]
```
Combining factors and affine types

\[2, 1, 0, 2, 3, 2, 3, 2, 1, 0, 2, 3\]

sage: W = WeylGroup(['D', 4, 1])
sage: PF = W.pieri_factors()
sage: sorted([w.reduced_word() for w in PF.maximal_elements()], key=str)

\[[0, 2, 4, 3, 2, 0], [1, 0, 2, 4, 3, 2], [1, 2, 4, 3, 2, 1],
[2, 1, 0, 2, 4, 3], [2, 4, 3, 2, 1, 0], [3, 2, 1, 0, 2, 3],
[4, 2, 1, 0, 2, 4], [4, 3, 2, 1, 0, 2]\]

class sage.combinat.root_system.pieri_factors.PieriFactors_finite_type

Bases: PieriFactors

The Pieri factors of finite type A are the restriction of the Pieri factors of affine type A to finite permutations (under the canonical embedding of finite type A into the affine Weyl group), and the Pieri factors of finite type B are the restriction of the Pieri factors of affine type C. The finite type D Pieri factors are (weakly) conjectured to be the restriction of the Pieri factors of affine type D.

maximal_elements()

The current algorithm uses the fact that the maximal Pieri factors of affine type A, B, C, or D either contain a finite Weyl group element, or contain an affine Weyl group element whose reflection by \(s_0\) gets a finite Weyl group element, and that either of these finite group elements will serve as a maximal element for finite Pieri factors. A better algorithm is desirable.

EXAMPLES:

sage: PF = WeylGroup(['A', 5]).pieri_factors()
sage: [v.reduced_word() for v in PF.maximal_elements()]
[[5, 4, 3, 2, 1]]

sage: WeylGroup(['B', 4]).pieri_factors().maximal_elements()

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

class sage.combinat.root_system.pieri_factors.PieriFactors_type_A(W)

Bases: PieriFactors_finite_type

The set of Pieri factors for finite type A.

This is the set of elements of the Weyl group that have a reduced word that is strictly decreasing. This may also be viewed as the restriction of affine type A Pieri factors to finite Weyl group elements.

maximal_elements_combinatorial()

Return the maximal Pieri factors, using the type A combinatorial description.

EXAMPLES:

sage: W = WeylGroup(['A', 4])
sage: PF = W.pieri_factors()
sage: PF.maximal_elements_combinatorial()[0].reduced_word()

\[4, 3, 2, 1\]
stanley_symm_poly_weight(w)

EXAMPLES:

```python
sage: W = WeylGroup(['A',4])
sage: PF = W.pieri_factors()
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([3,1]))
0
```

class sage.combinat.root_system.pieri_factors.PieriFactors_type_A_affine(W, min_length, max_length, min_support, max_support)

Bases: PieriFactors_affine_type

The set of Pieri factors for type A affine, that is the set of elements of the Weyl Group which are cyclically decreasing.

Those are used for constructing (affine) Stanley symmetric functions.

The Pieri factors are in bijection with the proper subsets of the index_set. The bijection is given by the support. Namely, let \( f \) be a Pieri factor, and \( \text{red} \) a reduced word for \( f \). No simple reflection appears twice in red, and the support \( S \) of \( \text{red} \) (that is the \( i \) such that \( s_i \) appears in \( \text{red} \)) does not depend on the reduced word.

cardinality()

Return the cardinality of self.

EXAMPLES:

```python
sage: WeylGroup(['A', 3, 1]).pieri_factors().cardinality()
15
```

generating_series(weight=None)

Return a length generating series for the elements of self.

EXAMPLES:

```python
sage: W = WeylGroup(['A', 3, 1])
sage: W.pieri_factors().cardinality()
15
sage: W.pieri_factors().generating_series()
4*z^3 + 6*z^2 + 4*z + 1
```

maximal_elements_combinatorial()

Return the maximal Pieri factors, using the affine type A combinatorial description.

EXAMPLES:

```python
sage: W = WeylGroup(['A',4,1])
sage: PF = W.pieri_factors()
sage: [w.reduced_word() for w in PF.maximal_elements_combinatorial()]
[[3, 2, 1, 0], [2, 1, 0, 4], [1, 0, 4, 3], [0, 4, 3, 2], [4, 3, 2, 1]]
```

stanley_symm_poly_weight(w)

Weight used in computing (affine) Stanley symmetric polynomials for affine type A.

EXAMPLES:
sage: W = WeylGroup(['A',5,1])
sage: PF = W.pieri_factors()
sage: PF.stanley_symm_poly_weight(W.one())
0
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([5,4,2,1,0]))
0

subset

Return the subset of the elements of self of length length.

INPUT:
• length – a non-negative integer

EXAMPLES:

sage: PF = WeylGroup(['A', 3, 1]).pieri_factors(); PF
Pieri factors for Weyl Group of type ['A', 3, 1] (as a matrix group acting on...
˓→the root space)
sage: PF3 = PF.subset(length = 2)
sage: PF3.cardinality()
6

class sage.combinat.root_system.pieri_factors.PieriFactors_type_B(W)

Bases: PieriFactors_finite_type

The type B finite Pieri factors are realized as the set of elements that have a reduced word that is a subword of 12...n-1n(n-1)...21. They are the restriction of the type C affine Pieri factors to the set of finite Weyl group elements under the usual embedding.

maximal_elements_combinatorial()

Return the maximal Pieri factors, using the type B combinatorial description.

EXAMPLES:

sage: PF = WeylGroup(['B',4]).pieri_factors()
sage: PF.maximal_elements_combinatorial()[0].reduced_word()
[1, 2, 3, 4, 3, 2, 1]

stanley_symm_poly_weight(w)

Weight used in computing Stanley symmetric polynomials of type B.

The weight for finite type B is the number of components of the support of an element minus the number of occurrences of n in a reduced word.

EXAMPLES:

sage: W = WeylGroup(['B',5])
sage: PF = W.pieri_factors()
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([3,1,5]))
2
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([3,4,5]))
0
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([1,2,3,4,5,4]))
0
class sage.combinat.root_system.pieri_factors.PieriFactors_type_B_affine(W):
    Bases: PieriFactors_affine_type

    The type B affine Pieri factors are realized as the order ideal (in Bruhat order) generated by the following elements:
    
    - cyclic rotations of the element with reduced word 234...\(n-1\)\(n-1\)...320, except for 123...n...320 and 023...n...321.
    - 123...\(n-1\)\(n-1\)...321
    - 023...\(n-1\)\(n-1\)...320

    EXAMPLES:

    ::

        sage: W = WeylGroup(['B',4,1])
        sage: PF = W.pieri_factors()
        sage: W.from_reduced_word([2,3,4,3,2,1,0]) in PF.maximal_elements()
        True
        sage: W.from_reduced_word([0,2,3,4,3,2,1]) in PF.maximal_elements()
        False
        sage: W.from_reduced_word([1,0,2,3,4,3,2]) in PF.maximal_elements()
        True
        sage: W.from_reduced_word([0,2,3,4,3,2,0]) in PF.maximal_elements()
        True
        sage: W.from_reduced_word([0,2,0]) in PF
        True

    maximal_elements_combinatorial()

    Return the maximal Pieri factors, using the affine type B combinatorial description.

    EXAMPLES:

    ::

        sage: W = WeylGroup(['B',4,1])
        sage: [u.reduced_word() for u in W.pieri_factors().maximal_elements_combinatorial()]
        [[1, 0, 2, 3, 4, 3, 2], [2, 1, 0, 2, 3, 4, 3], [3, 2, 1, 0, 2, 3, 4], [4, 3, 2, 1, 0, 2, 3, 4], [3, 4, 3, 2, 1, 0, 2, 3], [2, 3, 4, 3, 2, 1, 0], [1, 0, 2, 3, 4, 3, 2, 1], [0, 2, 3, 4, 3, 2, 1, 0]]

    stanley_symm_poly_weight(w)

    Return the weight of a Pieri factor to be used in the definition of Stanley symmetric functions.

    For type B, this weight involves the number of components of the complement of the support of an element, where we consider 0 and 1 to be one node – if 1 is in the support, then we pretend 0 in the support, and vice versa. We also consider 0 and 1 to be one node for the purpose of counting components of the complement (as if the Dynkin diagram were that of type C). Let \(n\) be the rank of the affine Weyl group in question (if type ['B',k,1] then we have \(n = k+1\)). Let \(\chi(v.\text{length}() < n-1)\) be the indicator function that is 1 if the length of \(v\) is smaller than \(n-1\), and 0 if the length of \(v\) is greater than or equal to \(n-1\). If we call \(c'(v)\) the number of components of the complement of the support of \(v\), then the type B weight is given by \(\text{weight} = c'(v) - \chi(v.\text{length}() < n-1)\).

    EXAMPLES:

    ::

        sage: W = WeylGroup(['B',5,1])
        sage: PF = W.pieri_factors()
        sage: PF.stanley_symm_poly_weight(W.from_reduced_word([0,3]))
        1

    (continues on next page)
class sage.combinat.root_system.pieri_factors.PieriFactors_type_C_affine(W)

Bases: PieriFactors_affine_type

The type C affine Pieri factors are realized as the order ideal (in Bruhat order) generated by cyclic rotations of the element with unique reduced word 123...(n − 1)n(n − 1)...3210.

EXAMPLES:

sage: W = WeylGroup(['C',3,1])
sage: PF = W.pieri_factors()
sage: sorted([u.reduced_word() for u in PF.maximal_elements()], key=str)
[[0, 1, 2, 3, 2, 1], [1, 0, 1, 2, 3, 2], [1, 2, 3, 2, 1, 0],
 [2, 1, 0, 1, 2, 3], [2, 3, 2, 1, 0, 1], [3, 2, 1, 0, 1, 2]]

maximal_elements_combinatorial()

Return the maximal Pieri factors, using the affine type C combinatorial description.

EXAMPLES:

sage: PF = WeylGroup(['C',3,1]).pieri_factors()
sage: [w.reduced_word() for w in PF.maximal_elements_combinatorial()]
[[0, 1, 2, 3, 2, 1], [1, 0, 1, 2, 3, 2], [2, 1, 0, 1, 2, 3], [3, 2, 1, 0, 1, 2],
 [2, 3, 2, 1, 0, 1], [1, 2, 3, 2, 1, 0]]

stanley_symm_poly_weight(w)

Return the weight of a Pieri factor to be used in the definition of Stanley symmetric functions.

For type C, this weight is the number of connected components of the support (the indices appearing in a reduced word) of an element.

EXAMPLES:

sage: W = WeylGroup(['C',5,1])
sage: PF = W.pieri_factors()
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([1,3]))
2
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([1,3,2,0]))
1
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([5,3,0]))
3
sage: PF.stanley_symm_poly_weight(W.one())
0
class sage.combinat.root_system.pieri_factors.PieriFactors_type_D_affine(W)

Bases: PieriFactors_affine_type

The type D affine Pieri factors are realized as the order ideal (in Bruhat order) generated by the following elements:

- cyclic rotations of the element with reduced word $234...((n-2)n(n-1)(n-2)...3210$ such that 1 and 0 are always adjacent and $(n-1)$ and $n$ are always adjacent.
- $123...((n-2)n(n-1)(n-2)...321$
- $023...((n-2)n(n-1)(n-2)...320$
- $n(n-2)...2102...(n-2)n$
- $(n-1)(n-2)...2102...(n-2)(n-1)$

EXAMPLES:

```python
sage: W = WeylGroup(['D', 5, 1])
sage: PF = W.pieri_factors()
sage: W.from_reduced_word([3, 2, 1, 0]) in PF
True
sage: W.from_reduced_word([0, 3, 2, 1]) in PF
False
sage: W.from_reduced_word([0, 1, 3, 2]) in PF
True
sage: W.from_reduced_word([2, 0, 1, 3]) in PF
True
sage: sorted([u.reduced_word() for u in PF.maximal_elements()], key=str)
[[0, 2, 3, 5, 4, 3, 2, 0], [1, 0, 2, 3, 5, 4, 3, 2], [1, 2, 3, 5, 4, 3, 2, 1],
 [2, 1, 0, 2, 3, 5, 4, 3], [2, 3, 5, 4, 3, 2, 1, 0], [3, 2, 1, 0, 2, 3, 5, 4],
 [3, 5, 4, 3, 2, 1, 0, 2], [4, 3, 2, 1, 0, 2, 3, 4], [5, 3, 2, 1, 0, 2, 3, 5],
 [5, 4, 3, 2, 1, 0, 2, 3]]
```

maximal_elements_combinatorial()

Return the maximal Pieri factors, using the affine type D combinatorial description.

EXAMPLES:

```python
sage: W = WeylGroup(['D', 5, 1])
sage: PF = W.pieri_factors()
sage: set(PF.maximal_elements_combinatorial()) == set(PF.maximal_elements())
True
```

stanley_symm_poly_weight(w)

Return the weight of $w$, to be used in the definition of Stanley symmetric functions.

INPUT:

- $w$ – a Pieri factor for this type

For type $D$, this weight involves the number of components of the complement of the support of an element, where we consider 0 and 1 to be one node – if 1 is in the support, then we pretend 0 in the support, and vice versa. Similarly with $n-1$ and $n$. We also consider 0 and 1, $n-1$ and $n$ to be one node for the purpose of counting components of the complement (as if the Dynkin diagram were that of type $C$).

Type D Stanley symmetric polynomial weights are still conjectural. The given weight comes from conditions on elements of the affine Fomin-Stanley subalgebra, but work is needed to show this weight is correct for affine Stanley symmetric functions – see [LSS2009, Pon2010] for details.
EXAMPLES:

```python
sage: W = WeylGroup(['D', 5, 1])
sage: PF = W.pieri_factors()
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([5,2,1]))
0
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([5,2,1,0]))
0
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([5,2]))
1
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([]))
0
sage: W = WeylGroup(['D',7,1])
sage: PF = W.pieri_factors()
sage: PF.stanley_symm_poly_weight(W.from_reduced_word([2,4,6]))
2
```

5.1.234 Tutorial: visualizing root systems

Root systems encode the positions of collections of hyperplanes in space, and form the fundamental combinatorial data underlying Coxeter and Weyl groups, Lie algebras and groups, etc. The theory can be a bit intimidating at first because of the many technical gadgets (roots, coroots, weights, ...). Visualizing them goes a long way toward building a geometric intuition.

This tutorial starts from simple plots and guides you all the way to advanced plots with your own combinatorial data drawn on top of it.

See also:
- **Root systems** – An overview of root systems in Sage
- **RootLatticeRealizations.ParentMethods.plot()** – the main plotting function, with pointers to all the subroutines

First plots

In this first plot, we draw the root system for type $A_2$ in the ambient space. It is generated from two hyperplanes at a 120 degree angle:

```python
sage: L = RootSystem(['A',2]).ambient_space()
sage: L.plot()
Graphics object consisting of 13 graphics primitives
```

Each of those hyperplane $H_{\alpha_i}$ is described by a linear form $\alpha_i^\vee$ called simple coroot. To each such hyperplane corresponds a reflection along a vector called root. In this picture, the reflections are orthogonal and the two simple roots $\alpha_1$ and $\alpha_2$ are vectors which are normal to the reflection hyperplanes. The same color code is used uniformly: blue for 1, red for 2, green for 3, ... (see `CartanType.color()`). The fundamental weights, $\Lambda_1$ and $\Lambda_2$ form the dual basis of the coroots.

The two reflections generate a group of order six which is nothing but the usual symmetric group $S_3$, in its natural action by permutations of the coordinates of the ambient space. Wait, but the ambient space should be of dimension 3 then? That’s perfectly right. Here is the full picture in 3D:
However in this space, the line \((1, 1, 1)\) is fixed by the action of the group. Therefore, the so called barycentric projection orthogonal to \((1, 1, 1)\) gives a convenient 2D picture which contains all the essential information. The same projection is used by default in type \(G_2\):

```
sage: L = RootSystem(['G',2]).ambient_space()
sage: L.plot(reflection_hyperplanes='all')
```
Graphics object consisting of 21 graphics primitives

The group is now the dihedral group of order 12, generated by the two reflections \(s_1\) and \(s_2\). The picture displays the hyperplanes for all 12 reflections of the group. Those reflections delimit 12 chambers which are in one to one correspondence with the elements of the group. The fundamental chamber, which is grayed out, is associated with the identity of the group.

**Warning:** The fundamental chamber is currently plotted as the cone generated by the fundamental weights. As can be seen on the previous 3D picture this is not quite correct if the fundamental weights do not span the space.

Another caveat is that some plotting features may require manipulating elements with rational coordinates which will fail if one is working in, say, the weight lattice. It is therefore recommended to use the root, weight, or ambient
spaces for plotting purposes rather than their lattice counterparts.

Coming back to the symmetric group, here is the picture in the weight space, with all roots and all reflection hyperplanes; remark that, unlike in the ambient space, a root is not necessarily orthogonal to its corresponding reflection hyperplane:

```
sage: L = RootSystem("A",2).weight_space()
sage: L.plot(roots="all", reflection_hyperplanes="all").show(figsize=15)
```

**Note:** Setting a larger figure size as above can help reduce the overlap between the text labels when the figure gets crowded.

One can further customize which roots to display, as in the following example showing the positive roots in the weight space for type ['G',2], labelled by their coordinates in the root lattice:

```
sage: Q = RootSystem(['G',2]).root_space()
sage: L = RootSystem(['G',2]).ambient_space()
sage: L.plot(roots=list(Q.positive_roots()), fundamental_weights=False)
```

```
Graphics object consisting of 17 graphics primitives
```

One can also customize the projection by specifying a function. Here, we display all the roots for type $E_8$ using the
projection from its eight dimensional ambient space onto 3D described on Wikipedia’s E8 3D picture:

```
sage: M = matrix([[0., -0.556793440452, 0.19694925177, -0.19694925177, 0.0805477263944, -0.385290876171, 0., 0.385290876171], 
               ....: [0.180913155536, 0., 0.160212955043, 0.160212955043, 0., 0.0990170516545, 0.766360424875, 0.0990170516545], 
               ....: [0.338261212718, 0, 0, -0.338261212718, 0.672816364803, 0.171502564281, 0, -0.171502564281]])
sage: L = RootSystem(["E",8]).ambient_space()
sage: L.plot(roots="all", reflection_hyperplanes=False, projection=lambda v: M*vector(v), labels=False) # long time
```

The projection function should be linear or affine, and return a vector with rational coordinates. The rationale for the later constraint is to allow for using the PPL exact library for manipulating polytopes. Indeed exact calculations give cleaner pictures (adjacent objects, intersection with the bounding box, ...). Besides the interface to PPL is indeed currently faster than that for CDD, and it is likely to become even more so.
Exercise

Draw all finite root systems in 2D, using the canonical projection onto their Coxeter plane. See Stembridge’s page.

Alcoves and chambers

We now draw the root system for type $G_2$, with its alcoves (in finite type, those really are the chambers) and the corresponding elements of the Weyl group. We enlarge a bit the bounding box to make sure everything fits in the picture:

```
sage: RootSystem(["G",2]).ambient_space().plot(alcoves=True, alcove_labels=True, bounding_box=5)
```

The same picture in 3D, for type $B_3$:

```
sage: RootSystem(["B",3]).ambient_space().plot(alcoves=True, alcove_labels=True)
```
Exercise

Can you spot the fundamental chamber? The fundamental weights? The simple roots? The longest element of the Weyl group?

Alcove pictures for affine types

We now draw the usual alcove picture for affine type $A_2^{(1)}$:

```
sage: L = RootSystem(["A",2,1]).ambient_space()
sage: L.plot()  # long time
```

Graphics object consisting of 160 graphics primitives

This picture is convenient because it is low dimensional and contains most of the relevant information. Beside, by choosing the ambient space, the elements of the Weyl group act as orthogonal affine maps. In particular, reflections are usual (affine) orthogonal reflections. However this is in fact only a slice of the real picture: the Weyl group actually acts by linear maps on the full ambient space. Those maps stabilize the so-called level $l$ hyperplanes, and we are visualizing here what’s happening at level 1. Here is the full picture in 3D:

```
sage: L.plot(bounding_box=[[-3,3],[-3,3],[-1,1]], affine=False)  # long time
```

Graphics3d Object
In fact, in type $A$, this really is a picture in 4D, but as usual the barycentric projection kills the boring extra dimension for us. It's usually more readable to only draw the intersection of the reflection hyperplanes with the level 1 hyperplane:

```
sage: L.plot(affine=False, level=1)  # long time
```

Such 3D pictures are useful to better understand technicalities, like the fact that the fundamental weights do not necessarily all live at level 1:

```
sage: L = RootSystem(["G",2,1]).ambient_space()
sage: L.plot(affine=False, level=1)
```

Note: Such pictures may tend to be a bit flat, and it may be helpful to play with the aspect_ratio and more generally with the various options of the `show()` method:

```
sage: p = L.plot(affine=False, level=1)
sage: p.show(aspect_ratio=[1,1,2], frame=False)
```
Exercise

Draw the alcove picture at level 1, and compare the position of the fundamental weights and the vertices of the fundamental alcove.

As for finite root systems, the alcoves are indexed by the elements of the Weyl group $W$. Two alcoves indexed by $u$ and $v$ respectively share a wall if $u$ and $v$ are neighbors in the right Cayley graph: $u = vs_i$; the color of that wall is given by $i$:

```
sage: L = RootSystem(['C', 2, 1]).ambient_space()
sage: L.plot(coroots='simple', alcove_labels=True)  # long time
```

Graphics object consisting of 216 graphics primitives

Even 2D pictures of the rank $1 + 1$ cases can give some food for thought. Here, we draw the root lattice, with the positive roots of small height in the root poset:

```
sage: L = RootSystem(['A', 1, 1]).root_lattice()
sage: seed = L.simple_roots()
sage: succ = attrcall("pred")
sage: positive_roots = RecursivelyEnumeratedSet(seed, succ, structure='graded')
sage: it = iter(positive_roots)
sage: first_positive_roots = [next(it) for i in range(10)]
sage: L.plot(roots=first_positive_roots, affine=False, alcoves=False)
```

Graphics object consisting of 24 graphics primitives

Exercises

1. Use the same trick to draw the reflection hyperplanes in the weight lattice for the coroots of small height. Add the indexing of the alcoves by elements of the Weyl group. See below for a solution.

2. Draw the positive roots in the weight lattice and in the extended weight lattice.

3. Draw the reflection hyperplanes in the root lattice

4. Recreate John Stembridge's "Sandwich" arrangement pictures by choosing appropriate coroots for the reflection hyperplanes.

Here is a polished solution for the first exercise:
sage: L = RootSystem(["A",1,1]).weight_space()
sage: seed = L.simple_coroots()
sage: succ = attrcall("pred")
sage: positive_coroots = RecursivelyEnumeratedSet(seed, succ, structure='graded')
sage: it = iter(positive_coroots)
sage: first_positive_coroots = [next(it) for i in range(20)]
sage: p = L.plot(fundamental_chamber=True, reflection_hyperplanes=first_positive_coroots,
....:            affine=False, alcove_labels=1,
....:            bounding_box=[[-9,9],[-1,1]],
....:            projection=lambda x: matrix([[1,-1],[1,1]])*vector(x))
sage: p.show(figsize=20)  # long time

Higher dimension affine pictures

We now do some plots for rank 4 affine types, at level 1. The space is tiled by the alcoves, each of which is a 3D simplex:

sage: L = RootSystem(["A",3,1]).ambient_space()
sage: L.plot(reflection_hyperplanes=False, bounding_box=85/100)  # long time
Graphics3d Object
It is recommended to use a small bounding box here, for otherwise the number of simplices grows quicker than what Sage can handle smoothly. It can help to specify explicitly which alcoves to visualize. Here is the fundamental alcove, specified by an element of the Weyl group:

```python
sage: W = L.weyl_group()
sage: L.plot(reflection_hyperplanes=False, alcoves=[W.one()], bounding_box=2)
Graphics3d Object
```

and the fundamental polygon, specified by the coordinates of its center in the root lattice:

```python
sage: W = L.weyl_group()
sage: L.plot(reflection_hyperplanes=False, alcoves=[[0,0]], bounding_box=2)
Graphics3d Object
```

Finally, we draw the alcoves in the classical fundamental chambers, using that those are indexed by the elements of the Weyl group having no other left descent than 0. In order to see the inner structure, we only draw the wireframe of the facets of the alcoves. Specifying the wireframe option requires a more flexible syntax for plots which will be explained later on in this tutorial:

```python
sage: L = RootSystem(["B",3,1]).ambient_space()
sage: W = L.weyl_group()
sage: alcoves = [-w for d in range(12) for w in W.affine_grassmannian_elements_of_given_length(d)]
```

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```python
sage: p = L.plot_fundamental_chamber("classical")
sage: p += L.plot_alcoves(alcoves=alcoves, wireframe=True)
sage: p += L.plot_fundamental_weights()
sage: p.show(frame=False)
```

Exercises

1. Draw the fundamental alcove in the ambient space, just by itself (no reflection hyperplane, root, ...). The automorphism group of the Dynkin diagram for $A^{(1)}_3$ (a cycle of length 4) is the dihedral group. Visualize the corresponding symmetries of the fundamental alcove.

2. Draw the fundamental alcoves for the other rank 4 affine types, and recover the automorphism groups of their Dynkin diagram from the pictures.

Drawing on top of a root system plot

The root system plots have been designed to be used as wallpaper on top of which to draw more information. In the following example, we draw an alcove walk, specified by a word of indices of simple reflections, on top of the weight lattice in affine type $A_{2,1}$:

```python
sage: L = RootSystem(["A",2,1]).ambient_space()
sage: w1 = [0,2,1,2,0,2,1,0,2,1,2,1,2,0,2,0,1,2,0]
sage: L.plot(alcove_walk=w1, bounding_box=6) # long time
Graphics object consisting of 535 graphics primitives
```

Now, what about drawing several alcove walks, and specifying some colors? A single do-it-all plot method would be cumbersome; so instead, it is actually built on top of many methods (see the list below) that can be called independently and combined at will:

```python
sage: L.plot_roots() + L.plot_reflection_hyperplanes()
Graphics object consisting of 12 graphics primitives
```

**Note:** By default the axes are disabled in root system plots since they tend to pollute the picture. Annoyingly they come back when combining them. Here is a workaround:

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In order to specify common information for all the pieces of a root system plot (choice of projection, bounding box, color code for the index set, ...), the easiest is to create an option object using `plot_parse_options()`, and pass it down to each piece. We use this to plot our two walks:

```python
sage: plot_options = L.plot_parse_options(bounding_box=[[-2,5],[-2,6]])
sage: w2 = [2,1,2,0,2,0,1,2,1,2,0,1,2,0,2,0,2,0,1,2,0,2,0,2]
sage: p = L.plot_alcoves(plot_options=plot_options) # long time
sage: p += L.plot_alcove_walk(w1, color="green", plot_options=plot_options) # long time
sage: p += L.plot_alcove_walk(w2, color="orange", plot_options=plot_options) # long time
sage: p # long time
Graphics object consisting of ... graphics primitives
```

And another with some foldings:

```python
sage: p += L.plot_alcove_walk([0,1,2,0,2,0,1,2,0,1],
....: foldings=[False, False, True, False, False, True, False, False, False, False, True, False, False, True, False, False, False, False, False, False, False],
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```

5.1. Comprehensive Module List
Here we show a weight at level 0 and the reduced word implementing the translation by this weight:

```python
 sage: L = RootSystem(['A',2,1]).ambient_space()
 sage: P = RootSystem(['A',2,1]).weight_space(extended=True)
 sage: Lambda = P.fundamental_weights()
 sage: walk = L.reduced_word_of_translation(L(t))
 sage: plot_options = L.plot_parse_options(bounding_box=[[-2,5],[-2,5]])
 sage: p = L.plot(plot_options=plot_options) # long time
 sage: p += L.plot_alcove_walk(walk, color="green", plot_options=plot_options) # long time
 sage: p += plot_options.family_of_vectors({t: L(t)}) # long time
 sage: plot_options.finalize(p) # long time
 Graphics object consisting of ... graphics primitives
```

Note that the coloring of the translated alcove does not match with that of the fundamental alcove: the translation
actually lives in the extended Weyl group and is the composition of the simple reflections indexed by the alcove walk together with a rotation implementing an automorphism of the Dynkin diagram.

We conclude with a rank $3+1$ alcove walk:

```
sage: L = RootSystem(["B",3,1]).ambient_space()
sage: w3 = [0,2,1,3,2,0,2,1,0,2,3,1,2,1,3,2,0,2,0,1,2,0]
sage: L.plot_fundamental_weights() + L.plot_reflection_hyperplanes(bounding_box=2) + L.plot_alcove_walk(w3)
Graphics3d Object
```

Exercise

1. Draw the tiling of 3D space by the fundamental polygons for types A,B,C,D. Hints: use the wireframe option of `RootLatticeRealizations.ParentMethods.plot_alcoves()` and the color option of `plot()` to only draw the alcove facets indexed by 0.

Solution

```
sage: L = RootSystem(["A",3,1]).ambient_space()
sage: alcoves = cartesian_product([[0,1],[0,1],[0,1]])
sage: color = lambda i: "black" if i==0 else None
sage: L.plot_alcoves(alcoves=alcoves, color=color, bounding_box=10,wireframe=True).
```

# long time
Hand drawing on top of a root system plot (aka Coxeter graph paper)

Taken from John Stembridge’s excellent data archive:

“If you’ve ever worked with affine reflection groups, you’ve probably wasted lots of time drawing the reflecting hyper-planes of the rank 2 groups on scraps of paper. You may also have wished you had pads of graph paper with these lines drawn in for you. If so, you’ve come to the right place. Behold! Coxeter graph paper!”

Now you can create your own customized color Coxeter graph paper:

```
sage: L = RootSystem(["C", 2, 1]).ambient_space()
sage: p = L.plot(bounding_box=[[-8, 9], [-5, 7]], coroots="simple")  # long time (10 s)
sage: p
# long time
Graphics object consisting of ... graphics primitives
```

By default Sage’s plot are bitmap pictures which would come out ugly if printed on paper. Instead, we recommend saving the picture in postscript or svg before printing it:

```
sage: p.save("C21paper.eps")  # not tested
```

**Note:** Drawing pictures with a large number of alcoves is currently somewhat ridiculously slow. This is due to the use of generic code that works uniformly in all dimension rather than taylor-made code for 2D. Things should improve
with the fast interface to the PPL library (see e.g. github issue #12553).

**Drawing custom objects on top of a root system plot**

So far so good. Now, what if one wants to draw, on top of a root system plot, some object for which there is no preexisting plot method? Again, the `plot_options` object come in handy, as it can be used to compute appropriate coordinates. Here we draw the permutohedron, that is the Cayley graph of the symmetric group \( W \), by positioning each element \( w \) at \( w(\rho) \), where \( \rho \) is in the fundamental alcove:

```python
sage: L = RootSystem(["A",2]).ambient_space()
sage: rho = L.rho()
sage: plot_options = L.plot_parse_options()
sage: W = L.weyl_group()
sage: g = W.cayley_graph(side="right")
sage: positions = {w: plot_options.projection(w.action(rho)) for w in W}
sage: p = L.plot_roots()
sage: p += g.plot(pos = positions, vertex_size=0, color_by_label=plot_options.color)
sage: p.axes(False)
sage: p
```

Graphics object consisting of 30 graphics primitives

---

**Todo:** Could we have nice \LaTeX{} labels in this graph?

The same picture for \( A_3 \) gives a nice 3D permutohedron:

```python
sage: L = RootSystem(["A",3]).ambient_space()
sage: rho = L.rho()
sage: plot_options = L.plot_parse_options()
sage: W = L.weyl_group()
sage: g = W.cayley_graph(side="right")
sage: positions = {w: plot_options.projection(w.action(rho)) for w in W}
sage: p = L.plot_roots()
sage: p += g.plot3d(pos3d=positions, color_by_label=plot_options.color)
```

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Exercises

1. Locate the identity element of $W$ in the previous picture.
2. Rotate the picture appropriately to highlight the various symmetries of the permutohedron.
3. Make a function out of the previous example, and explore the Cayley graphs of all rank 2 and 3 Weyl groups.
4. Draw the root poset for type $B_2$ and $B_3$.
5. Draw the root poset for type $E_8$ to recover the picture from Wikipedia article [File:E8_3D.png](File:E8_3D.png).

Similarly, we display a crystal graph by positioning each element according to its weight:

```python
sage: C = crystals.Tableaux("A",2, shape=[4,2])
sage: L = C.weight_lattice_realization()
sage: plot_options = L.plot_parse_options()
sage: g = C.digraph()
sage: positions = {x: plot_options.projection(x.weight()) for x in C}
sage: p = L.plot()
sage: p += g.plot(pos=positions,
....: color_by_label=plot_options.color, vertex_size=0)
sage: p.axes(False)
sage: p.show(figsize=15)
```

Note: In the above picture, many pairs of tableaux have the same weight and are thus superposed (look for example near the center). Some more layout logic would be needed to separate those nodes properly, but the foundations are laid firmly and uniformly across all types of root systems for writing such extensions.

Here is an analogue picture in 3D:
Exercise

Explore the previous picture and notice how the edges of the crystal graph are parallel to the simple roots.

Enjoy and please post your best pictures on the Sage-Combinat wiki.

```python
class sage.combinat.root_system.plot.PlotOptions(space, projection=True, bounding_box=3,
          color=<bound method CartanTypeFactory.color of <class
          'sage.combinat.root_system.cartan_type.CartonTypeFactory'>>,
          labels=True, level=None, affine=None, arrowsize=5)

Bases: object
A class for plotting options for root lattice realizations.

See also:

- `RootLatticeRealizations.ParentMethods.plot()` for a description of the plotting options
- `Tutorial: visualizing root systems` for a tutorial on root system plotting

```
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See also:

index_of_object()

EXAMPLES:

```
sage: L = RootSystem("A",2).root_lattice()
sage: options = L.plot_parse_options(labels=False)
sage: alpha = L.simple_roots()
sage: options.color(1)
'blue'
sage: options.color(2)
'red'
sage: for alpha in L.roots():
....:   print("{} {}\nalpha[1] blue
alpha[2] red
-alpha[1] black
-alpha[2] black
```

cone(rays=[], lines=[], color='black', thickness=1, alpha=1, wireframe=False, label=None, draw_degenerate=True, as_polyhedron=False)

Return the cone generated by the given rays and lines.

INPUT:

- rays, lines – lists of elements of the root lattice realization (default: [])
- color – a color (default: "black")
- alpha – a number in the interval [0, 1] (default: 1) the desired transparency
- label – an object to be used as for this cone. The label itself will be constructed by calling latex() or repr() on the object depending on the graphics backend.
- draw_degenerate – a boolean (default: True) whether to draw cones with a degenerate intersection with the bounding box
- as_polyhedron – a boolean (default: False) whether to return the result as a polyhedron, without clipping it to the bounding box, and without making a plot out of it (for testing purposes)

OUTPUT:

A graphic object, a polyhedron, or 0.

EXAMPLES:

```
sage: L = RootSystem("A",2).root_lattice()
sage: options = L.plot_parse_options()
sage: alpha = L.simple_roots()
sage: p = options.cone(rays=[alpha[1]], lines=[alpha[2]], color='green', label=2)
sage: p
Graphics object consisting of 2 graphics primitives
sage: list(p)
[Polygon defined by 4 points,
Text '$2$' at the point (3.15...,3.15...)]
```
sage: options.cone(rays=[alpha[1]], lines=[alpha[2]], color='green', label=2, \n      as_polyhedron=True)
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex, 1 \n      ray, 1 line

An empty result, being outside of the bounding box:
sage: options = L.plot_parse_options(labels=True, bounding_box=[[−10,−9]]*2)
sage: options.cone(rays=[alpha[1]], lines=[alpha[2]], color='green', label=2)  \n    \empty

Test that the options are properly passed down:
sage: L = RootSystem(["A",2]).root_lattice()
sage: options = L.plot_parse_options()
sage: p = options.cone(rays=[alpha[1]+alpha[2]], color='green', label=2, \n      thickness=4, alpha=.5)
sage: list(p)
[Line defined by 2 points, Text '$2$' at the point (3.15...,3.15...)]
sage: sorted(p[0].options().items())
[('alpha', 0.500000000000000), ('legend_color', None),  \n    ('legend_label', None), ('rgbcolor', 'green'), ('thickness', 4), \n    ('zorder', 1)]

This method is tested indirectly but extensively by the various plot methods of root lattice realizations.

empty(*args)
Return an empty plot.

EXAMPLES:
sage: L = RootSystem(["A",2]).root_lattice()
sage: options = L.plot_parse_options(labels=True)
This currently returns int(0):
sage: options.empty()
0

This is not a plot, so may cause some corner cases. On the other hand, 0 behaves as a fast neutral element, which is important given the typical idioms used in the plotting code:
sage: p = point([0,0])
sage: p + options.empty() is p
True

family_of_vectors(vectors)
Return a plot of a family of vectors.

INPUT:
• vectors – family or vectors in self

The vectors are labelled by their index.

EXAMPLES:
```python
sage: L = RootSystem(['A',2]).root_lattice()
sage: options = L.plot_parse_options()
sage: alpha = L.simple_roots()
sage: p = options.family_of_vectors(alpha); p
Graphics object consisting of 4 graphics primitives
sage: list(p)
[Arrow from (0.0,0.0) to (1.0,0.0),
 Text '$1$' at the point (1.05,0.0),
 Arrow from (0.0,0.0) to (0.0,1.0),
 Text '$2$' at the point (0.0,1.05)]
```

Handling of colors and labels:

```python
sage: color=lambda i: "purple" if i==1 else None
sage: options = L.plot_parse_options(labels=False, color=color)
sage: p = options.family_of_vectors(alpha)
sage: list(p)
[Arrow from (0.0,0.0) to (1.0,0.0)]
sage: p[0].options()['rgbcolor']
'purple'
```

Matplotlib emits a warning for arrows of length 0 and draws nothing anyway. So we do not draw them at all:

```python
sage: L = RootSystem(['A',2,1]).ambient_space()
sage: options = L.plot_parse_options()
sage: Lambda = L.fundamental_weights()
sage: p = options.family_of_vectors(Lambda); p
Graphics object consisting of 5 graphics primitives
sage: list(p)
[Text '$0$' at the point (0.0,0.0),
 Arrow from (0.0,0.0) to (0.5,0.86602451838...),
 Text '$1$' at the point (0.525,0.909325744308...),
 Arrow from (0.0,0.0) to (-0.5,0.86602451838...),
 Text '$2$' at the point (-0.525,0.909325744308...)]
```

`finalize(G)`

Finalize a root system plot.

INPUT:

- `G` – a root system plot or `None`

This sets the aspect ratio to 1 and remove the axes. This should be called by all the user-level plotting methods of root systems. This will become mostly obsolete when customization options won’t be lost anymore upon addition of graphics objects and there will be a proper empty object for 2D and 3D plots.

EXAMPLES:

```python
sage: L = RootSystem(['B',2,1]).ambient_space()
sage: options = L.plot_parse_options()
sage: p = L.plot_roots(plot_options=options)
sage: p += L.plot_coroots(plot_options=options)
sage: p.axes()
True
```

(continues on next page)
sage: p = options.finalize(p)
sage: p.axes()
False
sage: p.aspect_ratio()
1.0

sage: options = L.plot_parse_options(affine=False)
sage: p = L.plot_roots(plot_options=options)
sage: p += point([[1,1,0]])
sage: p = options.finalize(p)
sage: p.aspect_ratio()
[1.0, 1.0, 1.0]

If the input is 0, this returns an empty graphics object:

sage: type(options.finalize(0))
<class 'sage.plot.plot3d.base.Graphics3dGroup'>

sage: options = L.plot_parse_options()
sage: type(options.finalize(0))
<class 'sage.plot.graphics.Graphics'>

sage: list(options.finalize(0))
[]

in_bounding_box(x)

Return whether x is in the bounding box.

INPUT:

• x – an element of the root lattice realization

This method is currently one of the bottlenecks, and therefore cached.

EXAMPLES:

sage: L = RootSystem(['A',2,1]).ambient_space()
sage: options = L.plot_parse_options()
sage: alpha = L.simple_roots()
sage: options.in_bounding_box(alpha[1])
True
sage: options.in_bounding_box(3*alpha[1])
False

index_of_object(i)

Try to return the node of the Dynkin diagram indexing the object i.

OUTPUT: a node of the Dynkin diagram or None

EXAMPLES:

sage: L = RootSystem(['A',3]).root_lattice()
sage: alpha = L.simple_roots()
sage: omega = RootSystem(['A',3]).weight_lattice().fundamental_weights()
sage: options = L.plot_parse_options(labels=False)
sage: options.index_of_object(3)
intersection_at_level_1(x)

Return x scaled at the appropriate level, if level is set; otherwise return x.

INPUT:

• x – an element of the root lattice realization

EXAMPLES:

```python
sage: L = RootSystem(['A',2,1]).weight_space()
sage: options = L.plot_parse_options()
sage: options.intersection_at_level_1(L.rho())
```

When level is not set, x is returned:

```python
sage: options = L.plot_parse_options(affine=False)
sage: options.intersection_at_level_1(L.rho())
Lambda[0] + Lambda[1] + Lambda[2]
```

projection(v)

Return the projection of v.

INPUT:

• x – an element of the root lattice realization

OUTPUT:

An immutable vector with integer or rational coefficients.

EXAMPLES:

```python
sage: L = RootSystem(['A',2,1]).ambient_space()
sage: options = L.plot_parse_options()
sage: options.projection(L.rho())
(0, 989/571)
```

reflection_hyperplane(coroot, as_polyhedron=False)

Return a plot of the reflection hyperplane indexed by this coroot.
• coroot – a coroot

EXAMPLES:

```python
sage: L = RootSystem(["B",2]).weight_space()
sage: alphacheck = L.simple_coroots()
sage: options = L.plot_parse_options()
sage: H = options.reflection_hyperplane(alphacheck[1]); H
Graphics object consisting of 2 graphics primitives
```

Todo: Display the periodic orientation by adding a + and a − sign close to the label. Typically by using the associated root to shift a bit from the vertex upon which the hyperplane label is attached.

```python
text(label, position, rgbcolor=(0, 0, 0))
```

Return text widget with label label at position position.

INPUT:

• label – a string, or a Sage object upon which latex will be called
• position – a position
• rgbcolor – the color as an RGB tuple

EXAMPLES:

```python
sage: L = RootSystem(["A",2]).root_lattice()
sage: options = L.plot_parse_options()
sage: list(options.text("coucou", [0,1]))
[Text 'coucou' at the point (0.0,1.0)]
sage: list(options.text(L.simple_root(1), [0,1]))
[Text '$\alpha_{1}$' at the point (0.0,1.0)]
sage: list(options.text(L.simple_root(2), [1,0], rgbcolor=(1,0.5,0)))
[Text '$\alpha_{2}$' at the point (1.0,0.0)]
sage: options = RootSystem(["A",2]).root_lattice().plot_parse_options().plot_parse_options(labels=False)
sage: options.text("coucou", [0,1])
0
sage: options = RootSystem(["B",3]).root_lattice().plot_parse_options()
sage: print(options.text("coucou", [0,1,2]).x3d_str())
<Transform translation='0 1 2'>
<Shape><Text string='coucou' solid='true'/><Appearance><Material diffuseColor='0.0 0.0 0.0' shininess='1.0' specularColor='0.0 0.0 0.0'/></Appearance></Shape>
</Transform>
```

```python
thickness(i)
```

Return the thickness to be used for lines indexed by i.

INPUT:

• i – an index
See also:

`index_of_object()`

EXAMPLES:

```python
sage: L = RootSystem(["A",2,1]).root_lattice()
sage: options = L.plot_parse_options(labels=False)
sage: alpha = L.simple_roots()
sage: options.thickness(0)
2
sage: options.thickness(1)
1
sage: options.thickness(2)
1
sage: for alpha in L.simple_roots():
    ....:    print("{} {}").format(alpha, options.thickness(alpha))
alpha[0] 2
alpha[1] 1
alpha[2] 1
```

```

```

```

```

```

```python
sage: from sage.combinat.root_system.root_lattice_realizations import barycentric_projection_matrix
sage: m = barycentric_projection_matrix(0); m
[]
```

```

```

```

```python
sage: matrix(QQ,0,1).nrows()
0
```

```

```

```

```python
sage: matrix(QQ,0,1).ncols()
1
```

```

```

```

```python
sage: barycentric_projection_matrix(1)
[1 -1]
```

```

```

```

```python
sage: barycentric_projection_matrix(2)
[1/2*sqrt(3) -1/2*sqrt(3) 0]
[1/2 1/2 -1]
```

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```
Four vectors in dimension 3:

```python
sage: m = barycentric_projection_matrix(3); m
[ 1/3*sqrt(3)*sqrt(2) -1/3*sqrt(3)*sqrt(2) 0 ]
[ 1/3*sqrt(2) 1/3*sqrt(2) -2/3*sqrt(2) ]
[ 1/3 1/3 1/3 -1 ]
```

The columns give four vectors that sum up to zero:

```python
sage: sum(m.columns())
(0, 0, 0)
```

and have regular mutual angles:

```python
sage: m.transpose()*m
[ 1 -1/3 -1/3 -1/3 ]
[-1/3 1 -1/3 -1/3 ]
[-1/3 -1/3 1 -1/3 ]
[-1/3 -1/3 -1/3 1 ]
```

Here is a plot of them:

```python
sage: sum(arrow((0,0,0),x) for x in m.columns())
Graphics3d Object
```

For 2D drawings of root systems, it is desirable to rotate the result to match with the usual conventions:

```python
sage: barycentric_projection_matrix(2, angle=2*pi/3)
[ 1/2 -1 1/2 ]
[ 1/2*sqrt(3) 0 -1/2*sqrt(3) ]
```

### 5.1.235 Finite complex reflection groups

Let \( V \) be a finite-dimensional complex vector space. A reflection of \( V \) is an operator \( r \in \text{GL}(V) \) that has finite order and fixes pointwise a hyperplane in \( V \).

For more definitions and classification types of finite complex reflection groups, see [Wikipedia article Complex reflection group](https://en.wikipedia.org/wiki/Complex_reflection_group).

The point of entry to work with reflection groups is `ReflectionGroup()` which can be used with finite Cartan-Killing types:

```python
sage: ReflectionGroup(['A',2])
Irreducible real reflection group of rank 2 and type A2
sage: ReflectionGroup(['F',4])
Irreducible real reflection group of rank 4 and type F4
sage: ReflectionGroup(['H',3])
Irreducible real reflection group of rank 3 and type H3
```

or with Shephard-Todd types:
sage: ReflectionGroup((1,1,3))  # optional - gap3
Irreducible real reflection group of rank 2 and type A2

sage: ReflectionGroup((2,1,3))  # optional - gap3
Irreducible real reflection group of rank 3 and type B3

sage: ReflectionGroup((3,1,3))  # optional - gap3
Irreducible complex reflection group of rank 3 and type G(3,1,3)

sage: ReflectionGroup((4,2,3))  # optional - gap3
Irreducible complex reflection group of rank 3 and type G(4,2,3)

sage: ReflectionGroup(4)  # optional - gap3
Irreducible complex reflection group of rank 2 and type ST4

sage: ReflectionGroup(31)  # optional - gap3
Irreducible complex reflection group of rank 4 and type ST31

Also reducible types are allowed using concatenation:

sage: ReflectionGroup(['A',3],(4,2,3))  # optional - gap3
Reducible complex reflection group of rank 6 and type A3 x G(4,2,3)

Some special cases also occur, among them are:

sage: W = ReflectionGroup((2,2,2)); W  # optional - gap3
Reducible real reflection group of rank 2 and type A1 x A1

sage: W = ReflectionGroup((2,2,3)); W  # optional - gap3
Irreducible real reflection group of rank 3 and type A3

Warning: Uses the GAP3 package Chevie which is available as an experimental package (installed by `sage -i gap3`) or to download by hand from Jean Michel's website.

A guided tour

We start with the example type $B_2$:

sage: W = ReflectionGroup(['B',2]); W  # optional - gap3
Irreducible real reflection group of rank 2 and type B2

Most importantly, observe that the group elements are usually represented by permutations of the roots:

sage: for w in W: print(w)  # optional - gap3
()
(1,3)(2,6)(5,7)
(1,5)(2,4)(6,8)
(1,7,5,3)(2,4,6,8)
(1,3,5,7)(2,8,6,4)
(2,8)(3,7)(4,6)
(1,7)(3,5)(4,8)
(1,5)(2,6)(3,7)(4,8)

This has the drawback that one can hardly see anything. Usually, one would look at elements with either of the following methods:
sage: for w in W: w.reduced_word()  # optional - gap3
[]
[2]
[1]
[1, 2]
[2, 1]
[2, 1, 2]
[1, 2, 1]
[1, 2, 1, 2]

sage: for w in W: w.reduced_word_in_reflections()  # optional - gap3
[]
[2]
[1]
[1, 2]
[1, 4]
[3]
[4]
[1, 3]

sage: for w in W: w.reduced_word(); w.to_matrix(); print("")  # optional - gap3
[]
[1 0]
[0 1]

[2]
[ 1 1]
[ 0 -1]

[1]
[-1 0]
[ 2 1]

[1, 2]
[-1 -1]
[ 2 1]

[2, 1]
[ 1 1]
[-2 -1]

[2, 1, 2]
[ 1 0]
[-2 -1]

[1, 2, 1]
[-1 -1]
[ 0 1]

[1, 2, 1, 2]
[-1 0]
[ 0 -1]

5.1. Comprehensive Module List
The standard references for actions of complex reflection groups have the matrices acting on the right, so:

```
  sage: W.simple_reflection(1).to_matrix()           # optional - gap3
  [-1  0]
  [ 2  1]
```

sends the simple root \(\alpha_0\), or \((1,0)\) in vector notation, to its negative, while sending \(\alpha_1\) to \(2\alpha_0 + \alpha_1\).

**Todo:**

- properly provide root systems for real reflection groups
- element class should be unique to be able to work with large groups without creating elements multiple times
- `is_shephard_group`, `is_generalized_coxeter_group`
- exponents and coexponents
- coinvariant ring:
  - fake degrees from Torsten Hoge
  - operation of linear characters on all characters
  - harmonic polynomials
- linear forms for hyperplanes
- field of definition
- intersection lattice and characteristic polynomial:

  ```
  X = [ alpha(t) for t in W.distinguished_reflections() ]
  X = Matrix(CF,X).transpose()
  Y = Matroid(X)
  ```

- linear characters
- permutation \(\pi\) on irreducibles
- hyperplane orbits (76.13 in Gap Manual)
- improve invariant_form with a code similar to the one in `reflection_group_real.py`
- add a method `reflection_to_root` or `distinguished_reflection_to_positive_root`
- diagrams in ASCII-art (76.15)
- standard (BMR) presentations
- character table directly from Chevie
- `GenericOrder` (76.20), `TorusOrder` (76.21)
- correct fundamental invariants for \(G_34\), check the others
- copy hardcoded data (degrees, invariants, braid relations...) into sage
- add other hardcoded data from the tables in chevie (location is SAGEDIR/local/gap3/gap-jm5-2015-02-01/gap3/pkg/chevie/tbl): basic derivations, discriminant, ...
- transfer code for `reduced_word_in_reflections` into Gap4 or Sage
- list of reduced words for an element
- list of reduced words in reflections for an element
Combinatorics, Release 10.0

• Hurwitz action?
• is_crystallographic() should be hardcoded

AUTHORS:
• Christian Stump (2015): initial version

```python
class sage.combinat.root_system.reflection_group_complex.ComplexReflectionGroup(W_types, index_set=None, hyperplane_index_set=None, reflection_index_set=None):
    Bases: UniqueRepresentation, PermutationGroup_generic
    A complex reflection group given as a permutation group.
    See also:
    ReflectionGroup()

class Element
    Bases: ComplexReflectionGroupElement

congruent_class()
    Return the conjugacy class of self.
    EXAMPLES:
    sage: W = ReflectionGroup((1,1,3))
    sage: for w in W:
    ....:     print(w.conjugacy_class())
    []
    [(1,3)(2,5)(4,6), (1,4)(2,3)(5,6), (1,5)(2,4)(3,6)]
    [(1,2,6)(3,4,5), (1,6,2)(3,5,4)]
    [(1,3)(2,5)(4,6), (1,4)(2,3)(5,6), (1,5)(2,4)(3,6)]

congruent_class_representative()
    Return a representative of the conjugacy class of self.
    EXAMPLES:
    sage: W = ReflectionGroup((1,1,3))
    sage: for w in W:
    ....:     print(w.conjugacy_class_representative().reduced_word())
    []
    [1]
    [2] [1]
    [1] [1]
    [1, 2] [1, 2]
    [2, 1] [1, 2]
    [1, 2, 1] [1, 2, 1]

reflection_length(in_unitary_group=False)
    Return the reflection length of self.
```

5.1. Comprehensive Module List 2459
This is the minimal numbers of reflections needed to obtain self.

INPUT:
• in_unitary_group – (default: False) if True, the reflection length is computed in the unitary group which is the dimension of the move space of self

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: sorted([t.reflection_length() for t in W])  # optional - gap3
[0, 1, 1, 2, 2]

sage: W = ReflectionGroup((2,1,2))  # optional - gap3
sage: sorted([t.reflection_length() for t in W])  # optional - gap3
[0, 1, 1, 2, 2, 2]

sage: W = ReflectionGroup((2,2,2))  # optional - gap3
sage: sorted([t.reflection_length() for t in W])  # optional - gap3
[0, 1, 2]

sage: W = ReflectionGroup((3,1,2))  # optional - gap3
sage: sorted([t.reflection_length() for t in W])  # optional - gap3
[0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
```

**apply_vector_field**(f, vf=None)

Returns a rational function obtained by applying the vector field vf to the rational function f.

If vf is not given, the primitive vector field is used.

EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: for x in W.primitive_vector_field()[0].parent().gens():  # optional - gap3
....:     print(W.apply_vector_field(x))
3*x1/(6*x0^2 - 6*x0*x1 - 12*x1^2)
1/(6*x0^2 - 6*x0*x1 - 12*x1^2)
```

**braid_relations()**

Return the braid relations of self.

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.braid_relations()  # optional - gap3
[[[1, 2, 1], [2, 1, 2]]]

sage: W = ReflectionGroup((2,1,3))  # optional - gap3
sage: W.braid_relations()  # optional - gap3
[[[1, 2, 1, 2], [2, 1, 2, 1]], [[1, 3], [3, 1]], [[2, 3, 2], [3, 2, 3]]]

sage: W = ReflectionGroup((2,2,3))  # optional - gap3
sage: W.braid_relations()  # optional - gap3
[[[1, 2, 1], [2, 1, 2]], [[1, 3], [3, 1]], [[2, 3, 2], [3, 2, 3]]]
```

**cartan_matrix()**

Return the Cartan matrix associated with self.
If \( \text{self} \) is crystallographic, the returned Cartan matrix is an instance of \( \text{CartanMatrix} \), and a normal matrix otherwise.

Let \( s_1, \ldots, s_n \) be a set of reflections which generate \( \text{self} \) with associated simple roots \( s_1, \ldots, s_n \) and simple coroots \( s_i^\vee \). Then the Cartan matrix \( C = (c_{ij}) \) is given by \( s_i^\vee(s_j) \). The Cartan matrix completely determines the reflection representation if the \( s_i \) are linearly independent.

**EXAMPLES:**

```python
sage: ReflectionGroup(['A',4]).cartan_matrix() # optional - gap3
[ 2 -1 0 0]
[ -1 2 -1 0]
[ 0 -1 2 -1]
[ 0 0 -1 2]
sage: ReflectionGroup(['H',4]).cartan_matrix() # optional - gap3
[ 2 E(5)^2 + E(5)^3 0 0]
[ E(5)^2 + E(5)^3 2 -1 0]
[ 0 -1 2 -1]
[ 0 0 -1 2]
sage: ReflectionGroup(4).cartan_matrix() # optional - gap3
[-2*E(3) - E(3)^2 E(3)^2]
[ -E(3)^2 -2*E(3) - E(3)^2]
sage: ReflectionGroup((4,2,2)).cartan_matrix() # optional - gap3
[ 2 -2*E(4) -2]
[ E(4) 2 1 - E(4)]
[ -1 1 + E(4) 2]
```

codegrees()  
Return the codegrees of \( \text{self} \) ordered within each irreducible component of \( \text{self} \).

**EXAMPLES:**

```python
sage: W = ReflectionGroup((1,1,4))
# optional - gap3
sage: W.codegrees() # optional - gap3
(2, 1, 0)
sage: W = ReflectionGroup((2,1,4))
# optional - gap3
sage: W.codegrees() # optional - gap3
(6, 4, 2, 0)
sage: W = ReflectionGroup((4,1,4))
# optional - gap3
sage: W.codegrees() # optional - gap3
(12, 8, 4, 0)
sage: W = ReflectionGroup((4,2,4))
# optional - gap3
sage: W.codegrees() # optional - gap3
(12, 8, 4, 0)
sage: W = ReflectionGroup((4,4,4))
# optional - gap3
sage: W.codegrees() # optional - gap3
(8, 8, 4, 0)
```

(continues on next page)
sage: W = ReflectionGroup((1,1,4), (3,1,2))  # optional - gap3
sage: W.codegrees()  # optional - gap3
(2, 1, 0, 3, 0)

sage: W = ReflectionGroup((1,1,4), (6,1,12), 23)  # optional - gap3
    # fails in GAP3
sage: W.codegrees()  # optional - gap3
(2, 1, 0, 66, 60, 54, 48, 42, 36, 30, 24, 18, 12, 6, 0, 8, 4, 0)

conjugacy_classes()
Return the conjugacy classes of self.

EXAMPLES:

sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: for C in W.conjugacy_classes(): sorted(C)  # optional - gap3
    [()]
    [(1,3)(2,5)(4,6), (1,4)(2,3)(5,6), (1,5)(2,4)(3,6)]
    [(1,2,6)(3,4,5), (1,6,2)(3,5,4)]

sage: W = ReflectionGroup((1,1,4))  # optional - gap3
sage: sum(len(C) for C in W.conjugacy_classes()) == W.cardinality()  # optional - gap3
True

sage: W = ReflectionGroup((3,1,2))  # optional - gap3
sage: sum(len(C) for C in W.conjugacy_classes()) == W.cardinality()  # optional - gap3
True

sage: W = ReflectionGroup(23)  # optional - gap3
sage: sum(len(C) for C in W.conjugacy_classes()) == W.cardinality()  # optional - gap3
True

conjugacy_classes_representatives()
Return the shortest representatives of the conjugacy classes of self.

EXAMPLES:

sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: [w.reduced_word() for w in W.conjugacy_classes_representatives()]  # optional - gap3
    [[]], [1], [1, 2]]

sage: W = ReflectionGroup((1,1,4))  # optional - gap3
sage: [w.reduced_word() for w in W.conjugacy_classes_representatives()]  # optional - gap3
    [[]], [1], [1, 3], [1, 2], [1, 3, 2]]

sage: W = ReflectionGroup((3,1,2))  # optional - gap3
sage: [w.reduced_word() for w in W.conjugacy_classes_representatives()]  # optional - gap3
    [[]], [1], [1, 3], [1, 2], [1, 3, 2]]

(continues on next page)
Sage:
W = ReflectionGroup(('B', 3))
  # optional - gap3
W.coxeter_number()  # optional - gap3
[30]

Sage:
W = ReflectionGroup((2, 1, 4))
  # optional - gap3
W.degrees()  # optional - gap3
(2, 4, 6, 8)

Sage:
W = ReflectionGroup((4, 1, 4))
  # optional - gap3
W.degrees()  # optional - gap3
(2, 4, 6, 8)
Examples of reducible types:

```
sage: W = ReflectionGroup((1,1,4), (3,1,2)); W
Reducible complex reflection group of rank 5 and type A3 x G(3,1,2)
sage: W.degrees()
(2, 3, 4, 3, 6)
sage: W = ReflectionGroup((1,1,4), (6,1,12), 23)
# fails in GAP3 #
sage: W.degrees()
(2, 3, 4, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 2, 6, 10)
```

discriminant()

Return the discriminant of `self` in the polynomial ring on which the group acts.

This is the product

$$\prod_{H} \alpha_H^{e_H},$$

where $\alpha_H$ is the linear form of the hyperplane $H$ and $e_H$ is its stabilizer order.

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: W.discriminant()  # optional - gap3
x0^6 - 3*x0^5*x1 - 3/4*x0^4*x1^2 + 13/2*x0^3*x1^3
- 3/4*x0^2*x1^4 - 3*x0*x1^5 + x1^6
sage: W = ReflectionGroup(['B',2])  # optional - gap3
sage: W.discriminant()  # optional - gap3
x0^6*x1^2 - 6*x0^5*x1^3 + 13*x0^4*x1^4 - 12*x0^3*x1^5 + 4*x0^2*x1^6
```

discriminant_in_invariant_ring(invariants=None)

Return the discriminant of `self` in the invariant ring.

This is the function $f$ in the invariants such that $f(F_1(x),\ldots,F_n(x))$ is the discriminant.

EXAMPLES:

```
sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: W.discriminant_in_invariant_ring()  # optional - gap3
6*t0^3*t1^2 - 18*t0^4*t2 + 9*t1^4 - 36*t0*t1^2*t2 + 24*t0^2*t2^2 - 8*t2^3
```
sage: W = ReflectionGroup(["B",3]) # optional - gap3
sage: W.discriminant_in_invariant_ring() # optional - gap3
-t0^2*t1^2*t2 + 16*t0^3*t2^2 + 2*t1^3*t2 - 36*t0*t1*t2^2 + 108*t2^3

sage: W = ReflectionGroup(["H",3]) # optional - gap3
sage: W.discriminant_in_invariant_ring() # long time # optional - gap3
(-829*E(5) - 1658*E(5)^2 - 1658*E(5)^3 - 829*E(5)^4)*t0^15
+ (213700*E(5) + 427400*E(5)^2 + 427400*E(5)^3 + 213700*E(5)^4)*t0^12*t1
+ (-22233750*E(5) - 44467500*E(5)^2 - 44467500*E(5)^3 - 22233750*E(5)^4)*t0^9*t1^2
+ (438750*E(5) + 877500*E(5)^2 + 877500*E(5)^3 + 438750*E(5)^4)*t0^10*t2
+ (1162187500*E(5) + 2324375000*E(5)^2 + 2324375000*E(5)^3 + 1162187500*E(5)^4)*t0^6*t1^3
+ (-74250000*E(5) - 148500000*E(5)^2 - 148500000*E(5)^3 - 742500000*E(5)^4)*t0^7*t1*t2
+ (-28369140625*E(5) - 5673821250*E(5)^2 - 5673821250*E(5)^3)*t0^4*t1^2*t2
+ (1191796875*E(5) + 2383593750*E(5)^2 + 2383593750*E(5)^3 + 1191796875*E(5)^4)*t0^5*t2^2
+ (175781250000*E(5) + 351562500000*E(5)^2 + 351562500000*E(5)^3 +
\quad 175781250000*E(5)^4)*t0^4*t1^4
+ (131835937500*E(5) + 263671875000*E(5)^2 + 263671875000*E(5)^3 +
\quad 131835937500*E(5)^4)*t0^5*t1^3*t2
+ (-100195312500*E(5) - 200390625000*E(5)^2 - 200390625000*E(5)^3 -
\quad 100195312500*E(5)^4)*t0^2*t1^2*t2
+ (395507812500*E(5) + 791015625000*E(5)^2 + 791015625000*E(5)^3 +
\quad 395507812500*E(5)^4)*t2^3

\textbf{distinguished_reflection}(i)

Return the \(i\)-th distinguished reflection of \(self\).

These are the reflections in \(self\) acting on the complement of the fixed hyperplane \(H\) as \(\exp(2\pi i/n)\), where \(n\) is the order of the reflection subgroup fixing \(H\).

**EXAMPLES:**

\begin{verbatim}
sage: W = ReflectionGroup((1,1,3)) # optional - gap3
dsage: W.distinguished_reflection(1) # optional - gap3
(1,4)(2,3)(5,6)
sage: W.distinguished_reflection(2) # optional - gap3
(1,3)(2,5)(4,6)
sage: W.distinguished_reflection(3) # optional - gap3
(1,5)(2,4)(3,6)
\end{verbatim}
distinguished_reflections()  
Return a finite family containing the distinguished reflections of self indexed by hyperplane_index_set().

These are the reflections in self acting on the complement of the fixed hyperplane $H$ as $\exp(2\pi i/n)$, where $n$ is the order of the reflection subgroup fixing $H$.

EXAMPLES:

```sage
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.distinguished_reflections()  # optional - gap3
Finite family {1: (1,4)(2,3)(5,6), 2: (1,3)(2,5)(4,6), 3: (1,5)(2,4)(3,6)}
```

```sage
sage: W = ReflectionGroup((1,1,3),hyperplane_index_set=['a','b','c'])  # optional - gap3
sage: W.distinguished_reflections()  # optional - gap3
Finite family {'a': (1,4)(2,3)(5,6), 'b': (1,3)(2,5)(4,6), 'c': (1,5)(2,4)(3,6)}
```

```sage
sage: W = ReflectionGroup((3,1,1))  # optional - gap3
sage: W.distinguished_reflections()  # optional - gap3
Finite family {1: (1,2,3)}
```

```sage
sage: W = ReflectionGroup((1,1,3),(3,1,2))  # optional - gap3
sage: W.distinguished_reflections()  # optional - gap3
Finite family {1: (1,6)(2,5)(7,8), 2: (1,5)(2,7)(6,8),
3: (3,9,15)(4,10,16)(12,17,23)(14,18,24)(20,25,29)(21,22,26)(27,28,30),
5: (1,7)(2,6)(5,8),
7: (4,21,27)(10,22,28)(11,13,19)(12,14,20)(16,26,30)(17,18,25)(23,24,29),
```

fake_degrees()  
Return the list of the fake degrees associated to self.

The fake degrees are $q$-versions of the degree of the character. In particular, they sum to Hilbert series of the coinvariant algebra of self.
Note: The ordering follows the one in Chevie and is not compatible with the current implementation of irreducible_characters().

EXAMPLES:

```python
sage: W = ReflectionGroup(12)  # optional - gap3
sage: W.fake_degrees()          # optional - gap3
[1, q^12, q^11 + q, q^8 + q^4, q^7 + q^5, q^6 + q^4 + q^2, 
 q^10 + q^8 + q^6, q^9 + q^7 + q^5 + q^3]

sage: W = ReflectionGroup(['H',4])  # optional - gap3
sage: W.cardinality()              # optional - gap3
14400
sage: sum(fdeg.subs(q=1)**2 for fdeg in W.fake_degrees()) # optional - gap3
14400
```

**fundamental_invariants()**

Return the fundamental invariants of self.

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.fundamental_invariants()     # optional - gap3
(-2*x0^2 + 2*x0*x1 - 2*x1^2, 6*x0^2*x1 - 6*x0*x1^2)

sage: W = ReflectionGroup((3,1,2))  # optional - gap3
sage: W.fundamental_invariants()     # optional - gap3
(x0^3 + x1^3, x0^3*x1^3)
```

**hyperplane_index_set()**

Return the index set of the hyperplanes of self.

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,4))  # optional - gap3
sage: W.hyperplane_index_set()       # optional - gap3
(1, 2, 3, 4, 5, 6)

sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=[1,3,'asdf',7,9,11])  # optional - gap3
sage: W.hyperplane_index_set()       # optional - gap3
(1, 3, 'asdf', 7, 9, 11)

sage: W = ReflectionGroup((1,1,4),hyperplane_index_set=('a','b','c','d','e','f'))  # optional - gap3
sage: W.hyperplane_index_set()       # optional - gap3
('a', 'b', 'c', 'd', 'e', 'f')
```

**independent_roots()**

Return a collection of simple roots generating the underlying vector space of self.

For well-generated groups, these are all simple roots. Otherwise, a linearly independent subset of the simple roots is chosen.

EXAMPLES:
Combinatorics, Release 10.0

```
sage: W = ReflectionGroup((1,1,3))       # optional - gap3
sage: W.independent_roots()              # optional - gap3
Finite family {1: (1, 0), 2: (0, 1)}

sage: W = ReflectionGroup((4,2,3))       # optional - gap3
sage: W.simple_roots()                   # optional - gap3
Finite family {1: (1, 0, 0), 2: (-E(4), 1, 0), 3: (-1, 1, 0), 4: (0, -1, 1)}

sage: W.independent_roots()              # optional - gap3
Finite family {1: (1, 0, 0), 2: (-E(4), 1, 0), 4: (0, -1, 1)}

index_set()

Return the index set of the simple reflections of self.

EXAMPLES:
```
sage: W = ReflectionGroup((1,1,4))       # optional - gap3
sage: W.index_set()                      # optional - gap3
(1, 2, 3)

sage: W = ReflectionGroup((1,1,4), index_set=[1,3,'asdf']) # optional - gap3
sage: W.index_set()                      # optional - gap3
(1, 3, 'asdf')

sage: W = ReflectionGroup((1,1,4), index_set=('a', 'b', 'c')) # optional - gap3
sage: W.index_set()                      # optional - gap3
('a', 'b', 'c')

invariant_form(brute_force=False)

Return the form that is invariant under the action of self.

This is unique only up to a global scalar on the irreducible components.

INPUT:

• brute_force – if True, the computation is done by applying the Reynolds operator; this is, the invariant form of $e_i$ and $e_j$ is computed as the sum $\langle w(e_i), w(e_j) \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product

EXAMPLES:
```
sage: W = ReflectionGroup(['A',3])       # optional - gap3
sage: F = W.invariant_form(); F          # optional - gap3
[ 1 -1/2 0]
[-1/2 1 -1/2]
[ 0 -1/2 1]

To check that this is indeed the invariant form, see:
```
sage: S = W.simple_reflections()          # optional - gap3
sage: all( F == S[i].matrix()*F*S[i].matrix().transpose() for i in W.index_set() ) # optional - gap3
True

sage: W = ReflectionGroup(['B',3])       # optional - gap3
sage: F = W.invariant_form(); F          # optional - gap3
[ 1 -1 0]
```
(continues on next page)
It also worked for badly generated groups:

```python
sage: W = ReflectionGroup(7)  # optional - gap3
sage: W.is_well_generated()  # optional - gap3
False
sage: F = W.invariant_form(); F  # optional - gap3
[1 0]
[0 1]
```

And also for reducible types:

```python
sage: W = ReflectionGroup(['B',3],(4,2,3),4,7); W  # optional - gap3
Reducible complex reflection group of rank 10 and type B3 x G(4,2,3) x ST4 x ST7
sage: F = W.invariant_form(); S = W.simple_reflections()  # optional - gap3
sage: all( F == S[i].matrix()*F*S[i].matrix().transpose().conjugate() for i in W.index_set() )  # optional - gap3
True
```

### invariant_form_standardization()  
Return the transformation of the space that turns the invariant form of self into the standard scalar product.

Let $I$ be the invariant form of a complex reflection group, and let $A$ be the Hermitian matrix such that $A^2 = I$. The matrix $A$ defines a change of basis such that the identity matrix is the invariant form. Indeed, we have

$$(A^{-1} x A) I (A^{-1} y A)^* = A^{-1} x I y^* A^{-1} = A^{-1} I A^{-1} = I,$$

where $I$ is the identity matrix.
**irreducible_components()**

Return a list containing the irreducible components of `self` as finite reflection groups.

**Examples:**

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.irreducible_components()  # optional - gap3
[Irreducible real reflection group of rank 2 and type A2]
```

```python
sage: W = ReflectionGroup((1,1,3),(2,1,3))  # optional - gap3
sage: W.irreducible_components()  # optional - gap3
[Irreducible real reflection group of rank 2 and type A2,
Irreducible real reflection group of rank 3 and type B3]
```

**is_crystallographic()**

Return `True` if `self` is crystallographic.

This is, if the field of definition is the rational field.

**Todo:** Make this more robust and do not use the matrix representation of the simple reflections.

**Examples:**

```python
sage: W = ReflectionGroup((1,1,3)); W  # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: W.is_crystallographic()  # optional - gap3
True
```

```python
sage: W = ReflectionGroup((2,1,3)); W  # optional - gap3
Irreducible real reflection group of rank 3 and type B3
sage: W.is_crystallographic()  # optional - gap3
True
```

```python
sage: W = ReflectionGroup(23); W  # optional - gap3
Irreducible real reflection group of rank 3 and type H3
sage: W.is_crystallographic()  # optional - gap3
False
```

```python
sage: W = ReflectionGroup((3,1,3)); W  # optional - gap3
Irreducible complex reflection group of rank 3 and type G(3,1,3)
sage: W.is_crystallographic()  # optional - gap3
False
```

```python
sage: W = ReflectionGroup((4,2,2)); W  # optional - gap3
Irreducible complex reflection group of rank 2 and type G(4,2,2)
(continues on next page)```
iteration_tracking_words()

Return an iterator going through all elements in self that tracks the reduced expressions. This can be much slower than using the iteration as a permutation group with strong generating set.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: for w in W.iteration_tracking_words(): w  # optional - gap3
(1,4)(2,3)(5,6)
(1,3)(2,5)(4,6)
(1,6,2)(3,5,4)
(1,2,6)(3,4,5)
(1,5)(2,4)(3,6)
```

jacobian_of_fundamental_invariants(invs=None)

Return the matrix \([\partial x_i F_j]\), where invs are are any polynomials \(F_1, \ldots, F_n\) in \(x_1, \ldots, x_n\).

INPUT:

- invs – (default: the fundamental invariants) the polynomials \(F_1, \ldots, F_n\)

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: W.fundamental_invariants()  # optional - gap3
(-2*x0^2 + 2*x0*x1 - 2*x1^2, 6*x0^2*x1 - 6*x0*x1^2)
sage: W.jacobian_of_fundamental_invariants()  # optional - gap3
[ -4*x0 + 2*x1 2*x0 - 4*x1]
[12*x0^2*x1 - 6*x1^2 6*x0^2 - 12*x0*x1]
```

number_of_irreducible_components()

Return the number of irreducible components of self.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.number_of_irreducible_components()  # optional - gap3
1
sage: W = ReflectionGroup((1,1,3),(2,1,3))  # optional - gap3
sage: W.number_of_irreducible_components()  # optional - gap3
2
```

primitive_vector_field(invs=None)

Return the primitive vector field of self is irreducible and well-generated.

The primitive vector field is given as the coefficients (being rational functions) in the basis \(\partial x_1, \ldots, \partial x_n\).

This is the partial derivation along the unique invariant of degree given by the Coxeter number. It can be computed as the row of the inverse of the Jacobian given by the highest degree.

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EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: W.primitive_vector_field()     # optional - gap3
(3*x1/(6*x0^2 - 6*x0*x1 - 12*x1^2), 1/(6*x0^2 - 6*x0*x1 - 12*x1^2))
```

**rank()**

Return the rank of `self`.

This is the dimension of the underlying vector space.

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.rank()                      # optional - gap3
2
sage: W = ReflectionGroup((2,1,3))  # optional - gap3
sage: W.rank()                      # optional - gap3
3
sage: W = ReflectionGroup((3,1,1))  # optional - gap3
sage: W.rank()                      # optional - gap3
3
```

**reflection(i)**

Return the `i`-th reflection of `self`.

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.reflection(1)               # optional - gap3
(1,4)(2,3)(5,6)
sage: W.reflection(2)               # optional - gap3
(1,3)(2,5)(4,6)
sage: W.reflection(3)               # optional - gap3
(1,5)(2,4)(3,6)
sage: W = ReflectionGroup((3,1,1),reflection_index_set=['a','b'])  # optional - gap3
sage: W.reflection('a')             # optional - gap3
(1,2,3)
sage: W.reflection('b')            # optional - gap3
(1,3,2)
```

**reflection_character()**

Return the reflection characters of `self`.

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.reflection_character()       # optional - gap3
[2, 0, -1]
```
\texttt{reflection_eigenvalues}(w, \text{is\_class\_representative}=False)

Return the reflection eigenvalue of \(w\) in self.

INPUT:

\begin{itemize}
  \item \text{is\_class\_representative} – boolean (default True) whether to compute instead on the conjugacy class representative.
\end{itemize}

See also:

\texttt{reflection_eigenvalues\_family()}

EXAMPLES:

\begin{verbatim}
sage: W = ReflectionGroup((1,1,3)) # optional - gap3 sage: for w in W: # optional - gap3 ....:   print('%s %s%(w.reduced_word(), W.reflection_eigenvalues(w))) [[] [0, 0] [2] [1/2, 0] [1] [1/2, 0] [1, 2] [1/3, 2/3] [2, 1] [1/3, 2/3] [1, 2, 1] [1/2, 0]
\end{verbatim}

\texttt{reflection_eigenvalues\_family()}

Return the reflection eigenvalues of self as a finite family indexed by the class representatives of self.

OUTPUT:

\begin{itemize}
  \item list with entries \(k/n\) representing the eigenvalue \(\zeta^k_n\).
\end{itemize}

EXAMPLES:

\begin{verbatim}
\end{verbatim}
```python
sage: W = ReflectionGroup(23)  # optional - gap3
sage: reflection_eigenvalues = W.reflection_eigenvalues_family()  # optional - gap3
sage: for elt in sorted(reflection_eigenvalues.keys()):  # optional - gap3
    print('%s %s
(%0, 0, 0)
(1, 8, 4) (2, 21, 3) (5, 10, 11) (6, 18, 17) (7, 9, 12) (13, 14, 15) (16, 23, 19) (20, 25, 26) (22, 24, 27) (28, 29, 30) [1/3, 2/3, 0]
(1, 16) (2, 5) (4, 7) (6, 9) (8, 10) (11, 13) (12, 14) (17, 20) (19, 22) (21, 24) (23, 25) (26, 28) (27, 29) [1/2, 0, 0]
(1, 16) (2, 9) (3, 18) (4, 10) (5, 6) (7, 8) (11, 14) (12, 13) (17, 24) (19, 25) (20, 21) (22, 23) (26, 29) (27, 28) [1/2, 1/2, 0]
(1, 16) (2, 17) (3, 18) (4, 19) (5, 20) (6, 21) (7, 22) (8, 23) (9, 24) (10, 25) (11, 26) (12, 27) (13, 28) (14, 29) (15, 30) [1/2, 1/2, 1/2]
(1, 19, 20, 2, 7) (3, 6, 11, 13, 9) (4, 5, 17, 22, 16) (8, 12, 15, 14, 10) (18, 21, 26, 28, 24) (23, 27, 30, 29, 25) [1/5, 4/5, 0]
(1, 20, 7, 19, 2) (3, 11, 9, 6, 13) (4, 17, 16, 5, 22) (8, 15, 10, 12, 14) (18, 26, 24, 21, 28) (23, 30, 25, 27, 29) [2/5, 3/5, 0]
(1, 23, 26, 29, 22, 16, 8, 11, 14, 7) (2, 10, 4, 9, 18, 17, 25, 19, 24, 3) (5, 21, 27, 30, 28, 20, 6, 12, 15, 13) [1/10, 1/2, 9/10]
(1, 24, 17, 16, 9, 2) (3, 12, 13, 18, 27, 28) (4, 21, 29, 19, 6, 14) (5, 25, 26, 20, 10, 11) (7, 23, 30, 22, 8, 15) [1/6, 1/2, 5/6]
(1, 29, 8, 7, 26, 16, 14, 23, 22, 11) (2, 9, 25, 3, 4, 17, 24, 10, 18, 19) (5, 30, 6, 13, 27, 20, 15, 21, 28, 12) [3/10, 1/2, 7/10]
```

**reflection_hyperplane**(i, as_linear_functional=False, with_order=False)

Return the i-th reflection hyperplane of self.

The i-th reflection hyperplane corresponds to the i distinguished reflection.

**INPUT:**

- i – an index in the index set
- as_linear_functionals – (default:False) flag whether to return the hyperplane or its linear functional in the basis dual to the given root basis

**EXAMPLES:**

```python
sage: W = ReflectionGroup((2,1,2))  # optional - gap3
sage: W.reflection_hyperplane(3)  # optional - gap3
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
```

One can ask for the result as a linear form:

```python
sage: W.reflection_hyperplane(3, True)  # optional - gap3
(0, 1)
```

**reflection_hyperplanes**(as_linear_functionals=False, with_order=False)

Return the list of all reflection hyperplanes of self, either as a codimension 1 space, or as its linear functional.

**INPUT:**
• as_linear_functionals – (default:False) flag whether to return the hyperplane or its linear functional in the basis dual to the given root basis

**EXAMPLES:**

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: for H in W.reflection_hyperplanes(): H  # optional - gap3
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 2]  
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 1/2]  
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 -1]
```

```python
sage: for H in W.reflection_hyperplanes(as_linear_functionals=True): H  # optional - gap3
(1, -1/2)
(1, -2)
(1, 1)
```

```python
sage: W = ReflectionGroup((2,1,2))  # optional - gap3
sage: for H in W.reflection_hyperplanes(): H  # optional - gap3
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 1]  
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 1/2]  
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 0]  
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]  
```

```python
sage: for H in W.reflection_hyperplanes(as_linear_functionals=True): H  # optional - gap3
(1, -1)
(1, -2)
(0, 1)
(1, 0)
```

```python
sage: for H in W.reflection_hyperplanes(as_linear_functionals=True, with_order=True): H  # optional - gap3
(((1, -1), 2)
(((1, -2), 2)
(((0, 1), 2)
(((1, 0), 2)
```

```python
reflection_index_set()
```
Return the index set of the reflections of self.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))  # optional - gap3
sage: W.reflection_index_set()  # optional - gap3
(1, 2, 3, 4, 5, 6)
sage: W = ReflectionGroup((1,1,4), reflection_index_set=[1,3,'asdf',7,9,11])  # optional - gap3
sage: W.reflection_index_set()  # optional - gap3
(1, 3, 'asdf', 7, 9, 11)
sage: W = ReflectionGroup((1,1,4), reflection_index_set=('a', 'b', 'c', 'd', 'e', 'f'))  # optional - gap3
sage: W.reflection_index_set()  # optional - gap3
('a', 'b', 'c', 'd', 'e', 'f')
```

reflections()    

Return a finite family containing the reflections of self, indexed by self.reflection_index_set().

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.reflections()  # optional - gap3
Finite family {1: (1,4)(2,3)(5,6), 2: (1,3)(2,5)(4,6), 3: (1,5)(2,4)(3,6)}
sage: W = ReflectionGroup((1,1,3),reflection_index_set=['a','b','c'])  # optional - gap3
sage: W.reflections()  # optional - gap3
Finite family {'a': (1,4)(2,3)(5,6), 'b': (1,3)(2,5)(4,6), 'c': (1,5)(2,4)(3,6)}
sage: W = ReflectionGroup((3,1,1))  # optional - gap3
sage: W.reflections()  # optional - gap3
Finite family {1: (1,2,3), 2: (1,3,2)}
sage: W = ReflectionGroup((1,1,3),(3,1,2))  # optional - gap3
sage: W.reflections()  # optional - gap3
```

roots()    

Return all roots corresponding to all reflections of self.
EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.roots()                      # optional - gap3
[(1, 0), (0, 1), (1, 1), (-1, 0), (0, -1), (-1, -1)]

sage: W = ReflectionGroup((3,1,2))  # optional - gap3
sage: W.roots()                      # optional - gap3
[(1, 0), (-1, 1), (E(3), 0), (-E(3), 1), (0, 1), (1, -1),
 (0, E(3)), (1, -E(3)), (E(3)^2, 0), (-E(3)^2, 1),
 (E(3), -1), (E(3), -E(3)), (0, E(3)^2), (1, -E(3)^2),
 (-1, E(3)), (-E(3), E(3)), (E(3)^2, -1), (E(3)^2, -E(3)),
 (E(3), -E(3)^2), (-E(3)^2, E(3)), (-1, E(3)^2),
 (-E(3), E(3)^2), (E(3)^2, -E(3)^2), (-E(3)^2, E(3)^2)]

sage: W = ReflectionGroup((4,2,2))  # optional - gap3
sage: W.roots()                      # optional - gap3
[(1, 0), (-E(4), 1), (-1, 1), (-1, 0), (E(4), 1), (1, 1),
 (0, -E(4)), (E(4), -1), (E(4), E(4)), (0, E(4)),
 (E(4), -E(4)), (0, 1), (1, -E(4)), (1, -1), (0, -1),
 (1, E(4)), (-E(4), 0), (-E(4), E(4)), (0, -E(4), E(4)),
 (-E(4), -1), (-E(4), -E(4)), (-1, -E(4)), (-1, -1)]

sage: W = ReflectionGroup((1,1,4), (3,1,2))  # optional - gap3
sage: W.roots()                      # optional - gap3
[(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0),
 (0, 0, 0, 1, 0), (0, 0, 0, -1, 1), (1, 1, 0, 0, 0),
 (0, 1, 1, 0, 0), (1, 1, 1, 0, 0), (-1, 0, 0, 0, 0),
 (0, -1, 0, 0, 0), (0, 0, -1, 0, 0), (-1, -1, 0, 0, 0),
 (0, 0, 0, -E(3), 1), (0, 0, 0, 0, 1), (0, 0, 0, 1, -1),
 (0, 0, 0, 0, E(3)), (0, 0, 0, 1, -E(3)), (0, 0, 0, E(3)^2, 0),
 (0, 0, 0, -E(3)^2, 1), (0, 0, 0, E(3), -1), (0, 0, 0, E(3), -E(3)),
 (0, 0, 0, 0, E(3)^2), (0, 0, 0, 1, -E(3)^2), (0, 0, 0, -1, E(3)),
 (0, 0, 0, -E(3), E(3)), (0, 0, 0, E(3)^2, -1),
 (0, 0, 0, E(3)^2, -E(3)), (0, 0, 0, E(3), -E(3)^2),
 (0, 0, 0, -E(3)^2, E(3)), (0, 0, 0, -1, E(3)^2),
 (0, 0, 0, -E(3), E(3)^2), (0, 0, 0, E(3)^2, -E(3)^2),
 (0, 0, 0, -E(3)^2, E(3)^2)]
```

**series()**

Return the series of the classification type to which `self` belongs.

For real reflection groups, these are the Cartan-Killing classification types “A”, “B”, “C”, “D”, “E”, “F”, “G”, “H”, “T”, and for complex non-real reflection groups these are the Shephard-Todd classification type “ST”.

**EXAMPLES:**

```python
sage: ReflectionGroup((1,1,3)).series()  # optional - gap3
['A']

sage: ReflectionGroup((3,1,3)).series()  # optional - gap3
['ST']
```

**set_reflection_representation**(refl_repr=None)

5.1. Comprehensive Module List 2477
Set the reflection representation of self.

INPUT:

- `refl_repr` – a dictionary representing the matrices of the generators of `self` with keys given by the index set, or `None` to reset to the default reflection representation

EXAMPLES:

```python
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: for w in W: w.to_matrix(); print("-----")  # optional - gap3
[1 0]
[0 1]
-----
[ 1 1]
[ 0 -1]
-----
[-1 0]
[ 1 1]
-----
[-1 -1]
[ 1 0]
-----
[ 0 1]
[-1 -1]
-----
[ 0 -1]
[-1 0]
-----

sage: W.set_reflection_representation({1: matrix([[0,1,0], [1,0,0], [0,0,1]]), 2:
matrix([[1,0,0], [0,0,1], [0,1,0]])})  # optional - gap3
sage: for w in W: w.to_matrix(); print("-----")  # optional - gap3
[1 0 0]
[0 1 0]
[0 0 1]
-----
[1 0 0]
[0 0 1]
[0 1 0]
-----
[0 1 0]
[1 0 0]
[0 0 1]
-----
[0 0 1]
[1 0 0]
[0 1 0]
-----
[0 0 1]
[0 0 1]
[1 0 0]
-----
[0 0 1]
```

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Combinatorics, Release 10.0

\[ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

-----

```python
sage: W.set_reflection_representation()                 # optional - gap3
```

**simple_coroot(i)**

Return the simple root with index \( i \).

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',3])                      # optional - gap3
sage: W.simple_coroot(1)                                 # optional - gap3
(2, -1, 0)
```

**simple_coroots()**

Return the simple coroots of `self`.

These are the coroots corresponding to the simple reflections.

**EXAMPLES:**

```python
sage: W = ReflectionGroup((1,1,3))                       # optional - gap3
sage: W.simple_coroots()                                 # optional - gap3
Finite family {1: (2, -1), 2: (-1, 2),}

sage: W = ReflectionGroup((1,1,4), (2,1,2))             # optional - gap3
sage: W.simple_coroots()                                 # optional - gap3
Finite family {1: (2, -1, 0, 0, 0), 2: (-1, 2, -1, 0, 0), 3: (0, -1, 2, 0, 0), 4: (0, 0, 0, 2, -2), 5: (0, 0, 0, -1, 2)}

sage: W = ReflectionGroup((3,1,2))                       # optional - gap3
sage: W.simple_coroots()                                 # optional - gap3
Finite family {1: (-2*E(3) - E(3)^2, 0), 2: (-1, 1)}

sage: W = ReflectionGroup((1,1,4), (3,1,2))             # optional - gap3
sage: W.simple_coroots()                                 # optional - gap3
Finite family {1: (2, -1, 0, 0, 0), 2: (-1, 2, -1, 0, 0), 3: (0, -1, 2, 0, 0), 4: (0, 0, 0, -2*E(3) - E(3)^2, 0), 5: (0, 0, 0, -1, 1)}
```

**simple_reflection(i)**

Return the \( i \)-th simple reflection of `self`.

**EXAMPLES:**

```python
sage: W = ReflectionGroup((1,1,3))                       # optional - gap3
sage: W.simple_reflection(1)                             # optional - gap3
(1,4)(2,3)(5,6)

sage: W.simple_reflections()                             # optional - gap3
Finite family {1: (1,4)(2,3)(5,6), 2: (1,3)(2,5)(4,6)}
```

**simple_root(i)**

Return the simple root with index \( i \).

**EXAMPLES:**

```python
```
Combinatorics, Release 10.0

```
sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: W.simple_root(1)               # optional - gap3
(1, 0, 0)
sage: W.simple_root(2)               # optional - gap3
(0, 1, 0)
sage: W.simple_root(3)               # optional - gap3
(0, 0, 1)
```

**simple_roots()**

Return the simple roots of self.

These are the roots corresponding to the simple reflections.

**EXAMPLES:**

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: W.simple_roots()               # optional - gap3
Finite family {1: (1, 0), 2: (0, 1)}

sage: W = ReflectionGroup((1,1,4), (2,1,2))  # optional - gap3
sage: W.simple_roots()               # optional - gap3
Finite family {1: (1, 0, 0, 0), 2: (0, 1, 0, 0), 3: (0, 0, 1, 0), 4: → (0, 0, 0, 1), 5: (0, 0, 0, 0)}

sage: W = ReflectionGroup((3,1,2))  # optional - gap3
sage: W.simple_roots()               # optional - gap3
Finite family {1: (1, 0), 2: (-1, 1)}

sage: W = ReflectionGroup((1,1,4), (3,1,2))  # optional - gap3
sage: W.simple_roots()               # optional - gap3
Finite family {1: (1, 0, 0, 0), 2: (0, 1, 0, 0), 3: (0, 0, 1, 0), 4: → (0, 0, 0, 1), 5: (0, 0, 0, -1, 1)}
```

**class** `sage.combinat.root_system.reflection_group_complex.IrreducibleComplexReflectionGroup(W_types, index_set=None, hyperplane_index_set=None, reflection_index_set=None)`

Bases: `ComplexReflectionGroup`

**class** `Element`

Bases: `Element`

**is_coxeter_element**(which_primitive=1, is_class_representative=False)

Return True if self is a Coxeter element.

This is, whether self has an eigenvalue that is a primitive h-th root of unity.

**INPUT:**

- **which_primitive** – (default:1) for which power of the first primitive h-th root of unity to look as a reflection eigenvalue for a regular element
• is_class_representative – boolean (default True) whether to compute instead on the conjugacy class representative

See also:

coxeter_element() coxeter_elements()

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: for w in W:  # optional - gap3
    ....:   print('%s %s' % (w.reduced_word(), w.is_coxeter_element()))
[() False
[2] False
[1] False
[1, 2] True
[2, 1] True
[1, 2, 1] False
```

**is_h_regular(is_class_representative=False)**

Return whether self is regular.

This is if self has an eigenvector with eigenvalue $h$ and which does not lie in any reflection hyperplane. Here, $h$ denotes the Coxeter number.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: for w in W:  # optional - gap3
    ....:   print('%s %s' % (w.reduced_word(), w.is_h_regular()))
[() False
[2] False
[1] False
[1, 2] True
[2, 1] True
[1, 2, 1] False
```

**is_regular(h, is_class_representative=False)**

Return whether self is regular.

This is, if self has an eigenvector with eigenvalue of order $h$ and which does not lie in any reflection hyperplane.

INPUT:

• $h$ – the order of the eigenvalue
• is_class_representative – boolean (default True) whether to compute instead on the conjugacy class representative

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))  # optional - gap3
sage: h = W.coxeter_number()  # optional - gap3
sage: for w in W:  # optional - gap3
    ....:   print("{} {}".format(w.reduced_word(), w.is_regular(h)))
[() False
[2] False
[1] False
[1, 2] True
```

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\[
\begin{align*}
[2, 1] & \text{ True} \\
[1, 2, 1] & \text{ False}
\end{align*}
\]

```python
sage: W = ReflectionGroup(23); h = W.coxeter_number()  # optional - gap3
sage: for w in W:  # optional - gap3
    ....: if w.is_regular(h):
    ....:     w.reduced_word()
[1, 2, 3]
[2, 1, 3]
[1, 3, 2]
[3, 2, 1]
[2, 1, 2, 3, 2]
[2, 3, 2, 1, 2]
[1, 2, 1, 2, 3, 2, 1]
[1, 2, 3, 2, 1, 2, 3]
[2, 1, 2, 1, 3, 2, 1, 2, 3]
[2, 1, 2, 3, 2, 1, 2, 1, 3]
[1, 2, 3, 2, 1, 2, 1, 3, 2]
[3, 2, 1, 2, 1, 3, 2, 1, 2]
[1, 2, 1, 2, 1, 3, 2, 1, 2]
[2, 3, 2, 1, 2, 1, 3, 2, 1]
[2, 1, 2, 1, 3, 2, 1, 2, 1]
[2, 3, 2, 1, 2, 1, 3, 2, 1, 2]
[1, 2, 1, 2, 1, 3, 2, 1, 2, 3]
[1, 2, 3, 2, 1, 2, 1, 3, 2, 1]
[2, 1, 2, 3, 2, 1, 2, 1, 3, 2, 1]
[2, 1, 2, 3, 2, 1, 2, 1, 3, 2, 1, 2]
[1, 2, 1, 2, 1, 3, 2, 1, 2, 1, 3, 2]
[1, 2, 1, 2, 1, 3, 2, 1, 2, 1, 3, 2, 1]
[2, 3, 2, 1, 2, 1, 3, 2, 1, 2, 1, 3, 2]
[1, 2, 1, 2, 1, 3, 2, 1, 2, 1, 3, 2, 1, 2]
[2, 3, 2, 1, 2, 1, 3, 2, 1, 2, 1, 3, 2, 1, 2]
```

Check that github issue #25478 is fixed:

```python
sage: W = ReflectionGroup(["A",5])  # optional - gap3
sage: w = W.from_reduced_word([1,2,3,5])  # optional - gap3
sage: w.is_regular(4)  # optional - gap3
False
sage: W = ReflectionGroup(["A",3])  # optional - gap3
sage: len([w for w in W if w.is_regular(w.order())])  # optional - gap3
18
```

\[
\text{sage.combinat.root_system.reflection_group_complex.multi_partitions}(n, S, i=\text{None})
\]

Return all vectors as lists of the same length as \(S\) whose standard inner product with \(S\) equals \(n\).

**EXAMPLES:**

```python
sage: from sage.combinat.root_system.reflection_group_complex import multi_partitions
sage: multi_partitions(10, [2,3,3,4])
[[[5, 0, 0, 0],
  [3, 0, 0, 1],
  [2, 2, 0, 0],
```

(continues on next page)
sage.combinat.root_system.reflection_group_complex.power(k)

Return $f^k$ and caching all intermediate results.

Speeds the computation if one has to compute $f^k$'s for many values of $k$.

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f = -2*x^2 + 2*x*y - 2*y^2 + 2*y*z - 2*z^2
sage: all( f^k == power(f,k) for k in range(20) )
True
```

5.1.236 Finite real reflection groups

Let $V$ be a finite-dimensional real vector space. A reflection of $V$ is an operator $r \in \text{GL}(V)$ that has order 2 and fixes pointwise a hyperplane in $V$. In the present implementation, finite real reflection groups are tied with a root system.

Finite real reflection groups with root systems have been classified according to finite Cartan-Killing types. For more definitions and classification types of finite complex reflection groups, see Wikipedia article Complex_reflection_group.

The point of entry to work with reflection groups is `ReflectionGroup()` which can be used with finite Cartan-Killing types:

```python
sage: ReflectionGroup(['A',2])  # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: ReflectionGroup(['F',4])  # optional - gap3
Irreducible real reflection group of rank 4 and type F4
sage: ReflectionGroup(['H',3])  # optional - gap3
Irreducible real reflection group of rank 3 and type H3
```

AUTHORS:

- Christian Stump (initial version 2011–2015)

**Warning:** Uses the GAP3 package Chevie which is available as an experimental package (installed by `sage -i gap3`) or to download by hand from Jean Michel’s website.
class sage.combinat.root_system.reflection_group_real.IrreducibleRealReflectionGroup(W_types, index_set=None, hyperplane_index_set=None, reflection_index_set=None)

Bases: RealReflectionGroup, IrreducibleComplexReflectionGroup

class Element

Bases: Element, Element

class sage.combinat.root_system.reflection_group_real.RealReflectionGroup(W_types, index_set=None, hyperplane_index_set=None, reflection_index_set=None)

Bases: ComplexReflectionGroup

A real reflection group given as a permutation group.

See also:

ReflectionGroup()

class Element

Bases: RealReflectionGroupElement, Element

left_coset_representatives()

Return the left coset representatives of self.

See also:

right_coset_representatives()

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])   # optional - gap3
sage: for w in W:                   # optional - gap3
    ....: lcr = w.left_coset_representatives()
    ....: print("%s %s"%(w.reduced_word(),
    ....: [v.reduced_word() for v in lcr]))
[] [] [] [2] [1, 2] [2, 1] [1, 2, 1]
[1] [] [1] [2, 1]
[1, 2] []
[2, 1] []
[1, 2, 1] [] [2] [1, 2]

right_coset_representatives()

Return the right coset representatives of self.

EXAMPLES:
```
sage: W = ReflectionGroup(['A',2])                      # optional - gap3
sage: for w in W:                                        # optional - gap3
....:     rcr = w.right_coset Representatives()
....:     print("%s %s"%(w.reduced_word(),
....:             [v.reduced_word() for v in rcr]))
[] [[1, 2], [1, 2, 1], [1, 2, 1]]
[2] [[2], [1, 2]]
[1] [[1], [1, 2]]
[1, 2] [[]]
[2, 1] [[]]
[1, 2, 1] [[2], [2, 1]]

almost_positive_roots()

Return the almost positive roots of self.

EXAMPLES:

sage: W = ReflectionGroup(['A',3], ['B',2])            # optional - gap3
sage: W.almost_positive_roots()                         # optional - gap3
[(-1, 0, 0, 0, 0),
 (0, -1, 0, 0, 0),
 (0, 0, -1, 0, 0),
 (0, 0, 0, -1, 0),
 (0, 0, 0, 0, -1),
 (1, 0, 0, 0, 0),
 (0, 1, 0, 0, 0),
 (0, 0, 1, 0, 0),
 (0, 0, 0, 1, 0),
 (1, 1, 0, 0, 0),
 (0, 0, 0, 1, 1),
 (1, 1, 1, 0, 0),
 (0, 0, 0, 2, 1)]

sage: W = ReflectionGroup(['A',3])                    # optional - gap3
sage: W.almost_positive_roots()                         # optional - gap3
[(-1, 0, 0),
 (0, -1, 0),
 (0, 0, -1),
 (1, 0, 0),
 (0, 1, 0),
 (0, 0, 1),
 (1, 1, 0),
 (0, 1, 1),
 (1, 1, 1)]

bipartite_index_set()

Return the bipartite index set of a real reflection group.

EXAMPLES:

sage: W = ReflectionGroup(['A',5])                    # optional - gap3
sage: W.bipartite_index_set()                          # optional - gap3
(continues on next page)
\[
[[1, 3, 5], [2, 4]]
\]

\[
\text{sage: } W = \text{ReflectionGroup(['A', 5], index_set=['a', 'b', 'c', 'd', 'e'])} \quad \# \text{optional - gap3}
\]

\[
\text{sage: } W_.bipartite_index_set() \quad \# \text{optional - gap3}
\]

\[
[[\text{'a'}, \text{'c'}, \text{'e'}], [\text{'b'}, \text{'d'}]]
\]

**bruhat_cone**(*x*, *y*, *side*='upper', *backend*='cdd')

Return the (upper or lower) Bruhat cone associated to the interval \([x, y]\).

To a cover relation \(v \prec w\) in strong Bruhat order you can assign a positive root \(\beta\) given by the unique reflection \(s_\beta\) such that \(s_\beta v = w\).

The upper Bruhat cone of the interval \([x, y]\) is the non-empty, polyhedral cone generated by the roots corresponding to \(x \prec a\) for all atoms \(a\) in the interval. The lower Bruhat cone of the interval \([x, y]\) is the non-empty, polyhedral cone generated by the roots corresponding to \(c \prec y\) for all coatoms \(c\) in the interval.

**INPUT:**

- *x* - an element in the group \(W\)
- *y* - an element in the group \(W\)
- *side* (default: 'upper') – must be one of the following:
  - 'upper' - return the upper Bruhat cone of the interval \([x, y]\)
  - 'lower' - return the lower Bruhat cone of the interval \([x, y]\)
- *backend* – string (default: 'cdd'); the backend to use to create the polyhedron

**EXAMPLES:**

\[
\text{sage: } W = \text{ReflectionGroup(['A', 2])} \quad \# \text{optional - gap3}
\]

\[
\text{sage: } W = \text{ReflectionGroup(['E', 6])} \quad \# \text{optional - gap3}
\]

\[
\text{sage: } W = \text{ReflectionGroup(['E', 6])} \quad \# \text{optional - gap3}
\]

**REFERENCES:**

- [Dy1994]
- [JS2021]

**cartan_type**()

Return the Cartan type of \(self\).

**EXAMPLES:**
sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: W.cartan_type()  # optional - gap3
['A', 3]

sage: W = ReflectionGroup(['A',3], ['B',3])  # optional - gap3
sage: W.cartan_type()  # optional - gap3
A3xB3 relabelled by {1: 3, 2: 2, 3: 1}

coxeter_diagram()

Return the Coxeter diagram associated to self.

EXAMPLES:

sage: G = ReflectionGroup(['B',3])  # optional - gap3
sage: G.coxeter_diagram().edges(labels=True, sort=True)  # optional - gap3
[(1, 2, 4), (2, 3, 3)]

coxeter_matrix()

Return the Coxeter matrix associated to self.

EXAMPLES:

sage: G = ReflectionGroup(['A',3])  # optional - gap3
sage: G.coxeter_matrix()  # optional - gap3
[1 3 2]
[3 1 3]
[2 3 1]

fundamental_weight(i)

Return the fundamental weight with index i.

EXAMPLES:

sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: [ W.fundamental_weight(i) for i in W.index_set() ]  # optional - gap3
[(3/4, 1/2, 1/4), (1/2, 1, 1/2), (1/4, 1/2, 3/4)]

fundamental_weights()

Return the fundamental weights of self in terms of the simple roots.

The fundamental weights are defined by $s_j(\omega_i) = \omega_i - \delta_{i=j}\alpha_j$ for the simple reflection $s_j$ with corresponding simple roots $\alpha_j$.

In other words, the transpose Cartan matrix sends the weight basis to the root basis. Observe again that the action here is defined as a right action, see the example below.

EXAMPLES:

sage: W = ReflectionGroup(['A',3], ['B',2])  # optional - gap3
sage: W.fundamental_weights()  # optional - gap3
Finite family {1: (3/4, 1/2, 1/4, 0, 0), 2: (1/2, 1, 1/2, 0, 0), 3: (1/4, 1/2, \ldots \cdot 3/4, 0, 0), 4: (0, 0, 0, 1, 1/2), 5: (0, 0, 0, 1, 1)}

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(continues on next page)
Finite family \{1: (3/4, 1/2, 1/4), 2: (1/2, 1, 1/2), 3: (1/4, 1/2, 3/4)\}

```
sage: W = ReflectionGroup(['A',3])             # optional - gap3
sage: S = W.simple_reflections()                # optional - gap3
sage: N = W.fundamental_weights()               # optional - gap3
sage: for i in W.index_set():                   # optional - gap3
    ....: for j in W.index_set():
    ....:     print("{} {} {} {}".format(i, j, N[i], N[i]*S[j].to_matrix()))
1 1 (3/4, 1/2, 1/4) (-1/4, 1/2, 1/4)
1 2 (3/4, 1/2, 1/4) (3/4, 1/2, 1/4)
1 3 (3/4, 1/2, 1/4) (3/4, 1/2, 1/4)
2 1 (1/2, 1, 1/2) (1/2, 1, 1/2)
2 2 (1/2, 1, 1/2) (1/2, 0, 1/2)
2 3 (1/2, 1, 1/2) (1/2, 1, 1/2)
3 1 (1/4, 1/2, 3/4) (1/4, 1/2, 3/4)
3 2 (1/4, 1/2, 3/4) (1/4, 1/2, 3/4)
3 3 (1/4, 1/2, 3/4) (1/4, 1/2, -1/4)
```

**iteration (algorithm='breadth', tracking_words=True)**

Return an iterator going through all elements in self.

**INPUT:**

- **algorithm** (default: 'breadth') – must be one of the following:
  - 'breadth' - iterate over in a linear extension of the weak order
  - 'depth' - iterate by a depth-first-search
  - 'parabolic' - iterate by using parabolic subgroups
- **tracking_words** (default: True) – whether or not to keep track of the reduced words and store them in _reduced_word

**Note:** The fastest iteration is the parabolic iteration and the depth first algorithm without tracking words is second. In particular, 'depth' is ~1.5x faster than 'breadth'.

**Note:** The 'parabolic' iteration does not track words and requires keeping the subgroup corresponding to \( I \setminus \{ i \} \) in memory (for each \( i \) individually).

**EXAMPLES:**

```
sage: W = ReflectionGroup(['B',2])             # optional - gap3
sage: for w in W.iteration("breadth",True):    # optional - gap3
    ....: print("%s %s%(w, w._reduced_word))
() []
(1,3)(2,6)(5,7) [1]
(1,5)(2,4)(6,8) [0]
(1,7,5,3)(2,4,6,8) [0, 1]
(1,3,5,7)(2,8,6,4) [1, 0]
(2,8)(3,7)(4,6) [1, 0, 1]
```

(continues on next page)
positive_roots()  
Return the positive roots of self.

EXAMPLES:

sage: W = ReflectionGroup(['A',3], ['B',2])  # optional - gap3
sage: W.positive_roots()  # optional - gap3
[(1, 0, 0, 0, 0),
 (0, 1, 0, 0, 0),
 (0, 0, 1, 0, 0),
 (0, 0, 0, 1, 0),
 (0, 0, 0, 0, 1),
 (1, 0, 1, 0, 0),
 (0, 1, 0, 1, 0),
 (0, 0, 0, 1, 1),
 (1, 1, 1, 0, 0),
 (0, 0, 0, 2, 1)]

sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: W.positive_roots()  # optional - gap3
[(1, 0, 0), (0, 1, 0), (0, 0, 1),
 (1, 1, 0), (0, 1, 1), (1, 1, 1)]

reflection_to_positive_root(r)  
Return the positive root orthogonal to the given reflection.

EXAMPLES:

sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: for r in W.reflections():  # optional - gap3
 ....:    print(W.reflection_to_positive_root(r))
 (1, 0)
 (0, 1)
 (1, 1)

right_coset_representatives(J)  
Return the right coset representatives of self for the parabolic subgroup generated by the simple reflections in J.

EXAMPLES:
sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: for J in Subsets([1,2,3]): W.right_coset_representatives(J)  # optional - gap3

[(), (2,5)(3,9)(4,6)(8,11)(10,12), (1,4)(2,8)(3,5)(7,10)(9,11),
 (1,7)(2,4)(5,6)(8,10)(11,12), (1,2,10)(3,6,5)(4,7,8)(9,12,11),
 (1,4,6)(2,3,11)(5,8,9)(7,10,12), (1,6,4)(2,11,3)(5,9,8)(7,12,10),
 (1,7)(2,6)(3,9)(4,5)(8,12)(10,11),
 (1,10,2)(3,5,6)(4,8,7)(9,11,12), (1,2,3,12)(4,5,10,11)(6,7,8,9),
 (1,5,9,10)(2,12,8,6)(3,4,7,11), (1,6)(2,9)(3,8)(5,11)(7,12),
 (1,8)(2,7)(3,6)(4,10)(9,12), (1,10,9,5)(2,6,8,12)(3,11,7,4),
 (1,12,3,2)(4,11,10,5)(6,9,8,7), (1,3)(2,12)(4,10)(5,11)(6,8)(7,9),
 (1,5,12)(2,9,4)(3,10,8)(6,7,11), (1,8,11)(2,5,7)(3,12,4)(6,10,9),
 (1,11,8)(2,7,5)(3,4,12)(6,9,10), (1,12,5)(2,4,9)(3,8,10)(6,11,7),
 (1,3,7,9)(2,11,6,10)(4,8,5,12), (1,9,7,3)(2,10,6,11)(4,12,5,8),
 (1,11)(3,10)(4,9)(5,7)(6,12), (1,9)(2,8)(3,7)(4,11)(5,10)(6,12)]

[(), (2,5)(3,9)(4,6)(8,11)(10,12), (1,4)(2,8)(3,5)(7,10)(9,11),
 (1,2,10)(3,6,5)(4,7,8)(9,12,11), (1,10,2)(3,6,5)(4,7,8)(9,12,11),
 (1,12,3,2)(4,11,10,5)(6,9,8,7), (1,3)(2,12)(4,10)(5,11)(6,8)(7,9),
 (1,5,12)(2,9,4)(3,10,8)(6,7,11), (1,8,11)(2,5,7)(3,12,4)(6,10,9),
 (1,11,8)(2,7,5)(3,4,12)(6,9,10), (1,12,5)(2,4,9)(3,8,10)(6,11,7),
 (1,11)(3,10)(4,9)(5,7)(6,12)]

[(), (1,4)(2,8)(3,5)(7,10)(9,11), (1,7)(2,4)(5,6)(8,10)(11,12),
 (1,2,10)(3,6,5)(4,7,8)(9,12,11), (1,6,4)(2,11,3)(5,9,8)(7,12,10),
 (1,10,2)(3,5,6)(4,8,7)(9,11,12), (1,5,9,10)(2,12,8,6)(3,4,7,11),
 (1,8)(2,7)(3,6)(4,10)(9,12), (1,12)(3,2)(4,11,10,5)(6,9,8,7),
 (1,11)(4,9)(5,7)(6,12), (1,9,7,3)(2,10,6,11)(4,12,5,8),
 (1,11)(3,10)(4,9)(5,7)(6,12)]

[(), (1,2,3)(5,6), (1,3)(2,5)(4,6), (1,5)(2,4)(3,6)]

root_to_reflection(root)

Return the reflection along the given root.

EXAMPLES:

sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: for beta in W.roots(): W.root_to_reflection(beta)  # optional - gap3

(1,4)(2,8)(3,5)
(1,3)(2,5)(4,6)
(1,5)(2,4)(3,6)
simple_root_index(i)

Return the index of the simple root $\alpha_i$.

This is the position of $\alpha_i$ in the list of simple roots.

EXAMPLES:

```
sage: W = ReflectionGroup(['A',3]) # optional - gap3
sage: [W.simple_root_index(i) for i in W.index_set()] # optional - gap3
[0, 1, 2]
```

sage.combinat.root_system.reflection_group_real.ReflectionGroup(*args, **kwds)

Construct a finite (complex or real) reflection group as a Sage permutation group by fetching the permutation representation of the generators from chevie’s database.

INPUT:

can be one or multiple of the following:

- a triple $(r, p, n)$ with $p$ divides $r$, which denotes the group $G(r, p, n)$
- an integer between 4 and 37, which denotes an exceptional irreducible complex reflection group
- a finite Cartan-Killing type

EXAMPLES:

Finite reflection groups can be constructed from

Cartan-Killing classification types:

```
sage: W = ReflectionGroup(['A',3]); W # optional - gap3
Irreducible real reflection group of rank 3 and type A3
sage: W = ReflectionGroup(['H',4]); W # optional - gap3
Irreducible real reflection group of rank 4 and type H4
sage: W = ReflectionGroup(['I',5]); W # optional - gap3
Irreducible real reflection group of rank 2 and type I2(5)
```

the complex infinite family $G(r, p, n)$ with $p$ divides $r$:

```
sage: W = ReflectionGroup((1,1,4)); W # optional - gap3
Irreducible real reflection group of rank 3 and type A3
sage: W = ReflectionGroup((2,1,3)); W # optional - gap3
Irreducible real reflection group of rank 3 and type B3
```

Chevalley-Shepard-Todd exceptional classification types:

```
sage: W = ReflectionGroup(23); W # optional - gap3
Irreducible real reflection group of rank 3 and type H3
```

Cartan types and matrices:
```python
sage: ReflectionGroup(CartanType(['A',2]))  # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: ReflectionGroup(CartanType([[ 'A',2],[ 'A',2]]))  # optional - gap3
Reducible real reflection group of rank 4 and type A2 x A2
sage: C = CartanMatrix(['A',2])  # optional - gap3
sage: ReflectionGroup(C)  # optional - gap3
Irreducible real reflection group of rank 2 and type A2

multiples of the above:

```python
sage: W = ReflectionGroup(['A',2],[ 'B',2]); W  # optional - gap3
Reducible real reflection group of rank 4 and type A2 x B2
sage: W = ReflectionGroup(['A',2],[4]); W  # optional - gap3
Reducible complex reflection group of rank 4 and type A2 x ST4
sage: W = ReflectionGroup((4,2,2),4); W  # optional - gap3
Reducible complex reflection group of rank 4 and type G(4,2,2) x ST4

```

sage.combinat.root_systemreflection_group_real.is_chevie_available()
Test whether the GAP3 Chevie package is available.

EXAMPLES:

```python
sage: from sage.combinat.root_systemreflection_group_realisatis Chevie_available
sage: is_chevie_available()  # random
False
sage: is_chevie_available() in [True, False]
True

5.1.237 Group algebras of root lattice realizations

```

```python
5.1.237 Group algebras of root lattice realizations

```

```python
```

```python
```
sage: L = RootSystem(["A",3]).ambient_space()
sage: W = L.weyl_group()
sage: M = L.algebra(QQ['q', 't'])
sage: m = M.an_element(); m  # TODO: investigate why we don't get something more interesting
B[(2, 2, 3, 0)]
sage: m = (m+1)^2; m
B[(0, 0, 0, 0)] + 2*B[(2, 2, 3, 0)] + B[(4, 4, 6, 0)]
sage: w = W.an_element(); w.reduced_word()
[1, 2, 3]
sage: m.acted_upon(w)
B[(0, 0, 0, 0)] + 2*B[(0, 2, 2, 3)] + B[(0, 4, 4, 6)]

expand(alphabet)

Expand self into variables in the alphabet.

INPUT:
  • alphabet – a non empty list/tuple of (invertible) variables in a ring to expand in

EXAMPLES:

sage: L = RootSystem(["A",2]).ambient_lattice()
sage: KL = L.algebra(QQ)
sage: p = KL.an_element() + KL.sum_of_monomials(L.some_elements()); p
B[(1, 0, 0)] + B[(1, -1, 0)] + B[(1, 1, 0)] + 2*B[(2, 2, 3)] + B[(0, 1, -1)]
sage: F = LaurentPolynomialRing(QQ, "x,y,z")
sage: p.expand(F.gens())
2*x^2*y^2*z^3 + x*y + x + y*z^-1 + x*y^-1

class ParentMethods

Bases: object

T0_check_on_basis(q1, q2, convention='antidominant')

Return the $T_0^\vee$ operator acting on the basis.

This implements the formula for $T_0^\vee$ in Section 6.12 of [Haiman06].

REFERENCES:

Warning: The current implementation probably returns just nonsense, if the convention is not "dominant".

EXAMPLES:

sage: K = QQ['q1,q2'].fraction_field()
sage: q1,q2 = K.gens()
sage: L = RootSystem(["A",1,1]).ambient_space()
sage: L0 = L.classical()
sage: KL = L.algebra(K)
sage: some_weights = L.fundamental_weights()
sage: f = KL.T0_check_on_basis(q1,q2, convention="dominant")
sage: f(L0.zero())
(q1+q2)*B[(0, 0)] + q1*B[(1, -1)]

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sage: L = RootSystem(['A',3,1]).ambient_space()
sage: L0 = L.classical()
sage: KL = L.algebra(K)
sage: some_weights = L0.fundamental_weights()
sage: f = KL.T0_check_on_basis(q1,q2, convention="dominant")
sage: f(L0.zero())
# not checked
(q1+q2)*B[(0, 0, 0, 0)] + q1^3/q2^2*B[(1, 0, 0, -1)]

The following results have not been checked:

sage: for x in some_weights:
....: print("{}: {}".format(x, f(x)))
(1, 0, 0, 0) : q1*B[(1, 0, 0, 0)]
(1, 1, 0, 0) : q1*B[(1, 1, 0, 0)]
(1, 1, 1, 0) : q1*B[(1, 1, 1, 0)]

Some examples for type $B_2^{(1)}$ dual:

sage: L = RootSystem("B2~*").ambient_space()
sage: L0 = L.classical()
sage: e = L.basis()
sage: K = QQ['q,u'].fraction_field()
sage: q,u = K.gens()
sage: q1 = u
sage: q2 = -1/u
sage: KL = L.algebra(K)
sage: KL0 = KL.classical()
sage: f = KL.T0_check_on_basis(q1,q2, convention="dominant")
sage: T = KL.twisted_demazure_lusztig_operators(q1,q2, convention="dominant")

Direct calculation:

sage: T.Tw(0)(KL0.monomial(L0([0,0])))
((u^2-1)/u)*B[(0, 0)] + u^3*B[(1, 1)]
sage: KL.T0_check_on_basis(q1,q2, convention="dominant")(L0([0,0]))
((u^2-1)/u)*B[(0, 0)] + u^3*B[(1, 1)]

Step by step calculation, comparing by hand with Mark Shimozono:

sage: res = T.Tw_inverse(1)(res); res
u*B[(0, 0)]
sage: res = res * KL0.monomial(L0([-1,1])); res
u*B[(-1, 1)]
sage: res = T.Tw_inverse(2)(res); res
((u^2-1)/u)*B[(0, 0)] + u^3*B[(1, 1)]

cartan_type()  

Return the Cartan type of self.

EXAMPLES:
```python
sage: A = RootSystem(['A',2,1]).ambient_space().algebra(QQ)
sage: A.cartan_type()
['A', 2, 1]
sage: A = RootSystem(['B',2]).weight_space().algebra(QQ)
sage: A.cartan_type()
['B', 2]
```

```python
classical()

Return the group algebra of the corresponding classical lattice.

EXAMPLES:
```n
```python
sage: KL = RootSystem(['A',2,1]).ambient_space().algebra(QQ)
sage: KL.classical()
Algebra of the Ambient space of the Root system of type ['A', 2] over Rational Field
```

```python
demazure_lusztig_operator_on_basis(weight, i, q1, q2, convention='antidominant')

Return the result of applying the \(i\)-th Demazure-Lusztig operator on weight.

INPUT:

• weight – an element \(\lambda\) of the weight lattice
• i – an element of the index set
• q1,q2 – two elements of the ground ring
• convention – “antidominant”, “bar”, or “dominant” (default: “antidominant”)

See `demazure_lusztig_operators()` for the details.

EXAMPLES:
```n
```python
sage: L = RootSystem(['A',1]).ambient_space()
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KL = L.algebra(K)
sage: KL.demazure_lusztig_operator_on_basis(L((2,2)), 1, q1, q2)
q1*B[(2, 2)]
sage: KL.demazure_lusztig_operator_on_basis(L((3,0)), 1, q1, q2)
(q1+q2)*B[(1, 2)] + (q1+q2)*B[(2, 1)] + (q1+q2)*B[(3, 0)] + q1*B[(0, 3)]
sage: KL.demazure_lusztig_operator_on_basis(L((0,3)), 1, q1, q2)
(-q1-q2)*B[(1, 2)] + (-q1-q2)*B[(2, 1)] + (-q2)*B[(3, 0)]
```

At \(q_1 = 1\) and \(q_2 = 0\) we recover the action of the isobaric divided differences \(\pi_i\):

```python
sage: KL.demazure_lusztig_operator_on_basis(L((2,2)), 1, 1, 0)
B[(2, 2)]
sage: KL.demazure_lusztig_operator_on_basis(L((3,0)), 1, 1, 0)
B[(1, 2)] + B[(2, 1)] + B[(3, 0)] + B[(0, 3)]
sage: KL.demazure_lusztig_operator_on_basis(L((0,3)), 1, 1, 0)
-B[(1, 2)] - B[(2, 1)]
```

Or \(1 - \pi_i\), for bar=True:

```python
sage: KL.demazure_lusztig_operator_on_basis(L((2,2)), 1, 1, 0, convention="bar")
0
```

(continues on next page)
sage: KL.demazure_lusztig_operator_on_basis(L((3,0)), 1, 1, 0, convention="bar")
-B[(1, 2)] - B[(2, 1)] - B[(0, 3)]

sage: KL.demazure_lusztig_operator_on_basis(L((0,3)), 1, 1, 0, convention="bar")
B[(1, 2)] + B[(2, 1)] + B[(0, 3)]

At $q_1 = 1$ and $q_2 = -1$ we recover the action of the simple reflection $s_i$:

sage: KL.demazure_lusztig_operator_on_basis(L((2,2)), 1, 1, -1)
B[(2, 2)]

sage: KL.demazure_lusztig_operator_on_basis(L((3,0)), 1, 1, -1)
B[(0, 3)]

sage: KL.demazure_lusztig_operator_on_basis(L((0,3)), 1, 1, -1)
B[(3, 0)]

demazure_lusztig_operator_on_classical_on_basis(weight, i, q, q1, q2, convention='antidominant')

Return the result of applying the $i$-th Demazure-Lusztig operator on the classical weight weight embedded at level 0.

INPUT:
• weight – a classical weight $\lambda$
• $i$ – an element of the index set
• $q_1,q_2$ – two elements of the ground ring
• convention – “antidominant”, “bar”, or “dominant” (default: “antidominant”)

See demazure_lusztig_operators() for the details.

Todo:
• Optimize the code to only do the embedding/projection for $T_0$
• Add an option to specify at which level one wants to work. Currently this is level 0.

EXAMPLES:

sage: L = RootSystem(['A',1,1]).ambient_space()
sage: L0 = L.classical()
sage: K = QQ['q,q1,q2']
sage: q, q1, q2 = K.gens()
sage: KL = L.algebra(K)
sage: KL0 = L0.algebra(K)

These operators coincide with the usual Demazure-Lusztig operators:

sage: KL.demazure_lusztig_operator_on_classical_on_basis(L0((2,2)), 1, q,␣
˓→q1, q2)
q1*B[(2, 2)]

sage: KL0.demazure_lusztig_operator_on_basis(L0((2,2)), 1, q1, q2)
q1*B[(2, 2)]

sage: KL.demazure_lusztig_operator_on_classical_on_basis(L0((3,0)), 1, q,␣
˓→q1, q2)
(q1+q2)*B[(1, 2)] + (q1+q2)*B[(2, 1)] + (q1+q2)*B[(3, 0)] + q1*B[(0, 3)]
except that we now have an action of $T_0$, which introduces some $q$ s:

```
sage: KL0.demazure_lusztig_operator_on_classical_on_basis(L0((2,2)), 0, q, \rightarrow q1, q2)
q1*B[(2, 2)]
sage: KL.demazure_lusztig_operator_on_classical_on_basis(L0((3,0)), 0, q, \rightarrow q1, q2)
(-q^2*q1-q^2*q2)*B[(1, 2)] + (-q*q1-q*q2)*B[(2, 1)] + (-q^3*q2)*B[(0, 3)]
```

### demazure_lusztig_operators($q1, q2, convention='antidominant'\')

Return the Demazure-Lusztig operators acting on `self`.

**INPUT:**
- `q1, q2` – two elements of the ground ring
- `convention` – “antidominant”, “bar”, or “dominant” (default: “antidominant”)

If $R$ is the parent weight ring, the Demazure-Lusztig operator $T_i$ is the linear map $R \rightarrow R$ obtained by interpolating between the isobaric divided difference operator $\pi_i$ (see `isobaric_divided_difference_on_basis()`) and the simple reflection $s_i$.

$$(q_1 + q_2)\pi_i - q_2 s_i$$

The Demazure-Lusztig operators give the usual representation of the operator $T_i$ of the (affine) Hecke algebra with eigenvalues $q_1$ and $q_2$ associated to the Weyl group.

Several variants are available to match with various conventions used in the literature:
- “bar” replaces $\pi_i$ in the formula above by $\pi_i = (1 - \pi_i)$.
- “dominant” conjugates the operator by $x\lambda \mapsto x^{-\lambda}$.

The names dominant and antidominant for the conventions were chosen with regards to the nonsymmetric Macdonald polynomials. The $Y$ operators for the Macdonald polynomials in the “dominant” convention satisfy $Y_\lambda = T_\lambda$ for $\lambda$ dominant. This is also the convention used in \cite{Haiman06}. For the “antidominant” convention, $Y_\lambda = T_\lambda$ with $\lambda$ antidominant.

**See also:**
- `demazure_lusztig_operator_on_basis()`.  
- `NonSymmetricMacdonaldPolynomials`.

**REFERENCES:**

**EXAMPLES:**

```
sage: L = RootSystem(['A',1]).ambient_space()
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KL = L.algebra(K)
sage: T = KL.demazure_lusztig_operators(q1, q2)
sage: Tbar = KL.demazure_lusztig_operators(q1, q2, convention='bar')
sage: Tdominant = KL.demazure_lusztig_operators(q1, q2, convention='dominant')
sage: x = KL.monomial(L((3,0)))
sage: T[1](x)
(q1+q2)*B[(1, 2)] + (q1+q2)*B[(2, 1)] + (q1+q2)*B[(3, 0)] + q1*B[(0, 3)]
```

(continues on next page)
We repeat similar computation in the affine setting:

```python
sage: L = RootSystem("A",2,1).ambient_space()
sage: K = QQ["q1,q2"].fraction_field()
sage: q1, q2 = K.gens()
sage: KL = L.algebra(K)
sage: T = KL.demazure_lusztig_operators(q1, q2)
sage: Tbar = KL.demazure_lusztig_operators(q1, q2, convention="bar")
sage: Tdominant = KL.demazure_lusztig_operators(q1, q2, convention="dominant")
sage: e = L.basis()
sage: x = KL.monomial(3*e[0])
sage: T[1](x)
(q1^2+2*q1*q2+q2^2)*B[e[0] + 2*e[1]] + (q1^2+2*q1*q2+q2^2)*B[2*e[0] + e[1]] + (q1^2+2*q1*q2+q2^2)*B[3*e[0]] + q1^2*B[3*e[2]]
```

and use that to check, for example, the braid relations:

```python
sage: T[1,2,1](x) - T[2,1,2](x)
0
```

The operators satisfy the relations of the affine Hecke algebra with parameters $q_1, q_2$:

```python
sage: T._test_relations()
sage: Tdominant._test_relations()
```
sage: Tbar._test_relations()  # -q2,q1+2*q2  # todo: not implemented: set the appropriate eigenvalues!

And the $\bar{T}$ are basically the inverses of the $T$ s:

```python
sage: Tinv = KL.demazure_lusztig_operators(2/q1+1/q2, -1/q1, convention="bar")
sage: [Tinv[1](T[1](x))-x for x in KL.some_elements()]
[0, 0, 0, 0, 0, 0, 0]
```

We check that $\Lambda_1 - \Lambda_0$ is an eigenvector for the $Y$ s in affine type:

```python
sage: K = QQ['q,q1,q2'].fraction_field()
sage: q,q1,q2=K.gens()
sage: L = RootSystem(["A",2,1]).ambient_space()
sage: L0 = L.classical()
sage: Lambda = L.fundamental_weights()
sage: alphacheck = L0.simple_coroots()
sage: KL = L.algebra(K)
sage: T = KL.demazure_lusztig_operators(q1, q2, convention="dominant")
sage: Y = T.Y()
sage: alphacheck = Y.keys().alpha()  # alpha of coroot lattice is alphacheck
sage: alphacheck
Finite family {0: alphacheck[0], 1: alphacheck[1], 2: alphacheck[2]}
sage: x = KL.monomial(Lambda[1]-Lambda[0]); x
B[e[0]]
```

In fact it is not exactly an eigenvector, but the extra ‘delta’ term is to be interpreted as a $q$ parameter:

```python
sage: Y[alphacheck[0]](KL.one())
q2^2/q1^2*B[0]
sage: Y[alphacheck[1]](x)
((-q2^2)/(-q1^2))*B[e[0]]-e['delta']
sage: Y[alphacheck[2]](x)
(q1/(-q2))*B[e[0]]
sage: KL.q_project(Y[alphacheck[1]](x),q)
((-q2^2)/(-q*q1^2))*B[(1, 0, 0)]
sage: KL.q_project(x, q)
B[(1, 0, 0)]
sage: KL.q_project(Y[alphacheck[0]](x),q)
((-q*q1)/q2)*B[(1, 0, 0)]
sage: KL.q_project(Y[alphacheck[1]](x),q)
((-q2^2)/(-q*q1^2))*B[(1, 0, 0)]
sage: KL.q_project(Y[alphacheck[2]](x),q)
(q1/(-q2))*B[(1, 0, 0)]
```

We now check systematically that the Demazure-Lusztig operators satisfy the relations of the Iwahori-Hecke algebra:

```python
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: for cartan_type in CartanType.samples(crystallographic=True): # long
   time 12s
```
Recall that the Demazure-Lusztig operators are only defined when all monomials belong to the weight lattice. Thus, in the group algebra of the ambient space, we need to specify explicitly the elements on which to run the tests:

```python
sage: for cartan_type in CartanType.samples(crystallographic=True):  # long
    # time 12s
    L = RootSystem(cartan_type).ambient_space()
    KL = L.algebra(K)
    weight_lattice = RootSystem(cartan_type).weight_lattice(extended=L.is_extended())
    elements = [ KL.monomial(L(weight)) for weight in weight_lattice.some_elements() ]
    T = KL.demazure_lusztig_operators(q1,q2)
    T._test_relations(elements=elements)
```

demazure_lusztig_operators_on_classical($q$, $q1$, $q2$, convention='antidominant')

Return the Demazure-Lusztig operators acting at level 1 on self.classical().

**INPUT:**
- $q, q1, q2$ – three elements of the ground ring
- convention – “antidominant”, “bar”, or “dominant” (default: “antidominant”)

Let $KL$ be the group algebra of an affine weight lattice realization $L$. The Demazure-Lusztig operators for $KL$ act on the group algebra of the corresponding classical weight lattice by embedding it at level 1, and projecting back.

**See also:**
- demazure_lusztig_operators().
- demazure_lusztig_operator_on_classical_on_basis().
- q_project()
For $T_1, \ldots$ we recover the usual Demazure-Lusztig operators:

\[
\text{sage: } T[1](x) \\
(q_1+q_2)*B[(1, 2)] + (q_1+q_2)*B[(2, 1)] + (q_1+q_2)*B[(3, 0)] + q_1*B[(0, 3)]
\]

For $T_0$, we can note that, in the projection, $\delta$ is mapped to $q$:

\[
\text{sage: } T[0](x) \\
(-q^2*q_1-q^2*q_2)*B[(1, 2)] + (-q*q_1-q*q_2)*B[(2, 1)] + (-q^3*q_2)*B[(0, 3)]
\]

Note that there is no translation part, and in particular 1 is an eigenvector for all $T_i$'s:

\[
\text{sage: } T[0](KL0.one()) \\
q_1*B[(0, 0)]
\]

\[
\text{sage: } T[1](KL0.one()) \\
q_1*B[(0, 0)]
\]

\[
\text{sage: } Y = T.Y() \\
\text{sage: } alphacheck=Y.keys().simple_roots() \\
\text{sage: } Y[alphacheck[0]](KL0.one()) \\
((-q^2)/(q*q_1))*B[(0, 0)]
\]

Matching with Ion Bogdan’s hand calculations from 3/15/2013:

\[
\text{sage: } L = \text{RootSystem(["A",1,1]).weight_space(extended=True)} \\
\text{sage: } K = \text{QQ[\{q,u\}].fraction_field()} \\
\text{sage: } q, u = K.gens() \\
\text{sage: } KL = L.algebra(K) \\
\text{sage: } KL0 = KL.classical() \\
\text{sage: } omega = KL0.basis().keys() \\
\text{sage: } omega = KL0.fundamental_weights() \\
\text{sage: } T = KL.demazure_lusztig_operators_on_classical(q, u, -1/u, convention="dominant") \\
\text{sage: } Y = T.Y() \\
\text{sage: } alphacheck = Y.keys().simple_roots() \\
\text{sage: } Y[alphacheck[0]].word, Y[alphacheck[0]].signs, Y[alphacheck[0]].scalar \\
((0, 0), (-1, -1), 1/q)
\]

\[
\text{sage: } Y[alphacheck[1]].word, Y[alphacheck[1]].signs, Y[alphacheck[1]].scalar \\
# This is $T_0 T_1 (T_1$ acts first, then $T_0$); Ion gets $T_1 T_0$ \\
((1, 0), (1, 1), 1) \\
\]

\[
\text{sage: } Y[0].word, Y[0].signs, Y[0].scalar \\
# This is $1/q T_1^{\ominus 1} T_0^{\ominus 1}$ \\
((0, 1), (-1, -1), 1/q)
\]

Note that the following computations use the “dominant” convention:
demazure_operators()

Return the Demazure operators acting on self.

The \( i \)-th Demazure operator is defined by:

\[
\pi_i = \frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}
\]

It acts on \( e^\lambda \), for \( \lambda \) a weight, by:

\[
\pi_i e^\lambda = \frac{e^\lambda - e^{-\alpha_i + s_i \lambda}}{1 - e^{-\alpha_i}}
\]

This matches with Lascoux’ definition \([\text{Lascoux2003}]\) of \( \pi_i \), and with the \( i \)-th Demazure operator of \([\text{Kumar1987}]\), which also works for general Kac-Moody types.

REFERENCES:

EXAMPLES:

We compute some Schur functions, as images of dominant monomials under the action of the maximal isobaric divided difference \( \Delta_{w_0} \):

\begin{verbatim}
sage: L = RootSystem(['A',2]).ambient_lattice()
sage: KL = L.algebra(QQ)
sage: w0 = tuple(L.weyl_group().long_element().reduced_word())
sage: pi = KL.demazure_operators()
sage: pi0 = pi[w0]
sage: pi0(KL.monomial(L((2,1))))
2*B[(1, 1, 1)] + B[(1, 2, 0)] + B[(1, 0, 2)] + B[(2, 1, 0)] + B[(2, 0, 1)]
˓→ B[(0, 1, 2)] + B[(0, 2, 1)]
\end{verbatim}

Let us make the result into an actual polynomial:

\begin{verbatim}
sage: P = QQ['x,y,z']
sage: pi0(KL.monomial(L((2,1)))).expand(P.gens())
x^2*z + x*y^2 + x^2*z + 2*x*y^*z + y^2*z + x*z^2 + y*z^2
\end{verbatim}

This is indeed a Schur function:

\begin{verbatim}
sage: s = SymmetricFunctions(QQ).s()
sage: s[2,1].expand(3, P.variable_names())
x^2*y + x*y^2 + x^2*z + 2*x*y*z + y^2*z + x*z^2 + y*z^2
\end{verbatim}

Let us check this systematically on Schur functions of degree 6:
We check systematically that these operators satisfy the Iwahori-Hecke algebra relations:

```python
sage: for cartan_type in CartanType.samples(crystallographic=True): # long˓→time
    L = RootSystem(cartan_type).weight_lattice()
    KL = L.algebra(QQ)
    T = KL.demazure_operators()
    T._test_relations()
```

```python
sage: L = RootSystem(['A',1,1]).weight_lattice()
```

```python
sage: KL = L.algebra(QQ)
```

```python
sage: T = KL.demazure_operators()
```

```
sage: T._test_relations()
```

**Warning:** The Demazure operators are only defined if all the elements in the support have integral scalar products with the coroots (basically, they are in the weight lattice). Otherwise an error is raised:

```python
sage: L = RootSystem(CartanType(['G',2]).dual()).ambient_space()
```

```python
sage: KL = L.algebra(QQ)
```

```python
sage: pi = KL.demazure_operators()
```

```python
sage: pi[1](KL.monomial(L([0,0,1])))
Traceback (most recent call last):
...
ValueError: the weight does not have an integral scalar product with the␣˓→coroot
```

**divided_difference_on_basis**(*weight, i*)

Return the result of applying the *i*-th divided difference on *weight*.

**INPUT:**
- *weight* – a weight
- *i* – an element of the index set

**Todo:** type free definition (Viviane’s definition uses that we are in the ambient space)

**EXAMPLES:**

```python
sage: L = RootSystem(["A",1]).ambient_space()
sage: KL = L.algebra(QQ)
sage: KL.divided_difference_on_basis(L((2,2)), 1) # todo: not implemented
```

In type $A$ and in the ambient lattice, we recover the usual action of divided differences polynomials:
sage: x, y = QQ['x,y'].gens()
sage: d = lambda p: (p - p(y,x)) / (x-y)
sage: d(x^2*y^2)
 0
sage: d(x^3)
x^2 + x*y + y^2
sage: d(y^3)
-x^2 - x*y - y^2

from_polynomial(p)

Construct an element of self from a polynomial p.

INPUT:
  • p – a polynomial

EXAMPLES:

sage: L = RootSystem(['A',2]).ambient_lattice()
sage: KL = L.algebra(QQ)
sage: x,y,z = QQ['x,y,z'].gens()
sage: KL.from_polynomial(x)
B[(1, 0, 0)]
sage: KL.from_polynomial(x^2*y + 2*y - z)
B[(2, 1, 0)] + 2*B[(0, 1, 0)] - B[(0, 0, 1)]

Todo: make this work for Laurent polynomials too

isobaric_divided_difference_on_basis(weight, i)

Return the result of applying the i-th isobaric divided difference on weight.

INPUT:
  • weight – a weight
  • i – an element of the index set

See also:

demazure_operators()

EXAMPLES:

sage: L = RootSystem(['A',1]).ambient_space()
sage: KL = L.algebra(QQ)
sage: KL.isobaric_divided_difference_on_basis(L((2,2)), 1)
B[(2, 2)]
sage: KL.isobaric_divided_difference_on_basis(L((3,0)), 1)
B[(1, 2)] + B[(2, 1)] + B[(3, 0)] + B[(0, 3)]
sage: KL.isobaric_divided_difference_on_basis(L((0,3)), 1)
-B[(1, 2)] - B[(2, 1)]

In type A and in the ambient lattice, we recover the usual action of divided differences on polynomials:

sage: x,y = QQ['x,y'].gens()
sage: d = lambda p: (x*p - (x*p)(y,x)) / (x-y)
sage: d(x^2*y^2)
x^2*y^2
(continues on next page)
sage: d(x^3)
x^3 + x*y + x*y^2 + y^3
sage: d(y^3)
-x^2*y - x*y^2

REFERENCES:

\textbf{\texttt{q_project}}(x, q)

Implement the \textit{q}-projection morphism from \texttt{self} to the group algebra of the classical space.

INPUT:

\cdot x – an element of the group algebra of \texttt{self}
\cdot q – an element of the ground ring

This is an algebra morphism mapping \(\delta\) to \(q\) and \(X^b\) to its classical counterpart for the other elements \(b\) of the basis of the realization.

EXAMPLES:

\begin{verbatim}
sage: K = QQ['q'].fraction_field()
sage: q = K.gen()
sage: KL = RootSystem(["A",2,1]).ambient_space().algebra(K)
sage: L = KL.basis().keys()
sage: e = L.basis()
sage: KL.q_project(x, q)
B[(2, 2, 3)] + 1/q^2*B[(0, 4, 3)]
\end{verbatim}

\begin{verbatim}
sage: KL = RootSystem(["BC",3,2]).ambient_space().algebra(K)
sage: L = KL.basis().keys()
sage: e = L.basis()
sage: KL.q_project(x, q)
B[(2, 2, 3)] + 1/q^2*B[(0, 4, 3)]
\end{verbatim}

\textbf{Warning:} Recall that the null root, usually denoted \(\delta\), is in fact \(a[0] \delta\) in Sage’s notation, in order to avoid half integer coefficients (this only makes a difference in type BC). Similarly, what’s usually denoted \(q\) is in fact \(q^a[0]\) in Sage’s notations, to avoid manipulating square roots:

\begin{verbatim}
sage: KL.q_project(KL.monomial(L.null_root()),q)
q^2*B[(0, 0, 0)]
\end{verbatim}

\textbf{\texttt{q_project_on_basis}}(l, q)

Return the monomial \(c * cl(l)\) in the group algebra of the classical lattice.

INPUT:

\cdot l – an element of the root lattice realization
\cdot q – an element of the ground ring

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Here, $c(l)$ is the projection of $l$ in the classical lattice, and $c$ is the coefficient of $l$ in $\delta$.

See also:

$q\_project\_on\_basis()$

EXAMPLES:

```
sage: K = QQ["q'].fraction_field()
sage: q = K.gen()
sage: KL = RootSystem("A",2,1).ambient_space().algebra(K)
sage: L = KL.basis().keys()
sage: e = L.basis()
sage: KL.q_project_on_basis( 4*e[1] + 3*e[2] + e['deltacheck'] - 2*e['delta'], q)
1/q^2*B[(0, 4, 3)]
```

some_elements()

Return some elements of the algebra self.

EXAMPLES:

```
sage: A = RootSystem("A",2,1).ambient_space().algebra(QQ)
sage: A.some_elements()
 B[-e[0] + e[2] + e['delta']],
 B[e[0] - e[1]],
 B[e[1] - e[2]],
 B[e['deltacheck']],
 B[e[0] + e['deltacheck']],
 B[e[0] + e[1] + e['deltacheck']]]
sage: A = RootSystem("B",2).weight_space().algebra(QQ)
sage: A.some_elements()
 B[-Lambda[1] + 2*Lambda[2]],
 B[Lambda[1]],
 B[Lambda[2]]]
```

twisted_demazure_lusztig_operator_on_basis(weight, i, q1, q2, convention='antidominant')

Return the twisted Demazure-Lusztig operator acting on the basis.

INPUT:

- weight – an element $\lambda$ of the weight lattice
- i – an element of the index set
- q1, q2 – two elements of the ground ring
- convention – “antidominant”, “bar”, or “dominant” (default: “antidominant”)

See also:

`twisted_demazure_lusztig_operators()`

EXAMPLES:

```
sage: L = RootSystem("A",3,1).ambient_space()
sage: e = L.basis()
sage: K = QQ["q1,q2'"].fraction_field()
```

(continues on next page)
sage: q1, q2 = K.gens()
sage: KL = L.algebra(K)
sage: Lambda = L.classical().fundamental_weights()
sage: KL.twisted_demazure_lusztig_operator_on_basis(Lambda[1]+2*Lambda[2], →1, q1, q2, convention="dominant")
(-q2)*B[(2, 3, 0, 0)]
sage: KL.twisted_demazure_lusztig_operator_on_basis(Lambda[1]+2*Lambda[2], →2, q1, q2, convention="dominant")
(-q1-q2)*B[(3, 1, 1, 0)] + (-q2)*B[(3, 0, 2, 0)]
sage: KL.twisted_demazure_lusztig_operator_on_basis(Lambda[1]+2*Lambda[2], →3, q1, q2, convention="dominant")
q1*B[(3, 2, 0, 0)]
sage: KL.twisted_demazure_lusztig_operator_on_basis(Lambda[1]+2*Lambda[2], →0, q1, q2, convention="dominant")
((q1*q2+q2^2)/q1)*B[(1, 2, 1, 1)] + ((q1*q2+q2^2)/q1)*B[(1, 2, 2, 0)] + q2^ →2/q1*B[(1, 2, 2, 0)] + ((q1*q2+q2^2)/q1)*B[(2, 1, 1, 1)] + ((q1^2+2*q1*q2+q2^2)/q1)*B[(2, →1, 2, 0)] + ((q1*q2+q2^2)/q1)*B[(2, 1, 0, 2)] + ((q1^2+2*q1*q2+q2^2)/q1)*B[(2, 2, 1, →0)] + ((q1*q2+q2^2)/q1)*B[(2, 2, 0, 1)]

twisted_demazure_lusztig_operators(q1, q2, convention='antidominant')

Return the twisted Demazure-Lusztig operators acting on self.

INPUT:
• q1, q2 – two elements of the ground ring
• convention – “antidominant”, “bar”, or “dominant” (default: “antidominant”)

Warning:
• the code is currently only tested for q1*q2 = -1
• only the “dominant” convention is functional for i = 0

For T₁, . . . , Tₙ, these operators are the usual Demazure-Lusztig operators. On the other hand, the operator T₀ is twisted:

sage: L = RootSystem(["A",3,1]).ambient_space()
sage: e = L.basis()
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KL = L.algebra(K)
sage: T = KL.twisted_demazure_lusztig_operators(q1, q2, convention="dominant 
...")
sage: T._test_relations()

Todo: Choose a good set of Cartan Type to run on. Rank >4 is too big. But C₁ and B₁ are boring.

We now check systematically that those operators satisfy the relations of the Iwahori-Hecke algebra:
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: for cartan_type in CartanType.samples(affine=True,crystallographic=True):  # long time 12s
    ....:     if cartan_type.rank() > 4: continue
    ....:     if cartan_type.type() == 'BC': continue
    ....:     KL = RootSystem(cartan_type).weight_lattice().algebra(K)
    ....:     T = KL.twisted_demazure_lusztig_operators(q1, q2, convention="dominant")
    ....:     T._test_relations()

todo: Investigate why $T_0^\vee$ currently does not satisfy the quadratic relation in type $BC$. This should hopefully be fixed when $T_0^\vee$ will have a more uniform implementation:

sage: cartan_type = CartanType(['BC',1,2])
sage: KL = RootSystem(cartan_type).weight_lattice().algebra(K)
sage: T = KL.twisted_demazure_lusztig_operators(q1,q2, convention="dominant")
sage: T._test_relations()
Traceback (most recent call last):
 ... tester.assertTrue(Ti(Ti(x,i,-q2),i,-q1).is_zero()) ... 
AssertionError: False is not true

Comparison with $T_0$:

sage: L = RootSystem(['A',2,1]).ambient_space()
sage: e = L.basis()
sage: K = QQ['t,q'].fraction_field()
sage: t,q = K.gens()
sage: q1 = t
sage: q2 = -1
sage: KL = L.algebra(K)
sage: L0 = L.classical()
sage: T = KL.demazure_lusztig_operators(q1,q2, convention="dominant")
sage: def T0(*l0): return KL.q_project(T[0].on_basis()(L.embed_at_level(L0(l0), 1)), q)
sage: def T0_check_on_basis = KL.T0_check_on_basis(q1, q2, convention="dominant")
sage: def T0c(*l0): return T0_check_on_basis(L0(l0))
sage: T0(0,0,1)  # not double checked
(-t+1)/q)*B[(1, 0, 0)] + 1/q^2*B[(2, 0, -1)]
sage: T0c(0,0,1) (t^2-t)*B[(1, 0, 0)] + (t^2-t)*B[(1, 1, -1)] + t^2*B[(2, 0, -1)] + (t-1)*B[(0, 0, 1)]
5.1.238 Root lattice realizations

```python
class sage.combinat.root_system.root_lattice_realizations.RootLatticeRealizations(base, name=None):
    Bases: Category_over_base_ring
    
    The category of root lattice realizations over a given base ring
    
    A root lattice realization \( L \) over a base ring \( R \) is a free module (or vector space if \( R \) is a field) endowed with an embedding of the root lattice of some root system.
    
    Typical root lattice realizations over \( \mathbb{Z} \) include the root lattice, weight lattice, and ambient lattice. Typical root lattice realizations over \( \mathbb{Q} \) include the root space, weight space, and ambient space.
    
    To describe the embedding, a root lattice realization must implement a method `simple_root()` returning for each \( i \) in the index set the image of the simple root \( \alpha_i \) under the embedding.
    
    A root lattice realization must further implement a method on elements `scalar()`, computing the scalar product with elements of the coroot lattice or coroot space.
    
    Using those, this category provides tools for reflections, roots, the Weyl group and its action, …
```

See also:

- `RootSystem`
- `WeightLatticeRealizations`
- `RootSpace`
- `WeightSpace`
- `AmbientSpace`

**EXAMPLES:**

Here, we consider the root system of type \( A_7 \), and embed the root lattice element \( x = \alpha_2 + 2\alpha_6 \) in several root lattice realizations:

```python
sage: R = RootSystem(['A',7])
sage: alpha = R.root_lattice().simple_roots()
sage: L = R.root_space()
sage: L(x)
sage: L = R.weight_lattice()
sage: L(x)
sage: L = R.ambient_space()
sage: L(x)
(0, 1, -1, 0, 2, -2, 0, 0)
```

We embed the root space element \( x = \alpha_2 + 1/2\alpha_6 \) in several root lattice realizations:
sage: alpha = R.root_space().simple_roots()

sage: L = R.weight_space()
sage: L(x)

sage: L = R.ambient_space()
sage: L(x)
(0, 1, -1, 0, 1/2, -1/2, 0, 0)

Of course, one can’t embed the root space in the weight lattice:

sage: L = R.weight_lattice()
sage: L(x)
Traceback (most recent call last):
  ...TypeError: do not know how to make x (= alpha[2] + 1/2*alpha[5]) an element of self
    (=Weight lattice of the Root system of type ['A', 7])

If $K_1$ is a subring of $K_2$, then one could in theory have an embedding from the root space over $K_1$ to any root lattice realization over $K_2$; this is not implemented:

sage: K1 = QQ
sage: K2 = QQ['q']
sage: L = R.weight_space(K2)

sage: alpha = R.root_space(K2).simple_roots()
sage: L(alpha[1])

sage: alpha = R.root_space(K1).simple_roots()
sage: L(alpha[1])
Traceback (most recent call last):
  ...TypeError: do not know how to make x (= alpha[1]) an element of self (=Weight space
    over the Univariate Polynomial Ring in q over Rational Field of the Root system
    of type ['A', 7])

By a slight abuse, the embedding of the root lattice is not actually required to be faithful. Typically for an affine root system, the null root of the root lattice is killed in the non extended weight lattice:

sage: R = RootSystem(['A', 3, 1])
sage: delta = R.root_lattice().null_root()
sage: L = R.weight_lattice()
sage: L(delta)
0

Algebras
   alias of Algebras

class ElementMethods
    Bases: object
affine_orbit()

The orbit of \texttt{self} under the dot or affine action of the Weyl group.

EXAMPLES:

```python
sage: L = RootSystem(['A', 2]).ambient_lattice()
sage: sorted(L.rho().dot_orbit())  # the output order is not specified
[(-2, 1, 4), (-2, 3, 2), (2, -1, 2), (2, 1, 0), (0, -1, 4), (0, 3, 0)]
```

```python
sage: L = RootSystem(['B', 2]).weight_lattice()
sage: sorted(L.fundamental_weights()[1].dot_orbit())  # the output order is not specified
[-4*Lambda[1], -4*Lambda[1] + 4*Lambda[2],
 Lambda[1], Lambda[1] - 6*Lambda[2],
```

We compare the dot action orbit to the regular orbit:

```python
sage: L = RootSystem(['A', 3]).weight_lattice()
sage: len(L.rho().dot_orbit())
24
sage: len((-L.rho()).dot_orbit())
1
sage: La = L.fundamental_weights()
sage: len(La[1].dot_orbit())
24
sage: len((-L.rho() + La[1]).dot_orbit())
4
sage: len((-L.rho() + La[1]).dot_orbit())
4
sage: len(La[2].dot_orbit())
24
sage: len(La[2].dot_orbit())
6
sage: len((-L.rho() + La[2]).dot_orbit())
6
```

associated_coroot()

Returns the coroot associated to this root

EXAMPLES:

```python
sage: alpha = RootSystem(['A', 3]).root_space().simple_roots()
sage: alpha[1].associated_coroot()
alphacheck[1]
```

associated_reflection()

Given a positive root \texttt{self}, returns a reduced word for the reflection orthogonal to \texttt{self}.

Since the answer is cached, it is a tuple instead of a list.

EXAMPLES:
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```
sage: RootSystem(['C',3]).root_lattice().simple_root(3).weyl_action([1,2]).
    ...associated_reflection()
(1, 2, 3, 2, 1)
sage: RootSystem(['C',3]).root_lattice().simple_root(2).associated_
    ...reflection()
(2,)
```

**descents**(index_set=None, positive=False)

Returns the descents of pt

**EXAMPLES:**

```
sage: space=RootSystem(['A',5]).weight_space()
sage: alpha=space.simple_roots()
[3, 5]
```

**dot_action**(w, inverse=False)

Act on self by w using the dot or affine action.

Let w be an element of the Weyl group. The dot action or affine action is given by:

\[ w \cdot \lambda = w(\lambda + \rho) - \rho, \]

where \( \rho \) is the sum of the fundamental weights.

**INPUT:**

• w – an element of a Coxeter or Weyl group of the same Cartan type, or a tuple or a list (such as a reduced word) of elements from the index set

• inverse – a boolean (default: False); whether to act by the inverse element

**EXAMPLES:**

```
sage: P = RootSystem(['B',3]).weight_lattice()
sage: La = P.fundamental_weights()
sage: mu.dot_action([1])
sage: mu.dot_action([3])
Lambda[1] + Lambda[3]
sage: mu.dot_action([1,2,3])
```

We check that the origin of this action is at \(-\rho\):

```
sage: all((-P.rho()).dot_action([i]) == -P.rho()
    ....:     for i in P.index_set())
True
```

**REFERENCES:**

• Wikipedia article Affine_action

**dot_orbit()**

The orbit of self under the dot or affine action of the Weyl group.

**EXAMPLES:**

```
We compare the dot action orbit to the regular orbit:

```python
sage: L = RootSystem(['A', 3]).weight_lattice()
sage: len(L.rho().dot_orbit())
24
sage: len((-L.rho()).dot_orbit())
1
sage: La = L.fundamental_weights()
sage: len(La[1].dot_orbit())
24
sage: len(La[1].orbit())
4
sage: len((-L.rho() + La[1]).dot_orbit())
4
sage: len(La[2].dot_orbit())
24
sage: len(La[2].orbit())
6
sage: len((-L.rho() + La[2]).dot_orbit())
6
```

---

e**xtraspecial_pair()**

Return the extraspecial pair of self under the ordering defined by `positive_roots_by_height()`.

The *extraspecial pair* of a positive root $\gamma$ with some total ordering $<$ of the root lattice that respects height is the pair of positive roots $(\alpha, \beta)$ such that $\gamma = \alpha + \beta$ and $\alpha$ is as small as possible.

**EXAMPLES:**

```python
sage: Q = RootSystem(['G', 2]).root_lattice()
sage: Q.highest_root().extraspecial_pair()
(alpha[2], 3*alpha[1] + alpha[2])
```

---

**first_descent**(index_set=None, positive=False)

Returns the first descent of pt

One can use the index_set option to restrict to the parabolic subgroup indexed by index_set.

**EXAMPLES:**
greater()  
Returns the elements in the orbit of self which are greater than self in the weak order.

EXAMPLES:

```
sage: L = RootSystem(['A',3]).ambient_lattice()
sage: e = L.basis()
sage: e[2].greater()
[(0, 0, 1, 0), (0, 0, 0, 1)]
sage: len(L.rho().greater())
24
sage: len((-L.rho()).greater())
1
sage: sorted([len(x.greater()) for x in L.rho().orbit()])
[1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, 8, 8, 8, 8, 12, 12, 12, 24]
```

has_descent($i$, positive=False)  
Test if self has a descent at position $i$, that is if self is on the strict negative side of the $i^{th}$ simple reflection hyperplane.

If positive if True, tests if it is on the strict positive side instead.

EXAMPLES:

```
sage: space=RootSystem(['A',5]).weight_space()
sage: alpha=space.simple_roots()
sage: [alpha[i].has_descent(1) for i in space.index_set()]
[False, True, False, False, False]
sage: [(-alpha[i]).has_descent(1) for i in space.index_set()]
[True, False, False, False, False]
sage: [alpha[i].has_descent(1, True) for i in space.index_set()]
[True, False, False, False, False]
sage: [(-alpha[i]).has_descent(1, True) for i in space.index_set()]
[False, True, False, False, False]
True
False
sage: (alpha[1]+alpha[2]+alpha[4]).has_descent(1, True)
True
```

height()  
Return the height of self.

The height of a root $\alpha = \sum_i a_i \alpha_i$ is defined to be $h(\alpha) := \sum_i a_i$.

EXAMPLES:
is_dominant(index_set=None, positive=True)

Returns whether self is dominant.

This is done with respect to the subrootsystem indicated by the subset of Dynkin nodes index_set. If index_set is None then the entire Dynkin node set is used. If positive is False then the dominance condition is replaced by antidominance.

EXAMPLES:

```
sage: L = RootSystem(['A',2]).ambient_lattice()
sage: Lambda = L.fundamental_weights()
sage: [x.is_dominant() for x in Lambda]
[True, True]
sage: [x.is_dominant(positive=False) for x in Lambda]
[False, False]
sage: (Lambda[1]-Lambda[2]).is_dominant()  # True
False
sage: (-Lambda[1]+Lambda[2]).is_dominant()  # False
False
sage: (Lambda[1]-Lambda[2]).is_dominant([1])  # True
True
sage: (Lambda[1]-Lambda[2]).is_dominant([2])  # False
False
sage: [x.is_dominant() for x in L.roots()]
[False, True, False, False, False, False]
sage: [x.is_dominant(positive=False) for x in L.roots()]
[False, True, False, False, True, False]
```

is_dominant_weight()

Test whether self is a dominant element of the weight lattice.

EXAMPLES:

```
sage: L = RootSystem(['A',2]).ambient_lattice()
sage: Lambda = L.fundamental_weights()
sage: [x.is_dominant() for x in Lambda]
[True, True]
sage: (3*Lambda[1]+Lambda[2]).is_dominant()  # True
True
sage: (Lambda[1]-Lambda[2]).is_dominant()  # False
False
sage: (-Lambda[1]+Lambda[2]).is_dominant()  # False
False
```

Tests that the scalar products with the coroots are all nonnegative integers. For example, if $x$ is the sum of a dominant element of the weight lattice plus some other element orthogonal to all coroots, then the implementation correctly reports $x$ to be a dominant weight:

```
sage: x = Lambda[1] + L([-1,-1,-1])
sage: x.is_dominant_weight()
True
```
is_imaginary_root()
Return True if self is an imaginary root.
A root $\alpha$ is imaginary if it is not $W$ conjugate to a simple root where $W$ is the corresponding Weyl group.

EXAMPLES:
```
sage: Q = RootSystem(['B',2,1]).root_lattice()
sage: alpha = Q.simple_roots()
sage: alpha[0].is_imaginary_root()
False
sage: elt.is_imaginary_root()
True
```

is_long_root()
Return True if self is a long (real) root.

EXAMPLES:
```
sage: Q = RootSystem(['B',2,1]).root_lattice()
sage: alpha = Q.simple_roots()
sage: alpha[0].is_long_root()
True
sage: alpha[1].is_long_root()
True
sage: alpha[2].is_long_root()
False
```

is_parabolic_root(index_set)
Supposing that self is a root, is it in the parabolic subsystem with Dynkin nodes index_set?

INPUT:
• index_set – the Dynkin node set of the parabolic subsystem.

Todo: This implementation is only valid in the root or weight lattice

EXAMPLES:
```
sage: alpha = RootSystem(['A',3]).root_lattice().from_vector(vector([1,1,-0]))
sage: alpha.is_parabolic_root([1,3])
False
sage: alpha.is_parabolic_root([1,2])
True
sage: alpha.is_parabolic_root([2])
False
```

is_real_root()
Return True if self is a real root.
A root $\alpha$ is real if it is $W$ conjugate to a simple root where $W$ is the corresponding Weyl group.

EXAMPLES:
is_short_root()

Return True if self is a short (real) root.

Returns False unless the parent is an irreducible root system of finite type having two root lengths and self is of the shorter length. There is no check of whether self is actually a root.

EXAMPLES:

```python
sage: Q = RootSystem(['B', 2, 1]).root_lattice()
sage: alpha = Q.simple_roots()
sage: alpha[0].is_short_root()
True
sage: elt.is_real_root()
False
```

An example in affine type:

```python
sage: Q = RootSystem(['B', 2, 1]).root_lattice()
sage: alpha = Q.simple_roots()
sage: alpha[0].is_short_root()
False
sage: alpha[1].is_short_root()
False
sage: alpha[2].is_short_root()
True
```

level()

EXAMPLES:

```python
sage: L = RootSystem(['A', 2, 1]).weight_lattice()
sage: L.rho().level()
3
```

norm_squared()

Return the norm squared of self with respect to the symmetric form.

EXAMPLES:

```python
sage: Q = RootSystem(['B', 2, 1]).root_lattice()
sage: alpha = Q.simple_roots()
sage: alpha[1].norm_squared()
(continues on next page)
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(continued from previous page)

4
sage: alpha[2].norm_squared()
2
sage: elt.norm_squared()
50
sage: elt.norm_squared()
0
sage: Q = RootSystem(CartanType(['A', 4, 2]).dual()).root_lattice()
sage: Qc = RootSystem(['A', 4, 2]).coroot_lattice()
sage: alpha = Q.simple_roots()
sage: alphac = Qc.simple_roots()
sage: elt.norm_squared()
0
sage: eltc.norm_squared()
0

orbit()

The orbit of self under the action of the Weyl group.

EXAMPLES:

\( \rho \) is a regular element whose orbit is in bijection with the Weyl group. In particular, it has 6 elements for the symmetric group \( S_3 \):

sage: L = RootSystem(['A', 2]).ambient_lattice()
sage: sorted(L.rho().orbit())

[(1, 2, 0), (1, 0, 2), (2, 1, 0),
 (2, 0, 1), (0, 1, 2), (0, 2, 1)]

sage: L = RootSystem(['A', 3]).weight_lattice()
sage: len(L.rho().orbit())
24
sage: len(L.fundamental_weights()[1].orbit())
4
sage: len(L.fundamental_weights()[2].orbit())
6

pred(index_set=None)

Return the immediate predecessors of self for the weak order.

INPUT:

- index_set - a subset (as a list or iterable) of the nodes of the Dynkin diagram; (default: None for all of them)

If index_set is specified, the successors for the corresponding parabolic subsystem are returned.

EXAMPLES:

sage: L = RootSystem(['A', 3]).weight_lattice()
sage: Lambda = L.fundamental_weights()
sage: Lambda[1].pred()
[]
sage: L.rho().pred()
[]
sage: (-L.rho()).pred()
sage: (-L.rho()).pred(index_set=[1])

reduced_word(index_set=None, positive=True)

Returns a reduced word for the inverse of the shortest Weyl group element that sends the vector self into the dominant chamber.

With the index_set optional parameter, this is done with respect to the corresponding parabolic subgroup.

If positive is False, use the antidominant chamber instead.

EXAMPLES:

sage: space=RootSystem(['A',5]).weight_space()
sage: alpha=RootSystem(['A',5]).weight_space().simple_roots()
sage: alpha[1].reduced_word()
[2, 3, 4, 5]
sage: alpha[1].reduced_word([1,2])
[2]

reflection(root, use_coroot=False)

Reflects self across the hyperplane orthogonal to root.

If use_coroot is True, root is interpreted as a coroot.

EXAMPLES:

sage: R = RootSystem(['C',4])
sage: weight_lattice = R.weight_lattice()
sage: mu = weight_lattice.from_vector(vector([0,0,1,2]))
sage: coroot_lattice = R.coroot_lattice()
sage: alphavee = coroot_lattice.from_vector(vector([0,0,1,1]))
sage: mu.reflection(alphavee, use_coroot=True)
sage: root_lattice = R.root_lattice()
sage: beta = root_lattice.from_vector(vector([0,1,1,0]))
sage: mu.reflection(beta)

scalar(lambdacheck)

Implement the natural pairing with the coroot lattice.

INPUT:
- self – an element of a root lattice realization
- lambdacheck – an element of the coroot lattice or coroot space

OUTPUT: the scalar product of self and lambdacheck

EXAMPLES:
sage: L = RootSystem(["A",4]).root_lattice()
sage: alpha = L.simple_roots()
sage: alphacheck = L.simple_coroots()
sage: alpha[1].scalar(alphacheck[1])
2
sage: alpha[1].scalar(alphacheck[2])
-1
sage: matrix([[ alpha[i].scalar(alphacheck[j])
        for i in L.index_set() ]
        for j in L.index_set() ])
[ 2 -1 0 0]
[-1 2 -1 0]
[ 0 -1 2 -1]
[ 0 0 -1 2]

\texttt{simple\_reflection}(i)

Returns the image of \texttt{self} by the \textit{i}-th simple reflection.

EXAMPLES:

\begin{verbatim}
sage: alpha = RootSystem(["A", 3]).root_lattice().alpha()
sage: alpha[1].simple_reflection(2)
sage: Q = RootSystem(["A", 3, 1]).weight_lattice(extended = \texttt{True})
sage: Lambda = Q.fundamental_weights()
sage: L = Lambda[0] + Q.null_root()
sage: L.simple_reflection(0)
-Lambda[0] + Lambda[1] + Lambda[3]
\end{verbatim}

\texttt{simple\_reflections}()

The images of \texttt{self} by all the simple reflections

EXAMPLES:

\begin{verbatim}
sage: alpha = RootSystem(["A", 3]).root_lattice().alpha()
sage: alpha[1].simple_reflections()
[-alpha[1], alpha[1] + alpha[2], alpha[1]]
\end{verbatim}

\texttt{smaller}()

Returns the elements in the orbit of \texttt{self} which are smaller than \texttt{self} in the weak order.

EXAMPLES:

\begin{verbatim}
sage: L = RootSystem(["A",3]).ambient_lattice()
sage: e = L.basis()
sage: e[2].smaller()
[(0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0)]
sage: len(L.rho().smaller())
1
sage: len((-L.rho()).smaller())
24
sage: sorted([len(x.smaller()) \texttt{ for } x \texttt{ in L.rho().orbit() }])
[1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, 8, 8, 8, 8, 12, 12, 12, 24]
\end{verbatim}
**succ(index_set=None)**

Return the immediate successors of *self* for the weak order.

**INPUT:**

- **index_set** - a subset (as a list or iterable) of the nodes of the Dynkin diagram; (default: None for all of them)

If **index_set** is specified, the successors for the corresponding parabolic subsystem are returned.

**EXAMPLES:**

```python
sage: L = RootSystem(['A',3]).weight_lattice()
sage: Lambda = L.fundamental_weights()
sage: Lambda[1].succ()
[-Lambda[1] + Lambda[2]]
sage: L.rho().succ()
sage: (-L.rho()).succ()
[]
sage: L.rho().succ(index_set=[1])
sage: L.rho().succ(index_set=[2])
```

**symmetric_form(alpha)**

Return the symmetric form of *self* with alpha.

Consider the simple roots \( \alpha_i \) and let \((b_{ij})_{ij}\) denote the symmetrized Cartan matrix \((a_{ij})_{ij}\), we have

\[
(\alpha_i | \alpha_j) = b_{ij}
\]

and extended bilinearly. See Chapter 6 in Kac, Infinite Dimensional Lie Algebras for more details.

**EXAMPLES:**

```python
sage: Q = RootSystem(['B',2,1]).root_lattice()
sage: alpha = Q.simple_roots()
sage: alpha[1].symmetric_form(alpha[0])
0
sage: alpha[1].symmetric_form(alpha[1])
4
sage: elt.symmetric_form(alpha[1])
-14
sage: elt.symmetric_form(alpha[0]+2*alpha[2])
14
sage: Q = RootSystem(CartanType(['A',4,2]).dual()).root_lattice()
sage: Qc = RootSystem(['A',4,2]).coroot_lattice()
sage: alpha = Q.simple_roots()
sage: alphac = Qc.simple_roots()
sage: elt.symmetric_form(alpha[1])
0
sage: eltc.symmetric_form(alphac[1])
0
```
**to_ambient()**

Map self to the ambient space.

**EXAMPLES:**

```plaintext
sage: alpha = CartanType(['B',4]).root_system().root_lattice().an_element();
    → alpha
sage: alpha.to_ambient()
(2, 0, 1, -3)
sage: mu = CartanType(['B',4]).root_system().weight_lattice().an_element();
    → mu
sage: mu.to_ambient()
(7, 5, 3, 0)
sage: v = CartanType(['B',4]).root_system().ambient_space().an_element(); v
(2, 2, 3, 0)
sage: v.to_ambient()
(2, 2, 3, 0)
sage: alphavee = CartanType(['B',4]).root_system().coroot_lattice().an_element(); alphavee
sage: alphavee.to_ambient()
(2, 0, 1, -3)
```

**to_classical()**

Map self to the classical lattice/space.

Only makes sense for affine type.

**EXAMPLES:**

```plaintext	sage: R = CartanType(['A',3,1]).root_system()
sage: alpha = R.root_lattice().an_element(); alpha
sage: alb = alpha.to_classical(); alb
sage: Alb.parent()
Root lattice of the Root system of type ['A', 3]
sage: v = R.ambient_space().an_element(); v
2*e[0] + 2*e[1] + 3*e[2]
sage: v.to_classical()
(2, 2, 3, 0)
```

**to_dominant_chamber(index_set=None, positive=True, reduced_word=False)**

Returns the unique dominant element in the Weyl group orbit of the vector self.

If **positive** is False, returns the antidominant orbit element.

With the **index_set** optional parameter, this is done with respect to the corresponding parabolic subgroup.

If **reduced_word** is True, returns the 2-tuple (weight, direction) where weight is the (anti)dominant orbit element and direction is a reduced word for the Weyl group element sending weight to self.
**Warning:** In infinite type, an orbit may not contain a dominant element. In this case the function may go into an infinite loop.

For affine root systems, errors are generated if the orbit does not contain the requested kind of representative. If the input vector is of positive (resp. negative) level, then there is a dominant (resp. antidominant) element in its orbit but not an antidominant (resp. dominant) one. If the vector is of level zero, then there are neither dominant nor antidominant orbit representatives, except for multiples of the null root, which are themselves both dominant and antidominant orbit representatives.

**EXAMPLES:**

```python
  ... ValueError: alpha[1] is not in the orbit of the fundamental chamber sage: mu.to_dominant_chamber(positive=False) Traceback (most recent call last):
  ... ValueError: alpha[1] is not in the orbit of the negative of the fundamental chamber
```

**to_dual_type_cospace()**

Map self to the dual type cospace.

For example, if `self` is in the root lattice of type ['B', 2], send it to the coroot lattice of type ['C', 2].

**EXAMPLES:**

```python
```
to_simple_root(reduced_word=False)

Return (the index of) a simple root in the orbit of the positive root self.

INPUT:

• self – a positive root
• reduced_word – a boolean (default: False)

OUTPUT:

• The index $i$ of a simple root $\alpha_i$. If reduced_word is True, this returns instead a pair ($i$, word), where word is a sequence of reflections mapping $\alpha_i$ up the root poset to self.

EXAMPLES:

```
sage: L = RootSystem(['A',3]).root_lattice()
sage: positive_roots = L.positive_roots()
sage: for alpha in sorted(positive_roots):
    ....:     print("{} {}").format(alpha, alpha.to_simple_root())
alpha[1] 1
alpha[2] 2
alpha[3] 3
```

```
sage: for alpha in sorted(positive_roots):
    ....:     print("{} {}".format(alpha, alpha.to_simple_root(reduced_word=True)))
alpha[1] (1, ())
alpha[2] (2, ())
alpha[3] (3, ())
```

ALGORITHM:

This method walks from self down to the antidominant chamber by applying successively the simple reflection given by the first descent. Since self is a positive root, each step goes down the root poset, and one must eventually cross a simple root $\alpha_i$.

See also:

• first_descent()
• to_dominant_chamber()

Warning: The behavior is not specified if the input is not a positive root. For a finite root system, this is currently caught (albeit with a not perfect message):

```
sage: alpha = L.simple_roots()
sage: (2*alpha[1]).to_simple_root()
Traceback (most recent call last):
```

For an infinite root system, this method may run into an infinite recursion if the input is not a positive root.
INPUT:
• self – an element \( t \) at level 0
• \( x \) – an element of the same space

Returns \( x \) translated by \( t \), that is \( x + \text{level}(x)t \)

EXAMPLES:

```python
sage: L = RootSystem(['A',2,1]).weight_lattice()
sage: alpha = L.simple_roots()
sage: Lambda = L.fundamental_weights()
sage: t = alpha[2]
```

Let us look at the translation of an element of level 1:

```python
sage: Lambda[1].level() 1
sage: t.translation(Lambda[1])
-Lambda[0] + 2*Lambda[2]
```

and of an element of level 0:

```python
sage: alpha[1].level() 0
sage: t.translation(alpha[1])
```

The arguments are given in this seemingly unnatural order to make it easy to construct the translation function:

```python
sage: f = t.translation
sage: f(Lambda[1])
-Lambda[0] + 2*Lambda[2]
```

`weyl_action(element, inverse=False)`

Act on self by an element of the Coxeter or Weyl group.

INPUT:
• element – an element of a Coxeter or Weyl group of the same Cartan type, or a tuple or a list (such as a reduced word) of elements from the index set
• inverse – a boolean (default: False); whether to act by the inverse element

EXAMPLES:

```python
sage: wl = RootSystem(['A',3]).weight_lattice()
sage: mu = wl.from_vector(vector([1,0,-2]))
sage: mudom, rw = mu.to_dominant_chamber(positive=False, reduced_word=True)
sage: mudom, rw
(-Lambda[2] - Lambda[3], [1, 2])
```
Acting by a (reduced) word:

```
sage: mudom.weyl_action(rw)
sage: mu.weyl_action(rw, inverse = True)
```

Acting by an element of the Coxeter or Weyl group on a vector in its own lattice of definition (implemented by matrix multiplication on a vector):

```
sage: w = wl.weyl_group().from_reduced_word([1, 2])
sage: mudom.weyl_action(w)
```

Acting by an element of an isomorphic Coxeter or Weyl group (implemented by the action of a corresponding reduced word):

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([1, 2])
sage: wl.weyl_group() == W
False
sage: mudom.weyl_action(w)
```

```
weyl_stabilizer(index_set=None)
```

Returns the subset of Dynkin nodes whose reflections fix self.

If index_set is not None, only consider nodes in this set. Note that if self is dominant or antidominant, then its stabilizer is the parabolic subgroup defined by the returned node set.

EXAMPLES:

```
sage: wl = RootSystem(['A',2,1]).weight_lattice(extended = True)
sage: al = wl.null_root()
sage: al.weyl_stabilizer()
[0, 1, 2]
sage: wl = RootSystem(['A',4]).weight_lattice()
sage: mu = wl.from_vector(vector([1,1,0,0]))
sage: mu.weyl_stabilizer()
[3, 4]
sage: mu.weyl_stabilizer(index_set = [1,2,3])
[3]
```

```
class ParentMethods
    Bases: object
```

```
a_long_simple_root()
```

Returns a long simple root, corresponding to the highest outgoing edge in the Dynkin diagram.

Caveat: this may be break in affine type $A_2^{(2)}$

Caveat: meaningful/broken for non irreducible?

TODO: implement CartanType.nodes_by_length as in MuPAD-Combinat (using CartanType.symmetrizer), and use it here.
almost_positive_roots()

Returns the almost positive roots of self

These are the positive roots together with the simple negative roots.

See also:

almost_positive_root_decomposition(), tau_plus_minus()

EXAMPLES:

```
sage: L = RootSystem(['A',2]).root_lattice()
sage: L.almost_positive_roots()
[-alpha[1], alpha[1], alpha[1] + alpha[2], -alpha[2], alpha[2]]
```

almost_positive_roots_decomposition()

Returns the decomposition of the almost positive roots of self

This is the list of the orbitsof the almost positive roots under the action of the dihedral group generated
by the operators \( \tau_+ \) and \( \tau_- \).

See also:

• almost_positive_roots()
• tau_plus_minus()

EXAMPLES:

```
sage: RootSystem(['A',2]).root_lattice().almost_positive_roots_decomposition()
[[[-alpha[1], alpha[1], alpha[1] + alpha[2], alpha[2], -alpha[2]]]]
sage: RootSystem(['B',2]).root_lattice().almost_positive_roots_decomposition()
[[[-alpha[1], alpha[1], alpha[1] + 2*alpha[2]], [-alpha[2], alpha[2]],
  -alpha[1] + alpha[2]]]
sage: RootSystem(['D',4]).root_lattice().almost_positive_roots_decomposition()
```

alpha()

Return the family \((\alpha_i)_{i \in I}\) of the simple roots, with the extra feature that, for simple irreducible root
systems, \(\alpha_0\) yields the opposite of the highest root.

EXAMPLES:

```
sage: alpha = RootSystem(['A',2]).root_lattice().alpha()
sage: alpha[1]
alpha[1]
sage: alpha[0]
```
alphacheck()

Return the family \((\alpha_i^\vee)_{i \in I}\) of the simple coroots, with the extra feature that, for simple irreducible root systems, \(\alpha_0^\vee\) yields the coroot associated to the opposite of the highest root (caveat: for non-simply-laced root systems, this is not the opposite of the highest coroot!).

EXAMPLES:

```python
sage: alphacheck = RootSystem(['A',2]).ambient_space().alphacheck()
sage: alphacheck
Finite family {1: (1, -1, 0), 2: (0, 1, -1)}
```

Here is now \(\alpha_0^\vee\):

\((-1, 0, 1)\)

Todo: add a non simply laced example

Finally, here is an affine example:

```python
sage: RootSystem(['A',2,1]).weight_space().alphacheck()
sage: alphacheck
Finite family {0: alphacheck[0], 1: alphacheck[1], 2: alphacheck[2]}
```

basic_imaginary_roots()

Return the basic imaginary roots of self.

The basic imaginary roots \(\delta\) are the set of imaginary roots in \(-C^\vee\) where \(C\) is the dominant chamber (i.e., \(\langle \beta, \alpha_i^\vee \rangle \leq 0\) for all \(i \in I\)). All imaginary roots are \(W\)-conjugate to a simple imaginary root.

EXAMPLES:

```python
sage: Q = RootSystem(['A',2,1]).root_lattice() s sage: delta = Q.basic_imaginary_roots()[0] sage: all(delta.scalar(Q.simple_coroot(i)) <= 0 for i in Q.index_set())
True
```
cartan_type()

EXAMPLES:

```python
sage: r = RootSystem(['A',4]).root_space() sage: r.cartan_type()
['A', 4]
```
classical()

Return the corresponding root/weight/ambient lattice/space.

EXAMPLES:

```python
sage: RootSystem(['A',4,1]).root_lattice().classical()
Root lattice of the Root system of type ['A', 4]
```
\begin{minted}{python}

sage: RootSystem(['A', 4, 1]).weight_lattice().classical()
Weight lattice of the Root system of type ['A', 4]
sage: RootSystem(['A', 4, 1]).ambient_space().classical()
Ambient space of the Root system of type ['A', 4]

\end{minted}

**cohighest_root()**

Returns the associated coroot of the highest root.

**Note:** this is usually not the highest coroot.

**EXAMPLES:**

\begin{minted}{python}

sage: RootSystem(['A', 3]).ambient_space().cohighest_root()
(1, 0, 0, -1)

\end{minted}

**coroot_lattice()**

Returns the coroot lattice.

**EXAMPLES:**

\begin{minted}{python}

sage: RootSystem(['A', 2]).root_lattice().coroot_lattice()
Coroot lattice of the Root system of type ['A', 2]

\end{minted}

**coroot_space** (*base_ring*=Rational Field)

Return the coroot space over *base_ring*.

**INPUT:**

- *base_ring* – a ring (default: Q)

**EXAMPLES:**

\begin{minted}{python}

sage: RootSystem(['A', 2]).root_lattice().coroot_space()
Coroot space over the Rational Field of the Root system of type ['A', 2]
sage: RootSystem(['A', 2]).root_lattice().coroot_space(QQ['q'])
Coroot space over the Univariate Polynomial Ring in q over Rational Field

\end{minted}

**dual_type_cospace()**

Returns the cospace of dual type.

For example, if invoked on the root lattice of type ['B', 2], returns the coroot lattice of type ['C', 2].

**Warning:** Not implemented for ambient spaces.

**EXAMPLES:**

\begin{minted}{python}

sage: CartanType(['B', 2]).root_system().root_lattice().dual_type_cospace()
Coroot lattice of the Root system of type ['C', 2]
sage: CartanType(['F', 4]).root_system().coweight_lattice().dual_type_cospace()
Weight lattice of the Root system of type ['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}

\end{minted}
**dynkin_diagram()**

**EXAMPLES:**

```python
sage: r = RootSystem(['A',4]).root_space()
sage: r.dynkin_diagram()
O---O---O---O
1 2 3 4
A4
```

**fundamental_weights_from_simple_roots()**

Return the fundamental weights.

This is computed from the simple roots by using the inverse of the Cartan matrix. This method is therefore only valid for finite types and if this realization of the root lattice is large enough to contain them.

**EXAMPLES:**

In the root space, we retrieve the inverse of the Cartan matrix:

```python
c sage: L = RootSystem(['B',3]).root_space()
sage: L.fundamental_weights_from_simple_roots()
tsage: ~L.cartan_type().cartan_matrix()
[ 1 1 1/2]
[ 1 2 1]
[ 1 2 3/2]
```

In the weight lattice and the ambient space, we retrieve the fundamental weights:

```python
c sage: L = RootSystem(['B',3]).weight_lattice()
sage: L.fundamental_weights_from_simple_roots()
```

```python
c Finite family {1: Lambda[1], 2: Lambda[2], 3: Lambda[3]}
sage: L = RootSystem(['B',3]).ambient_space()
sage: L.fundamental_weights()
Finite family {1: (1, 0, 0), 2: (1, 1, 0), 3: (1/2, 1/2, 1/2)}
sage: L.fundamental_weights_from_simple_roots()
```

```python
c Finite family {1: (1, 0, 0), 2: (1, 1, 0), 3: (1/2, 1/2, 1/2)}
```

However the fundamental weights do not belong to the root lattice:

```python
c sage: L = RootSystem(['B',3]).root_lattice()
sage: L.fundamental_weights_from_simple_roots()
Traceback (most recent call last):
  ...:
ValueError: The fundamental weights do not live in this realization of the...
```

 Beware of the usual $GL_n$ vs $SL_n$ catch in type $A$:

```python
c sage: L = RootSystem(['A',3]).ambient_space()
sage: L.fundamental_weights()
```

(continues on next page)
Finite family \{1: (1, 0, 0, 0), 2: (1, 1, 0, 0), 3: (1, 1, 1, 0)\}

```python
sage: L.fundamental_weights_from_simple_roots()
```

Finite family \{1: (3/4, -1/4, -1/4, -1/4), 2: (1/2, 1/2, -1/2, -1/2), 3: (1/4, 1/4, 1/4, -3/4)\}

```python
sage: L = RootSystem(['A',3]).ambient_lattice()
sage: L.fundamental_weights_from_simple_roots()
```

Traceback (most recent call last):
...
ValueError: The fundamental weights do not live in this realization of the root lattice

`generalized_nonnesting_partition_lattice(m, facade=False)`

Return the lattice of \(m\)-nonnesting partitions

This has been defined by Athanasiadis, see chapter 5 of [Arm06].

**INPUT:**

- \(m\) – integer

**See also:**

`nonnesting_partition_lattice()`

**EXAMPLES:**

```python
sage: R = RootSystem(['A', 2]).ambient_lattice()
sage: RS = R.root_lattice()
sage: P = RS.generalized_nonnesting_partition_lattice(2); P
Finite lattice containing 12 elements
sage: P.coxeter_transformation()**20 == 1
True
```

`highest_root()`

Returns the highest root (for an irreducible finite root system)

**EXAMPLES:**

```python
sage: RootSystem(['A',4]).ambient_space().highest_root()
(1, 0, 0, 0, -1)
sage: RootSystem(['E',6]).weight_space().highest_root()
Lambda[2]
```

`index_set()`

**EXAMPLES:**

```python
sage: r = RootSystem(['A',4]).root_space()
sage: r.index_set()
(1, 2, 3, 4)
```

`long_roots()`

Return a list of the long roots of `self`.

**EXAMPLES:**

```python
```
negative_roots()

Returns the negative roots of self.

EXAMPLES:

```python
sage: L = RootSystem(['B',3]).root_lattice()
sage: sorted(L.long_roots())
  alpha[1], alpha[1] + alpha[2],
```

Algorithm: negate the positive roots

nonnesting_partition_lattice(facade=False)

Return the lattice of nonnesting partitions

This is the lattice of order ideals of the root poset.

This has been defined by Postnikov, see Remark 2 in [Reiner97].

See also:

generalized_nonnesting_partition_lattice(), root_poset()

EXAMPLES:

```python
sage: R = RootSystem(['A', 3])
sage: RS = R.root_lattice()
sage: P = RS.nonnesting_partition_lattice(); P
Finite lattice containing 14 elements
sage: P.coxeter_transformation()**10 == 1
True
sage: R = RootSystem(['B', 3])
sage: RS = R.root_lattice()
sage: P = RS.nonnesting_partition_lattice(); P
Finite lattice containing 20 elements
sage: P.coxeter_transformation()**7 == 1
True
```

REFERENCES:

nonparabolic_positive_root_sum(index_set=None)

Return the sum of positive roots not in a parabolic subsystem.

The conventions for index_set are as in nonparabolic_positive_roots().

EXAMPLES:
sage: Q = RootSystem(['A',3]).root_lattice()
sage: Q.nonparabolic_positive_root_sum((1,2))
sage: Q.nonparabolic_positive_root_sum()
[]
sage: Q.nonparabolic_positive_root_sum((()))

nonparabolic_positive_roots(index_set=None)

Return the positive roots of self that are not in the parabolic subsystem indicated by index_set.

If index_set is None, as in positive_roots() it is assumed to be the entire Dynkin node set. Then
the parabolic subsystem consists of all positive roots and the empty list is returned.

EXAMPLES:

sage: L = RootSystem(['A',3]).root_lattice()
sage: L.nonparabolic_positive_roots()
[]
sage: sorted(L.nonparabolic_positive_roots((1,2)))
sage: sorted(L.nonparabolic_positive_roots(()))

null_coroot()

Returns the null coroot of self.

The null coroot is the smallest non trivial positive coroot which is orthogonal to all simple roots. It exists for any affine root system.

EXAMPLES:

sage: RootSystem(['C',2,1]).root_lattice().null_coroot()
sage: RootSystem(['D',4,1]).root_lattice().null_coroot()
sage: RootSystem(['F',4,1]).root_lattice().null_coroot()

null_root()

Returns the null root of self. The null root is the smallest non trivial positive root which is orthogonal to all simple coroots. It exists for any affine root system.

EXAMPLES:

sage: RootSystem(['C',2,1]).root_lattice().null_root()
sage: RootSystem(['D',4,1]).root_lattice().null_root()
sage: RootSystem(['F',4,1]).root_lattice().null_root()
plot(roots='simple', coroots=False, reflection_hyperplanes='simple', fundamental_weights=None, fundamental_chamber=None, alcoves=None, alcove_labels=False, alcove_walk=None, **options)

Return a picture of this root lattice realization.

**INPUT:**

- **roots** – which roots to display, if any. Can be one of the following:
  - "simple" – The simple roots (the default)
  - "classical" – Not yet implemented
  - "all" – Only works in the finite case
  - A list or tuple of roots
  - False
- **coroots** – which coroots to display, if any. Can be one of the following:
  - "simple" – The simple coroots (the default)
  - "classical" – Not yet implemented
  - "all" – Only works in the finite case
  - A list or tuple of coroots
  - False
- **fundamental_weights** – a boolean or None (default: None) whether to display the fundamental weights. If None, the fundamental weights are drawn if available.
- **reflection_hyperplanes** – which reflection hyperplanes to display, if any. Can be one of the following:
  - "simple" – The simple roots
  - "classical" – Not yet implemented
  - "all" – Only works in the finite case
  - A list or tuple of roots
  - False (the default)
- **fundamental_chamber** – whether and how to draw the fundamental chamber. Can be one of the following:
  - A boolean – Set to True to draw the fundamental chamber
  - "classical" – Draw the classical fundamental chamber
  - None – (the default) The fundamental chamber is drawn except in the root lattice where this is not yet implemented. For affine types the classical fundamental chamber is drawn instead.
- **alcoves** – one of the following (default: True):
  - A boolean – Whether to display the alcoves
  - A list of alcoves – The alcoves to be drawn. Each alcove is specified by the coordinates of its center in the root lattice (affine type only). Otherwise the alcoves that intersect the bounding box are drawn.
- **alcove_labels** – one of the following (default: False):
  - A boolean – Whether to display the elements of the Weyl group indexing the alcoves. This currently requires to also set the alcoves option.
  - A number $l$ – The label is drawn at level $l$ (affine type only), which only makes sense if affine is False.
- **bounding_box** – a rational number or a list of pairs thereof (default: 3)

Specifies a bounding box, in the coordinate system for this plot, in which to plot alcoves and other infinite objects. If the bounding box is a number $a$, then the bounding box is of the form $[-a, a]$ in all directions. Beware that there can be some border effects and the returned graphic is not necessarily strictly contained in the bounding box.

- **alcove_walk** – an alcove walk or None (default: None)

The alcove walk is described by a list (or iterable) of vertices of the Dynkin diagram which specifies which wall is crossed at each step, starting from the fundamental alcove.

- **projection** – one of the following (default: True):
  - True – The default projection for the root lattice realization is used.
- False – No projection is used.
- barycentric – A barycentric projection is used.
- A function – If a function is specified, it should implement a linear (or affine) map taking as input an element of this root lattice realization and returning its desired coordinates in the plot, as a vector with rational coordinates.

- color – a function mapping vertices of the Dynkin diagram to colors (default: "black" for 0, "blue" for 1, "red" for 2, "green" for 3)
  
  This is used to set the color for the simple roots, fundamental weights, reflection hyperplanes, alcove facets, etc. If the color is None, the object is not drawn.

- labels – a boolean (default: True) whether to display labels on the simple roots, fundamental weights, etc.

**EXAMPLES:**

```python
sage: L = RootSystem(['A',2,1]).ambient_space().plot()  # long time
```

**See also:**
- `plot_parse_options()`
- `plot_roots()`, `plot_coroots()`
- `plot_fundamental_weights()`
- `plot_fundamental_chamber()`
- `plot_reflection_hyperplanes()`
- `plot_alcoves()`
- `plot_alcove_walk()`
- `plot_ls_paths()`
- `plot_mv_polytope()`
- `plot_crystal()`

`plot_alcove_walk`(word, start=None, foldings=None, color='orange', **options)

Plot an alcove walk.

**INPUT:**
- word – a list of elements of the index set
- foldings – a list of booleans or None (default: None)
- start – an element of this space (default: None for \( \rho \))
- **options – plotting options

**See also:**
- `plot()` for a description of the plotting options
- *Tutorial: visualizing root systems* for a tutorial on root system plotting

**EXAMPLES:**

An alcove walk of type \( A_2^{(1)} \):

```python
sage: L = RootSystem(['A',2,1]).ambient_space()
sage: w1 = [0,2,1,2,0,2,1,0,2,1,2,1,0,2,0,1,2,0]
sage: p = L.plot_alcoves(bounding_box=5)  # long time (5s)
sage: p += L.plot_alcove_walk(w1)  # long time
sage: p  # long time
```

Graphics object consisting of 375 graphics primitives

The same plot with another alcove walk:

```python
sage: w2 = [2,1,2,0,2,0,2,1,2,0,1,2,1,2,1,0,1,2,0,2,0,1,2,0]
sage: p += L.plot_alcove_walk(w2, color="orange")  # long time
```
And another with some foldings:

```
sage: pic = L.plot_alcoves(bounding_box=3)  # long time
sage: pic += L.plot_alcove_walk([0,1,2,0,2,0,1,2,0,1],  # long time (3s)
    foldings = [False, False, True, False, False, False, True, False, False],
    color="green"); pic
```

```
Graphics object consisting of 155 graphics primitives
```

```
plot_alcoves(alcoves=True, alcove_labels=False, wireframe=False, **options)
```

Plot the alcoves and optionally their labels.

**INPUT:**
- `alcoves` – a list of alcoves or `True` (default: `True`)
- `alcove_labels` – a boolean or a number specifying at which level to put the label (default: `False`)
- `**options` – Plotting options

**See also:**
- `plot()` for a description of the plotting options
- `Tutorial: visualizing root systems` for a tutorial on root system plotting, and in particular how the alcoves can be specified.

**EXAMPLES:**

2D plots:

```
sage: RootSystem("A",2,1).ambient_space().plot_alcoves()  # long time (3s)
```

```
Graphics object consisting of 228 graphics primitives
```

3D plots:

```
sage: RootSystem("A",2,1).weight_space().plot_alcoves(affine=False)  # long time (3s)
```

```
Graphics3d Object
```

```
sage: RootSystem("G",2,1).ambient_space().plot_alcoves(affine=False,
    level=1)  # long time (3s)
```

```
Graphics3d Object
```

Here we plot a single alcove:

```
sage: L = RootSystem("A",3,1).ambient_space()
sage: W = L.weyl_group()
sage: L.plot(alcoves=[W.one()], reflection_hyperplanes=False, bounding_ 
   box=2)
```

```
Graphics3d Object
```

```
plot_bounding_box(**options)
```

Plot the bounding box.

**INPUT:**
- `**options` – Plotting options
This is mostly for testing purposes.

**See also:**
- `plot()` for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```
sage: L = RootSystem(['A',2,1]).ambient_space()
sage: L.plot_bounding_box()
Graphics object consisting of 1 graphics primitive
```

`plot_coroots(collection='simple', **options)`

Plot the (simple/classical) coroots of this root lattice.

**INPUT:**

- `collection` – which coroots to display. Can be one of the following:
  - "simple" (the default)
  - "classical"
  - "all"
- `**options` – Plotting options

**See also:**

- `plot()` for a description of the plotting options
- Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```
sage: RootSystem(['B',3]).ambient_space().plot_coroots()
Graphics3d Object
```

`plot_crystal(crystal, plot_labels=True, label_color='black', edge_labels=False, circle_size=0.06, circle_thickness=1.6, **options)`

Plot a finite crystal.

**INPUT:**

- `crystal` – the finite crystal to plot
- `plot_labels` – (default: True) can be one of the following:
  - True - use the latex labels
  - 'circles' - use circles for multiplicity up to 4; if the multiplicity is larger, then it uses the multiplicity
  - 'multiplicities' - use the multiplicities
- `label_color` – (default: 'black') the color of the labels
- `edge_labels` – (default: False) if True, then draw in the edge label
- `circle_size` – (default: 0.06) the size of the circles
- `circle_thickness` – (default: 1.6) the thinkness of the extra rings of circles
- `**options` – plotting options

**See also:**

- `plot()` for a description of the plotting options
- Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```
sage: L = RootSystem(['A',2]).ambient_space()
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: L.plot_crystal(C, plot_labels='multiplicities')
Graphics object consisting of 15 graphics primitives
sage: C = crystals.Tableaux(['A',2], shape=[8,4])
sage: p = L.plot_crystal(C, plot_labels='circles')
sage: p.show(figsize=15)
```
A 3-dimensional example:

```python
sage: L = RootSystem(['B',3]).ambient_space()
sage: C = crystals.Tableaux(['B',3], shape=[2,1])
sage: L.plot_crystal(C, plot_labels='circles', edge_labels=True)  # long time
```

**plot_fundamental_chamber**(style='normal', **options)

Plot the (classical) fundamental chamber.

**INPUT:**

- `style` – "normal" or "classical" (default: "normal")
- **options** – Plotting options

**See also:**

- `plot()` for a description of the plotting options
- *Tutorial: visualizing root systems* for a tutorial on root system plotting

**EXAMPLES:**

2D plots:

```python
sage: RootSystem(['B',2]).ambient_space().plot_fundamental_chamber()
Graphics object consisting of 1 graphics primitive
sage: RootSystem(['B',2,1]).ambient_space().plot_fundamental_chamber()
Graphics object consisting of 1 graphics primitive
sage: RootSystem(['B',2,1]).ambient_space().plot_fundamental_chamber(style='classical')
Graphics object consisting of 1 graphics primitive
```

3D plots:

```python
sage: RootSystem(['A',3,1]).weight_space().plot_fundamental_chamber()
Graphics3d Object
sage: RootSystem(['B',3,1]).ambient_space().plot_fundamental_chamber()
Graphics3d Object
```

This feature is currently not available in the root lattice/space:

```python
sage: list(RootSystem(['A',2]).root_lattice().plot_fundamental_chamber())
Traceback (most recent call last):
...
TypeError: classical fundamental chamber not yet available in the root_lattice
```

**plot_fundamental_weights**( **options**)

Plot the fundamental weights of this root lattice.

**INPUT:**

- **options** – Plotting options

**See also:**

- `plot()` for a description of the plotting options
- *Tutorial: visualizing root systems* for a tutorial on root system plotting

**EXAMPLES:**
plot_hedron(**options)
Plot the polyhedron whose vertices are given by the orbit of ρ.

In type $A$, this is the usual permutohedron.

See also:
• plot() for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```python
sage: RootSystem(['B',3]).ambient_space().plot_hedron()
Graphics3d Object
```

Surprise: polyhedra of large dimension know how to project themselves nicely:

```python
sage: RootSystem(['F',4]).ambient_space().plot_hedron()  # long time
Graphics3d Object
```

plot_ls_paths(paths, plot_labels=None, colored_labels=True, **options)
Plot LS paths.

INPUT:
• paths – a finite crystal or list of LS paths
• plot_labels – (default: None) the distance to plot the LS labels from the endpoint of the path;
  set to None to not display the labels
• colored_labels – (default: True) if True, then color the labels the same color as the LS path
• **options – plotting options

See also:
• plot() for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```python
sage: B = crystals.LSPaths(['A',2], [1,1])
sage: L = RootSystem(['A',2]).ambient_space()
sage: L.plot_fundamental_weights() + L.plot_ls_paths(B)
Graphics object consisting of 14 graphics primitives
```

This also works in 3 dimensions:

```python
sage: B = crystals.LSPaths(['B',3], [2,0,0])
sage: L = RootSystem(['B',3]).ambient_space()
```
sage: L.plot_ls_paths(B)
Graphics3d Object

plot_mv_polytope(mv_polytope, mark_endpoints=True, circle_size=0.06, circle_thickness=1.6, wireframe='blue', fill='green', alpha=1, **options)

Plot an MV polytope.

INPUT:
• mv_polytope – an MV polytope
• mark_endpoints – (default: True) mark the endpoints of the MV polytope
• circle_size – (default: 0.06) the size of the circles
• circle_thickness – (default: 1.6) the thickness of the extra rings of circles
• wireframe – (default: 'blue') color to draw the wireframe of the polytope with
• fill – (default: 'green') color to fill the polytope with
• alpha – (default: 1) the alpha value (opacity) of the fill
• **options – plotting options

See also:
• plot() for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

sage: B = crystals.infinity.MVPolytopes(['C',2])
sage: L = RootSystem(['C',2]).ambient_space()
sage: p = B.highest_weight_vector().f_string([1,2,1,2])
sage: L.plot_fundamental_weights() + L.plot_mv_polytope(p)
Graphics object consisting of 14 graphics primitives

This also works in 3 dimensions:

sage: B = crystals.infinity.MVPolytopes(['A',3])
sage: L = RootSystem(['A',3]).ambient_space()
sage: p = B.highest_weight_vector().f_string([2,1,3,2])
sage: L.plot_mv_polytope(p)
Graphics3d Object

plot_parse_options(**args)

Return an option object to be used for root system plotting.

EXAMPLES:

sage: L = RootSystem(['A',2,1]).ambient_space()
sage: options = L.plot_parse_options()
sage: options
<sage.combinat.root_system.plot.PlotOptions object at ...>

See also:
• plot() for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

plot_reflection_hyperplanes(collection='simple', **options)

Plot the simple reflection hyperplanes.

INPUT:
• collection – which reflection hyperplanes to display. Can be one of the following:
  – "simple" (the default)
  – "classical"
  – "all"
• **options – Plotting options

See also:

• plot() for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```sage
sage: RootSystem(["A",2,1]).ambient_space().plot_reflection_hyperplanes()
Graphics object consisting of 6 graphics primitives
sage: RootSystem(["G",2,1]).ambient_space().plot_reflection_hyperplanes()
Graphics object consisting of 6 graphics primitives
sage: RootSystem(["A",3]).weight_space().plot_reflection_hyperplanes()
Graphics3d Object
sage: RootSystem(["B",3]).ambient_space().plot_reflection_hyperplanes()
Graphics3d Object
sage: RootSystem(["A",3,1]).weight_space().plot_reflection_hyperplanes()
Graphics3d Object
sage: RootSystem(["B",3,1]).ambient_space().plot_reflection_hyperplanes()
Graphics3d Object
sage: RootSystem(["A",2,1]).weight_space().plot_reflection_hyperplanes(affine=False, level=1)
Graphics3d Object
sage: RootSystem(["A",2]).root_lattice().plot_reflection_hyperplanes()
Graphics object consisting of 4 graphics primitives
```

Todo: Provide an option for transparency?

---

**plot_roots**(collection='simple', **options)

Plot the (simple/classical) roots of this root lattice.

INPUT:

• collection – which roots to display can be one of the following:
  – "simple" (the default)
  – "classical"
  – "all"
• **options – Plotting options

See also:

• plot() for a description of the plotting options
• Tutorial: visualizing root systems for a tutorial on root system plotting

EXAMPLES:

```sage
sage: RootSystem(["B",3]).ambient_space().plot_roots()
Graphics3d Object
sage: RootSystem(["B",3]).ambient_space().plot_roots("all")
Graphics3d Object
```

**positive_imaginary_roots()**

Return the positive imaginary roots of self.
EXAMPLES:

```
sage: L = RootSystem(['A',3]).root_lattice()
sage: L.positive_imaginary_roots()
()

sage: L = RootSystem(['A',3,1]).root_lattice()
sage: PIR = L.positive_imaginary_roots(); PIR
Positive imaginary roots of type ['A', 3, 1]
sage: [PIR.unrank(i) for i in range(5)]
  3*alpha[0] + 3*alpha[1] + 3*alpha[2] + 3*alpha[3],
```

The `positive_real_roots()` method returns the positive real roots of the root system.

EXAMPLES:

```
sage: L = RootSystem(['A',3]).root_lattice()
sage: sorted(L.positive_real_roots())
  alpha[2], alpha[2] + alpha[3], alpha[3]]

sage: L = RootSystem(['A',3,1]).root_lattice()
sage: PRR = L.positive_real_roots(); PRR
Positive real roots of type ['A', 3, 1]
sage: [PRR.unrank(i) for i in range(10)]
[alpha[1],
  alpha[2],
  alpha[3],
  alpha[1] + alpha[2],
  alpha[2] + alpha[3],

sage: Q = RootSystem(['A',4,2]).root_lattice()
sage: PR = Q.positive_roots()
sage: [PR.unrank(i) for i in range(5)]
[alpha[1],
  alpha[2],
  alpha[1] + alpha[2],
  2*alpha[1] + alpha[2],
  alpha[0] + alpha[1] + alpha[2]]

sage: Q = RootSystem(['D',3,2]).root_lattice()
sage: PR = Q.positive_roots()
sage: [PR.unrank(i) for i in range(5)]
[alpha[1],
```
alpha[2],
alpha[1] + 2*alpha[2],
alpha[1] + alpha[2],
alpha[0] + alpha[1] + 2*alpha[2]]

**positive_roots(index_set=None)**

Return the positive roots of self.

If index_set is not None, returns the positive roots of the parabolic subsystem with simple roots in index_set.

Algorithm for finite type: generate them from the simple roots by applying successive reflections toward the positive chamber.

**EXAMPLES:**

```python
sage: L = RootSystem(['A',3]).root_lattice()
sage: sorted(L.positive_roots())
[alpha[1], alpha[1] + alpha[2],
 alpha[2] + alpha[3], alpha[3]]
sage: sorted(L.positive_roots((1,2)))
[alpha[1], alpha[1] + alpha[2], alpha[2]]
sage: sorted(L.positive_roots(()))
[]
sage: L = RootSystem(['A',3,1]).root_lattice()
sage: PR = L.positive_roots(); PR
Disjoint union of Family (Positive real roots of type ['A', 3, 1],
 Positive imaginary roots of type ['A', 3, 1])
sage: [PR.unrank(i) for i in range(10)]
[alpha[1],
 alpha[2],
 alpha[3],
 alpha[1] + alpha[2],
 alpha[2] + alpha[3],
```

**positive_roots_by_height**(increasing=True)

Returns a list of positive roots in increasing order by height.

If increasing is False, returns them in decreasing order.

**Warning:** Raise an error if the Cartan type is not finite.

**EXAMPLES:**

```python
sage: L = RootSystem(['C',2]).root_lattice()
sage: L.positive_roots_by_height()
```
\[ [\alpha_2, \alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2] \]

\texttt{sage: L.positive_roots_by_height(increasing = False)}
\[ [2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2, \alpha_1] \]

\texttt{sage: L = RootSystem([\text{'A'}, 2, 1]).root_lattice()}
\texttt{sage: L.positive_roots_by_height()}
Traceback (most recent call last):
... 
NotImplementedError: Only implemented for finite Cartan type

\textbf{positive_roots_nonparabolic}(\textit{index_set=None})

Returns the set of positive roots outside the parabolic subsystem with Dynkin node set \textit{index_set}.

\textbf{INPUT:}

- \textit{index_set} – (default:None) the Dynkin node set of the parabolic subsystem. It should be a tuple. The default value implies the entire Dynkin node set

\textbf{EXAMPLES:}

\texttt{sage: lattice = RootSystem([\text{'A'}, 3]).root_lattice()}
\texttt{sage: sorted(lattice.positive_roots_nonparabolic((1, 3)), key=str)}
\[ [\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3] \]
\texttt{sage: sorted(lattice.positive_roots_nonparabolic((2, 3)), key=str)}
\[ [\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3] \]
\texttt{sage: lattice.positive_roots_nonparabolic()}
\[ [] \]
\texttt{sage: lattice.positive_roots_nonparabolic((1, 2, 3))}
\[ [] \]

\textbf{Warning:} This returns an error if the Cartan type is not finite.

\textbf{positive_roots_nonparabolic_sum}(\textit{index_set=None})

Returns the sum of positive roots outside the parabolic subsystem with Dynkin node set \textit{index_set}.

\textbf{INPUT:}

- \textit{index_set} – (default:None) the Dynkin node set of the parabolic subsystem. It should be a tuple. The default value implies the entire Dynkin node set

\textbf{EXAMPLES:}

\texttt{sage: lattice = RootSystem([\text{'A'}, 3]).root_lattice()}
\texttt{sage: lattice.positive_roots_nonparabolic_sum((1, 3))}
\[ 2\alpha_1 + 4\alpha_2 + 2\alpha_3 \]
\texttt{sage: lattice.positive_roots_nonparabolic_sum((2, 3))}
\[ 3\alpha_1 + 2\alpha_2 + \alpha_3 \]
\texttt{sage: lattice.positive_roots_nonparabolic_sum(\texttt{})}
\[ 3\alpha_1 + 4\alpha_2 + 3\alpha_3 \]
\texttt{sage: lattice.positive_roots_nonparabolic_sum()}
\[ 0 \]
\texttt{sage: lattice.positive_roots_nonparabolic_sum((1, 2, 3))}
\[ 0 \]
**Warning:** This returns an error if the Cartan type is not finite.

**positive_roots_parabolic(index_set=None)**
Return the set of positive roots for the parabolic subsystem with Dynkin node set `index_set`.

**INPUT:**
- `index_set` – (default:None) the Dynkin node set of the parabolic subsystem. It should be a tuple.

The default value implies the entire Dynkin node set

**EXAMPLES:**

```python
sage: lattice = RootSystem(['A',3]).root_lattice()
sage: sorted(lattice.positive_roots_parabolic((1,3)), key=str)
[alpha[1], alpha[3]]
sage: sorted(lattice.positive_roots_parabolic((2,3)), key=str)
[alpha[2], alpha[2] + alpha[3], alpha[3]]
sage: sorted(lattice.positive_roots_parabolic(), key=str)
```

**Warning:** This returns an error if the Cartan type is not finite.

**projection(root, coroot=None, to_negative=True)**
Returns the projection along the root, and across the hyperplane define by coroot, as a function from self to self. \(\pi\) is a half-linear map which stabilizes the negative half space, and acts by reflection on the positive half space.

If to_negative is False, then this project onto the positive half space instead.

**EXAMPLES:**

```python
sage: space = RootSystem(['A',2]).weight_lattice()
sage: x = space.simple_roots()[1]
sage: y = space.simple_coroots()[1]
sage: pi = space.projection(x, y)
sage: x
sage: pi(x)
sage: pi(-x)
sage: pi = space.projection(x, y, False)
sage: pi(-x)
```

**reflection(root, coroot=None)**
Returns the reflection along the root, and across the hyperplane define by coroot, as a function from self to self.

**EXAMPLES:**

```python
sage: space = RootSystem(['A',2]).weight_lattice()
sage: x = space.simple_roots()[1]
```

(continues on next page)
sage: y = space.simple_coroots()[1]
sage: s = space.reflection(x, y)
sage: x
sage: s(x)
sage: s(-x)

**root_poset** *(restricted=False, facade=False)*

Returns the (restricted) root poset associated to `self`.

The elements are given by the positive roots (resp. non-simple, positive roots), and \( \alpha \leq \beta \) iff \( \beta - \alpha \) is a non-negative linear combination of simple roots.

**INPUT:**
- `restricted` – (default:False) if True, only non-simple roots are considered.
- `facade` – (default:False) passes facade option to the poset generator.

**EXAMPLES:**

```python
sage: Phi = RootSystem(['A', 1]).root_poset(); Phi
Finite poset containing 1 elements
sage: Phi.cover_relations()
[]
sage: Phi = RootSystem(['A', 2]).root_poset(); Phi
Finite poset containing 3 elements
sage: sorted(Phi.cover_relations(), key=str)
sage: Phi = RootSystem(['A', 3]).root_poset(restricted=True); Phi
Finite poset containing 3 elements
sage: sorted(Phi.cover_relations(), key=str)
sage: Phi = RootSystem(['B', 2]).root_poset(); Phi
Finite poset containing 4 elements
sage: sorted(Phi.cover_relations(), key=str)
```

**roots()**

Return the roots of `self`.

**EXAMPLES:**

```python
sage: RootSystem(['A', 2]).ambient_lattice().roots()
[(1, -1, 0), (1, 0, -1), (0, 1, -1), (-1, 1, 0), (-1, 0, 1), (0, -1, 1)]
```

This matches with Wikipedia article Root systems:
```python
sage: for T in CartanType.samples(finite = True, crystallographic = True):
    ....:     print("%s %3s %3s"%(T, len(RootSystem(T).root_lattice().roots()),
    ....:     len(RootSystem(T).weight_lattice().roots())))
[A', 1] 2 2
[A', 5] 30 30
[B', 1] 2 2
[B', 5] 50 50
[C', 1] 2 2
[C', 5] 50 50
[D', 2] 4 4
[D', 3] 12 12
[D', 5] 40 40
[E', 6] 72 72
[E', 7] 126 126
[E', 8] 240 240
[F', 4] 48 48
[G', 2] 12 12
```

**Todo:** The result should be an enumerated set, and handle infinite root systems.

`s()`  
Return the family \((s_i)_{i \in I}\) of the simple reflections of this root system.

**EXAMPLES:**

```python
sage: r = RootSystem(['A', 2]).root_lattice()
sage: s = r.simple_reflections()
sage: s[1]( r.simple_root(1) )
-alpha[1]
```

`short_roots()`  
Return a list of the short roots of `self`.

**EXAMPLES:**

```python
sage: L = RootSystem(['B',3]).root_lattice()
sage: sorted(L.short_roots())
 -alpha[2] - alpha[3],
 alpha[2] + alpha[3],
 -alpha[3],
 alpha[3]]
```

`simple_coroot(i)`  
Returns the \(i\)th simple coroot.

**EXAMPLES:**

```python
sage: RootSystem(['A',2]).root_lattice().simple_coroot(1)
alphacheck[1]
```

5.1. Comprehensive Module List
simple_coroots()

Returns the family \((\alpha_i')_{i \in I}\) of the simple coroots.

EXAMPLES:

```python
sage: alphacheck = RootSystem(['A',3]).root_lattice().simple_coroots()
sage: [alphacheck[i] for i in [1, 2, 3]]
[alphacheck[1], alphacheck[2], alphacheck[3]]
```

simple_projection\((i, to_negative=True)\)

Returns the projection along the \(i\)th simple root, and across the hyperplane define by the \(i\)th simple coroot, as a function from self to self.

INPUT:

• \(i\) - \(i\) is in self’s index set

EXAMPLES:

```python
sage: space = RootSystem(['A',2]).weight_lattice()
sage: x = space.simple_roots()[1]
sage: pi = space.simple_projection(1)
sage: x
sage: pi(x)
sage: pi(-x)
sage: pi = space.simple_projection(1,False)
sage: pi(-x)
```

simple_projections\((to_negative=True)\)

Returns the family \((s_i)\) of the simple projections of this root system

EXAMPLES:

```python
sage: space = RootSystem(['A',2]).weight_lattice()
sage: pi = space.simple_projections()
sage: x = space.simple_roots()
sage: pi[1](x[2])
```

simple_reflection\((i)\)

Returns the \(i\)th simple reflection, as a function from self to self.

INPUT:

• \(i\) - \(i\) is in self’s index set

EXAMPLES:

```python
sage: space = RootSystem(['A',2]).ambient_lattice()
sage: s = space.simple_reflection(1)
sage: x = space.simple_roots()[1]
sage: x
(1, -1, 0)
sage: s(x)
(-1, 1, 0)
```
**simple_reflections()**

Return the family \((s_i)_{i \in I}\) of the simple reflections of this root system.

**EXAMPLES:**

```python
sage: r = RootSystem(['A', 2]).root_lattice()
sage: s = r.simple_reflections()
sage: s[1]( r.simple_root(1) )
-alpha[1]
```

**simple_root(i)**

Returns the \(i^{th}\) simple root.

This should be overridden by any subclass, and typically implemented as a cached method for efficiency.

**EXAMPLES:**

```python
sage: r = RootSystem(['A',3]).root_lattice()
sage: r.simple_root(1)
alpha[1]
```

**simple_roots()**

Returns the family \((\alpha_i)_{i \in I}\) of the simple roots.

**EXAMPLES:**

```python
sage: alpha = RootSystem(['A',3]).root_lattice().simple_roots()
sage: [alpha[i] for i in [1,2,3]]
[alpha[1], alpha[2], alpha[3]]
```

**simple_roots_tilde()**

Return the family \((\tilde{\alpha}_i)_{i \in I}\) of the simple roots.

**INPUT:**

- `self` – an affine root lattice realization

The \(\tilde{\alpha}_i\) give the embedding of the root lattice of the other affinization of the same classical root lattice into this root lattice (space?).

This uses the fact that \(\alpha_i = \tilde{\alpha}_i\) for \(i\) not a special node, and that

\[
\delta = \sum \alpha_i \alpha_i = \sum b_i \tilde{\alpha}_i
\]

**EXAMPLES:**

In simply laced cases, this is boring:

```python
sage: RootSystem(['A',3, 1]).root_lattice().simple_roots_tilde()
Finite family {0: alpha[0], 1: alpha[1], 2: alpha[2], 3: alpha[3]}
```

This was checked by hand:

```python
sage: RootSystem(['C',2,1]).coroot_lattice().simple_roots_tilde()
Finite family {0: alphacheck[0] - alphacheck[2], 1: alphacheck[1], 2: alphacheck[2]}
sage: RootSystem(['B',2,1]).coroot_lattice().simple_roots_tilde()
Finite family {0: alphacheck[0] - alphacheck[1], 1: alphacheck[1], 2: alphacheck[2]}
```
some_elements()

Return some elements of this root lattice realization

EXAMPLES:

```
sage: L = RootSystem(['A',2]).weight_lattice()
sage: L.some_elements()
sage: L = RootSystem(['A',2]).root_lattice()
sage: L.some_elements()
```

tau_epsilon_operator_on_almost_positive_roots(J)

The $\tau_\epsilon$ operator on almost positive roots

Given a subset $J$ of non adjacent vertices of the Dynkin diagram, this constructs the operator on the almost positive roots which fixes the negative simple roots $\alpha_i$ for $i$ not in $J$, and acts otherwise by:

$$
\tau_+(\beta) = \prod_{i \in J} s_i(\beta)
$$

See Equation (1.2) of [CFZ2002].

EXAMPLES:

```
sage: L = RootSystem(['A',4]).root_lattice()
sage: tau = L.tau_epsilon_operator_on_almost_positive_roots([1,3])
sage: alpha = L.simple_roots()
```

The action on a negative simple root not in $J$:

```
sage: tau(-alpha[2])
-alpha[2]
```

The action on a negative simple root in $J$:

```
sage: tau(-alpha[1])
alpha[1]
```

The action on all almost positive roots:

```
sage: for root in L.almost_positive_roots():
    ....:     print('tau({:<41}) = {}'.format(str(root), tau(root)))
tag(-alpha[1]) = alpha[1]
tag(alpha[1]) = -alpha[1]
tag(alpha[4])
tag(-alpha[2]) = -alpha[2]
tag(alpha[2]) = alpha[1] + alpha[2] + ...
tag(alpha[3])
```

(continues on next page)

\tau(-\alpha[3]) = \alpha[3]
\tau(\alpha[3]) = -\alpha[3]
\tau(\alpha[3] + \alpha[4]) = \alpha[4]
\tau(-\alpha[4]) = -\alpha[4]
\tau(\alpha[4]) = \alpha[3] + \alpha[4]

This method works on any root lattice realization:

```python
sage: L = RootSystem(['B',3]).ambient_space()
sage: tau = L.tau_epsilon_operator_on_almost_positive_roots([1,3])
sage: for root in L.almost_positive_roots():
    print('tau({:<41}) = {}'.format(str(root), tau(root)))
tau((-1, 1, 0)) = (1, -1, 0)
tau((1, 0, 0)) = (0, 1, 0)
tau((1, -1, 0)) = (-1, 1, 0)
tau((1, 1, 0)) = (1, 1, 0)
tau((1, 0, -1)) = (0, 1, 1)
tau((1, 0, 1)) = (0, 1, -1)
tau((0, -1, 1)) = (0, -1, 1)
tau((0, 1, 0)) = (1, 0, 0)
tau((0, 1, -1)) = (1, 0, 1)
tau((0, 1, 1)) = (1, 0, -1)
tau((0, 0, -1)) = (0, 0, 1)
tau((0, 0, 1)) = (0, 0, -1)
```

See also:

tau_plus_minus()

tau_plus_minus()

Returns the \(\tau^+\) and \(\tau^-\) piecewise linear operators on self

Those operators are induced by the bipartition \(\{L, R\}\) of the simple roots of self, and stabilize the almost positive roots. Namely, \(\tau^+\) fixes the negative simple roots \(\alpha_i\) for \(i\) in \(R\), and acts otherwise by:

\[
\tau_+(\beta) = \left(\prod_{i \in L} s_i\right)\beta
\]

\(\tau^-\) acts analogously, with \(L\) and \(R\) interchanged.

Those operators are used to construct the associahedron, a polytopal realization of the cluster complex (see Associahedron).

See also:

tau_epsilon_operator_on_almost_positive_roots()

EXAMPLES:

We explore the example of [CFZ2002] Eq.(1.3):

```python
sage: S = RootSystem(['A',2]).root_lattice()
sage: taup, taum = S.tau_plus_minus()
sage: for beta in S.almost_positive_roots():
    print("\{} , \{} , \{}").format(beta, taup(beta), taum(beta))
```
to_ambient_space_morphism()

Return the morphism to the ambient space.

EXAMPLES:

```python
sage: CartanType(['B',2]).root_system().root_lattice().to_ambient_space_morphism()
Generic morphism:
  From: Root lattice of the Root system of type ['B', 2]
  To: Ambient space of the Root system of type ['B', 2]
```

```python
sage: CartanType(['B',2]).root_system().coroot_lattice().to_ambient_space_morphism()
Generic morphism:
  From: Coroot lattice of the Root system of type ['B', 2]
  To: Ambient space of the Root system of type ['B', 2]
```

```python
sage: CartanType(['B',2]).root_system().weight_lattice().to_ambient_space_morphism()
Generic morphism:
  From: Weight lattice of the Root system of type ['B', 2]
  To: Ambient space of the Root system of type ['B', 2]
```

weyl_group(prefix=None)

Returns the Weyl group associated to self.

EXAMPLES:

```python
sage: RootSystem(['F',4]).ambient_space().weyl_group()
Weyl Group of type ['F', 4] (as a matrix group acting on the ambient space)
```

```python
sage: RootSystem(['F',4]).root_space().weyl_group()
Weyl Group of type ['F', 4] (as a matrix group acting on the root space)
```

super_categories()

EXAMPLES:

```python
sage: from sage.combinat.root_system.root_lattice_realizations import RootLatticeRealizations
sage: RootLatticeRealizations(QQ).super_categories()
[Category of vector spaces with basis over Rational Field]
```
5.1.239 Root lattices and root spaces

```python
class sage.combinat.root_system.root_space.RootSpace(root_system, base_ring):
    Bases: CombinatorialFreeModule
    
The root space of a root system over a given base ring
    
    INPUT:
    
    • root_system - a root system
    • base_ring: a ring \( R \)
    
    The root space (or lattice if base_ring is \( Z \)) of a root system is the formal free module \( \bigoplus_i R \alpha_i \) generated by
    
    the simple roots \( (\alpha_i)_{i \in I} \) of the root system.
    
    This class is also used for coroot spaces (or lattices).
    
    See also:
    
    • RootSystem()
    • RootSystem.root_lattice() and RootSystem.root_space()
    • RootLatticeRealizations()
    
Todo: standardize the variable used for the root space in the examples (P?)

Element
    alias of RootSpaceElement

simple_root()
    Return the basis element indexed by \( i \).
    
    INPUT:
    
    • \( i \) – an element of the index set
    
    EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.monomial('a')
B['a']
```

F.monomial is in fact (almost) a map:

```
sage: F.monomial
Term map from {a', b', c'} to Free module generated by {a', b', c'} over Rational Field
```

to_ambient_space_morphism()
    The morphism from self to its associated ambient space.
    
    EXAMPLES:

```
sage: CartanType(['A',2]).root_system().root_lattice().to_ambient_space_morphism()
Generic morphism:
From: Root lattice of the Root system of type ['A', 2]
To:  Ambient space of the Root system of type ['A', 2]
```
```
to_coroot_space_morphism()

Returns the nu map to the coroot space over the same base ring, using the symmetrizer of the Cartan matrix

It does not map the root lattice to the coroot lattice, but has the property that any root is mapped to some scalar multiple of its associated coroot.

EXAMPLES:

```python
sage: R = RootSystem(['A',3]).root_space()
sage: alpha = R.simple_roots()
sage: f = R.to_coroot_space_morphism()
sage: f(alpha[1])
alphacheck[1]
sage: f(alpha[1]+alpha[2])

sage: R = RootSystem(['A',3]).root_lattice()
sage: alpha = R.simple_roots()
sage: f = R.to_coroot_space_morphism()
sage: f(alpha[1])
alphacheck[1]
sage: f(alpha[1]+alpha[2])

sage: S = RootSystem(['G',2]).root_space()
sage: alpha = S.simple_roots()
sage: f = S.to_coroot_space_morphism()
sage: f(alpha[1])
alphacheck[1]
sage: f(alpha[1]+alpha[2])
```

class sage.combinat.root_system.root_space.RootSpaceElement

Bases: IndexedFreeModuleElement

associated_coroot()

Returns the coroot associated to this root

OUTPUT:

An element of the coroot space over the same base ring; in particular the result is in the coroot lattice whenever self is in the root lattice.

EXAMPLES:

```python
sage: L = RootSystem(['B',3]).root_space()
sage: alpha = L.simple_roots()
sage: alpha[1].associated_coroot()
alphacheck[1]
sage: alpha[1].associated_coroot().parent()
Coroot space over the Rational Field of the Root system of type ['B', 3]

sage: L.highest_root()
sage: L.highest_root().associated_coroot()
```
Combinatorics, Release 10.0

(continued from previous page)

```
sage: alpha = RootSystem(['B', 3]).root_lattice().simple_roots()
sage: alpha[1].associated_coroot()
alphacheck[1]
sage: alpha[1].associated_coroot().parent()
Coroot lattice of the Root system of type ['B', 3]
```

**is_positive_root()**

Checks whether an element in the root space lies in the nonnegative cone spanned by the simple roots.

EXEMPLARY:

```
sage: R=RootSystem(['A',3,1]).root_space()
sage: B=R.basis()
sage: w=B[0]+B[3]
sage: w.is_positive_root()
True
sage: w.is_positive_root()
False
```

**max_coroot_le()**

Returns the highest positive coroot whose associated root is less than or equal to self.

**INPUT:**

- • self – an element of the nonnegative integer span of simple roots.

Returns None for the zero element.

Really self is an element of a coroot lattice and this method returns the highest root whose associated coroot is <= self.

**Warning:** This implementation only handles finite Cartan types

EXEMPLARY:

```
sage: root_lattice = RootSystem(['C',2]).root_lattice()
sage: root_lattice.from_vector(vector([1,1])).max_coroot_le()
sage: root_lattice.from_vector(vector([2,1])).max_coroot_le()
sage: root_lattice = RootSystem(['B',2]).root_lattice()
sage: root_lattice.from_vector(vector([1,1])).max_coroot_le()
sage: root_lattice.from_vector(vector([1,2])).max_coroot_le()
sage: root_lattice.zero().max_coroot_le() is None
True
sage: root_lattice.from_vector(vector([-1,0])).max_coroot_le()
Traceback (most recent call last):
...```
ValueError: -alpha[1] is not in the positive cone of roots

```python
sage: root_lattice = RootSystem(['A',2,1]).root_lattice()
sage: root_lattice.simple_root(1).max_coroot_le()
Traceback (most recent call last):
  ... 
NotImplementedError: Only implemented for finite Cartan type
```

**max_quantum_element()**

Returns a reduced word for the longest element of the Weyl group whose shortest path in the quantum Bruhat graph to the identity, has sum of quantum coroots at most `self`.

**INPUT:**

- `self` – an element of the nonnegative integer span of simple roots.

Really `self` is an element of a coroot lattice.

**Warning:** This implementation only handles finite Cartan types

**EXAMPLES:**

```python
sage: Qvee = RootSystem(['C',2]).coroot_lattice()
sage: Qvee.from_vector(vector([1,2])).max_quantum_element()
[2, 1, 2, 1]
sage: Qvee.from_vector(vector([1,1])).max_quantum_element()
[1, 2, 1]
sage: Qvee.from_vector(vector([0,2])).max_quantum_element()
[2]
```

**quantum_root()**

Returns True if `self` is a quantum root and False otherwise.

**INPUT:**

- `self` – an element of the nonnegative integer span of simple roots.

A root $\alpha$ is a quantum root if $\ell(s_\alpha) = \langle 2\rho, \alpha' \rangle - 1$ where $\ell$ is the length function, $s_\alpha$ is the reflection across the hyperplane orthogonal to $\alpha$, and $2\rho$ is the sum of positive roots.

**Warning:** This implementation only handles finite Cartan types and assumes that `self` is a root.

**Todo:** Rename to `is_quantum_root`

**EXAMPLES:**

```python
sage: Q = RootSystem(['C',2]).root_lattice()
sage: positive_roots = Q.positive_roots()
```

...: print("{} {} ".format(x, x.quantum_root()))
alpha[1] True

(continues on next page)
alpha[2] True

**scalar(lambdacheck)**

The scalar product between the root lattice and the coroot lattice.

**EXAMPLES:**

```
sage: L = RootSystem(['B',4]).root_lattice()
sage: alpha = L.simple_roots()
sage: alphacheck = L.simple_coroots()
sage: alpha[1].scalar(alphacheck[1])
2
sage: alpha[1].scalar(alphacheck[2])
-1
```

The scalar products between the roots and coroots are given by the Cartan matrix:

```
sage: matrix([[ alpha[i].scalar(alphacheck[j])
................ for i in L.index_set() ]
................ for j in L.index_set() ])
[ 2 -1 0 0]
[-1 2 -1 0]
[ 0 -1 2 -1]
[ 0 0 -2 2]
sage: L.cartan_type().cartan_matrix()
[ 2 -1 0 0]
[-1 2 -1 0]
[ 0 -1 2 -1]
[ 0 0 -2 2]
```

**to_ambient()**

Map self to the ambient space.

**EXAMPLES:**

```
sage: alpha = CartanType(['B',2]).root_system().root_lattice().an_element();
    ... alpha
sage: alpha.to_ambient()
(2, 0)
sage: alphavee = CartanType(['B',2]).root_system().coroot_lattice().an_element(); alphavee
sage: alphavee.to_ambient()
(2, 2)
```
5.1.240 Root systems

See Root Systems for an overview.

class sage.combinat.root_system.root_system.RootSystem(cartan_type, as_dual_of=None)
    Bases: UniqueRepresentation, SageObject
    A class for root systems.

EXAMPLES:

We construct the root system for type $B_3$:

```
sage: R=RootSystem(['B',3]); R
Root system of type ['B', 3]
```

$R$ models the root system abstractly. It comes equipped with various realizations of the root and weight lattices, where all computations take place. Let us play first with the root lattice:

```
sage: space = R.root_lattice()
sage: space
Root lattice of the Root system of type ['B', 3]
```

This is the free $\mathbb{Z}$-module $\bigoplus_i \mathbb{Z} \alpha_i$ spanned by the simple roots:

```
sage: space.base_ring()
Integer Ring
sage: list(space.basis())
[alpha[1], alpha[2], alpha[3]]
```

Let us do some computations with the simple roots:

```
sage: alpha = space.simple_roots()
```

There is a canonical pairing between the root lattice and the coroot lattice:

```
sage: R.coroot_lattice()
Coroot lattice of the Root system of type ['B', 3]
```

We construct the simple coroots, and do some computations (see comments about duality below for some caveat):

```
sage: alphacheck = space.simple_coroots()
sage: list(alphacheck)
[alphacheck[1], alphacheck[2], alphacheck[3]]
```

We can carry over the same computations in any of the other realizations of the root lattice, like the root space $\bigoplus_i \mathbb{Q} \alpha_i$, the weight lattice $\bigoplus_i \mathbb{Z} \Lambda_i$, the weight space $\bigoplus_i \mathbb{Q} \Lambda_i$. For example:

```
sage: space = R.weight_space()
sage: space
Weight space over the Rational Field of the Root system of type ['B', 3]
```
Combinatorics, Release 10.0

```
sage: space.base_ring()
Rational Field
sage: list(space.basis())
[Lambda[1], Lambda[2], Lambda[3]]

sage: alpha = space.simple_roots()
```

The fundamental weights are the dual basis of the coroots:

```
sage: Lambda = space.fundamental_weights()
sage: Lambda[1]
```

```
sage: alphacheck = space.simple_coroots()
sage: list(alphacheck)
[alphacheck[1], alphacheck[2], alphacheck[3]]
```

```
sage: [Lambda[i].scalar(alphacheck[1]) for i in space.index_set()]
[1, 0, 0]
sage: [Lambda[i].scalar(alphacheck[2]) for i in space.index_set()]
[0, 1, 0]
sage: [Lambda[i].scalar(alphacheck[3]) for i in space.index_set()]
[0, 0, 1]
```

Let us use the simple reflections. In the weight space, they work as in the number game: firing the node \( i \) on an element \( x \) adds \( c \) times the simple root \( \alpha_i \), where \( c \) is the coefficient of \( i \) in \( x \):

```
sage: s = space.simple_reflections()
sage: Lambda[1].simple_reflection(1)
sage: Lambda[2].simple_reflection(1)
Lambda[2]
sage: Lambda[3].simple_reflection(1)
Lambda[3]
```

It can be convenient to manipulate the simple reflections themselves:

```
sage: s = space.simple_reflections()
sage: s[1](Lambda[1])
sage: s[1](Lambda[2])
Lambda[2]
sage: s[1](Lambda[3])
Lambda[3]
```
Ambient spaces

The root system may also come equipped with an ambient space. This is a $\mathbb{Q}$-module, endowed with its canonical Euclidean scalar product, which admits simultaneous embeddings of the (extended) weight and the (extended) coweight lattice, and therefore the root and the coroot lattice. This is implemented on a type by type basis for the finite crystallographic root systems following Bourbaki’s conventions and is extended to the affine cases. Coefficients permitting, this is also available as an ambient lattice.

See also:
ambient_space() and ambient_lattice() for details

In finite type $A$, we recover the natural representation of the symmetric group as group of permutation matrices:

```python
sage: RootSystem(["A",2]).ambient_space().weyl_group().simple_reflections()
Finite family {1: [0 1 0]
    [1 0 0]
    [0 0 1],
    2: [1 0 0]
    [0 0 1]
    [0 1 0]}
```

In type $B$, $C$, and $D$, we recover the natural representation of the Weyl group as groups of signed permutation matrices:

```python
sage: RootSystem(["B",3]).ambient_space().weyl_group().simple_reflections()
Finite family {1: [0 1 0]
    [1 0 0]
    [0 0 1],
    2: [1 0 0]
    [0 0 1]
    [0 1 0],
    3: [ 1 0 0]
    [ 0 1 0]
    [ 0 0 -1]}
```

In (untwisted) affine types $A$, ..., $D$, one can recover from the ambient space the affine permutation representation, in window notation. Let us consider the ambient space for affine type $A$:

```python
sage: L = RootSystem(["A",2,1]).ambient_space(); L
Ambient space of the Root system of type ['A', 2, 1]
```

Define the “identity” by an appropriate vector at level $-3$:

```python
sage: e = L.basis(); Lambda = L.fundamental_weights()
sage: id = e[0] + 2*e[1] + 3*e[2] - 3*Lambda[0]
```

The corresponding permutation is obtained by projecting it onto the classical ambient space:

```python
sage: L.classical()
Ambient space of the Root system of type ['A', 2]
sage: L.classical()(id)
(1, 2, 3)
```

Here is the orbit of the identity under the action of the finite group:
sage: W = L.weyl_group()
sage: S3 = [ w.action(id) for w in W.classical() ]
sage: [L.classical()(x) for x in S3]
[(1, 2, 3), (3, 1, 2), (2, 3, 1), (2, 1, 3), (1, 3, 2), (3, 2, 1)]

And the action of $s_0$ on these yields:

sage: s = W.simple_reflections()
sage: [L.classical()(s[0].action(x)) for x in S3]
[(0, 2, 4), (-1, 1, 6), (-2, 3, 5), (0, 1, 5), (-1, 3, 4), (-2, 2, 6)]

We can also plot various components of the ambient spaces:

sage: L = RootSystem(['A',2]).ambient_space()
sage: L.plot()
Graphics object consisting of 13 graphics primitives

For more on plotting, see Tutorial: visualizing root systems.

**Dual root systems**

The root system is aware of its dual root system:

sage: R.dual
Dual of root system of type ['B', 3]

R.dual is really the root system of type $C_3$:

sage: R.dual.cartan_type()
['C', 3]

And the coroot lattice that we have been manipulating before is really implemented as the root lattice of the dual root system:

sage: R.dual.root_lattice()
Coroot lattice of the Root system of type ['B', 3]

In particular, the coroots for the root lattice are in fact the roots of the coroot lattice:

sage: list(R.root_lattice().simple_coroots())
[alphacheck[1], alphacheck[2], alphacheck[3]]
sage: list(R.coroot_lattice().simple_roots())
[alphacheck[1], alphacheck[2], alphacheck[3]]
sage: list(R.dual.root_lattice().simple_roots())
[alphacheck[1], alphacheck[2], alphacheck[3]]

The coweight lattice and space are defined similarly. Note that, to limit confusion, all the output have been tweaked appropriately.

See also:

- sage.combinat.root_system
- RootSpace
ambient_lattice()
Return the ambient lattice for this root_system.
This is the ambient space, over $\mathbb{Z}$.

See also:

- ambient_space()
- root_lattice()
- weight_lattice()

EXAMPLES:

```
sage: RootSystem(['A',4]).ambient_lattice()
Ambient lattice of the Root system of type ['A', 4]
sage: RootSystem(['A',4,1]).ambient_lattice()
Ambient lattice of the Root system of type ['A', 4, 1]
```

Except in type A, only an ambient space can be realized:

```
sage: RootSystem(['B',4]).ambient_lattice()
sage: RootSystem(['C',4]).ambient_lattice()
sage: RootSystem(['D',4]).ambient_lattice()
sage: RootSystem(['E',6]).ambient_lattice()
sage: RootSystem(['F',4]).ambient_lattice()
sage: RootSystem(['G',2]).ambient_lattice()
```

ambient_space(base_ring=Rational Field)
Return the usual ambient space for this root_system.

INPUT:

- base_ring – a base ring (default: $\mathbb{Q}$)

This is a base_ring-module, endowed with its canonical Euclidean scalar product, which admits simultaneous embeddings into the weight and the coweight lattice, and therefore the root and the coroot lattice, and preserves scalar products between elements of the coroot lattice and elements of the root or weight lattice (and dually).

There is no mechanical way to define the ambient space just from the Cartan matrix. Instead it is constructed from hard coded type by type data, according to the usual Bourbaki conventions. Such data is provided for all the finite (crystallographic) types. From this data, ambient spaces can be built as well for dual types, reducible types and affine types. When no data is available, or if the base ring is not large enough, None is returned.

**Warning:** for affine types

See also:
• The section on ambient spaces in \texttt{RootSystem}
  
  \texttt{ambient_lattice()}
  
  \texttt{AmbientSpace}
  
  \texttt{AmbientSpace}
  
  \texttt{root_space()}
  
  \texttt{weight:space()}
  
EXAMPDLES:

\begin{verbatim}
sage: RootSystem(['A',4]).ambient_space()
Ambient space of the Root system of type ['A', 4]
sage: RootSystem(['B',4]).ambient_space()
Ambient space of the Root system of type ['B', 4]
sage: RootSystem(['C',4]).ambient_space()
Ambient space of the Root system of type ['C', 4]
sage: RootSystem(['D',4]).ambient_space()
Ambient space of the Root system of type ['D', 4]
sage: RootSystem(['E',6]).ambient_space()
Ambient space of the Root system of type ['E', 6]
sage: RootSystem(['F',4]).ambient_space()
Ambient space of the Root system of type ['F', 4]
sage: RootSystem(['G',2]).ambient_space()
Ambient space of the Root system of type ['G', 2]
\end{verbatim}

An alternative base ring can be provided as an option:

\begin{verbatim}
sage: e = RootSystem(['B',3]).ambient_space(RR)
sage: TestSuite(e).run()
\end{verbatim}

It should contain the smallest ring over which the ambient space can be defined (\texttt{Z} in type \texttt{A} or \texttt{Q} otherwise). Otherwise \texttt{None} is returned:

\begin{verbatim}
sage: RootSystem(['B',2]).ambient_space(ZZ)
\end{verbatim}

The base ring should also be totally ordered. In practice, only \texttt{Z} and \texttt{Q} are really supported at this point, but you are welcome to experiment:

\begin{verbatim}
sage: e = RootSystem(['G',2]).ambient_space(RR)
sage: TestSuite(e).run()
Failure in _test_root_lattice_realization:
Traceback (most recent call last):
  ...
AssertionError: 2.000000000000000 != 2.000000000000000
\end{verbatim}
The following tests failed: _test_root_lattice_realization

```
cartan_matrix()
EXAMPLES:
sage: RootSystem(['A',3]).cartan_matrix()
[ 2 -1  0]
[-1  2 -1]
[ 0 -1  2]
```

cartan_type()
Returns the Cartan type of the root system.

```
EXAMPLES:
sage: R = RootSystem(['A',3])
sage: R.cartan_type()
['A', 3]
```

coambient_space(base_ring=Rational Field)
Return the coambient space for this root system.
This is the ambient space of the dual root system.

See also:

• ambient_space()

```
EXAMPLES:
sage: L = RootSystem(['B',2]).ambient_space(); L
Ambient space of the Root system of type ['B', 2]
sage: coL = RootSystem(['B',2]).coambient_space(); coL
Coambient space of the Root system of type ['B', 2]
```
The roots and coroots are interchanged:

```
sage: coL.simple_roots()
Finite family {1: (1, -1), 2: (0, 2)}
sage: L.simple_coroots()
Finite family {1: (1, -1), 2: (0, 2)}
```

coroot_lattice()
Returns the coroot lattice associated to self.

```
EXAMPLES:
sage: RootSystem(['A',3]).coroot_lattice()
Coroot lattice of the Root system of type ['A', 3]
```
coroot_space\(\text{base\_ring=Rational Field}\)

Returns the coroot space associated to self.

EXAMPLES:

```
sage: RootSystem(['A',3]).coroot_space()
Coroot space over the Rational Field of the Root system of type ['A', 3]
```

coweight_lattice\(\text{extended=False}\)

Returns the coweight lattice associated to self.

This is the weight lattice of the dual root system.

See also:

- \text{coweight\_space()}
- \text{weight\_space(), weight\_lattice()}
- WeightSpace

EXAMPLES:

```
sage: RootSystem(['A',3]).coweight_lattice()
Coweight lattice of the Root system of type ['A', 3]
sage: RootSystem(['A',3,1]).coweight_lattice(extended = True)
Extended coweight lattice of the Root system of type ['A', 3, 1]
```

coweight_space\(\text{base\_ring=Rational Field, extended=False}\)

Returns the coweight space associated to self.

This is the weight space of the dual root system.

See also:

- \text{coweight\_lattice()}
- \text{weight\_space(), weight\_lattice()}
- WeightSpace

EXAMPLES:

```
sage: RootSystem(['A',3]).coweight_space()
Coweight space over the Rational Field of the Root system of type ['A', 3]
sage: RootSystem(['A',3,1]).coweight_space(extended=True)
Extended coweight space over the Rational Field of the Root system of type ['A', 3, 1]
```
dynkin_diagram()

Returns the Dynkin diagram of the root system.

EXAMPLES:
```python
sage: R = RootSystem(['A',3])
sage: R.dynkin_diagram()
O---O---O
  1   2   3
A3
```

**index_set()**

EXAMPLES:

```python
sage: RootSystem(['A',3]).index_set()
(1, 2, 3)
```

**is_finite()**

Returns True if self is a finite root system.

EXAMPLES:

```python
sage: RootSystem(['A',3]).is_finite()
True
sage: RootSystem(['A',3,1]).is_finite()
False
```

**is_irreducible()**

Returns True if self is an irreducible root system.

EXAMPLES:

```python
sage: RootSystem(['A', 3]).is_irreducible()
True
sage: RootSystem('A2xB2').is_irreducible()
False
```

**root_lattice()**

Returns the root lattice associated to self.

EXAMPLES:

```python
sage: RootSystem(['A',3]).root_lattice()
Root lattice of the Root system of type ['A', 3]
```

**root_poset(restricted=False, facade=False)**

Returns the (restricted) root poset associated to self.

The elements are given by the positive roots (resp. non-simple, positive roots), and \( \alpha \leq \beta \) iff \( \beta - \alpha \) is a non-negative linear combination of simple roots.

INPUT:

- restricted – (default:False) if True, only non-simple roots are considered.
- facade – (default:False) passes facade option to the poset generator.

EXAMPLES:

```python
sage: Phi = RootSystem(['A',2]).root_poset(); Phi
Finite poset containing 3 elements
```

(continues on next page)
sage: sorted(Phi.cover_relations(), key=str)

sage: Phi = RootSystem(['A',3]).root_poset(restricted=True); Phi
Finite poset containing 3 elements
sage: sorted(Phi.cover_relations(), key=str)

sage: Phi = RootSystem(['B',2]).root_poset(); Phi
Finite poset containing 4 elements
sage: Phi.cover_relations()

root_space(base_ring=Rational Field)
Returns the root space associated to self.

EXAMPLES:

sage: RootSystem(['A',3]).root_space()
Root space over the Rational Field of the Root system of type ['A', 3]

weight_lattice(extended=False)
Returns the weight lattice associated to self.

See also:

- weight_space()
- coweight_space(), coweight_lattice()
- WeightSpace

EXAMPLES:

sage: RootSystem(['A',3]).weight_lattice()
Weight lattice of the Root system of type ['A', 3]

sage: RootSystem(['A',3,1]).weight_space(extended = True)
Extended weight space over the Rational Field of the Root system of type ['A', 3, 1]

weight_space(base_ring=Rational Field, extended=False)
Returns the weight space associated to self.

See also:

- weight_lattice()
- coweight_space(), coweight_lattice()
- WeightSpace

EXAMPLES:
sage: RootSystem(['A',3]).weight_space()
Weight space over the Rational Field of the Root system of type ['A', 3]

sage: RootSystem(['A',3,1]).weight_space(extended = True)
Extended weight space over the Rational Field of the Root system of type ['A',3, 1]

sage.combinat.root_system.root_system.WeylDim(ct, coeffs)
The Weyl Dimension Formula.

INPUT:

• type - a Cartan type
• coeffs - a list of nonnegative integers

The length of the list must equal the rank type[1]. A dominant weight hwv is constructed by summing the
fundamental weights with coefficients from this list. The dimension of the irreducible representation of the
semisimple complex Lie algebra with highest weight vector hwv is returned.

EXAMPLES:

For \(SO(7)\), the Cartan type is \(B_3\), so:

\[
\begin{align*}
\text{sage: WeylDim(['B',3],[1,0,0])} & \quad \# \text{ standard representation of } SO(7) \\
& \quad 7 \\
\text{sage: WeylDim(['B',3],[0,1,0])} & \quad \# \text{ exterior square} \\
& \quad 21 \\
\text{sage: WeylDim(['B',3],[0,0,1])} & \quad \# \text{ spin representation of } spin(7) \\
& \quad 8 \\
\text{sage: WeylDim(['B',3],[1,0,1])} & \quad \# \text{ sum of the first and third fundamental weights} \\
& \quad 48 \\
\text{sage: [WeylDim(['F',4],x) for x in ([1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1])]} \\
& \quad [52, 1274, 273, 26] \\
\text{sage: [WeylDim(['E', 6], x) for x in ([0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, \ldots, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [2, 0, 0, 0, 0, 0])]} \\
& \quad [1, 78, 27, 351, 351, 351, 27, 650, 351]
\end{align*}
\]

5.1.241 Root system data for super type A

class sage.combinat.root_system.type_super_A.AmbientSpace(root_system, base_ring, index_set=None)

Bases: AmbientSpace

The ambient space for (super) type \(A(m|n)\).

EXAMPLES:

\[
\begin{align*}
\text{sage: R = RootSystem(['A', [2,1]])} \\
\text{sage: AL = R.ambient_space(); AL} \\
\text{Ambient space of the Root system of type ['A', [2, 1]]} \\
\text{sage: AL.basis()} \\
\text{Finite family {-3: (1, 0, 0, 0, 0),} \\
& \quad -2: (0, 1, 0, 0, 0), \ldots}
\end{align*}
\]
class Element
   Bases: AmbientSpaceElement

   associated_coroot()
       Return the coroot associated to self.

   EXAMPLES:

   sage: L = RootSystem(['A', [3,2]]).ambient_space()
sage: al = L.simple_roots()
sage: al[-1].associated_coroot()
(0, 0, 1, -1, 0, 0, 0)
sage: al[0].associated_coroot()
(0, 0, 0, 1, -1, 0, 0)
sage: al[1].associated_coroot()
(0, 0, 0, 0, -1, 1, 0)
sage: a = al[-1] + al[0] + al[1]; a
(0, 0, 1, 0, 0, -1, 0)
sage: a.associated_coroot()
(0, 0, 1, 0, -2, 1, 0)
sage: h = L.simple_coroots()
sage: h[-1] + h[0] + h[1]
(0, 0, 1, 0, -2, 1, 0)
sage: (al[-1] + al[0] + al[2]).associated_coroot()
(0, 0, 1, 0, -1, -1, 0)

dot_product(lambdacheck)
   The scalar product with elements of the coroot lattice embedded in the ambient space.

   EXAMPLES:

   sage: L = RootSystem(['A', [2,1]]).ambient_space()
sage: a = L.simple_roots()
sage: matrix([[a[i].inner_product(a[j]) for j in L.index_set()] for i in L.
   .....index_set()]])
   [ 2 -1  0  0]
   [-1  2 -1  0]
   [ 0 -1  0  1]
   [ 0  0  1 -2]

   has_descent(i, positive=False)
   Test if self has a descent at position i, that is if self is on the strict negative side of the ith simple reflection hyperplane.

   If positive is True, tests if it is on the strict positive side instead.

   EXAMPLES:
```python
sage: L = RootSystem(['A', [2,1]]).ambient_space()
sage: al = L.simple_roots()
sage: [al[i].has_descent(1) for i in L.index_set()]
[False, False, True, False]
sage: [(-al[i]).has_descent(1) for i in L.index_set()]
[False, False, False, True]
sage: [al[i].has_descent(1, True) for i in L.index_set()]
[False, False, False, True]
sage: [(-al[i]).has_descent(1, True) for i in L.index_set()]
[False, False, True, False]
sage: (al[-2] + al[0] + al[1]).has_descent(-1)
True
sage: (al[-2] + al[0] + al[1]).has_descent(1)
False
sage: (al[-2] + al[0] + al[1]).has_descent(1, positive=True)
True
sage: all(all(not la.has_descent(i) for i in L.index_set()) for la in L.fundamental_weights())
True
```

**inner_product** *(lambdacheck)*

The scalar product with elements of the coroot lattice embedded in the ambient space.

**EXAMPLES:**

```python
sage: L = RootSystem(['A', [2,1]]).ambient_space()
sage: a = L.simple_roots()
sage: matrix([[a[i].inner_product(a[j]) for j in L.index_set()] for i in L.index_set()])
[ 2 -1 0 0]
[-1 2 -1 0]
[ 0 -1 0 1]
[ 0 0 1 -2]
```

**is_dominant_weight**

Test whether self is a dominant element of the weight lattice.

**EXAMPLES:**

```python
sage: L = RootSystem(['A',2]).ambient_lattice()
sage: Lambda = L.fundamental_weights()
sage: [x.is_dominant() for x in Lambda]
[True, True]
sage: (3*Lambda[1]+Lambda[2]).is_dominant()
True
sage: (Lambda[1]-Lambda[2]).is_dominant()
False
sage: (-Lambda[1]+Lambda[2]).is_dominant()
False
```

Tests that the scalar products with the coroots are all nonnegative integers. For example, if \( x \) is the sum of a dominant element of the weight lattice plus some other element orthogonal to all coroots, then the implementation correctly reports \( x \) to be a dominant weight:
sage: x = Lambda[1] + L([-1,-1,-1])
sage: x.is_dominant_weight()
True

canonical_form()

The result of applying the projection to the ambient space and then
retracting.

EXAMPLES:

sage: L = RootSystem(['A', [2,1]]).ambient_space()
sage: a = L.simple_roots()
sage: matrix([[a[i].inner_product(a[j]) for j in L.index_set()] for i in L.index_set()])

\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & 1 & -2
\end{pmatrix}
\]

dimension()

Return the dimension of this ambient space.

EXAMPLES:

sage: e = RootSystem(['A', [4,2]]).ambient_space()
sage: e.dimension()
8

fundamental_weight(i)

Return the fundamental weight \( \Lambda_i \) of self.

EXAMPLES:

sage: L = RootSystem(['A', [3,2]]).ambient_space()
sage: L.fundamental_weight(-1)
(1, 1, 0, 0, 0, 0, 0)
sage: L.fundamental_weight(0)
(1, 1, 1, 0, 0, 0, 0)
sage: L.fundamental_weight(2)
(1, 1, 1, -1, -1, -2)
sage: list(L.fundamental_weights())
[(1, 0, 0, 0, 0, 0, 0),
 (1, 1, 0, 0, 0, 0, 0),
 (1, 1, 1, 0, 0, 0, 0),
 (1, 1, 1, 1, 0, 0, 0),
 (1, 1, 1, 1, -1, -2),
 (1, 1, 1, 1, -1, -1),
 (1, 1, 1, 1, -1, -2)]

sage: L = RootSystem(['A', [2,3]]).ambient_space()
sage: La = L.fundamental_weights()
sage: al = L.simple_roots()
sage: I = L.index_set()
sage: matrix([[al[i].scalar(La[j]) for j in I] for i in I])

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
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(continued from previous page)

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

**highest_root()**
Return the highest root of `self`.

**EXAMPLES:**

```
sage: e = RootSystem(['A', [4,2]]).ambient_lattice()
sage: e.highest_root()
(1, 0, 0, 0, 0, 0, 0, -1)
```

**negative_even_roots()**
Return the negative even roots of `self`.

**EXAMPLES:**

```
sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: e.negative_even_roots()
[(0, -1, 1, 0, 0), (-1, 0, 1, 0, 0),
 (-1, 1, 0, 0, 0), (0, 0, 0, -1, 1)]
```

**negative_odd_roots()**
Return the negative odd roots of `self`.

**EXAMPLES:**

```
sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: e.negative_odd_roots()
[(0, 0, -1, 1, 0),
 (0, 0, -1, 0, 1),
 (0, -1, 0, 1, 0),
 (-1, 0, 0, 1, 0),
 (-1, 0, 0, 0, 1)]
```

**negative_roots()**
Return the negative roots of `self`.

**EXAMPLES:**

```
sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: e.negative_roots()
[(0, -1, 1, 0, 0),
 (-1, 0, 1, 0, 0),
 (-1, 1, 0, 0, 0),
 (0, 0, 0, -1, 1),
 (0, 0, -1, 1, 0),
 (0, -1, 0, 1, 0),
 (0, -1, 0, 0, 1),
 (0, -1, 0, 0, 0),
```

(continues on next page)
(-1, 0, 0, 1, 0),
(-1, 0, 0, 0, 1)]

**positive_even_roots()**
Return the positive even roots of self.

EXAMPLES:
```
sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: e.positive_even_roots()
[(0, 1, -1, 0, 0), (1, 0, -1, 0, 0),
(1, -1, 0, 0, 0), (0, 0, 0, 1, -1)]
```

**positive_odd_roots()**
Return the positive odd roots of self.

EXAMPLES:
```
sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: e.positive_odd_roots()
[(0, 0, 1, -1, 0),
(0, 0, 1, 0, -1),
(0, 1, 0, -1, 0),
(0, 1, 0, 0, -1),
(1, 0, 0, -1, 0),
(1, 0, 0, 0, -1)]
```

**positive_roots()**
Return the positive roots of self.

EXAMPLES:
```
sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: e.positive_roots()
[(0, 1, -1, 0, 0),
(1, 0, -1, 0, 0),
(1, -1, 0, 0, 0),
(0, 0, 0, 1, -1),
(0, 0, 1, -1, 0),
(0, 0, 1, 0, -1),
(0, 1, 0, -1, 0),
(0, 1, 0, 0, -1),
(1, 0, 0, -1, 0),
(1, 0, 0, 0, -1)]
```

**simple_coroot(i)**
Return the simple coroot $h_i$ of self.

EXAMPLES:
```
sage: L = RootSystem(['A', [3,2]]).ambient_space()
sage: L.simple_coroot(-2)
(0, 1, -1, 0, 0, 0, 0)
sage: L.simple_coroot(0)
```

(continues on next page)
sage: L.simple_coroot(2)
(0, 0, 0, 0, -1, 1)

sage: list(L.simple_coroots())
[(1, -1, 0, 0, 0, 0),
 (0, 1, -1, 0, 0, 0),
 (0, 0, 1, -1, 0, 0),
 (0, 0, 0, 1, -1, 0),
 (0, 0, 0, 0, -1, 1),
 (0, 0, 0, 0, 0, -1)]

simple_root(i)
Return the i-th simple root of self.

EXAMPLES:

sage: e = RootSystem(['A', [2,1]]).ambient_lattice()
sage: list(e.simple_roots())
[(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, -1), (0, 0, 0, -1)]

class method smallest_base_ring(cartan_type=None)
Return the smallest base ring the ambient space can be defined upon.

See also:
smallest_base_ring()

EXAMPLES:

sage: e = RootSystem(['A', [3,1]]).ambient_space()
sage: e.smallest_base_ring()
Integer Ring

class sage.combinat.root_system.type_super_A.CartanType(m, n)

Bases: SuperCartanType_standard

Cartan Type A(m|n).

See also:
CartanType()

AmbientSpace
alias of AmbientSpace

ascii_art(label=<function CartanType.<lambda> at 0x7f39823f39a0>, node=None)
Return an ascii art representation of the Dynkin diagram.

EXAMPLES:

sage: t = CartanType(['A', [3,2]])
sage: t.ascii_art()
O---O---O---X---O---O
-3 -2 -1 0 1 2

(continues on previous page)
cartan_matrix()

Return the Cartan matrix associated to self.

EXAMPLES:

```python
sage: ct = CartanType(['A', [2,3]])
sage: ct.cartan_matrix()
[ 2 -1  0  0  0  0]
[-1  2 -1  0  0  0]
[ 0 -1  0  1  0  0]
[ 0  0 -1  2 -1  0]
[ 0  0  0 -1  2 -1]
[ 0  0  0  0 -1  2]
```

dual()

Return dual of self.

EXAMPLES:

```python
sage: CartanType(['A', [2,3]]).dual()
['A', [2, 3]]
```

dynkin_diagram()

Return the Dynkin diagram of super type A.

EXAMPLES:

```python
sage: a = CartanType(['A', [4,2]]).dynkin_diagram()
sage: a
O---O---O---O---X---O---O
-4  -3  -2  -1  0  1  2
A4|2  
sage: a.edges(sort=True)
[(-4, -3, 1), (-3, -4, 1), (-3, -2, 1), (-2, -3, 1),
 (-2, -1, 1), (-1, -2, 1), (-1, 0, 1), (0, -1, 1),
 (0, 1, 1), (1, 0, -1), (1, 2, 1), (2, 1, 1)]
```
index_set()
Return the index set of self.

EXAMPLES:

```
sage: CartanType(['A', [2,3]]).index_set()
(-2, -1, 0, 1, 2, 3)
```

is_affine()
Return whether self is affine or not.

EXAMPLES:

```
sage: CartanType(['A', [2,3]]).is_affine()
False
```

is_finite()
Return whether self is finite or not.

EXAMPLES:

```
sage: CartanType(['A', [2,3]]).is_finite()
True
```

is_irreducible()
Return whether self is irreducible, which is True.

EXAMPLES:

```
sage: CartanType(['A', [3,4]]).is_irreducible()
True
```

relabel(relabelling)
Return a relabelled copy of this Cartan type.

INPUT:

• relabelling – a function (or a list or dictionary)

OUTPUT:

an isomorphic Cartan type obtained by relabelling the nodes of the Dynkin diagram. Namely, the node with label i is relabelled f(i) (or, by f[i] if f is a list or dictionary).

EXAMPLES:

```
sage: ct = CartanType(['A', [1,2]])
sage: ct.dynkin_diagram()
O---X---O---O
-1 0 1 2
A1|2
sage: f={1:2,2:1,0:0,-1:-1}
sage: ct.relabel(f)
['A', [1, 2]] relabelled by {-1: -1, 0: 0, 1: 2, 2: 1}
sage: ct.relabel(f).dynkin_diagram()
O---X---O---O
-1 0 2 1
A1|2 relabelled by {-1: -1, 0: 0, 1: 2, 2: 1}
```
**root_system()**
Return root system of self.

**EXAMPLES:**

```
sage: CartanType(['A', [2,3]]).root_system()
Root system of type ['A', [2, 3]]
```

**symmetrizer()**
Return symmetrizing matrix for self.

**EXAMPLES:**

```
sage: CartanType(['A', [2,3]]).symmetrizer()
Finite family {-2: 1, -1: 1, 0: 1, 1: -1, 2: -1, 3: -1}
```

**type()**
Return type of self.

**EXAMPLES:**

```
sage: CartanType(['A', [2,3]]).type()
'A'
```

### 5.1.242 Root system data for type A

**class** `sage.combinat.root_system.type_A.AmbientSpace(root_system, base_ring, index_set=None)`

**Bases:** `AmbientSpace`

**EXAMPLES:**

```
sage: R = RootSystem(['A',3])
sage: e = R.ambient_space(); e
Ambient space of the Root system of type ['A', 3]
sage: TestSuite(e).run()
```

By default, this ambient space uses the barycentric projection for plotting:

```
sage: L = RootSystem(['A',2]).ambient_space()
sage: e = L.basis()
sage: L._plot_projection(e[0])
(1/2, 989/1142)
sage: L._plot_projection(e[1])
(-1, 0)
sage: L._plot_projection(e[2])
(1/2, -989/1142)
sage: L = RootSystem(['A',3]).ambient_space()
sage: l = L.an_element(); l
(2, 2, 3, 0)
sage: L._plot_projection(l)
(0, -1121/1189, 7/3)
```

See also:


- `sage.combinat.root_system.root_lattice_realizations.RootLatticeRealizations.ParentMethods._plot_projection()

`det(k=1)`

returns the vector (1,...,1) which in the ['A',r] weight lattice, interpreted as a weight of GL(r+1,CC) is the determinant. If the optional parameter k is given, returns (k,...,k), the k-th power of the determinant.

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_space()
sage: e.det(1/2)
(1/2, 1/2, 1/2, 1/2)
```

dimensions()

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_space()
sage: e.dimension()
4
```

`fundamental_weight(i)`

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: e.fundamental_weights()
Finite family {1: (1, 0, 0, 0), 2: (1, 1, 0, 0), 3: (1, 1, 1, 0)}
```

`highest_root()`

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: e.highest_root()
(1, 0, 0, -1)
```

`negative_roots()`

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: e.negative_roots()
[(-1, 1, 0, 0),
 (-1, 0, 1, 0),
 (-1, 0, 0, 1),
 (0, -1, 1, 0),
 (0, -1, 0, 1),
 (0, 0, -1, 1)]
```

`positive_roots()`

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: e.positive_roots()
[(1, -1, 0, 0),
 (1, 0, -1, 0),
 (1, 0, 0, -1),
 ...]
```
root(i, j)

Note that indexing starts at 0.

EXAMPLES:

```
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: e.root(0,1)
(1, -1, 0, 0)
```

simple_root(i)

EXAMPLES:

```
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: e.simple_roots()
Finite family {1: (1, -1, 0, 0), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1)}
```

classmethod smallest_base_ring(cartan_type=None)

Returns the smallest base ring the ambient space can be defined upon

See also:

smallest_base_ring()

EXAMPLES:

```
sage: e = RootSystem(['A',3]).ambient_space()
sage: e.smallest_base_ring()
Integer Ring
```

class sage.combinat.root_system.type_A.CartanType(n)

Cartan Type $A_n$

See also:

CartanType()

AmbientSpace

alias of AmbientSpace

PieriFactors

alias of PieriFactors_type_A

ascii_art(label=<function CartanType.<lambda> at 0x7f39823f0f70>, node=None)

Return an ascii art representation of the Dynkin diagram.

EXAMPLES:

```
sage: print(CartanType(['A',0]).ascii_art())
sage: print(CartanType(['A',1]).ascii_art())
```

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```python
sage: print(CartanType(['A',3]).ascii_art())
O---O---O
1 2 3
sage: print(CartanType(['A',12]).ascii_art())
1 2 3 4 5 6 7 8 9 10 11 12
sage: print(CartanType(['A',5]).ascii_art(label = lambda x: x+2))
O---O---O---O---O
3 4 5 6 7
sage: print(CartanType(['A',5]).ascii_art(label = lambda x: x-2))
O---O---O---O---O
-1 0 1 2 3
```

`coxeter_number()`

Return the Coxeter number associated with `self`.

EXAMPLES:

```python
sage: CartanType(['A',4]).coxeter_number()
5
```

`dual_coxeter_number()`

Return the dual Coxeter number associated with `self`.

EXAMPLES:

```python
sage: CartanType(['A',4]).dual_coxeter_number()
5
```

`dynkin_diagram()`

Returns the Dynkin diagram of type A.

EXAMPLES:

```python
sage: a = CartanType(['A',3]).dynkin_diagram()
sage: a
O---O---O
1 2 3
A3
dynkin_diagram().edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 2, 1)]
```
5.1.243 Root system data for (untwisted) type A affine

```python
class sage.combinat.root_system.type_A_affine.CartanType(n):
    Bases: CartanType_standard_untwisted_affine

    EXAMPLES:

    sage: ct = CartanType(['A', 4, 1])
    sage: ct
    ['A', 4, 1]
    sage: ct._repr_(compact = True)
    'A4~'
    sage: ct.is_irreducible()
    True
    sage: ct.is_finite()
    False
    sage: ct.is_affine()
    True
    sage: ct.is_untwisted_affine()
    True
    sage: ct.is_crystallographic()
    True
    sage: ct.is_simply_laced()
    True
    sage: ct.classical()
    ['A', 4]
    sage: ct.dual()
    ['A', 4, 1]
    sage: ct = CartanType(['A', 1, 1])
    sage: ct.is_simply_laced()
    False
    sage: ct.dual()
    ['A', 1, 1]
```

PieriFactors

alias of `PieriFactors_type_A_affine`

```python
ascii_art(label=<function CartanType.<lambda> at 0x7f39823d75b0>, node=None)

Return an ascii art representation of the extended Dynkin diagram.

EXAMPLES:

```
sage: print(CartanType(['A',3,1]).ascii_art())
          0
         /|
        / |
       /  |
    0---1---2---3
sage: print(CartanType(['A',5,1]).ascii_art(label = lambda x: x+2))
        2
   0--2--4--6--8
```
```
(continues on next page)
sage: print(CartanType(['A',1,1]).ascii_art())
O<=>O
0 1

sage: print(CartanType(['A',1,1]).ascii_art(label = lambda x: x+2))
O<=>O
2 3
dual()
Type $A_1^\vee$ is self dual despite not being simply laced.

EXAMPLES:

sage: CartanType(['A',1,1]).dual()
['A', 1, 1]
dynkin_diagram()
Returns the extended Dynkin diagram for affine type A.

EXAMPLES:

sage: a = CartanType(['A',3,1]).dynkin_diagram()
sage: a
0
O-------+
| | 3
| | 1
0---0---0---0
1 2 3
A3~
sage: a.edges(sort=True)
[(0, 1, 1), (0, 3, 1), (1, 0, 1), (1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 0, 1), (3, 2, 1)]
sage: a = DynkinDiagram(['A',1,1])
sage: a
0<=>0
0 1
A1~
sage: a.edges(sort=True)
[(0, 1, 2), (1, 0, 2)]
5.1.244 Root system data for type A infinity

class sage.combinat.root_system.type_A_infinity.CartanType(index_set)

Bases: CartanType_standard, CartanType_simple

The Cartan type $A_\infty$.

We use $\mathbb{N}$ and $\mathbb{Z}$ to explicitly differentiate between the $A_{+\infty}$ and $A_\infty$ root systems, respectively. While $\infty$ is the same as $+\text{Infinity}$ in Sage, it is used as an alias for $\mathbb{Z}$.

ascii_art(label=<function CartanType.<lambda> at 0x7f39823f3d00>, node=None)

Return an ascii art representation of the extended Dynkin diagram.

EXAMPLES:

```
sage: print(CartanType(['A', ZZ]).ascii_art())
          .---O---O---O---O---O---O---O---...
          -3 -2 -1  0  1  2  3

sage: print(CartanType(['A', NN]).ascii_art())
           O---O---O---O---O---O---O---...
            0  1  2  3  4  5  6
```

dual()

Simply laced Cartan types are self-dual, so return self.

EXAMPLES:

```
sage: CartanType(['A', NN]).dual()
['A', NN]

sage: CartanType(['A', ZZ]).dual()
['A', ZZ]
```

index_set()

Return the index set for the Cartan type $\text{self}$. The index set for all standard finite Cartan types is of the form $\{1, \ldots, n\}$. (See type_I for a slight abuse of this).

EXAMPLES:

```
sage: CartanType(['A', NN]).index_set()
Non negative integer semiring

sage: CartanType(['A', ZZ]).index_set()
Integer Ring
```

is_affine()

Return False because $\text{self}$ is not (untwisted) affine.

EXAMPLES:

```
sage: CartanType(['A', NN]).is_affine()
False

sage: CartanType(['A', ZZ]).is_affine()
False
```
is_crystallographic()
Return False because self is not crystallographic.
EXAMPLES:

```python
sage: CartanType(['A', NN]).is_crystallographic()
True
sage: CartanType(['A', ZZ]).is_crystallographic()
True
```

is_finite()
Return True because self is not finite.
EXAMPLES:

```python
sage: CartanType(['A', NN]).is_finite()
False
sage: CartanType(['A', ZZ]).is_finite()
False
```

is_simply_laced()
Return True because self is simply laced.
EXAMPLES:

```python
sage: CartanType(['A', NN]).is_simply_laced()
True
sage: CartanType(['A', ZZ]).is_simply_laced()
True
```

is_untwisted_affine()
Return False because self is not (untwisted) affine.
EXAMPLES:

```python
sage: CartanType(['A', NN]).is_untwisted_affine()
False
sage: CartanType(['A', ZZ]).is_untwisted_affine()
False
```

rank()
Return the rank of self which for type $X_n$ is $n$.
EXAMPLES:

```python
sage: CartanType(['A', NN]).rank()
+Infinity
sage: CartanType(['A', ZZ]).rank()
+Infinity
```

As this example shows, the rank is slightly ambiguous because the root systems of type $['A', NN]$ and type $['A', ZZ]$ have the same rank. Instead, it is better to use index_set() to differentiate between these two root systems.

type()
Return the type of self.
EXAMPLES:
**5.1.245 Root system data for type B**

```python
sage: CartanType(['A', NN]).type()
'A'
sage: CartanType(['A', ZZ]).type()
'A'
```

### AmbientSpace

**class** `sage.combinat.root_system.type_B.AmbientSpace(root_system, base_ring, index_set=None)`

Bases: `AmbientSpace`

**dimension()**

EXAMPLES:

```python
sage: e = RootSystem(['B',3]).ambient_space()
sage: e.dimension()
3
```

**fundamental_weight(i)**

EXAMPLES:

```python
sage: RootSystem(['B',3]).ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0), 2: (1, 1, 0), 3: (1/2, 1/2, 1/2)}
```

**negative_roots()**

EXAMPLES:

```python
sage: RootSystem(['B',3]).ambient_space().negative_roots()
[(-1, 1, 0),
 (-1, -1, 0),
 (-1, 0, 1),
 (-1, 0, -1),
 (0, -1, 1),
 (0, -1, -1),
 (-1, 0, 0),
 (0, -1, 0),
 (0, 0, -1)]
```

**positive_roots()**

EXAMPLES:

```python
sage: RootSystem(['B',3]).ambient_space().positive_roots()
[(1, -1, 0),
 (1, 1, 0),
 (1, 0, -1),
 (1, 0, 1),
 (0, 1, -1),
 (0, 1, 1),
 (1, 0, 0),
 (0, 1, 0),
 (0, 0, 1)]
```
\textbf{root}\((i, j)\)

Note that indexing starts at 0.

EXAMPLES:

```
sage: e = RootSystem(['B',3]).ambient_space()
sage: e.root(0,1)
(1, -1, 0)
```

\textbf{simple_root}\((i)\)

EXAMPLES:

```
sage: e = RootSystem(['B',4]).ambient_space()
sage: e.simple_roots()
Finite family {1: (1, -1, 0, 0), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1), 4: (0, 0, 0, 1)}
sage: e.positive_roots()
[(1, -1, 0, 0),
 (1, 1, 0, 0),
 (1, 0, -1, 0),
 (1, 0, 1, 0),
 (1, 0, 0, -1),
 (1, 0, 0, 1),
 (0, 1, -1, 0),
 (0, 1, 1, 0),
 (0, 1, 0, -1),
 (0, 1, 0, 1),
 (0, 0, 1, -1),
 (0, 0, 1, 1),
 (0, 0, 1, 1),
 (0, 1, 0, 0),
 (0, 1, 0, 1),
 (0, 0, 0, 1)]
```

\textbf{class}\ sage.combinat.root_system.type_B.CartanType\((n)\)

Bases: CartanType_standard_finite, CartanType_simple, CartanType_crystallographic

EXAMPLES:

```
sage: ct = CartanType(['B',4])
sage: ct
['B', 4]
sage: ct._repr_(compact = True)
'B4'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
True
sage: ct.is_affine()
False
```
(continues on next page)
sage: ct = CartanType(['B',1])
sage: ct.is_simply_laced()
True
sage: ct.affine()
['B', 1, 1]

AmbientSpace

alias of AmbientSpace

PieriFactors

alias of PieriFactors_type_B

ascii_art(label=<function CartanType.<lambda> at 0x7f39823f1750>, node=None)

Return an ascii art representation of the Dynkin diagram.

EXAMPLES:

```python
sage: print(CartanType(['B',1]).ascii_art())
O
1
sage: print(CartanType(['B',2]).ascii_art())
0=>=0
1 2
sage: print(CartanType(['B',5]).ascii_art(label = lambda x: x+2))
0---0---0---0---0=>=0
3 4 5 6 7
```

coxeter_number()

Return the Coxeter number associated with self.

EXAMPLES:

```python
sage: CartanType(['B',4]).coxeter_number()
8
```

dual()

Types B and C are in duality:

EXAMPLES:

```python
sage: CartanType(['C',3]).dual()
['B', 3]
```

dual_coxeter_number()

Return the dual Coxeter number associated with self.
EXAMPLES:

```python
sage: CartanType(['B',4]).dual_coxeter_number()
7
```

dynkin_diagram()  
Returns a Dynkin diagram for type B.

EXAMPLES:

```python
sage: b = CartanType(['B',3]).dynkin_diagram()  
sage: b
O---O=>=O
1 2 3
B3
sage: b.edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 2), (3, 2, 1)]

sage: b = CartanType(['B',1]).dynkin_diagram()  
sage: b
O
1
B1
sage: b.edges(sort=True)
[]
```

### 5.1.246 Root system data for type BC affine

class `sage.combinat.root_system.type_BC_affine.CartanType(n)`

Bases: `CartanType_standard_affine`

EXAMPLES:

```python
sage: ct = CartanType(['BC',4,2])  
sage: ct
['BC', 4, 2]
sage: ct._repr_(compact = True)
'BC4~'

sage: ct.dynkin_diagram()  
O=<=O---O---O=<=O
0 1 2 3 4
BC4~

sage: ct.is_irreducible()
True
sage: ct.is_finite()
False
sage: ct.is_affine()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
False
```

(continues on next page)
sage: ct.classical()
['C', 4]
sage: dual = ct.dual()
sage: dual.dynkin_diagram()
0=>=O---O---O=>=O
0 1 2 3 4
BC4~*

sage: dual.special_node()
0
sage: dual.classical().dynkin_diagram()
O---O---O=>=O
1 2 3 4
B4

sage: CartanType(['BC',1,2]).dynkin_diagram()
4
0=<<=0
0 1
BC1~

**ascii_art**(label=<function CartanType.<lambda> at 0x7f3982408700>, node=None)

Return a ascii art representation of the extended Dynkin diagram.

**EXAMPLES:**

```python
sage: print(CartanType(['BC',2,2]).ascii_art())
O=<=O=<=O
0 1 2
sage: print(CartanType(['BC',3,2]).ascii_art())
O=<=O---O=<=O
0=<<=0--0=<<=0
0 1 2 3
sage: print(CartanType(['BC',5,2]).ascii_art(label = lambda x: x+2))
O=<=O---O---O---O=<<=0
2 3 4 5 6 7
sage: print(CartanType(['BC',1,2]).ascii_art(label = lambda x: x+2))
4
0=<<=0
0 1
```
### Cartan Type

**classical()**

Returns the classical Cartan type associated with self

```python
sage: CartanType(['BC', 3, 2]).classical()
['C', 3]
```

**dynkin_diagram()**

Returns the extended Dynkin diagram for affine type BC.

**Examples:**

```python
sage: c = CartanType(['BC', 3, 2]).dynkin_diagram()
sage: c
O---O=<=O
0 1 2 3
BC3~
sage: c.edges(sort=True)
[(0, 1, 1), (1, 0, 2), (1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 2, 2)]
```

```python
sage: c = CartanType(['A', 6, 2]).dynkin_diagram() # should be the same as above; did fail at some point!
sage: c
O---O---O
0 1 2
BC2~
sage: c.edges(sort=True)
[(0, 1, 1), (1, 0, 2), (1, 2, 1), (2, 1, 2)]
```

```python
sage: c = CartanType(['BC', 1, 2]).dynkin_diagram()
sage: c
O=<=O
0 1
BC1~
sage: c.edges(sort=True)
[(0, 1, 1), (1, 0, 4)]
```
### 5.1.247 Root system data for (untwisted) type B affine

```python
class sage.combinat.root_system.type_B_affine.CartanType(n):
    Bases: CartanType_standard_untwisted_affine

EXAMPLES:

```sage```
c = CartanType(['B',4,1])
c
['B', 4, 1]
c._repr_(compact = True)
'B4~'
c.is_irreducible()
True
c.is_finite()
False
c.is_affine()
True
c.is_untwisted_affine()
True
c.is_crystallographic()
True
c.is_simply_laced()
False
c.classical()
['B', 4]
c.dual()
['B', 4, 1]^*
c.dual().is_untwisted_affine()
False
```

**PieriFactors**

alias of *PieriFactors_type_B_affine*

```python
ascii_art(label=<function CartanType.<lambda> at 0x7f39823d7ac0>, node=None)

Return an ascii art representation of the extended Dynkin diagram.

EXAMPLES:

```sage```
print(CartanType(['B',3,1]).ascii_art())
0 0
| |
0---0=>=0
1 2 3
print(CartanType(['B',5,1]).ascii_art(label = lambda x: x+2))
0 2
| |
0---0---0---0=>=0
3 4 5 6 7
```

(continues on next page)
Combinatorics, Release 10.0

\begin{verbatim}
sage: print(CartanType(['B',2,1]).ascii_art(label = lambda x: x+2))
O=>=O=<=O
2 4 3
sage: print(CartanType(['B',1,1]).ascii_art(label = lambda x: x+2))
O<=>O
2 3
\end{verbatim}

dynkin_diagram()

Return the extended Dynkin diagram for affine type $B$.

EXAMPLES:

\begin{verbatim}
sage: b = CartanType(['B',3,1]).dynkin_diagram()
sage: b
O 0
| |
0---O=>=O=<=O
1 2 3
B3~
sage: b.edges(sort=True)
[(0, 2, 1), (1, 2, 1), (2, 0, 1), (2, 1, 1), (2, 3, 2), (3, 2, 1)]
sage: b = CartanType(['B',2,1]).dynkin_diagram(); b
O=>=O=<=O
0 2 1
B2~
sage: b.edges(sort=True)
[(0, 2, 2), (1, 2, 2), (2, 0, 1), (2, 1, 1)]
sage: b = CartanType(['B',1,1]).dynkin_diagram(); b
O<=>O
0 1
B1~
sage: b.edges(sort=True)
[(0, 1, 2), (1, 0, 2)]
\end{verbatim}

5.1.248 Root system data for type C

\begin{verbatim}
class sage.combinat.root_system.type_C.AmbientSpace(root_system, base_ring, index_set=None)
Bases: AmbientSpace

EXAMPLES:

sage: e = RootSystem(['C',2]).ambient_space(); e
Ambient space of the Root system of type ['C', 2]

One cannot construct the ambient lattice because the fundamental coweights have rational coefficients:

sage: e.smallest_base_ring()
Rational Field
\end{verbatim}

sage: RootSystem(['B',2]).ambient_space().fundamental_weights()
Finite family {1: (1, 0), 2: (1/2, 1/2)}

dimension()

EXAMPLES:

sage: e = RootSystem(['C',3]).ambient_space()
sage: e.dimension()
3

fundamental_weight(i)

EXAMPLES:

sage: RootSystem(['C',3]).ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0), 2: (1, 1, 0), 3: (1, 1, 1)}

negative_roots()

EXAMPLES:

sage: RootSystem(['C',3]).ambient_space().negative_roots()
[(-1, 1, 0),
 (-1, 0, 1),
 (0, -1, 1),
 (-1, -1, 0),
 (-1, 0, -1),
 (0, -1, -1),
 (-2, 0, 0),
 (0, -2, 0),
 (0, 0, -2)]

positive_roots()

EXAMPLES:

sage: RootSystem(['C',3]).ambient_space().positive_roots()
[(1, 1, 0),
 (1, 0, 1),
 (0, 1, 1),
 (1, -1, 0),
 (1, 0, -1),
 (0, 1, -1),
 (2, 0, 0),
 (0, 2, 0),
 (0, 0, 2)]

root(i, j, p1, p2)

Note that indexing starts at 0.

EXAMPLES:

sage: e = RootSystem(['C',3]).ambient_space()
sage: e.root(0, 1, 1, 1)
(-1, -1, 0)
simple_root(i)

EXAMPLES:

```
sage: RootSystem(['C',3]).ambient_space().simple_roots()
Finite family {1: (1, -1, 0), 2: (0, 1, -1), 3: (0, 0, 2)}
```

class sage.combinat.root_system.type_C.CartanType(n)

Bases: CartanType_standard_finite, CartanType_simple, CartanType_crystallographic

EXAMPLES:

```
sage: ct = CartanType(['C',4])
sage: ct
['C', 4]
sage: ct._repr_(compact = True)
'C4'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
False
sage: ct.affine()
['C', 4, 1]
sage: ct.dual()
['B', 4]
sage: ct = CartanType(['C',1])
sage: ct.is_simply_laced()
True
sage: ct.affine()
['C', 1, 1]
```

AmbientSpace

alias of AmbientSpace

ascii_art(label=<function CartanType.<lambda> at 0x7f39824092d0>, node=None)

Return a ascii art representation of the extended Dynkin diagram.

EXAMPLES:

```
sage: print(CartanType(['C',1]).ascii_art())
O
1
sage: print(CartanType(['C',2]).ascii_art())
0<=0
1 2
sage: print(CartanType(['C',3]).ascii_art())
0--0<=0
1 2 3
sage: print(CartanType(['C',5]).ascii_art(label = lambda x: x+2))
(continues on next page)
```
coxeter_number()
    Return the Coxeter number associated with self.

    EXAMPLES:

    sage: CartanType(['C',4]).coxeter_number()
    8

dual()
    Types B and C are in duality:

    EXAMPLES:

    sage: CartanType(['C', 3]).dual()
    ['B', 3]

dual_coxeter_number()
    Return the dual Coxeter number associated with self.

    EXAMPLES:

    sage: CartanType(['C',4]).dual_coxeter_number()
    5
dynkin_diagram()
    Returns a Dynkin diagram for type C.

    EXAMPLES:

    sage: c = CartanType(['C',3]).dynkin_diagram()
    sage: c
    O---O=<=O
    1 2 3
    C3
    sage: c.edges(sort=True)
    [(1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 2, 2)]
    sage: b = CartanType(['C',1]).dynkin_diagram()
    sage: b
    O
    1
    C1
    sage: b.edges(sort=True)
    []
5.1.249 Root system data for (untwisted) type C affine

class sage.combinat.root_system.type_C_affine.CartanType(n)

Bases: CartanType_standard_untwisted_affine

EXAMPLES:

```
sage: ct = CartanType(['C',4,1])
sage: ct
['C', 4, 1]
sage: ct._repr_(compact = True)
'C4~'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
False
sage: ct.is_affine()
True
sage: ct.is_untwisted_affine()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
False
sage: ct.classical()
['C', 4]
sage: ct.dual()
['C', 4, 1]^*
sage: ct.dual().is_untwisted_affine()
False
```

**PieriFactors**

alias of *PieriFactors_type_C_affine*

*ascii_art* (label=<function CartanType.<lambda> at 0x7f39823d7f40>, node=None)

Return a ascii art representation of the extended Dynkin diagram.

EXAMPLES:

```
sage: print(CartanType(['C',5,1]).ascii_art(label = lambda x: x+2))
0=>=O---O---O---O<=<=O
2 3 4 5 6 7
sage: print(CartanType(['C',3,1]).ascii_art())
0=>=O---O<=<=0
\emptyset 1 2 3
sage: print(CartanType(['C',2,1]).ascii_art())
0=>=0<=<=0
\emptyset 1 2
sage: print(CartanType(['C',1,1]).ascii_art())
0<=<=0
\emptyset 1
```
**dynkin_diagram()**

Returns the extended Dynkin diagram for affine type C.

**EXAMPLES:**

```
sage: c = CartanType(['C',3,1]).dynkin_diagram()
sage: c
O=>=O---O=<=O
0 1 2 3
C3~
sage: c.edges(sort=True)
[(0, 1, 2), (1, 0, 1), (1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 2, 2)]
```

### 5.1.250 Root system data for type D

**class** `sage.combinat.root_system.type_D.AmbientSpace(root_system, base_ring, index_set=None)`

**Bases:** `AmbientSpace`

**dimension()**

**EXAMPLES:**

```
sage: e = RootSystem(['D',3]).ambient_space()
sage: e.dimension()
3
```

**fundamental_weight(i)**

**EXAMPLES:**

```
sage: RootSystem(['D',4]).ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0, 0), 2: (1, 1, 0, 0), 3: (1/2, 1/2, 1/2, -1/2), 4:...
    (1/2, 1/2, 1/2, 1/2)}
```

**negative_roots()**

**EXAMPLES:**

```
sage: RootSystem(['D',4]).ambient_space().negative_roots()
[(-1, 1, 0, 0),
 (-1, 0, 1, 0),
 (0, -1, 1, 0),
 (-1, 0, 0, 1),
 (0, -1, 0, 1),
 (0, 0, -1, 1),
 (-1, -1, 0, 0),
 (-1, 0, -1, 0),
 (0, -1, -1, 0),
 (-1, 0, 0, -1),
 (0, -1, 0, -1),
 (0, 0, -1, -1)]
```

**positive_roots()**

**EXAMPLES:**
sage: RootSystem(['D',4]).ambient_space().positive_roots()
[(1, 1, 0, 0),
 (1, 0, 1, 0),
 (0, 1, 1, 0),
 (1, 0, 0, 1),
 (0, 1, 0, 1),
 (0, 0, 1, 1),
 (1, -1, 0, 0),
 (1, 0, -1, 0),
 (0, 1, -1, 0),
 (1, 0, 0, -1),
 (0, 1, 0, -1),
 (0, 0, 1, -1)]

root(i, j, p1, p2)

Note that indexing starts at 0.

EXAMPLES:

sage: e = RootSystem(['D',3]).ambient_space()
sage: e.root(0, 1, 1, 1)
(-1, -1, 0)
sage: e.root(0, 0, 1, 1)
(-1, 0, 0)

simple_root(i)

EXAMPLES:

sage: RootSystem(['D',4]).ambient_space().simple_roots()
Finite family {1: (1, -1, 0, 0), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1), 4: (0, 0, 1, 1)}

class sage.combinat.root_system.type_D.CartanType(n)

Bases: CartanType_standard_finite, CartanType_simply_laced

EXAMPLES:

sage: ct = CartanType(['D',4])
sage: ct
['D', 4]
sage: ct._repr_(compact = True)
'D4'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
True
sage: ct.dual()
['D', 4]
sage: ct.affine()

(continues on next page)
['D', 4, 1]
sage: ct = CartanType(['D', 2])
sage: ct.is_irreducible()
False
sage: ct.dual()
['D', 2]
sage: ct.affine()
Traceback (most recent call last):
  ...
ValueError: ['D', 2, 1] is not a valid Cartan type

\texttt{AmbientSpace}  
alias of \texttt{AmbientSpace}

\texttt{ascii_art}(\texttt{label=\langle\texttt{function CartanType.<lambda> at 0x7f3982409cf0\rangle, node=None}})

Return a ascii art representation of the extended Dynkin diagram.

\textbf{EXAMPLES:}

\begin{verbatim}sage: print(CartanType(['D',3]).ascii_art())
0 3
  |
0---0
1 2
\end{verbatim}

\begin{verbatim}sage: print(CartanType(['D',4]).ascii_art())
0 4
  |
0---0---0
1 2 3
\end{verbatim}

\begin{verbatim}sage: print(CartanType(['D',4]).ascii_art(label = lambda x: x+2))
0 6
  |
0---0---0
3 4 5
\end{verbatim}

\begin{verbatim}sage: print(CartanType(['D',6]).ascii_art(label = lambda x: x+2))
0 8
  |
0---0---0---0---0---0---0
3 4 5 6 7
\end{verbatim}

\texttt{coxeter_number}()

Return the Coxeter number associated with \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}sage: CartanType(['D',4]).coxeter_number()
6
\end{verbatim}
dual_coxeter_number()

Return the dual Coxeter number associated with self.

EXAMPLES:

```
sage: CartanType(['D',4]).dual_coxeter_number()
sage: 6
```

dynkin_diagram()

Returns a Dynkin diagram for type D.

EXAMPLES:

```
sage: d = CartanType(['D',5]).dynkin_diagram(); d
O 5
|  
|  
O---O---O---O
1 2 3 4
D5
sage: d.edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 1), (3, 2, 1), (3, 4, 1), (3, 5, 1), (4, 3, 1), (5, 3, 1)]
sage: d = CartanType(['D',4]).dynkin_diagram(); d
O 4
|  
|  
O---O---O
1 2 3
D4
sage: d.edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 1), (2, 4, 1), (3, 2, 1), (4, 2, 1)]
sage: d = CartanType(['D',3]).dynkin_diagram(); d
O 3
|  
|  
O---0
1 2
D3
sage: d.edges(sort=True)
[(1, 2, 1), (1, 3, 1), (2, 1, 1), (3, 1, 1)]
sage: d = CartanType(['D',2]).dynkin_diagram(); d
O O
1 2
D2
sage: d.edges(sort=True)
[]
```

is_atomic()

Implements CartanType_abstract.is_atomic()
$D_2$ is atomic, like all $D_n$, despite being non irreducible.

**EXAMPLES:**

```python
sage: CartanType(['D',2]).is_atomic()
True
sage: CartanType(['D',2]).is_irreducible()
False
```

### 5.1.251 Root system data for (untwisted) type D affine

**class** `sage.combinat.root_system.type_D_affine.CartanType(n)`

Bases: `CartanType_standard_untwisted_affine, CartanType_simply_laced`

**EXAMPLES:**

```python
sage: ct = CartanType(['D',4,1])
sage: ct
['D', 4, 1]
sage: ct._repr_(compact = True)
'D4~'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
False
sage: ct.is_affine()
True
sage: ct.is_untwisted_affine()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
True
sage: ct.classical()
['D', 4]
sage: ct.dual()
['D', 4, 1]
```

**ascii_art**

alias of `PieriFactors_type_D_affine`

**Return an ascii art representation of the extended Dynkin diagram.**

**dynkin_diagram()**

Returns the extended Dynkin diagram for affine type D.

**EXAMPLES:**

```python
sage: d = CartanType(['D',6,1]).dynkin_diagram()
sage: d
|
0 0 0 6 |
```

(continues on next page)
5.1.252 Root system data for type E

class sage.combinat.root_system.type_E.AmbientSpace(root_system, baseRing)

Bases: AmbientSpace

The lattice behind E6, E7, or E8. The computations are based on Bourbaki, Groupes et Algèbres de Lie, Ch. 4,5,6 (planche V-VII).

dimension()

EXAMPLES:
Combinatorics, Release 10.0

sage: e = RootSystem(['E',6]).ambient_space()
sage: e.dimension()
8

fundamental_weights()

EXAMPLES:

sage: e = RootSystem(['E',6]).ambient_space()
sage: e.fundamental_weights()
Finite family {1: (0, 0, 0, 0, 0, -2/3, -2/3, 2/3), 2: (1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2), 3: (-1/2, 1/2, 1/2, 1/2, 1/2, -5/6, -5/6, 5/6), 4: (0, 0, -1, 1, 1, -1, -1, 1), 5: (0, 0, 1, 1, -2/3, -2/3, 2/3), 6: (0, 0, 0, 1, -1/3, -1/3, 1/3)}

negative_roots()

The negative roots.

EXAMPLES:

sage: e = RootSystem(['E',6]).ambient_space()
sage: e.negative_roots()
[(-1, -1, 0, 0, 0, 0, 0, 0),
 (-1, 0, -1, 0, 0, 0, 0, 0),
 (-1, 0, 0, -1, 0, 0, 0, 0),
 (-1, 0, 0, 0, -1, 0, 0, 0),
 (0, -1, -1, 0, 0, 0, 0, 0),
 (0, -1, 0, -1, 0, 0, 0, 0),
 (0, -1, 0, 0, -1, 0, 0, 0),
 (0, 0, -1, -1, 0, 0, 0, 0),
 (0, 0, -1, 0, -1, 0, 0, 0),
 (0, 0, 0, -1, -1, 0, 0, 0),
 (1, -1, 0, 0, 0, 0, 0, 0),
 (1, 0, -1, 0, 0, 0, 0, 0),
 (1, 0, 0, -1, 0, 0, 0, 0),
 (1, 0, 0, 0, -1, 0, 0, 0),
 (0, 1, -1, 0, 0, 0, 0, 0),
 (0, 1, 0, -1, 0, 0, 0, 0),
 (0, 0, 1, -1, 0, 0, 0, 0),
 (0, 0, 1, 0, -1, 0, 0, 0),
 (0, 0, 0, 1, -1, 0, 0, 0),
 (0, 0, 0, 0, -1, 0, 0, 0),
 (-1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (-1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2),
 (-1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
 (-1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
 (-1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
 (-1/2, 1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
 (-1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2, -1/2),
 (-1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2, -1/2),
 (-1/2, 1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
 (1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2),
 (1/2, -1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
 (1/2, 1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
 (1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
 (1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
 (1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
 (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2),
 (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2),

(continues on next page)
These are the roots positive w.r. to lexicographic ordering of the basis elements (e1<...<e4).

**EXAMPLES:**

```python
sage: e = RootSystem(['E',6]).ambient_space()
sage: e.positive_roots()
[(1, 1, 0, 0, 0, 0, 0, 0),
 (1, 0, 1, 0, 0, 0, 0, 0),
 (1, 0, 0, 1, 0, 0, 0, 0),
 (0, 1, 1, 0, 0, 0, 0, 0),
 (0, 1, 0, 1, 0, 0, 0, 0),
 (0, 0, 1, 1, 0, 0, 0, 0),
 (0, 0, 1, 0, 1, 0, 0, 0),
 (0, 0, 0, 1, 1, 0, 0, 0),
 (-1, 1, 0, 0, 0, 0, 0, 0),
 (-1, 0, 1, 0, 0, 0, 0, 0),
 (-1, 0, 0, 1, 0, 0, 0, 0),
 (-1, 0, 0, 0, 1, 0, 0, 0),
 (0, -1, 1, 0, 0, 0, 0, 0),
 (0, -1, 0, 1, 0, 0, 0, 0),
 (0, 0, -1, 1, 0, 0, 0, 0),
 (0, 0, 0, -1, 1, 0, 0, 0),
 (0, 0, 0, 0, -1, 1, 0, 0),
 (0, 0, 0, 0, 0, -1, 1, 0),
 (0, 0, 0, 0, 0, 0, -1, 1),
 (0, 0, 0, 0, 0, 0, 0, -1),
 (1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
 (1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
 (1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
 (1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
 (1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2, -1/2),
 (1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2),
 (1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2),
 (0, -1, 0, 1, 0, 0, 0, 0),
 (0, -1, 0, 0, 1, 0, 0, 0),
 (0, 0, 0, 0, -1, 1, 0, 0),
 (0, 0, 0, 0, -1, 0, 1, 0),
 (0, 0, 0, -1, 0, 1, 0, 0),
 (0, 0, 0, 0, -1, 0, 1, 0),
 (0, 0, 0, 0, 0, 0, -1, 1),
 (0, 0, 0, 0, 0, 0, -1, 1),
 (0, 0, 0, 0, 0, 0, 0, -1),
 (-1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
 (-1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (-1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
 (-1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (-1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (-1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
 (-1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
 (-1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2, -1/2),
 (-1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2),
 (-1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
 (-1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2)]
```

```python
sage: e.rho()
(0, 1, 2, 3, 4, -4, -4, 4)
sage: E8 = RootSystem(['E',8])
sage: e = E8.ambient_space()
```

(continues on next page)
sage: e.negative_roots()
[(-1, -1, 0, 0, 0, 0, 0, 0),
 (-1, 0, -1, 0, 0, 0, 0, 0),
 (-1, 0, 0, -1, 0, 0, 0, 0),
 (-1, 0, 0, 0, -1, 0, 0, 0),
 (-1, 0, 0, 0, 0, -1, 0, 0),
 (-1, 0, 0, 0, 0, 0, -1, 0),
 (-1, 0, 0, 0, 0, 0, 0, -1),
 (0, -1, -1, 0, 0, 0, 0, 0),
 (0, -1, 0, -1, 0, 0, 0, 0),
 (0, 0, -1, 0, 0, 0, 0, -1),
 (0, -1, 0, 0, -1, 0, 0, 0),
 (0, 0, -1, 0, 0, -1, 0, 0),
 (0, -1, 0, 0, 0, -1, 0, 0),
 (0, 0, -1, 0, 0, 0, -1, 0),
 (0, -1, 0, 0, 0, 0, -1, 0),
 (0, 0, -1, 0, 0, 0, 0, -1),
 (0, 0, 0, -1, 0, -1, 0, 0),
 (0, 0, 0, -1, 0, 0, -1, 0),
 (0, 0, 0, 0, -1, 0, -1, 0),
 (0, 0, 0, 0, 0, -1, 0, -1),
 (0, 0, 0, 0, 0, -1, -1, 0),
 (1, -1, 0, 0, 0, 0, 0, 0),
 (1, 0, -1, 0, 0, 0, 0, 0),
 (1, 0, 0, -1, 0, 0, 0, 0),
 (1, 0, 0, 0, -1, 0, 0, 0),
 (1, 0, 0, 0, 0, -1, 0, 0),
 (1, 0, 0, 0, 0, 0, -1, 0),
 (0, 1, -1, 0, 0, 0, 0, 0),
 (0, 1, 0, -1, 0, 0, 0, 0),
 (0, 0, -1, 0, 0, -1, 0, 0),
 (0, 0, 0, -1, 0, 0, -1, 0),
 (0, 0, 0, 0, -1, 0, -1, 0),
 (0, 0, 0, 0, 0, -1, 0, -1),
 (0, 0, 0, 0, 0, 0, -1, 0),
 (0, 0, 1, 0, 0, -1, 0, 0),
 (0, 0, 0, 1, 0, 0, -1, 0),
 (0, 0, 0, 0, 1, 0, 0, -1),
 (0, 0, 0, 0, 0, 1, -1, 0, 0),
 (0, 0, 0, 0, 0, 0, 1, -1),
 (0, 0, 0, 0, 0, 0, 0, 1),
 (0, 0, 0, 0, 0, 1, 0, 0),
 (0, 0, 0, 0, 0, 0, 1, 0),
 (0, 0, 0, 0, 0, 0, 0, 1),
 (0, 0, 0, 0, 0, 0, 0, 0),

(continues on next page)
(0, 0, 0, 0, 1, 0, -1, 0),
(0, 0, 0, 0, 1, 0, 0, -1),
(0, 0, 0, 0, 1, -1, 0),
(0, 0, 0, 0, 0, 1, 0, -1),
(0, 0, 0, 0, 0, 0, 1, -1),
(-1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2),
(-1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2, -1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2),
(-1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2, -1/2),
(-1/2, -1/2, -1/2, 1/2, -1/2, 1/2, -1/2, -1/2),
(-1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
(-1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, 1/2, -1/2, -1/2),
(-1/2, 1/2, -1/2, -1/2, 1/2, -1/2, -1/2, -1/2),
(-1/2, 1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2),
(-1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2),
(-1/2, 1/2, -1/2, 1/2, -1/2, 1/2, 1/2, -1/2),
(-1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
(-1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2),
(-1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2),
(-1/2, 1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
(-1/2, 1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2),
(-1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, 1/2, -1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2),
(1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2),
(1/2, -1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, -1/2, 1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2, -1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2),
(1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2),
\[
\begin{align*}
(1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2), \\
(1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2), \\
(1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2, -1/2), \\
(1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2), \\
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2).
\end{align*}
\]

```
sage: e.\texttt{rho()}
(0, 1, 2, 3, 4, 5, 6, 23)
```

```python
def root(i1, i2=None, i3=None, i4=None, i5=None, i6=None, i7=None, i8=None, p1=0, p2=0, p3=0, p4=0, p5=0, p6=0, p7=0, p8=0):
    # Compute an element of the underlying lattice, using the specified elements
```

EXAMPLES:

```
sage: e = RootSystem(['E',6]).ambient_space()
sage: [ e.root(i, j, p3=1) for i in range(e.n) for j in range(i+1, e.n) ]
```

(continues on next page)
simple_root\((i)\)

There are computed as what Bourbaki calls the Base:
\[
a_1 = e_2 - e_3, a_2 = e_3 - e_4, a_3 = e_4, a_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)
\]

EXAMPLES:

```
sage: LE6 = RootSystem(['E', 6]).ambient_space()
sage: LE6.simple_roots()
Finite family {1: (1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2), 2: (1, 1, 0, 0, 0, 0, 0), 3: (-1, 1, 0, 0, 0, 0, 0), 4: (0, -1, 1, 0, 0, 0, 0), 5: (0, 0, -1, 1, 0, 0, 0), 6: (0, 0, 0, -1, 1, 0, 0)}
```

class `sage.combinat.root_system.type_E.CartanType`\(\(n\)\)

Bases: `CartanType_standard_finite`, `CartanType_simple`, `CartanType_simply_laced`

EXAMPLES:

```
sage: ct = CartanType(['E', 6])
sage: ct
['E', 6]
sage: ct._repr_(compact = True)
'E6'
sage: ct.is_irreducible()
True
tsage: ct.is_finite()
True
tsage: ct.is_affine()
False
tsage: ct.is_crystallographic()
True
tsage: ct.is_simply_laced()
True
tsage: ct.affine()
['E', 6, 1]
tsage: ct.dual()
['E', 6]
```

`AmbientSpace`

alias of `AmbientSpace`

`ascii_art`

`label=<function CartanType.<lambda> at 0x7f398240a710>, node=None`

Return a ascii art representation of the extended Dynkin diagram.

EXAMPLES:
sage: print(CartanType(['E',6]).ascii_art(label = lambda x: x+2))
 0 4
   |
   |
0---0---0---0---0---0
3 5 6 7 8
sage: print(CartanType(['E',7]).ascii_art(label = lambda x: x+2))
 0 4
   |
   |
0---0---0---0---0---0---0
3 5 6 7 8 9
sage: print(CartanType(['E',8]).ascii_art(label = lambda x: x+1))
 0 3
   |
   |
0---0---0---0---0---0---0---0
2 4 5 6 7 8 9

coxeter_number()
Return the Coxeter number associated with self.
EXAMPLES:

sage: CartanType(['E',6]).coxeter_number()
12
sage: CartanType(['E',7]).coxeter_number()
18
sage: CartanType(['E',8]).coxeter_number()
30
dual_coxeter_number()
Return the dual Coxeter number associated with self.
EXAMPLES:

sage: CartanType(['E',6]).dual_coxeter_number()
12
sage: CartanType(['E',7]).dual_coxeter_number()
18
sage: CartanType(['E',8]).dual_coxeter_number()
30
dynkin_diagram()
Returns a Dynkin diagram for type E.
EXAMPLES:

sage: e = CartanType(['E',6]).dynkin_diagram()
sage: e
0 2
   |
   |
0---0---0---0---0
(continues on next page)
1 3 4 5 6

E6

sage: e.edges(sort=True)
[(1, 3, 1), (2, 4, 1), (3, 1, 1), (3, 4, 1), (4, 2, 1), (4, 3, 1), (4, 5, 1), (5, 4, 1), (5, 6, 1), (6, 5, 1)]

sage: e = CartanType(['E',7]).dynkin_diagram()
sage: e
O 2
|   1 3 4 5 6
O---O---O---O---O---O
1 2

E7

sage: e.edges(sort=True)
[(1, 3, 1), (2, 4, 1), (3, 1, 1), (3, 4, 1), (4, 2, 1), (4, 3, 1), (4, 5, 1), (5, 4, 1), (5, 6, 1), (6, 5, 1), (6, 7, 1), (7, 6, 1)]

sage: e = CartanType(['E',8]).dynkin_diagram()
sage: e
O 2
|   1 3 4 5 6 7 8
O---O---O---O---O---O---O
1 2

E8

sage: e.edges(sort=True)
[(1, 3, 1), (2, 4, 1), (3, 1, 1), (3, 4, 1), (4, 2, 1), (4, 3, 1), (4, 5, 1), (5, 4, 1), (5, 6, 1), (6, 5, 1), (6, 7, 1), (7, 6, 1), (7, 8, 1), (8, 7, 1)]

5.1.253 Root system data for (untwisted) type E affine

class sage.combinat.root_system.type_E_affine.CartanType(n)

    Bases: CartanType_standard_untwisted_affine, CartanType_simply_laced

    EXAMPLES:

    sage: ct = CartanType(['E',6,1])
sage: ct
['E', 6, 1]
sage: ct._repr_(compact = True)
'E6~'

    sage: ct.is_irreducible()
    True
    sage: ct.is_finite()
    False
    sage: ct.is_affine()
    True
    sage: ct.is_untwisted_affine()
    True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
True
sage: ct.classical()
['E', 6]
sage: ct.dual()
['E', 6, 1]

ascii_art(label=<function CartanType.<lambda> at 0x7f398240ab90>, node=None)

Return an ascii art representation of the extended Dynkin diagram.

EXAMPLES:

sage: print(CartanType(['E',6,1]).ascii_art(label = lambda x: x+2))
 0 2
 |  
 |  
 0---0---0---0---0
 3 5 6 7 8
sage: print(CartanType(['E',7,1]).ascii_art(label = lambda x: x+2))
 0 4
 |  
 |  
 0---0---0---0---0---0---0
 2 3 5 6 7 8 9
sage: print(CartanType(['E',8,1]).ascii_art(label = lambda x: x-3))
 0 -1
 |  
 |  
 0---0---0---0---0---0---0---0
 -2 0 1 2 3 4 5 -3

dynkin_diagram()

Returns the extended Dynkin diagram for affine type E.

EXAMPLES:

sage: e = CartanType(['E', 6, 1]).dynkin_diagram()
sage: e

(continues on next page)
5.1.254 Root system data for type F

class sage.combinat.root_system.type_F.AmbientSpace(root_system, base_ring)

Bases: AmbientSpace

The lattice behind $F_4$. The computations are based on Bourbaki, Groupes et Algèbres de Lie, Ch. 4.5.6 (planche VIII).

dimension()

Return the dimension of self.

EXAMPLES:
sage: e = RootSystem(['F',4]).ambient_space()
sage: e.dimension()
4

fundamental_weights()

Return the fundamental weights of self.

EXAMPLES:

sage: e = RootSystem(['F',4]).ambient_space()
sage: e.fundamental_weights()
Finite family {1: (1, 1, 0, 0), 2: (2, 1, 1, 0), 3: (3/2, 1/2, 1/2, 1/2), 4: (1, 0, 0, 0)}

negative_roots()

Return the negative roots.

EXAMPLES:

sage: e = RootSystem(['F',4]).ambient_space()
sage: e.negative_roots()
[(-1, 0, 0, 0),
 (0, -1, 0, 0),
 (0, 0, -1, 0),
 (-1, -1, 0, 0),
 (-1, 0, -1, 0),
 (-1, 0, 0, -1),
 (-1, -1, 0, -1),
 (-1, 1, 0, 0),
 (-1, 0, 1, 0),
 (-1, 0, 0, 1),
 (0, -1, 1, 0),
 (0, 0, -1, 1),
 (0, 0, 0, -1),
 (-1/2, -1/2, 1/2, 1/2),
 (-1/2, -1/2, 1/2, -1/2),
 (-1/2, 1/2, -1/2, -1/2),
 (-1/2, 1/2, 1/2, 1/2),
 (-1/2, 1/2, -1/2, 1/2),
 (-1/2, -1/2, -1/2, 1/2),
 (-1/2, -1/2, -1/2, -1/2)]

positive_roots()

Return the positive roots.

These are the roots which are positive with respect to the lexicographic ordering of the basis elements \(\epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4\).

EXAMPLES:
sage: e = RootSystem(['F',4]).ambient_space()
sage: e.positive_roots()
[(1, 0, 0, 0),
(0, 1, 0, 0),
(0, 0, 1, 0),
(0, 0, 0, 1),
(1, 1, 0, 0),
(1, 0, 1, 0),
(1, 0, 0, 1),
(0, 1, 1, 0),
(0, 1, 0, 1),
(0, 0, 1, 1),
(1, -1, 0, 0),
(1, 0, -1, 0),
(1, 0, 0, -1),
(0, 1, -1, 0),
(0, 1, 0, -1),
(0, 0, 1, -1),
(1/2, 1/2, 1/2, 1/2),
(1/2, 1/2, 1/2, -1/2),
(1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, -1/2)]
sage: e.rho()
(11/2, 5/2, 3/2, 1/2)

root(i, j=None, k=None, l=None, p1=0, p2=0, p3=0, p4=0)

Compute a root from base elements of the underlying lattice. The arguments specify the basis elements
and the signs. Sadly, the base elements are indexed zero-based. We assume that if one of the indices is not
given, the rest are not as well.

EXAMPLES:

sage: e = RootSystem(['F',4]).ambient_space()
sage: [ e.root(i,j,p2=1) for i in range(e.n) for j in range(i+1,e.n) ]
[(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1), (0, 1, -1, 0), (0, 1, 0, -1), (0, 1, 0, -1)]

simple_root(i)

Return the $i$-th simple root.

It is computed according to what Bourbaki calls the Base:

$$\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4, \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

EXAMPLES:

sage: e = RootSystem(['F',4]).ambient_space()
sage: e.simple_roots()
Finite family {1: (0, 1, -1, 0), 2: (0, 0, 1, -1), 3: (0, 0, 0, 1), 4: (1/2, -1/2, -1/2, -1/2)}
class sage.combinat.root_system.type_F.CartanType
    Bases: CartanType_standard_finite, CartanType_simple, CartanType_crystallographic

EXAMPLES:

    sage: ct = CartanType(['F',4])
    sage: ct
    ['F', 4]
    sage: ct._repr_(compact = True)
    'F4'
    sage: ct.is_irreducible()
    True
    sage: ct.is_finite()
    True
    sage: ct.is_crystallographic()
    True
    sage: ct.is_simply_laced()
    False
    sage: ct.dual() ['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
    sage: ct.affine()
    ['F', 4, 1] 

AmbientSpace
    alias of AmbientSpace

ascii_art(label=<function CartanType.<lambda> at 0x7f398240b640>, node=None)
    Return an ascii art representation of the extended Dynkin diagram.

EXAMPLES:

    sage: print(CartanType(['F',4]).ascii_art(label = lambda x: x+2))
    O---O=>=O---O
    0  1  2  3  4
    sage: print(CartanType(['F',4]).ascii_art(label = lambda x: x-2))
    O---O=>=O---O
    -2  -1  0  1

coxeter_number()
    Return the Coxeter number associated with self.

EXAMPLES:

    sage: CartanType(['F',4]).coxeter_number()
    12

dual()
    Return the dual Cartan type.

    This uses that \( F_4 \) is self-dual up to relabelling.

EXAMPLES:

    sage: F4 = CartanType(['F',4])
    sage: F4.dual()
Combinatorics, Release 10.0

[Continued from previous page]

\[
\begin{bmatrix}
'relabeled by {1: 4, 2: 3, 3: 2, 4: 1}
sage: F4.dynkin_diagram()
O---O=>=O---O
1 2 3 4
F4
sage: F4.dual().dynkin_diagram()
O---O=>=O---O
4 3 2 1
F4 relabeled by {1: 4, 2: 3, 3: 2, 4: 1}

dual_coxeter_number()

Return the dual Coxeter number associated with self.

EXAMPLES:
sage: CartanType(['F',4]).dual_coxeter_number()
9
dynkin_diagram()

Returns a Dynkin diagram for type F.

EXAMPLES:
sage: f = CartanType(['F',4]).dynkin_diagram()
sage: f
O---O=>=O---O
1 2 3 4
F4
sage: f.edges(sort=True)
[(1, 2, 1), (2, 1, 1), (2, 3, 2), (3, 2, 1), (3, 4, 1), (4, 3, 1)]

5.1.255 Root system data for (untwisted) type F affine

class sage.combinat.root_system.type_F_affine.CartanType

Bases: CartanType_standard_untwisted affine

EXAMPLES:
sage: ct = CartanType(['F',4,1])
sage: ct
['F', 4, 1]
sage: ct._repr_(compact = True)
'F4~'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
False
sage: ct.is_affine()
True
sage: ct.is_untwisted_affine()
True
\texttt{sage: } ct.\text{is\_crystallographic}()
\texttt{True}
\texttt{sage: } ct.\text{is\_simply\_laced}()
\texttt{False}
\texttt{sage: } ct.\text{classical}()
\texttt{[}'F', 4\texttt{]}
\texttt{sage: } ct.\text{dual}()
\texttt{[}'F', 4, 1\texttt{]^*}
\texttt{sage: } ct.\text{dual}().\text{is\_untwisted\_affine}()
\texttt{False}

\texttt{ascii\_art}(\texttt{label=\{function\ CartanType.\_<lambda>\ at 0x7f398240bb50\}, node=None})

Returns a ascii art representation of the extended Dynkin diagram

\texttt{EXAMPLES:}
\texttt{sage: print(CartanType([}'F', 4, 1\texttt{]). ascii\_art(\texttt{label = lambda x: x+2})}
\begin{verbatim}
O---O---O=>=O---O
2 3 4 5 6
\end{verbatim}

\texttt{dynkin\_diagram()}

Returns the extended Dynkin diagram for affine type F.

\texttt{EXAMPLES:}
\texttt{sage: } f = CartanType([}'F', 4, 1\texttt{]).\text{dynkin\_diagram}()
\texttt{sage: } f
\begin{verbatim}
O---O---O=>=O---O
\end{verbatim}
\texttt{\theta 1 2 3 4}
\texttt{F4~}
\texttt{sage: } f.\text{edges}(\texttt{sort=\{True\})}
\begin{verbatim}
[(0, 1, 1), (1, 0, 1), (1, 2, 1), (2, 1, 1), (2, 3, 2), (3, 2, 1), (3, 4, 1), \sim
(4, 3, 1)]
\end{verbatim}

\section{5.1.256 Root system data for type G}

\texttt{class sage.combinat.root_system.type_G.AmbientSpace(root\_system, base\_ring, index\_set=None)}

Bases: \texttt{AmbientSpace}

\texttt{EXAMPLES:}
\texttt{sage: } e = RootSystem([}'G', 2\texttt{]).\text{ambient\_space}(); e
\begin{verbatim}
Ambient space of the Root system of type ['G', 2]
\end{verbatim}

One can not construct the ambient lattice because the simple coroots have rational coefficients:
\texttt{sage: } e.\text{simple\_coroots}()
Finite family \{1: (0, 1, -1), 2: (1/3, -2/3, 1/3}\}
\texttt{sage: } e.\text{smallest\_base\_ring}()
\begin{verbatim}
Rational Field
\end{verbatim}

By default, this ambient space uses the barycentric projection for plotting:
Combinatorics, Release 10.0

```python
sage: L = RootSystem(['G',2]).ambient_space()
sage: e = L.basis()
sage: L._plot_projection(e[0])
(1/2, 989/1142)
sage: L._plot_projection(e[1])
(-1, 0)
sage: L._plot_projection(e[2])
(1/2, -989/1142)
sage: L = RootSystem(['A',3]).ambient_space()
sage: l = L.an_element(); l
(2, 2, 3, 0)
sage: L._plot_projection(l)
(0, -1121/1189, 7/3)
```

See also:

- sage.combinat.root_system.root_lattice_realizations.RootLatticeRealizations.
  ParentMethods._plot_projection()

**dimension()**

```python
sage: e = RootSystem(['G',2]).ambient_space()
sage: e.dimension()
3
```

**fundamental_weights()**

```python
sage: CartanType(['G',2]).root_system().ambient_space().fundamental_weights()
Finite family {1: (1, 0, -1), 2: (2, -1, -1)}
```

**negative_roots()**

```python
sage: CartanType(['G',2]).root_system().ambient_space().negative_roots()
[(0, -1, 1), (-1, 2, -1), (-1, 1, 0), (-1, 0, 1), (-1, -1, 2), (-2, 1, 1)]
```

**positive_roots()**

```python
sage: CartanType(['G',2]).root_system().ambient_space().positive_roots()
[(0, 1, -1), (1, -2, 1), (1, -1, 0), (1, 0, -1), (1, 1, -2), (2, -1, -1)]
```

**simple_root(i)**

```python
sage: CartanType(['G',2]).root_system().ambient_space().simple_roots()
Finite family {1: (0, 1, -1), 2: (1, -2, 1)}
```

```
sage: ct = CartanType(['G',2])
sage: ct
['G', 2]
sage: ct._repr_(compact = True)
'G2'
sage: ct.is_irreducible()
True
sage: ct.is_finite()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
False
sage: ct.dual()
['G', 2] relabelled by {1: 2, 2: 1}
sage: ct.affine()
['G', 2, 1]

AmbientSpace

alias of AmbientSpace

ascii_art(label=<function CartanType.<lambda> at 0x7f39824184c0>, node=None)

Return an ascii art representation of the Dynkin diagram.

EXAMPLES:

sage: print(CartanType(['G',2]).ascii_art(label=lambda x: x+2))
  3
0<=0
  3  4

coxeter_number()

Return the Coxeter number associated with self.

EXAMPLES:

sage: CartanType(['G',2]).coxeter_number()
6
dual()

Return the dual Cartan type.

This uses that $G_2$ is self-dual up to relabelling.

EXAMPLES:

sage: G2 = CartanType(['G',2])
sage: G2.dual()
['G', 2] relabelled by {1: 2, 2: 1}
sage: G2.dynkin_diagram()
  3
0<=0
  1  2

(continues on next page)
G2
sage: G2.dual().dynkin_diagram()
3
| 0<=<0 |
| 2   | 1 |
G2 relabelled by {1: 2, 2: 1}

dual_coxeter_number()
Return the dual Coxeter number associated with self.

EXAMPLES:

sage: CartanType(['G',2]).dual_coxeter_number()
4

dynkin_diagram()
Returns a Dynkin diagram for type G.

EXAMPLES:

sage: g = CartanType(['G',2]).dynkin_diagram()
sage: g
3
| 0<=<0 |
| 1   | 2 |

sage: g.edges(sort=True)
[(1, 2, 1), (2, 1, 3)]

5.1.257 Root system data for (untwisted) type G affine

class sage.combinat.root_system.type_G_affine.CartanType
    Bases: CartanType_standard_untwisted_affine

EXAMPLES:

sage: ct = CartanType(['G',2,1])
sage: ct
['G', 2, 1]
sage: ct._repr_(compact = True)
'G2~'

sage: ct.is_irreducible()
True
sage: ct.is_finite()
False
sage: ct.is_affine()
True
sage: ct.is_untwisted_affine()
True
sage: ct.is_crystallographic()
True
sage: ct.is_simply_laced()
False
sage: ct.classical()
['G', 2]
sage: ct.dual()
['G', 2, 1]^*
sage: ct.dual().is_untwisted_affine()
False

ascii_art(label=<function CartanType.<lambda> at 0x7f39824189d0>, node=None)

Returns an ascii art representation of the Dynkin diagram

EXAMPLES:

sage: print(CartanType(['G',2,1]).ascii_art(label = lambda x: x+2))
3
0=<=O---O
3 4 2

dynkin_diagram()

Returns the extended Dynkin diagram for type G.

EXAMPLES:

sage: g = CartanType(['G',2,1]).dynkin_diagram()
sage: g
3
0=<=O---O
1 2 0
G2~
sage: g.edges(sort=True)
[(0, 2, 1), (1, 2, 1), (2, 0, 1), (2, 1, 3)]

5.1.258 Root system data for type H

class sage.combinat.root_system.type_H.CartanType(n)

Bases: CartanType_standard_finite, CartanType_simple

EXAMPLES:

sage: ct = CartanType(['H',3])
sage: ct
['H', 3]
sage: ct._repr_(compact = True)
'H3'
sage: ct.rank()
3
sage: ct.is_irreducible()
True
sage: ct.is_finite()
True
sage: ct.is_affine()
False
sage: ct.is_crystallographic()
False
sage: ct.is_simply_laced()
False

coxeter_diagram()

Returns a Coxeter diagram for type H.

EXAMPLES:

```
sage: ct = CartanType(['H',3])
sage: ct.coxeter_diagram()
Graph on 3 vertices
sage: ct.coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 5)]
sage: ct.coxeter_matrix()
[1 3 2]
[3 1 5]
[2 5 1]

sage: ct = CartanType(['H',4])
sage: ct.coxeter_diagram()
Graph on 4 vertices
sage: ct.coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 3), (3, 4, 5)]
sage: ct.coxeter_matrix()
[1 3 2 2]
[3 1 3 2]
[2 3 1 5]
[2 2 5 1]
```

coxeter_number()

Return the Coxeter number associated with self.

EXAMPLES:

```
sage: CartanType(['H',3]).coxeter_number()
10
sage: CartanType(['H',4]).coxeter_number()
30
```
5.1.259 Root system data for type I

class sage.combinat.root_system.type_I.CartanType(n)
   Bases: CartanType_standard_finite, CartanType_simple

EXAMPLES:

sage: ct = CartanType(['I',5])
sage: ct
['I', 5]
sage: ct._repr_(compact = True)
'I5'
sage: ct.rank()
2
sage: ct.index_set()
(1, 2)
sage: ct.is_irreducible()
True
sage: ct.is_finite()
True
sage: ct.is_affine()
False
sage: ct.is_crystallographic()
False
sage: ct.is_simply_laced()
False

coxeter_diagram()
   Returns the Coxeter matrix for this type.

   EXAMPLES:

sage: ct = CartanType(['I', 4])
sage: ct.coxeter_diagram()
Graph on 2 vertices
sage: ct.coxeter_diagram().edges(sort=True)
[(1, 2, 4)]
sage: ct.coxeter_matrix()
[1 4]
[4 1]

coxeter_number()
   Return the Coxeter number associated with self.

   EXAMPLES:

sage: CartanType(['I',3]).coxeter_number()
3
sage: CartanType(['I',12]).coxeter_number()
12

index_set()
   Type I_2(p) is indexed by \{1, 2\}.

   EXAMPLES:
sage: CartanType(['I', 5]).index_set()
(1, 2)

rank()
Type $I_2(p)$ is of rank 2.

EXAMPLES:

sage: CartanType(['I', 5]).rank()
2

5.1.260 Root system data for type Q

class sage.combinat.root_system.type_Q.CartanType(m)
Bases: CartanType_standard_finite

Cartan Type $Q_n$

See also:
CartanType()

dual()
Return dual of self.

EXAMPLES:

sage: Q = CartanType(['Q',3])
sage: Q.dual()
['Q', 3]

index_set()
Return the index set for Cartan type $Q$.
The index set for type $Q$ is of the form \{-n, \ldots, -1, 1, \ldots, n\}.

EXAMPLES:

sage: CartanType(['Q', 3]).index_set()
(1, 2, -2, -1)

is_irreducible()
Return whether this Cartan type is irreducible.

EXAMPLES:

sage: Q = CartanType(['Q',3])
sage: Q.is_irreducible()
True

is_simply_laced()
Return whether this Cartan type is simply-laced.

EXAMPLES:
sage: Q = CartanType(['Q',3])
sage: Q.is_simply_laced()
True

root_system()

Return the root system of self.

EXAMPLES:

sage: Q = CartanType(['Q',3])
sage: Q.root_system()
Root system of type ['A', 2]

5.1.261 Root system data for affine Cartan types

class sage.combinat.root_system.type_affine.AmbientSpace(root_system, base_ring)

Bases: CombinatorialFreeModule

Ambient space for affine types.

This is constructed from the data in the corresponding classical ambient space. Namely, this space is obtained by adding two elements $\delta$ and $\delta^\vee$ to the basis of the classical ambient space, and by endowing it with the canonical scalar product.

The coefficient of an element in $\delta^\vee$, thus its scalar product with $\delta^\vee$ gives its level, and dually for the colevel. The canonical projection onto the classical ambient space (by killing $\delta$ and $\delta^\vee$) maps the simple roots (except $\alpha_0$) onto the corresponding classical simple roots, and similarly for the coroots, fundamental weights,... Altogether, this uniquely determines the embedding of the root, coroot, weight, and coweight lattices. See simple_root() and fundamental_weight() for the details.

Warning: In type $BC$, the null root is in fact:

sage: R = RootSystem(['BC',3,2]).ambient_space()
sage: R.null_root()
2*e['delta']

Warning: In the literature one often considers a larger affine ambient space obtained from the classical ambient space by adding four dimensions, namely for the fundamental weight $\Lambda_0$ the fundamental coweight $\Lambda_0^\vee$, the null root $\delta$, and the null coroot $c$ (aka central element). In this larger ambient space, the scalar product is degenerate: $\langle \delta, \delta \rangle = 0$ and similarly for the null coroot.

In the current implementation, $\Lambda_0$ and the null coroot are identified:

sage: L = RootSystem(['A',3,1]).ambient_space()
sage: Lambda = L.fundamental_weights()
sage: Lambda[0]
e['deltacheck']
sage: L.null_coroot()
e['deltacheck']

Therefore the scalar product of the null coroot with itself differs from the larger ambient space:

5.1. Comprehensive Module List
In general, scalar products between two elements that do not live on “opposite sides” won’t necessarily match.

EXAMPLES:

```python
sage: R = RootSystem(['A',3,1])
sage: e = R.ambient_space(); e
Ambient space of the Root system of type ['A', 3, 1]
sage: TestSuite(e).run()
```

Systematic checks on all affine types:

```python
sage: for ct in CartanType.samples(affine=True, crystallographic=True):
    ....:     if ct.classical().root_system().ambient_space() is not None:
    ....:         print(ct)
    ....:         L = ct.root_system().ambient_space()
    ....:         assert L
    ....:         TestSuite(L).run()
['A', 1, 1]
['A', 5, 1]
['B', 1, 1]
['B', 5, 1]
['C', 1, 1]
['C', 5, 1]
['D', 3, 1]
['D', 5, 1]
['E', 6, 1]
['E', 7, 1]
['E', 8, 1]
['F', 4, 1]
['G', 2, 1]
['BC', 1, 2]
['BC', 5, 2]
['B', 5, 1]^*
['C', 4, 1]^*
['F', 4, 1]^*
['G', 2, 1]^*
['BC', 1, 2]^*
['BC', 5, 2]^*
```

class Element

    Bases: IndexedFreeModuleElement

    associated_coroot()

    Return the coroot associated to self.

    INPUT:
    • self – a root

    EXAMPLES:
sage: alpha = RootSystem(['C',2,1]).ambient_space().simple_roots()
sage: alpha
Finite family {0: -2*e[0] + e['delta'], 1: e[0] - e[1], 2: 2*e[1]}
sage: alpha[0].associated_coroot()
-e[0] + e['deltacheck']
sage: alpha[1].associated_coroot()
e[0] - e[1]
sage: alpha[2].associated_coroot()
e[1]

inner_product\(\text{other}\)
Implement the canonical inner product of \texttt{self} with \texttt{other}.

EXAMPLES:

```sage
sage: e = RootSystem(['B',3,1]).ambient_space()
sage: B = e.basis()
sage: matrix([[x.inner_product(y) for x in B] for y in B])
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
sage: x = e.an_element(); x
2*e[0] + 2*e[1] + 3*e[2]
sage: x.inner_product(x)
17
```

scalar() is an alias for this method:

```sage
sage: x.scalar(x)
17
```

Todo: Lift to CombinatorialFreeModule.Element as canonical_inner_product

scalar\(\text{other}\)
Implement the canonical inner product of \texttt{self} with \texttt{other}.

EXAMPLES:

```sage
sage: e = RootSystem(['B',3,1]).ambient_space()
sage: B = e.basis()
sage: matrix([[x.inner_product(y) for x in B] for y in B])
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
sage: x = e.an_element(); x
2*e[0] + 2*e[1] + 3*e[2]
sage: x.inner_product(x)
17
```
Combinatorics, Release 10.0

**scalar()** is an alias for this method:

```
sage: x.scalar(x)
17
```

**Todo:** Lift to CombinatorialFreeModule.Element as canonical_inner_product

coroot_lattice()

**EXAMPLES:**

```
sage: RootSystem(['A',3,1]).ambient_lattice().coroot_lattice()
Ambient lattice of the Root system of type ['A', 3, 1]
```

**Todo:** Factor out this code with the classical ambient space.

**fundamental_weight(i)**

Return the fundamental weight \( \Lambda_i \) in this ambient space.

It is constructed by taking the corresponding fundamental weight of the classical ambient space (or 0 for \( \Lambda_0 \)) and raising it to the appropriate level by adding a suitable multiple of \( \delta^\vee \).

**EXAMPLES:**

```
sage: RootSystem(['A',3,1]).ambient_space().fundamental_weight(2)
e[0] + e[1] + e['deltacheck']
sage: RootSystem(['A',3,1]).ambient_space().fundamental_weights()
Finite family {0: e['deltacheck'], 1: e[0] + e['deltacheck'],
...e['deltacheck']}
sage: RootSystem(['A',3]).ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0, 0), 2: (1, 1, 0, 0), 3: (1, 1, 1, 0)}
sage: RootSystem(['A',3,1]).weight_lattice().fundamental_weights().
  ...map(attrcall("level"))
Finite family {0: 1, 1: 1, 2: 1, 3: 1}

sage: RootSystem(['B',3,1]).ambient_space().fundamental_weights()
Finite family {0: e['deltacheck'], 1: e[0] + e['deltacheck'],
2: e[0] + e[1] + 2*e['deltacheck'], 3: 1/2*e[0] + 1/
sage: RootSystem(['B',3]).ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0, 0), 2: (1, 1, 0, 0), 3: (1/2, 1/2, 1/2, 1/2)}
sage: RootSystem(['B',3,1]).weight_lattice().fundamental_weights().
  ...map(attrcall("level"))
Finite family {0: 1, 1: 1, 2: 2, 3: 1}
```

In type \( BC \) dual, the coefficient of 'delta^vee' is the level divided by 2 to take into account that the null coroot is \( 2\delta^\vee \):

```
sage: R = CartanType(['BC',3,2]).dual().root_system()
sage: R.ambient_space().fundamental_weights()
Finite family {0: e['deltacheck'], 1: e[0] + e['deltacheck'],
```

(continues on next page)
2: $e[0] + e[1] + e[\text{deltacheck}]$,  

sage: R.weight_lattice().fundamental_weights().map(attrcall("level"))
Finite family {0: 2, 1: 2, 2: 2, 3: 1}
sage: R.ambient_space().null_coroot()
2*e[\text{deltacheck}]

By a slight naming abuse this function also accepts "delta" as input so that it can be used to implement the embedding from the extended weight lattice::

sage: RootSystem(['A',3,1]).ambient_space().fundamental_weight("delta")
e[\text{delta}]

.. _is_extended:

is_extended()
Return whether this is a realization of the extended weight lattice: yes!

See also:

- :class:`sage.combinat.root_system.weight_space.WeightSpace`
- :class:`sage.combinat.root_system.weight_lattice_realizations.WeightLatticeRealizations.ParentMethods.is_extended`

EXAMPLES:

sage: RootSystem(['A',3,1]).ambient_space().is_extended()
True

simple_coroot($i$)

Return the $i$-th simple coroot $\alpha_i^\vee$ of this affine ambient space.

EXAMPLES:

sage: RootSystem(['A',3,1]).ambient_space().simple_coroot(1)
e[0] - e[1]

It is built as the coroot associated to the simple root $\alpha_i$:

sage: RootSystem(['B',3,1]).ambient_space().simple_roots()
Finite family {0: -e[0] - e[1] + e[\text{deltacheck}], 1: e[0] - e[1], 2: e[1] - e[2], 3: ...
  \rightarrow e[2]}
sage: RootSystem(['B',3,1]).ambient_space().simple_coroots()

Todo: Factor out this code with the classical ambient space.

simple_root($i$)

Return the $i$-th simple root of this affine ambient space.

EXAMPLES:

It is built straightforwardly from the corresponding simple root $\alpha_i$ in the classical ambient space:
sage: RootSystem(['A',3,1]).ambient_space().simple_root(1)
e[0] - e[1]

For the special node (typically $i = 0$), $\alpha_0$ is built from the other simple roots using the column annihilator
of the Cartan matrix and adding $\delta$, where $\delta$ is the null root:

sage: RootSystem(['A',3]).ambient_space().simple_roots()
Finite family {1: (1, -1, 0, 0), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1)}

sage: RootSystem(['A',3,1]).ambient_space().simple_roots()

Here is a twisted affine example:

sage: RootSystem(CartanType(['B',3,1]).dual()).ambient_space().simple_roots()

In fact $\delta$ is really $1/\alpha_0$ times the null root (see the discussion in WeightSpace) but this only makes a
difference in type $BC$:

sage: L = RootSystem(CartanType(['BC',3,2])).ambient_space()
sage: L.simple_roots()
Finite family {0: -e[0] + e['delta'], 1: e[0] - e[1], 2: e[1] - e[2], 3: 2*e[2]}
sage: L.null_root()
2*e['delta']

Note: An alternative would have been to use the default implementation of the simple roots as linear
combinations of the fundamental weights. However, as in type $A_n$ it is preferable to take a slight variant to
avoid rational coefficient (the usual $GL_n$ vs $SL_n$ issue).

See also:

• simple_root()
• WeightSpace
• CartanType.col_annihilator()
• null_root()

classmethod smallest_base_ring(cartan_type)

Return the smallest base ring the ambient space can be defined on.

This is the smallest base ring for the associated classical ambient space.

See also:

smallest_base_ring()

EXAMPLES:

sage: cartan_type = CartanType(['A',3,1])
sage: cartan_type.AmbientSpace.smallest_base_ring(cartan_type)
Integer Ring
5.1.262 Root system data for dual Cartan types

class sage.combinat.root_system.type_dual.AmbientSpace(root_system, base_ring, index_set=None)
Bases: AmbientSpace

Ambient space for a dual finite Cartan type.

It is constructed in the canonical way from the ambient space of the original Cartan type by switching the roles of simple roots, fundamental weights, etc.

Note: Recall that, for any finite Cartan type, and in particular the a simply laced one, the dual Cartan type is constructed as another preexisting Cartan type. Furthermore the ambient space for an affine type is constructed from the ambient space for its classical type. Thus this code is not actually currently used.

It is kept for cross-checking and for reference in case it could become useful, e.g., for dual of general Kac-Moody types.

For the doctests, we need to explicitly create a dual type. Subsequently, since reconstruction of the dual of type $F_4$ is the relabelled Cartan type, pickling fails on the TestSuite run.

EXAMPLES:

sage: ct = sage.combinat.root_system.type_dual.CartanType(CartanType(['F',4]))

sage: L = ct.root_system().ambient_space(); L
Ambient space of the Root system of type ['F', 4]^*

sage: TestSuite(L).run(skip=['_test_elements','_test_pickling'])

dimension()

Return the dimension of this ambient space.

See also:
sage.combinat.root_system.ambient_space.AmbientSpace.dimension()

EXAMPLES:

sage: ct = sage.combinat.root_system.type_dual.CartanType(CartanType(['F',4]))

sage: L = ct.root_system().ambient_space()

sage: L.dimension()
4

fundamental_weights()

Return the fundamental weights.

They are computed from the simple roots by inverting the Cartan matrix. This is acceptable since this is only about ambient spaces for finite Cartan types. Also, we do not have to worry about the usual $GL_n$ vs $SL_n$ catch because type $A$ is self dual.

An alternative would have been to start from the fundamental coweights in the dual ambient space, but those are not yet implemented.
EXAMPLES:

```python
sage: ct = sage.combinat.root_system.type_dual.CartanType(CartanType(['F',4]))
sage: L = ct.root_system().ambient_space()
sage: L.fundamental_weights()
Finite family {1: (1, 1, 0, 0), 2: (2, 1, 1, 0), 3: (3, 1, 1, 1), 4: (2, 0, 0, 0)}
```

Note that this ambient space is isomorphic, but not equal, to that obtained by constructing $F_4$ dual by relabelling:

```python
sage: ct = CartanType(['F',4]).dual(); ct
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
sage: ct.root_system().ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0, 0), 2: (3/2, 1/2, 1/2, 1/2), 3: (2, 1, 1, 0), 4: (1, -1, 0, 0)}
```

**simple_root(i)**

Return the $i$-th simple root.

It is constructed by looking up the corresponding simple coroot in the ambient space for the dual Cartan type.

EXAMPLES:

```python
sage: ct = sage.combinat.root_system.type_dual.CartanType(CartanType(['F',4]))
sage: ct.root_system().ambient_space().simple_root(1)
(0, 1, -1, 0)
sage: ct.root_system().ambient_space().simple_roots()
Finite family {1: (0, 1, -1, 0), 2: (0, 0, 0, 1), 3: (0, 0, 1, -1), 4: (1, -1, -1, -1)}
sage: ct.dual().root_system().ambient_space().simple_coroots()
Finite family {1: (0, 1, -1, 0), 2: (0, 0, 0, 1), 3: (0, 0, 1, -1), 4: (1, -1, -1, -1)}
```

Note that this ambient space is isomorphic, but not equal, to that obtained by constructing $F_4$ dual by relabelling:

```python
sage: ct = CartanType(['F',4]).dual(); ct
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
sage: ct.root_system().ambient_space().simple_roots()
Finite family {1: (1/2, -1/2, -1/2, -1/2), 2: (0, 0, 0, 1), 3: (0, 0, 1, -1), 4: (0, 1, -1, 0)}
```

class sage.combinat.root_system.type_dual.CartanType(type)

Bases: CartanType_decorator, CartanType_crystallographic

A class for dual Cartan types.

The dual of a (crystallographic) Cartan type is a Cartan type with the same index set, but all arrows reversed in the Dynkin diagram (otherwise said, the Cartan matrix is transposed). It shares a lot of properties in common with its dual. In particular, the Weyl group is isomorphic to that of the dual as a Coxeter group.

EXAMPLES:
For all finite Cartan types, and in particular the simply laced ones, the dual Cartan type is given by another preexisting Cartan type:

\texttt{sage: CartanType(['A',4]).dual() [\texttt{'A'}, 4]}
\texttt{sage: CartanType(['B',4]).dual() [\texttt{'C'}, 4]}
\texttt{sage: CartanType(['C',4]).dual() [\texttt{'B'}, 4]}
\texttt{sage: CartanType(['F',4]).dual() [\texttt{'F'}, 4] relabelled by \{1: 4, 2: 3, 3: 2, 4: 1\}}

So to exercise this class we consider some non simply laced affine Cartan types and also create explicitly $F_4^*$ as a dual cartan type:

\texttt{sage: from sage.combinat.root_system.type_dual import CartanType as CartanTypeDual}
\texttt{sage: F4d = CartanTypeDual(CartanType(['F',4])); F4d [\texttt{'F'}, 4]^*}
\texttt{sage: G21d = CartanType(['G',2,1]).dual(); G21d [\texttt{'G'}, 2, 1]^*}

They share many properties with their original Cartan types:

\texttt{sage: F4d.is_irreducible() True}
\texttt{sage: F4d.is_crystallographic() True}
\texttt{sage: F4d.is_simply_laced() False}
\texttt{sage: F4d.is_finite() True}
\texttt{sage: F4d.is_affine() False}
\texttt{sage: G21d.is_finite() True}
\texttt{sage: G21d.is_affine() True}

\textbf{Note:} F4d is pickled by construction as F4.dual() hence the above failure.

\textbf{ascii_art}(label=\texttt{<function CartanType.<lambda> at 0x7f398241a3b0>, node=None})

Return an ascii art representation of this Cartan type (by hacking the ascii art representation of the dual Cartan type)

\textbf{EXAMPLES:}

\texttt{sage: print(CartanType(['B', 3, 1]).dual().ascii_art()) 0 0 |
| 0---0==<=0 1 2 3 (continues on next page)
sage: print(CartanType(['C', 4, 1]).dual().ascii_art())
O=<=O---O---O=>=O
0 1 2 3 4
sage: print(CartanType(['G', 2, 1]).dual().ascii_art())
3
O=>=O---O
1 2 0
sage: print(CartanType(['F', 4, 1]).dual().ascii_art())
O---O---O=<=O---O
0 1 2 3 4
sage: print(CartanType(['BC', 4, 2]).dual().ascii_art())
O=>=O---O---O=>=O
0 1 2 3 4
dual()
EXAMPLES:
sage: ct = CartanType(['F', 4, 1]).dual()
sage: ct.dual()
['F', 4, 1]
dynkin_diagram()
EXAMPLES:
sage: ct = CartanType(['F', 4, 1]).dual()
sage: ct.dynkin_diagram()
O---O---O=<=O---O
0 1 2 3 4
F4~
class sage.combinat.root_system.type_dual.CartanType_affine(type)
    Bases: CartanType, CartanType_affine

basic_untwisted()
    Return the basic untwisted Cartan type associated with this affine Cartan type.

    Given an affine type \( X_n^{(r)} \), the basic untwisted type is \( X_n \). In other words, it is the classical Cartan type that is twisted to obtain self.

    EXAMPLES:

    sage: CartanType(['A', 7, 2]).basic_untwisted()
    ['A', 7]
sage: CartanType(['E', 6, 2]).basic_untwisted()
    ['E', 6]
sage: CartanType(['D', 4, 3]).basic_untwisted()
    ['D', 4]
classical()
    Return the classical Cartan type associated with self (which should be affine).

    EXAMPLES:
sage: CartanType(['A', 3, 1]).dual().classical()
[\'A\', 3]
sage: CartanType(['B', 3, 1]).dual().classical()
[\'C\', 3]
sage: CartanType(['F', 4, 1]).dual().classical()
[\'F\', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
sage: CartanType(['BC', 4, 2]).dual().classical()
[\'B\', 4]

special_node()

Implement CartanType_affine.special_node()

The special node of the dual of an affine type $T$ is the special node of $T$.

EXAMPLES:

sage: CartanType(['A', 3, 1]).dual().special_node()
0
sage: CartanType(['B', 3, 1]).dual().special_node()
0
sage: CartanType(['F', 4, 1]).dual().special_node()
0
sage: CartanType(['BC', 4, 2]).dual().special_node()
0

class sage.combinat.root_system.type_dual.CartanType_finite(type)

Bases: CartanType, CartanType_finite

AmbientSpace

alias of AmbientSpace

5.1.263 Extended Affine Weyl Groups

AUTHORS:

- Daniel Bump (2012): initial version
- Daniel Orr (2012): initial version
- Anne Schilling (2012): initial version
- Mark Shimozono (2012): initial version
- Nicolas M. Thiery (2012): initial version
- Mark Shimozono (2013): twisted affine root systems, multiple realizations, GL_n

sage.combinat.root_system.extended_affine_weyl_group.ExtendedAffineWeylGroup(cartan_type, 
general_linear=None, 
**print_options)

The extended affine Weyl group.

INPUT:

- cartan_type – An affine or finite Cartan type (a finite Cartan type is an abbreviation for its untwisted affinization)
• `general_linear` – (default: None) If True and `cartan_type` indicates untwisted type A, returns the universal central extension
• `print_options` – Special instructions for printing elements (see below)

**Mnemonics**

- “P” – subgroup of translations
- “Pv” – subgroup of translations in a dual form
- “W0” – classical Weyl group
- “W” – affine Weyl group
- “F” – fundamental group of length zero elements

There are currently six realizations: “PW0”, “W0P”, “WF”, “FW”, “PvW0”, and “W0Pv”.

“PW0” means the semidirect product of “P” with “W0” acting from the right. “W0P” is similar but with “W0” acting from the left. “WF” is the semidirect product of “W” with “F” acting from the right, etc.

Recognized arguments for `print_options` are:

• `print_tuple` – True or False (default: False) If True, elements are printed \((a, b)\), otherwise as \(a * b\)
• `affine` – Prefix for simple reflections in the affine Weyl group
• `classical` – Prefix for simple reflections in the classical Weyl group
• `translation` – Prefix for the translation elements
• `fundamental` – Prefix for the elements of the fundamental group

These options are not mutable.

The *extended affine Weyl group* was introduced in the following references.

**REFERENCES:**

• [Ka1990]

**Notation**

• \(R\) – An irreducible affine root system
• \(I\) – Set of nodes of the Dynkin diagram of \(R\)
• \(R_0\) – The classical subsystem of \(R\)
• \(I_0\) – Set of nodes of the Dynkin diagram of \(R_0\)
• \(E\) – Extended affine Weyl group of type \(R\)
• \(W\) – Affine Weyl group of type \(R\)
• \(W_0\) – finite (classical) Weyl group (of type \(R_0\))
• \(M\) – translation lattice for \(W\)
• \(L\) – translation lattice for \(E\)
• \(F\) – Fundamental subgroup of \(E\) (the length zero elements)
• \(P\) – Finite weight lattice
• $Q$ – Finite root lattice
• $P^\vee$ – Finite coweight lattice
• $Q^\vee$ – Finite coroot lattice

Translation lattices

The styles “PW0” and “W0P” use the following lattices:
• Untwisted affine: $L = P^\vee$, $M = Q^\vee$
• Dual of untwisted affine: $L = P$, $M = Q$
• $BC_n(A^{(2)}): L = M = P$
• Dual of $BC_n(A^{(2)\dagger})$: $L = M = P^\vee$

The styles “PvW0” and “W0Pv” use the following lattices:
• Untwisted affine: The weight lattice of the dual finite Cartan type.
• Dual untwisted affine: The same as for “PW0” and “W0P”.

For mixed affine type ($A^{(2)}_n$, aka $BC_n$, and their affine duals) the styles “PvW0” and “W0Pv” are not implemented.

Finite and affine Weyl groups $W_0$ and $W$

The finite Weyl group $W_0$ is generated by the simple reflections $s_i$ for $i \in I_0$ where $s_i$ is the reflection across a suitable hyperplane $H_i$ through the origin in the real span $V$ of the lattice $M$.

$R$ specifies another (affine) hyperplane $H_0$. The affine Weyl group $W$ is generated by $W_0$ and the reflection $S_0$ across $H_0$.

Extended affine Weyl group $E$

The complement in $V$ of the set $H$ of hyperplanes obtained from the $H_i$ by the action of $W$, has connected components called alcoves. $W$ acts freely and transitively on the set of alcoves. After the choice of a certain alcove (the fundamental alcove), there is an induced bijection from $W$ to the set of alcoves under which the identity in $W$ maps to the fundamental alcove.

Then $L$ is the largest sublattice of $V$, whose translations stabilize the set of alcoves.

There are isomorphisms

$$W \cong M \rtimes W_0 \cong W_0 \ltimes M$$
$$E \cong L \rtimes W_0 \cong W_0 \ltimes L$$
Combinatorics, Release 10.0

Fundamental group of affine Dynkin automorphisms

Since $L$ acts on the set of alcoves, the group $F = L/M$ may be viewed as a subgroup of the symmetries of the fundamental alcove or equivalently the symmetries of the affine Dynkin diagram. $F$ acts on the set of alcoves and hence on $W$. Conjugation by an element of $F$ acts on $W$ by permuting the indices of simple reflections.

There are isomorphisms

$$E \cong F \ltimes W \cong W \ltimes F$$

An affine Dynkin node is special if it is conjugate to the zero node under some affine Dynkin automorphism.

There is a bijection $i \mapsto \pi_i$ from the set of special nodes to the group $F$, where $\pi_i$ is the unique element of $F$ that sends $0$ to $i$. When $L = P$ (resp. $L = P^\vee$) the element $\pi_i$ is induced (under the isomorphism $F \cong L/M$) by addition of the coset of the $i$-th fundamental weight (resp. coweight).

The length function of the Coxeter group $W$ may be extended to $E$ by $\ell(w\pi) = \ell(w)$ where $w \in W$ and $\pi \in F$. This is the number of hyperplanes in $H$ separating the fundamental alcove from its image by $w\pi$ (or equivalently $w$).

It is known that if $G$ is the compact Lie group of adjoint type with root system $R_0$ then $F$ is isomorphic to the fundamental group of $G$, or to the center of its simply-connected covering group. That is why we call $F$ the fundamental group.

In the future we may want to build an element of the group from an appropriate linear map $f$ on some of the root lattice realizations for this Cartan type: $W$.from_endomorphism(f).

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',2,1]); E
Extended affine Weyl group of type ['A', 2, 1]
sage: type(E)
<class 'sage.combinat.root_system.extended_affine_weyl_group.ExtendedAffineWeylGroup_class_with_category'>
sage: PW0 = E.PW0(); PW0
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of
   Multiplicative form of Coweight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)
sage: W0P = E.W0P(); W0P
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of
   Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice) acting on Multiplicative form of Coweight lattice of the Root system of type ['A', 2]
sage: PvW0 = E.PvW0(); PvW0
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of
   Multiplicative form of Weight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice)
sage: W0Pv = E.W0Pv(); W0Pv
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of
   Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice) acting on Multiplicative form of Weight lattice of the Root system of type ['A', 2]
```
When the realizations are constructed from each other as above, there are built-in coercions between them.

The translation lattice and its distinguished basis are obtained from $\mathbb{E}$:

```python
sage: L = E.lattice(); L
Coweight lattice of the Root system of type ['A', 2]
sage: b = E.lattice_basis(); b
Finite family {1: Lambda[1], 2: Lambda[2]}
```

Translation lattice elements can be coerced into any realization:

```python
sage: FW(b[1]-b[2])
pi[2] * S0*S1
sage: PW0(b[1]-b[2])
t[Lambda[1] - Lambda[2]]
```

The dual form of the translation lattice and its basis are similarly obtained:

```python
sage: Lv = E.dual_lattice(); Lv
Weight lattice of the Root system of type ['A', 2]
sage: bv = E.dual_lattice_basis(); bv
Finite family {1: Lambda[1], 2: Lambda[2]}
sage: FW(bv[1]-bv[2])
pi[2] * S0*S1
```

The abstract fundamental group is accessed from $\mathbb{E}$:

```python
sage: F = E.fundamental_group(); F
Fundamental group of type ['A', 2, 1]
```
Its elements are indexed by the set of special nodes of the affine Dynkin diagram:

\[
\text{sage: } E\text{.cartan_type()\text{.special_nodes()}}
\]
\[
(0, 1, 2)
\]

\[
\text{sage: } F\text{.special_nodes()}
\]
\[
(0, 1, 2)
\]

\[
\text{sage: } [F(i) \text{ for i in F\text{.special_nodes()}}]
\]
\[
[\pi[0], \pi[1], \pi[2]]
\]

There is a coercion from the fundamental group into each realization:

\[
\text{sage: } F(2)
\]
\[
\pi[2]
\]

\[
\text{sage: } WF(F(2))
\]
\[
\pi[2]
\]

\[
\text{sage: } W0P(F(2))
\]
\[
s2*s1 \ast t[-\Lambda[1]]
\]

\[
\text{sage: } W0v(F(2))
\]
\[
s2*s1 \ast t[-\Lambda[1]]
\]

Using \(E\) one may access the classical and affine Weyl groups and their morphisms into each realization:

\[
\text{sage: } W0 = E\text{.classical_weyl()}; W0
\]
Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)

\[
\text{sage: } v = W0\text{.from_reduced_word([1,2,1])}; v
\]
\[
s1*s2*s1
\]

\[
\text{sage: } PW0(v)
\]
\[
s1*s2*s1
\]

\[
\text{sage: } WF(v)
\]
\[
S1*S2*S1
\]

\[
\text{sage: } W = E\text{.affine_weyl()}; W
\]
Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root lattice)

\[
\text{sage: } w = W\text{.from_reduced_word([2,1,0])}; w
\]
\[
S2*S1*S0
\]

\[
\text{sage: } WF(w)
\]
\[
S2*S1*S0
\]

\[
\text{sage: } PW0(w)
\]
\[
t[\Lambda[1] - 2*\Lambda[2]] \ast s1
\]

Note that for untwisted affine type, the dual form of the classical Weyl group is isomorphic to the usual one, but acts on a different lattice and is therefore different to \(sage\):

\[
\text{sage: } W0v = E\text{.dual_classical_weyl()}; W0v
\]
Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice)

\[
\text{sage: } v = W0v\text{.from_reduced_word([1,2])}
\]

\[
\text{sage: } x = Pw0(v); x
\]
\[
s1*s2
\]

\[
\text{sage: } y = Pw0(v); y
\]
\[
s1*s2
\]

\[
\text{sage: } x\text{.parent()} == y\text{.parent()}
\]
False

However, because there is a coercion from \(Pw0\) to \(Pw0\), the elements \(x\) and \(y\) compare as equal:
An element can be created directly from a reduced word:

```python
sage: PW0.from_reduced_word([2,1,0])
t[\Lambda^1 - 2\Lambda^2] * s1
```

Here is a demonstration of the printing options:

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1], affine="sx", classical="Sx", translation="x", fundamental="pix")
sage: PW0 = E.PW0()
sage: y = PW0(E.lattice_basis()[1])
sage: y
x[\Lambda^1]

sage: FW = E.FW()
sage: FW(y)
pix[1] * sx2*sx1

sage: PW0.an_element()
x[2*\Lambda^1 + 2*\Lambda^2] * Sx1*Sx2
```

**Todo:**

- Implement a “slow” action of $E$ on any affine root or weight lattice realization.
- Implement the level $m$ actions of $E$ and $W$ on the lattices of finite type.
- Implement the relevant methods from the usual affine Weyl group
- Implementation by matrices: style “M”.
- Use case: implement the Hecke algebra on top of this

The semidirect product construction in Sage currently only admits multiplicative groups. Therefore for the styles involving “P” and “Pv”, one must convert the additive group of translations $L$ into a multiplicative group by applying the `sage.groups.group_exp.GroupExp` functor.

**The general linear case**

The general linear group is not semisimple. Sage can build its extended affine Weyl group:

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1], general_linear=True); E
Extended affine Weyl group of GL(3)
```

If the Cartan type is $['A', n-1, 1]$ and the parameter `general_linear` is not True, the extended affine Weyl group that is built will be for $SL_n$, not $GL_n$. But if `general_linear` is True, let $W_a$ and $W_e$ be the affine and extended affine Weyl groups. We make the following nonstandard definition: the extended affine Weyl group $W_e(GL_n)$ is defined by

$$W_e(GL_n) = P(GL_n) \rtimes W$$

where $W$ is the finite Weyl group (the symmetric group $S_n$) and $P(GL_n)$ is the weight lattice of $GL_n$, which is usually identified with the lattice $\mathbb{Z}^n$ of $n$-tuples of integers.
There is an isomorphism

\[ W_{\mathbb{Z}}(GL_n) = \mathbb{Z} \rtimes W_a \]

where the group of integers \( \mathbb{Z} \) (with generator \( \pi \)) acts on \( W_a \) by

\[ \pi s_i \pi^{-1} = s_{i+1} \]

and the indices of the simple reflections are taken modulo \( n \):

We regard \( \mathbb{Z} \) as the fundamental group of affine type \( GL_n \):

\[ \text{sage: } F = E.fundamental_group(); F \]

Fundamental group of \( GL(3) \)

\[ \text{sage: } F.special_nodes() \]

Integer Ring

\[ \text{sage: } x = FW.from_fundamental(F(10)); x \]

\[ \text{sage: } x*x \]

\[ \text{sage: } E.PVW0()(x*x) \]

\[ t[(7, 7, 6)] \]

\[ \text{sage: } \text{ExtendedAffineWeylGroupFW}(E) \]

Bases: UniqueRepresentation, Parent

The parent-with-realization class of an extended affine Weyl group.

\[ \text{class } \text{ExtendedAffineWeylGroupFW}(E) \]

Bases: GroupSemidirectProduct, BindableClass

Extended affine Weyl group, realized as the semidirect product of the affine Weyl group by the fundamental group.

INPUT:

- \( E \) – A parent with realization in \( \text{ExtendedAffineWeylGroupClass} \)
EXAMPLES:

```python
sage: ExtendedAffineWeylGroup(['A',2,1]).FW()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Fundamental group of type ['A', 2, 1] acting on Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root lattice)
```

**Element**

**alias of** `ExtendedAffineWeylGroupFWElement`

**from_affine_weyl** (`w`)

Return the image of `w` under the map of the affine Weyl group into the right (affine Weyl group) factor in the “FW” style.

**EXAMPLES:**

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1],print_tuple=True)
sage: E.FW().from_affine_weyl(E.affine_weyl().from_reduced_word([0,2,1]))
(pi[0], S0*S2*S1)
```

**from_fundamental** (`f`)

Return the image of the fundamental group element `f` into self.

**EXAMPLES:**

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1],print_tuple=True)
sage: E.FW().from_fundamental(E.fundamental_group()(2))
(pi[2], 1)
```

**simple_reflections**()

Return the family of simple reflections of self.

**EXAMPLES:**

```python
sage: ExtendedAffineWeylGroup(['A',2,1],print_tuple=True).FW().simple_reflections()
Finite family {0: (pi[0], S0), 1: (pi[0], S1), 2: (pi[0], S2)}
```

**class ExtendedAffineWeylGroupFWElement**

**Bases:** `GroupSemidirectProductElement`

The element class for the “FW” realization.

**action_on_affine_roots** (`beta`)

Act by self on the affine root lattice element `beta`.

**EXAMPLES:**

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1],affine="s")
sage: x = E.FW().an_element(); x
pi[2] * s0*s1*s2
sage: v = RootSystem(['A',2,1]).root_lattice().an_element(); v
sage: x.action_on_affine_roots(v)
alpha[0] + alpha[1]
```
**has_descent**(*i*, **side='right', positive=False*)

Return whether *self* has descent at *i*.

**INPUT:**
- *i* – an affine Dynkin index.

**OPTIONAL:**
- **side** – ‘left’ or ‘right’ (default: ‘right’)
- **positive** – True or False (default: False)

**EXAMPLES:**

```
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.FW().an_element(); x
pi[2] * S0*S1*S2
sage: [(i, x.has_descent(i)) for i in E.cartan_type().index_set()]
[(0, False), (1, False), (2, True)]
```

**to_affine_weyl_right()**

Project *self* to the right (affine Weyl group) factor in the “FW” style.

**EXAMPLES:**

```
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.FW().from_translation(E.lattice_basis()[1]); x
pi[1] * S2*S1
sage: x.to_affine_weyl_right()
S2*S1
```

**to_fundamental_group()**

Return the projection of *self* to the fundamental group in the “FW” style.

**EXAMPLES:**

```
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.FW().from_translation(E.lattice_basis()[2]); x
pi[2] * S1*S2
sage: x.to_fundamental_group()
pi[2]
```

**class ExtendedAffineWeylGroupPW0**(*E*)

**Bases:** *GroupSemidirectProduct, BindableClass*

Extended affine Weyl group, realized as the semidirect product of the translation lattice by the finite Weyl group.

**INPUT:**
- *E* – A parent with realization in *ExtendedAffineWeylGroup_Class*

**EXAMPLES:**

```
sage: ExtendedAffineWeylGroup(['A',2,1]).PW0()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Multiplicative form of Coweight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)
```
Element
alias of ExtendedAffineWeylGroupPW0Element

\texttt{S0()}
Return the affine simple reflection.

EXAMPLES:
\begin{verbatim}
sage: ExtendedAffineWeylGroup(['B',2]).PW0().S0()
t[Lambda_2] * s2*s1*s2
\end{verbatim}

\texttt{from_classical_weyl(w)}
Return the image of \(w\) under the homomorphism of the classical Weyl group into self.

EXAMPLES:
\begin{verbatim}
sage: E = ExtendedAffineWeylGroup("A3",print_tuple=True)
sage: E.PW0().from_classical_weyl(E.classical_weyl().from_reduced_word([1,2]))
(t[0], s1*s2)
\end{verbatim}

\texttt{from_translation(la)}
Map the translation lattice element \(la\) into self.

EXAMPLES:
\begin{verbatim}
sage: E = ExtendedAffineWeylGroup(['A',2,1], translation="tau", print_tuple=True)
sage: la = E.lattice().an_element(); la
2*Lambda_1 + 2*Lambda_2
sage: E.PW0().from_translation(la)
(tau[2*Lambda_1 + 2*Lambda_2], 1)
\end{verbatim}

\texttt{simple_reflection(i)}
Return the \(i\)-th simple reflection in self.

EXAMPLES:
\begin{verbatim}
sage: E = ExtendedAffineWeylGroup("G2")
sage: [(i, E.PW0().simple_reflection(i)) for i in E.cartan_type().index_set()]
[(0, t[Lambda_2] * s2*s1*s2*s1*s2), (1, s1), (2, s2)]
\end{verbatim}

\texttt{simple_reflections()}
Return a family for the simple reflections of self.

EXAMPLES:
\begin{verbatim}
sage: ExtendedAffineWeylGroup("A3").PW0().simple_reflections()
Finite family {0: t[Lambda_1 + Lambda_2] * s1*s2*s3*s2*s1, 1: \ldots
             s1, 2: s2, 3: s3}
\end{verbatim}

\textbf{class ExtendedAffineWeylGroupPW0Element}
Bases: GroupSemidirectProductElement
The element class for the “PW0” realization.
action($la$)

Return the action of self on an element $la$ of the translation lattice.

**EXAMPLES:**

```
sage: E = ExtendedAffineWeylGroup(['A',2,1]); PW0=E.PW0()
sage: x = PW0.an_element(); x
sage: la = E.lattice().an_element(); la
sage: x.action(la)
```

has_descent($i$, side='right', positive=False)

Return whether self has $i$ as a descent.

**INPUT:**

• $i$ – an affine Dynkin node

**OPTIONAL:**

• side – ‘left’ or ‘right’ (default: ‘right’)

• positive – True or False (default: False)

**EXAMPLES:**

```
sage: w = ExtendedAffineWeylGroup(['A',4,2]).PW0().from_reduced_word([0,1]); ...
  w
  t[Lambda[1]] * s1*s2
sage: w.has_descent(0, side='left')
True
```

to_classical_weyl()

Return the image of self under the homomorphism that projects to the classical Weyl group factor after rewriting it in either style “PW0” or “W0P”.

**EXAMPLES:**

```
sage: s = ExtendedAffineWeylGroup(['A',2,1]).PW0().S0(); s
t[Lambdacheck[1] + Lambda[2]] * s1*s2*s1
sage: s.to_classical_weyl()
s1*s2*s1
```

to_translation_left()

The image of self under the map that projects to the translation lattice factor after factoring it to the left as in style “PW0”.

**EXAMPLES:**

```
sage: s = ExtendedAffineWeylGroup(['A',2,1]).PW0().S0(); s
t[Lambdacheck[1] + Lambda[2]] * s1*s2*s1
sage: s.to_translation_left()
```

class ExtendedAffineWeylGroupPvW0($E$)

**Bases:** GroupSemidirectProduct, BindableClass

Extended affine Weyl group, realized as the semidirect product of the dual form of the translation lattice by the finite Weyl group.
INPUT:
• \( E \) – A parent with realization in \( \text{ExtendedAffineWeylGroup}_\text{Class} \)

EXAMPLES:

```python
sage: ExtendedAffineWeylGroup(['A',2,1]).PvW0()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Multiplicative form of Weight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice)
```

**Element**

alias of \( \text{ExtendedAffineWeylGroupPvW0Element} \)

**from_dual_classical_weyl(\( w \))**

Return the image of \( w \) under the homomorphism of the dual form of the classical Weyl group into self.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A','3',1],print_tuple=True)
sage: E.PvW0().from_dual_classical_weyl(E.dual_classical_weyl().from_reduced_word([1,2]))
t([0], s1*s2)
```

**from_dual_translation(\( la \))**

Map the dual translation lattice element \( la \) into self.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1], translation='tau', print_tuple=True)
sage: la = E.dual_lattice().an_element(); la
sage: E.PvW0().from_dual_translation(la)
```

**simple_reflections()**

Return a family for the simple reflections of self.

EXAMPLES:

```python
sage: ExtendedAffineWeylGroup(['A',3,1]).PvW0().simple_reflections()
Finite family {0: t[Lambda[1] + Lambda[3]] * s1*s2*s3*s2*s1, 1: s1, 2: s2, 3: s3}
```

**class ExtendedAffineWeylGroupPvW0Element**

Bases: \( \text{GroupSemidirectProductElement} \)

The element class for the “PvW0” realization.

**dual_action(\( la \))**

Return the action of self on an element \( la \) of the dual version of the translation lattice.

EXAMPLES:
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.PvW0().an_element(); x
sage: la = E.dual_lattice().an_element(); la
sage: x.dual_action(la)

has_descent($i$, side='right', positive=False)

Return whether self has $i$ as a descent.

INPUT:

• $i$ - an affine Dynkin index

OPTIONAL:

• side – 'left' or 'right' (default: 'right')

• positive – True or False (default: False)

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',4,2])
sage: w = E.PvW0().from_reduced_word([0,1]); w
t[ Lambda[1]] * s1*s2
sage: [(i, w.has_descent(i, side='left')) for i in E.cartan_type().index_set()]
[(0, True), (1, False), (2, False)]

to_dual_classical_weyl()

Return the image of self under the homomorphism that projects to the dual classical Weyl group factor after rewriting it in either style “PvW0” or “W0Pv”.

EXAMPLES:

sage: s = ExtendedAffineWeylGroup(['A',2,1]).PvW0().simple_reflection(0); s
t[Lambda[1] + Lambda[2]] * s1*s2*s1
sage: s.to_dual_classical_weyl()
s1*s2*s1

to_dual_translation_left()

The image of self under the map that projects to the dual translation lattice factor after factoring it to the left as in style “PvW0”.

EXAMPLES:

sage: s = ExtendedAffineWeylGroup(['A',2,1]).PvW0().simple_reflection(0); s
t[Lambda[1] + Lambda[2]] * s1*s2*s1
sage: s.to_dual_translation_left()

class ExtendedAffineWeylGroupW0P($E$)

Bases: GroupSemidirectProduct, BindableClass

Extended affine Weyl group, realized as the semidirect product of the finite Weyl group by the translation lattice.

INPUT:

• $E$ – A parent with realization in ExtendedAffineWeylGroup_Class
EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',2,1]).W0P()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice) acting on Multiplicative form of Coweight lattice of the Root system of type ['A', 2]
```

**Element**

alias of `ExtendedAffineWeylGroupW0PElement`

**S0()**

Return the zero-th simple reflection in style “W0P”.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',3,1]).W0P().S0()
s1*s2*s3*s2*s1 * t[-Lambdacheck[1] - Lambdacheck[3]]
```

**from_classical_weyl(w)**

Return the image of the classical Weyl group element \( w \) in self.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',2,1],print_tuple=True)
sage: E.W0P().from_classical_weyl(E.classical_weyl().from_reduced_word([2,1]))
(s2*s1, t[0])
```

**from_translation(\( la \))**

Return the image of the lattice element \( la \) in self.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',2,1],print_tuple=True)
sage: E.W0P().from_translation(E.lattice().an_element())
```

**simple_reflection(\( i \))**

Return the \( i \)-th simple reflection in self.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',3,1]); W0P = E.W0P()
sage: [(i, W0P.simple_reflection(i)) for i in E.cartan_type().index_set()]
[(0, s1*s2*s3*s2*s1 * t[-Lambdacheck[1] - Lambdacheck[3]]), (1, s1), (2, s2), (3, s3)]
```

**simple_reflections()**

Return the family of simple reflections.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',3,1]).W0P().simple_reflections()
Finite family {0: s1*s2*s3*s2*s1 * t[-Lambdacheck[1] - Lambdacheck[3]], 1: s1, 2: s2, 3: s3}
```
class ExtendedAffineWeylGroupW0PElement
Bases: GroupSemidirectProductElement
The element class for the W0P realization.

has_descent(i, side='right', positive=False)
Return whether self has i as a descent.

INPUT:
• i - an index.
OPTIONAL:
• side - 'left' or 'right' (default: 'right')
• positive - True or False (default: False)

EXAMPLES:

```python
sage: W0P = ExtendedAffineWeylGroup(['A',4,2]).W0P()
sage: w = W0P.from_reduced_word([0,1]); w
s1*s2 * t[Lambda[1] - Lambda[2]]
sage: w.has_descent(0, side='left')
True
```

to_classical_weyl()
Project self into the classical Weyl group.

EXAMPLES:

```python
sage: x = ExtendedAffineWeylGroup(['A',2,1]).W0P().simple_reflection(0); x
s1*s2*s1 * t[-Lambdacheck[1] - Lambdacheck[2]]
sage: x.to_classical_weyl()
s1*s2*s1
```

to_translation_right()
Project onto the right (translation) factor in the “W0P” style.

EXAMPLES:

```python
sage: x = ExtendedAffineWeylGroup(['A',2,1]).W0P().simple_reflection(0); x
s1*s2*s1 * t[-Lambdacheck[1] - Lambdacheck[2]]
sage: x.to_translation_right()
```

class ExtendedAffineWeylGroupW0Pv(E)
Bases: GroupSemidirectProduct, BindableClass
Extended affine Weyl group, realized as the semidirect product of the finite Weyl group, acting on the dual form of the translation lattice.

INPUT:
• E - A parent with realization in ExtendedAffineWeylGroup_Class

EXAMPLES:

```python
sage: ExtendedAffineWeylGroup(['A',2,1]).W0Pv()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice) acting on Multiplicative form of Weight lattice of the Root system of type ['A', 2]
```
Combinatorics, Release 10.0

Element
alias of ExtendedAffineWeylGroupW0PvElement

from_dual_classical_weyl(w)
Return the image of w under the homomorphism of the dual form of the classical Weyl group into self.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',3,1],print_tuple=True)
sage: E.W0Pv().from_dual_classical_weyl(E.dual_classical_weyl().from_reduced_word([1,2]))
(s1*s2, t[0])
```

from_dual_translation(la)
Map the dual translation lattice element la into self.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',2,1], translation="tau", print_tuple=True)
sage: la = E.dual_lattice().an_element(); la
sage: E.W0Pv().from_dual_translation(la)
```

simple_reflections()
Return a family for the simple reflections of self.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',3,1]).W0Pv().simple_reflections()
Finite family {0: s1*s2*s3*s2*s1 * t[-Lambda[1] - Lambda[3]], 1: s1, 2: s2, ...
˓→3: s3}
```

class ExtendedAffineWeylGroupW0PvElement
Bases: GroupSemidirectProductElement
The element class for the “W0Pv” realization.

dual_action(la)
Return the action of self on an element la of the dual version of the translation lattice.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.W0Pv().an_element(); x
sage: la = E.dual_lattice().an_element(); la
sage: x.dual_action(la)
```

has_descent(i, side='right', positive=False)
Return whether self has i as a descent.

INPUT:
• $i$ - an affine Dynkin index

OPTIONAL:
• side - 'left' or 'right' (default: 'right')
• positive - True or False (default: False)

EXAMPLES:

```python
sage: w = ExtendedAffineWeylGroup(['A',4,2]).W0Pv().from_reduced_word([0, -1]); w
s1*s2 * t[Lambda[1] - Lambda[2]]
sage: w.has_descent(0, side='left')
True
```

to_dual_classical_weyl()

Return the image of self under the homomorphism that projects to the dual classical Weyl group factor after rewriting it in either style “PvW0” or “W0Pv”.

EXAMPLES:

```python
sage: s = ExtendedAffineWeylGroup(['A',2,1]).W0Pv().simple_reflection(0); s
s1*s2*s1 * t[-Lambda[1] - Lambda[2]]
sage: s.to_dual_classical_weyl()
s1*s2*s1
```

to_dual_translation_right()

The image of self under the map that projects to the dual translation lattice factor after factoring it to the right as in style “W0Pv”.

EXAMPLES:

```python
sage: s = ExtendedAffineWeylGroup(['A',2,1]).W0Pv().simple_reflection(0); s
s1*s2*s1 * t[-Lambda[1] - Lambda[2]]
sage: s.to_dual_translation_right()
```

class ExtendedAffineWeylGroupWF(E)

Bases: GroupSemidirectProduct, BindableClass

Extended affine Weyl group, realized as the semidirect product of the affine Weyl group by the fundamental group.

INPUT:

• $E$ – A parent with realization in ExtendedAffineWeylGroup.Class

EXAMPLES:

```python
sage: ExtendedAffineWeylGroup(['A',2,1]).WF()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect Product of Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root lattice) acted upon by Fundamental group of type ['A', 2, 1]
```

Element

alias of ExtendedAffineWeylGroupWFElem

from_affine_weyl(w)

Return the image of the affine Weyl group element $w$ in self.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: E = ExtendedAffineWeylGroup(['C',2,1],print_tuple=True)
sage: E.WF().from_affine_weyl(E.affine_weyl().from_reduced_word([1,2,1,0]))
(S1*S2*S1*S0, pi[0])
```

**from_fundamental(f)**

Return the image of \( f \) under the homomorphism from the fundamental group into the right (fundamental group) factor in “WF” style.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['E',6,1],print_tuple=True); WF = E.WF();
˓→ F = E.fundamental_group()
sage: [(x,WF.from_fundamental(x)) for x in F]
[(pi[0], (1, pi[0])), (pi[1], (1, pi[1])), (pi[6], (1, pi[6]))]
```

**simple_reflections()**

Return the family of simple reflections.

EXAMPLES:

```python
sage: ExtendedAffineWeylGroup(['A',3,1],affine="r").WF().simple_reflections()
Finite family {0: r0, 1: r1, 2: r2, 3: r3}
```

**class ExtendedAffineWeylGroupWFElement**

Bases: GroupSemidirectProductElement

Element class for the “WF” realization.

**bruhat_le(x)**

Return whether \( \text{self} \) is less than or equal to \( x \) in the Bruhat order.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1],affine="s", print_tuple=True);
˓→ WF=E.WF()
sage: r = E.affine_weyl().from_reduced_word
sage: v = r([1,0])
sage: w = r([1,2,0])
sage: v.bruhat_le(w)
True
sage: vv = WF.from_affine_weyl(v); vv
(s1*s0, pi[0])
```

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**has_descent**\( (i, \text{side}='right', \text{positive}=False) \)

Return whether self has \( i \) as a descent.

**INPUT:**
- \( i \) – an affine Dynkin index

**OPTIONAL:**
- \( \text{side} \) – ‘left’ or ‘right’ (default: ‘right’)
- \( \text{positive} \) – True or False (default: False)

**EXAMPLES:**

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.WF().an_element(); x
S0*S1*S2 * pi[2]
sage: [(i, x.has_descent(i)) for i in E.cartan_type().index_set()]
[(0, True), (1, False), (2, False)]
```

**to_affine_weyl_left()**

Project self to the left (affine Weyl group) factor in the “WF” style.

**EXAMPLES:**

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.WF().from_translation(E.lattice_basis()[1]); x
S0*S2 * pi[1]
sage: x.to_affine_weyl_left()
S0*S2
```

**to_fundamental_group()**

Project self to the right (fundamental group) factor in the “WF” style.

**EXAMPLES:**

```python
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: x = E.WF().from_translation(E.lattice_basis()[1]); x
S0*S2 * pi[1]
sage: x.to_fundamental_group()
pi[1]
```

**FW()**

Realizes self in “FW”-style.

**EXAMPLES:**

```python
sage: ExtendedAffineWeylGroup(['A',2,1]).FW()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Fundamental group of type ['A', 2, 1] acting on Weyl Group of type ['A', 2, → 1] (as a matrix group acting on the root lattice)
```

**PW0()**

Realizes self in “PW0”-style.

**EXAMPLES:**

```python
sage: ExtendedAffineWeylGroup(['A',2,1]).PW0()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product...
```
of Multiplicative form of Coweight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)

\[ \text{PW0_to_WF_func}(x) \]

Implements coercion from style “PW0” to “WF”.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A', 2, 1])
sage: x = E.PW0().an_element(); x
sage: E.PW0_to_WF_func(x)
S0*S1*S2*S0*S1*S0
```

Warning: This function cannot use coercion, because it is used to define the coercion maps.

\[ \text{PvW0()} \]

Realizes self in “PvW0”-style.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A', 2, 1]).PvW0()
```

Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Multiplicative form of Weight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice)

**class Realizations**

Bases: `Category_realization_of_parent`

The category of the realizations of an extended affine Weyl group

**class ElementMethods**

Bases: object

\[ \text{action}(la) \]

Action of self on a lattice element la.

INPUT:
- self – an element of the extended affine Weyl group
- la – an element of the translation lattice of the extended affine Weyl group, the lattice denoted by the mnemonic “P” in the documentation for `ExtendedAffineWeylGroup()`.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A', 2, 1], affine="s")
sage: x = E.FW().an_element(); x
pi[2] * s0*s1*s2
sage: la = E.lattice().an_element(); la
sage: x.action(la)
```
sage: E = ExtendedAffineWeylGroup(['C',2,1],affine="s")
sage: x = E.PW0().from_translation(E.lattice_basis()[1])
sage: x.action(E.lattice_basis()[2])

Warning: Must be implemented by style “PW0”.

action_on_affine_roots(beta)
Act by self on the affine root lattice element beta.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: beta = E.cartan_type().root_system().root_lattice().an_element();
    → beta
sage: x = E.FW().an_element(); x
pi[2] * S0*S1*S2
sage: x.action_on_affine_roots(beta)
alpha[0] + alpha[1]

Warning: Must be implemented by style “FW”.

alcove_walk_signs()
Return a signed alcove walk for self.

INPUT:
• An element self of the extended affine Weyl group.
OUTPUT:
• A 3-tuple (g, rw, signs).
ALGORITHM:
The element self can be uniquely written self = g * w where g has length zero and w is an
element of the nonextended affine Weyl group. Let w have reduced word rw. Starting with g
and applying simple reflections from rw, one obtains a sequence of extended affine Weyl group
elements (that is, alcoves) and simple roots. The signs give the sequence of sides on which the
alcoves lie, relative to the face indicated by the simple roots.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',3,1]); FW=E.FW()
sage: w = FW.from_reduced_word([0,2,1,3,0])*FW.from_fundamental(1); w
pi[1] * S3*S1*S2*S0*S3
sage: w.alcove_walk_signs()
(pi[1], [3, 1, 2, 0, 3], [-1, 1, -1, -1, 1])

apply_simple_projection(i, side='right', length_increasing=True)
Return the product of self by the simple reflection s_i if that product is of greater length than
self and otherwise return self.

INPUT:
• `self` – an element of the extended affine Weyl group
• `i` – a Dynkin node (index of a simple reflection $s_i$)
• `side` – 'right' or 'left' (default: 'right') according to which side of `self` the reflection $s_i$ should be multiplied
• `length_increasing` – True or False (default True). If False do the above with the word “greater” replaced by “less”.

**EXAMPLES**:

```sage
sage: x = ExtendedAffineWeylGroup(['A',3,1]).WF().an_element(); x
S0*S1*S2*S3 * pi[3]
sage: x.apply_simple_projection(1)
S0*S1*S2*S3*S0 * pi[3]
sage: x.apply_simple_projection(1, length_increasing=False)
S0*S1*S2*S3 * pi[3]
```

### `apply_simple_reflection(i, side='right')`

Apply the $i$-th simple reflection to `self`.

**EXAMPLES**:

```sage
sage: x = ExtendedAffineWeylGroup(['A',3,1]).WF().an_element(); x
S0*S1*S2*S3 * pi[3]
sage: x.apply_simple_reflection(1)
S0*S1*S2*S3*S0 * pi[3]
sage: x.apply_simple_reflection(0, side='left')
S1*S2*S3 * pi[3]
```

### `bruhat_le(x)`

Return whether `self <= x` in Bruhat order.

**INPUT:**
• `self` – an element of the extended affine Weyl group
• `x` – another element with the same parent as `self`

**EXAMPLES**:

```sage
sage: E = ExtendedAffineWeylGroup(['A',2,1],print_tuple=True); WF=E.WF()
sage: W = E.affine_weyl()
sage: v = W.from_reduced_word([2,1,0])
sage: w = W.from_reduced_word([2,0,1,0])
sage: v.bruhat_le(w)
True
sage: vx = WF.from_affine_weyl(v); vx
(S2*S1*S0, pi[0])
sage: wx = WF.from_affine_weyl(w); wx
(S2*S0*S1*S0, pi[0])
sage: vx.bruhat_le(wx)
True
sage: F = E.fundamental_group()
sage: f = WF.from_fundamental(F(2))
sage: vx.bruhat_le(wx*f)
False
sage: (vx*f).bruhat_le(wx*f)
True
```
Warning: Must be implemented by “WF”.

coset_representative(index_set, side='right')

Return the minimum length representative in the coset of self with respect to the subgroup generated by the reflections given by index_set.

INPUT:
• self – an element of the extended affine Weyl group
• index_set – a subset of the set of Dynkin nodes
• side – ‘right’ or ‘left’ (default: ‘right’) the side on which the subgroup acts

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',3,1]); WF = E.WF()
sage: b = E.lattice_basis()
sage: I0 = E.cartan_type().classical().index_set()
sage: [WF.from_translation(x).coset_representative(index_set=I0) for x in b]
```

```
[pi[1], pi[2], pi[3]]
```

dual_action(la)

Action of self on a dual lattice element la.

INPUT:
• self – an element of the extended affine Weyl group
• la – an element of the dual translation lattice of the extended affine Weyl group, the lattice denoted by the mnemonic “Pv” in the documentation for ExtendedAffineWeylGroup().

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A',2,1], affine="s")
sage: x = E.FW().an_element(); x
pi[2] * s0*s1*s2
sage: la = E.dual_lattice().an_element(); la
sage: x.dual_action(la)
sage: E = ExtendedAffineWeylGroup(['C',2,1], affine="s")
sage: x = E.PvW0().from_dual_translation(E.dual_lattice_basis()[1])
sage: x.dual_action(E.dual_lattice_basis()[2])
```

Warning: Must be implemented by style “PvW0”.

face_data(i)

Return a description of one of the bounding hyperplanes of the alcove of an extended affine Weyl group element.

INPUT:
• self – An element of the extended affine Weyl group
• i – an affine Dynkin node

OUTPUT:
• A 2-tuple \((m, \beta)\) defined as follows.

ALGORITHM:
Each element of the extended affine Weyl group corresponds to an alcove, and each alcove has a face for each affine Dynkin node. Given the data of \( \textbf{self} \) and \( i \), let the extended affine Weyl group element \( \textbf{self} \) act on the affine simple root \( \alpha_i \), yielding a real affine root, which can be expressed uniquely as

\[
^{\textbf{self}} \cdot \alpha_i = m\delta + \beta
\]

where \( m \) is an integer (the height of the \( i \)-th bounding hyperplane of the alcove of \( \textbf{self} \)) and \( \beta \) is a classical root (the normal vector for the hyperplane which points towards the alcove).

**EXAMPLES:**

```plaintext
sage: x = ExtendedAffineWeylGroup(['A',2,1]).PW0().an_element(); x
sage: x.face_data(0)
(-1, alpha[1])
```

```plaintext
def first_descent(side='right', positive=False, index_set=None):
    Return the first descent of \( \textbf{self} \).

    INPUT:
    • side – ‘left’ or ‘right’ (default: ‘right’)
    • positive – True or False (default: False)
    • index_set – an optional subset of Dynkin nodes

    If \( \text{index_set} \) is not None, then the descent must be in \( \text{index_set} \).

    EXAMPLES:

    ```
sage: x = ExtendedAffineWeylGroup(['A',3,1]).WF().an_element(); x
S0*S1*S2*S3 * pi[3]
sage: x.first_descent()
0
sage: x.first_descent(side='left')
0
sage: x.first_descent(positive=True)
1
sage: x.first_descent(side='left', positive=True)
1
```
```
sage: I = E.cartan_type().index_set()
sage: [(i, x.has_descent(i)) for i in I]
[(0, True), (1, False), (2, False), (3, False)]
sage: [(i, x.has_descent(i, side='left')) for i in I]
[(0, True), (1, False), (2, False), (3, False)]
sage: [(i, x.has_descent(i, positive=True)) for i in I]
[(0, False), (1, True), (2, True), (3, True)]

Warning: This method is abstract because it is used in the recursive coercions between “PW0” and “WF” and other methods use this coercion.

is_affine_grassmannian()

Return whether self is affine Grassmannian.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',2,1]); PW0=E.PW0()
sage: F = E.fundamental_group()
sage: [(x,PW0.from_fundamental(x).is_affine_grassmannian()) for x in F]
[(pi[0], True), (pi[1], True), (pi[2], True)]
sage: b = E.lattice_basis()
sage: [(-x,PW0.from_translation(-x).is_affine_grassmannian()) for x in b]
[(-Lambdacheck[1], True), (-Lambdacheck[2], True)]

is_grassmannian(index_set, side='right')

Return whether self is of minimum length in its coset with respect to the subgroup generated by the reflections of index_set.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',3,1]); PW0=E.PW0()
sage: x = PW0.from_translation(E.lattice_basis()[1]); x
\t[Lambdacheck[1]]
sage: I = E.cartan_type().index_set()
sage: [(i, x.is_grassmannian(index_set=[i])) for i in I]
[(0, True), (1, False), (2, True), (3, True)]
sage: [(i, x.is_grassmannian(index_set=[i], side='left')) for i in I]
[(0, False), (1, True), (2, True), (3, True)]

is_translation()

Return whether self is a translation element or not.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',2,1]); FW=E.FW()
sage: F = E.fundamental_group()
sage: FW.from_affine_weyl(E.affine_weyl().from_reduced_word([1,2,1,0])).is_translation()
True
sage: FW.from_translation(E.lattice_basis()[1]).is_translation()
True
sage: FW.simple_reflection(0).is_translation()
False

length()

Return the length of self in the Coxeter group sense.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A',3,1]); PW0=E.PW0()
sage: I0 = E.cartan_type().classical().index_set()
sage: [PW0.from_translation(E.lattice_basis()[i]).length() for i in I0]
[3, 4, 3]
```

to_affine_grassmannian()

Return the unique affine Grassmannian element in the same coset of self with respect to the finite Weyl group acting on the right.

EXAMPLES:

```python
sage: elts = ExtendedAffineWeylGroup(['A',2,1]).PW0().some_elements()
sage: [(x, x.to_affine_grassmannian()) for x in elts]

```

to_affine_weyl_left()

Return the projection of self to the affine Weyl group on the left, after factorizing using the style “WF”.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A',3,1]); PW0=E.PW0()
sage: b = E.lattice_basis()
sage: [(x,PW0.from_translation(x).to_affine_weyl_left()) for x in b]

[(Lambdacheck[1], S0*S3*S2), (Lambdacheck[2], S0*S3*S1*S0), (Lambdacheck[3], S0*S1*S2)]
```

Warning: Must be implemented in style “WF”.

to_affine_weyl_right()

Return the projection of self to the affine Weyl group on the right, after factorizing using the style “FW”.

EXAMPLES:

```python
sage: E = ExtendedAffineWeylGroup(['A',3,1]); PW0=E.PW0()
sage: b = E.lattice_basis()
sage: [(x,PW0.from_translation(x).to_affine_weyl_right()) for x in b]

[(Lambdacheck[1], S3*S2*S1), (Lambdacheck[2], S2*S3*S1*S2), (Lambdacheck[3], S1*S2*S3)]
```
Warning: Must be implemented in style “FW”.

to_classical_weyl()
Return the image of self under the homomorphism to the classical Weyl group.
EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',3,1]).WF().simple_reflection(0).to_classical_weyl()
s1*s2*s3*s2*s1
```

Warning: Must be implemented in style “PW0”.

to_dual_classical_weyl()
Return the image of self under the homomorphism to the dual form of the classical Weyl group.
EXAMPLES:

```
sage: x = ExtendedAffineWeylGroup(['A',3,1]).WF().simple_reflection(0).to_dual_classical_weyl(); x
s1*s2*s3*s2*s1
sage: x.parent()
Weyl Group of type ['A', 3] (as a matrix group acting on the weight lattice)
```

Warning: Must be implemented in style “PvW0”.

to_dual_translation_left()
Return the projection of self to the dual translation lattice after factorizing it to the left using the style “PvW0”.
EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',3,1]).PvW0().simple_reflection(0).to_dual_translation_left()
Lambda[1] + Lambda[3]
```

Warning: Must be implemented in style “PvW0”.

to_dual_translation_right()
Return the projection of self to the dual translation lattice after factorizing it to the right using the style “W0Pv”.
EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',3,1]).PW0().simple_reflection(0).to_dual_translation_right()
```
Warning: Must be implemented in style “W0Pv”.

**to_fundamental_group()**
Return the image of self under the homomorphism to the fundamental group.
EXAMPLES:
```python
to_translation_left()
Return the projection of self to the translation lattice after factorizing it to the left using the style “PW0”.
EXAMPLES:
```python
to_translation_right()
Return the projection of self to the translation lattice after factorizing it to the right using the style “W0P”.
EXAMPLES:
```python
class ParentMethods
Bases: object
from_affine_weyl(w)
Return the image of w under the homomorphism from the affine Weyl group into self.
EXAMPLES:
sage: E = ExtendedAffineWeylGroup(['A',3,1]); PW0=E.PW0()
sage: W = E.affine_weyl()
sage: w = W.from_reduced_word([2,1,3,0])
sage: x = PW0.from_affine_weyl(w); x
sage: FW = E.FW()
sage: y = FW.from_affine_weyl(w); y
S2*S3*S1*S0
sage: FW(x) == y
True

Warning: Must be implemented in style “WF” and “FW”.

from_classical_weyl(w)
Return the image of w from the finite Weyl group into self.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',3,1]); PW0=E.PW0()
sage: W0 = E.classical_weyl()
sage: w = W0.from_reduced_word([2,1,3])
sage: y = PW0.from_classical_weyl(w); y
s2*s3*s1
sage: y.parent() == PW0
True
sage: y.to_classical_weyl() == w
True
sage: W0P = E.W0P()
sage: z = W0P.from_classical_weyl(w); z
s2*s3*s1
sage: z.parent() == W0P
True
sage: W0P(y) == z
True
sage: FW = E.FW()
sage: x = FW.from_classical_weyl(w); x
S2*S3*S1
sage: x.parent() == FW
True
sage: FW(y) == x
True
sage: FW(z) == x
True

Warning: Must be implemented in style “PW0” and “W0P”.

from_dual_classical_weyl(w)
Return the image of w from the finite Weyl group of dual form into self.

EXAMPLES:
sage: E = ExtendedAffineWeylGroup(['A',3,1]); PvW0 = E.PvW0()
sage: W0v = E.dual_classical_weyl()
sage: w = W0v.from_reduced_word([2,1,3])
sage: y = PvW0.from_dual_classical_weyl(w); y
s2*s3*s1
sage: y.parent() == PvW0
True
sage: y.to_dual_classical_weyl() == w
True
sage: x = E.FW().from_dual_classical_weyl(w); x
S2*S3*S1
sage: PvW0(x) == y
True

Warning: Must be implemented in style “PvW0” and “W0Pv”.

from_dual_translation(la)
Return the image of la under the homomorphism of the dual version of the translation lattice into self.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',2,1]); PvW0 = E.PvW0()
sage: bv = E.dual_lattice_basis(); bv
Finite family {1: Lambda[1], 2: Lambda[2]}
sage: x = PvW0.from_dual_translation(2*bv[1]-bv[2]); x
sage: FW = E.FW()
sage: y = FW.from_dual_translation(2*bv[1]-bv[2]); y
S0*S2*S0*S1
sage: FW(x) == y
True

from_fundamental(x)
Return the image of x under the homomorphism from the fundamental group into self.

EXAMPLES:

sage: E = ExtendedAffineWeylGroup(['A',3,1])
sage: PW0=E.PW0()
sage: F = E.fundamental_group()
sage: Is = F.special_nodes()
sage: [(i, PW0.from_fundamental(F(i))) for i in Is]
[(0, 1), (1, t[Lambdacheck[1]] * s1*s2*s3), (2, t[Lambdacheck[2]] * s2*s3*s1*s2), (3, t[Lambdacheck[3]] * s3*s2*s1)]
sage: [(i, E.WP().from_fundamental((F(i)))) for i in Is]
[(0, 1), (1, s1*s2*s3 * t[-Lambdacheck[1]]), (2, s2*s3*s1*s2 * t[-Lambdacheck[2]]), (3, s3*s2*s1 * t[-Lambdacheck[3]])]
sage: [(i, E.WF().from_fundamental(F(i))) for i in Is]
[(0, 1), (1, pi[1]), (2, pi[2]), (3, pi[3])]
**Warning:** This method must be implemented by the “WF” and “FW” realizations.

**from_reduced_word**(word)

Converts an affine or finite reduced word into a group element.

**EXAMPLES:**

```sage
sage: ExtendedAffineWeylGroup(['A',2,1]).PW0().from_reduced_word([1,0,1,2])
t[-Lambda[1] + 2*Lambda[2]]
```

**from_translation**(la)

Return the element of translation by la in self.

**INPUT:**

- **self** – a realization of the extended affine Weyl group
- **la** – an element of the translation lattice

In the notation of the documentation for `ExtendedAffineWeylGroup()`, la must be an element of ‘P’.

**EXAMPLES:**

```sage
sage: E = ExtendedAffineWeylGroup(['A',2,1]); PW0=E.PW0()
sage: b = E.lattice_basis(); b
Finite family {1: Lambda[1], 2: Lambda[2]}
sage: x = PW0.from_translation(2*b[1]-b[2]); x
sage: FW = E.FW()
sage: y = FW.from_translation(2*b[1]-b[2]); y
S0*S2*S0*S1
sage: FW(x) == y
True
```

Since the implementation as a semidirect product requires wrapping the lattice group to make it multiplicative, we cannot declare that this map is a morphism for `sage Groups()`.

**Warning:** This method must be implemented by the “PW0” and “W0P” realizations.

**simple_reflection**(i)

Return the i-th simple reflection in self.

**INPUT:**

- **self** – a realization of the extended affine Weyl group
- **i** – An affine Dynkin node

**EXAMPLES:**

```sage
sage: ExtendedAffineWeylGroup(['A',3,1]).PW0().simple_reflection(0)
t[Lambda[1] + Lambda[3]] * s1*s2*s3*s2*s1
sage: ExtendedAffineWeylGroup(['C',2,1]).WF().simple_reflection(0)
S0
sage: ExtendedAffineWeylGroup(['D',3,2]).PvW0().simple_reflection(1)
s1
```
**simple_reflections()**

Return a family from the set of affine Dynkin nodes to the simple reflections in the realization of the extended affine Weyl group.

**EXAMPLES:**

```
sage: ExtendedAffineWeylGroup(['A',3,1]).W0P().simple_reflections()
Finite family {0: s1*s2*s3*s2*s1 * t[-Lambdacheck[1] - Lambdacheck[3]],
  1: s1, 2: s2, 3: s3}
sage: ExtendedAffineWeylGroup(['A',3,1]).WF().simple_reflections()
Finite family {0: S0, 1: S1, 2: S2, 3: S3}
sage: ExtendedAffineWeylGroup(['A',3,1], print_tuple=True).FW().simple_reflections()
Finite family {0: (pi[0], S0), 1: (pi[0], S1), 2: (pi[0], S2), 3: (pi[0], S3)}
sage: ExtendedAffineWeylGroup(['A',3,1], fundamental="f",print_tuple=True).FW().simple_reflections()
Finite family {0: (f[0], S0), 1: (f[0], S1), 2: (f[0], S2), 3: (f[0], S3)}
sage: ExtendedAffineWeylGroup(['A',3,1]).PvW0().simple_reflections()
Finite family {0: t[Lambda[1] + Lambda[3]] * s1*s2*s3*s2*s1, 1: s1, 2: s2, 3: s3}
```

**super_categories()**

**EXAMPLES:**

```
sage: R = ExtendedAffineWeylGroup(['A',2,1]).Realizations(); R
Category of realizations of Extended affine Weyl group of type ['A', 2, 1]
sage: R.super_categories()
[Category of associative inverse realizations of unital magmas]
```

**W0P()**

Realizes self in “W0P”-style.

**EXAMPLES:**

```
sage: ExtendedAffineWeylGroup(['A',2,1]).W0P()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice) acting on Multiplicative form of Coweight lattice of the Root system of type ['A', 2]
```

**W0Pv()**

Realizes self in “W0Pv”-style.

**EXAMPLES:**

```
sage: ExtendedAffineWeylGroup(['A',2,1]).W0Pv()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice) acting on Multiplicative form of Weight lattice of the Root system of type ['A', 2]
```

**WF()**

Realizes self in “WF”-style.
EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A', 2, 1]).WF()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root lattice) acted upon by Fundamental group of type ['A', 2, 1] realized by Semidirect product of Multiplicative form of Coweight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)
```

**WF_to_PW0_func**

Coercion from style “WF” to “PW0”.

EXAMPLES:

```
sage: E = ExtendedAffineWeylGroup(['A', 2, 1])
sage: x = E.WF().an_element(); x
S0*S1*S2 * pi[2]
sage: E.WF_to_PW0_func(x)
t[Lambdacheck[1] + 2*Lambdacheck[2]] * s1*s2*s1
```

Warning: Since this is used to define some coercion maps it cannot itself use coercion.

**a_realization()**

Return the default realization of an extended affine Weyl group.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',2,1]).a_realization()
Extended affine Weyl group of type ['A', 2, 1] realized by Semidirect product of Multiplicative form of Coweight lattice of the Root system of type ['A', 2] acted upon by Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)
```

**affine_weyl()**

Return the affine Weyl group of self.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',2,1]).affine_weyl()
Weyl Group of type ['A', 2, 1] (as a matrix group acting on the root lattice)
sage: ExtendedAffineWeylGroup(['A',5,2]).affine_weyl()
Weyl Group of type ['B', 3, 1]^* (as a matrix group acting on the root lattice)
sage: ExtendedAffineWeylGroup(['A',4,2]).affine_weyl()
Weyl Group of type ['B', 2, 2] (as a matrix group acting on the root lattice)
sage: ExtendedAffineWeylGroup(WeylGroup(['C', 2, 1]).dual()).affine_weyl()
Weyl Group of type ['C', 2, 1]^* (as a matrix group acting on the root lattice)
```

**cartan_type()**

The Cartan type of self.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['D',3,2]).cartan_type()
['C', 2, 1]**
**classical_weyl()**

Return the classical Weyl group of `self`.

**EXAMPLES:**

```
sage: ExtendedAffineWeylGroup(['A',2,1]).classical_weyl()
Weyl Group of type ['A', 2] (as a matrix group acting on the coweight lattice)
sage: ExtendedAffineWeylGroup(['A',5,2]).classical_weyl()
Weyl Group of type ['A', 5] (as a matrix group acting on the weight lattice)
sage: ExtendedAffineWeylGroup(['A',4,2]).classical_weyl()
Weyl Group of type ['C', 2] (as a matrix group acting on the weight lattice)
sage: ExtendedAffineWeylGroup(CartanType(['A',4,2]).dual()).classical_weyl()
Weyl Group of type ['C', 2] (as a matrix group acting on the coweight lattice)
```

**classical_weyl_to_affine(w)**

The image of `w` under the homomorphism from the classical Weyl group into the affine Weyl group.

**EXAMPLES:**

```
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: W0 = E.classical_weyl()
sage: w = W0.from_reduced_word([1,2]); w
s1*s2
sage: v = E.classical_weyl_to_affine(w); v
S1*S2
```

**dual_classical_weyl()**

Return the dual version of the classical Weyl group of `self`.

**EXAMPLES:**

```
sage: ExtendedAffineWeylGroup(['A',2,1]).dual_classical_weyl()
Weyl Group of type ['A', 2] (as a matrix group acting on the weight lattice)
sage: ExtendedAffineWeylGroup(['A',5,2]).dual_classical_weyl()
Weyl Group of type ['C', 3] (as a matrix group acting on the weight lattice)
```

**dual_classical_weyl_to_affine(w)**

The image of `w` under the homomorphism from the dual version of the classical Weyl group into the affine Weyl group.

**EXAMPLES:**

```
sage: E = ExtendedAffineWeylGroup(['A',2,1])
sage: W0v = E.dual_classical_weyl()
sage: w = W0v.from_reduced_word([1,2]); w
s1*s2
sage: v = E.dual_classical_weyl_to_affine(w); v
S1*S2
```

**dual_lattice()**

Return the dual version of the translation lattice for `self`.

**EXAMPLES:**

```
```
sage: ExtendedAffineWeylGroup(['A',2,1]).dual_lattice()
Weight lattice of the Root system of type ['A', 2]
sage: ExtendedAffineWeylGroup(['A',5,2]).dual_lattice()
Weight lattice of the Root system of type ['C', 3]

dual_lattice_basis()

Return the distinguished basis of the dual version of the translation lattice for self.

EXAMPLES:

sage: ExtendedAffineWeylGroup(['A',2,1]).dual_lattice_basis()
Finite family {1: Lambda[1], 2: Lambda[2]}
sage: ExtendedAffineWeylGroup(['A',5,2]).dual_lattice_basis()
Finite family {1: Lambda[1], 2: Lambda[2], 3: Lambda[3]}

exp_dual_lattice()

Return the multiplicative version of the dual version of the translation lattice for self.

EXAMPLES:

sage: ExtendedAffineWeylGroup(['A',2,1]).exp_dual_lattice()
Multiplicative form of Weight lattice of the Root system of type ['A', 2]

exp_lattice()

Return the multiplicative version of the translation lattice for self.

EXAMPLES:

sage: ExtendedAffineWeylGroup(['A',2,1]).exp_lattice()
Multiplicative form of Coweight lattice of the Root system of type ['A', 2]

fundamental_group()

Return the abstract fundamental group.

EXAMPLES:

sage: F = ExtendedAffineWeylGroup(['D',5,1]).fundamental_group(); F
Fundamental group of type ['D', 5, 1]
sage: [a for a in F]
[pi[0], pi[1], pi[4], pi[5]]

group_generators()

Return a set of generators for the default realization of self.

EXAMPLES:

sage: ExtendedAffineWeylGroup(['A',2,1]).group_generators()
(t[Lambdacheck[1]], t[Lambdacheck[2]], s1, s2)

lattice()

Return the translation lattice for self.

EXAMPLES:
sage: ExtendedAffineWeylGroup(['A',2,1]).lattice()
Coweight lattice of the Root system of type ['A', 2]
sage: ExtendedAffineWeylGroup(['A',5,2]).lattice()
Weight lattice of the Root system of type ['C', 3]
sage: ExtendedAffineWeylGroup(['A',4,2]).lattice()
Weight lattice of the Root system of type ['C', 2]
sage: ExtendedAffineWeylGroup(CartanType(['A',4,2]).dual()).lattice()
Coweight lattice of the Root system of type ['B', 2]
sage: ExtendedAffineWeylGroup(CartanType(['A',2,1]), general_linear=True).
˓→lattice()
Ambient space of the Root system of type ['A', 2]

```
lattice_basis()
```

Return the distinguished basis of the translation lattice for self.

EXAMPLES:

```
sage: ExtendedAffineWeylGroup(['A',2,1]).lattice_basis()
Finite family {1: Lambdacheck[1], 2: Lambdacheck[2]}
sage: ExtendedAffineWeylGroup(['A',5,2]).lattice_basis()
Finite family {1: Lambda[1], 2: Lambda[2], 3: Lambda[3]}
sage: ExtendedAffineWeylGroup(['A',4,2]).lattice_basis()
Finite family {1: Lambda[1], 2: Lambda[2]}
sage: ExtendedAffineWeylGroup(CartanType(['A',4,2]).dual()).lattice_basis()
Finite family {1: Lambdacheck[1], 2: Lambdacheck[2]}
```

### 5.1.264 Fundamental Group of an Extended Affine Weyl Group

**AUTHORS:**

- Mark Shimozono (2013) initial version

**class** `sage.combinat.root_system.fundamental_group.FundamentalGroupElement(parent, x)`

Bases: `MultiplicativeGroupElement`

This should not be called directly

**EXAMPLES:**

```
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
˓→FundamentalGroupOfExtendedAffineWeylGroup
sage: x = FundamentalGroupOfExtendedAffineWeylGroup(['A',4,1], prefix="f").an_
˓→element()
sage: TestSuite(x).run()
```

```
act_on_affine_lattice(wt)
```

Act by self on the element `wt` of an affine root/weight lattice realization.

**EXAMPLES:**

```
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
˓→FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',3,1])
sage: wt = RootSystem(F.cartan_type()).weight_lattice().an_element(); wt
(continues on next page)```
\[2\lambda_0 + 2\lambda_1 + 3\lambda_2\]

\texttt{sage: F(3).act_on_affine_lattice(wt)}
\[2\lambda_0 + 3\lambda_1 + 2\lambda_3\]

\textbf{Warning:} Doesn't work on ambient spaces.

\textbf{act\_on\_affine\_weyl}(\textit{w})

Act by \texttt{self} on the element \textit{w} of the affine Weyl group.

\textbf{EXAMPLES:}

\texttt{sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup}
\texttt{sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',3,1])}
\texttt{sage: W = WeylGroup(F.cartan_type(),prefix='s')}
\texttt{sage: w = W.from_reduced_word([2,3,0])}
\texttt{sage: F(1).act_on_affine_weyl(w).reduced_word()}
\[[3, 0, 1]\]

\textbf{value()}

Return the special node which indexes the special automorphism \texttt{self}.

\textbf{EXAMPLES:}

\texttt{sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup}
\texttt{sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',4,1], prefix='f')}
\texttt{sage: F.special_nodes()}
\((0, 1, 2, 3, 4)\)
\texttt{sage: x = F(4); x}
\texttt{f[4]}
\texttt{sage: x.value()}
\[4\]

\textbf{class} \texttt{sage.combinat.root_system.fundamental_group.FundamentalGroupGL}(\textit{cartan\_type}, \textit{prefix}='\texttt{pi}')

\textbf{Bases:} \texttt{FundamentalGroupOfExtendedAffineWeylGroup\_Class}

Fundamental group of \textit{GL}_n. It is just the integers with extra privileges.

\textbf{Element}

alias of \texttt{FundamentalGroupGLElement}

\textbf{action(\textit{i})}

The action of the \textit{i}-th automorphism on the affine Dynkin node set.

\textbf{EXAMPLES:}

\texttt{sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup}
\texttt{sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_=\texttt{True})}
\texttt{sage: F.action(4)(2)}
Combinatorics, Release 10.0

sage: F.action(-4)(2)
1

an_element()
An element of self.

EXAMPLES:

sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True).
an_element()
pi[5]

dual_node(i)
The node whose special automorphism is inverse to that of i.

EXAMPLES:

sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True)
sage: F.dual_node(2)
-2

family()
The family associated with the set of special nodes.

EXAMPLES:

sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: fam = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True).family() # indirect doctest
sage: fam
Lazy family (<lambda>(i))_{i in Integer Ring}
sage: fam[-3]
-3

group_generators()
Return group generators for self.

EXAMPLES:

sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True).
group_generators()
(pi[1],)

one()
Return the identity element of the fundamental group.
EXAMPLES:

```python
def from sage.combinat.root_system.fundamental_group import ~ FundamentalGroupOfExtendedAffineWeylGroup
sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True).
def one()
pi[0]
```

**product** \((x, y)\)

Return the product of \(x\) and \(y\).

EXAMPLES:

```python
def from sage.combinat.root_system.fundamental_group import ~ FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True)
sage: F.special_nodes()
Integer Ring
sage: F(2)*F(3)
pi[5]
sage: F(1)*F(3)^(-1)
pi[-2]
```

**reduced_word** \((i)\)

A reduced word for the finite permutation part of the special automorphism indexed by \(i\).

More precisely, return a reduced word for the finite Weyl group element \(u\) where \(i\)-th automorphism (expressed in the extended affine Weyl group) has the form \(tu\) where \(t\) is a translation element.

EXAMPLES:

```python
def from sage.combinat.root_system.fundamental_group import ~ FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True)
sage: F.reduced_word(10)
(1, 2)
```

**some_elements** ()

Return some elements of \(self\).

EXAMPLES:

```python
def from sage.combinat.root_system.fundamental_group import ~ FundamentalGroupOfExtendedAffineWeylGroup
sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True).
some_elements()
[pi[-2], pi[2], pi[5]]
```

**class** `sage.combinat.root_system.fundamental_group.FundamentalGroupGLElement` \((parent, x)\)

Bases: `FundamentalGroupElement`

**act_on_classical_ambient** \((wt)\)

Act by \(self\) on the classical ambient weight vector \(wt\).

EXAMPLES:
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup

sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1], general_linear=True)

sage: f = F.an_element(); f

pi[5]

sage: la = F.cartan_type().classical().root_system().ambient_space().an_element(); la

(2, 2, 3)

sage: f.act_on_classical_ambient(la)

(2, 3, 2)

sage.combinat.root_system.fundamental_group.FundamentalGroupOfExtendedAffineWeylGroup(cartan_type, prefix='pi', general_linear=None)

Factory for the fundamental group of an extended affine Weyl group.

INPUT:

• cartan_type – a Cartan type that is either affine or finite, with the latter being a shorthand for the untwisted affinization

• prefix (default: ‘pi’) – string that labels the elements of the group

• general_linear – (default: None, meaning False) In untwisted type A, if True, use the universal central extension

Fundamental group

Associated to each affine Cartan type $\tilde{X}$ is an extended affine Weyl group $E$. Its subgroup of length-zero elements is called the fundamental group $F$. The group $F$ can be identified with a subgroup of the group of automorphisms of the affine Dynkin diagram. As such, every element of $F$ can be viewed as a permutation of the set $I$ of affine Dynkin nodes.

Let $0 \in I$ be the distinguished affine node; it is the one whose removal produces the associated finite Cartan type (call it $X$). A node $i \in I$ is called special if some automorphism of the affine Dynkin diagram, sends $0$ to $i$. The node $0$ is always special due to the identity automorphism. There is a bijection of the set of special nodes with the fundamental group. We denote the image of $i$ by $\pi_i$. The structure of $F$ is determined as follows.

• $\tilde{X}$ is untwisted – $F$ is isomorphic to $P^\vee/Q^\vee$ where $P^\vee$ and $Q^\vee$ are the coweight and coroot lattices of type $X$. The group $P^\vee/Q^\vee$ consists of the cosets $\omega_i^\vee + Q^\vee$ for special nodes $i$, where $\omega_0^\vee = 0$ by convention. In this case the special nodes $i$ are the cominuscule nodes, the ones such that $\langle \alpha_j^\vee, \omega_i^\vee \rangle$ is 0 or 1 for all $j \in I_0 = I \setminus \{0\}$. For $i$ special, addition by $\omega_i^\vee + Q^\vee$ permutes $P^\vee/Q^\vee$ and therefore permutes the set of special nodes. This permutation extends uniquely to an automorphism of the affine Dynkin diagram.

• $\tilde{X}$ is dual untwisted – (that is, the dual of $\tilde{X}$ is untwisted) $F$ is isomorphic to $P/Q$ where $P$ and $Q$ are the weight and root lattices of type $X$. The group $P/Q$ consists of the cosets $\omega_i + Q$ for special nodes $i$, where $\omega_0 = 0$ by convention. In this case the special nodes $i$ are the minuscule nodes, the ones such that $\langle \alpha_j^\vee, \omega_i \rangle$ is 0 or 1 for all $j \in I_0$. For $i$ special, addition by $\omega_i + Q$ permutes $P/Q$ and therefore permutes the set of special nodes. This permutation extends uniquely to an automorphism of the affine Dynkin diagram.

• $\tilde{X}$ is mixed – (that is, not of the above two types) $F$ is the trivial group.

EXAMPLES:
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup

sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',3,1]); F
Fundamental group of type ['A', 3, 1]
sage: F.cartan_type().dynkin_diagram()
\[\begin{array}{c}
\circ & \circ & \circ \\
0 & 1 & 2
\end{array}\]
A3~
sage: F.special_nodes()
(0, 1, 2, 3)
sage: F(1)^2
\pi[2]
sage: F(1)*F(2)
\pi[3]
sage: F(3)^(-1)
\pi[1]

sage: F = FundamentalGroupOfExtendedAffineWeylGroup("B3"); F
Fundamental group of type ['B', 3, 1]
sage: F.cartan_type().dynkin_diagram()
\[\begin{array}{c}
\circ \quad \circ \quad \circ \\
0 & 1 & 2
\end{array}\]
B3~
sage: F.special_nodes()
(0, 1)

sage: F = FundamentalGroupOfExtendedAffineWeylGroup("C2"); F
Fundamental group of type ['C', 2, 1]
sage: F.cartan_type().dynkin_diagram()
\[\begin{array}{c}
\circ \quad \circ \quad \circ \\
0 & 1 & 2
\end{array}\]
C2~
sage: F.special_nodes()
(0, 2)

sage: F = FundamentalGroupOfExtendedAffineWeylGroup("D4"); F
Fundamental group of type ['D', 4, 1]
sage: F.cartan_type().dynkin_diagram()
\[\begin{array}{c}
\circ & \circ & \circ & \circ \\
0 & 1 & 2 & 3
\end{array}\]
D4~
sage: F.special_nodes()
(0, 1, 3, 4)
sage: (F(4), F(4)^2)
(pi[4], pi[0])

sage: F = FundamentalGroupOfExtendedAffineWeylGroup(\"D5\")
Fundamental group of type ['D', 5, 1]
sage: F.cartan_type().dynkin_diagram()
0 O O 5
| |
| |
0---O---O---O
1 2 3 4
D5~

sage: F.special_nodes()
(0, 1, 4, 5)
sage: (F(5), F(5)^2, F(5)^3, F(5)^4)
(pi[5], pi[1], pi[4], pi[0])

sage: F = FundamentalGroupOfExtendedAffineWeylGroup(\"E6\")
Fundamental group of type ['E', 6, 1]
sage: F.cartan_type().dynkin_diagram()
0 O 2
| |
| |
O---O---O---O---O
1 3 4 5 6
E6~

sage: F.special_nodes()
(0, 1, 6)
sage: F(1)^2
pi[6]

sage: F = FundamentalGroupOfExtendedAffineWeylGroup([\"A",2,1], general_linear=True)
Fundamental group of GL(3)
sage: x = F.an_element(); x
pi[5]

We also implement a fundamental group for $GL_n$. It is defined to be the group of integers, which is the covering group of the fundamental group $\mathbb{Z}/n\mathbb{Z}$ for affine $SL_n$:

sage: F = FundamentalGroupOfExtendedAffineWeylGroup(\"A", 2, 1, general_linear=True);
\rightarrow F
Fundamental group of GL(3)
sage: x = F.an_element(); x
pi[5]
Combinatorics, Release 10.0

(continued from previous page)

```python
sage: x*x
pi[10]
sage: x.inverse()
pi[-5]
sage: wt = F.cartan_type().classical().root_system().ambient_space().an_element(); wt
(2, 2, 3)
sage: x.act_on_classical_ambient(wt)
(2, 3, 2)
sage: w = WeylGroup(F.cartan_type(),prefix="s").an_element(); w
s0*s1*s2
sage: x.act_on_affine_weyl(w)
s2*s0*s1
```

class sage.combinat.root_system.fundamental_group.FundamentalGroupOfExtendedAffineWeylGroup_class(cartan_type, prefix, finite=True):
    Bases: UniqueRepresentation, Parent

The group of length zero elements in the extended affine Weyl group.

**Element**

alias of `FundamentalGroupElement`

**action(i)**

Return a function which permutes the affine Dynkin node set by the $i$-th special automorphism.

EXAMPLES:

```python
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1])
sage: [[(i, j, F.action(i)(j)) for j in F.index_set()] for i in F.special_nodes()]
```

```python
sage: G = FundamentalGroupOfExtendedAffineWeylGroup(['D',4,1])
sage: [[(i, j, G.action(i)(j)) for j in G.index_set()] for i in G.special_nodes()]
```

**an_element()**

Return an element of `self`.

EXAMPLES:

```python
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',4,1],prefix="f").an_
```

(continues on next page)
element()  
\[ f[4] \]

cartan_type()  
The Cartan type of self.  
EXAMPLES:  
```python  
```

dual_node(i)  
Return the node that indexes the inverse of the \(i\)-th element.  
EXAMPLES:  
```python  
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup  sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',4,1])  sage: [(i, F.dual_node(i)) for i in F.special_nodes()]  [(0, 0), (1, 4), (2, 3), (3, 2), (4, 1)]  sage: G = FundamentalGroupOfExtendedAffineWeylGroup(['E',6,1])  sage: [(i, G.dual_node(i)) for i in G.special_nodes()]  [(0, 0), (1, 6), (6, 1)]  sage: H = FundamentalGroupOfExtendedAffineWeylGroup(['D',5,1])  sage: [(i, H.dual_node(i)) for i in H.special_nodes()]  [(0, 0), (1, 1), (4, 5), (5, 4)]  
```

group_generators()  
Return a tuple of generators of the fundamental group.  

**Warning:** This returns the entire group, a necessary behavior because it is used in `__iter__`().  

EXAMPLES:  
```python  
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup  sage: FundamentalGroupOfExtendedAffineWeylGroup(['E',6,1],prefix="f").group_generators()  Finite family {0: f[0], 1: f[1], 6: f[6]}  
```

index_set()  
The node set of the affine Cartan type of `self`.  
EXAMPLES:  
```python  
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup  sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1]).index_set()  (0, 1, 2)  
```
one()

Return the identity element of the fundamental group.

EXAMPLES:

```
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',3,1])
sage: F.one()
pi[0]
```

product(x, y)

Return the product of \( x \) and \( y \).

EXAMPLES:

```
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',3,1])
sage: F.special_nodes()
(0, 1, 2, 3)
sage: F(2)*F(3)
pi[1]
sage: F(1)*F(3)^(-1)
pi[2]
```

reduced_word(i)

Return a reduced word for the finite Weyl group element associated with the \( i \)-th special automorphism.

More precisely, for each special node \( i \), self.reduced_word(i) is a reduced word for the element \( v \) in the finite Weyl group such that in the extended affine Weyl group, the \( i \)-th special automorphism is equal to \( tv \) where \( t \) is a translation element.

EXAMPLES:

```
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: F = FundamentalGroupOfExtendedAffineWeylGroup(['A',3,1])
sage: [(i, F.reduced_word(i)) for i in F.special_nodes()]
[(0, ()), (1, (1, 2, 3)), (2, (2, 1, 3, 2)), (3, (3, 2, 1))]
```

special_nodes()

Return the special nodes of self.

See sage.combinat.root_system.cartan_type.special_nodes().

EXAMPLES:

```
sage: from sage.combinat.root_system.fundamental_group import FundamentalGroupOfExtendedAffineWeylGroup
sage: FundamentalGroupOfExtendedAffineWeylGroup(['D',4,1]).special_nodes()
(0, 1, 3, 4)
sage: FundamentalGroupOfExtendedAffineWeylGroup(['A',2,1]).special_nodes()
(0, 1, 2)
sage: FundamentalGroupOfExtendedAffineWeylGroup(['C',3,1]).special_nodes()
(0, 3)
```

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5.1.265 Root system data for folded Cartan types

AUTHORS:

- Travis Scrimshaw (2013-01-12) - Initial version

**class** `sage.combinat.root_system.type_folded.CartanTypeFolded(cartan_type, folding_of, orbit)`

Bases: `UniqueRepresentation, SageObject`

A Cartan type realized from a (Dynkin) diagram folding.

Given a Cartan type $X$, we say $\hat{X}$ is a folded Cartan type of $X$ if there exists a diagram folding of the Dynkin diagram of $\hat{X}$ onto $X$.

A folding of a simply-laced Dynkin diagram $D$ with index set $I$ is an automorphism $\sigma$ of $D$ where all nodes any orbit of $\sigma$ are not connected. The resulting Dynkin diagram $\hat{D}$ is induced by $I/\sigma$ where we identify edges in $\hat{D}$ which are not incident and add a $k$-edge if we identify $k$ incident edges and the arrow is pointing towards the incident node. We denote the index set of $\hat{D}$ by $\hat{I}$ and by abuse of notation, we denote the folding by $\sigma$.

We also have scaling factors $\gamma_i$ for $i \in \hat{I}$ and defined as the unique numbers such that the map $\Lambda_j \mapsto \gamma_j \sum_{i \in \sigma^{-1}(j)} \Lambda_i$ is the smallest proper embedding of the weight lattice of $X$ to $\hat{X}$.

If the Cartan type is simply laced, the default folding is the one induced from the identity map on $D$.

If $X$ is affine type, the default embeddings we consider here are:

\[ C_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)}, D_n^{(2)} \mapsto A_{2n-1}^{(1)}, \]
\[ B_n^{(1)} \mapsto D_{n+1}^{(1)}, \]
\[ E_6^{(2)}, F_4^{(1)} \mapsto E_6^{(1)}, \]
\[ D_4^{(3)}, G_2^{(1)} \mapsto D_4^{(1)}, \]

and were chosen based on virtual crystals. In particular, the diagram foldings extend to crystal morphisms and gives a realization of Kirillov-Reshetikhin crystals for non-simply-laced types as simply-laced types. See [OSShimo03] and [FOS2009] for more details. Here we can compute $\gamma_i = \max(c_i/c_i)$ where $(c_i)$ are the translation factors of the root system. In a more type-dependent way, we can define $\gamma_i$ as follows:

1. There exists a unique arrow (multiple bond) in $X$.
   a. Suppose the arrow points towards 0. Then $\gamma_i = 1$ for all $i \in \hat{I}$.
   b. Otherwise $\gamma_i$ is the order of $\sigma$ for all $i$ in the connected component of 0 after removing the arrow, else $\gamma_i = 1$.

2. There is not a unique arrow. Thus $\hat{X} = A_{2n-1}^{(1)}$ and $\gamma_i = 1$ for all $1 \leq i \leq n - 1$. If $i \in \{0, n\}$, then $\gamma_i = 2$ if the arrow incident to $i$ points away and is 1 otherwise.

We note that $\gamma_i$ only depends upon $X$.
If the Cartan type is finite, then we consider the classical foldings/embeddings induced by the above affine foldings/embeddings:

\[
C_n \rightarrow A_{2n-1}, \\
B_n \rightarrow D_{n+1}, \\
P_4 \rightarrow E_6, \\
G_2 \rightarrow D_4.
\]

For more information on Cartan types, see `sage.combinat.root_system.cartan_type`.

Other foldings may be constructed by passing in an optional `folding_of` second argument. See below.

**INPUT:**

- `cartan_type` – the Cartan type \( X \) to create the folded type
- `folding_of` – the Cartan type \( \hat{X} \) which \( X \) is a folding of
- `orbit` – the orbit of the Dynkin diagram automorphism \( \sigma \) given as a list of lists where the \( a \)-th list corresponds to the \( a \)-th entry in \( I \) or a dictionary with keys in \( I \) and values as lists

**Note:** If \( X \) is an affine type, we assume the special node is fixed under \( \sigma \).

**EXAMPLES:**

```sage
fct = CartanType(['C',4,1]).as_folding(); fct
['C', 4, 1] as a folding of ['A', 7, 1]
sage: fct.scaling_factors()
Finite family {0: 2, 1: 1, 2: 1, 3: 1, 4: 2}
sage: fct.folding_orbit()
Finite family {0: (0,), 1: (1, 7), 2: (2, 6), 3: (3, 5), 4: (4,)}
```

A simply laced Cartan type can be considered as a virtual type of itself:

```sage
fct = CartanType(['A',4,1]).as_folding(); fct
['A', 4, 1] as a folding of ['A', 4, 1]
sage: fct.scaling_factors()
Finite family {0: 1, 1: 1, 2: 1, 3: 1, 4: 1}
sage: fct.folding_orbit()
Finite family {0: (0,), 1: (1,), 2: (2,), 3: (3,), 4: (4,)}
```

Finite types:

```sage
fct = CartanType(['C',4]).as_folding(); fct
['C', 4] as a folding of ['A', 7]
sage: fct.scaling_factors()
Finite family {1: 1, 2: 1, 3: 1, 4: 2}
sage: fct.folding_orbit()
Finite family {1: (1, 7), 2: (2, 6), 3: (3, 5), 4: (4,)}
```

```sage
fct = CartanType(['F',4]).dual().as_folding(); fct
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1} as a folding of ['E', 6]
sage: fct.scaling_factors()
Finite family {1: 1, 2: 1, 3: 2, 4: 2}
sage: fct.folding_orbit()
Finite family {1: (1, 6), 2: (3, 5), 3: (4,), 4: (2,)}
```
REFERENCES:

- Wikipedia article Dynkin_diagram#Folding

cartan_type()

Return the Cartan type of self.

EXAMPLES:

```python
sage: fct = CartanType(['C', 4, 1]).as_folding()
sage: fct.cartan_type()
['C', 4, 1]
```

folding_of()

Return the Cartan type of the virtual space.

EXAMPLES:

```python
sage: fct = CartanType(['C', 4, 1]).as_folding()
sage: fct.folding_of()
['A', 7, 1]
```

folding_orbit()

Return the orbits under the automorphism $\sigma$ as a dictionary (of tuples).

EXAMPLES:

```python
sage: fct = CartanType(['C', 4, 1]).as_folding()
sage: fct.folding_orbit()
Finite family {0: (0,), 1: (1, 7), 2: (2, 6), 3: (3, 5), 4: (4,)}
```

scaling_factors()

Return the scaling factors of self.

EXAMPLES:

```python
sage: fct = CartanType(['C', 4, 1]).as_folding()
sage: fct.scaling_factors()
Finite family {0: 2, 1: 1, 2: 1, 3: 1, 4: 2}

sage: fct = CartanType(['BC', 4, 2]).as_folding()
sage: fct.scaling_factors()
Finite family {0: 1, 1: 1, 2: 1, 3: 1, 4: 2}

sage: fct = CartanType(['BC', 4, 2]).dual().as_folding()
sage: fct.scaling_factors()
Finite family {0: 2, 1: 1, 2: 1, 3: 1, 4: 1}

sage: CartanType(['BC', 4, 2]).relabel({0:4, 1:3, 2:2, 3:1, 4:0}).as_folding().
scaling_factors()
Finite family {0: 2, 1: 1, 2: 1, 3: 1, 4: 1}
```
5.1.266 Root system data for Cartan types with marked nodes

```python
class sage.combinat.root_system.type_marked.AmbientSpace(root_system, base_ring, index_set=None):
    Bases: AmbientSpace

    Ambient space for a marked finite Cartan type.
    It is constructed in the canonical way from the ambient space of the original Cartan type.

    EXAMPLES:
    >>> L = CartanType(['F',4]).marked_nodes([1,3]).root_system().ambient_space(); L
    Ambient space of the Root system of type ['F', 4] with nodes (1, 3) marked
    >>> TestSuite(L).run()

    dimension()
    Return the dimension of this ambient space.
    See also:
    sage.combinat.root_system.ambient_space.AmbientSpace.dimension()

    EXAMPLES:
    >>> L = CartanType(['F',4]).marked_nodes([1,3]).root_system().ambient_space()
    >>> L.dimension()
    4

    fundamental_weight(i)
    Return the i-th fundamental weight.
    It is constructed by looking up the corresponding simple coroot in the ambient space for the original Cartan type.

    EXAMPLES:
    >>> L = CartanType(['F',4]).marked_nodes([1,3]).root_system().ambient_space()
    >>> L.fundamental_weight(1)
    (1, 1, 0, 0)
    >>> L.fundamental_weights()
    Finite family {1: (1, 1, 0, 0), 2: (2, 1, 1, 0),
    3: (3/2, 1/2, 1/2, 1/2), 4: (1, 0, 0, 0)}

    simple_root(i)
    Return the i-th simple root.
    It is constructed by looking up the corresponding simple coroot in the ambient space for the original Cartan type.

    EXAMPLES:
    >>> L = CartanType(['F',4]).marked_nodes([1,3]).root_system().ambient_space()
    >>> L.simple_root(1)
    (0, 1, -1, 0)
    >>> L.simple_roots()
    Finite family {1: (0, 1, -1, 0), 2: (0, 0, 1, -1),
    3: (0, 0, 0, 1), 4: (1/2, -1/2, -1/2, -1/2)}
```

(continues on next page)
Finite family {1: (0, 1, -1, 0), 2: (0, 0, 1, -1), 3: (0, 0, 0, 2), 4: (1, -1, -1, -1)}

```python
class sage.combinat.root_system.type_marked.CartanType(ct, marked_nodes):
    Bases: CartanType_decorator

    A class for Cartan types with marked nodes.

    INPUT:
    • ct – a Cartan type
    • marked_nodes – a list of marked nodes

    EXAMPLES:
    We take the Cartan type $B_4$:

    ```
sage: T = CartanType(['B', 4])
sage: T.dynkin_diagram()
O---O---O=>=O
1 2 3 4
B4
    ```

    And mark some of its nodes:

    ```
sage: T = T.marked_nodes([2, 3])
sage: T.dynkin_diagram()
O---X---X=>=O
1 2 3 4
B4 with nodes (2, 3) marked
    ```

    Markings are not additive:

    ```
sage: T.marked_nodes([1, 4]).dynkin_diagram()
X---O---O=>=X
1 2 3 4
B4 with nodes (1, 4) marked
    ```

    And trivial relabelling are honoured nicely:

    ```
sage: T = T.marked_nodes([])
sage: T.dynkin_diagram()
O---O---O=>=O
1 2 3 4
B4
    ```

class ascii_art(label=<function CartanType.<lambda> at 0x7f3982447520>, node=None)

    Return an ascii art representation of this Cartan type.

    EXAMPLES:

    ```
sage: print(CartanType(['G', 2]).marked_nodes([2]).ascii_art())
3
0<=X
1 2
    ```
```

(continues on next page)
```python
sage: print(CartanType(['B', 3, 1]).marked_nodes([0, 3]).ascii_art())
\[\begin{array}{c}
X 0 \\
| \\
| \\
0--0=>=X \\
1 2 3
\end{array}\]
sage: print(CartanType(['F', 4, 1]).marked_nodes([0, 2]).ascii_art())
\[\begin{array}{cccc}
X---O---X=>=O---O \\
0 1 2 3 4
\end{array}\]
```

dual()

Implements `sage.combinat.root_system.cartan_type.CartanType_abstract.dual()`, using that taking the dual and marking nodes are commuting operations.

EXAMPLES:

```python
sage: T = CartanType(['BC',3, 2])
sage: T.marked_nodes([1,3]).dual().dynkin_diagram()
\[\begin{array}{c}
O=>=X---O=>=X \\
0 1 2 3
\end{array}\]
BC3~* with nodes (1, 3) marked
sage: T.dual().marked_nodes([1,3]).dynkin_diagram()
\[\begin{array}{c}
O=>=X---O=>=X \\
0 1 2 3
\end{array}\]
BC3~* with nodes (1, 3) marked
```

dynkin_diagram()

Return the Dynkin diagram for this Cartan type.

EXAMPLES:

```python
sage: CartanType(['G', 2]).marked_nodes([2]).dynkin_diagram()
\[\begin{array}{cc}
3 \\
O=<=X \\
1 2
\end{array}\]
G2 with node 2 marked
```

marked_nodes(marked_nodes)

Return self with nodes marked_nodes marked.

EXAMPLES:

```python
sage: ct = CartanType(['A',12])
sage: m = ct.marked_nodes([1,4,6,7,8,12]); m
['A', 12] with nodes (1, 4, 6, 7, 8, 12) marked
sage: m.marked_nodes([2])
['A', 12] with node 2 marked
sage: m.marked_nodes([]) is ct
True
```

relabel(relabelling)

Return the relabelling of self.

EXAMPLES:
```python
sage: T = CartanType(['BC', 3, 2])
sage: T.marked_nodes([1, 3]).relabel(lambda x: x+2).dynkin_diagram()
O=<=X---O=<=X
2 3 4 5
BC3~ relabelled by {0: 2, 1: 3, 2: 4, 3: 5} with nodes (3, 5) marked
sage: T.relabel(lambda x: x+2).marked_nodes([3, 5]).dynkin_diagram()
O=<=X---O=<=X
2 3 4 5
BC3~ relabelled by {0: 2, 1: 3, 2: 4, 3: 5} with nodes (3, 5) marked
```

### type()

Return the type of `self` or `None` if unknown.

**EXAMPLES:**

```python
sage: ct = CartanType(['F', 4]).marked_nodes([1, 3])
sage: ct.type()
'F'
```

### class sage.combinat.root_system.type_marked.CartanType_affine(ct, marked_nodes)

**Bases:** `CartanType`, `CartanType_affine`

#### basic_untwisted()

Return the basic untwisted Cartan type associated with this affine Cartan type.

Given an affine type $\Phi_n^{(r)}$, the basic untwisted type is $X_n$. In other words, it is the classical Cartan type that is twisted to obtain `self`.

**EXAMPLES:**

```python
sage: CartanType(['A', 7, 2]).marked_nodes([1, 3]).basic_untwisted()
['A', 7] with nodes (1, 3) marked
sage: CartanType(['D', 4, 3]).marked_nodes([0, 2]).basic_untwisted()
['D', 4] with node 2 marked
```

#### classical()

Return the classical Cartan type associated with `self`.

**EXAMPLES:**

```python
sage: T = CartanType(['A', 4, 1]).marked_nodes([0, 2, 4])
sage: T.dynkin_diagram()
0
X--
| |
| |
0---X---0---X
1 2 3 4
A4~ with nodes (0, 2, 4) marked
sage: T0 = T.classical()
sage: T0
['A', 4] with nodes (2, 4) marked
sage: T0.dynkin_diagram()
0---X---0---X
```

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**is_untwisted_affine()**

Implement `CartanType_affine.is_untwisted_affine()`.

A marked Cartan type is untwisted affine if the original is.

**EXAMPLES:**

```python
sage: CartanType(['B', 3, 1]).marked_nodes([1,3]).is_untwisted_affine()
True
```

**special_node()**

Return the special node of the Cartan type.

**See also:**

`special_node()`

It is the special node of the non-marked Cartan type.

**EXAMPLES:**

```python
sage: CartanType(['B', 3, 1]).marked_nodes([1,3]).special_node()
0
```

---

**class** `sage.combinat.root_system.type_marked.CartanType_finite(ct, marked_nodes)`

**Bases:** `CartanType`, `CartanType_finite`

**AmbientSpace**

alias of `AmbientSpace`

**affine()**

Return the affine Cartan type associated with `self`.

**EXAMPLES:**

```python
sage: B4 = CartanType(['B',4]).marked_nodes([1,3])
sage: B4.dynkin_diagram()
X---O---X=>=O
1 2 3 4
B4 with nodes (1, 3) marked
sage: B4.affine().dynkin_diagram()
  O 0
 /|
X---O---X=>=O
1 2 3 4
B4~ with nodes (1, 3) marked
```
5.1.267 Root system data for reducible Cartan types

```python
class sage.combinat.root_system.type_reducible.AmbientSpace(root_system, base_ring, index_set=None):

    Bases: AmbientSpace

    EXAMPLES:

sage: RootSystem("A2xB2").ambient_space()
Ambient space of the Root system of type A2xB2

ambient_spaces()
    Returns a list of the irreducible Cartan types of which the given reducible Cartan type is a product.

    EXAMPLES:

sage: RootSystem("A2xB2").ambient_space().ambient_spaces()
[Ambient space of the Root system of type ['A', 2], Ambient space of the Root system of type ['B', 2]]

cartan_type()
    EXAMPLES:

sage: RootSystem("A2xB2").ambient_space().cartan_type()
A2xB2

component_types()
    EXAMPLES:

sage: RootSystem("A2xB2").ambient_space().component_types()
[[['A', 2], ['B', 2]]

dimension()
    EXAMPLES:

sage: RootSystem("A2xB2").ambient_space().dimension()
5

fundamental_weights()
    EXAMPLES:

sage: RootSystem("A2xB2").ambient_space().fundamental_weights()
Finite family {1: (1, 0, 0, 0, 0), 2: (1, 1, 0, 0, 0), 3: (0, 0, 0, 1, 0), 4: (0, 0, 0, 1/2, 1/2)}

inject_weights(i, v)
    Produces the corresponding element of the lattice.

    INPUT:

    • i - an integer in range(self.components)
    • v - a vector in the i-th component weight lattice

    EXAMPLES:
```
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```python
sage: V = RootSystem("A2xB2").ambient_space()
sage: [V.inject_weights(i,V.ambient_spaces()[i].fundamental_weights()[1]) for i in range(2)]
[(1, 0, 0, 0, 0), (0, 0, 0, 1, 0)]
sage: [V.inject_weights(i,V.ambient_spaces()[i].fundamental_weights()[2]) for i in range(2)]
[(1, 1, 0, 0, 0), (0, 0, 0, 1/2, 1/2)]
```

negative_roots()
EXAMPLES:
```python
sage: RootSystem("A1xA2").ambient_space().negative_roots()
[(-1, 1, 0, 0, 0), (0, 0, -1, 1, 0), (0, 0, -1, 0, 1), (0, 0, 0, -1, 1)]
```

positive_roots()
EXAMPLES:
```python
sage: RootSystem("A1xA2").ambient_space().positive_roots()
[(1, -1, 0, 0, 0), (0, 0, 1, -1, 0), (0, 0, 1, 0, -1), (0, 0, 0, 1, -1)]
```

simple_coroot(i)
EXAMPLES:
```python
sage: A = RootSystem("A1xB2").ambient_space()
sage: A.simple_coroot(2)
(0, 0, 1, -1)
sage: A.simple_coroots()
Finite family {1: (1, -1, 0, 0), 2: (0, 0, 1, -1), 3: (0, 0, 0, 2)}
```

simple_root(i)
EXAMPLES:
```python
sage: A = RootSystem("A1xB2").ambient_space()
sage: A.simple_root(2)
(0, 0, 1, -1)
sage: A.simple_roots()
Finite family {1: (1, -1, 0, 0), 2: (0, 0, 1, -1), 3: (0, 0, 0, 1)}
```

class sage.combinat.root_system.type_reducible.CartanType(types)
Bases: SageObject, CartanType_abstract
A class for reducible Cartan types.

Reducible root systems are ones that can be factored as direct products. Strictly speaking type $D_2$ (corresponding to orthogonal groups of degree 4) is reducible since it is isomorphic to $A_1 \times A_1$. However type $D_2$ is not built using this class for our purposes.

INPUT:

* types – a list of simple Cartan types

EXAMPLES:
```python
sage: t1, t2 = [CartanType(x) for x in (["A",1], ["B",2])]
sage: CartanType([t1, t2])
(continues on next page)
```
A reducible Cartan type is finite (resp. crystallographic, simply laced) if all its components are:

```
sage: t = CartanType("A2xB2")
sage: t.is_finite()
True
sage: t.is_crystallographic()
True
sage: t.is_simply_laced()
False
```

This is implemented by inserting the appropriate abstract super classes (see `_add_abstract_superclass()`):

```
sage: t.__class__.mro()
[<class 'sage.combinat.root_system.type_reducible.CartanType_with_superclass'>, ...
    <class 'sage.combinat.root_system.type_reducible.CartanType'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
    <class 'sage.combinat.root_system.type_reducible.type_reducible'>, ...
]
```

The index set of the reducible Cartan type is obtained by relabelling successively the nodes of the Dynkin diagrams of the components by 1,2,...:

```
sage: t = CartanType(["A",4], ["BC",5,2], ["C",3])
sage: t.index_set()
(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)
sage: t.dynkin_diagram()
O---O---O---O
  1 2 3 4
O=<=O---O---O=<=O
  5 6 7 8 9 10
O---O=<=O
  11 12 13
A4xBC5~xC3
```

**AmbientSpace**

alias of **AmbientSpace**

**ascii_art** *(label=<function CartanType.<lambda> at 0x7f39824849d0>, node=None)*

Return an ascii art representation of this reducible Cartan type.

EXAMPLES:

```
sage: print(CartanType("F4xA2").ascii_art(label = lambda x: x+2))
0---0=>=0---0
  3 4 5 6
0---0
  7 8
sage: print(CartanType(["BC",5,2], ["A",4]).ascii_art())
```

(continues on next page)
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sage: print(CartanType(["A",4], ["BC",5,2], ["C",3]).ascii_art())
O=<=O---O---O---O=<=O
1 2 3 4 5 6
O---O---O---O
7 8 9 10
sage: print(CartanType(["A",4], ["BC",5,2], ["C",3]).ascii_art())
O---O---O---O
1 2 3 4
O=<=O---O---O---O=<=O
5 6 7 8 9 10
O---O=<=O
11 12 13

cartan_matrix(subdivide=True)

Return the Cartan matrix associated with self. By default the Cartan matrix is a subdivided block matrix showing the reducibility but the subdivision can be suppressed with the option subdivide = False.

EXAMPLES:

sage: ct = CartanType("A2","B2")
sage: ct.cartan_matrix()  
[ 2 -1  0  0]
[-1  2  0  0]
[-       -]
[ 0  0  2 -1]
[ 0  0  2  2]
sage: ct.cartan_matrix(subdivide=False)
[ 2 -1  0  0]
[-1  2  0  0]
[ 0  0  2 -1]
[ 0  0  2  2]
sage: ct.index_set() == ct.cartan_matrix().index_set()
True

component_types()

A list of Cartan types making up the reducible type.

EXAMPLES:

sage: CartanType(["A",2],["B",2]).component_types()
[["A", 2], ["B", 2]]

coxeter_diagram()

Return the Coxeter diagram for self.

EXAMPLES:

sage: cd = CartanType("A2xB2xF4").coxeter_diagram()
sage: cd
Graph on 8 vertices
sage: cd.edges(sort=True)
[(1, 2, 3), (3, 4, 4), (5, 6, 3), (6, 7, 4), (7, 8, 3)]
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(continued from previous page)

```python
sage: CartanType("F4xA2").coxeter_diagram().edges(sort=True)
[(1, 2, 3), (2, 3, 4), (3, 4, 3), (5, 6, 3)]

sage: cd = CartanType("A1xH3").coxeter_diagram(); cd
Graph on 4 vertices
sage: cd.edges(sort=True)
[(2, 3, 3), (3, 4, 5)]

dual()

EXAMPLES:

```python
sage: CartanType("A2xB2").dual()
A2xC2
```

dynkin_diagram()

Returns a Dynkin diagram for type reducible.

EXAMPLES:

```python
sage: dd = CartanType("A2xB2xF4").dynkin_diagram()
sage: dd
O---O
1 2
O=>=O
3 4
O---O=>=O---O
5 6 7 8
A2xB2xF4
sage: dd.edges(sort=True)
[(1, 2, 1), (2, 1, 1), (3, 4, 2), (4, 3, 1), (5, 6, 1), (6, 5, 1), (6, 7, 2), ...
   (7, 6, 1), (7, 8, 1), (8, 7, 1)]

sage: CartanType("F4xA2").dynkin_diagram()
O---O
1 2
3 4
O---O
5 6
F4xA2
```

index_set()

Implements `CartanType_abstract.index_set()`.

For the moment, the index set is always of the form `{1, ..., n}`.

EXAMPLES:

```python
sage: CartanType("A2","A1").index_set()
(1, 2, 3)
```

is_affine()

Report that this reducible Cartan type is not affine

EXAMPLES:
5.1.268 Root system data for relabelled Cartan types

```python
sage: CartanType(['A',2],['B',2]).is_affine()
False

is_finite()
EXAMPLES:
```
```python
sage: cti = CartanType(['A',2],['B',2])
sage: cti.is_finite()
True
sage: ct2 = CartanType(['A',2],['B',2,1])
sage: ct2.is_finite()
False
```

is_irreducible()
Report that this Cartan type is not irreducible.
EXAMPLES:
```
```
```python
sage: ct = CartanType(['A',2],['B',2])
sage: ct.is_irreducible()
False
```

rank()
Returns the rank of self.
EXAMPLES:
```
```
```python
sage: CartanType("A2","A1").rank()
3
```

type()
Returns "reducible" since the type is reducible.
EXAMPLES:
```
```
```python
sage: CartanType(['A',2],['B',2]).type()
'reducible'
```

5.1.268 Root system data for relabelled Cartan types

```python
class sage.combinat.root_system.type_relabel.AmbientSpace(root_system, base_ring, index_set=None)
Bases: AmbientSpace
Ambient space for a relabelled finite Cartan type.
It is constructed in the canonical way from the ambient space of the original Cartan type, by relabelling the simple roots, fundamental weights, etc.
EXAMPLES:
```
```python
sage: cycle = {1:2, 2:3, 3:4, 4:1}
sage: L = CartanType("F",4).relabel(cycle).root_system().ambient_space(); L
Ambient space of the Root system of type ['F', 4] relabelled by {1: 2, 2: 3, 3: 4, 4: 1}
```
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4: 1}
sage: TestSuite(L).run()

**dimension()**
Return the dimension of this ambient space.

See also:
sage.combinat.root_system.ambient_space.AmbientSpace.dimension()

**fundamental_weight(i)**
Return the i-th fundamental weight.

It is constructed by looking up the corresponding simple coroot in the ambient space for the original Cartan type.

**simple_root(i)**
Return the i-th simple root.

It is constructed by looking up the corresponding simple coroot in the ambient space for the original Cartan type.

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(1/2, -1/2, -1/2, -1/2)

```
sage: L.simple_roots()
Finite family {1: (1/2, -1/2, -1/2, -1/2), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1),
        4: (0, 0, 0, 1)}
```

```
sage: L.simple_coroots()
Finite family {1: (1, -1, -1, -1), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1), 4: (0, 0, 0, 2)}
```

class sage.combinat.root_system.type_relabel.CartanType(type, relabelling)

Bases: CartanType_decorator

A class for relabelled Cartan types.

```
ascii_art(label=<function CartanType.<lambda> at 0x7f3982485750>, node=None)

Return an ascii art representation of this Cartan type.

EXAMPLES:
```
```
sage: print(CartanType(['G', 2]).relabel({1:2,2:1}).ascii_art())

3

O=<=O

2 1
```
```
sage: print(CartanType(['B', 3, 1]).relabel([1,3,2,0]).ascii_art())

O 1

| |

O---O=>=O

3 2 0
```
```
sage: print(CartanType(['F', 4, 1]).relabel(lambda n: 4-n).ascii_art())

O---O---O=>=O---O

4 3 2 1 0
```

coxeter_diagram()

Return the Coxeter diagram for self.

EXAMPLES:
```
sage: ct = CartanType(['H', 3]).relabel({1:3,2:2,3:1})
sage: G = ct.coxeter_diagram(); G
Graph on 3 vertices
sage: G.edges(sort=True)
[(1, 2, 5), (2, 3, 3)]
```

dual()

Implements sage.combinat.root_system.cartan_type.CartanType_abstract.dual(), using that taking the dual and relabelling are commuting operations.

EXAMPLES:
```
sage: T = CartanType(['BC',3, 2])
sage: cycle = {1:2, 2:3, 3:0, 0:1}
sage: T.relabel(cycle).dual().dynkin_diagram()
```

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```
O=>=O---O=>=O
1  2  3  0
BC3~* relabelled by {0: 1, 1: 2, 2: 3, 3: 0}
sage: T.dual().relabel(cycle).dynkin_diagram()
O=>=O---O=>=O
1  2  3  0
BC3~* relabelled by {0: 1, 1: 2, 2: 3, 3: 0}
```

dynkin_diagram()

Returns the Dynkin diagram for this Cartan type.

EXAMPLES:

```
sage: CartanType(['G', 2]).relabel({1:2,2:1}).dynkin_diagram()
3
O=<=O
2  1
G2 relabelled by {1: 2, 2: 1}
```

index_set()

EXAMPLES:

```
sage: ct = CartanType(['G', 2]).relabel({1:2,2:1})
sage: ct.index_set()
(1, 2)
```

type()

Return the type of self or None if unknown.

EXAMPLES:

```
sage: ct = CartanType(['G', 2]).relabel({1:2,2:1})
sage: ct.type()
'G'
```

class sage.combinat.root_system.type_relabel.CartanType_affine(type, relabelling)

Bases: CartanType, CartanType_affine

basic_untwisted()

Return the basic untwisted Cartan type associated with this affine Cartan type.

Given an affine type $X_n^{(r)}$, the basic untwisted type is $X_n$. In other words, it is the classical Cartan type that is twisted to obtain self.

EXAMPLES:

```
sage: ct = CartanType(['A', 5, 2]).relabel({0:1, 1:0, 2:2, 3:3})
sage: ct.basic_untwisted()
['A', 5]
```

classical()

Return the classical Cartan type associated with self.

EXAMPLES:
sage: A41 = CartanType(['A',4,1])
sage: A41.dynkin_diagram()
O-----------+
| |
| |
0--0---0---0
1 2 3 4
A4~
sage: T = A41.relabel({0:1, 1:2, 2:3, 3:4, 4:0})
sage: T
['A', 4, 1] relabelled by {0: 1, 1: 2, 2: 3, 3: 4, 4: 0}
sage: T.dynkin_diagram()
1
O-----------+
| |
| |
0--0---0---0
2 3 4 0
A4~ relabelled by {0: 1, 1: 2, 2: 3, 3: 4, 4: 0}
sage: T0 = T.classical()
sage: T0
['A', 4] relabelled by {1: 2, 2: 3, 3: 4, 4: 0}
sage: T0.dynkin_diagram()
O---O---O---O
2 3 4 0
A4 relabelled by {1: 2, 2: 3, 3: 4, 4: 0}

is_untwisted_affine()
Implement CartanType_affine.is_untwisted_affine()
A relabelled Cartan type is untwisted affine if the original is.

EXAMPLES:
sage: CartanType(['B', 3, 1]).relabel({1:2, 2:3, 3:0, 0:1}).is_untwisted_affine()
True

special_node()
Returns a special node of the Dynkin diagram

See also:
special_node()
It is obtained by relabelling of the special node of the non relabelled Dynkin diagram.

EXAMPLES:
sage: CartanType(['B', 3, 1]).special_node()
0
sage: CartanType(['B', 3, 1]).relabel({1:2, 2:3, 3:0, 0:1}).special_node()
1
class sage.combinat.root_system.type_relabel.CartanType_finite(type, relabelling)

Bases: CartanType, CartanType_finite

AmbientSpace
  alias of AmbientSpace

affine()

Return the affine Cartan type associated with self.

EXAMPLES:

```
sage: B4 = CartanType(['B',4])
sage: B4.dynkin_diagram()
O---O---O=>=O
1 2 3 4
B4
sage: B4.affine().dynkin_diagram()
   0
  |
  |
0---0---0=>=0
1 2 3 4
B4~
```

If possible, this reuses the original label for the special node:

```
sage: T = B4.relabel({1:2, 2:3, 3:4, 4:1}); T.dynkin_diagram()
O---O---O=>=O
2 3 4 1
B4 relabelled by {1: 2, 2: 3, 3: 4, 4: 1}
sage: T.affine().dynkin_diagram()
   0
  |
  |
0---0---0=>=0
2 3 4 1
B4~ relabelled by {0: 0, 1: 2, 2: 3, 3: 4, 4: 1}
```

Otherwise, it chooses a label for the special node in 0, 1, ...:

```
sage: T = B4.relabel({1:0, 2:1, 3:2, 4:3}); T.dynkin_diagram()
O---O---O=>=O
0 1 2 3
B4 relabelled by {1: 0, 2: 1, 3: 2, 4: 3}
sage: T.affine().dynkin_diagram()
   0
  |
  |
0---0---0=>=0
0 1 2 3
B4~ relabelled by {0: 4, 1: 0, 2: 1, 3: 2, 4: 3}
```

This failed before github issue #13724:
sage: ct = CartanType(['G',2]).dual(); ct
['G', 2] relabelled by {1: 2, 2: 1}
sage: ct.affine()
['G', 2, 1] relabelled by {0: 0, 1: 2, 2: 1}

sage: ct = CartanType(['F',4]).dual(); ct
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}
sage: ct.affine()
['F', 4, 1] relabelled by {0: 0, 1: 4, 2: 3, 3: 2, 4: 1}

Check that we don’t inadvertently change the internal relabelling of ct:

sage: ct
['F', 4] relabelled by {1: 4, 2: 3, 3: 2, 4: 1}

5.1.269 Weight lattice realizations

class sage.combinat.root_system.weight_lattice_realizations.WeightLatticeRealizations(base, name=None)

Bases: Category_over_base_ring

The category of weight lattice realizations over a given base ring

A weight lattice realization $L$ over a base ring $R$ is a free module (or vector space if $R$ is a field) endowed with an embedding of the root lattice of some root system. By restriction, this embedding defines an embedding of the root lattice of this root system, which makes $L$ a root lattice realization.

Typical weight lattice realizations over $\mathbb{Z}$ include the weight lattice, and ambient lattice. Typical weight lattice realizations over $\mathbb{Q}$ include the weight space, and ambient space.

To describe the embedding, a weight lattice realization must implement a method fundamental_weight($i$) returning for each $i$ in the index set the image of the fundamental weight $\Lambda_i$ under the embedding.

In order to be a proper root lattice realization, a weight lattice realization should also implement the scalar product with the coroot lattice; on the other hand, the embedding of the simple roots is given for free.

See also:
- RootSystem
- RootLatticeRealizations
- WeightSpace
- AmbientSpace

EXAMPLES:
Here, we consider the root system of type $A_7$, and embed the weight lattice element $x = \Lambda_1 + 2\Lambda_3$ in several root lattice realizations:

sage: R = RootSystem(['A',7])
sage: Lambda = R.weight_lattice().fundamental_weights()
sage: L = R.weight_space()
We embed the weight space element $x = \Lambda_1 + \frac{1}{2}\Lambda_3$ in the ambient space:

```python
sage: Lambda = R.weight_space().fundamental_weights()
sage: x = Lambda[2] + \frac{1}{2} * Lambda[5]
sage: L = R.ambient_space()
sage: L(x)
(3/2, 3/2, 1/2, 1/2, 1/2, 0, 0, 0)
```

Of course, one can't embed the weight space in the ambient lattice:

```python
sage: L = R.ambient_lattice()
sage: L(x)
Traceback (most recent call last):
...
TypeError: do not know how to make x (= Lambda[2] + 1/2*Lambda[5]) an element of...
˓→self (=Ambient lattice of the Root system of type ['A', 7])
```

If $K_1$ is a subring of $K_2$, then one could in theory have an embedding from the weight space over $K_1$ to any weight lattice realization over $K_2$; this is not implemented:

```python
sage: K1 = QQ
sage: K2 = QQ['q']
sage: L = R.ambient_space(K2)
sage: Lambda = R.weight_space(K2).fundamental_weights()
sage: L(Lambda[1])
(1, 0, 0, 0, 0, 0, 0, 0)
```

```python
sage: Lambda = R.weight_space(K1).fundamental_weights()
sage: L(Lambda[1])
Traceback (most recent call last):
...
TypeError: do not know how to make x (= Lambda[1]) an element of self (=Ambient...
˓→space of the Root system of type ['A', 7])
```

```python
class ElementMethods
    Bases: object

    symmetric_form(la)
        Return the symmetric form of self with la.

        Return the pairing $\langle \rangle$ on the weight lattice. See Chapter 6 in Kac, Infinite Dimensional Lie Algebras for more details.
```
**Warning:** For affine root systems, if you are not working in the extended weight lattice/space, this may return incorrect results.

**EXAMPLES:**

```python
sage: P = RootSystem(['C',2]).weight_lattice()
sage: al = P.simple_roots()
sage: al[1].symmetric_form(al[1])
2
sage: al[1].symmetric_form(al[2])
-2
sage: al[2].symmetric_form(al[1])
-2
sage: Q = RootSystem(['C',2]).root_lattice()
sage: alQ = Q.simple_roots()
sage: all(al[i].symmetric_form(al[j]) == alQ[i].symmetric_form(alQ[j])
....:     for i in P.index_set() for j in P.index_set())
True

sage: P = RootSystem(['C',2,1]).weight_lattice(extended=True)
sage: al = P.simple_roots()
sage: al[1].symmetric_form(al[1])
2
sage: al[1].symmetric_form(al[2])
-2
sage: al[1].symmetric_form(al[0])
-2
sage: al[0].symmetric_form(al[1])
-2
sage: Q = RootSystem(['C',2,1]).root_lattice()
sage: alQ = Q.simple_roots()
sage: all(al[i].symmetric_form(al[j]) == alQ[i].symmetric_form(alQ[j])
....:     for i in P.index_set() for j in P.index_set())
True
sage: La = P.basis()
sage: [La['delta'].symmetric_form(al) for al in P.simple_roots()]
[0, 0, 0]
sage: [La[0].symmetric_form(al) for al in P.simple_roots()]
[1, 0, 0]

sage: P = RootSystem(['C',2,1]).weight_lattice()
sage: Q = RootSystem(['C',2,1]).root_lattice()
sage: al = P.simple_roots()
sage: alQ = Q.simple_roots()
sage: all(al[i].symmetric_form(al[j]) == alQ[i].symmetric_form(alQ[j])
....:     for i in P.index_set() for j in P.index_set())
True
```

The result of \((\Lambda_0 | \alpha_0)\) should be 1, however we get 0 because we are not working in the extended weight lattice:

```python
sage: La = P.basis()
```

(continues on next page)
\begin{verbatim}
sage: [La[0].symmetric_form(al) for al in P.simple_roots()][0, 0, 0]

to_weight_space(base_ring=None)
Map self to the weight space.

Warning: Implemented for finite Cartan type.

EXAMPLES:
sage: b = CartanType(['B',2]).root_system().ambient_space().from_
     ...vector(vector([1,-2])); b
(1, -2)
sage: b.to_weight_space()
sage: b = CartanType(['B',2]).root_system().ambient_space().from_
     ...vector(vector([1/2,0])); b
(1/2, 0)
sage: b.to_weight_space()
1/2*Lambda[1]
sage: b.to_weight_space(ZZ)
Traceback (most recent call last): ...
TypeError: no conversion of this rational to integer
sage: b = CartanType(['G',2]).root_system().ambient_space().from_
     ...vector(vector([4,-5,1])); b
(4, -5, 1)
sage: b.to_weight_space()
\end{verbatim}

\begin{verbatim}
class ParentMethods
    Bases: object

dynkin_diagram_automorphism_of_alcove_morphism(f)
Return the Dynkin diagram automorphism induced by an alcove morphism

INPUT:
• f - a linear map from self to self which preserves alcoves
This method returns the Dynkin diagram automorphism for the decomposition
\( f = dw \) (see reduced_word_of_alcove_morphism()), as a dictionary mapping elements
of the index set to itself.

EXAMPLES:
sage: R = RootSystem(['A',2,1]).weight_lattice()
sage: alpha = R.simple_roots()
sage: Lambda = R.fundamental_weights()

Translations by elements of the root lattice induce a trivial Dynkin diagram automorphism:
sage: R.dynkin_diagram_automorphism_of_alcove_morphism(alpha[0].translation)
{0: 0, 1: 1, 2: 2}
\end{verbatim}
This is no more the case for translations by general elements of the (classical) weight lattice at level 0:

```
sage: omega1 = Lambda[1] - Lambda[0]
sage: omega2 = Lambda[2] - Lambda[0]

sage: R = RootSystem(['C',2,1]).weight_lattice()
sage: alpha = R.simple_roots()
sage: R.dynkin_diagram_automorphism_of_alcove_morphism(omega1.translation)
{0: 1, 1: 0, 2: 2}
sage: R.dynkin_diagram_automorphism_of_alcove_morphism(omega2.translation)
{0: 2, 1: 0, 2: 1}
```

Algorithm: computes $w$ of the decomposition, and see how $f \circ w^{-1}$ permutes the simple roots.

```
embed_at_level(x, level=1)
```

Embed the classical weight $x$ in the level $level$ hyperplane

This is achieved by translating the straightforward embedding of $x$ by $c\Lambda_0$ for $c$ some appropriate scalar.

INPUT:

- $x$ – an element of the corresponding classical weight/ambient lattice
- $level$ – an integer or element of the base ring (default: 1)

EXAMPLES:

```
sage: L = RootSystem(['B',3,1]).weight_space()
sage: L0 = L.classical()
sage: alpha = L0.simple_roots()
sage: omega = L0.fundamental_weights()
sage: L.embed_at_level(omega[1], 1)
Lambda[1]
sage: L.embed_at_level(omega[2], 1)
-Lambda[0] + Lambda[2]
sage: L.embed_at_level(omega[3], 1)
Lambda[3]
sage: L.embed_at_level(alpha[1], 1)
```
**fundamental_weight(i)**

Returns the $i^{th}$ fundamental weight

INPUT:
- $i$ – an element of the index set

By a slight notational abuse, for an affine type this method should also accept "delta" as input, and return the image of $\delta$ of the extended weight lattice in this realization.

This should be overridden by any subclass, and typically be implemented as a cached method for efficiency.

**EXAMPLES:**

```python
sage: L = RootSystem(['A',3]).ambient_lattice()
sage: L.fundamental_weight(1)
(1, 0, 0, 0)
sage: L = RootSystem(['A',3,1]).weight_lattice(extended=True)
sage: L.fundamental_weight(1)
Lambda[1]
sage: L.fundamental_weight("delta")
delta
```

**fundamental_weights()**

Returns the family $(\Lambda_i)_{i \in I}$ of the fundamental weights.

**EXAMPLES:**

```python
sage: e = RootSystem(['A',3]).ambient_lattice()
sage: f = e.fundamental_weights()
sage: [f[i] for i in [1,2,3]]
[(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0)]
```

**is_extended()**

Return whether this is a realization of the extended weight lattice

**See also:**
`sage.combinat.root_system.weight_space.WeightSpace`

**EXAMPLES:**

```python
sage: RootSystem(['A',3,1]).weight_lattice().is_extended()
False
sage: RootSystem(['A',3,1]).weight_lattice(extended=True).is_extended()
True
```

This method is irrelevant for finite root systems, since the weight lattice need not be extended to ensure that the root lattice embeds faithfully:

```python
sage: RootSystem(['A',3]).weight_lattice().is_extended()
False
```

**reduced_word_of_alcove_morphism(f)**

Return the reduced word of an alcove morphism.

**INPUT:**
- $f$ – a linear map from self to self which preserves alcoves

---

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Let $A$ be the fundamental alcove. This returns a reduced word $i_1, \ldots, i_k$ such that the affine Weyl group element $w = s_{i_1} \circ \cdots \circ s_{i_k}$ maps the alcove $f(A)$ back to $A$. In other words, the alcove walk $i_1, \ldots, i_k$ brings the fundamental alcove to the corresponding translated alcove.

Let us throw in a bit of context to explain the main use case. It is customary to realize the alcove picture in the coroot or coweight lattice $R^\vee$. The extended affine Weyl group is then the group of linear maps on $R^\vee$ which preserve the alcoves. By [Kac “Infinite-dimensional Lie algebra”, Proposition 6.5] the affine Weyl group is the semidirect product of the associated finite Weyl group and the group of translations in the coroot lattice (the extended affine Weyl group uses the coweight lattice instead). In other words, an element of the extended affine Weyl group admits a unique decomposition of the form:

$$f = dw,$$

where $w$ is in the Weyl group, and $d$ is a function which maps the fundamental alcove to itself. As $d$ permutes the walls of the fundamental alcove, it permutes accordingly the corresponding simple roots, which induces an automorphism of the Dynkin diagram.

This method returns a reduced word for $w$, whereas the method `dynkin_diagram_automorphism_of_alcove_morphism()` returns $d$ as a permutation of the nodes of the Dynkin diagram.

Nota bene: recall that the coroot (resp. coweight) lattice is implemented as the root (resp weight) lattice of the dual root system. Hence, this method is implemented for weight lattice realizations, but in practice is most of the time used on the dual side.

EXAMPLES:

We start with type $A$ which is simply laced; hence we do not have to worry about the distinction between the weight and coweight lattice:

```python
sage: R = RootSystem(["A",2,1]).weight_lattice()
sage: alpha = R.simple_roots()
sage: Lambda = R.fundamental_weights()
```

We consider first translations by elements of the root lattice:

```python
sage: R.reduced_word_of_alcove_morphism(alpha[0].translation) [1, 2, 1, 0]
sage: R.reduced_word_of_alcove_morphism(alpha[1].translation) [0, 2, 0, 1]
sage: R.reduced_word_of_alcove_morphism(alpha[2].translation) [0, 1, 0, 2]
```

We continue with translations by elements of the classical weight lattice, embedded at level 0:

```python
```

```python
sage: R.reduced_word_of_alcove_morphism(omega1.translation) [0, 2] sage: R.reduced_word_of_alcove_morphism(omega1.translation) [0, 1]
```

The following tests ensure that the code agrees with the tables in Kashiwara’s private notes on affine quantum algebras (2008).

`reduced_word_of_translation(t)`

Given an element of the root lattice, this returns a reduced word $i_1, \ldots, i_k$ such that the Weyl group element $s_{i_1} \circ \cdots \circ s_{i_k}$ implements the “translation” where $x$ maps to $x + \text{level}(x) \ast t$. In other words, the alcove walk $i_1, \ldots, i_k$ brings the fundamental alcove to the corresponding translated alcove.

**Note:** There are some technical conditions for $t$ to actually be a translation; those are not tested
(TODO: detail).

EXAMPLES:

```
sage: R = RootSystem(['A',2,1]).weight_lattice()
sage: alpha = R.simple_roots()
sage: R.reduced_word_of_translation(alpha[1])
[0, 2, 0, 1]
sage: R.reduced_word_of_translation(alpha[2])
[0, 1, 0, 2]
sage: R.reduced_word_of_translation(alpha[0])
[1, 2, 1, 0]

sage: R = RootSystem(['D',5,1]).weight_lattice()
sage: Lambda = R.fundamental_weights()
sage: omega1 = Lambda[1] - Lambda[0]
sage: R.reduced_word_of_translation(omega1)
[0, 2, 3, 4, 5, 3, 2, 0]
sage: R.reduced_word_of_translation(omega2)
[0, 2, 1, 3, 2, 4, 3, 5, 3, 2, 1, 4, 3, 2]
```

A non simply laced case:

```
sage: R = RootSystem(['C',2,1]).weight_lattice()
sage: Lambda = R.fundamental_weights()
sage: c = R.cartan_type().translation_factors()
sage: c
Finite family {0: 1, 1: 2, 2: 1}
sage: R.reduced_word_of_translation((Lambda[1]-Lambda[0]) * c[1])
[0, 1, 2, 1]
sage: R.reduced_word_of_translation((Lambda[2]-Lambda[0]) * c[2])
[0, 1, 0]
```

See also _test_reduced_word_of_translation().

Todo:
- Add a picture in the doc
- Add a method which, given an element of the classical weight lattice, constructs the appropriate value for $t$

**rho()**

EXAMPLES:

```
sage: RootSystem(['A',3]).ambient_lattice().rho()
(3, 2, 1, 0)
```

**rho_classical()**

Return the embedding at level 0 of $\rho$ of the classical lattice.

EXAMPLES:
signs_of_alcovewalk(walk)

Let \( \text{walk} = [i_1, \ldots, i_n] \) denote an alcove walk starting from the fundamental alcove \( y_0 \), crossing at step 1 the wall \( i_1 \), and so on.

For each \( k \), set \( w_k = s_{i_1} \circ s_{i_k} \), and denote by \( y_k = w_k(y_0) \) the alcove reached after \( k \) steps. Then, \( y_k \) is obtained recursively from \( y_{k-1} \) by applying the following reflection:

\[
y_k = s_{w_{k-1}^{-1} \alpha_{i_k}} y_{k-1}.
\]

The step is said positive if \( w_{k-1}^{-1} \alpha_{i_k} \) is a negative root (considering \( w_{k-1} \) as element of the classical Weyl group and \( \alpha_{i_k} \) as a classical root) and negative otherwise. The algorithm implemented here use the equivalent property:

\[
\langle w_{k-1}^{-1} \rho_0, \alpha^\vee_{i_k} \rangle > 0
\]

Where \( \rho_0 \) is the sum of the classical fundamental weights embedded at level 0 in this space (see \( \text{rho_classical()} \)), and \( \alpha^\vee_{i_k} \) is the simple coroot associated to \( \alpha_{i_k} \).

This function returns a list of the form \([+1, +1, -1, \ldots]\), where the \( k^{th} \) entry denotes whether the \( k^{th} \) step was positive or negative.

See equation 3.4, of Ram: Alcove walks ..., arXiv math/0601343v1

EXAMPLES:

```python
sage: L = RootSystem(['C',2,1]).weight_lattice()
sage: L.signs_of_alcovewalk([1,2,0,1,2,1,2,0,1,2])
[-1, -1, 1, 1, 1, 1, 1, 1]
```
**Warning:** This method currently does not work in the weight lattice for type BC dual because $\rho_0$ does not live in this lattice (but an integral multiple of it would do the job as well).

**simple_root($i$)**

Returns the $i$-th simple root

This default implementation takes the $i$-th simple root in the weight lattice and embeds it in `self`.

**EXAMPLES:**

Since all the weight lattice realizations in Sage currently implement a `simple_root` method, we have to call this one by hand:

```python
sage: from sage.combinat.root_system.weight_lattice_realizations import...
...
WeightLatticeRealizations
sage: simple_root = WeightLatticeRealizations(QQ).parent_class.simple_root.f
sage: L = RootSystem("A3").ambient_space()
sage: simple_root(L, 1)
(1, -1, 0, 0)
sage: simple_root(L, 2)
(0, 1, -1, 0)
sage: simple_root(L, 3)
(1, 1, 2, 0)
```

Note that this last root differs from the one implemented in `L` by a multiple of the vector $(1, 1, 1, 1)$:

```python
sage: L.simple_roots()
Finite family {1: (1, -1, 0, 0), 2: (0, 1, -1, 0), 3: (0, 0, 1, -1)}
```

This is a harmless artefact of the $SL$ versus $GL$ interpretation of type $A$; see the thematic tutorial on Lie Methods and Related Combinatorics in Sage for details.

**weyl_dimension($highest_weight$)**

Return the dimension of the highest weight representation of highest weight `highest_weight`.

**EXAMPLES:**

```python
sage: RootSystem(['A',3]).ambient_lattice().weyl_dimension([2,1,0,0])
20
sage: P = RootSystem(['C',2]).weight_lattice()
sage: La = P.basis()
sage: P.weyl_dimension(La[1]+La[2])
16
sage: type(RootSystem(['A',3]).ambient_lattice().weyl_dimension([2,1,0,0]))
<class 'sage.rings.integer.Integer'>
```

**super_categories()**

**EXAMPLES:**

```python
sage: from sage.combinat.root_system.weight_lattice_realizations import...
...
WeightLatticeRealizations
sage: WeightLatticeRealizations(QQ).super_categories()
[Category of root lattice realizations over Rational Field]
```
5.1.270 Weight lattices and weight spaces

class sage.combinat.root_system.weight_space.WeightSpace(root_system, base_ring, extended)

Bases: CombinatorialFreeModule

INPUT:

• root_system – a root system
• base_ring – a ring $R$
• extended – a boolean (default: False)

The weight space (or lattice if base_ring is $\mathbb{Z}$) of a root system is the formal free module $\bigoplus_i R\Lambda_i$ generated by the fundamental weights $(\Lambda_i)_{i \in I}$ of the root system.

This class is also used for coweight spaces (or lattices).

See also:

• RootSystem()
• RootSystem.weight_lattice() and RootSystem.weight_space()
• WeightLatticeRealizations()

EXAMPLES:

sage: Q = RootSystem(['A', 3]).weight_lattice(); Q
Weight lattice of the Root system of type ['A', 3]
sage: Q.simple_roots()
sage: Q = RootSystem(['A', 3, 1]).weight_lattice(); Q
Weight lattice of the Root system of type ['A', 3, 1]
sage: Q.simple_roots()
Finite family {0: 2*Lambda[0] - Lambda[1] - Lambda[3],
1: -Lambda[0] + 2*Lambda[1] - Lambda[2],

For infinite types, the Cartan matrix is singular, and therefore the embedding of the root lattice is not faithful:

sage: sum(Q.simple_roots())
0

In particular, the null root is zero:

sage: Q.null_root()
0

This can be compensated by extending the basis of the weight space and slightly deforming the simple roots to make them linearly independent, without affecting the scalar product with the coroots. This feature is currently only implemented for affine types. In that case, if extended is set, then the basis of the weight space is extended by an element $\delta$: 

\[ \Lambda_i + \delta \]
```python
sage: Q = RootSystem(['A', 3, 1]).weight_lattice(extended = True); Q
Extended weight lattice of the Root system of type ['A', 3, 1]
sage: Q.basis().keys()
{0, 1, 2, 3, 'delta'}
```

And the simple root $\alpha_0$ associated to the special node is deformed as follows:

```python
sage: Q.simple_roots()
   1: -Lambda[0] + 2*Lambda[1] - Lambda[2],
```

Now, the null root is nonzero:

```python
sage: Q.null_root()
delta
```

**Warning:** By a slight notational abuse, the extra basis element used to extend the fundamental weights is called $\delta$ in the current implementation. However, in the literature, $\delta$ usually denotes instead the null root. Most of the time, those two objects coincide, but not for type $BC$ (aka. $A_{2n}^{(2)}$). Therefore we currently have:

```python
sage: Q = RootSystem(['A',4,2]).weight_lattice(extended = True)
sage: Q.simple_root(0)
2*Lambda[0] - Lambda[1] + delta
sage: Q.null_root()
2*delta
```

whereas, with the standard notations from the literature, one would expect to get respectively $2\Lambda_0 - \Lambda_1 + 1/2\delta$ and $\delta$.

Other than this notational glitch, the implementation remains correct for type $BC$.

The notations may get improved in a subsequent version, which might require changing the index of the extra basis element. To guarantee backward compatibility in code not included in Sage, it is recommended to use the following idiom to get that index:

```python
sage: F = Q.basis_extension(); F
Finite family {'delta': delta}
sage: index = F.keys()[0]; index
'delta'
```

Then, for example, the coefficient of an element of the extended weight lattice on that basis element can be recovered with:

```python
sage: Q.null_root()[index]
2
```

**Element**

alias of `WeightSpaceElement`

**basis_extension()**

Return the basis elements used to extend the fundamental weights
EXAMPLES:

```python
sage: Q = RootSystem(\"A\",3,1).weight_lattice()
sage: Q.basis_extension()
Family ()
sage: Q = RootSystem(\"A\",3,1).weight_lattice(extended=True)
sage: Q.basis_extension()
Finite family {\'delta\': \delta}
```

This method is irrelevant for finite types:

```python
sage: Q = RootSystem(\"A\",3).weight_lattice()
sage: Q.basis_extension()
Family ()
```

**fundamental_weight(i)**

Returns the $i$-th fundamental weight

**INPUT:**

- $i$ – an element of the index set or "delta"

By a slight notational abuse, for an affine type this method also accepts "delta" as input, and returns the image of $\delta$ of the extended weight lattice in this realization.

**See also:**

fundamental_weight()

**EXAMPLES:**

```python
sage: Q = RootSystem(\"A\",3).weight_lattice()
sage: Q.fundamental_weight(1)
Lambda[1]
sage: Q = RootSystem(\"A\",3,1).weight_lattice(extended=True)
sage: Q.fundamental_weight(1)
Lambda[1]
sage: Q.fundamental_weight("delta")
delta
```

**is_extended()**

Return whether this is an extended weight lattice.

**See also:**

is_extended()

**EXAMPLES:**

```python
sage: RootSystem(\"A\",3,1).weight_lattice().is_extended()
False
sage: RootSystem(\"A\",3,1).weight_lattice(extended=True).is_extended()
True
```

**simple_root(j)**

Returns the $j$-th simple root
EXAMPLES:

```python
sage: L = RootSystem(["C",4]).weight_lattice()
sage: L.simple_root(3)
```

Its coefficients are given by the corresponding column of the Cartan matrix:

```python
sage: L.cartan_type().cartan_matrix()[:,2]
[ 0]
[-1]
[ 2]
[-1]
```

Here are all simple roots:

```python
sage: L.simple_roots()
Finite family {1: 2*Lambda[1] - Lambda[2],
```

For the extended weight lattice of an affine type, the simple root associated to the special node is deformed by adding $\delta$, where $\delta$ is the null root:

```python
sage: L = RootSystem(["C",4,1]).weight_lattice(extended=True)
sage: L.simple_root(0)
```

In fact $\delta$ is really $1/a_0$ times the null root (see the discussion in `WeightSpace`) but this only makes a difference in type $BC$:

```python
sage: L = RootSystem(CartanType(["BC",4,2])).weight_lattice(extended=True)
sage: L.simple_root(0)
sage: L.null_root()
2*delta
```

See also:

- `simple_root()`
- `CartanType.col_annihilator()`
- `to_ambient_space_morphism()`

The morphism from `self` to its associated ambient space.

EXAMPLES:

```python
sage: CartanType(["A",2]).root_system().weight_lattice().to_ambient_space_morphism()
Generic morphism:
From: Weight lattice of the Root system of type ['A', 2]
To:    Ambient space of the Root system of type ['A', 2]
```
Warning: Implemented only for finite Cartan type.

class sage.combinat.root_system.weight_space.WeightSpaceElement
Bases: IndexedFreeModuleElement

is_dominant()
Checks whether an element in the weight space lies in the positive cone spanned by the basis elements (fundamental weights).

EXAMPLES:

```
sage: W = RootSystem(['A',3]).weight_space()
sage: Lambda = W.basis()
sage: w = Lambda[1]+Lambda[3]
sage: w.is_dominant()
True
sage: w = Lambda[1]-Lambda[2]
sage: w.is_dominant()
False
```

In the extended affine weight lattice, ‘delta’ is orthogonal to the positive coroots, so adding or subtracting it should not effect dominance

```
sage: P = RootSystem(['A',2,1]).weight_lattice(extended=true)
sage: Lambda = P.fundamental_weights()
sage: delta = P.null_root()
sage: w = Lambda[1]-delta
sage: w.is_dominant()
True
```

scalar(lambdacheck)
The canonical scalar product between the weight lattice and the coroot lattice.

Todo:
- merge with_apply_multi_module_morphism
- allow for any root space / lattice
- define properly the return type (depends on the base rings of the two spaces)
- make this robust for extended weight lattices (i might be “delta”)

EXAMPLES:

```
sage: L = RootSystem(['C',4,1]).weight_lattice()
sage: Lambda = L.fundamental_weights()
sage: alphacheck = L.simple_coroots()
sage: Lambda[1].scalar(alphacheck[1])
1
sage: Lambda[1].scalar(alphacheck[2])
0
```

The fundamental weights and the simple coroots are dual bases:
sage: matrix([ [ Lambda[i].scalar(alphacheck[j]) 
......: for i in L.index_set() ] 
......: for j in L.index_set() ])
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]

Note that the scalar product is not yet implemented between the weight space and the coweight space; in any cases, that won’t be the job of this method:

sage: R = RootSystem(['A',3])
sage: alpha = R.weight_space().roots()
sage: alphacheck = R.coweight_space().roots()
sage: alpha[1].scalar(alphacheck[1])
Traceback (most recent call last):
˓→coroot space

to_ambient()
Maps self to the ambient space.

EXAMPLES:

sage: mu = CartanType(['B',2]).root_system().weight_lattice().an_element(); mu
sage: mu.to_ambient()
(3, 1)

Warning: Only implemented in finite Cartan type. Does not work for coweight lattices because there is no implemented map from the coweight lattice to the ambient space.

to_weight_space()
Map self to the weight space.

Since self.parent() is the weight space, this map just returns self. This overrides the generic method in WeightSpaceRealizations.

EXAMPLES:

sage: mu = CartanType(['A',2]).root_system().weight_lattice().an_element(); mu
sage: mu.to_weight_space()
5.1.271 Weyl Character Rings

class sage.combinat.root_system.weyl_characters.WeightRing(parent, prefix)

Bases: CombinatorialFreeModule

The weight ring, which is the group algebra over a weight lattice.

A Weyl character may be regarded as an element of the weight ring. In fact, an element of the weight ring is an element of the Weyl character ring if and only if it is invariant under the action of the Weyl group.

The advantage of the weight ring over the Weyl character ring is that one may conduct calculations in the weight ring that involve sums of weights that are not Weyl group invariant.

EXAMPLES:

```
sage: A2 = WeylCharacterRing(['A',2])
sage: a2 = WeightRing(A2)
sage: wd = prod(a2(x/2)-a2(-x/2) for x in a2.space().positive_roots()); wd
a2(-1,1,0) - a2(-1,0,1) - a2(1,-1,0) + a2(1,0,-1) + a2(0,1,-1)
sage: chi = A2([5,3,0]); chi
A2(5,3,0)
sage: a2(chi)
a2(1,2,5) + 2*a2(1,3,4) + 2*a2(1,4,3) + a2(1,5,2) + a2(2,1,5) + 2*a2(2,2,4) + 3*a2(2,3,3) + 2*a2(2,4,2) + a2(2,5,1) + 2*a2(3,1,4) + 3*a2(3,2,3) + 3*a2(3,3,2) + 2*a2(3,4,1) + a2(3,5,0) + a2(3,0,5) + 2*a2(4,1,3) + 2*a2(4,2,2) + 2*a2(4,3,1) + a2(4,4,0) + a2(4,0,4) + a2(5,1,2) + a2(5,2,1) + a2(5,3,0) + a2(5,0,3) + a2(0,3,5) + a2(0,4,4) + a2(0,5,3)
sage: a2(chi)*wd
-a2(-1,3,6) + a2(-1,6,3) + a2(3,-1,6) - a2(3,6,-1) - a2(6,-1,3) + a2(6,3,-1)
sage: sum((-1)^w.length()*a2([6,3,-1]).weyl_group_action(w) for w in a2.space().weyl_group())
-a2(-1,3,6) + a2(-1,6,3) + a2(3,-1,6) - a2(3,6,-1) - a2(6,-1,3) + a2(6,3,-1)
sage: a2(chi)*wd == sum((-1)^w.length()*a2([6,3,-1]).weyl_group_action(w) for w in a2.space().weyl_group())
True
```

class Element

Bases: IndexedFreeModuleElement

A class for weight ring elements.

cartan_type()

Return the Cartan type.

EXAMPLES:

```
sage: A2 = WeylCharacterRing("A2")
sage: a2 = WeightRing(A2)
sage: a2([0,1,0]).cartan_type()
['A', 2]
```

character()

Assuming that self is invariant under the Weyl group, this will express it as a linear combination of characters. If self is not Weyl group invariant, this method will not terminate.

EXAMPLES:
Combinatorics, Release 10.0

```
sage: A2 = WeylCharacterRing(['A',2])
sage: a2 = WeightRing(A2)
sage: W = a2.space().weyl_group()
sage: mu = a2(2,1,0)
sage: nu = sum(mu.weyl_group_action(w) for w in W); nu
a2(1,2,0) + a2(1,0,2) + a2(2,1,0) + a2(2,0,1) + a2(0,1,2) + a2(0,2,1)
sage: nu.character()
-2*A2(1,1,1) + A2(2,1,0)
demazure(w, debug=False)
Return the result of applying the Demazure operator $\partial_w$ to self.

INPUT:
• w – a Weyl group element, or its reduced word
If $w = s_i$ is a simple reflection, the operation $\partial_w$ sends the weight $\lambda$ to
$$\lambda - s_i \cdot \lambda + \alpha_i \over 1 + \alpha_i,$$
where the numerator is divisible the denominator in the weight ring. This is extended by multiplicativity to all $w$ in the Weyl group.

EXAMPLES:
```
sage: B2 = WeylCharacterRing("B2",style="coroots")
sage: b2 = WeightRing(B2)
sage: b2(1,0).demazure([1])
b2(1,0) + b2(-1,2)
sage: b2(1,0).demazure([2])
b2(1,0)
sage: r = b2(1,0).demazure([1,2]); r
b2(1,0) + b2(-1,2)
sage: r.demazure([1])
b2(1,0) + b2(-1,2)
sage: r.demazure([2])
b2(0,0) + b2(1,0) + b2(1,-2) + b2(-1,2)
```

demazure_lusztig(i, v)
Return the result of applying the Demazure-Lusztig operator $T_i$ to self.

INPUT:
• i – an element of the index set (or a reduced word or Weyl group element)
• v – an element of the base ring
If $R$ is the parent WeightRing, the Demazure-Lusztig operator $T_i$ is the linear map $R \to R$ that sends (for a weight $\lambda$) $R(\lambda)$ to
$$(R(\alpha_i) - 1)^{-1} \left( R(\lambda) - R(s_i \lambda) - v(R(\lambda) - R(\alpha_i + s_i \lambda)) \right)$$
where the numerator is divisible by the denominator in $R$. The Demazure-Lusztig operators give a representation of the Iwahori–Hecke algebra associated to the Weyl group. See
In the examples, we confirm the braid and quadratic relations for type $B_2$.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: P.<v> = PolynomialRing(QQ)
sage: B2 = WeylCharacterRing("B2", style="coroots", base_ring=P); b2 = B2.ambient()
sage: def T1(f): return f.demazure_lusztig(1,v)
sage: def T2(f): return f.demazure_lusztig(2,v)
sage: T1(T2(T1(b2(1,1))))
(v^2-v)*b2(0,-1) + v^2*b2(-1,1)
sage: [T1(T2(T1(T2(b2(i,j)))))) == T2(T1(T2(T1(b2(i,j)))))) for i in [-2..2] for j in [-1,1]
[True, True, True, True, True, True, True, True, True, True]
```

Instead of an index \(i\) one may use a reduced word or Weyl group element:

```python
sage: b2(1,0).demazure_lusztig([2,1],v)==T2(T1(b2(1,0)))
True
sage: W = B2.space().weyl_group(prefix="s")
sage: [s1,s2]=W.simple_reflections()
sage: b2(1,0).demazure_lusztig(s2*s1,v)==T2(T1(b2(1,0)))
True
```

**scale\(k\)**

Multiply a weight by \(k\).
The operation is extended by linearity to the weight ring.

**INPUT:**
- \(k\) – a nonzero integer

**EXAMPLES:**

```python
sage: g2 = WeylCharacterRing("G2", style="coroots").ambient()
sage: g2(2,3).scale(2)
g2(4,6)
```

**shift\(\mu\)**

Add \(\mu\) to any weight.
Extended by linearity to the weight ring.

**INPUT:**
- \(\mu\) – a weight

**EXAMPLES:**

```python
sage: g2 = WeylCharacterRing("G2", style="coroots").ambient()
sage: [g2(1,2).shift(fw) for fw in g2.fundamental_weights()]
[g2(2,2), g2(1,3)]
```

**weyl_group_action\(w\)**

Return the action of the Weyl group element \(w\) on \(self\).

**EXAMPLES:**

```python
sage: G2 = WeylCharacterRing(["G",2])
sage: g2 = WeightRing(G2)
```
```python
sage: L = g2.space()
sage: [fw1, fw2] = L.fundamental_weights()
sage: sum(g2(fw2).weyl_group_action(w) for w in L.weyl_group())
2*g2(-2,1,1) + 2*g2(-1,-1,2) + 2*g2(-1,2,-1) + 2*g2(1,-2,1) + 2*g2(1,1,-2) + 2*g2(2,-1,-1)
```

**cartan_type()**

Return the Cartan type.

**EXAMPLES:**

```python
sage: A2 = WeylCharacterRing("A2")
sage: WeightRing(A2).cartan_type()
['A', 2]
```

**fundamental_weights()**

Return the fundamental weights.

**EXAMPLES:**

```python
sage: WeightRing(WeylCharacterRing("G2")):fundamental_weights()
Finite family {1: (1, 0, -1), 2: (2, -1, -1)}
```

**one_basis()**

Return the index of 1.

**EXAMPLES:**

```python
sage: A3 = WeylCharacterRing("A3")
sage: WeightRing(A3).one_basis()
(0, 0, 0, 0)
sage: WeightRing(A3).one()
a3(0,0,0,0)
```

**parent()**

Return the parent Weyl character ring.

**EXAMPLES:**

```python
sage: A2 = WeylCharacterRing("A2")
sage: a2 = WeightRing(A2)
sage: a2.parent()
The Weyl Character Ring of Type A2 with Integer Ring coefficients
sage: a2.parent() == A2
True
```

**positive_roots()**

Return the positive roots.

**EXAMPLES:**

```python
sage: WeightRing(WeylCharacterRing("G2")):positive_roots()
[(0, 1, -1), (1, -2, 1), (1, -1, 0), (1, 0, -1), (1, 1, -2), (2, -1, -1)]
```
product_on_basis$(a, b)$
Return the product of basis elements indexed by $a$ and $b$.

EXAMPLES:
```
sage: A2 = WeylCharacterRing("A2")
sage: a2 = WeightRing(A2)
sage: a2(1,0,0) * a2(0,1,0) # indirect doctest
a2(1,1,0)
```

simple_roots()
Return the simple roots.

EXAMPLES:
```
sage: WeightRing(WeylCharacterRing("G2")).simple_roots()
Finite family {1: (0, 1, -1), 2: (1, -2, 1)}
```

some_elements()
Return some elements of self.

EXAMPLES:
```
sage: A3 = WeylCharacterRing("A3")
sage: a3 = WeightRing(A3)
sage: a3.some_elements()
[a3(1,0,0,0), a3(1,1,0,0), a3(1,1,1,0)]
```

space()
Return the weight space realization associated to self.

EXAMPLES:
```
sage: E8 = WeylCharacterRing(['E',8])
sage: e8 = WeightRing(E8)
sage: e8.space()
Ambient space of the Root system of type ['E', 8]
```

weyl_character_ring()
Return the parent Weyl Character Ring.
A synonym for self.parent().

EXAMPLES:
```
sage: A2 = WeylCharacterRing("A2")
sage: a2 = WeightRing(A2)
sage: a2.weyl_character_ring()
The Weyl Character Ring of Type A2 with Integer Ring coefficients
```

wt_repr(wt)
Return a string representing the irreducible character with highest weight vector $wt$.

Uses coroot notation if the associated Weyl character ring is defined with style="coroots".

EXAMPLES:
class sage.combinat.root_system.weyl_characters.WeylCharacterRing(ct, base_ring=Integer Ring, prefix=None, style='lattice', k=None, conjugate=False, cyclotomic_order=None, fusion_labels=None, inject_variables=False)

Bases: CombinatorialFreeModule

A class for rings of Weyl characters.

Let $K$ be a compact Lie group, which we assume is semisimple and simply-connected. Its complexified Lie algebra $L$ is the Lie algebra of a complex analytic Lie group $G$. The following three categories are equivalent: finite-dimensional representations of $K$; finite-dimensional representations of $L$; and finite-dimensional analytic representations of $G$. In every case, there is a parametrization of the irreducible representations by their highest weight vectors. For this theory of Weyl, see (for example):

- Adams, *Lectures on Lie groups*
- Broecker and Tom Dieck, *Representations of Compact Lie Groups*
- Bump, *Lie Groups*
- Fulton and Harris, *Representation Theory*
- Goodman and Wallach, *Representations and Invariants of the Classical Groups*
- Hall, *Lie Groups, Lie Algebras and Representations*
- Humphreys, *Introduction to Lie Algebras and their representations*
- Procesi, *Lie Groups*
- Samelson, *Notes on Lie Algebras*
- Varadarajan, *Lie groups, Lie algebras, and their representations*
- Zhelobenko, *Compact Lie Groups and their Representations*.

Computations that you can do with these include computing their weight multiplicities, products (thus decomposing the tensor product of a representation into irreducibles) and branching rules (restriction to a smaller group).

There is associated with $K$, $L$ or $G$ as above a lattice, the weight lattice, whose elements (called weights) are characters of a Cartan subgroup or subalgebra. There is an action of the Weyl group $W$ on the lattice, and elements of a fixed fundamental domain for $W$, the positive Weyl chamber, are called dominant. There is for each representation a unique highest dominant weight that occurs with nonzero multiplicity with respect to a certain partial order, and it is called the highest weight vector.

**EXAMPLES:**

```python
g2 = WeylCharacterRing("G2")
g2 = WeylCharacterRing("G2", style="coroots")
```
Here $R(1)$, $R(fw1)$, and $R(fw2)$ are irreducible representations with highest weight vectors $0$, $\Lambda_1$, and $\Lambda_2$ respectively (the first two fundamental weights).

For type $A$ (also $G_2$, $F_4$, $E_6$ and $E_7$) we will take as the weight lattice not the weight lattice of the semisimple group, but for a larger one. For type $A$, this means we are concerned with the representation theory of $K = U(n)$ or $G = GL(n, \mathbb{C})$ rather than $SU(n)$ or $SU(n, \mathbb{C})$. This is useful since the representation theory of $GL(n)$ is ubiquitous, and also since we may then represent the fundamental weights (in sage.combinat.root_system.root_system) by vectors with integer entries. If you are only interested in $SL(3)$, say, use WeylCharacterRing(['A',2]) as above but be aware that $R([a,b,c])$ and $R([a+1,b+1,c+1])$ represent the same character of $SL(3)$ since $R([1,1,1])$ is the determinant.

For more information, see the thematic tutorial Lie Methods and Related Combinatorics in Sage, available at: https://doc.sagemath.org/html/en/thematic_tutorials/lie.html

```python
class Element

    Bases: IndexedFreeModuleElement

    A class for Weyl characters.

    adams_operation(r)

        Return the $r$-th Adams operation of self.

        INPUT:

            • $r$ – a positive integer

        This is a virtual character, whose weights are the weights of self, each multiplied by $r$.

        EXAMPLES:

            sage: A2 = WeylCharacterRing("A2")
            sage: A2(1,1,0).adams_operator(3)
            A2(2,2,2) - A2(3,2,1) + A2(3,3,0)

    adams_operator(r)

        Return the $r$-th Adams operation of self.

        INPUT:

            • $r$ – a positive integer

        This is a virtual character, whose weights are the weights of self, each multiplied by $r$.

        EXAMPLES:

            sage: A2 = WeylCharacterRing("A2")
            sage: A2(1,1,0).adams_operator(3)
            A2(2,2,2) - A2(3,2,1) + A2(3,3,0)

    branch(S, rule='default')

        Return the restriction of the character to the subalgebra.

        If no rule is specified, we will try to specify one.

        INPUT:

            • $S$ – a Weyl character ring for a Lie subgroup or subalgebra
            • rule – a branching rule

        See branch_weyl_character() for more information about branching rules.

        EXAMPLES:
```

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sage: B3 = WeylCharacterRing(['B',3])
sage: A2 = WeylCharacterRing(['A',2])
sage: [B3(w).branch(A2,rule='levi') for w in B3.fundamental_weights()]
[A2(0,0,0) + A2(1,0,0) + A2(0,0,-1),
 A2(0,0,0) + A2(1,0,0) + A2(1,1,0) + A2(0,-1,-1) + A2(0,0,-1),
 A2(-1/2,-1/2,-1/2) + A2(1/2,-1/2,-1/2) + A2(1/2,1/2,-1/2) + A2(1/2,1/2,1/2)]

```
cartan_type()

Return the Cartan type of self.

EXAMPLES:

    sage: A2 = WeylCharacterRing("A2")
    sage: A2([1,0,0]).cartan_type()
    ['A', 2]
```

degree()

Return the degree of self.

This is the dimension of the associated module.

EXAMPLES:

    sage: B3 = WeylCharacterRing(['B',3])
    sage: [B3(x).degree() for x in B3.fundamental_weights()]
    [7, 21, 8]

dual()

The involution that replaces a representation with its contragredient. (For Fusion rings, this is the
conjugation map.)

EXAMPLES:

    sage: A3 = WeylCharacterRing("A3", style="coroots")
    sage: (A3(1,0,0)^2).dual()
    A3(0,1,0) + A3(0,0,2)

exterior_power(k)

Return the \( k \)-th exterior power of self.

INPUT:

- \( k \) – a nonnegative integer

The algorithm is based on the identity \( ke_k = \sum_{r=1}^{k} (-1)^{k-1} p_k e_{k-r} \) relating the power-sum and elementary symmetric polynomials. Applying this to the eigenvalues of an element of the parent Lie group in the representation self, the \( e_k \) become exterior powers and the \( p_k \) become Adams operations, giving an efficient recursive implementation.

EXAMPLES:

    sage: B3 = WeylCharacterRing("B3",style="coroots")
    sage: spin = B3(0,0,1)
    sage: spin.exterior_power(6)
    B3(1,0,0) + B3(0,1,0)
**exterior_square()**

Return the exterior square of the character.

**EXAMPLES:**

```python
sage: A2 = WeylCharacterRing("A2", style="coroots")
sage: A2(1,0).exterior_square()
A2(0,1)
```

**frobenius_schur_indicator()**

Return:

- 1 if the representation is real (orthogonal)
- -1 if the representation is quaternionic (symplectic)
- 0 if the representation is complex (not self dual)

The Frobenius-Schur indicator of a character $\chi$ of a compact group $G$ is the Haar integral over the group of $\chi(g^2)$. Its value is 1, -1 or 0. This method computes it for irreducible characters of compact Lie groups by checking whether the symmetric and exterior square characters contain the trivial character.

**Todo:** Try to compute this directly without actually calculating the full symmetric and exterior squares.

**EXAMPLES:**

```python
sage: B2 = WeylCharacterRing("B2", style="coroots")
sage: B2(1,0).frobenius_schur_indicator()
1
sage: B2(0,1).frobenius_schur_indicator()
-1
```

**highest_weight()**

Return the parametrizing dominant weight of an irreducible character.

This method is only available for basis elements.

**EXAMPLES:**

```python
sage: G2 = WeylCharacterRing("G2", style="coroots")
sage: [x.highest_weight() for x in [G2(1,0),G2(0,1)]]
[(1, 0, -1), (2, -1, -1)]
```

**inner_product(other)**

Compute the inner product with another character.

The irreducible characters are an orthonormal basis with respect to the usual inner product of characters, interpreted as functions on a compact Lie group, by Schur orthogonality.

**INPUT:**

- other – another character

**EXAMPLES:**

```python
sage: A2 = WeylCharacterRing("A2")
sage: [f1,f2] = A2.fundamental_weights()
sage: r1 = A2(f1)*A2(f2); r1
A2(1,1,1) + A2(2,1,0)
```
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(continued from previous page)

```python
sage: r2 = A2(f1)^3; r2
A2(1,1,1) + 2*A2(2,1,0) + A2(3,0,0)
sage: r1.inner_product(r2)
3
```

**invariant_degree()**

Return the multiplicity of the trivial representation in self.

Multiplicities of other irreducibles may be obtained using `multiplicity()`.

**EXAMPLES:**

```python
sage: A2 = WeylCharacterRing("A2",style="coroots")
sage: rep = A2(1,0)^2*A2(0,1)^2; rep
2*A2(0,0) + A2(0,3) + 4*A2(1,1) + A2(3,0) + A2(2,2)
sage: rep.invariant_degree()
2
```

**is_irreducible()**

Return whether self is an irreducible character.

**EXAMPLES:**

```python
sage: B3 = WeylCharacterRing(["B",3])
sage: [B3(x).is_irreducible() for x in B3.fundamental_weights()]
[True, True, True]
sage: sum(B3(x) for x in B3.fundamental_weights()).is_irreducible()
False
```

**multiplicity(other)**

Return the multiplicity of the irreducible other in self.

**INPUT:**

- `other` – an irreducible character

**EXAMPLES:**

```python
sage: B2 = WeylCharacterRing("B2",style="coroots")
sage: rep = B2(1,1)^2; rep
B2(0,0) + B2(1,0) + 2*B2(0,2) + B2(2,0) + 2*B2(1,2) + B2(0,4) + B2(3,0) + B2(2,2)
sage: rep.multiplicity(B2(0,2))
2
```

**symmetric_power(k)**

Return the k-th symmetric power of self.

**INPUT:**

- `k` – a nonnegative integer

The algorithm is based on the identity \( k h_k = \sum_{r=1}^{k} p_k h_{k-r} \) relating the power-sum and complete symmetric polynomials. Applying this to the eigenvalues of an element of the parent Lie group in the representation self, the \( h_k \) become symmetric powers and the \( p_k \) become Adams operations, giving an efficient recursive implementation.

**EXAMPLES:**
sage: B3 = WeylCharacterRing("B3",style="coroots")
sage: spin = B3(0,0,1)
sage: spin.symmetric_power(6)
B3(0,0,0) + B3(0,0,2) + B3(0,0,4) + B3(0,0,6)

symmetric_square()
Return the symmetric square of the character.
EXAMPLES:

sage: A2 = WeylCharacterRing("A2",style="coroots")
sage: A2(1,0).symmetric_square()
A2(2,0)

weight_multiplicities()
Return the dictionary of weight multiplicities for the Weyl character self.
The character does not have to be irreducible.
EXAMPLES:

sage: B2 = WeylCharacterRing("B2",style="coroots")
sage: B2(0,1).weight_multiplicities()
{(-1/2, -1/2): 1, (-1/2, 1/2): 1, (1/2, -1/2): 1, (1/2, 1/2): 1}

adjoint_representation()
Return the adjoint representation as an element of the WeylCharacterRing.
EXAMPLES:

sage: G2 = WeylCharacterRing("G2",style="coroots")
sage: G2.adjoint_representation()
G2(0,1)

affine_reflect(wt, k=0)
Return the reflection of wt in the hyperplane \( \theta \).
Optionally, this also shifts by a multiple \( k \) of \( \theta \).
INPUT:
• wt – a weight
• k – (optional) a positive integer
EXAMPLES:

sage: B22 = FusionRing("B2",2)
sage: fw = B22.fundamental_weights(); fw
Finite family {1: (1, 0), 2: (1/2, 1/2)}
sage: [B22.affine_reflect(x,2) for x in fw]
[(2, 1), (3/2, 3/2)]

ambient()
Return the weight ring of self.
EXAMPLES:
Combinatorics, Release 10.0

```python
sage: WeylCharacterRing("A2").ambient()
The Weight ring attached to The Weyl Character Ring of Type A2 with Integer
Ring coefficients
```

**base_ring()**

Return the base ring of self.

**EXAMPLES:**

```python
sage: R = WeylCharacterRing(["A", 3], base_ring = CC); R.base_ring()
Complex Field with 53 bits of precision
```

**cartan_type()**

Return the Cartan type of self.

**EXAMPLES:**

```python
sage: WeylCharacterRing("A2").cartan_type()
["A", 2]
```

**char_from_weights** *(mdict)*

Construct a Weyl character from an invariant linear combination of weights.

**INPUT:**

- **mdict** – a dictionary mapping weights to coefficients, and representing a linear combination of weights which shall be invariant under the action of the Weyl group

**OUTPUT:** the corresponding Weyl character

**EXAMPLES:**

```python
sage: A2 = WeylCharacterRing("A2")
sage: v = A2._space([3,1,0]); v
(3, 1, 0)
sage: d = dict([(x,1) for x in v.orbit()]); d
{(1, 3, 0): 1,
 (1, 0, 3): 1,
 (3, 1, 0): 1,
 (3, 0, 1): 1,
 (0, 1, 3): 1,
 (0, 3, 1): 1}
sage: A2.char_from_weights(d)
-A2(2,1,1) - A2(2,2,0) + A2(3,1,0)
```

**demazure_character** *(hwv, word, debug=False)*

Compute the Demazure character.

**INPUT:**

- **hwv** – a (usually dominant) weight
- **word** – a Weyl group word

Produces the Demazure character with highest weight hwv and word as an element of the weight ring. Only available if **style=coroots**. The Demazure operators are also available as methods of `WeightRing` elements, and as methods of crystals. Given a `CrystalOfTableaux` with given highest weight vector, the

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Demazure method on the crystal will give the equivalent of this method, except that the Demazure character of the crystal is given as a sum of monomials instead of an element of the \texttt{WeightRing}.

See \texttt{WeightRing.Element.demazure()} and \texttt{sage.categories.classical_crystals.ClassicalCrystals.ParentMethods.demazure_character()}

EXAMPLES:

```python
sage: A2 = WeylCharacterRing("A2", style="coroots")
sage: h = sum(A2.fundamental_weights()); h
(2, 1, 0)
sage: A2.demazure_character(h, word=[1,2])
a2(0,0) + a2(-2,1) + a2(2,-1) + a2(1,1) + a2(-1,2)
sage: A2.demazure_character((1,1), word=[1,2])
a2(0,0) + a2(-2,1) + a2(2,-1) + a2(1,1) + a2(-1,2)
```

\texttt{dot_reduce(a)}

Auxiliary function for \texttt{product_on_basis()}. Return a pair \([\epsilon, b]\) where \(b\) is a dominant weight and \(\epsilon\) is 0, 1 or -1. To describe \(b\), let \(w\) be an element of the Weyl group such that \(w(a + \rho)\) is dominant. If \(w(a + \rho) - \rho\) is dominant, then \(\epsilon\) is the sign of \(w\) and \(b\) is \(w(a + \rho) - \rho\). Otherwise, \(\epsilon\) is zero.

INPUT:

- \(a\) – a weight

EXAMPLES:

```python
sage: A2 = WeylCharacterRing("A2")
sage: weights = sorted(A2(2,1,0).weight_multiplicities().keys(), key=str);
weights
[(0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 1, 1), (1, 2, 0), (2, 0, 1), (2, 1, 0)]
sage: [A2.dot_reduce(x) for x in weights]
[[(0, 0, 0), [-1, (1, 1, 1)], [-1, (1, 1, 1)], [1, (1, 1, 1)], [0, (0, 0, -


\texttt{dynkin_diagram()} Return the Dynkin diagram of self.

EXAMPLES:

```python
sage: WeylCharacterRing("E7").dynkin_diagram()
O 2
|   0---0---0---0---0---0---0
|   1 3 4 5 6 7
E7
```

\texttt{extended_dynkin_diagram()} Return the extended Dynkin diagram, which is the Dynkin diagram of the corresponding untwisted affine type.

EXAMPLES:

```python
**fundamental_weights()**

Return the fundamental weights.

EXAMPLES:

```python
sage: WeylCharacterRing("G2").fundamental_weights()
Finite family {1: (1, 0, -1), 2: (2, -1, -1)}
```

**highest_root()**

Return the highest root.

EXAMPLES:

```python
sage: WeylCharacterRing("G2").highest_root()
(2, -1, -1)
```

**irr_repr(hwv)**

Return a string representing the irreducible character with highest weight vector `hwv`.

EXAMPLES:

```python
sage: B3 = WeylCharacterRing("B3")
sage: [B3.irr_repr(v) for v in B3.fundamental_weights()]
["B3(1,0,0)", "B3(1,1,0)", "B3(1/2,1/2,1/2)"]
sage: B3 = WeylCharacterRing("B3", style="coroots")
sage: [B3.irr_repr(v) for v in B3.fundamental_weights()]
["B3(1,0,0)", "B3(0,1,0)", "B3(0,0,1)"
```

**level(wt)**

Return the level of the weight, defined to be the value of the weight on the coroot associated with the highest root.

EXAMPLES:

```python
sage: R = FusionRing("F4",2); [R.level(x) for x in R.fundamental_weights()]
[2, 3, 2, 1]
sage: [cartan_type("F4-2").a()[x] for x in [1..4]]
[2, 3, 2, 1]
```

**lift()**

The embedding of `self` into its weight ring.

EXAMPLES:

```python
sage: A2 = WeylCharacterRing("A2")
sage: A2.lift
Generic morphism:
```

(continues on next page)
From: The Weyl Character Ring of Type A2 with Integer Ring coefficients
To: The Weight ring attached to The Weyl Character Ring of Type A2 with Integer Ring coefficients

sage: x = -A2(2,1,1) - A2(2,2,0) + A2(3,1,0)
sage: A2.lift(x)
\[a2(1,3,0) + a2(1,0,3) + a2(3,1,0) + a2(3,0,1) + a2(0,1,3) + a2(0,3,1)\]

As a shortcut, you may also do:

sage: x.lift()
\[a2(1,3,0) + a2(1,0,3) + a2(3,1,0) + a2(3,0,1) + a2(0,1,3) + a2(0,3,1)\]

Or even:

sage: a2 = WeightRing(A2)
sage: a2(x)
\[a2(1,3,0) + a2(1,0,3) + a2(3,1,0) + a2(3,0,1) + a2(0,1,3) + a2(0,3,1)\]

**lift_on_basis**(irr)

Expand the basis element indexed by the weight `irr` into the weight ring of `self`.

**INPUT:**

- `irr` – a dominant weight

This is used to implement `lift()`.

**EXAMPLES:**

sage: A2 = WeylCharacterRing("A2")
sage: v = A2._space([2,1,0]); v
\((2, 1, 0)\)
sage: A2.lift_on_basis(v)
\(2*a2(1,1,1) + a2(1,2,0) + a2(1,0,2) + a2(2,1,0) + a2(2,0,1) + a2(0,1,2) + a2(0,2,1)\)

This is consistent with the analogous calculation with symmetric Schur functions:

sage: s = SymmetricFunctions(QQ).s()
sage: s[2,1].expand(3)
x0^2*x1 + x0*x1^2 + x0^2*x2 + 2*x0*x1*x2 + x1^2*x2 + x0*x2^2 + x1*x2^2

**maximal_subgroup**(ct)

Return a branching rule or a list of branching rules.

**INPUT:**

- `ct` – the Cartan type of a maximal subgroup of `self`.

In rare cases where there is more than one maximal subgroup (up to outer automorphisms) with the given Cartan type, the function returns a list of branching rules.

**EXAMPLES:**
For more information, see the related method `maximal_subgroups()`.

### maximal_subgroups()

This method is only available if the Cartan type of `self` is irreducible and of rank no greater than 8. This method produces a list of the maximal subgroups of `self`, up to (possibly outer) automorphisms. Each line in the output gives the Cartan type of a maximal subgroup followed by a command that creates the branching rule.

**EXAMPLES:**

```python
sage: WeylCharacterRing("E6").maximal_subgroups()  
D5:branching_rule("E6","D5","levi")  
C4:branching_rule("E6","C4","symmetric")  
F4:branching_rule("E6","F4","symmetric")  
A2:branching_rule("E6","A2","miscellaneous")  
G2:branching_rule("E6","G2","miscellaneous")  
A2xG2:branching_rule("E6","A2xG2","miscellaneous")  
A1xA5:branching_rule("E6","A1xA5","extended")  
A2xA2xA2:branching_rule("E6","A2xA2xA2","extended")  
```

Note that there are other embeddings of (for example $A_2$ into $E_6$ as nonmaximal subgroups. These embeddings may be constructed by composing branching rules through various subgroups.

Once you know which maximal subgroup you are interested in, to create the branching rule, you may either paste the command to the right of the colon from the above output onto the command line, or alternatively invoke the related method `maximal_subgroup()`:

```python
sage: branching_rule("E6","G2","miscellaneous")  
miscellaneous branching rule E6 => G2  
sage: WeylCharacterRing("E6").maximal_subgroup("G2")  
miscellaneous branching rule E6 => G2  
```

It is believed that the list of maximal subgroups is complete, except that some subgroups may be not be invariant under outer automorphisms. It is reasonable to want a list of maximal subgroups that is complete up to conjugation, but to obtain such a list you may have to apply outer automorphisms. The group of outer automorphisms modulo inner automorphisms is isomorphic to the group of symmetries of the Dynkin diagram, and these are available as branching rules. The following example shows that while a branching rule from $D_4$ to $A_1 \times C_2$ is supplied, another different one may be obtained by composing it with the triality automorphism of $D_4$:

```python
sage: [D4,A1xC2]=[WeylCharacterRing(x,style="coroots") for x in ["D4","A1xC2"]]  
sage: fw = D4.fundamental_weights()  
sage: b = D4.maximal_subgroup("A1xC2")  
sage: [D4(fw).branch(A1xC2,rule=b) for fw in D4.fundamental_weights()]  
[A1xC2(1,1,0), A1xC2(2,0,0) + A1xC2(2,0,1) + A1xC2(0,2,0), A1xC2(1,1,0), A1xC2(2,0,0) + A1xC2(0,0,1)]  
sage: b1 = branching_rule("D4","D4","triality")^b  
```

(continues on next page)
sage: [D4(fw).branch(A1xC2,rule=b1) for fw in D4.fundamental_weights()]
[A1xC2(1,1,0),
 A1xC2(2,0,0) + A1xC2(2,0,1) + A1xC2(0,2,0),
 A1xC2(2,0,0) + A1xC2(0,0,1),
 A1xC2(1,1,0)]

one_basis()
Return the index of 1 in self.

EXAMPLES:

sage: WeylCharacterRing("A3").one_basis()
(0, 0, 0, 0)

sage: WeylCharacterRing("A3").one()
A3(0,0,0,0)

positive_roots()
Return the positive roots.

EXAMPLES:

sage: WeylCharacterRing("G2").positive_roots()
[(0, 1, -1), (1, -2, 1), (1, -1, 0), (1, 0, -1), (1, 1, -2), (2, -1, -1)]

product_on_basis(a, b)
Compute the tensor product of two irreducible representations a and b.

EXAMPLES:

sage: D4 = WeylCharacterRing(["D",4])
sage: spin_plus = D4(1/2,1/2,1/2,1/2)
sage: spin_minus = D4(1/2,1/2,1/2,-1/2)
sage: spin_plus * spin_minus # indirect doctest
D4(1,0,0,0) + D4(1,1,1,0)
sage: spin_minus * spin_plus
D4(1,0,0,0) + D4(1,1,1,0)

Uses the Brauer-Klimyk method.

rank()
Return the rank.

EXAMPLES:

sage: WeylCharacterRing("G2").rank()
2

retract()
The partial inverse map from the weight ring into self.

EXAMPLES:

sage: A2 = WeylCharacterRing("A2")
sage: a2 = WeightRing(A2)
sage: A2.retract

(continues on next page)
Generic morphism:
From: The Weight ring attached to The Weyl Character Ring of Type A2 with Integer Ring coefficients
To: The Weyl Character Ring of Type A2 with Integer Ring coefficients

\texttt{sage: v = A2._space([3,1,0]); v}
(3, 1, 0)
\texttt{sage: chi = a2.sum_of_monomials(v.orbit()); chi}
a2(1,3,0) + a2(1,0,3) + a2(3,1,0) + a2(3,0,1) + a2(0,1,3) + a2(0,3,1)
\texttt{sage: A2.retract(chi)}
-A2(2,1,1) - A2(2,2,0) + A2(3,1,0)

The input should be invariant:
\texttt{sage: A2.retract(a2.monomial(v))}
Traceback (most recent call last):
  ...  
ValueError: multiplicity dictionary may not be Weyl group invariant

As a shortcut, you may use conversion:
\texttt{sage: A2(chi)}
-A2(2,1,1) - A2(2,2,0) + A2(3,1,0)
\texttt{sage: A2(a2.monomial(v))}
Traceback (most recent call last):
  ...  
ValueError: multiplicity dictionary may not be Weyl group invariant

\textbf{simple_coroots()}
Return the simple coroots.

\textbf{EXAMPLES:}
\texttt{sage: WeylCharacterRing("G2").simple_coroots()}
Finite family {1: (0, 1, -1), 2: (1/3, -2/3, 1/3)}

\textbf{simple_roots()}
Return the simple roots.

\textbf{EXAMPLES:}
\texttt{sage: WeylCharacterRing("G2").simple_roots()}
Finite family {1: (0, 1, -1), 2: (1, -2, 1)}

\textbf{some_elements()}
Return some elements of self.

\textbf{EXAMPLES:}
\texttt{sage: WeylCharacterRing("A3").some_elements()}
[A3(1,0,0,0), A3(1,1,0,0), A3(1,1,1,0)]

\textbf{space()}
Return the weight space associated to self.
EXAMPLES:

```python
sage: WeylCharacterRing('[E,8]').space()
Ambient space of the Root system of type ['E', 8]
```

```

sage.combinat.root_system.weyl_characters.irreducible_character_freudenthal(hwv, debug=False)
```

Return the dictionary of multiplicities for the irreducible character with highest weight λ.

The weight multiplicities are computed by the Freudenthal multiplicity formula. The algorithm is based on recursion relation that is stated, for example, in Humphrey’s book on Lie Algebras. The multiplicities are invariant under the Weyl group, so to compute them it would be sufficient to compute them for the weights in the positive Weyl chamber. However after some testing it was found to be faster to compute every weight using the recursion, since the use of the Weyl group is expensive in its current implementation.

INPUT:

- hwv – a dominant weight in a weight lattice.
- L – the ambient space

EXAMPLES:

```python
sage: WeylCharacterRing("A2")(2,1,0).weight_multiplicities()
# indirect doctest
{(1, 1, 1): 2, (1, 2, 0): 1, (1, 0, 2): 1, (2, 1, 0): 1, (2, 0, 1): 1, (0, 1, 2): 1, (0, 2, 1): 1}
```

### 5.1.272 Weyl Groups

AUTHORS:

- Daniel Bump (2008): initial version
- Mike Hansen (2008): initial version
- Anne Schilling (2008): initial version
- Nicolas Thiéry (2008): initial version
- Volker Braun (2013): LibGAP-based matrix groups

EXAMPLES:

The Cayley graph of the Weyl Group of type ['A', 3]:

```python
sage: w = WeylGroup(['A',3])
sage: d = w.cayley_graph(); d
Digraph on 24 vertices
sage: d.show3d(color_by_label=True, edge_size=0.01, vertex_size=0.03)
```

The Cayley graph of the Weyl Group of type ['D', 4]:

```python
sage: w = WeylGroup(['D',4])
sage: d = w.cayley_graph(); d
Digraph on 192 vertices
sage: d.show3d(color_by_label=True, edge_size=0.01, vertex_size=0.03) #long time (less than one minute)
```
Todo: More examples on Weyl Groups should be added here.

```python
class sage.combinat.root_system.weyl_group.ClassicalWeylSubgroup(domain, prefix)
    Bases: WeylGroup_gens
    A class for Classical Weyl Subgroup of an affine Weyl Group
    EXAMPLES:

    sage: G = WeylGroup(['A',3,1]).classical()
    sage: G
    Parabolic Subgroup of the Weyl Group of type ['A', 3, 1] (as a matrix group acting on the root space)
    sage: G.category()
    Category of finite irreducible weyl groups
    sage: G.cardinality()
    24
    sage: G.index_set()
    (1, 2, 3)
    sage: TestSuite(G).run()
```

Todo: implement:
- Parabolic subrootsystems
- Parabolic subgroups with a set of nodes as argument

```python
cartan_type()
    EXAMPLES:

    sage: WeylGroup(['A',3,1]).classical().cartan_type()
    ['A', 3]
    sage: WeylGroup(['A',3,1]).classical().index_set()
    (1, 2, 3)

    Note: won't be needed, once the lattice will be a parabolic sub root system
```

```python
simple_reflections()
    EXAMPLES:

    sage: WeylGroup(['A',2,1]).classical().simple_reflections()
    Finite family {1: [ 1 0 0]
    [ 1 -1 1]
    [ 0 0 1],
    2: [ 1 0 0]
    [ 0 1 0]
    [ 1 1 -1]}

    Note: won't be needed, once the lattice will be a parabolic sub root system
```

```python
weyl_group(prefix='hereditary')
    Return the Weyl group associated to the parabolic subgroup.
    EXAMPLES:
```
sage: WeylGroup(["A",4,1]).classical().weyl_group()
Weyl Group of type ['A', 4, 1] (as a matrix group acting on the root space)
sage: WeylGroup(["C",4,1]).classical().weyl_group()
Weyl Group of type ['C', 4, 1] (as a matrix group acting on the root space)
sage: WeylGroup(["E",8,1]).classical().weyl_group()
Weyl Group of type ['E', 8, 1] (as a matrix group acting on the root space)

sage.combinat.root_system.weyl_group.WeylGroup(x, prefix=None, implementation='matrix')

Return the Weyl group of the root system defined by the Cartan type (or matrix) ct.

INPUT:

• x – a root system or a Cartan type (or matrix)

OPTIONAL:

• prefix – changes the representation of elements from matrices to products of simple reflections
• implementation – one of the following:
  * 'matrix' - as matrices acting on a root system
  * "permutation" - as a permutation group acting on the roots

EXAMPLES:

The following constructions yield the same result, namely a weight lattice and its corresponding Weyl group:

sage: G = WeylGroup(['F',4])
sage: L = G.domain()

or alternatively and equivalently:

sage: L = RootSystem(['F',4]).ambient_space()
sage: G = L.weyl_group()
sage: W = WeylGroup(L)

Either produces a weight lattice, with access to its roots and weights.

sage: G = WeylGroup(['F',4])
sage: G.order()
1152
sage: [s1,s2,s3,s4] = G.simple_reflections()
sage: w = s1*s2*s3*s4; w
[ 1/2  1/2  1/2  1/2]
[-1/2  1/2  1/2 -1/2]
[ 1/2  1/2 -1/2 -1/2]
[ 1/2 -1/2  1/2 -1/2]
sage: type(w) == G.element_class
True
sage: w.order()
12
sage: w.length() # length function on Weyl group
4

The default representation of Weyl group elements is as matrices. If you prefer, you may specify a prefix, in which case the elements are represented as products of simple reflections.

sage: W=WeylGroup("C3",prefix="s")
sage: [s1,s2,s3]=W.simple_reflections() # lets Sage parse its own output

(continues on next page)
sage: s2*s1*s2*s3
s1*s2*s3*s1
sage: s2*s1*s2*s3 == s1*s2*s3*s1
True
sage: (s2*s3)^2==(s3*s2)^2
True
sage: (s1*s2*s3*s1).matrix()
\[
\begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]
sage: L = G.domain()
sage: fw = L.fundamental_weights(); fw
Finite family {1: (1, 1, 0, 0), 2: (2, 1, 1, 0), 3: (3/2, 1/2, 1/2, 1/2), 4: (1, 0, 0, 0)}
sage: rho = sum(fw); rho
(11/2, 5/2, 3/2, 1/2)
sage: w.action(rho) # action of G on weight lattice
(5, -1, 3, 2)

We can also do the same for arbitrary Cartan matrices:

sage: cm = CartanMatrix([[2,-5,0],[-2,2,-1],[0,-1,2]])
sage: W = WeylGroup(cm)
sage: W.gens()
\[
\begin{bmatrix}
-1 & 5 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
2 & -1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & -1 \\
\end{bmatrix}
\]
sage: s0,s1,s2 = W.gens()
sage: s1*s2*s1
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 0 & -1 \\
2 & -1 & 0 \\
\end{bmatrix}
\]
sage: s2*s1*s2
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 0 & -1 \\
2 & -1 & 0 \\
\end{bmatrix}
\]
sage: s0*s1*s0*s2*s0
\[
\begin{bmatrix}
9 & 0 & -5 \\
2 & 0 & -1 \\
0 & 1 & -1 \\
\end{bmatrix}
\]

Same Cartan matrix, but with a prefix to display using simple reflections:

sage: W = WeylGroup(cm, prefix='s')
sage: s0,s1,s2 = W.gens()
sage: s0*s2*s1
s2*s0*s1
sage: (s1*s2)^3
1

(continues on next page)
sage: (s0*s1)^5
s0*s1*s0*s1*s0*s1*s0*s1*s0*s1
sage: s0*s1*s2*s1*s2
s2*s0*s1
sage: s0*s1*s2*s0*s2
s0*s1*s0

class sage.combinat.root_system.weyl_group.WeylGroupElement(parent, g, check=False)

Bases: MatrixGroupElement_gap

Class for a Weyl Group elements

action(v)

Return the action of self on the vector v.

EXAMPLES:

sage: W = WeylGroup(['A',2])
sage: s = W.simple_reflections()
sage: v = W.domain()([1,0,0])
sage: s[1].action(v)
(0, 1, 0)

apply_simple_reflection(i, side='right')

domain()

Return the ambient lattice associated with self.

EXAMPLES:

sage: W = WeylGroup(['A',2])
sage: s1 = W.simple_reflection(1)
sage: s1.domain()
Ambient space of the Root system of type ['A', 2]

has_descent(i, positive=False, side='right')

Test if self has a descent at position i.

An element \( w \) has a descent in position \( i \) if \( w \) is on the strict negative side of the \( i^{th} \) simple reflection hyperplane.

If positive is True, tests if it is on the strict positive side instead.
EXAMPLES:

```python
sage: W = WeylGroup(['A',3])
sage: s = W.simple_reflections()
sage: [W.one().has_descent(i) for i in W.domain().index_set()]
[False, False, False]
sage: [s[1].has_descent(i) for i in W.domain().index_set()]
[True, False, False]
sage: [s[2].has_descent(i) for i in W.domain().index_set()]
[False, True, False]
sage: [s[3].has_descent(i) for i in W.domain().index_set()]
[False, False, True]
sage: [s[3].has_descent(i, True) for i in W.domain().index_set()]
[True, True, False]
```

```python
sage: W = WeylGroup(['A',3,1])
sage: s = W.simple_reflections()
sage: [W.one().has_descent(i) for i in W.domain().index_set()]
[False, False, False, False]
sage: [s[0].has_descent(i) for i in W.domain().index_set()]
[True, False, False, False]
sage: w = s[0] * s[1]
sage: [w.has_descent(i) for i in W.domain().index_set()]
[False, True, False, False]
sage: [w.has_descent(i, side = "left") for i in W.domain().index_set()]
[True, False, False, False]
sage: w = s[0] * s[2]
sage: [w.has_descent(i) for i in W.domain().index_set()]
[True, False, True, False]
sage: [w.has_descent(i, side = "left") for i in W.domain().index_set()]
[True, False, True, False]
```

```python
sage: W = WeylGroup(['A',3])
sage: W.one().has_descent(0)
True
sage: W.w0.has_descent(0)
False
```

`has_left_descent(i)`
Test if self has a left descent at position i.

EXAMPLES:

```python
sage: W = WeylGroup(['A',3])
sage: s = W.simple_reflections()
sage: [W.one().has_left_descent(i) for i in W.domain().index_set()]
[False, False, False]
sage: [s[1].has_left_descent(i) for i in W.domain().index_set()]
[True, False, False]
sage: [s[2].has_left_descent(i) for i in W.domain().index_set()]
[False, True, False]
sage: [s[3].has_left_descent(i) for i in W.domain().index_set()]
[False, False, True]
sage: [(s[3]*s[2]).has_left_descent(i) for i in W.domain().index_set()]
[False, False, True]
```
has_right_descent(i)
    Test if self has a right descent at position i.

    EXAMPLES:
    sage: W = WeylGroup(['A',3])
    sage: s = W.simple_reflections()
    sage: [W.one().has_right_descent(i) for i in W.domain().index_set()]
    [False, False, False]
    sage: [s[1].has_right_descent(i) for i in W.domain().index_set()]
    [True, False, False]
    sage: [s[2].has_right_descent(i) for i in W.domain().index_set()]
    [False, True, False]
    sage: [s[3].has_right_descent(i) for i in W.domain().index_set()]
    [False, False, True]
    sage: [(s[3]*s[2]).has_right_descent(i) for i in W.domain().index_set()]
    [False, True, False]

to_matrix()
    Return self as a matrix.

    EXAMPLES:
    sage: G = WeylGroup(['A',2])
    sage: s1 = G.simple_reflection(1)
    sage: s1.to_matrix() == s1.matrix()
    True

to_permutation()
    A first approximation of to_permutation ...
    This assumes types A,B,C,D on the ambient lattice
    This further assume that the basis is indexed by 0,1,... and returns a permutation of (5,4,2,3,1) (beuargl),
    as a tuple

to_permutation_string()
    EXAMPLES:
    sage: W = WeylGroup(['A',3])
    sage: s = W.simple_reflections()
    sage: (s[1]*s[2]*s[3]).to_permutation_string()
    '2341'

class sage.combinat.root_system.weyl_group.WeylGroup_gens(domain, prefix)
    Bases: UniqueRepresentation, FinitelyGeneratedMatrixGroup_gap

    EXAMPLES:
    sage: G = WeylGroup(['B',3])
    sage: TestSuite(G).run()
    sage: cm = CartanMatrix([[2,-5,0],[-2,2,-1],[0,-1,2]])
    sage: W = WeylGroup(cm)
    sage: TestSuite(W).run() # long time
Element

alias of WeylGroupElement

cartan_type()

Return the CartanType associated to self.

EXAMPLES:

```sage
G = WeylGroup(['F',4])
sage: G.cartan_type()
['F', 4]
```

classical()

If self is a Weyl group from an affine Cartan Type, this give the classical parabolic subgroup of self.

Caveat: we assume that 0 is a special node of the Dynkin diagram

TODO: extract parabolic subgroup method

EXAMPLES:

```sage
G = WeylGroup(['A',3,1])
sage: G.classical()
Parabolic Subgroup of the Weyl Group of type ['A', 3, 1] (as a matrix group acting on the root space)
sage: WeylGroup(['A',3]).classical()
Traceback (most recent call last):
  ... ValueError: classical subgroup only defined for affine types
```

domain()

Return the domain of the element of self, that is the root lattice realization on which they act.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: G = WeylGroup(['F',4])
sage: G.domain()
Ambient space of the Root system of type ['F', 4]
sage: G = WeylGroup(['A',3,1])
sage: G.domain()
Root space over the Rational Field of the Root system of type ['A', 3, 1]

from_morphism(f)

index_set()

Return the index set of self.

EXAMPLES:

sage: G = WeylGroup(['F',4])
sage: G.index_set()
(1, 2, 3, 4)
sage: G = WeylGroup(['A',3,1])
sage: G.index_set()
(0, 1, 2, 3)

long_element_hardcoded()

Return the long Weyl group element (hardcoded data).

Do we really want to keep it? There is a generic implementation which works in all cases. The hardcoded
should have a better complexity (for large classical types), but there is a cache, so does this really matter?

EXAMPLES:


sage: [WeylGroup(t).long_element().length() for t in types]

[15, 9, 9, 12, 6, 24, 36]

sage: all(WeylGroup(t).long_element() == WeylGroup(t).long_element_hardcoded() for t in types) # long time (17s on sage.math, 2011)
True

morphism_matrix(f)

one()

Return the unit element of the Weyl group.

EXAMPLES:

sage: W = WeylGroup(['A',3])
sage: e = W.one(); e

[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]

sage: type(e) == W.element_class
True

reflections()

Return the reflections of self.
```
The reflections of a Coxeter group $W$ are the conjugates of the simple reflections. They are in bijection with the positive roots, for given a positive root, we may have the reflection in the hyperplane orthogonal to it. This method returns a family indexed by the positive roots taking values in the reflections. This requires self to be a finite Weyl group.

**Note:** Prior to github issue #20027, the reflections were the keys of the family and the values were the positive roots.

**EXAMPLES:**

```python
sage: W = WeylGroup("B2", prefix="s")
sage: refdict = W.reflections(); refdict
Finite family {(1, -1): s1, (0, 1): s2, (1, 1): s2*s1*s2, (1, 0): s1*s2*s1}
sage: [r-refdict[r].action(r) for r in refdict.keys()]
[(0, 0), (0, 0), (0, 0), (0, 0)]
sage: W = WeylGroup(['A',2,1], prefix="s")
sage: W.reflections()
Lazy family (real root to reflection(i))_{i in Positive real roots of type ['A', 2, 1]}
```

**simple_reflection(i)**
Return the $i^{th}$ simple reflection.

**EXAMPLES:**

```python
sage: G = WeylGroup(['F',4])
sage: G.simple_reflection(1)
[1 0 0 0]
[0 0 1 0]
[0 1 0 0]
[0 0 0 1]
sage: W=WeylGroup(['A',2,1])
sage: W.simple_reflection(1)
[ 1 0 0]
[ 1 -1 1]
[ 0 0 1]
```

**simple_reflections()**
Return the simple reflections of self, as a family.

**EXAMPLES:**

There are the simple reflections for the symmetric group:

```python
sage: W=WeylGroup(['A',2])
sage: s = W.simple_reflections(); s
Finite family {1: [0 1 0]
[1 0 0]
[0 0 1], 2: [1 0 0]
[0 0 1]
[0 1 0])}
```

As a special feature, for finite irreducible root systems, $s[0]$ gives the reflection along the highest root:
We now look at some further examples:

```
sage: W = WeylGroup(['A',2,1])
sage: W.simple_reflections()
Finite family {0: [-1 1 1]
[ 0 1 0]
[ 0 0 1], 1: [ 1 0 0]
[ 1 -1 1]
[ 0 0 1], 2: [ 1 0 0]
[ 0 1 0]
[ 1 1 -1]}
sage: W = WeylGroup(['F',4])
sage: [s1,s2,s3,s4] = W.simple_reflections()
sage: w = s1*s2*s3*s4; w
[ 1/2 1/2 1/2 1/2]
[-1/2 1/2 1/2 -1/2]
[ 1/2 1/2 -1/2 -1/2]
[ 1/2 -1/2 1/2 -1/2]
sage: s4^2 == W.one()
True
sage: type(w) == W.element_class
True
```

```
unit()

Return the unit element of the Weyl group.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: e = W.one(); e
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage: type(e) == W.element_class
True
```
```
```
sage: W = WeylGroup(['A',4], implementation="permutation")
sage: W.cartan_type()
['A', 4]

**distinguished_reflections()**
Return the reflections of self.

**EXAMPLES:**

```python
sage: W = WeylGroup(['B',2], implementation="permutation")
sage: W.distinguished_reflections()
Finite family {1: (1,5)(2,4)(6,8), 2: (1,3)(2,6)(5,7),
    3: (2,8)(3,7)(4,6), 4: (1,7)(3,5)(4,8)}
```

**independent_roots()**
Return the simple roots of self.

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',4], implementation="permutation")
sage: W.simple_roots()
Finite family {1: (1, 0, 0, 0), 2: (0, 1, 0, 0),
    3: (0, 0, 1, 0), 4: (0, 0, 0, 1)}
```

**index_set()**
Return the index set of self.

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',4], implementation="permutation")
sage: W.index_set()
(1, 2, 3, 4)
```

**iteration(algorithm='breadth', tracking_words=True)**
Return an iterator going through all elements in self.

**INPUT:**

- **algorithm** (default: 'breadth') – must be one of the following:
  - 'breadth' - iterate over in a linear extension of the weak order
  - 'depth' - iterate by a depth-first-search
- **tracking_words** (default: True) – whether or not to keep track of the reduced words and store them in _reduced_word

**Note:** The fastest iteration is the depth first algorithm without tracking words. In particular, 'depth' is ~1.5x faster.

**EXAMPLES:**

```python
sage: W = WeylGroup(['B',2], implementation="permutation")
sage: for w in W.iteration("breadth", True):
    (continues on next page)
```
....:    print("%s %s"%(w, w._reduced_word))
() []
(1,3)(2,6)(5,7) [1]
(1,5)(2,4)(6,8) [0]
(1,7,5,3)(2,4,6,8) [0, 1]
(1,3,5,7)(2,8,6,4) [1, 0]
(2,8)(3,7)(4,6) [1, 0, 1]
(1,7)(3,5)(4,8) [0, 1, 0]
(1,5)(2,6)(3,7)(4,8) [0, 1, 0, 1]

sage: for w in W.iteration("depth", False): w
()
(1,3)(2,6)(5,7)
(1,5)(2,4)(6,8)
(1,3,5,7)(2,8,6,4)
(1,7)(3,5)(4,8)
(1,7,5,3)(2,4,6,8)
(2,8)(3,7)(4,6)
(1,5)(2,6)(3,7)(4,8)

number_of_reflections()  
Return the number of reflections in self.

EXAMPLES:

sage: W = WeylGroup(['D',4], implementation="permutation")
sage: W.number_of_reflections()  
12

positive_roots()  
Return the positive roots of self.

EXAMPLES:

sage: W = WeylGroup(['C',3], implementation="permutation")
sage: W.positive_roots()  
((1, 0, 0),
 (0, 1, 0),
 (0, 0, 1),
 (1, 1, 0),
 (0, 1, 1),
 (0, 2, 1),
 (1, 2, 1),
 (1, 0, 1))

rank()  
Return the rank of self.

EXAMPLES:

sage: W = WeylGroup(['A',4], implementation="permutation")
sage: W.rank()  
4
reflection_index_set()
Return the index set of reflections of self.

EXAMPLES:

```python
sage: W = WeylGroup(['A',3], implementation="permutation")
sage: W.reflection_index_set()
(1, 2, 3, 4, 5, 6)
```

reflections()
Return the reflections of self.

EXAMPLES:

```python
sage: W = WeylGroup(['B',2], implementation="permutation")
sage: W.distinguished_reflections()
Finite family {1: (1,5)(2,4)(6,8), 2: (1,3)(2,6)(5,7),
            3: (2,8)(3,7)(4,6), 4: (1,7)(3,5)(4,8)}
```

roots()
Return the roots of self.

EXAMPLES:

```python
sage: W = WeylGroup(['G',2], implementation="permutation")
sage: W.roots()
((1, 0),
 (0, 1),
 (1, 1),
 (3, 1),
 (2, 1),
 (3, 2),
 (-1, 0),
 (0, -1),
 (-1, -1),
 (-3, -1),
 (-2, -1),
 (-3, -2))
```

simple_reflection(i)
Return the i-th simple reflection of self.

EXAMPLES:

```python
sage: W = WeylGroup(['A',4], implementation="permutation")
sage: W.simple_reflection(1)
(1,11)(2,5)(6,8)(9,10)(12,15)(16,18)(19,20)
sage: W.simple_reflections()
Finite family {1: (1,11)(2,5)(6,8)(9,10)(12,15)(16,18)(19,20),
               2: (1,5)(2,12)(3,6)(7,9)(11,15)(13,16)(17,19),
               3: (2,6)(3,13)(4,7)(5,8)(12,16)(14,17)(15,18),
               4: (3,7)(4,14)(6,9)(8,10)(13,17)(16,19)(18,20)}
```

simple_root_index(i)
Return the index of the simple root \(\alpha_i\).
This is the position of $\alpha_i$ in the list of simple roots.

EXAMPLES:

```python
sage: W = WeylGroup(['A',3], implementation="permutation")
sage: [W.simple_root_index(i) for i in W.index_set()]
[0, 1, 2]
```

```python
simple_roots()
```

Return the simple roots of self.

EXAMPLES:

```python
sage: W = WeylGroup(['A',4], implementation="permutation")
sage: W.simple_roots()
Finite family {1: (1, 0, 0, 0), 2: (0, 1, 0, 0), 3: (0, 0, 1, 0), 4: (0, 0, 0, 1)}
```

### 5.1.273 Rooted (Unordered) Trees

**AUTHORS:**

- Florent Hivert (2011): initial version

**class** `sage.combinat.rooted_tree.LabelledRootedTree(parent, children, label=None, check=True)`

Bases: `AbstractLabelledClonableTree, RootedTree`

Labelled rooted trees.

A labelled rooted tree is a rooted tree with a label attached at each node.

More formally: The *labelled rooted trees* are an inductive datatype defined as follows: A labelled rooted tree is a multiset of labelled rooted trees, endowed with a label (which can be any object, including `None`). The trees that belong to this multiset are said to be the *children* of the tree. (Notice that the labels of these children may and may not be of the same type as the label of the tree). A labelled rooted tree which has no children (so the only information it carries is its label) is said to be a *leaf*.

Every labelled rooted tree gives rise to an unlabelled rooted tree (`RootedTree`) by forgetting the labels. (This is implemented as a conversion.)

**INPUT:**

- `children` – a list or tuple or more generally any iterable of trees or objects convertible to trees
- `label` – any hashable Sage object (default is `None`)

**EXAMPLES:**

```python
sage: x = LabelledRootedTree([], label = 3); x
3[]
sage: LabelledRootedTree([x, x, x], label = 2)
2[3[], 3[], 3[]]
sage: LabelledRootedTree((x, x, x), label = 2)
2[3[], 3[], 3[]]
sage: LabelledRootedTree([[],[[], []]], label = 3)
3[None[], None[None[], None[]]]
```

Children are reordered using the value of the `sort_key()` method:
sage: y = LabelledRootedTree([], label = 5); y
5[]
sage: xyy2 = LabelledRootedTree((x, y, y), label = 2); xyy2
2[3[], 5[], 5[]]
sage: yxy2 = LabelledRootedTree((y, x, y), label = 2); yxy2
2[3[], 5[], 5[]]
sage: xyy2 == yxy2
True

Converting labelled into unlabelled rooted trees by forgetting the labels, and back (the labels are initialized as None):

sage: xyy2crude = RootedTree(yxy2); xyy2crude
[[], [], []]
sage: LabelledRootedTree(xyy2crude)
None[None[], None[], None[]]

sort_key()

Return a tuple of nonnegative integers encoding the labelled rooted tree self.

The first entry of the tuple is a pair consisting of the number of children of the root and the label of the root. Then the rest of the tuple is obtained as follows: List the tuples corresponding to all children (we are regarding the children themselves as trees). Order this list (not the tuples!) in lexicographically increasing order, and flatten it into a single tuple.

This tuple characterizes the labelled rooted tree uniquely, and can be used to sort the labelled rooted trees provided that the labels belong to a type which is totally ordered.

Note: The tree self must be normalized before calling this method (see normalize()). This doesn’t matter unless you are inside the clone() context manager, because outside of it every rooted tree is already normalized.

Note: This method overrides RootedTree.sort_key() and returns a result different from what the latter would return, as it wants to encode the whole labelled tree including its labelling rather than just the unlabelled tree. Therefore, be careful with using this method on subclasses of RootedOrderedTree; under some circumstances they could inherit it from another superclass instead of from RootedTree, which would cause the method to forget the labelling. See the docstrings of RootedTree.sort_key() and sage.combinat.ordered_tree.OrderedTree.sort_key().

EXAMPLES:

sage: LRT = LabelledRootedTrees(); LRT
Labelled rooted trees
sage: x = LRT([], label = 3); x
3[]
sage: x.sort_key()
((0, 3),)
sage: y = LRT([x, x, x], label = 2); y
2[3[], 3[], 3[]]
sage: y.sort_key()
((3, 2), (0, 3), (0, 3), (0, 3))
(continues on next page)
sage: LRT.an_element().sort_key()
((3, 'alpha'), (0, 3), (1, 5), (0, None), (2, 42), (0, 3), (0, 3))
sage: lb = RootedTrees([[], [[], []]]).canonical_labelling()
sage: lb.sort_key()
((2, 1), (0, 2), (2, 3), (0, 4), (0, 5))

class sage.combinat.rooted_tree.LabelledRootedTrees
    Bases: UniqueRepresentation, Parent

This is a parent stub to serve as a factory class for labelled rooted trees.

EXAMPLES:

sage: LRT = LabelledRootedTrees(); LRT
Labelled rooted trees
sage: x = LRT([], label = 3); x
3[]
sage: x.parent() is LRT
True
sage: y = LRT([x, x, x], label = 2); y
2[3[], 3[], 3[]]
sage: y.parent() is LRT
True

Todo: Add the possibility to restrict the labels to a fixed set.

class sage.combinat.rooted_tree.LabelledRootedTrees_all(category=None)
    Bases: LabelledRootedTrees

Class of all (unordered) labelled rooted trees.

See LabelledRootedTree for a definition.

Element
    alias of LabelledRootedTree

labelled_trees()
    Return the set of labelled trees associated to self.

    EXAMPLES:

    sage: LabelledRootedTrees().labelled_trees()
    Labelled rooted trees

unlabelled_trees()
    Return the set of unlabelled trees associated to self.

    EXAMPLES:

    sage: LabelledRootedTrees().unlabelled_trees()
    Rooted trees

class sage.combinat.rooted_tree.RootedTree(parent=None, children=[], check=True)
    Bases: AbstractClonableTree, NormalizedClonableList
The class for unordered rooted trees.

The **unordered rooted trees** are an inductive datatype defined as follows: An unordered rooted tree is a multiset of unordered rooted trees. The trees that belong to this multiset are said to be the *children* of the tree. The tree that has no children is called a *leaf*.

The *labelled rooted trees* (*LabelledRootedTree*) form a subclass of this class; they carry additional data.

One can create a tree from any list (or more generally iterable) of trees or objects convertible to a tree.

**EXAMPLES:**

```python
sage: RootedTree([])
[]
sage: RootedTree([[], [[]]])
[[[], [[]]]]
sage: RootedTree([[], []])
[[[], []]]
sage: O = OrderedTree([[], []]); O
[[[], []]]
sage: RootedTree(O)  # this is O with the ordering forgotten
[[[], []]]
```

One can also enter any small rooted tree (“small” meaning that no vertex has more than 15 children) by using a simple numerical encoding of rooted trees, namely, the *from_hexacode()* function. (This function actually parametrizes ordered trees, and here we make it parametrize unordered trees by forgetting the ordering.)

```python
sage: from sage.combinat.abstract_tree import from_hexacode
sage: RT = RootedTrees()
sage: from_hexacode('32001010', RT)
[[[], [[]], [], []]]
```

**Note:** Unlike an ordered tree, an (unordered) rooted tree is a multiset (rather than a list) of children. That is, two ordered trees which differ from each other by switching the order of children are equal to each other as (unordered) rooted trees. Internally, rooted trees are encoded as `sage.structure.list_clone.NormalizedClonableList` instances, and instead of storing their children as an actual multiset, they store their children as a list which is sorted according to their `sort_key()` value. This is as good as storing them as multisets, since the `sort_key()` values are sortable and distinguish different (unordered) trees. However, if you wish to define a subclass of `RootedTree` which implements rooted trees with extra structure (say, a class of edge-colored rooted trees, or a class of rooted trees with a cyclic order on the list of children), then the inherited `sort_key()` method will no longer distinguish different trees (and, as a consequence, equal trees will be regarded as distinct). Thus, you will have to override the method by one that does distinguish different trees.

### `graft_list(other)`

Return the list of trees obtained by grafting *other* on *self*.

Here grafting means that one takes the disjoint union of *self* and *other*, chooses a node of *self*, and adds the root of *other* to the list of children of this node. The root of the resulting tree is the root of *self*. (This can be done for each node of *self*; this method returns the list of all results.)

This is useful for free pre-Lie algebras.

**EXAMPLES:**

```python
sage: RT = RootedTree
sage: x = RT([])
```
sage: y = RT([x, x])
sage: x.graft_list(x)
[[]]
sage: l = y.graft_list(x); l
[[[], []], [[], [[]]], [[], []]]
sage: [parent(i) for i in l]
[Rooted trees, Rooted trees, Rooted trees]


\textbf{graft\_on\_root}(\texttt{other})

Return the tree obtained by grafting \texttt{other} on the root of \texttt{self}.

Here grafting means that one takes the disjoint union of \texttt{self} and \texttt{other}, and adds the root of \texttt{other} to the list of children of \texttt{self}. The root of the resulting tree is the root of \texttt{self}.

This is useful for free Nap algebras.

\textbf{EXAMPLES:}

sage: RT = RootedTree
sage: x = RT(())
sage: y = RT([x, x])
sage: x.graft_on_root(x)
[[]]
sage: y.graft_on_root(x)
[[[], []], [[], [[]]], [[], []]]
sage: x.graft_on_root(y)
[[[], []], [[], []]]


\textbf{is\_empty}()

Return if \texttt{self} is the empty tree.

For rooted trees, this always returns \texttt{False}.

\textbf{Note:} This is not the same as \texttt{bool(t)}, which returns whether \texttt{t} has some child or not.

\textbf{EXAMPLES:}

sage: t = RootedTrees(4)([[[], [[]]]])
sage: t.is_empty()
False
sage: bool(t)
True
sage: t = RootedTrees(1)([])
sage: t.is_empty()
False
sage: bool(t)
False


\textbf{normalize}()

Normalize \texttt{self}.

This function is at the core of the implementation of rooted (unordered) trees. The underlying structure is provided by ordered rooted trees. Every rooted tree is represented by a normalized element in the set of its planar embeddings.
There should be no need to call `normalize` directly as it is called automatically upon creation and cloning or modification (by `NormalizedClonableList`).

The normalization has a recursive definition. It means first that every sub-tree is itself normalized, and also that sub-trees are sorted. Here the sort is performed according to the values of the `sort_key()` method.

**EXAMPLES:**

```python
sage: RT = RootedTree
sage: RT([[[]], [[]]]) == RT([[[]], [[]]])  # indirect doctest
True
sage: rt1 = RT([[[]], [[]]])
sage: rt2 = RT([[[]], [[]]])
sage: rt1 is rt2
False
sage: rt1 == rt2
True
sage: rt1._get_list() == rt2._get_list()
True
```

**single_graft**(x, grafting_function, path_prefix=())

Graft subtrees of x on self using the given function.

Let \(x_1, x_2, \ldots, x_p\) be the children of the root of x. For each \(i\), the subtree of x comprising all descendants of \(x_i\) is joined by a new edge to the vertex of self specified by the \(i\)-th path in the grafting function (i.e., by the path `grafting_function[i]`).

The number of vertices of the result is the sum of the numbers of vertices of self and x minus one, because the root of x is not used.

This is used to define the product of the Grossman-Larson algebras.

**INPUT:**

- x – a rooted tree
- grafting_function – a list of paths in self
- path_prefix – optional tuple (default ()

The `path_prefix` argument is only used for internal recursion.

**EXAMPLES:**

```python
sage: LT = LabelledRootedTrees()
sage: y = LT([LT([], label='b'),], label='a')
sage: x = LT([LT([], label='d'),], label='c')
sage: y.single_graft(x,[(0,),])
a[b[d[]]]
sage: t = LT([LT([], label='b'),LT([], label='c')], label='a')
sage: s = LT([LT([], label='d'),LT([], label='e')], label='f')
sage: t.single_graft(s,[(0,),(1,)])
a[b[d[]], c[e[]]]
```

**sort_key()**

Return a tuple of nonnegative integers encoding the rooted tree self.

The first entry of the tuple is the number of children of the root. Then the rest of the tuple is obtained as follows: List the tuples corresponding to all children (we are regarding the children themselves as trees). Order this list (not the tuples!) in lexicographically increasing order, and flatten it into a single tuple.
This tuple characterizes the rooted tree uniquely, and can be used to sort the rooted trees.

**Note:** The tree `self` must be normalized before calling this method (see `normalize()`). This doesn't matter unless you are inside the `clone()` context manager, because outside of it every rooted tree is already normalized.

**Note:** By default, this method does not encode any extra structure that `self` might have. If you have a subclass inheriting from `RootedTree` which allows for some extra structure, you need to override `sort_key()` in order to preserve this structure (for example, the `LabelledRootedTree` class does this in `LabelledRootedTree.sort_key()`). See the note in the docstring of `sage.combinat.ordered_tree.OrderedTree.sort_key()` for a pitfall.

**EXAMPLES:**

```python
sage: RT = RootedTree
sage: RT([[[]],[]]).sort_key()
(2, 0, 1, 0)
sage: RT([[[],[]]]).sort_key()
(2, 0, 1, 0)
```

```python
class sage.combinat.rooted_tree.RootedTrees
    Bases: UniqueRepresentation, Parent
    Factory class for rooted trees.

    INPUT:
    • size – (optional) an integer

    OUTPUT:
    the set of all rooted trees (of the given size `size` if specified)

    EXAMPLES:

    ```python
    sage: RootedTrees()
    Rooted trees
    sage: RootedTrees(2)
    Rooted trees with 2 nodes
    ```

```python
class sage.combinat.rooted_tree.RootedTrees_all
    Bases: DisjointUnionEnumeratedSets, RootedTrees
    Class of all (unordered, unlabelled) rooted trees.

    See `RootedTree` for a definition.

    **Element**
    alias of `RootedTree`

    **labelled_trees()**
    Return the set of labelled trees associated to `self`.

    **EXAMPLES:**
```
Labelled rooted trees

As a consequence:

```python
sage: lb = RootedTrees()([[[]],[[], [[]]]]).canonical_labelling()
sage: lb
1[[2[]], 3[4[], 5[]]]
sage: lb.__class__
<class 'sage.combinat.rooted_tree.LabelledRootedTrees_all_with_category.element_class'>
sage: lb.parent()
Labelled rooted trees
```

leaf()

Return a leaf tree with self as parent.

EXAMPLES:

```python
sage: RootedTrees().leaf()
[]
```

unlabelled_trees()

Return the set of unlabelled trees associated to self.

EXAMPLES:

```python
sage: RootedTrees().unlabelled_trees()
Rooted trees
```

class sage.combinat.rooted_tree.RootedTrees.size(n)

Bases: RootedTrees

The enumerated set of rooted trees with a given number of nodes.

The number of nodes of a rooted tree is defined recursively: The number of nodes of a rooted tree with $a$ children is $a$ plus the sum of the number of nodes of each of these children.

cardinality()

Return the cardinality of self.

EXAMPLES:

```python
sage: RootedTrees(1).cardinality()
1
sage: RootedTrees(3).cardinality()
2
```

check_element(el, check=True)

Check that a given tree actually belongs to self.

This just checks the number of vertices.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: RT3 = RootedTrees(3)
sage: RT3([[[]],[[]]])  # indirect doctest
[[[]], [[]]]
sage: RT3([[[]],[[]],[[]]])  # indirect doctest
Traceback (most recent call last):
... ValueError: wrong number of nodes
```

```
element_class()
```

```python
sage.combinat.rooted_tree.number_of_rooted_trees()

Return the number of rooted trees with \( n \) nodes.

Compute the number \( a(n) \) of rooted trees with \( n \) nodes using the recursive formula ([SL000081]):

\[
a(n + 1) = \frac{1}{n} \sum_{k=1}^{n} \left( \sum_{d|k} da(d) \right) a(n - k + 1)
\]

EXAMPLES:

```python
sage: from sage.combinat.rooted_tree import number_of_rooted_trees
sage: [number_of_rooted_trees(i) for i in range(10)]
[0, 1, 1, 2, 4, 9, 20, 48, 115, 286]
```

REFERENCES:

5.1.274 Robinson-Schensted-Knuth correspondence

AUTHORS:

- Travis Scrimshaw (2012-12-07): Initial version
- Chaman Agrawal (2019-06-24): Refactoring on the Rule class
- Matthew Lancellotti (2018): initial version of super RSK
- Jianping Pan, Wencin Poh, Anne Schilling (2020-08-31): initial version of RuleStar

Introduction

The Robinson-Schensted-Knuth (RSK) correspondence is most naturally stated as a bijection between generalized permutations (also known as two-line arrays, biwords, ...) and pairs of semi-standard Young tableaux \((P, Q)\) of identical shape.

The basic operation in the RSK correspondence is a row insertion \( P \leftarrow k \) (where \( P \) is a given semi-standard Young tableau, and \( k \) is an integer). Different insertion algorithms have been implemented for the RSK correspondence and can be specified as an argument in the function call.

EXAMPLES:

We can perform RSK and its inverse map on a variety of objects:

```python
sage: p = Tableau([[[1,2,2],[2]]]); q = Tableau([[[1,3,3],[2]]])
sage: gp = RSK_inverse(p, q); gp
[[[1, 2, 3, 3], [2, 1, 2, 2]]]
```
sage: RSK("gp")  # RSK of a biword
[[[1, 2, 2], [2]], [[1, 3, 3], [2]]]
sage: RSK([2,3,2,1,2,3])  # Robinson-Schensted of a word
[[[1, 2, 3], [2], [3]], [[1, 2, 5, 6], [3], [4]]]
sage: RSK([2,3,2,1,2,3], insertion=RSK.rules.EG)  # Edelman-Greene
[[[1, 2, 3], [2, 3], [3]], [[1, 2, 6], [3, 5], [4]]]
sage: m = RSK_inverse(p, q, 'matrix'); m  # output as matrix
[[0 1]
 [1 0]
[0 2]]
sage: RSK(m)  # RSK of a matrix
[[[1, 2, 2], [2]], [[1, 3, 3], [2]]]

Insertions currently available

The following insertion algorithms for RSK correspondence are currently available:

- RSK insertion (RuleRSK).
- Edelman-Greene insertion (RuleEG), an algorithm defined in [EG1987] Definition 6.20 (where it is referred to as Coxeter-Knuth insertion).
- Hecke RSK algorithm (RuleHecke), defined using the Hecke insertion studied in [BKSTY06] (but using rows instead of columns).
- Dual RSK insertion (RuleDualRSK).
- CoRSK insertion (RuleCoRSK), defined in [GR2018v5sol].
- Super RSK insertion (RuleSuperRSK), a combination of row and column insertions defined in [Muth2019].
- Star insertion (RuleStar), defined in [MPPS2020].

Implementing your own insertion rule

The functions RSK() and RSK_inverse() are written so that it is easy to implement insertion algorithms you come across in your research.

To implement your own insertion algorithm, you first need to import the base class for a rule:

sage: from sage.combinat.rsk import Rule

Using the Rule class as parent class for your insertion rule, first implement the insertion and the reverse insertion algorithm for RSK() and RSK_inverse() respectively (as methods forward_rule and backward_rule). If your insertion algorithm uses the same forward and backward rules as RuleRSK, differing only in how an entry is inserted into a row, then this is not necessary, and it suffices to merely implement the insertion and reverse_insertion methods.

For more information, see Rule.

REFERENCES:

class sage.combinat.rsk.InsertionRules
    Bases: object

    Catalog of rules for RSK-like insertion algorithms.
Perform the Robinson-Schensted-Knuth (RSK) correspondence.

The Robinson-Schensted-Knuth (RSK) correspondence (also known as the RSK algorithm) is most naturally stated as a bijection between generalized permutations (also known as two-line arrays, biwords, ...) and pairs of semi-standard Young tableaux \((P, Q)\) of identical shape. The tableau \(P\) is known as the insertion tableau, and \(Q\) is known as the recording tableau.

The basic operation is known as row insertion \(P \leftarrow k\) (where \(P\) is a given semi-standard Young tableau, and \(k\) is an integer). Row insertion is a recursive algorithm which starts by setting \(k_0 = k\), and in its \(i\)-th step inserts the number \(k_i\) into the \(i\)-th row of \(P\) (we start counting the rows at 0) by replacing the first integer greater than \(k_i\) in the row by \(k_i\) and defines \(k_i+1\) as the integer that has been replaced. If no integer greater than \(k_i\) exists in the \(i\)-th row, then \(k_i\) is simply appended to the row and the algorithm terminates at this point.

A generalized permutation (or biword) is a list \(((j_0, k_0), (j_1, k_1), \ldots, (j_{\ell-1}, k_{\ell-1}))\) of pairs such that the letters \(j_0, j_1, \ldots, j_{\ell-1}\) are weakly increasing (that is, \(j_0 \leq j_1 \leq \cdots \leq j_{\ell-1}\), whereas the letters \(k_i\) satisfy \(k_i \leq k_{i+1}\) whenever \(j_i = j_{i+1}\). The \(\ell\)-tuple \((j_0, j_1, \ldots, j_{\ell-1})\) is called the top line of this generalized permutation, whereas the \(\ell\)-tuple \((k_0, k_1, \ldots, k_{\ell-1})\) is called its bottom line.

Now the RSK algorithm, applied to a generalized permutation \(p = ((j_0, k_0), (j_1, k_1), \ldots, (j_{\ell-1}, k_{\ell-1}))\) (encoded as a lexicographically sorted list of pairs) starts by initializing two semi-standard tableaux \(P_0\) and \(Q_0\) as empty tableaux. For each nonnegative integer \(t\) starting at 0, take the pair \((j_t, k_t)\) from \(p\) and set \(P_{t+1} = P_t \leftarrow k_t\), and define \(Q_{t+1}\) by adding a new box filled with \(j_t\) to the tableau \(Q_t\) at the same location the row insertion on \(P_t\) ended (that is to say, adding a new box with entry \(j_t\) such that \(P_{t+1}\) and \(Q_{t+1}\) have the same shape). The iterative process stops when \(t\) reaches the size of \(p\), and the pair \((P_t, Q_t)\) at this point is the image of \(p\) under the Robinson-Schensted-Knuth correspondence.

This correspondence has been introduced in [Knu1970], where it has been referred to as “Construction A”.

For more information, see Chapter 7 in [Sta-EC2].
INPUT:

- **obj1, obj2** – can be one of the following:
  - a word in an ordered alphabet (in this case, **obj1** is said word, and **obj2** is **None**)
  - an integer matrix
  - two lists of equal length representing a generalized permutation (namely, the lists \((j_0, j_1, \ldots, j_{\ell-1})\) and \((k_0, k_1, \ldots, k_{\ell-1})\) represent the generalized permutation \(((j_0, k_0), (j_1, k_1), \ldots, (j_{\ell-1}, k_{\ell-1}))\)
  - any object which has a method \_rsk\_iter() which returns an iterator over the object represented as generalized permutation or a pair of lists (in this case, **obj1** is said object, and **obj2** is **None**).

- **insertion** – (default: RSK.rules.RSK) the following types of insertion are currently supported:
  - RSK.rules.RSK (or 'RSK') – Robinson-Schensted-Knuth insertion (RuleRSK)
  - RSK.rules.EG (or 'EG') – Edelman-Greene insertion (only for reduced words of permutations/elements of a type ACoxeter group) (RuleEG)
  - RSK.rules.Hecke (or 'hecke') – Hecke insertion (only guaranteed for generalized permutations whose top row is strictly increasing) (RuleHecke)
  - RSK.rules.dualRSK (or 'dualRSK') – Dual RSK insertion (only for strict biwords) (RuleDualRSK)
  - RSK.rules.coRSK (or 'coRSK') – CoRSK insertion (only for strict cobiwords) (RuleCoRSK)
  - RSK.rules.superRSK (or 'super') – Super RSK insertion (only for restricted super biwords) (RuleSuperRSK)
  - RSK.rules.Star (or 'Star') – Star-insertion (only for fully commutative words in the 0-Hecke monoid) (RuleStar)

- **check_standard** – (default: False) check if either of the resulting tableaux is a standard tableau, and if so, typecast it as such.

For precise information about constraints on the input and output, as well as the definition of the algorithm (if it is not standard RSK), see the particular Rule class.

EXAMPLES:

If we only input one row, it is understood that the top row should be \((1, 2, \ldots, n)\):

```python
sage: RSK([3,3,2,4,1])
[[[1, 3, 4], [2], [3]], [[1, 2, 4], [3], [5]]]
sage: RSK(Word([3,3,2,4,1]))
[[[1, 3, 4], [2], [3]], [[1, 2, 4], [3], [5]]]
sage: RSK(Word([2,3,3,2,1,3,2,3]))
[[[1, 2, 2, 3, 3], [2, 3], [3]], [[1, 2, 3, 6, 8], [4, 7], [5]]]
```

We can provide a generalized permutation:

```python
sage: RSK([1,2,2,2], [2,1,1,2])
[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]
sage: RSK(Word([1,1,3,4,4]), [1,4,2,1,3])
[[[1, 1, 3], [2], [4]], [[1, 1, 4], [3], [4]]]
sage: RSK([1,3,3,4,4], Word([6,2,2,1,7]))
[[[1, 2, 7], [2], [6]], [[1, 3, 4], [3], [4]]]
```

We can provide a matrix:
We can also provide something looking like a matrix:

```
sage: RSK([[0,1],[2,1]])

[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]
```

There is also `RSK_inverse()` which performs the inverse of the bijection on a pair of semistandard tableaux. We note that the inverse function takes 2 separate tableaux as inputs, so to compose with `RSK()`, we need to use the python `*` on the output:

```
sage: RSK_inverse("RSK([[1, 2, 2, 2], [2, 1, 1, 2]])

[[[1, 2, 2, 2], [2, 1, 1, 2]], [[1, 2, 2, 2], [2, 1, 1, 2]]]
  sage: P,Q = RSK([1, 2, 2, 2], [2, 1, 1, 2])
  sage: RSK_inverse(P, Q)

[[1, 2, 2, 2], [2, 1, 1, 2]]
```

`sage.combinat.rsk.RSK_inverse(p, q, output='array', insertion=<class 'sage.combinat.rsk.RuleRSK'>)

Return the generalized permutation corresponding to the pair of tableaux \((p, q)\) under the inverse of the Robinson-Schensted-Knuth correspondence.

For more information on the bijection, see `RSK()`.

**INPUT:**

- \(p, q\) – two semi-standard tableaux of the same shape, or (in the case when Hecke insertion is used) an increasing tableau and a set-valued tableau of the same shape (see the note below for the format of the set-valued tableau)
- **output** – (default: 'array') if \(q\) is semi-standard:
  - 'array' – as a two-line array (i.e. generalized permutation or biword)
  - 'matrix' – as an integer matrix
and if \(q\) is standard, we can also have the output:
- 'word' – as a word
and additionally if \(p\) is standard, we can also have the output:
- 'permutation' – as a permutation
- **insertion** – (default: `RSK.rules.RSK`) the insertion algorithm used in the bijection. Currently the following are supported:
  - `RSK.rules.RSK` (or 'RSK') – Robinson-Schensted-Knuth insertion (`RuleRSK`)
  - `RSK.rules.EG` (or 'EG') – Edelman-Greene insertion (only for reduced words of permutations/elements of a type \(A\) Coxeter group) (`RuleEG`)
  - `RSK.rules.Hecke` (or 'hecke') – Hecke insertion (only guaranteed for generalized permutations whose top row is strictly increasing) (`RuleHecke`)
  - `RSK.rules.dualRSK` (or 'dualRSK') – Dual RSK insertion (only for strict biwords) (`RuleDualRSK`)
  - `RSK.rules.coRSK` (or 'coRSK') – CoRSK insertion (only for strict cobiwords) (`RuleCoRSK`)
  - `RSK.rules.superRSK` (or 'super') – Super RSK insertion (only for restricted super biwords) (`RuleSuperRSK`)
- \texttt{RSK.rules.Star} (or ‘Star’) – ⋆-insertion (only for fully commutative words in the 0-Hecke monoid) \textit{(RuleStar)}

For precise information about constraints on the input and output, see the particular \textit{Rule} class.

\textbf{Note}: In the case of Hecke insertion, the input variable $q$ should be a set-valued tableau, encoded as a tableau whose entries are strictly increasing tuples of positive integers. Each such tuple encodes the set of its entries.

\textbf{EXAMPLES}:

If both $p$ and $q$ are standard:

```python
sage: t1 = Tableau([[1, 2, 5], [3], [4]])
sage: t2 = Tableau([[1, 2, 3], [4], [5]])
sage: RSK_inverse(t1, t2)
[[1, 2, 3, 4, 5], [1, 4, 5, 3, 2]]
sage: RSK_inverse(t1, t2, 'word')
word: 14532
```

```python
sage: RSK_inverse(t1, t2, 'matrix')
[[1 0 0 0 0]
 [0 0 1 0]
 [0 0 0 1]
 [0 0 1 0]
 [0 1 0 0]]
```

```python
sage: RSK_inverse(t1, t2, 'permutation')
[1, 4, 5, 3, 2]
```

If the first tableau is semistandard:

```python
sage: p = Tableau([[1,2,2], [3]])
```

```python
sage: q = Tableau([[1,2,4], [3]])
```

```python
sage: ret = RSK_inverse(p, q); ret
[[1, 2, 3, 4], [1, 3, 2, 2]]
```

```python
sage: RSK_inverse(p, q, 'word')
word: 1322
```

In general:

```python
sage: p = Tableau([[1,2,2], [2]])
```

```python
sage: q = Tableau([[1,3,3], [2]])
```

```python
sage: RSK_inverse(p, q)
[[1, 2, 3, 3], [2, 1, 2, 2]]
```

```python
sage: RSK_inverse(p, q, 'matrix')
[0 1]
[1 0]
[0 2]
```

Using Hecke insertion:
sage: w = [5, 4, 3, 1, 4, 2, 5, 5]
sage: pq = RSK(w, insertion=RSK.rules.Hecke)
sage: RSK_inverse(*pq, insertion=RSK.rules.Hecke, output='list')
[5, 4, 3, 1, 4, 2, 5, 5]

Note: The constructor of Tableau accepts not only semistandard tableaux, but also arbitrary lists that are fillings of a partition diagram. (And such lists are used, e.g., for the set-valued tableau q that is passed to RSK_inverse(p, q, insertion='hecke').) The user is responsible for ensuring that the tableaux passed to RSK_inverse are of the right types (semistandard, standard, increasing, set-valued as needed).

class sage.combinat.rsk.Rule
Bases: UniqueRepresentation

Generic base class for an insertion rule for an RSK-type correspondence.

An instance of this class should implement a method insertion() (which can be applied to a letter j and a list r, and modifies r in place by “bumping” j into it appropriately; it then returns the bumped-out entry or None if no such entry exists) and a method reverse_insertion() (which does the same but for reverse bumping). It may also implement _backward_format_output() and _forward_format_output() if the RSK correspondence should return something other than (semi)standard tableaux (in the forward direction) and matrices or biwords (in the backward direction). The to_pairs() method should also be overridden if the input for the (forward) RSK correspondence is not the usual kind of biwords (i.e., pairs of two n-tuples \([a_1, a_2, \ldots, a_n]\) and \([b_1, b_2, \ldots, b_n]\) satisfying \((a_1, b_1) \leq (a_2, b_2) \leq \cdots \leq (a_n, b_n)\) in lexicographic order). Finally, if forward_rule() and backward_rule() have to be overridden if the overall structure of the RSK correspondence differs from that of classical RSK (see, e.g., the case of Hecke insertion, in which a letter bumped into a row may change a different row).

backward_rule(p, q, output)

Return the generalized permutation obtained by applying reverse insertion to a pair of tableaux \((p, q)\).

INPUT:

- p, q – two tableaux of the same shape.
- output – (default: 'array') if q is semi-standard:
  - 'array' – as a two-line array (i.e. generalized permutation or biword)
  - 'matrix' – as an integer matrix
and if q is standard, we can also have the output:
- 'word' – as a word
  and additionally if p is standard, we can also have the output:
- 'permutation' – as a permutation

EXAMPLES:

sage: from sage.combinat.rsk import RuleRSK
sage: t1 = Tableau([[1, 3, 4], [2], [3]])
sage: t2 = Tableau([[1, 2, 4], [3], [5]])
sage: RuleRSK().backward_rule(t1, t2, 'array')
[[1, 2, 3, 4, 5], [3, 3, 2, 4, 1]]
sage: t1 = Tableau([[1, 1, 1, 3, 7]])
sage: t2 = Tableau([[1, 2, 3, 4, 5]])
sage: RuleRSK().backward_rule(t1, t2, 'array')
[[1, 2, 3, 4, 5], [1, 1, 1, 3, 7]]
sage: t1 = Tableau([[1, 3], [3], [6], [7]])
sage: t2 = Tableau([[1, 4], [2], [3], [5]])
sage: RuleRSK().backward_rule(t1, t2, 'array')
[[1, 2, 3, 4, 5], [7, 6, 3, 3, 1]]

forward_rule(obj1, obj2, check_standard=False, check=True)

Return a pair of tableaux obtained by applying forward insertion to the generalized permutation [obj1, obj2].

INPUT:

• obj1, obj2 – can be one of the following ways to represent a generalized permutation (or, equivalently, biword):
  – two lists obj1 and obj2 of equal length, to be interpreted as the top row and the bottom row of the biword
  – a matrix obj1 of nonnegative integers, to be interpreted as the generalized permutation in matrix form (in this case, obj2 is None)
  – a word obj1 in an ordered alphabet, to be interpreted as the bottom row of the biword (in this case, obj2 is None; the top row of the biword is understood to be (1, 2, ..., n) by default)
  – any object obj1 which has a method _rsk_iter(), as long as this method returns an iterator yielding pairs of numbers, which then are interpreted as top entries and bottom entries in the biword (in this case, obj2 is None)
• check_standard – (default: False) check if either of the resulting tableaux is a standard tableau, and if so, typecast it as such
• check – (default: True) whether to check that obj1 and obj2 actually define a valid biword

EXAMPLES:

sage: from sage.combinat.rsk import RuleRSK
sage: RuleRSK().forward_rule([3,3,2,4,1], None)
[[[1, 3, 4], [2], [3]], [[1, 2, 4], [3], [5]]]
sage: RuleRSK().forward_rule([1, 1, 1, 3, 7], None)
[[[1, 1, 1, 3, 7]], [[1, 2, 3, 4, 5]]]
sage: RuleRSK().forward_rule([7, 6, 3, 3, 1], None)
[[[1, 3], [3], [6], [7]], [[1, 4], [2], [3], [5]]]

to_pairs(obj1=None, obj2=None, check=True)

Given a valid input for the RSK algorithm, such as two n-tuples obj1 = [a1, a2, ..., an] and obj2 = [b1, b2, ..., bn] forming a biword (i.e., satisfying a1 ≤ a2 ≤ ··· ≤ an, and if ai = ai+1, then bi ≤ bi+1), or a matrix (“generalized permutation”), or a single word, return the array [(a1, b1), (a2, b2), ..., (an, bn)].

INPUT:

• obj1, obj2 – anything representing a biword (see the doc of forward_rule() for the encodings accepted).
• check – (default: True) whether to check that obj1 and obj2 actually define a valid biword.

EXAMPLES:
```python
sage: from sage.combinat.rsk import Rule
sage: list(Rule().to_pairs([1, 2, 2, 2], [2, 1, 1, 2]))
[(1, 2), (2, 1), (2, 1), (2, 2)]
```

```python
sage: m = Matrix(ZZ, 3, 2, [0, 1, 1, 0, 0, 2]) ; m
[0 1]
[1 0]
[0 2]
sage: list(Rule().to_pairs(m))
[(1, 2), (2, 1), (3, 2), (3, 2)]
```

**class** `sage.combinat.rsk.RuleCoRSK`

Bases: `RuleRSK`

Rule for coRSK insertion.

CoRSK insertion differs from classical RSK insertion in the following ways:

- The input (in terms of biwords) is no longer a biword, but rather a strict cobiword — i.e., a pair of two lists \([a_1, a_2, \ldots, a_n]\) and \([b_1, b_2, \ldots, b_n]\) that satisfy the strict inequalities \((a_1, b_1) \prec (a_2, b_2) \prec \cdots \prec (a_n, b_n)\), where the binary relation \(\prec\) on pairs of integers is defined by having \((u_1, v_1) \prec (u_2, v_2)\) if and only if either \(u_1 < u_2\) or \((u_1 = u_2\) and \(v_1 > v_2\)). In terms of matrices, this means that the input is not an arbitrary matrix with nonnegative integer entries, but rather a \(\{0, 1\}\)-matrix (i.e., a matrix whose entries are 0’s and 1’s).

- The output still consists of two tableaux \((P, Q)\) of equal shapes, but rather than both of them being semistandard, now \(Q\) is row-strict (i.e., its transpose is semistandard) while \(P\) is semistandard.

Bumping proceeds in the same way as for RSK insertion.

The RSK and coRSK algorithms agree for permutation matrices.

For more information, see Section A.4 in [Ful1997] (specifically, construction (1d)) or the second solution to Exercise 2.7.12(a) in [GR2018v5sol].

**EXAMPLES:**

```python
sage: RSK([[1, 2, 5, 3, 1]], insertion = RSK.rules.coRSK)
[[[1, 1, 3], [2], [5]], [[1, 2, 3], [4], [5]]]
sage: RSK(Word([2, 3, 3, 2, 1, 3, 2, 3]), insertion = RSK.rules.coRSK)
[[[1, 2, 2, 3, 3], [2, 3], [3]], [[1, 2, 3, 6, 8], [4, 7], [5]]]
sage: RSK(Word([3, 3, 2, 4, 1]), insertion = RSK.rules.coRSK)
[[[1, 3, 4], [2], [3]], [[1, 2, 4], [3], [5]]]
sage: from sage.combinat.rsk import to_matrix
sage: RSK(to_matrix([[1, 1, 3, 3, 4], [3, 2, 2, 1, 3]]), insertion = RSK.rules.coRSK)
[[[1, 2, 3], [2], [3]], [[1, 3, 4], [1], [3]]]
```

Using coRSK insertion with a \(\{0, 1\}\)-matrix:

```python
sage: RSK(matrix([[0, 1], [1, 0]]), insertion = RSK.rules.coRSK)
[[[1], [2]], [[1], [2]]]
```

We can also give it something looking like a matrix:

```python
sage: RSK([[0, 1], [1, 0]], insertion = RSK.rules.coRSK)
[[[1], [2]], [[1], [2]]]
```

We can also use the inverse correspondence:
When applied to two standard tableaux, backwards coRSK insertion behaves identically to the usual backwards RSK insertion:

```
sage: t1 = Tableau([[1, 2, 5], [3], [4]])
sage: t2 = Tableau([[1, 2, 3], [4], [5]])
sage: RSK_inverse(t1, t2, insertion=RSK.rules.coRSK)
[[1, 2, 3, 4, 5], [1, 4, 5, 3, 2]]
sage: RSK_inverse(t1, t2, 'word', insertion=RSK.rules.coRSK)
word: 14532
sage: RSK_inverse(t1, t2, 'matrix', insertion=RSK.rules.coRSK)
[1 0 0 0 0]
[0 0 1 0 0]
[0 0 0 0 1]
[0 0 1 0 0]
[0 1 0 0 0]
sage: RSK_inverse(t1, t2, 'permutation', insertion=RSK.rules.coRSK)
[1, 4, 5, 3, 2]
sage: RSK_inverse(t1, t1, 'permutation', insertion=RSK.rules.coRSK)
[1, 4, 3, 2, 5]
sage: RSK_inverse(t2, t2, 'permutation', insertion=RSK.rules.coRSK)
[1, 2, 5, 4, 3]
sage: RSK_inverse(t2, t1, 'permutation', insertion=RSK.rules.coRSK)
[1, 5, 4, 2, 3]
```

For coRSK, the first tableau is semistandard while the second tableau is transpose semistandard:

```
sage: p = Tableau([[1,2,2],[5]]); q = Tableau([[1,2,4],[3]])
sage: ret = RSK_inverse(p, q, insertion=RSK.rules.coRSK); ret
[[1, 2, 3, 4], [1, 5, 2, 2]]
sage: RSK_inverse(p, q, 'word', insertion=RSK.rules.coRSK)
word: 1522
```

For coRSK, the first tableau is semistandard while the second tableau is transpose semistandard:

```
sage: RSK_inverse("RSK([1, 2, 2, 2], [2, 3, 2, 1],
....: insertion=RSK.rules.coRSK),insertion=RSK.rules.coRSK)
[[[1, 2, 2, 2], [2, 3, 2, 1]]
sage: P,Q = RSK([1, 2, 2, 2], [2, 3, 2, 1],insertion=RSK.rules.coRSK)
sage: RSK_inverse(P, Q, insertion=RSK.rules.coRSK)
[[1, 2, 2, 2], [2, 3, 2, 1]]
```

```
5.1. Comprehensive Module List 2765
```

backward_rule(p, q, output)

Return the strict cobiword obtained by applying reverse coRSK insertion to a pair of tableaux (p, q).

INPUT:

- p, q – two tableaux of the same shape
- output – (default: 'array') if q is row-strict:
  - 'array' – as a two-line array (i.e. strict cobiword)
  - 'matrix' – as a \{0, 1\}-matrix
- if q is standard, we can have the output:
  - 'word' – as a word
- if p is standard, we can also have the output:
'permutation' – as a permutation

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleCoRSK
sage: t1 = Tableau([[1, 1, 2], [2, 3], [4]])
sage: t2 = Tableau([[1, 4, 5], [1, 4], [2]])
sage: RuleCoRSK().backward_rule(t1, t2, 'array')
[[1, 1, 2, 4, 4, 5], [4, 2, 1, 3, 1, 2]]
```

to_pairs(obj1=None, obj2=None, check=True)

Given a valid input for the coRSK algorithm, such as two \(n\)-tuples \(\text{obj1} = [a_1, a_2, \ldots, a_n]\) and \(\text{obj2} = [b_1, b_2, \ldots, b_n]\) forming a strict cobiword (i.e., satisfying \(a_1 \leq a_2 \leq \cdots \leq a_n\), and if \(a_i = a_{i+1}\), then \(b_i > b_{i+1}\)), or a \(\{0, 1\}\)-matrix (“rook placement”), or a single word, return the array \([(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)]\).

INPUT:

- obj1, obj2 – anything representing a strict cobiword (see the doc of forward_rule() for the encodings accepted)
- check – (default: True) whether to check that obj1 and obj2 actually define a valid strict cobiword

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleCoRSK
sage: list(RuleCoRSK().to_pairs([1, 2, 2, 2], [2, 3, 2, 1]))
[(1, 2), (2, 3), (2, 2), (2, 1)]
```

class sage.combinat.rsk.RuleDualRSK

Bases: Rule

Rule for dual RSK insertion.

Dual RSK insertion differs from classical RSK insertion in the following ways:

- The input (in terms of biwords) is no longer an arbitrary biword, but rather a strict biword (i.e., a pair of two lists \([a_1, a_2, \ldots, a_n]\) and \([b_1, b_2, \ldots, b_n]\) that satisfy the strict inequalities \((a_1, b_1) < (a_2, b_2) < \cdots < (a_n, b_n)\) in lexicographic order). In terms of matrices, this means that the input is not an arbitrary matrix with nonnegative integer entries, but rather a \(\{0, 1\}\)-matrix (i.e., a matrix whose entries are 0’s and 1’s).
The output still consists of two tableaux \((P, Q)\) of equal shapes, but rather than both of them being semistandard, now \(P\) is row-strict (i.e., its transpose is semistandard) while \(Q\) is semistandard.

The main difference is in the way bumping works. Namely, when a number \(k_i\) is inserted into the \(i\)-th row of \(P\), it bumps out the first integer greater or equal to \(k_i\) in this row (rather than greater than \(k_i\)).

The RSK and dual RSK algorithms agree for permutation matrices.

For more information, see Chapter 7, Section 14 in [Sta-EC2] (where dual RSK is called RSK*) or the third solution to Exercise 2.7.12(a) in [GR2018v5sol].

**EXAMPLES:**

```sage
sage: RSK([3,3,2,4,1], insertion=RSK.rules.dualRSK)
[[[1, 4], [2], [3], [3]], [[1, 4], [2], [3], [5]]]
```

```sage
sage: RSK(Word([3,3,2,4,1]), insertion=RSK.rules.dualRSK)
[[[1, 4], [2], [3], [3]], [[1, 4], [2], [3], [5]]]
```

```sage
sage: RSK(Word([2,3,3,2,1,3,2,3]), insertion=RSK.rules.dualRSK)
[[[1, 2, 3], [2, 3], [2, 3], [3]], [[1, 2, 8], [3, 6], [4, 7], [5]]]
```

Using dual RSK insertion with a strict biword:

```sage
sage: RSK([[1,1,2,4,4,5],[2,4,1,1,3,2]],insertion=RSK.rules.dualRSK)
[[[1, 2], [1, 3], [2, 4]], [[1, 1], [2, 4], [4, 5]]]
```

```sage
sage: RSK(Word([1,1,2,3,3,4,5]), [1,3,2,1,3,2], insertion=RSK.rules.dualRSK)
[[[1, 2, 3], [1, 2], [3], [3]], [[1, 1, 3], [2, 4], [3], [5]]]
```

```sage
sage: RSK([[1, 2, 2, 2], [2, 1, 2, 4], insertion=RSK.rules.dualRSK)
[[[1, 2, 4], [2]], [[1, 2, 2], [2]]]
```

```sage
sage: RSK([1, 2, 2, 2], Word([1,3,3,4,4]), insertion=RSK.rules.dualRSK)
[[[1, 2, 7], [1], [6]], [[1, 3, 4], [3], [4]]]
```

Using dual RSK insertion with a \(\{0, 1\}\)-matrix:

```sage
sage: RSK(matrix([[[0,1],[1,1]]], insertion=RSK.rules.dualRSK)
[[[1, 2], [2]], [[1, 2], [2]]]
```

We can also give it something looking like a matrix:

```sage
sage: RSK([[0,1],[1,1]], insertion=RSK.rules.dualRSK)
[[[1, 2], [2]], [[1, 2], [2]]]
```

Let us now call the inverse correspondence:

```sage
sage: RSK_inverse(*RSK([[1, 2, 2, 2], [2, 1, 2, 4], insertion=RSK.rules.dualRSK))
[[1, 2, 2, 2], [2, 1, 2, 4]]
```

```sage
sage: P,Q = RSK([[1, 2, 2, 2], [2, 1, 2, 4],insertion=RSK.rules.dualRSK)
```

```sage
sage: RSK_inverse(P, Q, insertion=RSK.rules.dualRSK)
[[1, 2, 2, 2], [2, 1, 2, 4]]
```

When applied to two standard tableaux, reverse dual RSK insertion behaves identically to the usual reverse RSK insertion:
Let us check that forward and backward dual RSK are mutually inverse when the first tableau is merely transpose semistandard:

```sage
sage: p = Tableau([[1,2,2],[1]]); q = Tableau([[1,2,4],[3]])
sage: ret = RSK_inverse(p, q, insertion=RSK.rules.dualRSK); ret
[[1, 2, 3, 4], [1, 2, 1, 2]]
sage: RSK_inverse(p, q, 'word', insertion=RSK.rules.dualRSK)
word: 1212
```

In general for dual RSK:

```sage
sage: p = Tableau([[1,1,2],[1]]); q = Tableau([[1,3,3],[2]])
sage: RSK_inverse(p, q, insertion=RSK.rules.dualRSK)
[[1, 2, 3, 3], [1, 1, 1, 2]]
sage: RSK_inverse(p, q, 'matrix', insertion=RSK.rules.dualRSK)
[1 0]
[1 0]
[1 1]
```

`insertion(j, r)`

Insert the letter $j$ from the second row of the biword into the row $r$ using dual RSK insertion, if there is bumping to be done.

The row $r$ is modified in place if bumping occurs. The bumped-out entry, if it exists, is returned.

EXAMPLES:

```sage
sage: from sage.combinat.rsk import RuleDualRSK
sage: r = [1, 3, 4, 5]
sage: j = RuleDualRSK().insertion(4, r); j
4
sage: r
[1, 3, 4, 5]
```
reverse_insertion(\(x, \text{row}\))

Reverse bump the row \(\text{row}\) of the current insertion tableau with the number \(x\) using dual RSK insertion.

The row \(\text{row}\) is modified in place. The bumped-out entry is returned.

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleDualRSK
sage: r = [1, 2, 4, 6, 7]
sage: x = RuleDualRSK().reverse_insertion(6, r); r
[1, 2, 4, 6, 7]
sage: x
6
sage: r = [1, 2, 4, 5, 7]
sage: x = RuleDualRSK().reverse_insertion(6, r); r
[1, 2, 4, 6, 7]
sage: x
5
```

\(\texttt{to_pairs}(\text{obj1=None, obj2=None, check=True})\)

Given a valid input for the dual RSK algorithm, such as two \(n\)-tuples \(\text{obj1} = [a_1, a_2, \ldots, a_n]\) and \(\text{obj2} = [b_1, b_2, \ldots, b_n]\) forming a strict biword (i.e., satisfying \(a_1 \leq a_2 \leq \cdots \leq a_n\), and if \(a_i = a_{i+1}\), then \(b_i < b_{i+1}\)) or a \(\{0, 1\}\)-matrix (“rook placement”), or a single word, return the array \([(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)]\).

INPUT:

- \(\text{obj1, obj2}\) – anything representing a strict biword (see the doc of \texttt{forward_rule()} for the encodings accepted)
- \(\text{check}\) – (default: True) whether to check that \(\text{obj1}\) and \(\text{obj2}\) actually define a valid strict biword

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleDualRSK
sage: list(RuleDualRSK().to_pairs([1, 2, 2, 2], [2, 1, 2, 3]))
[(1, 2), (2, 1), (2, 2), (2, 3)]
sage: RuleDualRSK().to_pairs([1, 2, 2, 2], [1, 2, 3, 3])
Traceback (most recent call last):
... ValueError: invalid strict biword
```
class sage.combinat.rsk.RuleEG

Bases: Rule

Rule for Edelman-Greene insertion.

For a reduced word of a permutation (i.e., an element of a type $A$ Coxeter group), one can use Edelman-Greene insertion, an algorithm defined in [EG1987] Definition 6.20 (where it is referred to as Coxeter-Knuth insertion). The Edelman-Greene insertion is similar to the standard row insertion except that (using the notations in the documentation of RSK()) if $k_i$ and $k_i + 1$ both exist in row $i$, we only set $k_i+1 = k_i + 1$ and continue.

**EXAMPLES:**

Let us reproduce figure 6.4 in [EG1987]:

```python
sage: RSK([2,3,2,1,2,3], insertion=RSK.rules.EG)
[[[1, 2, 3], [2, 3]], [[1, 2, 6], [3, 5], [4]]]
```

Some more examples:

```python
sage: a = [2, 1, 2, 3, 2]
sage: pq = RSK(a, insertion=RSK.rules.EG); pq
[[[1, 2, 3], [2, 3]], [[1, 3, 4], [2, 5]]]
sage: RSK(RSK_inverse(*pq, insertion=RSK.rules.EG, output='matrix'),
......: insertion=RSK.rules.EG)
[[[1, 2, 3], [2, 3]], [[1, 3, 4], [2, 5]]]
sage: RSK_inverse(*pq, insertion=RSK.rules.EG)
[[1, 2, 3, 4, 5], [2, 1, 2, 3, 2]]
```

The RSK algorithm (RSK()) built using the Edelman-Greene insertion rule RuleEG is a bijection from reduced words of permutations/elements of a type $A$ Coxeter group to pairs consisting of an increasing tableau and a standard tableau of the same shape (see [EG1987] Theorem 6.25). The inverse of this bijection is obtained using RSK_inverse(). If the optional parameter output = 'permutation' is set in RSK_inverse(), then the function returns not the reduced word itself but the permutation (of smallest possible size) whose reduced word it is (although the order of the letters is reverse to the usual Sage convention):

```python
sage: w = RSK_inverse(*pq, insertion=RSK.rules.EG, output='permutation'); w
[4, 3, 1, 2]
sage: list(reversed(a)) in w.reduced_words()
True
```

**insertion($j, r$)**

Insert the letter $j$ from the second row of the biword into the row $r$ using Edelman-Greene insertion, if there is bumping to be done.
The row \( r \) is modified in place if bumping occurs. The bumped-out entry, if it exists, is returned.

**EXAMPLES:**

```python
sage: from sage.combinat.rsk import RuleEG
sage: qr, r = [1,2,3,4,5], [3,3,2,4,8]
sage: j = RuleEG().insertion(9, r)
sage: j is None
True
sage: qr, r = [1,2,3,4,5], [2,3,4,5,8]
sage: j = RuleEG().insertion(3, r); r
[2, 3, 4, 5, 8]
sage: j
4
sage: qr, r = [1,2,3,4,5], [2,3,5,5,8]
sage: j = RuleEG().insertion(3, r); r
[2, 3, 3, 5, 8]
sage: j
5
```

### reverse_insertion\((x, \text{row})\)

Reverse bump the row \( \text{row} \) of the current insertion tableau with the number \( x \).

The row \( \text{row} \) is modified in place. The bumped-out entry is returned.

**EXAMPLES:**

```python
sage: from sage.combinat.rsk import RuleEG
sage: r = [1,1,1,2,3,3]
sage: x = RuleEG().reverse_insertion(3, r); r
[1, 1, 1, 2, 3, 3]
sage: x
2
```

### class sage.combinat.rsk.RuleHecke

**Bases:** Rule

Rule for Hecke insertion.

The Hecke RSK algorithm is similar to the classical RSK algorithm, but is defined using the Hecke insertion introduced in [BKSTY06] (but using rows instead of columns). It is not clear in what generality it works; thus, following [BKSTY06], we shall assume that our biword \( p \) has top row \((1, 2, \ldots, n)\) (or, at least, has its top row strictly increasing).

The Hecke RSK algorithm returns a pair of an increasing tableau and a set-valued standard tableau. If \( p = ((j_0, k_0), (j_1, k_1), \ldots, (j_{t-1}, k_{t-1})) \), then the algorithm recursively constructs pairs \((P_0, Q_0), (P_1, Q_1), \ldots, (P_t, Q_t)\) of tableaux. The construction of \( P_{t+1} \) and \( Q_{t+1} \) from \( P_t, Q_t, j_t \) and \( k_t \) proceeds as follows: Set \( i = j_t, x = k_t \). If \( P_t = P_t \) and \( Q_t = Q_t \), we are going to insert \( x \) into the increasing tableau \( P \) and update the set-valued “recording tableau” \( Q \) accordingly. As in the classical RSK algorithm, we first insert \( x \) into row 1 of \( P \), then into row 2 of the resulting tableau, and so on, until the construction terminates. The details are different: Suppose we are inserting \( x \) into row \( R \) of \( P \). If (Case 1) there exists an entry \( y \) in row \( R \) such that \( x < y \), then let \( y \) be the minimal such entry. We replace this entry \( y \) with \( x \) if the result is still an increasing tableau; in either subcase, we then continue recursively, inserting \( y \) into the next row of \( P \). If, on the other hand, (Case 2) no such \( y \) exists, then we append \( x \) to the end of \( R \) if the result is an increasing tableau (Subcase 2.1), and otherwise (Subcase 2.2) do nothing. Furthermore, in Subcase 2.1, we add the box that we have just filled with \( x \) in \( P \) to the shape of \( Q \), and fill it with the one-element set \{\( i \}\}. In Subcase 2.2, we find the bottommost box of the column containing the rightmost box of row \( R \), and add \( i \) to the entry of \( Q \) in this
box (this entry is a set, since $Q$ is set-valued). In either subcase, we terminate the recursion, and set $P_{t+1} = P$ and $Q_{t+1} = Q$.

Notice that set-valued tableaux are encoded as tableaux whose entries are tuples of positive integers; each such tuple is strictly increasing and encodes a set (namely, the set of its entries).

**EXAMPLES:**

As an example of Hecke insertion, we reproduce Example 2.1 in arXiv 0801.1319v2:

```python
sage: w = [5, 4, 1, 3, 4, 2, 5, 1, 2, 1, 4, 2, 4]
sage: P,Q = RSK(w, insertion=RSK.rules.Hecke); [P,Q]

[[[1, 2, 4, 5], [2, 4, 5], [3, 5], [4], [5]],
 [[[1, ), (4, ), (5, ), (7, )],
  [(2, ), (9, ), (11, 13)],
  [(3, ), (12, )],
  [(6, )],
  [(8, 10)]]]
sage: wp = RSK_inverse(P, Q, insertion=RSK.rules.Hecke, 
sage: ....: output='list'); wp
[5, 4, 1, 3, 4, 2, 5, 1, 2, 1, 4, 2, 4]
sage: wp == w
True
```

**backward_rule** ($p$, $q$, output)

Return the generalized permutation obtained by applying reverse Hecke insertion to a pair of tableaux $(p, q)$.

**INPUT:**

- $p, q$ – two tableaux of the same shape
- output – (default: 'array') if $q$ is semi-standard:
  - 'array' – as a two-line array (i.e. generalized permutation or biword)
  - 'word' – as a word
  - 'list' – as a list

**EXAMPLES:**

```python
sage: from sage.combinat.rsk import RuleHecke
sage: t1 = Tableau([[1, 4], [2], [3]])

sage: t2 = Tableau([[1, 2], [4, ]], [[3, ]], [[5, ]])

sage: RuleHecke().backward_rule(t1, t2, 'array')
[[1, 2, 3, 4, 5], [3, 3, 2, 4, 1]]

sage: t1 = Tableau([[1, 4], [2, 3]])

sage: t2 = Tableau([[1, 2], [4, ]], [[3, ]], [[5, ]])

sage: RuleHecke().backward_rule(t1, t2, 'array')
Traceback (most recent call last):
...
ValueError: p(=[[[1, 4], [2, 3]]) and
q(=[[[1, 2], [4, ]], [[3, ]], [[5, ]]]) must have the same shape
```
\textbf{forward\_rule}(obj1, obj2, check\_standard=False)

Return a pair of tableaux obtained by applying Hecke insertion to the generalized permutation \([obj1, obj2]\).

INPUT:

- \texttt{obj1, obj2} – can be one of the following ways to represent a generalized permutation (or, equivalently, biword):
  - two lists \texttt{obj1} and \texttt{obj2} of equal length, to be interpreted as the top row and the bottom row of the biword
  - a word \texttt{obj1} in an ordered alphabet, to be interpreted as the bottom row of the biword (in this case, \texttt{obj2} is \texttt{None}; the top row of the biword is understood to be \((1, 2, \ldots, n)\) by default)

- \texttt{check\_standard} – (default: \texttt{False}) check if either of the resulting tableaux is a standard tableau, and if so, typecast it as such

**EXAMPLES:**

```python
sage: from sage.combinat.rsk import RuleHecke
sage: p, q = RuleHecke().forward_rule([3,3,2,4,1], None); p
[[1, 4], [2], [3]]
sage: q
[[[1, 2], [4,]], [[3,]], [[5,]]]
sage: isinstance(p, SemistandardTableau)
True
sage: isinstance(q, Tableau)
True
```

\textbf{insertion}(j, ir, r, p)

Insert the letter \texttt{j} from the second row of the biword into the row \texttt{r} of the increasing tableau \texttt{p} using Hecke insertion, provided that \texttt{r} is the \texttt{ir}-th row of \texttt{p}, and provided that there is bumping to be done.

The row \texttt{r} is modified in place if bumping occurs. The bumped-out entry, if it exists, is returned.

**EXAMPLES:**

```python
sage: from sage.combinat.rsk import RuleHecke
sage: from bisect import bisect_right
sage: p, q, r = [], [], [3,3,8,8,8,9]
sage: j, ir = 8, 1
sage: j1 = RuleHecke().insertion(j, ir, r, p)
sage: j1 == r[bisect_right(r, j)]
True
```

\textbf{reverse\_insertion}(i, x, row, p)

Reverse bump the row \texttt{row} of the current insertion tableau \texttt{p} with the number \texttt{x}, provided that \texttt{row} is the \texttt{i}-th row of \texttt{p}.

The row \texttt{row} is modified in place. The bumped-out entry is returned.

**EXAMPLES:**

```python
sage: from sage.combinat.rsk import RuleHecke
sage: from bisect import bisect_left
sage: r = [2,3,3,4,8,9]
sage: x, i, p = 9, 1, [1, 2]
``` (continues on next page)
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(continued from previous page)

```
sage: x1 = RuleHecke().reverse_insertion(i, x, r, p)
sage: x1 == r[bisect_left(r,x) - 1]
True
```

class sage.combinat.rsk.RuleRSK

Bases: Rule

Rule for the classical Robinson-Schensted-Knuth insertion.

See RSK() for the definition of this operation.

EXAMPLES:

```
sage: RSK([1, 2, 2, 2], [2, 1, 1, 2], insertion=RSK.rules.RSK)
[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]
sage: p = Tableau([[1,2,2], [2]]); q = Tableau([[1,3,3], [2]])
sage: RSK_inverse(p, q, insertion=RSK.rules.RSK)
[[1, 2, 3, 3], [2, 1, 2, 2]]
```

insertion(j, r)

Insert the letter \(j\) from the second row of the biword into the row \(r\) using classical Schensted insertion, if there is bumping to be done.

The row \(r\) is modified in place if bumping occurs. The bumped-out entry, if it exists, is returned.

EXAMPLES:

```
sage: from sage.combinat.rsk import RuleRSK
sage: qr, r = [1,2,3,4,5], [3,3,2,4,8]
sage: j = RuleRSK().insertion(9, r)
sage: j is None
True
sage: qr, r = [1,2,3,4,5], [3,3,2,4,8]
sage: j = RuleRSK().insertion(3, r)
sage: j
4
```

reverse_insertion(x, row)

Reverse bump the row \(row\) of the current insertion tableau with the number \(x\).

The row \(row\) is modified in place. The bumped-out entry is returned.

EXAMPLES:

```
sage: from sage.combinat.rsk import RuleRSK
sage: r = [2,3,3,4,8]
sage: x = RuleRSK().reverse_insertion(4, r); r
[2, 3, 4, 4, 8]
sage: x
3
```

class sage.combinat.rsk.RuleStar

Bases: Rule

Rule for \(\star\)-insertion.
The $\star$-insertion is similar to the classical RSK algorithm and is defined in [MPPS2020]. The bottom row of the increasing Hecke biword is a word in the 0-Hecke monoid that is fully commutative. When inserting a letter $x$ into a row $R$, there are three cases:

- **Case 1:** If $R$ is empty or $x > \max(R)$, append $x$ to row $R$ and terminate.
- **Case 2:** Otherwise if $x$ is not in $R$, locate the smallest $y$ in $R$ with $y > x$. Bump $y$ with $x$ and insert $y$ into the next row.
- **Case 3:** Otherwise, if $x$ is in $R$, locate the smallest $y$ in $R$ with $y \leq x$ and interval $[y, x]$ contained in $R$. Row $R$ remains unchanged and $y$ is to be inserted into the next row.

The $\star$-insertion returns a pair consisting a conjugate of a semistandard tableau and a semistandard tableau. It is a bijection from the collection of all increasing Hecke biwords whose bottom row is a fully commutative word to pairs $(P, Q)$ of tableaux of the same shape such that $P$ is conjugate semistandard, $Q$ is semistandard and the row reading word of $P$ is fully commutative [MPPS2020].

**EXAMPLES:**

As an example of $\star$-insertion, we reproduce Example 28 in [MPPS2020]:

```python
sage: from sage.combinat.rsk import RuleStar
sage: p, q = RuleStar().forward_rule([1,1,2,2,4,4], [1,3,2,4,2,4])
sage: ascii_art(p, q)
1 2 4 1 1 2
1 4 2 4
3 4
sage: line1, line2 = RuleStar().backward_rule(p, q)
([1, 1, 2, 2, 4, 4], [1, 3, 2, 4, 2, 4])
sage: RSK_inverse(p, q, output='DecreasingHeckeFactorization', insertion='Star')
(4, 2)(4, 2)(3, 1)
```

```python
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: h = DecreasingHeckeFactorization([[4, 2], [], [3, 1]]);
h
(4, 2)(4, 2)(3, 1)
sage: p, q = RSK(h, insertion='Star')
sage: ascii_art(p, q)
1 2 4 1 1 2
1 4 2 4
3 4
sage: RSK_inverse(p, q, insertion='Star')
[[1, 1, 2, 2, 4, 4], [1, 3, 2, 4, 2, 4]]
sage: f = RSK_inverse(p, q, output='DecreasingHeckeFactorization', insertion='Star')
sage: f == h
True
```

**Warning:** When output is set to 'DecreasingHeckeFactorization', the inverse of $\star$-insertion of $(P, Q)$ returns a decreasing factorization whose number of factors is the maximum entry of $Q$:

```python
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: h1 = DecreasingHeckeFactorization([[], [3, 1], [1]]); h1
```

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```
(3, 1)(1)
sage: P, Q = RSK(h1, insertion='Star')
sage: ascii_art(P, Q)
  1 3 1 2
  1 2
sage: h2 = RSK_inverse(P, Q, insertion='Star',
...: output='DecreasingHeckeFactorization'); h2
(3, 1)(1)
```

**backward_rule** *(p, q, output='array')*

Return the increasing Hecke biword obtained by applying reverse \star-insertion to a pair of tableaux \((p, q)\).

**INPUT:**

- \(p, q\) – two tableaux of the same shape, where \(p\) is the conjugate of a semistandard tableau, whose reading word is fully commutative and \(q\) is a semistandard tableau.
- \(output\) – (default: 'array') if \(q\) is semi-standard:
  - 'array' – as a two-line array (i.e. generalized permutation or biword) that is an increasing Hecke biword
  - 'DecreasingHeckeFactorization' – as a decreasing factorization in the 0-Hecke monoid and if \(q\) is standard:
  - 'word' – as a (possibly non-reduced) word in the 0-Hecke monoid

**Warning:** When output is 'DecreasingHeckeFactorization', the number of factors in the output is the largest number in \(obj1\).

**EXAMPLES:**

```
sage: from sage.combinat.rsk import RuleStar
sage: p, q = RuleStar().forward_rule([1,1,2,2,4,4], [1,3,2,4,2,4])
sage: ascii_art(p, q)
  1 2 4 1 1 2
  1 4 2 4
  3 4
sage: line1, line2 = RuleStar().backward_rule(p, q); line1, line2
([1, 1, 2, 2, 4, 4], [1, 3, 2, 4, 2, 4])
sage: RuleStar().backward_rule(p, q, output = 'DecreasingHeckeFactorization')
(4, 2)(4, 2)(3, 1)
```

**forward_rule** *(obj1, obj2=None, check_braid=True)*

Return a pair of tableaux obtained by applying forward insertion to the increasing Hecke biword \([obj1, obj2]\).

**INPUT:**

- \(obj1, obj2\) – can be one of the following ways to represent a biword (or, equivalently, an increasing 0-Hecke factorization) that is fully commutative:
  - two lists \(obj1\) and \(obj2\) of equal length, to be interpreted as the top row and the bottom row of the biword.
– a word \( \text{obj1} \) in an ordered alphabet, to be interpreted as the bottom row of the biword (in this case, \( \text{obj2} = \text{None} \); the top row of the biword is understood to be \((1, 2, \ldots, n)\) by default).

– a DecreasingHeckeFactorization \( \text{obj1} \), the whose increasing Hecke biword will be interpreted as the bottom row; the top row is understood to be the indices of the factors for each letter in this biword.

• \text{check_braid} – (default: \text{True}) indicator to validate that input is associated to a fully commutative word in the 0-Hecke monoid, validation is performed if set to \text{True}; otherwise, this validation is ignored.

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleStar
sage: p, q = RuleStar().forward_rule([1,1,2,3,3], [2,3,3,1,3]); p, q
([[1, 3]], [[2, 3]], [[1],[1],[2, 3], [3]])
```

```python
sage: p, q = RuleStar().forward_rule([2,3,3,1,3]); p, q
([[1, 3], [2, 3], [1]], [[1, 2], [3, 5], [4]])
```

```python
sage: p, q = RSK([1,1,2,3,3], [2,3,3,1,3], insertion=RSK.rules.Star); p, q
([[1, 3]], [[2, 3]], [[1],[1],[2, 3], [3]])
```

```python
sage: from sage.combinat.crystals.fully_commutative_stable_grothendieck import DecreasingHeckeFactorization
sage: h = DecreasingHeckeFactorization([[3, 1], [3], [3, 2]])
```

```python
sage: p, q = RSK(h, insertion=RSK.rules.Star); p, q
([[1, 3], [2, 3], [2]], [[1, 1], [2, 3], [3]])
```

insertion \((b, r)\)

Insert the letter \( b \) from the second row of the biword into the row \( r \) using \( \star \)-insertion defined in [MPPS2020].

The row \( r \) is modified in place if bumping occurs and \( b \) is not in row \( r \). The bumped-out entry, if it exists, is returned.

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleStar
sage: RuleStar().insertion(3, [1,2,4,5])
4
sage: RuleStar().insertion(3, [1,2,3,5])
1
sage: RuleStar().insertion(6, [1,2,3,5]) \text{is None}
True
```

reverse_insertion \((x, r)\)

Reverse bump the row \( r \) of the current insertion tableau \( p \) with number \( x \), provided that \( r \) is the \( i \)-th row of \( p \).

The row \( r \) is modified in place. The bumped-out entry is returned.

EXAMPLES:

```python
sage: from sage.combinat.rsk import RuleStar
sage: RuleStar().reverse_insertion(4, [1,2,3,5])
3
sage: RuleStar().reverse_insertion(1, [1,2,3,5])
3
```
class sage.combinat.rsk.RuleSuperRSK

Bases: RuleRSK

Rule for super RSK insertion.

Super RSK is based on $\epsilon$-insertion, a combination of row and column classical RSK insertion. Super RSK insertion differs from the classical RSK insertion in the following ways:

- The input (in terms of biwords) is no longer an arbitrary biword, but rather a restricted super biword (i.e., a pair of two lists $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ that contains entries with even and odd parity and pairs with mixed parity entries do not repeat).
- The output still consists of two tableaux $(P, Q)$ of equal shapes, but rather than both of them being semistandard, now they are semistandard super tableaux.
- The main difference is in the way bumping works. Instead of having only row bumping super RSK uses $\epsilon$-insertion, a combination of classical RSK bumping along the rows and a dual RSK like bumping (i.e. when a number $k_i$ is inserted into the $i$-th row of $P$, it bumps out the first integer greater or equal to $k_i$ in the column) along the column.

EXAMPLES:

```
sage: RSK([1], [1], insertion='superRSK')
[[[1]], [[[1]]]]
sage: RSK([1, 2], [1, 3], insertion='superRSK')
[[[1], [2]], [[[1], [2]]]]
sage: RSK([1, 2, 3], [1, 3, "3p"], insertion='superRSK')
[[[1], [2], [2]], [[[1], [2], [3]]]]
sage: RSK([1, 3, "3p", "2p"], insertion='superRSK')
[[[1], [3], [3'], [2']], [[[1], [1], [2, 3]], [2]]]
sage: RSK(["1p", "2p", 2, 2, "3p", "3p", 3, 3],
      insertion='superRSK')
[[[1], [2], [3'], [3]], [[[1], [1], [2, 3', 3]], [2', 3, 3]], [2, [3]]]
sage: P = SemistandardSuperTableau([[1, '3p', 3], ['2p']])
sage: Q = SemistandardSuperTableau([["1p", '2p'], [2]])
sage: RSK_inverse(P, Q, insertion=RSK.rules.superRSK)
[[[1', 1, 2', 2], [1, 3, 3', 2']]]
```

We apply super RSK on Example 5.1 in [Muth2019]:

```
sage: P, Q = RSK(["1p", "2p", 2, 2, "3p", "3p", 3, 3],
          insertion='superRSK')
sage: P, Q
([[1', 2', 3', 3], [1, 2, 3'], [3']], [[1', 2, 2', 3], [2, 3, 3], [2']])
sage: ascii_art((P, Q))
( 1' 2' 3' 3 1' 2 2 3'
( 1 2 3' 2' 3 3
( 3' , 3')
sage: RSK_inverse(P, Q, insertion=RSK.rules.superRSK)
[[[1', 2', 2, 2, 3', 3', 3], [3', 1, 2, 3, 3', 3', 2', 1']]]
```

Example 6.1 in [Muth2019]:
Let us now call the inverse correspondence:

```python
sage: P, Q = RSK(["1p", 1, 2, 2, 2], [2, 1, 2, 3],
   ...:       insertion=RSK.rules.RSK)
```

```python
sage: RSK_inverse(P, Q, insertion=RSK.rules.RSK)
[[1, 2, 2, 2], [2, 1, 2, 3]]
```

When applied to two tableaux with only even parity elements, reverse super RSK insertion behaves identically
to the usual reverse RSK insertion:

```python
sage: P, Q = RSK(["1p", 1, 2, 2, 2], [2, 1, 2, 3],
   ...:       insertion=RSK.rules.RSK)
```

```python
sage: RSK_inverse(P, Q, insertion=RSK.rules.RSK)
[[1, 2, 2, 2], [2, 1, 2, 3]]
```

```python
sage: t1 = Tableau([[1, 2, 5], [3], [4]])
```

```python
sage: t2 = Tableau([[1, 2, 3], [4], [5]])
```

```python
sage: RSK_inverse(t1, t2, insertion=RSK.rules.RSK)
[[1, 2, 3, 4, 5], [1, 4, 5, 3, 2]]
```

```python
sage: t1 = SemistandardSuperTableau([[1, 2, 5], [3], [4]])
```

```python
sage: t2 = SemistandardSuperTableau([[1, 2, 3], [4], [5]])
```

```python
sage: RSK_inverse(t1, t2, insertion=RSK.rules.RSK)
[[1, 2, 3, 4, 5], [1, 4, 5, 3, 2]]
```

**backward_rule**(p, q, output='array')

Return the restricted super biword obtained by applying reverse super RSK insertion to a pair of tableaux
(p, q).

**INPUT:**

- p, q – two tableaux of the same shape
- output – (default: 'array') if q is row-strict:
  - 'array' – as a two-line array (i.e. restricted super biword)
  and if q is standard, we can have the output:
  - 'word' – as a word

**EXAMPLES:**
sage: from sage.combinat.rsk import RuleSuperRSK
sage: t1 = SemistandardSuperTableau([['1p', '3p', '4p'], [2], [3]])

sage: t2 = SemistandardSuperTableau([[1, 2, 4], [3], [5]])

sage: RuleSuperRSK().backward_rule(t1, t2, 'array')
[[1, 2, 3, 4, 5], [4', 3, 3', 2, 1']]

sage: t1 = SemistandardSuperTableau([[1, 3], [3p]])

sage: t2 = SemistandardSuperTableau([[1, 2], [3]])

sage: RuleSuperRSK().backward_rule(t1, t2, 'array')
[[1, 2, 3], [1, 3, 3']]

forward_rule(obj1, obj2, check_standard=False, check=True)

Return a pair of tableaux obtained by applying forward insertion to the restricted super biword [obj1, obj2].

INPUT:

• obj1, obj2 – can be one of the following ways to represent a generalized permutation (or, equivalently, biword):
  – two lists obj1 and obj2 of equal length, to be interpreted as the top row and the bottom row of the biword
  – a word obj1 in an ordered alphabet, to be interpreted as the bottom row of the biword (in this case, obj2 is None; the top row of the biword is understood to be (1, 2, ..., n) by default)
  – any object obj1 which has a method _rsk_iter(), as long as this method returns an iterator yielding pairs of numbers, which then are interpreted as top entries and bottom entries in the biword (in this case, obj2 is None)

• check_standard – (default: False) check if either of the resulting tableaux is a standard super tableau, and if so, typecast it as such

• check – (default: True) whether to check that obj1 and obj2 actually define a valid restricted super biword

EXAMPLES:

sage: from sage.combinat.rsk import RuleSuperRSK
sage: p, q = RuleSuperRSK().forward_rule([[1, 2], [1, 3]]); p
[[1, 3]]

sage: q
[[1, 2]]

sage: isinstance(p, SemistandardSuperTableau)
True

sage: isinstance(q, SemistandardSuperTableau)
True

insertion(j, r, epsilon=0)

Insert the letter j from the second row of the biword into the row r using dual RSK insertion or classical Schensted insertion depending on the value of epsilon, if there is bumping to be done.

The row r is modified in place if bumping occurs. The bumped-out entry, if it exists, is returned.

EXAMPLES:

sage: from sage.combinat.rsk import RuleSuperRSK

sage: from bisect import bisect_left, bisect_right

(continues on next page)
\[
\text{reverse\_insertion}(x, \text{row}, \text{epsilon}=0)
\]
Reverse bump the row \text{row} of the current insertion tableau with the number \(x\) using dual RSK insertion or classical Schensted insertion depending on the value of \(\text{epsilon}\).

The row \text{row} is modified in place. The bumped-out entry is returned along with the bumped position.

EXAMPLES:

\[
\text{sage: from sage.combinat.rsk import RuleSuperRSK}
\text{sage: from bisect import bisect_left, bisect_right}
\text{sage: r = [1, 3, 3, 3, 4]}
\text{sage: j = 2}
\text{sage: j, y = RuleSuperRSK().reverse_insertion(j, r, epsilon=0); r}
[2, 3, 3, 3, 4]
\text{sage: j}
1
\text{sage: y}
0
\text{sage: r = [1, 3, 3, 3, 4]}
\text{sage: j = 3}
\text{sage: j, y = RuleSuperRSK().reverse_insertion(j, r, epsilon=0); r}
[3, 3, 3, 3, 4]
\text{sage: j}
1
\text{sage: y}
0
\text{sage: r = [1, 3, 3, 3, 4]}
\text{sage: j = (3)}
\text{sage: j, y = RuleSuperRSK().reverse_insertion(j, r, epsilon=1); r}
[1, 3, 3, 3, 4]
\text{sage: j}
3
\text{sage: y}
3
\]

\[
\text{to\_pairs}(\text{obj1}=\text{None}, \text{obj2}=\text{None}, \text{check}=\text{True})
\]
Given a valid input for the super RSK algorithm, such as two \(n\)-tuples \(\text{obj1} = [a_1, a_2, \ldots, a_n]\) and \(\text{obj2} \ldots\)
whenever the number is an integer). Row insertion is a recursive algorithm which starts by setting 

\[ A \]

Now the RSK algorithm, applied to a generalized permutation \( P \) and \( Q \) (where \( P \) and \( Q \) are semi-standard Young tableaux of identical shape). The tableau \( P \) is known as the insertion tableau, and \( Q \) is known as the recording tableau.

The basic operation is known as row insertion \( P \leftarrow k \) (where \( P \) is a given semi-standard Young tableau, and \( k \) is an integer). Row insertion is a recursive algorithm which starts by setting \( k_0 = k \), and in its \( i \)-th step inserts the number \( k_i \) into the \( i \)-th row of \( P \) (we start counting the rows at 0) by replacing the first integer greater than \( k_i \) in the row by \( k_i \) and defines \( k_{i+1} \) as the integer that has been replaced. If no integer greater than \( k_i \) exists in the \( i \)-th row, then \( k_{i} \) is simply appended to the row and the algorithm terminates at this point.

A generalized permutation (or biword) is a list \( \{(j_0, k_0), (j_1, k_1), \ldots, (j_{\ell-1}, k_{\ell-1})\} \) of pairs such that the letters \( j_0, j_1, \ldots, j_{\ell-1} \) are weakly increasing (that is, \( j_0 \leq j_1 \leq \cdots \leq j_{\ell-1} \)), whereas the letters \( k_1, k_2, \ldots, k_{\ell-1} \) satisfy \( k_1 \leq k_2 \leq \cdots \leq k_{\ell-1} \) whenever \( j_i = j_{i+1} \). The \( \ell \)-tuple \( (j_0, j_1, \ldots, j_{\ell-1}) \) is called the top line of this generalized permutation, whereas the \( \ell \)-tuple \( (k_0, k_1, \ldots, k_{\ell-1}) \) is called its bottom line.

Now the RSK algorithm, applied to a generalized permutation \( p = \{(j_0, k_0), (j_1, k_1), \ldots, (j_{\ell-1}, k_{\ell-1})\} \) (encoded as a lexicographically sorted list of pairs) starts by initializing two semi-standard tableaux \( P_0 \) and \( Q_0 \) as empty tableaux. For each nonnegative integer \( t \) starting at 0, take the pair \( (j_t, k_t) \) from \( p \) and set \( P_{t+1} = P_t \leftarrow k_t \), and define \( Q_{t+1} \) by adding a new box filled with \( j_t \) to the tableau \( Q_t \) at the same location the row insertion on \( P_t \) ended (that is to say, adding a new box with entry \( j_t \) such that \( P_{t+1} \) and \( Q_{t+1} \) have the same shape). The iterative process stops when \( t \) reaches the size of \( p \), and the pair \( (P_t, Q_t) \) at this point is the image of \( p \) under the Robinson-Schensted-Knuth correspondence.

This correspondence has been introduced in [Knu1970], where it has been referred to as “Construction A”.

For more information, see Chapter 7 in [Sta-EC2].

We also note that integer matrices are in bijection with generalized permutations. Furthermore, we can convert any word \( w \) (and, in particular, any permutation) to a generalized permutation by considering the top row to be \( (1, 2, \ldots, n) \) where \( n \) is the length of \( w \).
The optional argument `insertion` allows to specify an alternative insertion procedure to be used instead of the standard Robinson-Schensted-Knuth insertion.

**INPUT:**

- `obj1, obj2` – can be one of the following:
  - a word in an ordered alphabet (in this case, `obj1` is said word, and `obj2` is `None`)
  - an integer matrix
  - two lists of equal length representing a generalized permutation (namely, the lists \((j_0, j_1, \ldots, j_{\ell-1})\) and \((k_0, k_1, \ldots, k_{\ell-1})\) represent the generalized permutation \(((j_0, k_0), (j_1, k_1), \ldots, (j_{\ell-1}, k_{\ell-1}))\)
  - any object which has a method `_rsk_iter()` which returns an iterator over the object represented as generalized permutation or a pair of lists (in this case, `obj1` is said object, and `obj2` is `None`).

- `insertion` – (default: `RSK.rules.RSK`) the following types of insertion are currently supported:
  - `RSK.rules.RSK` (or 'RSK') – Robinson-Schensted-Knuth insertion (`RuleRSK`)
  - `RSK.rules.EG` (or 'EG') – Edelman-Greene insertion (only for reduced words of permutations/elements of a type \(A\) Coxeter group) (`RuleEG`)
  - `RSK.rules.Hecke` (or 'hecke') – Hecke insertion (only guaranteed for generalized permutations whose top row is strictly increasing) (`RuleHecke`)
  - `RSK.rules.dualRSK` (or 'dualRSK') – Dual RSK insertion (only for strict biwords) (`RuleDualRSK`)
  - `RSK.rules.coRSK` (or 'coRSK') – CoRSK insertion (only for strict cobiwords) (`RuleCoRSK`)
  - `RSK.rules.superRSK` (or 'super') – Super RSK insertion (only for restricted super biwords) (`RuleSuperRSK`)
  - `RSK.rules.Star` (or 'Star') – ⋆-insertion (only for fully commutative words in the 0-Hecke monoid) (`RuleStar`)

- `check_standard` – (default: `False`) check if either of the resulting tableaux is a standard tableau, and if so, typecast it as such

For precise information about constraints on the input and output, as well as the definition of the algorithm (if it is not standard RSK), see the particular `Rule` class.

**EXAMPLES:**

If we only input one row, it is understood that the top row should be \((1, 2, \ldots, n)\):

```
sage: RSK([3,3,2,4,1])
[[[1, 3, 4], [2], [3]], [[1, 2, 4], [3], [5]]]
sage: RSK(Word([3,3,2,4,1]))
[[[1, 3, 4], [2], [3]], [[1, 2, 4], [3], [5]]]
sage: RSK(Word([2,3,3,2,1,3,2,3]))
[[[1, 2, 2, 3, 3], [2, 3], [3]], [[1, 2, 3, 6, 8], [4, 7], [5]]]
```

We can provide a generalized permutation:

```
sage: RSK([[1, 2, 2, 2], [2, 1, 1, 2]])
[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]
sage: RSK(Word([1,1,3,4,4]), [1,4,2,1,3])
[[[1, 1, 3], [2], [4]], [[1, 4], [3], [4]]]
sage: RSK([[1,3,3,4,4], Word([6,2,2,1,7])])
[[[1, 2, 7], [2], [6]], [[1, 3, 4], [3], [4]]]
```
We can provide a matrix:

```python
sage: RSK(matrix([[0,1],[2,1]]))
[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]
```

We can also provide something looking like a matrix:

```python
sage: RSK([[0,1],[2,1]])
[[[1, 1, 2], [2]], [[1, 2, 2], [2]]]
```

There is also `RSK_inverse()` which performs the inverse of the bijection on a pair of semistandard tableaux. We note that the inverse function takes 2 separate tableaux as inputs, so to compose with `RSK()`, we need to use the python * on the output:

```python
sage: RSK_inverse(*RSK([1, 2, 2, 2], [2, 1, 1, 2]))
[[1, 2, 2, 2], [2, 1, 1, 2]]
sage: P,Q = RSK([1, 2, 2, 2], [2, 1, 1, 2])
sage: RSK_inverse(P, Q)
[[1, 2, 2, 2], [2, 1, 1, 2]]
```

```python
sage.combinat.rsk.robinson_schensted_knuth_inverse(p, q, output='array', insertion=<class 'sage.combinat.rsk.RuleRSK'>)
```

Return the generalized permutation corresponding to the pair of tableaux \((p, q)\) under the inverse of the Robinson-Schensted-Knuth correspondence.

For more information on the bijection, see `RSK()`.

**INPUT:**

- `p, q` – two semi-standard tableaux of the same shape, or (in the case when Hecke insertion is used) an increasing tableau and a set-valued tableau of the same shape (see the note below for the format of the set-valued tableau)
- `output` – (default: 'array') if \(q\) is semi-standard:
  - 'array' – as a two-line array (i.e. generalized permutation or biword)
  - 'matrix' – as an integer matrix
and if \(q\) is standard, we can also have the output:
- 'word' – as a word
and additionally if \(p\) is standard, we can also have the output:
- 'permutation' – as a permutation
- `insertion` – (default: `RSK.rules.RSK`) the insertion algorithm used in the bijection. Currently the following are supported:
  - `RSK.rules.RSK` (or 'RSK') – Robinson-Schensted-Knuth insertion (`RuleRSK`)
  - `RSK.rules.EG` (or 'EG') – Edelman-Greene insertion (only for reduced words of permutations/elements of a type \(A\) Coxeter group) (`RuleEG`)
  - `RSK.rules.Hecke` (or 'hecke') – Hecke insertion (only guaranteed for generalized permutations whose top row is strictly increasing) (`RuleHecke`)
  - `RSK.rules.dualRSK` (or 'dualRSK') – Dual RSK insertion (only for strict biwords) (`RuleDualRSK`)
  - `RSK.rules.coRSK` (or 'coRSK') – CoRSK insertion (only for strict cobiwords) (`RuleCoRSK`)
- `RSK.rules.superRSK` (or 'super') – Super RSK insertion (only for restricted super biwords) 
  *(RuleSuperRSK)*
- `RSK.rules.Star` (or 'Star') – *-insertion (only for fully commutative words in the 0-Hecke monoid) *(RuleStar)*

For precise information about constraints on the input and output, see the particular *Rule* class.

**Note:** In the case of Hecke insertion, the input variable `q` should be a set-valued tableau, encoded as a tableau whose entries are strictly increasing tuples of positive integers. Each such tuple encodes the set of its entries.

**EXAMPLES:**

If both `p` and `q` are standard:

```sage
sage: t1 = Tableau([[1, 2, 5], [3], [4]])
sage: t2 = Tableau([[1, 2, 3], [4], [5]])
sage: RSK_inverse(t1, t2)
[[1, 2, 3, 4, 5], [1, 4, 5, 3, 2]]
sage: RSK_inverse(t1, t2, 'word')
word: 14532
sage: RSK_inverse(t1, t2, 'matrix')
[1 0 0 0 0]
[0 0 0 1 0]
[0 0 1 0 0]
[0 1 0 0 0]
sage: RSK_inverse(t1, t2, 'permutation')
[1, 4, 5, 3, 2]
sage: RSK_inverse(t1, t1, 'permutation')
[1, 4, 3, 2, 5]
sage: RSK_inverse(t2, t2, 'permutation')
[1, 2, 5, 4, 3]
sage: RSK_inverse(t2, t1, 'permutation')
[1, 5, 4, 2, 3]
```

If the first tableau is semistandard:

```sage
sage: p = Tableau([[1,2,2],[3]]); q = Tableau([[1,2,4],[3]])
sage: ret = RSK_inverse(p, q); ret
[[1, 2, 3, 4], [2, 1, 2, 2]]
sage: RSK_inverse(p, q, 'word')
word: 1322
```

In general:

```sage
sage: p = Tableau([[1,2,2],[2]]); q = Tableau([[1,3,3],[2]])
sage: RSK_inverse(p, q)
[[1, 2, 3, 3], [2, 1, 2, 2]]
sage: RSK_inverse(p, q, 'matrix')
[0 1]
[1 0]
[0 2]
```

Using Hecke insertion:
sage: w = [5, 4, 3, 1, 4, 2, 5, 5]
sage: pq = RSK(w, insertion=RSK.rules.Hecke)
sage: RSK_inverse(*pq, insertion=RSK.rules.Hecke, output='list')
[5, 4, 3, 1, 4, 2, 5, 5]

Note: The constructor of Tableau accepts not only semistandard tableaux, but also arbitrary lists that are fillings of a partition diagram. (And such lists are used, e.g., for the set-valued tableau q that is passed to RSK_inverse(p, q, insertion='hecke').) The user is responsible for ensuring that the tableaux passed to RSK_inverse are of the right types (semistandard, standard, increasing, set-valued as needed).

g.sage.combinat.rsk.to_matrix(t, b)

Return the integer matrix corresponding to a two-line array.

INPUT:

• t – the top row of the array
• b – the bottom row of the array

OUTPUT:

An \(m \times n\)-matrix (where \(m\) and \(n\) are the maximum entries in \(t\) and \(b\) respectively) whose \((i, j)\)-th entry, for any \(i\) and \(j\), is the number of all positions \(k\) satisfying \(t_k = i\) and \(b_k = j\).

EXAMPLES:

sage: from sage.combinat.rsk import to_matrix
sage: to_matrix([[1, 1, 3, 3, 4], [2, 3, 1, 1, 3]])
\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
2 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

5.1.275 Schubert Polynomials

See Wikipedia article Schubert_polynomial and SymmetricFunctions.com. Schubert polynomials are representatives of cohomology classes in flag varieties. In \(n\) variables, they are indexed by permutations \(w \in S_n\). They also form a basis for the coinvariant ring of the \(S_n\) action on \(\mathbb{Z}[x_1, x_2, \ldots, x_n]\).

EXAMPLES:

sage: X = SchubertPolynomialRing(ZZ)
sage: w = [1,2,5,4,3]; # a list representing an element of 'S_5'
sage: X(w)
X[1, 2, 5, 4, 3]

This can be expanded in terms of polynomial variables:

sage: X(w).expand()
x0^2*x1 + x0*x1^2 + x0^2*x2 + 2*x0*x1*x2 + x1^2*x2 +
+ x0*x2^2 + x1*x2^2 + x0^2*x3 + x0*x1*x3 + x1^2*x3 +
+ x0*x2*x3 + x1*x2*x3 + x2^2*x3

We can also convert back from polynomial variables. For example, the longest permutation is a single term. In \(S_5\), this is the element (in one line notation) \(w_0 = 54321\):
The polynomials also have the property that if the indexing permutation \( w \) is multiplied by a simple transposition \( s_i = (i, i+1) \) such that the length of \( w \) is more than the length of \( ws_i \), then the Schubert polynomial of the permutation \( ws_i \) is computed by applying the divided difference operator `divided_difference()` to the polynomial indexed by \( w \). For example, applying the divided difference operator \( \partial_2 \) to the Schubert polynomial \( S_{w_0} \):

```
sage: Sw0.divided_difference(2)
X[5, 3, 4, 2, 1]
```

We can also check the properties listed in the Wikipedia article Schubert_polynomial:

```
sage: X([1,2,3,4,5])  # the identity in one-line notation
X[1]
sage: X([1,3,2,4,5]).expand()  # the transposition swapping 2 and 3
x0 + x1
sage: X([2,4,5,3,1]).expand()
x0^2*x1^2*x2*x3 + x0^2*x1*x2^2*x3 + x0*x1^2*x2^2*x3
sage: w = [4,5,1,2,3]
sage: s = SymmetricFunctions(QQ).schur()
sage: s[3,3].expand(2)
x0^3*x1^3
sage: X(w).expand()
x0^3*x1^3
```

```
sage.combinat.schubert_polynomial.SchubertPolynomialRing(R)
```

Return the Schubert polynomial ring over \( R \) on the \( X \) basis.

This is the basis made of the Schubert polynomials.

```
EXAMPLES:
```

```
sage: X = SchubertPolynomialRing(ZZ); X
Schubert polynomial ring with X basis over Integer Ring
sage: TestSuite(X).run()
sage: X([1])
X[1]
sage: X([1,2,3])*X([2,1,3])
X[2, 1]
sage: X([2,1,3])*X([2,1,3])
X[3, 1, 2]
sage: X([2,1,3])+X([3,1,2,4])
X[2, 1] + X[3, 1, 2]
sage: a = X([2,1,3])+X([3,1,2,4])
sage: a^2
X[3, 1, 2] + 2*X[4, 1, 2, 3] + X[5, 1, 2, 3, 4]
```

```
class sage.combinat.schubert_polynomial.SchubertPolynomialRing_xbasis(R)
```

Bases: `CombinatorialFreeModule`

```
EXAMPLES:
```

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sage: X = SchubertPolynomialRing(QQ)
sage: X == loads(dumps(X))
True

Element
alias of SchubertPolynomial_class

one_basis()
Return the index of the unit of this algebra.

EXAMPLES:

sage: X = SchubertPolynomialRing(QQ)
sage: X.one() # indirect doctest
X[1]

product_on_basis(left, right)
EXAMPLES:

sage: p1 = Permutation([3,2,1])
sage: p2 = Permutation([2,1,3])
sage: X = SchubertPolynomialRing(QQ)
sage: X.product_on_basis(p1,p2)
X[4, 2, 1, 3]

some_elements()
Return some elements.

EXAMPLES:

sage: X = SchubertPolynomialRing(QQ)
sage: X.some_elements()
[X[1], X[1] + 2*X[2, 1], -X[3, 2, 1] + X[4, 2, 1, 3]]

class sage.combinat.schubert_polynomial.SchubertPolynomial_class
Bases: IndexedFreeModuleElement

divided_difference(i, algorithm='sage')
Return the i-th divided difference operator, applied to self.

Here, i can be either a permutation or a positive integer.

INPUT:

• i – permutation or positive integer
• algorithm – (default: 'sage') either 'sage' or 'symmetrica'; this determines which software is
called for the computation

OUTPUT:

The result of applying the i-th divided difference operator to self.

If i is a positive integer, then the i-th divided difference operator δ_i is the linear operator sending each
polynomial \( f = f(x_1, x_2, \ldots, x_n) \) \( (n \geq i + 1 \text{ variables}) \) to the polynomial
\[
\frac{f - f_i}{x_i - x_{i+1}},
\]
where \( f_i = f(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+1}, \ldots, x_n) \).
If $\sigma$ is a permutation in the $n$-th symmetric group, then the $\sigma$-th divided difference operator $\delta_\sigma$ is the composition $\delta_{\sigma_1} \delta_{\sigma_2} \ldots \delta_{\sigma_k}$, where $\sigma = s_{\sigma_1} \circ s_{\sigma_2} \circ \ldots \circ s_{\sigma_k}$ is any reduced expression for $\sigma$ (the precise choice of reduced expression is immaterial).

Note: The `expand()` method results in a polynomial in $n$ variables named $x^0$, $x^1$, $x^2$, ..., $x^{(n-1)}$ rather than $x_1, x_2, \ldots, x_n$. The variable named $x_i$ corresponds to $x_{i+1}$. Thus, `self.divided_difference(i)` involves the variables $x(i-1)$ and $x_i$ getting switched (in the numerator).

**EXAMPLES:**

```python
sage: X = SchubertPolynomialRing(ZZ)
sage: a = X([3,2,1])
sage: a.divided_difference(1)
X[2, 3, 1]
sage: a.divided_difference([3,2,1])
X[1]
sage: a.divided_difference(5)
0
```

Any divided difference of 0 is 0:

```python
sage: X.zero().divided_difference(2)
0
```

This is compatible when a permutation is given as input:

```python
sage: a = X([3,2,4,1])
sage: a.divided_difference([2,3,1])
0
sage: a.divided_difference(1).divided_difference(2)
0
```

```python
sage: a = X([4,3,2,1])
sage: a.divided_difference([2,3,1])
X[3, 2, 4, 1]
sage: a.divided_difference(1).divided_difference(2)
X[3, 2, 4, 1]
sage: a.divided_difference([4,1,3,2])
X[1, 4, 2, 3]
sage: b = X([4, 1, 3, 2])
sage: b.divided_difference(1).divided_difference(2)
X[1, 3, 4, 2]
sage: b.divided_difference(1).divided_difference(2).divided_difference(3)
X[1, 3, 2]
sage: b.divided_difference(1).divided_difference(2).divided_difference(3).divided_difference(2)
X[1]
sage: b.divided_difference(1).divided_difference(2).divided_difference(3).divided_difference(3)
0
```

sage: b.divided_difference(1).divided_difference(2).divided_difference(3).divided_difference(1)
0

5.1. Comprehensive Module List
**expand()**

**EXAMPLES:**

```python
sage: X = SchubertPolynomialRing(ZZ)
sage: X([2,1,3]).expand()
x0
sage: [X(p).expand() for p in Permutations(3)]
[1, x0 + x1, x0, x0*x1, x0^2, x0^2*x1]
```

**multiply_variable(i)**

Return the Schubert polynomial obtained by multiplying `self` by the variable $x_i$.

**EXAMPLES:**

```python
sage: X = SchubertPolynomialRing(ZZ)
sage: a = X([3,2,4,1])
sage: a.multiply_variable(0)
X[4, 2, 3, 1]
sage: a.multiply_variable(1)
X[3, 4, 2, 1]
sage: a.multiply_variable(2)
sage: a.multiply_variable(3)
X[3, 2, 4, 5, 1]
```

**scalar_product(x)**

Return the standard scalar product of `self` and `x`.

**EXAMPLES:**

```python
sage: X = SchubertPolynomialRing(ZZ)
sage: a = X([3,2,4,1])
sage: a.scalar_product(a)
0
sage: b = X([4,3,2,1])
sage: b.scalar_product(a)
X[1, 3, 4, 6, 2, 5]
sage: Permutation([1, 3, 4, 6, 2, 5, 7]).to_lehmer_code()
[0, 1, 1, 2, 0, 0, 0]
sage: s = SymmetricFunctions(ZZ).schur()
sage: c = s([2,1,1])
sage: b.scalar_product(a).expand()  # Long output
x0^2*x1*x2 + x0*x1^2*x2 + x0*x1*x2^2 + x0^2*x1*x3 + x0*x1^2*x3 + x0^2*x2*x3 + ...
   ... + 3*x0*x1*x2*x3 + x1^2*x2*x3 + x0*x2^2*x3 + x1*x2^2*x3 + x0*x1*x3^2 + x0*x2*x3^2 + ...
   ... + 2 + x1*x2*x3^2
sage: c.expand(4)  # Long output
x0^2*x1*x2 + x0*x1^2*x2 + x0*x1*x2^2 + x0^2*x1*x3 + x0*x1^2*x3 + x0^2*x2*x3 + ...
   ... + 3*x0*x1*x2*x3 + x1^2*x2*x3 + x0*x2^2*x3 + x1*x2^2*x3 + x0*x1*x3^2 + x0*x2*x3^2 + ...
   ... + 2 + x1*x2*x3^2
```
5.1.276 Set Partitions

AUTHORS:

• Mike Hansen
• MuPAD-Combinat developers (for algorithms and design inspiration).
• Travis Scrimshaw (2013-02-28): Removed CombinatorialClass and added entry point through SetPartition.
• Martin Rubey (2017-10-10): Cleanup, add crossings and nestings, add random generation.

This module defines a class for immutable partitioning of a set. For mutable version see DisjointSet().

class sage.combinat.set_partition.AbstractSetPartition

Methods of set partitions which are independent of the base set

base_set()

Return the base set of self, which is the union of all parts of self.

EXAMPLES:

sage: SetPartition([[1], [2,3], [4]]).base_set()
{1, 2, 3, 4}
sage: SetPartition([[1,2,3,4]]).base_set()
{1, 2, 3, 4}
sage: SetPartition([]).base_set()
{}

base_set_cardinality()

Return the cardinality of the base set of self, which is the sum of the sizes of the parts of self.

This is also known as the size (sometimes the weight) of a set partition.

EXAMPLES:

sage: SetPartition([[1], [2,3], [4]]).base_set_cardinality()
4
sage: SetPartition([[1,2,3,4]]).base_set_cardinality()
4

coarsenings()

Return a list of coarsenings of self.

See also:

refinements()

EXAMPLES:

sage: SetPartition([[1,3],[2,4]]).coarsenings()
[[(1, 2, 3, 4), {1, 3}, {2, 4}]]
sage: SetPartition([[1],[2,4],[3]]).coarsenings()
[[(1, 2, 3, 4),
 {1, 2, 4}, {3},
 {1, 3}, {2, 4},
 {1}, {2, 3, 4},]
Combinatorics, Release 10.0

(continued from previous page)

```python
{{1}, {2, 4}, {3}}
sage: SetPartition([]).coarsenings()
[[]]
```

**conjugate()**

An involution exchanging singletons and circular adjacencies.

This method implements the definition of the conjugate of a set partition defined in [Cal2005].

**INPUT:**

- `self` – a set partition of an ordered set

**OUTPUT:**

a set partition

**EXAMPLES:**

```python
sage: SetPartition([[1,6,7],[2,8],[3,4,5]]).conjugate()
{{1, 4, 7}, {2, 8}, {3}, {5}, {6}}
sage: all(sp.conjugate().conjugate()==sp for sp in SetPartitions([1,3,5,7]))
True
sage: SetPartition([]).conjugate()
{}```

**inf(other)**

The product of the set partitions `self` and `other`.

The product of two set partitions $B$ and $C$ is defined as the set partition whose parts are the nonempty intersections between each part of $B$ and each part of $C$. This product is also the infimum of $B$ and $C$ in the classical set partition lattice (that is, the coarsest set partition which is finer than each of $B$ and $C$). Consequently, `inf` acts as an alias for this method.

**See also:**

`sup()`

**EXAMPLES:**

```python
sage: x = SetPartition([[1,2], [3,5,4]])
sage: y = SetPartition(( (3,1,2), (5,4) ))
sage: x * y
{{1, 2}, {3}, {4, 5}}
sage: S = SetPartitions(4)
sage: sp1 = S([2,3,4], [1])
sage: sp2 = S([1,3], [2,4])
sage: sp1.inf(sp2) == sp1
True
```

**max_block_size()**

The maximum block size of the diagram.

**EXAMPLES:**
sage: from sage.combinat.diagram_algebras import PartitionDiagram,

sage: pd = PartitionDiagram([[1,-3,-5],[2,4],[3,-1,-2],[5],[-4]])
sage: pd.max_block_size()
3
sage: sorted(d.max_block_size() for d in PartitionDiagrams(2))
[1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4]
sage: sorted(sp.max_block_size() for sp in SetPartitions(3))
[1, 2, 2, 2, 3]

standard_form()

Return self as a list of lists.

When the ground set is totally ordered, the elements of each block are listed in increasing order.

This is not related to standard set partitions (which simply means set partitions of \([n] = \{1, 2, \ldots, n\}\) for some integer \(n\)) or standardization (standardization()).

EXAMPLES:

sage: [x.standard_form() for x in SetPartitions(4, [2,2])]

[[[1, 2], [3, 4]], [[1, 4], [2, 3]], [[1, 3], [2, 4]]]

sup(t)

Return the supremum of self and t in the classical set partition lattice.

The supremum of two set partitions \(B\) and \(C\) is obtained as the transitive closure of the relation which relates \(i\) to \(j\) if and only if \(i\) and \(j\) are in the same part in at least one of the set partitions \(B\) and \(C\).

See also:

__mul__()

EXAMPLES:

sage: S = SetPartitions(4)
sage: sp1 = S([[2,3,4], [1]])
sage: sp2 = S([[1,3], [2,4]])
sage: s = S([[1,2,3,4]])
sage: sp1.sup(sp2) == s
True

class sage.combinat.set_partition.SetPartition(parent, s, check=True)

Bases: AbstractSetPartition

A partition of a set.

A set partition \(p\) of a set \(S\) is a partition of \(S\) into subsets called parts and represented as a set of sets. By extension, a set partition of a nonnegative integer \(n\) is the set partition of the integers from 1 to \(n\). The number of set partitions of \(n\) is called the \(n\)-th Bell number.

There is a natural integer partition associated with a set partition, namely the nonincreasing sequence of sizes of all its parts.

There is a classical lattice associated with all set partitions of \(n\). The infimum of two set partitions is the set partition obtained by intersecting all the parts of both set partitions. The supremum is obtained by transitive closure of the relation \(i\) related to \(j\) if and only if they are in the same part in at least one of the set partitions.
We will use terminology from partitions, in particular the length of a set partition $A = \{A_1, \ldots, A_k\}$ is the number of parts of $A$ and is denoted by $|A| := k$. The size of $A$ is the cardinality of $S$. We will also sometimes use the notation $[n] := \{1, 2, \ldots, n\}$.

EXAMPLES:

There are 5 set partitions of the set $\{1, 2, 3\}$:

```
sage: SetPartitions(3).cardinality()
sage: 5
```

Here is the list of them:

```
sage: SetPartitions(3).list()
sage: [\{(1, 2, 3), \{(1, 2), (3)\}, \{(1, 3), (2)\}, \{(1), (2, 3)\}, \{(1), (2), (3)\}]
```

There are 6 set partitions of $\{1, 2, 3, 4\}$ whose underlying partition is $[2, 1, 1]$:

```
sage: SetPartitions(4, [2,1,1]).list()
sage: [\{(1), \{2, 4\}, \{3\}\}, \{(1), \{2\}, \{3, 4\}\}, \{(1, 4), \{2\}, \{3\}\}, \{(1, 3), \{2\}, \{4\}\}, \{(1, 2), \{3\}, \{4\}\}, \{(1), \{2, 3\}, \{4\}\}]
```

Since github issue #14140, we can create a set partition directly by `SetPartition`, which creates the base set by taking the union of the parts passed in:

```
sage: s = SetPartition([[1,3], [2,4]]); s
sage: SetPartitions
```

```
sage: s.parent()
sage: Set partitions
```

```
sage: apply_permutation(p)
sage: Apply p to the underlying set of self.
```

INPUT:
- *p* – a permutation

EXAMPLES:

```
sage: x = SetPartition([[1,2], [3,5,4]])
sage: p = Permutation([2,1,4,5,3])
sage: x.apply_permutation(p)
\{(1, 2), \{3, 4, 5\}\}
sage: q = Permutation([3,2,1,5,4])
sage: x.apply_permutation(q)
\{(1, 4, 5), \{2, 3\}\}
sage: m = PerfectMatching([(1,4),(2,6),(3,5)])
```

```
sage: m.apply_permutation(Permutation([4,1,5,6,3,2]))
[\{(1, 2), \{(3, 5), (4, 6)\}\]
```

```
sage: arcs()
sage: Return self as a list of arcs.
```
Assuming that the blocks are sorted, the arcs are the pairs of consecutive elements in the blocks.

EXAMPLES:

```python
sage: A = SetPartition([[1],[2,3],[4]])
sage: A.arcs()
[(2, 3)]
sage: B = SetPartition([[1,3,6,7],[2,5],[4]])
sage: B.arcs()
[(1, 3), (3, 6), (6, 7), (2, 5)]
```

**cardinality()**

Return the len of self

EXAMPLES:

```python
sage: from sage.structure.list_clone_demo import IncreasingArrays
sage: len(IncreasingArrays()([1,2,3]))
3
```

**check()**

Check that we are a valid set partition.

EXAMPLES:

```python
sage: S = SetPartitions(4)
sage: s = S([[1, 3], [2, 4]])
sage: s.check()
```

**closers()**

Return the maximal elements of the blocks.

EXAMPLES:

```python
sage: P = SetPartition([[1,2,4,7],[3,9],[5,6,10,11,13],[8],[12]])
sage: P.closers()
[7, 8, 9, 12, 13]
```

**crossings()**

Return the crossing arcs of a set partition on a totally ordered set.

OUTPUT:

We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns a list of the pairs of crossing lines (as a line correspond to a pair, it returns a list of pairs of pairs).

EXAMPLES:

```python
sage: p = SetPartition([[1,2,4,7],[3,9],[5,6,10,11,13],[8],[12]])
sage: p.crossings()
[((1, 4), (2, 5)), ((1, 4), (3, 6)), ((2, 5), (3, 6)), ((3, 6), (5, 7))]```

**crossings_iterator()**

Return the crossing arcs of a set partition on a totally ordered set.

OUTPUT:
We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns an iterator over the pairs of crossing lines (as a line correspond to a pair, the iterator produces pairs of pairs).

EXAMPLES:

```python
sage: p = SetPartition([[1,4],[2,5,7],[3,6]])
sage: next(p.crossings_iterator())
((1, 4), (2, 5))
```

is_atomic()  
Return if self is an atomic set partition.  

A (standard) set partition $A$ can be split if there exist $j < i$ such that $\max(A_j) < \min(A_i)$ where $A$ is ordered by minimal elements. This means we can write $A = B|C$ for some nonempty set partitions $B$ and $C$. We call a set partition atomic if it cannot be split and is nonempty. Here, the pipe symbol $|$ is as defined in method pipe().

EXAMPLES:

```python
sage: SetPartition([[1,3], [2]]).is_atomic()
True
sage: SetPartition([[1,3], [2], [4]]).is_atomic()
False
sage: SetPartition([[1], [2,4], [3]]).is_atomic()
False
sage: SetPartition([[1,2,3,4]]).is_atomic()
True
sage: SetPartition([[1, 4], [2], [3]]).is_atomic()
True
sage: SetPartition([]).is_atomic()
False
```

is_noncrossing()  
Check if self is noncrossing.

OUTPUT:

We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns True if the picture obtained this way has no crossings.

EXAMPLES:

```python
sage: p = SetPartition([[1,4],[2,5,7],[3,6]])
sage: p.is_noncrossing()
False
sage: n = PerfectMatching([3,8,1,7,6,5,4,2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: n.is_noncrossing()
False
sage: PerfectMatching([(1, 4), (2, 3), (5, 6)]).is_noncrossing()
True
```

is_nonnesting()  
Return if self is nonnesting or not.
OUTPUT:

We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns True if the picture obtained this way has no nestings.

EXAMPLES:

```
sage: n = PerfectMatching([3,8,1,7,6,5,4,2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: n.is_nonnesting()
False
sage: PerfectMatching([(1, 3), (2, 5), (4, 6)]).is_nonnesting()
True
```

latex_options()  
Return the latex options for use in the _latex_ function as a dictionary. The default values are set using the global options.

Options can be found in set_latex_options()

EXAMPLES:

```
sage: SP = SetPartition([[1,6], [3,5,4]]); SP.latex_options()
{'angle': 0, 'color': 'black', 'fill': False, 'plot': None, 'radius': '1cm', 'show_labels': True, 'tikz_scale': 1}
```

nestings()  
Return the nestings of self.

OUTPUT:

We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns the list of the pairs of nesting lines (as a line correspond to a pair, it returns a list of pairs of pairs).

EXAMPLES:

```
sage: m = PerfectMatching([(1, 6), (2, 7), (3, 5), (4, 8)])
sage: m.nestings()
[[(1, 6), (3, 5)], [(2, 7), (3, 5)]]

sage: n = PerfectMatching([3,8,1,7,6,5,4,2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: n.nestings()
[[(2, 8), (4, 7)], [(2, 8), (5, 6)], [(4, 7), (5, 6)]]
```

nestings_iterator()  
Iterate over the nestings of self.

OUTPUT:
We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns an iterator over the pairs of nesting lines (as a line correspond to a pair, the iterator produces pairs of pairs).

EXAMPLES:

```
sage: n = PerfectMatching([[1, 6], [2, 7], [3, 5], [4, 8]])
sage: it = n.nestings_iterator()
sage: next(it)
((1, 6), (3, 5))
sage: next(it)
((2, 7), (3, 5))
sage: next(it)
Traceback (most recent call last):
  ... StopIteration
```

```
number_of_crossings()
Return the number of crossings.

OUTPUT:
We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns the number the pairs of crossing lines.

EXAMPLES:

```
sage: p = SetPartition([[1,4],[2,5,7],[3,6]])
sage: p.number_of_crossings()
4
sage: n = PerfectMatching([3,8,1,7,6,5,4,2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: n.number_of_crossings()
1
```

```
number_of_nestings()
Return the number of nestings of self.

OUTPUT:
We place the elements of the ground set in order on a line and draw the set partition by linking consecutive elements of each block in the upper half-plane. This function returns the number the pairs of nesting lines.

EXAMPLES:

```
sage: n = PerfectMatching([3,8,1,7,6,5,4,2]); n
[(1, 3), (2, 8), (4, 7), (5, 6)]
sage: n.number_of_nestings()
3
```

```
openers()
Return the minimal elements of the blocks.

EXAMPLES:
```
ordered_set_partition_action(s)

Return the action of an ordered set partition s on self.

Let \( A = \{A_1, A_2, \ldots, A_k\} \) be a set partition of some set \( S \) and \( s \) be an ordered set partition (i.e., set composition) of a subset of \([k]\). Let \( A^\downarrow \) denote the standardization of \( A \), and \( A_{i_1,i_2,\ldots,i_m} \) denote the sub-partition \( \{A_{i_1}, A_{i_2}, \ldots, A_{i_m}\} \) for any subset \( \{i_1, \ldots, i_m\} \) of \( \{1, \ldots, k\} \). We define the set partition \( s(A) \) by

\[
s(A) = A_{s_1}^\downarrow | A_{s_2}^\downarrow | \cdots | A_{s_q}^\downarrow,
\]

where \( s = (s_1, s_2, \ldots, s_q) \). Here, the pipe symbol \(|\) is as defined in method pipe().

This is \( s[A] \) in section 2.3 in [LM2011].

INPUT:

- \( s \) – an ordered set partition with base set a subset of \( \{1, \ldots, k\} \)

EXAMPLES:

```sage
sage: A = SetPartition([[1], [2,4], [3]])
sage: s = OrderedSetPartition([[1,3], [2]])
sage: A.ordered_set_partition_action(s)
{\{1\}, {2}, {3}}
sage: s = OrderedSetPartition([[2,3], [1]])
sage: A.ordered_set_partition_action(s)
{\{1, 2\}, {3, 4}}
```

We create Figure 1 in [LM2011] (we note that there is a typo in the lower-left corner of the table in the published version of the paper, whereas the arXiv version gives the correct partition):

```sage
sage: A = SetPartition([[1,3], [2,9], [4,5,8], [7]])
sage: B = SetPartition([[1,3], [2,8], [4,5,6], [7]])
sage: C = SetPartition([[1,5], [2,8], [3,4,6], [7]])
sage: s = OrderedSetPartition([[1,3], [2]])
sage: t = OrderedSetPartition([[2], [3,4]])
sage: u = OrderedSetPartition([[1],[2,3,4]])
sage: A.ordered_set_partition_action(s)
{\{1, 2\}, {3, 4, 5}, {6, 7}}
sage: A.ordered_set_partition_action(t)
{\{1, 2\}, {3, 4, 6}, {5}}
sage: A.ordered_set_partition_action(u)
{\{1, 2\}, {3, 8}, {4, 5, 7}, {6}}
```

(continues on next page)
pipe(other)
Return the pipe of the set partitions self and other.

The pipe of two set partitions is defined as follows:

For any integer $k$ and any subset $I$ of $\mathbb{Z}$, let $I + k$ denote the subset of $\mathbb{Z}$ obtained by adding $k$ to every element of $k$.

If $B$ and $C$ are set partitions of $[n]$ and $[m]$, respectively, then the pipe of $B$ and $C$ is defined as the set partition

$$\{B_1, B_2, \ldots, B_b, C_1 + n, C_2 + n, \ldots, C_c + n\}$$

of $[n + m]$, where $B = \{B_1, B_2, \ldots, B_b\}$ and $C = \{C_1, C_2, \ldots, C_c\}$. This pipe is denoted by $B|C$.

EXAMPLES:

```python
sage: SetPartition([[1,3],[2,4]]).pipe(SetPartition([[1,3],[2]]))
{{1, 3}, {2, 4}, {5, 7}, {6}}
```

plot(angle=None, color='black', base_set_dict=None)
Return a plot of self.

INPUT:

- `angle` – (default: $\pi/4$) the angle at which the arcs take off (if angle is negative, the arcs are drawn below the horizontal line)
- `color` – (default: 'black') color of the arcs
- `base_set_dict` – (optional) dictionary with keys elements of `base_set()` and values as integer or float

EXAMPLES:

```python
sage: p = SetPartition([[1,10,11],[2,3,7],[4,5,6],[8,9]])
sage: p.plot()
Graphics object consisting of 29 graphics primitives
```

(continues on next page)
5.1. Comprehensive Module List
Point set defined by 1 point(s): \[(2.0, 0.0)\]
Point set defined by 1 point(s): \[(3.0, 0.0)\]
Point set defined by 1 point(s): \[(4.0, 0.0)\]

Text '
 at the point (0.0,-0.1)

Text '
 at the point (1.0,-0.1)

Text '
 at the point (2.0,-0.1)

Text '
 at the point (3.0,-0.1)

Text '
 at the point (4.0,-0.1)

Arc with center (1.0,-1.0) radii (1.41421356237...,1.41421356237...)
angle 0.0 inside the sector (0.785398163397...,2.35619449019...)

Arc with center (2.5,-0.5) radii (0.70710678118...,0.70710678118...)
angle 0.0 inside the sector (0.785398163397...,2.35619449019...)

Arc with center (2.5,-1.5) radii (2.1213203435...,2.1213203435...)
angle 0.0 inside the sector (0.785398163397...,2.35619449019...)

refinements()  
Return a list of refinements of self.
See also:

cosenings()

EXAMPLES:

```python
sage: SetPartition([[1,3],[2,4]]).refinements()
[[[1, 3], [2, 4]],
 [[[1, 3], {2}, {4}],
 [[[1], {2}, {3}],
 [[[1], {2}, {3}, {4}]]

sage: SetPartition([[1],[2,4],[3]]).refinements()
[[{1}, {2, 4}, {3}],
 [{1}, {2}, {3}, {4}]]

sage: SetPartition([]).refinements()
[{}]

restriction(I)

Return the restriction of self to a subset I (which is given as a set or list or any other iterable).

EXAMPLES:

```python
sage: A = SetPartition([[1], [2,3]])

sage: A.restriction([1,2])
{{1}, {2}}

sage: A.restriction([2,3])
{{2, 3}}

sage: A.restriction([])
{}

sage: A.restriction([4])
{}
```

set_latex_options(**kwargs)

Set the latex options for use in the _latex_ function

- **tikz_scale** – (default: 1) scale for use with tikz package
- **plot** – (default: None) None returns the set notation, linear returns a linear plot, cyclic returns a cyclic plot
- **color** – (default: 'black') the arc colors
- **fill** – (default: False) if True then fills color, else you can pass in a color to alter the fill color - only works with cyclic plot
- **show_labels** – (default: True) if True shows labels - only works with plots
- **radius** – (default: "1cm") radius of circle for cyclic plot - only works with cyclic plot
- **angle** – (default: 0) angle for linear plot

EXAMPLES:

```python
sage: SP = SetPartition([[1,6], [3,5,4]])

sage: SP.set_latex_options(tikz_scale=2,plot='linear',fill=True,color='blue',
->angle=45)

sage: SP.set_latex_options(plot='cyclic')

sage: SP.latex_options()

{'}angle': 45,
 'color': 'blue',
```
shape()

Return the integer partition whose parts are the sizes of the sets in self.

EXAMPLES:

```
sage: S = SetPartitions(5)
sage: x = S([[1,2], [3,5,4]])
sage: x.shape()
[3, 2]
sage: y = S([[2], [3,1], [5,4]])
sage: y.shape()
[2, 2, 1]
```

shape_partition()

Return the integer partition whose parts are the sizes of the sets in self.

EXAMPLES:

```
sage: S = SetPartitions(5)
sage: x = S([[1,2], [3,5,4]])
sage: x.shape_partition()
[3, 2]
sage: y = S([[2], [3,1], [5,4]])
sage: y.shape_partition()
[2, 2, 1]
```

size()

Return the cardinality of the base set of self, which is the sum of the sizes of the parts of self.

This is also known as the size (sometimes the weight) of a set partition.

EXAMPLES:

```
sage: SetPartition([[1], [2,3], [4]]).base_set_cardinality()
4
sage: SetPartition([[1,2,3,4]]).base_set_cardinality()
4
```

standardization()

Return the standardization of self.

Given a set partition \( A = \{A_1, \ldots, A_n\} \) of an ordered set \( S \), the standardization of \( A \) is the set partition of \( \{1, 2, \ldots, |S|\} \) obtained by replacing the elements of the parts of \( A \) by the integers \( 1, 2, \ldots, |S| \) in such a way that their relative order is preserved (i.e., the smallest element in the whole set partition is replaced by 1, the next-smallest by 2, and so on).

EXAMPLES:
strict_coarsenings()  
Return all strict coarsenings of self.

Strict coarsening is the binary relation on set partitions defined as the transitive-and-reflexive closure of the relation ≺ defined as follows: For two set partitions $A$ and $B$, we have $A \prec B$ if there exist parts $A_i, A_j$ of $A$ such that $\max(A_i) < \min(A_j)$ and $B = A \setminus \{A_i, A_j\} \cup \{A_i \cup A_j\}$.

EXAMPLES:

```plaintext
sage: A = SetPartition([[1], [2, 3], [4]])
sage: A.strict_coarsenings()
[[[1], [2, 3], [4]], [[1, 2, 3], [4]], [[1, 4], [2, 3]], [[1], [2, 3, 4]], [[1, 2, 3, 4]]]
```

```plaintext
sage: S = SetPartitions(5)
sage: x = S([[1, 2], [3, 5, 4]])
sage: x.shape()
[3, 2]
sage: y = S([[2], [3, 1], [5, 4]])
sage: y.shape()
[2, 2, 1]
```

to_permutation()  
Convert a set partition of $\{1, \ldots, n\}$ to a permutation by considering the blocks of the partition as cycles.

The cycles are such that the number of excedences is maximised, that is, each cycle is of the form $(a_1, a_2, \ldots, a_k)$ with $a_1 < a_2 < \ldots < a_k$.

EXAMPLES:

```plaintext
sage: s = SetPartition([[1, 3], [2, 4]])
sage: s.to_permutation()
[3, 4, 1, 2]
```

to_restricted_growth_word(bijection='blocks')  
Convert a set partition of $\{1, \ldots, n\}$ to a word of length $n$ with letters in the non-negative integers such that each letter is at most 1 larger than all the letters before.

```plaintext
sage: SetPartition([[c, b], [d, f], [e, a]]).standardization()
{{1, 5}, {2, 3}, {4, 6}}
```
INPUT:

- **bijection** (default: blocks) – defines the map from set partitions to restricted growth functions. These are currently:
  - **blocks**: `to_restricted_growth_word_blocks()`.
  - **intertwining**: `to_restricted_growth_word_intertwining()`.

OUTPUT:

A restricted growth word.

See also: `SetPartitions.from_restricted_growth_word()`

EXAMPLES:

```python
sage: P = SetPartition([(1,4),[2,8],[3,5,6,9],[7]])
sage: P.to_restricted_growth_word()
[0, 1, 2, 0, 2, 2, 3, 1, 2]
sage: P.to_restricted_growth_word("intertwining")
[0, 1, 2, 2, 1, 0, 3, 3, 2]
sage: P = SetPartition([(1,2,4,7),[3,9],[5,6,10,11,13],[8],[12]])
sage: P.to_restricted_growth_word()
[0, 0, 1, 0, 2, 2, 0, 3, 1, 2, 2, 4, 2]
sage: P.to_restricted_growth_word("intertwining")
[0, 0, 1, 1, 2, 0, 1, 3, 3, 3, 0, 4, 1]
```

to_restricted_growth_word_blocks()

Convert a set partition of \(\{1, ..., n\}\) to a word of length \(n\) with letters in the non-negative integers such that each letter is at most 1 larger than all the letters before.

The word is obtained by sorting the blocks by their minimal element and setting the letters at the positions of the elements in the \(i\)-th block to \(i\).

OUTPUT:

a restricted growth word.

See also: `to_restricted_growth_word()` `SetPartitions.from_restricted_growth_word()`

EXAMPLES:

```python
sage: P = SetPartition([[1,4],[2,8],[3,5,6,9],[7]])
sage: P.to_restricted_growth_word_blocks()
[0, 1, 2, 0, 2, 2, 3, 1, 2]
```

to_restricted_growth_word_intertwining()

Convert a set partition of \(\{1, ..., n\}\) to a word of length \(n\) with letters in the non-negative integers such that each letter is at most 1 larger than all the letters before.

The \(i\)-th letter of the word is the numbers of crossings of the arc (or half-arc) in the extended arc diagram ending at \(i\), with arcs (or half-arcs) beginning at a smaller element and ending at a larger element.

OUTPUT:
a restricted growth word.

See also:

to_restricted_growth_word() SetPartitions.from_restricted_growth_word()

EXAMPLES:

```python
sage: P = SetPartition([[1,4],[2,8],[3,5,6,9],[7]])
sage: P.to_restricted_growth_word_intertwining()
[0, 1, 2, 2, 1, 0, 3, 3, 2]
```

to_rook_placement(bijection='arcs')

Return a set of pairs defining a placement of non-attacking rooks on a triangular board.

The cells of the board corresponding to a set partition of \{1, \ldots, n\} are the pairs \((i, j)\) with \(0 < i < j < n + 1\).

INPUT:

- **bijection** (default: arcs) – defines the bijection from set partitions to rook placements. These are currently:
  - arcs: arcs()
  - gamma: to_rook_placement_gamma()
  - rho: to_rook_placement_rho()
  - psi: to_rook_placement_psi()

See also:

SetPartitions.from_rook_placement()

EXAMPLES:

```python
sage: P = SetPartition([[1,2,4,7],[3,9],[5,6,10,11,13],[8],[12]])
sage: P.to_rook_placement()
[(1, 2), (2, 4), (4, 7), (3, 9), (5, 6), (6, 10), (10, 11), (11, 13)]
sage: P.to_rook_placement("gamma")
[(1, 4), (3, 5), (4, 6), (5, 8), (7, 11), (8, 9), (10, 12), (12, 13)]
sage: P.to_rook_placement("rho")
[(1, 2), (2, 6), (3, 4), (4, 10), (5, 9), (6, 7), (10, 11), (11, 13)]
sage: P.to_rook_placement("psi")
[(1, 2), (2, 6), (3, 4), (5, 9), (6, 7), (7, 10), (9, 11), (11, 13)]
```

to_rook_placement_gamma()

Return the rook diagram obtained by placing rooks according to Wachs and White's bijection gamma.

Note that our index convention differs from the convention in [WW1991]: regarding the rook board as a lower-right triangular grid, we refer with \((i, j)\) to the cell in the \(i\)-th column from the right and the \(j\)-th row from the top.

The algorithm proceeds as follows: non-attacking rooks are placed beginning at the left column. If \(n+1-i\) is an opener, column \(i\) remains empty. Otherwise, we place a rook into column \(i\), such that the number of cells below the rook, which are not yet attacked by another rook, equals the index of the block to which \(n+1-i\) belongs.

OUTPUT:

A list of coordinates.
See also:

- `to_rook_placement()`
- `SetPartitions.from_rook_placement()`
- `SetPartitions.from_rook_placement_gamma()`

EXAMPLES:

```python
sage: P = SetPartition([[1,4],[2,8],[3,5,6,9],[7]])
sage: P.to_rook_placement_gamma()
[(1, 3), (2, 7), (4, 5), (5, 6), (6, 9)]
```

Figure 5 in [WW1991]:

```python
sage: P = SetPartition([[1,2,4,7],[3,9],[5,6,10,11,13],[8],[12]])
sage: r = P.to_rook_placement_gamma(); r
[(1, 4), (3, 5), (4, 6), (5, 8), (7, 11), (8, 9), (10, 12), (12, 13)]
```

`to_rook_placement_psi()`

Return the rook diagram obtained by placing rooks according to Yip’s bijection psi.

OUTPUT:

A list of coordinates.

See also:

- `to_rook_placement()`
- `SetPartitions.from_rook_placement()`
- `SetPartitions.from_rook_placement_psi()`

EXAMPLES:

Example 36 (arXiv version: Example 4.5) in [Yip2018]:

```python
sage: P = SetPartition([[1, 5], [2], [3, 8, 9], [4], [6, 7]])
sage: P.to_rook_placement_psi()
[(1, 7), (3, 8), (4, 5), (7, 9)]
```

Note that the columns corresponding to the minimal elements of the blocks remain empty.

`to_rook_placement_rho()`

Return the rook diagram obtained by placing rooks according to Wachs and White’s bijection rho.

Note that our index convention differs from the convention in [WW1991]: regarding the rook board as a lower-right triangular grid, we refer with \((i, j)\) to the cell in the \(i\)-th column from the right and the \(j\)-th row from the top.

The algorithm proceeds as follows: non-attacking rooks are placed beginning at the top row. The columns corresponding to the closers of the set partition remain empty. Let \(rs_j\) be the number of closers which are larger than \(j\) and whose block is before the block of \(j\).

We then place a rook into row \(j\), such that the number of cells to the left of the rook, which are not yet attacked by another rook and are not in a column corresponding to a closer, equals \(rs_j\), unless there are not enough cells in this row available, in which case the row remains empty.

One can show that the precisely those rows which correspond to openers of the set partition remain empty.
OUTPUT:
A list of coordinates.

See also:

• to_rook_placement()
• SetPartitions.from_rook_placement()
• SetPartitions.from_rook_placement_rho()

EXAMPLES:

```
sage: P = SetPartition([[1,4],[2,8],[3,5,6,9],[7]])
sage: P.to_rook_placement_rho()
[(1, 5), (2, 6), (3, 4), (5, 9), (6, 8)]
```

Figure 6 in [WW1991]:

```
sage: P = SetPartition([[1,2,4,7],[3,9],[5,6,10,11,13],[8],[12]])
sage: r = P.to_rook_placement_rho(); r
[(1, 2), (2, 6), (3, 4), (4, 10), (5, 9), (6, 7), (10, 11), (11, 13)]
sage: sorted(P.closers() + [i for i, _ in r]) == list(range(1,14))
True
sage: sorted(P.openers() + [j for _, j in r]) == list(range(1,14))
True
```

class sage.combinat.set_partition.SetPartitions

Bases: UniqueRepresentation, Parent

An (unordered) partition of a set \( S \) is a set of pairwise disjoint nonempty subsets with union \( S \), and is represented by a sorted list of such subsets.

SetPartitions(s) returns the class of all set partitions of the set \( s \), which can be given as a set or a string; if a string, each character is considered an element.

SetPartitions(n), where \( n \) is an integer, returns the class of all set partitions of the set \( \{1, 2, \ldots, n\} \).

You may specify a second argument \( k \). If \( k \) is an integer, SetPartitions returns the class of set partitions into \( k \) parts; if it is an integer partition, SetPartitions returns the class of set partitions whose block sizes correspond to that integer partition.

The Bell number \( B_n \), named in honor of Eric Temple Bell, is the number of different partitions of a set with \( n \) elements.

EXAMPLES:

```
sage: S = [1,2,3,4]
sage: SetPartitions(S, 2)
Set partitions of {1, 2, 3, 4} with 2 parts
sage: SetPartitions([1,2,3,4], [3,1]).list()
[[(1), (2, 3, 4)], [(1, 2), (3), (4)], [(1, 2, 4), {3}], [(1, 3, 4), {2}]]
sage: SetPartitions(7, [3,3,1]).cardinality()
70
```

In strings, repeated letters are not considered distinct as of github issue #14140:
\texttt{sage: SetPartitions('abcde').cardinality()}
52
\texttt{sage: SetPartitions('aabcd').cardinality()}
15

REFERENCES:

- Wikipedia article \texttt{Partition\_of\_a\_set}

\textbf{Element}

alias of \texttt{SetPartition}

\texttt{from\_arcs}(arcs, n)

Return the coarsest set partition of \{1, ..., n\} such that any two elements connected by an arc are in the same block.

INPUT:

- n – an integer specifying the size of the set partition to be produced.
- arcs – a list of pairs specifying which elements are in the same block.

See also:

- \texttt{from\_rook\_placement()}
- \texttt{SetPartition.to\_rook\_placement()}
- \texttt{SetPartition.arcs()}

EXAMPLES:

\texttt{sage: SetPartitions().from\_arcs([(2,3)], 5)}
\{\{1\}, \{2, 3\}, \{4\}, \{5\}\}

\texttt{from\_restricted\_growth\_word}(w, bijection='blocks')

Convert a word of length \(n\) with letters in the non-negative integers such that each letter is at most 1 larger than all the letters before to a set partition of \{1, ..., n\}.

INPUT:

- w – a restricted growth word.
- bijection (default: blocks) – defines the map from restricted growth functions to set partitions. These are currently:
  - blocks: .
  - intertwining: \texttt{from\_restricted\_growth\_word\_intertwining()}

OUTPUT:

A set partition.

See also:

\texttt{SetPartition.to\_restricted\_growth\_word()}

EXAMPLES:
Combinatorics, Release 10.0

```
sage: SetPartitions().from_restricted_growth_word([0, 1, 2, 0, 2, 2, 3, 1, 2])
{{1, 4}, {2, 8}, {3, 5, 6, 9}, {7}}
sage: SetPartitions().from_restricted_growth_word([0, 0, 1, 0, 2, 2, 0, 3, 1, 2, →2, 4, 2])
{{1, 2, 4, 7}, {3, 9}, {5, 6, 10, 11, 13}, {8}, {12}}
sage: SetPartitions().from_restricted_growth_word([0, 0, 1, 0, 2, 2, 0, 3, 1, 2, →2, 4, 2], "intertwining")
{{1, 2, 6, 7, 9}, {3, 4}, {5, 10, 13}, {8, 11}, {12}}
```

**from_restricted_growth_word_blocks**(*w*)

Convert a word of length *n* with letters in the non-negative integers such that each letter is at most 1 larger than all the letters before to a set partition of \{1, ..., *n*\}.

*w*[1] is the index of the block containing *i+1* when sorting the blocks by their minimal element.

**INPUT:**

- *w* – a restricted growth word.

**OUTPUT:**

A set partition.

**See also:**

*from_restricted_growth_word()*  *SetPartition.to_restricted_growth_word()*

**EXAMPLES:**

```
sage: SetPartitions().from_restricted_growth_word_blocks([0, 0, 1, 0, 2, →2, 3, 1, 2, 2, 4, 2])
{{1, 2, 4, 7}, {3, 9}, {5, 6, 10, 11, 13}, {8}, {12}}
```

**from_restricted_growth_word_intertwining**(*w*)

Convert a word of length *n* with letters in the non-negative integers such that each letter is at most 1 larger than all the letters before to a set partition of \{1, ..., *n*\}.

The *i*-th letter of the word is the numbers of crossings of the arc (or half-arc) in the extended arc diagram ending at *i*, with arcs (or half-arcs) beginning at a smaller element and ending at a larger element.

**INPUT:**

- *w* – a restricted growth word.

**OUTPUT:**

A set partition.

**See also:**

*from_restricted_growth_word()*  *SetPartition.to_restricted_growth_word()*

**EXAMPLES:**

```
sage: SetPartitions().from_restricted_growth_word_intertwining([0, 0, 1, 0, 2, →2, 0, 3, 1, 2, 2, 4, 2])
{{1, 2, 6, 7, 9}, {3, 4}, {5, 10, 13}, {8, 11}, {12}}
```
from_rook_placement(rooks, bijection='arcs', n=None)

Convert a rook placement of the triangular grid to a set partition of \{1, ..., n\}.

If n is not given, it is first checked whether it can be determined from the parent, otherwise it is the maximal occurring integer in the set of rooks.

INPUT:

- rooks – a list of pairs \((i, j)\) satisfying \(0 < i < j < n + 1\).
- bijection (default: arcs) – defines the map from rook placements to set partitions. These are currently:
  - arcs: from_arcs().
  - gamma: from_rook_placement_gamma().
  - rho: from_rook_placement_rho().
  - psi: from_rook_placement_psi().
- n – (optional) the size of the ground set.

See also:

SetPartition.to_rook_placement()

EXAMPLES:

```sage
SetPartitions(9).from_rook_placement([[1,4],[2,8],[3,5],[5,6],[6,9]])
{{1, 4}, {2, 8}, {3, 5, 6, 9}, {7}}

sage: SetPartitions(13).from_rook_placement([[12,13],[10,12],[8,9],[7,11],[5,8],
                                           [4,6],[3,5],[1,4]], "gamma")
{{1, 2, 4, 7}, {3, 9}, {5, 6, 10, 11, 13}, {8}, {12}}
```

from_rook_placement_gamma(rooks, n)

Return the set partition of \{1, ..., n\} corresponding to the given rook placement by applying Wachs and White’s bijection gamma.

Note that our index convention differs from the convention in [WW1991]: regarding the rook board as a lower-right triangular grid, we refer with \((i, j)\) to the cell in the \(i\)-th column from the right and the \(j\)-th row from the top.

INPUT:

- n – an integer specifying the size of the set partition to be produced.
- rooks – a list of pairs \((i, j)\) such that \(0 < i < j < n + 1\).

OUTPUT:

A set partition.

See also:

- from_rook_placement()
- SetPartition.to_rook_placement()
- SetPartition.to_rook_placement_gamma()

EXAMPLES:

Figure 5 in [WW1991] concerns the following rook placement:
\texttt{sage: } r = [(1, 4), (3, 5), (4, 6), (5, 8), (7, 11), (8, 9), (10, 12), (12, 13)]

Note that the rook \((1, 4)\), translated into Wachs and White’s convention, is a rook in row 4 from the top and column 13 from the left. The corresponding set partition is:

\texttt{sage: } SetPartitions().from_rook_placement_gamma(r, 13)

\[
\{\{1, 2, 4, 7\}, \{3, 9\}, \{5, 6, 10, 11, 13\}, \{8\}, \{12\}\}
\]

\textbf{from_rook_placement_psi}(\texttt{rooks}, \texttt{n})

Return the set partition of \(\{1, \ldots, n\}\) corresponding to the given rook placement by applying Yip’s bijection \(\psi\).

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{n} – an integer specifying the size of the set partition to be produced.
  \item \texttt{rooks} – a list of pairs \((i, j)\) such that \(0 < i < j < n + 1\).
\end{itemize}

\textbf{OUTPUT:}

A set partition.

\textbf{See also:}

\begin{itemize}
  \item \texttt{from_rook_placement()}
  \item \texttt{SetPartition.to_rook_placement()}
  \item \texttt{SetPartition.to_rook_placement_psi()}
\end{itemize}

\textbf{EXAMPLES:}

Example 36 (arXiv version: Example 4.5) in [Yip2018] concerns the following rook placement:

\texttt{sage: } r = [(4, 5), (1, 7), (3, 8), (7, 9)]
\texttt{sage: } SetPartitions().from_rook_placement_psi(r, 9)

\[
\{\{1, 5\}, \{2\}, \{3, 8, 9\}, \{4\}, \{6, 7\}\}
\]

\textbf{from_rook_placement_rho}(\texttt{rooks}, \texttt{n})

Return the set partition of \(\{1, \ldots, n\}\) corresponding to the given rook placement by applying Wachs and White’s bijection \(\rho\).

Note that our index convention differs from the convention in [WW1991]: regarding the rook board as a lower-right triangular grid, we refer with \((i, j)\) to the cell in the \(i\)-th column from the right and the \(j\)-th row from the top.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{n} – an integer specifying the size of the set partition to be produced.
  \item \texttt{rooks} – a list of pairs \((i, j)\) such that \(0 < i < j < n + 1\).
\end{itemize}

\textbf{OUTPUT:}

A set partition.

\textbf{See also:}

\begin{itemize}
  \item \texttt{from_rook_placement()}
  \item \texttt{SetPartition.to_rook_placement()}
\end{itemize}
• **SetPartition.to_rook_placement_rho()**

**EXAMPLES:**

Figure 5 in [WW1991] concerns the following rook placement:

```
sage: r = [(1, 2), (2, 6), (3, 4), (4, 10), (5, 9), (6, 7), (10, 11), (11, 13)]
```

Note that the rook (1, 2), translated into Wachs and White’s convention, is a rook in row 2 from the top and column 13 from the left. The corresponding set partition is:

```
sage: SetPartitions().from_rook_placement_rho(r, 13)
{{1, 2, 4, 7}, {3, 9}, {5, 6, 10, 11, 13}, {8}, {12}}
```

**is_less_than(s, t)**

Check if $s < t$ in the refinement ordering on set partitions.

This means that $s$ is a refinement of $t$ and satisfies $s \neq t$.

A set partition $s$ is said to be a refinement of a set partition $t$ of the same set if and only if each part of $s$ is a subset of a part of $t$.

**EXAMPLES:**

```
sage: S = SetPartitions(4)
sage: s = S([[1,3],[2,4]])
sage: t = S([[1],[2],[3],[4]])
sage: S.is_less_than(t, s)
True
sage: S.is_less_than(s, t)
False
sage: S.is_less_than(s, s)
False
```

**is_strict_refinement(s, t)**

Return True if $s$ is a strict refinement of $t$ and satisfies $s \neq t$.

A set partition $s$ is said to be a strict refinement of a set partition $t$ of the same set if and only if one can obtain $t$ from $s$ by repeatedly combining pairs of parts whose convex hulls don’t intersect (i.e., whenever we are combining two parts, the maximum of each of them should be smaller than the minimum of the other).

**EXAMPLES:**

```
sage: S = SetPartitions(4)
sage: s = S([[1],[2],[3],[4]])
sage: t = S([[1,3],[2,4]])
sage: u = S([[1,2,3,4]])
sage: S.is_strict_refinement(s, t)
True
sage: S.is_strict_refinement(t, u)
False
sage: A = SetPartition([[1,3],[2,4]])
sage: B = SetPartition([[1,2,3,4]])
sage: S.is_strict_refinement(s, A)
True
```

(continues on next page)
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sage: S.is_strict_refinement(t, B)
False

lt(s, t)
Check if \( s < t \) in the refinement ordering on set partitions.
This means that \( s \) is a refinement of \( t \) and satisfies \( s \neq t \).
A set partition \( s \) is said to be a refinement of a set partition \( t \) of the same set if and only if each part of \( s \) is a subset of a part of \( t \).

EXAMPLES:

```
sage: S = SetPartitions(4)
sage: s = S([[1,3],[2,4]])
sage: t = S([[1],[2],[3],[4]])
sage: S.is_less_than(t, s)
True
sage: S.is_less_than(s, t)
False
sage: S.is_less_than(s, s)
False
```

class sage.combinat.set_partition.SetPartitions_all
Bases: SetPartitions
All set partitions.

subset(size=None, **kwargs)
Return the subset of set partitions of a given size and additional keyword arguments.

EXAMPLES:

```
sage: P = SetPartitions()
sage: P.subset(4)
Set partitions of {1, 2, 3, 4}
```

class sage.combinat.set_partition.SetPartitions_set(s)
Bases: SetPartitions
Set partitions of a fixed set \( S \).

base_set()
Return the base set of self.

EXAMPLES:

```
sage: SetPartitions(3).base_set()
{1, 2, 3}
sage: sorted(SetPartitions(['a', 'b', 'c']).base_set())
['a', 'b', 'c']
```

base_set_cardinality()
Return the cardinality of the base set of self.

EXAMPLES:
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sage: SetPartitions(3).base_set_cardinality()
3

cardinality()

Return the number of set partitions of the set \( S \).

The cardinality is given by the \( n \)-th Bell number where \( n \) is the number of elements in the set \( S \).

EXAMPLES:

sage: SetPartitions([1,2,3,4]).cardinality()
15
sage: SetPartitions(3).cardinality()
5
sage: SetPartitions(3,2).cardinality()
3
sage: SetPartitions([]).cardinality()
1

random_element()

Return a random set partition.

This is a very naive implementation of Knuth’s outline in F3B, 7.2.1.5.

EXAMPLES:

sage: S = SetPartitions(10)
sage: s = S.random_element()
sage: s.parent() is S
True
sage: assert s in S, s
sage: S = SetPartitions(['a', 'b', 'c'])
sage: s = S.random_element()
sage: s.parent() is S
True
sage: assert s in S, s

class sage.combinat.set_partition.SetPartitions_setn(s, k)

Bases: SetPartitions_set

Set partitions with a given number of blocks.

cardinality()

The Stirling number of the second kind is the number of partitions of a set of size \( n \) into \( k \) blocks.

EXAMPLES:

sage: SetPartitions(5, 3).cardinality()
25
sage: stirling_number2(5,3)
25

number_of_blocks()

Return the number of blocks of the set partitions in self.

EXAMPLES:
sage: SetPartitions(5, 3).number_of_blocks()
3

random_element()

Return a random set partition of self.

See https://mathoverflow.net/questions/141999.

EXAMPLES:

sage: S = SetPartitions(10, 4)
sage: s = S.random_element()
sage: s.parent() is S
True
sage: assert s in S, s

sage: S = SetPartitions(["a", "b", "c"], 2)
sage: s = S.random_element()
sage: s.parent() is S
True
sage: assert s in S, s

class sage.combinat.set_partition.SetPartitions_setparts(s, parts)

Bases: SetPartitions_set

Set partitions with fixed partition sizes corresponding to an integer partition \( \lambda \).

cardinality()

Return the cardinality of self.

This algorithm counts for each block of the partition the number of ways to fill it using values from the set. Then, for each distinct value \( v \) of block size, we divide the result by the number of ways to arrange the blocks of size \( v \) in the set partition.

For example, if we want to count the number of set partitions of size 13 having [3,3,3,2,2] as underlying partition we compute the number of ways to fill each block of the partition, which is \( \binom{13}{3} \binom{10}{3} \binom{7}{3} \binom{4}{2} \binom{2}{2} \) and as we have three blocks of size 3 and two blocks of size 2, we divide the result by 3!2! which gives us 600600.

EXAMPLES:

sage: SetPartitions(3, [2,1]).cardinality()
3
sage: SetPartitions(13, Partition([3,3,3,2,2])).cardinality()
600600

random_element()

Return a random set partition of self.

ALGORITHM:

Based on the cardinality method. For each block size \( k_i \), we choose a uniformly random subset \( X_i \subseteq S_i \) of size \( k_i \) of the elements \( S_i \) that have not yet been selected. Thus, we define \( S_{i+1} = S_i \setminus X_i \) with \( S_i = S \) being the defining set. This is not yet proven to be uniformly distributed, but numerical tests show this is likely uniform.

EXAMPLES:
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sage: S = SetPartitions(10, [4,3,2,1])
sage: s = S.random_element()
sage: assert s in S, s

sage: S = SetPartitions(["a", "b", "c", "d"], [2,2])
sage: s = S.random_element()
sage: assert s in S, s

shape()
Return the partition of block sizes of the set partitions in self.

EXAMPLES:

sage: SetPartitions(5, [2,2,1]).shape()

2, 2, 1

sage.combinat.set_partition.cyclic_permutations_of_set_partition(set_part)
Return all combinations of cyclic permutations of each cell of the set partition.

AUTHORS:
• Robert L. Miller

EXAMPLES:

sage: from sage.combinat.set_partition import cyclic_permutations_of_set_partition
sage: cyclic_permutations_of_set_partition([[1,2,3,4],[5,6,7]])

[[[1, 2, 3, 4], [5, 6, 7]],
 [[1, 2, 4, 3], [5, 6, 7]],
 [[1, 3, 2, 4], [5, 6, 7]],
 [[1, 3, 4, 2], [5, 6, 7]],
 [[1, 4, 2, 3], [5, 6, 7]],
 [[1, 4, 3, 2], [5, 6, 7]],
 [[1, 2, 3, 4], [5, 7, 6]],
 [[1, 2, 4, 3], [5, 7, 6]],
 [[1, 3, 2, 4], [5, 7, 6]],
 [[1, 3, 4, 2], [5, 7, 6]],
 [[1, 4, 2, 3], [5, 7, 6]],
 [[1, 4, 3, 2], [5, 7, 6]]]

sage.combinat.set_partition.cyclic_permutations_of_set_partition_iterator(set_part)
Iterates over all combinations of cyclic permutations of each cell of the set partition.

AUTHORS:
• Robert L. Miller

EXAMPLES:

sage: from sage.combinat.set_partition import cyclic_permutations_of_set_partition_iterator
sage: list(cyclic_permutations_of_set_partition_iterator([[1,2,3,4],[5,6,7]]))

(continues on next page)
5.1.277 Ordered Set Partitions

AUTHORS:

• Mike Hansen
• MuPAD-Combinat developers (for algorithms and design inspiration)
• Travis Scrimshaw (2013-02-28): Removed CombinatorialClass and added entry point through OrderedSetPartition.

class sage.combinat.set_partition_ordered.OrderedSetPartition(parent, s, check=True)
Bases: ClonableArray

An ordered partition of a set.

An ordered set partition \( p \) of a set \( s \) is a list of pairwise disjoint nonempty subsets of \( s \) such that the union of these subsets is \( s \). These subsets are called the parts of the partition.

We represent an ordered set partition as a list of sets. By extension, an ordered set partition of a nonnegative integer \( n \) is the set partition of the integers from 1 to \( n \). The number of ordered set partitions of \( n \) is called the \( n \)-th ordered Bell number.

There is a natural integer composition associated with an ordered set partition, that is the sequence of sizes of all its parts in order.

The number \( T_n \) of ordered set partitions of \( \{1, 2, \ldots, n\} \) is the so-called \( n \)-th Fubini number (also known as the \( n \)-th ordered Bell number; see Wikipedia article Ordered Bell number). Its exponential generating function is

\[
\sum_n \frac{T_n}{n!} x^n = \frac{1}{2 - e^x}.
\]

(See sequence OEIS sequence A000670 in OEIS.)

INPUT:

• parts – an object or iterable that defines an ordered set partition (e.g., a list of pairwise disjoint sets) or a packed word (e.g., a list of letters on some alphabet). If there is ambiguity and if the input should be treated as a packed word, the keyword from_word should be used.

EXAMPLES:

There are 13 ordered set partitions of \( \{1, 2, 3\} \):

\[
\begin{align*}
&[[1, 2, 3, 4], [5, 6, 7]], \\
&[[1, 2, 4, 3], [5, 6, 7]], \\
&[[1, 3, 2, 4], [5, 6, 7]], \\
&[[1, 3, 4, 2], [5, 6, 7]], \\
&[[1, 4, 2, 3], [5, 6, 7]], \\
&[[1, 4, 3, 2], [5, 6, 7]], \\
&[[1, 2, 3, 4], [5, 7, 6]], \\
&[[1, 2, 4, 3], [5, 7, 6]], \\
&[[1, 3, 2, 4], [5, 7, 6]], \\
&[[1, 3, 4, 2], [5, 7, 6]], \\
&[[1, 4, 2, 3], [5, 7, 6]], \\
&[[1, 4, 3, 2], [5, 7, 6]].
\end{align*}
\]
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```python
sage: OrderedSetPartitions(3).cardinality()
13
```

Here is the list of them:

```python
sage: OrderedSetPartitions(3).list()
[[\{1\}, \{2\}, \{3\}],
 [\{1\}, \{3\}, \{2\}],
 [\{2\}, \{1\}, \{3\}],
 [\{3\}, \{1\}, \{2\}],
 [\{2\}, \{3\}, \{1\}],
 [\{3\}, \{2\}, \{1\}],
 [\{1\}, \{2, 3\}],
 [\{2\}, \{1, 3\}],
 [\{3\}, \{1, 2\}],
 [\{1, 2\}, \{3\}],
 [\{1, 3\}, \{2\}],
 [\{2, 3\}, \{1\}],
 [\{1, 2, 3\}]]
```

There are 12 ordered set partitions of \{1, 2, 3, 4\} whose underlying composition is \([1, 2, 1]\):

```python
sage: OrderedSetPartitions(4,[1,2,1]).list()
[[\{1\}, \{2, 3\}, \{4\}],
 [\{1\}, \{2, 4\}, \{3\}],
 [\{1\}, \{3, 4\}, \{2\}],
 [\{2\}, \{1, 3\}, \{4\}],
 [\{2\}, \{1, 4\}, \{3\}],
 [\{3\}, \{1, 2\}, \{4\}],
 [\{4\}, \{1, 2\}, \{3\}],
 [\{3\}, \{1, 4\}, \{2\}],
 [\{4\}, \{1, 3\}, \{2\}],
 [\{2\}, \{3, 4\}, \{1\}],
 [\{3\}, \{2, 4\}, \{1\}],
 [\{4\}, \{2, 3\}, \{1\}]]
```

Since [github issue #14140](https://github.com/sagemath/sage/issues/14140), we can create an ordered set partition directly by `OrderedSetPartition` which creates the parent object by taking the union of the partitions passed in. However it is recommended and (marginally) faster to create the parent first and then create the ordered set partition from that.

```python
sage: s = OrderedSetPartition([[1,3],[2,4]]); s
\[[1, 3], [2, 4]\]

sage: s.parent()
Ordered set partitions of \{1, 2, 3, 4\}
```

We can construct the ordered set partition from a word, which we consider as packed:

```python
sage: OrderedSetPartition([2,4,1,2])
\[[1, 4], [2]\]

sage: OrderedSetPartition(from_word=[2,4,1,2])
\[[1, 4], [2]\]

sage: OrderedSetPartition(from_word='bdab')
\[[1, 4], [2]\]
```

Chapter 5. Comprehensive Module List
**Warning:** The elements of the underlying set should be hashable.

**REFERENCES:**
Wikipedia article Ordered_partition_of_a_set

**base_set()**
Return the base set of `self`.
This is the union of all parts of `self`.

**EXAMPLES:**
```sage
sage: OrderedSetPartition([[1], [2,3], [4]]).base_set()
frozenset({1, 2, 3, 4})
sage: OrderedSetPartition([[1,2,3,4]]).base_set()
frozenset({1, 2, 3, 4})
sage: OrderedSetPartition([]).base_set()
frozenset()
```

**base_set_cardinality()**
Return the cardinality of the base set of `self`.
This is the sum of the sizes of the parts of `self`.
This is also known as the size (sometimes the weight) of an ordered set partition.

**EXAMPLES:**
```sage
sage: OrderedSetPartition([[1], [2,3], [4]]).base_set_cardinality()
4
sage: OrderedSetPartition([[1,2,3,4]]).base_set_cardinality()
4
```

**static bottom_up_osp(X, comp)**
Return the ordered set partition obtained by listing the elements of the set `X` in increasing order, and placing bars between some of them according to the integer composition `comp` (namely, the bars are placed in such a way that the lengths of the resulting blocks are exactly the entries of `comp`).

**INPUT:**
- `X` – a finite set (or list or tuple)
- `comp` – a composition whose sum is the size of `X` (can be given as a list or tuple or composition)

**EXAMPLES:**
```sage
sage: buo = OrderedSetPartition.bottom_up_osp
sage: buo(Set([1, 4, 7, 9]), [2, 1, 1])
[[1, 4], {7}, {9}]
sage: buo(Set([1, 4, 7, 9]), [1, 3])
[[1], {4, 7, 9}]
sage: buo(Set([1, 4, 7, 9]), [1, 1, 1])
[[1], {4}, {7}, {9}]
sage: buo(range(8), [1, 4, 2, 1])
[[0], {1, 2, 3, 4}, {5, 6}, {7}]
sage: buo([], [])
[]
```
check()

Check that we are a valid ordered set partition.

EXAMPLES:

```
sage: OS = OrderedSetPartitions(4)
sage: s = OS([[1, 3], [2, 4]])
sage: s.check()
```

complement()

Return the complement of the ordered set partition `self`.

This assumes that `self` is an ordered set partition of an interval of \( \mathbb{Z} \).

Let \((P_1, P_2, \ldots, P_k)\) be an ordered set partition of some interval \( I \) of \( \mathbb{Z} \). Let \( \omega \) be the unique strictly decreasing bijection \( I \rightarrow I \). Then, the complement of \((P_1, P_2, \ldots, P_k)\) is defined to be the ordered set partition \((\omega(P_1), \omega(P_2), \ldots, \omega(P_k))\).

EXAMPLES:

```
sage: OrderedSetPartition([[1, 2], [3]]).complement()=[[2, 3], {1}]
sage: OrderedSetPartition([[1, 3], [2]]).complement()=[[1, 3], {2}]
sage: OrderedSetPartition([[2, 3]]).complement()=[[2, 3]]
sage: OrderedSetPartition([[1, 5], [2, 3], [4]]).complement()=[[1, 5], {3, 4}, {2}]
sage: OrderedSetPartition([[1, 3, 4]]).complement()=[[1, 3, 4], [-1, -2, -3, -4, -5]]
sage: OrderedSetPartition([]).complement()[]
```

fatten(grouping)

Return the ordered set partition fatter than `self`, obtained by grouping together consecutive parts according to the integer composition `grouping`.

See finer() for the definition of “fatter”.

INPUT:

- `grouping` – a composition whose sum is the length of `self`

EXAMPLES:

Let us start with the ordered set partition:

```
sage: c = OrderedSetPartition([[2, 5], [1], [3, 4]])
```

With `grouping` equal to \((1, \ldots, 1)\), \(c\) is left unchanged:

```
sage: c.fatten(Composition([1,1,1]))
[[2, 5], {1}, {3, 4}]
```

With `grouping` equal to \((\ell)\) where \(\ell\) is the length of \(c\), this yields the coarsest ordered set partition above \(c\):

```
sage: c.fatten(Composition([3]))
[[1, 2, 3, 4, 5]]
```
Other values for grouping yield (all the) other ordered set partitions coarser than $c$:

```
sage: c.fatten(Composition([2,1]))
[[1, 2, 5], {3, 4}]
sage: c.fatten(Composition([1,2]))
[[2, 5], {1, 3, 4}]
```

**fatter()**

Return the set of ordered set partitions which are fatter than `self`.

See **finer()** for the definition of “fatter”.

**EXAMPLES:**

```
sage: C = OrderedSetPartition([[2, 5], [1], [3, 4]]).fatter()
sage: C.cardinality()
4
sage: sorted(C)
[[[2, 5], {1}, {3, 4}],
 [[2, 5], {1, 3, 4}],
 [[1, 2, 5], {3, 4}],
 [[1, 2, 3, 4, 5]]]
```

Some extreme cases:

```
sage: list(OrderedSetPartition([[]]).fatter())
[[[]]]
sage: list(Composition([]).fatter())
[[[]]]
sage: sorted(OrderedSetPartition([[1], [2], [3], [4]]).fatter())
[[[1], {2}, {3}, {4}],
 [[1], {2}, {3, 4}],
 [[1], {2, 3}, {4}],
 [[1], {2, 3, 4}],
 [[1, 2], {3}, {4}],
 [[1, 2], {3, 4}],
 [[1, 2, 3}, {4}],
 [[1, 2, 3, 4}]]
```

**finer()**

Return the set of ordered set partitions which are finer than `self`.

See **is_finer()** for the definition of “finer”.

**EXAMPLES:**

```
sage: C = OrderedSetPartition([[1, 3], [2]]).finer()
sage: C.cardinality()
3
sage: C.list()
[[[1], {3}, {2}], [[3], {1}, {2}], [[1, 3], {2}]]
```
{[[]]}

sage: W = OrderedSetPartition([[4, 9], [-1, 2]])
sage: W.finer().list()
[[{9}, {4}, {2}, {-1}],
 [{9}, {4}, {-1}, {2}],
 [{9}, {4}, {-1, 2}],
 [{4}, {9}, {2}, {-1}],
 [{4}, {9}, {-1}, {2}],
 [{4}, {9}, {-1, 2}],
 [{4, 9}, {2}, {-1}],
 [{4, 9}, {-1}, {2}],
 [{4, 9}, {-1, 2}]]

is_finer(co2)

Return True if the ordered set partition self is finer than the ordered set partition co2; otherwise, return False.

If A and B are two ordered set partitions of the same set, then A is said to be finer than B if B can be obtained from A by (repeatedly) merging consecutive parts. In this case, we say that B is fatter than A.

EXAMPLES:

sage: A = OrderedSetPartition([[1, 3], [2]])
sage: B = OrderedSetPartition([[1], [3], [2]])
sage: A.is_finer(B)
False
sage: B.is_finer(A)
True
sage: C = OrderedSetPartition([[3], [1], [2]])
sage: A.is_finer(C)
False
sage: C.is_finer(A)
True
sage: OrderedSetPartition([[2], [5], [1], [4]]).is_finer(OrderedSetPartition([[2, 5], [1, 4]]))
True
sage: OrderedSetPartition([[5], [2], [1], [4]]).is_finer(OrderedSetPartition([[2, 5], [1, 4]]))
True
sage: OrderedSetPartition([[2], [1], [5], [4]]).is_finer(OrderedSetPartition([[2, 5], [1, 4]]))
False
sage: OrderedSetPartition([[2, 5, 1], [4]]).is_finer(OrderedSetPartition([[2, 5], [1, 4]]))
False

is_strongly_finer(co2)

Return True if the ordered set partition self is strongly finer than the ordered set partition co2; otherwise, return False.

If A and B are two ordered set partitions of the same set, then A is said to be strongly finer than B if B can be obtained from A by (repeatedly) merging consecutive parts, provided that every time we merge two consecutive parts C_i and C_{i+1}, we have max C_i < min C_{i+1}. In this case, we say that B is strongly fatter.
than \( A \).

**EXAMPLES:**

```python
sage: A = OrderedSetPartition([[1, 3], [2]])
sage: B = OrderedSetPartition([[1], [3], [2]])
sage: A.is_strongly_finer(B)
False
sage: B.is_strongly_finer(A)
True
sage: C = OrderedSetPartition([[3], [1], [2]])
sage: A.is_strongly_finer(C)
False
sage: C.is_strongly_finer(A)
False
sage: OrderedSetPartition([[2], [5], [1], [4]]).is_strongly_finer(OrderedSetPartition([[2, 5], [1, 4]]))
True
sage: OrderedSetPartition([[5], [2], [1], [4]]).is_strongly_finer(OrderedSetPartition([[2, 5], [1, 4]]))
False
sage: OrderedSetPartition([[2], [1], [5], [4]]).is_strongly_finer(OrderedSetPartition([[2, 5], [1, 4]]))
False
sage: OrderedSetPartition([[2, 5, 1], [4]]).is_strongly_finer(OrderedSetPartition([[2, 5], [1, 4]]))
False
```

**length()**

Return the number of parts of \( self \).

**EXAMPLES:**

```python
sage: OS = OrderedSetPartitions(4)
sage: s = OS([[1, 3], [2, 4]])
sage: s.length()
2
```

**number_of_inversions()**

Return the number of inversions in \( self \).

An inversion of an ordered set partition with blocks \([B_1, B_2, \ldots, B_k]\) is a pair of letters \( i \) and \( j \) with \( i < j \) such that \( i \) is minimal in \( B_m \), \( j \in B_l \), and \( l < m \).

**REFERENCES:**

- [Wilson2016]

**EXAMPLES:**

```python
sage: OrderedSetPartition([[2,5],[4,6],[1,3]]).number_of_inversions()
5
sage: OrderedSetPartition([[1,3,8],[2,4],[5,6,7]]).number_of_inversions()
3
```

**reversed()**

Return the reversal of the ordered set partition \( self \).
The reversal of an ordered set partition \((P_1, P_2, \ldots, P_k)\) is defined to be the ordered set partition \((P_k, P_{k-1}, \ldots, P_1)\).

**EXAMPLES:**

```
sage: OrderedSetPartition([[1, 3], [2]]).reversed()
[[2], [1, 3]]
sage: OrderedSetPartition([[1, 5], [2, 4]]).reversed()
[[2, 4], [1, 5]]
sage: OrderedSetPartition([[-1], [-2], [3, 4], [0]]).reversed()
[[0], [3, 4], [-2], [-1]]
sage: OrderedSetPartition([]).reversed()
[]
```

**size()**

Return the cardinality of the base set of \(self\).

This is the sum of the sizes of the parts of \(self\).

This is also known as the size (sometimes the weight) of an ordered set partition.

**EXAMPLES:**

```
sage: OrderedSetPartition([[1], [2,3], [4]]).base_set_cardinality()
4
sage: OrderedSetPartition([[1,2,3,4]]).base_set_cardinality()
4
```

**strongly_fatter()**

Return the set of ordered set partitions which are strongly fatter than \(self\).

See **strongly_finer()** for the definition of “strongly fatter”.

**EXAMPLES:**

```
sage: C = OrderedSetPartition([[2, 5], [1], [3, 4]]).strongly_fatter()
sage: C.cardinality()
2
sage: sorted(C)
[[[2, 5], {1}, {3, 4}], [[2, 5], {1, 3, 4}]]

sage: OrderedSetPartition([[4, 9], [-1, 2]]).strongly_fatter().list()
[[[4, 9], {-1, 2}]]
```

Some extreme cases:

```
sage: list(OrderedSetPartition([[5]]).strongly_fatter())
[[[5]]]
sage: list(OrderedSetPartition([]).strongly_fatter())
[[[]]]
sage: sorted(OrderedSetPartition([[1], [2], [3], [4]]).strongly_fatter())
[[[1], {2}, {3}, {4}],
[[1], {2}, {3, 4}],
[[1], {2, 3}, {4}],
[[1], {2, 3, 4}],
[[1, 2], {3}, {4}],
[[1, 2, 3}, {4}],
```

(continues on next page)
\[
\{[1, 2], \{3, 4\}, \\
[1, 2, 3], \{4\}, \\
[1, 2, 3, 4]\}
\]

\textbf{sage: sorted(OrderedSetPartition([\{1\}, \{3\}, \{2\}, \{4\}]).strongly_fatter())}
\[
[\{\{1\}, \{3\}, \{2\}, \{4\}\}, \\
[\{1\}, \{3\}, \{2\}, \{4\}\}, \\
[\{1\}, \{3\}, \{2\}, \{4\}\}, \\
[\{1\}, \{3\}, \{2\}, \{4\}\}]
\]

\textbf{sage: sorted(OrderedSetPartition([\{4\}, \{1\}, \{5\}, \{3\}]).strongly_fatter())}
\[
[\{\{4\}, \{1\}, \{5\}, \{3\}\}, \{\{4\}, \{1\}, \{5\}, \{3\}\}]
\]

\textbf{strongly_finer()}

Return the set of ordered set partitions which are strongly finer than self.

See \textit{is_strongly_finer()} for the definition of “strongly finer”.

\textbf{EXAMPLES:}

\textbf{sage: C = OrderedSetPartition([\{1, 3\}, \{2\}]).strongly_finer()}
\textbf{sage: C.cardinality()}
2
\textbf{sage: C.list()}
[\{\{1\}, \{3\}, \{2\}\}, \{\{1\}, \{3\}, \{2\}\}]

\textbf{sage: OrderedSetPartition([\{\}]).strongly_finer()}
[\{\}]

\textbf{sage: W = OrderedSetPartition([\{4, 9\}, \{-1, 2\}])}
\textbf{sage: W.strongly_finer().list()}
[\{\{4\}, \{9\}, \{-1\}, \{2\}\}, \\
\{\{4\}, \{9\}, \{-1, 2\}\}, \\
\{\{4\}, \{9\}, \{-1\}, \{2\}\}, \\
\{\{4\}, \{-1\}, \{2\}\}]

\textbf{static sum(osps)}

Return the concatenation of the given ordered set partitions osps (provided they have no elements in common).

\textbf{INPUT:}

- osps – a list (or iterable) of ordered set partitions

\textbf{EXAMPLES:}

\textbf{sage: OrderedSetPartition.sum([OrderedSetPartition([\{4, 1\}, \{3\}]), \ldots OrderedSetPartition([\{7\}, \{2\}]), OrderedSetPartition([\{5, 6\}])])}
\[
[\{1\}, \{3\}, \{7\}, \{2\}, \{5, 6\}]
\]

Any iterable can be provided as input:

\textbf{sage: OrderedSetPartition.sum([OrderedSetPartition([\{2*i, 2*i+1\}] for i in [4,1, \ldots 3]})}
\[
[\{8\}, \{2\}, \{3\}, \{6\}, \{7\}]
\]

Empty inputs are handled gracefully:
to_composition()
   Return the integer composition whose parts are the sizes of the sets in self.

   EXAMPLES:

   sage: S = OrderedSetPartitions(5)
sage: x = S([[3,5,4], [1, 2]])
sage: x.to_composition()
   [3, 2]
sage: y = S([[3,1], [2], [5,4]])
sage: y.to_composition()
   [2, 1, 2]

to_packed_word()
   Return the packed word on alphabet \{1, 2, 3, \ldots\} corresponding to self.

   A packed word on alphabet \{1, 2, 3, \ldots\} is any word whose maximum letter is the same as its total number
   of distinct letters. Let \(P\) be an ordered set partition of a set \(X\). The corresponding packed word
   \(w_1w_2\cdots w_n\) is constructed by having letter \(w_i = j\) if the \(i\)-th smallest entry in \(X\) occurs in the \(j\)-th block of \(P\).

   See also:
   Word.to_ordered_set_partition()

   Warning:  This assumes there is a total order on the underlying set.

   EXAMPLES:

   sage: S = OrderedSetPartitions()
sage: x = S([[3,5], [2], [1,4,6]])
sage: x.to_packed_word()
   word: 321313
sage: x = S([['a', 'c', 'e'], ['b', 'd']])
sage: x.to_packed_word()
   word: 12121

class sage.combinat.set_partition_ordered.OrderedSetPartitions(s)
   Bases: UniqueRepresentation, Parent

   Return the combinatorial class of ordered set partitions of \(s\).

   The optional argument \(c\), if specified, restricts the parts of the partition to have certain sizes (the entries of \(c\)).

   EXAMPLES:

   sage: OS = OrderedSetPartitions([1,2,3,4]); OS
   Ordered set partitions of \{1, 2, 3, 4\}
sage: OS.cardinality()
   75
sage: OS.first()
   [[1], {2}, {3}, {4}]

(continues on next page)
sage: OS.last()
[[1, 2, 3, 4]]
sage: OS.random_element().parent() == OS
True

sage: OS = OrderedSetPartitions([1,2,3,4], [2,2]); OS
Ordered set partitions of {1, 2, 3, 4} into parts of size [2, 2]
sage: OS.cardinality()
6
sage: OS.first()
[[1, 2], [3, 4]]
sage: OS.last()
[[3, 4], [1, 2]]
sage: OS.list()
[[[1, 2], [3, 4]],
 [[1, 3], [2, 4]],
 [[1, 4], [2, 3]],
 [[2, 3], [1, 4]],
 [[2, 4], [1, 3]],
 [[3, 4], [1, 2]]]

sage: OS = OrderedSetPartitions("cat")
sage: OS  # random
Ordered set partitions of {'a', 't', 'c'}
sage: sorted(OS.list(), key=str)
[[{'a'}, {'c'}, {'t'}],
 [{'a'}, {'c'}, {'t'}],
 [{'a'}, {'a'}, {'t'}],
 [{'a'}, {'c'}, {'t'}],
 [{'c'}, {'a'}, {'t'}],
 [{'c'}, {'t'}, {'a'}],
 [{'t'}, {'a'}, {'c'}],
 [{'t'}, {'c'}, {'a'}],
 [{'t'}, {'a'}, {'c'}]]

Element
does not return

from_finite_word(w, check=True)
Return the unique ordered set partition of \{1, 2, \ldots, n\} corresponding to a word \(w\) of length \(n\).

See also:

Word.to_ordered_set_partition()

EXAMPLES:

sage: A = OrderedSetPartitions().from_finite_word('abcabcbd'); A
[[[1, 4, 7], [2, 5, 8], [3, 6], [9]]]
sage: B = OrderedSetPartitions().from_finite_word([1,2,3,1,2,3,1,2,4])
sage: A == B
True

class sage.combinat.set_partition_ordered.OrderedSetPartitions_all
    Bases: OrderedSetPartitions

    Ordered set partitions of \{1, \ldots, n\} for all \(n \in \mathbb{Z}_{\geq 0}\).

    class Element(parent, s, check=True)
        Bases: OrderedSetPartition

    subset(size=None, **kwargs)
        Return the subset of ordered set partitions of a given size and additional keyword arguments.

        EXAMPLES:
        sage: P = OrderedSetPartitions()
sage: P.subset(4)
        Ordered set partitions of \{1, 2, 3, 4\}

class sage.combinat.set_partition_ordered.OrderedSetPartitions_s(s)
    Bases: OrderedSetPartitions

    Class of ordered partitions of a set \(S\).

    cardinality()

    EXAMPLES:
    sage: OrderedSetPartitions(0).cardinality()
    1
    sage: OrderedSetPartitions(1).cardinality()
    1
    sage: OrderedSetPartitions(2).cardinality()
    3
    sage: OrderedSetPartitions(3).cardinality()
    13
    sage: OrderedSetPartitions([1,2,3]).cardinality()
    13
    sage: OrderedSetPartitions(4).cardinality()
    75
    sage: OrderedSetPartitions(5).cardinality()
    541

class sage.combinat.set_partition_ordered.OrderedSetPartitions_scomp(s, comp)
    Bases: OrderedSetPartitions

    cardinality()

    Return the cardinality of self.

    The number of ordered set partitions of a set of length \(k\) with composition shape \(\mu\) is equal to

    \[
    \frac{k!}{\prod_{\mu_i \neq 0} \mu_i !}.
    \]

    EXAMPLES:
sage: OrderedSetPartitions(5,[2,3]).cardinality()
10
sage: OrderedSetPartitions(0, []).cardinality()
1
sage: OrderedSetPartitions(0, [0]).cardinality()
1
sage: OrderedSetPartitions(0, [0,0]).cardinality()
1
sage: OrderedSetPartitions(5, [2,0,3]).cardinality()
10

class sage.combinat.set_partition_ordered.OrderedSetPartitions_sn(s, n)
    Bases: OrderedSetPartitions

cardinality()    

    Return the cardinality of self.

    The number of ordered partitions of a set of size \( n \) into \( k \) parts is equal to \( k!S(n, k) \) where \( S(n, k) \) denotes the Stirling number of the second kind.

    EXAMPLES:

sage: OrderedSetPartitions(4,2).cardinality()
14
sage: OrderedSetPartitions(4,1).cardinality()
1

class sage.combinat.set_partition_ordered.SplitNK(s, comp)
    Bases: OrderedSetPartitions_scomp

sage.combinat.set_partition_ordered.multiset_permutation_next_lex(l)
    Return the next multiset permutation after \( l \).

    EXAMPLES:

sage: from sage.combinat.set_partition_ordered import multiset_permutation_next_lex
sage: l = [0, 0, 1, 1, 2]
sage: while multiset_permutation_next_lex(l):
    ....:   print(l)
[0, 0, 1, 2, 1]
[0, 0, 2, 1, 1]
[0, 1, 0, 1, 2]
[0, 1, 0, 2, 1]
[0, 1, 1, 0, 2]
[0, 1, 1, 2, 0]
...
[1, 1, 2, 0, 0]
[1, 2, 0, 0, 1]
[1, 2, 0, 1, 0]
[1, 2, 1, 0, 0]
[2, 0, 0, 1, 1]
[2, 0, 1, 0, 1]
[2, 0, 1, 1, 0]
[2, 1, 0, 1, 1]
sage.combinat.set_partition_ordered.multiset_permutation_to_ordered_set_partition(l, m)

Convert a multiset permutation to an ordered set partition.

INPUT:

- l – a multiset permutation
- m – number of parts

EXAMPLES:

```python
sage: from sage.combinat.set_partition_ordered import multiset_permutation_to_ordered_set_partition
sage: l = [0, 0, 1, 1, 2]
sage: multiset_permutation_to_ordered_set_partition(l, 3)
[[0, 1], [2, 3], [4]]
```

### 5.1.278 Symmetric Functions

- Introduction to Symmetric Functions
- Symmetric Functions
  - Symmetric functions, with their multiple realizations
  - Classical symmetric functions
  - Schur symmetric functions
  - Monomial symmetric functions
  - Multiplicative symmetric functions
  - Elementary symmetric functions
  - Homogeneous symmetric functions
  - Power sum symmetric functions
  - Characters of the symmetric group as bases of the symmetric functions
  - Orthogonal Symmetric Functions
  - Symplectic Symmetric Functions
  - Generic dual bases symmetric functions
  - Symmetric functions defined by orthogonality and triangularity
  - Kostka-Foulkes Polynomials
  - Hall-Littlewood Polynomials
  - Hecke Character Basis
  - Jack Symmetric Functions
  - k-Schur Functions
  - Quotient of symmetric function space by ideal generated by Hall-Littlewood symmetric functions
• **LLT symmetric functions**
• **Macdonald Polynomials**
• **Non-symmetric Macdonald Polynomials**
• **Witt symmetric functions**

### 5.1.279 Characters of the symmetric group as bases of the symmetric functions

Just as the Schur functions are the irreducible characters of $GL_n$ and form a basis of the symmetric functions, the irreducible symmetric group character basis are the irreducible characters of $S_n$ when the group is realized as the permutation matrices.

**REFERENCES:**

```python
class sage.combinat.sf.character.generic_character(Sym, basis_name=None, prefix=None, graded=True)

    Bases: SymmetricFunctionAlgebra_generic

class sage.combinat.sf.character.induced_trivial_character_basis(Sym, prefix)

    Bases: generic_character
```

This basis is introduced in [OZ2015].

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: h = Sym.h()
sage: ht = SymmetricFunctions(QQ).ht()
sage: st = SymmetricFunctions(QQ).st()
sage: ht(s[2,1])
sage: s(ht[2,1])
sage: ht(h[2,1])
ht[1] + 2^ht[1, 1] + ht[2, 1]
sage: s(h[2,1])
h[1] - 2*s[1, 1] + h[2, 1]
sage: st(h[2,1])
sage: ht(st[2,1])
sage: ht[2]^ht[1,1]
ht[1, 1] + 2^ht[1, 1, 1] + ht[2, 1, 1]
sage: h[4,2].kronecker_product(h[4,1,1])
h[2, 2, 1, 1] + 2*h[3, 1, 1, 1] + h[4, 1, 1]
sage: s(st[2,1])
```

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(continued from previous page)

\[
3{s}[1] - 2{s}[1, 1] - 2{s}[2] + {s}[2, 1]
\]

```
sage: st(st(2[1])
sage: st(2)*st[1]
sage: s[4,2].kronecker_product(s[5,1])
```

```
class sage.combinat.sf.character.irreducible_character_basis(Sym, pfix)
    Bases: generic_character

The irreducible symmetric group character basis of the symmetric functions.

This is a basis of the symmetric functions that has the property that self(la).
character_to_frobenius_image(n) is equal to s([n-sum(la)]+la).

It should also have the property that the (outer) structure constants are the analogue of the stable Kronecker
coefficients on the Schur basis.

This basis is introduced in [OZ2015].

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: h = Sym.h()
sage: ht = SymmetricFunctions(QQ).ht()
sage: st = SymmetricFunctions(QQ).st()
sage: st(ht(2[1])
sage: ht(st(2[1])
sage: s(st(2[1])
3{s}[1] - 2{s}[1, 1] - 2{s}[2] + {s}[2, 1]
sage: st(st(2[1])
sage: st(2)*st[1]
sage: s[4,2].kronecker_product(s[5,1])
sage: st(1,1,1).counit()
-1
```

```
sage: all(sum(c*st(la)*st(mu).antipode() for
.....: (la, mu, c) in st(ga).coproduct())==st(st(ga).counit())
.....: for ga in Partitions(3))
True
```
5.1.280 Classical symmetric functions

```python
class sage.combinat.sf.classical.SymmetricFunctionAlgebra_classical(Sym, basis_name=None, prefix=None, graded=True)
```

Bases: `SymmetricFunctionAlgebra_generic`
The class of classical symmetric functions.

**Todo:** delete this class once all coercions will be handled by Sage's coercion model

```python
class Element
    Bases: `SymmetricFunctionAlgebra_generic.Element`
```

A symmetric function.

`sage.combinat.sf.classical.init()`
Set up the conversion functions between the classical bases.

**EXAMPLES:**

```python
sage: from sage.combinat.sf.classical import init
sage: sage.combinat.sf.classical.conversion_functions = {}
```

The following checks if the bug described in github issue #15312 is fixed.

```python
sage: change = sage.combinat.sf.classical.conversion_functions[('powersum', 'Schur ->')]
sage: hideme = change({Partition([1]*47):ZZ(1)})  # long time
sage: change({Partition([2,2]):QQ(1)})
```

5.1.281 Generic dual bases symmetric functions

```python
class sage.combinat.sf.dual.SymmetricFunctionAlgebra_dual(dual_basis, scalar, scalar_name='', basis_name=None, prefix=None)
```

Bases: `SymmetricFunctionAlgebra_classical`
Generic dual basis of a basis of symmetric functions.

**INPUT:**

- `dual_basis` – a basis of the ring of symmetric functions
- `scalar` – A function \( z \) on partitions which determines the scalar product on the power sum basis by \( \langle p_\mu , p_\nu \rangle = z(\mu) \). (Independently on the function chosen, the power sum basis will always be orthogonal; the function scalar only determines the norms of the basis elements.) This defaults to the function `zee` defined in `sage.combinat.sf.sfa`, that is, the function is defined by:

\[
\lambda \mapsto \prod_{i=1}^{\infty} m_i(\lambda)! t^{m_i(\lambda)},
\]
where $m_i(\lambda)$ means the number of times $i$ appears in $\lambda$. This default function gives the standard Hall scalar product on the ring of symmetric functions.

- **scalar_name** – (default: the empty string) a string giving a description of the scalar product specified by the parameter `scalar`
- **basis_name** – (optional) a string to serve as name for the basis to be generated (such as “forgotten” in “the forgotten basis”); don’t set it to any of the already existing basis names (such as homogeneous, monomial, forgotten, etc.).
- **prefix** – (default: 'd' and the prefix for dual_basis) a string to use as the symbol for the basis

**OUTPUT:**

The basis of the ring of symmetric functions dual to the basis `dual_basis` with respect to the scalar product determined by `scalar`.

**EXAMPLES:**

```python
sage: e = SymmetricFunctions(QQ).e()
sage: f = e.dual_basis(prefix = "m", basis_name="Forgotten symmetric functions"); f
Symmetric Functions over Rational Field in the Forgotten symmetric functions basis
sage: TestSuite(f).run(elements = [f[1,1]+2*f[2], f[1]+3*f[1,1]])
sage: TestSuite(f).run()  # long time (11s on sage.math, 2011)
```

This class defines canonical coercions between `self` and `self^*`, as follow:

Lookup for the canonical isomorphism from `self` to $P$ (powersum), and build the adjoint isomorphism from $P^*$ to `self^*`. Since $P$ is self-adjoint for this scalar product, derive an isomorphism from $P$ to `self^*`, and by composition with the above get an isomorphism from `self` to `self^*` (and similarly for the isomorphism `self^*` to `self`).

This should be stripped down to just (auto?) defining canonical isomorphism by adjunction (as in MuPAD-Combinat), and let the coercion handle the rest.

Inversions may not be possible if the base ring is not a field:

```python
sage: m = SymmetricFunctions(ZZ).m()
sage: h = m.dual_basis(lambda x: 1)
sage: h[2,1]
Traceback (most recent call last):
  ...
TypeError: no conversion of this rational to integer
```

By transitivity, this defines indirect coercions to and from all other bases:

```python
sage: s = SymmetricFunctions(QQ['t'].fraction_field()).s()
sage: t = QQ['t'].fraction_field().gen()
sage: zee_hl = lambda x: x.centralizer_size(t=t)
sage: S = s.dual_basis(zee_hl)
sage: S(s([2,1]))
(-t/(t^5-2*t^4+t^3-t^2+2*t-1))*d_s[1, 1, 1] + ((-t^2-1)/(t^5-2*t^4+t^3-t^2+2*t-1))*d_s[2, 1] + (-t/(t^5-2*t^4+t^3-t^2+2*t-1))*d_s[3]
```

**class Element**

Bases: `Element`

An element in the dual basis.
At least one of the following must be specified. The one (if any) which is not provided will be computed.

- **dictionary** – an internal dictionary for the monomials and coefficients of `self`
- **dual** – `self` as an element of the dual basis.

### `dual()`

Return `self` in the dual basis.

**OUTPUT:**
- the element `self` expanded in the dual basis to `self.parent()`

**EXAMPLES:**

```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(scalar=zee)
sage: a = h([2,1])
sage: a.parent()  # Dual basis to Symmetric Functions over Rational Field in the monomial basis
sage: a.dual()  # 3*m[1, 1, 1] + 2*m[2, 1] + m[3]
```

### `expand(n, alphabet='x')`

Expand the symmetric function `self` as a symmetric polynomial in `n` variables.

**INPUT:**
- `n` – a nonnegative integer
- `alphabet` – (default: `'x'`) a variable for the expansion

**OUTPUT:**
A monomial expansion of `self` in the `n` variables labelled by `alphabet`.

**EXAMPLES:**

```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(zee)
sage: a = h([2,1])+h([3])
sage: a.expand(2)  # 2*x0^3 + 3*x0^2*x1 + 3*x0*x1^2 + 2*x1^3
sage: a.dual().expand(2)  # 2*x0^3 + 3*x0^2*x1 + 3*x0*x1^2 + 2*x1^3
sage: a.expand(2, alphabet='y')  # 2*y0^3 + 3*y0^2*y1 + 3*y0*y1^2 + 2*y1^3
sage: a.expand(2, alphabet='x,y')  # 2*x^3 + 3*x^2*y + 3*x*y^2 + 2*y^3
sage: h([1]).expand(0)  # h([1]).expand(0)
0
sage: (3*h([])).expand(0)  # (3*h([])).expand(0)
3
```

### `omega()`

Return the image of `self` under the omega automorphism.

The **omega automorphism** is defined to be the unique algebra endomorphism \( \omega \) of the ring of symmetric functions that satisfies \( \omega(e_k) = h_k \) for all positive integers \( k \) (where \( e_k \) stands for the \( k \)-th elementary symmetric function, and \( h_k \) stands for the \( k \)-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the...
omega involution. It sends the power-sum symmetric function $p_k$ to $(-1)^{k-1} p_k$ for every positive integer $k$.

The images of some bases under the omega automorphism are given by

$$
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
$$

where $\lambda$ is any partition, where $\ell(\lambda)$ denotes the length (length()) of the partition $\lambda$, where $\lambda'$ denotes the conjugate partition (conjugate()) of $\lambda$, and where the usual notations for bases are used ($e =$ elementary, $h =$ complete homogeneous, $p =$ powersum, $s =$ Schur).

omega_involution() is a synonym for the omega() method.

OUTPUT:
- the result of applying omega to self

EXAMPLES:
```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(zee)
sage: hh = SymmetricFunctions(QQ).homogeneous()
sage: hh([2,1]).omega()  # h[1, 1, 1] - h[2, 1]
sage: h([2,1]).omega()   # d_m[1, 1, 1] - d_m[2, 1]
```

omega_involution()
Return the image of self under the omega automorphism.

The omega automorphism is defined to be the unique algebra endomorphism $\omega$ of the ring of symmetric functions that satisfies $\omega(e_k) = h_k$ for all positive integers $k$ (where $e_k$ stands for the $k$-th elementary symmetric function, and $h_k$ stands for the $k$-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the omega involution. It sends the power-sum symmetric function $p_k$ to $(-1)^{k-1} p_k$ for every positive integer $k$.

The images of some bases under the omega automorphism are given by

$$
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
$$

where $\lambda$ is any partition, where $\ell(\lambda)$ denotes the length (length()) of the partition $\lambda$, where $\lambda'$ denotes the conjugate partition (conjugate()) of $\lambda$, and where the usual notations for bases are used ($e =$ elementary, $h =$ complete homogeneous, $p =$ powersum, $s =$ Schur).

omega_involution() is a synonym for the omega() method.

OUTPUT:
- the result of applying omega to self

EXAMPLES:
```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(zee)
sage: hh = SymmetricFunctions(QQ).homogeneous()
sage: hh([2,1]).omega()  # h[1, 1, 1] - h[2, 1]
sage: h([2,1]).omega()   # d_m[1, 1, 1] - d_m[2, 1]
```
**scalar(x)**

Return the standard scalar product of self and x.

**INPUT:**
- x – element of the symmetric functions

**OUTPUT:**
- the scalar product between x and self

**EXAMPLES:**

```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(scalar=zee)
sage: a = h([2,1])
sage: a.scalar(a)
2
```

**scalar_hl(x)**

Return the Hall-Littlewood scalar product of self and x.

**INPUT:**
- x – element of the same dual basis as self

**OUTPUT:**
- the Hall-Littlewood scalar product between x and self

**EXAMPLES:**

```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(scalar=zee)
sage: a = h([2,1])
sage: a.scalar_hl(a)
(-t - 2)/(t^4 - 2*t^3 + 2*t - 1)
```

**product(left, right)**

Return product of left and right.

Multiplication is done by performing the multiplication in the dual basis of self and then converting back to self.

**INPUT:**
- left, right – elements of self

**OUTPUT:**
- the product of left and right in the basis self

**EXAMPLES:**

```python
sage: m = SymmetricFunctions(QQ).monomial()
sage: zee = sage.combinat.sf.sfa.zee
sage: h = m.dual_basis(scalar=zee)
sage: a = h([2,1])
sage: a*a; b # indirect doctest
d_m[2, 2]
sage: b.dual()
6*m[1, 1, 1, 1] + 4*m[2, 1, 1] + 3*m[2, 2] + 2*m[3, 1] + m[4]
```
transition_matrix(basis, n)

Returns the transition matrix between the \( n^{th} \) homogeneous components of self and basis.

INPUT:

- basis – a target basis of the ring of symmetric functions
- n – nonnegative integer

OUTPUT:

- A transition matrix from self to basis for the elements of degree n. The indexing order of the rows and columns is the order of Partitions(n).

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: e = Sym.elementary()
sage: f = e.dual_basis()
sage: f.transition_matrix(s, 5)
[ 1 -1 0 1 0 -1 1]
[-2 2 -1 -1 1 0 0]
[-2 2 -1 -1 1 0 0]
[ 3 -1 -1 1 0 0 0]
[ 3 -2 1 0 0 0 0]
[-4 1 0 0 0 0 0]
[ 1 0 0 0 0 0 0]
sage: Partitions(5).list()
[[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]]
sage: s(f[2,2,1])
sage: e.transition_matrix(s, 5).inverse().transpose()
[ 1 -1 0 1 0 -1 1]
[-2 2 -1 -1 1 0 0]
[-2 2 -1 -1 1 0 0]
[ 3 -1 -1 1 0 0 0]
[ 3 -2 1 0 0 0 0]
[-4 1 0 0 0 0 0]
[ 1 0 0 0 0 0 0]
```

5.1.282 Elementary symmetric functions

class sage.combinat.sf.elementary.SymmetricFunctionAlgebra_elementary(Sym)

Bases: SymmetricFunctionAlgebra_multiplicative

A class for methods for the elementary basis of the symmetric functions.

INPUT:

- self – an elementary basis of the symmetric functions
- Sym – an instance of the ring of symmetric functions

class Element

Bases: Element
\texttt{expand(n, alphabet='x')} \\
Expand the symmetric function \texttt{self} as a symmetric polynomial in \( n \) variables.

\textbf{INPUT:}

- \( n \) – a nonnegative integer
- \texttt{alphabet} – (default: 'x') a variable for the expansion

\textbf{OUTPUT:}

A monomial expansion of \texttt{self} in the \( n \) variables labelled by \texttt{alphabet}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: e = SymmetricFunctions(QQ).e()
sage: e([2,1]).expand(3)
x0^2*x1 + x0*x1^2 + 3*x0^2*x1 + x1^2*x2 + x0*x2^2 + x1*x2^2
sage: e([1,1,1]).expand(2)
x0^3 + 3*x0^2*x1 + 3*x0*x1^2 + x1^3
sage: e([3]).expand(2)
0
sage: e([2]).expand(3)
x0^2*x1 + x0*x2 + x1*x2
sage: e([3]).expand(4,alphabet='x,y,z,t')
x*y*z + x*y*t + x*z*t + y*z*t
sage: e([3]).expand(4,alphabet='y')
y0*y1*y2 + y0*y1*y3 + y0*y2*y3 + y1*y2*y3
sage: e([]).expand(2)
1
sage: e([]).expand(0)
1
sage: (3*e([])).expand(0)
3
\end{verbatim}
The limit of $e^x q$ as $q \to 1$ is $e^x$.

**INPUT:**
- $t$ (default: None) – the value to use for $t$; the default is to create a ring of polynomials in $t$.
- $q$ (default: 1) – the value to use for $q$. If $q$ is None, then a ring (or fraction field) of polynomials in $q$ is created.

**EXAMPLES:**

```sage
e = SymmetricFunctions(QQ).e()
x = e[[3,2]]
x.exponential_specialization()
1/12*t^5
x = 5*e[[2]] + 3*e[[1]] + 1
x.exponential_specialization(t=var("t"), q=var("q"))
5*q*t^2/(q + 1) + 3*t + 1
```

**omega()**

Return the image of self under the omega automorphism.

The **omega automorphism** is defined to be the unique algebra endomorphism $\omega$ of the ring of symmetric functions that satisfies $\omega(e_k) = h_k$ for all positive integers $k$ (where $e_k$ stands for the $k$-th elementary symmetric function, and $h_k$ stands for the $k$-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the **omega involution**. It sends the power-sum symmetric function $p_k$ to $(-1)^{k-1}p_k$ for every positive integer $k$.

The images of some bases under the omega automorphism are given by

$$
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)}p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
$$

where $\lambda$ is any partition, where $\ell(\lambda)$ denotes the length (length()) of the partition $\lambda$, where $\lambda'$ denotes the conjugate partition (conjugate()) of $\lambda$, and where the usual notations for bases are used ($e =$ elementary, $h =$ complete homogeneous, $p =$ powersum, $s =$ Schur).

**omega_involution()** is a synonym for the **omega()** method.

**EXAMPLES:**

```sage
e = SymmetricFunctions(QQ).e()
a = e([2,1]); a
e[[2, 1]]
a.omega()
e[[1, 1, 1]] - e[[2, 1]]
```

**omega_involution()**

Return the image of self under the omega automorphism.

The **omega automorphism** is defined to be the unique algebra endomorphism $\omega$ of the ring of symmetric functions that satisfies $\omega(e_k) = h_k$ for all positive integers $k$ (where $e_k$ stands for the $k$-th elementary symmetric function, and $h_k$ stands for the $k$-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the **omega involution**. It sends the power-sum symmetric function $p_k$ to $(-1)^{k-1}p_k$ for every positive integer $k$. 

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The images of some bases under the omega automorphism are given by

\[ \omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'}, \]

where \( \lambda \) is any partition, where \( \ell(\lambda) \) denotes the length of \( \lambda \), where \( \lambda' \) denotes the conjugate partition of \( \lambda \), and where the usual notations for bases are used (\( e = \) elementary, \( h = \) complete homogeneous, \( p = \) powersum, \( s = \) Schur).

\texttt{omega_involution()} is a synonym for the \texttt{omega()} method.

**EXAMPLES:**

```python
sage: e = SymmetricFunctions(QQ).e()
sage: a = e([2,1]); a
\texttt{e[2, 1]}
sage: a.omega()
\texttt{e[1, 1, 1] - e[2, 1]}
```

**principal_specialization\((n=+Infinity, q=None)\)**

Return the principal specialization of a symmetric function.

The principal specialization of order \( n \) at \( q \) is the ring homomorphism \( ps_{n,q} \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{-1} \) for \( i \in \{1, \ldots, n\} \) and \( x_i \mapsto 0 \) for \( i > n \). Here, \( q \) is a given element of \( R \), and we assume that the variables of our symmetric functions are \( x_1, x_2, x_3, \ldots \). (To be more precise, \( ps_{n,q} \) is a \( K \)-algebra homomorphism, where \( K \) is the base ring.) See Section 7.8 of [EnumComb2].

The stable principal specialization at \( q \) is the ring homomorphism \( ps_q \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for all \( i \). This is well-defined only if the resulting infinite sums converge; thus, in particular, setting \( q = 1 \) in the stable principal specialization is an invalid operation.

**INPUT:**
- \( n \) (default: \texttt{infinity}) – a nonnegative integer or \texttt{infinity}, specifying whether to compute the principal specialization of order \( n \) or the stable principal specialization.
- \( q \) (default: \texttt{None}) – the value to use for \( q \); the default is to create a ring of polynomials in \( q \) (or a field of rational functions in \( q \)) over the given coefficient ring.

We use the formulas from Proposition 7.8.3 of [EnumComb2] (using Gaussian binomial coefficients \( \binom{u}{v}_q \)):

\[ ps_{n,q}(e_\lambda) = \prod_i q^{\lambda_i}(n) \binom{n}{\lambda_i}_q, \]
\[ ps_{n,1}(e_\lambda) = \prod_i \binom{n}{\lambda_i}, \]
\[ ps_q(e_\lambda) = \prod_i q^{\lambda_i}/\prod_{j=1}^{\lambda_i} (1 - q^j). \]

**EXAMPLES:**

```
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```
By default, we return a rational functions in $q$. Sometimes it is better to obtain an element of the symbolic ring:

```
sage: x.principal_specialization(q=var("q"))
-3*q/((q^2 - 1)*(q - 1)^2) - 5/(q - 1)^3 + 1
```

verschiebung($n$)

Return the image of the symmetric function self under the $n$-th Verschiebung operator.

The $n$-th Verschiebung operator $V_n$ is defined to be the unique algebra endomorphism $V$ of the ring of symmetric functions that satisfies $V(h_r) = h_{r/n}$ for every positive integer $r$ divisible by $n$, and satisfies $V(h_r) = 0$ for every positive integer $r$ not divisible by $n$. This operator $V_n$ is a Hopf algebra endomorphism. For every nonnegative integer $r$ with $n \mid r$, it satisfies

$$V_n(h_r) = h_{r/n}, \quad V_n(p_r) = np_{r/n}, \quad V_n(e_r) = (-1)^{r-r/n}e_{r/n}$$

(where $h$ is the complete homogeneous basis, $p$ is the powersum basis, and $e$ is the elementary basis). For every nonnegative integer $r$ with $n \nmid r$, it satisfies

$$V_n(h_r) = V_n(p_r) = V_n(e_r) = 0.$$

The $n$-th Verschiebung operator is also called the $n$-th Verschiebung endomorphism. Its name derives from the Verschiebung (German for “shift”) endomorphism of the Witt vectors.

The $n$-th Verschiebung operator is adjoint to the $n$-th Frobenius operator (see frobenius() for its definition) with respect to the Hall scalar product (scalar()).

The action of the $n$-th Verschiebung operator on the Schur basis can also be computed explicitly. The following (probably clumsier than necessary) description can be obtained by solving exercise 7.61 in Stanley [STA].

Let $\lambda$ be a partition. Let $n$ be a positive integer. If the $n$-core of $\lambda$ is nonempty, then $V_n(s_\lambda) = 0$. Otherwise, the following method computes $V_n(s_\lambda)$: Write the partition $\lambda$ in the form $(\lambda_1, \lambda_2, \ldots, \lambda_{ns})$ for some nonnegative integer $s$. (If $n$ does not divide the length of $\lambda$, then this is achieved by adding trailing zeroes to $\lambda$.) Set $\beta_i = \lambda_i + ns - i$ for every $s \in \{1, 2, \ldots, ns\}$. Then, $(\beta_1, \beta_2, \ldots, \beta_{ns})$ is a strictly decreasing sequence of nonnegative integers. Stably sort the list $(1, 2, \ldots, ns)$ in order of (weakly) increasing remainder of $-1 - \beta_i$ modulo $n$. Let $\xi$ be the sign of the permutation that is used for this sorting. Let $\psi$ be the sign of the permutation that is used to stably sort the list $(1, 2, \ldots, ns)$ in order of (weakly) increasing remainder of $i - 1$ modulo $n$. (Notice that $\psi = (-1)^{n(n-1)s(s-1)/4}$.) Then, $V_n(s_\lambda) = \xi\psi \prod_{i=0}^{n-1} s_{\lambda(i)}$, where $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)})$ is the $n$-quotient of $\lambda$.

**INPUT:**
- $n$ - a positive integer

**OUTPUT:**

The result of applying the $n$-th Verschiebung operator (on the ring of symmetric functions) to self.

**EXAMPLES:**
The Verschiebung endomorphisms are multiplicative:

```
sage: all( all( e(lam).verschiebung(2) * e(mu).verschiebung(2) 
....: == (e(lam) * e(mu)).verschiebung(2) 
....: for mu in Partitions(4) ) 
....: for lam in Partitions(4) )
```
```
True
```

**coproduct_on_generators**(*i*)

Returns the coproduct on self[*i*].

**INPUT:**

- self – an elementary basis of the symmetric functions
- i – a nonnegative integer

**OUTPUT:**

- returns the coproduct on the elementary generator \( e(i) \)

**EXAMPLES:**

```
sage: Sym = SymmetricFunctions(QQ)
sage: e = Sym.elementary()
sage: e.coproduct_on_generators(2)
sage: e.coproduct_on_generators(0)
e[] # e[]
```

### 5.1.283 Hall-Littlewood Polynomials

Notation used in the definitions follows mainly [Mac1995].

**class** \`sage.combinat.sf.hall_littlewood.HallLittlewood\*(Sym, \( t=t' \))

**Bases:** \`UniqueRepresentation\`

The family of Hall-Littlewood symmetric function bases.

The Hall-Littlewood symmetric functions are a family of symmetric functions that depend on a parameter \( t \).

**INPUT:**

By default the parameter for these functions is \( t \), and whatever the parameter is, it must be in the base ring.

**EXAMPLES:**

```
sage: SymmetricFunctions(QQ).hall_littlewood(1)
Hall-Littlewood polynomials with \( t=1 \) over Rational Field
sage: SymmetricFunctions(QQ['t'].fraction_field()).hall_littlewood()
```

(continues on next page)
Hall-Littlewood polynomials over Fraction Field of Univariate Polynomial Ring in t → over Rational Field

P()

Return the algebra of symmetric functions in the Hall-Littlewood P basis. This is the same as the HL basis in John Stembridge’s SF examples file.

INPUT:

• self – a class of Hall-Littlewood symmetric function bases

OUTPUT:

The class of the Hall-Littlewood P basis.

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P(); HLP
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Hall-Littlewood P basis
sage: SP = Sym.hall_littlewood(t=-1).P(); SP
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Hall-Littlewood P with t=-1 basis
sage: s = Sym.schur()
sage: s(HLP([2,1]))
(-t^2-t)*s[1, 1, 1] + s[2, 1]
```

The Hall-Littlewood polynomials in the P basis at $t = 0$ are the Schur functions:

```
sage: Sym = SymmetricFunctions(QQ)
sage: HLP = Sym.hall_littlewood(t=0).P()
sage: s = Sym.schur()
sage: s(HLP([2,1])) == s([2,1])
True
```

The Hall-Littlewood polynomials in the P basis at $t = 1$ are the monomial symmetric functions:

```
sage: Sym = SymmetricFunctions(QQ)
sage: HLP = Sym.hall_littlewood(t=1).P()
sage: m = Sym.monomial()
sage: m(HLP([2,2,1])) == m([2,2,1])
True
```

We end with some examples of coercions between:

1. Hall-Littlewood P basis.
2. Hall-Littlewood polynomials in the Q basis
3. Hall-Littlewood polynomials in the $Q'$ basis (via the Schurs)
4. Classical symmetric functions

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P()
sage: HLQ = Sym.hall_littlewood().Q()
```
sage: HLQp = Sym.hall_littlewood().Qp()
sage: s = Sym.schur()
sage: p = Sym.power()
sage: HLP(HLP([2])) # indirect doctest
(-t+1)*HLP[2]
sage: HLP(HLP([2]))
t*HLP[1, 1] + HLP[2]
sage: HLP(s([2]))
t*HLP[1, 1] + HLP[2]
sage: HLP(p([2]))
(t-1)*HLP[1, 1] + HLP[2]
sage: s = HLQp.symmetric_function_ring().s()
sage: HLQp.transition_matrix(s,3)
\[
\begin{bmatrix}
1 & 0 & 0 \\
t & 1 & 0 \\
t^3 & t^2 + t & 1
\end{bmatrix}
\]
sage: s.transition_matrix(HLP,3)
\[
\begin{bmatrix}
1 & t & t^3 \\
0 & 1 & t^2 + t \\
0 & 0 & 1
\end{bmatrix}
\]

The method `sage.combinat.sf.sfa.SymmetricFunctionAlgebra_generic_Element.hl_creation_operator()` is a creation operator for the \(Q\) basis:

sage: HLQp[1].hl_creation_operator([3]).hl_creation_operator([3])
HLQp[3, 3, 1]

Transitions between bases with the parameter \(t\) specialized:

sage: Sym = SymmetricFunctions(FractionField(QQ['y','z']))
sage: (y,z) = Sym.base_ring().gens()
sage: HLy = Sym.hall_littlewood(t=y)
sage: HLz = Sym.hall_littlewood(t=z)
sage: Qpy = HLy.Qp()
sage: Qpz = HLz.Qp()
sage: s = Sym.schur()
sage: s( Qpy[3,1] + z*Qpy[2,2] )
z*s[2, 2] + (y*z+1)*s[3, 1] + (y^2*z+y)*s[4]
sage: s( Qpy[3,1] + y*Qpz[2,2] )
y*s[2, 2] + (y*z+1)*s[3, 1] + (y*z^2+y)*s[4]
sage: s( Qpy[3,1] + y*Qpy[2,2] )
y*s[2, 2] + (y^2+1)*s[3, 1] + (y^3+y)*s[4]

sage: Qy = HLy.Q()
sage: Qz = HLz.Q()
sage: Py = HLy.P()
sage: Pz = HLz.P()
sage: Pz(Qpy[2,1])
(y*z^3+z^2)*HLP[1, 1, 1] + (y*z+1)*HLP[2, 1] + y*HLP[3]
sage: Pz(Qz[2,1])
(z^2-2*z+1)*HLP[2, 1]
sage: Qz(Py[2])

(continues on next page)
((-y+z)/(z^3-z^2-z+1))*HLQ[1, 1] + (1/(-z+1))*HLQ[2]
sage: Qy(Pz[2])
((y-z)/(y^3-y^2-y+1))*HLQ[1, 1] + (1/(-y+1))*HLQ[2]
sage: Qy.hall_littlewood_family() == HLY
True
sage: Qy.hall_littlewood_family() == HLz
False
sage: Qz.symmetric_function_ring() == Qy.symmetric_function_ring()
True

sage: Sym = SymmetricFunctions(FractionField(QQ['q']))
sage: q = Sym.base_ring().gen()
sage: HL = Sym.hall_littlewood(t=q)
sage: HLQp = HL.Qp()
sage: HLQ = HL.Q()
sage: HLP = HL.P()
sage: s = Sym.schur()
sage: s(HLQp[3,2]).plethysm((1-q)*s[1])/(1-q)^2
(-q^5-q^4)*s[1, 1, 1, 1, 1] + (q^3+q^2)*s[2, 1, 1, 1] - q*s[2, 2, 1] - q*s[3, 1, 1] + s[3, 2]
sage: s(HLP[3,2])
(-q^5-q^4)*s[1, 1, 1, 1, 1] + (q^3+q^2)*s[2, 1, 1, 1] - q*s[2, 2, 1] - q*s[3, 1, 1] + s[3, 2]

The $P$ and $Q$-Schur at $t = -1$ indexed by strict partitions are a basis for the space algebraically generated by the odd power sum symmetric functions:

sage: Sym = SymmetricFunctions(FractionField(QQ['q']))
sage: SP = Sym.hall_littlewood(t=-1).P()
sage: SQ = Sym.hall_littlewood(t=-1).Q()
sage: p = Sym.power()
sage: SP(SQ[3,2,1])
8*HLP[3, 2, 1]
sage: SP(SQ[2,2,1])
0
sage: p(SP[3,2,1])
1/45*p[1, 1, 1, 1, 1] - 1/9*p[3, 1, 1, 1] - 1/9*p[3, 3] + 1/5*p[5, 1]
sage: SP(p[3,3])
HLQ[3, 1] + 2*q*HLQ[4]

Returns the algebra of symmetric functions in Hall-Littlewood $Q$ basis. This is the same as the $Q$ basis in John Stembridge’s SF examples file.

More extensive examples can be found in the documentation for the Hall-Littlewood $P$ basis.

INPUT:

• self – a class of Hall-Littlewood symmetric function bases

OUTPUT:

• returns the class of the Hall-Littlewood $Q$ basis
EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLQ = Sym.hall_littlewood().Q(); HLQ
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Hall-Littlewood Q basis
sage: SQ = SymmetricFunctions(QQ).hall_littlewood(t=-1).Q(); SQ
Symmetric Functions over Rational Field in the Hall-Littlewood Q with t=-1 basis
```

**Qp()**

Returns the algebra of symmetric functions in Hall-Littlewood $Q'$ (Qp) basis. This is dual to the Hall-Littlewood $P$ basis with respect to the standard scalar product.

More extensive examples can be found in the documentation for the Hall-Littlewood P basis.

**INPUT:**

- `self` – a class of Hall-Littlewood symmetric function bases

**OUTPUT:**

- returns the class of the Hall-Littlewood Qp-basis

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLQp = Sym.hall_littlewood().Qp(); HLQp
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Hall-Littlewood Qp basis
```

**base_ring()**

Returns the base ring of the symmetric functions where the Hall-Littlewood symmetric functions live.

**INPUT:**

- `self` – a class of Hall-Littlewood symmetric function bases

**OUTPUT:**

The base ring of the symmetric functions.

**EXAMPLES:**

```python
sage: HL = SymmetricFunctions(QQ['t'].fraction_field()).hall_littlewood(t=1)
sage: HL.base_ring()
Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

**symmetric_function_ring()**

The ring of symmetric functions associated to the class of Hall-Littlewood symmetric functions.

**INPUT:**

- `self` – a class of Hall-Littlewood symmetric function bases

**OUTPUT:**

- returns the ring of symmetric functions

**EXAMPLES:**
sage: HL = SymmetricFunctions(FractionField(QQ['t'])).hall_littlewood()
sage: HL.symmetric_function_ring()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field

class sage.combinat.sf.hall_littlewood.HallLittlewood_generic(hall_littlewood)
Bases: SymmetricFunctionAlgebra_generic

A class with methods for working with Hall-Littlewood symmetric functions which are common to all bases.

INPUT:
• self – a Hall-Littlewood symmetric function basis
• hall_littlewood – a class of Hall-Littlewood bases

class Element
Bases: SymmetricFunctionAlgebra_generic_Element

Methods for elements of a Hall-Littlewood basis that are common to all bases.

expand(n, alphabet='x')
Expands the symmetric function as a symmetric polynomial in n variables.

INPUT:
• self – an element of a Hall-Littlewood basis
• n – a positive integer
• alphabet – a string representing a variable name (default: 'x')

OUTPUT:
• returns a symmetric polynomial of self in n variables

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P()
sage: HLQ = Sym.hall_littlewood().Q()
sage: HLQp = Sym.hall_littlewood().Qp()
sage: HLP([2]).expand(2)
x0^2 + (-t + 1)*x0*x1 + x1^2
sage: HLQ([2]).expand(2)
(-t + 1)*x0^2 + (t^2 - 2*t + 1)*x0*x1 + (-t + 1)*x1^2
sage: HLQp([2]).expand(2)
x0^2 + x0*x1 + x1^2
sage: HLQp([2]).expand(2, 'y')
y0^2 + y0*y1 + y1^2
sage: HLQp([2]).expand(1)
x^2

scalar(x, zee=None)

Returns standard scalar product between self and x.

This is the default implementation that converts both self and x into Schur functions and performs the scalar product that basis.

The Hall-Littlewood $P$ basis is dual to the $Q_p$ basis with respect to this scalar product.

INPUT:
• self – an element of a Hall-Littlewood basis
• x – another symmetric element of the symmetric functions
OUTPUT:

• returns the scalar product between self and x

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P()
sage: HLQ = Sym.hall_littlewood().Q()
sage: HLQp = Sym.hall_littlewood().Qp()
sage: HLP([2]).scalar(HLQp([2]))
1
sage: HLP([2]).scalar(HLQp([1,1]))
0
sage: HLP([2]).scalar(HLQ([2]), lambda mu: mu.centralizer_size(t = HLP.t))
1
sage: HLP([2]).scalar(HLQ([1,1]), lambda mu: mu.centralizer_size(t = HLP.t))
0
```

```python
scalar_hl(x, t=None)
```

Returns the Hall-Littlewood (with parameter t) scalar product of self and x.

The Hall-Littlewood scalar product is defined in Macdonald's book [Mac1995]. The power sum basis is orthogonal and $\langle p_\mu, p_\nu \rangle = z_\mu \prod_i 1/(1-t^{\mu_i})$

The Hall-Littlewood $P$ basis is dual to the $Q$ basis with respect to this scalar product.

INPUT:

• self – an element of a Hall-Littlewood basis
• x – another symmetric element of the symmetric functions
• t – an optional parameter, if this parameter is not specified then the value of the t from the basis is used in the calculation

OUTPUT:

• returns the Hall-Littlewood scalar product between self and x

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P()
sage: HLQ = Sym.hall_littlewood().Q()
sage: HLP([2]).scalar_hl(HLQ([2]))
1
sage: HLP([2]).scalar_hl(HLQ([1,1]))
0
sage: HLQ([2]).scalar_hl(HLQ([2]), lambda mu: mu.centralizer_size(t = HLP.t))
1
sage: HLQ([2]).scalar_hl(HLQ([1,1]), lambda mu: mu.centralizer_size(t = HLP.t))
0
```

```python
hall_littlewood_family()
```

The family of Hall-Littlewood bases associated to self

INPUT:

• self – a Hall-Littlewood symmetric function basis

OUTPUT:
returns the class of Hall-Littlewood bases

EXAMPLES:

```python
sage: HLP = SymmetricFunctions(FractionField(QQ['t'])).hall_littlewood().P()
sage: HLP.hall_littlewood_family()
Hall-Littlewood polynomials with t=1 over Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

`product(left, right)`

Multiply an element of the Hall-Littlewood symmetric function basis `self` and another symmetric function. Convert to the Schur basis, do the multiplication there, and convert back to `self` basis.

INPUT:

• `self` – a Hall-Littlewood symmetric function basis
• `left` – an element of the basis `self`
• `right` – another symmetric function

OUTPUT:

the product of `left` and `right` expanded in the basis `self`

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P()
sage: HLP([2])^2 # indirect doctest
(t+1)*HLP[2, 2] + (-t+1)*HLP[3, 1] + HLP[4]
```

`transition_matrix(basis, n)`

Returns the transitions matrix between `self` and `basis` for the homogeneous component of degree `n`.

INPUT:

• `self` – a Hall-Littlewood symmetric function basis
• `basis` – another symmetric function basis
• `n` – a non-negative integer representing the degree

OUTPUT:

• Returns a \( r \times r \) matrix of elements of the base ring of `self` where \( r \) is the number of partitions of \( n \).
  The entry corresponding to row \( \mu \), column \( \nu \) is the coefficient of `basis`(\( \nu \)) in `self`(\( \mu \))

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HLP = Sym.hall_littlewood().P()
```
\begin{verbatim}
sage: s = Sym.schur()
sage: HLP.transition_matrix(s, 4)
\begin{bmatrix}
  1 & -t & 0 & t^2 & -t^3 \\
  0 & 1 & -t & -t & t^3 + t^2 \\
  0 & 0 & 1 & -t & t^3 \\
  0 & 0 & 0 & 1 & -t^3 - t^2 - t \\
  0 & 0 & 0 & 0 & 1
\end{bmatrix}
sage: HLQ = Sym.hall_littlewood().Q()
sage: HLQ.transition_matrix(s, 3)
\begin{bmatrix}
  -t + 1 & t^2 - t & \cdot \\
  0 & t^2 - 2*t + 1 & -t^4 + \cdot \\
  -t^3 + t^2 - t & 0 & -t^6 + t^5 + t^4 \cdot \\
  -t^2 - t + 1 & \cdot & \cdot
\end{bmatrix}
sage: HLQp = Sym.hall_littlewood().Qp()
sage: HLQp.transition_matrix(s, 3)
\begin{bmatrix}
  1 & 0 & 0 \\
  t & 1 & 0 \\
  t^3 & t^2 + t & 1
\end{bmatrix}
\end{verbatim}

```python
class sage.combinat.sf.hall_littlewood.HallLittlewood_p(hall_littlewood):
    Bases: HallLittlewood_generic

    A class representing the Hall-Littlewood \( P \) basis of symmetric functions

class Element
    Bases: Element

class sage.combinat.sf.hall_littlewood.HallLittlewood_q(hall_littlewood):
    Bases: HallLittlewood_generic

    The \( Q \) basis is defined as a normalization of the \( P \) basis.

    INPUT:

    * self – an instance of the Hall-Littlewood \( P \) basis
    * hall_littlewood – a class for the family of Hall-Littlewood bases

    EXAMPLES:

    ```python
    sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
    sage: Q = Sym.hall_littlewood().Q()
    sage: TestSuite(Q).run(skip=['_test_associativity', '_test_distributivity', '_test_prod']) # products are too expensive, long time (3s on sage.math, 2012)
    sage: TestSuite(Q).run(elements = [Q.t^Q[1,1]+Q[2], Q[1]+(1+Q.t)*Q[1,1]]) # long time (depends on previous)
    ```
    ```python
    sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
    sage: HLP = Sym.hall_littlewood().P()
    sage: HLQ = Sym.hall_littlewood().Q()
    sage: HLQp = Sym.hall_littlewood().Qp()
    sage: s = Sym.schur(); p = Sym.power()
    sage: HLP( HLP([2,1]) + HLP([3]) )
    ```
```
\[ \frac{1}{(t^2-2^t+1)} \text{HLQ}[2, 1] - \frac{1}{(t-1)^2} \text{HLQ}[3] \]

\[ \frac{1}{(t^3-t^2-t+1)} \text{HLQ}[1, 1] - \frac{1}{(t-1)^2} \text{HLQ}[2] \]

\[ \frac{1}{(t^2-1)} \text{HLQ}[1, 1] - \frac{1}{(t-1)^2} \text{HLQ}[2] \]

```
sage: HLQ(HLQP([2]))  # indirect doctest
\frac{t}{(t^3-t^2-t+1)} \text{HLQ}[1, 1] - \frac{1}{(t-1)^2} \text{HLQ}[2]
```

```
sage: HLQ(S([2]))
\frac{t}{(t^3-t^2-t+1)} \text{HLQ}[1, 1] - \frac{1}{(t-1)^2} \text{HLQ}[2]
```

```
sage: HLQ(p([2]))
\frac{1}{(t^2-1)} \text{HLQ}[1, 1] - \frac{1}{(t-1)^2} \text{HLQ}[2]
```

```
class Element
  Bases: Element

class sage.combinat.sf.hall_littlewood.HallLittlewood_qp(hall_littlewood)
  Bases: HallLittlewood_generic

  The Hall-Littlewood \( Q_p \) basis is calculated through the symmetrica library (see the function HallLittlewood_qp._to_s()).
  
  INPUT:
  
  - self – an instance of the Hall-Littlewood \( P \) basis
  
  - hall_littlewood – a class for the family of Hall-Littlewood bases

  EXAMPLES:
  
  ```python
  sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
  sage: Qp = Sym.hall_littlewood().Q()
  sage: TestSuite(Qp).run(skip=['_test_passociativity', '_test_distributivity', '_test_prod']) # products are too expensive, long time (3s on sage.math, 2012)
  sage: TestSuite(Qp).run(elements = [Qp.t*Qp[1,1]+Qp[2], Qp[1]+(1+Qp.t)*Qp[1,1]])  # long time (depends on previous)
  ```
  
  ```python
  sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
  sage: HLP = Sym.hall_littlewood().P()
  sage: HLQ = Sym.hall_littlewood().Q()
  sage: HLQP = Sym.hall_littlewood().QP()
  sage: s = Sym.schur(); p = Sym.power()
  sage: HLQP(HLP([2]))  # indirect doctest
  -t*\text{HLQP}[1, 1] + (t^2+1)*\text{HLQP}[2]
  sage: HLQP(s(HLQ([2])))  # work around bug reported in issue #12969
  (t^2-t)*\text{HLQP}[1, 1] + (-t^3+t^2-t+1)*\text{HLQP}[2]
  sage: HLQP(s([2]))
  \text{HLQP}[2]
  ```
  
  ```python
  sage: HLQP(p([2]))
  -HLQP[1, 1] + (t+1)*HLQP[2]
  ```
  
  ```python
  sage: s = HLQP.symmetric_function_ring().s()
  sage: HLQP.transition_matrix(s,3)
  [ 1 \ 0 \ 0]
  [ t \ 1 \ 0]
  [ t^3 t^2 + t \ 1]
  ```
  ```python
  sage: s.transition_matrix(HLP,3)
  [ 1 \ t \ t^3]
  [ 0 \ 1 \ t^2 + t]
  [ 0 \ 0 \ 1]
  ```
5.1.284 Hecke Character Basis

The basis of symmetric functions given by characters of the Hecke algebra (of type $A$).

AUTHORS:

• Travis Scrimshaw (2017-08): Initial version

class sage.combinat.sf.hecke.HeckeCharacter(sym, q=q')
Bases: SymmetricFunctionAlgebra_multipartitive

Basis of the symmetric functions that gives the characters of the Hecke algebra in analogy to the Frobenius formula for the symmetric group.

Consider the Hecke algebra $H_n(q)$ with quadratic relations

$$T_i^2 = (q - 1)T_i + q.$$ 

Let $\mu$ be a partition of $n$ with length $\ell$. The character $\chi$ of a $H_n(q)$-representation is completely determined by the elements $T_{\gamma_\mu}$, where

$$\gamma_\mu = (\mu_1, \ldots, 1)(\mu_2 + \mu_1, \ldots, 1 + \mu_1) \cdots (n, \ldots, 1 + \sum_{i<\ell} \mu_i),$$

(written in cycle notation). We define a basis of the symmetric functions by

$$q_{\mu} = \sum_{\lambda \vdash n} \chi^\lambda(T_{\gamma_\mu}) s_\lambda.$$ 

INPUT:

• sym – the ring of symmetric functions

EXAMPLES:

```sage
sage: q = ZZ['q'].fraction_field().gen()
sage: Sym = SymmetricFunctions(q.parent())
sage: qbar = Sym.hecke_character(q)
sage: s = Sym.s()
sage: s(qbar([2]))
-s[1, 1] + q^2*s[2]
sage: s(qbar([4]))
-s[1, 1, 1, 1] + q^3*s[2, 1, 1] - q^2*s[3, 1] + q^4*s[4]
sage: qbar(s[2])
(1/(q+1))*qbar[1, 1] + (1/(q+1))*qbar[2]
sage: qbar(s[1,1])
(q/(q+1))*qbar[1, 1] - (1/(q+1))*qbar[2]
sage: s(qbar[2,1])
```

(continues on next page)
-s[1, 1, 1] + (q-1)*s[2, 1] + q*s[3]

\text{\texttt{sage}}: \texttt{qbar(s[2, 1])}
(q/(q^2+q+1))*qbar[1, 1, 1] + ((q-1)/(q^2+q+1))*qbar[2, 1]
- (1/(q^2+q+1))*qbar[3]

We compute character tables for Hecke algebras, which correspond to the transition matrix from the \( q \) basis to the Schur basis:

\text{\texttt{sage}}: \texttt{qbar.transition_matrix(s, 1)}
[1]
\text{\texttt{sage}}: \texttt{qbar.transition_matrix(s, 2)}
[ q -1]
[ 1 1]
\text{\texttt{sage}}: \texttt{qbar.transition_matrix(s, 3)}
[ q^2 -q 1]
[ q q -1 -1]
[ 1 2 1]
\text{\texttt{sage}}: \texttt{qbar.transition_matrix(s, 4)}
[ q^3 -q^2 0 q -1]
[ q^2 q^2 -q -q +1 1]
[ q^2 q^2 -2*q q^2 +1 -2*q +1 1]
[ q 2*q -1 q -1 q -2 -1]
[ 1 3 2 3 1]

We can do computations with a specialized \( q \) to a generic element of the base ring. We compute some examples with \( q = 2 \):

\text{\texttt{sage}}: \texttt{qbar = Sym.qbar(q=2)}
\text{\texttt{sage}}: \texttt{s = Sym.schur()}
\text{\texttt{sage}}: \texttt{qbar(s[2,1])}
2/7*qbar[1, 1, 1] + 1/7*qbar[2, 1] - 1/7*qbar[3]
\text{\texttt{sage}}: \texttt{s(qbar[2,1])}
-s[1, 1, 1] + s[2, 1] + 2*s[3]

REFERENCES:
• [Ram1991]
• [RR1997]

\texttt{coproduct_on_generators}(r)

Return the coproduct on the generator \( \bar{q}_r \) of \texttt{self}.

Define the coproduct on \( \bar{q}_r \) by

\[ \Delta(\bar{q}_r) = \bar{q}_0 \otimes \bar{q}_r + (q-1) \sum_{j=1}^{r-1} \bar{q}_j \otimes \bar{q}_{r-j} + \bar{q}_r \otimes \bar{q}_0. \]

EXAMPLES:

\text{\texttt{sage}}: \texttt{q = ZZ['q'].fraction_field().gen()}
\texttt{sage}: \texttt{Sym = SymmetricFunctions(q.parent())}
\texttt{sage}: \texttt{qbar = Sym.hecke_character()}
\texttt{sage}: \texttt{s = Sym.s()}
```
sage: qbar[2].coproduct()
```

### 5.1.285 Homogeneous symmetric functions

By this we mean the basis formed of the complete homogeneous symmetric functions $h_\lambda$, not an arbitrary graded basis.

```python
class sage.combinat.sf.homogeneous.SymmetricFunctionAlgebra_homogeneous(Sym)
    Bases: SymmetricFunctionAlgebra_multiplicative

A class of methods specific to the homogeneous basis of symmetric functions.

INPUT:

- `self` – a homogeneous basis of symmetric functions
- `Sym` – an instance of the ring of symmetric functions

class Element
    Bases: Element

expand(n, alphabet='x')

Expand the symmetric function `self` as a symmetric polynomial in `n` variables.

INPUT:

- `n` – a nonnegative integer
- `alphabet` – (default: 'x') a variable for the expansion

OUTPUT:

A monomial expansion of `self` in the `n` variables labelled by `alphabet`.

EXAMPLES:

```
sage: h = SymmetricFunctions(QQ).h()
sage: h([3]).expand(2)
x0^3 + x0*x1^2 + x0^2*x1 + x1^3
sage: h([1,1,1]).expand(2)
x0^3 + 3*x0*x1^2 + 3*x0^2*x1 + x1^3
sage: h([2,1]).expand(3)
x0^3 + 2*x0*2*x1 + 2*x0*x1^2 + x1^3 + 2*x0^2*x2 + 3*x0*x1*x2 + 2*x1^2*x2 + 2*x0*x2^2 + 2*x1*x2^2 + x2^3
sage: h([3]).expand(2,alphabet='y')
y0^3 + y0*y1^2 + y0*y1 + y1^3
sage: h([3]).expand(2,alphabet='x,y')
x^3 + x^2*y + x*y^2 + y^3
sage: h([3]).expand(3,alphabet='x,y,z')
x^3 + x^2*y + x^2*z + x*y^2 + x*y*z + y^2*z + x*y*z + y^3 + z^3
sage: (h([]) + 2*h([1])).expand(3)
2*x0 + 2*x1 + 2*x2 + 1
sage: h([1]).expand(0)
0
sage: (3*h([])).expand(0)
3
```
```
**exponential_specialization**(\(\text{t} = \text{None}, \text{q} = 1\))

Return the exponential specialization of a symmetric function (when \(\text{q} = 1\)), or the \(\text{q}\)-exponential specialization (when \(\text{q} \neq 1\)).

The exponential specialization \(ex\) at \(t\) is a \(K\)-algebra homomorphism from the \(K\)-algebra of symmetric functions to another \(K\)-algebra \(R\). It is defined whenever the base ring \(K\) is a \(Q\)-algebra and \(t\) is an element of \(R\). The easiest way to define it is by specifying its values on the powersum symmetric functions to be \(p_1 = t\) and \(p_n = 0\) for \(n > 1\). Equivalently, on the homogeneous functions it is given by \(ex(h_n) = t^n / n!\); see Proposition 7.8.4 of [EnumComb2].

By analogy, the \(\text{q}\)-exponential specialization is a \(K\)-algebra homomorphism from the \(K\)-algebra of symmetric functions to another \(K\)-algebra \(R\) that depends on two elements \(t\) and \(q\) of \(R\) for which the elements \(1 - q^i\) for all positive integers \(i\) are invertible. It can be defined by specifying its values on the complete homogeneous symmetric functions to be

\[
ex_q(h_n) = t^n / [n]_q!,
\]

where \([n]_q!\) is the \(q\)-factorial. Equivalently, for \(q \neq 1\) and a homogeneous symmetric function \(f\) of degree \(n\), we have

\[
ex_q(f) = (1 - q)^n t^n p_{\text{st}}(f),
\]

where \(p_{\text{st}}(f)\) is the stable principal specialization of \(f\) (see `principal_specialization()`). (See (7.29) in [EnumComb2].)

The limit of \(ex_q\) as \(q \to 1\) is \(ex\).

**INPUT:**

- \(\text{t}\) (default: \(\text{None}\)) – the value to use for \(t\); the default is to create a ring of polynomials in \(t\).
- \(\text{q}\) (default: \(1\)) – the value to use for \(q\). If \(q\) is \(\text{None}\), then a ring (or fraction field) of polynomials in \(q\) is created.

**EXAMPLES:**

```python
sage: h = SymmetricFunctions(QQ).h()
sage: x = h[5,3]
sage: x.exponential_specialization()
1/720^*t^8
sage: factorial(5)^*factorial(3)
720
sage: x = 5*h[1,1,1] + 3*h[2,1] + 1
sage: x.exponential_specialization()
13/2^*t^3 + 1
```

We also support the \(\text{q}\)-exponential_specialization:

```python
sage: factor(h[3].exponential_specialization(q=var("q"), t=var("t")))
t^3/((q^2 + q + 1)^*q + 1)
```

**omega()**

Return the image of self under the omega automorphism.

The omega automorphism is defined to be the unique algebra endomorphism \(\omega\) of the ring of symmetric functions that satisfies \(\omega(e_k) = h_k\) for all positive integers \(k\) (where \(e_k\) stands for the \(k\)-th elementary symmetric function, and \(h_k\) stands for the \(k\)-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the
omega involution. It sends the power-sum symmetric function $p_k$ to $(-1)^k p_k$ for every positive integer $k$.

The images of some bases under the omega automorphism are given by

\[
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
\]

where $\lambda$ is any partition, where $\ell(\lambda)$ denotes the length (\texttt{length}()) of the partition $\lambda$, where $\lambda'$ denotes the conjugate partition (\texttt{conjugate}()) of $\lambda$, and where the usual notations for bases are used ($e =$ elementary, $h =$ complete homogeneous, $p = $ powersum, $s = $ Schur).

\texttt{omega_involution()} is a synonym for the \texttt{omega()} method.

**OUTPUT:**

- the image of \texttt{self} under the omega automorphism

**EXAMPLES:**

```python
sage: h = SymmetricFunctions(QQ).h()
sage: a = h([2,1]); a
h[2, 1]
sage: a.omega()
h[1, 1, 1] - h[2, 1]
sage: e = SymmetricFunctions(QQ).e()
sage: e(h([2,1]).omega())
e[2, 1]
```

\texttt{omega_involution()} is a synonym for the \texttt{omega()} method.

**OUTPUT:**

- the image of \texttt{self} under the omega automorphism

**EXAMPLES:**

```python
sage: h = SymmetricFunctions(QQ).h()
sage: a = h([2,1]); a
h[2, 1]
sage: a.omega()
h[1, 1, 1] - h[2, 1]
sage: e = SymmetricFunctions(QQ).e()
sage: e(h([2,1]).omega())
e[2, 1]
```
Return the principal specialization of a symmetric function.

The principal specialization of order \( n \) at \( q \) is the ring homomorphism \( ps_{n,q} \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for \( i \in \{1, \ldots, n\} \) and \( x_i \mapsto 0 \) for \( i > n \). Here, \( q \) is a given element of \( R \), and we assume that the variables of our symmetric functions are \( x_1, x_2, x_3, \ldots \). (To be more precise, \( ps_{n,q} \) is a \( K \)-algebra homomorphism, where \( K \) is the base ring.) See Section 7.8 of [EnumComb2].

The stable principal specialization at \( q \) is the ring homomorphism \( ps_q \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for all \( i \). This is well-defined only if the resulting infinite sums converge; thus, in particular, setting \( q = 1 \) in the stable principal specialization is an invalid operation.

**INPUT:**
- \( n \) (default: infinity) – a nonnegative integer or infinity, specifying whether to compute the principal specialization of order \( n \) or the stable principal specialization.
- \( q \) (default: None) – the value to use for \( q \); the default is to create a ring of polynomials in \( q \) (or a field of rational functions in \( q \)) over the given coefficient ring.

We use the formulas from Proposition 7.8.3 of [EnumComb2] (using Gaussian binomial coefficients \( \left( \binom{n}{i} \right)_q \)):

\[
ps_{n,q}(h_\lambda) = \prod_i \left( \frac{n + \lambda_i - 1}{\lambda_i} \right)_q,
\]
\[
ps_{n,1}(h_\lambda) = \prod_i \left( \frac{n + \lambda_i - 1}{\lambda_i} \right),
\]
\[
ps_q(h_\lambda) = 1 / \prod_i \lambda_i \prod_j (1 - q^j).
\]

**EXAMPLES:**

```
sage: h = SymmetricFunctions(QQ).h()
sage: x = h[2,1]
sage: x.principal_specialization(3)
q^6 + 2*q^5 + 4*q^4 + 4*q^3 + 4*q^2 + 2*q + 1
sage: x = 3*h[2] + 2*h[1] + 1
sage: x.principal_specialization(3, q=var("q"))
2*(q^3 - 1)/(q - 1) + 3*(q^4 - 1)/(q^2 - 1) + 1
```

Return the coproduct on \( h_i \).

**INPUT:**
- self – a homogeneous basis of symmetric functions
- i – a nonnegative integer

**OUTPUT:**
- the sum \( \sum_{r=0}^{i} h_r \otimes h_{i-r} \)

**EXAMPLES:**

```
sage: Sym = SymmetricFunctions(QQ)
sage: h = Sym.homogeneous()
```
5.1.286 Jack Symmetric Functions

Jack’s symmetric functions appear in [Ma1995] Chapter VI, section 10. Zonal polynomials are the subject of [Ma1995] Chapter VII. The parameter $\alpha$ in that reference is the parameter $t$ in this implementation in Sage.

REFERENCES:

```python
sage: h.coproduct_on_generators(2)
sage: h.coproduct_on_generators(0)
# h[]
```

class `sage.combinat.sf.jack.Jack`(Sym, $t$='t')

Bases: `UniqueRepresentation`

The family of Jack symmetric functions including the $P$, $Q$, $J$, $Q_p$ bases. The default parameter is $t$.

INPUT:
- `self` – the family of Jack symmetric function bases
- `Sym` – a ring of symmetric functions
- `t` – an optional parameter (default: ‘$t$’)

EXAMPLES:

```python
sage: SymmetricFunctions(FractionField(QQ['t'])).jack()
Jack polynomials over Fraction Field of Univariate Polynomial Ring in t over Rational Field
sage: SymmetricFunctions(QQ).jack(1)
Jack polynomials with $t=1$ over Rational Field
```

$J()$

Returns the algebra of Jack polynomials in the $J$ basis.

INPUT:
- `self` – the family of Jack symmetric function bases

OUTPUT: the $J$ basis of the Jack symmetric functions

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: J = Sym.jack().J(); J
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field
sage: SymmetricFunctions(QQ).jack(t=-1)
Symmetric Functions over Rational Field in the Jack $J$ with $t=-1$ basis
```

At $t = 1$, the Jack polynomials in the $J$ basis are scalar multiples of the Schur functions with the scalar given by a Partition’s `hook_product()` method at 1:
At \( t = 2 \), the Jack polynomials in the \( J \) basis are scalar multiples of the zonal polynomials with the scalar given by a Partition’s \texttt{hook_product()} method at 2.

\begin{verbatim}
sage: Sym = SymmetricFunctions(QQ)
sage: JJ = Sym.jack(t=2).J()
sage: Z = Sym.zonal()
sage: p = Partition([2,2,1])
sage: Z(JJ(p)) == p.hook_product(2)*Z(p)
True
\end{verbatim}

\[ \text{P()} \]

Returns the algebra of Jack polynomials in the \( P \) basis.

INPUT:
- self – the family of Jack symmetric function bases

OUTPUT:
- the \( P \) basis of the Jack symmetric functions

EXAMPLES:

\begin{verbatim}
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: JP = Sym.jack().P(); JP
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack P basis
sage: Sym.jack(t=-1).P()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack P with t=-1 basis
\end{verbatim}

At \( t = 1 \), the Jack polynomials in the \( P \) basis are the Schur symmetric functions.
At $t = 2$, the Jack polynomials in the $P$ basis are the zonal polynomials.
Combinatorics, Release 10.0

sage: JP(JJ([3]))
(2*t^2+3*t+1)*JackP[3]
sage: JP(JJ([2,1]))
(t+2)*JackP[2, 1]
sage: JP(JJ([1,1,1]))
6*JackP[1, 1, 1]

sage: s = Sym.schur()
sage: JP(s([2,1]))
((2*t-2)/(t+2))*JackP[1, 1, 1] + JackP[2, 1]
sage: s(_)
s[2, 1]

Q()

Returns the algebra of Jack polynomials in the $Q$ basis.

INPUT:

• self – the family of Jack symmetric function bases

OUTPUT:

• the $Q$ basis of the Jack symmetric functions

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: JQ = Sym.jack().Q(); JQ
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack Q basis
sage: Sym = SymmetricFunctions(QQ)
sage: Sym.jack(t=-1).Q()
Symmetric Functions over Rational Field in the Jack Q with t=-1 basis

sage: JQ(sum(JP(p) for p in Partitions(3)))
(1/6*t^3+1/2*t^2+1/3*t)*JackQ[1, 1, 1] + ((2*t^3+t^2)/(t+2))*JackQ[2, 1] + (3*t^3/(t^2+3/2*t+1/2))*JackQ[3]

sage: s = Sym.schur()
sage: JQ(s([3])) # indirect doctest
(1/6*t^3-1/2*t^2+1/3*t)*JackQ[1, 1, 1] + ((2*t^3-2*t^2)/(t+2))*JackQ[2, 1] + (3*t^3/(t^2+3/2*t+1/2))*JackQ[3]

Qp()

Returns the algebra of Jack polynomials in the $Qp$, which is dual to the $P$ basis with respect to the standard scalar product.

INPUT:
• \texttt{self} -- the family of Jack symmetric function bases

OUTPUT:

• the $Q'$ basis of the Jack symmetric functions

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: JP = Sym.jack().P()
sage: JQp = Sym.jack().Qp(); JQp
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack Qp basis
sage: a = JQp([2])
sage: a.scalar(JP([2]))
1
sage: a.scalar(JP([1,1]))
0
sage: JP(JQp([2]))
((t-1)/(t+1))*JackP[1, 1] + JackP[2]
sage: JP._normalize(JP(JQp([2])))
((t-1)/(t+1))*JackP[1, 1] + JackP[2]
```

\texttt{base\_ring}()

Returns the base ring of the symmetric functions in which the Jack symmetric functions live

INPUT:

• \texttt{self} -- the family of Jack symmetric function bases

OUTPUT:

• the base ring of the symmetric functions ring of \texttt{self}

EXAMPLES:

```python
sage: J2 = SymmetricFunctions(QQ).jack(t=2)
sage: J2.base_ring()
Rational Field
```

\texttt{symmetric\_function\_ring}()

Returns the base ring of the symmetric functions of the Jack symmetric function bases

INPUT:

• \texttt{self} -- the family of Jack symmetric function bases

OUTPUT:

• the symmetric functions ring of \texttt{self}

EXAMPLES:

```python
sage: Jacks = SymmetricFunctions(FractionField(QQ['t'])).jack()
sage: Jacks.symmetric_function_ring()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field
```
class sage.combinat.sf.jack.JackPolynomials_generic(jack)

Bases: SymmetricFunctionAlgebra_generic

A class of methods which are common to all Jack bases of the symmetric functions

INPUT:

• self – a Jack basis of the symmetric functions
• jack – a family of Jack symmetric function bases

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: JP = Sym.jack().P(); JP.base_ring()
Fraction Field of Univariate Polynomial Ring in t over Rational Field
sage: Sym = SymmetricFunctions(QQ)
sage: JP = Sym.jack(t=2).P(); JP.base_ring()
Rational Field
```

class Element

Bases: SymmetricFunctionAlgebra_generic_Element

scalar_jack(x, t=None)

A scalar product where the power sums are orthogonal and $\langle p_\mu, p_\mu \rangle = z_\mu t^{\text{length}(\mu)}$

INPUT:

• self – an element of a Jack basis of the symmetric functions
• x – an element of the symmetric functions
• t – an optional parameter (default: uses the parameter from the basis)

OUTPUT:

• returns the Jack scalar product between x and self

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: JP = Sym.jack().P()
sage: JQ = Sym.jack().Q()
sage: p = Partitions(3).list()
sage: matrix([[JP(a).scalar_jack(JQ(b)) for a in p] for b in p])
[1 0 0]
[0 1 0]
[0 0 1]
```

c1(part)

Returns the $t$-Jack scalar product between $J(\text{part})$ and $P(\text{part})$.

INPUT:

• self – a Jack basis of the symmetric functions
• part – a partition
• t – an optional parameter (default: uses the parameter $t$ from the Jack basis)

OUTPUT:

• a polynomial in the parameter $t$ which is equal to the scalar product of $J(\text{part})$ and $P(\text{part})$

EXAMPLES:
```python
sage: JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: JP.c1(Partition([2,1]))
t + 2
```

### `c2(part)`

Returns the $t$-Jack scalar product between $J(part)$ and $Q(part)$.

**INPUT:**
- `self` – a Jack basis of the symmetric functions
- `part` – a partition
- `t` – an optional parameter (default: uses the parameter $t$ from the Jack basis)

**OUTPUT:**
- a polynomial in the parameter $t$ which is equal to the scalar product of $J(part)$ and $Q(part)$

**EXAMPLES:**

```python
sage: JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: JP.c2(Partition([2,1]))
2*t^3 + t^2
```

```python
sage: JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: JP.c2(Partition([2,1]))
2*t^3 + t^2
```

### `coproduct_by_coercion(elt)`

Returns the coproduct of the element `elt` by coercion to the Schur basis.

**INPUT:**
- `self` – a Jack symmetric function basis
- `elt` – an instance of this basis

**OUTPUT:**
- The coproduct acting on `elt`, the result is an element of the tensor squared of the Jack symmetric function basis

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: Sym.jack().P()[:,2].coproduct() #indirect doctest
```

### `jack_family()`

Returns the family of Jack bases associated to the basis `self`

**INPUT:**
- `self` – a Jack basis of the symmetric functions

**OUTPUT:**
- the family of Jack symmetric functions associated to `self`

**EXAMPLES:**

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sage: JackP = SymmetricFunctions(QQ).jack(t=2).P()
sage: JackP.jack_family()
Jack polynomials with t=2 over Rational Field

product(left, right)

The product of two Jack symmetric functions is done by multiplying the elements in the P basis and then expressing the elements in the basis self.

INPUT:
• self – a Jack basis of the symmetric functions
• left, right – symmetric function elements

OUTPUT:
the product of left and right expanded in the basis self

EXAMPLES:

```python
sage: JJ = SymmetricFunctions(FractionField(QQ['t'])).jack().J()
sage: JJ[[1]]^2  # indirect doctest
(t/(t+1))*JackJ[1, 1] + (1/(t+1))*JackJ[2]
sage: JJ[[2]]^2
(t^2/(t^2+3/2*t+1/2))*JackJ[2, 2] + (4/3*t/(t^2+4/3*t+1/3))*JackJ[3, 1] + ((1/6*t+1/6)/(t^2+5/6*t+1/6))*JackJ[4]
sage: JQ = SymmetricFunctions(FractionField(QQ['t'])).jack().Q()
sage: JQ[[1]]^2  # indirect doctest
JackQ[1, 1] + (2/(t+1))*JackQ[2]
sage: JQ[[2]]^2
JackQ[2, 2] + (2/(t+1))*JackQ[3, 1] + ((t+1)/(t^2+5/6*t+1/6))*JackQ[4]
```

class sage.combinat.sf.jack.JackPolynomials_j(jack)

Bases: JackPolynomials_generic

The J basis is defined as a normalized form of the P basis

INPUT:
• self – an instance of the Jack P basis of the symmetric functions
• jack – a family of Jack symmetric function bases

EXAMPLES:

```python
sage: J = SymmetricFunctions(FractionField(QQ['t'])).jack().J()
sage: TestSuite(J).run(skip=['_test_associativity', '_test_distributivity', '_test_prod'])  # products are too expensive
```

class Element

Bases: Element

class sage.combinat.sf.jack.JackPolynomials_p(jack)

Bases: JackPolynomials_generic

The P basis is uni-triangularly related to the monomial basis and orthogonal with respect to the Jack scalar product.
INPUT:

- `self` – an instance of the Jack $P$ basis of the symmetric functions
- `jack` – a family of Jack symmetric function bases

EXAMPLES:

```python
sage: P = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: TestSuite(P).run(skip=['_test_associativity', '_test_distributivity', '_test_→prod']) # products are too expensive
sage: TestSuite(P).run(elements = [P.t*P[1,1]+P[2], P[1]+(1+t)*P[1,1]])
```

class Element

Bases: `Element`

`scalar_jack(x, t=None)`

The scalar product on the symmetric functions where the power sums are orthogonal and $\langle p_{\mu}, p_{\mu} \rangle = z_{\mu} t^{\text{length}(\mu)}$ where the $t$ parameter from the Jack symmetric function family.

INPUT:

- `self` – an element of the Jack $P$ basis
- `x` – an element of the $P$ basis

EXAMPLES:

```python
sage: JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: l = [JP(p) for p in Partitions(3)]
sage: matrix([[a.scalar_jack(b) for a in l] for b in l])
```

```
\begin{bmatrix}
3t^3/(t^2 + 3/2t + 1/2) & 0 & \cdots \\
0 & (2t^3 + t^2)/(t + 2) & \cdots \\
0 & 0 & 1/6t^3 + 1/2t^2 + 1/3t \\
\end{bmatrix}
```

`product(left, right)`

The product of two Jack symmetric functions is done by multiplying the elements in the monomial basis and then expressing the elements the basis `self`.

INPUT:

- `self` – a Jack basis of the symmetric functions
- `left, right` – symmetric function elements

OUTPUT:

the product of `left` and `right` expanded in the basis `self`

EXAMPLES:

```python
sage: JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: m = JP.symmetric_function_ring().m()
sage: JP([1])^2 # indirect doctest
(2t/(t+1))*JackP[1, 1] + JackP[2]
sage: m(_)
2*m[1, 1] + m[2]
sage: JP = SymmetricFunctions(QQ).jack(t=2).P()
sage: JP([2,1])^2
```

(continues on next page)

`sage: m(_)`
45*m[1, 1, 1, 1, 1] + 51/2*m[2, 1, 1, 1, 1] + 29/2*m[2, 2, 1, 1] + 33/4*m[2, 2, 2] + 9*m[3, 1, 1, 1] + 5*m[3, 2, 1] + 2*m[3, 3] + 2*m[4, 1, 1] + m[4, 2]

**scalar_jack_basis**(part1, part2=None)

Returns the scalar product of \(P(part1)\) and \(P(part2)\).

This is equation (10.16) of [Mc1995] on page 380.

**INPUT:**

- `self` – an instance of the Jack \(P\) basis of the symmetric functions
- `part1` – a partition
- `part2` – an optional partition (default : None)

**OUTPUT:**

- the scalar product between \(P(part1)\) and \(P(part2)\) (or itself if \(part2\) is None)

**REFERENCES:**

**EXAMPLES:**

```sage:
JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: JP.scalar_jack_basis(Partition([2,1]), Partition([1,1,1]))
0
sage: JP._normalize_coefficients(JP.scalar_jack_basis(Partition([3,2,1]), Partition([3,2,1])))
(6*t^6 + 10*t^5 + 11/2*t^4 + t^3)/(t^3 + 11/2*t^2 + 10*t + 6)
sage: JJ(JP[3,2,1]).scalar_jack(JP[3,2,1])
(6*t^6 + 10*t^5 + 11/2*t^4 + t^3)/(t^3 + 11/2*t^2 + 10*t + 6)
```

With a single argument, takes \(part2 = part1\):

```sage:
JP.scalar_jack_basis(Partition([2,1]), Partition([2,1]))
(2*t^3 + t^2)/(t + 2)
sage: JJ(JP[2,1]).scalar_jack(JP[2,1])
(2*t^3 + t^2)/(t + 2)
```

**class** `sage.combinat.sf.jack.JackPolynomials_q(jack)`

**Bases:** `JackPolynomials_generic`

The \(Q\) basis is defined as a normalized form of the \(P\) basis

**INPUT:**

- `self` – an instance of the Jack \(Q\) basis of the symmetric functions
- `jack` – a family of Jack symmetric function bases

**EXAMPLES:**
class Element
Bases: Element
class sage.combinat.sf.jack.JackPolynomials_qp(jack)
Bases: JackPolynomials_generic
The \(Q_p\) basis is the dual basis to the \(P\) basis with respect to the standard scalar product

INPUT:
- \(self\) – an instance of the Jack \(Q_p\) basis of the symmetric functions
- \(jack\) – a family of Jack symmetric function bases

EXAMPLES:

sage: Qp = SymmetricFunctions(FractionField(QQ['t'])).jack().Qp()
sage: TestSuite(Qp).run(elements = [Qp.t*Qp[1,1]+Qp[2], Qp[1]+(1+Qp.t)*Qp[1,1]]) # long time (3s on sage.math, 2012)

class Element
Bases: Element
coproduct_by_coercion(elt)
Returns the coproduct of the element \(elt\) by coercion to the Schur basis.

INPUT:
- \(elt\) – an instance of the \(Q_p\) basis

OUTPUT:
- The coproduct acting on \(elt\), the result is an element of the tensor squared of the \(Q_p\) symmetric function basis

EXAMPLES:

sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: JQp = Sym.jack().Qp()
sage: JQp[2,2].coproduct() # indirect doctest
→JackQp[1, 1] + ((2*t^3+4*t^2)/(t^3+5/2*t^2+2*t+1/2))*JackQp[2] # JackQp[2] +

product(left, right)
The product of two Jack symmetric functions is done by multiplying the elements in the monomial basis and then expressing the elements the basis \(self\).

INPUT:
- \(self\) – an instance of the Jack \(Q_p\) basis of the symmetric functions

5.1. Comprehensive Module List
• left, right – symmetric function elements

OUTPUT:
the product of left and right expanded in the basis self

EXAMPLES:

```
sage: JQp = SymmetricFunctions(FractionField(QQ['t'])).jack().Qp()
sage: h = JQp.symmetric_function_ring().h()
sage: JQp([1])^2  # indirect doctest
JackQp[1, 1] + (2/(t+1))*JackQp[2]
sage: h(_)
JackQp[1, 1] - 6/5*JackQp[3, 1]
```

class sage.combinat.sf.jack.SymmetricFunctionAlgebra_zonal(Sym)

Bases: SymmetricFunctionAlgebra_generic

Returns the algebra of zonal polynomials.

INPUT:

• self – a zonal basis of the symmetric functions
• Sym – a ring of the symmetric functions

EXAMPLES:

```
sage: Z = SymmetricFunctions(QQ).zonal()
sage: Z([2])^2
sage: Z = SymmetricFunctions(QQ).zonal()
sage: TestSuite(Z).run(skip=['_test_associativity', '_test_distributivity', '_test_prod'])  # products are too expensive
sage: TestSuite(Z).run(elements = [Z[1,1]*Z[2], Z[1]*2*Z[1,1]])
```

class Element

Bases: SymmetricFunctionAlgebra_generic.Element

scalar_zonal(x)

The zonal scalar product has the power sum basis and the zonal symmetric functions are orthogonal.
In particular, \( \langle p_\mu, p_\nu \rangle = z_\mu 2^{\text{length}(|\mu|)} \).

INPUT:

• self – an element of the zonal basis
• x – an element of the symmetric function

OUTPUT:

• the scalar product between self and x

EXAMPLES:
```python
sage: Sym = SymmetricFunctions(QQ)
sage: Z = Sym.zonal()
sage: parts = Partitions(3).list()
sage: matrix([[Z(a).scalar_zonal(Z(b)) for a in parts] for b in parts])
[[16/5  0   0]
 [  0  5   0]
 [  0   0   4]]
sage: p = Z.symmetric_function_ring().power()
sage: matrix([[Z(p(a)).scalar_zonal(p(b)) for a in parts] for b in parts])
[[  6   0   0]
 [  0   8   0]
 [  0   0  48]]
```

**product** *(left, right)*

The product of two zonal symmetric functions is done by multiplying the elements in the monomial basis and then expressing the elements in the basis *self*.

**INPUT:**

- *self* – a zonal basis of the symmetric functions
- *left, right* – symmetric function elements

**OUTPUT:**

the product of *left* and *right* expanded in the basis *self*

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: Z = Sym.zonal()
sage: JP = Sym.jack(t=1).P()
sage: Z([2])*Z([3])
sage: Z([2])*JP([2])
```

```python
sage.combinat.sf.jack.c1(part, t)
```

Returns the $t$-Jack scalar product between $\text{J}(\text{part})$ and $\text{P}(\text{part})$.

**INPUT:**

- *part* – a partition
- *t* – an optional parameter (default: uses the parameter $t$ from the Jack basis)

**OUTPUT:**

- a polynomial in the parameter $t$ which is equal to the scalar product of $\text{J}(\text{part})$ and $\text{P}(\text{part})$

**EXAMPLES:**

```python
sage: from sage.combinat.sf.jack import c1
sage: t = QQ['t'].gen()
sage: [c1(p,t) for p in Partitions(3)]
[2^t*t^2 + 3*t + 1, t + 2, 6]
```
sage.combinat.sf.jack.c2(part, t)

Returns the t-Jack scalar product between J(part) and Q(part).

INPUT:

- self – a Jack basis of the symmetric functions
- part – a partition
- t – an optional parameter (default: uses the parameter t from the Jack basis)

OUTPUT:

- a polynomial in the parameter t which is equal to the scalar product of J(part) and Q(part)

EXAMPLES:

```python
sage: from sage.combinat.sf.jack import c2
sage: t = QQ['t'].gen()
sage: [c2(p,t) for p in Partitions(3)]
[6*t^3, 2*t^3 + t^2, t^3 + 3*t^2 + 2*t]
```

sage.combinat.sf.jack.normalize_coefficients(self, c)

If our coefficient ring is the field of fractions over a univariate polynomial ring over the rationals, then we should clear both the numerator and denominator of the denominators of their coefficients.

INPUT:

- self – a Jack basis of the symmetric functions
- c – a coefficient in the base ring of self

OUTPUT:

- divide numerator and denominator by the greatest common divisor

EXAMPLES:

```python
sage: JP = SymmetricFunctions(FractionField(QQ['t'])).jack().P()
sage: t = JP.base_ring().gen()
sage: a = 2/(1/2*t+1/2)
sage: JP._normalize_coefficients(a)
4/(t + 1)
sage: a = 1/(1/3+1/6^t)
sage: JP._normalize_coefficients(a)
6/(t + 2)
sage: a = 24/(4*t^2 + 12*t + 8)
sage: JP._normalize_coefficients(a)
6/(t^2 + 3*t + 2)
```

sage.combinat.sf.jack.part_scalar_jack(part1, part2, t)

Returns the Jack scalar product between p(part1) and p(part2) where p is the power-sum basis.

INPUT:

- part1, part2 – two partitions
- t – a parameter

OUTPUT:

- returns the scalar product between the power sum indexed by part1 and part2
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EXAMPLES:

```
sage: Q.<t> = QQ[]
sage: from sage.combinat.sf.jack import part_scalar_jack
sage: matrix([[part_scalar_jack(p1,p2,t) for p1 in Partitions(4)] for p2 in Partitions(4)])
\[
\begin{bmatrix}
4*t & 0 & 0 & 0 & 0 \\
0 & 3*t^2 & 0 & 0 & 0 \\
0 & 0 & 8*t^2 & 0 & 0 \\
0 & 0 & 0 & 4*t^3 & 0 \\
0 & 0 & 0 & 0 & 24*t^4
\end{bmatrix}
```

5.1.287 Quotient of symmetric function space by ideal generated by Hall-Littlewood symmetric functions

The quotient of symmetric functions by the ideal generated by the Hall-Littlewood P symmetric functions indexed by partitions with first part greater than \( k \). When \( t = 1 \) this space is the quotient of the symmetric functions by the ideal generated by the monomial symmetric functions indexed by partitions with first part greater than \( k \).

AUTHORS:

- Chris Berg (2012-12-01)
- Mike Zabrocki - \( k \)-bounded Hall Littlewood P and dual \( k \)-Schur functions (2012-12-02)

class sage.combinat.sf.k_dual.AffineSchurFunctions(kBoundedRing)

Bases: KBoundedQuotientBasis

This basis is dual to the \( k \)-Schur functions at \( t = 1 \). This realization follows the monomial expansion given by Lam [Lam2006].

REFERENCES:

class sage.combinat.sf.k_dual.DualkSchurFunctions(kBoundedRing)

Bases: KBoundedQuotientBasis

This basis is dual to the \( k \)-Schur functions. The expansion is given in Section 4.12 of [LLMSSZ]. When \( t = 1 \) this basis is equal to the AffineSchurFunctions and that basis is more efficient in this case.

REFERENCES:

class sage.combinat.sf.k_dual.KBoundedQuotient(Sym, k, t='t')

Bases: UniqueRepresentation, Parent

Initialization of the ring of Symmetric functions modulo the ideal of monomial symmetric functions which are indexed by partitions whose first part is greater than \( k \).

INPUT:

- \( \text{Sym} \) – an element of class sage.combinat.sf.sf.SymmetricFunctions
- \( k \) – a positive integer
- \( R \) – a ring

EXAMPLES:
sage: Sym = SymmetricFunctions(QQ)
sage: Q = Sym.kBoundedQuotient(3,t=1)
sage: Q
3-Bounded Quotient of Symmetric Functions over Rational Field with t=1
sage: km = Q.km()
sage: km
3-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the 3-
bounded monomial basis
sage: F = Q.affineSchur()
sage: F(km(F[3,1,1])) == F[3,1,1]
True
sage: km(F(km([3,2]))) == km[3,2]
True
sage: F[3,2].lift()
m[1, 1, 1, 1, 1] + m[2, 1, 1, 1, 1] + m[2, 2, 1] + m[3, 1, 1, 1] + m[3, 2]
sage: F[F[2,1]*F[2,1]]
2*F3[1, 1, 1, 1, 1, 1] + 4*F3[2, 1, 1, 1, 1] + 4*F3[2, 2, 1, 1] + 4*F3[2, 2, 2] + ...
sage: km(F[1,2])
Traceback (most recent call last):
...  ValueError: [1, 2] is not an element of 3-Bounded Partitions
sage: F[4,2]
Traceback (most recent call last):
...  ValueError: [4, 2] is not an element of 3-Bounded Partitions
sage: km[2,1]*km[2,1]
4*m3[2, 1, 1, 1] + 6*m3[2, 2, 1] + 2*m3[3, 1, 1] + 2*m3[3, 2]
sage: HLPk = Q.kHallLittlewoodP()
sage: HLPk[HLPk[2,1]*HLPk[2,1]]
sage: dks = Q.dual_k_Schur()
sage: dks[2,1]*dks[2,1]
2*dks3[1, 1, 1, 1, 1, 1] + 4*dks3[2, 1, 1, 1, 1, 1] + 4*dks3[2, 2, 1, 1, 1] + 4*dks3[2, 2, 2, 1] + 4*dks3[2, 2, 2] + 2*dks3[3, 1, 1, 1, 1] + 4*dks3[3, 2, 1, 1] + 2*dks3[3, 3, 1] + 2*dks3[3, 3, 2]

sage: Q = Sym.kBoundedQuotient(3)
Traceback (most recent call last):
...  TypeError: unable to convert 't' to a rational
sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: Q = Sym.kBoundedQuotient(3)
sage: km = Q.km()
sage: F = Q.affineSchur()
sage: F(km(F[3,1,1])) == F[3,1,1]
True
sage: km(F(km([3,2]))) == km[3,2]
True
sage: dks = Q.dual_k_Schur()
sage: HLPk = Q.kHallLittlewoodP()
sage: dks(HLPk(dks[3,1,1])) == dks[3,1,1]
True

(continues on next page)
sage: km(dks(km([3,2]))) == km[3,2]
True
sage: dks[2,1]*dks[2,1]
(t^3+t^2)*dks3[1, 1, 1, 1, 1, 1] + (t^2+2*t+1)*dks3[2, 2, 2] + (t+1)*dks3[3, 1, 1, 1] + (2*t+2)*dks3[3, 2, 1] + (t+1)*dks3[3, 3]

AffineGrothendieckPolynomial(\(\lambda a\), \(m\))

Returns the affine Grothendieck polynomial indexed by the partition \(\lambda a\). Because this belongs to the completion of the algebra, and hence is an infinite sum, it computes only up to those symmetric functions of degree at most \(m\). See \_AffineGrothendieckPolynomial() for the code.

INPUT:

- \(\lambda a\) – A \(k\)-bounded partition
- \(m\) – An integer

EXAMPLES:

sage: Q = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: Q.AffineGrothendieckPolynomial([2,1],4)
2*m3[1, 1, 1] - 8*m3[1, 1, 1, 1] + m3[2, 1] - 3*m3[2, 1, 1] - m3[2, 2]

F() 

The affine Schur basis of the \(k\)-bounded quotient of symmetric functions, indexed by \(k\)-bounded partitions. This is also equal to the affine Stanley symmetric functions (see WeylGroups.ElementMethods.stanley_symmetric_function()) indexed by an affine Grassmannian permutation.

EXAMPLES:

sage: Sym = SymmetricFunctions(QQ)
sage: Q = Sym.kBoundedQuotient(3,t=1)
sage: Q.F() 2-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the 2-bounded affine Schur basis

a_realization() 

Returns a particular realization of self (the basis of \(k\)-bounded monomials if \(t\) = 1 and the basis of \(k\)-bounded Hall-Littlewood functions otherwise).

EXAMPLES:

sage: Sym = SymmetricFunctions(QQ)
sage: Q = Sym.kBoundedQuotient(3,t=1)
sage: Q.a_realization() 3-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the 3-bounded monomial basis
sage: Q = Sym.kBoundedQuotient(3,t=2)
sage: Q.a_realization() 3-Bounded Quotient of Symmetric Functions over Rational Field with t=2 in the 3-bounded Hall-Littlewood P basis

affineSchur() 

The affine Schur basis of the \(k\)-bounded quotient of symmetric functions, indexed by \(k\)-bounded partitions. This is also equal to the affine Stanley symmetric functions (see WeylGroups.ElementMethods.stanley_symmetric_function()) indexed by an affine Grassmannian permutation.

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EXAMPLES:

```
sage: SymmetricFunctions(QQ).kBoundedQuotient(2,t=1).affineSchur()
2-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the 2-
bounded affine Schur basis
```

ambient()

Returns the Symmetric Functions over the same ring as self. This is needed to realize our ring as a quotient.

an_element()

Returns an element of the quotient ring of $k$-bounded symmetric functions. This method is here to make the TestSuite run properly.

EXAMPLES:

```
sage: Q = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: Q.an_element()
2*m3[] + 2*m3[1] + 3*m3[2]
```

dks()

The dual $k$-Schur basis of the $k$-bounded quotient of symmetric functions, indexed by $k$-bounded partitions. At $t = 1$ this is also equal to the affine Schur basis and calculations will be faster using elements in the `affineSchur()` basis.

EXAMPLES:

```
sage: SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(2).dual_k_Schur()
2-Bounded Quotient of Symmetric Functions over Fraction Field of UnivariatePolynomial Ring in t over Rational Field in the dual 2-Schur basis
```

dual_k_Schur()

The dual $k$-Schur basis of the $k$-bounded quotient of symmetric functions, indexed by $k$-bounded partitions. At $t = 1$ this is also equal to the affine Schur basis and calculations will be faster using elements in the `affineSchur()` basis.

EXAMPLES:

```
sage: SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(2).dual_k_Schur()
2-Bounded Quotient of Symmetric Functions over Fraction Field of UnivariatePolynomial Ring in t over Rational Field in the dual 2-Schur basis
```

kHLP()

The Hall-Littlewood P basis of the $k$-bounded quotient of symmetric functions, indexed by $k$-bounded partitions. At $t = 1$ this basis is equal to the $k$-bounded monomial basis and calculations will be faster using elements in the $k$-bounded monomial basis (see `kmonomial()`).

EXAMPLES:

```
sage: SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(2).kHallLittlewoodP()
2-Bounded Quotient of Symmetric Functions over Fraction Field of UnivariatePolynomial Ring in t over Rational Field in the 2-bounded Hall-Littlewood P basis
```
### kHallLittlewoodP()

The Hall-Littlewood P basis of the $k$-bounded quotient of symmetric functions, indexed by $k$-bounded partitions. At $t = 1$ this basis is equal to the $k$-bounded monomial basis and calculations will be faster using elements in the $k$-bounded monomial basis (see `kmonomial()`).

**EXAMPLES:**

```python
sage: SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(2).kHallLittlewoodP()
2-Bounded Quotient of Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the 2-bounded Hall-Littlewood P basis
```

### km()

The monomial basis of the $k$-bounded quotient of symmetric functions, indexed by $k$-bounded partitions.

**EXAMPLES:**

```python
sage: SymmetricFunctions(QQ).kBoundedQuotient(2,t=1).kmonomial()
2-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the 2-bounded monomial basis
```

### kmonomial()

The monomial basis of the $k$-bounded quotient of symmetric functions, indexed by $k$-bounded partitions.

**EXAMPLES:**

```python
sage: SymmetricFunctions(QQ).kBoundedQuotient(2,t=1).kmonomial()
2-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the 2-bounded monomial basis
```

### lift(la)

Gives the lift map from the quotient ring of $k$-bounded symmetric functions to the symmetric functions. This method is here to make the TestSuite run properly.

**INPUT:**
- `la` – A $k$-bounded partition

**OUTPUT:**
- The monomial element or a Hall-Littlewood P element of the symmetric functions indexed by the partition `la`.

**EXAMPLES:**

```python
sage: Q = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: Q.lift([2,1])
m[2, 1]
sage: Q = SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(3)
sage: Q.lift([2,1])
HLP[2, 1]
```

### one()

Returns the unit of the quotient ring of $k$-bounded symmetric functions. This method is here to make the TestSuite run properly.

**EXAMPLES:**
```python
sage: Q = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: Q.one()
m3[]
```

**realizations()**

A list of realizations of the $k$-bounded quotient.

**EXAMPLES:**

```python
sage: kQ = SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(3)
sage: kQ.realizations()
[3-Bounded Quotient of Symmetric Functions over Fraction Field of Univariate
 Polynomial Ring in t over Rational Field in the 3-bounded monomial basis, 3-
 Bounded Quotient of Symmetric Functions over Fraction Field of Univariate
 Polynomial Ring in t over Rational Field in the 3-bounded Hall-Littlewood P
 basis, 3-Bounded Quotient of Symmetric Functions over Fraction Field of
 Univariate Polynomial Ring in t over Rational Field in the 3-bounded affine
 Schur basis, 3-Bounded Quotient of Symmetric Functions over Fraction Field of
 Univariate Polynomial Ring in t over Rational Field in the dual 3-Schur basis]
sage: HLP = kQ.ambient().hall_littlewood().P()
sage: all( rzn(HLP[3,2,1]).lift() == HLP[3,2,1] for rzn in kQ.realizations())
True
sage: kQ = SymmetricFunctions(QQ).kBoundedQuotient(3,1)
sage: kQ.realizations()
[3-Bounded Quotient of Symmetric Functions over Rational Field with t=1 in the
 3-bounded monomial basis, 3-Bounded Quotient of Symmetric Functions over
 Rational Field with t=1 in the 3-bounded Hall-Littlewood P basis, 3-Bounded,
 Quotient of Symmetric Functions over Rational Field with t=1 in the 3-bounded
 affine Schur basis, 3-Bounded Quotient of Symmetric Functions over Rational,
 Field with t=1 in the dual 3-Schur basis]
sage: m = kQ.ambient().m()
sage: all( rzn(m[3,2,1]).lift() == m[3,2,1] for rzn in kQ.realizations())
True
```

**retract(λa)**

Gives the retract map from the symmetric functions to the quotient ring of $k$-bounded symmetric functions. This method is here to make the TestSuite run properly.

**INPUT:**

- `λa` – A partition

**OUTPUT:**

- The monomial element of the $k$-bounded quotient indexed by $λa$.

**EXAMPLES:**

```python
sage: Q = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: Q.retract([2,1])
m3[2, 1]
```

```python
class sage.combinat.sf.k_dual.KBoundedQuotientBases(base)
Bases: Category_realization_of_parent

The category of bases for the $k$-bounded subspace of symmetric functions.
```
class ElementMethods
Bases: object

class ParentMethods
Bases: object

ambient()
Returns the symmetric functions.

EXAMPLES:
sage: km = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).km()
sage: km.ambient()
Symmetric Functions over Rational Field

antipode(element)
Return the antipode of element via lifting to the symmetric functions and then retracting into the $k$-bounded quotient basis.

INPUT:
• element – an element in a basis of the ring of symmetric functions

EXAMPLES:
sage: dks3 = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).dual_k_Schur()
sage: dks3[3,2].antipode()
-dks3[1, 1, 1, 1]
sage: km = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).km()
sage: km[3,2].antipode()
m3[3, 2]
sage: km.antipode(km[3,2])
m3[3, 2]
sage: m = SymmetricFunctions(QQ).m()
sage: m[3,2].antipode()
m[3, 2] + 2*m[5]
sage: km = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3).
˓→km()
sage: km[1,1,1,1].antipode()
(t^3-3*t^2+3*t)*m3[1, 1, 1] + (-t^2+2*t)*m3[2, 1, 1] + t*m3[2, 2] +
˓→t*m3[3, 1]
sage: kHP = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3).
˓→kHP()
sage: kHP[2,2].antipode()
(t^9-t^6-t^5+t^2)*HLP3[1, 1, 1] + (t^6-t^3-t^2+t)*HLP3[2, 1, 1] + (t^5-t^ ˓→2+1)*HLP3[2, 2] + (t^4-t)*HLP3[3, 1]
sage: dks = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3).
˓→dks()
sage: dks[2,2].antipode()
dks3[2, 2]
sage: dks[3,2].antipode()
-t^2*dks3[1, 1, 1, 1] + (t^2-1)*dks3[2, 2, 1] + (-t^5+t)*dks3[3, 2]

coproduct(element)
Return the coproduct of element via lifting to the symmetric functions and then returning to the $k$-bounded quotient basis. This method is implemented for all $t$ but is (weakly) conjectured to not be
the correct operation for arbitrary \( t \) because the coproduct on dual-\( k \)-Schur functions does not have a positive expansion.

**INPUT:**

- `element` – an element in a basis of the ring of symmetric functions

**EXAMPLES:**

```
sage: Q3 = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: km = Q3.km()
sage: km[3,2].coproduct()
```

```
```

```
sage: dks3 = Q3.dual_k_Schur()
sage: dks3[2,2].coproduct()
```

```
dks3[] # dks3[2, 2] + dks3[1] # dks3[2, 1] + dks3[1, 1] # dks3[1, 1] +
```

```
```

```
sage: Q3t = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3)
sage: km = Q3t.km()
sage: km[3,2].coproduct()
```

```
```

```
sage: dks = Q3t.dks()
sage: dks[2,1,1].coproduct()
```

```
dks3[] # dks3[2, 1, 1] + (-t+1)*dks3[1] # dks3[2, 1, 1] + dks3[1, 1] # dks3[2, 1, 1] +
```

```
...dks3[2] # dks3[2, 1, 1] + dks3[2, 1, 1] # dks3[2, 1, 1] +
```

```
```

```
sage: kHLP = Q3t.kHLP()
sage: kHLP[2,1].coproduct()
```

```
```

```
...HLP3[1, 1] # HLP3[2, 1] + HLP3[2] # HLP3[2, 1] + (-t^2+1)*HLP3[1, 1] # HLP3[1, 1] +
```

```
...HLP3[1, 1] # HLP3[1, 1] + HLP3[2, 1] # HLP3[2, 1] # HLP3[]
```

```
sage: km.coproduct(km[3,2])
```

```
```

counit(element)

Return the counit of `element`.

The counit is the constant term of `element`.

**INPUT:**

- `element` – an element in a basis

**EXAMPLES:**

```
sage: km = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3).
```

```
...km()
```

```
sage: f = 2*km[2,1] - 3*km([])
```

```
sage: f.counit()
```

```
-3
```

```
sage: km.counit(f)
```

```
-3
```

degree_on_basis(b)

Return the degree of the basis element indexed by `b`.

**INPUT:**

- `b` – a partition

**EXAMPLES:**

```
sage: km = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3).
```

```
...km()
```

```
sage: f = 2*km[2,1] - 3*km([])
```

```
sage: f.counit()
```

```
-3
```

```
sage: km.counit(f)
```

```
-3
```
Combinatorics, Release 10.0

```python
sage: F = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).affineSchur()
sage: F.degree_on_basis(Partition([3,2]))
5
```

**indices()**

The set of \( k \)-bounded partitions of all non-negative integers.

**EXAMPLES:**

```python
sage: km = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).km()
sage: km.indices()
3-Bounded Partitions
```

**lift(la)**

Implements the lift map from the basis \( self \) to the monomial basis of symmetric functions.

**INPUT:**
- \( la \) – A \( k \)-bounded partition.

**OUTPUT:**
- A symmetric function in the monomial basis.

**EXAMPLES:**

```python
sage: F = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).affineSchur()
sage: F.lift([3,1])
m[1, 1, 1, 1] + m[2, 1, 1] + m[2, 2] + m[3, 1]
sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: dks = Sym.kBoundedQuotient(3).dual_k_Schur()
sage: dks.lift([3,1])
t^5*HLP[1, 1, 1, 1] + t^2*HLP[2, 1, 1] + t*HLP[2, 2] + HLP[3, 1]
sage: dks = Sym.kBoundedQuotient(3,t=1).dual_k_Schur()
sage: dks.lift([3,1])
m[1, 1, 1, 1] + m[2, 1, 1] + m[2, 2] + m[3, 1]
```

**one_basis()**

Return the basis element indexing 1.

**EXAMPLES:**

```python
sage: F = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).affineSchur()
sage: F.one()  # indirect doctest
F3[]
```

**product(x, y)**

Returns the product of two elements \( x \) and \( y \).

**INPUT:**
- \( x, y \) – Elements of the \( k \)-bounded quotient of symmetric functions.

**OUTPUT:**
- A \( k \)-bounded symmetric function in the dual \( k \)-Schur function basis

**EXAMPLES:**

```python
sage: dks3 = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).dual_k_Schur()
sage: dks3.product(dks3[2,1],dks3[1,1])
2^5*dks3[1, 1, 1, 1] + 2^5*dks3[2, 1, 1, 1] + 2^5*dks3[2, 2, 1] + dks3[3, 1, 1]
```
Combinatorics, Release 10.0

(continued from previous page)

\[
\begin{align*}
\text{sage: } & \text{dks3.product(dks3[2,1]+dks3[1], dks3[1,1])} \\
& \text{dks3[1, 1, 1] + t^2*dks3[2, 1, 1, 1, 1] + (t+1)*dks3[2, 1, 1, 1] + (t+1)*dks3[2, 2, 1] + dks3[3, 2]} \\
\text{sage: } & \text{dks3.product(dks3[2,1]+dks3[1], dks3([]))} \\
& \text{dks3[1] + dks3[2, 1]} \\
\text{sage: } & \text{dks3.product(dks3([], dks3([4,1])))} \\
& \text{Traceback (most recent call last):} \\
& \ldots \\
& \text{TypeError: do not know how to make x (= [4, 1]) an element of self (=3-} \\
\text{bounded Quotient of Symmetric Functions over Rational Field with t=1 in} \\
\text{the dual 3-Schur basis)}
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{F = SymmetricFunctions(QQ['t']).fraction_field()}. \text{kBoundedQuotient(3,t=1).affineSchur()} \\
\text{sage: } & \text{F.product(F[2,1],F[1,1])} \\
\text{sage: } & \text{F.product(F[2,1]+F[1], F[1,1])} \\
\text{sage: } & \text{F.product(F[2,1]+F[1], F([]))} \\
\text{sage: } & \text{F.product(F([], F([4,1])))} \\
& \text{Traceback (most recent call last):} \\
& \ldots \\
& \text{TypeError: do not know how to make x (= [4, 1]) an element of self (=3-} \\
\text{bounded Quotient of Symmetric Functions over Rational Field with t=1 in} \\
\text{the 3-bounded affine Schur basis)}
\end{align*}
\]

(continues on next page)
sage: F.product(F[2,1],F[2])
(t^4+t^3-2*t^2+1)*F3[1, 1, 1, 1] + (-t^2+t+1)*F3[2, 1, 1, 1] + (-t^2+t+2)*F3[2, 2, 1] + (t+1)*F3[3, 1, 1] + (t+1)*F3[3, 2]

sage: F.product(F[2,1]+F[1], F[1,1])

sage: F.product(F[2,1]+F[1], F([]))
F3[1] + F3[2, 1]

sage: F.product(F([]), F([]))
F3[[]]

sage: km = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).km()

sage: km.product(km[2,1],km[2,1])
4*m3[2, 2, 1, 1] + 6*m3[2, 2, 2] + 2*m3[3, 2, 1] + 2*m3[3, 3]

sage: Q3 = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3)

sage: km = Q3.km()

sage: km.product(km[2,1],km[2,1])
(t^5+7*t^4-8*t^3-28*t^2+47*t-19)*m3[1, 1, 1, 1, 1] + (t^4-3*t^3-9*t^2+23*t-12)*m3[2, 1, 1, 1, 1] + (-t^3-3*t^2+11*t-3)*m3[2, 2, 1, 1] + (-t^2+5*t+2)*m3[2, 2, 2] + (6*t-6)*m3[3, 1, 1, 1] + (3*t-1)*m3[3, 2, 1] + (t+1)*m3[3, 3]

sage: dks = Q3.dual_k_Schur()

sage: km.product(dks[2,1],dks[1,1])
20*m3[1, 1, 1, 1, 1] + 9*m3[2, 1, 1, 1] + 4*m3[2, 2, 1] + 2*m3[3, 1, 1] + m3[3, 2]

\textbf{retract}(la)

Gives the retract map from the symmetric functions to the quotient ring of $k$-bounded symmetric functions. This method is here to make the TestSuite run properly.

INPUT:
\begin{itemize}
\item la – A partition
\end{itemize}

OUTPUT:
\begin{itemize}
\item The monomial element of the $k$-bounded quotient indexed by la.
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: Q = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1)
sage: Q.retract([2,1])
m3[2, 1]
\end{verbatim}

\textbf{super_categories}()

The super categories of self.

EXAMPLES:

\begin{verbatim}
sage: Sym = SymmetricFunctions(QQ['t'])
sage: from sage.combinat.sf.k_dual import KBoundedQuotientBases
sage: Q = Sym.kBoundedQuotient(3,t=1)
sage: KQB = KBoundedQuotientBases(Q)
sage: KQB.super_categories()
[Category of realizations of 3-Bounded Quotient of Symmetric Functions over Univariate Polynomial Ring in t over Rational Field with t=1, Join of Category of graded hopf algebras with basis over Univariate Polynomial ring in t over Rational Field with t=1, Category of graded modules over Univariate Polynomial Ring in t over Rational Field with t=1]
\end{verbatim}
class sage.combinat.sf.k_dual.KBoundedQuotientBasis(kBoundedRing, prefix)
    Bases: CombinatorialFreeModule
    Abstract base class for the bases of the $k$-bounded quotient.

class sage.combinat.sf.k_dual.kMonomial(kBoundedRing)
    Bases: KBoundedQuotientBasis
    The basis of monomial symmetric functions indexed by partitions with first part less than or equal to $k$.

lift(la)
    Implements the lift function on the monomial basis. Given a $k$-bounded partition $la$, the lift will return the corresponding monomial basis element.

INPUT:
    • $la$ – A $k$-bounded partition

OUTPUT:
    • A monomial symmetric function.

EXAMPLES:

sage: km = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).km()
sage: km.lift(Partition([3,1]))
m[3, 1]
sage: km.lift([])
m[]
sage: km.lift(Partition([4,1]))
Traceback (most recent call last):
  ...TypeError: do not know how to make x (= [4, 1]) an element of self (=3-Bounded˓
  →Quotient of Symmetric Functions over Rational Field with t=1 in the 3-bounded˓
  →monomial basis)

retract(la)
    Implements the retract function on the monomial basis. Given a partition $la$, the retract will return the corresponding $k$-bounded monomial basis element if $la$ is $k$-bounded; zero otherwise.

INPUT:
    • $la$ – A partition

OUTPUT:
    • A $k$-bounded monomial symmetric function in the $k$-quotient of symmetric functions.

EXAMPLES:
```python
sage: km = SymmetricFunctions(QQ).kBoundedQuotient(3,t=1).km()
sage: km.retract(Partition([3,1]))
m3[3, 1]
sage: km.retract(Partition([4,1]))
0
sage: km.retract([])
m3[

sage: km = SymmetricFunctions(FractionField(QQ['t'])).kBoundedQuotient(3).km()
sage: km.retract(Partition([3,1]))
m3[3, 1]
sage: km.retract(Partition([4,1]))
(t^4+t^3-9*t^2+11*t-4)*m3[1, 1, 1, 1, 1] + (-3*t^2+6*t-3)*m3[2, 1, 1, 1] + (-t^2+3*t-2)*m3[2, 2, 1] + (2*t-2)*m3[3, 1, 1] + (t-1)*m3[3, 2]
sage: km.retract([])
m3[

class sage.combinat.sf.k_dual.kbounded_HallLittlewoodP(kBoundedRing)
Bases: KBoundedQuotientBasis

The basis of P Hall-Littlewood symmetric functions indexed by partitions with first part less than or equal to \(k\).

**lift(la)**

Implements the lift function on the Hall-Littlewood P basis. Given a \(k\)-bounded partition \(la\), the lift will return the corresponding Hall-Littlewood P basis element.

**INPUT:**
- \(la\) – A \(k\)-bounded partition

**OUTPUT:**
- A Hall-Littlewood symmetric function.

**EXAMPLES:**

```python
sage: kHLP = SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(3).kHallLittlewoodP()
sage: kHLP.lift(Partition([3,1]))
HLP[3, 1]
sage: kHLP.lift([])
HLP[]
sage: kHLP.lift(Partition([4,1]))
Traceback (most recent call last):
  ...
TypeError: do not know how to make x (= [4, 1]) an element of self (=3-Bounded_...)
```
retract(\(la\))

Implements the retract function on the Hall-Littlewood P basis. Given a partition \(la\), the retract will return the corresponding \(k\)-bounded Hall-Littlewood P basis element if \(la\) is \(k\)-bounded; zero otherwise.

**INPUT:**

- \(la\) – A partition

**OUTPUT:**

- A \(k\)-bounded Hall-Littlewood P symmetric function in the \(k\)-quotient of symmetric functions.

**EXAMPLES:**

```python
sage: kHLP = SymmetricFunctions(QQ['t'].fraction_field()).kBoundedQuotient(3).kBoundedLittlewoodP()
sage: kHLP.retract(Partition([3,1]))
HLP3[3, 1]
sage: kHLP.retract(Partition([4,1]))
0
sage: kHLP.retract([])
HLP3[]
sage: m = kHLP.realization_of().ambient().m()
sage: kHLP(m[[2,2]])
(t^4-t^3-t+1)*HLP3[1, 1, 1, 1] + (t-1)*HLP3[2, 1, 1] + HLP3[2, 2]
```

### 5.1.288 Kostka-Foulkes Polynomials

Based on the algorithms in John Stembridge’s SF package for Maple which can be found at [http://www.math.lsa.umich.edu/~jrs/maple.html](http://www.math.lsa.umich.edu/~jrs/maple.html).

`sage.combinat.sf.kfpoly.KostkaFoulkesPolynomial` \((\mu, \nu, t=None)\)

Returns the Kostka-Foulkes polynomial \(K_{\mu, \nu}(t)\).

**INPUT:**

- \(\mu, \nu\) – partitions
- \(t\) – an optional parameter (default: None)

**OUTPUT:**

- the Kostka-Foulkes polynomial indexed by partitions \(\mu\) and \(\nu\) and evaluated at the parameter \(t\). If \(t\) is `None` the resulting polynomial is in the polynomial ring \(Z[t']\).

**EXAMPLES:**

```python
sage: KostkaFoulkesPolynomial([2,2],[2,2])
1
sage: KostkaFoulkesPolynomial([2,2],[4])
0
sage: KostkaFoulkesPolynomial([2,2],[1,1,1,1])
```

(continues on next page)
\[ t^4 + t^2 \]

```python
sage: KostkaFoulkesPolynomial([2,2],[2,1,1])
t
sage: q = PolynomialRing(QQ,'q').gen()
sage: KostkaFoulkesPolynomial([2,2],[2,1,1],q)
```

```python
sage.combinat.sf.kfpoly.compat(n, mu, nu)

Generate all possible partitions of \( n \) that can precede \( \mu, \nu \) in a rigging sequence.

**INPUT:**
- \( n \) – a positive integer
- \( \mu, \nu \) – partitions

**OUTPUT:**
- a list of partitions

**EXAMPLES:**

```python
sage: from sage.combinat.sf.kfpoly import *
sage: compat(4, [1], [2,1])
[[1, 1, 1, 1], [2, 1, 1], [2, 2], [3, 1], [4]]
sage: compat(3, [1], [2,1])
[[1, 1, 1], [2, 1], [3]]
sage: compat(2, [1], [])
[[2]]
sage: compat(3, [1], [])
[[2, 1], [3]]
sage: compat(2, [1], [4])
[[4]]
sage: compat(3, [2], [1])
[[3]]
sage: compat(4, [1,1], [])
[[2, 2], [3, 1], [4]]
sage: compat(4, [2], [])
[[4]]
```

```python
sage.combinat.sf.kfpoly.dom(mup, snu)

Return True if \( \sum(\mu[:i+1]) \geq \text{snu}[i] \) for all \( 0 \leq i < \text{len(snu)} \); otherwise, it returns False.

**INPUT:**
- \( \text{mup} \) – a partition conjugate to \( \mu \)
- \( \text{snu} \) – a sequence of positive integers

**OUTPUT:**
- a boolean value

**EXAMPLES:**

```python
sage: from sage.combinat.sf.kfpoly import *
sage: dom([3,2,1],[2,4,5])
True
sage: dom([3,2,1],[2,4,7])
False
```
Combinatorics, Release 10.0

sage: dom([3,2,1],[2,6,5])
False
sage: dom([3,2,1],[4,4,4])
False

sage.combinat.sf.kfpoly.kfpoly(mu, nu, t=None)
Return the Kostka-Foulkes polynomial $K_{\mu,\nu}(t)$ by generating all rigging sequences for the shape $\mu$, and then selecting those of content $\nu$.

INPUT:

• $\mu$, $\nu$ – partitions
• $t$ – an optional parameter (default: None)

OUTPUT:

• the Koskta-Foulkes polynomial indexed by partitions $\mu$ and $\nu$ and evaluated at the parameter $t$. If $t$ is None the resulting polynomial is in the polynomial ring $\mathbb{Z}[t']$.

EXAMPLES:

```python
sage: from sage.combinat.sf.kfpoly import kfpoly
sage: kfpoly([2,2], [2,1,1])
t
sage: kfpoly([4], [2,1,1])
t^3
sage: kfpoly([4], [2,2])
t^2
sage: kfpoly([1,1,1,1], [2,2])
0
```

sage.combinat.sf.kfpoly.riggings(part)
Generate all possible rigging sequences for a fixed partition $part$.

INPUT:

• $part$ – a partition

OUTPUT:

• a list of riggings associated to the partition $part$

EXAMPLES:

```python
sage: from sage.combinat.sf.kfpoly import *
sage: riggings([3])
[[[1, 1, 1], [[2, 1]], [[3]]]
sage: riggings([2,1])
[[[2, 1], [1]], [[3], [1]]]
sage: riggings([1,1,1])
[[[3], [2], [1]]]
sage: riggings([2,2])
[[[2, 2], [1, 1]], [[3, 1], [1, 1]], [[4], [1, 1]], [[4], [2]]]
sage: riggings([2,2,2])
[[[3, 3], [2, 2], [1, 1]], [[4, 2], [2, 2], [1, 1]]]
```

(continues on next page)
sage.combinat.sf.kfpoly.schur_to_hl(%(mu)s, t=None)

Return a dictionary corresponding to $s_{\mu}$ in Hall-Littlewood $P$ basis.

INPUT:

- $\mu$ – a partition
- $t$ – an optional parameter (default: the generator from $\mathbb{Z}[t']$)

OUTPUT:

- a dictionary with the coefficients $K_{\mu\nu}(t)$ for $\nu$ smaller in dominance order than $\mu$

EXAMPLES:

```python
sage: from sage.combinat.sf.kfpoly import *

sage: a = schur_to_hl([1,1,1])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1], t^2 + t)

sage: a = schur_to_hl([2,1])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1], t^3)
([2, 1], t)

sage: a = schur_to_hl([3])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1], t^3)
([2, 1], t)
([3], 1)

sage: a = schur_to_hl([4])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1, 1], t^6)
([2, 1, 1], t^3)
([2, 2], t^2)
([3, 1], t)
([4], 1)

sage: a = schur_to_hl([3,1])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1, 1], t^5 + t^4 + t^3)
([2, 1, 1], t^3)
([2, 2], t)
([3, 1], 1)

sage: a = schur_to_hl([2,2])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1, 1], t^4 + t^2)
([2, 1, 1], t)
([2, 2], 1)

sage: a = schur_to_hl([2,1,1])
sage: for mc in sorted(a.items()): print(mc)
([1, 1, 1, 1], t^3 + t^2 + t)
([2, 1, 1], 1)
```

(continues on next page)
sage: a = schur_to_hl([1,1,1,1])  
sage: for mc in sorted(a.items()): print(mc)  
([1, 1, 1, 1], 1)

sage: a = schur_to_hl([2,2,2])  
sage: for mc in sorted(a.items()): print(mc)  
([1, 1, 1, 1, 1, 1], t^9 + t^7 + t^6 + t^5 + t^3)  
([2, 1, 1, 1, 1, 1], t^4 + t^2)  
([2, 2, 1, 1, 1], t)  
([2, 2, 2, 1, 1], 1)

sage.combinat.sf.kfpoly.weight(rg, t=None)

Return the weight of a rigging.

INPUT:

• rg – a rigging, a list of partitions
• t – an optional parameter, (default: the generator from \(\mathbb{Z}[t']\))

OUTPUT:

• a polynomial in the parameter t

EXAMPLES:

sage: from sage.combinat.sf.kfpoly import weight  
sage: weight([[2,1], [1]])  
1
sage: weight([[3], [1]])  
t^2 + t
sage: weight([[2,1], [3]])  
t^4
sage: weight([[2, 2], [1, 1]])  
1
sage: weight([[3, 1], [1, 1]])  
t
sage: weight([[4], [1, 1], [2]])  
16
sage: weight([[4], [2]], t=2)  
4

5.1.289 LLT symmetric functions

REFERENCES:

class sage.combinat.sf.llt.LLT_class(Sym, k, t='t')

Bases: UniqueRepresentation

A class for working with LLT symmetric functions.

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['t']))  
sage: L3 = Sym.llt(3); L3  
level 3 LLT polynomials over Fraction Field of Univariate Polynomial Ring in t over \(\mathbb{Q}[t']\)
We require that the parameter $t$ must be in the base ring:

```python
sage: Symxt = SymmetricFunctions(QQ['x','t'].fraction_field())
sage: (x,t) = Symxt.base_ring().gens()
sage: LLT3x = Symxt.llt(3,t=x)
sage: LLT3 = Symxt.llt(3)
sage: HS3x = LLT3x.hspin()
sage: HS3t = LLT3.hspin()
sage: s = Symxt.schur()
sage: s(HS3x[2,1])
s[2, 1] + x*s[3]
sage: s(HS3t[2,1])
s[2, 1] + t*s[3]
sage: HS3x(HS3t[2,1])
HSp3[2, 1] + (-x+t)*HSp3[3]
sage: s(HS3x(HS3t[2,1]))
s[2, 1] + t*s[3]
sage: LLT3t2 = Symxt.llt(3,t=2)
sage: HC3t2 = LLT3t2.hcospin()
sage: HS3x(HC3t2[3,1])
2*HSp3[3, 1] + (-2*x+1)*HSp3[4]
```

**base_ring()**

Returns the base ring of `self`.

**INPUT:**

- `self` – a family of LLT symmetric functions bases

**OUTPUT:**

- returns the base ring of the symmetric function ring associated to `self`

**EXAMPLES:**

```python
sage: SymmetricFunctions(FractionField(QQ['t'])).llt(3).base_ring()
Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

**cospin(skp)**

Calculate a single instance of the cospin symmetric functions.

These are the functions defined in [LLT1997] equation (26).

**INPUT:**

- `self` – a family of LLT symmetric functions bases
• skp – a partition or a list of partitions or a list of skew partitions

OUTPUT:
the monomial expansion of the LLT symmetric function cospin functions indexed by skp

EXAMPLES:

```sage
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: L3 = Sym.llt(3)
sage: L3.cospin([2,1])
m[1]
sage: L3.cospin([3,2,1])
(t+1)*m[1, 1] + m[2]
sage: s = Sym.schur()
sage: s(L3.cospin([[2],[1],[2]]))
t^4*s[2, 2, 1] + t^3*s[3, 1, 1] + (t^3+t^2)*s[3, 2] + (t^2+t)*s[4, 1] + s[5]
```

```
hcospin()

Returns the HCospin basis. This basis is defined [LLT1997] equation (27).

INPUT:
• self – a family of LLT symmetric functions bases

OUTPUT:
• returns the h-cospin basis of the LLT symmetric functions

EXAMPLES:

```sage
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HCosp3 = Sym.llt(3).hcospin(); HCosp3
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over \text{Rational Field in the level 3 LLT cospin basis}
sage: HCosp3([1])^2
1/t*HCosp3[1, 1] + ((t-1)/t)*HCosp3[2]
sage: s = Sym.schur()
sage: HCosp3(s([2]))
HCosp3[2]
sage: HCosp3(s([1,1]))
1/t*HCosp3[1, 1] - 1/t*HCosp3[2]
sage: s(HCosp3([2,1]))
t*s[2, 1] + s[3]
```

```
hspin()

Returns the HSpin basis. This basis is defined [LLT1997] equation (28).

INPUT:
• self – a family of LLT symmetric functions bases

OUTPUT:
• returns the h-spin basis of the LLT symmetric functions

EXAMPLES:

```sage
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HCosp3 = Sym.llt(3).hcospin(); HCosp3
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over \text{Rational Field in the level 3 LLT cospin basis}
sage: HCosp3([1])^2
1/t*HCosp3[1, 1] + ((t-1)/t)*HCosp3[2]
sage: s = Sym.schur()
sage: HCosp3(s([2]))
HCosp3[2]
sage: HCosp3(s([1,1]))
1/t*HCosp3[1, 1] - 1/t*HCosp3[2]
sage: s(HCosp3([2,1]))
t*s[2, 1] + s[3]
```
```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: HSp3 = Sym.llt(3).hspin(); HSp3
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the level 3 LLT spin basis

sage: HSp3([1])^2
HSp3[1, 1] + (-t+1)*HSp3[2]
sage: s = Sym.schur()
sage: HSp3(s([2]))
HSp3[2]
sage: HSp3(s([1,1]))
HSp3[1, 1] - t*HSp3[2]
sage: s(HSp3([2,1]))
s[2, 1] + t*s[3]
```

**level()**

Returns the level of `self`.

**INPUT:**

- `self` – a family of LLT symmetric functions bases

**OUTPUT:**

- the level is the parameter of \( k \) in the basis

**EXAMPLES:**

```python
sage: SymmetricFunctions(FractionField(QQ['t'])).llt(3).level()
3
```

**spin_square**(skp)

Calculate a single instance of a spin squared LLT symmetric function associated with a partition, list of partitions, or a list of skew partitions.

This family of symmetric functions is defined in [LT2000] equation (43).

**INPUT:**

- `self` – a family of LLT symmetric functions bases
- `skp` – a partition or a list of partitions or a list of skew partitions

**OUTPUT:**

the monomial expansion of the LLT symmetric function spin-square functions indexed by `skp`

**EXAMPLES:**

```python
sage: L3 = SymmetricFunctions(FractionField(QQ['t'])).llt(3)
sage: L3.spin_square([2,1])
t*m[1]
sage: L3.spin_square([3,2,1])
(t^3+t)*m[1, 1] + t^3*m[2]
sage: L3.spin_square([[1],[1],[1]])
(t+t^3+t^5+t^6)*m[1, 1, 1] + (t^3+t^4+t^5)*m[2, 1] + t^6*m[3]
sage: L3.spin_square([[2,2],[1]],[[2,1],[1]])
(2*t^4+3*t^2+1)*m[1, 1, 1, 1] + (t^4+t^2)*m[2, 1, 1] + t^4*m[2, 2]
```

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The symmetric function algebra associated to the family of LLT symmetric function bases

INPUT:
• self – a family of LLT symmetric functions bases

OUTPUT:
• returns the symmetric function ring associated to self.

EXAMPLES:

```python
sage: L3 = SymmetricFunctions(FractionField(QQ['t'])).llt(3)
sage: L3.symmetric_function_ring()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

A class of methods for the h-cospin LLT basis of the symmetric functions.

INPUT:
• self – an instance of the LLT h-cospin basis
• llt – a family of LLT symmetric function bases

```python
class Element
    Bases: SymmetricFunctionAlgebra_generic_Element

class sage.combinat.sf.llt.LLT_generic(llt)
    Bases: LLT_generic

A class of methods which are common to both the hspin and hcospin of the LLT symmetric functions.

INPUT:
• self – an instance of the LLT hspin or hcospin basis
• llt – a family of LLT symmetric functions

EXAMPLES:

```python
sage: SymmetricFunctions(FractionField(QQ['t'])).llt(3).hspin()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the level 3 LLT spin basis
sage: SymmetricFunctions(QQ).llt(3,t=2).hspin()
Symmetric Functions over Rational Field in the level 3 LLT spin with t=2 basis
```

```python
sage: QQz = FractionField(QQ['z']); z = QQz.gen()
sage: SymmetricFunctions(QQz).llt(3,t=z).hspin()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in z over Rational Field in the level 3 LLT spin with t=z basis
```

```python
class Element
    Bases: SymmetricFunctionAlgebra_generic_Element

level()

Returns the level of self.

INPUT:
• self – an instance of the LLT hspin or hcospin basis

**OUTPUT:**

• returns the level associated to the basis self.

**EXAMPLES:**

```
sage: HSp3 = SymmetricFunctions(FractionField(QQ['t'])).llt(3).hspin()
sage: HSp3.level()
3
```

**llt_family()**

The family of the llt bases of the symmetric functions.

**INPUT:**

• self – an instance of the LLT hspin or hcospin basis

**OUTPUT:**

• returns an instance of the family of LLT bases associated to self.

**EXAMPLES:**

```
sage: HSp3 = SymmetricFunctions(FractionField(QQ['t'])).llt(3).hspin()
sage: HSp3.llt_family()
level 3 LLT polynomials over Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

**product(left, right)**

Convert to the monomial basis, do the multiplication there, and convert back to the basis self.

**INPUT:**

• self – an instance of the LLT hspin or hcospin basis

• left, right – elements of the symmetric functions

**OUTPUT:**

the product of left and right expanded in the basis self

**EXAMPLES:**

```
sage: HSp3 = SymmetricFunctions(FractionField(QQ['t'])).llt(3).hspin()
sage: HSp3.product(HSp3([1]), HSp3([2]))
HSp3[2, 1] + (-t+1)*HSp3[3]
sage: HCosp3 = SymmetricFunctions(FractionField(QQ['t'])).llt(3).hcospin()
sage: HCosp3.product(HCosp3([1]), HSp3([2]))
1/t*HCosp3[2, 1] + ((t-1)/t)*HCosp3[3]
```

**class** `sage.combinat.sf.llt.LLT_spin(llt)`

**Bases:** `LLT_generic`

A class of methods for the h-spin LLT basis of the symmetric functions.

**INPUT:**

• self – an instance of the LLT hcospin basis

• llt – a family of LLT symmetric function bases


class Element
Bases: Element

5.1.290 Macdonald Polynomials

Notation used in the definitions follows mainly [Mac1995].

The integral forms of the bases \( H \) and \( H_t \) do not appear in Macdonald’s book. They correspond to the two bases

\[
H_\mu[X;q,t] = \sum_\nu K_{\nu\mu}(q,t) s_\nu[X]
\]

and \( \tilde{H}_\mu[X;q,t] = t^{\lambda(\mu)} \sum_\nu K_{\nu\mu}(q,1/t) s_\nu[X] \) where \( K_{\mu\nu}(q,t) \) are the Macdonald \( q,t \)-Kostka coefficients.

The \( H_t \) in this case is short for \( \tilde{H} \) and is the basis which is the graded Frobenius image of the Garsia-Haiman modules [GH1993].

REFERENCES:

- [Mac1995]

class sage.combinat.sf.macdonald.Macdonald(Sym, q='q', t='t')
Bases: UniqueRepresentation

Macdonald Symmetric functions including \( P, Q, J, H, H_t \) bases also including the \( S \) basis which is the plethystic transformation of the Schur basis (that which is dual to the Schur basis with respect to the Macdonald \( q, t \)-scalar product)

INPUT:

- \texttt{self} – a family of Macdonald symmetric function bases

EXAMPLES:

sage: t = QQ['t'].gen(); SymmetricFunctions(QQ['t'].fraction_field()).macdonald(q=t, \texttt{t}=1)
Macdonald polynomials with \( q=t \) and \( t=1 \) over Fraction Field of Univariate Polynomial Ring in \( t \) over Rational Field
sage: Sym = SymmetricFunctions(FractionField(QQ['t'])).macdonald()
Traceback (most recent call last):
... TypeError: unable to evaluate \textquoteleft q' in Fraction Field of Univariate Polynomial Ring in \( t \) over Rational Field

\( H() \)

Returns the Macdonald polynomials on the \( H \) basis. When the \( H \) basis is expanded on the Schur basis, the coefficients are the \( qt \)-Kostka numbers.

INPUT:

- \texttt{self} – a family of Macdonald symmetric function bases

OUTPUT:

- returns the \( H \) Macdonald basis of symmetric functions

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: H = Sym.macdonald() \texttt{.H(); H}
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in \( q, t \) over Rational Field in the Macdonald H basis

(continues on next page)
Coercions to/from the Schur basis are implemented:

```python
sage: H = Sym.macdonald().H()
sage: s = Sym.schur()
sage: H(s([2]))
(q/(q*t-1))*McdH[1, 1] - (1/(q*t-1))*McdH[2]
```

`Ht()`

Returns the Macdonald polynomials on the $H_t$ basis. The elements of the $H_t$ basis are eigenvectors of the \( \nabla \) operator. When expanded on the Schur basis, the coefficients are the modified \( q_t \)-Kostka numbers.

**INPUT:**

- `self` – a family of Macdonald symmetric function bases

**OUTPUT:**

- returns the $H_t$ Macdonald basis of symmetric functions

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['q', 't']))
sage: Ht = Sym.macdonald().Ht(); Ht
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field in the Macdonald Ht basis
sage: [Ht(p).nabla() for p in Partitions(3)]
[q^3*McdHt[3], q*t*McdHt[2, 1], t^3*McdHt[1, 1, 1]]
```

Coercions to/from the Schur basis are implemented:

```python
sage: Ht = Sym.macdonald().Ht()
sage: s = Sym.schur()
sage: Ht(s([2,1]))
((-q)/(-q+t))*McdHt[1, 1] + (t/(-q+t))*McdHt[2, 1]
```

5.1. Comprehensive Module List 2899
\texttt{J()}

Returns the Macdonald polynomials on the $J$ basis also known as the integral form of the Macdonald polynomials. These are scalar multiples of both the $P$ and $Q$ bases. When expressed in the $P$ or $Q$ basis, the scaling coefficients are polynomials in $q$ and $t$ rather than rational functions.

The $J$ basis is calculated using determinantal formulas of Lapointe-Lascoux-Morse giving the action on the $S$-basis [LLM1998].

\textbf{INPUT:}

- \texttt{self} – a family of Macdonald symmetric function bases

\textbf{OUTPUT:}

- returns the $J$ Macdonald basis of symmetric functions

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: J = Sym.macdonald().J(); J
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field in the Macdonald J basis
sage: P = Sym.macdonald().P()
sage: Q = Sym.macdonald().Q()
sage: P(J([2]))
(q^2*t^2-q*t-t+1)*McdP[2]
sage: P(J([1,1]))
t^3-t^2-t+1)*McdP[1, 1]
sage: Q(J([2]))
(q^3-q^2-q+1)*McdQ[2]
sage: Q(J([1,1]))
(q^2*t-q*t-q+1)*McdQ[1, 1]
sage: P(J([2]))
(q^2*t^2-q^2*t-1)*McdJ[2]
sage: J(Q([2]))
(1/(q^3-q^2-q+1))*McdJ[2]
sage: Q(s([2]))
(q-t)/(q^4-t^4)*s[1] + (q^2*t^2-q^2*t-1)*s[2]
sage: s(J([2]))
((q-t)/(q^4-q^3*t^3+q^2*t^2+t+1))*McdJ[1, 1] + (1/(q^t^2-q^t-1))*McdJ[2]
\end{verbatim}

\textbf{P()}

Returns Macdonald polynomials in $P$ basis. The $P$ basis is defined here as a normalized form of the $J$ basis.

\textbf{INPUT:}
• self – a family of Macdonald symmetric function bases

OUTPUT:
• returns the $P$ Macdonald basis of symmetric functions

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P(); P
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t → over Rational Field in the Macdonald P basis
sage: P[2]
McdP[2]
```

The $P$ Macdonald basis is upper triangularly related to the monomial symmetric functions and are orthogonal with respect to the $qt$-Hall scalar product:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P(); P
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t → over Rational Field in the Macdonald P basis
sage: m = Sym.monomial()
sage: P.transition_matrix(m,2)
[ 1 (q*t - q + t - 1)/(q*t - 1)]
[ 0 1]
sage: P([1,1]).scalar_qt(P([2]))
0
sage: P([2]).scalar_qt(P([2]))
(-q^3 + q^2 + q - 1)/(-q*t^2 + q*t + t - 1)
sage: P([1,1]).scalar_qt(P([1,1]))
(-q^2*t + q*t + q - 1)/(-t^3 + t^2 + t - 1)
```

When $q = 0$, the Macdonald polynomials on the $P$ basis are the same as the Hall-Littlewood polynomials on the $P$ basis.

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: P = Sym.macdonald(q=0).P(); P
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Macdonald P with q=0 basis
sage: P([2])^2
sage: HLP = Sym.hall_littlewood().P()
sage: HLP([2])^2
(t+1)*HLP[2, 2] + (-t+1)*HLP[3, 1] + HLP[4]
```

Coercions from the $Q$ and $J$ basis (proportional) are implemented:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P()
sage: Q = Sym.macdonald().Q()
sage: J = Sym.macdonald().J()
sage: s = Sym.schur()
```
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sage: P(Q([2]))
((q*t^2-q*t-t+1)/(q^3-q^2-q+1))*McdP[2]
sage: P(Q([2,1]))
((-q^t^4+2*q^t^3-q^t^2+t^2-2*t+1)/(-q^4*t+2*q^3*t-q^2*t+q^2-2*q+1))*McdP[2, 1]

sage: P(J([2]))
(q*t^2-q*t-t+1)*McdP[2]
sage: P(J([2,1]))
(-q^t^4+2*q^t^3-q^t^2+t^2-2*t+1)*McdP[2, 1]

By transitivity, one get coercions from the classical bases:

sage: P(s([2]))
((q-t)/(q*t-1))*McdP[1, 1] + McdP[2]
sage: P(s([2,1]))
((q*t-t^2+q-t)/(q*t^2-1))*McdP[1, 1, 1] + McdP[2, 1]

sage: Sym = SymmetricFunctions(QQ['x', 'y', 'z']).fraction_field()
sage: (x,y,z) = Sym.base_ring().gens()
sage: Macxy = Sym.macdonald(q=x,t=y)
sage: Macyz = Sym.macdonald(q=y,t=z)
sage: Maczx = Sym.macdonald(q=z,t=x)
sage: P1 = Macxy.P()
sage: P2 = Macyz.P()
sage: P3 = Maczx.P()
sage: m(P1[2,1])
((-2*x*y^2+x*y-y^2+x-y+2)/(-x*y^2+1))*m[1, 1, 1] + m[2, 1]
sage: m(P2[2,1])
((-2*y*z^2+y*z-z^2+y-z+2)/(-y*z^2+1))*m[1, 1, 1] + m[2, 1]
sage: m(P1(P2(P3[2,1])))
((-2*x^2*z-x^2*x-z+x+z+2)/(-x^2*z+1))*m[1, 1, 1] + m[2, 1]
sage: P1(P2[2])
((-x*y^2+2*x*y*z-y^2*z+x^2*y-z)/(x*y^2*z-x*y-y*z+1))*McdP[1, 1] + McdP[2]
sage: m(z*P1[2]+x*P2[2])
((x^2*y^2*z+x*y^2*z^2-x^2*y^2+x^2*y^2*z-x*y^2*z^2+y^2*z^2-x^2*y^2*z-y^2*z^2+x*y-
→y^2*z+x+z)/(x*y^2*z-x^2*y-y*z+1))*m[1, 1] + (x+z)*m[2]

QO

Returns the Macdonald polynomials on the Q basis. These are dual to the Macdonald polynomials on the P basis with respect to the qt-Hall scalar product. The Q basis is defined to be a normalized form of the J basis.

INPUT:

• self – a family of Macdonald symmetric function bases

OUTPUT:

• returns the Q Macdonald basis of symmetric functions

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['q', 't']))
sage: Q = Sym.macdonald().Q(); Q
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t,
over Rational Field in the Macdonald Q basis

\begin{verbatim}
sage: P = Sym.macdonald().P()
sage: Q([2]).scalar_qt(P([2]))
1
sage: Q([2]).scalar_qt(P([1,1]))
0
sage: Q([1,1]).scalar_qt(P([2]))
0
sage: Q([1,1]).scalar_qt(P([1,1]))
1
sage: Q(P([2]))
((q^3-q^2-q+1)/(q*t^2-q*t-t+1))*McdQ[2]
sage: Q(P([1,1]))
((q^2*t-q*t-q+1)/(t^3-t^2-t+1))*McdQ[1, 1]
\end{verbatim}

Coercions from the $P$ and $J$ basis (proportional) are implemented:

\begin{verbatim}
sage: P = Sym.macdonald().P()
sage: Q = Sym.macdonald().Q()
sage: J = Sym.macdonald().J()
sage: s = Sym.schur()
sage: Q(J([2]))
(q^3-q^2-q+1)*McdQ[2]
sage: Q(P([2]))
((q^3-q^2-q+1)/(q*t^2-q*t-t+1))*McdQ[2]
sage: P(Q(P([2])))
McdP[2]
sage: Q(P(Q([2])))
McdQ[2]
\end{verbatim}

By transitivity, one gets coercions from the classical bases:

\begin{verbatim}
sage: Q(s([2]))
((q^2*t-q*t+1)/(t^3-t^2-t+1))*McdQ[1, 1] + ((q^3-q^2-q+1)/(q^2*t^2-q^2*t+1))*McdQ[2]
\end{verbatim}

The function \texttt{S()} returns the modified Schur functions defined by the plethystic substitution $S_\mu = s_\mu[X(1-t)/(1-q)]$. When the Macdonald polynomials in the $J$ basis are expressed in terms of the modified Schur functions at $q = 0$, the coefficients are $ql$-Kostka numbers.

**INPUT:**
- \texttt{self} – a family of Macdonald symmetric function bases

**OUTPUT:**
- returns the $S$ Macdonald basis of symmetric functions

**EXAMPLES:**
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: S = Sym.macdonald().S(); S
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field in the Macdonald S basis
sage: p = Sym.power()
sage: p(S[2,1])
((-1/3*t^3+1/3)/(q^3-1))*p[3]
sage: J = Sym.macdonald().J()
sage: S(J([2]))
(q^3-q^2-q+1)*McdS[2]
sage: S(J([1,1]))
(q^2*t-q*t-q+1)*McdS[1, 1] + (q^2-q*t+q)*McdS[2]
sage: S = Sym.macdonald(q=0).S()
sage: S(J([1,1]))
McdS[1, 1] + t*McdS[2]
sage: S(J([2]))
(q^3-q^2-q+1)*McdS[2]
sage: p(S[2,1])
(-1/3*t^3+t^2-t+1/3)*p[1, 1, 1] + (1/3*t^3-1/3)*p[3]
sage: from sage.combinat.sf.macdonald import qt_kostka
sage: qt_kostka([2],[1,1])
t
sage: qt_kostka([1,1],[2])
q

Coercions to/from the Schur basis are implemented:

sage: S = Sym.macdonald().S()
sage: s = Sym.schur()
sage: S(s([2]))
((q^3-q*t+q*t-q-t)/(t^3-t^2-t+1))*s[1, 1] + ((-q^3+q^2+q-1)/(-t^3+t^2+t-1))*s[2]
sage: s(S([2]))
((q^2-q*t-q+t)/(t^3-t^2-t+1))*s[1, 1] + ((-q^2*t+q*t+q-1)/(-t^3+t^2+t-1))*s[2]

base_ring()

Returns the base ring of the symmetric functions where the Macdonald symmetric functions live

INPUT:
• self – a family of Macdonald symmetric function bases

OUTPUT:
• the base ring associated to the corresponding symmetric function ring

EXAMPLES:

sage: Sym = SymmetricFunctions(QQ['q']).fraction_field()
sage: Mac = Sym.macdonald(t=0)
sage: Mac.base_ring()
Fraction Field of Univariate Polynomial Ring in q over Rational Field
**symmetric_function_ring()**

Returns the base ring of the symmetric functions where the Macdonald symmetric functions live

INPUT:

• self – a family of Macdonald symmetric function bases

OUTPUT:

• the symmetric function ring associated to the Macdonald bases

EXAMPLES:

```python
sage: Mac = SymmetricFunctions(QQ['q'].fraction_field()).macdonald(t=0)
sage: Mac.symmetric_function_ring()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in q over Rational Field
```

class sage.combinat.sf.macdonald.MacdonaldPolynomials_generic(macdonald)

Bases: SymmetricFunctionAlgebra_generic

A class for methods for one of the Macdonald bases of the symmetric functions

INPUT:

• self – a Macdonald basis

• macdonald – a family of Macdonald symmetric function bases

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t'])); Sym.rename("Sym"); Sym
Sym
sage: Sym.macdonald().P()
Sym in the Macdonald P basis
sage: Sym.macdonald(t=2).P()
Sym in the Macdonald P with t=2 basis
sage: Sym.rename()```

class Element

Bases: SymmetricFunctionAlgebra_generic_Element

```
nabla(q=None, t=None, power=1)
```

Return the value of the nabla operator applied to self.

The eigenvectors of the nabla operator are the Macdonald polynomials in the $Ht$ basis. For more information see: [BGHT1999].

The operator nabla acts on symmetric functions and has the Macdonald $Ht$ basis as eigenfunctions and the eigenvalues are $q^{n(\mu) t^\mu} n(\mu) = \sum_i (i-1) \mu_i$ and $\mu'$ is the conjugate shape of $\mu$.

If the parameter power is an integer then it calculates nabla to that integer. The default value of power is 1.

INPUT:

• self – an element of a Macdonald basis

• q, t – optional parameters to specialize

• power – an integer (default: 1)

OUTPUT:

• returns the symmetric function of $\nabla$ acting on self

EXAMPLES:
Combinatorics, Release 10.0

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P()
sage: P[[1,1]].nabla()
((q^2*t+q^3*t^2-2*t)/(q^t-1))*McdP[1, 1] + McdP[2]
sage: P[[1,1]].nabla(t=1)
((q^2*t+q^3*t^2-2*t)/(q^t-1))*McdP[1, 1] + McdP[2]
sage: H = Sym.macdonald().H()
sage: H[[1,1]].nabla()
t*McdH[1, 1] + (-t^2+1)*McdH[2]
sage: H[[1,1]].nabla(q=1)
((t^2+q-t-1)/(q^t-1))*McdH[1, 1] + ((-t^3+t^2+t-1)/(q^t-1))*McdH[2]
sage: H(0).nabla()
0
sage: H[[2,2,1]].nabla(t=1/H.t)
((-q^2)/(-t^4))*McdH[2, 2, 1]
sage: H[[2,2,1]].nabla(t=1/H.t,power=-1)
((-t^4)/(-q^2))*McdH[2, 2, 1]
```

c1(part)

Returns the qt-Hall scalar product between J(part) and P(part).

INPUT:

- self – a Macdonald basis
- part – a partition

OUTPUT:

- returns the qt-Hall scalar product between J(part) and P(part)

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P()
sage: P.c1(Partition([2,1]))
-q^4*t + 2*q^3*t - q^2*t + q + 1
```

c2(part)

Returns the qt-Hall scalar product between J(part) and Q(part).

INPUT:

- self – a Macdonald basis
- part – a partition

OUTPUT:

- returns the qt-Hall scalar product between J(part) and Q(part)

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P()
sage: P.c2(Partition([2,1]))
-q*t^4 + 2*q^3*t^2 - q^2*t^2 + t^2 - 2*t + 1
```
macdonald_family()
Returns the family of Macdonald bases associated to the basis self

INPUT:
• self – a Macdonald basis

OUTPUT:
• the family of Macdonald symmetric functions associated to self

EXAMPLES:

```
sage: MacP = SymmetricFunctions(QQ['q'].fraction_field()).macdonald(t=0).P()
sage: MacP.macdonald_family()
Macdonald polynomials with t=0 over Fraction Field of Univariate Polynomial
˓→Ring in q over Rational Field
```

product(left, right)
Multiply an element of the Macdonald symmetric function basis self and another symmetric function
Convert to the Schur basis, do the multiplication there, and convert back to self basis.

INPUT:
• self – a Macdonald symmetric function basis
  • left – an element of the basis self
  • right – another symmetric function

OUTPUT:
the product of left and right expanded in the basis self

EXAMPLES:

```
sage: Mac = SymmetricFunctions(FractionField(QQ['q','t'])).macdonald()
sage: H = Mac.H()
sage: J = Mac.J()
sage: P = Mac.P()
sage: Q = Mac.Q()
sage: Ht = Mac.Ht()
sage: J([1])^2  #indirect doctest
((-q^2+1)/(-q^2*t+1))*McdJ[2, 1] + ((-t+1)/(-q^2*t+1))*McdJ[3]
sage: J.product(J[1], J[2])
((-q^2+1)/(-q^2*t+1))*McdJ[2, 1] + ((-t+1)/(-q^2*t+1))*McdJ[3]
sage: H.product(H[1], H[2])
((q^2-1)/(q^2*t-1))*McdH[2, 1] + ((-t+1)/(-q^2*t+1))*McdH[3]
sage: P.product(P[1], P[2])
((-q^3*t^2+q*t^2+q^2*t-q^2+t)/(q^2*t^2+q^2*t+q*t-1))*McdP[2, 1] + McdP[3]
sage: Q.product(Q[1],Q[2])
McdQ[2, 1] + ((q^2*t-3*(q^2*t^2+q^2*t+q*t-1))/(q^3*t^2+q^2*t-q+t-1))*McdQ[3]
sage: Ht.product(Ht[1],Ht[2])
((q^2-1)/(q^2*t))*McdHt[2, 1] + ((t-1)/(-q^2*t+1))*McdHt[3]
```

class sage.combinat.sf.macdonald.MacdonaldPolynomials_h(macdonald)
Bases: MacdonaldPolynomials_generic

The $H$ basis is defined as $H_\mu = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda$ where $K_{\lambda\mu}(q,t)$ are the Macdonald Kostka coefficients.
In this implementation, it is calculated by using the Macdonald $H_t$ basis and substituting $t \rightarrow 1/t$ and multiplying by $t^{n(\mu)}$.

**INPUT:**
- `self` – a Macdonald $H$ basis
- `macdonald` – a family of Macdonald bases

**class Element**

**Bases:** `Element`

```python
class sage.combinat.sf.macdonald.MacdonaldPolynomials_ht(macdonald):
    Bases: MacdonaldPolynomials_generic
```

The $H_t$ basis is defined as $\tilde{H}_\mu = t^{n(\mu)} \sum_\lambda K_{\lambda \mu}(q,t) s_\lambda$ where $K_{\lambda \mu}(q,t)$ are the Macdonald $(q,t)$-Kostka coefficients and $n(\mu) = \sum_i (i-1) \mu_i$.

It is implemented here by using a Pieri formula due to F. Bergeron and M. Haiman [BH2013].

**INPUT:**
- `self` – a Macdonald $H_t$ basis
- `macdonald` – a family of Macdonald bases

**class sage.combinat.sf.macdonald.MacdonaldPolynomials_ht**

**Bases:** `MacdonaldPolynomials_generic`

The operator `nabla` acts on symmetric functions and has the Macdonald $H_t$ basis as eigenfunctions and the eigenvalues are $q^{n(\mu')} t^{n(\mu)}$ where $n(\mu) = \sum_i (i-1) \mu_i$.

If the parameter `power` is an integer then it calculates `nabla` to that integer. The default value of `power` is 1.

**INPUT:**
- `self` – an element of the Macdonald $H_t$ basis
- `q, t` – optional parameters to specialize
- `power` – an integer (default: 1)

**OUTPUT:**
- returns the symmetric function of $\nabla$ acting on `self`

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: Ht = Sym.macdonald().Ht()
sage: t = Ht.t; q = Ht.q
sage: s = Sym.schur()
sage: a = sum(Ht(p) for p in Partitions(3))
sage: Ht(0).nabla()
0
sage: a.nabla() == t^3*Ht([1,1,1])+q*t*Ht([2,1]) + q^3*Ht([3])
True
sage: a.nabla(t=3) == 27*Ht([1,1,1])+3*q*Ht([2,1]) + q^3*Ht([3])
True
sage: a.nabla(q=3) == t^3*Ht([1,1,1])+3*t*Ht([2,1]) + 27*Ht([3])
True
```

(continues on next page)
```python
sage: Ht[2,1].nabla(power=-1)
1/(q*t)*McdHt[2, 1]
sage: Ht[2,1].nabla(power=4)
q^4*t^4*McdHt[2, 1]
sage: s(a.nabla(q=3))
(t^6+27*q^3+3*q*t^2)*s[1, 1, 1] + (t^5*t+3*t^2+27*q)*s[2, 1] + (t^3+3*t+27)*s[3]
sage: Ht = Sym.macdonald(q=3).Ht()
sage: a = sum(Ht(p) for p in Partitions(3))
sage: s(a.nabla())
(t^6+9*t^2+729)*s[1, 1, 1] + (t^5+t^4+3*t^2+9*t+324)*s[2, 1] + (t^3+3*t+27)*s[3]
```

**class** `sage.combinat.sf.macdonald.MacdonaldPolynomials_j(macdonald)`

Bases: `MacdonaldPolynomials_generic`

The $J$ basis is calculated using determinantal formulas of Lapointe-Lascoux-Morse giving the action on the $S$-basis.

**INPUT:**

- **self** – a Macdonald $J$ basis
- **macdonald** – a family of Macdonald bases

**class** `Element`

Bases: `Element`

**class** `sage.combinat.sf.macdonald.MacdonaldPolynomials_p(macdonald)`

Bases: `MacdonaldPolynomials_generic`

The $P$ basis is defined here as the $J$ basis times a normalizing coefficient $c2$.

**INPUT:**

- **self** – a Macdonald $P$ basis
- **macdonald** – a family of Macdonald bases

**class** `Element`

Bases: `Element`

**scalar_qt_basis(part1, part2=None)**

Returns the scalar product of $P(part1)$ and $P(part2)$ This scalar product formula is given in equation (4.11) p.323 and (6.19) p.339 of Macdonald’s book [Mac1995].

**INPUT:**

- **self** – a Macdonald $P$ basis
- **part1, part2** – partitions

**OUTPUT:**

- returns the scalar product of $P(part1)$ and $P(part2)$

**EXAMPLES:**

5.1. Comprehensive Module List
With a single argument, takes \( part2 = part1 \):

\[
\text{sage: } P.\text{scalar_qt_basis}(\text{Partition([2,1]), Partition([2,1])}) \\
\text{sage: } (-q^4*t + 2*q^3*t - q^2*t + q^2 - 2*q + 1)/(-q^4*t^4 + 2*q^4*t^3 - q^4*t^2 + t^2 - 2*t^2 + 1)
\]

class \text{sage.combinat.sf.macdonald.MacdonaldPolynomials_q(macdonald)}

Bases: \text{MacdonaldPolynomials_generic}

The \( Q \) basis is defined here as the \( J \) basis times a normalizing coefficient.

INPUT:

\begin{itemize}
  \item self – a Macdonald \( Q \) basis
  \item macdonald – a family of Macdonald bases
\end{itemize}

class \text{Element}

Bases: \text{Element}

class \text{sage.combinat.sf.macdonald.MacdonaldPolynomials_s(macdonald)}

Bases: \text{MacdonaldPolynomials_generic}

An implementation of the basis \( s_\lambda[(1-t)X/(1-q)] \)

This is perhaps misnamed as a ‘Macdonald’ basis for the symmetric functions but is used in the calculation of the Macdonald \( J \) basis (see method ‘creation’ below) but does use both of the two parameters and can be specialized to \( s_\lambda[(1-t)X] \) and \( s_\lambda[X/(1-t)] \).

INPUT:

\begin{itemize}
  \item self – a Macdonald \( S \) basis
  \item macdonald – a family of Macdonald bases
\end{itemize}

class \text{Element}

Bases: \text{Element}

\text{creation(k)}

This function is a creation operator for the \( J \)-basis for which the action is known on the ‘Macdonald’ \( S \)-basis by formula from [LLM1998].

INPUT:

\begin{itemize}
  \item self – an element of the Macdonald \( S \) basis
  \item k – a positive integer
\end{itemize}

OUTPUT:

\begin{itemize}
  \item returns the column adding operator on the \( J \) basis on \text{self}
\end{itemize}

EXAMPLES:
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: S = Sym.macdonald().S()
sage: a = S(1)
sage: a.creation(1)
(-q+1)*McdS[1]
sage: a.creation(2)
(q^2*t-q*t-q+1)*McdS[1, 1] + (q^2-q^t-q+t)*McdS[2]

product(left, right)

The multiplication of the modified Schur functions behaves the same as the multiplication of the Schur functions.

INPUT:

• self – a Macdonald S basis
• left, right – a symmetric functions

OUTPUT:

the product of left and right

EXAMPLES:

sage.combinat.sf.macdonald.c1(part, q, t)

This function returns the qt-Hall scalar product between J(part) and P(part).

This coefficient is $c_\lambda$ in equation (8.1') p. 352 of Macdonald’s book [Mac1995].

INPUT:

• part – a partition
• q, t – parameters

OUTPUT:

• returns a polynomial of the scalar product between the J and P bases

EXAMPLES:

sage.combinat.sf.macdonald.c2(part, q, t)

This function returns the qt-Hall scalar product between J(part) and Q(part).

This coefficient is $c_\lambda$ in equation (8.1) p. 352 of Macdonald’s book [Mac1995].

INPUT:

• part – a partition
• q, t – parameters

OUTPUT:
• returns a polynomial of the scalar product between the J and P bases

EXAMPLES:
```
sage: from sage.combinat.sf.macdonald import c2
sage: R.<q,t> = QQ[]
sage: c2(Partition([1,1]),q,t)
t^3 - t^2 - t + 1
sage: c2(Partition([2,1]),q,t)
-q^4*t^4 + 2*q^3*t^3 - q^2*t^2 + t^2 - 2*t + 1
```

```
sage.combinat.sf.macdonald.cmunu(nu)

Return the coefficient of \( \tilde{H}_\nu \) in \( h_t^{-1} \tilde{H}_\mu \).

Proposition 5 of F. Bergeron and M. Haiman [BH2013] states
\[
c_{\mu\nu} = \sum_{\alpha \vdash \nu} c_{\mu\alpha} c_{\alpha\nu} B_{\alpha/\nu} / B_{\mu/\nu}
\]
where \( c_{\mu\nu} \) is the coefficient of \( \tilde{H}_\nu \) in \( h_t^{-1} \tilde{H}_\mu \) and \( B_{\mu/\nu} \) is the bi-exponent generator implemented in the function `sage.combinat.sf.macdonald.Bmu()`.

INPUT:
• \( \mu, \nu \) – partitions with \( \nu \) contained in \( \mu \)

OUTPUT:
• an element of the fraction field of polynomials in \( q \) and \( t \)

EXAMPLES:
```
sage: from sage.combinat.sf.macdonald import cmunu
sage: cmunu(Partition([2,1]),Partition([1]))
q + t + 1
sage: cmunu(Partition([2,2]),Partition([1,1]))
(-q^3 - q^2 + q*t + t)/(-q + t)
```

```
sage.combinat.sf.macdonald.cmunu1(nu)

Return the coefficient of \( \tilde{H}_\nu \) in \( h_t^{-1} \tilde{H}_\mu \).

INPUT:
• \( \mu, \nu \) – partitions with \( \nu \) precedes \( \mu \)

OUTPUT:
• an element of the fraction field of polynomials in \( q \) and \( t \)

EXAMPLES:
```
sage: from sage.combinat.sf.macdonald import cmunu1
sage: cmunu1(Partition([2,2]),Partition([2]))
(-q^3 - q^2 + q*t + t)/(-q + t)
esage: Ht = Sym.macdonald().Ht()
esage: all(Ht[2,2].skew_by(h[r]).coefficient(nu)
    == cmunu1(Partition([2,2]),nu)
    for r in range(1,5) for nu in Partitions(4-r))
True
```
sage: from sage.combinat.sf.macdonald import cmunu1
sage: cmunu1(Partition([2,1]),Partition([2]))
(-t^2 + q)/(q - t)
sage: cmunu1(Partition([2,1]),Partition([1,1]))
(-q^2 + t)/(-q + t)
sage: Sym = SymmetricFunctions(QQ['q','t'].fraction_field())
sage: h = Sym.h()
sage: Ht = Sym.macdonald().Ht()
sage: all(Ht[3,2,1].skew_by(h[1]).coefficient(nu)
....: == cmunu1(Partition([3,2,1]),nu)
....: for nu in Partition([3,2,1]).down_list())
True

sage.combinat.sf.macdonald.qt_kostka(lam, mu)

Returns the $K_{\lambda\mu}(q,t)$ by computing the change of basis from the Macdonald H basis to the Schurs.

INPUT:

- lam, mu – partitions of the same size

OUTPUT:

- returns the $q,t$-Kostka polynomial indexed by the partitions lam and mu

EXAMPLES:

sage: from sage.combinat.sf.macdonald import qt_kostka
sage: qt_kostka([2,1,1],[1,1,1,1])
t^3 + t^2 + t
sage: qt_kostka([1,1,1,1],[2,1,1])
q
sage: qt_kostka([1,1,1,1],[3,1])
q^3
sage: qt_kostka([1,1,1,1],[1,1,1,1])
1
sage: qt_kostka([2,1,1],[2,2])
q^2*t + q*t + q
sage: qt_kostka([2,2],[2,2])
q^2*t^2 + 1
sage: qt_kostka([4],[3,1])
t
sage: qt_kostka([2,2],[3,1])
q^2*t + q
sage: qt_kostka([3,1],[2,1,1])
q^3*t + t^2 + t
sage: qt_kostka([2,1,1],[2,1,1])
q^2*t^2 + q*t + 1
sage: qt_kostka([2,1],[1,1,1,1])
0
5.1.291 Monomial symmetric functions

class sage.combinat.sf.monomial.SymmetricFunctionAlgebra_monomial(Sym)

Bases: SymmetricFunctionAlgebra_classical

A class for methods related to monomial symmetric functions

INPUT:

• self – a monomial symmetric function basis
• Sym – an instance of the ring of the symmetric functions

class Element

Bases: Element

expand(n, alphabet='x')

Expand the symmetric function self as a symmetric polynomial in n variables.

INPUT:

• n – a nonnegative integer
• alphabet – (default: 'x') a variable for the expansion

OUTPUT:

A monomial expansion of self in the n variables labelled by alphabet.

EXAMPLES:

```
sage: m = SymmetricFunctions(QQ).m()
sage: m([2,1]).expand(3)
x0^2*x1 + x0*x1^2 + x0^2*x2 + x0^2*x3 + x1^2*x2 + x1^2*x3
sage: m([1,1,1]).expand(2)
0
sage: m([2,1]).expand(3,alphabet='z')
z0^2*z1 + z0*z1^2 + z0^2*z2 + z0*z2^2 + z1^2*z2 + z1*z2^2
sage: m([2,1]).expand(3,alphabet='x,y,z')
x^2*y + x*y^2 + x^2*z + y^2*z + x*z^2 + y*z^2
sage: m([1]).expand(0)
0
sage: (3*m([])).expand(0)
3
```

exponential_specialization(t=None, q=1)

Return the exponential specialization of a symmetric function (when q = 1), or the q-exponential specialization (when q \neq 1).

The exponential specialization $ex$ at t is a $K$-algebra homomorphism from the $K$-algebra of symmetric functions to another $K$-algebra $R$. It is defined whenever the base ring $K$ is a $Q$-algebra and $t$ is an element of $R$. The easiest way to define it is by specifying its values on the powersum symmetric functions to be $p_1 = t$ and $p_n = 0$ for $n > 1$. Equivalently, on the homogeneous functions it is given by $ex(h_n) = t^n/n!$; see Proposition 7.8.4 of [EnumComb2].

By analogy, the $q$-exponential specialization is a $K$-algebra homomorphism from the $K$-algebra of symmetric functions to another $K$-algebra $R$ that depends on two elements $t$ and $q$ of $R$ for which the elements $1 - q^i$ for all positive integers $i$ are invertible. It can be defined by specifying its values on the complete homogeneous symmetric functions to be

$$ex_q(h_n) = t^n/[n]_q!,$$
where \([n]_q\) is the \(q\)-factorial. Equivalently, for \(q \neq 1\) and a homogeneous symmetric function \(f\) of degree \(n\), we have

\[
ex_q(f) = (1 - q)^n t^n ps_q(f),
\]

where \(ps_q(f)\) is the stable principal specialization of \(f\) (see \texttt{principal_specialization()}). (See (7.29) in [EnumComb2].)

The limit of \(ex_q\) as \(q \to 1\) is \(ex\).

**INPUT:**
- \(t\) (default: \(None\)) – the value to use for \(t\); the default is to create a ring of polynomials in \(t\).
- \(q\) (default: \(1\)) – the value to use for \(q\). If \(q\) is \(None\), then a ring (or fraction field) of polynomials in \(q\) is created.

**EXAMPLES:**

```python
sage: m = SymmetricFunctions(QQ).m()
sage: (m[3]+m[2,1]+m[1,1,1]).exponential_specialization()
1/6*t^3
```

We also support the \(q\)-exponential_specialization:

```python
sage: factor(m[3].exponential_specialization(q=var("q"), t=var("t")))
(q - 1)^2*t^3/(q^2 + q + 1)
```

**principal_specialization** \((n=+Infinity, q=None)\)

Return the principal specialization of a symmetric function.

The principal specialization of order \(n\) at \(q\) is the ring homomorphism \(ps_{n,q}\) from the ring of symmetric functions to another commutative ring \(R\) given by \(x_i \mapsto q^{i-1}\) for \(i \in \{1, \ldots, n\}\) and \(x_i \mapsto 0\) for \(i > n\). Here, \(q\) is a given element of \(R\), and we assume that the variables of our symmetric functions are \(x_1, x_2, x_3, \ldots\). (To be more precise, \(ps_{n,q}\) is a \(K\)-algebra homomorphism, where \(K\) is the base ring.) See Section 7.8 of [EnumComb2].

The stable principal specialization at \(q\) is the ring homomorphism \(ps_q\) from the ring of symmetric functions to another commutative ring \(R\) given by \(x_i \mapsto q^{i-1}\) for all \(i\). This is well-defined only if the resulting infinite sums converge; thus, in particular, setting \(q = 1\) in the stable principal specialization is an invalid operation.

**INPUT:**
- \(n\) (default: \(infinity\)) – a nonnegative integer or \(infinity\), specifying whether to compute the principal specialization of order \(n\) or the stable principal specialization.
- \(q\) (default: \(None\)) – the value to use for \(q\); the default is to create a ring of polynomials in \(q\) (or a field of rational functions in \(q\)) over the given coefficient ring.

For \(q=1\) and finite \(n\) we use the formula from Proposition 7.8.3 of [EnumComb2]:

\[
ps_{n,1}(m_\lambda) = \binom{n}{\ell(\lambda)} \left( m_1(\lambda), m_2(\lambda), \ldots \right),
\]

where \(\ell(\lambda)\) denotes the length of \(\lambda\).

In all other cases, we convert to complete homogeneous symmetric functions.

**EXAMPLES:**

```python

```
\texttt{sage: m = SymmetricFunctions(QQ).m()}
\texttt{sage: x = m[3,1]}
\texttt{sage: x.principal_specialization(3)}
\texttt{q^7 + q^6 + q^5 + q^3 + q^2 + q}
\texttt{sage: x = 5*m[2] + 3*m[1] + 1}
\texttt{sage: x.principal_specialization(3, q=var("q"))}
\texttt{-10*(q^3 - 1)*q/(q - 1) + 5*(q^3 - 1)^2/(q - 1)^2 + 3*(q^3 - 1)/(q - 1) + 1}

\texttt{antipode\_by\_coercion(element)}

The antipode of \texttt{element} via coercion to and from the power-sum basis or the Schur basis (depending on whether the power sums really form a basis over the given ground ring).

INPUT:

- \texttt{element} – element in a basis of the ring of symmetric functions

EXAMPLES:

\texttt{sage: Sym = SymmetricFunctions(QQ)}
\texttt{sage: m = Sym.monomial()}
\texttt{sage: m[3,2].antipode()}\n\texttt{m[3, 2] + 2*m[5]}
\texttt{sage: m.antipode\_by\_coercion(m[3,2])}
\texttt{m[3, 2] + 2*m[5]}

\texttt{sage: Sym = SymmetricFunctions(ZZ)}
\texttt{sage: m = Sym.monomial()}
\texttt{sage: m[3,2].antipode()}\n\texttt{m[3, 2] + 2*m[5]}
\texttt{sage: m.antipode\_by\_coercion(m[3,2])}
\texttt{m[3, 2] + 2*m[5]}

\underline{Todo}: Is there a not too difficult way to get the power-sum computations to work over any ring, not just one with coercion from \texttt{Q}?

\texttt{from\_polynomial(f, check=True)}

Return the symmetric function in the monomial basis corresponding to the polynomial \texttt{f}.

INPUT:

- \texttt{self} – a monomial symmetric function basis
- \texttt{f} – a polynomial in finitely many variables over the same base ring as \texttt{self}. It is assumed that this polynomial is symmetric.
- \texttt{check} – boolean (default: \texttt{True}), checks whether the polynomial is indeed symmetric

OUTPUT:

- This function converts a symmetric polynomial \texttt{f} in a polynomial ring in finitely many variables to a symmetric function in the monomial basis of the ring of symmetric functions over the same base ring.

EXAMPLES:
from polynomial_exp(p)

Conversion from polynomial in exponential notation

INPUT:

• self – a monomial symmetric function basis
• p – a polynomial over the same base ring as self

OUTPUT:

• This returns a symmetric function by mapping each monomial of p with exponents exp into m_\lambda where \lambda is the partition with exponential notation exp.

EXAMPLES:

```python
sage: m = SymmetricFunctions(QQ).m()
sage: P = PolynomialRing(QQ, 'x', 5)
sage: x = P.gens()
sage: m.from_polynomial_exp(f)
m[5, 5, 5, 3, 1, 1]
```

The exponential notation of the partition (5, 5, 5, 3, 1, 1) is:

```python
sage: Partition(([5, 5, 5, 3, 1, 1])).to_exp()
[2, 0, 1, 0, 3]
```

Therefore, the monomial:

```python
```

is mapped to:

```python
sage: m.from_polynomial_exp(f)
m[5, 5, 5, 3, 1, 1]
```

Furthermore, this function is linear:
sage: \( f = 3 \cdot x[3] + 2 \cdot x[0]^2 \cdot x[2] \cdot x[4]^3 \)
sage: m.from_polynomial_exp(f)
\( 3^*m[4] + 2^*m[5, 5, 5, 3, 1, 1] \)

See also:

\texttt{Partition()}, \texttt{Partition.to.exp()}

\texttt{product(left, right)}

Return the product of \texttt{left} and \texttt{right}.

- \texttt{left, right} – symmetric functions written in the monomial basis \texttt{self}.

\textbf{OUTPUT:}

- the product of \texttt{left} and \texttt{right}, expanded in the monomial basis, as a dictionary whose keys are partitions and whose values are the coefficients of these partitions (more precisely, their respective monomial symmetric functions) in the product.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: m = SymmetricFunctions(QQ).m()
sage: a = m([2,1])
sage: a^2
4^*m[2, 2, 1, 1] + 6^*m[2, 2, 2] + 2^*m[3, 2, 1] + 2^*m[3, 3] + 2^*m[4, 1, 1] + m[4, 2]
\end{verbatim}

\begin{verbatim}
sage: QQx.<x> = QQ['x']
sage: m = SymmetricFunctions(QQx).m()
sage: a = m([2,1])+x
sage: 2*a
2^*x*m[] + 2^*m[2, 1]
sage: a^2
x^2^*m[] + 2^*x^*m[2, 1] + 4^*m[2, 2, 1] + 6^*m[2, 2, 2] + 2^*m[3, 2, 1] + 2^*m[3, 3] + 2^*m[4, 1, 1] + m[4, 2]
\end{verbatim}

\section*{5.1.292 Multiplicative symmetric functions}

A realization \( h \) of the ring of symmetric functions is multiplicative if for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) we have \( h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots \).

\textbf{class} \texttt{sage.combinat.sf.multiplicative.SymmetricFunctionAlgebra_multiplicative(Sym, basis_name=None, prefix=None, graded=True)}

\textbf{Bases:} \texttt{SymmetricFunctionAlgebra_classical}

The class of multiplicative bases of the ring of symmetric functions.

A realization \( q \) of the ring of symmetric functions is multiplicative if for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) we have \( q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots \) (with \( q_0 \) meaning 1).

Examples of multiplicative realizations are the elementary symmetric basis, the complete homogeneous basis, the powersum basis (if the base ring is a \( \mathbb{Q} \)-algebra), and the Witt basis (but not the Schur basis or the monomial basis).
**coproduct_on_basis** (*mu*)

Return the coproduct on a basis element for multiplicative bases.

**INPUT:**

- *mu* – a partition

**OUTPUT:**

- the image of self[*mu*] under comultiplication; this is an element of the tensor square of self

**EXAMPLES:**

```
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.powersum()
sage: p.coproduct_on_basis([2,1])
```

```
sage: e = Sym.elementary()
sage: e.coproduct_on_basis([3,1])
```

```
sage: h = Sym.homogeneous()
sage: h.coproduct_on_basis([3,1])
```

**product_on_basis** (*left, right*)

Return the product of *left* and *right*.

**INPUT:**

- *left, right* – partitions

**OUTPUT:**

- an element of self

**EXAMPLES:**

```
sage: e = SymmetricFunctions(QQ).e()
sage: e([2,1])^2 # indirect doctest
e[2, 2, 1, 1]
```

```
sage: h = SymmetricFunctions(QQ).h()
sage: h([2,1])^2
h[2, 2, 1, 1]
```

```
sage: p = SymmetricFunctions(QQ).p()
sage: p([2,1])^2
p[2, 2, 1, 1]
```

```
sage: QQx.<x> = QQ[]
sage: p = SymmetricFunctions(QQx).p()
sage: (x*p([2]))^2
x^2*p[2, 2]
```

(continues on next page)
5.1.293 \(k\)-Schur Functions

```python
sage: TestSuite(p).run() # to silence sage -coverage
```

```python
5.1.293 \(k\)-Schur Functions
```

```python
class sage.combinat.sf.new_kschur.KBoundedSubspace(Sym, k, t='t')
    Bases: UniqueRepresentation, Parent

This class implements the subspace of the ring of symmetric functions spanned by \(\{s_\lambda[X/(1-t)]\}_{\lambda_1 \leq k}\). When \(t = 1\), this space is in fact a subring of the ring of symmetric functions generated by the complete homogeneous symmetric functions \(h_i\) for \(1 \leq i \leq k\).

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: KB = Sym.kBoundedSubspace(3,1); KB
3-bounded Symmetric Functions over Rational Field with t=1
sage: Sym = SymmetricFunctions(QQ['t'])
sage: KB = Sym.kBoundedSubspace(3); KB
3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field
```

The \(k\)-Schur function basis can be constructed as follows:

```python
sage: ks = KB.kschur(); ks
3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field in the 3-Schur basis
```

**K_kschur()**

Return the \(k\)-bounded basis called the K-\(k\)-Schur basis.

See also:

- `khomogeneous()`

**khomogeneous()**

The homogeneous basis of this algebra.

See also:

- `kHomogeneous()`
EXAMPLES:

```python
sage: kh3 = SymmetricFunctions(QQ).kBoundedSubspace(3,1).khomogeneous()
sage: TestSuite(kh3).run()
```

**kschur()**

The $k$-Schur basis of this algebra.

See also:

**kSchur()**

EXAMPLES:

```python
sage: ks3 = SymmetricFunctions(QQ).kBoundedSubspace(3,1).kschur()
sage: TestSuite(ks3).run()
```

**ksplit()**

The $k$-split basis of this algebra.

See also:

**kSplit()**

EXAMPLES:

```python
sage: ksp3 = SymmetricFunctions(QQ).kBoundedSubspace(3,1).ksplit()
sage: TestSuite(ksp3).run()
```

**realizations()**

A list of realizations of this algebra.

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).kBoundedSubspace(3,1).realizations()
[3-bounded Symmetric Functions over Rational Field with t=1 in the 3-Schur basis,
3-bounded Symmetric Functions over Rational Field with t=1 in the 3-split basis,
3-bounded Symmetric Functions over Rational Field with t=1 in the 3-bounded homogeneous basis,
3-bounded Symmetric Functions over Rational Field with t=1 in the K-3-Schur basis]
sage: SymmetricFunctions(QQ['t']).kBoundedSubspace(3).realizations()
[3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field in the 3-Schur basis,
3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field in the 3-split basis]
```

**retract**(*sym*)

Return the retract of *sym* from the ring of symmetric functions to *self*.

INPUT:

- *sym* – a symmetric function

OUTPUT:

- the analogue of the symmetric function in the $k$-bounded subspace (if possible)
EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: KB = Sym.kBoundedSubspace(3,1); KB
3-bounded Symmetric Functions over Rational Field with t=1
sage: KB.retract(s[2]+s[3])
sage: KB.retract(s[2,1,1])
Traceback (most recent call last):
... ValueError: s[2, 1, 1] is not in the image
```

class sage.combinat.sf.new_kschur.KBoundedSubspaceBases(base, t='t')

Bases: Category_realization_of_parent

The category of bases for the $k$-bounded subspace of symmetric functions.

class ElementMethods

Bases: object

```python
expand(*args, **kwargs)
```

Return the monomial expansion of `self` in $n$ variables.

INPUT:

- `n` – positive integer

OUTPUT: monomial expansion of `self` in $n$ variables

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: ks = Sym.kschur(3,1)
sage: ks[3,1].expand(2)
x0^4 + 2*x0^3*x1 + 2*x0^2*x1^2 + 2*x0*x1^3 + x1^4
sage: s = Sym.schur()
sage: ks[3,1].expand(2) == s(ks[3,1]).expand(2)
True
```

```python
hl_creation_operator(nu, t=None)
```

This is the vertex operator that generalizes Jing’s operator.

It is a linear operator that raises the degree by $|\nu|$. This creation operator is a $t$-analogue of multiplication by $s(\nu)$.

See also:

Proposition 5 in [SZ2001].

INPUT:

- `nu` – a partition or a list of integers
- `t` – (default: `None`, in which case `t` is used) an element of the base ring
EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: ks = Sym.kschur(4)
sage: s = Sym.schur()
sage: s(ks([3,1,1]).hl_creation_operator([1]))
(t-1)*s[2, 2, 1, 1] + t^2*s[3, 1, 1, 1] + (t^3+t^2-t)*s[3, 2, 1] + (t^3-t^2)*s[3, 3] + (t^4+t^3)*s[4, 1, 1] + t^4*s[4, 2] + t^5*s[5, 1]
sage: ks([3,1,1]).hl_creation_operator([1])
(t-1)*ks4[2, 2, 1, 1] + t^2*ks4[3, 1, 1, 1] + t^3*ks4[3, 2, 1] + (t^3-t^2)*ks4[3, 3] + t^4*ks4[4, 1, 1]
sage: Sym = SymmetricFunctions(QQ)
sage: ks = Sym.kschur(4,t=1)
sage: ks([3,1,1]).hl_creation_operator([1])
ks4[3, 1, 1, 1] + ks4[3, 2, 1] + ks4[4, 1, 1]
```

**is_schur_positive(*args, **kwargs)**

Return whether self is Schur positive.

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: ks = Sym.kschur(3,1)
sage: f = ks[3,2]+ks[1]
sage: f.is_schur_positive()
True
sage: f = ks[3,2]-ks[1]
sage: f.is_schur_positive()
False
sage: Sym = SymmetricFunctions(QQ['t'])
sage: ks = Sym.kschur(3)
sage: f = ks[3,2]+ks[1]
sage: f.is_schur_positive()
True
sage: f = ks[3,2]-ks[1]
sage: f.is_schur_positive()
False
```

**omega()**

Return the \( \omega \) operator on self.

At \( t = 1 \), \( \omega \) maps the \( k \)-Schur function \( s_{\lambda}^{(k)} \) to \( s_{\lambda'}^{(k)} \), where \( \lambda' \) is the \( k \)-conjugate of the partition \( \lambda \).

See also:

- **k_conjugate()**.

For generic \( t \), \( \omega \) sends \( s_{\lambda}^{(k)}[X;t] \) to \( t^d s_{\lambda'}^{(k)}[X;1/t] \), where \( d \) is the size of the core of \( \lambda \) minus the size of \( \lambda \). Most of the time, this result is not in the \( k \)-bounded subspace.

See also:

- **omega_t_inverse()**.

EXAMPLES:
sage: Sym = SymmetricFunctions(QQ)
sage: ks = Sym.kschur(3,1)
sage: ks[2,2,1,1].omega()
sage: ks3[2, 2, 2]
sage: kh = Sym.khomogeneous(3)
sage: kh[3].omega()
h3[1, 1, 1] - 2*h3[2, 1] + h3[3]
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: ks = Sym.kschur(3)
sage: ks[3,1,1].omega_t_inverse()
1/t*ks3[2, 1, 1, 1]
sage: ks[3,2].omega_t_inverse()
1/t^2*ks3[1, 1, 1, 1, 1]

omega_t_inverse()

Return the map $t \rightarrow 1/t$ composed with $\omega$ on self.

Unlike the map $\omega()$, the result of $\omega_t^{-1}()$ lives in the $k$-bounded subspace and hence will return an element even for generic $t$. For $t = 1$, $\omega()$ and $\omega_t^{-1}()$ return the same result.

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: ks3 = Sym.kschur(3)
sage: 1/t*ks3[2, 1, 1, 1]
sage: ks3[3,2].omega_t_inverse()
1/t^2*ks3[1, 1, 1, 1, 1]

scalar($x$, $zee=None$)

Return standard scalar product between self and $x$.

INPUT:

- $x$ – element of the ring of symmetric functions over the same base ring as self
- $zee$ – an optional function on partitions giving the value for the scalar product between $p_\mu$ and $p_\mu$ (default is to use the standard $zee()$ function)

See also:

scalar()

EXAMPLES:

sage: Sym = SymmetricFunctions(QQ['t'])
sage: ks3 = Sym.kschur(3)
sage: ks3[3,2,1].scalar( ks3[2,2,2] )
t^3 + t
dks3 = Sym.kBoundedQuotient(3).dks()
sage: [ks3[3,2,1].scalar(dks3(la)) for la in Partitions(6, max_part=3)]
[0, 1, 0, 0, 0, 0]
dks3 = Sym.kBoundedQuotient(3,t=1).dks()
sage: [ks3[2,2,2].scalar(dks3(la)) for la in Partitions(6, max_part=3)]
[0, t - 1, 0, 1, 0, 0]
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(continued from previous page)

```python
sage: [ks3[2,2,2].scalar(dks3(la)) for la in Partitions(6, max_part=3)]
[0, 0, 0, 1, 0, 0, 0]
sage: kH = Sym.khomogeneous(4)
sage: kH([2,2,1]).scalar(ks3[2,2,1])
3
```

class ParentMethods
   Bases: object

   an_element()
   Return an element of self.
   EXAMPLES:

   ```python
   sage: SymmetricFunctions(QQ['t']).kschur(3).an_element()
   2*ks3[] + 2*ks3[1] + 3*ks3[2]
   ```

   antipode(element)
   Return the antipode on self by lifting to the space of symmetric functions, computing the antipode, and then converting to self.parent(). This is only the antipode for \( t = 1 \) and for other values of \( t \) the result may not be in the space where the \( k \)-Schur functions live.
   INPUT:
   • element – an element in a basis of the ring of symmetric functions
   EXAMPLES:

   ```python
   sage: Sym = SymmetricFunctions(QQ)
sage: ks3 = Sym.kschur(3,1)
sage: ks3[3,2].antipode()
-ks3[1, 1, 1, 1, 1]
sage: ks3.antipode(ks3[3,2])
-ks3[1, 1, 1, 1, 1]
```

coprodut(element)
Return the coproduct operation on element.
The coproduct is first computed on the homogeneous basis if \( t = 1 \) and on the Hall-Littlewood \( Qp \) basis otherwise. The result is computed then converted to the tensor squared of self.parent().
INPUT:
• element – an element in a basis of the ring of symmetric functions
EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: ks3 = Sym.kschur(3,1)
sage: ks3[2,1].coproduct()
sage: h3 = Sym.khomogeneous(3)
sage: h3[2,1].coproduct()
sage: ks3t = SymmetricFunctions(FractionField(QQ['t'])).kschur(3)
sage: ks3t[2,1].coproduct()
```

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sage: ks3t[3,1].coproduct()
sage: h3.coproduct(h3[2, 1])

counit\(\text{element}\)

Return the counit of \text{element}.

The counit is the constant term of \text{element}.

INPUT:
• \text{element} – an element in a basis of the ring of symmetric functions

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: ks3 = Sym.kschur(3,1)
sage: f = 2*ks3[2,1] + 3*ks3[]
sage: f.counit()
3
sage: ks3.counit(f)
3
```

degree_on_basis\(\text{b}\)

Return the degree of the basis element indexed by \text{b}.

INPUT:
• \text{b} – a partition

EXAMPLES:

```
sage: ks3 = SymmetricFunctions(QQ).kschur(3,1)
sage: ks3.degree_on_basis(Partition([3,2]))
5
```

one_basis()

Return the basis element indexing 1.

EXAMPLES:

```
sage: ks3 = SymmetricFunctions(QQ).kschur(3,1)
sage: ks3.one()  # indirect doctest
ks3[]
```

transition_matrix\(\text{other}, n\)

Return the degree \text{n} transition matrix between \self and \text{other}.

INPUT:
• \text{other} – a basis in the ring of symmetric functions
• \text{n} – a positive integer
The entry in the \(i^{th}\) row and \(j^{th}\) column is the coefficient obtained by writing the \(i^{th}\) element of the basis of \texttt{self} in terms of the basis \texttt{other}, and extracting the \(j^{th}\) coefficient.

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ); s = Sym.schur()
sage: ks3 = Sym.kschur(3,1)
sage: ks3.transition_matrix(s,5)
[1 1 0 0 0 0]
[0 1 0 1 0 0]
[0 0 1 0 1 0]
[0 0 0 1 0 1]
[0 0 0 0 1 1]
```

```python
sage: Sym = SymmetricFunctions(QQ['t'])
sage: s = Sym.schur()
sage: ks = Sym.kschur(3)
sage: ks.transition_matrix(s,5)
[t^2  t  1  0  0  0]
[   0  t  0  1  0  0]
[   0  0  t  0  1  0]
[   0  0  0  t  0  1]
[   0  0  0  0  t^2  t]
```

**super_categories()**

The super categories of \texttt{self}.

**EXAMPLES:**

```python
sage: from sage.combinat.sf.new_kschur import KBoundedSubspaceBases
sage: KB = Sym.kBoundedSubspace(3)
sage: KBB = KBoundedSubspaceBases(KB); KBB
Category of k bounded subspace bases of 3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field
sage: KBB.super_categories()
```

**class** \texttt{sage.combinat.sf.new_kschur.K_kSchur}(kBoundedRing)

This class implements the basis of the \(k\)-bounded subspace called the K-\(k\)-Schur basis.

See [Morse2011], [LamSchillingShimozono2010].

**REFERENCES:**

**K_k_Schur_non_commutative_variables(la)**

Return the K-\(k\)-Schur function, as embedded inside the affine zero Hecke algebra.

**INPUT:**

- \(la\) – A \(k\)-bounded Partition
OUTPUT:

• An element of the affine zero Hecke algebra.

EXAMPLES:

```python
sage: g = SymmetricFunctions(QQ).kBoundedSubspace(3,1).K_kschur()
sage: g.K_k_Schur_non_commutative_variables([2,1])
T[3,1,0] + T[1,2,0] + T[3,2,0] + T[0,1,0] + T[2,0,1] + T[0,3,0] + T[2,0,3] +
sage: g.K_k_Schur_non_commutative_variables([])
1
sage: g.K_k_Schur_non_commutative_variables([4,1])
Traceback (most recent call last):
  ...
ValueError: Partition should be 3-bounded
```

`homogeneous_basis_noncommutative_variables_zero_Hecke(la)`

Return the homogeneous basis element indexed by \( la \), viewed as an element inside the affine zero Hecke algebra. For the code, see method `_homogeneous_basis`.

INPUT:

• \( la \) – A \( k \)-bounded partition

OUTPUT:

• An element of the affine zero Hecke algebra.

EXAMPLES:

```python
sage: g = SymmetricFunctions(QQ).kBoundedSubspace(3,1).K_kschur()
sage: g.homogeneous_basis_noncommutative_variables_zero_Hecke([2,1])
T[2,1,0] + T[3,1,0] + T[1,2,0] + T[3,2,0] + T[0,1,0] + T[2,0,1] + T[0,3,0] + T[2,0,3] +
sage: g.homogeneous_basis_noncommutative_variables_zero_Hecke([])
1
```

`lift(x)`

Return the lift of a \( k \)-bounded symmetric function.

INPUT:

• \( x \) – An expression in the K-\( k \)-Schur basis. Equivalently, \( x \) can be a
  \( k \)-bounded partition (then \( x \) corresponds to the basis element indexed by \( x \))

OUTPUT:

• A symmetric function.

EXAMPLES:

```python
sage: g = SymmetricFunctions(QQ).kBoundedSubspace(3,1).K_kschur()
sage: g.lift([2,1])
sage: g.lift([])
h[]
sage: g.lift([4,1])
```

(continues on next page)
Traceback (most recent call last):
...
TypeError: do not know how to make x (= [4, 1]) an element of self (=3-bounded, Symmetric Functions over Rational Field with t=1 in the K-3-Schur basis)

**product** *(x, y)*

Return the product of the two K-*k*-Schur functions.

**INPUT:**
- *x, y* – elements of the *k*-bounded subspace, in the K-*k*-Schur basis.

**OUTPUT:**
- An element of the *k*-bounded subspace, in the K-*k*-Schur basis

**EXAMPLES:**

```python
sage: g = SymmetricFunctions(QQ).kBoundedSubspace(3,1).K_kschur()
sage: g.product(g([2,1]), g[1])
sage: g.product(g([2,1]), g([]))
Kks3[2, 1]
```

**retract** *(x)*

Return the retract of a symmetric function.

**INPUT:**
- *x* – A symmetric function.

**OUTPUT:**
- A *k*-bounded symmetric function in the K-*k*-Schur basis.

**EXAMPLES:**

```python
sage: g = SymmetricFunctions(QQ).kBoundedSubspace(3,1).K_kschur()
sage: m = SymmetricFunctions(QQ).m()
sage: g.retract(m[2,1])
sage: g.retract(m([]))
Kks3[]
```

**class** *sage.combinat.sf.new_kschur.kHomogeneous*(kBoundedRing)

**Bases:** *CombinatorialFreeModule*

Space of *k*-bounded homogeneous symmetric functions.

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: kH = Sym.khomogeneous(3)
sage: kH[2]
h3[2]
sage: kH[2].lift()
h[2]
```
class sage.combinat.sf.new_kschur.KSchur(kBoundedRing)

Bases: CombinatorialFreeModule

Space of \(k\)-Schur functions.

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ['t'])
sage: KB = Sym.kBoundedSubspace(3); KB
3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational
˓→Field
```

The \(k\)-Schur function basis can be constructed as follows:

```python
sage: ks3 = KB.kschur(); ks3
3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational
˓→Field in the 3-Schur basis
```

We can convert to any basis of the ring of symmetric functions and, whenever it makes sense, also the other way round:

```python
sage: s = Sym.schur()
sage: s(ks3([3,2,1]))
s[3, 2, 1] + t*s[4, 1, 1] + t*s[4, 2] + t^2*s[5, 1]
sage: t = Sym.base_ring().gen()
sage: ks3(s([3,2,1]) + t*s([4,1,1]) + t*s([4,2]) + t^2*s([5,1]))
ks3[3, 2, 1]
sage: s(ks3([2,1,1]))
s[2, 1, 1] + t*s[3, 1]
sage: ks3(s([2,1,1]) + t*s([3,1]))
ks3[2, 1, 1]
```

\(k\)-Schur functions are indexed by partitions with first part \(\leq k\). Constructing a \(k\)-Schur function for a larger partition raises an error:

```python
sage: ks3([4,3,2,1]) #
Traceback (most recent call last):
...
TypeError: do not know how to make x (= [4, 3, 2, 1]) an element of self (=3-
˓→bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational
˓→Field in the 3-Schur basis)
```

Similarly, attempting to convert a function that is not in the linear span of the \(k\)-Schur functions raises an error:

```python
sage: ks3(s([4]))
Traceback (most recent call last):
...
ValueError: s[4] is not in the image
```

Note that the product of \(k\)-Schur functions is not guaranteed to be in the space spanned by the \(k\)-Schurs. In general, we only have that a \(k\)-Schur times a \(j\)-Schur function is in the \((k+j)\)-bounded subspace. The multiplication of two \(k\)-Schur functions thus generally returns the product of the lift of the functions to the ambient symmetric function space. If the result happens to lie in the \(k\)-bounded subspace, then the result is cast into the \(k\)-Schur basis:
Because the target space of the product of a $k$-Schur and a $j$-Schur has several possibilities, the product of a $k$-Schur and $j$-Schur function is not implemented for distinct $k$ and $j$. Let us show how to get around this ‘manually’:

```python
sage: ks3 = Sym.kBoundedSubspace(3).kschur()
sage: ks2([2,1]) * ks3([3,1])
Traceback (most recent call last):
  ...TypeError: unsupported operand parent(s) for *: '2-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field in the 2-Schur basis' and '3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field in the 3-Schur basis'
```

The workaround:

```python
sage: f = s(ks2([2,1])) * s(ks3([3,1])); f
s[3, 2, 1, 1] + s[3, 2, 2] + (t+1)*s[3, 3, 1] + s[4, 1, 1, 1]
+ (2*t+2)*s[4, 2, 1] + (t^2+t+1)*s[4, 3] + (2*t+1)*s[5, 1, 1]
+ (t^2+2*t+1)*s[5, 2] + (t^2+2*t)*s[6, 1] + t^2*s[7]
```

or:

```python
sage: f = ks2[2,1].lift() * ks3[3,1].lift()
sage: ks5 = Sym.kBoundedSubspace(5).kschur()
sage: ks5(f) # The product of a 'ks2' with a 'ks3' is a 'ks5'.
ks[3, 2, 1, 1] + ks5[3, 2, 2] + (t+1)*ks5[3, 3, 1] + ks5[4, 1, 1, 1]
+ (t+2)*ks5[4, 2, 1] + (t^2+t+1)*ks5[4, 3] + (t+1)*ks5[5, 1, 1] + ks5[5, 2]
```

For other technical reasons, taking powers of $k$-Schur functions is not implemented, even when the answer is still in the $k$-bounded subspace:

```python
sage: ks2([1])^2
Traceback (most recent call last):
  ...TypeError: unsupported operand parent(s) for ^: '2-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field in the 2-Schur basis' and 'Integer Ring'
```

**Todo:** Get rid of said technical “reasons”.

However, at $t = 1$, the product of $k$-Schur functions is in the span of the $k$-Schur functions always. Below are some examples at $t = 1$

```python
sage: ks3 = Sym.kBoundedSubspace(3, t=1).kschur(); ks3
3-bounded Symmetric Functions over Univariate Polynomial Ring in t over Rational Field
```
Field with t=1 in the 3-Schur basis

```sage
s = SymmetricFunctions(ks3.base_ring()).schur()
sage: ks3(s([3]))
ks3[3]
sage: ks3([3,2,1])
s[3, 2, 1] + s[4, 1, 1] + s[4, 2] + s[5, 1]
sage: ks3([2,1])^2  # taking powers works for t=1
ks3[2, 2, 1, 1] + ks3[2, 2, 2] + ks3[3, 1, 1, 1]
```

**product_on_basis** *(left, right)*

Take the product of two \(k\)-Schur functions.

If \(t \neq 1\), then take the product by lifting to the Schur functions and then retracting back into the \(k\)-bounded subspace (if possible).

If \(t = 1\), then the product calls `_product_on_basis_via_rectangles()`.

**INPUT:**

- `left, right` – partitions

**OUTPUT:**

- an element of the \(k\)-Schur functions

**EXAMPLES:**

```sage
sage: Sym = SymmetricFunctions(QQ['t'])
sage: ks3 = Sym.kschur(3,1)
sage: kH = Sym.khomogeneous(3)
sage: ks3([2,1,1]) + ks3[2, 2] + ks3[3, 1]
sage: ks3([3,3,2,2,1,1,1])^2
ks3[3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

**class**  

```sage
sage.combinat.sf.new_kschur.kSplit(kBoundedRing)
```

Bases: `CombinatorialFreeModule`

The \(k\)-split basis of the space of \(k\)-bounded-symmetric functions

Fix \(k\) a positive integer and \(t\) an element of the base ring.

The \(k\)-split functions are a basis for the space of \(k\)-bounded symmetric functions that also have the bases

\[
\{Q^{k}_{\lambda}(X; t)\}_{\lambda_1 \leq k} = \{s^{(k)}_{\lambda}(X; t)\}_{\lambda_1 \leq k}
\]
where $Q'_\lambda[X; t]$ are the Hall-Littlewood symmetric functions (using the notation of [MAC]) and $s^{(k)}_\lambda[X; t]$ are the $k$-Schur functions. If $t$ is not a root of unity, then

$$\{s_\lambda[X/(1-t)]\}_{\lambda \leq k}$$

is also a basis of this space.

The $k$-split basis has the property that $Q'_\lambda[X; t]$ expands positively in the $k$-split basis and the $k$-split basis conjecturally expands positively in the $k$-Schur functions. See [LLMSSZ] p. 81.

The $k$-split basis is defined recursively using the Hall-Littlewood creation operator defined in [SZ2001]. If a partition $\mu\nu$ is the concatenation (as lists) of a partition $\mu$ and $\nu$ where $\mu$ has maximal hook length equal to $k$ then $ksp(\mu\nu) = ksp(\mu).hl_creation_operator(\mu)$. If the hook length of $\mu\nu$ is less than or equal to $k$, then $ksp(\mu\nu)$ is equal to the Schur function indexed by $\mu\nu$.

**EXAMPLES:**

```python
sage: Symt = SymmetricFunctions(QQ['t'].fraction_field())
sage: kBS3 = Symt.kBoundedSubspace(3)
sage: ks3 = kBS3.kschur()
sage: ksp3 = kBS3.ksplit()
sage: ks3(ksp3[2,1,1])
ks3[2, 1, 1] + t*ks3[2, 2]
sage: ksp3(ks3[2,1,1])
ksp3[2, 1, 1] - t*ksp3[2, 2]
sage: ksp3[2,1]*ksp3[1]
s[2, 1, 1] + s[2, 2] + s[3, 1]
sage: ksp3[2,1].hl_creation_operator([1])
t*ksp3[2, 1, 1] + (-t^2+t)*ksp3[2, 2]
```

```python
sage: Qp = Symt.hall_littlewood().Qp()
sage: ksp4 = kBS4.ksplit()
sage: ksp4(ksp3([3,2,1]))
ksp4[3, 2, 2, 1] - t*ksp4[3, 3, 1, 1] + t*ksp4[3, 3, 2]
```

---

**5.1.294 Non-symmetric Macdonald Polynomials**

```python
class sage.combinat.sf.ns_macdonald.AugmentedLatticeDiagramFilling(l, pi=None):
    Bases: CombinatorialObject

    EXAMPLES:
    ```
```
sage: a == loads.dumps(a)
True

are_attacking(i, j, ii, jj)

Return True if the boxes (i, j) and (ii, jj) in self are attacking.

EXAMPLES:

```sage
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[],[],[5,5]]]
sage: all( a.are_attacking(i,j,ii,jj) for (i,j),(ii,jj) in a.attacking_boxes())
True
sage: a.are_attacking(1,1,3,2)
False
```

attacking_boxes()

Return a list of pairs of boxes in self that are attacking.

EXAMPLES:

```sage
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[],[],[5,5]]]

[[(1, 1), (2, 1)),
 ((1, 1), (3, 1)),
 ((1, 1), (6, 1)),
 ((1, 1), (2, 0)),
 ((1, 1), (3, 0))]
```

boxes()

Return a list of the coordinates of the boxes of self, including the ‘basement row’.

EXAMPLES:

```sage
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[],[],[5,5]]]

[(1, 1),
 (1, 2),
 (2, 1),
 (3, 1),
 (3, 2),
 (3, 3),
 (6, 1),
 (6, 2),
 (1, 0),
 (2, 0),
 (3, 0),
 (4, 0),
 (5, 0),
 (6, 0)]
```

coeff(q, t)

Return the coefficient in front of self in the HHL formula for the expansion of the non-symmetric Macdonald polynomial E(self.shape()).

EXAMPLES:
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: q,t = var('q,t')
sage: a.coeff(q,t)
(t - 1)^4/((q^2*t^3 - 1)^2*(q*t^2 - 1)^2)

**coeff_integral**(q, t)

Return the coefficient in front of `self` in the HHL formula for the expansion of the integral non-symmetric Macdonald polynomial E(self.shape())

EXAMPLES:

```python
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: q,t = var('q,t')
sage: a.coeff_integral(q,t)
(q^2*t^3 - 1)^2*(q*t^2 - 1)^2*(t - 1)^4
```

coinv()

Return `self`’s co-inversion statistic.

EXAMPLES:

```python
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: a.coinv()
2
```

descents()

Return a list of the descents of `self`.

EXAMPLES:

```python
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: a.descents()
[(1, 2), (3, 2)]
```

inv()

Return `self`’s inversion statistic.

EXAMPLES:

```python
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: a.inv()
15
```

inversions()

Return a list of the inversions of `self`.

EXAMPLES:

```python
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: a.inversions()[:5]
[((6, 2), (3, 2)),
 ((1, 2), (6, 1)),
 ((1, 2), (3, 1)),
 ((1, 2), (2, 1)),
 ((6, 1), (3, 1))]
```
is_non_attacking()

Return True if self is non-attacking.

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[1],[1],[5,5]])
sage: a.is_non_attacking()
True
sage: a = AugmentedLatticeDiagramFilling([[1, 1, 1], [2, 3], [3]])
sage: a.is_non_attacking()
False
sage: a = AugmentedLatticeDiagramFilling([[2,2],[1]])
sage: a.is_non_attacking()
False
sage: pi = Permutation([2,1]).to_permutation_group_element()
sage: a = AugmentedLatticeDiagramFilling([[2,2],[1]],pi)
sage: a.is_non_attacking()
True
```

maj()

Return the major index of self.

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[1],[1],[5,5]])
sage: a.maj()
3
```

permuted_filling(sigma)

EXAMPLES:

```
sage: pi=Permutation([2,1,4,3]).to_permutation_group_element()
sage: fill=[[2],[1,2,3],[1],[3,1]]
sage: AugmentedLatticeDiagramFilling(fill).permuted_filling(pi)
[[2, 1], [1, 2, 1, 4], [4], [3, 4, 2]]
```

reading_order()

Return a list of coordinates of the boxes in self, starting from the top right, and reading from right to left. Note that this includes the ‘basement row’ of self.

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[1],[1],[5,5]])
sage: a.reading_order()
[(3, 3),
 (6, 2),
 (3, 2),
 (1, 2),
 (6, 1),
 (3, 1),]```
reading_word()

Return the reading word of self, obtained by reading the boxes entries of self from right to left, starting in the upper right.

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[]],[[]],[5,5]])
sage: a.reading_word()
word: 25465321
```

shape()

Return the shape of self.

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[]],[[]],[5,5]])
sage: a.shape()
[2, 1, 3, 0, 0, 2]
```

weight()

Return the weight of self.

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[]],[[]],[5,5]])
sage: a.weight()
[1, 2, 1, 1, 2, 1]
```

```
from sage.combinat.sf.ns_macdonald import E

E(mu, q=None, t=None, pi=None)

Return the non-symmetric Macdonald polynomial in type A corresponding to a shape mu, with basement permuted according to pi.

Note that if both q and t are specified, then they must have the same parent.

REFERENCE:


See also:

NonSymmetricMacdonaldPolynomials for a type free implementation where the polynomials are constructed recursively by the application of intertwining operators.

EXAMPLES:

```
```

5.1. Comprehensive Module List
sage: from sage.combinat.sf.ns_macdonald import E
sage: E([0,0,0])
1
sage: E([1,0,0])
x0
sage: E([0,1,0])
(q^t - 1)/(q^t - 1)*x0 + (t - 1)/(q^t - 1)*x1 + x2
sage: E([1,1,0])
x0*x1
sage: E([1,0,1])
(t - 1)/(q^t^2 - 1)*x0^2 + x0*x2
sage: E([0,1,1])
(q^t^2 - 1)*x0^2 + (q^t^2 - 1)*x0*x1 + x1^2 + (q^t^2 - 2*q^t + q)/(q^t^3 - q^t^2 - q^t + 1)*x0*x2 + (t^2 - 2*t + 1)*x0*x1
sage: E([2,0,0])
(q^t^3 - q^t^2 - q^t + 1)*x0^2 + (q^t^2 - 2*q^t + q)/(q^t^3 - q^t^2 - q^t + 1)*x0*x1 + (q^t^2 - 2*q^t + q)/(q^t^3 - q^t^2 - q^t + 1)*x0*x2
sage: E([0,2,0])
(q^t^3 - q^t^2 - q^t + 1)*x0^2 + (q^t^4 - q^t^3 - q^t^2 - q^t + 1)*x0*x1 + (q^t^2 - 2*q^t + q)/(q^t^3 - q^t^2 - q^t + 1)*x0*x2 + (q^t^2 - 2*q^t + q)/(q^t^3 - q^t^2 - q^t + 1)*x0*x2
sage: E_integral(mu, q=None, t=None, pi=None)
Return the integral form for the non-symmetric Macdonald polynomial in type A corresponding to a shape mu.
Note that if both q and t are specified, then they must have the same parent.
REFERENCE:
EXAMPLES:
\[+ 1)^\ast x_0^\ast x_1 + (t^2 - 2^\ast t + 1)^\ast x_1^\ast x_2 + (q^\ast t^3 - q^\ast t^2 - q^\ast t + q)^\ast x_0^\ast x_2 + (q^\ast t^2 - q^\ast t + q)^\ast x_1^\ast x_2\]

`sage.combinat.sf.ns_macdonald.Ht(mu, q=None, t=None, pi=None)`

Return the symmetric Macdonald polynomial using the Haiman, Haglund, and Loehr formula.

Note that if both \( q \) and \( t \) are specified, then they must have the same parent.

**REFERENCE:**


**EXAMPLES:**

```python
sage: from sage.combinat.sf.ns_macdonald import Ht
sage: Ht = SymmetricFunctions(QQ['q','t'].fraction_field()).macdonald().Ht()
sage: Ht([0,0,1])
x0 + x1 + x2
sage: Ht([1]).expand(3)
x0 + x1 + x2
sage: Ht([0,0,2])
x0^2 + (q + 1)*x0*x1 + x1^2 + (q + 1)*x0*x2 + (q + 1)*x1*x2 + x2^2
sage: Ht([2]).expand(3)
x0^2 + (q + 1)*x0*x1 + x1^2 + (q + 1)*x0*x2 + (q + 1)*x1*x2 + x2^2
```

**class** `sage.combinat.sf.ns_macdonald.LatticeDiagram(l, copy=True)`

**Bases:** `CombinatorialObject`

**a(i,j)**

Return the length of the arm of the box \((i,j)\) in `self`.

**EXAMPLES:**

```python
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.a(5,2)
3
```

**arm(i,j)**

Return the arm of the box \((i,j)\) in `self`.

**EXAMPLES:**

```python
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.arm(5,2)
[(1, 2), (3, 2), (8, 1)]
```

**arm_left(i,j)**

Return the left arm of the box \((i,j)\) in `self`.

**EXAMPLES:**

```python
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.arm_left(5,2)
[(1, 2), (3, 2)]
```
```
arm_right(i, j)

Return the right arm of the box (i, j) in self.

EXAMPLES:

```
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.arm_right(5,2)
[(8, 1)]
```

```
boxes()

EXAMPLES:

```
sage: a = LatticeDiagram([3,0,2])
sage: a.boxes()
[(1, 1), (1, 2), (1, 3), (3, 1), (3, 2)]
sage: a = LatticeDiagram([2, 1, 3, 0, 0, 2])
sage: a.boxes()
[(1, 1), (1, 2), (2, 1), (3, 1), (3, 2), (3, 3), (6, 1), (6, 2)]
```

```
boxes_same_and_lower_right(ii, jj)

Return a list of the boxes of self that are in row jj but not identical with (ii, jj), or lie in the row jj - 1 (the row directly below jj; this might be the basement) and strictly to the right of (ii, jj).

EXAMPLES:

```
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[[]],[[]],[5,5]])
sage: a = a.shape()
sage: a.boxes_same_and_lower_right(1,1)
[(2, 1), (3, 1), (6, 1), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)]
sage: a.boxes_same_and_lower_right(1,2)
[(3, 2), (6, 2), (2, 1), (3, 1), (6, 1)]
sage: a.boxes_same_and_lower_right(3,3)
[(6, 2)]
sage: a.boxes_same_and_lower_right(2,3)
[(3, 3), (3, 2), (6, 2)]
```

```
flip()

Return the flip of self, where flip is defined as follows. Let r = max(self). Then self.flip()[i] = r - self[i].

EXAMPLES:

```
sage: a = LatticeDiagram([3,0,2])
sage: a.flip()
[0, 3, 1]
```

```
l(i, j)

Return self[i] - j.

EXAMPLES:

```
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.l(5,2)
1
```
**leg**(i, j)
Return the leg of the box (i, j) in self.

**EXAMPLES:**

```
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.leg(5,2)
[(5, 3)]
```

**size()**
Return the number of boxes in self.

**EXAMPLES:**

```
sage: a = LatticeDiagram([3,1,2,4,3,0,4,2,3])
sage: a.size()
22
```

class **sage.combinat.sf.ns_macdonald.NonattackingBacktracker**(shape, pi=None)
Bases: **GenericBacktracker**

**EXAMPLES:**

```
sage: from sage.combinat.sf.ns_macdonald import NonattackingBacktracker
sage: n = NonattackingBacktracker(LatticeDiagram([0,1,2]))
sage: n._ending_position
(3, 2)
sage: n._initial_state
(2, 1)
```

**get_next_pos**(ii, jj)

**EXAMPLES:**

```
sage: from sage.combinat.sf.ns_macdonald import NonattackingBacktracker
sage: a = AugmentedLatticeDiagramFilling([[1,6],[2],[3,4,2],[],[],[5,5]])
sage: n = NonattackingBacktracker(a.shape())
sage: n.get_next_pos(1, 1)
(2, 1)
sage: n.get_next_pos(6, 1)
(1, 2)
sage: n = NonattackingBacktracker(LatticeDiagram([2,2,2]))
sage: n.get_next_pos(3, 1)
(1, 2)
```

**sage.combinat.sf.ns_macdonald.NonattackingFillings**(shape, pi=None)

Returning the finite set of nonattacking fillings of a given shape.

**EXAMPLES:**

```
sage: NonattackingFillings([0,1,2])
Nonattacking fillings of [0, 1, 2]
sage: NonattackingFillings([0,1,2]).list()
[[[1], [2, 1], [3, 2, 1]],
 [[[1], [2, 1], [3, 2, 2]],
 [[1], [2, 1], [3, 2, 3]],

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```

---

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class sage.combinat.sf.ns_macdonald.NonattackingFillings_shape(shape, pi=None)

Bases: Parent, UniqueRepresentation

EXAMPLES:

```sage
n = NonattackingFillings([0,1,2])
n == loads(dumps(n))
True
```

flip()

Return the nonattacking fillings of the flipped shape.

EXAMPLES:

```sage
NonattackingFillings([0,1,2]).flip()
Nonattacking fillings of [2, 1, 0]
```

5.1.295 Orthogonal Symmetric Functions

AUTHORS:

• Travis Scrimshaw (2013-11-10): Initial version

class sage.combinat.sf.orthogonal.SymmetricFunctionAlgebra_orthogonal(Sym)

Bases: SymmetricFunctionAlgebra_generic

The orthogonal symmetric function basis (or orthogonal basis, to be short).

The orthogonal basis \( \{ o_\lambda \} \) where \( \lambda \) is taken over all partitions is defined by the following change of basis with the Schur functions:

\[
s_\lambda = \sum_{\mu} \left( \sum_{\nu \in H} c^\lambda_{\mu\nu} \right) o_\mu
\]

where \( H \) is the set of all partitions with even-width rows and \( c^\lambda_{\mu\nu} \) is the usual Littlewood-Richardson (LR) coefficients. By the properties of LR coefficients, this can be shown to be a upper unitriangular change of basis.

Note: This is only a filtered basis, not a \( \mathbb{Z} \)-graded basis. However this does respect the induced \( (\mathbb{Z}/2\mathbb{Z}) \)-grading.

INPUT:

• Sym – an instance of the ring of the symmetric functions

REFERENCES:
EXAMPLES:

Here are the first few orthogonal symmetric functions, in various bases:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: o = Sym.o()
sage: e = Sym.e()
sage: h = Sym.h()
sage: p = Sym.p()
sage: s = Sym.s()
sage: m = Sym.m()

sage: p(o([1]))
p[1]
sage: m(o([1]))
m[1]
sage: e(o([1]))
e[1]
sage: h(o([1]))
h[1]
sage: s(o([1]))
s[1]

sage: p(o([2]))
-p[] + 1/2*p[1, 1] + 1/2*p[2]
sage: m(o([2]))
-m[] + m[1, 1] + m[2]
sage: e(o([2]))
-e[] + e[1, 1] - e[2]
sage: h(o([2]))
-h[] + h[2]
sage: s(o([2]))
-s[] + s[2]

sage: p(o([3]))
sage: m(o([3]))
-m[1] + m[1, 1, 1] + m[2, 1] + m[3]
sage: e(o([3]))
sage: h(o([3]))
-h[1] + h[3]
sage: s(o([3]))
-s[1] + s[3]
```

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Some examples of conversions the other way:

\begin{verbatim}
  sage: o(h[3])
o[1] + o[3]
  sage: o(e[3])
o[1, 1, 1]
  sage: o(m[2,1])
o[1] - 2*o[1, 1, 1] + o[2, 1]
  sage: o(p[3])
o[1, 1, 1] - o[2, 1] + o[3]
\end{verbatim}

Some multiplication:

\begin{verbatim}
  sage: o([2]) * o([1,1])
  sage: o([2,1,1]) * o([2])
\end{verbatim}

Examples of the Hopf algebra structure:

\begin{verbatim}
  sage: o([1]).antipode()
o[1]
  sage: o([1,1]).antipode()
o[1]
  sage: o([1]).coproduct()
o[] # o[1] + o[1] # o[]
  sage: o([2]).coproduct()
  sage: o([1]).counit()
 0
  sage: o.one().counit()
1
\end{verbatim}
5.1.296 Symmetric functions defined by orthogonality and triangularity

One characterization of Schur functions is that they are upper triangularly related to the monomial symmetric functions and orthogonal with respect to the Hall scalar product. We can use the class `SymmetricFunctionAlgebra_orthotriang` to obtain the Schur functions from this definition.

```plaintext
sage: from sage.combinat.sf.sfa import zee
sage: from sage.combinat.sf.orthotriang import SymmetricFunctionAlgebra_orthotriang
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.m()
sage: s = SymmetricFunctionAlgebra_orthotriang(Sym, m, zee, 's', 'Schur functions')
sage: s([2,1,1])^2
sage: s2 = SymmetricFunctions(QQ).s()
sage: s2([2,1,1])^2
```

class `sage.combinat.sf.orthotriang.SymmetricFunctionAlgebra_orthotriang`(
    Sym, base, scalar, prefix, basis_name, leading_coeff=None)

Initialization of the symmetric function algebra defined via orthotriangular rules.

INPUT:

- `self` – a basis determined by an orthotriangular definition
- `Sym` – ring of symmetric functions
- `base` – an instance of a basis of the ring of symmetric functions (e.g. the Schur functions)
- `scalar` – a function zee on partitions. The function zee determines the scalar product on the power sum basis with normalization \(<p_\mu, p_\nu> = zee(\mu)\).
- `prefix` – the prefix used to display the basis
- `basis_name` – the name used for the basis

Note: The base ring is required to be a \(\mathbb{Q}\)-algebra for this method to be usable, since the scalar product is defined by its values on the power sum basis.

EXAMPLES:

```plaintext
sage: from sage.combinat.sf.sfa import zee
sage: from sage.combinat.sf.orthotriang import SymmetricFunctionAlgebra_orthotriang
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.m()
sage: s = SymmetricFunctionAlgebra_orthotriang(Sym, m, zee, 's', 'Schur'); s
Symmetric Functions over Rational Field in the Schur basis
```

class `Element`

    Bases: `SymmetricFunctionAlgebra_generic.Element`
**product**(*left*, *right*)

Return *left* * right by converting both to the base and then converting back to *self*.

**INPUT:**
- *self* – a basis determined by an orthotriangular definition
- *left*, *right* – elements in *self*

**OUTPUT:**
- the expansion of the product of *left* and *right* in the basis *self*.

**EXAMPLES:**
```python
sage: from sage.combinat.sf.sfa import zee
sage: from sage.combinat.sf.orthotriang import SymmetricFunctionAlgebra_.orthotriang
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.m()
sage: s = SymmetricFunctionAlgebra_orthotriang(Sym, m, zee, 's', 'Schur',
                                          'functions')
sage: s([1])*s([2,1])  # indirect doctest
s[2, 1, 1] + s[2, 2] + s[3, 1]
```

### 5.1.297 Power sum symmetric functions

**class** `sage.combinat.sf.powersum.SymmetricFunctionAlgebra_power(Sym)`

**Bases:** `SymmetricFunctionAlgebra_multiplicative`

A class for methods associated to the power sum basis of the symmetric functions

**INPUT:**
- *self* – the power sum basis of the symmetric functions
- *Sym* – an instance of the ring of symmetric functions

**class** `Element`

**Bases:** `Element`

**adams_operation**(*n*)

Return the image of the symmetric function *self* under the *n*-th Frobenius operator.

The *n*-th Frobenius operator *f*ₙ is defined to be the map from the ring of symmetric functions to itself that sends every symmetric function *P(x₁, x₂, x₃, ...)* to *P(x₁ⁿ, x₂ⁿ, x₃ⁿ, ...)*. This operator *f*ₙ is a Hopf algebra endomorphism, and satisfies

\[ fₙm(λ₁, λ₂, λ₃, ...) = m(nλ₁, nλ₂, nλ₃, ...) \]

for every partition \((λ₁, λ₂, λ₃, ...)\) (where *m* means the monomial basis). Moreover, *f*ₙ\((p_r) = p_{nr}\) for every positive integer *r* (where *p_k* denotes the *k*-th powersum symmetric function).

The *n*-th Frobenius operator is also called the *n*-th Frobenius endomorphism. It is not related to the Frobenius map which connects the ring of symmetric functions with the representation theory of the symmetric group.

The *n*-th Frobenius operator is also the *n*-th Adams operator of the Λ-ring of symmetric functions over the integers.
The $n$-th Frobenius operator can also be described via plethysm: Every symmetric function $P$ satisfies
\[ f_n(P) = p_n \circ P = P \circ p_n, \]
where $p_n$ is the $n$-th powersum symmetric function, and $\circ$ denotes (outer) plethysm.

**INPUT:**
- $n$ – a positive integer

**OUTPUT:**

The result of applying the $n$-th Frobenius operator (on the ring of symmetric functions) to `self`.

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: p = Sym.p()
sage: p[3].frobenius(2)
p[6]
sage: p[4,2,1].frobenius(3)
p[12, 6, 3]
sage: p([]).frobenius(4)
p[]
sage: p[3].frobenius(1)
p[3]
sage: (p([3]) - p([2]) + p([])).frobenius(3)
```

See also:

`plethysm()`

`eval_at_permutation_roots(rho)`
Evaluate at eigenvalues of a permutation matrix.
Evaluate an element of the power sum basis at the eigenvalues of a permutation matrix with cycle structure $\rho$.

This function evaluates an element at the roots of unity
\[ \Xi_{\rho_1}, \Xi_{\rho_2}, \ldots, \Xi_{\rho_\ell} \]
where
\[ \Xi_m = 1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1} \]
and $\zeta_m$ is an $m$ root of unity. These roots of unity represent the eigenvalues of permutation matrix with cycle structure $\rho$.

**INPUT:**
- $\rho$ – a partition or a list of non-negative integers

**OUTPUT:**
- an element of the base ring

**EXAMPLES:**

```python
sage: p = SymmetricFunctions(QQ).p()
sage: p([3,3]).eval_at_permutation_roots([6])
0
sage: p([3,3]).eval_at_permutation_roots([3])
9
sage: p([3,3]).eval_at_permutation_roots([1])
```

(continues on next page)
sage: p([3,3]).eval_at_permutation_roots([3,3])
36
sage: p([3,3]).eval_at_permutation_roots([1,1,1,1])
25
5

\textbf{expand}(n, alphabet='x')

Expand the symmetric function \texttt{self} as a symmetric polynomial in \(n\) variables.

INPUT:
- \(n\) – a nonnegative integer
- \(alphabet\) – (default: 'x') a variable for the expansion

OUTPUT:
A monomial expansion of \texttt{self} in the \(n\) variables labelled by \(alphabet\).

EXAMPLES:

\begin{verbatim}
sage: p = SymmetricFunctions(QQ).p()
sage: a = p([2])
sage: a.expand(2)
x0^2 + x1^2
sage: a.expand(3, alphabet=['a','b','c'])
a^2 + b^2 + c^2
sage: p([2,1,1]).expand(2)
x0^4 + 2*x0*x1^3 + 2*x0^2*x1^2 + 2*x0^3*x1 + x1^4
sage: p([7]).expand(4)
x0^7 + x1^7 + x2^7 + x3^7
sage: p([7]).expand(4, alphabet='t')
t0^7 + t1^7 + t2^7 + t3^7
sage: p([7]).expand(4, alphabet='x,y,z,t')
x^7 + y^7 + z^7 + t^7
sage: p(1).expand(4)
1
sage: p(0).expand(4)
0
sage: (p([]) + 2*p([1])).expand(3)
2*x0 + 2*x1 + 2*x2 + 1
sage: p([1]).expand(0)
0
sage: (3*p([])).expand(0)
3
\end{verbatim}

\textbf{exponential_specialization}(t=None, q=1)

Return the exponential specialization of a symmetric function (when \(q = 1\)), or the \(q\)-exponential specialization (when \(q \neq 1\)).

The exponential specialization \(ex\) at \(t\) is a \(K\)-algebra homomorphism from the \(K\)-algebra of symmetric functions to another \(K\)-algebra \(R\). It is defined whenever the base ring \(K\) is a \(Q\)-algebra and \(t\) is an element of \(R\). The easiest way to define it is by specifying its values on the powersum symmetric functions to be \(p_1 = t\) and \(p_n = 0\) for \(n > 1\). Equivalently, on the homogeneous functions it is given by \(ex(h_n) = t^n/n!\); see Proposition 7.8.4 of [EnumComb2].
By analogy, the $q$-exponential specialization is a $K$-algebra homomorphism from the $K$-algebra of symmetric functions to another $K$-algebra $R$ that depends on two elements $t$ and $q$ of $R$ for which the elements $1 - q^i$ for all positive integers $i$ are invertible. It can be defined by specifying its values on the complete homogeneous symmetric functions to be

$$ex_q(h_n) = t^n/[n]_q!,$$

where $[n]_q!$ is the $q$-factorial. Equivalently, for $q \neq 1$ and a homogeneous symmetric function $f$ of degree $n$, we have

$$ex_q(f) = (1 - q)^n t^n ps_q(f),$$

where $ps_q(f)$ is the stable principal specialization of $f$ (see principal_specialization()). (See (7.29) in [EnumComb2].)

The limit of $ex_q$ as $q \to 1$ is $ex$.

**INPUT:**

- $t$ (default: None) – the value to use for $t$; the default is to create a ring of polynomials in $t$.
- $q$ (default: 1) – the value to use for $q$. If $q$ is None, then a ring (or fraction field) of polynomials in $q$ is created.

**EXAMPLES:**

```python
sage: p = SymmetricFunctions(QQ).p()
sage: x = p[8,7,3,1]
sage: x.exponential_specialization()
0
sage: x.exponential_specialization(t=var("t"))
5*t^2 + 2*t + 1
```

We also support the $q$-exponential_specialization:

```python
sage: factor(p[3].exponential_specialization(q=var("q"), t=var("t")))
(q - 1)^2*t^3/(q^2 + q + 1)
```

### frobenius($n$)

Return the image of the symmetric function self under the $n$-th Frobenius operator.

The $n$-th Frobenius operator $f_n$ is defined to be the map from the ring of symmetric functions to itself that sends every symmetric function $P(x_1, x_2, x_3, \ldots)$ to $P(x_1^n, x_2^n, x_3^n, \ldots)$. This operator $f_n$ is a Hopf algebra endomorphism, and satisfies

$$f_n m(\lambda_1, \lambda_2, \lambda_3, \ldots) = m(n\lambda_1, n\lambda_2, n\lambda_3, \ldots)$$

for every partition $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ (where $m$ means the monomial basis). Moreover, $f_n(p_r) = p_{nr}$ for every positive integer $r$ (where $p_k$ denotes the $k$-th powersum symmetric function).

The $n$-th Frobenius operator is also called the $n$-th Frobenius endomorphism. It is not related to the Frobenius map which connects the ring of symmetric functions with the representation theory of the symmetric group.

The $n$-th Frobenius operator is also the $n$-th Adams operator of the $\Lambda$-ring of symmetric functions over the integers.

The $n$-th Frobenius operator can also be described via plethysm: Every symmetric function $P$ satisfies $f_n(P) = p_n \circ P = P \circ p_n$, where $p_n$ is the $n$-th powersum symmetric function, and $\circ$ denotes (outer) plethysm.

**INPUT:**
• \( n \) – a positive integer

OUTPUT:

The result of applying the \( n \)-th Frobenius operator (on the ring of symmetric functions) to \( \text{self} \).

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: p = Sym.p()
sage: p[3].frobenius(2)
p[6]
sage: p[4,2,1].frobenius(3)
p[12, 6, 3]
sage: p([]).frobenius(4)
p[]
sage: p[3].frobenius(1)
p[3]
sage: (p([3]) - p([2]) + p([])).frobenius(3)
```

See also:

`plethysm()`

`omega()`

Return the image of \( \text{self} \) under the omega automorphism.

The **omega automorphism** is defined to be the unique algebra endomorphism \( \omega \) of the ring of symmetric functions that satisfies \( \omega(e_k) = h_k \) for all positive integers \( k \) (where \( e_k \) stands for the \( k \)-th elementary symmetric function, and \( h_k \) stands for the \( k \)-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the **omega involution**. It sends the power-sum symmetric function \( p_k \) to \( (-1)^{k-1} p_k \) for every positive integer \( k \).

The images of some bases under the omega automorphism are given by

\[
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
\]

where \( \lambda \) is any partition, where \( \ell(\lambda) \) denotes the length \( \text{(length())} \) of the partition \( \lambda \), where \( \lambda' \) denotes the conjugate partition \( \text{(conjugate())} \) of \( \lambda \), and where the usual notations for bases are used \( (e = \text{elementary}, h = \text{complete homogeneous}, p = \text{powersum}, s = \text{Schur}) \).

`omega_involution()` is a synonym for the `omega()` method.

OUTPUT:

• the image of \( \text{self} \) under the omega automorphism

EXAMPLES:

```python
sage: p = SymmetricFunctions(QQ).p()
sage: a = p([2,1]); a
p[2, 1]
sage: a.omega()
-p[2, 1]
sage: p([]).omega()
p[]
sage: p([]).omega()

(continues on next page)
```
omega_involution()  
Return the image of self under the omega automorphism.

The omega automorphism is defined to be the unique algebra endomorphism \( \omega \) of the ring of symmetric functions that satisfies \( \omega(e_k) = h_k \) for all positive integers \( k \) (where \( e_k \) stands for the \( k \)-th elementary symmetric function, and \( h_k \) stands for the \( k \)-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the omega involution. It sends the power-sum symmetric function \( p_k \) to \((-1)^{k-1} p_k\) for every positive integer \( k \).

The images of some bases under the omega automorphism are given by

\[
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
\]

where \( \lambda \) is any partition, where \( \ell(\lambda) \) denotes the length (\( \text{length}() \)) of the partition \( \lambda \), where \( \lambda' \) denotes the conjugate partition (\( \text{conjugate}() \)) of \( \lambda \), and where the usual notations for bases are used (\( e = \text{elementary}, h = \text{complete homogeneous}, p = \text{powersum}, s = \text{Schur} \)).

omega_involution() is a synonym for the omega() method.

OUTPUT:
• the image of self under the omega automorphism

EXAMPLES:

```sage
sage: p = SymmetricFunctions(QQ).p()
sage: a = p([2,1]); a
p[2, 1]
sage: a.omega()
-p[2, 1]
sage: p([]).omega()
p[]
sage: p(0).omega()
0
```

principal_specialization(n=\(\infty\), q=None)  
Return the principal specialization of a symmetric function.

The principal specialization of order \( n \) at \( q \) is the ring homomorphism \( \text{ps}_{n,q} \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for \( i \in \{1, \ldots, n\} \) and \( x_i \mapsto 0 \) for \( i > n \). Here, \( q \) is a given element of \( R \), and we assume that the variables of our symmetric functions are \( x_1, x_2, x_3, \ldots \). (To be more precise, \( \text{ps}_{n,q} \) is a \( K \)-algebra homomorphism, where \( K \) is the base ring.) See Section 7.8 of [EnumComb2].

The stable principal specialization at \( q \) is the ring homomorphism \( \text{ps}_q \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for all \( i \). This is well-defined only if the resulting infinite sums converge; thus, in particular, setting \( q = 1 \) in the stable principal specialization is an invalid operation.

INPUT:
• \( n \) (default: \(\infty\)) – a nonnegative integer or \(\infty\), specifying whether to compute the principal specialization of order \( n \) or the stable principal specialization.
• q (default: None) – the value to use for \( q \); the default is to create a ring of polynomials in \( q \) (or a field of rational functions in \( q \)) over the given coefficient ring.

We use the formulas from Proposition 7.8.3 of [EnumComb2]:

\[
\begin{align*}
\psi_{n,q}(\pi) & = \prod_{i} (1 - q^{n\lambda_i})/(1 - q^{\lambda_i}), \\
\psi_{n,1}(\pi) & = \ell(\lambda), \\
\psi_{q}(\pi) & = 1/ \prod_{i} (1 - q^{\lambda_i}),
\end{align*}
\]

where \( \ell(\lambda) \) denotes the length of \( \lambda \), and where the products range from \( i = 1 \) to \( i = \ell(\lambda) \).

**EXAMPLES:**

```sage
cp = SymmetricFunctions(QQ).p()
x = cp[8,7,3,1]
x.principal_specialization(3, q=var("q"))
(q^24 - 1)*(q^21 - 1)*(q^9 - 1)/((q^8 - 1)*(q^7 - 1)*(q - 1))
x = 5*cp[1,1,1] + 3*cp[2,1] + 1
x.principal_specialization(3, q=var("q"))
5*(q^3 - 1)^3/(q - 1)^3 + 3*(q^6 - 1)*(q^3 - 1)/((q^2 - 1)*(q - 1)) + 1
```

By default, we return a rational function in \( q \):

```sage
x.principal_specialization(3)
8*q^6 + 18*q^5 + 36*q^4 + 38*q^3 + 36*q^2 + 18*q + 9
```

If \( n \) is not given we return the stable principal specialization:

```sage
x.principal_specialization(q=var("q"))
3/((q^2 - 1)*(q - 1)) - 5/(q - 1)^3 + 1
```

**scalar**(*x, zee=None*)

Return the standard scalar product of \( \text{self} \) and \( x \).

**INPUT:**

• \( x \) – a power sum symmetric function
• \( zee \) – (default: uses standard \( zee \) function) optional input specifying the scalar product on the power sum basis with normalization \( \langle p_\mu, p_\nu \rangle = zee(\mu) \). \( zee \) should be a function on partitions.

Note that the power-sum symmetric functions are orthogonal under this scalar product. With the default value of \( zee \), the value of \( \langle p_\lambda, p_\lambda \rangle \) is given by the size of the centralizer in \( S_n \) of a permutation of cycle type \( \lambda \).

**OUTPUT:**

• the standard scalar product between \( \text{self} \) and \( x \), or, if the optional parameter \( zee \) is specified, then the scalar product with respect to the normalization \( \langle p_\mu, p_\nu \rangle = zee(\mu) \) with the power sum basis elements being orthogonal

**EXAMPLES:**

```sage
cp = SymmetricFunctions(QQ).p()
p = Partitions(4)
matrix([ [cp(a).scalar(cp(b)) for a in p] for b in p])
[ [4 0 0 0] [0 3 0 0]]
```
verschiebung$(n)$

Return the image of the symmetric function $\text{self}$ under the $n$-th Verschiebung operator.

The $n$-th Verschiebung operator $V_n$ is defined to be the unique algebra endomorphism $V$ of the ring of symmetric functions that satisfies $V(h_r) = h_{r/n}$ for every positive integer $r$ divisible by $n$, and satisfies $V(h_r) = 0$ for every positive integer $r$ not divisible by $n$. This operator $V_n$ is a Hopf algebra endomorphism. For every nonnegative integer $r$ with $n \mid r$, it satisfies

$$V_n(h_r) = V_n(p_r) = V_n(e_r) = (-1)^{r-r/n}e_{r/n}$$

(where $h$ is the complete homogeneous basis, $p$ is the power sum basis, and $e$ is the elementary basis). For every nonnegative integer $r$ with $n \nmid r$, it satisfies

$$V_n(h_r) = V_n(p_r) = V_n(e_r) = 0.$$

The $n$-th Verschiebung operator is also called the $n$-th Verschiebung endomorphism. Its name derives from the Verschiebung (German for “shift”) endomorphism of the Witt vectors.

The $n$-th Verschiebung operator is adjoint to the $n$-th Frobenius operator (see $\text{frobenius()}$ for its definition) with respect to the Hall scalar product ($\text{scalar()}$).

The action of the $n$-th Verschiebung operator on the Schur basis can also be computed explicitly. The following (probably clumsier than necessary) description can be obtained by solving exercise 7.61 in Stanley’s [STA].

Let $\lambda$ be a partition. Let $n$ be a positive integer. If the $n$-core of $\lambda$ is nonempty, then $V_n(s_\lambda) = 0$. Otherwise, the following method computes $V_n(s_\lambda)$: Write the partition $\lambda$ in the form $(\lambda_1, \lambda_2, \ldots, \lambda_{ns})$ for some nonnegative integer $s$. (If $n$ does not divide the length of $\lambda$, then this is achieved by adding trailing zeroes to $\lambda$.) Set $\beta_i = \lambda_i + ns - i$ for every $s \in \{1, 2, \ldots, ns\}$. Then, $(\beta_1, \beta_2, \ldots, \beta_{ns})$ is a strictly decreasing sequence of nonnegative integers. Stably sort the list $(1, 2, \ldots, ns)$ in order of (weakly) increasing remainder of $i - 1$ modulo $n$. Let $\xi$ be the sign of the permutation that is used for this sorting. Let $\psi$ be the sign of the permutation that is used to stably sort the list $(1, 2, \ldots, ns)$ in order of (weakly) increasing remainder of $i - 1$ modulo $n$. (Notice that $\psi = (-1)^{n(n-1)s(s-1)/4}$.) Then, $V_n(s_\lambda) = \xi \psi \prod_{i=0}^{n-1} s_{\lambda(i)}$, where $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)})$ is the $n$-quotient of $\lambda$.

**INPUT:**
- $n$ – a positive integer

**OUTPUT:**
The result of applying the $n$-th Verschiebung operator (on the ring of symmetric functions) to $\text{self}$.

**EXAMPLES:**
The Verschiebung endomorphisms are multiplicative:

```
sage: all( all( p(lam).verschiebung(2) * p(mu).verschiebung(2)
.....:     == (p(lam) * p(mu)).verschiebung(2)
.....:     for mu in Partitions(4) )
.....:     for lam in Partitions(4) )
True
```

Testing the adjointness between the Frobenius operators $f_n$ and the Verschiebung operators $V_n$:

```
sage: all( all( p(lam).verschiebung(2).scalar(p(mu))
.....:     == p(lam).scalar(p(mu).frobenius(2))
.....:     for mu in Partitions(2) )
.....:     for lam in Partitions(4) )
True
```

**antipode_on_basis**(partition)

Return the antipode of self[partition].

The antipode on the generator $p_i$ (for $i > 0$) is $-p_i$, and the antipode on $p_{\mu}$ is $(-1)^{\text{length}(\mu)} p_\mu$.

INPUT:

- self – the power sum basis of the symmetric functions
- partition – a partition

OUTPUT:

- the result of the antipode on self(partition)

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.p()
sage: p.antipode_on_basis([2])
-p[2]
sage: p.antipode_on_basis([3])
-p[3]
sage: p.antipode_on_basis([2,2])
p[2, 2]
sage: p.antipode_on_basis([[]])
p[]
```

**bottom_schur_function**(partition, degree=None)

Return the least-degree component of $s[partition]$, where $s$ denotes the Schur basis of the symmetric functions, and the grading is not the usual grading on the symmetric functions but rather the grading which gives every $p_i$ degree 1.
This least-degree component has its degree equal to the Frobenius rank of partition, while the degree with respect to the usual grading is still the size of partition.

This method requires the base ring to be a (commutative) $\mathbb{Q}$-algebra. This restriction is unavoidable, since the least-degree component (in general) has noninteger coefficients in all classical bases of the symmetric functions.

The optional keyword degree allows taking any homogeneous component rather than merely the least-degree one. Specifically, if degree is set, then the degree-th component will be returned.

REFERENCES:

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.p()
sage: p.bottom_schur_function([2,2,1])
-1/6*p[3, 2] + 1/4*p[4, 1]
sage: p.bottom_schur_function([2,1])
-1/3*p[3]
sage: p.bottom_schur_function([3])
1/3*p[3]
sage: p.bottom_schur_function([1,1,1])
1/3*p[3]
sage: p.bottom_schur_function(Partition([1,1,1]))
1/3*p[3]
sage: p.bottom_schur_function([2,1], degree=1)
-1/3*p[3]
sage: p.bottom_schur_function([2,1], degree=2)
0
sage: p.bottom_schur_function([2,1], degree=3)
1/3*p[1, 1, 1]
sage: p.bottom_schur_function([2,2,1], degree=3)
1/8*p[2, 2, 1] - 1/6*p[3, 1, 1]
```

coproduct_on_generators($i$)

Return coproduct on generators for power sums $p_i$ (for $i > 0$).

The elements $p_i$ are primitive elements.

INPUT:

• self – the power sum basis of the symmetric functions

• i – a positive integer

OUTPUT:

• the result of the coproduct on the generator $p(i)$

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.powersum()
sage: p.coproduct_on_generators(2)
```

eval_at_permutation_roots_on_generators($k$, $\rho$)

Evaluate $p_k$ at eigenvalues of permutation matrix.
This function evaluates a symmetric function $p([k])$ at the eigenvalues of a permutation matrix with cycle structure $\rho$.

This function evaluates a $p_k$ at the roots of unity

$$\Xi_{\rho_1}, \Xi_{\rho_2}, \ldots, \Xi_{\rho_\ell}$$

where

$$\Xi_m = 1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1}$$

and $\zeta_m$ is an $m$ root of unity. This is characterized by $p_k[A, B] = p_k[\zeta_m A] + p_k[B]$ and $p_k[\Xi_m] = 0$ unless $m$ divides $k$ and $p_{rm}[\Xi_m] = m$.

INPUT:

- k – a non-negative integer
- rho – a partition or a list of non-negative integers

OUTPUT:

- an element of the base ring

EXAMPLES:

```python
sage: p = SymmetricFunctions(QQ).p()
sage: p.eval_at_permutation_roots_on_generators(3, [6])
0
sage: p.eval_at_permutation_roots_on_generators(3, [3])
3
sage: p.eval_at_permutation_roots_on_generators(3, [1])
1
sage: p.eval_at_permutation_roots_on_generators(3, [3,3])
6
sage: p.eval_at_permutation_roots_on_generators(3, [1,1,1,1,1])
5
```

### 5.1.298 Schur symmetric functions

```python
class sage.combinat.sf.schur.SymmetricFunctionAlgebra_schur(Sym):
    Bases: SymmetricFunctionAlgebra_classical

    A class for methods related to the Schur symmetric function basis

    INPUT:

    - self – a Schur symmetric function basis
    - Sym – an instance of the ring of the symmetric functions

class Element
    Bases: Element

    expand(n, alphabet='x')
        Expand the symmetric function self as a symmetric polynomial in n variables.

    INPUT:

    - n – a nonnegative integer
```
• alphabet – (default: ‘x’) a variable for the expansion

OUTPUT:

A monomial expansion of self in the n variables labelled by alphabet.

EXAMPLES:

```python
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: a.expand(2)
x0^2*x1 + x0*x1^2
sage: a.expand(3)
x0^2*x1 + x0*x1^2 + x0^2*x2 + 2*x0*x1*x2 + x1^2*x2 + x0^2*x2 + x1*x2^2 + x0^2*x3 + x0^2*x1*x3 + x1^2*x3 + 2*x0*x2*x3 + 2*x1*x2*x3 + x2^2*x3 + x0^2*x3 + 2*x3 + 2*x0*x1*x3 + x1^2*x3 + 2*x0*x2*x3 + 2*x1*x2*x3 + x2^2*x3 + x0^2*x3^2 + x1^2*x3^2 + x2*x3^2
sage: a.expand(2, alphabet='y')
y0^2*y1 + y0*y1^2
sage: a.expand(2, alphabet=['a','b'])
a^2*b + a*b^2
sage: s([1,1,1,1]).expand(3)
\emptyset
sage: (s([]) + 2*s([1])).expand(3)
2*x0 + 2*x1 + 2*x2 + 1
sage: s([1]).expand(0)
\emptyset
sage: (3*s([])).expand(0)
3
```

exponential_specialization(t=None, q=1)

Return the exponential specialization of a symmetric function (when q = 1), or the q-exponential specialization (when q ≠ 1).

The exponential specialization \( ex \) at \( t \) is a \( K \)-algebra homomorphism from the \( K \)-algebra of symmetric functions to another \( K \)-algebra \( R \). It is defined whenever the base ring \( K \) is a \( Q \)-algebra and \( t \) is an element of \( R \). The easiest way to define it is by specifying its values on the powersum symmetric functions to be \( p_1 = t \) and \( p_n = 0 \) for \( n > 1 \). Equivalently, on the homogeneous functions it is given by \( ex(h_n) = t^n/n! \); see Proposition 7.8.4 of [EnumComb2].

By analogy, the q-exponential specialization is a \( K \)-algebra homomorphism from the \( K \)-algebra of symmetric functions to another \( K \)-algebra \( R \) that depends on two elements \( t \) and \( q \) of \( R \) for which the elements \( 1 - q^i \) for all positive integers \( i \) are invertible. It can be defined by specifying its values on the complete homogeneous symmetric functions to be

\[
ex_q(h_n) = t^n/[n]_q!,
\]

where \([n]_q!\) is the \( q \)-factorial. Equivalently, for \( q \neq 1 \) and a homogeneous symmetric function \( f \) of degree \( n \), we have

\[
ex_q(f) = (1 - q^n) t^n ps_q(f),
\]

where \( ps_q(f) \) is the stable principal specialization of \( f \) (see principal_specialization()). (See (7.29) in [EnumComb2].)

The limit of \( ex_q \) as \( q \to 1 \) is \( ex \).

INPUT:
Combinatorics, Release 10.0

- \( t \) (default: `None`) – the value to use for \( t \); the default is to create a ring of polynomials in \( t \).
- \( q \) (default: `1`) – the value to use for \( q \). If \( q \) is `None`, then a ring (or fraction field) of polynomials in \( q \) is created.

We use the formula in the proof of Corollary 7.21.6 of [EnumComb2]

\[
ex_q(s_\lambda) = t^{\ell(\lambda)} q^{\sum_i (i-1) \lambda_i} \prod_{u \in \lambda} (1 + q + q^2 + \cdots + q^{h(u)-1})
\]

where \( h(u) \) is the hook length of a cell \( u \) in \( \lambda \).

As a limit case, we obtain a formula for \( q = 1 \)

\[
ex_1(s_\lambda) = f^\lambda |\lambda|! / |\lambda|!
\]

where \( f^\lambda \) is the number of standard Young tableaux of shape \( \lambda \).

**EXAMPLES:**

```
sage: s = SymmetricFunctions(QQ).s()
sage: x = s[5,3]
sage: x.exponential_specialization()
1/1440*t^8
sage: x = 5*s[1,1,1] + 3*s[2,1] + 1
sage: x.exponential_specialization()
11/6*t^3 + 1
```

We also support the \( q \)-exponential_specialization:

```
sage: factor(s[3].exponential_specialization(q=var("q"), t=var("t")))
t^3/((q^2 + q + 1)*(q + 1))
```

**omega()**

Return the image of `self` under the omega automorphism.

The **omega automorphism** is defined to be the unique algebra endomorphism \( \omega \) of the ring of symmetric functions that satisfies \( \omega(e_k) = h_k \) for all positive integers \( k \) (where \( e_k \) stands for the \( k \)-th elementary symmetric function, and \( h_k \) stands for the \( k \)-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the **omega involution**. It sends the power-sum symmetric function \( p_k \) to \((-1)^{k-1}p_k \) for every positive integer \( k \).

The images of some bases under the omega automorphism are given by

\[
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
\]

where \( \lambda \) is any partition, where \( \ell(\lambda) \) denotes the length (**length()**) of the partition \( \lambda \), where \( \lambda' \) denotes the conjugate partition (**conjugate()**) of \( \lambda \), and where the usual notations for bases are used (\( e = \) elementary, \( h = \) complete homogeneous, \( p = \) powersum, \( s = \) Schur).

**omega_involution()** is a synonym for the **omega()** method.

**OUTPUT:**
- the image of `self` under the omega automorphism

**EXAMPLES:**
omega_involution()  
Return the image of self under the omega automorphism.

The omega automorphism is defined to be the unique algebra endomorphism \( \omega \) of the ring of symmetric functions that satisfies \( \omega(\epsilon_k) = h_k \) for all positive integers \( k \) (where \( \epsilon_k \) stands for the \( k \)-th elementary symmetric function, and \( h_k \) stands for the \( k \)-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the omega involution. It sends the power-sum symmetric function \( p_k \) to \((-1)^{k-1} p_k \) for every positive integer \( k \).

The images of some bases under the omega automorphism are given by

\[
\omega(\epsilon_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda, \quad \omega(s_\lambda) = s_{\lambda'},
\]

where \( \lambda \) is any partition, where \( \ell(\lambda) \) denotes the length (length()) of the partition \( \lambda \), where \( \lambda' \) denotes the conjugate partition (conjugate()) of \( \lambda \), and where the usual notations for bases are used (\( e \) = elementary, \( h \) = complete homogeneous, \( p \) = powersum, \( s \) = Schur).

omega_involution() is a synonym for the omega() method.

OUTPUT:
- the image of self under the omega automorphism

EXAMPLES:

```python
sage: s = SymmetricFunctions(QQ).s()
sage: s([2,1]).omega()
s[2, 1]
sage: s([2,1,1]).omega()
s[3, 1]
```

principal_specialization(n=+Infinity, q=None)

Return the principal specialization of a symmetric function.

The principal specialization of order \( n \) at \( q \) is the ring homomorphism \( ps_{n,q} \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for \( i \in \{1, \ldots, n\} \) and \( x_i \mapsto 0 \) for \( i > n \). Here, \( q \) is a given element of \( R \), and we assume that the variables of our symmetric functions are \( x_1, x_2, x_3, \ldots \). (To be more precise, \( ps_{n,q} \) is a \( K \)-algebra homomorphism, where \( K \) is the base ring.) See Section 7.8 of [EnumComb2].

The stable principal specialization at \( q \) is the ring homomorphism \( ps_q \) from the ring of symmetric functions to another commutative ring \( R \) given by \( x_i \mapsto q^{i-1} \) for all \( i \). This is well-defined only if the resulting infinite sums converge; thus, in particular, setting \( q = 1 \) in the stable principal specialization is an invalid operation.

INPUT:
- \( n \) (default: infinity) -- a nonnegative integer or infinity, specifying whether to compute the principal specialization of order \( n \) or the stable principal specialization.
- \( q \) (default: None) -- the value to use for \( q \); the default is to create a ring of polynomials in \( q \) (or a field of rational functions in \( q \)) over the given coefficient ring.
For \( q = 1 \) we use the formula from Corollary 7.21.4 of [EnumComb2]:

\[
    p_{s_n,1} (s_\lambda) = \prod_{u \in \lambda} (n + c(u))/h(u),
\]

where \( h(u) \) is the hook length of a cell \( u \) in \( \lambda \), and where \( c(u) \) is the content of a cell \( u \) in \( \lambda \). For \( n = \text{infinity} \) we use the formula from Corollary 7.21.3 of [EnumComb2]

\[
    p_{s_q} (s_\lambda) = q^{\sum_i (i-1) \lambda_i} \prod_{u \in \lambda} (1 - q^{h(u)}).
\]

Otherwise, we use the formula from Theorem 7.21.2 of [EnumComb2],

\[
    p_{s_n,q} (s_\lambda) = q^{\sum_i (i-1) \lambda_i} \prod_{u \in \lambda} (1 - q^{n+c(u)})/(1 - q^{h(u)}).
\]

EXAMPLES:

```python
sage: s = SymmetricFunctions(QQ).s()
sage: x = s[2]
sage: x.principal_specialization(3)
q^4 + q^3 + 2*q^2 + q + 1
sage: x = 3*s[2,2] + 2*s[1] + 1
sage: x.principal_specialization(3, q=var("q"))
3*(q^4 - 1)*(q^3 - 1)*q^2/((q^2 - 1)*(q - 1)) + 2*(q^3 - 1)/(q - 1) + 1
sage: x.principal_specialization(q=var("q"))
-2/(q - 1) + 3*q^2/((q^3 - 1)*q^2/1) + 1
```

**scalar**

Return the standard scalar product between \( \text{self} \) and \( x \).

Note that the Schur functions are self-dual with respect to this scalar product. They are also lower-triangularly related to the monomial symmetric functions with respect to this scalar product.

**INPUT:**

- \( x \) – element of the ring of symmetric functions over the same base ring as \( \text{self} \)
- \( \text{zee} \) – an optional function on partitions giving the value for the scalar product between the power-sum symmetric function \( p_\mu \) and itself (the default value is the standard \( \text{zee}() \) function)

**OUTPUT:**

- the scalar product between \( \text{self} \) and \( x \)

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(ZZ).s()
sage: a = s([2,1])
sage: b = s([1,1,1])
sage: c = 2*s([1,1,1])
sage: d = a + b
sage: a.scalar(a)
1
sage: b.scalar(b)
1
sage: b.scalar(a)
0
```
sage: b.scalar(c) 2
sage: c.scalar(c) 4
sage: d.scalar(a) 1
sage: d.scalar(b) 1
sage: d.scalar(c) 2
sage: m = SymmetricFunctions(ZZ).monomial()
sage: p4 = Partitions(4)
sage: l = [ [s(p).scalar(m(q)) for q in p4] for p in p4]
sage: matrix(l)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 \\
-1 & 2 & 1 & -3 & 1 \\
\end{bmatrix}
\]

verschiebung\((n)\)

Return the image of the symmetric function \(\text{self}\) under the \(n\)-th Verschiebung operator.

The \(n\)-th Verschiebung operator \(V_n\) is defined to be the unique algebra endomorphism \(V\) of the ring of symmetric functions that satisfies \(V(h_r) = h_{r/n}\) for every positive integer \(r\) divisible by \(n\), and satisfies \(V(h_r) = 0\) for every positive integer \(r\) not divisible by \(n\). This operator \(V_n\) is a Hopf algebra endomorphism. For every nonnegative integer \(r\) with \(n \mid r\), it satisfies

\[
V_n(h_r) = h_{r/n}, \quad V_n(p_r) = np_{r/n}, \quad V_n(e_r) = (-1)^{r-r/n}e_{r/n}
\]

(where \(h\) is the complete homogeneous basis, \(p\) is the powersum basis, and \(e\) is the elementary basis). For every nonnegative integer \(r\) with \(n \notmid r\), it satisfies

\[
V_n(h_r) = V_n(p_r) = V_n(e_r) = 0.
\]

The \(n\)-th Verschiebung operator is also called the \(n\)-th Verschiebung endomorphism. Its name derives from the Verschiebung (German for ‘shift’) endomorphism of the Witt vectors.

The \(n\)-th Verschiebung operator is adjoint to the \(n\)-th Frobenius operator (see \texttt{frobenius()} for its definition) with respect to the Hall scalar product (\texttt{scalar()}).

The action of the \(n\)-th Verschiebung operator on the Schur basis can also be computed explicitly. The following (probably clumsier than necessary) description can be obtained by solving exercise 7.61 in Stanley’s [STA].

Let \(\lambda\) be a partition. Let \(n\) be a positive integer. If the \(n\)-core of \(\lambda\) is nonempty, then \(V_n(s_\lambda) = 0\). Otherwise, the following method computes \(V_n(s_\lambda)\): Write the partition \(\lambda\) in the form \((\lambda_1, \lambda_2, \ldots, \lambda_{ns})\) for some nonnegative integer \(s\). (If \(n\) does not divide the length of \(\lambda\), then this is achieved by adding trailing zeroes to \(\lambda\).) Set \(\beta_i = \lambda_i + ns - i\) for every \(s \in \{1, 2, \ldots, ns\}\). Then, \((\beta_1, \beta_2, \ldots, \beta_{ns})\) is a strictly decreasing sequence of nonnegative integers. Stably sort the list \((1, 2, \ldots, ns)\) in order of (weakly) increasing remainder of \(-1 - \beta_i\) modulo \(n\). Let \(\xi\) be the sign of the permutation that is used for this sorting. Let \(\psi\) be the sign of the permutation that is used to stably sort the list \((1, 2, \ldots, ns)\) in order of (weakly) increasing remainder of \(i - 1\) modulo \(n\). (Notice that \(\psi = (-1)^{n(n-1)s(s-1)/4}\).) Then, \(V_n(s_\lambda) = \xi \psi \prod_{i=0}^{ns-1} s_{\lambda(i)}\), where \((\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)})\) is the \(n\)-quotient of \(\lambda\).
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INPUT:

• \( n \) – a positive integer

OUTPUT:

The result of applying the \( n \)-th Verschiebung operator (on the ring of symmetric functions) to \( \text{self} \).

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: s = Sym.s()
sage: s[5].verschiebung(2)
0
sage: s[6].verschiebung(6)
s[1]
sage: s[6,3].verschiebung(3)
s[2, 1] + s[3]
sage: s[6,3,1].verschiebung(2)
-s[3, 2]
sage: s[3,2,1].verschiebung(1)
s[3, 2, 1]
sage: s([]).verschiebung(1)
s[]
sage: s([]).verschiebung(4)
s[]
```

coproduct_on_basis(\( \mu \))

Returns the coproduct of \( \text{self}(\mu) \).

Here \( \text{self} \) is the basis of Schur functions in the ring of symmetric functions.

INPUT:

• \( \text{self} \) – a Schur symmetric function basis
• \( \mu \) – a partition

OUTPUT:

• the image of the \( \mu \)-th Schur function under the comultiplication of the Hopf algebra of symmetric functions; this is an element of the tensor square of the Schur basis

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: s.coproduct_on_basis([2])
```

product_on_basis(\( \text{left}, \text{right} \))

Return the product of \( \text{left} \) and \( \text{right} \).

INPUT:

• \( \text{self} \) – a Schur symmetric function basis
• \( \text{left}, \text{right} \) – partitions

OUTPUT:

• an element of the Schur basis, the product of \( \text{left} \) and \( \text{right} \)
5.1.299 Symplectic Symmetric Functions

AUTHORS:

- Travis Scrimshaw (2013-11-10): Initial version

```python
class sage.combinat.sf.symplectic.SymmetricFunctionAlgebra_symplectic(Sym):
    Bases: SymmetricFunctionAlgebra_generic
```

The symplectic symmetric function basis (or symplectic basis, to be short).

The symplectic basis $\{sp_\lambda\}$ where $\lambda$ is taken over all partitions is defined by the following change of basis with the Schur functions:

$$s_\lambda = \sum_{\mu} \left( \sum_{\nu \in V} c_{\mu \nu}^{\lambda} \right) sp_\mu$$

where $V$ is the set of all partitions with even-height columns and $c_{\mu \nu}^{\lambda}$ is the usual Littlewood-Richardson (LR) coefficients. By the properties of LR coefficients, this can be shown to be a upper unitriangular change of basis.

**Note:** This is only a filtered basis, not a $\mathbb{Z}$-graded basis. However this does respect the induced $(\mathbb{Z}/2\mathbb{Z})$-grading.

**INPUT:**

- `Sym` – an instance of the ring of the symmetric functions

**REFERENCES:**

**EXAMPLES:**

Here are the first few symplectic symmetric functions, in various bases:

```
sage: Sym = SymmetricFunctions(QQ)
sage: sp = Sym.sp()
sage: e = Sym.e()
sage: h = Sym.h()
sage: p = Sym.p()
sage: s = Sym.s()
sage: m = Sym.m()
sage: p(sp([1]))
p[1]
sage: m(sp([1]))
m[1]
sage: e(sp([1]))
e[1]
sage: h(sp([1]))
h[1]
sage: s(sp([1]))
s[1]
sage: p(sp([2]))
1/2*p[1, 1] + 1/2*p[2]
sage: m(sp([2]))
m[1, 1] + m[2]
sage: e(sp([2]))
```

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Combinatorics, Release 10.0

(continued from previous page)

\[ e[1, 1] - e[2] \]
\[ \text{sage: } h(sp([2])) \]
\[ h[2] \]
\[ \text{sage: } s(sp([2])) \]
\[ s[2] \]
\[ \text{sage: } p(sp([3])) \]
\[ 1/6*p[1, 1, 1] + 1/2*p[2, 1] + 1/3*p[3] \]
\[ \text{sage: } m(sp([3])) \]
\[ m[1, 1, 1] + m[2, 1] + m[3] \]
\[ \text{sage: } e(sp([3])) \]
\[ \text{sage: } h(sp([3])) \]
\[ h[3] \]
\[ \text{sage: } s(sp([3])) \]
\[ s[3] \]

Some examples of conversions the other way:

\[ \text{sage: } sp(h[3]) \]
\[ sp[3] \]
\[ \text{sage: } sp(e[3]) \]
\[ sp[1] + sp[1, 1, 1] \]
\[ \text{sage: } sp(m([2, 1])) \]
\[ -sp[1] - 2*sp[1, 1, 1] + sp[2, 1] \]
\[ \text{sage: } sp(p([3])) \]

Some multiplication:

\[ \text{sage: } sp([2]) * sp([1,1]) \]
\[ \text{sage: } sp([2,1,1]) * sp([2]) \]
\[ sp[1, 1] + sp[1, 1, 1] + 2*sp[2, 1, 1] + sp[2, 2] + sp[2, 2, 1, 1] \]
\[ \text{sage: } sp([1,1]) * sp([2,1]) \]

(continues on next page)

Examples of the Hopf algebra structure:

```python
sage: sp([1]).antipode()
-sp[1]
sage: sp([2]).antipode()
sp[] + sp[1, 1]
sage: sp([1]).coproduct()
sp[] # sp[1] + sp[1] # sp[]
sage: sp([2]).coproduct()
sage: sp([1]).counit()
0
sage: sp.one().counit()
1
```

5.1.300 Symmetric functions, with their multiple realizations

```python
class sage.combinat.sf.sf.SymmetricFunctions(R)
    Bases: UniqueRepresentation, Parent

The abstract algebra of commutative symmetric functions

Symmetric Functions in Sage

Author: Jason Bandlow, Anne Schilling, Nicolas M. Thiery, Mike Zabrocki

This document is an introduction to working with symmetric function theory in Sage. It is not intended to be an introduction to the theory of symmetric functions ([MAC] and [STA], Chapter 7, are two excellent references.) The reader is also expected to be familiar with Sage.

The algebra of symmetric functions

The algebra of symmetric functions is the unique free commutative graded connected algebra over the given ring, with one generator in each degree. It can also be thought of as the inverse limit (in the category of graded algebras) of the algebra of symmetric polynomials in \( n \) variables as \( n \to \infty \). Sage allows us to construct the algebra of symmetric functions over any ring. We will use a base ring of rational numbers in these first examples:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: Sym
Symmetric Functions over Rational Field
```

Sage knows certain categorical information about this algebra:

```python
sage: Sym.category()
Join of Category of hopf algebras over Rational Field
    and Category of unique factorization domains
    and Category of graded algebras over Rational Field
```

(continues on next page)
and Category of commutative algebras over Rational Field
and Category of monoids with realizations
and Category of graded coalgebras over Rational Field
and Category of coalgebras over Rational Field with realizations
and Category of cocommutative coalgebras over Rational Field

Notice that \( \text{Sym} \) is an abstract algebra. This reflects the fact that there are multiple natural bases. To work with specific elements, we need a realization of this algebra. In practice, this means we need to specify a basis.

**An example basis - power sums**

Here is an example of how one might use the power sum realization:

```
sage: p = Sym.powersum()
sage: p
Symmetric Functions over Rational Field in the powersum basis
```

\( p \) now represents the realization of the symmetric function algebra on the power sum basis. The basis itself is accessible through:

```
sage: p.basis()
Lazy family (Term map from Partitions to Symmetric Functions over Rational Field in␣˓→the powersum basis(i))_{i in Partitions}
sage: p.basis().keys()
Partitions
```

This last line means that \( p.basis() \) is an association between the set of Partitions and the basis elements of the algebra \( p \). To construct a specific element one can therefore do:

```
sage: p.basis()[Partition([2,1,1])]
p[2, 1, 1]
```

As this is rather cumbersome, realizations of the symmetric function algebra allow for the following abuses of notation:

```
sage: p[Partition([2, 1, 1])]
p[2, 1, 1]
sage: p[[2, 1, 1]]
p[2, 1, 1]
sage: p[2, 1, 1]
p[2, 1, 1]
```

or even:

```
sage: p[(i for i in [2, 1, 1])]
p[2, 1, 1]
```

In the special case of the empty partition, due to a limitation in Python syntax, one cannot use:

```
sage: p[]  # todo: not implemented
```

Please use instead:
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\begin{verbatim}
  sage: p[[]]
p[]

  Note:  When elements are constructed using the p[something ] syntax , an error will be raised if the input
cannot be interpreted as a partition. This is not the case when p.basis() is used:

  sage: p['something']
  Traceback (most recent call last):
  ...
  ValueError: all parts of 'something' should be nonnegative integers
  sage: p.basis()['something']
p'something'

Elements of p are linear combinations of such compositions:

  sage: p.an_element()

\end{verbatim}

Algebra structure

Algebraic combinations of basis elements can be entered in a natural way:

\begin{verbatim}
  3*p[2, 1, 1] + 2*p[4, 1]

Let us explore the other operations of p. We can ask for the mathematical properties of p:

\begin{verbatim}
  sage: p.categories()
  [Category of graded bases of Symmetric Functions over Rational Field,
   Category of filtered bases of Symmetric Functions over Rational Field,
   Category of bases of Symmetric Functions over Rational Field,
   Category of graded hopf algebras with basis over Rational Field,
   ...]

To start with, p is a graded algebra, the grading being induced by the size of the partitions. Due to this, the one
is the basis element indexed by the empty partition:

\begin{verbatim}
  sage: p.one()
p[]

The p basis is multiplicative; that is, multiplication is induced by linearity from the (nonincreasingly sorted)
catetnation of partitions:

\begin{verbatim}
  sage: p[3,1] * p[2,1]
p[3, 2, 1, 1]
  sage: (p.one() + 2 * p[3,1]) * p[4, 2]
p[4, 2] + 2*p[4, 3, 2, 1]
\end{verbatim}

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The classical bases

In addition to the power sum basis, other classical bases of the symmetric function algebra include the elementary, complete homogeneous, monomial, and Schur bases:

```
sage: e = Sym.elementary()
sage: h = Sym.homogeneous()
sage: m = Sym.monomial()
sage: s = Sym.schur()
```

These and others can be defined all at once with the single command:

```
sage: Sym.inject_shorthands()
Defining e as shorthand for Symmetric Functions over Rational Field in the elementary basis
Defining f as shorthand for Symmetric Functions over Rational Field in the forgotten basis
Defining h as shorthand for Symmetric Functions over Rational Field in the homogeneous basis
Defining m as shorthand for Symmetric Functions over Rational Field in the monomial basis
Defining p as shorthand for Symmetric Functions over Rational Field in the powersum basis
Defining s as shorthand for Symmetric Functions over Rational Field in the Schur basis
```

We can then do conversions from one basis to another:

```
sage: s(p[2,1])
-s[1, 1, 1] + s[3]
sage: m(p[3])
m[3]
sage: m(p[3,2])
m[3, 2] + m[5]
```

For computations which mix bases, Sage will return a result with respect to a single (not necessarily predictable) basis:

```
p[1] + 1/12*p[1, 1, 1, 1, 1, 1] - 1/6*p[2, 1, 1, 1, 1] - 1/4*p[2, 2, 1, 1] + 1/
6*p[3, 1, 1, 1] + 1/6*p[3, 2, 1]
```

The one for different bases such as the power sum and Schur function is the same:

```
sage: s.one() == p.one()
True
```
Basic computations

In this section, we explore some of the many methods that can be applied to an arbitrary symmetric function:

```
sage: f = s[2]^2; f
```

For more methods than discussed here, create a symmetric function as above, and use \( f.<\text{tab}>\).

Representation theory of the symmetric group

The Schur functions \( s_\lambda \) can also be interpreted as irreducible characters of the symmetric group \( S_n \), where \( n \) is the size of the partition \( \lambda \). Since the Schur functions of degree \( n \) form a basis of the symmetric functions of degree \( n \), it follows that an arbitrary symmetric function (homogeneous of degree \( n \)) may be interpreted as a function on the symmetric group. In this interpretation the power sum symmetric function \( p_\lambda \) is the characteristic function of the conjugacy class with shape \( \lambda \), multiplied by the order of the centralizer of an element. Hence the irreducible characters can be computed as follows:

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: p = Sym.power()
sage: P = Partitions(5).list()
sage: P = [P[i] for i in range(len(P)-1,-1,-1)]
sage: M = matrix([[s[P[i]].scalar(p[P[j]]) for j in range(len(P))] for i in range(len(P))])
sage: M
[ 1 -1 1 1 -1 -1 1]
[ 4 -2 0 1 1 0 -1]
[ 5 -1 1 -1 -1 1 0]
[ 6 0 -2 0 0 0 1]
[ 5 1 1 -1 1 -1 0]
[ 4 2 0 1 -1 0 -1]
[ 1 1 1 1 1 1 1]
```

We can indeed check that this agrees with the character table of \( S_5 \):

```
sage: SymmetricGroup(5).character_table() == M
True
```

In this interpretation of symmetric functions as characters on the symmetric group, the multiplication and comultiplication are interpreted as induction (from \( S_n \times S_m \) to \( S_{n+m} \)) and restriction, respectively. The Schur functions can also be interpreted as characters of \( GL_n \), see Partitions and Schur functions.

The omega involution

The \( \omega \) involution is the linear extension of the map which sends \( e_\lambda \) to \( h_\lambda \):

```
sage: h(f)
h[2, 2]
sage: e(f.omega())
e[2, 2]
```
The Hall scalar product

The Hall scalar product on the algebra of symmetric functions makes the Schur functions into an orthonormal basis:

```
sage: f.scalar(f)
3
```

Skewing

Skewing is the adjoint operation to multiplication with respect to this scalar product:

```
sage: f.skew_by(s[1])
2*s[2, 1] + 2*s[3]
```

In general, \( s[\lambda] \text{.skew_by}(s[\mu]) \) is the symmetric function typically denoted \( s_{\lambda \setminus \mu} \) or \( s_{\lambda/\mu} \).

Expanding into variables

We can expand a symmetric function into a symmetric polynomial in a specified number of variables:

```
sage: f.expand(2)
x0^4 + 2*x0^3*x1 + 3*x0^2*x1^2 + 2*x0*x1^3 + x1^4
```

See the documentation for `expand` for more examples.

The Kronecker product

As in the section on the Representation theory of the symmetric group, a symmetric function may be considered as a class function on the symmetric group where the elements \( p_\mu/z_\mu \) are the indicators of a permutation having cycle structure \( \mu \). The Kronecker product of two symmetric functions corresponds to the pointwise product of these class functions.

Since the Schur functions are the irreducible characters of the symmetric group under this identification, the Kronecker product of two Schur functions corresponds to the internal tensor product of two irreducible symmetric group representations.

Under this identification, the Kronecker product of \( p_\mu/z_\mu \) and \( p_\nu/z_\nu \) is \( p_\mu/z_\mu \) if \( \mu = \nu \), and the result is equal to 0 otherwise.

`internal_product`, `kronecker_product`, `inner_tensor` and `itensor` are different names for the same function.

```
sage: f.kronecker_product(f)
s[1, 1, 1, 1] + 3*s[2, 1, 1] + 4*s[2, 2] + 5*s[3, 1] + 3*s[4]
```
Plethysm

The plethysm of symmetric functions is the operation corresponding to composition of representations of the general linear group. See [STA] Chapter 7, Appendix 2 for details.

\[
\text{sage: } s[2].\text{plethysm}(s[2])
\]
\[
s[2, 2] + s[4]
\]

Plethysm can also be written as a composition of functions:

\[
\text{sage: } s[2](s[2])
\]
\[
s[2, 2] + s[4]
\]

If the coefficient ring contains degree 1 elements, these are handled properly by plethysm:

\[
\text{sage: } \text{R.<t> = QQ[]; } s = \text{SymmetricFunctions(R).schur()}
\]
\[
\text{sage: } s[2]( (1-t)*s[1] )
\]
\[
(t^2-t)*s[1, 1] + (-t+1)*s[2]
\]

See the documentation for \text{plethysm} for more information.

Inner plethysm

The operation of inner plethysm \( f \text{.inner_plethysm}(g) \) models the composition of the \( S_n \) representation represented by \( g \) with the \( GL_m \) representation whose character is \( f \). See the documentation of \text{inner_plethysm}, [ST94] or [STA], exercise 7.74 solutions for more information:

\[
\text{sage: } s = \text{SymmetricFunctions(QQ).schur()}
\]
\[
\text{sage: } f = s[2]^2
\]
\[
\text{sage: } f \text{.inner_plethysm}(s[2])
\]
\[
s[2]
\]

Hopf algebra structure

The ring of symmetric functions is further endowed with a coalgebra structure. The coproduct is an algebra morphism, and therefore determined by its values on the generators; the power sum generators are primitive:

\[
\text{sage: } p[1].\text{coproduct()}
\]
\[
\]
\[
\text{sage: } p[2].\text{coproduct()}
\]
\[
\]

The coproduct, being cocommutative on the generators, is cocommutative everywhere:

\[
\text{sage: } p[2, 1].\text{coproduct()}
\]
\[
\]

This coproduct, along with the counit which sends every symmetric function to its 0-th homogeneous component, makes the ring of symmetric functions into a graded connected bialgebra. It is known that every graded connected bialgebra has an antipode. For the ring of symmetric functions, the antipode can be characterized explicitly: The antipode is an anti-algebra morphism (thus an algebra morphism, since our algebra is commutative) which sends \( p_\lambda \) to \((-1)^{\text{length}(\lambda)} p_\lambda\) for every partition \( \lambda \). Thus, in particular, it sends the generators on the \( p \) basis to their opposites:
The graded connected bialgebra of symmetric functions over a \( Q \)-algebra has a rather simply-understood structure: It is (isomorphic to) the symmetric algebra of its space of primitives (which is spanned by the power-sum symmetric functions).

Here are further examples:

```
sage: f = s[2]^2
sage: f.antipode()
s[1, 1, 1, 1] + s[2, 1, 1] + s[2, 2]
sage: f.coproduct()
+ s[3, 1] # s[] + s[4] # s[]
sage: f.coproduct().apply_multilinear_morphism( lambda x,y: x*y.antipode() )
0
```

**Transformations of symmetric functions**

There are many methods in Sage which make it easy to manipulate symmetric functions. For example, if we have some function which acts on partitions (say, conjugation), it is a simple matter to apply it to the support of a symmetric function. Here is an example:

```
sage: conj = lambda mu: mu.conjugate()
sage: f.map_support(conj)
h[1, 1, 1, 1] + 2*h[2, 1, 1]
```

We can also easily modify the coefficients:

```
sage: def foo(mu, coeff):
   return mu.conjugate(), -coeff
sage: f.map_item(foo)
-h[1, 1, 1, 1] - 2*h[2, 1, 1]
```

See also `map_coefficients`.

There are also methods for building functions directly:

```
sage: s.sum_of_monomials(mu for mu in Partitions(3))
s[1, 1, 1] + s[2, 1] + s[3]
sage: s.sum_of_monomials(Partitions(3))
s[1, 1, 1] + s[2, 1] + s[3]
sage: s.sum_of_terms( (mu, mu[0]) for mu in Partitions(3))
s[1, 1, 1] + 2*s[2, 1] + 3*s[3]
```

These are the preferred way to build elements within a program; the result will usually be faster than using `sum()`.

It also guarantees that empty sums yields the zero of \( s \) (see also `s.sum`).

Note also that it is a good idea to use:
instead of $s(1)$ and $s(0)$ within programs where speed is important, in order to prevent unnecessary coercions.

**Different base rings**

Depending on the base ring, the different realizations of the symmetric function algebra may not span the same space:

```
sage: SZ = SymmetricFunctions(ZZ)
sage: p = SZ.power(); s = SZ.schur()
sage: p(s[1,1,1])
Traceback (most recent call last):
...  
TypeError: no conversion of this rational to integer
```

Because of this, some functions may not behave as expected when working over the integers, even though they make mathematical sense:

```
sage: s[1,1,1].plethysm(s[1,1,1])
Traceback (most recent call last):
...  
TypeError: no conversion of this rational to integer
```

It is possible to work over different base rings simultaneously:

```
sage: s = SymmetricFunctions(QQ).schur()
sage: p = SymmetricFunctions(QQ).power()
sage: sz = SymmetricFunctions(ZZ).schur(); sz._prefix = 'sz'
sage: pz = SymmetricFunctions(ZZ).power(); pz._prefix = 'pz'
sage: p(sz[1,1,1])
1/6*p[1, 1, 1] - 1/2*p[2, 1] + 1/3*p[3]
sz[1, 1, 1]
```

As shown in this example, if you are working over multiple base rings simultaneously, it is a good idea to change the prefix in some cases, so that you can tell from the output which realization your result is in.

Let us change the notation back for the remainder of this tutorial:

```
sage: sz._prefix = 's'
sage: pz._prefix = 'p'
```

One can also use the Sage standard renaming idiom to get shorter outputs:

```
sage: Sym = SymmetricFunctions(QQ)
sage: Sym.rename("Sym")
sage: Sym 
Sym
 sage: Sym.rename() 
```

---

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And we name it back:

```
sage: Sym.rename("Symmetric Functions over Rational Field"); Sym
Symmetric Functions over Rational Field
```

### Other bases

There are two additional basis of the symmetric functions which are not considered as classical bases:

- forgotten basis
- Witt basis

The forgotten basis is the dual basis of the elementary symmetric functions basis with respect to the Hall scalar product. The Witt basis can be constructed by

\[
\prod_{d=1}^{\infty} \left(1 - w_d t^d\right)^{-1} = \sum_{n=0}^{\infty} h_n t^n
\]

where \( t \) is a formal variable.

There are further bases of the ring of symmetric functions, in general over fields with parameters such as \( q \) and \( t \):

- Hall-Littlewood bases
- Jack bases
- Macdonald bases
- \( k \)-Schur functions
- Hecke character basis

We briefly demonstrate how to access these bases. For more information, see the documentation of the individual bases.

The Jack polynomials can be obtained as:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: Jack = Sym.jack()
sage: J(P[2,1])
(1/(t+2))*JackJ[2, 1]
```

The parameter \( t \) can be specialized as follows:

```
sage: Sym = SymmetricFunctions(QQ)
sage: Jack = Sym.jack(t = 1)
sage: J(P[2,1])
1/3*JackJ[2, 1]
```

Similarly one can access the Hall-Littlewood and Macdonald polynomials, etc:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: Mcd = Sym.macdonald()
```

(continues on next page)
We can also construct the $\bar{q}$ basis that can be used to determine character tables for Hecke algebras (with quadratic relation $T_i^2 = (1 - q)T_i + q$):

\begin{verbatim}
sage: Sym = SymmetricFunctions(ZZ['q'].fraction_field())
sage: qbar = Sym.hecke_character()
sage: s = Sym.s()
sage: s(qbar[2,1])
-s[1, 1, 1] + (q-1)*s[2, 1] + q*s[3]
\end{verbatim}

$k$-Schur functions

The $k$-Schur functions live in the $k$-bounded subspace of the ring of symmetric functions. It is possible to compute in the $k$-bounded subspace directly:

\begin{verbatim}
sage: Sym = SymmetricFunctions(QQ)
sage: ks = Sym.kschur(3,1)
sage: f = ks[2,1]*ks[2,1]; f
ks3[2, 2, 1, 1] + ks3[2, 2, 2] + ks3[3, 1, 1, 1]
\end{verbatim}

or to lift to the ring of symmetric functions:

\begin{verbatim}
sage: f.lift()
s[2, 2, 1, 1] + s[2, 2, 2] + s[3, 1, 1, 1] + 2*s[3, 2, 1] + s[4, 1, 1] + s[4, 2]
\end{verbatim}

However, it is not always possible to convert a symmetric function to the $k$-bounded subspace:

\begin{verbatim}
sage: s = Sym.schur()
sage: ks(s[2,1,1])
Traceback (most recent call last):
  ...
ValueError: s[2, 1, 1] is not in the image
\end{verbatim}

The $k$-Schur functions are more generally defined with a parameter $t$ and they are a basis of the subspace spanned by the Hall-Littlewood $Q_p$ symmetric functions indexed by partitions whose first part is less than or equal to $k$:

\begin{verbatim}
sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: SymS3 = Sym.kBoundedSubspace(3)   # default t='t'
sage: ks = SymS3.kschur()
sage: Qp = Sym.hall_littlewood().Qp()
sage: ks(Qp[2,1,1,1])
ks3[2, 1, 1, 1] + (t^2+t)*ks3[2, 2, 1] + (t^3+t^2)*ks3[3, 1, 1] + t^4*ks3[3, 2]
\end{verbatim}

The subspace spanned by the $k$-Schur functions with a parameter $t$ are not known to form a natural algebra. However it is known that the product of a $k$-Schur function and an $\ell$-Schur function is in the linear span of the $k + \ell$-Schur functions:

\begin{verbatim}
sage: ks(ks[2,1]*ks[1,1])
Traceback (most recent call last):
\end{verbatim}
... ValueError: s[2, 1, 1, 1] + s[2, 2, 1] + s[3, 1, 1] + s[3, 2] is not in the image

sage: ks[2,1]*ks[1,1]
s[2, 1, 1, 1] + s[2, 2, 1] + s[3, 1, 1] + s[3, 2]
sage: ks6 = Sym.kBoundedSubspace(6).kschur()
sage: ks6(ks6[3,1,1]*ks[3])

The $k$-split basis is a second basis of the ring spanned by the $k$-Schur functions with a parameter $t$. The $k$-split basis has the property that $Q'_k[X; t]$ expands positively in the $k$-split basis and the $k$-split basis conjecturally expands positively in the $k$-Schur functions. The definition can be found in [LLMSSZ] p. 81.:

sage: ksp3 = Symm3.ksplit()
sage: ksp3(Qp[2,1,1,1])
ksp3[2, 1, 1, 1] + t^2*ksp3[2, 2, 1] + (t^3+t^2)*ksp3[3, 1, 1] + t^4*ksp3[3, 2]

The dual space to the subspace spanned by the $k$-Schur functions is most naturally realized as a quotient of the ring of symmetric functions by an ideal. When $t = 1$ the ideal is generated by the monomial symmetric functions indexed by partitions whose first part is greater than $k$.:

sage: Sym = SymmetricFunctions(QQ)
sage: SymQ3 = Sym.kBoundedQuotient(3,t=1)
sage: km = SymQ3.kmonomial()
sage: km[2,1]*km[2,1]
4*m3[2, 2, 1, 1] + 6*m3[2, 2, 2] + 2*m3[3, 2, 1] + 2*m3[3, 3]

When $t$ is not equal to 1, the subspace spanned by the $k$-Schur functions is realized as a quotient of the ring of symmetric functions by the ideal generated by the Hall-Littlewood symmetric functions in the P basis indexed by partitions with first part greater than $k$.:

sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: SymQ3 = Sym.kBoundedQuotient(3)
sage: kHLP = SymQ3.kHallLittlewoodP()
sage: kHLP[2,1]*kHLP[2,1]
(t^2+2*t+1)*HLP[2, 2, 1, 1] + (t^3+2*t^2+2*t+1)*HLP[2, 2, 2] + (-t^4-t^3+t+1)*HLP[3, 1, 1, 1] + (-t^2+t+2)*HLP[3, 2, 1] + (t+1)*HLP[3, 3]
In this space, the basis which is dual to the $k$-Schur functions conjecturally expands positively in the $k$-bounded Hall-Littlewood functions and has positive structure coefficients:

```
sage: dks = SymQ3.dual_k_Schur()
sage: kHLP(dks[2,2])
(t^4+t^2)*HLP3[1, 1, 1, 1] + t*HLP3[2, 1, 1] + HLP3[2, 2]
sage: dks[2,1]*dks[1,1]
(t^2+t)*dks3[1, 1, 1, 1, 1] + (t+1)*dks3[2, 1, 1, 1] + (t+1)*dks3[2, 2, 1] + dks3[3, 1, 1] + dks3[3, 2]
```

At $t = 1$ the $k$-bounded Hall-Littlewood basis is equal to the $k$-bounded monomial basis and the dual $k$-Schur elements are equal to the affine Schur basis. The $k$-bounded monomial basis and affine Schur functions are faster and should be used instead of the $k$-bounded Hall-Littlewood P basis and dual $k$-Schur functions when $t = 1$:

```
sage: SymQ3 = Sym.kBoundedQuotient(3,t=1)
sage: dks = SymQ3.dual_k_Schur()
sage: F = SymQ3.affineSchur()
sage: F[3,1]==dks[3,1]
True
```

**Implementing new bases**

In order to implement a new symmetric function basis, Sage will need to know at a minimum how to change back and forth between at least one other basis (although they do not necessarily have to be the same basis). All of the standard functions associated with the basis will have a default implementation (although a more specific implementation may be more efficient).

To present an idea of how this is done, we will create here the example of how to implement the basis $s_{\mu}[X(1-t)]$.

To begin, we import the class `sage.combinat.sf.sfa.SymmetricFunctionAlgebra_generic()`. Our new basis will inherit all of the default methods from this class:

```
sage: from sage.combinat.sf.sfa import SymmetricFunctionAlgebra_generic as SFA_
  ˓→generic
```

Now the basis we are creating has a parameter $t$ which is possible to specialize. In this example we will convert to and from the Schur basis. For this we implement methods `_self_to_s` and `_s_to_self`. By registering these two functions as coercions, Sage then knows automatically how it possible to change between any two bases for which there is a path of changes of bases.

```
sage: from sage.categories.morphism import SetMorphism
sage: class SFA_st(SFA_generic):
    ...:     def __init__(self, Sym, t):
    ...:         SFA_generic.__init__(self, Sym, basis_name=
    ...:             "Schur functions with a plethystic substitution of X -> X(1-t)",
    ...:             prefix='st')
    ...:         self._s = Sym.s()
    ...:         self.t = Sym.base_ring()(t)
    ...:         cat = HopfAlgebras(Sym.base_ring()).WithBasis()
    ...:         self.register_coercion(
    ...:             SetMorphism(Hom(self._s, self, cat), self._s_to_self))
    ...:         self._s.register_coercion(
    ...:             SetMorphism(Hom(self, self._s, cat), self._self_to_s))
    ...:     def _s_to_self(self, f):
```

(continues on next page)
An instance of this basis is created by calling it with a symmetric function ring $\text{Sym}$ and a parameter $t$ which is in the base ring of $\text{Sym}$. The $\text{Element}$ class inherits all of the methods from $\text{sage.combinat.sf.sfa.SymmetricFunctionAlgebra_generic_Element}$. In the reference [MAC] on page 354, this basis is denoted $S_\lambda(x; t)$ and the change of basis coefficients of the Macdonald $J$ basis are the coefficients $K_{\lambda \mu}(q, t)$. Here is an example of its use:

```python
sage: QQqt = QQ['q','t'].fraction_field()
sage: (q,t) = QQqt.gens()
sage: st = SFA_st(SymmetricFunctions(QQqt),t)
sage: st
Symmetric Functions over Fraction Field of Multivariate Polynomial
Ring in q, t over Rational Field in the Schur functions with a
plethystic substitution of X -> X(1-t) basis
sage: st[2]).coproduct()
sage: J = st.symmetric_function_ring().macdonald().J()
sage: st(J[2,1])
    q*st[1, 1, 1] + (q*t+1)*st[2, 1] + t*st[3]
```

**Acknowledgements**

The design is heavily inspired from the implementation of symmetric functions in MuPAD-Combinat (see [HT04] and [FD06]).

**REFERENCES:**

**Further tests**

**Todo:**

- Introduce fields with degree 1 elements as in MuPAD-Combinat, to get proper plethysm.
- Use UniqueRepresentation to get rid of all the manual cache handling for the bases
- Devise a mechanism so that pickling bases of symmetric functions pickles the coercions which have a cache.

**Schur()**

The Schur basis of the Symmetric Functions

**EXAMPLES:**
```
sage: SymmetricFunctions(QQ).schur()
Symmetric Functions over Rational Field in the Schur basis

Witt(coerce_h=True, coerce_e=False, coerce_p=False)
The Witt basis of the symmetric functions.

EXAMPLES:
```
sage: SymmetricFunctions(QQ).witt()
Symmetric Functions over Rational Field in the Witt basis
```
sage: SymmetricFunctions(QQ).witt(coerce_p=True)
Symmetric Functions over Rational Field in the Witt basis
```
sage: SymmetricFunctions(QQ).witt(coerce_h=False, coerce_e=True, coerce_p=True)
Symmetric Functions over Rational Field in the Witt basis

```
a_realization()
Return a particular realization of self (the Schur basis).

EXAMPLES:
```
sage: Sym = SymmetricFunctions(QQ)
sage: Sym.a_realization()
Symmetric Functions over Rational Field in the Schur basis

```
complete()
The complete basis of the Symmetric Functions

EXAMPLES:
```
sage: SymmetricFunctions(QQ).complete()
Symmetric Functions over Rational Field in the homogeneous basis

```
e()
The elementary basis of the Symmetric Functions

EXAMPLES:
```
sage: SymmetricFunctions(QQ).elementary()
Symmetric Functions over Rational Field in the elementary basis

```
elementary()
The elementary basis of the Symmetric Functions

EXAMPLES:
```
sage: SymmetricFunctions(QQ).elementary()
Symmetric Functions over Rational Field in the elementary basis

```
f()
The forgotten basis of the Symmetric Functions (or the basis dual to the elementary basis with respect to the Hall scalar product).

EXAMPLES:
```
sage: SymmetricFunctions(QQ).forgotten()
Symmetric Functions over Rational Field in the forgotten basis
```
forgotten()

The forgotten basis of the Symmetric Functions (or the basis dual to the elementary basis with respect to the Hall scalar product).

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).forgotten()
Symmetric Functions over Rational Field in the forgotten basis
```

from_polynomial(f)

Converts a symmetric polynomial f to a symmetric function.

INPUT:

• f – a symmetric polynomial

This function converts a symmetric polynomial f in a polynomial ring in finitely many variables to a symmetric function in the monomial basis of the ring of symmetric functions over the same base ring.

EXAMPLES:

```python
sage: P = PolynomialRing(QQ, 'x', 3)
sage: x = P.gens()
sage: f = x[0] + x[1] + x[2]
sage: S = SymmetricFunctions(QQ)
sage: S.from_polynomial(f)
m[1]
sage: f = x[0] + 2*x[1] + x[2]
sage: S.from_polynomial(f)
Traceback (most recent call last):
  ... ValueError: x0 + 2*x1 + x2 is not a symmetric polynomial
```

h()

The complete basis of the Symmetric Functions

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).complete()
Symmetric Functions over Rational Field in the homogeneous basis
```

hall_littlewood(t=t')

Returns the entry point for the various Hall-Littlewood bases.

INPUT:

• t – parameter

Hall-Littlewood symmetric functions including bases P, Q, Qp. The Hall-Littlewood P and Q functions at $t = -1$ are the Schur-P and Schur-Q functions when indexed by strict partitions.

The parameter $t$ must be in the base ring of parent.

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: P = Sym.hall_littlewood().P(); P
```

Symmetric Functions over Fraction Field of Univariate Polynomial Ring in \( t \) over Rational Field in the Hall-Littlewood P basis

\[
\text{sage}: \text{P}[2]
\]

\[
\text{sage}: \text{HLP}[2]
\]

Symmetric Functions over Fraction Field of Univariate Polynomial Ring in \( t \) over Rational Field in the Hall-Littlewood Q basis

\[
\text{sage}: \text{Q}[2]
\]

\[
\text{sage}: \text{HLQ}[2]
\]

Symmetric Functions over Fraction Field of Univariate Polynomial Ring in \( t \) over Rational Field in the Hall-Littlewood Qp basis

\[
\text{sage}: \text{Qp}[2]
\]

\[
\text{sage}: \text{HLQp}[2]
\]

**hecke_character** \((q=q')\)

The basis of symmetric functions that determines the character tables for Hecke algebras.

**EXAMPLES:**

\[
\text{sage}: \text{SymmetricFunctions(ZZ['q'].fraction_field()).hecke_character()}
\]

Symmetric Functions over Fraction Field of Univariate Polynomial Ring in \( q \) over Integer Ring in the Hecke character with \( q=q \) basis

\[
\text{sage}: \text{SymmetricFunctions(QQ).hecke_character(1/2)}
\]

Symmetric Functions over Rational Field in the Hecke character with \( q=1/2 \) basis

**homogeneous**()

The complete basis of the Symmetric Functions

**EXAMPLES:**

\[
\text{sage}: \text{SymmetricFunctions(QQ).complete()}
\]

Symmetric Functions over Rational Field in the homogeneous basis

**ht**()

The induced trivial character basis of the Symmetric Functions.

The trivial character of

\[
S_{n-|\lambda|} \times S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell(\lambda)}
\]

induced to the group \( S_n \) is a symmetric function in the eigenvalues of a permutation matrix. This basis is that character.

It has the property that if the element indexed by the partition \( \lambda \) is evaluated at the roots of a permutation of cycle structure \( \rho \) then the value is the coefficient \( \langle h_{(n-|\lambda|,\lambda)}, p_\rho \rangle \).

In terms of methods that are implemented in Sage, if \( n \) is a sufficiently large integer, then \( \text{ht}(\text{lam}) \).

\text{character_to_frobenius_image}(n) \) is equal to the complete function indexed by \( [n-\text{sum}(\lambda)]+\lambda \).

This basis is introduced in [OZ2015].

**See also:**

\text{character_to_frobenius_image()}, \text{eval_at_permutation_roots}()

**EXAMPLES:**

5.1. Comprehensive Module List
induced_trivial_character()

The induced trivial character basis of the Symmetric Functions.

The trivial character of

$$S_{n-|\lambda|} \times S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell(\lambda)}$$

induced to the group $S_n$ is a symmetric function in the eigenvalues of a permutation matrix. This basis is that character.

It has the property that if the element indexed by the partition $\lambda$ is evaluated at the roots of a permutation of cycle structure $\rho$ then the value is the coefficient $\langle h_{(n-|\lambda|,\lambda)}, p_\rho \rangle$.

In terms of methods that are implemented in Sage, if $n$ is a sufficiently large integer, then $ht(lam).character_to_frobenius_image(n)$ is equal to the complete function indexed by $[n-\text{sum(lam)}]+lam$.

This basis is introduced in [OZ2015].

See also:

character_to_frobenius_image().eval_at_permutation_roots()

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).induced_trivial_character()
Symmetric Functions over Rational Field in the induced trivial symmetric group...
˓→-character basis
sage: ht = SymmetricFunctions(QQ).ht()
sage: h = SymmetricFunctions(QQ).h()
sage: h(ht([3,2]).character_to_frobenius_image(9))
h[4, 3, 2]
sage: h(ht([3,2]).character_to_frobenius_image(7))
h[3, 2, 2]
sage: h(ht([3,2]).character_to_frobenius_image(5))
h[3, 2]
sage: h(ht([3,2]).character_to_frobenius_image(4))
0
sage: p = SymmetricFunctions(QQ).p()
sage: [h([4,1]).scalar(p(rho)) for rho in Partitions(5)]
[0, 1, 0, 2, 1, 3, 5]
sage: [ht([1]).eval_at_permutation_roots(rho) for rho in Partitions(5)]
[0, 1, 0, 2, 1, 3, 5]
```

(continues on next page)
sage: p = SymmetricFunctions(QQ).p()
sage: [h([4,1]).scalar(p(rho)) for rho in Partitions(5)]
[0, 1, 0, 2, 1, 3, 5]
sage: [ht([1]).eval_at_permutation_roots(rho) for rho in Partitions(5)]
[0, 1, 0, 2, 1, 3, 5]

irreducible_symmetric_group_character()

The irreducible $S_n$ character basis of the Symmetric Functions.

This basis has the property that if the element indexed by the partition $\lambda$ is evaluated at the roots of a permutation of cycle structure $\rho$ then the value is the irreducible character $\chi^{(|\rho|-|\lambda|,\lambda)}(\rho)$.

In terms of methods that are implemented in Sage, if $n$ is a sufficiently large integer, then $st(lam)$.character_to_frobenius_image(n) is equal the Schur function indexed by $[n-sum(lam)]+lam$.

This basis is introduced in [OZ2015].

See also:

character_to_frobenius_image(), eval_at_permutation_roots()

EXAMPLES:

sage: SymmetricFunctions(QQ).irreducible_symmetric_group_character()
Symmetric Functions over Rational Field in the irreducible symmetric group character basis
sage: st = SymmetricFunctions(QQ).st()
sage: s = SymmetricFunctions(QQ).s()
sage: s(st([3,2]).character_to_frobenius_image(9))
s[4, 3, 2]
sage: s(st([3,2]).character_to_frobenius_image(7))
0
sage: s(st([3,2]).character_to_frobenius_image(6))
-s[2, 2, 2]
sage: list(SymmetricGroup(5).character_table()[-2])
[4, 2, 0, 1, -1, 0, -1]
sage: list(reversed([st([1]).eval_at_permutation_roots(rho) for rho in Partitions(5)]))
[4, 2, 0, 1, -1, 0, -1]

jack($t'$)

Returns the entry point for the various Jack bases.

INPUT:

* $t$ – parameter

Jack symmetric functions including bases $P, Q, Qp$.

The parameter $t$ must be in the base ring of parent.

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: JP = Sym.jack().P(); JP
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack P basis

(continues on next page)
sage: JQ = Sym.jack().Q(); JQ
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack Q basis

sage: JJ = Sym.jack().J(); JJ
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack J basis

sage: JQp = Sym.jack().Qp(); JQp
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field in the Jack Qp basis

kBoundedQuotient($k, t'='t)$

Returns the $k$-bounded quotient space of the ring of symmetric functions.

**INPUT:**

- $k$ - a positive integer

The quotient of the ring of symmetric functions ...

**See also:**

`sage.combinat.sf.k_dual.KBoundedQuotient()`

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: KQ = Sym.kBoundedQuotient(3); KQ
Traceback (most recent call last):
... TypeError: unable to convert 't' to a rational
sage: KQ = Sym.kBoundedQuotient(3,t=1); KQ
3-Bounded Quotient of Symmetric Functions over Rational Field with t=1
sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: KQ = Sym.kBoundedQuotient(3); KQ
3-Bounded Quotient of Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

kBoundedSubspace($k, t'='t)$

Return the $k$-bounded subspace of the ring of symmetric functions.

**INPUT:**

- $k$ - a positive integer
- $t$ a formal parameter; $t = 1$ yields a subring

The subspace of the ring of symmetric functions spanned by $\{ s_{\lambda}[X/(1 - t)] \}_{\lambda_1 \leq k} = \{ s_{\lambda}^{(k)}[X,t] \}_{\lambda_1 \leq k}$ over the base ring $\mathbb{Q}[t]$. When $t = 1$, this space is in fact a subalgebra of the ring of symmetric functions generated by the complete homogeneous symmetric functions $h_i$ for $1 \leq i \leq k$.

**See also:**

`sage.combinat.sf.new kschur.KBoundedSubspace()`

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: KB = Sym.kBoundedSubspace(3,1); KB
```
3-bounded Symmetric Functions over Rational Field with $t=1$

```python
sage: Sym = SymmetricFunctions(QQ['t'])
sage: Sym.kBoundedSubspace(3)
```

3-bounded Symmetric Functions over Univariate Polynomial Ring in $t$ over Rational Field

```python
sage: Sym = SymmetricFunctions(QQ['z'])
sage: z = Sym.base_ring().gens()[0]
sage: Sym.kBoundedSubspace(3, t=z)
```

3-bounded Symmetric Functions over Univariate Polynomial Ring in $z$ over Rational Field with $t=z$

### `khomogeneous($k$)`

Returns the homogeneous symmetric functions in the $k$-bounded subspace.

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: kh = Sym.khomogeneous(4)
sage: kh[3]*kh[4]
```

$h_4[4, 3]$

```python
sage: kh[4].lift()
```

$h_4$

### `kschur($k$, $t='t'$)`

Returns the $k$-Schur functions.

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: ks = Sym.kschur(3,1)
sage: ks[2]*ks[2]
```

$ks_3[2, 2] + ks_3[3, 1]$

```python
sage: ks[2,1,1].lift()
```

$s_2, 1, 1] + s_3[3, 1]$

### `ksplit($k$, $t='t'$)`

Returns the $k$-split basis of the $k$-bounded subspace.

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: ksp = Sym.ksplit(3,1)
sage: ksp[2]*ksp[2]
```

$ksp_3[2, 2] + ksp_3[3, 1]$

```python
sage: ksp[2,1,1].lift()
```

$s_2, 1, 1] + s_3[3, 1]$

(continues on next page)
sage: Sym = SymmetricFunctions(QQ['t'])
sage: ksp = Sym.ksplit(3)
sage: ksp[2,1,1].lift()
\[s[2, 1, 1] + t^2 s[2, 2] + t^2 s[3, 1]\]

The LLT symmetric functions.

INPUT:

- k – a positive integer indicating the level
- t – a parameter (default: t)

LLT polynomials in $h_{spin}$ and $hcospin$ bases.

EXAMPLES:

sage: llt3 = SymmetricFunctions(QQ['t'].fraction_field()).llt(3); llt3
level 3 LLT polynomials over Fraction Field of Univariate Polynomial Ring in t \rightarrow over Rational Field
sage: llt3.hspin()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over \rightarrow Rational Field in the level 3 LLT spin basis
sage: llt3.hcospin()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in t over \rightarrow Rational Field in the level 3 LLT cospin basis

The monomial basis of the Symmetric Functions

EXAMPLES:

sage: SymmetricFunctions(QQ).monomial()
Symmetric Functions over Rational Field in the monomial basis

macdonald($q=q', t=t'$)

Returns the entry point for the various Macdonald bases.

INPUT:

- q, t – parameters

Macdonald symmetric functions including bases $P, Q, J, H, Ht$. This also contains the $S$ basis which is dual to the Schur basis with respect to the $q, t$ scalar product.

The parameters $q$ and $t$ must be in the base_ring of parent.

EXAMPLES:

sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: P = Sym.macdonald().P(); P
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t, \rightarrow
over Rational Field in the Macdonald P basis

```python
sage: P[2]
McdP[2]
```

over Rational Field in the Macdonald Q basis

```python
sage: Q = Sym.macdonald().Q(); Q
Symmetric Functions over Fraction Field of Multivariate Polynomial Ring in q, t
over Rational Field in the Macdonald Q basis
```

```python
sage: S = Sym.macdonald().S()
sage: s = Sym.schur()
sage: matrix([[(S(la).scalar_qt(s(mu)) for la in Partitions(3)] for mu in Partitions(3)])
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
sage: H = Sym.macdonald().H()
sage: s(H[2,2])
q^2*s[1, 1, 1] + (q^2*t+q*t+q)*s[2, 1, 1] + (q^2*t^2+1)*s[2, 2] + (q*t^2+q*t+t)*s[3, 1] + t^2*s[4]
```

```python
sage: Sym = SymmetricFunctions(QQ['z','q']).fraction_field()
sage: (z,q) = Sym.base_ring().gens()
sage: Hzq = Sym.macdonald(q=z,t=q).H()
sage: H1z = Sym.macdonald(q=1,t=z).H()
sage: s = Sym.schur()
sage: s(H1z([2,2]))
s[1, 1, 1] + (2*z+1)*s[2, 1, 1] + (z^2+1)*s[2, 2] + (z^2+2*z)*s[3, 1] + z^2*s[4]
sage: s(Hzq[2,2])
z^2*s[1, 1, 1] + (z^2*q+z*q+z)*s[2, 1, 1] + (z^2*q^2+1)*s[2, 2] + (z^2*q^2+z*q+q)*s[3, 1] + q^2*s[4]
sage: s(H1z(Hzq[2,2]))
z^2*s[1, 1, 1] + (z^2*q+z*q+z)*s[2, 1, 1] + (z^2*q^2+1)*s[2, 2] + (z^2*q^2+z*q+q)*s[3, 1] + q^2*s[4]
```

```
mynomial()

The monomial basis of the Symmetric Functions

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).monomial()
Symmetric Functions over Rational Field in the monomial basis
```

```
o()

The orthogonal basis of the symmetric functions.

See also:

SymmetricFunctionAlgebra_orthogonal

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).orthogonal()
Symmetric Functions over Rational Field in the orthogonal basis
```

```
othogonal()

The orthogonal basis of the symmetric functions.
```
See also:

`SymmetricFunctionAlgebra_orthogonal`

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).orthogonal()
Symmetric Functions over Rational Field in the orthogonal basis
```

\(p()\)

The power sum basis of the Symmetric Functions

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).powersum()
Symmetric Functions over Rational Field in the powersum basis
```

\(\text{power}()\)

The power sum basis of the Symmetric Functions

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).powersum()
Symmetric Functions over Rational Field in the powersum basis
```

\(\text{powersum}()\)

The power sum basis of the Symmetric Functions

EXAMPLES:

```python
sage: SymmetricFunctions(QQ).powersum()
Symmetric Functions over Rational Field in the powersum basis
```

\(qbar(q='q')\)

The basis of symmetric functions that determines the character tables for Hecke algebras.

EXAMPLES:

```python
sage: SymmetricFunctions(ZZ['q'].fraction_field()).hecke_character()
Symmetric Functions over Fraction Field of Univariate Polynomial Ring in q over Integer Ring in the Hecke character with q=q basis
sage: SymmetricFunctions(QQ).hecke_character(1/2)
Symmetric Functions over Rational Field in the Hecke character with q=1/2 basis
```

\(\text{register_isomorphism}(\text{morphism}, only\_conversion=False)\)

Register an isomorphism between two bases of self, as a canonical coercion (unless the optional keyword only\_conversion is set to True, in which case the isomorphism is registered as conversion only).

EXAMPLES:

We override the canonical coercion from the Schur basis to the powersum basis by a (stupid!) map \(s_\lambda \mapsto 2p_\lambda\).

```python
sage: Sym = SymmetricFunctions(QQ[['zorglub']]) # make sure we are not going to screw up later tests
sage: s = Sym.s(); p = Sym.p().dual_basis()
```

(continues on next page)
sage: phi = s.module_morphism(diagonal = lambda t: 2, codomain = p)
sage: phi(s[2, 1])
2*d_p[2, 1]
sage: Sym.register_isomorphism(phi)
sage: p(s[2,1])
2*d_p[2, 1]

The map is supposed to implement the canonical isomorphism between the two bases. Otherwise, the results will be mathematically wrong, as above. Use with care!

s()

The Schur basis of the Symmetric Functions

EXAMPLES:

```sage```
SymmetricFunctions(QQ).schur()
Symmetric Functions over Rational Field in the Schur basis
```

schur()

The Schur basis of the Symmetric Functions

EXAMPLES:

```sage```
SymmetricFunctions(QQ).schur()
Symmetric Functions over Rational Field in the Schur basis
```

sp()

The symplectic basis of the symmetric functions.

See also:

SymmetricFunctionAlgebra_symplectic

EXAMPLES:

```sage```
SymmetricFunctions(QQ).symplectic()
Symmetric Functions over Rational Field in the symplectic basis
```

st()

The irreducible $S_n$ character basis of the Symmetric Functions.

This basis has the property that if the element indexed by the partition $\lambda$ is evaluated at the roots of a permutation of cycle structure $\rho$ then the value is the irreducible character $\chi^{(|\rho| - |\lambda|, \lambda)}(\rho)$.

In terms of methods that are implemented in Sage, if $n$ is a sufficiently large integer, then $st(lam).character_to_frobenius_image(n)$ is equal the Schur function indexed by $[n-\text{sum(lam)}]+lam$.

This basis is introduced in [OZ2015].

See also:

character_to_frobenius_image(), eval_at_permutation_roots()

EXAMPLES:

```sage```
SymmetricFunctions(QQ).irreducible_symmetric_group_character()
Symmetric Functions over Rational Field in the irreducible symmetric group character basis
```
sage: st = SymmetricFunctions(QQ).st()
sage: s = SymmetricFunctions(QQ).s()
sage: s(st([3,2]).character_to_frobenius_image(9))
[4, 3, 2]
sage: s(st([3,2]).character_to_frobenius_image(7))
0
sage: s(st([3,2]).character_to_frobenius_image(6))
-s[2, 2, 2]
sage: list(SymmetricGroup(5).character_table()[-2])
[4, 2, 0, 1, -1, 0, -1]

**symplectic()**

The symplectic basis of the symmetric functions.

See also:

```
SymmetricFunctionAlgebra_symplectic
```

EXAMPLES:

```sage
sage: SymmetricFunctions(QQ).symplectic()
Symmetric Functions over Rational Field in the symplectic basis
```

w(coerce_h=True, coerce_e=False, coerce_p=False)

The Witt basis of the symmetric functions.

EXAMPLES:

```sage
sage: SymmetricFunctions(QQ).witt()
Symmetric Functions over Rational Field in the Witt basis
```

witt(coerce_h=True, coerce_e=False, coerce_p=False)

The Witt basis of the symmetric functions.

EXAMPLES:

```sage
sage: SymmetricFunctions(QQ).witt()
Symmetric Functions over Rational Field in the Witt basis
```

zonal()

The zonal basis of the Symmetric Functions

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: Sym = SymmetricFunctions(QQ['x'])
sage: p = Sym.p(); s = Sym.s()
sage: def t(x) : [(p,c)] = x; return [(p,2*c), (p.conjugate(), c)]
sage: f = sage.combinat.sf.sf.SymmetricaConversionOnBasis(t, p, s)
sage: f(Partition([3,1]))
s[2, 1, 1] + 2*s[3, 1]
```

5.1.301 Symmetric Functions

For a comprehensive tutorial on how to use symmetric functions in Sage

See also:

SymmetricFunctions()

We define the algebra of symmetric functions in the Schur and elementary bases:

```python
sage: s = SymmetricFunctions(QQ).schur()
sage: e = SymmetricFunctions(QQ).elementary()
```

Each is actually a graded Hopf algebra whose basis is indexed by integer partitions:

```python
sage: s.category()
Category of graded bases of Symmetric Functions over Rational Field
sage: s.basis().keys()
Partitions
```

Let us compute with some elements in different bases:

```python
sage: f1 = s([2,1]); f1
s[2, 1]
sage: f2 = e(f1); f2 # basis conversion
e[2, 1] - e[3]
sage: f1 == f2
True
```
Combinatorics, Release 10.0

sage: f1.expand(3, alphabet=['x','y','z'])
x^2*y + x*y^2 + x^2*z + 2*x*y*z + y^2*z + x*z^2 + y*z^2

sage: f2.expand(3, alphabet=['x','y','z'])
x^2*y + x*y^2 + x^2*z + 2*x*y*z + y^2*z + x*z^2 + y*z^2

sage: m = SymmetricFunctions(QQ).monomial()
sage: m([3,1])
m[3, 1]
sage: m(4) # This is the constant 4, not the partition 4.
4*m[]
sage: m([4]) # This is the partition 4.
m[4]
sage: 3*m([3,1]) - 1/2*m([4])
3*m[3, 1] - 1/2*m[4]

sage: p = SymmetricFunctions(QQ).power()
sage: f = p(3)
sage: f
3*p[]
sage: f.parent()
Symmetric Functions over Rational Field in the powersum basis
sage: f + p([3,2])
3*p[] + p[3, 2]

One can convert symmetric functions to symmetric polynomials and vice versa:

sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: h = Sym.homogeneous()
sage: p = Sym.p()
sage: e = Sym.e()
sage: m = Sym.m()
sage: a = s([3,1])
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
Here are further examples:

```
sage: h(m([1]))
h[1]
sage: h( m([2]) + m([1, 1]) )
h[2]
sage: h( m([3]) + m([2, 1]) + m([1, 1, 1]) )
h[3]
sage: h( m([4]) + m([3, 1]) + m([2, 2]) + m([2, 1, 1]) + m([1, 1, 1, 1]) )
h[4]
sage: k = 5
sage: h( sum([ m(part) for part in Partitions(k) ]))
h[5]
sage: k = 10
sage: h( sum([ m(part) for part in Partitions(k) ]))
h[10]
```

```python
def P3 = Partitions(3)
sage: P3.list()
[[3], [2, 1], [1, 1, 1]]
sage: m = SymmetricFunctions(QQ).monomial()
sage: f = sum([m(p) for p in P3])
sage: m.get_print_style()
'lex'
sage: f
m[1, 1, 1] + m[2, 1] + m[3]
sage: m.set_print_style('length')
sage: f
m[3] + m[2, 1] + m[1, 1, 1]
sage: m.set_print_style('maximal_part')
sage: f
m[1, 1, 1] + m[2, 1] + m[3]
sage: m.set_print_style('lex')
```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: m = Sym.m()
sage: m([3])*s([2,1])
2*m[3, 1, 1, 1] + m[3, 2, 1] + 2*m[4, 1, 1] + m[4, 2] + m[5, 1]
sage: s(m([3])*s([2,1])))
s[2, 1, 1, 1, 1] - s[2, 2, 2] - s[3, 3] + s[5, 1]
sage: e = Sym.e()
sage: e([4])*e([3])*e([1])
e[4, 3, 1]

AUTHORS:

- Mike Hansen (2007-06-15)
- Nicolas M. Thiery (partial refactoring)
- Mike Zabrocki, Anne Schilling (2012)
- Darij Grinberg (2013) Sym over rings that are not characteristic 0

class sage.combinat.sf.sfa.FilteredSymmetricFunctionsBases(parent_with_realization)

Bases: Category_realization_of_parent

The category of filtered bases of the ring of symmetric functions.

super_categories()

The super categories of self.

EXAMPLES:

```
sage: from sage.combinat.sf.sfa import FilteredSymmetricFunctionsBases
sage: Sym = SymmetricFunctions(QQ)
sage: bases = FilteredSymmetricFunctionsBases(Sym)
sage: bases.super_categories()
[Category of bases of Symmetric Functions over Rational Field, Category of commutative filtered hopf algebras with basis over Rational Field]
```

class sage.combinat.sf.sfa.GradedSymmetricFunctionsBases(parent_with_realization)

Bases: Category_realization_of_parent

The category of graded bases of the ring of symmetric functions.

These are further required to have the property that the basis element indexed by the empty partition is 1.
class ElementMethods
    Bases: object

degree_negation()
    Return the image of self under the degree negation automorphism of the ring of symmetric functions.
    The degree negation is the automorphism which scales every homogeneous element of degree \( k \) by \((-1)^k\) (for all \( k \)). Calling `degree_negation(self)` is equivalent to calling `self.parent().degree_negation(self)`.

    EXAMPLES:
    sage: Sym = SymmetricFunctions(ZZ)
    sage: m = Sym.monomial()
    sage: f = 2*m[2,1] + 4*m[1,1] - 5*m[1] - 3*m[[]]
    sage: f.degree_negation()
    -3*m[[]] + 5*m[1] + 4*m[1, 1] - 2*m[2, 1]
    sage: x = m.zero().degree_negation(); x
    0
    sage: parent(x) is m
    True

degree_zero_coefficient()
    Return the degree zero coefficient of self.

    EXAMPLES:
    sage: Sym = SymmetricFunctions(QQ)
    sage: m = Sym.monomial()
    sage: f = 2*m[2,1] + 3*m[[]]
    sage: f.degree_zero_coefficient()
    3

is_unit()
    Return whether this element is a unit in the ring.

    EXAMPLES:
    sage: m = SymmetricFunctions(ZZ).monomial()
    sage: (2*m[2,1] + m[[]]).is_unit()
    False
    sage: (3/2*m([])).is_unit()
    True

class ParentMethods
    Bases: object

antipode_by_coercion(element)
    The antipode of element.

    INPUT:
    • element – element in a basis of the ring of symmetric functions

    EXAMPLES:
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.p()
sage: s = Sym.s()
sage: e = Sym.e()
sage: h = Sym.h()
sage: (h([]) + h([1])).antipode()  # indirect doctest
h[1] - h[]
sage: (s([]) + s([1]) + s[2]).antipode()
s[] - s[1] + s[1, 1]
sage: (p([2]) + p([3])).antipode()
sage: (e([2]) + e([3])).antipode()
sage: f = Sym.f()
sage: f([3, 2, 1]).antipode()

The antipode is an involution:

sage: Sym = SymmetricFunctions(ZZ)
sage: s = Sym.s()
sage: all( s[u].antipode().antipode() == s[u] for u in Partitions(4) )
True

The antipode is an algebra homomorphism:

sage: Sym = SymmetricFunctions(FiniteField(23))
sage: h = Sym.h()
sage: all( all( (s[u] * s[v]).antipode() == s[u].antipode() * s[v].antipode() )
      for u in Partitions(3) )
      for v in Partitions(3) )
True

counit(element)

Return the counit of element.

The counit is the constant term of element.

INPUT:
  • element – element in a basis of the ring of symmetric functions

EXAMPLES:

sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.monomial()
sage: f = 2*m[2,1] + 3*m[]
sage: f.counit()
3
degree_negation(element)

Return the image of element under the degree negation automorphism of the ring of symmetric functions.

The degree negation is the automorphism which scales every homogeneous element of degree \( k \) by \((-1)^k\) for all \( k \).
INPUT:

- **element** – symmetric function written in **self**

EXAMPLES:

```
sage: Sym = SymmetricFunctions(ZZ)
sage: m = Sym.monomial()
sage: f = 2*m[2,1] + 4*m[1,1] - 5*m[1] - 3*m[]
sage: m.degree_negation(f)
-3*m[] + 5*m[1] + 4*m[1, 1] - 2*m[2, 1]
```

**super_categories()**

The super categories of **self**.

EXAMPLES:

```
sage: from sage.combinat.sf.sfa import GradedSymmetricFunctionsBases
sage: Sym = SymmetricFunctions(QQ)
sage: bases = GradedSymmetricFunctionsBases(Sym)
sage: bases.super_categories()
[Category of filtered bases of Symmetric Functions over Rational Field,  
 Category of commutative graded hopf algebras with basis over Rational Field]
```

**class sage.combinat.sf.sfa.SymmetricFunctionAlgebra_generic(Sym, basis_name=None, prefix=None, graded=True)**

Bases: **CombinatorialFreeModule**

Abstract base class for symmetric function algebras.

**Todo:** Most of the methods in this class are generic (manipulations of morphisms,...) and should be generalized (or removed)

**Element**

alias of **SymmetricFunctionAlgebra_generic_Element**

**basis_name()**

Return the name of the basis of **self**.

This is used for output and, for the classical bases of symmetric functions, to connect this basis with Sym- metrica.

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
'schur'
sage: p = Sym.p()
'powersum'
sage: h = Sym.h()
sage: h.basis_name()
'homogeneous'
sage: e = Sym.e()
sage: e.basis_name()
```

(continues on next page)
'elementary'
sage: m = Sym.m()
sage: m.basis_name()
'monomial'
sage: f = Sym.f()
sage: f.basis_name()
'forgotten'

coproduct_by_coercion(elt)
Return the coproduct of the element elt by coercion to the Schur basis.

INPUT:
• elt – an instance of this basis

OUTPUT:
• The image of elt under the comultiplication (=coproduct) of the coalgebra of symmetric functions.
The result is an element of the tensor squared of the basis self.

EXAMPLES:
sage: m = SymmetricFunctions(QQ).m()
sage: m[3,1,1].coproduct()
```
```
sage: m.coproduct_by_coercion(m[2,1])
sage: m.coproduct_by_coercion(m[2,1]) == m([2,1]).coproduct()
True
sage: McdH = SymmetricFunctions(QQ['q','t'].fraction_field()).macdonald().H()
sage: McdH[2,1].coproduct()
```
```
sage: HLQp = SymmetricFunctions(QQ['t'].fraction_field()).hall_littlewood().Qp()
sage: HLQp[2,1].coproduct()
```
```
sage: Sym = SymmetricFunctions(FractionField(QQ['t']))
sage: LLT = Sym.llt(3)
sage: LLT.cospin([3,2,1]).coproduct()
```
```
sage: f = SymmetricFunctions(ZZ).f()
sage: f[3].coproduct()
```
```
sage: f[3,2,1].coproduct()
```
```

dual_basis(scalar=None, scalar_name='', basis_name=None, prefix=None)
Return the dual basis of self with respect to the scalar product scalar.

INPUT:
• scalar – A function zee from partitions to the base ring which specifies the scalar product by
\langle p_\lambda, p_\mu \rangle = zee(\lambda). (Independently on the function chosen, the power sum basis will always be or-
thogonal; the function scalar only determines the norms of the basis elements.) If scalar is None,
then the standard (Hall) scalar product is used.

• scalar_name – name of the scalar function

• prefix – prefix used to display the basis

EXAMPLES:

The duals of the elementary symmetric functions with respect to the Hall scalar product are the forgotten
symmetric functions.

```python
sage: e = SymmetricFunctions(QQ).e()
sage: f = e.dual_basis(prefix='f'); f
Dual basis to Symmetric Functions over Rational Field in the elementary basis
˓→ with respect to the Hall scalar product
sage: f([2,1])**2
sage: f([2,1]).scalar(e([2,1]))
1
sage: f([2,1]).scalar(e([1,1,1]))
0
```

Since the power-sum symmetric functions are orthogonal, their duals with respect to the Hall scalar product
are scalar multiples of themselves.

```python
sage: p = SymmetricFunctions(QQ).p()
sage: q = p.dual_basis(prefix='q'); q
Dual basis to Symmetric Functions over Rational Field in the powersum basis
˓→ with respect to the Hall scalar product
sage: q([2,1])**2
4*q[2, 2, 1, 1]
sage: p([2,1]).scalar(q([2,1]))
1
sage: p([2,1]).scalar(q([1,1,1]))
0
```

\textbf{from\_polynomial}(\textit{poly, check=True})

Convert polynomial to a symmetric function in the monomial basis and then to the basis \textit{self}.

INPUT:

• \textit{poly} – a symmetric polynomial

• \textit{check} – (default: True) boolean, specifies whether the computation checks that the polynomial is
indeed symmetric

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: h = Sym.homogeneous()
sage: f = (h([]) + h([2,1]) + h([3])).expand(3)
sage: h.from_polynomial(f)
\text{h[]} + \text{h[2, 1]} + \text{h[3]}
```
sage: s = Sym.s()
sage: g = (s([]) + s([2,1])).expand(3); g
x0^2*x1 + x0*x1^2 + 2*x0*x1*x2 + x1^2*x2 + x0^2*x2 + x1*x2^2 + 1
sage: s.from_polynomial(g)
s[] + s[2, 1]

get_print_style()
Return the value of the current print style for self.

EXAMPLES:

sage: s = SymmetricFunctions(QQ).s()
sage: s.get_print_style()
'lex'
sage: s.set_print_style('length')
sage: s.get_print_style()
'length'
sage: s.set_print_style('lex')

prefix()
Return the prefix on the elements of self.

EXAMPLES:

sage: schur = SymmetricFunctions(QQ).schur()
sage: schur([3,2,1])
s[3, 2, 1]
sage: schur.prefix()
's'

product_by_coercion(left, right)
Return the product of elements left and right by coercion to the Schur basis.

INPUT:

• left, right – instances of this basis

OUTPUT:

• the product of left and right expressed in the basis self

EXAMPLES:

sage: p = SymmetricFunctions(QQ).p()
sage: p.product_by_coercion(p[3,1,1], p[2,2])
p[3, 2, 2, 1, 1]
sage: m = SymmetricFunctions(QQ).m()
sage: m.product_by_coercion(m[2,1],m[1,1]) == m[2,1]*m[1,1]
True

set_print_style(ps)
Set the value of the current print style to ps.

INPUT:

• ps – a string specifying the printing style
EXAMPLES:

```sage
s = SymmetricFunctions(QQ).s()
sage: s.get_print_style()
'lex'
sage: s.set_print_style('length')
sage: s.get_print_style()
'length'
sage: s.set_print_style('lex')
```

**symmetric_function_ring()**

Return the family of symmetric functions associated to the basis `self`.

**OUTPUT:**

- returns an instance of the ring of symmetric functions

**EXAMPLES:**

```sage
schur = SymmetricFunctions(QQ).schur()
sage: schur.symmetric_function_ring()
Symmetric Functions over Rational Field

power = SymmetricFunctions(QQ['t']).power()
sage: power.symmetric_function_ring()
Symmetric Functions over Univariate Polynomial Ring in t over Rational Field
```

**transition_matrix(basis, n)**

Return the transition matrix between `self` and `basis` for the homogeneous component of degree `n`.

**INPUT:**

- `basis` – a basis of the ring of symmetric functions
- `n` – a nonnegative integer

**OUTPUT:**

- a matrix of coefficients giving the expansion of the homogeneous degree-\(n\) elements of `self` in the degree-\(n\) elements of `basis`

**EXAMPLES:**

```sage
s = SymmetricFunctions(QQ).s()
sage: m = SymmetricFunctions(QQ).m()
sage: s.transition_matrix(m,5)

[[1 1 1 1 1 1]
 [0 1 1 2 3 4]
 [0 0 1 1 2 3 5]
 [0 0 0 1 1 3 6]
 [0 0 0 0 1 2 5]
 [0 0 0 0 0 1 4]
 [0 0 0 0 0 0 1]]
sage: s.transition_matrix(m,1)

[[1]]
sage: s.transition_matrix(m,0)

[[1]]
```
```python
sage: p = SymmetricFunctions(QQ).p()
sage: s.transition_matrix(p, 4)  
[ 1/4  1/3  1/8  1/4  1/24]  
[ 0 -1/3  1/4  1/8  0  1/12]  
[ 1/4  0 -1/8 -1/4  1/8  0]  
[-1/4  1/3  1/8 -1/4  1/24]  
sage: StoP = s.transition_matrix(p, 4)
sage: a = s([3,1])+5*s([1,1,1,1])-s([4])
sage: a  
5*s[1, 1, 1, 1] + s[3, 1] - s[4]  
sage: mon = sorted(a.support())
sage: coeffs = [a[i] for i in mon]
sage: coeffs  
[5, 1, -1]  
sage: mon  
[[1, 1, 1, 1], [3, 1], [4]]  
sage: cm = matrix([[-1,1,0,0,5]])
sage: cm * StoP  
[-7/4  4/3  3/8 -5/4  7/24]
sage: p(a)  
sage: h = SymmetricFunctions(QQ).h()
sage: e = SymmetricFunctions(QQ).e()
sage: s.transition_matrix(m,7) == h.transition_matrix(s,7).transpose()  
True  
sage: h.transition_matrix(m, 7) == h.transition_matrix(m, 7).transpose()  
True  
sage: h.transition_matrix(e, 7) == e.transition_matrix(h, 7)  
True  
sage: p.transition_matrix(s, 5)  
[ 1 -1  0  1  0 -1  1]  
[ 1  0 -1  0  1  1 -1]  
[ 1 -1  1  0 -1 -1  1]  
[ 1  1 -1  0  1  1  1]  
[ 1  0  1 -2  1  0  1]  
[ 1  2  1  0 -1 -2 -1]  
[ 1  4  5  6  5  4  1]  
sage: e.transition_matrix(m,7) == e.transition_matrix(m,7).transpose()  
True  
```

**class sage.combinat.sf.sfa.SymmetricFunctionAlgebra_generic_Element**

Bases: IndexedFreeModuleElement

Class of generic elements for the symmetric function algebra.

**arithmetic_product(x)**

Return the arithmetic product of self and x in the basis of self.
The arithmetic product is a binary operation $\mathcal{A}$ on the ring of symmetric functions which is bilinear in its two arguments and satisfies

$$p_\lambda \mathcal{A} p_\mu = \prod_{i \geq 1, j \geq 1} \gcd(\lambda_i, \mu_j) \frac{\text{lcm}(\lambda_i, \mu_j)}{p_i \lambda_i \cdot p_j \mu_j}$$

for any two partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ (where $p_\nu$ denotes the power-sum symmetric function indexed by the partition $\nu$, and $p_i$ denotes the $i$-th power-sum symmetric function).

This is enough to define the arithmetic product if the base ring is torsion-free as a $\mathbb{Z}$-module; for all other cases the arithmetic product is uniquely determined by requiring it to be functorial in the base ring. See http://mathoverflow.net/questions/138148/ for a discussion of this arithmetic product.

If $f$ and $g$ are two symmetric functions which are homogeneous of degrees $a$ and $b$, respectively, then $f \mathcal{A} g$ is homogeneous of degree $ab$.

The arithmetic product is commutative and associative and has unity $e_1 = p_1 = h_1$.

INPUT:

• $x$ – element of the ring of symmetric functions over the same base ring as self

OUTPUT:

Arithmetic product of self with x; this is a symmetric function over the same base ring as self.

EXAMPLES:

```sage```
```python
sage: s = SymmetricFunctions(QQ).s()
sage: s([2]).arithmetic_product(s([2]))
s[1, 1, 1, 1] + 2*s[2, 2] + s[4]
sage: s([2]).arithmetic_product(s([1,1]))
s[2, 1, 1] + s[3, 1]
```

The symmetric function $e[1]$ is the unity for the arithmetic product:

```sage```
```python
sage: e = SymmetricFunctions(ZZ).e()
sage: all( e([1]).arithmetic_product(e(q)) == e(q) for q in Partitions(4) )
True
```

The arithmetic product is commutative:

```sage```
```python
sage: e = SymmetricFunctions(FiniteField(19)).e()
sage: m = SymmetricFunctions(FiniteField(19)).m()
sage: all( all( e(p).arithmetic_product(m(q)) == m(q).arithmetic_product(e(p)) for p in Partitions(4) ) for q in Partitions(4) )
True
```

Note: The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a $\mathbb{Q}$-algebra (in which case the arithmetic product can be easily computed using the power sum basis) from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over $\mathbb{Q}$ (using the corresponding_basis_over hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

5.1. Comprehensive Module List 3003
bernstein_creation_operator($n$)

Return the image of self under the $n$-th Bernstein creation operator.

Let $n$ be an integer. The $n$-th Bernstein creation operator $B_n$ is defined as the endomorphism of the space $Sym$ of symmetric functions which sends every $f$ to

$$
\sum_{i \geq 0} (-1)^i h_{n+i} e_i^\perp,
$$

where usual notations are in place ($h$ stands for the complete homogeneous symmetric functions, $e$ for the elementary ones, and $e_i^\perp$ means skewing ($\text{skew_by()}$) by $e_i$).

This has been studied in [BBSSZ2012], section 2.2, where the following rule is given for computing $B_n$ on a Schur function: If $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an $n$-tuple of integers (positive or not), then

$$
B_n s_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} = s_{(n, \alpha_1, \alpha_2, \ldots, \alpha_n)}.
$$

Here, $s_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}$ is the “Schur function” associated to the $n$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and defined by literally applying the Jacobi-Trudi identity, i.e., by

$$
s_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} = \det ((h_{\alpha_i - i + j})_{i,j=1,2,\ldots,n}).
$$

This notion of a Schur function clearly extends the classical notion of Schur function corresponding to a partition, but is easily reduced to the latter (in fact, for any $n$-tuple $\alpha$ of integers, one easily sees that $s_{\alpha}$ is either 0 or minus-plus a Schur function corresponding to a partition; and it is easy to determine which of these is the case and find the partition by a combinatorial algorithm).

EXAMPLES:

Let us check that what this method computes agrees with the definition:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: e = Sym.e()
sage: h = Sym.h()
sage: s = Sym.s()
sage: def bernstein_creation_by_def(n, f):
    """n-th Bernstein creation operator applied to f"
    # computed according to its definition.
    res = f.parent().zero()
    if not f:
        return res
    max_degree = max(sum(m) for m, c in f)
    for i in range(max_degree + 1):
        if n + i >= 0:
            res += (-1) ** i * h[n + i] * f.skew_by(e[i])
    return res
sage: all( bernstein_creation_by_def(n, s[l]) == s[l].bernstein_creation_operator(n) for n in range(-2, 3) for l in Partitions(4) )
True
sage: all( bernstein_creation_by_def(n, s[l]) == s[l].bernstein_creation_operator(n) for n in range(-3, 4) for l in Partitions(3) )
True
sage: all( bernstein_creation_by_def(n, e[l]) == e[l].bernstein_creation_operator(n) for n in range(-3, 4) for k in range(3) for l in Partitions(k) )
True
```
Some examples:

```
sage: s[3,2].bernstein_creation_operator(3)
    +s[3,3,2]
sage: s[3,2].bernstein_creation_operator(1)
    -s[2,2,2]
sage: h[3,2].bernstein_creation_operator(-2)
    +h[2,1]
sage: h[3,2].bernstein_creation_operator(-1)
    -h[2,1,1] - h[2,2] - h[3,1]
sage: h[3,2].bernstein_creation_operator(0)
    -h[3,1,1] + h[3,2]
sage: h[3,2].bernstein_creation_operator(1)
    -h[2,2,2] + h[3,2,1]
sage: h[3,2].bernstein_creation_operator(2)
    -h[3,3,1] + h[4,2,1]
```

**character_to_frobenius_image(n)**

Interpret `self` as a $GL_n$ character and then take the Frobenius image of this character of the permutation matrices $S_n$ which naturally sit inside of $GL_n$.

To know the value of this character at a permutation of cycle structure $\rho$ the symmetric function `self` is evaluated at the eigenvalues of a permutation of cycle structure $\rho$. The Frobenius image is then defined as $\sum_{\rho \vdash n} f[\Xi_{\rho}] p_{\rho}/z_{\rho}$.

See also:

`eval_at_permutation_roots()`

**INPUT:**

- `n` – a non-negative integer to interpret `self` as a character of $GL_n$

**OUTPUT:**

- a symmetric function of degree $n$

**EXAMPLES:**

```
sage: s = SymmetricFunctions(QQ).s()
sage: s([1,1]).character_to_frobenius_image(5)
    +s[3,1,1] + s[4,1]
sage: s([2,1]).character_to_frobenius_image(5)
sage: s([2,2,2]).character_to_frobenius_image(3)
    +s[3]
sage: s([2,2,2]).character_to_frobenius_image(4)
sage: s([2,2,2]).character_to_frobenius_image(5)
```

**degree()**

Return the degree of `self` (which is defined to be 0 for the zero element).

**EXAMPLES:**

```
sage: s = SymmetricFunctions(QQ).s()
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1]) + 3
```
sage: z.degree()
4
sage: s(1).degree()
0
sage: s(0).degree()
0

\texttt{derivative\_with\_respect\_to\_p1}(n=1)

Return the symmetric function obtained by taking the derivative of \texttt{self} with respect to the power-sum symmetric function $p_1$ when the expansion of \texttt{self} in the power-sum basis is considered as a polynomial in $p_k$'s (with $k \geq 1$).

This is the same as skewing \texttt{self} by the first power-sum symmetric function $p_1$.

INPUT:

\begin{itemize}
\item $n$ – (default: 1) nonnegative integer which determines which power of the derivative is taken
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: p = SymmetricFunctions(QQ).p()
sage: a = p([1,1,1])
sage: a.derivative_with_respect_to_p1()
3*p[1, 1]
sage: a.derivative_with_respect_to_p1(1)
3*p[1, 1]
sage: a.derivative_with_respect_to_p1(2)
6*p[1]
sage: a.derivative_with_respect_to_p1(3)
6*p[]
sage: s = SymmetricFunctions(QQ).s()
sage: s([3]).derivative_with_respect_to_p1()
s[2]
sage: s([2,1]).derivative_with_respect_to_p1()
s[1, 1] + s[2]
sage: s([1,1,1]).derivative_with_respect_to_p1()
s[1, 1]
sage: s(0).derivative_with_respect_to_p1()
0
sage: s(1).derivative_with_respect_to_p1()
0
sage: s([1]).derivative_with_respect_to_p1()
s[]
\end{verbatim}

Let us check that taking the derivative with respect to $p[1]$ is equivalent to skewing by $p[1]$:

\begin{verbatim}
sage: p1 = s([1])
sage: all( s(lam).derivative_with_respect_to_p1() == s(lam).skew_by(p1) for lam in Partitions(4) )
True
\end{verbatim}

\texttt{eval\_at\_permutation\_roots}(\rho)

Evaluate at eigenvalues of a permutation matrix.
Evaluate a symmetric function at the eigenvalues of a permutation matrix whose cycle structure is \( \rho \).
This computation is computed by coercing to the power sum basis where the value may be computed on the generators.

This function evaluates an element at the roots of unity
\[ \Xi_{\rho_1}, \Xi_{\rho_2}, \ldots, \Xi_{\rho_\ell} \]
where
\[ \Xi_m = 1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{m-1} \]
and \( \zeta_m \) is an \( m \) root of unity. These roots of unity represent the eigenvalues of permutation matrix with cycle structure \( \rho \).

**INPUT:**

- \( \rho \) – a partition or a list of non-negative integers

**OUTPUT:**

- an element of the base ring

**EXAMPLES:**

```
sage: s = SymmetricFunctions(QQ).s()
sage: s([3,3]).eval_at_permutation_roots([6])
0
sage: s([3,3]).eval_at_permutation_roots([3])
1
sage: s([3,3]).eval_at_permutation_roots([1])
0
sage: s([3,3]).eval_at_permutation_roots([3,3])
4
sage: s([3,3]).eval_at_permutation_roots([1,1,1,1,1])
175
sage: (s[1]+s[2]+s[3]).eval_at_permutation_roots([3,2])
2
```

**expand** \((n, \text{alphabet}='x')\)

Expand the symmetric function \( \text{self} \) as a symmetric polynomial in \( n \) variables.

**INPUT:**

- \( n \) – a nonnegative integer
- \( \text{alphabet} \) – (default: 'x') a variable for the expansion

**OUTPUT:**

A monomial expansion of \( \text{self} \) in the \( n \) variables labelled \( x_0, x_1, \ldots, x_{n-1} \) (or just \( x \) if \( n = 1 \)), where \( x \) is \( \text{alphabet} \).

**EXAMPLES:**

```
sage: J = SymmetricFunctions(QQ).jack(t=2).J()
sage: J([2,1]).expand(3)
4*x0^2*x1 + 4*x0*x1^2 + 4*x0^2*x2 + 6*x0*x1*x2 + 4*x1^2*x2 + 4*x0*x2^2 + 4*x1*x2^2
sage: (2*J([2])).expand(0)
```

(continues on next page)
exponential_specialization($r$=None, $q$=1)

Return the exponential specialization of a symmetric function (when $q = 1$), or the $q$-exponential specialization (when $q \neq 1$).

The exponential specialization $ex$ at $t$ is a $K$-algebra homomorphism from the $K$-algebra of symmetric functions to another $K$-algebra $R$. It is defined whenever the base ring $K$ is a $Q$-algebra and $t$ is an element of $R$. The easiest way to define it is by specifying its values on the powersum symmetric functions to be $p_1 = t$ and $p_n = 0$ for $n > 1$. Equivalently, on the homogeneous functions it is given by $ex(h_n) = t^n/n!$; see Proposition 7.8.4 of [EnumComb2].

By analogy, the $q$-exponential specialization is a $K$-algebra homomorphism from the $K$-algebra of symmetric functions to another $K$-algebra $R$ that depends on two elements $t$ and $q$ of $R$ for which the elements $1 - q^i$ for all positive integers $i$ are invertible. It can be defined by specifying its values on the complete homogeneous symmetric functions to be

$$ex_q(h_n) = t^n/[n]_q!,$$

where $[n]_q!$ is the $q$-factorial. Equivalently, for $q \neq 1$ and a homogeneous symmetric function $f$ of degree $n$, we have

$$ex_q(f) = (1-q)^n t^n ps_q(f),$$

where $ps_q(f)$ is the stable principal specialization of $f$ (see principal_specialization()). (See (7.29) in [EnumComb2].)

The limit of $ex_q$ as $q \to 1$ is $ex$.

INPUT:

- $t$ (default: None) – the value to use for $t$; the default is to create a ring of polynomials in $t$.
- $q$ (default: 1) – the value to use for $q$. If $q$ is None, then a ring (or fraction field) of polynomials in $q$ is created.

EXAMPLES:

```python
sage: m = SymmetricFunctions(QQ).m()
sage: (m[2,1]+m[1,1]).exponential_specialization()
1/2*t^2
sage: (m[2,1]+m[1,1]).exponential_specialization(q=1)
1/2*t^2
sage: m[1,1].exponential_specialization(q=None)
(q/(q + 1))*t^2
sage: Qq = PolynomialRing(QQ, "q"); q = Qq.gen()
sage: m[1,1].exponential_specialization(q=q)
(q/(q + 1))*t^2
sage: Qt = PolynomialRing(QQ, "t"); t = Qt.gen()
sage: m[1,1].exponential_specialization(t=t)
1/2*t^2
sage: Qqt = PolynomialRing(QQ, ["q", "t"]); q, t = Qqt.gens()
sage: m[1,1].exponential_specialization(q=q, t=t)
q*t^2/(q + 1)
```
Combinatorics, Release 10.0

sage: x = m[3]+m[2,1]+m[1,1,1]
sage: d = x.homogeneous_degree()
sage: var("q t")
(q, t)
sage: factor((x.principal_specialization()*(1-q)^d*t^d))
t^3/((q^2 + q + 1)*(q + 1))
sage: factor(x.exponential_specialization(q=q, t=t))
t^3/((q^2 + q + 1)*(q + 1))

factor()

Return the factorization of this symmetric function.

EXAMPLES:

sage: e = SymmetricFunctions(QQ).e()
sage: R.<x, y> = QQ[]
sage: s = SymmetricFunctions(R.fraction_field()).s()
sage: factor((-x)*s[2, 1] - s[3]) * ((-x^2*y)*s[2] - s[4, 1])

frobenius(n)

Return the image of the symmetric function self under the n-th Frobenius operator.

The n-th Frobenius operator f_n is defined to be the map from the ring of symmetric functions to itself that sends every symmetric function P(x_1, x_2, x_3, ...) to P(x_1^n, x_2^n, x_3^n, ...). This operator f_n is a Hopf algebra endomorphism, and satisfies

f_n m(\lambda_1, \lambda_2, \lambda_3, ...) = m(n\lambda_1, n\lambda_2, n\lambda_3, ...)

for every partition (\lambda_1, \lambda_2, \lambda_3, ...) (where m means the monomial basis). Moreover, f_n(p_r) = p_{nr} for every positive integer r (where p_k denotes the k-th powersum symmetric function).

The n-th Frobenius operator is also called the n-th Frobenius endomorphism. It is not related to the Frobenius map which connects the ring of symmetric functions with the representation theory of the symmetric group.

The n-th Frobenius operator is also the n-th Adams operator of the Lambda-ring of symmetric functions over the integers.

The n-th Frobenius operator can also be described via plethysm: Every symmetric function P satisfies f_n(P) = p_n \circ P = P \circ p_n, where p_n is the n-th powersum symmetric function, and \circ denotes (outer) plethysm.

INPUT:

* n – a positive integer

OUTPUT:

The result of applying the n-th Frobenius operator (on the ring of symmetric functions) to self.

EXAMPLES:
Combinatorics, Release 10.0

```
sage: Sym = SymmetricFunctions(ZZ)
sage: p = Sym.p()
sage: h = Sym.h()
sage: s = Sym.s()
sage: m = Sym.m()
sage: s[3].frobenius(2)
sage: m[4,2,1].frobenius(3)
m[12, 6, 3]
sage: p[4,2,1].frobenius(3)
p[12, 6, 3]
sage: h[4].frobenius(2)
```

The Frobenius endomorphisms are multiplicative:

```
sage: all( all( s(lam).frobenius(3) * s(mu).frobenius(3) # long time
.....: == (s(lam) * s(mu)).frobenius(3)
.....: for mu in Partitions(3) )
.....: for lam in Partitions(3) )
True
```

```
sage: all( all( m(lam).frobenius(2) * m(mu).frobenius(2)
.....: == (m(lam) * m(mu)).frobenius(2)
.....: for mu in Partitions(4) )
.....: for lam in Partitions(4) )
True
```

```
sage: all( all( p(lam).frobenius(2) * p(mu).frobenius(2)
.....: == (p(lam) * p(mu)).frobenius(2)
.....: for mu in Partitions(3) )
.....: for lam in Partitions(4) )
True
```

Being Hopf algebra endomorphisms, the Frobenius operators commute with the antipode:

```
sage: all( p(lam).frobenius(4).antipode()
.....: == p(lam).antipode().frobenius(4)
.....: for lam in Partitions(3) )
True
```

Testing the \( f_n(P) = p_n \circ P = P \circ p_n \) equality (over \( \mathbb{Q} \), since plethysm is currently not defined over \( \mathbb{Z} \) in Sage):

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: p = Sym.p()
sage: all( s(lam).frobenius(3) == s(lam).plethysm(p[3])
.....: == s(p[3].plethysm(s(lam)))
.....: for lam in Partitions(4) )
True
```

By Exercise 7.61 in Stanley’s EC2 [STA] (see the errata on his website), \( f_n(h_m) \) is a linear combination of Schur polynomials (of straight shapes) using coefficients 0, 1 and \(-1\) only; moreover, all partitions whose Schur polynomials occur with coefficient \( \neq 0 \) in this combination have empty \( n \)-cores. Let us check this on examples:
Combinatorics, Release 10.0

```python
sage: all( all( all( (coeff == -1 or coeff == 1)
........and lam.core(n) == Partition([])
........for lam, coeff in s([m]).frobenius(n) )
........for n in range(2, 4) )
........for m in range(4) )
True
```

**See also:**

`plethysm()`

**Todo:**

This method is fast on the monomial and the powersum bases, while all other bases get converted to the monomial basis. For most bases, this is probably the quickest way to do, but at least the Schur basis should have a better option. (Quoting from Stanley’s EC2 [STA]: “D. G. Duncan, J. London Math. Soc. 27 (1952), 235-236, or Y. M. Chen, A. M. Garsia, and J. B. Remmel, Contemp. Math. 34 (1984), 109-153”.)

---

**gcd(other)**

Return the greatest common divisor with other.

**INPUT:**

- `other` – the other symmetric function

**EXAMPLES:**

```python
sage: e = SymmetricFunctions(ZZ).e()
sage: B = 7*e[2] + e[5,1]
sage: C = 3*e[1,1] + e[2]
sage: gcd(A*B^2, B*C)
7*e[2] + e[5, 1]
sage: p = SymmetricFunctions(ZZ).p()
sage: gcd(e[2,1], p[1,1]-p[2])
e[2]
sage: gcd(p[2,1], p[3,2]-p[2,1])
p[2]
```

---

**hl_creation_operator(nu, t=None)**

This is the vertex operator that generalizes Jing’s operator.

It is a linear operator that raises the degree by $|\nu|$. This creation operator is a t-analogue of multiplication by $s(\nu)$.

**See also:**

Proposition 5 in [SZ2001].

**INPUT:**
- `nu` – a partition or a list of integers
- `t` – (default: `None`, in which case `t` is used) an element of the base ring

**REFERENCES:**

**EXAMPLES:**
inner_plethysm(x)

Return the inner plethysm of self with x.

Whenever $R$ is a $\mathbb{Q}$-algebra, and $f$ and $g$ are two symmetric functions over $R$ such that the constant term of $f$ is zero, the inner plethysm of $f$ with $g$ is a symmetric function over $R$, and the degree of this symmetric function is the same as the degree of $g$. We will denote the inner plethysm of $f$ with $g$ by $f \{ g \}$ (in contrast to the notation of outer plethysm which is generally denoted $f[ g ]$); in Sage syntax, it is $f$. inner_plethysm(g).

First we describe the axiomatic definition of the operation; see below for a representation-theoretic interpretation. In the following equations, we denote the outer product (i.e., the standard product on the ring of symmetric functions, product()) by $\cdot$ and the Kronecker product (itensor()) by $\ast$.

\[
(f + g)\{h\} = f\{h\} + g\{h\}
\]
\[
(f \cdot g)\{h\} = (f\{h\}) \ast (g\{h\})
\]
\[
p_k\{f + g\} = p_k\{f\} + p_k\{g\}
\]

where $p_k$ is the $k$-th power-sum symmetric function for every $k > 0$.

Let $\sigma$ be a permutation of cycle type $\mu$ and let $\mu^k$ be the cycle type of $\sigma^k$. Then,

\[
p_k\{p_\mu / z_\mu\} = \sum_{\nu: \nu^k = \mu} p_\nu / z_\nu
\]
Since \((p_\mu/z_\mu)_\mu\) is a basis for the symmetric functions, these four formulas define the symmetric function operation \(f\{g\}\) for any symmetric functions \(f\) and \(g\) (where \(f\) has constant term 0) by expanding \(f\) in the power sum basis and \(g\) in the dual basis \(p_\mu/z_\mu\).

**See also:** itensor(), partition_power(), plethysm()

This operation admits a representation-theoretic interpretation in the case where \(f\) is a Schur function \(s_\lambda\) and \(g\) is a homogeneous degree \(n\) symmetric function with nonnegative integral coefficients in the Schur basis. The symmetric function \(f\{g\}\) is the Frobenius image of the \(S_n\)-representation constructed as follows. The assumptions on \(g\) imply that \(g\) is the Frobenius image of a representation \(\rho\) of the symmetric group \(S_n\):

\[
\rho : S_n \to GL_N.
\]

If the degree \(N\) of this representation is greater than or equal to the number of parts of \(\lambda\), then \(f\), which denotes \(s_\lambda\), corresponds to the character of some irreducible \(GL_N\)-representation, say

\[
\sigma : GL_N \to GL_M.
\]

The composition \(\sigma \circ \rho : S_n \to GL_M\) is a representation of \(S_n\) whose Frobenius image is precisely \(f\{g\}\). If \(N\) is less than the number of parts of \(\lambda\), then \(f\{g\}\) is 0 by definition.

When \(f\) is a symmetric function with constant term \(\neq 0\), the inner plethysm \(f\{g\}\) isn’t well-defined in the ring of symmetric functions. Indeed, it is not clear how to define \(1\{g\}\). The most sensible way to get around this probably is defining it as the infinite sum \(h_0 + h_1 + h_2 + \cdots\) (where \(h_i\) means the \(i\)-th complete homogeneous symmetric function) in the completion of this ring with respect to its grading. This is how [SchaThi1994] defines \(1\{g\}\). The present method, however, sets it to be the sum of \(h_i\) over all \(i\) for which the \(i\)-th homogeneous component of \(g\) is nonzero. This is rather a hack than a reasonable definition. Use with caution!

**Note:** If a symmetric function \(g\) is written in the form \(g = g_0 + g_1 + g_2 + \cdots\) with each \(g_i\) homogeneous of degree \(i\), then \(f\{g\} = f\{g_0\} + f\{g_1\} + f\{g_2\} + \cdots\) for every \(f\) with constant term 0. But in general, inner plethysm is not linear in the second variable.

**REFERENCES:**

**INPUT:**

- \(x\) – element of the ring of symmetric functions over the same base ring as \(self\)

**OUTPUT:**

- an element of symmetric functions in the parent of \(self\)

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: p = Sym.power()
sage: h = Sym.complete()
sage: s([2,1]).inner_plethysm(s([1,1,1]))
0
sage: s([2]).inner_plethysm(s([2,1]))
s[2, 1] + s[3]
sage: s([1,1]).inner_plethysm(s([2,1]))
```
inner_tensor(x)

Return the internal (tensor) product of self and x in the basis of self.

The internal tensor product can be defined as the linear extension of the definition on power sums \( p_\lambda \ast p_\mu = \delta_{\lambda, \mu} z_\lambda \), where \( z_\lambda = (r_1^r_1!)(r_2^r_2!)(\cdots) \) for \( \lambda = (r_1^r_2 \cdots) \) and where \( \ast \) denotes the internal tensor product. The internal tensor product is also known as the Kronecker product, or as the second multiplication on the ring of symmetric functions.

Note that the internal product of any two homogeneous symmetric functions of equal degrees is a homogeneous symmetric function of the same degree. On the other hand, the internal product of two homogeneous symmetric functions of distinct degrees is 0.

Note: The internal product is sometimes referred to as “inner product” in the literature, but unfortunately this name is shared by a different operation, namely the Hall inner product (see scalar()).

INPUT:

• x – element of the ring of symmetric functions over the same base ring as self

OUTPUT:
• the internal product of \texttt{self} with \texttt{x} (an element of the ring of symmetric functions in the same basis as \texttt{self})

The methods \texttt{itensor()}, \texttt{internal_product()}, \texttt{kronecker_product()}, \texttt{inner_tensor()} are all synonyms.

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: b = s([3])
sage: a.itensor(b)
s[2, 1]
sage: c = s([3,2,1])
sage: c.itensor(c)
```

There are few quantitative results pertaining to Kronecker products in general, which makes their computation so difficult. Let us test a few of them in different bases.

The Kronecker product of any homogeneous symmetric function $f$ of degree $n$ with the $n$-th complete homogeneous symmetric function $h[n]$ (a.k.a. $s[n]$) is $f$:

```python
sage: h = SymmetricFunctions(ZZ).h()
sage: all( h([5]).itensor(h(p)) == h(p) for p in Partitions(5) )
True
```

The Kronecker product of a Schur function $s_{\lambda}$ with the $n$-th elementary symmetric function $e[n]$, where $n = |\lambda|$, is $s_{\lambda'}$ (where $\lambda'$ is the conjugate partition of $\lambda$):

```python
sage: F = CyclotomicField(12)
sage: s = SymmetricFunctions(F).s()
sage: e = SymmetricFunctions(F).e()
sage: all( e([5]).itensor(s(p)) == s(p.conjugate()) for p in Partitions(5) )
True
```

The Kronecker product is commutative:

```python
sage: e = SymmetricFunctions(FiniteField(19)).e()
sage: m = SymmetricFunctions(FiniteField(19)).m()
sage: all( all( e(p).itensor(m(q)) == m(q).itensor(e(p)) for q in Partitions(4) ) for p in Partitions(4) )
True
```

```python
sage: F = FractionField(QQ['q','t'])
sage: mq = SymmetricFunctions(F).macdonald().Q()
sage: mh = SymmetricFunctions(F).macdonald().H()
sage: all( all( mq(p).itensor(mh(r)) == mh(r).itensor(mq(p)) # long time
....:     for r in Partitions(4) )
....:     for p in Partitions(3) )
True
```
Let us check (on examples) Proposition 5.2 of Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon, “Non-commutative symmetric functions”, arXiv hep-th/9407124, for \( r = 2 \):

```python
sage: e = SymmetricFunctions(FiniteField(29)).e()
sage: s = SymmetricFunctions(FiniteField(29)).s()
sage: m = SymmetricFunctions(FiniteField(29)).m()
sage: def tensor_copr(u, v, w):
    # computes \( \mu((u \otimes v) \ast \Delta(w)) \) with
    # * meaning Kronecker product and \( \mu \) meaning
    # the usual multiplication.
    result = w.parent().zero()
    for partition_pair, coeff in w.coproduct():
        result += coeff * w.parent()(u).itensor(partition_pair[0]) * w.parent()(v).itensor(partition_pair[1])
    return result
sage: all(all(all(tensor_copr(e[u], s[v], m[w]) == (e[u] * s[v]).itensor(m[w]) for w in Partitions(5)) for v in Partitions(2)) for u in Partitions(3))
True
```

Some examples from Briand, Orellana, Rosas, “The stability of the Kronecker products of Schur functions.” arXiv 0907.4652:

```python
sage: s = SymmetricFunctions(ZZ).s()
sage: s[2,2].itensor(s[2,2])
s[1, 1, 1, 1] + s[2, 2] + s[4]
sage: s[3,2].itensor(s[3,2])
sage: s[4,2].itensor(s[4,2])
```

An example from p. 220 of Thibon, “Hopf algebras of symmetric functions and tensor products of symmetric group representations”, International Journal of Algebra and Computation, 1991:

```python
sage: s = SymmetricFunctions(QQbar).s()
sage: s[2,1].itensor(s[2,1])
s[1, 1, 1] + s[2, 1] + s[3]
```

**Note:** The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a \( \mathbb{Q} \)-algebra (in which case the Kronecker product can be easily computed using the power sum basis) from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over \( \mathbb{Q} \) (using the corresponding basis over hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

**internal_coproduct()**

Return the inner coproduct of \( \text{self} \) in the basis of \( \text{self} \).

The inner coproduct (also known as the Kronecker coproduct, as the internal coproduct, or as the second comultiplication on the ring of symmetric functions) is a ring homomorphism \( \Delta^* \) from the ring of symmet-
ric functions to the tensor product (over the base ring) of this ring with itself. It is uniquely characterized by the formula

$$\Delta^\times(h_n) = \sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda = \sum_{\lambda \vdash n} h_\lambda \otimes m_\lambda = \sum_{\lambda \vdash n} m_\lambda \otimes h_\lambda,$$

where $\lambda \vdash n$ means $\lambda$ is a partition of $n$, and $n$ is any nonnegative integer. It also satisfies

$$\Delta^\times(p_n) = p_n \otimes p_n$$

for any positive integer $n$. If the base ring is a $Q$-algebra, it also satisfies

$$\Delta^\times(h_n) = \sum_{\lambda \vdash n} z_{\lambda}^{-1} p_\lambda \otimes p_\lambda,$$

where

$$z_\lambda = \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!$$

with $m_i(\lambda)$ meaning the number of appearances of $i$ in $\lambda$ (see $zee()$).

The method $kronecker_coproduct()$ is a synonym of $internal_coproduct()$.

EXAMPLES:

```python
sage: s = SymmetricFunctions(ZZ).s()
sage: a = s([2,1])
sage: a.internal_coproduct()

sage: e = SymmetricFunctions(QQ).e()
sage: b = e([2])
sage: b.internal_coproduct()
```

The internal coproduct is adjoint to the internal product with respect to the Hall inner product: Any three symmetric functions $f, g$ and $h$ satisfy

$$\langle f * g, h \rangle = \sum_i \langle f, h' \rangle \langle g, h'' \rangle,$$

where we rewrite $\Delta^\times(h)$ as $\sum_i h'_i \otimes h''_i$.

Let us check this in degree 4:

```python
sage: e = SymmetricFunctions(FiniteField(29)).e()
sage: s = SymmetricFunctions(FiniteField(29)).s()
sage: m = SymmetricFunctions(FiniteField(29)).m()
sage: def tensor_incorp(f, g, h):
    # computes \sum_i \langle f, h' \rangle \langle g, h'' \rangle
    result = h.base_ring().zero()
    for partition_pair, coeff in h.internal_coproduct():
        result += coeff * h.parent()(f).scalar(partition_pair[0]) * h.parent()(g).scalar(partition_pair[1])
    return result
sage: all(all(all(tensor_incorp(e[u], s[v], m[w]) == (e[u].itensor(s[v])).
    →scalar(m[w])) # long time (10s on sage.math, 2013)
    →for w in Partitions(5) )
    →for v in Partitions(2) )
    →for u in Partitions(3) )
    True
```

5.1. Comprehensive Module List
Let us check the formulas for $\Delta^\times(h_n)$ and $\Delta^\times(p_n)$ given in the description of this method:

```
sage: e = SymmetricFunctions(QQ).e()
sage: p = SymmetricFunctions(QQ).p()
sage: h = SymmetricFunctions(QQ).h()
sage: s = SymmetricFunctions(QQ).s()
sage: all( s(h([n])).internal_coproduct() == sum([tensor([s(lam), s(lam)]) for lam in Partitions(n)]) for n in range(6) )
True
sage: all( h([n]).internal_coproduct() == sum([tensor([h(lam), h(m(lam)))] for lam in Partitions(n)]) for n in range(6) )
True
sage: all( factorial(n) * h([n]).internal_coproduct() == sum([lam.conjugacy_class_size() * tensor([h(p(lam)), h(p(lam)))] for lam in Partitions(n)]) for n in range(6) )
True
```

### internal_product(x)

Return the internal (tensor) product of `self` and `x` in the basis of `self`.

The internal tensor product can be defined as the linear extension of the definition on power sums $p_\lambda \ast p_\mu = \delta_{\lambda,\mu} z_\lambda p_\lambda$, where $z_\lambda = (1^{r_1} r_1!)(2^{r_2} r_2!)$ for $\lambda = (1^{r_1} 2^{r_2} \cdots$ and where $\ast$ denotes the internal tensor product. The internal tensor product is also known as the Kronecker product, or as the second multiplication on the ring of symmetric functions.

Note that the internal product of any two homogeneous symmetric functions of equal degrees is a homogeneous symmetric function of the same degree. On the other hand, the internal product of two homogeneous symmetric functions of distinct degrees is 0.

**Note:** The internal product is sometimes referred to as “inner product” in the literature, but unfortunately this name is shared by a different operation, namely the Hall inner product (see `scalar()`).

**INPUT:**

- `x` – element of the ring of symmetric functions over the same base ring as `self`

**OUTPUT:**

- the internal product of `self` with `x` (an element of the ring of symmetric functions in the same basis as `self`)

The methods `itensor()`, `internal_product()`, `kronecker_product()`, `inner_tensor()` are all synonyms.

**EXAMPLES:**

```
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: b = s([3])
sage: a.itensor(b)
s[2, 1]
sage: c = s([3,2,1])
sage: c.itensor(c)
```

(continues on next page)
s[1, 1, 1, 1, 1, 1] + 2*s[2, 1, 1, 1, 1] + 3*s[2, 2, 1, 1, 1] + 2*s[2, 2, 2] + 4*s[3, 1, 1, 1, 1] + 5*s[3, 2, 1, 1] + 2*s[3, 2, 2] + 4*s[3, 3, 1] + 2*s[4, 1, 1, 1, 1] + 3*s[4, 2, 1, 1] + 2*s[5, 1, 1, 1] + s[6]

There are few quantitative results pertaining to Kronecker products in general, which makes their computation so difficult. Let us test a few of them in different bases.

The Kronecker product of any homogeneous symmetric function $f$ of degree $n$ with the $n$-th complete homogeneous symmetric function $h[n]$ (a.k.a. $s[n]$) is $f$:

```python
sage: h = SymmetricFunctions(ZZ).h()
sage: all( h([n]).itensor(h(p)) == h(p) for p in Partitions(n) )
True
```

The Kronecker product of a Schur function $s_\lambda$ with the $n$-th elementary symmetric function $e[n]$, where $n = |\lambda|$, is $s_{\lambda'}$ (where $\lambda'$ is the conjugate partition of $\lambda$):

```python
sage: F = CyclotomicField(12)
sage: s = SymmetricFunctions(F).s()
sage: e = SymmetricFunctions(F).e()
sage: all( e([n]).itensor(s(p)) == s(p.conjugate()) for p in Partitions(n) )
True
```

The Kronecker product is commutative:

```python
sage: e = SymmetricFunctions(FiniteField(19)).e()
sage: m = SymmetricFunctions(FiniteField(19)).m()
sage: all( all( e(p).itensor(m(q)) == m(q).itensor(e(p)) for q in Partitions(4) )
    ....: for p in Partitions(4) )
True
```

Let us check (on examples) Proposition 5.2 of Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon, “Non-commutative symmetric functions”, arXiv hep-th/9407124, for $r = 2$:

```python
sage: e = SymmetricFunctions(FiniteField(29)).e()
sage: s = SymmetricFunctions(FiniteField(29)).s()
sage: m = SymmetricFunctions(FiniteField(29)).m()
sage: def tensor_copr(u, v, w):
    # computes \mu ((u \otimes v) * \Delta(w)) with
    # * meaning Kronecker product and \mu meaning usual multiplication.
    result = w.parent().zero()
    for partition_pair, coeff in w.coproduct():
        result += coeff * w.parent()(u).itensor(partition_pair[0]) * v.
    return result
sage: tensor_copr(e([2]), e([3]), e([1]))
```

(continues on next page)
Some examples from Briand, Orellana, Rosas, “The stability of the Kronecker products of Schur functions.”
arXiv 0907.4652:

```python
sage: s = SymmetricFunctions(ZZ).s()
sage: s[2,2].itensor(s[2,2])
s[1, 1, 1, 1] + s[2, 2] + s[4]
sage: s[3,2].itensor(s[3,2])
sage: s[4,2].itensor(s[4,2])
s[2, 2, 2] + s[3, 1, 1, 1] + 2*s[3, 2, 1] + s[4, 1, 1] + 2*s[4, 2] + s[5, 1] + ...
sage: s[6]
```

An example from p. 220 of Thibon, “Hopf algebras of symmetric functions and tensor products of symmetric group representations”, International Journal of Algebra and Computation, 1991:

```python
sage: s = SymmetricFunctions(QQbar).s()
sage: s[2,1].itensor(s[2,1])
s[1, 1, 1] + s[2, 1] + s[3]
```

**Note:** The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a \( \mathbb{Q} \)-algebra (in which case the Kronecker product can be easily computed using the power sum basis) from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over \( \mathbb{Q} \) (using the corresponding basis over hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

**is_schur_positive()**
Return True if and only if self is Schur positive.

If \( s \) is the space of Schur functions over self’s base ring, then this is the same as self._is_positive(s).

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1]) + s([3])
sage: a.is_schur_positive()
True
sage: a = s([2,1]) - s([3])
sage: a.is_schur_positive()
False
```

```python
sage: QQx = QQ['x']
sage: s = SymmetricFunctions(QQx).s()
(continues on next page)```
\textbf{itensor}(x)

Return the internal (tensor) product of self and x in the basis of self.

The internal tensor product can be defined as the linear extension of the definition on power sums $p_\lambda \ast p_\mu = \delta_{\lambda, \mu} z_\lambda$, where $z_\lambda = (1^{r_1} r_1!)(2^{r_2} r_2! \cdots$ for $\lambda = (1^{r_1} 2^{r_2} \cdots$ and where $\ast$ denotes the internal tensor product. The internal tensor product is also known as the Kronecker product, or as the second multiplication on the ring of symmetric functions.

Note that the internal product of any two homogeneous symmetric functions of equal degrees is a homogeneous symmetric function of the same degree. On the other hand, the internal product of two homogeneous symmetric functions of distinct degrees is 0.

\textbf{Note:} The internal product is sometimes referred to as “inner product” in the literature, but unfortunately this name is shared by a different operation, namely the Hall inner product (see \textit{scalar()}).

\textbf{INPUT:}

\begin{itemize}
  \item $x$ – element of the ring of symmetric functions over the same base ring as \textit{self}
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item the internal product of \textit{self} with $x$ (an element of the ring of symmetric functions in the same basis as \textit{self})
\end{itemize}

The methods \textit{itensor()}, \textit{internal_product()}, \textit{kronecker_product()}, \textit{inner_tensor()} are all synonyms.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: b = s([3])
sage: a.itensor(b) s[2, 1]
sage: c = s([3,2,1])
sage: c.itensor(c) s[1, 1, 1, 1, 1] + 2*s[2, 1, 1, 1, 1] + 3*s[2, 2, 1, 1] + 2*s[2, 2, 2] + 4*s[3, 1, 1, 1] + 5*s[3, 2, 1] + 2*s[3, 3] + 4*s[4, 1, 1] + 3*s[4, 2] + 2*s[5, 1] + s[6]
\end{verbatim}

There are few quantitative results pertaining to Kronecker products in general, which makes their computation so difficult. Let us test a few of them in different bases.
The Kronecker product of any homogeneous symmetric function \( f \) of degree \( n \) with the \( n \)-th complete homogeneous symmetric function \( h[n] \) (a.k.a. \( s[n] \)) is \( f \):

```python
sage: h = SymmetricFunctions(ZZ).h()
sage: all( h([5]).itensor(h(p)) == h(p) for p in Partitions(5) )
True
```

The Kronecker product of a Schur function \( s_\lambda \) with the \( n \)-th elementary symmetric function \( e[n] \), where \( n = |\lambda| \), is \( s_\lambda' \) (where \( \lambda' \) is the conjugate partition of \( \lambda \)):

```python
sage: F = CyclotomicField(12)
sage: s = SymmetricFunctions(F).s()
sage: e = SymmetricFunctions(F).e()
sage: all( e([5]).itensor(s(p)) == s(p.conjugate()) for p in Partitions(5) )
True
```

The Kronecker product is commutative:

```python
sage: e = SymmetricFunctions(FiniteField(19)).e()
sage: m = SymmetricFunctions(FiniteField(19)).m()
sage: all( all( e(p).itensor(m(q)) == m(q).itensor(e(p)) for q in Partitions(4) ) for p in Partitions(4) )
True
```

Let us check (on examples) Proposition 5.2 of Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon, “Non-commutative symmetric functions”, arXiv hep-th/9407124, for \( r = 2 \):

```python
sage: e = SymmetricFunctions(FiniteField(29)).e()
sage: s = SymmetricFunctions(FiniteField(29)).s()
sage: m = SymmetricFunctions(FiniteField(29)).m()
sage: def tensor_copr(u, v, w):
....: # computes \mu ((u \otimes v) * \Delta(w)) with
....: # * meaning Kronecker product and \mu meaning usual multiplication.
....: result = w.parent().zero()
....: for partition_pair, coeff in w.coproduct():
....:     result += coeff * w.parent()(u).itensor(partition_pair[0]) * w.parent()(v).itensor(partition_pair[1])
....: return result
sage: all( all( all( tensor_copr(e[u], s[v], m[w]) == (e[u] * s[v]).itensor(m[w]) for w in Partitions(5) ) for v in Partitions(2) ) for u in Partitions(3) )
True
```
Some examples from Briand, Orellana, Rosas, “The stability of the Kronecker products of Schur functions.” arXiv 0907.4652:

```python
sage: s = SymmetricFunctions(ZZ).s()
sage: s[2,2].itensor(s[2,2])
s[1, 1, 1, 1] + s[2, 2] + s[4]
sage: s[3,2].itensor(s[3,2])
sage: s[4,2].itensor(s[4,2])
```

An example from p. 220 of Thibon, “Hopf algebras of symmetric functions and tensor products of symmetric group representations”, International Journal of Algebra and Computation, 1991:

```python
sage: s = SymmetricFunctions(QQbar).s()
sage: s[2,1].itensor(s[2,1])
s[1, 1, 1] + s[2, 1] + s[3]
```

**Note:** The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a $\mathbb{Q}$-algebra (in which case the Kronecker product can be easily computed using the power sum basis) from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over $\mathbb{Q}$ (using the `corresponding_basis_over` hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

### kronecker_coproduct()

Return the inner coproduct of `self` in the basis of `self`.

The inner coproduct (also known as the Kronecker coproduct, as the internal coproduct, or as the second comultiplication on the ring of symmetric functions) is a ring homomorphism $\Delta^\times$ from the ring of symmetric functions to the tensor product (over the base ring) of this ring with itself. It is uniquely characterized by the formula

$$
\Delta^\times(h_n) = \sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda = \sum_{\lambda \vdash n} h_\lambda \otimes m_\lambda = \sum_{\lambda \vdash n} m_\lambda \otimes h_\lambda,
$$

where $\lambda \vdash n$ means $\lambda$ is a partition of $n$, and $n$ is any nonnegative integer. It also satisfies

$$
\Delta^\times(p_n) = p_n \otimes p_n
$$

for any positive integer $n$. If the base ring is a $\mathbb{Q}$-algebra, it also satisfies

$$
\Delta^\times(h_n) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda \otimes p_\lambda,
$$

where

$$
z_\lambda = \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!
$$

with $m_i(\lambda)$ meaning the number of appearances of $i$ in $\lambda$ (see `zee()`).

The method `kronecker_coproduct()` is a synonym of `internal_coproduct()`.

**EXAMPLES:**
The internal coproduct is adjoint to the internal product with respect to the Hall inner product: Any three symmetric functions \( f, g \) and \( h \) satisfy \( \langle f \ast g, h \rangle = \sum_i \langle f, h' \rangle \langle g, h'' \rangle \), where we write \( \Delta \times (h^n) \) as \( \sum_i h'_i \otimes h''_i \).

Let us check this in degree 4:

\[
\begin{align*}
\text{sage: } & e = \text{SymmetricFunctions}(\text{FiniteField}(29)).e() \\
\text{sage: } & s = \text{SymmetricFunctions}(\text{FiniteField}(29)).s() \\
\text{sage: } & m = \text{SymmetricFunctions}(\text{FiniteField}(29)).m() \\
\text{sage: } & \text{all( all( all( tensor_incopr(e[u], s[v], m[w]) == (e[u].itensor(s[v])).scalar(m[w])) for w in Partitions(5) ) for v in Partitions(2) ) for u in Partitions(3) )} \\
& \text{True}
\end{align*}
\]

Let us check the formulas for \( \Delta \times (h_n) \) and \( \Delta \times (p_n) \) given in the description of this method:

\[
\begin{align*}
\text{sage: } & e = \text{SymmetricFunctions}(\text{QQ}).e() \\
\text{sage: } & p = \text{SymmetricFunctions}(\text{QQ}).p() \\
\text{sage: } & h = \text{SymmetricFunctions}(\text{QQ}).h() \\
\text{sage: } & s = \text{SymmetricFunctions}(\text{QQ}).s() \\
\text{sage: } & \text{all( s(h([n])).internal_coproduct() == sum([tensor([s(lam), s(lam)]) for lam in Partitions(n)]) for n in range(6) )} \\
& \text{True} \\
\text{sage: } & \text{all( h([n]).internal_coproduct() == sum([tensor([h(lam), h(m(lam)))] for lam in Partitions(n)]) for n in range(6) )} \\
& \text{True} \\
\text{sage: } & \text{all( factorial(n) * h([n]).internal_coproduct() == sum([lam.conjugacy_class_size() * tensor([h(p(lam)), h(p(lam)))] for lam in Partitions(n)]) for n in range(6) )} \\
& \text{True}
\end{align*}
\]

**kronecker_product**

Return the internal (tensor) product of \( \text{self} \) and \( x \) in the basis of \( \text{self} \).
The internal tensor product can be defined as the linear extension of the definition on power sums $p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\lambda p_\lambda$, where $z_\lambda = (1^{r_1} r_1!) (2^{r_2} r_2!) \cdots$ for $\lambda = (1^{r_1} 2^{r_2} \cdots)$ and where $*$ denotes the internal tensor product. The internal tensor product is also known as the Kronecker product, or as the second multiplication on the ring of symmetric functions.

Note that the internal product of any two homogeneous symmetric functions of equal degrees is a homogeneous symmetric function of the same degree. On the other hand, the internal product of two homogeneous symmetric functions of distinct degrees is 0.

**Note:** The internal product is sometimes referred to as “inner product” in the literature, but unfortunately this name is shared by a different operation, namely the Hall inner product (see `scalar()`).

**INPUT:**
- $x$ – element of the ring of symmetric functions over the same base ring as `self`

**OUTPUT:**
- the internal product of `self` with $x$ (an element of the ring of symmetric functions in the same basis as `self`)

The methods `itensor()`, `internal_product()`, `kronecker_product()`, `inner_tensor()` are all synonyms.

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: b = s([3])
sage: a.itensor(b)
s[2, 1]
sage: c = s([3,2,1])
sage: c.itensor(c)
s[1, 1, 1, 1, 1, 1] + 2*s[2, 1, 1, 1] + 3*s[2, 2, 1] + 2*s[2, 2, 2]
+ 4*s[3, 1, 1] + 5*s[3, 2, 1] + 2*s[3, 3] + 4*s[4, 1, 1]
+ 3*s[4, 2] + 2*s[5, 1] + s[6]
```

There are few quantitative results pertaining to Kronecker products in general, which makes their computation so difficult. Let us test a few of them in different bases.

The Kronecker product of any homogeneous symmetric function $f$ of degree $n$ with the $n$-th complete homogeneous symmetric function $h[n]$ (a.k.a. $s[n]$) is $f$:

```python
sage: h = SymmetricFunctions(ZZ).h()
sage: all( h([5]).itensor(h(p)) == h(p) for p in Partitions(5) )
True
```

The Kronecker product of a Schur function $s_\lambda$ with the $n$-th elementary symmetric function $e[n]$, where $n = |\lambda|$, is $s_{\lambda'}$ (where $\lambda'$ is the conjugate partition of $\lambda$):

```python
sage: F = CyclotomicField(12)
sage: s = SymmetricFunctions(F).s()
sage: e = SymmetricFunctions(F).e()
sage: all( e([5]).itensor(s(p)) == s(p.conjugate()) for p in Partitions(5) )
True
```

The Kronecker product is commutative:
Let us check (on examples) Proposition 5.2 of Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon, “Non-commutative symmetric functions”, arXiv hep-th/9407124, for $r = 2$:

```python
sage: e = SymmetricFunctions(FiniteField(29)).e()
sage: s = SymmetricFunctions(FiniteField(29)).s()
sage: m = SymmetricFunctions(FiniteField(29)).m()
sage: def tensor_copr(u, v, w):
    # computes $\mu ((u \otimes v) * \Delta(w))$ with
    # * meaning Kronecker product and $\mu$ meaning
    # usual multiplication.
    result = w.parent().zero()
    for partition_pair, coeff in w.coproduct():
        result += coeff * w.parent()(u).itensor(partition_pair[0]) * w.
        return result
sage: all( all( tensor_copr(e[u], s[v], m[w])
             == (e[u] * s[v]).itensor(m[w])
             for w in Partitions(5) )
             for v in Partitions(2) )
...: for u in Partitions(3) )
True
```

Some examples from Briand, Orellana, Rosas, “The stability of the Kronecker products of Schur functions.” arXiv 0907.4652:

```python
sage: e = SymmetricFunctions(ZZ).e()
sage: s = SymmetricFunctions(ZZ).s()
sage: s[2,2].itensor(s[2,2])
s[1, 1, 1, 1] + s[2, 2] + s[4]
sage: s[3,2].itensor(s[3,2])
sage: s[4,2].itensor(s[4,2])
```

An example from p. 220 of Thibon, “Hopf algebras of symmetric functions and tensor products of symmetric group representations”, International Journal of Algebra and Computation, 1991:

```python
sage: s = SymmetricFunctions(QQbar).s()
```
Combinatorics, Release 10.0

---

Note: The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a \( \mathbb{Q} \)-algebra (in which case the Kronecker product can be easily computed using the power sum basis) from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over \( \mathbb{Q} \) (using the corresponding_basis_over hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

left_padded_kronecker_product(x)

Return the left-padded Kronecker product of self and x in the basis of self.

The left-padded Kronecker product is a bilinear map mapping two symmetric functions to another, not necessarily preserving degree. It can be defined as follows: Let \(*\) denote the Kronecker product \( \text{itensor()} \) on the space of symmetric functions. For any partitions \( \alpha, \beta, \gamma \), let \( g^{\gamma}_{\alpha,\beta} \) denote the coefficient of the complete homogeneous symmetric function \( h_\gamma \) in the Kronecker product \( h_\alpha * h_\beta \). For every partition \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) and every integer \( n > |\lambda| + \lambda_1 \) denote the \( n \)-completion of \( \lambda \) (this is the partition \( (n - |\lambda|, \lambda_1, \lambda_2, \lambda_3, \ldots) \); see \( \text{t_completion()} \)). Then, for any partitions \( \alpha \) and \( \beta \) and every integer \( n \geq |\alpha| + |\beta| + \alpha_1 + \beta_1 \), we can write the Kronecker product \( h_{\alpha[n]} * h_{\beta[n]} \) in the form

\[
h_{\alpha[n]} * h_{\beta[n]} = \sum_{\gamma} g^{\gamma}_{\alpha[n],\beta[n]} h_{\gamma[n]}
\]

with \( \gamma \) ranging over all partitions. The coefficients \( g^{\gamma}_{\alpha[n],\beta[n]} \) are independent on \( n \). These coefficients \( g^{\gamma}_{\alpha[n],\beta[n]} \) are denoted by \( \overline{g}^{\gamma}_{\alpha,\beta} \), and the symmetric function

\[
\sum_{\gamma} \overline{g}^{\gamma}_{\alpha,\beta} h_{\gamma}
\]

is said to be the left-padded Kronecker product of \( h_\alpha \) and \( h_\beta \). By bilinearity, this extends to a definition of a left-padded Kronecker product of any two symmetric functions.

This notion of left-padded Kronecker product can be lifted to the non-commutative symmetric functions \( \left( \text{left_padded_kronecker_product()} \right) \).

Warning: Do not mistake this product for the reduced Kronecker product \( \left( \text{reduced_kronecker_product()} \right) \), which uses the Schur functions instead of the complete homogeneous functions in its definition.

INPUT:

- \( x \) – element of the ring of symmetric functions over the same base ring as self

OUTPUT:

- the left-padded Kronecker product of self with x (an element of the ring of symmetric functions in the same basis as self)

EXAMPLES:
```python
sage: Sym = SymmetricFunctions(QQ)
sage: h = Sym.h()
sage: h[2,1].left_padded_kronecker_product(h[3])
h[1, 1, 1] + h[2, 1] + h[2, 1, 1] + h[2, 2, 1] + h[3, 2, 1]
sage: h[2,1].left_padded_kronecker_product(h[1])
h[1, 1, 1] + h[2, 1] + h[2, 1, 1]
sage: h[1].left_padded_kronecker_product(h[2,1])
h[1, 1, 1] + h[2, 1] + h[2, 1, 1]
sage: h[1,1].left_padded_kronecker_product(h[2])
h[1, 1] + 2*h[1, 1, 1] + h[2, 1, 1]
sage: h[1].left_padded_kronecker_product(h[2,1,1])
h[1, 1, 1] + 2*h[2, 1, 1] + h[2, 1, 1, 1]
sage: h[2].left_padded_kronecker_product(h[3])
h[2, 1] + h[2, 1, 1] + h[3, 2]
```

Taking the left-padded Kronecker product with $1 = h_{\emptyset}$ is the identity map on the ring of symmetric functions:

```python
sage: all( h[Partition([])].left_padded_kronecker_product(h[lam])
...: == h[lam] for i in range(4)
...: for lam in Partitions(i) )
True
```

Here is a rule for the left-padded Kronecker product of $h_1$ (this is the same as $h_{(1)}$) with any complete homogeneous function: Let $\lambda$ be a partition. Then, the left-padded Kronecker product of $h_1$ and $h_\lambda$ is $\sum_\mu a_\mu h_\mu$, where the sum runs over all partitions $\mu$, and the coefficient $a_\mu$ is defined as the number of ways to obtain $\mu$ from $\lambda$ by one of the following two operations:

- Insert a 1 into $\lambda$.
- Subtract 1 from one of the entries of $\lambda$ (and remove the entry if it thus becomes 0), and insert a 1 into $\lambda$.

We check this for partitions of size $\leq 4$:

```python
sage: def mults1(I):
...:     # Left-padded Kronecker multiplication by h[1].
...:     res = h[I[:] + [1]]
...:     for k in range(len(I)):
...:         I2 = I[:]
...:         if I2[k] == 1:
...:             I2 = I2[:k] + I2[k+1:]
...:         else:
...:             I2[k] -= 1
...:         res += h[sorted(I2 + [1], reverse=True)]
...:     return res
sage: all( mults1(I) == h[1].left_padded_kronecker_product(h[I])
...:       == h[I].left_padded_kronecker_product(h[1])
...:       for i in range(5) for I in Partitions(i) )
True
```

The left-padded Kronecker product is commutative:

```python
sage: all( h[lam].left_padded_kronecker_product(h[mu])
...:       == h[mu].left_padded_kronecker_product(h[lam])
```
..: for lam in Partitions(3) for mu in Partitions(3) )
    True

:nabla$(q=\text{None}, t=\text{None}, \text{power}=1)$

Return the value of the nabla operator applied to self.

The eigenvectors of the nabla operator are the Macdonald polynomials in the Ht basis.

If the parameter power is an integer then it calculates nabla to that integer. The default value of power is 1.

INPUT:

• q, t – optional parameters (default: None, in which case q and t are used)

• power – (default: 1) an integer indicating how many times to apply the operator ∇. Negative values of power indicate powers of $\nabla^{-1}$.

EXAMPLES:

```
sage: Sym = SymmetricFunctions(FractionField(QQ['q','t']))
sage: p = Sym.power()
sage: p([1,1]).nabla()
(-1/2*q*t+1/2*q+1/2*t+1/2)*p[1, 1] + (1/2*q^2-1/2*q-1/2*t+1/2)*p[2]
sage: p([2,1]).nabla(q=1)
(-1)*p[1, 1, 1] + t*p[2, 1]
sage: p([2]).nabla(q=1)*p([1]).nabla(q=1)
(-1+t)*p[1, 1, 1] + t*p[2, 1]
sage: s = Sym.schur()
sage: s([2,1]).nabla()
(-q^3*t-q^2*t^2-q*t^3)*s[1, 1, 1] + (-q^2*t-q^3)*s[2, 1]
sage: s([1,1,1]).nabla()
(q^3+q^2*t+q*t^2+q^2*t)*s[1, 1, 1] + (q^2+q*t+q^2)*s[2, 1] + s[3]
sage: s([1,1,1]).nabla(t=1)
(q^3+q^2*t+q)*s[1, 1, 1] + (q^2+q^2)*s[2, 1] + s[3]
sage: s([0]).nabla()

```

:omega$(\text{None})$

Return the image of self under the omega automorphism.

The omega automorphism is defined to be the unique algebra endomorphism $\omega$ of the ring of symmetric functions that satisfies $\omega(e_k) = h_k$ for all positive integers $k$ (where $e_k$ stands for the $k$-th elementary symmetric function, and $h_k$ stands for the $k$-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the omega involution. It sends the power-sum symmetric function $p_k$ to $(-1)^{k-1}p_k$ for every positive integer $k$.

The images of some bases under the omega automorphism are given by

\[
\omega(e_\lambda) = h_\lambda, \quad \omega(h_\lambda) = e_\lambda, \quad \omega(p_\lambda) = (-1)^{|\lambda|}p_\lambda, \quad \omega(s_\lambda) = s_\lambda.
\]
where $\lambda$ is any partition, where $\ell(\lambda)$ denotes the length \(\text{length}(\lambda)\) of the partition $\lambda$, where $\lambda'$ denotes the conjugate partition \(\text{conjugate}(\lambda)\) of $\lambda$, and where the usual notations for bases are used ($e = \text{elementary}, h = \text{complete homogeneous}, p = \text{powersum}, s = \text{Schur}$).

The default implementation converts to the Schur basis, then performs the automorphism and changes back.

omega_involution() is a synonym for the omega() method.

**EXAMPLES:**

```python
sage: J = SymmetricFunctions(QQ).jack(t=1).P()
sage: a = J([2,1]) + J([1,1,1])
sage: a.omega()
sage: J(0).omega()
0
sage: J(1).omega()
JackP[]
```

The forgotten symmetric functions are the images of the monomial symmetric functions under omega:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: m = Sym.m()
sage: f = Sym.f()
sage: all( f(lam) == m(lam).omega() for lam in Partitions(3) )
True
sage: all( m(lam) == f(lam).omega() for lam in Partitions(3) )
True
```

**omega_involution()**

Return the image of self under the omega automorphism.

The omega automorphism is defined to be the unique algebra endomorphism $\omega$ of the ring of symmetric functions that satisfies $\omega(e_k) = h_k$ for all positive integers $k$ (where $e_k$ stands for the $k$-th elementary symmetric function, and $h_k$ stands for the $k$-th complete homogeneous symmetric function). It furthermore is a Hopf algebra endomorphism and an involution, and it is also known as the omega involution. It sends the power-sum symmetric function $p_k$ to $(-1)^{k-1}p_k$ for every positive integer $k$.

The images of some bases under the omega automorphism are given by

$$\omega(e_{\lambda}) = h_{\lambda}, \quad \omega(h_{\lambda}) = e_{\lambda}, \quad \omega(p_{\lambda}) = (-1)^{\ell(\lambda)} p_{\lambda}, \quad \omega(s_{\lambda}) = s_{\lambda'},$$

where $\lambda$ is any partition, where $\ell(\lambda)$ denotes the length \(\text{length}(\lambda)\) of the partition $\lambda$, where $\lambda'$ denotes the conjugate partition \(\text{conjugate}(\lambda)\) of $\lambda$, and where the usual notations for bases are used ($e = \text{elementary}, h = \text{complete homogeneous}, p = \text{powersum}, s = \text{Schur}$).

The default implementation converts to the Schur basis, then performs the automorphism and changes back.

omega_involution() is a synonym for the omega() method.

**EXAMPLES:**

```python
sage: J = SymmetricFunctions(QQ).jack(t=1).P()
sage: a = J([2,1]) + J([1,1,1])
sage: a.omega()
sage: J(0).omega()
0
sage: J(1).omega()
JackP[]
```
The forgotten symmetric functions are the images of the monomial symmetric functions under omega:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: m = Sym.m()
sage: f = Sym.f()
sage: all( f(lam) == m(lam).omega() for lam in Partitions(3) )
True
sage: all( m(lam) == f(lam).omega() for lam in Partitions(3) )
True
```

**omega_qt**

Return the image of self under the $q, t$-deformed omega automorphism which sends $p_k$ to $(-1)^{k-1} \cdot \frac{1 - q^k}{1 - t^k} \cdot p_k$ for all positive integers $k$.

In general, this is well-defined outside of the powersum basis only if the base ring is a $\mathbb{Q}$-algebra.

If $q = t$, then this is the omega automorphism (**omega()**).

**INPUT:**
- $q, t$ – parameters (default: None, in which case 'q' and 't' are used)

**EXAMPLES:**

```python
sage: QQqt = QQ['q,t'].fraction_field()
sage: q,t = QQqt.gens()
sage: p = SymmetricFunctions(QQqt).p()
sage: p[5].omega_qt()
((-q^5+1)/(-t^5+1))*p[5]
sage: p[5].omega_qt(q,t)
((-q^5+1)/(-t^5+1))*p[5]
sage: p([2]).omega_qt(q,t)
((q^2-1)/(-t^2+1))*p[2]
sage: p([2,1]).omega_qt(q,t)
((-q^3+q^2+q-1)/(t^3-t^2-t+1))*p[2, 1]
sage: p([3,2]).omega_qt(5,q)
-(2976/(q^5-q^3-q^2+1))*p[3, 2]
sage: p([0]).omega_qt()
0
sage: p([1]).omega_qt()
p[]
```

5.1. Comprehensive Module List
plethysm(x, include=None, exclude=None)

Return the outer plethysm of self with x.

This is implemented only over base rings which are \( \mathbb{Q} \)-algebras. (To compute outer plethysms over general binomial rings, change bases to the fraction field.)

The outer plethysm of \( f \) with \( g \) is commonly denoted by \( f \triangleright g \) or by \( f \circ g \). It is an algebra map in \( f \), but not (generally) in \( g \).

By default, the degree one elements are taken to be the generators for the self’s base ring. This setting can be modified by specifying the include and exclude keywords.

INPUT:

- \( x \) – a symmetric function over the same base ring as self
- \( \text{include} \) – a list of variables to be treated as degree one elements instead of the default degree one elements
- \( \text{exclude} \) – a list of variables to be excluded from the default degree one elements

OUTPUT:

An element in the parent of \( x \) or the base ring \( R \) of self when \( x \) is in \( R \).

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: h = Sym.h()
sage: h3h2 = h[3](h[2]); h3h2
sage: s(h3h2)
sage: p = Sym.p()
sage: p3s21 = p[3](s[2,1]); p3s21
sage: p(p3s21)
1/3*p[3, 3, 3] - 1/3*p[9]
sage: e = Sym.e()
sage: e[3](e[2])
```

Note that the output is in the basis of the input \( x \):

```
sage: s[2,1](h[3])
```
Combinatorics, Release 10.0

Examples over a polynomial ring:

```python
sage: R.<t> = QQ[]
sage: s = SymmetricFunctions(R).s()
sage: a = s([3])
sage: f = t * s([2])
sage: a(f)
t^3*s[2, 2, 2] + t^3*s[4, 2] + t^3*s[6]
sage: a(0).plethysm(s([1]))
0
sage: a(1).plethysm(s([0]))
s[]
```

When `x` is a constant, then it is returned as an element of the base ring:

```python
sage: s[3](2).parent() is R
True
```

Sage also handles plethysm of tensor products of symmetric functions:

```python
sage: s = SymmetricFunctions(QQ).s()
sage: X = tensor([s[1],s[[[]]])
sage: Y = tensor([s[[[]]],s[1]])
sage: s[1,1,1](X+Y)
```

One can use this to work with symmetric functions in two sets of commuting variables. For example, we verify the Cauchy identities (in degree 5):

```python
sage: m = SymmetricFunctions(QQ).m()
sage: P5 = Partitions(5)
sage: sum(s[mu](X)*s[mu](Y) for mu in P5) == sum(m[mu](X)*h[mu](Y) for mu in P5)
True
sage: sum(s[mu](X)*s[mu.conjugate()](Y) for mu in P5) == sum(m[mu](X)*e[mu](Y) for mu in P5)
True
```

Sage can also do the plethysm with an element in the completion:
sage: g = L(s[1]) / (1 - L(s[1])); g
s[1] + (s[1,1]+s[2]) + (s[1,1,1]+2*s[2,1]+s[3])
+ (s[1,1,1,1]+3*s[2,1,1]+2*s[2,2]+3*s[3,1]+s[4])
+ (s[1,1,1,1,1]+4*s[2,1,1,1]+5*s[2,2,1]+6*s[3,1,1]+5*s[3,2]+4*s[4,1]+s[5])
+ ... + O^8
sage: fog = f(g)
sage: fog[:8]
[s[2, 1],
s[1, 1, 1, 1] + 3*s[2, 1, 1] + 2*s[2, 2] + 3*s[3, 1] + s[4],
2*s[1, 1, 1, 1] + 8*s[2, 1, 1, 1] + 10*s[2, 2, 1]
+ 12*s[3, 1, 1] + 10*s[3, 2, 1] + 2*s[4, 1],
3*s[1, 1, 1, 1, 1] + 17*s[2, 1, 1, 1, 1] + 30*s[2, 2, 1, 1] + 30*s[2, 2, 2]
+ 16*s[2, 2, 2] + 33*s[3, 1, 1, 1] + 54*s[3, 2, 1] + 16*s[3, 3],
+ 33*s[4, 1, 1] + 30*s[4, 2, 1] + 17*s[5, 1] + 3*s[6],
5*s[1, 1, 1, 1, 1, 1] + 30*s[2, 1, 1, 1, 1] + 70*s[2, 2, 1, 1, 1]
+ 70*s[2, 2, 2, 1] + 75*s[3, 1, 1, 1, 1] + 175*s[3, 2, 1, 1]
+ 105*s[3, 2, 2] + 175*s[3, 3, 1] + 100*s[4, 1, 1, 1] + 175*s[4, 2, 1]
+ 70*s[4, 3] + 75*s[5, 1, 1] + 70*s[5, 2, 30*s[6, 1] + 5*s[7]],
]
sage: parent(fog)
Lazy completion of Symmetric Functions over Rational Field in the Schur basis

See also:

frobenius()

Todo: The implementation of plethysm in sage.data_structures.stream.Stream_plethysm seems to be faster. This should be investigated.

principal_specialization(n=+Infinity, q=None)

Return the principal specialization of a symmetric function.

The principal specialization of order $n$ at $q$ is the ring homomorphism $p_{s_{n,q}}$ from the ring of symmetric functions to another commutative ring $R$ given by $x_i \mapsto q^{i-1}$ for $i \in \{1, \ldots, n\}$ and $x_i \mapsto 0$ for $i > n$. Here, $q$ is a given element of $R$, and we assume that the variables of our symmetric functions are $x_1, x_2, x_3, \ldots$. (To be more precise, $p_{s_{n,q}}$ is a $K$-algebra homomorphism, where $K$ is the base ring.) See Section 7.8 of [EnumComb2].

The stable principal specialization at $q$ is the ring homomorphism $p_s$, from the ring of symmetric functions to another commutative ring $R$ given by $x_i \mapsto q^{i-1}$ for all $i$. This is well-defined only if the resulting infinite sums converge; thus, in particular, setting $q = 1$ in the stable principal specialization is an invalid operation.

INPUT:

- **n** (default: infinity) – a nonnegative integer or infinity, specifying whether to compute the principal specialization of order $n$ or the stable principal specialization.

- **q** (default: None) – the value to use for $q$; the default is to create a ring of polynomials in $q$ (or a field of rational functions in $q$) over the given coefficient ring.

EXAMPLES:

sage: m = SymmetricFunctions(QQ).m()
sage: x = m[1,1]
sage: x.principal_specialization(3)
q^3 + q^2 + q
By default we return a rational function in $q$. Sometimes it is better to obtain an element of the symbolic ring:

```python
sage: h = SymmetricFunctions(QQ).h()
sage: (h[3]+h[2]).principal_specialization(q=var("q"))
1/((q^2 - 1)*(q - 1)) - 1/((q^3 - 1)*(q^2 - 1)*(q - 1))
```

In case $q$ is in the base ring, it must be passed explicitly:

```python
sage: R = QQ['q,t']
sage: Ht = SymmetricFunctions(R).macdonald().Ht()
sage: Ht[2].principal_specialization()
Traceback (most recent call last):
... ValueError: the variable q is in the base ring, pass it explicitly
sage: Ht[2].principal_specialization(q=R("q"))
(q^2 + 1)/(q^3 - q^2 - q + 1)
```

Note that the principal specialization can be obtained as a plethysm:

```python
sage: R = QQ['q'].fraction_field()
sage: s = SymmetricFunctions(R).s()
sage: one = s.one()
sage: q = R("q")
sage: f = s[3,2,2]
sage: f.principal_specialization(q=q) == f(one/(1-q)).coefficient([])
True
sage: f.principal_specialization(n=4, q=q) == f(one*(1-q^4)/(1-q)).→coefficient([])
True
```

### reduced_kronecker_product($x$)

Return the reduced Kronecker product of $s$ and $x$ in the basis of $s$.

The reduced Kronecker product is a bilinear map mapping two symmetric functions to another, not necessarily preserving degree. It can be defined as follows: Let $\ast$ denote the Kronecker product ($\itensor()$) on the space of symmetric functions. For any partitions $\alpha, \beta, \gamma$, let $g_{\alpha, \beta}^{\gamma}$ denote the coefficient of the Schur function $s_\gamma$ in the Kronecker product $s_\alpha \ast s_\beta$ (this is called a Kronecker coefficient). For every partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and every integer $n > |\lambda| + \lambda_1$, let $\lambda[n]$ denote the $n$-completion of $\lambda$ (this is the partition $(n - |\lambda|, \lambda_1, \lambda_2, \lambda_3, \ldots)$; see $t\_completion()$). Then, Theorem 1.2 of [BOR2009] shows that for any partitions $\alpha$ and $\beta$ and every integer $n \geq |\alpha| + |\beta| + \alpha_1 + \beta_1$, we can write the Kronecker product $s_\alpha[n] \ast s_\beta[n]$ in the form

$$s_\alpha[n] \ast s_\beta[n] = \sum_{\gamma} g_{\alpha[n], \beta[n]}^{\gamma[n]} s_\gamma[n]$$

with $\gamma$ ranging over all partitions. The coefficients $g_{\alpha[n], \beta[n]}^{\gamma[n]}$ are independent on $n$. These coefficients $g_{\alpha[n], \beta[n]}^{\gamma[n]}$ are denoted by $g_{\alpha, \beta}^{\gamma}$, and the symmetric function

$$\sum_{\gamma} g_{\alpha, \beta}^{\gamma}s_\gamma$$

is said to be the reduced Kronecker product of $s_\alpha$ and $s_\beta$. By bilinearity, this extends to a definition of a reduced Kronecker product of any two symmetric functions.
The definition of the reduced Kronecker product goes back to Murnaghan, and has recently been studied in [BOR2009], [BdVO2012] and other places (our notation $\gamma^{\alpha,\beta}_{\alpha,\beta}$ appears in these two sources).

**INPUT:**

- $x$ – element of the ring of symmetric functions over the same base ring as `self`

**OUTPUT:**

- the reduced Kronecker product of `self` with $x$ (an element of the ring of symmetric functions in the same basis as `self`)

**EXAMPLES:**

The example from page 2 of [BOR2009]:

```sage
Sym = SymmetricFunctions(QQ)
s = Sym.schur()
s[2].reduced_kronecker_product(s[2])
```

Taking the reduced Kronecker product with $1 = s_{\emptyset}$ is the identity map on the ring of symmetric functions:

```sage
all( s[Partition([i])].reduced_kronecker_product(s[lam]) == s[lam] for i in range(4) for lam in Partitions(i) )
```

True

While reduced Kronecker products are hard to compute in general, there is a rule for taking reduced Kronecker products with $s_1$. Namely, for every partition $\lambda$, the reduced Kronecker product of $s_\lambda$ with $s_1$ is $\sum \mu a_\mu s_\mu$, where the sum runs over all partitions $\mu$, and the coefficient $a_\mu$ is defined as the number of ways to obtain $\mu$ from $\lambda$ by one of the following three operations:

- Add an addable cell (`addable_cells()`) to $\lambda$.
- Remove a removable cell (`removable_cells()`) from $\lambda$.
- First remove a removable cell from $\lambda$, then add an addable cell to the resulting Young diagram.

This is, in fact, Proposition 5.15 of [CO2010] in an elementary wording. We check this for partitions of size $\leq 4$:

```sage
def mults1(lam):
    # Reduced Kronecker multiplication by s[1], according to [CO2010]_
    res = s.zero()
    for mu in lam.up_list():
        res += s(mu)
    for mu in lam.down_list():
        res += s(mu)
    for nu in mu.up_list():
        res += s(nu)
    return res
```

```sage
all( mults1(lam) == s[1].reduced_kronecker_product(s[lam]) for i in range(5) for lam in Partitions(i) )
```

True

Here is the example on page 3 of Christian Gutschwager’s arXiv 0912.4411v3.

\[
\text{sage: } \text{s}[3].\text{reduced_kronecker_product}(\text{s}[2,1])\\
\text{s}[1] + 2^*\text{s}[1, 1] + 3*\text{s}[2] + 5*\text{s}[2, 1] + 4^*\text{s}[2, 1, 1] + 3^*\text{s}[3, 1, 1] + 3^*\text{s}[3, 2] + \text{s}[3, 2, 1] + 2^*\text{s}[4] + 3^*\text{s}[4, 1] + \text{s}[4, 1, 1] + \text{s}[4, 2] + \text{s}[5] + \text{s}[5, 1]
\]

Todo: This implementation of the reduced Kronecker product is painfully slow.

**restrict_degree**\((d, \text{exact}=\text{True})\)
Return the degree \(d\) component of \(\text{self}\).

**INPUT:**
- \(d\) – positive integer, degree of the terms to be returned
- \(\text{exact}\) – boolean, if \(\text{True}\), returns the terms of degree exactly \(d\), otherwise returns all terms of degree less than or equal to \(d\)

**OUTPUT:**
- the homogeneous component of \(\text{self}\) of degree \(d\)

**EXAMPLES:**

\[
\text{sage: } s = \text{SymmetricFunctions(QQ).s()}\\
\text{sage: } z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])\\
\text{sage: } z.\text{restrict_degree}(2)\\
\text{0}\\
\text{sage: } z.\text{restrict_degree}(1)\\
\text{s}[1]\\
\text{sage: } z.\text{restrict_degree}(3)\\
\text{s}[1, 1, 1] + \text{s}[2, 1]\\
\text{sage: } z.\text{restrict_degree}(3, \text{exact}=\text{False})\\
\text{s}[1] + \text{s}[1, 1, 1] + \text{s}[2, 1]\\
\text{sage: } z.\text{restrict_degree}(0)\\
\text{0}
\]

**restrict_partition_lengths**\((l, \text{exact}=\text{True})\)
Return the terms of \(\text{self}\) labelled by partitions of length \(l\).

**INPUT:**
- \(l\) – nonnegative integer
- \(\text{exact}\) – boolean, defaulting to \(\text{True}\)

**OUTPUT:**
- if \(\text{True}\), returns the terms labelled by partitions of length precisely \(l\); otherwise returns all terms labelled by partitions of length less than or equal to \(l\)
EXAMPLES:

```
sage: s = SymmetricFunctions(QQ).s()
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])
sage: z.restrict_partition_lengths(2)
s[2, 1]
sage: z.restrict_partition_lengths(0)
0
sage: z.restrict_partition_lengths(2, exact = False)
```

`restrict_parts(n)`

Return the terms of self labelled by partitions \( \lambda \) with \( \lambda_1 \leq n \).

INPUT:

- `n` – positive integer, to restrict the parts of the partitions of the terms to be returned

EXAMPLES:

```
sage: s = SymmetricFunctions(QQ).s()
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])
sage: z.restrict_parts(2)
s[1] + s[1, 1, 1] + s[2, 1]
sage: z.restrict_parts(1)
s[1] + s[1, 1, 1]
```

`scalar(x, zee=None)`

Return the standard scalar product between self and x.

This is also known as the “Hall inner product” or the “Hall scalar product”.

INPUT:

- `x` – element of the ring of symmetric functions over the same base ring as self
- `zee` – an optional function on partitions giving the value for the scalar product between \( p_\mu \) and \( p_\mu \)
  (default is to use the standard zee() function)

This is the default implementation that converts both self and x into either Schur functions (if zee is not specified) or power-sum functions (if zee is specified) and performs the scalar product in that basis.

EXAMPLES:

```
sage: e = SymmetricFunctions(QQ).e()
sage: h = SymmetricFunctions(QQ).h()
sage: m = SymmetricFunctions(QQ).m()
sage: p4 = Partitions(4)
sage: matrix([[e(a).scalar(h(b)) for a in p4] for b in p4])
[ 0 0 0 1]
[ 0 0 1 4]
[ 0 1 2 6]
[ 1 4 6 12]
sage: matrix([[h(a).scalar(e(b)) for a in p4] for b in p4])
[ 0 0 0 1]
[ 0 0 1 4]
[ 0 1 2 6]
```

(continues on next page)
\[\begin{bmatrix}
0 & 1 & 2 & 5 & 12 \\
1 & 4 & 6 & 12 & 24
\end{bmatrix}\]

```
sage: matrix([[m(a).scalar(e(b)) for a in p4] for b in p4])
\[-1 & 2 & 1 & -3 & 1 \\
 0 & 1 & 0 & -2 & 1 \\
 0 & 0 & 1 & -2 & 1 \\
 0 & 0 & 0 & -1 & 1 \\
 0 & 0 & 0 & 0 & 1
\]
```

```
sage: matrix([[m(a).scalar(h(b)) for a in p4] for b in p4])
\[1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1
\]
```

```
sage: p = SymmetricFunctions(QQ).p()
sage: m(p[3,2]).scalar(p[3,2], zee=lambda mu: 2**mu.length())
4
```

```
sage: m(p[3,2]).scalar(p[2,2,1], lambda mu: 1)
0
```

```
sage: m[3,2].scalar(h[3,2], zee=lambda mu: 2**mu.length())
2/3
```

\section*{scalar_hl(x, t=None)}

Return the \(t\)-deformed standard Hall-Littlewood scalar product of \texttt{self} and \(x\).

**INPUT:**

- \(x\) – element of the ring of symmetric functions over the same base ring as \texttt{self}
- \(t\) – parameter (default: None, in which case \(t\) is used)

**EXAMPLES:**

```
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: sp = a.scalar_t(a); sp
(-t^2 - 1)/(t^5 - 2*t^4 + t^3 - t^2 + 2*t - 1)
sage: sp.parent()
Fraction Field of Univariate Polynomial Ring in t over Rational Field
```

\section*{scalar_jack(x, t=None)}

Return the Jack-scalar product between \texttt{self} and \(x\).

This scalar product is defined so that the power sum elements \(p_\mu\) are orthogonal and \(\langle p_\mu, p_\mu \rangle = z_\mu t^{\ell(\mu)}\), where \(\ell(\mu)\) denotes the length of \(\mu\).

**INPUT:**

- \(x\) – element of the ring of symmetric functions over the same base ring as \texttt{self}
- \(t\) – an optional parameter (default: None in which case \(t\) is used)

**EXAMPLES:**
sage: p = SymmetricFunctions(QQ['t']).power()
sage: matrix([[p(mu).scalar_jack(p(nu)) for nu in Partitions(4)] for mu in Partitions(4)])

\[
\begin{bmatrix}
4t & 0 & 0 & 0 & 0 \\
0 & 3t^2 & 0 & 0 & 0 \\
0 & 0 & 8t^2 & 0 & 0 \\
0 & 0 & 0 & 4t^3 & 0 \\
0 & 0 & 0 & 0 & 24t^4
\end{bmatrix}
\]

sage: matrix([[p(mu).scalar_jack(p(nu),2) for nu in Partitions(4)] for mu in Partitions(4)])

\[
\begin{bmatrix}
8 & 0 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 & 0 \\
0 & 0 & 32 & 0 & 0 \\
0 & 0 & 0 & 32 & 0 \\
0 & 0 & 0 & 0 & 384
\end{bmatrix}
\]

sage: JQ = SymmetricFunctions(QQ['t'].fraction_field()).jack().Q()
sage: matrix([[JQ(mu).scalar_jack(JQ(nu)) for nu in Partitions(3)] for mu in Partitions(3)])

\[
\begin{bmatrix}
(1/3t^2 + 1/2t + 1/6)/t^3 & 0 & 0 \\
0 & (1/2t + 1)/(t^3 + 1/2t^2) & 0 \\
0 & 0 & 6/(t^3 + 3t^2 + 2t)
\end{bmatrix}
\]

**scalar_qt**(*x, q=None, t=None*)

Return the $q, t$-deformed standard Hall-Littlewood scalar product of *self* and *x*.

**INPUT:**

- *x* – element of the ring of symmetric functions over the same base ring as *self*
- *q, t* – parameters (default: None in which case *q* and *t* are used)

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: sp = a.scalar_qt(a); factor(sp)
(t - 1)^{-3} * (q - 1) * (t^2 + t + 1)^{-1} * (q^2*t^2 - q*t^2 + q^2 - 2*q*t + t^2 - q + 1)

sage: sp.parent()
Fraction Field of Multivariate Polynomial Ring in q, t over Rational Field
```

```python
sage: a.scalar_qt(a,q=0)
-t^2 - 1)/(t^5 - 2*t^4 + t^3 - t^2 + 2*t - 1)
```

```python
sage: a.scalar_qt(a,t=0)
-q^3 + 2*q^2 - 2*q + 1
```

```python
sage: a.scalar_qt(a,5,7)  # q=5 and t=7
490/1539
```

```python
sage: (x,y) = var('x,y')
sage: a.scalar_qt(a,q=x,t=y)
1/3q(x^3 - 1)/(y^3 - 1) + 2/3q(x - 1)^3/(y - 1)^3
```

```python
sage: Rn = QQ['q','t','y','z'].fraction_field()
sage: (q,t,y,z) = Rn.gens()
sage: Mac = SymmetricFunctions(Rn).macdonald(q=q,t=t)
```
sage: a = Mac._sym.schur()([2,1])
sage: factor(Mac.P()(a).scalar_qt(Mac.Q()(a),q,t))
(t - 1)^-3 * (q - 1) * (t^2 + t + 1)^-1 * (q^2*t^2 - q*t^2 + q^2 - 2*q*t + t^2 - t + 1) * (q - 1)
sage: factor(Mac.P()(a).scalar_qt(Mac.Q()(a)))
(z - 1)^-3 * (y - 1) * (z^2 + z + 1)^-1 * (y^2*z^2 - y*z^2 + y^2 - 2*y*z + z^2 - y + 1)

scalar_t(x, t=None)

Return the t-deformed standard Hall-Littlewood scalar product of self and x.

INPUT:

- x – element of the ring of symmetric functions over the same base ring as self
- t – parameter (default: None, in which case t is used)

EXAMPLES:

sage: s = SymmetricFunctions(QQ).s()
sage: a = s([2,1])
sage: sp = a.scalar_t(a); sp
(-t^2 - 1)/(t^5 - 2*t^4 + t^3 - t^2 + 2*t - 1)
sage: sp.parent()
Fraction Field of Univariate Polynomial Ring in t over Rational Field

skew_by(x)

Return the result of skewing self by x. (Skewing by x is the endomorphism (as additive group) of the ring of symmetric functions adjoint to multiplication by x with respect to the Hall inner product.)

INPUT:

- x – element of the ring of symmetric functions over the same base ring as self

EXAMPLES:

sage: s = SymmetricFunctions(QQ).s()
sage: s([3,2]).skew_by(s([2]))
s[2, 1] + s[3]
sage: s([3,2]).skew_by(s([1,1,1]))
0
sage: s([3,2,1]).skew_by(s([2,1]))
s[1, 1, 1] + 2*s[2, 1] + s[3]

sage: p = SymmetricFunctions(QQ).powersum()
sage: p([4,3,3,2,2,1]).skew_by(p([2,1]))
4*p[4, 3, 3, 2, 2]
sage: zee = sage.combinat.sf.sfa.zee
sage: zee([4,3,3,2,2,1])/zee([4,3,3,2])
4
sage: s([0]).skew_by(s([1]))
0
sage: s([1]).skew_by(s([1]))
0
sage: s([0]).skew_by(s([0]))
0
\[ s[] \]
\[ \textbf{sage: } s([]).skew_by(s[1]) \]
\[ \theta \]

**\texttt{theta}(a)**

Return the image of \texttt{self} under the theta endomorphism which sends \( p_k \) to \( a \cdot p_k \) for every positive integer \( k \).

In general, this is well-defined outside of the powersum basis only if the base ring is a \( \mathbb{Q} \)-algebra.

**INPUT:**

- \( a \) – an element of the base ring

**EXAMPLES:**

\[ \textbf{sage: } s = \text{SymmetricFunctions}(\mathbb{Q}).s() \]
\[ \textbf{sage: } s([2, 1]).theta(2) \]
\[ 2*s[1, 1, 1] + 6*s[2, 1] + 2*s[3] \]
\[ \textbf{sage: } p = \text{SymmetricFunctions}(\mathbb{Q}).p() \]
\[ \textbf{sage: } p([2]).theta(2) \]
\[ 2*p[2] \]
\[ \textbf{sage: } p(0).theta(2) \]
\[ \theta \]
\[ \textbf{sage: } p(1).theta(2) \]
\[ p[] \]

**\texttt{theta\_qt}(q=None, t=None)**

Return the image of \texttt{self} under the \( q, t \)-deformed theta endomorphism which sends \( p_k \) to \( \frac{1}{1-t^k} \cdot p_k \) for all positive integers \( k \).

In general, this is well-defined outside of the powersum basis only if the base ring is a \( \mathbb{Q} \)-algebra.

**INPUT:**

- \( q, t \) – parameters (default: None, in which case 'q' and 't' are used)

**EXAMPLES:**

\[ \textbf{sage: } QQqt = \mathbb{Q}[\{q, t\}].fraction_field() \]
\[ \textbf{sage: } q, t = QQqt.gens() \]
\[ \textbf{sage: } p = \text{SymmetricFunctions}(\mathbb{Q}qt).p() \]
\[ \textbf{sage: } p([2]).theta_qt(q, t) \]
\[ ((-q^2+1)/(-t^2+1))*p[2] \]
\[ \textbf{sage: } p([2, 1]).theta_qt(q, t) \]
\[ ((q^3-q^2-q+1)/(t^3-t^2-t+1))*p[2, 1] \]
\[ \textbf{sage: } p(0).theta_qt(q=1, t=3) \]
\[ \theta \]
\[ \textbf{sage: } p([2, 1]).theta_qt(q=2, t=3) \]
\[ 3/16*p[2, 1] \]
\[ \textbf{sage: } s = p.realization_of().schur() \]
\[ \textbf{sage: } s([3]).theta_qt(q=0)*(1-t)^*(1-t^2)^*(1-t^3) \]
\[ t^3*s[1, 1, 1] + (t^2+t)*s[2, 1] + s[3] \]
\[ \textbf{sage: } p(1).theta_qt() \]
\[ p[] \]
verschiebung(n)

Return the image of the symmetric function self under the n-th Verschiebung operator.

The n-th Verschiebung operator \(V_n\) is defined to be the unique algebra endomorphism \(V\) of the ring of symmetric functions that satisfies \(V(h_r) = h_{r/n}\) for every positive integer \(r\) divisible by \(n\), and satisfies \(V(h_r) = 0\) for every positive integer \(r\) not divisible by \(n\). This operator \(V_n\) is a Hopf algebra endomorphism. For every nonnegative integer \(r\) with \(n \mid r\), it satisfies

\[ V_n(h_r) = h_{r/n}, \quad V_n(p_r) = np_{r/n}, \quad V_n(e_r) = (-1)^{r-r/n}e_{r/n} \]

(continues on next page)
sage: p[3,2,1].verschiebung(3)
0
sage: h[4].verschiebung(2)
h[2]
sage: p[2].verschiebung(2)
2*p[1]
sage: m[3,2,1].verschiebung(6)
12*m[1]

The Verschiebung endomorphisms are multiplicative:

sage: all( all( s(lam).verschiebung(2) * s(mu).verschiebung(2)
....:   == (s(lam) * s(mu)).verschiebung(2)
....:   for mu in Partitions(4) )
....:   for lam in Partitions(4) )
True

Being Hopf algebra endomorphisms, the Verschiebung operators commute with the antipode:

sage: all( p(lam).verschiebung(3).antipode()
....:   == p(lam).antipode().verschiebung(3)
....:   for lam in Partitions(6) )
True

Testing the adjointness between the Frobenius operators $f_n$ and the Verschiebung operators $V_n$:

sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: p = Sym.p()
sage: all( all( s(lam).verschiebung(2).scalar(p(mu))
....:   == s(lam).scalar(p(mu).frobenius(2))
....:   for mu in Partitions(3) )
....:   for lam in Partitions(6) )
True

class sage.combinat.sf.sfa.SymmetricFunctionsBases(parent_with_realization)
Bases: Category_realization_of_parent

The category of bases of the ring of symmetric functions.

INPUT:

• self – a category of bases for the symmetric functions
• base – ring of symmetric functions

class ParentMethods
Bases: object

Eulerian($n$, $j$, $k$=None)

Return the Eulerian symmetric function $Q_{n,j}$ (with $n$ either an integer or a partition) or $Q_{n,j,k}$ (if the optional argument $k$ is specified) in terms of the basis $self$.

It is known that the Eulerian quasisymmetric functions are in fact symmetric functions [SW2010]. For more information, see QuasiSymmetricFunctions.Fundamental.Eulerian(), which accepts the same syntax as this method.
**INPUT:**

- *n* – the nonnegative integer *n* or a partition
- *j* – the number of excedances
- *k* – (optional) if specified, determines the number of fixed points of the permutations which are being summed over

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.m()
sage: m.Eulerian(3, 1)
4*m[1, 1, 1] + 3*m[2, 1] + 2*m[3]
sage: h = Sym.h()
sage: h.Eulerian(4, 2)
sage: s = Sym.s()
sage: s.Eulerian(5, 2)
sage: s.Eulerian([2,2,1], 2)
sage: s.Eulerian(5, 2, 2)
sage: h.Eulerian([6], 3)
h[3, 2, 1] - h[4, 1, 1] + 2*h[4, 2] + h[5, 1]
sage: s.Eulerian([6], 3)
```

We check Equation (5.4) in [SW2010]:

```python
sage: h.Eulerian([6], 3)
h[3, 2, 1] - h[4, 1, 1] + 2*h[4, 2] + h[5, 1]
sage: s.Eulerian([6], 3)
```

### carlitz_shareshian_wachs(*n*, *d*, *s*, comparison=None)

Return the Carlitz-Shareshian-Wachs symmetric function $X_{n,d,s}$ (if comparison is None), or $U_{n,d,s}$ (if comparison is -1), or $V_{n,d,s}$ (if comparison is 0), or $W_{n,d,s}$ (if comparison is 1) written in the basis self. These functions are defined below.

The Carlitz-Shareshian-Wachs symmetric functions have been introduced in [GriRei18], Exercise 2.9.11, as refinements of a certain particular case of chromatic quasisymmetric functions defined by Shareshian and Wachs. Their definitions are as follows:

Let *n*, *d* and *s* be three nonnegative integers. Let $W(n,d,s)$ denote the set of all *n*-tuples $(w_1, w_2, \ldots, w_n)$ of positive integers having the property that there exist precisely *d* elements *i* of \{1, 2, \ldots, *n* - 1\} satisfying $w_i > w_{i+1}$, and precisely *s* elements *i* of \{1, 2, \ldots, *n* - 1\} satisfying $w_i = w_{i+1}$. For every $w = (w_1, w_2, \ldots, w_n) \in W(n,d,s)$, let $x_w$ be the monomial $x_{w_1}x_{w_2}\cdots x_{w_n}$. We then define the power series $X_{n,d,s}$ by

$$X_{n,d,s} = \sum_{w \in W(n,d,s)} x_w.$$ 

This is a symmetric function (according to [GriRei18], Exercise 2.9.11(b)), and for *s* = 0 equals the $t^d$-coefficient of the descent enumerator of Smirnov words of length *n* (an example of a chromatic quasisymmetric function which happens to be symmetric – see [ShaWach2014], Example 2.5).

Assume that *n* > 0. Then, we can define three further power series as follows:

$$U_{n,d,s} = \sum_{w_1 < w_n} x_w; \quad V_{n,d,s} = \sum_{w_1 = w_n} x_w; \quad W_{n,d,s} = \sum_{w_1 > w_n} x_w,$$
where all three sums range over $w = (w_1, w_2, \ldots, w_n) \in W(n, d, s)$. These three power series $U_{n,d,s}, V_{n,d,s}$ and $W_{n,d,s}$ are symmetric functions as well ([GriRei18], Exercise 2.9.11(c)). Their sum is $X_{n,d,s}$.

REFERENCES:

INPUT:

- $n$ – a nonnegative integer
- $d$ – a nonnegative integer
- $s$ – a nonnegative integer
- comparison (default: None) – a variable which can take the forms None, -1, 0 and 1

OUTPUT:

The Carlitz-Shareshian-Wachs symmetric function $X_{n,d,s}$ (if comparison is None), or $U_{n,d,s}$ (if comparison is -1), or $V_{n,d,s}$ (if comparison is 0), or $W_{n,d,s}$ (if comparison is 1) written in the basis self.

EXAMPLES:

The power series $X_{n,d,s}$:

```
sage: Sym = SymmetricFunctions(ZZ)
sage: m = Sym.m()
sage: m.carlitz_shareshian_wachs(3, 2, 1)
0
sage: m.carlitz_shareshian_wachs(3, 1, 1)
m[2, 1]
sage: m.carlitz_shareshian_wachs(3, 2, 0)
m[1, 1, 1]
sage: m.carlitz_shareshian_wachs(3, 0, 2)
m[3]
sage: m.carlitz_shareshian_wachs(3, 1, 0)
4*m[1, 1, 1] + m[2, 1]
sage: m.carlitz_shareshian_wachs(3, 0, 1)
m[2, 1]
sage: m.carlitz_shareshian_wachs(3, 0, 0)
m[1, 1, 1]
sage: m.carlitz_shareshian_wachs(5, 2, 2)
m[2, 2, 1] + m[3, 1, 1]
sage: m.carlitz_shareshian_wachs(1, 0, 0)
m[1]
sage: m.carlitz_shareshian_wachs(0, 0, 0)
m[]
```

The power series $U_{n,d,s}$:

```
sage: m.carlitz_shareshian_wachs(3, 2, 1, comparison=-1)
0
sage: m.carlitz_shareshian_wachs(3, 1, 1, comparison=-1)
0
sage: m.carlitz_shareshian_wachs(3, 2, 0, comparison=-1)
0
sage: m.carlitz_shareshian_wachs(3, 0, 2, comparison=-1)
0
sage: m.carlitz_shareshian_wachs(3, 1, 0, comparison=-1)
2*m[1, 1, 1]
```

(continues on next page)
The power series $V_{n,d,s}$:

```python
sage: m.carlitz_shareshian_wachs(3, 2, 1, comparison=0)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 1, 1, comparison=0)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 2, 0, comparison=0)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 0, 2, comparison=0)
m[3]
sage: m.carlitz_shareshian_wachs(3, 1, 0, comparison=0)
m[2, 1]
sage: m.carlitz_shareshian_wachs(3, 0, 1, comparison=0)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 0, 0, comparison=0)
\emptyset
sage: m.carlitz_shareshian_wachs(5, 2, 2, comparison=0)
\emptyset
sage: m.carlitz_shareshian_wachs(4, 2, 0, comparison=0)
m[2, 1, 1]
sage: m.carlitz_shareshian_wachs(1, 0, 0, comparison=0)
m[1]
```

The power series $W_{n,d,s}$:

```python
sage: m.carlitz_shareshian_wachs(3, 2, 1, comparison=1)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 1, 1, comparison=1)
m[2, 1]
sage: m.carlitz_shareshian_wachs(3, 2, 0, comparison=1)
m[1, 1, 1]
sage: m.carlitz_shareshian_wachs(3, 0, 2, comparison=1)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 1, 0, comparison=1)
2*m[1, 1, 1]
sage: m.carlitz_shareshian_wachs(3, 0, 1, comparison=1)
\emptyset
sage: m.carlitz_shareshian_wachs(3, 0, 0, comparison=1)
\emptyset
sage: m.carlitz_shareshian_wachs(5, 2, 2, comparison=1)
m[2, 2, 1] + m[3, 1, 1]
```

(continues on next page)
corresponding_basis_over(R)

Return the realization of symmetric functions corresponding to self but over the base ring \( R \). Only works when self is one of the classical bases, not one of the \( q, t \)-dependent ones. In the latter case, None is returned instead.

INPUT:

\* \( R \) – a commutative ring

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.monomial()
sage: m.corresponding_basis_over(ZZ)
Symmetric Functions over Integer Ring in the monomial basis
sage: Sym = SymmetricFunctions(CyclotomicField())
sage: s = Sym.schur()
sage: s.corresponding_basis_over(Integers(13))
Symmetric Functions over Ring of integers modulo 13 in the Schur basis
sage: P = ZZ['q','t']
sage: Sym = SymmetricFunctions(P)
sage: mj = Sym.macdonald().J()
sage: mj.corresponding_basis_over(Integers(13))
```

Todo: This function is an ugly hack using strings. It should be rewritten as soon as the bases of SymmetricFunctions are put on a more robust and systematic footing.

degree_on_basis(b)

Return the degree of the basis element indexed by \( b \).

INPUT:

\* self – a basis of the symmetric functions
\* b – a partition

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ['q','t'].fraction_field())
sage: m = Sym.monomial()
sage: m.degree_on_basis(Partition([3,2]))
5
sage: P = Sym.macdonald().P()
sage: P.degree_on_basis(Partition([[]])
```

formal_series_ring()

Return the completion of all formal linear combinations of self with finite linear combinations in each homogeneous degree (computed lazily).

EXAMPLES:
sage: s = SymmetricFunctions(ZZ).s()
sage: L = s.formal_series_ring()
sage: L
Lazy completion of Symmetric Functions over Integer Ring in the Schur basis

gessel_reutenauer(lam)

Return the Gessel-Reutenauer symmetric function corresponding to the partition lam written in the basis self.

Let λ be a partition. The Gessel-Reutenauer symmetric function $GR_\lambda$ corresponding to $\lambda$ is the symmetric function denoted $L_\lambda$ in [GR1993] and in Exercise 7.89 of [STA] and denoted $GR_\lambda$ in Definition 6.6.34 of [GriRei18]. It is also called the higher Lie character, for instance in [Sch2003b]. It can be defined in several ways:

- It is the sum of the monomials $x_w$ over all words $w$ over the alphabet $\{1, 2, 3, \ldots\}$ which have CFL type $\lambda$. Here, the monomial $x_w$ for a word $w = (w_1, w_2, \ldots, w_k)$ is defined as $x_{w_1} x_{w_2} \cdots x_{w_k}$, and the CFL type of a word $w$ is defined as the partition obtained by sorting (in decreasing order) the lengths of the factors in the Lyndon factorization (lyndon_factorization()) of $w$. The fact that this power series $GR_\lambda$ is symmetric is not obvious.

- It is the sum of the fundamental quasisymmetric functions $F_{\text{Des} \sigma}$ over all permutations $\sigma$ that have cycle type $\lambda$. See sage.combinat.ncsf_qsym.qsym.QuasiSymmetricFunctions. Fundamental for the definition of fundamental quasisymmetric functions, and cycle_type() for that of cycle type. For a permutation $\sigma$, we use $\text{Des} \sigma$ to denote the descent composition (descents_composition()) of $\sigma$. Again, this definition does not make the symmetry of $GR_\lambda$ obvious.

- For every positive integer $n$, we have

$$GR_{(n)} = \frac{1}{n} \sum_{d|n} \mu(d)p_d^{n/d},$$

where $p_d$ denotes the $d$-th power-sum symmetric function. This $GR_{(n)}$ is also denoted by $L_n$, and is called the Lie character. Now, the higher Lie character $GR_\lambda$ is defined as the product:

$$h_{m_1} [L_1] \cdot h_{m_2} [L_2] \cdot h_{m_3} [L_3] \cdots,$$

where $m_i$ denotes the multiplicity of the part $i$ in $\lambda$, and where the square brackets stand for plethysm (plethysm()). This definition makes the symmetry (but not the integrality!) of $GR_\lambda$ obvious.

The equivalences of these three definitions are proven in [GR1993] Sections 2-3. (See also [GriRei18] Subsection 6.6.2 for the equivalence of the first two definitions and further formulas.)

$GR_\lambda$ has further significance in representations afforded by the tensor algebra $T(V)$ of a finite dimensional vector space. The Poincaré-Birkhoff-Witt theorem describes the universal enveloping algebra of a Lie algebra. It gives a decomposition of the degree-$n$ component $T_n(V)$ of $T(V)$ into $GL(V)$ representations indexed by partitions. The higher Lie characters are the symmetric group $S_n$ characters corresponding to this decomposition via Schur-Weyl duality.

Another important question, Thrall’s problem (see e.g. [Sch2003b]) asks, for $\lambda$ a partition of $n$, can we combinatorially interpret the coefficients $\alpha_{\mu}^\lambda$ in the Schur-expansion of $GR_\lambda$:

$$GR_\lambda = \sum_{\mu|\nu} \alpha_{\mu}^\lambda s_\mu.$$  

INPUT:

- lam – a partition or a positive integer (in the latter case, it is understood to mean the partition [lam])

5.1. Comprehensive Module List
OUTPUT:

The Gessel-Reutenauer symmetric function $\text{GR}_\lambda$, where $\lambda$ is a partition, expanded in the basis `self`.

EXAMPLES:

The first few values of $\text{GR}_{(n)} = L_n$:

```sage
sage: Sym = SymmetricFunctions(ZZ)
sage: h = Sym.h()
sage: h.gessel_reutenauer(1)
  h[1]
sage: h.gessel_reutenauer(2)
  h[1, 1] - h[2]
sage: h.gessel_reutenauer(3)
  h[2, 1] - h[3]
sage: h.gessel_reutenauer(4)
  h[2, 1, 1] - h[2, 2]
sage: h.gessel_reutenauer(5)
sage: h.gessel_reutenauer(6)
```

Gessel-Reutenauer functions indexed by partitions:

```sage
sage: h.gessel_reutenauer([2, 1])
  h[1, 1, 1] - h[2, 1]
sage: h.gessel_reutenauer([2, 2])
```

The Gessel-Reutenauer functions are Schur-positive:

```sage
sage: s = Sym.s()
sage: s.gessel_reutenauer([2, 1])
  s[1, 1, 1] + s[2, 1]
sage: s.gessel_reutenauer([2, 2])
  s[1, 1, 1, 1] + s[2, 1, 1] + s[2, 2] + s[3, 2]
```

They do not form a basis, as the following example (from [GR1993] p. 201) shows:

```sage
sage: s.gessel_reutenauer([4]) == s.gessel_reutenauer([2, 1, 1])
True
```

They also go by the name higher Lie character:

```sage
sage: s.higher_lie_character([2, 2, 1]) == s.gessel_reutenauer([2, 2, 1])
True
```

Of the above three equivalent definitions of $\text{GR}_\lambda$, we use the third one for computations. Let us check that the second one gives the same results:

```sage
sage: QSym = QuasiSymmetricFunctions(ZZ)
sage: F = QSym.F()  # fundamental basis
sage: def GR_def2(lam):
  # \textbf{GR}_\lambda \lambda
```
\textbf{Note:} The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a \( \mathbb{Q} \)-algebra from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over \( \mathbb{Q} \) (using the \texttt{corresponding\_basis\_over} hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

\textbf{higher\_lie\_character}(\textit{lam})

Return the Gessel-Reutenauer symmetric function corresponding to the partition \textit{lam} written in the basis \texttt{self}.

Let \( \lambda \) be a partition. The \textit{Gessel-Reutenauer symmetric function} \( \texttt{GR}_\lambda \) corresponding to \( \lambda \) is the symmetric function denoted \( L_\lambda \) in [GR1993] and in Exercise 7.89 of [STA] and denoted \( \texttt{GR}_\lambda \) in Definition 6.6.34 of [GriRei18]. It is also called the \textit{higher Lie character}, for instance in [Sch2003b]. It can be defined in several ways:

- It is the sum of the monomials \( x_w \) over all words \( w \) over the alphabet \( \{1, 2, 3, \ldots \} \) which have CFL type \( \lambda \). Here, the monomial \( x_w \) for a word \( w = (w_1, w_2, \ldots, w_k) \) is defined as \( x_{w_1}x_{w_2} \cdots x_{w_k} \), and the \textit{CFL type} of a word \( w \) is defined as the partition obtained by sorting (in decreasing order) the lengths of the factors in the Lyndon factorization (\texttt{lyndon\_factorization()}) of \( w \). The fact that this power series \( \texttt{GR}_\lambda \) is symmetric is not obvious.
- It is the sum of the fundamental quasisymmetric functions \( F_{\text{Des}} \) over all permutations \( \sigma \) that have cycle type \( \lambda \). See \texttt{sage.combinat.ncsf_qsym.qsym.QuasiSymmetricFunctions}.

\begin{code}
\begin{verbatim}
....:       n = lam.size()
....:       r = F.sum_of_monomials([sigma.descents_composition()
....:       for sigma in Permutations(n)
....:       if sigma.cycle_type() == lam])
....:       return r.to_symmetric_function()

sage: all( GR_def2(lam) == h.gessel_reutenauer(lam)
....:     for n in range(5) for lam in Partitions(n) )
True

And the first one, too (assuming symmetry):

\begin{verbatim}
\begin{verbatim}
....:       all( GR_def1(lam) == h.gessel_reutenauer(lam)
....:     for n in range(5) for lam in Partitions(n) )
True
\end{verbatim}
\end{verbatim}
\end{verbatim}
\end{code}
Fundamental for the definition of fundamental quasisymmetric functions, and cycle_type() for that of cycle type. For a permutation $\sigma$, we use Des $\sigma$ to denote the descent composition (descents_composition()) of $\sigma$. Again, this definition does not make the symmetry of $GR_\lambda$ obvious.

- For every positive integer $n$, we have

$$GR_{(n)} = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d},$$

where $p_d$ denotes the $d$-th power-sum symmetric function. This $GR_{(n)}$ is also denoted by $L_n$, and is called the Lie character. Now, the higher Lie character $GR_\lambda$ is defined as the product:

$$h_{m_1} [L_1] \cdot h_{m_2} [L_2] \cdot h_{m_3} [L_3] \cdots,$$

where $m_i$ denotes the multiplicity of the part $i$ in $\lambda$, and where the square brackets stand for plethysm (plethysm()). This definition makes the symmetry (but not the integrality!) of $GR_\lambda$ obvious.

The equivalences of these three definitions are proven in [GR1993] Sections 2-3. (See also [GriRei18] Subsection 6.6.2 for the equivalence of the first two definitions and further formulas.)

$GR_\lambda$ has further significance in representations afforded by the tensor algebra $T(V)$ of a finite dimensional vector space. The Poincaré-Birkhoff-Witt theorem describes the universal enveloping algebra of a Lie algebra. It gives a decomposition of the degree-$n$ component $T_n(V)$ of $T(V)$ into $GL(V)$ representations indexed by partitions. The higher Lie characters are the symmetric group $S_n$ characters corresponding to this decomposition via Schur-Weyl duality.

Another important question, Thrall’s problem (see e.g. [Sch2003b]) asks, for $\lambda$ a partition of $n$, can we combinatorially interpret the coefficients $\alpha_\mu^\lambda$ in the Schur-expansion of $GR_\lambda$:

$$GR_\lambda = \sum_{\mu \vdash n} \alpha_\mu^\lambda s_\mu.$$ 

INPUT:
- $\lambda$ – a partition or a positive integer (in the latter case, it is understood to mean the partition $[\lambda]$)

OUTPUT:

The Gessel-Reutenauer symmetric function $GR_\lambda$, where $\lambda = \lambda$, expanded in the basis $self$.

EXAMPLES:

The first few values of $GR_{(n)} = L_n$:

```sage
sage: Sym = SymmetricFunctions(ZZ)
sage: h = Sym.h()
sage: h.gessel_reutenauer(1)
  h[1]
sage: h.gessel_reutenauer(2)
  h[1, 1] - h[2]
sage: h.gessel_reutenauer(3)
  h[2, 1] - h[3]
sage: h.gessel_reutenauer(4)
  h[2, 1, 1] - h[2, 2]
sage: h.gessel_reutenauer(5)
sage: h.gessel_reutenauer(6)
  h[2, 1, 1, 1, 1] - h[2, 2, 1, 1] - h[2, 2, 2]
```

(continues on next page)
Gessel-Reutenauer functions indexed by partitions:

\[
\]

\[
\text{sage: } h.gessel_reutenauer([2, 1]) \\
h[1, 1, 1] - h[2, 1]
\]

\[
\text{sage: } h.gessel_reutenauer([2, 2]) \\
\]

The Gessel-Reutenauer functions are Schur-positive:

\[
\text{sage: } s = \text{Sym}.s() \\
\text{sage: } s.gessel_reutenauer([2, 1]) \\
s[1, 1, 1] + s[2, 1]
\]

\[
\text{sage: } s.gessel_reutenauer([2, 2, 1]) \\
s[1, 1, 1, 1] + s[2, 1, 1] + s[2, 2, 1] + s[3, 2]
\]

They do not form a basis, as the following example (from [GR1993] p. 201) shows:

\[
\text{sage: } s.gessel_reutenauer([4]) == s.gessel_reutenauer([2, 1, 1]) \Rightarrow True
\]

They also go by the name higher Lie character:

\[
\text{sage: } s.higher_lie_character([2, 2, 1]) == s.gessel_reutenauer([2, 2, 1]) \Rightarrow True
\]

Of the above three equivalent definitions of $\text{GR}_\lambda$, we use the third one for computations. Let us check that the second one gives the same results:

\[
\text{sage: } QSym = \text{QuasiSymmetricFunctions}(\text{ZZ}) \\
\text{sage: } F = QSym.F() \# fundamental basis \\
\text{sage: } F\text{.def2}(':math:GR_\lambda') \Rightarrow \text{\# lambdabla}' \\
...... n = lam.size() \\
...... r = F.sum_of_monomials([sigma.descents_composition() \\
...... for sigma in Permutations(n) \\
...... if sigma.cycle_type() == lam]) \\
...... return r.to_symmetric_function() \\
\text{sage: } all( F\text{.def2}(lam) == h.gessel_reutenauer(lam) \\
...... for n in range(5) for lam in Partitions(n) ) \Rightarrow True
\]

And the first one, too (assuming symmetry):

\[
\text{sage: } m = \text{Sym}.m() \\
\text{sage: } F\text{.def1}(':math:GR_\lambda') \Rightarrow \text{\# lambdabla}' \\
...... n = lam.size() \\
...... Permuls_mset = sage.combinat.permutation.Permutations.mset \\
...... def coeff_of_m_mu_in_result(mu): \\
...... words_to_check = Permuls_mset([i for (i, l) in enumerate(mu) \\
...... for _ in range(l)]) \\
...... return sum((1 for w in words_to_check if
\]

(continues on next page)
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(continued from previous page)

```python
....:       Partition(list(reversed(sorted([len(v) for v in _Word(w).lyndon_factorization()]))))
....:       r = m.sum_of_terms([(mu, coeff_of_m_mu_in_result(mu))
....:                               for mu in Partitions(n)],
....:                               distinct=True)
....:       return r
```

```
sage: all( GR_def1(lam) == h.gessel_reutenauer(lam) for n in range(5) for lam in Partitions(n) )
True
```

Note: The currently existing implementation of this function is technically unsatisfactory. It distinguishes the case when the base ring is a $\mathbb{Q}$-algebra from the case where it isn’t. In the latter, it does a computation using universal coefficients, again distinguishing the case when it is able to compute the “corresponding” basis of the symmetric function algebra over $\mathbb{Q}$ (using the `corresponding_basis_over` hack) from the case when it isn’t (in which case it transforms everything into the Schur basis, which is slow).

### is_commutative()

Return whether this symmetric function algebra is commutative.

**INPUT:**

- `self` – a basis of the symmetric functions

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: s.is_commutative()
True
```

### is_field(`proof=True`)

Return whether `self` is a field. (It is not.)

**INPUT:**

- `self` – a basis of the symmetric functions
- `proof` – an optional argument (default value: `True`)

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: s.is_field()
False
```

### is_integral_domain(`proof=True`)

Return whether `self` is an integral domain. (It is if and only if the base ring is an integral domain.)

**INPUT:**

- `self` – a basis of the symmetric functions
- `proof` – an optional argument (default value: `True`)

**EXAMPLES:**

```python
sage: s = SymmetricFunctions(QQ).s()
sage: s.is_integral_domain()
True
```
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```python
sage: s = SymmetricFunctions(Zmod(14)).s()
sage: s.is_integral_domain()
False
```

**lehrer_solomon(lam)**

Return the Lehrer-Solomon symmetric function (also known as the Whitney homology character) corresponding to the partition lam written in the basis self.

Let \( \lambda \vdash n \) be a partition. The Lehrer-Solomon symmetric function \( \text{LS}_\lambda \) corresponding to \( \lambda \) is the Frobenius characteristic of the representation denoted \( \text{Ind}_{S_n}^{S_\lambda} (\xi_\lambda) \) in Theorem 4.5 of [LS1986] or \( W_\lambda \) in Theorem 2.7 of [HR2017]. It was first computed as a symmetric function in [Sun1994].

It is the symmetric group representation corresponding to a summand of the Whitney homology of the set partition lattice. The summand comes from the orbit of set partitions with block sizes corresponding to \( \lambda \) (after reordering appropriately).

It can be computed using Sundaram’s plethystic formula (see [Sun1994] Theorem 1.8):

\[
\text{LS}_\lambda = \prod_{\text{odd } j \geq 1} h_{m_j}[\pi_j] \prod_{\text{even } j \geq 2} e_{m_j}[\pi_j],
\]

where \( h_{m_j} \) are complete homogeneous symmetric functions, \( e_{m_j} \) are elementary symmetric functions, and \( \pi_j \) are the images of the Gessel-Reutenauer symmetric function \( \text{GR}_{(j)} \) (see \texttt{gessel_reutenauer()}) under the involution \( \omega \) (i.e. \texttt{omega_involution()})

```python
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: pi_2 = (s.gessel_reutenauer(2)).omega_involution()
sage: pi_1 = (s.gessel_reutenauer(1)).omega_involution()
sage: s.lehrer_solomon([2,1]) == pi_2 * pi_1  # since h_1, e_1 are ...plethystic identities
True
```

Note that this also gives the \( S_n \)-equivariant structure of the Orlik-Solomon algebra of the braid arrangement (also known as the type-\( A \) reflection arrangement).

The representation corresponding to \( \text{LS}_\lambda \) exhibits representation stability [Chu2012], and a sharp bound is given in [HR2017].

**INPUT:**

- \( \lambda \) – a partition or a positive integer (in the latter case, it is understood to mean the partition \( [\lambda] \))

**OUTPUT:**

The Lehrer-Solomon symmetric function \( \text{LS}_\lambda \), where \( \lambda = \lambda \), expanded in the basis self.

**EXAMPLES:**

The first few values of \( \text{LS}_{(n)} \):

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: h = Sym.h()
sage: h.lehrer_solomon(1)
h[1]
sage: h.lehrer_solomon(2)
h[2]
```

(continues on next page)
The *whitney_homology_character()* method is an alias:

```python
sage: Sym = SymmetricFunctions(ZZ)
sage: s = Sym.schur()
sage: s.lehrer_solomon([2, 2, 1]) == s.whitney_homology_character([2, 2, 1])
True
```

Lehrer-Solomon functions indexed by partitions:

```python
sage: h.lehrer_solomon([2, 1])
h[2, 1]
sage: h.lehrer_solomon([2, 2])
```

The Lehrer-Solomon functions are Schur-positive:

```python
sage: s = Sym.s()
sage: s.lehrer_solomon([2, 1])
s[2, 1] + s[3]
sage: s.lehrer_solomon([2, 2, 1])
s[3, 1, 1] + s[3, 2] + s[4, 1]
sage: s.lehrer_solomon([4, 1])
s[2, 1, 1, 1] + s[2, 2, 1] + 2*s[3, 1, 1] + s[3, 2] + s[4, 1]
```

`one_basis()`

Return the empty partition, as per `AlgebrasWithBasis.ParentMethods.one_basis`

**INPUT:**
- `self` – a basis of the ring of symmetric functions

**EXAMPLES:**

```python
sage: Sym = SymmetricFunctions(QQ['t'].fraction_field())
sage: s = Sym.s()
sage: s.one_basis()
[]
sage: Q = Sym.hall_littlewood().Q()
sage: Q.one_basis()
[]
```

**Todo:** generalize to `Modules.Graded_CONNECTED.PARENT METHODS`

`skew_schur(x)`

Return the skew Schur function indexed by `x` in `self`.

**INPUT:**
- `x` – a skew partition
EXAMPLES:

```python
sage: sp = SkewPartition([[5,3,3,1], [3,2,1]])
sage: s = SymmetricFunctions(QQ).s()
sage: s.skew_schur(sp)
sage: e = SymmetricFunctions(QQ).e()
sage: ess = e.skew_schur(sp)
```

```python
e[2, 1, 1, 1, 1] - e[2, 2, 1, 1] - e[3, 1, 1, 1] + e[3, 2, 1]
sage: ess == e(s.skew_schur(sp))
```

```python
True
```

whitney_homology_character(lam)

Return the Lehrer-Solomon symmetric function (also known as the Whitney homology character) corresponding to the partition lam written in the basis self.

Let \( \lambda \vdash n \) be a partition. The Lehrer-Solomon symmetric function \( \text{LS}_\lambda \) corresponding to \( \lambda \) is the Frobenius characteristic of the representation denoted \( \text{Ind}_{S_n}^{S_\lambda}(\chi_\lambda) \) in Theorem 4.5 of [LS1986] or \( W_\lambda \) in Theorem 2.7 of [HR2017]. It was first computed as a symmetric function in [Sun1994].

It is the symmetric group representation corresponding to a summand of the Whitney homology of the set partition lattice. The summand comes from the orbit of set partitions with block sizes corresponding to \( \lambda \) (after reordering appropriately).

It can be computed using Sundaram’s plethystic formula (see [Sun1994] Theorem 1.8):

\[
\text{LS}_\lambda = \prod_{\text{odd } j \geq 1} h_{m_j}[\pi_j] \prod_{\text{even } j \geq 2} e_{m_j}[\pi_j],
\]

where \( h_{m_j} \) are complete homogeneous symmetric functions, \( e_{m_j} \) are elementary symmetric functions, and \( \pi_j \) are the images of the Gessel-Reutenauer symmetric function \( \text{GR}_{(j)} \) (see gessel_reutenauer()) under the involution \( \omega \) (i.e. omega_involution()):

```python
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.s()
sage: pi_2 = (s.gessel_reutenauer(2)).omega_involution()
sage: pi_1 = (s.gessel_reutenauer(1)).omega_involution()
sage: s.lehrer_solomon([2,1]) == pi_2 * pi_1
```

```python
# since h_1, e_1 are plethystic identities
```

True

Note that this also gives the \( S_n \)-equivariant structure of the Orlik-Solomon algebra of the braid arrangement (also known as the type-\( A \) reflection arrangement).

The representation corresponding to \( \text{LS}_\lambda \) exhibits representation stability [Chu2012], and a sharp bound is given in [HR2017].

INPUT:
- \( \lambda \) – a partition or a positive integer (in the latter case, it is understood to mean the partition \([\lambda]\))

OUTPUT:
The Lehrer-Solomon symmetric function \( \text{LS}_\lambda \), where \( \lambda \) is \( \lambda \), expanded in the basis self.

EXAMPLES:
The first few values of \( \text{LS}_{(n)} \):
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```
sage: Sym = SymmetricFunctions(ZZ)
sage: h = Sym.h()
sage: h.lehrer_solomon(1)
  h[1]
sage: h.lehrer_solomon(2)
  h[2]
sage: h.lehrer_solomon(3)
  h[2, 1] - h[3]
sage: h.lehrer_solomon(4)
  h[2, 1, 1] - h[2, 2]
sage: h.lehrer_solomon(5)
```

The *whitney_homology_character()* method is an alias:

```
sage: Sym = SymmetricFunctions(ZZ)
sage: s = Sym.schur()
sage: s.lehrer_solomon([2, 2, 1]) == s.whitney_homology_character([2, 2, 1])
  True
```

Lehrer-Solomon functions indexed by partitions:

```
sage: h.lehrer_solomon([2, 1])
  h[2, 1]
sage: h.lehrer_solomon([2, 2])
```

The Lehrer-Solomon functions are Schur-positive:

```
sage: s = Sym.s()
sage: s.lehrer_solomon([2, 1])
  s[2, 1] + s[3]
sage: s.lehrer_solomon([2, 2, 1])
  s[3, 1, 1] + s[3, 2] + s[4, 1]
sage: s.lehrer_solomon([4, 1])
  s[2, 1, 1, 1] + s[2, 2, 1] + 2*s[3, 1, 1] + s[3, 2] + s[4, 1]
```

```
sage: super_categories()
  The super categories of self.
  EXAMPLES:
  sage: from sage.combinat.sf.sfa import SymmetricFunctionsBases
  sage: Sym = SymmetricFunctions(QQ)
sage: bases = SymmetricFunctionsBases(Sym)
sage: bases.super_categories()
  [Category of realizations of Symmetric Functions over Rational Field, 
  Category of commutative hopf algebras with basis over Rational Field, 
  Join of Category of realizations of hopf algebras over Rational Field 
  and Category of graded algebras over Rational Field 
  and Category of graded coalgebras over Rational Field, 
  Category of unique factorization domains]
  sage: Sym = SymmetricFunctions(ZZ["x"])```
sage: bases = SymmetricFunctionsBases(Sym)
sage: bases.super_categories()
[Category of realizations of Symmetric Functions over Univariate Polynomial
 → Ring in x over Integer Ring,
 Category of commutative hopf algebras with basis over Univariate Polynomial
 → Ring in x over Integer Ring,
 Join of Category of realizations of hopf algebras over Univariate Polynomial
 → Ring in x over Integer Ring
 and Category of graded algebras over Univariate Polynomial Ring in x over
 → Integer Ring
 and Category of graded coalgebras over Univariate Polynomial Ring in x,
 → over Integer Ring]

sage.combinat.sf.sfa.is_SymmetricFunction(x)
Checks whether x is a symmetric function.

EXAMPLES:

sage: from sage.combinat.sf.sfa import is_SymmetricFunction
sage: s = SymmetricFunctions(QQ).s()
sage: is_SymmetricFunction(2)
False
sage: is_SymmetricFunction(s(2))
True
sage: is_SymmetricFunction(s([2,1]))
True

sage.combinat.sf.sfa.is_SymmetricFunctionAlgebra(x)
Checks whether x is a symmetric function algebra.

EXAMPLES:

sage: from sage.combinat.sf.sfa import is_SymmetricFunctionAlgebra
sage: is_SymmetricFunctionAlgebra(5)
False
sage: is_SymmetricFunctionAlgebra(ZZ)
False
sage: is_SymmetricFunctionAlgebra(SymmetricFunctions(ZZ).schur())
True
sage: is_SymmetricFunctionAlgebra(SymmetricFunctions(QQ).e())
True
sage: is_SymmetricFunctionAlgebra(SymmetricFunctions(QQ).macdonald(q=1,t=1).P())
True
sage: is_SymmetricFunctionAlgebra(SymmetricFunctions(FractionField(QQ['q','t'])).
 → macdonald().P())
True

sage.combinat.sf.sfa.zee(part)
Return the size of the centralizer of any permutation of cycle type part.

Note that the size of the centralizer is the inner product between p(part) and itself, where p is the power-sum symmetric functions.

INPUT:
• part – an integer partition (for example, [2, 1, 1])

OUTPUT:
• the integer \[ \prod_i i^{m_i(part)} m_i(part)! \] where \( m_i(part) \) is the number of parts in the partition part equal to \( i \)

EXAMPLES:
sage: from sage.combinat.sf.sfa import zee
sage: zee([2,1,1])
4

5.1.302 Witt symmetric functions

class sage.combinat.sf.witt.SymmetricFunctionAlgebra_witt(Sym, coerce_h=True, coerce_e=False, coerce_p=False)

Bases: SymmetricFunctionAlgebra_multiplicative

The Witt symmetric function basis (or Witt basis, to be short).

The Witt basis of the ring of symmetric functions is denoted by \( (x_\lambda) \) in [HazWitt1], section 9.63, and by \( (q_\lambda) \) in [DoranIV1996]. We will denote this basis by \( (w_\lambda) \) (which is precisely how it is denoted in [GriRei18], Exercise 2.9.3(d)). It is a multiplicative basis (meaning that \( w_\emptyset = 1 \) and that every partition \( \lambda \) satisfies \( w_\lambda = w_{\lambda_1} w_{\lambda_2} w_{\lambda_3} \cdots \), where \( w_i \) means \( w(i) \) for every nonnegative integer \( i \)).

This basis can be defined in various ways. Probably the most well-known one is using the equation

\[
\prod_{d=1}^{\infty} (1 - w_dt^d)^{-1} = \sum_{n=0}^{\infty} h_n t^n
\]

where \( t \) is a formal variable and \( h_n \) are the complete homogeneous symmetric functions, extended to 0 by \( h_0 = 1 \).

This equation allows one to uniquely determine the functions \( w_1, w_2, w_3, \ldots \) by recursion; one consequently extends the definition to all \( w_\lambda \) by requiring multiplicativity.

A way to rewrite the above equation without power series is:

\[
h_n = \sum_{\lambda \vdash n} w_\lambda
\]

for all nonnegative integers \( n \), where \( \lambda \vdash n \) means that \( \lambda \) is a partition of \( n \).

A similar equation (which is easily seen to be equivalent to the former) is

\[
e_n = \sum_\lambda (-1)^{n - \ell(\lambda)} w_\lambda,
\]

with the sum running only over strict partitions \( \lambda \) of \( n \) this time. This equation can also be used to recursively define the \( w_n \). Furthermore, every positive integer \( n \) satisfies

\[
p_n = \sum_{d|n} d w_n^{n/d},
\]

and this can be used to define the \( w_n \) recursively over any ring which is torsion-free as a \( \mathbb{Z} \)-module. While these equations all yield easy formulas for classical bases of the ring of symmetric functions in terms of the Witt symmetric functions, it seems difficult to obtain explicit formulas in the other direction.

The Witt symmetric functions owe their name to the fact that the ring of symmetric functions can be viewed as the coordinate ring of the group scheme of Witt vectors, and the Witt symmetric functions are the functions that
send a Witt vector to its components (whereas the powersum symmetric functions send a Witt vector to its ghost components). Details can be found in [HazWitt1] or section 3.2 of [BorWi2004].

INPUT:

- Sym – an instance of the ring of the symmetric functions.
- coerce_h – (default: True) a boolean that determines whether the transition maps between the Witt basis and the complete homogeneous basis will be cached and registered as coercions.
- coerce_e – (default: False) a boolean that determines whether the transition maps between the Witt basis and the elementary symmetric basis will be cached and registered as coercions.
- coerce_p – (default: False) a boolean that determines whether the transition maps between the Witt basis and the powersum basis will be cached and registered as coercions (or conversions, if the base ring is not a $\mathbb{Q}$-algebra).

REFERENCES:

EXAMPLES:

Here are the first few Witt symmetric functions, in various bases:

```
sage: Sym = SymmetricFunctions(QQ)
sage: w = Sym.w()
sage: e = Sym.e()
sage: h = Sym.h()
sage: p = Sym.p()
sage: s = Sym.s()
sage: m = Sym.m()

sage: p(w([1]))
p[1]
sage: m(w([1]))
m[1]
sage: e(w([1]))
e[1]
sage: h(w([1]))
h[1]
sage: s(w([1]))
s[1]

sage: p(w([2]))
-1/2*p[1, 1] + 1/2*p[2]
sage: m(w([2]))
-m[1, 1]
sage: e(w([2]))
e[2]
sage: h(w([2]))
-h[1, 1] + h[2]
sage: s(w([2]))
-s[1, 1]

sage: p(w([3]))
-1/3*p[1, 1, 1] + 1/3*p[3]
sage: m(w([3]))
-2*m[1, 1, 1] - m[2, 1]
```

sage: e(w([3]))
-e[2, 1] + e[3]
sage: h(w([3]))
-h[2, 1] + h[3]
sage: s(w([3]))
-s[2, 1]

sage: Sym = SymmetricFunctions(ZZ)
sage: w = Sym.w()
sage: e = Sym.e()
sage: h = Sym.h()
sage: s = Sym.s()
sage: m = Sym.m()
sage: p = Sym.p()

sage: m(w([4]))
-9*m[1, 1, 1, 1] - 4*m[2, 1, 1] - 2*m[2, 2] - m[3, 1]
sage: e(w([4]))
sage: h(w([4]))
sage: s(w([4]))
-s[1, 1, 1, 1] - s[2, 1, 1] - s[2, 2] - s[3, 1]

Some examples of conversions the other way:

sage: w(h[3])
w[1, 1, 1] + w[2, 1] + w[3]
sage: w(e[3])
-w[2, 1] + w[3]
sage: w(m[2,1])
2*w[2, 1] - 3*w[3]
sage: w(p[3])
w[1, 1, 1] + 3*w[3]

Antipodes:

sage: w([1]).antipode()
-w[1]
sage: w([2]).antipode()
-w[1, 1] - w[2]

The following holds for all odd \(i\) and is easily proven by induction:

sage: all( w([i]).antipode() == -w([i]) for i in range(1, 10, 2) )
True

The Witt basis does not allow for simple expressions for comultiplication and antipode in general (this is related to the fact that the sum of two Witt vectors isn't easily described in terms of the components). Therefore, most computations with Witt symmetric functions, as well as conversions and coercions, pass through the complete homogeneous symmetric functions by default. However, one can also use the elementary symmetric functions instead, or (if the base ring is a \(Q\)-algebra) the powersum symmetric functions. This is what the optional keyword variables \(\text{coerce}_e, \text{coerce}_h\) and \(\text{coerce}_p\) are for. These variables do not affect the results of the (non-underscored) methods of \(\text{self}\), but they affect the speed of the computations (the more of these variables are set
to True, the faster these are) and the size of the cache (the more of these variables are set to True, the bigger the cache). Let us check that the results are the same no matter to what the variables are set:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.p()
sage: wh = Sym.w()
sage: we = Sym.w(coerce_h=False, coerce_e=True)
sage: wp = Sym.w(coerce_h=False, coerce_p=True)
sage: all( p(wh(lam)) == p(we(lam)) == p(wp(lam)) for lam in Partitions(4) )
True
sage: all ( wh(p(lam)).monomial_coefficients()
....: == we(p(lam)).monomial_coefficients()
....: == wp(p(lam)).monomial_coefficients() for lam in Partitions(4) )
True
```

coproduct(elt)

Return the coproduct of the element elt.

INPUT:

• elt – a symmetric function written in this basis

OUTPUT:

• The coproduct acting on elt; the result is an element of the tensor squared of the basis self

EXAMPLES:

```python
sage: w = SymmetricFunctions(QQ).w()
sage: w[2].coproduct()
sage: w.coproduct(w[2])
sage: w[2,1].coproduct()
→w[2, 1] # w[]
sage: w.coproduct(w[2,1])
→w[2, 1] # w[]
```

from_other_uncached(u)

Return an element u of another basis of the ring of symmetric functions, expanded in the Witt basis self. The result is the same as self(u), but the from_other_uncached method does not precompute a cache with transition matrices. Thus, from_other_uncached is faster when u is sparse.

INPUT:

• u – an element of self.realization_of()

OUTPUT:

• the expansion of u in the Witt basis self

EXAMPLES:

```python
sage: Sym = SymmetricFunctions(QQ)
sage: p = Sym.p()
sage: w = Sym.w()
```
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Here’s a verification of an obvious fact that would take long with regular coercion:

```python
sage: fouc = w.from_other_uncached
sage: fouc(p([15]))
w[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] + 3*w[3, 3, 3, 3, 3] + 5*w[5, 5] + 15*w[15]

sage: fouc(p([15])) * fouc(p([14])) == fouc(p([15, 14]))
True
```

Other bases:

```python
sage: e = Sym.e()
sage: h = Sym.h()
sage: s = Sym.s()
sage: all( fouc(e(lam)) == w(e(lam)) for lam in Partitions(5) )
True
sage: all( fouc(h(lam)) == w(h(lam)) for lam in Partitions(5) )
True
sage: all( fouc(p(lam)) == w(p(lam)) for lam in Partitions(5) )
True
sage: all( fouc(s(lam)) == w(s(lam)) for lam in Partitions(5) )
True
```

This is a versatile operator:

```python
sage: x = w([2, 2, 2, 2, 2, 2, 2])
sage: x
w[2, 2, 2, 2, 2, 2, 2]
```

```python
sage: x = w([1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1])
sage: x
w[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

This is a basis-related operator:

```python
sage: x = w([1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1])
sage: x
w[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

```python
sage: all( fouc(s(lam)) == w(s(lam)) for lam in Partitions(5) )
True
```

```

verschiebung\( (n) \)

Return the image of the symmetric function \( \text{self} \) under the \( n \)-th Verschiebung operator.

The \( n \)-th Verschiebung operator \( V_n \) is defined to be the unique algebra endomorphism \( V \) of the ring of symmetric functions that satisfies \( V(h_r) = h_{r/n} \) for every positive integer \( r \) divisible by \( n \), and satisfies \( V(h_r) = 0 \) for every positive integer \( r \) not divisible by \( n \). This operator \( V_n \) is a Hopf algebra endomorphism. For every nonnegative integer \( r \) with \( n \mid r \), it satisfies

\[
V_n(h_r) = h_{r/n}, \quad V_n(p_r) = np_{r/n}, \quad V_n(e_r) = (-1)^{r-r/n}e_{r/n}, \quad V_n(w_r) = w_{r/n},
\]

(where \( h \) is the complete homogeneous basis, \( p \) is the powersum basis, \( e \) is the elementary basis, and \( w \) is the Witt basis). For every nonnegative integer \( r \) with \( n \nmid r \), it satisfies

\[
V_n(h_r) = V_n(p_r) = V_n(e_r) = V_n(w_r) = 0.
\]

The \( n \)-th Verschiebung operator is also called the \( n \)-th Verschiebung endomorphism. Its name derives from the Verschiebung (German for “shift”) endomorphism of the Witt vectors.

The \( n \)-th Verschiebung operator is adjacent to the \( n \)-th Frobenius operator (see \texttt{frobenius()} for its definition) with respect to the Hall scalar product (\texttt{scalar()}).

The action of the \( n \)-th Verschiebung operator on the Schur basis can also be computed explicitly. The following (probably clumsier than necessary) description can be obtained by solving exercise 7.61 in Stanley’s [STA].

Let \( \lambda \) be a partition. Let \( n \) be a positive integer. If the \( n \)-core of \( \lambda \) is nonempty, then \( V_n(s_\lambda) = 0 \).
Otherwise, the following method computes \( V_n(s_\lambda) \): Write the partition \( \lambda \) in the form \( (\lambda_1, \lambda_2, \ldots, \lambda_{ns}) \) for some nonnegative integer \( s \). If \( n \) does not divide the length of \( \lambda \), then this is achieved by adding
trailing zeroes to \( \lambda \).) Set \( \beta_i = \lambda_i + ns - i \) for every \( s \in \{1, 2, \ldots, ns\} \). Then, \( (\beta_1, \beta_2, \ldots, \beta_{ns}) \) is a strictly decreasing sequence of nonnegative integers. Stably sort the list \( (1, 2, \ldots, ns) \) in order of (weakly) increasing remainder of \(-1 - \beta_i \mod n\). Let \( \xi \) be the sign of the permutation that is used for this sorting. Let \( \psi \) be the sign of the permutation that is used to stably sort the list \( (1, 2, \ldots, ns) \) in order of (weakly) increasing remainder of \( i - 1 \mod n \). (Notice that \( \psi = \frac{1}{n}n(n-1)(s-1)/4 \).)

Then, \( V_n(s\lambda) = \xi\psi \prod_{i=0}^{n-1} \lambda^{(i)} \), where \( (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)}) \) is the \( n \)-quotient of \( \lambda \).

INPUT:

\* \( n \) – a positive integer

OUTPUT:

The result of applying the \( n \)-th Verschiebung operator (on the ring of symmetric functions) to \( \text{self} \).

EXAMPLES:

```sage
sage: Sym = SymmetricFunctions(ZZ)
sage: w = Sym.w()
sage: w[3].verschiebung(2)
\emptyset
sage: w[4].verschiebung(4)
w[1]
```

5.1.303 Shard intersection order

This file builds a combinatorial version of the shard intersection order of type A (in the classification of finite Coxeter groups). This is a lattice on the set of permutations, closely related to noncrossing partitions and the weak order.

For technical reasons, the elements of the posets are not permutations, but can be easily converted from and to permutations:

```sage
from sage.combinat.shard_order import ShardPosetElement
sage: p0 = Permutation([1,3,4,2])
sage: e0 = ShardPosetElement(p0); e0
(1, 3, 4, 2)
sage: Permutation(list(e0)) == p0
True
```

See also:

A general implementation for all finite Coxeter groups is available as `shard_poset()`

REFERENCES:

**class** `sage.combinat.shard_order.ShardPosetElement(p)`

Bases: `tuple`

An element of the shard poset.

This is basically a permutation with extra stored arguments:

\* \( p \) – the permutation itself as a tuple
\* \( \text{runs} \) – the decreasing runs as a tuple of tuples
\* \( \text{run_indices} \) – a list integer \( \rightarrow \) index of the run
\* \( \text{dpg} \) – the transitive closure of the shard preorder graph
\* \( \text{spg} \) – the transitive reduction of the shard preorder graph
These elements can easily be converted from and to permutations:

```python
sage: from sage.combinat.shard_order import ShardPosetElement
sage: p0 = Permutation([1,3,4,2])
sage: e0 = ShardPosetElement(p0); e0
(1, 3, 4, 2)
sage: Permutation(list(e0)) == p0
True
```

`sage.combinat.shard_order.shard_poset(n)`

Return the shard intersection order on permutations of size $n$.

This is defined on the set of permutations. To every permutation, one can attach a pre-order, using the descending runs and their relative positions.

The shard intersection order is given by the implication (or refinement) order on the set of pre-orders defined from all permutations.

This can also be seen in a geometrical way. Every pre-order defines a cone in a vector space of dimension $n$. The shard poset is given by the inclusion of these cones.

See also:

`sage.combinat.shard_order.shard_preorder_graph()`

EXAMPLES:

```python
sage: P = posets.ShardPoset(4); P  # indirect doctest
Finite poset containing 24 elements
sage: P.chain_polynomial()
34*q^4 + 90*q^3 + 79*q^2 + 24*q + 1
sage: P.characteristic_polynomial()
q^3 - 11*q^2 + 23*q - 13
sage: P.zeta_polynomial()
17/3*q^3 - 6*q^2 + 4/3*q
sage: P.is_self_dual()
False
```

`sage.combinat.shard_order.shard_preorder_graph(runs)`

Return the preorder attached to a tuple of decreasing runs.

This is a directed graph, whose vertices correspond to the runs.

There is an edge from a run $R$ to a run $S$ if $R$ is before $S$ in the list of runs and the two intervals defined by the initial and final indices of $R$ and $S$ overlap.

This only depends on the initial and final indices of the runs. For this reason, this input can also be given in that shorten way.

INPUT:

- a tuple of tuples, the runs of a permutation, or
- a tuple of pairs $(i, j)$, each one standing for a run from $i$ to $j$.

OUTPUT:

a directed graph, with vertices labelled by integers

EXAMPLES:
sage: from sage.combinat.shard_order import shard_preorder_graph
sage: s = Permutation([2,8,3,9,6,4,5,1,7])
sage: def cut(lr):
       ....:     return tuple((r[0], r[-1]) for r in lr)
sage: shard_preorder_graph(cut(s.decreasing_runs()))
Digraph on 5 vertices
sage: s = Permutation([9,4,3,2,8,6,5,1,7])
sage: P = shard_preorder_graph(s.decreasing_runs())
sage: P.is_isomorphic(digraphs.TransitiveTournament(3))
True

5.1.304 Shifted primed tableaux

AUTHORS:

• Kirill Paramonov (2017-08-18): initial implementation
• Chaman Agrawal (2019-08-12): add parameter to allow primed diagonal entry

class sage.combinat.shifted_primed_tableau.CrystalElementShiftedPrimedTableau(parent, T, skew=None, check=True, processed=False)

Bases: ShiftedPrimedTableau

Class for elements of crystals.ShiftedPrimedTableau.

e(ind)

Compute the action of the crystal operator $e_i$ on a shifted primed tableau using cases from the papers [HPS2017] and [AO2018].

INPUT:

• ind – an element in the index set of the crystal

OUTPUT:

Primed tableau or None.

EXAMPLES:

sage: SPT = ShiftedPrimedTableaux([5,4,2])
sage: t = SPT([[1,1,1,'2p','3p'], [2,'3p',3,3],[3,4]])
sage: t.pp()
1 1 1 2' 3'
2 3' 3 3
3 4
sage: s = t.e(2)
sage: s.pp()
1 1 1 2' 3'
2 2 3 3
3 4
sage: t == s.f(2)
True

(continues on next page)
sage: SPT = ShiftedPrimedTableaux([2,1])
sage: t = SPT([[2, '3p'], [3]])
sage: t.e(-1).pp()
  1  3'
    3
sage: t.e(1).pp()
  1  3'
    3
sage: t.e(2).pp()
  2  2
    3

sage: r = SPT([[2, 2], [3]])
sage: r.e(-1).pp()
  1  2
    3
sage: r.e(1).pp()
  1  2
    3
sage: r.e(2) is None
True
sage: r = SPT([[1, '3p'], [3]])
sage: r.e(-1)
is None
True
sage: r.e(1)
is None
True
sage: r.e(2).pp()
  1  2'
    3
sage: r = SPT([[1, '2p'], [3]])
sage: r.e(-1).pp()
  1  1
    3
sage: r.e(1)
is None
True
sage: r.e(2).pp()
  1  2'
    3
sage: t = SPT([[2, '3p'], [3]])
sage: t.e(-1).e(2).e(2).e(-1) == t.e(2).e(1).e(1).e(2)
True
sage: t.e(-1).e(2).e(2).e(-1).pp()
  1  1
    2
sage: all(t.e(-1).e(2).e(2).e(-1).e(i) is None for i in {-1, 1, 2})
True

sage: SPT = ShiftedPrimedTableaux([4])
sage: t = SPT([[2,2,2,2]])
sage: t.e(-1).pp()
  1  2  2  2
sage: t.e(1).pp()
1 2 2 2
sage: t.e(-1).e(-1) is None
True
sage: t.e(1).e(1).pp()
1 1 2 2

\textit{f}(\textit{ind})

Compute the action of the crystal operator $f_i$ on a shifted primed tableau using cases from the papers [HPS2017] and [AO2018].

INPUT:

• \textit{ind} – element in the index set of the crystal

OUTPUT:

Primed tableau or None.

EXAMPLES:

\begin{verbatim}
sage: SPT = ShiftedPrimedTableaux([5,4,2])
sage: t = SPT([[1,1,1,1,'3p'],[2,2,2,'3p'],[3,3]])
sage: t.pp()
1 1 1 3'  
2 2 2 3'  
3 3
sage: s = t.f(2)
sage: s is None
True

sage: t = SPT([[1,1,1,'2p','3p'],[2,2,3,3],[3,4]])
sage: t.pp()
1 1 2' 3'  
2 3 3  
3 4
sage: s = t.f(2)
sage: s.pp()
1 1 2' 3'  
2 3' 3 3  
3 4

sage: SPT = ShiftedPrimedTableaux([2,1])
sage: t = SPT([[1,1],[2]])
sage: t.f(-1).pp()
1 2'  
2
sage: t.f(1).pp()
1 2'  
2
sage: t.f(2).pp()
1 1  
3
\end{verbatim}
sage: r = SPT([[1, '2p'], [2]])
sage: r.f(-1) is None
True
sage: r.f(1) is None
True
sage: r.f(2).pp()
1 2'
 3

sage: r = SPT([[1,1],[3]])
sage: r.f(-1).pp()
1 2'
 3
sage: r.f(1).pp()
1 2
 3
sage: r.f(2) is None
True

sage: r = SPT([[1,2],[3]])
sage: r.f(-1).pp()
2 2
 3
sage: r.f(1).pp()
2 2
 3
sage: r.f(2) is None
True

sage: t = SPT([[1,1],[2]])
sage: t.f(-1).f(2).f(2).f(-1) == t.f(2).f(1).f(-1).f(2)
True
sage: t.f(-1).f(2).f(2).f(-1).f(i) is None for i in {-1, 1, 2})
True

sage: SPT = ShiftedPrimedTableaux([4])
sage: t = SPT([[1,1,1,1]])
sage: t.f(-1).pp()
1 1 1 2'
sage: t.f(1).pp()
1 1 1 2
sage: t.f(-1).f(-1) is None
True
sage: t.f(1).f(-1).pp()
1 2'
 2
sage: t.f(1).f(1).pp()
1 1 2 2
sage: t.f(1).f(1).f(-1).pp()
1 2' 2 2

(continues on next page)
```python
sage: t.f(1).f(1).f(1).pp()
1 2 2 2
sage: t.f(1).f(1).f(1).f(-1).pp()
2 2 2 2
sage: t.f(1).f(1).f(1).f(1).pp()
2 2 2 2
sage: t.f(1).f(1).f(1).f(1).f(-1)
is None
True
```

**is_highest_weight** *(index_set=None)*

Return whether `self` is a highest weight element of the crystal.

An element is highest weight if it vanishes under all crystal operators $e_i$.

**EXAMPLES:**

```python
sage: SPT = ShiftedPrimedTableaux([5,4,2])
sage: t = SPT([(1, 1, 1, 1, 1), (2, 2, 2, "3p"), (3, 3)])
sage: t.is_highest_weight()
True
sage: SPT = ShiftedPrimedTableaux([5,4])
sage: s = SPT([(1, 1, 1, 1, 1), (2, 2, "3p", 3)])
sage: s.is_highest_weight(index_set=[1])
True
```

**reading_word()**

Return the reading word of `self`.

The reading word of a shifted primed tableau is constructed as follows:

1. List all primed entries in the tableau, column by column, in decreasing order within each column, moving from the rightmost column to the left, and with all the primes removed (i.e. all entries are increased by half a unit).

2. Then list all unprimed entries, row by row, in increasing order within each row, moving from the bottommost row to the top.

**EXAMPLES:**

```python
sage: SPT = ShiftedPrimedTableaux([4,2])
sage: t = SPT([[1, '2p', 2, 2], [2, '3p']])
sage: t.reading_word()
[3, 2, 2, 1, 2, 2]
```

**weight()**

Return the weight of `self`.

The weight of a shifted primed tableau is defined to be the vector with $i$-th component equal to the number of entries $i$ and $i'$ in the tableau.

**EXAMPLES:**

```python
sage: t = ShiftedPrimedTableau([[1, '2p', 2, 2], [2, '3p']])
sage: t.weight()
(1, 4, 1)
```
class sage.combinat.shifted_primed_tableau.PrimedEntry(entry=None, double=None)
    Bases: SageObject

    The class of entries in shifted primed tableaux.
    An entry in a shifted primed tableau is an element in the alphabet \( \{1' < 1 < 2' < 2 < \cdots < n' < n \} \).
    The difference between two elements \( i \) and \( i - 1 \) counts as a whole unit, whereas the difference between \( i \) and \( i' \) counts as half a unit. Internally, we represent an unprimed element \( x \) as \( 2x \) and the primed elements as the corresponding odd integer that respects the total order.

    INPUT:
    • entry – a half integer or a string of an integer possibly ending in \( 'p \) or \( 'r \)
    • double – the doubled value

decrease_half()
    Decrease self by half a unit.

decrease_one()
    Decrease self by one unit.

increase_half()
    Increase self by half a unit.

increase_one()
    Increase self by one unit.

integer()
    Return the corresponding integer \( i \) for primed entries of the form \( i \) or \( i' \).

is_primed()
    Checks if self is a primed element.

is_unprimed()
    Checks if self is an unprimed element.

primed()
    Prime self if it is an unprimed element.

unprimed()
    Unprime self if it is a primed element.

class sage.combinat.shifted_primed_tableau.ShiftedPrimedTableau(parent, T, skew=None, check=True, preprocessed=False)
    Bases: ClonableArray

    A shifted primed tableau.
    A primed tableau is a tableau of shifted shape in the alphabet \( X' = \{1' < 1 < 2' < 2 < \cdots < n' < n \} \) such that
    1. the entries are weakly increasing along rows and columns;
    2. a row cannot have two repeated primed elements, and a column cannot have two repeated non-primed elements;

    Skew shape of the shifted primed tableaux is specified either with an optional argument skew or with None entries.
Primed entries in the main diagonal can be allowed with the optional boolean parameter `primed_diagonal` (default: `False`).

**EXAMPLES:**

```python
sage: T = ShiftedPrimedTableaux([4,2])
sage: T([[1,"1''","3''"],[2,"3''"]])[1]
(2, 3')
sage: t = ShiftedPrimedTableau([[1,"2p",2.5,3],[2,2.5]])
sage: t[1]
(2, 3')
sage: ShiftedPrimedTableau([["2p",2,3],["2p","3p"],[2]], skew=[2,1])
[(None, None, 2', 3), (None, 2', 3'), (2,)]
sage: ShiftedPrimedTableau([[None,None,"2p"],[None,"2p"]])
[(None, None, 2'), (None, 2')]sage: T = ShiftedPrimedTableaux([4,2], primed_diagonal=True)
sage: T([[1,"2''","3''"],["2''","3''"]])[1] # With primed diagonal entry
(2', 3')
```

**check()**

Check that `self` is a valid primed tableau.

**EXAMPLES:**

```python
sage: T = ShiftedPrimedTableaux([4,2])
sage: t = T([[1,'2p',2,2],[2,'3p']])
sage: t.check()
sage: s = ShiftedPrimedTableau([["2p",2,3],["2p"],[2]], skew=[2,1])
sage: s.check()
sage: t = T([["1p","2p",2,2],[2,'3p']])
Traceback (most recent call last):
... ValueError: [['1p', '2p', 2, 2], [2, '3p']] is not an element of
Shifted Primed Tableaux of shape [4, 2]
sage: T = ShiftedPrimedTableaux([4,2], primed_diagonal=True)
sage: t = T([["1p","2p",2,2],[2,'3p']]) # primed_diagonal allowed
sage: t.check()
sage: t = T([["1p","1p",2,2],[2,'3p']])
Traceback (most recent call last):
... ValueError: [['1p', '1p', 2, 2], [2, '3p']] is not an element of
Shifted Primed Tableaux of shape [4, 2] and maximum entry 6
```

**is_standard()**

Return True if the entries of `self` are in bijection with positive primed integers $1', 1'', 2', \ldots, n$.

**EXAMPLES:**

```python
sage: ShiftedPrimedTableau([["1''", 1, "2'"], [2, "3'"]]).is_standard() 
(...: primed_diagonal=True).is_standard()
True
sage: ShiftedPrimedTableau([["1''", 1, 2], ["2'", "3'"]]),
(...: primed_diagonal=True).is_standard()
True
```
sage: ShiftedPrimedTableau([['1''', 1, 1], ['2''', 2]],
    primed_diagonal=True).is_standard()
False
sage: ShiftedPrimedTableau([[1, "2"'], [2]]).is_standard()
False
sage: s = ShiftedPrimedTableau([[None, None,"1p","2p",2], [None,"1"]])

max_entry()
Return the minimum unprimed letter $x > y$ for all $y$ in self.

EXAMPLES:

sage: Tab = ShiftedPrimedTableau([(1,1,'2p','3p'),(2,2)])
sage: Tab.max_entry()
3

pp()
Pretty print self.

EXAMPLES:

sage: t = ShiftedPrimedTableau([[1,'2p',2,2],[2,'3p']])
sage: t.pp()
1  2'  2  2

restrict(n)
Return the restriction of the shifted tableau to all the numbers less than or equal to n.

Note: If only the outer shape of the restriction, rather than the whole restriction, is needed, then the faster method restriction_outer_shape() is preferred. Similarly if only the skew shape is needed, use restriction_shape().

EXAMPLES:

sage: t = ShiftedPrimedTableau([[1,'2p',2,2],[2,'3p']])
sage: t.restrict(2).pp()
1  2'  2  2

(continues on next page)
sage: s = ShiftedPrimedTableau(["2p", 2, 3], ["2p"], skew=[2, 1])
sage: s.restrict(2).pp()
 . . 2' 2
 . 2'
sage: s.restrict(1.5).pp()
 . . 2'
 . 2'

restriction_outer_shape\( (n) \)

Return the outer shape of the restriction of the shifted tableau \self{} to \( n \).

If \( T \) is a (skew) shifted tableau and \( n \) is a half-integer, then the restriction of \( T \) to \( n \) is defined as the (skew) shifted tableau obtained by removing all cells filled with entries greater than \( n \) from \( T \).

This method computes merely the outer shape of the restriction. For the restriction itself, use \texttt{restrict()}.

EXAMPLES:

sage: s = ShiftedPrimedTableau(["2p", 2, 3], ["2p"], skew=[2, 1])
sage: s.pp()
 . . 2' 2 3
 . 2'
sage: s.restriction_outer_shape(2)
[4, 2]
sage: s.restriction_outer_shape("2p")
[3, 2]

restriction_shape\( (n) \)

Return the skew shape of the restriction of the skew tableau \self{} to \( n \).

If \( T \) is a shifted tableau and \( n \) is a half-integer, then the restriction of \( T \) to \( n \) is defined as the (skew) shifted tableau obtained by removing all cells filled with entries greater than \( n \) from \( T \).

This method computes merely the skew shape of the restriction. For the restriction itself, use \texttt{restrict()}.

EXAMPLES:

sage: s = ShiftedPrimedTableau(["2p", 2, 3], ["2p"], skew=[2, 1])
sage: s.pp()
 . . 2' 2 3
 . 2'
sage: s.restriction_shape(2)
[4, 2] / [2, 1]

shape()

Return the shape of the underlying partition of \self{}.

EXAMPLES:

sage: t = ShiftedPrimedTableau([[1, '2p', 2, 2], [2, '3p']])
sage: t.shape()
[4, 2]
sage: s = ShiftedPrimedTableau(["2p", 2, 3], ["2p"], skew=[2, 1])

(continues on next page)
to_chain()
Return the chain of partitions corresponding to the (skew) shifted tableau self, interlaced by one of the colours 1 if the added cell is on the diagonal, 2 if an ordinary entry is added and 3 if a primed entry is added.

EXAMPLES:

```python
sage: s = ShiftedPrimedTableau([(1, 2, 3.5, 5, 6.5), (3, 5.5)])
sage: s.pp()
  1  2  4  5  7
  3  6
sage: s.to_chain()
[[], 1, [1], 2, [2], 1, [2, 1], 3, [3, 1], 2, [4, 1], 3, [4, 2], 3, [5, 2]]

sage: s = ShiftedPrimedTableau([(1, 3.5), (2.5,), (6,)], skew=[2,1])
sage: s.pp()
  .  .  1  4
  3
  6
sage: s.to_chain()
[[2, 1], 2, [3, 1], 0, [3, 1], 3, [3, 2], 3, [4, 2], 0, [4, 2], 1, [4, 2, 1]]
```

weight()
Return the weight of self.

The weight of a shifted primed tableau is defined to be the vector with i-th component equal to the number of entries i and i' in the tableau.

EXAMPLES:

```python
sage: t = ShiftedPrimedTableau(['2p',2,2],[2,'3p'], skew=[1])
sage: t.weight()
(0, 4, 1)
```

class sage.combinat.shifted_primed_tableau.ShiftedPrimedTableaux(skew=None, primed_diagonal=False)

Bases: UniqueRepresentation, Parent

Returns the combinatorial class of shifted primed tableaux subject to the constraints given by the arguments.

A primed tableau is a tableau of shifted shape on the alphabet $X' = \{1' < 1 < 2' < 2 < \cdots < n' < n\}$ such that

1. the entries are weakly increasing along rows and columns
2. a row cannot have two repeated primed entries, and a column cannot have two repeated non-primed entries

INPUT:

Valid optional keywords:
• shape – the (outer skew) shape of tableaux
• weight – the weight of tableaux
• max_entry – the maximum entry of tableaux
• skew – the inner skew shape of tableaux
• primed_diagonal – allow primed entries in main diagonal of tableaux

The weight of a tableau is defined to be the vector with $i$-th component equal to the number of entries $i$ and $i'$ in the tableau. The sum of the coordinates in the weight vector must be equal to the number of entries in the partition.

The shape and skew must be strictly decreasing partitions. The primed_diagonal is a boolean (default: False).

EXAMPLES:

```
sage: SPT = ShiftedPrimedTableaux(weight=(1,2,2), shape=[3,2]); SPT
Shifted Primed Tableaux of weight (1, 2, 2) and shape [3, 2]
sage: SPT.list()
[[[1, 2, 2), (3, 3)],
 [[1, 2', 3'), (2, 3)],
 [[1, 2', 3'), (2, 3')],
 [[1, 2', 3'), (2', 3')],
 [[1, 2', 3'), (2', 3')],
 [[1, 2', 3'), (2', 3')],
 [[1, 2', 2], (3, 3)],
 [[1, 2', 2], (3', 3)],
 [[1, 2, 2), (3, 3)],
 [[1, 1, 2, 2), (3', 3)],
 [[1, 1, 2', 3'), (2, 3)],
 [[1, 1', 2, 3'), (2, 3')],
 [[1, 1', 2', 3'), (2', 3')],
 [[1, 1', 2', 3'), (2', 3')],
 [[1, 1', 2', 2), (3, 3)],
 [[1, 1', 2', 2), (3', 3)],
 sage: SPT = ShiftedPrimedTableaux(weight=(1,2), shape=[3,2],
 ....: primed_diagonal=True); SPT
Shifted Primed Tableaux of weight (1, 2, 2) and shape [3, 2]
sage: SPT.list()
[[[1, 2, 2), (3, 3)],
 [[1, 2', 2), (2, 3)],
 [[1, 1', 2, 2), (3', 3)],
 [[1, 1', 2, 2), (3', 3)]
```

(continues on next page)
\[
\begin{bmatrix}
(1, 2'), (2'), \\
(1', 2'), (2), \\
(1', 2'), (2'), 
\end{bmatrix}
\]

\[
\begin{bmatrix}
(1, 1), (2, 2), \\
(1, 1), (2'), 2) \\
(1, 1), (2), 2) \\
(1', 1), (2', 2) \\
(1, 1), (2'), 2) \\
(1, 1), (2', 2) \\
(1', 1), (2'), 2) \\
(1', 1), (2', 2) \\
(1', 1), (2', 2) \\
(1', 1), (2', 2) \\
(1', 1), (2', 2) \
\end{bmatrix}
\]

\[
\text{sage: } SPT = \text{ShiftedPrimedTableaux([3, 2], max_entry=2); SPT}
\]

\[
\text{Shifted Primed Tableaux of shape [3, 2] and maximum entry 2}
\]

\[
\text{sage: list(SPT)}
\]

\[
\text{[[[1, 1, 1], (2, 2)], [[1, 1, 2'), (2, 2)]],}
\]

\[
\text{sage: SPT = ShiftedPrimedTableaux([3, 2], max_entry=2, }
\]

\[
\text{primed_diagonal=True)}
\]

\[
\text{sage: list(SPT)}
\]

\[
\text{[[[1, 1, 1], (2, 2)], [[1, 1, 1), (2, 2)],}
\]

\[
\text{• ShiftedPrimedTableau}
\]

Element

alias of ShiftedPrimedTableau

options = Current options for Tableaux - ascii_art: repr - convention: English - display: list - latex: diagram

class sage.combinat.shifted_primed_tableau.ShiftedPrimedTableaux_all

Bases: ShiftedPrimedTableaux

The class of all shifted primed tableaux.

class sage.combinat.shifted_primed_tableau.ShiftedPrimedTableaux_shape

Bases: ShiftedPrimedTableaux

Shifted primed tableaux of a fixed shape.

Shifted primed tableaux admit a type \( A_n \) classical crystal structure with highest weights corresponding to a given shape.

The list of module generators consists of all elements of the crystal with nonincreasing weight entries.

The crystal is constructed following operations described in [HPS2017] and [AO2018].

The optional primed_diagonal allows primed entries in the main diagonal of all the Shifted primed tableaux of a fixed shape. If the max_entry is None then max_entry is set to the total number of entries in the tableau if primed_diagonal is True.

EXAMPLES:
We compute some of the crystal structure:

```python
sage: SPTC = crystals.ShiftedPrimedTableaux([3,2], 3)
sage: T = SPTC.module_generators[-1]
sage: T
[(1, 1, 2'), (2, 3')]
sage: T.f(2)
[(1, 1, 3'), (2, 3')]
sage: len(SPTC.module_generators)
7
sage: SPTC[0]
[(1, 1, 1), (2, 2)]
sage: SPTC.cardinality()
24
```

We compare this implementation with the \(q(n)\)-crystal on (tensor products) of letters:

```python
sage: tableau_crystal = crystals.ShiftedPrimedTableaux([4,1], 3)
sage: tableau_digraph = tableau_crystal.digraph()
sage: c = crystals.Letters(['Q', 3])
sage: tensor_crystal = tensor([c]*5)
sage: u = tensor_crystal(c(1), c(1), c(1), c(2), c(1))
sage: subcrystal = tensor_crystal.subcrystal(generators=[u],
.....: index_set=[1,2,-1])
sage: tensor_digraph = subcrystal.digraph()
sage: tensor_digraph.is_isomorphic(tableau_digraph, edge_labels=True)
True
```

If we allow primed entries in the main diagonal:

```python
sage: ShiftedPrimedTableaux([4,3,1], max_entry=4,
.....: primed_diagonal=True)
Shifted Primed Tableaux of shape [4, 3, 1] and maximum entry 4
sage: ShiftedPrimedTableaux([4,3,1], max_entry=4,
.....: primed_diagonal=True).cardinality()
3072
sage: SPTC = ShiftedPrimedTableaux([3,2], max_entry=3,
.....: primed_diagonal=True)
sage: T = SPTC[-1]
sage: T
[(1', 2', 2), (3', 3)]
sage: SPTC[0]
[(1, 1, 1), (2, 2)]
sage: SPTC.cardinality()
96
```

module_generators()

Return the generators of self as a crystal.

5.1. Comprehensive Module List
shape()
Return the shape of the shifted tableaux self.

class sage.combinat.shifted_primed_tableau.ShiftedPrimedTableaux_weight(weight, skew=None,
primed_diagonal=False)

Bases: ShiftedPrimedTableaux
Shifted primed tableaux of fixed weight.

EXAMPLES:

sage: ShiftedPrimedTableaux(weight=(2,3,1))
Shifted Primed Tableaux of weight (2, 3, 1)
sage: ShiftedPrimedTableaux(weight=(2,3,1)).cardinality()
17
sage: SPT = ShiftedPrimedTableaux(weight=(2,3,1), primed_diagonal=True)
sage: SPT.cardinality()
64
sage: T = ShiftedPrimedTableaux(weight=(3,2), primed_diagonal=True)
sage: T[:5]
[[1, 1, 1, 2, 2), (2, 3'], (3,)],
[(1, 1, 1, 2', 2'), (2, 3'), (3',)],
[(1, 1, 1, 2', 2'), (2, 3'), (3,)],
[(1, 1, 1, 2', 2), (2, 3'), (3',)],
[(1, 1, 1, 2), (2, 2), (3,)]
sage: T.cardinality()
16

class sage.combinat.shifted_primed_tableau.ShiftedPrimedTableaux_weight_shape(weight, shape,
skew=None,
primed_diagonal=False)

Bases: ShiftedPrimedTableaux
Shifted primed tableaux of the fixed weight and shape.

EXAMPLES:

sage: ShiftedPrimedTableaux([4,2,1], weight=(2,3,2))
Shifted Primed Tableaux of weight (2, 3, 2) and shape [4, 2, 1]
sage: ShiftedPrimedTableaux([4,2,1], weight=(2,3,2)).cardinality()
4
sage: T = ShiftedPrimedTableaux([4,2,1], weight=(2,3,2),
....: primed_diagonal=True)
sage: T[:6]
[[1, 1, 2, 2), (2, 3'), (3,)],
[(1, 1, 2, 2), (2, 3'), (3'),],
[(1, 1, 2, 2), (2', 3'), (3,)],
[(1, 1, 2, 2), (2, 3'), (3',)],
[(1, 1, 2', 3), (2, 2), (3,)],
[(1, 1, 2', 3), (2, 2), (3',)]
sage: T.cardinality()
32
5.1.305 Shuffle product of iterables

The shuffle product of two sequences of lengths $m$ and $n$ is a sum over the $\binom{m+n}{m}$ ways of interleaving the two sequences. That could be defined inductively by:

$$(a_n)_{n \geq 0} \shuffle (b_m)_{m \geq 0} = a_0 \cdot ((a_n)_{n \geq 1} \shuffle (b_m)_{m \geq 0}) + b_0 \cdot ((a_n)_{n \geq 0} \shuffle (b_m)_{m \geq 1})$$

with $(a_n)$ and $(b_m)$ two non-empty sequences and if one of them is empty then the product is equals to the other.

The shuffle product has been introduced by S. Eilenberg and S. Mac Lane in 1953 [EilLan53].

EXAMPLES:

```python
sage: from sage.combinat.shuffle import ShuffleProduct
sage: list(ShuffleProduct([1,2], ['a', 'b', 'c']))
[[1, 2, 'a', 'b', 'c'], ['a', 1, 2, 'b', 'c'], [1, 'a', 2, 'b', 'c'], ['a', 'b', 1, 2, 'c'], ['a', 1, 'b', 2, 'c'], [1, 'a', 'b', 2, 'c'], ['a', 'b', 'c', 1, 2], ['a', 'b', 1, 'c', 2], ['a', 1, 'b', 'c', 2], [1, 'a', 'b', 'c', 2]]
```

References:

Author:
- Jean-Baptiste Priez

class sage.combinat.shuffle.SetShuffleProduct($l1$, $l2$, element_constructor=None)

Bases: ShuffleProduct_abstract

The union of all possible shuffle products of two sets of iterables.

EXAMPLES:

```python
sage: from sage.combinat.shuffle import SetShuffleProduct
sage: sorted(SetShuffleProduct({(1,), (2,3)}, {(4,5), (6,)}))
[[1, 4, 5], [1, 6], [2, 3, 4, 5], [2, 3, 6], [2, 4, 3, 5], [2, 4, 5, 3], [2, 6, 3], [4, 1, 5], [4, 2, 3, 5], [4, 2, 5, 3], [4, 5, 1], [4, 5, 2, 3], [6, 1], [6, 2, 3]]
```
class sage.combinat.shuffle.ShuffleProduct(\(ll, 1, 2, \text{element_constructor}=\text{None}\))

Bases: ShuffleProduct_abstract

Shuffle product of two iterables.

EXAMPLES:

```python
sage: from sage.combinat.shuffle import ShuffleProduct
sage: list(ShuffleProduct("abc", "de", element_constructor="".join))
[['abcde', 'adbec', 'adbec', 'daebc', 'daebc', 'adebc', 'adebc', 'abdce', 'abdce'], ['dabce', 'dabce', 'adbce', 'adbce', 'abcde', 'abcde']]
```

cardinality()

Return the number of shuffles of \(l_1\) and \(l_2\), respectively of lengths \(m\) and \(n\), which is \(^{m+n} \binom{m}{n}\).

class sage.combinat.shuffle.ShuffleProduct_abstract(\(ll, 1, 2, \text{element_constructor}=\text{None}\))

Bases: Parent

Abstract base class for shuffle products.

class sage.combinat.shuffle.ShuffleProduct_overlapping(\(w1, w2, \text{element_constructor}=\text{None}, \text{add}=\text{<built-in function add}>\))

Bases: ShuffleProduct_abstract

The overlapping shuffle product of the two words \(w1\) and \(w2\).

If \(u\) and \(v\) are two words whose letters belong to an additive monoid or to another kind of alphabet on which addition is well-defined, then the overlapping shuffle product of \(u\) and \(v\) is a certain multiset of words defined as follows: Let \(a\) and \(b\) be the lengths of \(u\) and \(v\), respectively. Let \(A\) be the set \(\{(0, 1), (0, 2), \cdots, (0, a)\}\), and let \(B\) be the set \(\{(1, 1), (1, 2), \cdots, (1, b)\}\). Notice that the sets \(A\) and \(B\) are disjoint. We can make \(A\) and \(B\) into posets by setting \((k, i) \leq (k, j)\) for all \(k \in \{0, 1\}\) and \(i \leq j\). Then, \(A \cup B\) becomes a poset by disjoint union (we don’t set \((0, i) \leq (1, i)\)). Let \(p\) be the map from \(A \cup B\) to the set of all letters which sends every \((0, i)\) to the \(i\)-th letter of \(u\), and every \((1, j)\) to the \(j\)-th letter of \(v\). For every nonnegative integer \(c\) and every surjective map \(f : A \cup B \to \{1, 2, \cdots, c\}\) for which both restrictions \(f \mid A\) and \(f \mid B\) are strictly increasing. let \(w(f)\) be the length-\(c\) word such that for every \(1 \leq k \leq c\), the \(k\)-th letter of \(w(f)\) equals \(\sum_{j \in f^{-1}(k)} p(j)\) (this sum always has either one or two addends). The overlapping shuffle product of \(u\) and \(v\) is then the multiset of all \(w(f)\) with \(c\) ranging over all nonnegative integers and \(f\) ranging over the surjective maps \(f : A \cup B \to \{1, 2, \cdots, c\}\) for which both restrictions \(f \mid A\) and \(f \mid B\) are strictly increasing.

If one restricts \(c\) to a particular fixed nonnegative integer, then the multiset is instead called the overlapping shuffle product with precisely \(a + b - c\)` overlaps. This is nonempty only if \(\max\{a, b\} \leq c \leq a + b\).

If \(c = a + b\), then the overlapping shuffle product with precisely \(a + b - c\) overlaps is plainly the shuffle product (ShuffleProduct_w1w2).
INPUT:

- \texttt{w1, w2} – iterables
- \texttt{element\_constructor} – (default: the parent of \texttt{w1}) the function used to construct the output
- \texttt{add} – (default: \texttt{+}) the addition function

EXAMPLES:

```python
sage: from sage.combinat.shuffle import ShuffleProduct_overlapping
sage: w, u = [[2, 9], [9, 1]]
sage: S = ShuffleProduct_overlapping(w, u)
sage: sorted(S)
[[2, 9, 1, 9],
 [2, 9, 9, 1],
 [2, 9, 10],
 [2, 18, 1],
 [9, 1, 2, 9],
 [9, 2, 1, 9],
 [9, 2, 9, 1],
 [9, 2, 10],
 [9, 3, 9],
 [11, 1, 9],
 [11, 9, 1],
 [11, 10]]
sage: A = [{1,2}, {3,4}]
sage: B = [{2,3}, {4,5,6}]
sage: S = ShuffleProduct_overlapping(A, B, add=lambda X,Y: X.union(Y))
sage: list(S)
[[{1, 2}, {3, 4}, {2, 3}, {4, 5, 6}],
 [{1, 2}, {2, 3}, {3, 4}, {4, 5, 6}],
 [{1, 2}, {2, 3}, {4, 5, 6}, {3, 4}],
 [{2, 3}, {1, 2}, {3, 4}, {4, 5, 6}],
 [{2, 3}, {1, 2}, {4, 5, 6}, {3, 4}],
 [{2, 3}, {4, 5, 6}, {1, 2}, {3, 4}],
 [{1, 2, 3}, {3, 4}, {4, 5, 6}],
 [{1, 2}, {2, 3, 4}, {4, 5, 6}],
 [{1, 2, 3}, {4, 5, 6}, {3, 4}],
 [{1, 2}, {2, 3}, {3, 4, 5, 6}],
 [{2, 3}, {1, 2, 4, 5, 6}, {3, 4}],
 [{2, 3}, {1, 2}, {3, 4, 5, 6}],
 [{1, 2, 3}, {3, 4, 5, 6}],
 [{1, 2, 3}, {3, 4, 5, 6}]]
```

```
class sage.combinat.shuffle.ShuffleProduct_overlapping_r(w1, w2, r, element_constructor=None, add=<built-in function add>)

Bases: ShuffleProduct_abstract

The overlapping shuffle product of the two words \texttt{w1} and \texttt{w2} with precisely \texttt{r} overlaps.

See \texttt{ShuffleProduct\_overlapping} for a definition.

EXAMPLES:

```python
sage: from sage.combinat.shuffle import ShuffleProduct_overlapping_r
sage: w, u = map(Words(range(20)), [[2, 9], [9, 1]])
```
5.1.306 Sidon sets and their generalizations, Sidon $g$-sets

AUTHORS:
• Martin Raum (07-25-2011)

sage.combinat.sidon_sets.sidon_sets($N$, $g=1$)

Return the set of all Sidon-$g$ sets that have elements less than or equal to $N$.

A Sidon-$g$ set is a set of positive integers $A \subset [1, N]$ such that any integer $M$ can be obtain at most $g$ times as sums of unordered pairs of elements of $A$ (the two elements are not necessary distinct):

$$\# \{(a_i, a_j) | a_i, a_j \in A, a_i + a_j = M, a_i \leq a_j \} \leq g$$

INPUT:
• $N$ – A positive integer.
• $g$ – A positive integer (default: 1).

OUTPUT:
• A Sage set with categories whose element are also set of integers.

EXAMPLES:

```
sage: S = sidon_sets(3, 2)
sage: sorted(S, key=str)
[\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{1\}, \{2, 3\}, \{2\}, \{3\}, {}]
sage: S.cardinality()
8
sage: S.category()
Category of finite enumerated sets
sage: sid = S.an_element()
sage: sid
\{2\}
sage: sid.category()
Category of finite enumerated sets
```

sage.combinat.sidon_sets.sidon_sets_rec($g=1$)

Return the set of all Sidon-$g$ sets that have elements less than or equal to $N$ without checking the arguments. This internal function should not be call directly by user.
5.1.307 Similarity class types of matrices with entries in a finite field

The notion of a matrix conjugacy class type was introduced by J. A. Green in [Green55], in the context of computing the irreducible characters of finite general linear groups. The class types are equivalence classes of similarity classes of square matrices with entries in a finite field which, roughly speaking, have the same qualitative properties.

For example, all similarity classes of the same class type have centralizers of the same cardinality and the same degrees of elementary divisors. Qualitative properties of similarity classes such as semisimplicity and regularity descend to class types.

The most important feature of similarity class types is that, for any \( n \), the number of similarity class types of \( n \times n \) matrices is independent of \( q \). This makes it possible to perform many combinatorial calculations treating \( q \) as a formal variable.

In order to define similarity class types, recall that similarity classes of \( n \times n \) matrices with entries in \( \mathbf{F}_q \) correspond to functions

\[
c : \text{Irr} \mathbf{F}_q[t] \rightarrow \Lambda
\]

such that

\[
\sum_{f \in \text{Irr} \mathbf{F}_q[t]} |c(f)| \deg f = n,
\]

where we denote the set of irreducible monic polynomials in \( \mathbf{F}_q[t] \) by \( \text{Irr} \mathbf{F}_q[t] \), the set of all partitions by \( \Lambda \), and the size of \( \lambda \in \Lambda \) by \( |\lambda| \).

Similarity classes indexed by functions \( c_1 \) and \( c_2 \) as above are said to be of the same type if there exists a degree-preserving self-bijection \( \sigma \) of \( \text{Irr} \mathbf{F}_q[t] \) such that \( c_2 = c_1 \circ \sigma \). Thus, the type of \( c \) remembers only the degrees of the polynomials (and not the polynomials themselves) for which \( c \) takes a certain value \( \lambda \). Replacing each irreducible polynomial of degree \( d \) for which \( c \) takes a non-trivial value \( \lambda \) by the pair \( (d, \lambda) \), we obtain a multiset of such pairs. Clearly, \( c_1 \) and \( c_2 \) have the same type if and only if these multisets are equal. Thus a similarity class type may be viewed as a multiset of pairs of the form \((d, \lambda)\).

For \( 2 \times 2 \) matrices there are four types:

```python
sage: for tau in SimilarityClassTypes(2):
....:     print(tau)
[[1, [1]], [1, [1]]]
[[1, [2]]]
[[1, [1, 1]]]
[[2, [1]]]
```

These four types correspond to the regular split semisimple matrices, the non-semisimple matrices, the central matrices and the irreducible matrices respectively.

For any matrix \( A \) in a given similarity class type, it is possible to calculate the number elements in the similarity class of \( A \), the dimension of the algebra of matrices in \( M_n(A) \) that commute with \( A \), and the cardinality of the subgroup of \( \text{GL}_n(\mathbf{F}_q) \) that commute with \( A \). For each similarity class type, it is also possible to compute the number of classes of that type (and hence, the total number of matrices of that type). All these calculations treat the cardinality \( q \) of the finite field as a formal variable:

```python
sage: M = SimilarityClassType([[1, [1]], [1, [1]]])
sage: M.class_card()
q^2 + q
sage: M.centralizer_algebra_dim()
2
```
We now describe two applications of similarity class types.

We say that an \( n \times n \) matrix has rational canonical form type \( \lambda \) for some partition \( \lambda \) of \( n \) if the diagonal blocks in the rational canonical form have sizes given by the parts of \( \lambda \). Thus the matrices with rational canonical type \( (n) \) are the regular ones, while the matrices with rational canonical type \( (1^n) \) are the central ones.

Using similarity class types, it becomes easy to get a formula for the number of matrices with a given rational canonical type:

\[
sage: def matrices_with_rcf(la):
    return sum([tau.number_of_matrices() for tau in filter(lambda tau: tau.rcf()==la, → SimilarityClassTypes(la.size()))])
\]

\[
sage: matrices_with_rcf(Partition([2,1]))
q^6 + q^5 + q^4 - q^3 - q^2 - q
\]

Similarity class types can also be used to calculate the number of simultaneous similarity classes of \( k \)-tuples of \( n \times n \) matrices with entries in \( \mathbb{F}_q \) by using Burnside’s lemma:

\[
sage: from sage.combinat.similarity_class_type import order_of_general_linear_group, → centralizer_algebra_dim
sage: q = ZZ['q'].gen()
sage: def simultaneous_similarity_classes(n,k):
    return SimilarityClassTypes(n).sum(lambda la: q**(k*centralizer_algebra_ → dim(la)), invertible = True)/order_of_general_linear_group(n)
\]

\[
sage: simultaneous_similarity_classes(3, 2)
q^{10} + q^8 + 2*q^7 + 2*q^6 + 2*q^5 + q^4
\]

Similarity class types can be used to compute the coefficients of generating functions coming from the cycle index type techniques of Kung and Stong (see Morrison [Morrison06]).

They can also be used to compute the number of invariant subspaces for a matrix over a finite field of any given dimension. For this we use the elegant recursive formula of Ramaré [R17] (see also [PR22]).

Along with the results of [PSS13], similarity class types can be used to calculate the number of similarity classes of matrices of order \( n \) with entries in a principal ideal local ring of length two with residue field of cardinality \( q \) with centralizer of any given cardinality up to \( n = 4 \). Among these, the classes which are selftranspose can also be counted:

\[
sage: from sage.combinat.similarity_class_type import matrix_centralizer_cardinalities_ → length_two
sage: list(matrix_centralizer_cardinalities_length_two(3))
[(q^6 - 3*q^5 + 3*q^4 - q^3, 1/6*q^6 - 1/2*q^5 + 1/3*q^4),
(q^6 - 2*q^5 + q^4, q^5 - q^4),
(q^8 - 3*q^7 + 3*q^6 - q^5, 1/2*q^5 - q^4 + 1/2*q^3),
(q^8 - 2*q^7 + q^6, q^4 - q^3),
(q^10 - 2*q^9 + 2*q^8 - q^7, q^4 - q^3),
(q^8 - q^7 - q^6 + q^5, 1/2*q^5 - q^4 + 1/2*q^3),
(q^6 - q^5 - q^4 + q^3, 1/2*q^6 - 1/2*q^5),
(q^6 - q^5, q^4),
\]

(continues on next page)
Continued from previous page:

\[(q^{10} - 2q^9 + q^8, q^3),\]
\[(q^8 - 2q^7 + q^6, q^4 - q^3),\]
\[(q^8 - q^7, q^3 + q^2),\]
\[(q^{12} - 3q^{11} + 3q^{10} - q^9, 1/6q^4 - 1/2q^3 + 1/3q^2),\]
\[(q^{12} - 2q^{11} + q^{10}, q^3 - q^2),\]
\[(q^{14} - 2q^{13} + 2q^{11} - q^{10}, q^3 - q^2),\]
\[(q^{12} - q^{11} - q^{10} + q^9, 1/2q^4 - 1/2q^3),\]
\[(q^{12} - q^{11}, q^2),\]
\[(q^{14} - 2q^{13} + q^{12}, q^2),\]
\[(q^{18} - q^{17} - q^{16} + q^{14} + q^{13} - q^{12}, q^2),\]
\[(q^{12} - q^9, 1/3q^4 - 1/3q^2),\]
\[(q^{6} - q^3, 1/3q^6 - 1/3q^4)].\]

REFERENCES:

AUTHOR:
- Amritanshu Prasad (2013-07-18): initial implementation
- Amritanshu Prasad (2013-09-09): added functions for similarity classes over rings of length two
- Amritanshu Prasad (2022-07-31): added computation of similarity class type of a given matrix and invariant subspace generating function

class sage.combinat.similarity_class_type.PrimarySimilarityClassType(parent, deg, par)

Bases: Element

A primary similarity class type is a pair consisting of a partition and a positive integer.

For a partition \( \lambda \) and a positive integer \( d \), the primary similarity class type \((d, \lambda)\) represents similarity classes of square matrices of order \(|\lambda| \cdot d\) with entries in a finite field of order \( q \) which correspond to the \( F_q[t] \)-module

\[
\frac{F_q[t]}{p(t)^{\lambda_1}} \oplus \frac{F_q[t]}{p(t)^{\lambda_2}} \oplus \cdots
\]

for some irreducible polynomial \( p(t) \) of degree \( d \).

centralizer_algebra_dim()

Return the dimension of the algebra of matrices which commute with a matrix of type \( \text{self} \).

For a partition \((d, \lambda)\) this dimension is given by \( d(\lambda_1 + 3\lambda_2 + 5\lambda_3 + \cdots) \).

EXAMPLES:

```python
sage: PT = PrimarySimilarityClassType(2, [3, 2, 1])
sage: PT.centralizer_algebra_dim()
sage: 28
```

centralizer_group_card\((q=\text{None})\)

Return the cardinality of the centralizer group of a matrix of type \( \text{self} \) in a field of order \( q \).

INPUT:
- \( q \) – an integer or an indeterminate

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: PT = PrimarySimilarityClassType(1, [])
sage: PT.centralizer_group_card()
1
sage: PT = PrimarySimilarityClassType(2, [1, 1])
sage: PT.centralizer_group_card()
q^8 - q^6 - q^4 + q^2
```

**degree()**

Return degree of self.

**EXAMPLES:**

```python
sage: PT = PrimarySimilarityClassType(2, [3, 2, 1])
sage: PT.degree()
2
```

**invariant_subspace_generating_function** *(q=None, t=None)*

Return the invariant subspace generating function of self.

**INPUT:**

- `q` – (optional) an integer or an indeterminate
- `t` – (optional) an indeterminate

**EXAMPLES:**

```python
sage: PrimarySimilarityClassType(1, [2, 2]).invariant_subspace_generating_function()
t^4 + (q + 1)*t^3 + (q^2 + q + 1)*t^2 + (q + 1)*t + 1
```

**partition()**

Return partition corresponding to self.

**EXAMPLES:**

```python
sage: PT = PrimarySimilarityClassType(2, [3, 2, 1])
sage: PT.partition()
[3, 2, 1]
```

**size()**

Return the size of self.

**EXAMPLES:**

```python
sage: PT = PrimarySimilarityClassType(2, [3, 2, 1])
sage: PT.size()
12
```

**statistic** *(func, q=None)*

Return \( n_\lambda(q^d) \) where \( n_\lambda \) is the value returned by `func` upon input \( \lambda \), if `self` is \((d, \lambda)\).

**EXAMPLES:**

```python
sage: PT = PrimarySimilarityClassType(2, [3, 1])
sage: q = ZZ['q'].gen()
```

(continues on next page)
class sage.combinat.similarity_class_type.PrimarySimilarityClassTypes(n, min)
Bases: UniqueRepresentation, Parent

All primary similarity class types of size $n$ whose degree is greater than that of $\min$ or whose degree is that of $\min$ and whose partition is less than of $\min$ in lexicographic order.

A primary similarity class type of size $n$ is a pair $(\lambda, d)$ consisting of a partition $\lambda$ and a positive integer $d$ such that $|\lambda|d = n$.

INPUT:

- $n$ – a positive integer
- $\min$ – a primary matrix type of size $n$

EXAMPLES:

If $\min$ is not specified, then the class of all primary similarity class types of size $n$ is created:

```python
sage: PTC = PrimarySimilarityClassTypes(2)
sage: for PT in PTC:
    print(PT)
[1, [2]]
[1, [1, 1]]
[2, [1]]
```

If $\min$ is specified, then the class consists of only those primary similarity class types whose degree is greater than that of $\min$ or whose degree is that of $\min$ and whose partition is less than of $\min$ in lexicographic order:

```python
sage: PTC = PrimarySimilarityClassTypes(2, min = PrimarySimilarityClassType(1, [1, [1]]))
sage: for PT in PTC:
    print(PT)
[1, [1, 1]]
[2, [1]]
```

Element

alias of PrimarySimilarityClassType

size()

Return size of elements of self.

The size of a primary similarity class type $(d, \lambda)$ is $d|\lambda|$.

EXAMPLES:

```python
sage: PTC = PrimarySimilarityClassTypes(2)
sage: PTC.size()
2
```

class sage.combinat.similarity_class_type.SimilarityClassType(parent, tau)
Bases: CombinatorialElement

A similarity class type.
A matrix type is a multiset of primary similarity class types.

**INPUT:**
- tau – a list of primary similarity class types or a square matrix over a finite field

**EXAMPLES:**

```python
sage: tau1 = SimilarityClassType([[3, [3, 2, 1]], [2, [2, 1]]]); tau1
[[2, [2, 1]], [3, [3, 2, 1]]]
sage: SimilarityClassType(Matrix(GF(2), [[1,1], [0,1]]))
[[1, [2]]]
```

**as_partition_dictionary()**

Return a dictionary whose keys are the partitions of types occurring in `self` and the value at the key $\lambda$ is the partition formed by sorting the degrees of primary types with partition $\lambda$.

**EXAMPLES:**

```python
sage: tau = SimilarityClassType([[1, [1]], [1, [1]]])
sage: tau.as_partition_dictionary()
{[1]: [1, 1]}
```

**centralizer_algebra_dim()**

Return the dimension of the algebra of matrices which commute with a matrix of type `self`.

**EXAMPLES:**

```python
sage: tau = SimilarityClassType([[1, [1]], [1, [1]]])
sage: tau.centralizer_algebra_dim()
2
```

**centralizer_group_card(q=None)**

Return the cardinality of the group of matrices in $GL_n(F_q)$ which commute with a matrix of type `self`.

**INPUT:**
- q – an integer or an indeterminate

**EXAMPLES:**

```python
sage: tau = SimilarityClassType([[1, [1]], [1, [1]]])
sage: tau.centralizer_group_card()
q^2 - 2*q + 1
```

**class_card(q=None)**

Return the number of matrices in each similarity class of type `self`.

**INPUT:**
- q – an integer or an indeterminate

**EXAMPLES:**

```python
sage: tau = SimilarityClassType([[1, [1], [1, 1], [1, 1]])
sage: tau.class_card()
1
sage: tau = SimilarityClassType([[1, [1]], [1, [1]]])
1
```
Combinatorics, Release 10.0

(continued from previous page)

sage: tau.class_card()
q^2 + q

invariant_subspace_generating_function\( (q=\text{None}, t=\text{None}) \)

Return the invariant subspace generating function of self.

The invariant subspace generating function is the function is the polynomial
\[
\sum_{j \geq 0} a_j(q) t^j,
\]
where \(a_j(q)\) denotes the number of \(j\)-dimensional invariant subspaces of dimension \(j\) for any matrix with the similarity class type self with entries in a field of order \(q\).

EXAMPLES:

sage: SimilarityClassType([[1, [2, 2]]]).invariant_subspace_generating_function()
t^4 + (q + 1)*t^3 + (q^2 + q + 1)*t^2 + (q + 1)*t + 1
sage: A = Matrix(GF(2),
[(0, 1, 0, 0), (0, 1, 1, 1), (1, 0, 1, 0), (1, 1, 0, 0)])
sage: SimilarityClassType(A).invariant_subspace_generating_function()
t^4 + 1

is_regular()

Return True if every primary type in self has partition with one part.

EXAMPLES:

sage: tau = SimilarityClassType([[2, [1, 1]], [1, [3]]])
sage: tau.is_regular()
True
sage: tau = SimilarityClassType([[2, [1, 1]], [1, [2]]])
sage: tau.is_regular()
False

is_semisimple()

Return True if every primary similarity class type in self has all parts equal to 1.

EXAMPLES:

sage: tau = SimilarityClassType([[2, [1, 1]], [1, [3]]])
sage: tau.is_semisimple()
True
sage: tau = SimilarityClassType([[2, [1, 1]], [1, [2]]])
sage: tau.is_semisimple()
False

number_of_classes\( (invertible=\text{False}, q=\text{None}) \)

Return the number of similarity classes of matrices of type self.

INPUT:

- invertible – Boolean; return number of invertible classes if set to True
- q – An integer or an indeterminate

EXAMPLES:
sage: tau = SimilarityClassType([[1, [1]], [1, [1]]])
sage: tau.number_of_classes()
1/2*q^2 - 1/2*q

\texttt{number\_of\_matrices}(invertible=False, q=None)

Return the number of matrices of type \texttt{self}.

INPUT:

\begin{itemize}
\item invertible – A boolean; return the number of invertible matrices if set
\end{itemize}

EXAMPLES:

sage: tau = SimilarityClassType([[1, [1]]])
sage: tau.number_of_matrices()
q
sage: tau.number_of_matrices(invertible = True)
q - 1
sage: tau = SimilarityClassType([[1, [1]], [1, [1]]])
sage: tau.number_of_matrices()
1/2*q^4 - 1/2*q^2

\texttt{rcf}()

Return the partition corresponding to the rational canonical form of a matrix of type \texttt{self}.

EXAMPLES:

sage: tau = SimilarityClassType([[2, [1, 1, 1]], [1, [3, 2]]])
sage: tau.rcf()
[5, 4, 2]

\texttt{size}()

Return the sum of the sizes of the primary parts of \texttt{self}.

EXAMPLES:

sage: tau = SimilarityClassType([[3, [3, 2, 1]], [2, [2, 1]]])
sage: tau.size()
24

\texttt{statistic}(func, q=None)

Return

\[ \prod_{(d, \lambda) \in \tau} n_{\lambda}(q^d) \]

where \( n_{\lambda}(q) \) is the value returned by \texttt{func} on the input \( \lambda \).

INPUT:

\begin{itemize}
\item func – a function that takes a partition to a polynomial in \( q \)
\item q – an integer or an indeterminate
\end{itemize}

EXAMPLES:
sage: tau = SimilarityClassType([[1, [1]], [1], [2], [1, [1]]])
sage: from sage.combinat.similarity_class_type import fq
sage: tau.statistic(lambda la: prod([fq(m) for m in la.to_exp()]))
(q^9 - 3*q^8 + 2*q^7 + 2*q^6 - 4*q^5 + 4*q^4 - 2*q^3 - 2*q^2 + 3*q - 1)/q^9
sage: q = ZZ['q'].gen()
sage: tau.statistic(lambda la: q**la.size(), q = q)
q^8

class sage.combinat.similarity_class_type.SimilarityClassTypes(n, min)
Bases: UniqueRepresentation, Parent

Class of all similarity class types of size n with all primary matrix types greater than or equal to the primary matrix type min.

A similarity class type is a multiset of primary matrix types.

INPUT:

• n – a non-negative integer
• min – a primary similarity class type

EXAMPLES:

If min is not specified, then the class of all matrix types of size n is constructed:

sage: M = SimilarityClassTypes(2)
sage: for tau in M:
     ....:     print(tau)
[[[3, [3, 2, 1]], [2], [1, [1]]]]
[[[2, [2, 1]]]]
[[[1, [1, 1]]]]
[[[2, [1]]]]

If min is specified, then the class consists of only those similarity class types which are multisets of primary matrix types which either have size greater than that of min, or if they have size equal to that of min, then they occur after min in the iterator for PrimarySimilarityClassTypes(n), where n is the size of min:

sage: M = SimilarityClassTypes(2, min = [1, [1, 1]])
sage: for tau in M:
     ....:     print(tau)
[[[1, [1, 1]]]]
[[[2, [1]]]]

Element

alias of SimilarityClassType

size()

Return size of self.

EXAMPLES:
sum\( (\text{stat}, \text{sumover}=\text{matrices}, \text{invertible}=\text{False}, q=\text{None}) \)

Return the sum of a local statistic over all types.

Given a set of functions \( n_\lambda(q) \) (these could be polynomials or rational functions in \( q \), for each similarity class type \( \tau \) define

\[
n_\tau(q) = \prod_{(d, \lambda) \in \tau} n_\lambda(q^d).
\]

This function returns

\[
\sum n_\tau(g)(q)
\]

where \( \tau(g) \) denotes the type of a matrix \( g \), and the sum is over all \( n \times n \) matrices if \text{sumover} is set to "matrices", is over all \( n \times n \) similarity classes if \text{sumover} is set to "classes", and over all \( n \times n \) types if \text{sumover} is set to "types". If \text{invertible} is set to \text{True}, then the sum is only over invertible matrices or classes.

INPUT:

- \text{stat} – a function which takes partitions and returns a function of \( q \)
- \text{sumover} – can be one of the following:
  - "matrices"
  - "classes"
  - "types"
- \( q \) – an integer or an indeterminate

OUTPUT:

A function of \( q \).

EXAMPLES:

```python
sage: M = SimilarityClassTypes(2)
sage: M.sum(lambda la:1)
q^4
sage: M.sum(lambda la:1, invertible = True)
q^4 - q^3 - q^2 + q
sage: M.sum(lambda la:1, sumover = "classes")
q^2 + q
sage: M.sum(lambda la:1, sumover = "classes", invertible = True)
q^2 - 1
```

Burside’s lemma can be used to calculate the number of similarity classes of matrices:

```python
sage: from sage.combinat.similarity_class_type import centralizer_algebra_dim,
    order_of_general_linear_group
sage: q = ZZ['q'].gen()
sage: M.sum(lambda la:q**centralizer_algebra_dim(la), invertible = True)/order_of_general_linear_group(2)
q^2 + q
```

sage.combinat.similarity_class_type.centralizer_algebra_dim()

Return the dimension of the centralizer algebra in \( M_n(F_q) \) of a nilpotent matrix whose Jordan blocks are given by \( la \).

EXAMPLES:
sage: from sage.combinat.similarity_class_type import centralizer_algebra_dim
sage: centralizer_algebra_dim(Partition([2, 1]))
5

**Note:** If it is a list, la is expected to be sorted in decreasing order.

```
sage.combinat.similarity_class_type.centralizer_group_cardinality(q=None)
```

Return the cardinality of the centralizer group in $GL_n(F_q)$ of a nilpotent matrix whose Jordan blocks are given by la.

**INPUT:**
- lambda – a partition
- q – an integer or an indeterminate

**OUTPUT:**
A polynomial function of $q$.

**EXAMPLES:**
```
sage: from sage.combinat.similarity_class_type import centralizer_group_cardinality
sage: q = ZZ['q'].gen()
sage: centralizer_group_cardinality(Partition([2, 1]))
q^5 - 2*q^4 + q^3
```

```
sage.combinat.similarity_class_type.dictionary_from_generator(gen)
```

Given a generator for a list of pairs $(c, f)$, construct a dictionary whose keys are the distinct values for $c$ and whose value at $c$ is the sum of $f$ over all pairs of the form $(c', f)$ such that $c = c'$.

**EXAMPLES:**
```
sage: from sage.combinat.similarity_class_type import dictionary_from_generator
sage: dictionary_from_generator(((x // 2, x) for x in range(10)))
{0: 1, 1: 5, 2: 9, 3: 13, 4: 17}
```

It also works with lists:
```
sage: dictionary_from_generator([(x // 2, x) for x in range(10)])
{0: 1, 1: 5, 2: 9, 3: 13, 4: 17}
```

**Note:** Since the generator is first converted to a list, memory usage could be high.

```
sage.combinat.similarity_class_type.ext_orbit_centralizers(input_data, q=None, selftranspose=False)
```

Generate pairs consisting of centralizer cardinalities of orbits in $Ext^1(M, M)$ for the action of $Aut(M, M)$, where $M$ is the $F_q[t]$-module constructed from input and their frequencies.

**INPUT:**
- input_data – input for `input_parsing()`
- q – (default: $q$) an integer or an indeterminate
- selftranspose – (default: False) boolean stating if we only want selftranspose type
sage.combinat.similarity_class_type.ext_orbits(input_data, q=None, selftranspose=False)

Return the number of orbits in $\text{Ext}^1(M, M)$ for the action of $\text{Aut}(M, M)$, where $M$ is the $F_q[t]$-module constructed from input_data.

**INPUT:**
- `input_data` – input for `input_parsing()`
- `q` – (default: $q$) an integer or an indeterminate
- `selftranspose` – (default: False) boolean stating if we only want selftranspose type

sage.combinat.similarity_class_type.fq(q=None)

Return $(1 - q^{-1})(1 - q^{-2})\cdots(1 - q^{-n})$.

**INPUT:**
- `n` – a non-negative integer
- `q` – an integer or an indeterminate

**OUTPUT:**
A rational function in $q$.

**EXAMPLES:**

```python
sage: from sage.combinat.similarity_class_type import fq
sage: fq(0)
1
sage: fq(3)
(q^6 - q^5 - q^4 + q^2 + q - 1)/q^6
```

sage.combinat.similarity_class_type.input_parsing(data)

Recognize and return the intended type of input.

sage.combinat.similarity_class_type.invariant_subspace_generating_function(la, q=None, t=None)

Return the invariant subspace generating function of a nilpotent matrix with Jordan block sizes given by la.

**INPUT:**
- `la` – a partition
- `q` – (optional) an integer or an indeterminate
- `t` – (optional) an indeterminate

**OUTPUT:**
A polynomial in $t$ whose coefficients are polynomials in $q$.

**EXAMPLES:**

```python
sage: from sage.combinat.similarity_class_type import invariant_subspace_generating_function
sage: invariant_subspace_generating_function([2,2])
t^4 + (q + 1)*t^3 + (q^2 + q + 1)*t^2 + (q + 1)*t + 1
```

sage.combinat.similarity_class_type.matrix_centralizer_cardinalities(n, q=None, invertible=False)

Generate pairs consisting of centralizer cardinalities of matrices over a finite field and their frequencies.
sage.combinat.similarity_class_type.matrix_centralizer_cardinalities_length_two\(n, q=None, selftranspose=False, invertible=False\)

Generate pairs consisting of centralizer cardinalities of matrices over a principal ideal local ring of length two with residue field of order \(q\) and their frequencies.

**INPUT:**

- \(n\) – the order
- \(q\) – (default: \(q\)) an integer or an indeterminate
- \(selftranspose\) – (default: \(False\)) boolean stating if we only want selftranspose type
- \(invertible\) – (default: \(False\)) boolean stating if we only want invertible type

sage.combinat.similarity_class_type.matrix_similarity_classes\(n, q=None, invertible=False\)

Return the number of matrix similarity classes over a finite field of order \(q\).

sage.combinat.similarity_class_type.matrix_similarity_classes_length_two\(n, q=None, selftranspose=False, invertible=False\)

Return the number of similarity classes of matrices of order \(n\) with entries in a principal ideal local ring of length two.

**INPUT:**

- \(n\) – the order
- \(q\) – (default: \(q\)) an integer or an indeterminate
- \(selftranspose\) – (default: \(False\)) boolean stating if we only want selftranspose type
- \(invertible\) – (default: \(False\)) boolean stating if we only want invertible type

**EXAMPLES:**

We can generate Table 6 of [PSS13]:

```
sage: from sage.combinat.similarity_class_type import matrix_similarity_classes_length_two
sage: matrix_similarity_classes_length_two(2)
q^4 + q^3 + q^2
sage: matrix_similarity_classes_length_two(2, invertible = True)
q^4 - q
sage: matrix_similarity_classes_length_two(3)
q^6 + q^5 + 2*q^4 + q^3 + 2*q^2
sage: matrix_similarity_classes_length_two(3, invertible = True)
q^6 - q^3 + 2*q^2 - 2*q
sage: matrix_similarity_classes_length_two(4)
q^8 + q^7 + 3*q^6 + 3*q^5 + 5*q^4 + 3*q^3 + 3*q^2
sage: matrix_similarity_classes_length_two(4, invertible = True)
q^8 + q^6 - q^5 + 2*q^4 - 2*q^3 + 2*q^2 - 3*q
```

And also Table 7:
sage: matrix_similarity_classes_length_two(2, selftranspose = True)
q^4 + q^3 + q^2
sage: matrix_similarity_classes_length_two(2, selftranspose = True, invertible = True)
q^4 - q
sage: matrix_similarity_classes_length_two(3, selftranspose = True)
q^6 + q^5 + 2*q^4 + q^3
sage: matrix_similarity_classes_length_two(3, selftranspose = True, invertible = True)
q^6 - q^3
sage: matrix_similarity_classes_length_two(4, selftranspose = True)
q^8 + q^7 + 3*q^6 + 3*q^5 + 3*q^4 + q^3 + q^2
sage: matrix_similarity_classes_length_two(4, selftranspose = True, invertible = True)
q^8 + q^6 - q^5 - q

sage.combinat.similarity_class_type.order_of_general_linear_group(q=None)
Return the cardinality of the group of \( n \times n \) invertible matrices with entries in a field of order \( q \).

INPUT:
- \( n \) – a non-negative integer
- \( q \) – an integer or an indeterminate

EXAMPLES:

sage: from sage.combinat.similarity_class_type import order_of_general_linear_group
sage: order_of_general_linear_group(0)
1
sage: order_of_general_linear_group(2)
q^4 - q^3 - q^2 + q

sage.combinat.similarity_class_type.primitives(invertible=False, q=None)
Return the number of similarity classes of simple matrices of order \( n \) with entries in a finite field of order \( q \). This is the same as the number of irreducible polynomials of degree \( d \).

If \( \text{invertible} \) is \( \text{True} \), then only the number of similarity classes of invertible matrices is returned.

Note: All primitive classes are invertible unless \( n \) is 1.

INPUT:
- \( n \) – a positive integer
- \( \text{invertible} \) – boolean; if set, only number of non-zero classes is returned
- \( q \) – an integer or an indeterminate

OUTPUT:
- a rational function of the variable \( q \)

EXAMPLES:

sage: from sage.combinat.similarity_class_type import primitives
sage: primitives(1)
5.1.308 sine-Gordon Y-system plotter

This class builds the triangulations associated to sine-Gordon and reduced sine-Gordon Y-systems as constructed in [NS].

AUTHORS:

• Salvatore Stella (2014-07-18): initial version

EXAMPLES:

A reduced sine-Gordon example with 3 generations:

```
sage: Y = SineGordonYsystem('A', (6, 4, 3)); Y
A sine-Gordon Y-system of type A with defining integer tuple (6, 4, 3)
sage: Y.plot()  # not tested
```

The same integer tuple but for the non-reduced case:

```
sage: Y = SineGordonYsystem('D', (6, 4, 3)); Y
A sine-Gordon Y-system of type D with defining integer tuple (6, 4, 3)
sage: Y.plot()  # not tested
```

Todo: The code for plotting is extremely slow.

REFERENCES:

class sage.combinat.sine_gordon.SineGordonYsystem(X, na)

Bases: SageObject

A class to model a (reduced) sine-Gordon Y-system

Note that the generations, together with all integer tuples, in this implementation are numbered from 0 while in [NS] they are numbered from 1

INPUT:

• X – the type of the Y-system to construct (either ‘A’ or ‘D’)
• na – the tuple of positive integers defining the Y-system with na[0] > 2

See [NS]

EXAMPLES:

```
sage: Y = SineGordonYsystem('A', (6, 4, 3)); Y
A sine-Gordon Y-system of type A with defining integer tuple (6, 4, 3)
```
sage: Y.intervals()
(((0, 0, 'R'),),
 (0, 17, 'L'),
 (17, 34, 'L'),
 ...
 (104, 105, 'R'),
 (105, 0, 'R'))

sage: Y.triangulation()
((17, 89),
 (17, 72),
 (34, 72),
 ...
 (102, 105),
 (103, 105))

sage: Y.plot() #not tested

\textbf{F()}

Return the number of generations in \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.F()
3
\end{verbatim}

\textbf{intervals()}

Return, divided by generation, the list of intervals used to construct the initial triangulation.

Each such interval is a triple \((p, q, X)\) where \(p\) and \(q\) are the two extremal vertices of the interval and \(X\) is the type of the interval (one of ‘L’, ‘R’, ‘NL’, ‘NR’).

\textbf{ALGORITHM:}

The algorithm used here is the one described in section 5.1 of [NS]. The only difference is that we get rid of the special case of the first generation by treating the whole disk as a type ‘R’ interval.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.intervals()
(((0, 0, 'R'),),
 (0, 17, 'L'),
 (17, 34, 'L'),
 ...
 (104, 105, 'R'),
 (105, 0, 'R'))
\end{verbatim}

\textbf{na()}

Return the sequence of the integers \(n_a\) defining \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.na()
(6, 4, 3)
\end{verbatim}
pa()  
Return the sequence of integers \( p_a \), i.e. the total number of intervals of types ‘NL’ and ‘NR’ in the \((a+1)\)-th generation.

EXAMPLES:

```
sage: Y = SineGordonYsystem('A', (6, 4, 3))
sage: Y.pa()  
(1, 6, 25)
```

plot(**kwds)  
Plot the initial triangulation associated to \texttt{self}.

INPUT:
- \texttt{radius} - the radius of the disk; by default the length of the circle is the number of vertices
- \texttt{points_color} - the color of the vertices; default ‘black’
- \texttt{points_size} - the size of the vertices; default 7
- \texttt{triangulation_color} - the color of the arcs; default ‘black’
- \texttt{triangulation_thickness} - the thickness of the arcs; default 0.5
- \texttt{shading_color} - the color of the shading used on neuter intervals; default ‘lightgray’
- \texttt{reflections_color} - the color of the reflection axes; default ‘blue’
- \texttt{reflections_thickness} - the thickness of the reflection axes; default 1

EXAMPLES:

```
sage: Y = SineGordonYsystem('A', (6, 4, 3))
sage: Y.plot()  
# long time 2s  
Graphics object consisting of 219 graphics primitives
```

qa()  
Return the sequence of integers \( q_a \), i.e. the total number of intervals of types ‘L’ and ‘R’ in the \((a+1)\)-th generation.

EXAMPLES:

```
sage: Y = SineGordonYsystem('A', (6, 4, 3))
sage: Y qa()  
(6, 25, 81)
```

r()  
Return the number of vertices in the polygon realizing \texttt{self}.

EXAMPLES:

```
sage: Y = SineGordonYsystem('A', (6, 4, 3))
sage: Y r()  
106
```

rk()  
Return the sequence of integers \( r^k \), i.e. the width of an interval of type ‘L’ or ‘R’ in the \(k\)-th generation.

EXAMPLES:
```python
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.rk()
(106, 17, 4)
```

**triangulation()**

Return the initial triangulation of the polygon realizing self as a tuple of pairs of vertices.

**Warning:** In type ‘D’ the returned triangulation does NOT contain the two radii.

**ALGORITHM:**

We implement the four cases described by Figure 14 in [NS].

**EXAMPLES:**

```python
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.triangulation()
((17, 89),
 (17, 72),
 ...,
 (102, 105),
 (103, 105))
```

```python
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.type()
'A'
```

**vertices()**

Return the vertices of the polygon realizing self as the ring of integers modulo self.r().

**EXAMPLES:**

```python
sage: Y = SineGordonYsystem('A',(6,4,3))
sage: Y.vertices()
Ring of integers modulo 106
```

5.1.309 Six Vertex Model

```python
class sage.combinat.six_vertex_model.SixVertexConfiguration
    Bases: ClonableArray
    A configuration in the six vertex model.
    check()
        Check if self is a valid 6 vertex configuration.
    EXAMPLES:
```
```python
sage: M = SixVertexModel(3, boundary_conditions='ice')
sage: M[0].check()
```

**energy(epsilon)**

Return the energy of the configuration.

The energy of a configuration $\nu$ is defined as

$$E(\nu) = n_0\epsilon_0 + n_1\epsilon_1 + \cdots + n_5\epsilon_5$$

where $n_i$ is the number of vertices of type $i$ and $\epsilon_i$ is the $i$-th energy constant.

**Note:** We number our configurations as:

0. LR
1. LU
2. LD
3. UD
4. UR
5. RD

which differs from Wikipedia article Ice-type_model.

**EXAMPLES:**

```python
sage: M = SixVertexModel(3, boundary_conditions='ice')
sage: nu = M[2]; nu
```

```
^ ^ ^
| | | --> # --> # <- # <--
^ | | --> # <- # <> # <--
| V | --> # <- # <> # <--
|-- # --> # <- # <=--
```

```python
>>> nu.energy([1,2,1,2,1,2])
15
```

A KDP energy:

```python
>>> nu.energy([1,1,0,1,0,1])
7
```

A Rys $F$ energy:

```python
>>> nu.energy([0,1,1,0,1,1])
4
```

The zero field assumption:
plot(color='sign')

Return a plot of self.

INPUT:

• color – can be any of the following:
  – 4 - use 4 colors: black, red, blue, and green with each corresponding to up, right, down, and left respectively
  – 2 - use 2 colors: red for horizontal, blue for vertical arrows
  – 'sign' - use red for right and down arrows, blue for left and up arrows
  – a list of 4 colors for each direction
  – a function which takes a direction and a boolean corresponding to the sign

EXAMPLES:

```python
sage: M = SixVertexModel(2, boundary_conditions='ice')
sage: print(M[0].plot().description())
Arrow from (-1.0,0.0) to (0.0,0.0)
Arrow from (-1.0,1.0) to (0.0,1.0)
Arrow from (0.0,0.0) to (0.0,-1.0)
Arrow from (0.0,1.0) to (0.0,0.0)
Arrow from (0.0,1.0) to (0.0,2.0)
Arrow from (1.0,0.0) to (1.0,-1.0)
Arrow from (1.0,0.0) to (1.0,1.0)
Arrow from (1.0,1.0) to (0.0,1.0)
Arrow from (1.0,1.0) to (1.0,2.0)
Arrow from (2.0,0.0) to (1.0,0.0)
Arrow from (2.0,1.0) to (1.0,1.0)
```

to_signed_matrix()

Return the signed matrix of self.

The signed matrix corresponding to a six vertex configuration is given by 0 if there is a cross flow, a 1 if the outward arrows are vertical and −1 if the outward arrows are horizontal.

EXAMPLES:

```python
sage: M = SixVertexModel(3, boundary_conditions='ice')
sage: [x.to_signed_matrix() for x in M]
[[1 0 0] [1 0 0] [ 0 1 0] [0 1 0] [0 0 1] [0 0 1] [0 0 1] [0 1 0] [0 0 1] [0 1 0] [0 0 1] [1 0 0] [0 0 1] [0 1 0] [1 0 0] [0 1 0] [0 0 1] [0 1 0] [1 0 0] [0 1 0] [0 0 1] [0 1 0] [0 0 1] [0 1 0] [0 0 1] [0 1 0] [1 0 0]
```

class sage.combinat.six_vertex_model.SixVertexModel(n, m, boundary_conditions)

Bases: UniqueRepresentation, Parent

The six vertex model.
We model a configuration by indicating which configuration by the following six configurations which are determined by the two outgoing arrows in the Up, Right, Down, Left directions:

1. LR:

```
|     
V
<-- # -->
^     
```

2. LU:

```
^     
|     
<-- # <--
^     
```

3. LD:

```
|     
V
<-- # <--
|     
V     
```

4. UD:

```
^     
|     
--> # <--
|     
V     
```

5. UR:

```
^     
|     
--> # -->
^     
```

6. RD:

```
|     
V
--> # -->
|     
V     
```

INPUT:

- \( n \) – the number of rows
- \( m \) – (optional) the number of columns, if not specified, then the number of columns is the number of rows

5.1. Comprehensive Module List
• boundary_conditions – (optional) a quadruple of tuples whose entries are either:
  – True for an inward arrow,
  – False for an outward arrow, or
  – None for no boundary condition.

There are also the following predefined boundary conditions:
  – 'ice' - The top and bottom boundary conditions are outward and the left and right boundary conditions are inward; this gives the square ice model. Also called domain wall boundary conditions.
  – 'domain wall' - Same as 'ice'.
  – 'alternating' - The boundary conditions alternate between inward and outward.
  – 'free' - There are no boundary conditions.

EXAMPLES:

Here are the six types of vertices that can be created:

```python
sage: M = SixVertexModel(1)
sage: list(M)
[[^ | ^ | ^ | ^],
 [V | V | | V],
 [ <-- # --> <-- # --> <-- # --> # --> # --> # --> ]
[^ | | | ^],
 [ | | V | V | ],
 [ <-- # --> # --> # --> # --> # --> ]
```

When using the square ice model, it is known that the number of configurations is equal to the number of alternating sign matrices:

```python
sage: M = SixVertexModel(1, boundary_conditions='ice')
sage: len(M)
1
sage: M = SixVertexModel(4, boundary_conditions='ice')
sage: len(M)
42
sage: all(len(SixVertexModel(n, boundary_conditions='ice')) == AlternatingSignMatrices(n).cardinality() for n in range(1, 7))
True
```

An example with a specified non-standard boundary condition and non-rectangular shape:

```python
sage: M = SixVertexModel(2, 1, [[None], [True,True], [None], [None,None]])
sage: list(M)
[^ | ^ | ^ | ^],
 [ | V | | V],
 [ <-- # --> <-- # --> <-- # --> <-- # --> <-- # --> # --> ]
[^ | | | ^],
 [ | | V | V | ],
 [ <-- # --> # --> # --> # --> ]
```

(continues on next page)
| , V , V , V |

REFERENCES:
- Wikipedia article Vertex_model
- Wikipedia article Ice-type_model

**Element**
alias of *SixVertexConfiguration*

**boundary_conditions()**
Return the boundary conditions of *self*.

EXAMPLES:
```sage`
M = SixVertexModel(2, boundary_conditions='ice')
sage: M.boundary_conditions()
((False, False), (True, True), (False, False), (True, True))
```

**partition_function(beta, epsilon)**
Return the partition function of *self*.

The partition function of a 6 vertex model is defined by:

\[ Z = \sum_{\nu} e^{-\beta E(\nu)} \]

where we sum over all configurations and \( E \) is the energy function. The constant \( \beta \) is known as the inverse temperature and is equal to \( 1/k_B T \) where \( k_B \) is Boltzmann’s constant and \( T \) is the system’s temperature.

INPUT:
- *beta* – the inverse temperature constant \( \beta \)
- *epsilon* – the energy constants, see *energy()*

EXAMPLES:
```sage`
M = SixVertexModel(3, boundary_conditions='ice')
sage: M.partition_function(2, [1,2,1,2,1,2])
e^(-24) + 2*e^(-28) + e^(-30) + 2*e^(-32) + e^(-36)
```

REFERENCES:
- Wikipedia article Partition_function_(statistical_mechanics)

**class** *sage.combinat.six_vertex_model.SquareIceModel(n)*

Bases: *SixVertexModel*

The square ice model.

The square ice model is a 6 vertex model on an \( n \times n \) grid with the boundary conditions that the top and bottom boundaries are pointing outward and the left and right boundaries are pointing inward. These boundary conditions are also called domain wall boundary conditions.

Configurations of the 6 vertex model with domain wall boundary conditions are in bijection with alternating sign matrices.
class Element
    Bases: SixVertexConfiguration
    An element in the square ice model.

to_alternating_sign_matrix()
    Return an alternating sign matrix of self.

See also:
    to_signed_matrix()

EXAMPLES:

    sage: M = SixVertexModel(4, boundary_conditions='ice')
    sage: M[6].to_alternating_sign_matrix()
    [[1 0 0 0]
     [0 0 0 1]
     [0 0 1 0]
     [0 1 0 0]]
    sage: M[7].to_alternating_sign_matrix()
    [[0 1 0 0]
     [1 -1 1 0]
     [0 1 -1 1]
     [0 0 1 0]]

from_alternating_sign_matrix(asm)
    Return a configuration from the alternating sign matrix asm.

EXAMPLES:

    sage: M = SixVertexModel(3, boundary_conditions='ice')
    sage: asm = AlternatingSignMatrix([[0,1,0],[1,-1,1],[0,1,0]])
    sage: M.from_alternating_sign_matrix(asm)
    \[
    \begin{array}{cccc}
    \uparrow & \uparrow & \uparrow \\
    | & | & | \\
    \rightarrow # \rightarrow # \leftarrow # \leftarrow --
    \end{array}
    \]
5.1.310 Skew Partitions

A skew partition \( \text{skp} \) of size \( n \) is a pair of partitions \( [p_1, p_2] \) where \( p_1 \) is a partition of the integer \( n_1 \), \( p_2 \) is a partition of the integer \( n_2 \), \( p_2 \) is an inner partition of \( p_1 \), and \( n = n_1 - n_2 \). We say that \( p_1 \) and \( p_2 \) are respectively the inner and outer partitions of \( \text{skp} \).

A skew partition can be depicted by a diagram made of rows of cells, in the same way as a partition. Only the cells of the outer partition \( p_1 \) which are not in the inner partition \( p_2 \) appear in the picture. For example, this is the diagram of the skew partition \([5,4,3,1],[3,3,1]\).

```
sage: print(SkewPartition([[5,4,3,1],[3,3,1]]).diagram())
  **
    *
  **
    *
```

A skew partition can be connected, which can easily be described in graphic terms: for each pair of consecutive rows, there are at least two cells (one in each row) which have a common edge. This is the diagram of the connected skew partition \([5,4,3,1],[3,1]\):

```
sage: print(SkewPartition([[5,4,3,1],[3,1]]).diagram())
  **
    ***
    ***
    *
sage: SkewPartition([[5,4,3,1],[3,1]]).is_connected()
True
```

The first example of a skew partition is not a connected one.

Applying a reflection with respect to the main diagonal yields the diagram of the conjugate skew partition, here \([4, 3, 3, 2, 1],[3, 3, 2]\):

```
sage: SkewPartition([[5,4,3,1],[3,3,1]]).conjugate()
[4, 3, 3, 2, 1] / [3, 2, 2]
sage: print(SkewPartition([[5,4,3,1],[3,3,1]]).conjugate().diagram())
  *
    *
    *
  **
    *
```

The outer corners of a skew partition are the corners of its outer partition. The inner corners are the internal corners of the outer partition when the inner partition is taken off. Shown below are the coordinates of the inner and outer corners.

```
sage: SkewPartition([[5,4,3,1],[3,3,1]]).outer_corners()
[(0, 4), (1, 3), (2, 2), (3, 0)]
sage: SkewPartition([[5,4,3,1],[3,3,1]]).inner_corners()
[(0, 3), (2, 1), (3, 0)]
```

EXAMPLES:

There are 9 skew partitions of size 3, with no empty row nor empty column:

```
sage: SkewPartitions(3).cardinality()
9
```

(continues on next page)
sage: SkewPartitions(3).list()
[[3], [], [2, 1], [], [3, 1], [1], [2, 2], [], [3, 2], [1, 1], [2, 2, 1], [1, 1], [3, 2, 1], [2, 1]]

There are 4 connected skew partitions of size 3:

sage: SkewPartitions(3, overlap=1).cardinality()
4
sage: SkewPartitions(3, overlap=1).list()
[[3], [], [2, 1], [], [2, 2], [1, 1], [1, 1, 1], []]

This is the conjugate of the skew partition [[4, 3, 1], [2]]

sage: SkewPartition([[4, 3, 1], [2]]).conjugate()
[3, 2, 2, 1] / [1, 1]

Geometrically, we just applied a reflection with respect to the main diagonal on the diagram of the partition. Of course, this operation is an involution:

sage: SkewPartition([[4, 3, 1], [2]]).conjugate().conjugate()

The `jacobi_trudi()` method computes the Jacobi-Trudi matrix. See [Mac1995] for a definition and discussion.

sage: SkewPartition([[4, 3, 1], [2]]).jacobi_trudi()
[h[2] h[] 0]

This example shows how to compute the corners of a skew partition.

sage: SkewPartition([[4, 3, 1], [2]]).inner_corners()
[(0, 2), (1, 0)]
sage: SkewPartition([[4, 3, 1], [2]]).outer_corners()
[(0, 3), (1, 2), (2, 0)]

AUTHORS:

• Mike Hansen: Initial version
• Travis Scrimshaw (2013-02-11): Factored out CombinatorialClass
• Trevor K. Karn (2022-08-03): Add outside_corners

class sage.combinat.skew_partition.SkewPartition(parent, skp)

Bases: CombinatorialElement

A skew partition.
A skew partition of shape $\lambda/\mu$ is the Young diagram from the partition $\lambda$ and removing the partition $\mu$ from the upper-left corner in English convention.

**cell_poset**(orientation='SE')

Return the Young diagram of self as a poset. The optional keyword variable orientation determines the order relation of the poset.

The poset always uses the set of cells of the Young diagram of self as its ground set. The order relation of the poset depends on the orientation variable (which defaults to "SE"). Concretely, orientation has to be specified to one of the strings "NW", "NE", "SW", and "SE", standing for “northwest”, “northeast”, “southwest” and “southeast”, respectively. If orientation is "SE", then the order relation of the poset is such that a cell $u$ is greater or equal to a cell $v$ in the poset if and only if $u$ lies weakly southeast of $v$ (this means that $u$ can be reached from $v$ by a sequence of south and east steps; the sequence is allowed to consist of south steps only, or of east steps only, or even be empty). Similarly the order relation is defined for the other three orientations. The Young diagram is supposed to be drawn in English notation.

The elements of the poset are the cells of the Young diagram of self, written as tuples of zero-based coordinates (so that $(3, 7)$ stands for the 8-th cell of the 4-th row, etc.).

**EXAMPLES:**

```python
sage: p = SkewPartition([[3,3,1], [2,1]])
sage: Q = p.cell_poset(); Q
Finite poset containing 4 elements
sage: sorted(Q)
[(0, 2), (1, 1), (1, 2), (2, 0)]
sage: sorted(Q.maximal_elements())
[(1, 2), (2, 0)]
sage: sorted(Q.minimal_elements())
[(0, 2), (1, 1), (2, 0)]
sage: sorted(Q.upper_covers((1, 1)))
[(1, 2)]
sage: sorted(Q.upper_covers((0, 2)))
[(1, 2)]

sage: P = p.cell_poset(orientation="NW"); P
Finite poset containing 4 elements
sage: sorted(P)
[(0, 2), (1, 1), (1, 2), (2, 0)]
sage: sorted(P.minimal_elements())
[(1, 2), (2, 0)]
sage: sorted(P.maximal_elements())
[(0, 2), (1, 1), (2, 0)]
sage: sorted(P.upper_covers((1, 2)))
[(0, 2), (1, 1)]

sage: R = p.cell_poset(orientation="NE"); R
Finite poset containing 4 elements
sage: sorted(R)
[(0, 2), (1, 1), (1, 2), (2, 0)]
sage: R.maximal_elements()
[(0, 2)]
sage: R.minimal_elements()
[(2, 0)]
sage: R.upper_covers((2, 0))
```

(continues on next page)
cells()
Return the coordinates of the cells of self. Coordinates are given as (row-index, column-index) and are 0 based.

EXAMPLES:

```sage
sage: SkewPartition([[4, 3, 1], [2]]).cells()
[(0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0)]
sage: SkewPartition([[4, 3, 1], []]).cells()
[(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0)]
sage: SkewPartition([[2], []]).cells()
[(0, 0), (0, 1)]
```

column_lengths()
Return the column lengths of self.

EXAMPLES:

```sage
sage: SkewPartition([[3,2,1],[1,1]]).column_lengths()
[1, 2, 1]
sage: SkewPartition([[5,2,2,2],[2,1]]).column_lengths()
[2, 3, 1, 1, 1]
```

columns_intersection_set()
Return the set of cells in the columns of the outer shape of self which columns intersect the skew diagram of self.

EXAMPLES:

```sage
sage: skp = SkewPartition([[3,2,1],[2,1]])
sage: cells = Set([(0,0), (0, 1), (0,2), (1, 0), (1, 1), (2, 0)])
sage: skp.columns_intersection_set() == cells
True
```

conjugate()
Return the conjugate of the skew partition skp.

EXAMPLES:

```sage
sage: SkewPartition([[3,2,1],[2]]).conjugate()
[3, 2, 1] / [1, 1]
```

diagram()
Return the Ferrers diagram of self.

EXAMPLES:

```sage
sage: print(SkewPartition([[5,4,3,1],[3,3,1]]).ferrers_diagram())
**
*  ```
**sage:** print(SkewPartition([[5,4,3,1],[3,1]]).diagram())

**sage:** SkewPartitions.options(diagram_str='#', convention="French")

**sage:** print(SkewPartition([[5,4,3,1],[3,1]]).diagram())

#
###
###
##

**sage:** SkewPartitions.options._reset()

---

**ferrers_diagram()**
Return the Ferrers diagram of self.

**EXAMPLES:**

**sage:** print(SkewPartition([[5,4,3,1],[3,3,1]]).ferrers_diagram())

**sage:** print(SkewPartition([[5,4,3,1],[3,1]]).diagram())

**sage:** SkewPartitions.options(diagram_str='#', convention="French")

**sage:** print(SkewPartition([[5,4,3,1],[3,1]]).diagram())

#
###
###
##

**sage:** SkewPartitions.options._reset()

---

**frobenius_rank()**
Return the Frobenius rank of the skew partition self.

The Frobenius rank of a skew partition $\lambda/\mu$ can be defined in various ways. The quickest one is probably the following: Writing $\lambda$ as $(\lambda_1, \lambda_2, \ldots, \lambda_N)$, and writing $\mu$ as $(\mu_1, \mu_2, \ldots, \mu_N)$, we define the Frobenius rank of $\lambda/\mu$ to be the number of all $1 \leq i \leq N$ such that

$$\lambda_i - i \notin \{\mu_1 - 1, \mu_2 - 2, \ldots, \mu_N - N\}.$$

In other words, the Frobenius rank of $\lambda/\mu$ is the number of rows in the Jacobi-Trudi matrix of $\lambda/\mu$ which don’t contain $h_0$. Further definitions have been considered in [Sta2002] (where Frobenius rank is just being called rank).

If $\mu$ is the empty shape, then the Frobenius rank of $\lambda/\mu$ is just the usual Frobenius rank of the partition $\lambda$ (see **frobenius_rank()**).

**EXAMPLES:**
Combinatorics, Release 10.0

```
sage: SkewPartition([[8,8,7,4], [4,1,1]]).frobenius_rank()
4
sage: SkewPartition([[2,1], [1]]).frobenius_rank()
2
sage: SkewPartition([[2,1,1], [1]]).frobenius_rank()
2
sage: SkewPartition([[2,1,1], [1,1]]).frobenius_rank()
2
sage: SkewPartition([[5,4,3,2], [2,1,1]]).frobenius_rank()
3
sage: SkewPartition([[4,2,1], [3,1,1]]).frobenius_rank()
2
sage: SkewPartition([[4,2,1], [3,2,1]]).frobenius_rank()
1
```

If the inner shape is empty, then the Frobenius rank of the skew partition is just the standard Frobenius rank of the partition:

```
sage: all( SkewPartition([lam, Partition([[]])]).frobenius_rank()
....:   == lam.frobenius_rank() for i in range(6)
....:     for lam in Partitions(i) )
True
```

If the inner and outer shapes are equal, then the Frobenius rank is zero:

```
sage: all( SkewPartition([lam, lam]).frobenius_rank() == 0
....:       for i in range(6) for lam in Partitions(i) )
True
```

**inner()**

Return the inner partition of **self**.

**EXAMPLES:**

```
sage: SkewPartition([[3,2,1],[1,1]]).inner()
[1, 1]
```

**inner_corners()**

Return a list of the inner corners of **self**.

**EXAMPLES:**

```
sage: SkewPartition([[4, 3, 1], [2]]).inner_corners()
[[0, 2], (1, 0)]
sage: SkewPartition([[4, 3, 1], [1]]).inner_corners()
[[0, 0]]
```

**is_connected()**

Return True if **self** is a connected skew partition.

A skew partition is said to be *connected* if for each pair of consecutive rows, there are at least two cells (one in each row) which have a common edge.

**EXAMPLES:**
**is_overlap**(n)

Return True if the overlap of self is at most n.

See also:

* overlap()

EXAMPLES:

```python
sage: SkewPartition([[5,4,3,1],[3,1]]).is_overlap(1)
True
```

**is_ribbon()**

Return True if and only if self is a ribbon.

This means that if it has exactly one cell in each of q consecutive diagonals for some nonnegative integer q.

EXAMPLES:

```python
sage: P = SkewPartition([[4,4,3,3],[3,2,2]])
```

```python
sage: P.pp()
```

```
*
**
***
```

```python
sage: P.is_ribbon()
True
```

```python
sage: P = SkewPartition([[4,3,3],[1,1]])
```

```python
sage: P.pp()
```

```
***
**
***
```

```python
sage: P.is_ribbon()
False
```

```python
sage: P = SkewPartition([[4,4,3,2],[3,2,2]])
```

```python
sage: P.pp()
```

```
*
**
*
**
```

```python
sage: P.is_ribbon()
False
```

```python
sage: P = SkewPartition([[4,4,3,3],[4,2,2,1]])
```

```python
sage: P.pp()
```

```
**
*
```

(continues on next page)
**
sage: P.is_ribbon()
True

sage: P = SkewPartition([[4,4,3,3],[4,2,2]])
sage: P.pp()

**
*
***
sage: P.is_ribbon()
True

sage: SkewPartition([[2,2,1],[2,2,1]]).is_ribbon()
True

jacobi_trudi()
Return the Jacobi-Trudi matrix of self.

EXAMPLES:

sage: SkewPartition([[3,2,1],[2,1]]).jacobi_trudi()

[  h[1]  0  0 ]

sage: SkewPartition([[4,3,2],[2,1]]).jacobi_trudi()


k_conjugate(k)
Return the \(k\)-conjugate of the skew partition.

EXAMPLES:

sage: SkewPartition([[3,2,1],[2,1]]).k_conjugate(3)

[2, 1, 1, 1, 1] / [2, 1]

sage: SkewPartition([[3,2,1],[2,1]]).k_conjugate(4)

[2, 2, 1, 1] / [2, 1]

sage: SkewPartition([[3,2,1],[2,1]]).k_conjugate(5)

[3, 2, 1] / [2, 1]

outer()
Return the outer partition of self.

EXAMPLES:

sage: SkewPartition([[3,2,1],[1,1]]).outer()

[3, 2, 1]

outer_corners()
Return a list of the outer corners of self.

These are corners that are contained inside of the shape. For the corners which are outside of the shape, use outside_corners().
Warning: In the case that self is an honest (rather than skew) partition, these are the corners() of the outer partition. In the language of [Sag2001] these would be the “inner corners” of the outer partition.

See also:

• sage.combinat.skew_partition.SkewPartition.outside_corners()
• sage.combinat.partition.Partition.outside_corners()

EXAMPLES:

```python
sage: SkewPartition([[4, 3, 1], [2]]).outside_corners()
[(0, 3), (1, 2), (2, 0)]
```

outside_corners()

Return the outside corners of self.

The outside corners are corners which are outside of the shape. This should not be confused with outer_corners() which consists of corners inside the shape. It returns a result analogous to the .outside_corners() method on (non-skew) Partitions.

See also:

• sage.combinat.skew_partition.SkewPartition.outer_corners()
• sage.combinat.partition.Partition.outside_corners()

EXAMPLES:

```python
sage: mu = SkewPartition([[3,2,1],[2,1]])
sage: mu.pp()
*
*
```

```python
sage: mu.outside_corners()
[(0, 3), (1, 2), (2, 1), (3, 0)]
```

overlap()

Return the overlap of self.

The overlap of two consecutive rows in a skew partition is the number of pairs of cells (one in each row) that share a common edge. This number can be positive, zero, or negative.

The overlap of a skew partition is the minimum of the overlap of the consecutive rows, or infinity in the case of at most one row. If the overlap is positive, then the skew partition is called connected.

EXAMPLES:

```python
sage: SkewPartition([],[]).overlap()
+Infinity
sage: SkewPartition([1,[]]).overlap()
+Infinity
sage: SkewPartition([10,[]]).overlap()
+Infinity
```

(continues on next page)
sage: SkewPartition([[10],[2]]).overlap()
+Infinity
sage: SkewPartition([[10,1],[2]]).overlap()
-1
sage: SkewPartition([[10,10],[1]]).overlap()
9

\textbf{pieri\_macdonald\_coeffs()}

Computation of the coefficients which appear in the Pieri formula for Macdonald polynomials given in his book (Chapter 6.6 formula 6.24(ii))

**EXAMPLES:**

sage: SkewPartition([[3,2,1],[2,1]]).pieri_macdonald_coeffs()
1
sage: SkewPartition([[3,2,1],[2,2]]).pieri_macdonald_coeffs()
(q^2*t^3 - q*t^2 - t^2 + 1)/(q^2*t^3 - q*t^2 - q*t + 1)

sage: SkewPartition([[3,2,1],[2,2,1]]).pieri_macdonald_coeffs()
(q^6*t^8 - q^6*t^6 - q^4*t^7 - q^5*t^5 + q^4*t^5 - q^3*t^6 + q^5*t^3 + 2*q^3*t^4 + q^2*t^4 - q^2*t - q*t^3 - q^2*t - t^2 + 1)/(q^6*t^8 - q^5*t^7 - q^3*t^6 + 2*q^3*t^4 + q^2*t^4 + q*t^3 - q*t^2 - q*t + 1)

sage: SkewPartition([[3,3,2,2],[3,2,2,1]]).pieri_macdonald_coeffs()
(q^5*t^6 - q^5*t^5 + q^4*t^6 - q^4*t^5 - q^4*t^3 + q^4*t^2 - q^3*t^3 - q^2*t^4 + q^3*t^2 + q*t^3 + q - q + t - 1)/(q^5*t^6 - q^4*t^5 - q^3*t^4 + q^3*t^3 + q^2*t^2 - q*t + 1)

\textbf{pp()}

Pretty-print self.

**EXAMPLES:**

sage: SkewPartition([[5,4,3,1],[3,3,1]]).pp()
  **  
    *  
    **  
    *  

\textbf{quotient(k)}

The quotient map extended to skew partitions.

**EXAMPLES:**

sage: SkewPartition([[3, 3, 2, 1], [2, 1]]).quotient(2)
[[3], [], [], []]

\textbf{row\_lengths()}

Return the row lengths of self.

**EXAMPLES:**

sage: SkewPartition([[3,2,1],[1,1]]).row_lengths()
[2, 1, 1]
**rows_intersection_set()**
Return the set of cells in the rows of the outer shape of self which rows intersect the skew diagram of self.

**EXAMPLES:**
```python
sage: skp = SkewPartition([[3,2,1],[2,1]])
sage: cells = Set([(0,0), (0, 1), (0,2), (1,0), (1,1), (2,0)])
sage: skp.rows_intersection_set() == cells
True```

**size()**
Return the size of self.

**EXAMPLES:**
```python
sage: SkewPartition([[3,2,1],[1,1]]).size()
4```

**specht_module(base_ring=None)**
Return the Specht module corresponding to self.

**EXAMPLES:**
```python
sage: mu = SkewPartition([[3,2,1], [2]])
sage: SM = mu.specht_module(QQ)
sage: s = SymmetricFunctions(QQ).s()
sage: s(SM.frobenius_image())
s[2, 1, 1] + s[2, 2] + s[3, 1]
```

We verify that the Frobenius image is the corresponding skew Schur function:
```python
sage: s[3,2,1].skew_by(s[2])
s[2, 1, 1] + s[2, 2] + s[3, 1]
```

```python
sage: mu = SkewPartition([[4,2,1], [2,1]])
sage: SM = mu.specht_module(QQ)
sage: s(SM.frobenius_image())
sage: s(mu)
```

**specht_module_dimension(base_ring=None)**
Return the dimension of the Specht module corresponding to self.

This is equal to the number of standard (skew) tableaux of shape self.

**EXAMPLES:**
```python
sage: mu = SkewPartition([[3,2,1], [2]])
sage: mu.specht_module_dimension()
8
sage: mu.specht_module_dimension(GF(2))
8
```
**to_dag**(*format='string'*)

Return a directed acyclic graph corresponding to the skew partition *self*.

The directed acyclic graph corresponding to a skew partition *p* is the digraph whose vertices are the cells of *p*, and whose edges go from each cell to its lower and right neighbors (in English notation).

**INPUT:**

- *format* – either 'string' or 'tuple' (default: 'string'); determines whether the vertices of the resulting dag will be strings or 2-tuples of coordinates

**EXAMPLES:**

```python
sage: dag = SkewPartition([[3, 3, 1], [1, 1]]).to_dag()
sage: dag.edges(sort=True)
[('0,1', '0,2', None),
 ('0,1', '1,1', None),
 ('0,2', '1,2', None),
 ('1,1', '1,2', None)]
sage: dag.vertices(sort=True)
['0,1', '0,2', '1,1', '1,2', '2,0']
sage: dag = SkewPartition([[3, 2, 1], [1, 1]]).to_dag(format="tuple")
sage: dag.edges(sort=True)
[((0, 1), (0, 2), None), ((0, 1), (1, 1), None)]
sage: dag.vertices(sort=True)
[(0, 1), (0, 2), (1, 1), (2, 0)]
```

**to_list()**

Return *self* as a list of lists.

**EXAMPLES:**

```python
sage: s = SkewPartition([[4,3,1],[2]])
sage: s.to_list()
[[4, 3, 1], [2]]
sage: type(s.to_list())
<class 'list'>
```

**class** *sage.combinat.skew_partition.SkewPartitions*(*is_infinite=False*)

**Bases:** *UniqueRepresentation, Parent*

Skew partitions.

**Warning:** The iterator of this class only yields skew partitions which are reduced, in the sense that there are no empty rows before the last nonempty row, and there are no empty columns before the last nonempty column.

**EXAMPLES:**

```python
sage: SkewPartitions(4)
Skew partitions of 4
sage: SkewPartitions(4).cardinality()
28
sage: SkewPartitions(row_lengths=[2,1,2])
Skew partitions with row lengths [2, 1, 2]
```
sage: SkewPartitions(4, overlap=2)
Skew partitions of 4 with a minimum overlap of 2
sage: SkewPartitions(4, overlap=2).list()

**Element**
alias of *SkewPartition*

**from_row_and_column_length**(rowL, colL)
Construct a partition from its row lengths and column lengths.

**INPUT:**
- rowL – A composition or a list of positive integers
- colL – A composition or a list of positive integers

**OUTPUT:**
- If it exists the unique skew-partitions with row lengths rowL and column lengths colL.
- Raise a ValueError if rowL and colL are not compatible.

**EXAMPLES:**

```python
sage: S = SkewPartitions()
sage: print(S.from_row_and_column_length([3,1,2,2],[2,3,1,1,1]).diagram())
***
**
**
sage: S.from_row_and_column_length([],[])
[] / []
sage: S.from_row_and_column_length([1],[1])
[1] / []
sage: S.from_row_and_column_length([2,1],[2,1])
[2, 1] / []
sage: S.from_row_and_column_length([1,2],[1,2])
[2, 2] / [1]
sage: S.from_row_and_column_length([1,2],[1,3])
Traceback (most recent call last):
...
ValueError: sum mismatch: [1, 2] and [1, 3]
sage: S.from_row_and_column_length([3,2,1,2],[2,3,1,1,1])
Traceback (most recent call last):
...
ValueError: incompatible row and column length : [3, 2, 1, 2] and [2, 3, 1, 1, 1]
```

**Warning:** If some rows and columns have length zero, there is no way to retrieve unambiguously the skew partition. We therefore raise a ValueError. For examples here are two skew partitions with the same row and column lengths:

```python
sage: skp1 = SkewPartition([[2,2],[2,2]])
sage: skp2 = SkewPartition([[2,1],[2,1]])
sage: skp1.row_lengths(), skp1.column_lengths()
```
Combinatorics, Release 10.0

\begin{verbatim}
([0, 0], [0, 0])
sage: sp2.row_lengths(), sp2.column_lengths()
([0, 0], [0, 0])
sage: SkewPartitions().from_row_and_column_length([0,0], [0,0])
Traceback (most recent call last):
...
ValueError: row and column length must be positive

options = Current options for SkewPartitions - convention: English - diagram_str:
* display: quotient - latex: young_diagram - latex_diagram_str: \ast -
latex_marking_str: X

class sage.combinat.skew_partition.SkewPartitions_all
    Bases: SkewPartitions
    Class of all skew partitions.

class sage.combinat.skew_partition.SkewPartitions_n(n, overlap)
    Bases: SkewPartitions
    The set of skew partitions of \( n \) with overlap at least \( \text{overlap} \) and no empty row.
    INPUT:
    * \( n \) – a non-negative integer
    * \( \text{overlap} \) – an integer (default: 0)
    Caveat: this set is stable under conjugation only for \( \text{overlap} \) equal to 0 or 1. What exactly happens for negative
    overlaps is not yet well specified and subject to change (we may want to introduce vertical overlap constraints as well).

Todo: As is, this set is essentially the composition of Compositions(n) (which give the row lengths) and
SkewPartition(n, row_lengths=...), and one would want to “inherit” list and cardinality from this com-
position.

cardinality()
    Return the number of skew partitions of the integer \( n \) (with given overlap, if specified; and with no empty
    rows before the last row).
    EXAMPLES:

    sage: SkewPartitions(0).cardinality()
    1
    sage: SkewPartitions(4).cardinality()
    28
    sage: SkewPartitions(5).cardinality()
    87
    sage: SkewPartitions(4, overlap=1).cardinality()
    9
    sage: SkewPartitions(5, overlap=1).cardinality()
    20
    sage: s = SkewPartitions(5, overlap=-1)
sage: s.cardinality() == len(s.list())
    True
\end{verbatim}
class sage.combinat.skew_partition.SkewPartitions_rowlengths(co, overlap)
    Bases: SkewPartitions
    All skew partitions with given row lengths.
sage.combinat.skew_partition.row_lengths_aux(skp)
    EXAMPLES:

    sage: from sage.combinat.skew_partition import row_lengths_aux
    sage: row_lengths_aux([[5,4,3,1],[3,3,1]])
    [2, 1, 2]
sage: row_lengths_aux([[5,4,3,1],[3,1]])
    [2, 3]

5.1.311 Skew Tableaux

AUTHORS:

- Mike Hansen: Initial version
- Travis Scrimshaw, Arthur Lubovsky (2013-02-11): Factored out CombinatorialClass
- Trevor K. Karn (2022-08-03): added backward, lide

class sage.combinat.skew_tableau.SemistandardSkewTableaux(category=None)
    Bases: SkewTableaux
    Semistandard skew tableaux.

    This class can be initialized with several optional variables: the size of the skew tableaux (as a nameless integer variable), their shape (as a nameless skew partition variable), their weight (weight(), as a nameless second variable after either the size or the shape) and their maximum entry (as an optional keyword variable called max_entry, unless the weight has been specified). If neither the weight nor the maximum entry is specified, the maximum entry defaults to the size of the tableau.

    Note that “maximum entry” does not literally mean the highest entry; instead it is just an upper bound that no entry is allowed to surpass.

    EXAMPLES:

    The (infinite) class of all semistandard skew tableaux:

    sage: SemistandardSkewTableaux()
    Semistandard skew tableaux

    The (still infinite) class of all semistandard skew tableaux with maximum entry 2:

    sage: SemistandardSkewTableaux(max_entry=2)
    Semistandard skew tableaux with maximum entry 2

    The class of all semistandard skew tableaux of given size 3 and maximum entry 3:

    sage: SemistandardSkewTableaux(3)
    Semistandard skew tableaux of size 3 and maximum entry 3

    To set a different maximum entry:
**Combinatorics, Release 10.0**

```python
sage: SemistandardSkewTableaux(3, max_entry = 7)
Semistandard skew tableaux of size 3 and maximum entry 7
```

Specifying a shape:

```python
sage: SemistandardSkewTableaux([[2,1],[1]])
Semistandard skew tableaux of shape [2, 1] / [1] and maximum entry 3
```

Specifying both a shape and a maximum entry:

```python
sage: S = SemistandardSkewTableaux([[2,1],[1]], max_entry = 3); S
Semistandard skew tableaux of shape [2, 1] / [1] and maximum entry 3
sage: S.list()
[[[None, 1], [1]],
 [[[None, 2], [1]],
 [[[None, 1], [2]],
 [[[None, 3], [1]],
 [[[None, 1], [3]],
 [[[None, 2], [2]],
 [[[None, 3], [2]],
 [[[None, 2], [3]],
 [[[None, 3], [3]]]]
```

```python
sage: for n in range(5):
    ....:     print("{} {}".format(n, len(SemistandardSkewTableaux([[2,1],[1]], max_entry = n))))
0 0
1 0
2 1
3 9
4 35
```

Specifying a shape and a weight:

```python
sage: SemistandardSkewTableaux([[2,1],[1]], [2,1])
Semistandard skew tableaux of shape [2, 1] / [1] and weight [2, 1]
```

(the maximum entry is redundant in this case and thus is ignored).

Specifying a size and a weight:

```python
sage: SemistandardSkewTableaux(3, [2,1])
Semistandard skew tableaux of size 3 and weight [2, 1]
```

**Warning:** If the shape is not specified, the iterator of this class yields only skew tableaux whose shape is reduced, in the sense that there are no empty rows before the last nonempty row, and there are no empty columns before the last nonempty column. (Otherwise it would go on indefinitely.)

**Warning:** This class acts as a factory. The resulting classes are mainly useful for iteration. Do not rely on their containment tests, as they are not correct, e. g.
Combinatorics, Release 10.0

```python
sage: SkewTableau([[None]]) in SemistandardSkewTableaux(2)
True
```

class sage.combinat.skew_tableau.SemistandardSkewTableaux_all(max_entry)

Bases: SemistandardSkewTableaux

Class of all semistandard skew tableaux, possibly with a given maximum entry.

class sage.combinat.skew_tableau.SemistandardSkewTableaux_shape(p, max_entry)

Bases: SemistandardSkewTableaux

Class of semistandard skew tableaux of a fixed skew shape \( \lambda/\mu \) with a given max entry.

A semi-standard skew tableau with max entry \( i \) is required to have all its entries less or equal to \( i \). It is not required to actually contain an entry \( i \).

INPUT:

- \( p \) – A skew partition
- \( \text{max\_entry} \) – The max entry; defaults to the size of \( p \).

**Warning:** Input is not checked; please use SemistandardSkewTableaux to ensure the options are properly parsed.

cardinality()

**EXAMPLES:**

```python
sage: SemistandardSkewTableaux([[2,1],[]]).cardinality()
8
sage: SemistandardSkewTableaux([[2,1],[]], max_entry=2).cardinality()
2
```

class sage.combinat.skew_tableau.SemistandardSkewTableaux_shape_weight(p, mu)

Bases: SemistandardSkewTableaux

Class of semistandard skew tableaux of a fixed skew shape \( \lambda/\nu \) and weight \( \mu \).

class sage.combinat.skew_tableau.SemistandardSkewTableaux_size(n, max_entry)

Bases: SemistandardSkewTableaux

Class of all semi-standard skew tableaux of a fixed size \( n \), possibly with a given maximum entry.

**cardinality()**

**EXAMPLES:**

```python
sage: SemistandardSkewTableaux(2).cardinality()
8
```

class sage.combinat.skew_tableau.SemistandardSkewTableaux_size_weight(n, mu)

Bases: SemistandardSkewTableaux

Class of semi-standard tableaux of a fixed size \( n \) and weight \( \mu \).
cardinality()

EXAMPLES:

```python
sage: SemistandardSkewTableaux(2,[1,1]).cardinality()
4
```

class sage.combinat.skew_tableau.SkewTableau(parent, st)
Bases: ClonableList

A skew tableau.

Note that Sage by default uses the English convention for partitions and tableaux. To change this, see Tableaux.options().

EXAMPLES:

```python
sage: st = SkewTableau([[None, 1],[2,3]]); st
[[None, 1], [2, 3]]
sage: st.inner_shape()
[1]
sage: st.outer_shape()
[2, 2]
```

The `expr` form of a skew tableau consists of the inner partition followed by a list of the entries in each row from bottom to top:

```python
sage: SkewTableau(expr=[[1,1],[[5],[3,4],[1,2]]])
[[None, 1, 2], [None, 3, 4], [5]]
```

The chain form of a skew tableau consists of a list of partitions $\lambda_1, \lambda_2, \ldots$, such that all cells in $\lambda_{i+1}$ that are not in $\lambda_i$ have entry $i$:

```python
sage: SkewTableau(chain=[[2], [2, 1], [3, 1], [4, 3, 2, 1]])
[[None, None, 2, 3], [1, 3, 3], [3, 3], [3]]
```

backward_slide(corner=None)

Apply a backward jeu de taquin slide on the specified outside corner of self.

Backward jeu de taquin slides are defined in Section 3.7 of [Sag2001].

**Warning:** The `inner_corners()` and `outer_corners()` are the `sage.combinat.partition.Partition.corners()` of the inner and outer partitions of the skew shape. They are different from the inner/outer corners defined in [Sag2001].

The “inner corners” of [Sag2001] may be found by calling `outer_corners()`. The “outer corners” of [Sag2001] may be found by calling `self.outer_shape().outside_corners()`.

EXAMPLES:

```python
sage: T = SkewTableaux([[2, 2], [4, 4], [5]])
sage: Tableaux.options.display='array'
sage: Q = T.backward_slide(); Q
 . 2 2
 4 4
5
```
sage: Q.backward_slide((1, 2))
 2 2
 4 4
5
sage: Q.reverse_slide((1, 2)) == Q.backward_slide((1, 2))
True
sage: T = SkewTableaux([[1, 3],[3],[5]]); T
1 3
3
5
sage: T.reverse_slide((1,1))
 1
3 3
5

\textbf{bender\_knuth\_involution}(k, rows=None, check=True)

Return the image of self under the $k$-th Bender–Knuth involution, assuming self is a skew semistandard tableau.

Let $T$ be a tableau, then a \textit{lower free ‘$k$’ in $T$} means a cell of $T$ which is filled with the integer $k$ and whose direct lower neighbor is not filled with the integer $k + 1$ (in particular, this lower neighbor might not exist at all). Let an \textit{upper free ‘$k + 1$’ in $T$} mean a cell of $T$ which is filled with the integer $k + 1$ and whose direct upper neighbor is not filled with the integer $k$ (in particular, this neighbor might not exist at all). It is clear that for any row $r$ of $T$, the lower free $k$’s and the upper free $k + 1$’s in $r$ together form a contiguous interval or $r$.

The \textit{‘$k$’-th Bender–Knuth switch at row $i$} changes the entries of the cells in this interval in such a way that if it used to have $a$ entries of $k$ and $b$ entries of $k + 1$, it will now have $b$ entries of $k$ and $a$ entries of $k + 1$. For fixed $k$, the $k$-th Bender–Knuth switches for different $i$ commute. The composition of the $k$-th Bender–Knuth switches for all rows is called the \textit{‘$k$’-th Bender–Knuth involution}. This is used to show that the Schur functions defined by semistandard (skew) tableaux are symmetric functions.

INPUT:

- $k$ – an integer
- rows – (Default None) When set to None, the method computes the $k$-th Bender–Knuth involution as defined above. When an iterable, this computes the composition of the $k$-th Bender–Knuth switches at row $i$ over all $i$ in rows. When set to an integer $i$, the method computes the $k$-th Bender–Knuth switch at row $i$. Note the indexing of the rows starts with 1.
- check – (Default: True) Check to make sure self is semistandard. Set to False to avoid this check.

OUTPUT:

The image of self under either the $k$-th Bender–Knuth involution, the $k$-th Bender–Knuth switch at a certain row, or the composition of such switches, as detailed in the INPUT section.

EXAMPLES:
sage: t = SkewTableau([[None, None, None, 4, 4, 5, 6, 7], [None, 2, 4, 6, 7, 7, 7],
......: [None, 4, 5, 8, 8, 9], [None, 6, 7, 10], [None, 8, 8, 11], [None], [4]])
sage: t
[[None, None, None, 4, 4, 5, 6, 7], [None, 2, 4, 6, 7, 7, 7],
[None, 4, 5, 8, 8, 9], [None, 6, 7, 10], [None, 8, 8, 11], [None], [4]]
The Bender–Knuth involution is an involution:

```python
sage: t = SkewTableau([[None,3,4,4],[None,6,10],[7,7,11],[18]])
sage: all(t.bender_knuth_involution(k).bender_knuth_involution(k) == t for k in range(1,5))
True
```

The same for the single switches:

```python
sage: all(t.bender_knuth_involution(k, j).bender_knuth_involution(k, j) == t for k in range(1,5) for j in range(1, 5))
True
```

Locality of the Bender–Knuth involutions:

```python
sage: all(t.bender_knuth_involution(k).bender_knuth_involution(l).bender_knuth_involution(l).bender_knuth_involution(k) == t.bender_knuth_involution(l).bender_knuth_involution(k).bender_knuth_involution(l).bender_knuth_involution(k) for k in range(1,5) for l in range(1,5) if abs(k - l) > 1)
True
```

AUTHORS:

- Darij Grinberg (2013-05-14)

cells()

Return the cells in self.

EXAMPLES:

```python
sage: s = SkewTableau([[None,1,2],[3],[6]])
sage: s.cells()
[(0, 1), (0, 2), (1, 0), (2, 0)]
```

cells_by_content(c)

Return the coordinates of the cells in self with content c.
EXAMPLES:

```
sage: s = SkewTableau([[None,1,2],[3,4,5],[6]])
sage: s.cells_by_content(0)
[(1, 1)]
sage: s.cells_by_content(1)
[(0, 1), (1, 2)]
sage: s.cells_by_content(2)
[(0, 2)]
sage: s.cells_by_content(-1)
[(1, 0)]
sage: s.cells_by_content(-2)
[(2, 0)]
```

**cells_containing(i)**

Return the list of cells in which the letter \(i\) appears in the tableau \(self\). The list is ordered with cells appearing from left to right.

Cells are given as pairs of coordinates \((a, b)\), where both rows and columns are counted from 0 (so \(a = 0\) means the cell lies in the leftmost column of the tableau, etc.).

EXAMPLES:

```
sage: t = SkewTableau([[None,None,3],[None,3,5],[4,5]])
sage: t.cells_containing(5)
[(2, 1), (1, 2)]
sage: t.cells_containing(4)
[(2, 0)]
sage: t.cells_containing(2)
[]
sage: t = SkewTableau([[None,None,None,None],[None,4,5],[None,5,6],[None,9],
                    None])
sage: t.cells_containing(2)
[]
sage: t.cells_containing(4)
[(1, 1)]
sage: t.cells_containing(5)
[(2, 1), (1, 2)]
```

**check()**

Check that \(self\) is a valid skew tableau. This is currently far too liberal, and only checks some trivial things.

EXAMPLES:

```
sage: t = SkewTableau([[None,1,1],[2]])
sage: t.check()
```

(continues on next page)
sage: t = SkewTableau([[None, None, 1], [2, 4], [], [3, 4, 5]])
Traceback (most recent call last): ...
TypeError: a skew tableau cannot have an empty list for a row

sage: s = SkewTableau([[1, None, None],[2, None],[3]])
Traceback (most recent call last): ...
TypeError: not a valid skew tableau

**conjugate()**

Return the conjugate of self.

EXAMPLES:

```sage
sage: SkewTableau([[None,1],[2,3]]).conjugate()
[[None, 2], [1, 3]]
```

**entries_by_content(c)**

Return the entries in self with content c.

EXAMPLES:

```sage
sage: s = SkewTableau([[None,1,2],[3,4,5],[6]])
sage: s.entries_by_content(0)
[4]
sage: s.entries_by_content(1)
[1, 5]
sage: s.entries_by_content(2)
[2]
sage: s.entries_by_content(-1)
[3]
sage: s.entries_by_content(-2)
[6]
```

**evaluation()**

Return the weight (aka evaluation) of the tableau self. Trailing zeroes are omitted when returning the weight.

The weight of a skew tableau $T$ is the sequence $(a_1, a_2, a_3, \ldots)$, where $a_k$ is the number of entries of $T$ equal to $k$. This sequence contains only finitely many nonzero entries.

The weight of a skew tableau $T$ is the same as the weight of the reading word of $T$, for any reading order. **evaluation()** is a synonym for this method.

EXAMPLES:

```sage
sage: SkewTableau([[1,2],[3,4]]).weight()
[1, 1, 1, 1]
sage: SkewTableau([[None,2],[None,4],[None,5],[None]]).weight()
[0, 1, 0, 1, 1]
sage: SkewTableau([]).weight()
```
Combinatorics, Release 10.0

(continued from previous page)

sage: SkewTableau([[None, None, None], [None]]).weight()
[]

sage: SkewTableau([[None, 3, 4], [None, 6, 7], [4, 8], [5, 13], [6], [7]]).weight()
[0, 0, 1, 2, 1, 2, 2, 1, 0, 0, 0, 0, 1]

filling()

Return a list of the non-empty entries in self.

EXAMPLES:

sage: t = SkewTableau([[None, 1], [2, 3]])
sage: t.filling()
[[1], [2, 3]]

inner_shape()

Return the inner shape of self.

EXAMPLES:

sage: SkewTableau([[None, 1, 2], [None, 3], [4]]).inner_shape()
[1, 1]
sage: SkewTableau([[1, 2], [3, 4], [7]]).inner_shape()
[]
sage: SkewTableau([[None, None, None, 2, 3], [None, 1], [None], [2]]).inner_shape()
[3, 1, 1]

inner_size()

Return the size of the inner shape of self.

EXAMPLES:

sage: SkewTableau([[None, 2, 4], [None, 3], [1]]).inner_size()
2
sage: SkewTableau([[None, 2], [1, 3]]).inner_size()
1

is_k_tableau(k)

Checks whether self is a valid skew weak \( k \)-tableau.

EXAMPLES:

sage: t = SkewTableau([[None, 2, 3], [2, 3], [3]])
sage: t.is_k_tableau(3)
True
sage: t = SkewTableau([[None, 1, 3], [2, 2], [3]])
sage: t.is_k_tableau(3)
False

is_ribbon()

Return True if and only if the shape of self is a ribbon, that is, if it has exactly one cell in each of \( q \) consecutive diagonals for some nonnegative integer \( q \).

5.1. Comprehensive Module List
EXAMPLES:

```python
sage: S = SkewTableau([[None, None, 1, 2],[None, None, 3],[1, 3, 4]])
sage: S.pp()
  1 2
  3
1 3 4
sage: S.is_ribbon()
True

sage: S = SkewTableau([[None, 1, 2],[None, 2, 3],[1, 3, 4]])
sage: S.pp()
  1 2
  2 3
1 3 4
sage: S.is_ribbon()
False

sage: S = SkewTableau([[None, None, 1, 2],[None, None, 3],[1]])
sage: S.pp()
  1 2
  3
1
sage: S.is_ribbon()
False

sage: S = SkewTableau([[None, None, 1, 2],[None, None, 3],[1, 2, 4]])
sage: S.pp()
  1 2
  3
1 2 4
sage: S.is_ribbon()
True

sage: S = SkewTableau([[None, None, None, None],[None, None, None],[None, 2, 4]])
sage: S.pp()
  3
  2 4
sage: S.is_ribbon()
True

sage: S = SkewTableau([[None, None],[None]])
sage: S.pp()
  .
  .
sage: S.is_ribbon()
True
```

**is_semistandard()**

Return True if self is a semistandard skew tableau and False otherwise.

EXAMPLES:
is_standard()
Return True if self is a standard skew tableau and False otherwise.

EXAMPLES:

```python
sage: SkewTableau([[None, 2], [1, 3]]).is_standard()
True
sage: SkewTableau([[None, 2], [2, 4]]).is_standard()
True
sage: SkewTableau([[None, 3], [2, 4]]).is_standard()
True
sage: SkewTableau([[None, 2], [1, 2]]).is_standard()
False
sage: SkewTableau([[None, 2, 3]]).is_standard()
True
sage: SkewTableau([[None, 3, 2]]).is_standard()
False
sage: SkewTableau([[None, 2, 3], [1, 4]]).is_standard()
False
sage: SkewTableau([[None, 2, 3], [None, None, 4]]).is_standard()
False
```

outer_shape()
Return the outer shape of self.

EXAMPLES:

```python
sage: SkewTableau([[None,1,2],[None,3],[4]]).outer_shape()
[3, 2, 1]
```

outer_size()
Return the size of the outer shape of self.

EXAMPLES:

```python
sage: SkewTableau([[None, 2, 4], [None, 3], [1]]).outer_size()
6
sage: SkewTableau([[None, 2], [1, 3]]).outer_size()
4
```

pp()
Return a pretty print string of the tableau.

EXAMPLES:
rectify(algorithm=None)

Return a StandardTableau, SemistandardTableau, or just Tableau formed by applying the jeu de
taquin process to self.

See page 15 of [Ful1997].

INPUT:

• algorithm – optional: if set to 'jdt', rectifies by jeu de taquin; if set to 'schensted', rectifies by
Schensted insertion of the reading word; otherwise, guesses which will be faster.

EXAMPLES:

sage: S = SkewTableau([[None,1],[2,3]])
sage: S.rectify()
[[1, 3], [2]]

sage: T = SkewTableau([[None, None, None, 4],[None, None, 1, 6],[None, None, 5],[2, ...
→ 3]])
sage: T.rectify()
[[1, 3, 4, 6], [2, 5]]
sage: T.rectify(algorithm='jdt')
[[1, 3, 4, 6], [2, 5]]
sage: T.rectify(algorithm='schensted')
[[1, 3, 4, 6], [2, 5]]
sage: T.rectify(algorithm='spaghetti')
Traceback (most recent call last):
... ValueError: algorithm must be 'jdt', 'schensted', or None

restrict(n)

Return the restriction of the (semi)standard skew tableau to all the numbers less than or equal to n.

Note: If only the outer shape of the restriction, rather than the whole restriction, is needed, then the
faster method restriction_outer_shape() is preferred. Similarly if only the skew shape is needed,
use restriction_shape().

EXAMPLES:

sage: SkewTableau([[None,1],[2],[3]]).restrict(2)
[[None, 1], [2]]
sage: SkewTableau([[None,1],[2],[3]]).restrict(1)
[[None, 1]]
sage: SkewTableau([[None,1],[1],[2]]).restrict(1)
[[None, 1], [1]]

restriction_outer_shape(n)

Return the outer shape of the restriction of the semistandard skew tableau self to n.

If T is a semistandard skew tableau and n is a nonnegative integer, then the restriction of T to n is defined
as the (semistandard) skew tableau obtained by removing all cells filled with entries greater than n from T.
This method computes merely the outer shape of the restriction. For the restriction itself, use `restrict()`.

**EXAMPLES:**

```python
sage: SkewTableau([[None, None], [2, 3], [3, 4]]).restriction_outer_shape(3)
[2, 2, 1]
sage: SkewTableau([[None, 2], [None], [4], [5]]).restriction_outer_shape(2)
[2, 1]
sage: T = SkewTableau([[None, None, 3, 5], [None, 4, 4], [17]])
sage: T.restriction_outer_shape(0)
[2, 1]
sage: T.restriction_outer_shape(2)
[2, 1]
sage: T.restriction_outer_shape(3)
[3, 1]
sage: T.restriction_outer_shape(4)
[3, 3]
sage: T.restriction_outer_shape(19)
[4, 3, 1]
```

**restriction_shape**

Return the skew shape of the restriction of the semistandard skew tableau `self` to `n`.

If `T` is a semistandard skew tableau and `n` is a nonnegative integer, then the restriction of `T` to `n` is defined as the (semistandard) skew tableau obtained by removing all cells filled with entries greater than `n` from `T`.

This method computes merely the skew shape of the restriction. For the restriction itself, use `restrict()`.

**EXAMPLES:**

```python
sage: SkewTableau([[None, None], [2, 3], [3, 4]]).restriction_shape(3)
sage: SkewTableau([[None, 2], [None], [4], [5]]).restriction_shape(2)
[2, 1] / [1, 1]
sage: T = SkewTableau([[None, None, 3, 5], [None, 4, 4], [17]])
sage: T.restriction_shape(0)
[2, 1] / [2, 1]
sage: T.restriction_shape(2)
[2, 1] / [2, 1]
sage: T.restriction_shape(3)
[3, 1] / [2, 1]
sage: T.restriction_shape(4)
[3, 3] / [2, 1]
```

**reverse_slide** *(corner=None)*

Apply a backward jeu de taquin slide on the specified outside `corner` of `self`.

Backward jeu de taquin slides are defined in Section 3.7 of [Sag2001].
EXAMPLES:

```python
sage: T = SkewTableaux([[2, 2], [4, 4], [5]])
sage: Tableaux.options.display='array'
sage: Q = T.backward_slide(); Q
[ . 2 2 ]
[ . 4 4 ]
[ . 5 ]
sage: Q.backward_slide((1, 2))
[ . 2 2 ]
[ . 4 4 ]
[ . 5 ]
sage: Q.reverse_slide((1, 2)) == Q.backward_slide((1, 2))
True
```

```python
sage: T = SkewTableaux([[1, 3],[3],[5]]); T
[1 3]
[3]
[5]
sage: T.reverse_slide((1,1))
[ . 1 ]
[ . 3 3 ]
[ . 5 ]
```

**shape()**

Return the shape of self.

EXAMPLES:

```python
sage: SkewTableau([[None,1,2],[None,3],[4]]).shape()
[3, 2, 1] / [1, 1]
```

**shuffle(t2)**

Shuffle the standard tableaux self and t2.

Let t1 = self. The shape of t2 must extend the shape of t1, that is, self.outer_shape() == t2.inner_shape(). Then this function computes the pair of tableaux (t2_new, t1_new) obtained by using jeu de taquin slides to move the boxes of t2 behind the boxes of self.

The entries of t2_new are obtained by performing successive inwards jeu de taquin slides on t2 in the order indicated by the entries of t1, from largest to smallest. The entries of t1 then slide outwards one by one and land in the squares vacated successively by t2, forming t1_new.

**Note:** Equivalently, the entries of t1_new are obtained by performing outer jeu de taquin slides on t1 in the order indicated by the entries of t2, from smallest to largest. In this case the entries of t2 slide backwards and fill the squares successively vacated by t1 and so form t2_new. (This is not how the algorithm is implemented.)

**INPUT:**

- `self, t2` – a pair of standard SkewTableaux with `self.outer_shape() == t2.inner_shape()`

**OUTPUT:**

- `t2_new, t1_new` – a pair of standard `SkewTableaux` with `t2_new.outer_shape() == t1_new.inner_shape()`
EXAMPLES:

```python
sage: t1 = SkewTableau([[None, 1, 2], [3, 4]])
sage: t2 = SkewTableau([[None, None, None, 3], [None, None, 4], [1, 2, 5]])
sage: (t2_new, t1_new) = t1.shuffle(t2)
sage: t1_new
[[None, None, None, 2], [None, None, 1], [None, 3, 4]]
sage: t2_new
[[None, 2, 3], [1, 4], [5]]
sage: t1_new.outer_shape() == t2.outer_shape()
True
sage: t2_new.inner_shape() == t1.inner_shape()
True
```

Shuffling is an involution:

```python
sage: t1 = SkewTableau([[None, 1, 2], [3, 4]])
sage: t2 = SkewTableau([[None, None, None, 3], [None, None, 4], [1, 2, 5]])
sage: sh = lambda x,y : x.shuffle(y)
sage: (t1, t2) == sh(sh(t1, t2))
True
```

Both tableaux must be standard:

```python
sage: t1 = SkewTableau([[None, 1, 2], [2, 4]])
sage: t2 = SkewTableau([[None, None, None, 3], [None, None, 4], [1, 2, 5]])
sage: t1.shuffle(t2)
Traceback (most recent call last):
 ... ValueError: the tableaux must be standard
sage: t1 = SkewTableau([[None, 1, 2], [3, 4]])
sage: t2 = SkewTableau([[None, None, None, 3], [None, None, 4], [1, 2, 6]])
sage: t1.shuffle(t2)
Traceback (most recent call last):
 ... ValueError: the tableaux must be standard
```

The shapes (not just the nonempty cells) must be adjacent:

```python
sage: t1 = SkewTableau([[None, None, None], [1]])
sage: t2 = SkewTableau([[None], [None], [1]])
sage: t1.shuffle(t2)
Traceback (most recent call last):
 ... ValueError: the shapes must be adjacent
```

**size()**

Return the number of cells in `self`.

**EXAMPLES:**

```python
sage: SkewTableau([[None, None, None, 3, 1]]).size()
4
sage: SkewTableau([[None, 2, [1, 3]]]).size()
3
```
**slide**(corner=None, return_vacated=False)

Apply a jeu de taquin slide to self on the specified inner corner and return the resulting tableau.

If no corner is given, the topmost inner corner is chosen.

The optional parameter return_vacated=True causes the output to be the pair (t, (i, j)) where t is the new tableau and (i, j) are the coordinates of the vacated square.


**EXAMPLES:**

```python
sage: st = SkewTableau([[None, None, None, None, 2], [None, 2, 4, 6], [2, 3, 5, 5]])
sage: st.slide((2, 0))
[[None, None, None, None, 2], [None, 2, 4, 6], [2, 2, 4, 4], [3, 5, 6], [5]]
sage: st2 = SkewTableau([[None, None, 3], [None, 2, 4], [1, 5]])
sage: st2.slide((1, 0), True)
([[None, None, 3], [1, 2, 4], [5]], (2, 1))
```

**standardization**(check=True)

Return the standardization of self, assuming self is a semistandard skew tableau.

The standardization of a semistandard skew tableau $T$ is the standard skew tableau $st(T)$ of the same shape as $T$ whose reversed reading word is the standardization of the reversed reading word of $T$.

The standardization of a word $w$ can be formed by replacing all 1’s in $w$ by 1, 2, ..., $k_1$ from left to right, all 2’s in $w$ by $k_1 + 1, k_1 + 2, ..., k_2$, and repeating for all letters that appear in $w$. See also Word.standard_permutation().

**INPUT:**

- check – (Default: True) Check to make sure self is semistandard. Set to False to avoid this check.

**EXAMPLES:**

```python
sage: t = SkewTableau([[None, None, 3, 4, 7, 19], [None, 4, 4, 8], [None, 5, 16, 17], [None], [2], [3]])
sage: t.standardization()
[[None, None, 3, 6, 8, 12], [None, 4, 5, 9], [None, 7, 10, 11], [None], [1], [2]]
```

Standard skew tableaux are fixed under standardization:

```python
sage: p = Partition([4, 3, 3, 2])
sage: q = Partitions(3).random_element()
sage: all((t == t.standardization() for t in StandardSkewTableaux([p, q])))
True
```

The reading word of the standardization is the standardization of the reading word:

```python
sage: t = SkewTableau([[None, 3, 4, 6], [None, 6, 10], [7, 7, 11], [18]])
sage: t.to_word().standard_permutation() == t.standardization().to_permutation()
True
```

**to_chain**(max_entry=None)

Return the chain of partitions corresponding to the (semi)standard skew tableau self.
The optional keyword parameter `max_entry` can be used to customize the length of the chain. Specifically, if this parameter is set to a nonnegative integer \( n \), then the chain is constructed from the positions of the letters \( 1, 2, \ldots, n \) in the tableau.

**EXAMPLES:**

```python
sage: SkewTableau([[None,1],[2],[3]]).to_chain()
[[1], [2, 1], [2, 1, 1]]
```

```python
sage: SkewTableau([[None,1],[1],[2]]).to_chain()
[[1], [2, 1], [2, 1, 1]]
```

```python
sage: SkewTableau([[None,1],[1],[2]]).to_chain(max_entry=2)
[[1], [2, 1], [2, 1, 1]]
```

```python
sage: SkewTableau([[None,1],[1],[2]]).to_chain(max_entry=3)
[[1], [2, 1], [2, 1, 1], [2, 1, 1]]
```

```python
sage: SkewTableau([[None,1],[1],[2]]).to_chain(max_entry=1)
[[1], [2, 1]]
```

```python
sage: SkewTableau([[None,None,2],[None,3],[None,5]]).to_chain(max_entry=6)
[[2, 1, 1], [2, 1, 1], [3, 1, 1], [3, 2, 1], [3, 2, 1], [3, 2, 2], [3, 2, 2]]
```

```python
sage: SkewTableau([]).to_chain()
[[], []]
```

```python
sage: SkewTableau([]).to_chain(max_entry=1)
[[], []]
```

**to_expr()**

The first list in a result corresponds to the inner partition of the skew shape. The second list is a list of the rows in the skew tableau read from the bottom up.

Provided for compatibility with MuPAD-Combinat. In MuPAD-Combinat, if \( t \) is a skew tableau, then `to_expr(t)` gives the same result as `expr(t)` would give in MuPAD-Combinat.

**EXAMPLES:**

```python
sage: SkewTableau([[None,1,1,3],[None,2,2],[1]]).to_expr()
[[1, 1], [[1], [2, 2], [1, 1, 3]]
```

```python
sage: SkewTableau([]).to_expr()
[[], []]
```

**to_list()**

Return a (mutable) list representation of `self`.

**EXAMPLES:**

```python
sage: stlist = [[None, None, 3], [None, 1, 3], [2, 2]]
sage: st = SkewTableau(stlist)
sage: st.to_list()
[[None, None, 3], [None, 1, 3], [2, 2]]
sage: st.to_list() == stlist
True
```

**to_permutation()**

Return a permutation with the entries of `self` obtained by reading `self` row by row, from the bottommost to the topmost row, with each row being read from left to right, in English convention. See `to_word_by_row()`.

**EXAMPLES:**
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```python
sage: SkewTableau([[None,2],[3,4],[None],[1]]).to_permutation()
[1, 3, 4, 2]
sage: SkewTableau([[None,2],[None,4],[1],[3]]).to_permutation()
[3, 1, 4, 2]
sage: SkewTableau([[None]]).to_permutation()
[]
```

to_ribbon(check_input=True)

Return `self` as a ribbon-shaped tableau (RibbonShapedTableau), provided that the shape of `self` is a ribbon.

INPUT:

- `check_input` – (default: True) whether or not to check that `self` indeed has ribbon shape

EXAMPLES:

```python
sage: SkewTableau([[None,1],[2,3]]).to_ribbon()
[[None, 1], [2, 3]]
```

to_tableau()

Returns a tableau with the same filling. This only works if the inner shape of the skew tableau has size zero.

EXAMPLES:

```python
sage: SkewTableau([[1,2],[3,4]]).to_tableau()
[[1, 2], [3, 4]]
```

to_word()

Return a word obtained from a row reading of `self`.

This is the word obtained by concatenating the rows from the bottommost one (in English notation) to the topmost one.

EXAMPLES:

```python
sage: s = SkewTableau([[None,1],[2,3]])
sage: s.pp()
1
2 3
sage: s.to_word_by_row()
word: 231
sage: s = SkewTableau([[None, 2, 4], [None, 3], [1]])
sage: s.pp()
2 4
3
1
sage: s.to_word_by_row()
word: 1324
```

to_word_by_column()

Return the word obtained from a column reading of the skew tableau.

This is the word obtained by concatenating the columns from the rightmost one (in English notation) to the leftmost one.

EXAMPLES:
sage: s = SkewTableau([[None,1],[2,3]])
sage: s.pp()
   . 1
   2 3
sage: s.to_word_by_column()
word: 132

sage: s = SkewTableau([[None, 2, 4], [None, 3], [1]])
sage: s.pp()
   . 2 4
   . 3
   1
sage: s.to_word_by_column()
word: 4231

to_word_by_row()

Return a word obtained from a row reading of self.

This is the word obtained by concatenating the rows from the bottommost one (in English notation) to the topmost one.

EXAMPLES:

sage: s = SkewTableau([[None,1],[2,3]])
sage: s.pp()
   . 1
   2 3
sage: s.to_word_by_row()
word: 231
sage: s = SkewTableau([[None, 2, 4], [None, 3], [1]])
sage: s.pp()
   . 2 4
   . 3
   1
sage: s.to_word_by_row()
word: 1324

weight()

Return the weight (aka evaluation) of the tableau self. Trailing zeroes are omitted when returning the weight.

The weight of a skew tableau $T$ is the sequence $(a_1, a_2, a_3, \ldots)$, where $a_k$ is the number of entries of $T$ equal to $k$. This sequence contains only finitely many nonzero entries.

The weight of a skew tableau $T$ is the same as the weight of the reading word of $T$, for any reading order. evaluation() is a synonym for this method.

EXAMPLES:

sage: SkewTableau([[1,2],[3,4]]).weight()
[1, 1, 1, 1]
sage: SkewTableau([[None,2],[None,4],[None,5],[None]]).weight()
[0, 1, 0, 1, 1]

(continues on next page)
class sage.combinat.skew_tableau.SkewTableau_class(parent, st)

Bases: SkewTableau

This exists solely for unpickling SkewTableau_class objects.

class sage.combinat.skew_tableau.SkewTableaux(category=None)

Bases: UniqueRepresentation, Parent

Class of all skew tableaux.

Element

alias of SkewTableau

from_chain(chain)

Return the tableau corresponding to the chain of partitions.

EXAMPLES:

sage: SkewTableaux().from_chain([[1,1],[2,1],[3,1],[3,2],[3,3],[3,3,1]])
[[None, 1, 2], [None, 3, 4], [5]]

from_expr(expr)

Return a SkewTableau from a MuPAD-Combinat expr for a skew tableau.

The first list in expr is the inner shape of the skew tableau. The second list are the entries in the rows of
the skew tableau from bottom to top.

Provided primarily for compatibility with MuPAD-Combinat.

EXAMPLES:

sage: SkewTableaux().from_expr([[1,1],[[5],[3,4],[1,2]])
[[None, 1, 2], [None, 3, 4], [5]]

from_shape_and_word(shape, word)

Return the skew tableau corresponding to the skew partition shape and the word word obtained from the row
reading.

EXAMPLES:

sage: t = SkewTableau([[None, 1, 3], [None, 2], [4]])
sage: shape = t.shape()
sage: word = t.to_word()
sage: SkewTableaux().from_shape_and_word(shape, word)
[[None, 1, 3], [None, 2], [4]]
options = Current options for Tableaux - ascii_art: repr - convention: English - display: list - latex: diagram

class sage.combinat.skew_tableau.StandardSkewTableaux(category=None)

Bases: SkewTableaux

Standard skew tableaux.

EXAMPLES:

sage: S = StandardSkewTableaux(); S
Standard skew tableaux
sage: S.cardinality()
+Infinity

sage: S = StandardSkewTableaux(2); S
Standard skew tableaux of size 2
sage: S.cardinality()
4

sage: StandardSkewTableaux([[3, 2, 1], [1, 1]]).list()
[[None, 1, 2], [None, 2], [3]],
[[None, 1, 2], [None, 4], [3]],
[[None, 1, 3], [None, 4], [2]],
[[None, 1, 4], [None, 3], [2]],
[[None, 1, 4], [None, 2], [3]],
[[None, 1, 3], [None, 2], [4]],
[[None, 2, 4], [None, 3], [1]]

class sage.combinat.skew_tableau.StandardSkewTableaux_all

Bases: StandardSkewTableaux

Class of all standard skew tableaux.

class sage.combinat.skew_tableau.StandardSkewTableaux_shape(skp)

Bases: StandardSkewTableaux

Standard skew tableaux of a fixed skew shape $\lambda/\mu$.

cardinality()

Return the number of standard skew tableaux with shape of the skew partition skp. This uses a formula due to Aitken (see Cor. 7.16.3 of [Sta-EC2]).

EXAMPLES:

sage: StandardSkewTableaux([[3, 2, 1], [1, 1]]).cardinality()
8

class sage.combinat.skew_tableau.StandardSkewTableaux_size(n)

Bases: StandardSkewTableaux

Standard skew tableaux of a fixed size $n$.

cardinality()

EXAMPLES:
5.1.312 Functions that compute some of the sequences in Sloane’s tables

EXAMPLES:
Type `sloane.[tab]` to see a list of the sequences that are defined.

```sage
sage: a = sloane.A000005; a
The integer sequence tau(n), which is the number of divisors of n.
sage: a(1)
1
sage: a(6)
4
sage: a(100)
9
```

Type `d._eval??` to see how the function that computes an individual term of the sequence is implemented.

The input must be a positive integer:

```sage
sage: a(0)
Traceback (most recent call last):
  ...  
ValueError: input n (=0) must be a positive integer
sage: a(1/3)
Traceback (most recent call last):
  ...  
TypeError: input must be an int or Integer
```

You can also change how a sequence prints:

```sage
sage: a = sloane.A000005; a
The integer sequence tau(n), which is the number of divisors of n.
sage: a.rename('(..., tau(n), ...)')
sage: a
(...) tau(n), ...
sage: a.reset_name()
sage: a
The integer sequence tau(n), which is the number of divisors of n.
```

See also:

- If you want to get more informations relative to a sequence (references, links, examples, programs, ...), you can use the On-Line Encyclopedia of Integer Sequences provided by the `OEIS` module.
- If you plan to do a lot of automatic searches for subsequences, you should consider installing `SloaneEncyclopedia`, a local partial copy of the OEIS.
AUTHORS:

• William Stein: framework
• Jaap Spies: most sequences
• Nick Alexander: updated framework

class sage.combinat.sloane_functions.A000001

Bases: SloaneSequence

Number of groups of order \( n \).

INPUT:

• \( n \) – positive integer

OUTPUT: integer

EXAMPLES:

```python
sage: a = sloane.A000001; a
Number of groups of order n.
sage: a(0)
Traceback (most recent call last):
  ... ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(2)
1
sage: a(9)
2
sage: a.list(16)
[1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14]
sage: a(60)
13
```

AUTHORS:

• Jaap Spies (2007-02-04)

class sage.combinat.sloane_functions.A000004

Bases: SloaneSequence

The zero sequence.

INPUT:

• \( n \) – non negative integer

EXAMPLES:

```python
sage: a = sloane.A000004; a
The zero sequence.
sage: a(1)
0
sage: a(2007)
0
sage: a.list(12)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```
AUTHORS:
- Jaap Spies (2006-12-10)

class sage.combinat.sloane_functions.A000005
    Bases: SloaneSequence

The sequence \( \tau(n) \), which is the number of divisors of \( n \).
This sequence is also denoted \( d(n) \) (also called \( \tau(n) \) or \( \sigma_0(n) \)), the number of divisors of \( n \).

INPUT:
- \( n \) - positive integer

EXAMPLES:

```sage
d = sloane.A000005; d
The integer sequence tau(n), which is the number of divisors of n.
sage: d(1)
1
sage: d(6)
4
sage: d(51)
4
sage: d(100)
9
sage: d(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
```

```
sage: d.list(10)
[1, 2, 2, 3, 2, 4, 2, 4, 3, 4]
```

AUTHORS:
- Jaap Spies (2006-12-10)
- William Stein (2007-01-08)

class sage.combinat.sloane_functions.A000007
    Bases: SloaneSequence

The characteristic function of 0: \( a(n) = 0^n \).

INPUT:
- \( n \) – non negative integer

OUTPUT:
- \( \text{integer} \) – function value

EXAMPLES:

```sage
a = sloane.A000007;a
The characteristic function of 0: a(n) = 0^n.
sage: a(0)
1
sage: a(2)
0
```

(continues on next page)
AUTHORS:

• Jaap Spies (2007-01-12)

class sage.combinat.sloane_functions.A000008

Bases: SloaneSequence

Number of ways of making change for n cents using coins of 1, 2, 5, 10 cents.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A000008;a
Number of ways of making change for n cents using coins of 1, 2, 5, 10 cents.
sage: a(0)
1
sage: a(1)
1
sage: a(13)
16
sage: a.list(10)
[1, 1, 2, 2, 3, 4, 5, 6, 7, 8, 11, 12, 15, 16]
```

AUTHOR:

• J. Gaski (2009-05-29)

class sage.combinat.sloane_functions.A000009

Bases: SloaneSequence

Number of partitions of n into odd parts.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A000009;a
Number of partitions of n into odd parts.
sage: a(0)
1
sage: a(1)
1
```

(continues on next page)
AUTHOR:

- Jaap Spies (2007-01-30)

cf()

EXAMPLES:

```
sage: it = sloane.A000009.cf()
sage: [next(it) for i in range(14)]
[1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 18]
```

list(n)

EXAMPLES:

```
sage: sloane.A000009.list(14)
[1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 18]
```

class sage.combinat.sloane_functions.A000010

Bases: SloaneSequence

The integer sequence A000010 is Euler’s totient function.

Number of positive integers $i < n$ that are relative prime to $n$. Number of totatives of $n$.

Euler totient function $\phi(n)$: count numbers $n$ and prime to $n$. euler_phi is a standard Sage function implemented in PARI

INPUT:

- $n$ – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```
sage: a = sloane.A000010; a
Euler's totient function
sage: a(1)
1
sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer
sage: a(11)
10
sage: a.list(12)
[1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4]
sage: a(1/3)
Traceback (most recent call last):
```
...  
TypeError: input must be an int or Integer

AUTHORS:

• Jaap Spies (2007-01-12)

class sage.combinat.sloane_functions.A000012
Bases: SloaneSequence
The all 1's sequence.
INPUT:
  • n – non negative integer
OUTPUT:
  • integer – function value
EXAMPLES:

```
sage: a = sloane.A000012; a
The all 1's sequence.
sage: a(1)
1
sage: a(2007)
1
sage: a.list(12)
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

AUTHORS:

• Jaap Spies (2007-01-12)

class sage.combinat.sloane_functions.A000015
Bases: SloaneSequence
Smallest prime power ≥ n (where 1 is considered a prime power).
INPUT:
  • n – non negative integer
OUTPUT:
  • integer – function value
EXAMPLES:

```
sage: a = sloane.A000015; a
Smallest prime power >= n.
sage: a(1)
1
sage: a(8)
8
sage: a(305)
307
sage: a(-4)
Traceback (most recent call last):
```

(continues on next page)
ValueError: input n (=4) must be a positive integer

```
sage: a.list(12)
[1, 2, 3, 4, 5, 7, 7, 8, 9, 11, 11, 13]
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
```

AUTHORS:

• Jaap Spies (2007-01-18)

class sage.combinat.sloane_functions.A000016
Bases: SloaneSequence

Sloane's A000016

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
Sloane's A000016.
sage: a = sloane.A000016; a
  Sloane's A000016.
sage: a(1)
  1
sage: a(0)
  1
sage: a(8)
  16
sage: a(75)
  251859545753048193000
sage: a(-4)
Traceback (most recent call last):
...
ValueError: input n (=4) must be an integer >= 0
sage: a.list(12)
[1, 1, 1, 2, 2, 4, 6, 10, 16, 30, 52, 94]
```

AUTHORS:

• Jaap Spies (2007-01-18)

class sage.combinat.sloane_functions.A000027
Bases: SloaneSequence

The natural numbers. Also called the whole numbers, the counting numbers or the positive integers.

The following examples are tests of SloaneSequence more than A000027.

EXAMPLES:
The natural numbers.

Index \( n \) is interpreted as \_eval(\( n \)):

\[
\text{sage: } s[10] \\text{\textbar} 10
\]

Slices are interpreted with absolute offsets, so the following returns the terms of the sequence up to but not including the third term:

\[
\text{sage: } s[:3] \\text{\textbar} [1, 2] \\
\text{sage: } s.list(5) \\text{\textbar} [1, 2, 3, 4, 5]
\]

\[
\text{link = 'http://oeis.org/classic/A000027'}
\]

```python
class sage.combinat.sloane_functions.A000030
    Bases: SloaneSequence
    Initial digit of \( n \).

    INPUT:
    • \( n \) – non negative integer

    OUTPUT:
    • integer – function value

    EXAMPLES:
```

\[
\text{sage: } a = \text{sloane.A000030}; a \\
\text{Initial digit of } n \\
\text{sage: } a(0) \\text{\textbar} 0 \\
\text{sage: } a(1) \\text{\textbar} 1 \\
\text{sage: } a(8) \\text{\textbar} 8 \\
\text{sage: } a(454) \\text{\textbar} 4 \\
\text{sage: } a(-4) \\
\text{Traceback (most recent call last):} \\
\text{...} \\
\text{ValueError: input } n (-4) \text{ must be an integer } \geq 0 \\
\text{sage: } a.list(12) \\text{\textbar} [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 1]
\]

```

AUTHORS:

• Jaap Spies (2007-01-18)
class sage.combinat.sloane_functions.A000032

Bases: SloaneSequence

Lucas numbers (beginning at 2): \( L(n) = L(n-1) + L(n-2) \).

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A000032; a
Lucas numbers (beginning at 2): \( L(n) = L(n-1) + L(n-2) \).
sage: a(0)
2
sage: a(1)
1
sage: a(8)
47
sage: a(200)
627376215338105766356982006981782561278127
sage: a(-4)
Traceback (most recent call last):
... ValueError: input \( n \) (=-4) must be an integer >= 0
sage: a.list(12)
[2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199]

AUTHORS:

• Jaap Spies (2007-01-18)

class sage.combinat.sloane_functions.A000035

Bases: SloaneSequence

A simple periodic sequence.

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A000035; a
A simple periodic sequence.
sage: a(0.0)
Traceback (most recent call last):
... TypeError: input must be an int or Integer
sage: a(1)
1
sage: a(2)
AUTHORS:

• Jaap Spies (2007-02-02)

class sage.combinat.sloane_functions.A000040

Bases: SloaneSequence

The prime numbers.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A000040; a
The prime numbers.

sage: a(1)
2

sage: a(8)
19

sage: a(305)
2011

sage: a.list(12)
[2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37]

sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer

AUTHORS:

• Jaap Spies (2007-01-17)

class sage.combinat.sloane_functions.A000041

Bases: SloaneSequence

\(a(n) = \) number of partitions of \(n\) (the partition numbers).

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:
sage: a = sloane.A000041;a
a(n) = number of partitions of n (the partition numbers).
sage: a(0)
1
sage: a(2)
2
sage: a(8)
22
sage: a(200)
3972999029388
sage: a.list(9)
[1, 1, 2, 3, 5, 7, 11, 15, 22]

AUTHORS:
• Jaap Spies (2007-01-18)

class sage.combinat.sloane_functions.A000043
Bases: SloaneSequence

Primes p such that $2^p - 1$ is prime. $2^p - 1$ is then called a Mersenne prime.

INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

sage: a = sloane.A000043;a
Primes p such that $2^p - 1$ is prime. $2^p - 1$ is then called a Mersenne prime.
sage: a(1)
2
sage: a(2)
3
sage: a(39)
13466917
sage: a(40)
Traceback (most recent call last):
...
IndexError: list index out of range
sage: a.list(12)
[2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127]

AUTHORS:
• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000045
Bases: SloaneSequence

Sequence of Fibonacci numbers, offset 0,4.

REFERENCES:
• S. Plouffe, Project Gutenberg, The First 1001 Fibonacci Numbers, http://ibiblio.org/pub/docs/books/gutenberg/etext01/fbncc10.txt
We have one more. Our first Fibonacci number is 0.

INPUT:
- \( n \) – non negative integer

OUTPUT:
- \( \text{integer} \) – function value

EXAMPLES:

```
sage: a = sloane.A000045; a
Fibonacci numbers with index \( n \geq 0 \)
sage: a(0)
0
sage: a(1)
1
sage: a.list(12)
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
sage: a(1/3)
Traceback (most recent call last):
...
TypeError: input must be an int or Integer
```

AUTHORS:
- Jaap Spies (2007-01-13)

```
fib()

Returns a generator over all Fibonacci numbers, starting with 0.

EXAMPLES:

```
sage: it = sloane.A000045.fib()
sage: [next(it) for i in range(10)]
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
```

```
list(n)

EXAMPLES:

```
sage: sloane.A000045.list(10)
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
```
sage: a = sloane.A000069; a
Odious numbers: odd number of 1's in binary expansion.
sage: a(0)
1
sage: a(2)
4
sage: a.list(9)
[1, 2, 4, 7, 8, 11, 13, 14, 16]

AUTHORS:

• Jaap Spies (2007-02-02)

class sage.combinat.sloane_functions.A000073

Bases: SloaneSequence

Tribonacci numbers: \(a(n) = a(n-1) + a(n-2) + a(n-3)\). Starting with 0, 0, 1, ...

INPUT:

• \(n\) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A000073; a
Tribonacci numbers: \(a(n) = a(n-1) + a(n-2) + a(n-3)\).
sage: a(0)
0
sage: a(1)
0
sage: a(2)
1
sage: a(11)
149
sage: a.list(12)
[0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149]

AUTHORS:

• Jaap Spies (2007-01-19)

list(n)

EXAMPLES:

sage: sloane.A000073.list(10)
[0, 0, 1, 1, 2, 4, 7, 13, 24, 44]

class sage.combinat.sloane_functions.A000079

Bases: SloaneSequence

Powers of 2: \(a(n) = 2^n\).

INPUT:

• \(n\) – non negative integer
OUTPUT:

- integer – function value

EXAMPLES:

```sage
sage: a = sloane.A000079;a
Powers of 2: a(n) = 2^n.
sage: a(0)
1
sage: a(2)
4
sage: a(8)
256
sage: a(100)
1267650600228229401496703205376
sage: a.list(9)
[1, 2, 4, 8, 16, 32, 64, 128, 256]
```

AUTHORS:

- Jaap Spies (2007-01-18)

```python
class sage.combinat.sloane_functions.A000085
```

Bases: `SloaneSequence`

Number of self-inverse permutations on \( n \) letters, also known as involutions; number of Young tableaux with \( n \) cells.

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```sage
sage: a = sloane.A000085;a
Number of self-inverse permutations on n letters.
sage: a(0)
1
sage: a(1)
1
sage: a(2)
2
sage: a(12)
140152
sage: a.list(13)
[1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048]
```

AUTHORS:

- Jaap Spies (2007-02-03)

```python
class sage.combinat.sloane_functions.A000100
```

Bases: `SloaneSequence`

INPUT:


• **n** – non negative integer

**OUTPUT:**

• **integer** – function value

**EXAMPLES:**

```python
sage: a = sloane.A000100;a
Number of compositions of n in which the maximum part size is 3.
sage: a(0)
0
sage: a(1)
0
sage: a(2)
0
sage: a(3)
1
sage: a(11)
360
sage: a.list(12)
[0, 0, 0, 1, 2, 5, 11, 23, 47, 94, 185, 360]
```

**AUTHORS:**

• Jaap Spies (2007-01-26)

```
class sage.combinat.sloane_functions.A000108
```

**class** sage.combinat.sloane_functions.A000108

**Bases:** SloaneSequence

Catalan numbers: $C_n = \binom{2n}{n}/(n+1)$, also called Segner numbers.

**INPUT:**

• **n** – non negative integer

**OUTPUT:**

• **integer** – function value

**EXAMPLES:**

```python
sage: a = sloane.A000108;a
Catalan numbers: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). Also called Segner numbers.
sage: a(0)
1
sage: a.offset
0
sage: a(8)
1430
sage: a(40)
2622127042276492108820
sage: a.list(9)
[1, 1, 2, 5, 14, 42, 132, 429, 1430]
```

**AUTHORS:**
class sage.combinat.sloane_functions.A000110

Bases: ExponentialNumbers

The sequence of Bell numbers.

The Bell number $B_n$ counts the number of ways to put $n$ distinguishable things into indistinguishable boxes such that no box is empty.

Let $S(n, k)$ denote the Stirling number of the second kind. Then

$$B_n = \sum_{k=0}^{n} k^n S(n, k).$$

INPUT:

• n – non negative integer

OUTPUT:

• integer – $B_n$

EXAMPLES:

```
sage: a = sloane.A000110; a
Sequence of Bell numbers
sage: a.offset
0
sage: a(0)
1
sage: a(100)
4758591276764833658790768841387207826363669686825611466616334637559114497892442622672724044217756306953557882560751
sage: a.list(10)
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
```

AUTHORS:

• Nick Alexander

class sage.combinat.sloane_functions.A000120

Bases: SloaneSequence

1's-counting sequence: number of 1's in binary expansion of $n$.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A000120;a
1's-counting sequence: number of 1's in binary expansion of n.
sage: a(0)
0
sage: a(2)
1
sage: a(12)
(continues on next page)
sage: a.list(12)
[0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3]

AUTHORS:
• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000124
Bases: SloaneSequence
Central polygonal numbers (the Lazy Caterer’s sequence): \(n(n + 1)/2 + 1\).
Or, maximal number of pieces formed when slicing a pancake with \(n\) cuts.

INPUT:
• \(n\) – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

sage: a = sloane.A000124;a
Central polygonal numbers (the Lazy Caterer’s sequence): \(n(n+1)/2 + 1\).
sage: a(0) 1
sage: a(1) 2
sage: a(2) 4
sage: a(9) 46
sage: a.list(10)
[1, 2, 4, 7, 11, 16, 22, 29, 37, 46]

AUTHORS:
• Jaap Spies (2007-01-25)

class sage.combinat.sloane_functions.A000129
Bases: RecurrenceSequence2
Pell numbers: \(a(0) = 0, a(1) = 1;\) for \(n > 1, a(n) = 2a(n-1) + a(n-2)\).

Denominators of continued fraction convergents to \(\sqrt{2}\).
See also A001333

INPUT:
• \(n\) – non negative integer
OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A000129;a
Pell numbers: a(0) = 0, a(1) = 1; for n > 1, a(n) = 2*a(n-1) + a(n-2).
sage: a(0)
0
sage: a(2)
2
sage: a(12)
13860
sage: a.list(12)
[0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741]
```

AUTHORS:

- Jaap Spies (2007-01-25)

```python
class sage.combinat.sloane_functions.A000142
Bases: SloaneSequence

Factorial numbers: \(n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n\) (order of symmetric group \(S_n\), number of permutations of \(n\) letters).

INPUT:

- \(n\) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A000142;a
Factorial numbers: \(n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n\) (order of symmetric group \(S_n\), number of permutations of \(n\) letters).

sage: a(0)
1
sage: a(8)
40320
sage: a(40)
815915283247897734345611269596115894272000000000
sage: a.list(9)
[1, 1, 2, 6, 24, 120, 720, 5040, 40320]
```

AUTHORS:

- Jaap Spies (2007-01-12)

```python
class sage.combinat.sloane_functions.A000153
Bases: ExtremesOfPermanentsSequence

\(a(n) = n \cdot a(n-1) + (n-2) \cdot a(n-2)\), with \(a(0) = 0, a(1) = 1\).

With offset 1, permanent of \((0,1)\)-matrix of size \(n \times (n + d)\) with \(d = 2\) and \(n\) zeros not on a line. This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of \((0,1)\)-matrices, p. 201-202.
```
INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000153; a
a(n) = n*a(n-1) + (n-2)*a(n-2), with a(0) = 0, a(1) = 1.
```
```
sage: a(0)
0
sage: a(1)
1
sage: a(8)
82508
sage: a(20)
10315043624498196944
sage: a.list(8)
[0, 1, 2, 7, 32, 181, 1214, 9403]
```

AUTHORS:
• Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A000165

Bases: SloaneSequence

Double factorial numbers: \((2n)!! = 2^n \cdot n!\).

INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000165;a
Double factorial numbers: \((2n)!! = 2^n \cdot n!\).
```
```
sage: a(0)
1
sage: a.offset
0
sage: a(8)
10321920
sage: a(20)
2551082656125828464640000
sage: a.list(9)
[1, 2, 8, 48, 384, 3840, 46080, 645120, 10321920]
```

AUTHORS:
• Jaap Spies (2007-01-24)
class sage.combinat.sloane_functions.A000166
Bases: SloaneSequence

Subfactorial or rencontres numbers, or derangements: number of permutations of $n$ elements with no fixed points.
With offset 1 also the permanent of a (0,1)-matrix of order $n$ with $n$ 0's not on a line.

INPUT:
- $n$ – non negative integer

OUTPUT:
- integer – function value

EXAMPLES:

```python
sage: a = sloane.A000166;a
Subfactorial or rencontres numbers, or derangements: number of permutations of $n$ elements with no fixed points.
sage: a(0)
1
sage: a(1)
0
sage: a(2)
1
sage: a.offset
0
sage: a(8)
14833
sage: a(20)
895014631192902121
sage: a.list(9)
[1, 0, 1, 2, 9, 44, 265, 1854, 14833]
```

AUTHORS:
- Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A000169
Bases: SloaneSequence

Number of labeled rooted trees with $n$ nodes: $n^{n-1}$.

INPUT:
- $n$ – non negative integer

OUTPUT:
- integer – function value

EXAMPLES:

```python
sage: a = sloane.A000169;a
Number of labeled rooted trees with n nodes: n^(n-1).
sage: a(0)
Traceback (most recent call last):
  ... ValueError: input n (=0) must be a positive integer
sage: a(1)
```

(continues on next page)
AUTHORS:
• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000203

Bases: SloaneSequence

The sequence $\sigma(n)$, where $\sigma(n)$ is the sum of the divisors of $n$. Also called $\sigma_1(n)$.

The function $\text{sigma}(n, k)$ implements $\sigma_k(n)$ in Sage.

INPUT:
• $n$ – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000203; a
sigma(n) = sum of divisors of n. Also called sigma_1(n).
sage: a(1)
1
sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer
sage: a(256)
511
sage: a.list(12)
[1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28]
sage: a(1/3)
Traceback (most recent call last):
... TypeError: input must be an int or Integer
```

AUTHORS:
• Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A000204

Bases: SloaneSequence

Lucas numbers (beginning with 1): $L(n) = L(n - 1) + L(n - 2)$ with $L(1) = 1, L(2) = 3$.

INPUT:
• $n$ – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000204; a
Lucas numbers (beginning at 1): L(n) = L(n-1) + L(n-2), L(2) = 3.
sage: a(1)
1
sage: a(8)
47
sage: a(200)
627376215338105766356982006981782561278127
sage: a(-4)
Traceback (most recent call last):
  ... ValueError: input n (=-4) must be a positive integer
sage: a.list(12)
[1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322]
sage: a(0)
Traceback (most recent call last):
  ... ValueError: input n (=0) must be a positive integer
```

AUTHORS:

• Jaap Spies (2007-01-18)

```python
class sage.combinat.sloane_functions.A000213
Bases: SloaneSequence

Tribonacci numbers: a(n) = a(n-1) + a(n-2) + a(n-3). Starting with 1, 1, 1, ...
```

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000213;a
Tribonacci numbers: a(n) = a(n-1) + a(n-2) + a(n-3).
sage: a(0)
1
sage: a(1)
1
sage: a(2)
1
sage: a(11)
355
sage: a.list(12)
[1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355]
```

AUTHORS:

• Jaap Spies (2007-01-19)
list($n$)

EXAMPLES:

```python
sage: sloane.A000213.list(10)
[1, 1, 1, 3, 5, 9, 17, 31, 57, 105]
```

class sage.combinat.sloane_functions.A000217

Bases: SloaneSequence

Triangular numbers: $a(n) = \frac{(n+1)}{2} = \frac{n(n+1)}{2}$.

INPUT:

• $n$ – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000217;a
Triangular numbers: $a(n) = C(n+1,2) = \frac{n(n+1)}{2} = 0+1+2+...+n.$
sage: a(0)
0
sage: a(2)
3
sage: a(8)
36
sage: a(2000)
2001000
sage: a.list(9)
[0, 1, 3, 6, 10, 15, 21, 28, 36]
```

AUTHORS:

• Jaap Spies (2007-01-25)

class sage.combinat.sloane_functions.A000225

Bases: SloaneSequence

$2^n - 1$.

INPUT:

• $n$ – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000225;a
2^n - 1.
sage: a(0)
0
sage: a(-1)
Traceback (most recent call last):
...
```
ValueError: input n (=1) must be an integer >= 0
sage: a(12)
4095
sage: a.list(12)
[0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047]

AUTHORS:

• Jaap Spies (2007-01-25)

class sage.combinat.sloane_functions.A000244

Bases: SloaneSequence

Powers of 3: \(a(n) = 3^n\).

INPUT:

• \(n\) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A000244;a
Powers of 3: \(a(n) = 3^n\).
sage: a(-1)
Traceback (most recent call last):
...
ValueError: input n (=1) must be an integer >= 0
sage: a(0)
1
sage: a(3)
27
sage: a(11)
177147
sage: a.list(12)
[1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147]

AUTHORS:

• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000255

Bases: ExtremesOfPermanentsSequence

\(a(n) = n \cdot a(n - 1) + (n - 1) \cdot a(n - 2)\), with \(a(0) = 1\), \(a(1) = 1\).

With offset 1, permanent of \((0,1)\)-matrix of size \(n \times (n + d)\) with \(d = 1\) and \(n\) zeros not on a line. This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of \((0,1)\)-matrices, p. 201-202.

INPUT:

• \(n\) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:
sage: a = sloane.A000255;a
a(n) = n*a(n-1) + (n-1)*a(n-2), a(0) = 1, a(1) = 1.
sage: a(0)
1
sage: a(1)
1
sage: a.offset
0
sage: a(8)
148329
sage: a(22)
9923922230666898717143
sage: a.list(9)
[1, 1, 3, 11, 53, 309, 2119, 16687, 148329]

AUTHORS:
• Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A000261
Bases: ExtremesOfPermanentsSequence

a(n) = n * a(n - 1) + (n - 3) * a(n - 2), with a(1) = 1, a(2) = 1.
With offset 1, permanent of (0,1)-matrix of size n \times (n + d) with d = 3 and n zeros not on a line. This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of (0,1)-matrices, p. 201-202.

INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

sage: a = sloane.A000261;a
a(n) = n*a(n-1) + (n-3)*a(n-2), a(1) = 0, a(2) = 1.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
0
sage: a.offset
1
sage: a(8)
30637
sage: a(22)
1801366114380914335441
sage: a.list(9)
[0, 1, 3, 13, 71, 465, 3539, 30637, 296967]

AUTHORS:
class sage.combinat.sloane_functions.A000272
   Bases: SloaneSequence
   Number of labeled rooted trees on \( n \) nodes: \( n^{(n-2)} \).
   INPUT:
   • \( n \) – integer
   OUTPUT:
   • integer – function value

   EXAMPLES:

   sage: a = sloane.A000272;a
   Number of labeled rooted trees with \( n \) nodes: \( n^{(n-2)} \).
   sage: a(0)
   1
   sage: a(1)
   1
   sage: a(2)
   1
   sage: a(10)
   100000000
   sage: a.list(12)
   [1, 1, 1, 3, 16, 125, 1296, 16807, 262144, 4782969, 100000000, 2357947691]

   AUTHORS:
   • Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000290
   Bases: SloaneSequence
   The squares: \( a(n) = n^2 \).
   INPUT:
   • \( n \) – non negative integer
   OUTPUT:
   • integer – function value

   EXAMPLES:

   sage: a = sloane.A000290;a
   The squares: \( a(n) = n^2 \).
   sage: a(0)
   0
   sage: a(-1)
   Traceback (most recent call last):
   ...    ValueError: input \( n \) (-1) must be an integer \( \geq 0 \)
   sage: a(16)
   256
   sage: a.list(17)
   [0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256]
AUTHORS:

• Jaap Spies (2007-01-25)

class sage.combinat.sloane_functions.A000292

    Bases: SloaneSequence

Tetrahedral (or pyramidal) numbers: \( \binom{n+2}{3} = \frac{n(n+1)(n+2)}{6} \).

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```sage
sage: a = sloane.A000292;
a
Tetrahedral (or pyramidal) numbers: \( \binom{n+2}{3} = \frac{n(n+1)(n+2)}{6} \).
sage: a(0)
0
sage: a(2)
4
sage: a(11)
286
sage: a.list(12)
[0, 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286]
```

AUTHORS:

• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000302

    Bases: SloaneSequence

Powers of 4: \( a(n) = 4^n \).

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```sage
sage: a = sloane.A000302;
a
Powers of 4: \( a(n) = 4^n \).
sage: a(0)
1
sage: a(1)
4
sage: a(2)
16
sage: a(10)
1048576
sage: a.list(12)
[1, 4, 16, 64, 256, 1024, 4096, 16384, 65536, 262144, 1048576, 4194304]
```
AUTHORS:
• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000312
Bases: SloaneSequence
Number of labeled mappings from \(n\) points to themselves (endofunctions): \(n^n\).

INPUT:
• \(n\) – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```
sage: a = sloane.A000312; a
Number of labeled mappings from \(n\) points to themselves (endofunctions): \(n^n\).
sage: a(-1)
Traceback (most recent call last):
... ValueError: input \(n\) (=1) must be an integer \(\geq 0\)
sage: a(0)
1
sage: a(1)
1
sage: a(9)
387420489
sage: a.list(11)
[1, 1, 4, 27, 256, 3125, 46656, 823543, 16777216, 387420489, 10000000000]
```

AUTHORS:
• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000326
Bases: SloaneSequence
Pentagonal numbers: \(\frac{n(3n-1)}{2}\).

INPUT:
• \(n\) – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```
sage: a = sloane.A000326; a
Pentagonal numbers: \(\frac{n(3n-1)}{2}\).
sage: a(0)
0
sage: a(1)
1
sage: a(2)
5
```

(continues on next page)
AUTHORS:
  • Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000330
    Bases: SloaneSequence

Square pyramidal numbers: \(0^2 + 1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\).

INPUT:
  • n – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:

```
sage: a = sloane.A000330;a
Square pyramidal numbers: \(0^2 + 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\).
sage: a(-1)
Traceback (most recent call last):
...
ValueError: input n (=1) must be an integer >= 0
```

```
sage: a(0)
0
sage: a(3)
14
sage: a(11)
506
sage: a.list(12)
[0, 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506]
```

AUTHORS:
  • Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A000396
    Bases: SloaneSequence

Perfect numbers: equal to sum of proper divisors.

INPUT:
  • n – non negative integer

OUTPUT:
  • integer – function value
EXAMPLES:

```python
sage: a = sloane.A000396; a
Perfect numbers: equal to sum of proper divisors.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
6
sage: a(2)
28
sage: a(7)
137438691328
sage: a.list(7)
[6, 28, 496, 8128, 33550336, 8589869056, 137438691328]
```

AUTHORS:

- Jaap Spies (2007-01-25)

```python
class sage.combinat.sloane_functions.A000578
Bases: SloaneSequence
The cubes: \(a(n) = n^3\).
```

INPUT:

- \(n\) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A000578; a
The cubes: \(n^3\)
sage: a(-1)
Traceback (most recent call last):
...
ValueError: input n (= -1) must be an integer \(\geq 0\)
sage: a(0)
0
sage: a(3)
27
sage: a(11)
1331
sage: a.list(12)
[0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331]
```

AUTHORS:

- Jaap Spies (2007-01-26)

```python
class sage.combinat.sloane_functions.A000583
Bases: SloaneSequence
Fourth powers: \(a(n) = n^4\).
```

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INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```
sage: a = sloane.A000583; a
Fourth powers: n^4.
sage: a(0.0)
Traceback (most recent call last):
...
TypeError: input must be an int or Integer
sage: a(1)
1
sage: a(2)
16
sage: a(9)
6561
sage: a.list(10)
[0, 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561]
```

AUTHORS:
• Jaap Spies (2007-02-04)

```python
class sage.combinat.sloane_functions.A000583
Bases: ExponentialNumbers

The sequence of Uppuluri-Carpenter numbers.

The Uppuluri-Carpenter number $C_n$ counts the imbalance in the number of ways to put $n$ distinguishable things into an even number of indistinguishable boxes versus into an odd number of indistinguishable boxes, such that no box is empty.

Let $S(n, k)$ denote the Stirling number of the second kind. Then

$$C_n = \sum_{k=0}^{n} k = 0^n(-1)^k S(n, k).$$

INPUT:
• n – non negative integer

OUTPUT:
• integer – $C_n$

EXAMPLES:
```
sage: a = sloane.A000583; a
Sequence of Uppuluri-Carpenter numbers
sage: a.offset
0
sage: a(0)
1
sage: a(100)
397577026456518507969762382254187048845620355238545130875069912944235105204434466095866237103212454552161
(continues on next page)```
AUTHORS:
• Nick Alexander

```python
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
Warning: a(39) has 4,053,946 digits!
```

INPUT:
• $n$ – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:
```python
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a(1)
3
sage: a(2)
7
sage: a(12)
170141183460469231731687303715884105727
Warning: a(39) has 4,053,946 digits!
```

```python
sage: a(40)
Traceback (most recent call last):
...  
IndexError: list index out of range
```
```python
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
a
Mersenne primes (of form $2^p - 1$ where $p$ is a prime).
(See A000043 for the values of $p$.)
sage: a = sloane.A000668;
```
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```
EXAMPLES:

```python
sage: a = sloane.A000670;a
Number of preferential arrangements of n labeled elements.
```

```python
sage: a(0)
1
sage: a(1)
1
sage: a(2)
3
sage: a(9)
7087261
sage: a.list(10)
[1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261]
```

AUTHORS:

- Jaap Spies (2007-02-03)

```python
class sage.combinat.sloane_functions.A000720
```

Bases: `SloaneSequence`

\( p_i(n) \), the number of primes \( \leq n \). Sometimes called \( \text{PrimePi}(n) \).

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A000720;a
pi(n), the number of primes \( \leq n \). Sometimes called \( \text{PrimePi}(n) \)
```

```python
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(2)
1
sage: a(8)
4
sage: a(1000)
168
sage: a.list(12)
[0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5]
```

AUTHORS:

- Jaap Spies (2007-01-25)

```python
class sage.combinat.sloane_functions.A000796
```

Bases: `SloaneSequence`

Decimal expansion of \( \pi \).

INPUT:
• \( n \) – positive integer

OUTPUT:

• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000796;a
Decimal expansion of Pi.
sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer
sage: a(1)
3
sage: a(13)
9
sage: a.list(14)
[3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7]
sage: a(100)
7
```

AUTHOR:

• Jaap Spies (2007-01-30)

```
list(n)
```

EXAMPLES:

```python
sage: sloane.A000796.list(10)
[3, 1, 4, 1, 5, 9, 2, 6, 5, 3]
```

```
pi()
```

Based on an algorithm of Lambert Meertens The ABC-programming language!!!

EXAMPLES:

```python
sage: it = sloane.A000796.pi()
sage: [next(it) for i in range(10)]
[3, 1, 4, 1, 5, 9, 2, 6, 5, 3]
```

```
class sage.combinat.sloane_functions.A000961
Bases: SloaneSequence
```

Prime powers

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```python
sage: a = sloane.A000961;a
Prime powers.
```
AUTHORS:

• Jaap Spies (2007-01-25)

EXAMPLES:

```
sage: a = sloane.A000984;a
Central binomial coefficients: C(2n,n) = (2n)!/(n!)^2
sage: a(0)
1
sage: a(2)
6
sage: a(8)
12870
sage: a.list(9)
[1, 2, 6, 20, 70, 252, 924, 3432, 12870]
```

AUTHORS:

• Jaap Spies (2007-01-26)
OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A001006;a
Motzkin numbers: number of ways of drawing any number of nonintersecting chords _
→ among n points on a circle.
sage: a(0)
1
sage: a(1)
1
sage: a(2)
2
sage: a(12)
15511
sage: a.list(13)
[1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511]
```

AUTHORS:

- Jaap Spies (2007-02-02)

```python
class sage.combinat.sloane_functions.A001006
```

Bases: `RecurrenceSequence2`

Jacobsthal sequence: \( a(n) = a(n-1) + 2a(n-2) \), \( a(0) = 0 \) and \( a(1) = 1 \).

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A001045;a
Jacobsthal sequence: \( a(n) = a(n-1) + 2a(n-2) \).
sage: a(0)
0
sage: a(1)
1
sage: a(2)
1
sage: a(11)
683
sage: a.list(12)
[0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683]
```

AUTHORS:

- Jaap Spies (2007-01-26)

```python
class sage.combinat.sloane_functions.A001045
```

Bases: `RecurrenceSequence2`

Jacobsthal sequence: \( a(n) = a(n-1) + 2a(n-2) \), \( a(0) = 0 \) and \( a(1) = 1 \).

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```python
sage: a = sloane.A001055;
Number of ways of factoring \( n \) with all factors 1.
```

AUTHORS:

- Jaap Spies (2007-01-26)

```python
class sage.combinat.sloane_functions.A001055
```

Bases: `SloaneSequence`

Number of ways of factoring \( n \) with all factors 1.
INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:
```
sage: a = sloane.A001055;a
Number of ways of factoring n with all factors >1.
sage: a(0)
Traceback (most recent call last):
  ...
ValueError: input n (=0) must be a positive integer
da(1)
1
da(2)
1
da(9)
2
da: a.list(16)
[1, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 4, 1, 2, 2, 5]
```

AUTHORS:
• Jaap Spies (2007-02-04)

\textbf{nwf}(n, m)

EXAMPLES:
```
sage: sloane.A001055.nwf(4,1)
0
sage: sloane.A001055.nwf(4,2)
1
sage: sloane.A001055.nwf(4,3)
1
sage: sloane.A001055.nwf(4,4)
2
```

\textbf{class} \texttt{sage.combinat.sloane_functions.A001109}

\textbf{Bases}: \texttt{RecurrenceSequence2}

\(a(n)^2\) is a triangular number: \(a(n) = 6 \times a(n-1) - a(n-2)\) with \(a(0) = 0, a(1) = 1\).

INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:
```
sage: a = sloane.A001109;a
a(n)^2 is a triangular number: a(n) = 6 \times a(n-1) - a(n-2) with a(0)=0, a(1)=1
sage: a(0)
```

(continues on next page)
AUTHORS:

- Jaap Spies (2007-01-24)

**class** `sage.combinat.sloane_functions.A001110`

Bases: `RecurrenceSequence`

Numbers that are both triangular and square: \( a(n) = 34a(n-1) - a(n-2) + 2 \).

**INPUT:**

- \( n \) – non negative integer

**OUTPUT:**

- integer – function value

**EXAMPLES:**

```python
sage: a = sloane.A001110; a
Numbers that are both triangular and square: a(n) = 34a(n-1) - a(n-2) + 2.
sage: a(0)
0
sage: a(1)
1
sage: a(8)
55420693056
sage: a(21)
4446390382511295358038307980025
sage: a.list(9)
[0, 1, 6, 35, 204, 1189, 6930, 40391, 235416]
```

AUTHORS:

- Jaap Spies (2007-01-19)

**g(k)**

**EXAMPLES:**

```python
sage: sloane.A001110.g(2)
2
sage: sloane.A001110.g(1)
0
```
```python
link = 'http://oeis.org/classic/A001110'

class sage.combinat.sloane_functions.A001147
Bases: SloaneSequence
Double factorial numbers: \((2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)\).

INPUT:
  • \(n\) – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:

```
sage: a = sloane.A001147;a
Double factorial numbers: (2n-1)!! = 1.3.5....(2n-1).
sage: a(0)
1
sage: a.offset
0
sage: a(8)
2027025
sage: a(20)
319830986772877770815625
sage: a.list(9)
[1, 1, 3, 15, 105, 945, 10395, 135135, 2027025]
```

AUTHORS:
  • Jaap Spies (2007-01-24)

class sage.combinat.sloane_functions.A001157
Bases: SloaneSequence
The sequence \(\sigma_2(n)\), sum of squares of divisors of \(n\).
The function sigma(n, k) implements \(\sigma_k\) in Sage.

INPUT:
  • \(n\) – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:

```
sage: a = sloane.A001157;a
sigma_2(n): sum of squares of divisors of n
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(2)
5
sage: a(8)
85
```

(continues on next page)
AUTHORS:

• Jaap Spies (2007-02-03)

class sage.combinat.sloane_functions.A001221

Bases: SloaneSequence

Number of different prime divisors of $n$

Also called omega($n$) or $\omega(n)$. Maximal number of terms in any factorization of $n$. Number of prime powers that divide $n$.

INPUT:

• $n$ – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A001221

Number of different prime divisors of $n$

sage: a(0)

Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer

sage: a(1)

0

sage: a(2)

1

sage: a(12)

140151

sage: a.list(13)

[0, 1, 3, 9, 25, 75, 231, 763, 2619, 9495, 35695, 140151, 568503]
sage: a = sloane.A001221; a
Number of distinct primes dividing n (also called omega(n)).
sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer
sage: a(1)
0
sage: a(8)
1
sage: a(41)
1
sage: a(84792)
3
sage: a.list(12)
[0, 1, 1, 1, 2, 1, 1, 1, 2, 1, 2, 1]

AUTHORS:
• Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A001222
Bases: SloaneSequence
Number of prime divisors of n (counted with multiplicity).
Also called bigomega(n) or Ω(n). Maximal number of terms in any factorization of n. Number of prime powers that divide n.
INPUT:
• n – non negative integer
OUTPUT:
• integer – function value
EXAMPLES:

sage: a = sloane.A001222; a
Number of prime divisors of n (counted with multiplicity).
sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer
sage: a(1)
0
sage: a(8)
3
sage: a(41)
1
sage: a(84792)
5
sage: a.list(12)
[0, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 3]

AUTHORS:
class sage.combinat.sloane_functions.A001227

Bases: SloaneSequence

Number of odd divisors of \( n \).

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A001227; a
Number of odd divisors of n
sage: a.offset
1
sage: a(1)
1
sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer
sage: a(100)
3
sage: a(256)
1
sage: a(29)
2
sage: a.list(20)
[1, 1, 2, 1, 2, 2, 2, 1, 3, 2, 2, 2, 2, 2, 4, 1, 2, 3, 2, 2]
sage: a(-1)
Traceback (most recent call last):
... ValueError: input n (=1) must be a positive integer
```

AUTHORS:

• Jaap Spies (2007-01-14)

class sage.combinat.sloane_functions.A001333

Bases: RecurrenceSequence2

Numerator of continued fraction convergents to \( \sqrt{2} \).

See also A000129

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:
sage: a = sloane.A001333; a
Numerators of continued fraction convergents to \sqrt{2}.

sage: a(0)
1
sage: a(1)
1
sage: a(2)
3
sage: a(3)
7
sage: a(11)
8119
sage: a.list(12)
[1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119]

AUTHORS:

- Jaap Spies (2007-02-01)

class sage.combinat.sloane_functions.A001358

Bases: SloaneSequence

Products of two primes.

These numbers have been called semiprimes (or semi-primes), bprimers or 2-almost primes.

INPUT:

- n – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

sage: a = sloane.A001358; a
Products of two primes.

sage: a(0)
Traceback (most recent call last):
... ValueError: input n (=0) must be a positive integer

sage: a(2)
6
sage: a(8)
22
sage: a(200)
669
sage: a.list(9)
[4, 6, 9, 10, 14, 15, 21, 22, 25]

AUTHORS:

- Jaap Spies (2007-01-25)

list(n)

    EXAMPLES:
class sage.combinat.sloane_functions.A001405
Bases: SloaneSequence
Central binomial coefficients: $\binom{n}{\frac{n}{2}}$.

INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

<sage>
a = sloane.A001405;a
Central binomial coefficients: C(n,floor(n/2)).
a(0)
1
a(2)
2
a(12)
924
a.list(12)
[1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462]
</sage>

AUTHORS:
• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A001477
Bases: SloaneSequence
The nonnegative integers.

INPUT:
• n – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

<sage>
a = sloane.A001477;a
The nonnegative integers.
a(-1)
Traceback (most recent call last):
...
ValueError: input n (-1) must be an integer >= 0
a(0)
0
a(3382789)
3382789
a(11)
</sage>

(continues on next page)
AUTHORS:

• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A001694

Bases: SloaneSequence

This function returns the \( n \)-th Powerful Number:

A positive integer \( n \) is powerful if for every prime \( p \) dividing \( n \), \( p^2 \) also divides \( n \).

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

\[
\begin{align*}
\text{sage: } a &= \text{sloane.A001694; a} \\
\text{Powerful Numbers (also called squarefull, square-full or 2-full numbers).} \\
\text{sage: } a.\text{offset} \\
& 1 \\
\text{sage: } a(1) \\
& 1 \\
\text{sage: } a(4) \\
& 9 \\
\text{sage: } a(100) \\
& 3136 \\
\text{sage: } a(156) \\
& 7225 \\
\text{sage: } a.\text{list}(19) \\
& [1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 72, 81, 100, 108, 121, 125, 128, 144] \\
\text{sage: } a(-1) \\
\text{Traceback (most recent call last):} \\
... \\
\text{ValueError: input n (=1) must be a positive integer}
\end{align*}
\]

AUTHORS:

• Jaap Spies (2007-01-14)

is_powerful\((n)\)

Return True if and only if \( n \) is a powerful number.

A positive integer \( n \) is powerful if for every prime \( p \) dividing \( n \), \( p^2 \) also divides \( n \).

See OEIS sequence A001694.

INPUT:

• \( n \) – integer
OUTPUT:
True if $n$ is a powerful number, else False

EXAMPLES:

```
sage: a = sloane.A001694
sage: a.is_powerful(2500)
True
sage: a.is_powerful(20)
False
```

AUTHORS:
- Jaap Spies (2006-12-07)

list($n$)

EXAMPLES:

```
sage: sloane.A001694.list(9)
[1, 4, 8, 9, 16, 25, 27, 32, 36]
```

```python
class sage.combinat.sloane_functions.A001836
    Bases: SloaneSequence

    Numbers $n$ such that $\phi(2^n - 1) < \phi(2^n)$, where $\phi$ is Euler's totient function.

    Euler's totient function is also known as euler_phi, euler_phi is a standard Sage function.

    INPUT:
    - $n$ – non negative integer

    OUTPUT:
    - integer – function value

    EXAMPLES:

    ```python
    sage: a = sloane.A001836; a
    Numbers n such that phi(2n-1) < phi(2n), where phi is Euler's totient function.
    ... A000010.
    sage: a.offset
    1
    sage: a(1)
    53
    sage: a(8)
    683
    sage: a(300)
    17798
    sage: a.list(12)
    [53, 83, 158, 263, 293, 368, 578, 683, 743, 788, 878, 893]
    sage: a(0)
    Traceback (most recent call last):
    ... ValueError: input n (=0) must be a positive integer
    ```

Compare: Searching Sloane's online database... Numbers n such that phi(2n-1) phi(2n), where phi is Euler's totient function A000010. [53, 83, 158, 263, 293, 368, 578, 683, 743, 788, 878, 893]
AUTHORS:

- Jaap Spies (2007-01-17)

list(n)

EXAMPLES:

```
sage: sloane.A001836.list(9)
[53, 83, 158, 263, 293, 368, 578, 683, 743]
```

class sage.combinat.sloane_functions.A001906

Bases: RecurrenceSequence2

\[ F(2n) = \text{bisection of Fibonacci sequence: } a(n) = 3a(n - 1) - a(n - 2). \]

INPUT:

- n – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```
sage: a = sloane.A001906; a
F(2n) = bisection of Fibonacci sequence: a(n)=3a(n-1)-a(n-2).
sage: a(0)
0
sage: a(1)
1
sage: a(8)
987
sage: a(22)
701408733
sage: a.list(12)
[0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711]
```

AUTHORS:

- Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A001909

Bases: ExtremesOfPermanentsSequence

\[ a(n) = n \cdot a(n - 1) + (n - 4) \cdot a(n - 2), \text{ with } a(2) = 0, a(3) = 1. \]

With offset 1, permanent of (0,1)-matrix of size \( n \times (n + d) \) with \( d = 4 \) and \( n \) zeros not on a line. This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of (0,1)-matrices, p. 201-202.


INPUT:

- n – positive integer >= 2

OUTPUT:

- integer – function value

EXAMPLES:
AUTHORS:

- Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A001910

Bases: ExtremesOfPermanentsSequence

\( a(n) = n * a(n-1) + (n-5) * a(n-2) \), with \( a(3) = 0, a(4) = 1 \).

With offset 1, permanent of \((0,1)\)-matrix of size \( n \times (n + d) \) with \( d = 5 \) and \( n \) zeros not on a line. This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of \((0,1)\)-matrices, p. 201-202.


INPUT:

- \( n \) – positive integer \( \geq 3 \)

OUTPUT:

- integer – function value

EXAMPLES:

\sage: a = sloane.A001910;a
\sage: a(n) = n*a(n-1) + (n-5)*a(n-2), a(3) = 0, a(4) = 1.
\sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be an integer \( \geq 3 \)
\sage: a(3)
0
\sage: a.offset
3
\sage: a(8)
1909
\sage: a(22)
9812532164110663023
\sage: a.list(9)
[0, 1, 5, 31, 227, 1909, 190435, 2203319, 2203319, 2203319, 2203319]
AUTHORS:

• Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A001969

    Bases: SloaneSequence

    Evil numbers: even number of 1’s in binary expansion.

    INPUT:
    • n – non negative integer

    OUTPUT:
    • integer – function value

    EXAMPLES:

    sage: a = sloane.A001969;a
    Evil numbers: even number of 1's in binary expansion.
    sage: a(0)
    0
    sage: a(1)
    3
    sage: a(2)
    5
    sage: a(12)
    24
    sage: a.list(13)
    [0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24]

AUTHORS:

• Jaap Spies (2007-02-02)

class sage.combinat.sloane_functions.A002110

    Bases: SloaneSequence

    Primorial numbers (first definition): product of first \( n \) primes. Sometimes written \( p \# \).

    INPUT:
    • n – non negative integer

    OUTPUT:
    • integer – function value

    EXAMPLES:

    sage: a = sloane.A002110;a
    Primorial numbers (first definition): product of first \( n \) primes. Sometimes written \( p \# \).
    sage: a(0)
    1
    sage: a(2)
    6
    sage: a(8)
    9699690
    sage: a(17)
    (continues on next page)
1922760350154212639070
`sage`: `a.list(9)`

| 1, 2, 6, 30, 210, 2310, 30030, 510510, 9699690 |

**AUTHORS:**

- Jaap Spies (2007-01-25)

```python
class sage.combinat.sloane_functions.A002113
Bases: SloaneSequence
Palindromes in base 10.

INPUT:
- `n` – non negative integer

OUTPUT:
- `integer` – function value

EXAMPLES:

```python
sage: a = sloane.A002113; a
Palindromes in base 10.
sage: a(0)
0
sage: a(1)
1
sage: a(2)
2
sage: a(12)
33
sage: a.list(13)
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33]
```
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```
Combinatorics, Release 10.0

```python
sage: a = sloane.A002275;a
Repunits: \((10^n - 1)/9\). Often denoted by \(R_n\).
sage: a(0)
0
sage: a(2)
11
sage: a(8)
11111111
sage: a(20)
11111111111111111111
sage: a.list(9)
[0, 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111]
```

AUTHORS:

• Jaap Spies (2007-01-25)

class sage.combinat.sloane_functions.A002378

Bases: SloaneSequence

Oblong (or pronic, or heteromecic) numbers: \(n(n+1)\).

INPUT:

• \(n\) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```python
sage: a = sloane.A002378;a
Oblong (or pronic, or heteromecic) numbers: \(n(n+1)\).
sage: a(-1)
Traceback (most recent call last):
...
ValueError: input n (=1) must be an integer >= 0
sage: a(0)
0
sage: a(1)
2
sage: a(11)
132
sage: a.list(12)
[0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132]
```

AUTHORS:

• Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A002620

Bases: SloaneSequence

Quarter-squares: \(\lfloor n/2\rfloor \times \lceil n/2\rceil\). Equivalently, \([n^2]/4\).

INPUT:

• \(n\) – non negative integer
OUTPUT:

- integer – function value

EXAMPLES:

```sage
a = sloane.A002620
Quarter-squares: floor(n/2)*ceiling(n/2). Equivalently, floor(n^2/4).
sage: a(0)
0
sage: a(1)
0
sage: a(2)
1
sage: a(10)
25
sage: a.list(12)
[0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30]
```

AUTHORS:

- Jaap Spies (2007-01-26)

```python
class sage.combinat.sloane_functions.A002808:

Bases: SloaneSequence

The composite numbers: numbers \(n\) of the form \(x\cdot y\) for \(x > 1\) and \(y > 1\).

INPUT:

- \(n\) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```sage
a = sloane.A002808
The composite numbers: numbers \(n\) of the form \(x\cdot y\) for \(x > 1\) and \(y > 1\).
sage: a(0)
Traceback (most recent call last):
...
ValueError: input \(n\) (=0) must be a positive integer
sage: a(2)
6
sage: a(11)
20
sage: a.list(12)
[4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21]
```

AUTHORS:

- Jaap Spies (2007-01-26)

```python
list(n)
```

EXAMPLES:

```sage
sloane.A002808.list(10)
[4, 6, 8, 9, 10, 12, 14, 15, 16, 18]
```
class sage.combinat.sloane_functions.A003418

Bases: SloaneSequence

Least common multiple (or lcm) of \(\{1, 2, \ldots, n\}\).

INPUT:

• \(n\) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```sage
a = sloane.A003418;
a
```
Least common multiple (or lcm) of \(\{1, 2, \ldots, n\}\).

```sage
a(0)
```
1

```sage
a(1)
```
1

```sage
a(13)
```
360360

```sage
a.list(14)
```
[1, 1, 2, 6, 60, 60, 420, 840, 2520, 2520, 27720, 27720, 360360]  

```sage:
a(20.0)
```
Traceback (most recent call last):
...
TypeError: input must be an int or Integer

AUTHOR:

• Jaap Spies (2007-01-31)

class sage.combinat.sloane_functions.A004086

Bases: SloaneSequence

Read \(n\) backwards (referred to as \(R(n)\) in many sequences).

INPUT:

• \(n\) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```sage
a = sloane.A004086;
a
```
Read \(n\) backwards (referred to as \(R(n)\) in many sequences).

```sage
a(0)
```
0

```sage
a(1)
```
1

```sage
a(2)
```
2

```sage
a(3333)
```
3333

```sage: a(12345)
```
(continues on next page)
AUTHORS:
  • Jaap Spies (2007-02-02)

class sage.combinat.sloane_functions.A004526
Bases: SloaneSequence
The nonnegative integers repeated.

INPUT:
  • n – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:

```
sage: a = sloane.A004526; a
The nonnegative integers repeated.
sage: a(0)
0
sage: a(1)
0
sage: a(2)
1
sage: a(10)
5
sage: a.list(12)
[0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5]
```

AUTHORS:
  • Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A005100
Bases: SloaneSequence
Deficient numbers: \(\sigma(n) < 2n\).

INPUT:
  • n – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:

```
sage: a = sloane.A005100; a
Deficient numbers: sigma(n) < 2n
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer

sage: a(1)
1
sage: a(2)
2
sage: a(12)
14
sage: a.list(12)
[1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14]

AUTHORS:

• Jaap Spies (2007-01-26)

list(n)

EXAMPLES:

sage: sloane.A005100.list(10)
[1, 2, 3, 4, 5, 7, 8, 9, 10, 11]

class sage.combinat.sloane_functions.A005101

Bases: SloaneSequence

Abundant numbers (sum of divisors of \( n \) exceeds \( 2n \)).

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A005101;
a
Abundant numbers (sum of divisors of \( n \) exceeds \( 2n \)).
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
12
sage: a(2)
18
sage: a(12)
60
sage: a.list(12)
[12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60]

AUTHORS:

• Jaap Spies (2007-01-26)

list(n)

EXAMPLES:
Combinatorics, Release 10.0

```
sage: sloane.A005101.list(10)
[12, 18, 20, 24, 30, 36, 40, 42, 48, 54]
```

```python
class sage.combinat.sloane_functions.A005117

Bases: SloaneSequence

Square-free numbers

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A005117;a
Square-free numbers.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(2)
2
sage: a(12)
17
sage: a.list(12)
[1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17]
```

AUTHORS:

• Jaap Spies (2007-01-25)

```
list(n)
```

EXAMPLES:

```
sage: sloane.A005117.list(10)
[1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15]
```

```
class sage.combinat.sloane_functions.A005408

Bases: SloaneSequence

The odd numbers a(n) = 2n + 1.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A005408;a
The odd numbers a(n) = 2n + 1.
sage: a(-1)
```
```
(continues on next page)
```
Traceback (most recent call last):
  ...
ValueError: input n (=1) must be an integer >= 0
sage: a(0)
  1
sage: a(4)
  9
sage: a(11)
  23
sage: a.list(12)
[1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23]

AUTHORS:
  • Jaap Spies (2007-01-26)

class sage.combinat.sloane_functions.A005843
Bases: SloaneSequence
The even numbers: $a(n) = 2n$.
INPUT:
  • n – non negative integer
OUTPUT:
  • integer – function value
EXAMPLES:

sage: a = sloane.A005843;a
The even numbers: a(n) = 2n.
sage: a(0.0)
Traceback (most recent call last):
  ...
TypeError: input must be an int or Integer
sage: a(1)
  2
sage: a(2)
  4
sage: a(9)
  18
sage: a.list(10)
[0, 2, 4, 6, 8, 10, 12, 14, 16, 18]

AUTHORS:
  • Jaap Spies (2007-02-03)

class sage.combinat.sloane_functions.A006318
Bases: SloaneSequence
Large Schroeder numbers.
INPUT:
  • n – non negative integer
OUTPUT:
   • integer – function value

EXAMPLES:

```
sage: a = sloane.A006318;a
Large Schroeder numbers.
sage: a(0)
1
sage: a(1)
2
sage: a(2)
6
sage: a(9)
206098
sage: a.list(10)
[1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098]
```

AUTHORS:
   • Jaap Spies (2007-02-03)

```python
class sage.combinat.sloane_functions.A006530
    Bases: SloaneSequence
    Largest prime dividing \( n \) (with \( a(1) = 1 \)).
    INPUT:
    • \( n \) – non negative integer
    OUTPUT:
    • integer – function value

EXAMPLES:

```
sage: a = sloane.A006530;a
Largest prime dividing \( n \) (with \( a(1) = 1 \)).
sage: a(0)
Traceback (most recent call last):
  ... ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(2)
2
sage: a(8)
2
sage: a(11)
11
sage: a.list(15)
[1, 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, 7, 5]
```

AUTHORS:
   • Jaap Spies (2007-01-25)
class sage.combinat.sloane_functions.A006882

    Bases: SloaneSequence
    
    Double factorials n!!: a(n) = n · a(n − 2).
    
    INPUT:
        • n – non negative integer
    
    OUTPUT:
        • integer – function value
    
    EXAMPLES:

    sage: a = sloane.A006882;a
    Double factorials n!!: a(n)=n*a(n-2).
    sage: a(0)
    1
    sage: a(2)
    2
    sage: a(8)
    384
    sage: a(20)
    3715891200
    sage: a.list(9)
    [1, 1, 2, 3, 8, 15, 48, 105, 384]

    AUTHORS:
        • Jaap Spies (2007-01-24)

df()
    
    Double factorials n!!: a(n)=n*a(n-2).
    
    EXAMPLES:

    sage: it = sloane.A006882.df()
    sage: [next(it) for i in range(10)]
    [1, 1, 2, 3, 8, 15, 48, 105, 384, 945]

list(n)
    
    EXAMPLES:

    sage: sloane.A006882.list(10)
    [1, 1, 2, 3, 8, 15, 48, 105, 384, 945]

class sage.combinat.sloane_functions.A007318

    Bases: SloaneSequence
    
    Pascal’s triangle read by rows: C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}, 0 ≤ k ≤ n.
    
    INPUT:
        • n – non negative integer
    
    OUTPUT:
        • integer – function value
    
    EXAMPLES:
AUTHORS:

• Jaap Spies (2007-01-31)

keyword = ['nonn', 'tabl', 'nice', 'easy', 'core', 'triangle']

class sage.combinat.sloane_functions.A008275

Bases: SloaneSequence

Triangle of Stirling numbers of first kind, \( s(n,k), n \geq 1, 1 \leq k \leq n \).

The unsigned numbers are also called Stirling cycle numbers:
\[ |s(n,k)| = \text{number of permutations of } n \text{ objects with exactly } k \text{ cycles.} \]

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A008275;
Triangle of Stirling numbers of first kind, \( s(n,k), n \geq 1, 1 \leq k \leq n \).
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(2)
-1
sage: a(3)
1
sage: a(11)
24
sage: a.list(12)
[1, -1, 1, 2, -3, 1, -6, 11, -6, 1, 24, -50]

AUTHORS:

• Jaap Spies (2007-02-02)

keyword = ['sign', 'tabl', 'nice', 'core', 'triangle']
Combinatorics, Release 10.0

\( s(n, k) \)

**EXAMPLES:**

```python
sage: sloane.A008275.s(4,2)
11
sage: sloane.A008275.s(5,2)
-50
sage: sloane.A008275.s(5,3)
35
```

```python
class sage.combinat.sloane_functions.A008277

Bases: SloaneSequence

Triangle of Stirling numbers of 2nd kind, \( S_2(n, k) \), \( n \geq 1 \), \( 1 \leq k \leq n \).

**INPUT:**

- \( n \) – non negative integer

**OUTPUT:**

- \( \text{integer} \) – function value

**EXAMPLES:**

```python
sage: a = sloane.A008277;a
Triangle of Stirling numbers of 2nd kind, \( S_2(n, k) \), \( n \geq 1 \), \( 1 \leq k \leq n \).

sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(2)
1
sage: a(3)
1
sage: a(4.0)
Traceback (most recent call last):
...
TypeError: input must be an int or Integer
sage: a.list(15)
[1, 1, 1, 1, 3, 1, 1, 7, 6, 1, 1, 15, 25, 10, 1]
```

**AUTHORS:**

- Jaap Spies (2007-01-31)

**keyword** = ['nonn', 'tabl', 'nice', 'core', 'triangle']

\( s_2(n, k) \)

Returns the Stirling number \( S_2(n, k) \) of the 2nd kind.

**EXAMPLES:**

```python
sage: sloane.A008277.s2(4,2)
7
```
class sage.combinat.sloane_functions.A008683

Bases: SloaneSequence

Möbius function $\mu(n)$.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A008683;a
Moebius function mu(n).
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(2)
-1
sage: a(12)
0
sage: a.list(12)
[1, -1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0]
```

AUTHORS:

• Jaap Spies (2007-01-13)

class sage.combinat.sloane_functions.A010060

Bases: SloaneSequence

Thue-Morse sequence.

Let $A_k$ denote the first $2^k$ terms; then $A_0 = 0$, and for $k \geq 0$, $A_{k+1} = A_k B_k$, where $B_k$ is obtained from $A_k$ by interchanging 0’s and 1’s.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A010060;a
Thue-Morse sequence.
sage: a(0)
0
sage: a(1)
1
sage: a(2)
1
sage: a(12)
0
```

(continues on next page)
sage: a.list(13)
[0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0]

AUTHORS:
• Jaap Spies (2007-02-02)

class sage.combinat.sloane_functions.A015521
Bases: RecurrenceSequence2
Linear 2nd order recurrence, \(a(0) = 0\), \(a(1) = 1\) and \(a(n) = 3a(n-1) + 4a(n-2)\).

INPUT:
• \(n\) – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

sage: a = sloane.A015521; a
Linear 2nd order recurrence, \(a(n) = 3 a(n-1) + 4 a(n-2)\).
sage: a(0)
0
sage: a(1)
1
sage: a(8)
13107
sage: a(41)
967140655691703339764941
sage: a.list(12)
[0, 1, 3, 13, 51, 205, 819, 3277, 13107, 52429, 209715, 838861]

AUTHORS:
• Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A015523
Bases: RecurrenceSequence2
Linear 2nd order recurrence, \(a(0) = 0\), \(a(1) = 1\) and \(a(n) = 3a(n-1) + 5a(n-2)\).

INPUT:
• \(n\) – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

sage: a = sloane.A015523; a
Linear 2nd order recurrence, \(a(n) = 3 a(n-1) + 5 a(n-2)\).
sage: a(0)
0
sage: a(1)
1

AUTHORS:
  • Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A015530

Bases: RecurrenceSequence2

Linear 2nd order recurrence, \( a(0) = 0, a(1) = 1 \) and \( a(n) = 4a(n-1) + 3a(n-2) \).

INPUT:
  • \( n \) – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:

```
sage: a = sloane.A015530;a
Linear 2nd order recurrence, a(n) = 4 a(n-1) + 3 a(n-2).
sage: a(0)
0
sage: a(1)
1
sage: a(2)
4
sage: a.offset
0
sage: a(8)
41008
sage: a.list(9)
[0, 1, 4, 19, 88, 409, 1900, 8827, 41008]
```

AUTHORS:
  • Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A015531

Bases: RecurrenceSequence2

Linear 2nd order recurrence, \( a(0) = 0, a(1) = 1 \) and \( a(n) = 4a(n-1) + 5a(n-2) \).

INPUT:
  • \( n \) – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:
Combinatorics, Release 10.0

sage: a = sloane.A015531;a
Linear 2nd order recurrence, \( a(n) = 4 \ a(n-1) + 5 \ a(n-2) \).
sage: a(0)
0
sage: a(1)
1
sage: a(2)
4
sage: a.offset
0
sage: a(8)
65104
sage: a(60)
144560289664733924534327040115992282190104
sage: a.list(9)
[0, 1, 4, 21, 104, 521, 2604, 13021, 65104]

AUTHORS:

• Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A015551
Bases: RecurrenceSequence2
 Linear 2nd order recurrence, \( a(0) = 0, \ a(1) = 1 \) and \( a(n) = 6a(n-1) + 5a(n-2) \).
 INPUT:
  • \( n \) – non negative integer
 OUTPUT:
  • integer – function value

EXAMPLES:

sage: a = sloane.A015551;a
Linear 2nd order recurrence, \( a(n) = 6 \ a(n-1) + 5 \ a(n-2) \).
sage: a(0)
0
sage: a(1)
1
sage: a(2)
6
sage: a.offset
0
sage: a(8)
570216
sage: a(60)
7110606606530059736761484557155863822531970573036
sage: a.list(9)
[0, 1, 6, 41, 276, 1861, 12546, 84581, 570216]

AUTHORS:

• Jaap Spies (2007-01-19)
class sage.combinat.sloane_functions.A018252
Bases: SloaneSequence

The nonprime numbers, starting with 1.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A018252;a
The nonprime numbers.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(2)
4
sage: a(9)
15
sage: a.list(10)
[1, 4, 6, 8, 9, 10, 12, 14, 15, 16]

AUTHORS:

• Jaap Spies (2007-02-04)

class sage.combinat.sloane_functions.A020639
Bases: SloaneSequence

Least prime dividing \( n \) with \( a(1) = 1 \).

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A020639;a
Least prime dividing \( n \) \((a(1)=1)\).
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(13)
13
sage: a.list(14)
[1, 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, 2]
class sage.combinat.sloane_functions.A046660
Bases: SloaneSequence

Excess of \( n \) = number of prime divisors (with multiplicity) - number of prime divisors (without multiplicity), \( \Omega(n) - \omega(n) \).

INPUT:
• \( n \) – positive integer

OUTPUT:
• integer – function value

EXAMPLES:

```python
sage: a = sloane.A046660; a
Excess of \( n \) = Bigomega (with multiplicity) - omega (without multiplicity).
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
0
sage: a(8)
2
sage: a(41)
0
sage: a(84792)
2
sage: a.list(12)
[0, 0, 0, 1, 0, 0, 0, 2, 1, 0, 0, 1]
```

AUTHORS:
• Jaap Spies (2007-01-19)

class sage.combinat.sloane_functions.A049310
Bases: SloaneSequence

Triangle of coefficients of Chebyshev’s \( S(n, x) \) : \( U(n, \frac{x}{2}) \) polynomials (exponents in increasing order).

INPUT:
• \( n \) – non negative integer

OUTPUT:
• integer – function value

EXAMPLES:

```python`
sage: a = sloane.A049310; a
Triangle of coefficients of Chebyshev's $S(n,x) := U(n,x/2)$ polynomials (exponents $\rightarrow$ in increasing order).
sage: a(0)
1
sage: a(1)
0
sage: a(13)
0
sage: a.list(15)
[1, 0, 1, -1, 0, 1, 0, -2, 0, 1, 1, 0, -3, 0, 1]
sage: a(200)
0
sage: a.keyword
['sign', 'tabl', 'nice', 'easy', 'core', 'triangle']

AUTHORS:
  • Jaap Spies (2007-01-31)

keyword = ['sign', 'tabl', 'nice', 'easy', 'core', 'triangle']

class sage.combinat.sloane_functions.A051959
Bases: RecurrenceSequence

Linear second order recurrence. A051959.

INPUT:
  • n – non negative integer

OUTPUT:
  • integer – function value

EXAMPLES:
sage: a = sloane.A051959; a
Linear second order recurrence. A051959.
sage: a(0)
1
sage: a(1)
10
sage: a(8)
9969
sage: a(41)
42834431872413650
sage: a.list(12)
[1, 10, 36, 104, 273, 686, 1688, 4112, 9969, 24114, 58268, 140728]

AUTHORS:
  • Jaap Spies (2007-01-19)
g(k)
  EXAMPLES:
class sage.combinat.sloane_functions.A055790

Bases: ExtremesOfPermanentsSequence2

\[ a(n) = n \cdot a(n-1) + (n-2) \cdot a(n-2) \] \[ a(0) = 0, a(1) = 2 \].

With offset 1, permanent of (0,1)-matrix of size \( n \times (n+d) \) with \( d=1 \) and \( n-1 \) zeros not on a line. This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of (0,1)-matrices, p. 201-202.

REFERENCES:


INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

sage: a = sloane.A055790;

\[ a(n) = n \cdot a(n-1) + (n-2) \cdot a(n-2) \] \[ a(0) = 0, a(1) = 2 \].

sage: a(0)
0

sage: a(1)
2

sage: a(2)
4

sage: a.offset
0

sage: a(8)
165016

sage: a(22)
10356214297533070441564

sage: a.list(9)
[0, 2, 4, 14, 64, 362, 2428, 18806, 165016]

AUTHORS:

- Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A061084

Bases: SloaneSequence

Fibonacci-type sequence based on subtraction: \( a(0) = 1, a(1) = 2 \) and \( a(n) = a(n-2) - a(n-1) \).

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value
EXAMPLES:

\begin{verbatim}
sage: a = sloane.A061084; a
Fibonacci-type sequence based on subtraction: a(0) = 1, a(1) = 2 and a(n) = a(n-2)-a(n-1).
sage: a(0)
1
sage: a(1)
2
sage: a(8)
-29
sage: a(22)
-24476
sage: a.list(12)
[1, 2, -1, 3, -4, 7, -11, 18, -29, 47, -76, 123]
sage: a.keyword
['sign', 'easy', 'nice']
\end{verbatim}

AUTHORS:

- Jaap Spies (2007-01-18)

\texttt{keyword = ['sign', 'easy', 'nice']}

\texttt{class \ sage.combinat.sloane_functions.A064553}

\texttt{Bases: SloaneSequence}

\begin{verbatim}
a(1) = 1, a(prime(i)) = i + 1 for i > 0 and a(u \cdot v) = a(u) \cdot a(v) for u, v > 0.
\end{verbatim}

INPUT:

- \texttt{n} -- non negative integer

OUTPUT:

- \texttt{integer} -- function value

EXAMPLES:

\begin{verbatim}
sage: a = sloane.A064553; a
a(1) = 1, a(prime(i)) = i + 1 for i > 0 and a(u \cdot v) = a(u) \cdot a(v) for u, v > 0
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
1
sage: a(2)
2
sage: a(9)
9
sage: a.list(16)
[1, 2, 3, 4, 4, 6, 5, 8, 9, 8, 6, 12, 7, 10, 12, 16]
\end{verbatim}

AUTHORS:

- Jaap Spies (2007-02-04)
class sage.combinat.sloane_functions.A079922(offset=1)

Bases: SloaneSequence

function returns solutions to the Dancing School problem with \( n \) girls and \( n + 3 \) boys.

The value is \( \text{per}(B) \), the permanent of the \((0,1)\)-matrix \( B \) of size \( n \times n + 3 \) with \( b(i,j) = 1 \) if and only if \( i \leq j \leq i + n \).

REFERENCES:

• Jaap Spies, Nieuw Archief voor Wiskunde, 5/7 nr 4, December 2006

INPUT:

• \( n \) – positive integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A079922; a
Solutions to the Dancing School problem with n girls and n+3 boys
sage: a.offset
1
sage: a(1)
4
sage: a(8)
2227
sage: a.list(8)
[4, 13, 36, 90, 212, 478, 1044, 2227]
```

Compare: Searching Sloane’s online database... Solution to the Dancing School Problem with n girls and n+3 boys: f(n,3). [4, 13, 36, 90, 212, 478, 1044, 2227]

```
sage: a(-1)
Traceback (most recent call last):
...
ValueError: input n (=1) must be a positive integer
```

AUTHORS:

• Jaap Spies (2007-01-14)

class sage.combinat.sloane_functions.A079923(offset=1)

Bases: SloaneSequence

function returns solutions to the Dancing School problem with \( n \) girls and \( n + 4 \) boys.

The value is \( \text{per}(B) \), the permanent of the \((0,1)\)-matrix \( B \) of size \( n \times n + 3 \) with \( b(i,j) = 1 \) if and only if \( i \leq j \leq i + n \).

REFERENCES:

• Jaap Spies, Nieuw Archief voor Wiskunde, 5/7 nr 4, December 2006

INPUT:

• \( n \) – positive integer

OUTPUT:
• integer – function value

EXAMPLES:

```
sage: a = sloane.A079923; a
Solutions to the Dancing School problem with n girls and n+4 boys
sage: a.offset
1
sage: a(1)
5
sage: a(8)
15458
sage: a.list(8)
[5, 21, 76, 246, 738, 2108, 5794, 15458]
```

Compare: Searching Sloane’s online database... Solution to the Dancing School Problem with n girls and n+4 boys: f(n,4). [5, 21, 76, 246, 738, 2108, 5794, 15458]

```
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
```

AUTHORS:

• Jaap Spies (2007-01-17)

```
class sage.combinat.sloane_functions.A082411
```

Bases: `RecurrenceSequence2`

Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).

\[ a(0) = 407389224418, a(1) = 76343678551 \]. This is the second-order linear recurrence sequence with \( a(0) \) and \( a(1) \) co-prime, that R. L. Graham in 1964 stated did not contain any primes.

INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A082411; a
Second-order linear recurrence sequence with a(n) = a(n-1) + a(n-2).
sage: a(1)
76343678551
sage: a(2)
483732902969
sage: a(3)
560076581520
sage: a(20)
2219759332689173
sage: a.list(4)
[407389224418, 76343678551, 483732902969, 560076581520]
```

AUTHORS:
class sage.combinat.sloane_functions.A083103

Bases: RecurrenceSequence2

Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).

\( a(0) = 1786772701928802632268715130455793, a(1) = 1059683225053915111058165141686995 \). This is the second-order linear recurrence sequence with \( a(0) \) and \( a(1) \) co-prime, that R. L. Graham in 1964 stated did not contain any primes. It has not been verified. Graham made a mistake in the calculation that was corrected by D. E. Knuth in 1990.

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A083103;a
Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).
sage: a(1)
1059683225053915111058165141686995
sage: a(2)
284645926982717743326880272142788
sage: a(3)
39613915203663285435045413829783
sage: a.offset
0
sage: a(8)
45481392851206651551714764671352204
sage: a(20)
14639253684254059531823985143948191708
sage: a.list(4)
[1786772701928802632268715130455793, 1059683225053915111058165141686995, 284645926982717743326880272142788, 39613915203663285435045413829783]

AUTHORS:

• Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A083104

Bases: RecurrenceSequence2

Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).

\( a(0) = 331635635998274737472200656430763, a(1) = 1510028911088401971189590305498785 \). This is the second-order linear recurrence sequence with \( a(0) \) and \( a(1) \) co-prime. It was found by Ronald Graham in 1990.

INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:
sage: a = sloane.A083104;a
Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).
sage: a(3)
3351693458175078679851381267428333
sage: a.offset
0
sage: a(8)
3602187040834012982120004949074404
sage: a(20)
11601914177621826012468849361236300628

AUTHORS:

- Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A083105

Bases: RecurrenceSequence2

Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).
\[ a(0) = 62638280004239857, a(1) = 49463435743205655. \] This is the second-order linear recurrence sequence with \( a(0) \) and \( a(1) \) co-prime. It was found by Donald Knuth in 1990.

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

sage: a = sloane.A083105;a
Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).
sage: a(1)
49463435743205655
sage: a(2)
112101715747445512
sage: a(3)
161565151490651167
sage: a.offset
0
sage: a(8)
1853029790662436896
sage: a(20)
596510791500513098192
sage: a.list(4)
[62638280004239857, 49463435743205655, 112101715747445512, 161565151490651167]

AUTHORS:

- Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A083216

Bases: RecurrenceSequence2

Second-order linear recurrence sequence with \( a(n) = a(n-1) + a(n-2) \).
Combinatorics, Release 10.0

\[ a(0) = 2061567420555510, \ a(1) = 3794765361567513. \] This is a second-order linear recurrence sequence with \( a(0) \) and \( a(1) \) co-prime that does not contain any primes. It was found by Herbert Wilf in 1990.

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```
sage: a = sloane.A083216; a
Second-order linear recurrence sequence with a(n) = a(n-1) + a(n-2).
sage: a(0)
2061567420555510
sage: a(1)
3794765361567513
sage: a(8)
24410439567123023
sage: a(41)
28205204928690536
sage: a.list(4)
[2061567420555510, 3794765361567513, 24410439567123023, 28205204928690536]
```

AUTHORS:

- Jaap Spies (2007-01-19)

class `sage.combinat.sloane_functions.A090010`

Bases: `ExtremesOfPermanentsSequence2`

Permanent of \((0,1)\)-matrix of size \( n \times (n+d) \) with \( d = 6 \) and \( n \) zeros not on a line.

\`a(n) = (n+5)*a(n-1) + (n-1)*a(n-2), a(1)=6, a(2)=43'.

This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of \((0,1)\)-matrices, p. 201-202.

REFERENCES:


INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

```
sage: a = sloane.A090010; a
Permanent of \((0,1)\)-matrix of size \( n \times (n+d) \) with \( d=6 \) and \( n \) zeros not on a line.
sage: a(0)
ValueError: input n (=0) must be a positive integer
```

(continues on next page)
AUTHORS:

• Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A090012

Bases: SloaneSequence

Permanent of \((0,1)\)-matrix of size \(n \times (n + d)\) with \(d = 2\) and \(n - 1\) zeros not on a line.

\[ a(n) = (n + 1) * a(n - 1) + (n - 2) * a(n - 2), \quad a(1) = 3 \text{ and } a(2) = 9 \]

This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of \((0,1)\)-matrices, p. 201-202.

REFERENCES:


INPUT:

• \(n\) – non negative integer

OUTPUT:

• \texttt{integer} – function value

EXAMPLES:

\begin{verbatim}
sage: a = sloane.A090012;a
Permanent of \((0,1)\)-matrix of size \(n \times (n+d)\) with \(d=2\) and \(n-1\) zeros not on a line.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
3
sage: a(2)
9
sage: a.offset
1
sage: a(8)
890901
sage: a(22)
129020386652297208795129
\end{verbatim}
AUTHORS:
  • Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A090013

Bases: SloaneSequence

Permanent of (0,1)-matrix of size \( n \times (n + d) \) with \( d = 3 \) and \( n - 1 \) zeros not on a line.

\[
a(n) = (n + 1) * a(n - 1) + (n - 2) * a(n - 2) \mid a(1) = 4, a(2) = 16
\]

This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of (0,1)-matrices, p. 201-202.

REFERENCES:

INPUT:
  • \( n \) – non negative integer

OUTPUT:
  • \texttt{integer} – function value

EXAMPLES:

\begin{verbatim}
sage: a = sloane.A090013;a
Permanent of (0,1)-matrix of size n X (n+d) with d=3 and n-1 zeros not on a line.
sage: a(0)
Traceback (most recent call last):
  ... ValueError: input n (=0) must be a positive integer
sage: a(1)
4
sage: a(2)
16
sage: a.offset
1
sage: a(8)
3481096
sage: a(22)
1112998577171142607670336
sage: a.list(9)
[4, 16, 84, 536, 4004, 34176, 327604, 3481096, 40585284]
\end{verbatim}

AUTHORS:
  • Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A090014

Bases: SloaneSequence

Permanent of (0,1)-matrix of size \( n \times (n + d) \) with \( d = 4 \) and \( n - 1 \) zeros not on a line.
\[ a(n) = (n + 1) \times a(n - 1) + (n - 2) \times a(n - 2) \quad [a(1) = 5, a(2) = 25] \]

This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of (0,1)-matrices, p. 201-202.

REFERENCES:


INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A090014;a
Permanent of (0,1)-matrix of size n X (n+d) with d=4 and n-1 zeros not on a line.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(1)
5
sage: a(2)
25
sage: a.offset
1
sage: a(8)
11016595
sage: a(22)
7469733600354446865509725
sage: a.list(9)
[5, 25, 155, 1135, 9545, 90445, 952175, 11016595, 138864365]
```

AUTHORS:

• Jaap Spies (2007-01-23)
• integer – function value

EXAMPLES:

```sage
a = sloane.A090015;
Permanent of (0,1)-matrix of size n X (n+d) with d=3 and n-1 zeros not on a line.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
```

```sage
a(1)
6
a(2)
36
a.offset
1
a(8)
29976192
a(22)
41552258517692116794936876
a.list(9)
[6, 36, 258, 2136, 19998, 208524, 2393754, 29976192, 406446774]
```

AUTHORS:

• Jaap Spies (2007-01-23)

class *sage.combinat.sloane_functions.A090016*

Bases: *SloaneSequence*

Permanent of (0,1)-matrix of size \( n \times (n + d) \) with \( d = 6 \) and \( n - 1 \) zeros not on a line.

\[
a(n) = (n + 1) * a(n - 1) + (n - 2) * a(n - 2) \quad a(1) = 7, a(2) = 49
\]

A090016a(n) = A090010(n - 1) + A090010(n), a(1) = 7

This is a special case of Theorem 2.3 of Seok-Zun Song et al. Extremes of permanents of (0,1)-matrices, p. 201-202.

REFERENCES:


INPUT:

• n – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```sage
a = sloane.A090016;
Permanent of (0,1)-matrix of size n X (n+d) with d=6 and n-1 zeros not on a line.
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
```
AUTHORS:

- Jaap Spies (2007-01-23)

class sage.combinat.sloane_functions.A109814

Bases: SloaneSequence

The \( n \)th term of the sequence \( a(n) \) is the largest \( k \) such that \( n \) can be written as sum of \( k \) consecutive positive integers. By definition, \( n \) is the sum of at most \( a(n) \) consecutive positive integers. Suppose \( n \) is to be written as sum of \( k \) consecutive integers starting with \( m \), then \( 2n = k(2m + k - 1) \). Only one of the factors is odd. For each odd divisor \( d \) of \( n \) there is a unique corresponding \( k = \text{min}(d, 2n/d) \). \( a(n) \) can be alternatively defined as the largest among those \( k \).

See also:

- Wikipedia article Polite_number
- An exercise sheet (with answers) about sums of consecutive integers.

INPUT:

- \( n \) – non negative integer

OUTPUT:

- integer – function value

EXAMPLES:

sage: a = sloane.A109814; a
a(n) is the largest \( k \) such that \( n \) can be written as sum of \( k \) consecutive positive integers.
sage: a(0)
Traceback (most recent call last):
  ...
ValueError: input \( n =0 \) must be a positive integer
sage: a(2)
1
sage: a.list(9)
[1, 1, 2, 1, 2, 3, 2, 1, 3]
class sage.combinat.sloane_functions.A111774

Bases: SloaneSequence

Sequence of numbers of the third kind, i.e., numbers that can be written as a sum of at least three consecutive positive integers.

Odd primes can only be written as a sum of two consecutive integers. Powers of 2 do not have a representation as a sum of \( k \) consecutive integers (other than the trivial \( n = n \) for \( k = 1 \)).


INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

sage: a = sloane.A111774; a
Numbers that can be written as a sum of at least three consecutive positive integers.
sage: a(1)
6
sage: a(0)
Traceback (most recent call last):
  ... ValueError: input n (=0) must be a positive integer
sage: a(100)
141
sage: a(156)
209
sage: a(302)
386
sage: a.list(12)
[6, 9, 10, 12, 14, 15, 18, 20, 21, 22, 24, 25]
sage: a(1/3)
Traceback (most recent call last):
  ... TypeError: input must be an int or Integer

AUTHORS:

• Jaap Spies (2007-01-13)

is_number_of_the_third_kind(n)

Return True if and only if \( n \) is a number of the third kind.

A number is of the third kind if it can be written as a sum of at least three consecutive positive integers. Odd primes can only be written as a sum of two consecutive integers. Powers of 2 do not have a representation as a sum of \( k \) consecutive integers (other than the trivial \( n = n \) for \( k = 1 \)).


INPUT:

• \( n \) – positive integer
OUTPUT:
True if \( n \) is not prime and not a power of 2

EXAMPLES:

```python
sage: a = sloane.A111774
sage: a.is_number_of_the_third_kind(6)
True
sage: a.is_number_of_the_third_kind(100)
True
sage: a.is_number_of_the_third_kind(16)
False
sage: a.is_number_of_the_third_kind(97)
False
```

AUTHORS:
- Jaap Spies (2006-12-09)

```
lst(n)
```

EXAMPLES:

```python
sage: sloane.A111774.list(12)
[6, 9, 10, 12, 14, 15, 18, 20, 21, 22, 24, 25]
```

class sage.combinat.sloane_functions.A111775

Bases: SloaneSequence

Number of ways \( n \) can be written as a sum of at least three consecutive integers. Powers of 2 and (odd) primes can not be written as a sum of at least three consecutive integers. \( a(n) \) strongly depends on the number of odd divisors of \( n \) (A001227): Suppose \( n \) is to be written as sum of \( k \) consecutive integers starting with \( m \), then \( 2n = k(2m + k - 1) \). Only one of the factors is odd. For each odd divisor of \( n \) there is a unique corresponding \( k \), \( k = 1 \) and \( k = 2 \) must be excluded.


INPUT:
- \( n \) – non negative integer

OUTPUT:
- integer – function value

EXAMPLES:

```python
sage: a = sloane.A111775; a
Number of ways \( n \) can be written as a sum of at least three consecutive integers.

sage: a(1)
0
sage: a(0)
0
```

We have \( a(15)=2 \) because \( 15 = 4+5+6 \) and \( 15 = 1+2+3+4+5 \). The number of odd divisors of 15 is 4.

```python
sage: a(15)
2
```
AUTHORS:

• Jaap Spies (2006-12-09)

class sage.combinat.sloane_functions.A111787

Bases: SloaneSequence

This function returns the \( n \)-th number of Sloane’s sequence A111787

\[ a(n) = 0 \text{ if } n \text{ is an odd prime or a power of } 2. \]

For numbers of the third kind (see A111774) we proceed as follows: suppose \( n \) is to be written as sum of \( k \) consecutive integers starting with \( m \), then \( 2n = k(2m + k - 1) \).

Let \( p \) be the smallest odd prime divisor of \( n \) then \( a(n) = \min(p, 2n/p) \).


INPUT:

• \( n \) – non negative integer

OUTPUT:

• integer – function value

EXAMPLES:

```
sage: a = sloane.A111787; a
a(n) is the least k >= 3 such that n can be written as sum of k consecutive integers. a(n)=0 if such a k does not exist.
sage: a.offset
1
sage: a(1)
0
sage: a(0)
Traceback (most recent call last):
...
ValueError: input n (=0) must be a positive integer
sage: a(100)
5
sage: a(256)
0
sage: a(29)
0
sage: a.list(20)
[0, 0, 0, 0, 3, 0, 0, 3, 4, 0, 3, 0, 4, 3, 0, 0, 3, 0, 5]
sage: a(-1)
(continues on next page)```
Combinatorics, Release 10.0

Traceback (most recent call last):
...
ValueError: input n (=1) must be a positive integer

AUTHORS:
• Jaap Spies (2007-01-14)

class sage.combinat.sloane_functions.ExponentialNumbers(a)
Bases: SloaneSequence
A sequence of Exponential numbers.
EXAMPLES:
sage: from sage.combinat.sloane_functions import ExponentialNumbers
sage: ExponentialNumbers(0)
Sequence of Exponential numbers around 0

class sage.combinat.sloane_functions.ExtremesOfPermanentsSequence(offset=1)
Bases: SloaneSequence
gen(a0, a1, d)
EXAMPLES:
sage: it = sloane.A000153.gen(0,1,2)
sage: [next(it) for i in range(5)]
[0, 1, 2, 7, 32]

list(n)
EXAMPLES:
sage: sloane.A000153.list(8)
[0, 1, 2, 7, 32, 181, 1214, 9403]

class sage.combinat.sloane_functions.ExtremesOfPermanentsSequence2(offset=1)
Bases: ExtremesOfPermanentsSequence
gen(a0, a1, d)
EXAMPLES:
sage: from sage.combinat.sloane_functions import ExtremesOfPermanentsSequence2
sage: e = ExtremesOfPermanentsSequence2()
sage: it = e.gen(6,43,6)
sage: [next(it) for i in range(5)]
[6, 43, 307, 2542, 23799]

class sage.combinat.sloane_functions.RecurrenceSequence(offset=1)
Bases: SloaneSequence
list(n)
EXAMPLES:
sage: sloane.A001110.list(8)
[0, 1, 36, 1225, 41616, 1413721, 48024900, 1631432881]

5.1. Comprehensive Module List
class sage.combinat.sloane_functions.RecurrenceSequence2(offset=1)

Bases: SloaneSequence

list(n)

EXAMPLES:

sage: sloane.A001906.list(10)
[0, 1, 3, 8, 21, 55, 144, 377, 987, 2584]

class sage.combinat.sloane_functions.Sloane

Bases: SageObject

A collection of Sloane generating functions.

This class inspects sage.combinat.sloane_functions, accumulating all the SloaneSequence classes starting with 'A'. These are listed for tab completion, but not instantiated until requested.

EXAMPLES:

Ensure we have lots of entries:

sage: len(sloane.__dir__()) > 100
True

Ensure none are being incorrectly returned:

sage: [ None for n in sloane.__dir__() if not n.startswith('A') ]
[]

Ensure we can access dynamic constructions and cache correctly:

sage: s = sloane.A000587
sage: s is sloane.A000587
True

Ensure that we can access other functions in parent classes:

sage: sloane.__class__
<class 'sage.combinat.sloane_functions.Sloane'>

AUTHORS:

• Nick Alexander

class sage.combinat.sloane_functions.SloaneSequence(offset=1)

Bases: SageObject

Base class for a Sloane integer sequence.

list(n)

Return n terms of the sequence:

sequence[offset], sequence[offset+1], ..., sequence[offset+n-1].

EXAMPLES:

sage: sloane.A000012.list(4)
[1, 1, 1, 1]
sage.combinat.sloane_functions.perm_mh(m, h)

This functions calculates $f(g, h)$ from Sloane’s sequences A079908-A079928

INPUT:

• $m$ – positive integer
• $h$ – non negative integer

OUTPUT: permanent of the $m \times (m + h)$ matrix, etc.

EXAMPLES:

```python
sage: from sage.combinat.sloane_functions import perm_mh
sage: perm_mh(3,3)
36
sage: perm_mh(3,4)
76
```

AUTHORS:

• Jaap Spies (2006)

sage.combinat.sloane_functions.recur_gen2(a0, a1, a2, a3)

homogeneous general second-order linear recurrence generator with fixed coefficients

$a(0) = a0, a(1) = a1, a(n) = a2*a(n-1) + a3*a(n-2)$

EXAMPLES:

```python
sage: from sage.combinat.sloane_functions import recur_gen2
sage: it = recur_gen2(1,1,1,1)
sage: [next(it) for i in range(10)]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
```

sage.combinat.sloane_functions.recur_gen2b(a0, a1, a2, a3, b)

Inhomogeneous second-order linear recurrence generator with fixed coefficients and $b = f(n)$

$a(0) = a0, a(1) = a1, a(n) = a2*a(n-1) + a3*a(n-2) + f(n).$

EXAMPLES:

```python
sage: from sage.combinat.sloane_functions import recur_gen2b
sage: it = recur_gen2b(1,1,1,1, lambda n: 0)
sage: [next(it) for i in range(10)]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
```

sage.combinat.sloane_functions.recur_gen3(a0, a1, a2, a3, a4, a5)

homogeneous general third-order linear recurrence generator with fixed coefficients

$a(0) = a0, a(1) = a1, a(2) = a2, a(n) = a3*a(n-1) + a4*a(n-2) + a5*a(n-3)$

EXAMPLES:

```python
sage: from sage.combinat.sloane_functions import recur_gen3
sage: it = recur_gen3(1,1,1,1,1,1)
sage: [next(it) for i in range(10)]
[1, 1, 1, 3, 5, 9, 17, 31, 57, 105]
```
5.1.313 Combinatorial species

Todo: Short blurb about species

Todo: Proofread / point to the main classes rather than the modules?

Introductory material

- Enumeration of trees using generating functions
- Species, decomposable combinatorial classes

Basic Species

- Combinatorial Species
- Empty Species
- Recursive Species
- Characteristic Species
- Cycle Species
- Partition Species
- Permutation species
- Linear-order Species
- Set Species
- Subset Species
- Examples of Combinatorial Species

Operations on Species

- Sum species
- Product species
- Composition species
- Functorial composition species
5.1.314 Characteristic Species

```python
sage: X = species.CharacteristicSpecies(1)
sage: X.structures([1]).list()
[1]
sage: X.structures([1,2]).list()
[]
sage: X.generating_series()[0:4]
[0, 1, 0, 0]
sage: X.isotype_generating_series()[0:4]
[0, 1, 0, 0]
sage: X.cycle_index_series()[0:4]
[0, p[1], 0, 0]
```

```python
sage: F = species.CharacteristicSpecies(3)
sage: c = F.generating_series()[0:4]
sage: F._check()
True
sage: F == loads(dumps(F))
True
```

```python
class sage.combinat.species.characteristic_species.CharacteristicSpeciesStructure(parent, labels, list)
Bases: GenericSpeciesStructure

automorphism_group()
Returns the group of permutations whose action on this structure leave it fixed. For the characteristic species, there is only one structure, so every permutation is in its automorphism group.

EXAMPLES:
```
**canonical_label()**

**EXAMPLES:**

```python
sage: F = species.CharacteristicSpecies(3)
sage: a = F.structures(['a', 'b', 'c']).random_element(); a
{'a', 'b', 'c'}
sage: a.canonical_label()
{'a', 'b', 'c'}
```

**transport**(perm)

Returns the transport of this structure along the permutation perm.

**EXAMPLES:**

```python
sage: F = species.CharacteristicSpecies(3)
sage: a = F.structures(['a', 'b', 'c']).random_element(); a
{'a', 'b', 'c'}
sage: p = PermutationGroupElement((1,2))
sage: a.transport(p)
{'a', 'b', 'c'}
```

---

**class** sage.combinat.species.characteristic_species.**EmptySetSpecies**(min=None, max=None, weight=None)

Bases: CharacteristicSpecies

Returns the empty set species.

This species has exactly one structure on the empty set. It is the same (and is implemented) as CharacteristicSpecies(0).

**EXAMPLES:**

```python
sage: X = species.EmptySetSpecies()
sage: X.structures([]).list()
[]
sage: X.structures([1,2]).list()
[]
sage: X.generating_series()[0:4]
[1, 0, 0, 0]
sage: X.isotype_generating_series()[0:4]
[1, 0, 0, 0]
sage: X.cycle_index_series()[0:4]
[p[], 0, 0, 0]
```

---

**class** sage.combinat.species.characteristic_species.**SingletonSpecies**(min=None, max=None, weight=None)

Bases: CharacteristicSpecies

Returns the species of singletons.
This species has exactly one structure on a set of size 1. It is the same (and is implemented) as \texttt{CharacteristicSpecies(1)}.

EXAMPLES:

```python
sage: X = species.SingletonSpecies()
sage: X.structures([1]).list()
[1]
sage: X.structures([1,2]).list()
[]
sage: X.generating_series()[0:4]
[0, 1, 0, 0]
sage: X.isotype_generating_series()[0:4]
[0, 1, 0, 0]
sage: X.cycle_index_series()[0:4]
[0, p[1], 0, 0]
```

\texttt{sage.combinat.species.characteristic_species.SingletonSpecies_class}
alias of \texttt{SingletonSpecies}

### 5.1.315 Composition species

**class** \texttt{sage.combinat.species.composition_species.CompositionSpecies}\(\textit{(F, G, min=None, max=None, weight=None)}\)

Bases: \texttt{GenericCombinatorialSpecies, UniqueRepresentation}

Returns the composition of two species.

EXAMPLES:

```python
sage: E = species.SetSpecies()
sage: C = species.CycleSpecies()
sage: S = E(C)
sage: S.generating_series()[0:5]
[1, 1, 1, 1, 1]
sage: E(C) is S
True
```

\texttt{weight\_ring()}

Returns the weight ring for this species. This is determined by asking Sage’s coercion model what the result is when you multiply (and add) elements of the weight rings for each of the operands.

EXAMPLES:

```python
sage: E = species.SetSpecies(); C = species.CycleSpecies()
sage: L = E(C)
sage: L.weight_ring()
Rational Field
```

**class** \texttt{sage.combinat.species.composition_species.CompositionSpeciesStructure}\(\textit{parent, labels, pi, f, gs)}

Bases: \texttt{GenericSpeciesStructure}
### change_labels(labels)

Return a relabelled structure.

**INPUT:**

- `labels`, a list of labels.

**OUTPUT:**

A structure with the i-th label of self replaced with the i-th label of the list.

**EXAMPLES:**

```python
sage: E = species.SetSpecies(); C = species.CycleSpecies()
sage: L = E(C)
sage: S = L.structures(['a','b','c']).list()
sage: a = S[2]; a
F-structure: {{'a', 'c'}, {'b'}}; G-structures: (('a', 'c'), ('b'))
sage: a.change_labels([1,2,3])
F-structure: {{1, 3}, {2}}; G-structures: [(1, 3), (2)]
```

### transport(perm)

**EXAMPLES:**

```python
sage: p = PermutationGroupElement((2,3))
sage: E = species.SetSpecies(); C = species.CycleSpecies()
sage: L = E(C)
sage: S = L.structures(['a','b','c']).list()
sage: a = S[2]; a
F-structure: {{'a', 'c'}, {'b'}}; G-structures: (('a', 'c'), ('b'))
sage: a.transport(p)
F-structure: {{'a', 'b'}, {'c'}}; G-structures: (('a', 'c'), ('b'))
```

```python
sage.combinat.species.composition_species.CompositionSpecies_class
alias of CompositionSpecies
```

#### 5.1.316 Cycle Species

**class** `sage.combinat.species.cycle_species.CycleSpecies(min=None, max=None, weight=None)`

**Bases:** `GenericCombinatorialSpecies, UniqueRepresentation`

Returns the species of cycles.

**EXAMPLES:**

```python
sage: C = species.CycleSpecies(); C
Cyclic permutation species
sage: C.structures([1,2,3,4]).list()
[(1, 2, 3, 4),
 (1, 2, 4, 3),
 (1, 3, 2, 4),
 (1, 3, 4, 2),
 (1, 4, 2, 3),
 (1, 4, 3, 2)]
```
class sage.combinat.species.cycle_species.CycleSpeciesStructure(parent, labels, list)

Bases: GenericSpeciesStructure

automorphism_group()

Returns the group of permutations whose action on this structure leave it fixed.

EXAMPLES:

```
sage: P = species.CycleSpecies()
sage: a = P.structures([1, 2, 3, 4])[0]; a
(1, 2, 3, 4)
sage: a.automorphism_group()
Permutation Group with generators [(1,2,3,4)]
```

```
sage: [a.transport(perm) for perm in a.automorphism_group()]
[(1, 2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 4)]
```

canonical_label()

EXAMPLES:

```
sage: P = species.CycleSpecies()
sage: P.structures(["a","b","c"]).random_element().canonical_label()
('a', 'b', 'c')
```

permutation_group_element()

Returns this cycle as a permutation group element.

EXAMPLES:

```
sage: F = species.CycleSpecies()
sage: a = F.structures(["a", "b", "c"]) [0]; a
('a', 'b', 'c')
sage: a.permutation_group_element()
(1,2,3)
```

transport(perm)

Returns the transport of this structure along the permutation perm.

EXAMPLES:

```
sage: F = species.CycleSpecies()
sage: a = F.structures(["a", "b", "c"]) [0]; a
('a', 'b', 'c')
sage: p = PermutationGroupElement((1,2))
sage: a.transport(p)
('a', 'c', 'b')
```

sage.combinat.species.cycle_species.CycleSpecies_class

alias of CycleSpecies
5.1.317 Empty Species

class sage.combinat.species.empty_species.EmptySpecies(min=None, max=None, weight=None)
Bases: GenericCombinatorialSpecies, UniqueRepresentation

Returns the empty species. This species has no structure at all. It is the zero of the semi-ring of species.

EXAMPLES:

```python
sage: X = species.EmptySpecies(); X
Empty species
sage: X.structures([]).list()
[]
sage: X.structures([1]).list()
[]
sage: X.structures([1,2]).list()
[]
sage: X.generating_series()[0:4]
[0, 0, 0, 0]
sage: X.isotype_generating_series()[0:4]
[0, 0, 0, 0]
sage: X.cycle_index_series()[0:4]
[0, 0, 0, 0]
```

The empty species is the zero of the semi-ring of species. The following tests that it is neutral with respect to addition:

```python
sage: Empt = species.EmptySpecies()
sage: S = species.CharacteristicSpecies(2)
sage: X = S + Empt
sage: X == S  # TODO: Not Implemented
True
sage: (X.generating_series()[0:4] ==
    ....: S.generating_series()[0:4])
True
sage: (X.isotype_generating_series()[0:4] ==
    ....: S.isotype_generating_series()[0:4])
True
sage: (X.cycle_index_series()[0:4] ==
    ....: S.cycle_index_series()[0:4])
True
```

The following tests that it is the zero element with respect to multiplication:

```python
sage: Y = Empt*S
sage: Y == Empt  # TODO: Not Implemented
True
sage: Y.generating_series()[0:4]
[0, 0, 0, 0]
sage: Y.isotype_generating_series()[0:4]
[0, 0, 0, 0]
sage: Y.cycle_index_series()[0:4]
[0, 0, 0, 0]
```
alias of *EmptySpecies*

5.1.318 **Functorial composition species**

```python
class sage.combinat.species.functorial_composition_species.FunctorialCompositionSpecies(F, G,
    min=None,
    max=None,
    weight=None)
```

**Bases:** *GenericCombinatorialSpecies*

Returns the functorial composition of two species.

**EXAMPLES:**

```python
sage: E = species.SetSpecies()
sage: E2 = species.SetSpecies(size=2)
sage: WP = species.SubsetSpecies()
sage: P2 = E2*E
sage: G = WP.functorial_composition(P2)
sage: G.isotype_generating_series()[0:5]
[1, 1, 2, 4, 11]
sage: G = species.SimpleGraphSpecies()
sage: c = G.generating_series()[0:2]
sage: type(G)
<class 'sage.combinat.species.functorial_composition_species.'
FunctorialCompositionSpecies'>
sage: G == loads(dumps(G))
True
sage: G._check()  # False due to isomorphism types not being implemented
False
```

**weight_ring()**

Returns the weight ring for this species. This is determined by asking Sage’s coercion model what the result is when you multiply (and add) elements of the weight rings for each of the operands.

**EXAMPLES:**

```python
sage: G = species.SimpleGraphSpecies()
sage: G.weight_ring()
Rational Field
```

sage.combinat.species.functorial_composition_species.FunctorialCompositionSpecies_class

alias of *FunctorialCompositionSpecies*

```python
class sage.combinat.species.functorial_composition_species.FunctorialCompositionStructure(parent,
    labels, list)
```

**Bases:** *GenericSpeciesStructure*
5.1.319 Generating Series

This file makes a number of extensions to lazy power series by endowing them with some semantic content for how they're to be interpreted.

This code is based on the work of Ralf Hemmecke and Martin Rubey’s Aldor-Combinat, which can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/aldor/combinat/index.html. In particular, the relevant section for this file can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/AldorCombinat/combinatse10.html. One notable difference is that we use power-sum symmetric functions as the coefficients of our cycle index series.

REFERENCES:

```python
class sage.combinat.species.generating_series.CycleIndexSeries(parent, coeff_stream)
    Bases: LazySymmetricFunction

coefficient_cycle_type(t)
    Return the coefficient of a cycle type t in self.

    EXAMPLES:
    sage: from sage.combinat.species.generating_series import CycleIndexSeriesRing
    sage: p = SymmetricFunctions(QQ).power()
    sage: CIS = CycleIndexSeriesRing(QQ)
    sage: f = CIS([0, p([1]), 2*p([1,1]),3*p([2,1])])
    sage: f.coefficient_cycle_type([1])
    1
    sage: f.coefficient_cycle_type([1,1])
    2
    sage: f.coefficient_cycle_type([2,1])
    3

count(t)
    Return the number of structures corresponding to a certain cycle type t.

    EXAMPLES:
    sage: from sage.combinat.species.generating_series import CycleIndexSeriesRing
    sage: p = SymmetricFunctions(QQ).power()
    sage: CIS = CycleIndexSeriesRing(QQ)
    sage: f = CIS([0, p([1]), 2*p([1,1]),3*p([2,1])])
    sage: f.count([1])
    1
    sage: f.count([1,1])
    4
    sage: f.count([2,1])
    6

derivative(n=1)
    Return the species-theoretic n-th derivative of self.

    For a cycle index series \( F(p_1, p_2, p_3, \ldots) \), its derivative is the cycle index series \( F' = D_{p_1} F \) (that is, the formal derivative of \( F \) with respect to the variable \( p_1 \)).

    If \( F \) is the cycle index series of a species \( S \) then \( F' \) is the cycle index series of an associated species \( S' \) of \( S \)-structures with a “hole”.

    EXAMPLES:
    The species \( E \) of sets satisfies the relationship \( E' = E \):
The species $C$ of cyclic orderings and the species $L$ of linear orderings satisfy the relationship $C' = L$:

```python
sage: C = species.CycleSpecies().cycle_index_series()
sage: L = species.LinearOrderSpecies().cycle_index_series()
sage: L[:8] == C.derivative()[:8]
True
```

**exponential()**

Return the species-theoretic exponential of self.

For a cycle index $Z_F$ of a species $F$, its exponential is the cycle index series $Z_E \circ Z_F$, where $Z_E$ is the `ExponentialCycleIndexSeries()`.

The exponential $Z_E \circ Z_F$ is then the cycle index series of the species $E \circ F$ of “sets of $F$-structures”.

**EXAMPLES:**

Let $BT$ be the species of binary trees, $BF$ the species of binary forests, and $E$ the species of sets. Then we have $BF = E \circ BT$:

```python
sage: BT = species.BinaryTreeSpecies().cycle_index_series()
sage: BF = species.BinaryForestSpecies().cycle_index_series()
sage: BT.exponential().isotype_generating_series()[:8] == BF.isotype_generating_series()[:8]
True
```

**generating_series()**

Return the generating series of self.

**EXAMPLES:**

```python
sage: P = species.PartitionSpecies()
sage: cis = P.cycle_index_series()
sage: f = cis.generating_series()
sage: f[:5]
[1, 1, 1, 5/6, 5/8]
```

**isotype_generating_series()**

Return the isotype generating series of self.

**EXAMPLES:**

```python
sage: P = species.PermutationSpecies()
sage: cis = P.cycle_index_series()
sage: f = cis.isotype_generating_series()
sage: f[:10]
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30]
```

**logarithm()**

Return the combinatorial logarithm of self.

For a cycle index $Z_F$ of a species $F$, its logarithm is the cycle index series $Z_\Omega \circ Z_F$, where $Z_\Omega$ is the `LogarithmCycleIndexSeries()`.

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The logarithm $Z_\Omega \circ Z_F$ is then the cycle index series of the (virtual) species $\Omega \circ F$ of “connected $F$-structures”. In particular, if $F = E^+ \circ G$ for $E^+$ the species of nonempty sets and $G$ some other species, then $\Omega \circ F = G$.

**EXAMPLES:**

Let $G$ be the species of nonempty graphs and $CG$ be the species of nonempty connected graphs. Then $G = E^+ \circ CG$, so $CG = \Omega \circ G$:

```python
sage: G = species.SimpleGraphSpecies().cycle_index_series() - 1
sage: from sage.combinat.species.generating_series import LogarithmCycleIndexSeries
sage: CG = LogarithmCycleIndexSeries()(G)
sage: CG.isotype_generating_series()[0:8]
[0, 1, 1, 2, 6, 21, 112, 853]
```

**pointing()**

Return the species-theoretic pointing of self.

For a cycle index $F$, its pointing is the cycle index series $F^* = p_1 \cdot F'$.

If $F$ is the cycle index series of a species $S$ then $F^*$ is the cycle index series of an associated species $S^*$ of $S$-structures with a marked “root”.

**EXAMPLES:**

The species $E^*$ of “pointed sets” satisfies $E^* = X \cdot E$:

```python
sage: E = species.SetSpecies().cycle_index_series()
sage: X = species.SingletonSpecies().cycle_index_series()
sage: E.pointing()[0:8] == (X*E)[0:8]
True
```

**class sage.combinat.species.generating_series.CycleIndexSeriesRing(base_ring, sparse=True)**

Bases: `LazySymmetricFunctions`

Return the ring of cycle index series over $R$.

This is the ring of formal power series $\Lambda[x]$, where $\Lambda$ is the ring of symmetric functions over $R$ in the $p$-basis. Its purpose is to house the cycle index series of species (in a somewhat nonstandard notation tailored to Sage): If $F$ is a species, then the **cycle index series** of $F$ is defined to be the formal power series

$$
\sum_{n \geq 0} \frac{1}{n!} \left( \sum_{\sigma \in S_n} \text{fix } F[\sigma] \prod_{z \text{ is a cycle of } \sigma} p_{\text{length of } z} x^n \right) \in \Lambda Q[x],
$$

where $\text{fix } F[\sigma]$ denotes the number of fixed points of the permutation $F[\sigma]$ of $F[n]$. We notice that this power series is “equigraded” (meaning that its $x^n$-coefficient is homogeneous of degree $n$). A more standard convention in combinatorics would be to use $x_i$ instead of $p_i$, and drop the $x$ (that is, evaluate the above power series at $x = 1$); but this would be more difficult to implement in Sage, as it would be an element of a power series ring in infinitely many variables.

Note that it is just a **LazyPowerSeriesRing** (whose base ring is $\Lambda$) whose elements have some extra methods.

**EXAMPLES:**

```python
sage: from sage.combinat.species.generating_series import CycleIndexSeriesRing
sage: R = CycleIndexSeriesRing(QQ); R
Cycle Index Series Ring over Rational Field
```

(continues on next page)
sage: p = SymmetricFunctions(QQ).p()
sage: R(lambda n: p[n])


**Element**

alias of *CycleIndexSeries*

`sage.combinat.species.generating_series.ExponentialCycleIndexSeries()`

Return the cycle index series of the species \( E \) of sets.

This cycle index satisfies

\[ Z_E = \sum_{n \geq 0} \sum_{\lambda \vdash n} \frac{p_{\lambda}}{z_\lambda}. \]

**EXAMPLES:**

```python
sage: from sage.combinat.species.generating_series import _
    _ExponentialCycleIndexSeries
sage: R = ExponentialCycleIndexSeries()[:5]
```

**class** `sage.combinat.species.generating_series.ExponentialGeneratingSeries(parent, coeff_stream)`

Bases: `LazyPowerSeries`

A class for ordinary generating series.

Note that it is just a `LazyPowerSeries` whose elements have some extra methods.

**EXAMPLES:**

```python
sage: from sage.combinat.species.generating_series import _
    _OrdinaryGeneratingSeriesRing
sage: R = OrdinaryGeneratingSeriesRing(QQ)
sage: f = R(lambda n: n)
sage: f
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + O(z^7)
```

**count** \((n)\)

Return the number of structures of size \( n \).

**EXAMPLES:**

```python
sage: from sage.combinat.species.generating_series import _
    _ExponentialGeneratingSeriesRing
sage: R = ExponentialGeneratingSeriesRing(QQ)
sage: f = R(lambda n: 1)
sage: [f.count(i) for i in range(7)]
[1, 1, 2, 6, 24, 120, 720]
```

**counts** \((n)\)

Return the number of structures on a set for size \( i \) for each \( i \) in `range(n)`.

**EXAMPLES:**

5.1. Comprehensive Module List
functorial_composition(y)

Return the exponential generating series which is the functorial composition of \texttt{self} with \texttt{y}.

If \( f = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \) and \( g = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!} \), then functorial composition \( f \square g \) is defined as

\[
 f \square g = \sum_{n=0}^{\infty} f_n \frac{g_{n}}{n!} .
\]

REFERENCES:

• Section 2.2 of [BLL].

EXAMPLES:

sage: G = species.SimpleGraphSpecies()
sage: g = G.generating_series()
sage: [g.coefficient(i) for i in range(10)]
[1, 1, 1, 4/3, 8/3, 128/15, 2048/45, 131072/315, 2097152/315, 536870912/2835]

sage: E = species.SetSpecies()
sage: E2 = E.restricted(min=2, max=3)
sage: WP = species.SubsetSpecies()
sage: P2 = E2*E
sage: g1 = WP.generating_series()
sage: g2 = P2.generating_series()
sage: g1.functorial_composition(g2)[:10]
[1, 1, 1, 4/3, 8/3, 128/15, 2048/45, 131072/315, 2097152/315, 536870912/2835]
Return the cycle index series of the virtual species $\Omega$, the compositional inverse of the species $E^+$ of nonempty sets.

The notion of virtual species is treated thoroughly in [BLL]. The specific algorithm used here to compute the cycle index of $\Omega$ is found in [Labelle2008].

**EXAMPLES:**

The virtual species $\Omega$ is ‘properly virtual’, in the sense that its cycle index has negative coefficients:

```plaintext
sage: from sage.combinat.species.generating_series import LogarithmCycleIndexSeries
sage: LogarithmCycleIndexSeries()[0:4]
(0, p[1], -1/2*p[1, 1] - 1/2*p[2], 1/3*p[1, 1, 1] - 1/3*p[3])
```

Its defining property is that $\Omega \circ E^+ = E^+ \circ \Omega = X$ (that is, that composition with $E^+$ in both directions yields the multiplicative identity $X$):

```plaintext
sage: Eplus = sage.combinat.species.set_species.SetSpecies(min=1).cycle_index_series()
sage: LogarithmCycleIndexSeries()(Eplus)[0:4]
(0, p[1], 0, 0)
```

### class sage.combinat.species.generating_series.OrdinaryGeneratingSeries(parent, coeff_stream)

**Bases:** LazyPowerSeries

A class for ordinary generating series.

Note that it is just a LazyPowerSeries whose elements have some extra methods.

**EXAMPLES:**

```plaintext
sage: from sage.combinat.species.generating_series import OrdinaryGeneratingSeriesRing
sage: R = OrdinaryGeneratingSeriesRing(QQ)
sage: f = R(lambda n: n)
sage: f
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + O(z^7)
```

**count**($n$)

Return the number of structures on a set of size $n$.

**INPUT:**

- *n* – the size of the set

**EXAMPLES:**

```plaintext
sage: from sage.combinat.species.generating_series import OrdinaryGeneratingSeriesRing
sage: R = OrdinaryGeneratingSeriesRing(QQ)
sage: f = R(range(20))
sage: f.count(10)
10
```

**counts**($n$)

Return the number of structures on a set for size $i$ for each $i$ in range($n$).

**EXAMPLES:**

```plaintext
sage: from sage.combinat.species.generating_series import OrdinaryGeneratingSeriesRing
sage: R = OrdinaryGeneratingSeriesRing(QQ)
sage: f = R(range(20))
sage: f.counts(10)
10
```
```python
sage: from sage.combinat.species.generating_series import OrdinaryGeneratingSeriesRing
sage: R = OrdinaryGeneratingSeriesRing(QQ)
sage: f = R(range(20))
sage: f.counts(10)
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

class sage.combinat.species.generating_series.OrdinaryGeneratingSeriesRing(base_ring)

Bases: LazyPowerSeriesRing

Return the ring of ordinary generating series over \( R \).

Note that it is just a LazyPowerSeriesRing whose elements have some extra methods.

EXAMPLES:

```python
sage: from sage.combinat.species.generating_series import OrdinaryGeneratingSeriesRing
sage: R = OrdinaryGeneratingSeriesRing(QQ); R
Lazy Taylor Series Ring in z over Rational Field
sage: [R(lambda n: 1).coefficient(i) for i in range(4)]
[1, 1, 1, 1]
sage: R(lambda n: 1).counts(4)
[1, 1, 1, 1]
sage: R == loads(dumps(R))
True
```

Element

alias of OrdinaryGeneratingSeries

5.1.320 Examples of Combinatorial Species

sage.combinat.species.library.BinaryForestSpecies()

Return the species of binary forests.

Binary forests are defined as sets of binary trees.

EXAMPLES:

```python
sage: F = species.BinaryForestSpecies()
sage: F.generating_series().counts(10)
[1, 1, 3, 19, 193, 2721, 49171, 1084483, 28245729, 848456353]
sage: F.isotype_generating_series().counts(10)
[1, 1, 2, 4, 10, 26, 77, 235, 758, 2504]
sage: F.cycle_index_series()[:7]
[p[],
p[1],
3/2*p[1, 1] + 1/2*p[2],
19/6*p[1, 1, 1] + 1/2*p[2, 1] + 1/3*p[3],
907/48*p[1, 1, 1, 1, 1] + 19/12*p[2, 1, 1, 1] + 5/8*p[2, 2, 1] + 1/2*p[3, 1, 1] +
1/6*p[3, 2] + 1/4*p[4, 1] + 1/5*p[5],
49171/720*p[1, 1, 1, 1, 1, 1] + 193/48*p[2, 1, 1, 1, 1] + 15/16*p[2, 2, 1, 1] + 61/
```

Combinatorics, Release 10.0


**sage.combinat.species.library.BinaryTreeSpecies()**

Return the species of binary trees on \(n\) leaves.

The species of binary trees \(B\) is defined by \(B = X + B \cdot B\), where \(X\) is the singleton species.

**EXAMPLES:**

```python
sage: B = species.BinaryTreeSpecies()
sage: B.generating_series().counts(10)
[0, 1, 2, 12, 120, 1680, 30240, 665280, 17297280, 518918400]
sage: B.isotype_generating_series().counts(10)
[0, 1, 1, 2, 5, 14, 42, 132, 429, 1430]
sage: B._check()
True
```

**sage.combinat.species.library.SimpleGraphSpecies()**

Return the species of simple graphs.

**EXAMPLES:**

```python
sage: S = species.SimpleGraphSpecies()
sage: S.generating_series().counts(10)
[1, 1, 2, 8, 64, 1024, 32768, 2097152, 268435456, 68719476736]
sage: S.cycle_index_series()[:5]
[p[],
p[1],
p[1, 1] + p[2],
4/3*p[1, 1, 1] + 2*p[2, 1] + 2/3*p[3],
sage: S.isotype_generating_series()[:6]
[1, 1, 2, 4, 11, 34]
```

### 5.1.321 Linear-order Species

**class** `sage.combinat.species.linear_order_species.LinearOrderSpecies`(min=None, max=None, weight=None)

Bases: `GenericCombinatorialSpecies`, `UniqueRepresentation`

Returns the species of linear orders.

**EXAMPLES:**
sage: L = species.LinearOrderSpecies()
sage: L.generating_series()[0:5]
[1, 1, 1, 1, 1]
sage: L = species.LinearOrderSpecies()
sage: L._check()
True
sage: L == loads(dumps(L))
True

class sage.combinat.species.linear_order_species.LinearOrderSpeciesStructure(parent, labels, list)

Bases: GenericSpeciesStructure

automorphism_group()

Returns the group of permutations whose action on this structure leave it fixed. For the species of linear orders, there is no non-trivial automorphism.

EXAMPLES:

.. code-block:: python

    sage: F = species.LinearOrderSpecies()
sage: a = F.structures(["a", "b", "c"])[0]; a
    ["a", "b", "c"]
sage: a.automorphism_group()
    Symmetric group of order 1! as a permutation group

canonical_label()

EXAMPLES:

.. code-block:: python

    sage: P = species.LinearOrderSpecies()
sage: s = P.structures(["a", "b", "c"]).random_element()
sage: s.canonical_label()
    ['a', 'b', 'c']

transport(perm)

Returns the transport of this structure along the permutation perm.

EXAMPLES:

.. code-block:: python

    sage: F = species.LinearOrderSpecies()
sage: a = F.structures(["a", "b", "c"])[0]; a
    ['a', 'b', 'c']
sage: p = PermutationGroupElement((1,2))
sage: a.transport(p)
    ['b', 'a', 'c']
sage.combinat.species.linear_order_species.LinearOrderSpecies_class

alias of LinearOrderSpecies
5.1.322 Miscellaneous Functions

sage.combinat.species.misc.accept_size(f)

The purpose of this decorator is to change calls like species.SetSpecies(size=1) to species.SetSpecies(min=1, max=2). This is to make caching species easier and to restrict the number of parameters that the lower level code needs to know about.

EXAMPLES:

```
sage: from sage.combinat.species.misc import accept_size
sage: def f(*args, **kwds):
    ....:     print("{} {}".format(args, sorted(kwds.items())))
sage: f = accept_size(f)
sage: f(min=1)
() [('min', 1)]
sage: f(size=2)
() [('max', 3), ('min', 2)]
```

sage.combinat.species.misc.change_support(perm, support, change_perm=None)

Changes the support of a permutation defined on [1, ..., n] to support.

EXAMPLES:

```
sage: from sage.combinat.species.misc import change_support
sage: p = PermutationGroupElement((1,2,3)); p
(1,2,3)
sage: change_support(p, [3,4,5])
(3,4,5)
```

5.1.323 Partition Species

class sage.combinat.species.partition_species.PartitionSpecies(min=None, max=None, weight=None)

Bases: GenericCombinatorialSpecies

Returns the species of partitions.

EXAMPLES:

```
sage: P = species.PartitionSpecies()
sage: P.generating_series()[:5]
[1, 1, 1, 5/6, 5/8]
sage: P.isotype_generating_series()[:5]
[1, 1, 2, 3, 5]
sage: P = species.PartitionSpecies()
sage: P._check()
True
sage: P == loads(dumps(P))
True
```

class sage.combinat.species.partition_species.PartitionSpeciesStructure(parent, labels, list)

Bases: GenericSpeciesStructure

EXAMPLES:
sage: from sage.combinat.species.partition_species import PartitionSpeciesStructure
sage: P = species.PartitionSpecies()
sage: s = PartitionSpeciesStructure(P, ['a', 'b', 'c'], [[1, 2], [3]]); s
{'a', 'b'}, {'c'}
sage: s == loads(dumps(s))
True

automorphism_group()

Returns the group of permutations whose action on this set partition leave it fixed.

EXAMPLES:

sage: p = PermutationGroupElement((2,3))
sage: a = PartitionSpeciesStructure(None, [2, 3, 4], [[1, 2], [3]]);
a
{{2, 3}, {4}}
sage: a.automorphism_group()
Permutation Group with generators [(1,2)]

canonical_label()

EXAMPLES:

sage: P = species.PartitionSpecies()
sage: S = P.structures(['a', 'b', 'c'])
sage: [s.canonical_label() for s in S]
[['a', 'b', 'c'],
 {'a', 'b'}, {'c'}],
 {'a', 'b'}, {'c'}],
 {'a', 'b'}, {'c'}],
 {'a', 'b'}, {'c'}],
 {'a', 'b'}, {'c'}]

change_labels(labels)

Return a relabelled structure.

INPUT:

* labels, a list of labels.

OUTPUT:

A structure with the i-th label of self replaced with the i-th label of the list.

EXAMPLES:

sage: a = PartitionSpeciesStructure(None, [2, 3, 4], [[1, 2], [3]]);
a
{{2, 3}, {4}}
sage: a.change_labels([1, 2, 3])
{{1, 2}, {3}}

transport(perm)

Returns the transport of this set partition along the permutation perm. For set partitions, this is the direct product of the automorphism groups for each of the blocks.
EXAMPLES:

```python
sage: p = PermutationGroupElement((2,3))
sage: from sage.combinat.species.partition_species import PartitionSpeciesStructure
sage: a = PartitionSpeciesStructure(None, [2,3,4], [[1,2],[3]]); a
{{2, 3}, {4}}
sage: a.transport(p)
{{2, 4}, {3}}
```

sage.combinat.species.partition_species.PartitionSpecies_class
alias of PartitionSpecies

5.1.324 Permutation species

class sage.combinat.species.permutation_species.PermutationSpecies(min=None, max=None, weight=None)

Bases: GenericCombinatorialSpecies, UniqueRepresentation

Returns the species of permutations.

EXAMPLES:

```python
sage: P = species.PermutationSpecies()
sage: P.generating_series()[0:5]
[1, 1, 1, 1, 1]
sage: P.isotype_generating_series()[0:5]
[1, 1, 2, 3, 5]
sage: P = species.PermutationSpecies()
sage: c = P.generating_series()[0:3]
sage: P._check()
True
sage: P == loads(dumps(P))
True
```

class sage.combinat.species.permutation_species.PermutationSpeciesStructure(parent, labels, list)

Bases: GenericSpeciesStructure

automorphism_group()

Returns the group of permutations whose action on this structure leave it fixed.

EXAMPLES:

```python
sage: set_random_seed(0)
sage: p = PermutationGroupElement((2,3,4))
sage: P = species.PermutationSpecies()
sage: a = P.structures(['a', 'b', 'c', 'd'])[2]; a
['a', 'c', 'b', 'd']
sage: a.automorphism_group()
Permutation Group with generators [(2,3), (1,4)]
```
sage: [a.transport(perm) for perm in a.automorphism_group()]
[['a', 'c', 'b', 'd'],
 ['a', 'c', 'b', 'd'],
 ['a', 'c', 'b', 'd'],
 ['a', 'c', 'b', 'd']]

canonical_label()

EXAMPLES:

sage: P = species.PermutationSpecies()
sage: S = P.structures(['a', 'b', 'c'])
sage: [s.canonical_label() for s in S]
[['a', 'b', 'c'],
 ['b', 'a', 'c'],
 ['b', 'c', 'a'],
 ['b', 'c', 'a']]

permutation_group_element()

Returns self as a permutation group element.

EXAMPLES:

sage: p = PermutationGroupElement((2,3,4))
sage: P = species.PermutationSpecies()
sage: a = P.structures(['a', 'b', 'c', 'd'])[2]; a
['a', 'c', 'b', 'd']
sage: a.permutation_group_element()
(2,3)

transport(perm)

Returns the transport of this structure along the permutation perm.

EXAMPLES:

sage: p = PermutationGroupElement((2,3,4))
sage: P = species.PermutationSpecies()
sage: a = P.structures(['a', 'b', 'c', 'd'])[2]; a
['a', 'c', 'b', 'd']
sage: a.transport(p)
['a', 'd', 'c', 'b']
5.1.325 Product species

class sage.combinat.species.product_species.ProductSpecies(F, G, min=None, max=None, weight=None)

Bases: GenericCombinatorialSpecies, UniqueRepresentation

EXAMPLES:

sage: X = species.SingletonSpecies()
sage: A = X*X
sage: A.generating_series()[0:4]
[0, 0, 1, 0]
sage: P = species.PermutationSpecies()
sage: F = P * P; F
Product of (Permutation species) and (Permutation species)
sage: F == loads(dumps(F))
True
sage: F._check()
True

left_factor()

Returns the left factor of this product.

EXAMPLES:

sage: P = species.PermutationSpecies()
sage: X = species.SingletonSpecies()
sage: F = P*X
sage: F.left_factor()
Permutation species

right_factor()

Returns the right factor of this product.

EXAMPLES:

sage: P = species.PermutationSpecies()
sage: X = species.SingletonSpecies()
sage: F = P*X
sage: F.right_factor()
Singleton species

weight_ring()

Returns the weight ring for this species. This is determined by asking Sage’s coercion model what the result is when you multiply (and add) elements of the weight rings for each of the operands.

EXAMPLES:

sage: S = species.SetSpecies()
sage: C = S*S
sage: C.weight_ring()
Rational Field
```python
sage: S = species.SetSpecies(weight=QQ['t'].gen())
sage: C = S*S
sage: C.weight_ring()
Univariate Polynomial Ring in t over Rational Field

sage: S = species.SetSpecies()
sage: C = (S*S).weighted(QQ['t'].gen())
sage: C.weight_ring()
Univariate Polynomial Ring in t over Rational Field
```

```
class sage.combinat.species.product_species.ProductSpeciesStructure(parent, labels, subset, left, right)
    Bases: GenericSpeciesStructure

    automorphism_group()
        EXAMPLES:
```
```python
sage: p = PermutationGroupElement((2,3))
sage: S = species.SetSpecies()
sage: F = S * S
sage: a = F.structures([1,2,3,4])[1]; a
{1}*{2, 3, 4}
sage: a.automorphism_group()
Permutation Group with generators [(2,3), (2,3,4)]
```
```
sage: [a.transport(g) for g in a.automorphism_group()]
[[1]*{2, 3, 4},
 {1}*{2, 3, 4},
 {1}*{2, 3, 4},
 {1}*{2, 3, 4},
 {1}*{2, 3, 4}]
```
```
sage: a = F.structures([1,2,3,4])[8]; a
{2, 3}*{1, 4}
sage: [a.transport(g) for g in a.automorphism_group()]
[[2, 3]*{1, 4}, {2, 3}*{1, 4}, {2, 3}*{1, 4}, {2, 3}*{1, 4}]
```
```
canonical_label()
    EXAMPLES:
```
```python
sage: S = species.SetSpecies()
sage: F = S * S
sage: S = F.structures(['a','b','c']).list(); S
[[['a', 'b', 'c'],
  ['a']*['b', 'c'],
  ['b']*['a', 'c'],
  ['c']*['a', 'b'],
  ['a', 'b']*['c'],
  ['a', 'c']*['b'],
  ['b', 'c']*['a'],
  ['a', 'b', 'c']*{}]
```
sage: F.isotypes(['a','b','c']).cardinality()
4
sage: [s.canonical_label() for s in S]
[{}*{'a', 'b', 'c'},
 {'a'}*{'b', 'c'},
 {'a'}*{'b', 'c'},
 {'a', 'b'}*{'c'},
 {'a', 'b'}*{'c'},
 {'a', 'b'}*{'c'},
 {'a', 'b', 'c'}*{}]  

**change_labels(labels)**

Return a relabelled structure.

**INPUT:**

* labels, a list of labels.

**OUTPUT:**

A structure with the i-th label of self replaced with the i-th label of the list.

**EXAMPLES:**

```python
sage: S = species.SetSpecies()
sage: F = S * S
sage: a = F.structures(['a','b','c'])[0]; a
{}*{'a', 'b', 'c'}
sage: a.change_labels([1,2,3])
{}*{1, 2, 3}
```

**transport(perm)**

**EXAMPLES:**

```python
sage: p = PermutationGroupElement((2,3))
sage: S = species.SetSpecies()
sage: F = S * S
sage: a = F.structures(['a','b','c'])[4]; a
{'a', 'b'}*{'c'}
sage: a.transport(p)
{'a', 'c'}*{'b'}
```

```
sage.combinat.species.product_species.ProductSpecies_class
alias of ProductSpecies
```
5.1.326 Recursive Species

class sage.combinat.species.recursive_species.CombinatorialSpecies(min=None)
    Bases: GenericCombinatorialSpecies

EXAMPLES:

```
sage: F = CombinatorialSpecies()
sage: loads(dumps(F))
Combinatorial species

sage: X = species.SingletonSpecies()
sage: E = species.EmptySetSpecies()
sage: L = CombinatorialSpecies()
sage: L.define(E+X*L)
sage: L.generating_series()[0:4]
[1, 1, 1, 1]
sage: LL = loads(dumps(L))
sage: LL.generating_series()[0:4]
[1, 1, 1, 1]
```

define(x)
    Define self to be equal to the combinatorial species x.
    This is used to define combinatorial species recursively. All of the real work is done by calling the .set() method for each of the series associated to self.

EXAMPLES: The species of linear orders L can be recursively defined by $L = 1 + X \ast L$ where 1 represents the empty set species and X represents the singleton species.

```
sage: X = species.SingletonSpecies()
sage: E = species.EmptySetSpecies()
sage: L = CombinatorialSpecies()
sage: L.define(E+X*L)
sage: L.generating_series()[0:4]
[1, 1, 1, 1]
sage: L.structures([1,2,3]).cardinality()
6
sage: L.structures([1,2,3]).list()
[[1*2*(3*{})),
  1*(3*(2*{})),
  2*(1*(3*{})),
  2*(3*(1*{})),
  3*(1*(2*{})),
  3*(2*(1*{}))]
```

```
sage: L = species.LinearOrderSpecies()
sage: L.generating_series()[0:4]
[1, 1, 1, 1]
sage: L.structures([1,2,3]).cardinality()
6
sage: L.structures([1,2,3]).list()
[[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]]```
weight_ring()
EXAMPLES:

```
sage: F = species.CombinatorialSpecies()
sage: F.weight_ring()
Rational Field

sage: X = species.SingletonSpecies()
sage: E = species.EmptySetSpecies()
sage: L = CombinatorialSpecies()
sage: L.define(E+X*L)
sage: L.weight_ring()
Rational Field
```

class sage.combinat.species.recursive_species.CombinatorialSpeciesStructure(parent, s, **options)

Bases: SpeciesStructureWrapper

5.1.327 Set Species

class sage.combinat.species.set_species.SetSpecies(min=None, max=None, weight=None)

Bases: GenericCombinatorialSpecies, UniqueRepresentation

Returns the species of sets.

EXAMPLES:

```
sage: E = species.SetSpecies()
sage: E.structures([1,2,3]).list()
[[1, 2, 3]]
sage: E.isotype_generating_series()[0:4]
[1, 1, 1, 1]
sage: S = species.SetSpecies()
sage: c = S.generating_series()[0:3]
sage: S._check()
True
sage: S == loads(dumps(S))
True
```

class sage.combinat.species.set_species.SetSpeciesStructure(parent, labels, list)

Bases: GenericSpeciesStructure

automorphism_group()

Returns the group of permutations whose action on this set leave it fixed. For the species of sets, there is only one isomorphism class, so every permutation is in its automorphism group.

EXAMPLES:

```
sage: F = species.SetSpecies()
sage: a = F.structures(['a', 'b', 'c']).random_element(); a
{'a', 'b', 'c'}
sage: a.automorphism_group()
Symmetric group of order 3! as a permutation group
```
canonical_label()

EXAMPLES:

```python
sage: S = species.SetSpecies()
sage: a = S.structures(['a', 'b', 'c']).random_element(); a
{'a', 'b', 'c'}
sage: a.canonical_label()
{'a', 'b', 'c'}
```

transport(perm)

Returns the transport of this set along the permutation perm.

EXAMPLES:

```python
sage: F = species.SetSpecies()
sage: a = F.structures(['a', 'b', 'c']).random_element(); a
{'a', 'b', 'c'}
sage: p = PermutationGroupElement((1,2))
sage: a.transport(p)
{'a', 'b', 'c'}
```

sage.combinat.species.set_species.SetSpecies_class
alias of SetSpecies

5.1.328 Combinatorial Species

This file defines the main classes for working with combinatorial species, operations on them, as well as some implementations of basic species required for other constructions.

This code is based on the work of Ralf Hemmecke and Martin Rubey’s Aldor-Combinat, which can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/aldor/combinat/index.html. In particular, the relevant section for this file can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/AldorCombinat/combinatse8.html.

Weighted Species:

As a first application of weighted species, we count unlabeled ordered trees by total number of nodes and number of internal nodes. To achieve this, we assign a weight of 1 to the leaves and of $q$ to internal nodes:

```python
sage: q = QQ['q'].gen()
sage: leaf = species.SingletonSpecies()
sage: internal_node = species.SingletonSpecies(weight=q)
sage: L = species.LinearOrderSpecies(min=1)
sage: T = species.CombinatorialSpecies(min=1)
sage: T.define(leaf + internal_node*L(T))
sage: T.isotype_generating_series()[0:6]
[0, 1, q, q^2 + q, q^3 + 3*q^2 + q, q^4 + 6*q^3 + 6*q^2 + q]
```

Consider the following:

```python
sage: T.isotype_generating_series().coefficient(4)
q^3 + 3*q^2 + q
```

This means that, among the trees on 4 nodes, one has a single internal node, three have two internal nodes, and one has three internal nodes.
class sage.combinat.species.species.GenericCombinatorialSpecies

Bases: SageObject

algebraic_equation_system()

Return a system of algebraic equations satisfied by this species.

The nodes are numbered in the order that they appear as vertices of the associated digraph.

EXAMPLES:

```sage
B = species.BinaryTreeSpecies()
sage: B.algebraic_equation_system()
[-node3^2 + node1, -node1 + node3 + (-z)]
```

```sage
sorted(B.digraph().vertex_iterator(), key=str)
[Combinatorial species with min=1,
 Product of (Combinatorial species with min=1) and (Combinatorial species with min=1),
 Singleton species,
 Sum of (Singleton species) and (Product of (Combinatorial species with min=1) and (Combinatorial species with min=1))]
```

```sage
B.algebraic_equation_system()[0].parent()
Multivariate Polynomial Ring in node0, node1, node2, node3 over Fraction Field of Univariate Polynomial Ring in z over Rational Field
```

composition(g)

EXAMPLES:

```sage
S = species.SetSpecies()
sage: S(S)
Composition of (Set species) and (Set species)
```

cycle_index_series(base_ring=None)

Return the cycle index series for this species.

The cycle index series is a sequence of symmetric functions.

EXAMPLES:

```sage
P = species.PermutationSpecies()
sage: g = P.cycle_index_series()
sage: g[0:4]
```

digraph()

Return a directed graph where the vertices are the individual species that make up this one.

EXAMPLES:

```sage
X = species.SingletonSpecies()
sage: B = species.CombinatorialSpecies()
sage: B.define(X+B*B)
sage: g = B.digraph(); g
```
Multi-digraph on 4 vertices

```python
sage: sorted(g, key=str)
[(Combinatorial species, Product of (Combinatorial species) and (Combinatorial species), Singleton species, Sum of (Singleton species) and (Product of (Combinatorial species) and (Combinatorial species)))]
```

```python
d = {sp: i for i, sp in enumerate(g)}
g.relabel(d)
g.canonical_label().edges(sort=True)
```

```
[(0, 3, None), (2, 0, None), (2, 0, None), (3, 1, None), (3, 2, None)]
```

**functorial_composition**

Return the functorial composition of self with g.

**generating_series** *(base_ring=None)*

Return the generating series for this species.

This is an exponential generating series so the \( n \)-th coefficient of the series corresponds to the number of labeled structures with \( n \) labels divided by \( n! \).

**is_weighted**

Return True if this species has a nontrivial weighting associated with it.
**isotype_generating_series** (*base_ring=None*)

Return the isotype generating series for this species.

The \(n\)-th coefficient of this series corresponds to the number of isomorphism types for the structures on \(n\) labels.

**EXAMPLES:**

```python
sage: P = species.PermutationSpecies()
sage: g = P.isotype_generating_series()
sage: g[0:4]
[1, 1, 2, 3]
sage: g.counts(4)
[1, 1, 2, 3]
sage: P.isotypes([1, 2, 3]).list()
[[2, 3, 1], [2, 1, 3], [1, 2, 3]]
sage: len(_)
3
```

**isotypes** (*labels, structure_class=None*)

**EXAMPLES:**

```python
sage: F = CombinatorialSpecies()
sage: F.isotypes([1, 2, 3]).list()
Traceback (most recent call last):
  ...
NotImplementedError
```

**product** (*g*)

Return the product of self and g.

**EXAMPLES:**

```python
sage: P = species.PermutationSpecies()
sage: F = P * P; F
Product of (Permutation species) and (Permutation species)
```

**restricted** (*args, **kwds*)

Return the restriction of the species.

**INPUT:**

- `min` – optional integer
- `max` – optional integer

**EXAMPLES:**

```python
sage: S = species.SetSpecies().restricted(min=3); S
Set species with min=3
sage: S.structures([1, 2]).list()
[]
sage: S.generating_series()[0:5]
[0, 0, 0, 1/6, 1/24]
```

**structures** (*labels, structure_class=None*)

**EXAMPLES:**
Combinatorics, Release 10.0

```python
sage: F = CombinatorialSpecies()
sage: F.structures([1,2,3]).list()
Traceback (most recent call last):
  ... 
NotImplementedError
```

**sum(g)**

Return the sum of self and g.

EXAMPLES:

```python
sage: P = species.PermutationSpecies()
sage: F = P + P; F
Sum of (Permutation species) and (Permutation species)
sage: F.structures([1,2]).list()
[[1, 2], [2, 1], [1, 2], [2, 1]]
```

**weight_ring()**

Return the ring in which the weights of this species occur.

By default, this is just the field of rational numbers.

EXAMPLES:

```python
sage: species.SetSpecies().weight_ring()
Rational Field
```

**weighted(weight)**

Return a version of this species with the specified weight.

EXAMPLES:

```python
sage: t = ZZ['t'].gen()
sage: C = species.CycleSpecies(); C
Cyclic permutation species
sage: C.weighted(t)
Cyclic permutation species with weight=t
```

### 5.1.329 Species structures

We will illustrate the use of the structure classes using the “balls and bars” model for integer compositions. An integer composition of 6 such as `[2, 1, 3]` can be represented in this model as ‘oooooo’ where the 6 o’s correspond to the balls and the 2 ’s correspond to the bars. If BB is our species for this model, the it satisfies the following recursive definition:

\[
BB = o + oBB + o\|BB
\]

Here we define this species using the default structures:

```python
sage: ball = species.SingletonSpecies(); o = var('o')
sage: bar = species.EmptySetSpecies()
sage: BB = CombinatorialSpecies()
sage: BB.define(ball + ball*BB + ball*bar*BB)
sage: BB.isotypes([o]*3).list()
[o*(o*o), o*((o*{})*o), (o*{})*(o*o), (o*{})*((o*{})*o)]
```
If we ignore the parentheses, we can read off that the integer compositions are [3], [2, 1], [1, 2], and [1, 1, 1].

```python
class sage.combinat.species.structure.GenericSpeciesStructure(parent, labels, list):
    Bases: CombinatorialObject
    This is a base class from which the classes for the structures inherit.
    EXAMPLES:
    sage: from sage.combinat.species.structure import GenericSpeciesStructure
    sage: a = GenericSpeciesStructure(None, [2,3,4], [1,2,3])
    sage: a
    [2, 3, 4]
    sage: a.parent() is None
    True
    sage: a == loads(dumps(a))
    True
```

```python
change_labels(labels)
    Return a relabelled structure.
    INPUT:
        • labels, a list of labels.
    OUTPUT:
        A structure with the i-th label of self replaced with the i-th label of the list.
    EXAMPLES:
    sage: P = species.SubsetSpecies()
    sage: S = P.structures(['a', 'b', 'c'])
    sage: [s.change_labels([1,2,3]) for s in S]
    [{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
```

```python
is_isomorphic(x)
    EXAMPLES:
    sage: S = species.SetSpecies()
    sage: a = S.structures([1,2,3]).random_element(); a
    {1, 2, 3}
    sage: b = S.structures(['a','b','c']).random_element(); b
    {'a', 'b', 'c'}
    sage: a.is_isomorphic(b)
    True
```

```python
labels()
    Returns the labels used for this structure.
    Note: This includes labels which may not “appear” in this particular structure.
    EXAMPLES:
    sage: P = species.SubsetSpecies()
    sage: s = P.structures(['a', 'b', 'c']).random_element()
```

sage: s.labels()
['a', 'b', 'c']

parent()

Returns the species that this structure is associated with.

EXAMPLES:

.. code-block::

    sage: L = species.LinearOrderSpecies()
sage: a,b = L.structures([1,2])sage: a.parent()
Linear order species

class sage.combinat.species.structure.IsotypesWrapper(species, labels, structure_class)

Bases: SpeciesWrapper

A base class for the set of isotypes of a species with given set of labels. An object of this type is returned when you call the isotypes() method of a species.

EXAMPLES:

.. code-block::

    sage: F = species.SetSpecies()sage: S = F.isotypes([1,2,3])sage: S == loads(dumps(S))
    True

class sage.combinat.species.structure.SimpleIsotypesWrapper(species, labels, structure_class)

Bases: SpeciesWrapper

Warning: This is deprecated and currently not used for anything.

EXAMPLES:

.. code-block::

    sage: F = species.SetSpecies()sage: S = F.isotypes([1,2,3])sage: S == loads(dumps(S))
    True

class sage.combinat.species.structure.SimpleStructuresWrapper(species, labels, structure_class)

Bases: SpeciesWrapper

Warning: This is deprecated and currently not used for anything.

EXAMPLES:

.. code-block::

    sage: F = species.SetSpecies()sage: S = F.isotypes([1,2,3])sage: S == loads(dumps(S))
    True
sage.combinat.species.structure.SpeciesStructure
alias of GenericSpeciesStructure
class sage.combinat.species.structure.SpeciesStructureWrapper(parent, s, **options)
Bases: GenericSpeciesStructure
This is a class for the structures of species such as the sum species that do not provide “additional” structure. For example, if you have the sum $C$ of species $A$ and $B$, then a structure of $C$ will either be either something from $A$ or $B$. Instead of just returning one of these directly, a “wrapper” is put around them so that they have their parent is $C$ rather than $A$ or $B$:

```python
sage: X = species.SingletonSpecies()
sage: X2 = X+X
sage: s = X2.structures([1]).random_element(); s
1
sage: s.parent()
Sum of (Singleton species) and (Singleton species)
sage: from sage.combinat.species.structure import SpeciesStructureWrapper
sage: issubclass(type(s), SpeciesStructureWrapper)
True
```

EXAMPLES:

```python
sage: E = species.SetSpecies(); B = E+E
sage: s = B.structures([1,2,3]).random_element()
sage: s.parent()
Sum of (Set species) and (Set species)
sage: s == loads(dumps(s))
True
```

canonical_label()

EXAMPLES:

```python
sage: P = species.PartitionSpecies()
sage: s = (P+P).structures([1,2,3])[1]; s
{{1, 3}, {2}}
sage: s.canonical_label()
{{1, 2}, {3}}
```

change_labels(labels)

Return a relabelled structure.

INPUT:

- labels, a list of labels.

OUTPUT:

A structure with the i-th label of self replaced with the i-th label of the list.

EXAMPLES:

```python
sage: X = species.SingletonSpecies()
sage: X2 = X+X
sage: s = X2.structures([1]).random_element(); s
1
(continues on next page)
```
Combinatorics, Release 10.0

sage: s.change_labels(['a'])
'a'

transport(perm)

EXAMPLES:

sage: P = species.PartitionSpecies()
sage: s = (P+P).structures([1,2,3])[1]; s
{{1, 3}, {2}}
sage: s.transport(PermutationGroupElement((2,3)))
{{1, 2}, {3}}

class sage.combinat.species.structure.SpeciesWrapper(species, labels, iterator, generating_series, name, structure_class)

Bases: CombinatorialClass

This is a abstract base class for the set of structures of a species as well as the set of isotypes of the species.

Note: One typically does not use SpeciesWrapper directly, but instead instantiates one of its subclasses: StructuresWrapper or IsotypesWrapper.

EXAMPLES:

sage: from sage.combinat.species.structure import SpeciesWrapper
sage: F = species.SetSpecies()
sage: S = SpeciesWrapper(F, [1,2,3], '_structures', 'generating_series', 'Structures \rightarrow', None)
sage: S
Structures for Set species with labels [1, 2, 3]
sage: S.list()
[[1, 2, 3]]
sage: S.cardinality()
1

cardinality()

Returns the number of structures in this set.

EXAMPLES:

sage: F = species.SetSpecies()
sage: F.structures([1,2,3]).cardinality()
1

labels()

Returns the labels used on these structures. If \(X\) is the species, then \(\text{labels()}\) returns the preimage of these structures under the functor \(X\).

EXAMPLES:

sage: F = species.SetSpecies()
sage: F.structures([1,2,3]).labels()
[1, 2, 3]
class sage.combinat.species.structure.StructuresWrapper(species, labels, structure_class)
    Bases: SpeciesWrapper

    A base class for the set of structures of a species with given set of labels. An object of this type is returned when
    you call the structures() method of a species.

    EXAMPLES:

    sage: F = species.SetSpecies()
    sage: S = F.structures([1,2,3])
    sage: S == loads(dumps(S))
    True

5.1.330 Subset Species

class sage.combinat.species.subset_species.SubsetSpecies(min=None, max=None, weight=None)
    Bases: GenericCombinatorialSpecies, UniqueRepresentation

    Return the species of subsets.

    EXAMPLES:

    sage: S = species.SubsetSpecies()
    sage: S.generating_series()[0:5]
    [1, 2, 2, 4/3, 2/3]
    sage: S.isotype_generating_series()[0:5]
    [1, 2, 3, 4, 5]
    sage: S = species.SubsetSpecies()
    sage: c = S.generating_series()[0:3]
    sage: S._check()
    True
    sage: S == loads(dumps(S))
    True

class sage.combinat.species.subset_species.SubsetSpeciesStructure(parent, labels, list)
    Bases: GenericSpeciesStructure

    automorphism_group()

        Return the group of permutations whose action on this subset leave it fixed.

        EXAMPLES:

        sage: F = species.SubsetSpecies()
        sage: a = F.structures([1,2,3,4])[6]; a
        {1, 3}
        sage: a.automorphism_group()
        Permutation Group with generators [(2,4), (1,3)]
        sage: [a.transport(g) for g in a.automorphism_group()]
        [{1, 3}, {1, 3}, {1, 3}, {1, 3}]

    canonical_label()

        Return the canonical label of self.

        EXAMPLES:
```python
sage: P = species.SubsetSpecies()
sage: S = P.structures(['a', 'b', 'c'])
sage: [s.canonical_label() for s in S]
[{}, {'a'}, {'a'}, {'a'}, {'a', 'b'}, {'a', 'b'}, {'a', 'b', 'c'}]
```

**complement()**

Return the complement of self.

**EXAMPLES:**

```python
sage: F = species.SubsetSpecies()
sage: a = F.structures(['a', 'b', 'c'])[5]; a
{'a', 'c'}
sage: a.complement()
{'b'}
```

**label_subset()**

Return a subset of the labels that “appear” in this structure.

**EXAMPLES:**

```python
sage: P = species.SubsetSpecies()
sage: S = P.structures(['a', 'b', 'c'])
sage: [s.label_subset() for s in S]
[[], ['a'], ['b'], ['c'], ['a', 'b'], ['a', 'c'], ['b', 'c'], ['a', 'b', 'c']]
```

**transport**(perm)

Return the transport of this subset along the permutation perm.

**EXAMPLES:**

```python
sage: F = species.SubsetSpecies()
sage: a = F.structures(['a', 'b', 'c'])[5]; a
{'a', 'c'}
sage: p = PermutationGroupElement((1,2))
sage: a.transport(p)
{'b', 'c'}
sage: p = PermutationGroupElement((1,3))
sage: a.transport(p)
{'a', 'c'}
```

sage.combinat.species.subset_species.SubsetSpecies_class

alias of **SubsetSpecies**

### 5.1.331 Sum species

```python
class sage.combinat.species.sum_species.SumSpecies(F, G, min=None, max=None, weight=None)
Bases: GenericCombinatorialSpecies, UniqueRepresentation

Returns the sum of two species.

**EXAMPLES:**
```
```python
sage: S = species.PermutationSpecies()
sage: A = S+S
sage: A.generating_series()[:5]
[2, 2, 2, 2, 2]

sage: P = species.PermutationSpecies()
sage: F = P + P
sage: F._check()
True
sage: F == loads(dumps(F))
True

left_summand()
Returns the left summand of this species.

EXAMPLES:

```python
sage: P = species.PermutationSpecies()
sage: F = P + P*P
sage: F.left_summand()
Permutation species
```

right_summand()
Returns the right summand of this species.

EXAMPLES:

```python
sage: P = species.PermutationSpecies()
sage: F = P + P*P
sage: F.right_summand()
Product of (Permutation species) and (Permutation species)
```

weight_ring()
Returns the weight ring for this species. This is determined by asking Sage’s coercion model what the result is when you add elements of the weight rings for each of the operands.

EXAMPLES:

```python
sage: S = species.SetSpecies()
sage: C = S+S
sage: C.weight_ring()
Rational Field

sage: S = species.SetSpecies(weight=QQ['t'].gen())
sage: C = S + S
sage: C.weight_ring()
Univariate Polynomial Ring in t over Rational Field
```

class sage.combinat.species.sum_species.SumSpeciesStructure(parent, s, **options)
Bases: SpeciesStructureWrapper

sage.combinat.species.sum_species.SumSpecies_class
alias of SumSpecies
```

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5.1.332 Specht Modules

AUTHORS:
- Travis Scrimshaw (2023-1-22): initial version

```python
class sage.combinat.specht_module.SpechtModule(SGA, D):
    Bases: SubmoduleWithBasis

    A Specht module.

    Let $S_n$ be the symmetric group on $n$ letters and $R$ be a commutative ring. The Specht module $S^D$ for a diagram $D$ is an $S_n$-module defined as follows. Let

    $$R(D) := \sum_{w \in R_D} w, \quad C(D) := \sum_{w \in C_D} (-1)^w w,$$

    where $R_D$ (resp. $C_D$) is the row (resp. column) stabilizer of $D$. Then, we construct the Specht module $S^D$ as the left ideal

    $$S^D = R[S_n]C(D)R(D),$$

    where $R[S_n]$ is the group algebra of $S_n$ over $R$.

INPUT:
- SGA – a symmetric group algebra
- D – a diagram

EXAMPLES:
We begin by constructing all irreducible Specht modules for the symmetric group $S_4$ and show that they give a full set of irreducible representations both by having distinct Frobenius characters and the sum of the square of their dimensions is equal to $4!$:

```python
sage: SP = [la.specht_module(QQ) for la in Partitions(4)]
sage: s = SymmetricFunctions(QQ).s()
[s(S.frobenius_image()) for S in SP]
[s[4], s[3, 1], s[2, 2], s[2, 1, 1], s[1, 1, 1, 1]]
sage: sum(S.dimension()^2 for S in SP)
24
```

Next, we compute the Specht module for a more general diagram for $S_5$ and compute its irreducible decomposition by using its Frobenius character:

```python
sage: D = [(0,0), (0,1), (1,1), (1,2), (0,3)]
sage: SGA = SymmetricGroupAlgebra(QQ, 5)
sage: SM = SGA.specht_module(D)
sage: SM.dimension()
9
sage: s(SM.frobenius_image())
s[3, 2] + s[4, 1]
```

This carries a natural (left) action of the symmetric group (algebra):

```python
sage: S5 = SGA.group()
sage: v = SM.an_element(); v
```
Combinatorics, Release 10.0

(continued from previous page)

```python
sage: S5([2,1,5,3,4]) * v
sage: x = SGA.an_element(); x
[1, 2, 3, 4, 5] + 2*[1, 2, 3, 5, 4] + 3*[1, 2, 4, 3, 5] + [5, 1, 2, 3, 4]
sage: x * v
```

See also:

*SpechtRepresentation* for an implementation of the representation by matrices.

```python
class Element

    Bases: IndexedFreeModuleElement

def frobenius_image()

    Return the Frobenius image of self.

    The Frobenius map is defined as the map to symmetric functions

    \[ F(\chi) = \frac{1}{n!} \sum_{w \in S_n} \chi(w)p_\rho(w), \]

    where \( \chi \) is the character of the \( S_n \)-module self, \( p_\lambda \) is the powersum symmetric function basis element indexed by \( \lambda \), and \( \rho(w) \) is partition of the cycle type of \( w \). Specifically, this map takes irreducible representations indexed by \( \lambda \) to the Schur function \( s_\lambda \).

    EXAMPLES:

```
sage: s = SymmetricFunctions(QQ).s()
sage: SM = Partition([2,2,1]).specht_module(QQ)
sage: s(SM.frobenius_image())
s[2, 2, 1]
sage: SM = Partition([4,1]).specht_module(CyclotomicField(5))
sage: s(SM.frobenius_image())
s[4, 1]
```

We verify the regular representation:

```python
sage: from sage.combinat.diagram import Diagram
sage: D = Diagram(((0,0), (1,1), (2,2), (3,3), (4,4)))
sage: F = s(D.specht_module(QQ).frobenius_image()); F
s[1, 1, 1, 1, 1] + 4*s[2, 1, 1, 1] + 5*s[2, 2, 1] + 6*s[3, 1, 1] + 5*s[3, 2] + 4*s[4, 1] + s[5]
sage: F == sum(StandardTableaux(la).cardinality() * s[la]....: for la in Partitions(5))
True
sage: all(s[la] == s(la.specht_module(QQ).frobenius_image())....: for n in range(1, 5) for la in Partitions(n))
True
sage: D = Diagram(((0,0), (1,1), (1,2), (2,3), (2,4)))
sage: SM = D.specht_module(QQ)
sage: s(SM.frobenius_image())
```

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**representation_matrix(elt)**

Return the matrix corresponding to the left action of the symmetric group (algebra) element `elt` on `self`.

See also:

*SpechtRepresentation*

**EXAMPLES:**

```python
sage: SM = Partition([3,1,1]).specht_module(QQ)
sage: SM.representation_matrix(Permutation([2,1,3,5,4]))
[-1 0 0 1 -1 0]
[ 0 0 1 0 -1 1]
[ 0 1 0 -1 0 1]
[ 0 0 0 0 -1 0]
[ 0 0 0 -1 0 0]
[ 0 0 0 0 0 -1]
sage: SGA = SymmetricGroupAlgebra(QQ, 5)
sage: SM.representation_matrix(SGA([3,1,5,2,4]))
[ 0 -1 0 1 0 -1]
[ 0 0 0 0 0 -1]
[ 0 0 0 -1 0 0]
[ 0 0 -1 0 1 -1]
[ 1 0 0 -1 1 0]
[ 0 0 0 0 1 0]
```

**sage.combinat.specht_module.specht_module_rank**(`D`, *base_ring=None*)

Return the rank of the Specht module of diagram `D`.

**EXAMPLES:**

```python
sage: from sage.combinat.specht_module import specht_module_rank
sage: specht_module_rank([[0,0], [1,1], [2,2]])
6
```

**sage.combinat.specht_module.specht_module_spanning_set**(`D`, *SGA=None*)

Return a spanning set of the Specht module of diagram `D`.

**INPUT:**

- `D` – a list of cells `(r, c)` for row `r` and column `c`
- `SGA` – optional; a symmetric group algebra

**EXAMPLES:**

```python
sage: from sage.combinat.specht_module import specht_module_spanning_set
sage: specht_module_spanning_set([[0,0], [1,1], [2,1]])
([1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1])
```

```python
sage: SGA = SymmetricGroup(3).algebra(QQ)
sage: specht_module_spanning_set([[0,0], [1,1], [2,1]], SGA)
(() - (2,3), -(1,2) + (1,3,2), (1,2,3) - (1,3),
-() + (2,3), -(1,2,3) + (1,3), (1,2) - (1,3,2))
```
5.1.333 Subsets

The set of subsets of a finite set. The set can be given as a list or a Set or else as an integer $n$ which encodes the set \{1, 2, ..., $n$\}. See Subsets for more information and examples.

AUTHORS:

- Mike Hansen: initial version
- Florent Hivert (2009/02/06): doc improvements + new methods

class sage.combinat.subset.SubMultiset_s($s$

Bases: Parent

The combinatorial class of the sub multisets of $s$.

EXAMPLES:

```
sage: S = Subsets([1,2,2,3], submultiset=True)
sage: S.cardinality()
12
sage: S.list()
[[], [1], [2], [3], [1, 2], [1, 3], [2, 2], [2, 3], [1, 2, 2], [1, 2, 3], [2, 2, 3], [1, 2, 2, 3]]
sage: S.first()
[]
sage: S.last()
[1, 2, 2, 3]
```

cardinality()

Return the cardinality of self.

EXAMPLES:

```
sage: S = Subsets([1,1,2,3], submultiset=True)
sage: S.cardinality()
12
sage: len(S.list())
12

sage: S = Subsets([1,1,2,2,3], submultiset=True)
sage: S.cardinality()
18
sage: len(S.list())
18

sage: S = Subsets([1,1,1,2,2,3], submultiset=True)
```

(continues on next page)
element_class
alias of list
generating_serie(variable='x')
Return the polynomial associated to the counting of the elements of self weighted by the number of element they contain.
EXAMPLES:

sage: Subsets([1,1], submultiset=True).generating_serie()
x^2 + x + 1
sage: Subsets([1,1,2,3], submultiset=True).generating_serie()
x^4 + 3*x^3 + 4*x^2 + 3*x + 1
sage: Subsets([1,1,1,2,2,3,3,4], submultiset=True).generating_serie()
x^8 + 4*x^7 + 9*x^6 + 14*x^5 + 16*x^4 + 14*x^3 + 9*x^2 + 4*x + 1
sage: S = Subsets([1,1,1,2,2,3,3,4], submultiset=True)
sage: S.cardinality()
72
sage: sum(S.generating_serie())
72

random_element()
Return a random element of self with uniform law.
EXAMPLES:

sage: S = Subsets([1,1,2,3], submultiset=True)
sage: s = S.random_element()
sage: s
in S
True

class sage.combinat.subset.SubMultiset_sk(s, k)
Bases: SubMultiset_s

The combinatorial class of the subsets of size k of a multiset s. Note that each subset is represented by a list of the elements rather than a set since we can have multiplicities (no multiset data structure yet in sage).

EXAMPLES:

sage: S = Subsets([1,2,3,3], 2, submultiset=True)
sage: S._k
2
sage: S.cardinality()
4
sage: S.first()
[1, 2]
sage: S.last()
[3, 3]
Combinatorics, Release 10.0

sage: [sub for sub in S]
[[1, 2], [1, 3], [2, 3], [3, 3]]

cardinality()
Return the cardinality of self.

EXAMPLES:

sage: S = Subsets([1,2,2,3,3,3],4,submultiset=True)
sage: S.cardinality()  
5
sage: len(list(S))  
5

sage: S = Subsets([1,2,2,3,3,3],3,submultiset=True)
sage: S.cardinality()  
6
sage: len(list(S))  
6

generating_serie(variable='x')
Return the polynomial associated to the counting of the elements of self weighted by the number of elements they contains

EXAMPLES:

sage: x = ZZ['x'].gen()  
sage: l = [1,1,1,1,2,2,3]
sage: for k in range(len(l)):
    ....:     S = Subsets(l,k,submultiset=True)
    ....:     print(S.generating_serie('x') == S.cardinality()*x**k)
True
True
True
True
True
True
True
True

random_element()
Return a random submultiset of given length

EXAMPLES:

sage: s = Subsets(7,3).random_element()
sage: s in Subsets(7,3)  
True

sage: s = Subsets(7,5).random_element()
sage: s in Subsets(7,5)  
True

sage.combinat.subset.Subsets(s, k=None, submultiset=False)
Return the combinatorial class of the subsets of the finite set s. The set can be given as a list, Set or any iterable
convertible to a set. Alternatively, a non-negative integer \( n \) can be provided in place of \( s \); in this case, the result is the combinatorial class of the subsets of the set \( \{1, 2, \ldots, n\} \) (i.e. of the Sage `range(1,n+1)`).

A second optional parameter \( k \) can be given. In this case, \texttt{Subsets} returns the combinatorial class of subsets of \( s \) of size \( k \).

**Warning:** The subsets are returned as Sets. Do not assume that these Sets are ordered; they often are not! (E.g., \texttt{Subsets(10).list()}[619] returns \{10, 4, 5, 6, 7\} on my system.) See \texttt{SubsetsSorted} for a similar class which returns the subsets as sorted tuples.

Finally the option `submultiset` allows one to deal with sets with repeated elements, usually called multisets. The method then returns the class of all multisets in which every element is contained at most as often as it is contained in \( s \). These multisets are encoded as lists.

**EXAMPLES:**

```
sage: S = Subsets([1, 2, 3]); S
Subsets of {1, 2, 3}
sage: S.cardinality()
8
sage: S.first()
{}
sage: S.last()
{1, 2, 3}
sage: S.random_element() in S
True
sage: S.list()
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
```

Here is the same example where the set is given as an integer:

```
sage: S = Subsets(3)
sage: S.list()
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
```

We demonstrate various the effect of the various options:

```
sage: S = Subsets(3, 2); S
Subsets of {1, 2, 3} of size 2
sage: S.list()
[[1, 2], [1, 3], [2, 3]]
sage: S = Subsets([1, 2, 2], submultiset=True); S
SubMultiset of [1, 2, 2]
sage: S.list()
[[], [1], [2], [1, 2], [2, 2], [1, 2, 2]]
sage: S = Subsets([1, 2, 2, 3], 3, submultiset=True); S
SubMultiset of [1, 2, 2, 3] of size 3
sage: S.list()
[[1, 2, 2], [1, 2, 3], [2, 2, 3]]
sage: S = Subsets(['a', 'b', 'a', 'b'], 2, submultiset=True); S.list()
[['a', 'a'], ['a', 'b'], ['b', 'b']]
```
And it is possible to play with subsets of subsets:

```python
sage: S = Subsets(3)
sage: S2 = Subsets(S); S2
Subsets of Subsets of \{1, 2, 3\}
sage: S2.cardinality()
256
sage: it = iter(S2)
[sage: [next(it) for _ in range(8)]
[\{}, \{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}]

sage: S2.random_element()  # random
\{2\}, \{1, 2, 3\}, \{}

sage: [S2.unrank(k) for k in range(256)] == S2.list()
True

sage: S3 = Subsets(S2)
sage: S3.cardinality()
11579208923731619542357698568687907853269984665640564039457584007913129639936
sage: S3.unrank(14123091480)  # random
\{\{2\}, \{1, 2, 3\}, \{1, 2\}, \{3\}, \{\}\},
\{\{1, 2, 3\}, \{2\}, \{1\}, \{1, 3\},
\{\}, \{2\}, \{2, 3\}, \{1, 2\},
\{\}, \{2\}, \{1, 2, 3\}, \{1, 2\},
\{\}, \{\}, \{1\}, \{1, 2, 3\}\}

sage: T = Subsets(S2, 10)
sage: T.cardinality()
278826214645841800
sage: T.unrank(1441231049)  # random
\{\{1, 2, 3\}, \{2\}, \{2, 3\}, \{\}\}, \{\}, \{3\}, \{1\}, \{\}, \{1, 3\}\}
```

class sage.combinat.subset.SubsetsSorted(S)

Bases: Subsets_s

Lightweight class of all subsets of some set S, with each subset being encoded as a sorted tuple.

Used to model indices of algebras given by subsets (so we don’t have to explicitly build all $2^n$ subsets in memory). For example, CliffordAlgebra.

**element_class**

alias of tuple

**first()**

Return the first element of self.

EXAMPLES:

```python
sage: from sage.combinat.subset import SubsetsSorted
sage: S = SubsetsSorted(range(3))
sage: S.first()
()
```

**last()**

Return the last element of self.

EXAMPLES:
sage: from sage.combinat.subset import SubsetsSorted
sage: S = SubsetsSorted(range(3))
sage: S.last()
(0, 1, 2)

random_element()
Return a random element of self.

EXAMPLES:

sage: from sage.combinat.subset import SubsetsSorted
sage: S = SubsetsSorted(range(3))
sage: isinstance(S.random_element(), tuple)
True

unrank(r)
Return the subset which has rank r.

EXAMPLES:

sage: from sage.combinat.subset import SubsetsSorted
sage: S = SubsetsSorted(range(3))
sage: S.unrank(4)
(0, 1)

class sage.combinat.subset.Subsets_s(s)
Bases: Parent
Subsets of a given set.

EXAMPLES:

sage: S = Subsets(4); S
Subsets of {1, 2, 3, 4}
sage: S.cardinality()
16
sage: Subsets(4).list()
[[], [1], [2], [3], [4],
 [1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4],
 [1, 2, 3], [1, 2, 4], [1, 3, 4], [2, 3, 4],
 [1, 2, 3, 4]]

sage: S = Subsets(Subsets(Subsets(Subsets(GF(3))))); S
Subsets of Subsets of Subsets of Finite Field of size 3
sage: S.cardinality()
1157920892373165268467933643455623002672708534579755264601458643792929240937
sage: S.unrank(3149254230)  # random
{{{1}, {0, 2}}, {{0, 1, 2}, {0, 1}, {1}, {1, 2}},
 {{2}, {1, 2}, {0, 1, 2}, {0, 2}, {1}, {}},
 {{1, 2}, {0}},
 {{{0}, 1, 2}, {0, 1}, {0, 2}, {1, 2}}}
EXAMPLES:

```
sage: Subsets(Set([1,2,3])).cardinality()
8
sage: Subsets([1,2,3,3]).cardinality()
8
sage: Subsets(3).cardinality()
8
```

**element_class**

alias of `Set_object_enumerated`

**first()**

Returns the first subset of `s`. Since we aren’t restricted to subsets of a certain size, this is always the empty set.

EXAMPLES:

```
sage: Subsets([1,2,3]).first()
{}
sage: Subsets(3).first()
{}
```

**last()**

Return the last subset of `s`. Since we aren’t restricted to subsets of a certain size, this is always the set `s` itself.

EXAMPLES:

```
sage: Subsets([1,2,3]).last()
{1, 2, 3}
sage: Subsets(3).last()
{1, 2, 3}
```

**lattice()**

Return the lattice of subsets ordered by containment.

EXAMPLES:

```
sage: X = Subsets([7,8,9])
sage: X.lattice()
Finite lattice containing 8 elements
sage: Y = Subsets(0)
sage: Y.lattice()
Finite lattice containing 1 elements
```

**random_element()**

Return a random element of the class of subsets of `s` (in other words, a random subset of `s`).

EXAMPLES:

```
sage: Subsets(3).random_element()  # random
{2}
sage: Subsets([4,5,6]).random_element()  # random
{5}
```
sage: S = Subsets(Subsets(Subsets([0,1,2])))
sage: S.cardinality()
115792089237316195423570985008687907853269984665640564039457584007913129639936
sage: s = S.random_element()
sage: s
# random
{{1, 2}, {2}, {0}, {1}}, {{1, 2}, {0, 1, 2}, {0, 2}, {0}, {0, 1}}, ..., {{1, 2} \rightarrow {2}, {1}}, {{2}, {0, 2}, {}, {1}}}
sage: s in S
True

\textbf{rank} \textit{(sub)}

Return the rank of \textit{sub} as a subset of \textbf{s}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Subsets(3).rank([])
0
sage: Subsets(3).rank([1,2])
4
sage: Subsets(3).rank([1,2,3])
7
sage: Subsets(3).rank([2,3,4])
Traceback (most recent call last):
...
ValueError: {2, 3, 4} is not a subset of {1, 2, 3}
\end{verbatim}

\textbf{underlying_set}()

Return the set of elements.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Subsets(GF(13)).underlying_set()
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
\end{verbatim}

\textbf{unrank} \textit{(r)}

Return the subset of \textbf{s} that has rank \textit{k}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Subsets(3).unrank(0)
{}
sage: Subsets([2,4,5]).unrank(1)
{2}
sage: Subsets([1,2,3]).unrank(257)
Traceback (most recent call last):
...
IndexError: index out of range
\end{verbatim}

\textbf{class} \textbf{sage.combinat.subset.Subsets sk}(s, k)

Bases: \textbf{Subsets s}

Subsets of fixed size of a set.

\textbf{EXAMPLES:}
sage: S = Subsets([0,1,2,5,7], 3); S
Subsets of {0, 1, 2, 5, 7} of size 3
sage: S.cardinality()
10
sage: S.first(), S.last()
({0, 1, 2}, {2, 5, 7})
sage: S.random_element()  # random
{0, 5, 7}
sage: S([0,2,7])
{0, 2, 7}
sage: S([0,3,5])
Traceback (most recent call last):
... ValueError: {0, 3, 5} not in Subsets of {0, 1, 2, 5, 7} of size 3
sage: S([0])
Traceback (most recent call last):
... ValueError: {0} not in Subsets of {0, 1, 2, 5, 7} of size 3

an_element()
Returns an example of subset.

EXAMPLES:

sage: Subsets(0,0).an_element()
{}
sage: Subsets(3,2).an_element()
{1, 3}
sage: Subsets([2,4,5],2).an_element()
{2, 5}

cardinality()
EXAMPLES:

sage: Subsets(Set([1,2,3]), 2).cardinality()
3
sage: Subsets([1,2,3,3], 2).cardinality()
3
sage: Subsets([1,2,3], 1).cardinality()
3
sage: Subsets([1,2,3], 3).cardinality()
1
sage: Subsets([1,2,3], 0).cardinality()
1
sage: Subsets([1,2,3], 4).cardinality()
0
sage: Subsets(3,2).cardinality()
3
sage: Subsets(3,4).cardinality()
0

first()
Return the first subset of s of size k.
EXAMPLES:

```python
sage: Subsets(Set([1,2,3]), 2).first()
{1, 2}
sage: Subsets([1,2,3,3], 2).first()
{1, 2}
sage: Subsets(3,2).first()
{1, 2}
sage: Subsets(3,4).first()
Traceback (most recent call last):
... EmptySetError
```

`last()`

Return the last subset of `s` of size `k`.

EXAMPLES:

```python
sage: Subsets(Set([1,2,3]), 2).last()
{2, 3}
sage: Subsets([1,2,3,3], 2).last()
{2, 3}
sage: Subsets(3,2).last()
{2, 3}
sage: Subsets(3,4).last()
Traceback (most recent call last):
... EmptySetError
```

`random_element()`

Return a random element of the class of subsets of `s` of size `k` (in other words, a random subset of `s` of size `k`).

EXAMPLES:

```python
sage: s = Subsets(3, 2).random_element()
sage: s in Subsets(3, 2)
True
sage: Subsets(3,4).random_element()
Traceback (most recent call last):
... EmptySetError
```

`rank(sub)`

Return the rank of `sub` as a subset of `s` of size `k`.

EXAMPLES:

```python
sage: Subsets(3,2).rank([1,2])
0
sage: Subsets([2,3,4],2).rank([3,4])
2
sage: Subsets([2,3,4],2).rank([2])
Traceback (most recent call last):
... EmptySetError
```
ValueError: {2} is not a subset of length 2 of {2, 3, 4}
sage: Subsets([2,3,4],4).rank([2,3,4,5])
Traceback (most recent call last):
  ... 
ValueError: {2, 3, 4, 5} is not a subset of length 4 of {2, 3, 4}

\texttt{unrank}(r)

Return the subset of $s$ of size $k$ that has rank $r$.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: Subsets(3,2).unrank(0) {1, 2}
sage: Subsets([2,4,5],2).unrank(0) {2, 4}
sage: Subsets([1,2,8],3).unrank(42)
Traceback (most recent call last):
  ... 
IndexError: index out of range
\end{verbatim}

\texttt{sage.combinat.subset.dict_to_list}(d)

Return a list whose elements are the elements of $i$ of $d$ repeated with multiplicity $d[i]$.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.combinat.subset import dict_to_list
sage: dict_to_list({'a':1, 'b':3}) ['a', 'b', 'b', 'b']
\end{verbatim}

\texttt{sage.combinat.subset.list_to_dict}(l)

Return a dictionary of multiplicities and the list of its keys.

\textbf{INPUT:}

a list $l$ with possibly repeated elements

The keys are the elements of $l$ (in the same order in which they appear) and values are the multiplicities of each element in $l$.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.combinat.subset import list_to_dict
sage: list_to_dict(['a', 'b', 'b', 'b']) ({'a': 1, 'b': 3}, ['a', 'b'])
\end{verbatim}
5.1.334 Subsets satisfying a hereditary property

sage.combinat.subsets_hereditary.subsets_with_hereditary_property(f, X, max_obstruction_size=None, ncpus=1)

Return all subsets $S$ of $X$ such that $f(S)$ is true.

The boolean function $f$ must be decreasing, i.e. $f(S) \Rightarrow f(S')$ if $S' \subseteq S$.

This function is implemented to call $f$ as few times as possible. More precisely, $f$ will be called on all sets $S$ such that $f(S)$ is true, as well as on all inclusionwise minimal sets $S$ such that $f(S)$ is false.

The problem that this function answers is also known as the learning problem on monotone boolean functions, or as computing the set of winning coalitions in a simple game.

INPUT:

- $f$ – a boolean function which takes as input a list of elements from $X$.
- $X$ – a list/iterable.
- max_obstruction_size (integer) – if you know that there is a $k$ such that $f(S)$ is true if and only if $f(S')$ is true for all $S' \subseteq S$ with $S' \leq k$, set max_obstruction_size=k. It may dramatically decrease the number of calls to $f$. Set to None by default, meaning $k = |X|$.
- ncpus – number of cpus to use for this computation. Note that changing the value from 1 (default) to anything different enables parallel computations which can have a cost by itself, so it is not necessarily a good move. In some cases, however, it is a great move. Set to None to automatically detect and use the maximum number of cpus available.

Note: Parallel computations are performed through the parallel() decorator. See its documentation for more information, in particular with respect to the memory context.

EXAMPLES:

Sets whose elements all have the same remainder mod 2:

```
sage: from sage.combinat.subsets_hereditary import subsets_with_hereditary_property
sage: f = lambda x: (not x) or all(xx%2 == x[0]%2 for xx in x)
sage: list(subsets_with_hereditary_property(f, range(4)))
```

One can use this function to compute the independent sets of a graph. We know, however, that in this case the maximum obstructions are the edges, and have size 2. We can thus set max_obstruction_size=2, which reduces the number of calls to $f$ from 91 to 56:

```
sage: num_calls=0
sage: g = graphs.PetersenGraph()
sage: def is_independent_set(S):
....:     global num_calls
....:     num_calls+=1
....:     return g.subgraph(S).size()==0
```

(continues on next page)
Combinatorics, Release 10.0

5.1.335 Subsets whose elements satisfy a predicate pairwise

```python
sage: l1=list(subsets_with_hereditary_property(is_independent_set, g.
˓→vertices(sort=False)))
sage: num_calls
91
sage: num_calls=0
sage: l2=list(subsets_with_hereditary_property(is_independent_set, g.
˓→vertices(sort=False), max_obstruction_size=2))
sage: num_calls
56
sage: l1==l2
True
```

**class** `sage.combinat.subsets_pairwise.PairwiseCompatibleSubsets`(ambient, predicate, 
maximal=False, 
`element_class=<class 'sage.sets.set.Set_object_enumerated'>)`

Bases: `RecursivelyEnumeratedSet_forest`

The set of all subsets of `ambient` whose elements satisfy `predicate` pairwise

**INPUT:**

- `ambient` – a set (or iterable)
- `predicate` – a binary predicate

**Assumptions:** `predicate` is symmetric (`predicate(x,y) == predicate(y,x)`) and reflexive (`predicate(x,x) == True`).

**Note:** in fact, `predicate(x,x)` is never called.

**Warning:** The current name is suboptimal and is subject to change. Suggestions for a good name, and a good user entry point are welcome. Maybe `Subsets(..., independent = predicate)`.

**EXAMPLES:**

We construct the set of all subsets of \{4, 5, 6, 8, 9\} whose elements are pairwise relatively prime:

```python
sage: from sage.combinat.subsets_pairwise import PairwiseCompatibleSubsets
sage: def predicate(x,y): return gcd(x,y) == 1
sage: P = PairwiseCompatibleSubsets( [4,5,6,8,9], predicate); P
An enumerated set with a forest structure
sage: P.list()
[{}, {4}, {4, 5}, {9, 4, 5}, {9}, {5}, {5, 6}, {8, 5}, {8, 9, 5}, {9, 5}, {6},
˓→{8}, {8, 9}, {9}]
sage: P.cardinality()
14
sage: P.category()
Category of finite enumerated sets
```
Here we consider only those subsets which are maximal for inclusion (not yet implemented):

```
sage: P = PairwiseCompatibleSubsets( [4,5,6,8,9], predicate, maximal = True); P
An enumerated set with a forest structure
sage: P.list()                           # todo: not implemented
[[9, 4, 5], [5, 6], [8, 9, 5]]
sage: P.cardinality()                  # todo: not implemented
14
sage: P.category()
Category of finite enumerated sets
```

**Algorithm**

In the following, we order the elements of the ambient set by order of apparition. The elements of `self` are generated by organizing them in a search tree. Each node of this tree is of the form `(subset, rest)`, where:

- `subset` represents an element of `self`, represented by an increasing tuple
- `rest` is the set of all `y`s such that `y` appears after `x` in the ambient set and `predicate(x,y)` holds, represented by a decreasing tuple

The root of this tree is `( (), ambient )`. All the other elements are generated by recursive depth first search, which gives lexicographic order.

- `children(subset_rest)`
  Returns the children of a node in the tree.
- `post_process(subset_rest)`

### 5.1.336 Subwords

A subword of a word `w` is a word obtained by deleting the letters at some (non necessarily adjacent) positions in `w`. It is not to be confused with the notion of factor where one keeps adjacent positions in `w`. Sometimes it is useful to allow repeated uses of the same letter of `w` in a “generalized” subword. We call this a subword with repetitions.

For example:

- “bnjr” is a subword of the word “bonjour” but not a factor;
- “njo” is both a factor and a subword of the word “bonjour”;
- “nr” is a subword of “bonjour”;
- “rn” is not a subword of “bonjour”;
- “nnu” is not a subword of “bonjour”;
- “nnu” is a subword with repetitions of “bonjour”;

A word can be given either as a string, as a list or as a tuple.

As repetition can occur in the initial word, in general subwords of a given word form an enumerated multiset rather than a set!

**Todo:**

- implement subwords with repetitions
• implement the category of `EnumeratedMultiset` and inheritate from it when needed (i.e. the initial word has repeated letters)

AUTHORS:

• Mike Hansen: initial version

• Florent Hivert (2009/02/06): doc improvements + new methods + bug fixes

`sage.combinat.subword.Subwords(w, k=None, element_constructor=None)`

Return the set of subwords of `w`.

INPUT:

• `w` – a word (can be a list, a string, a tuple or a word)

• `k` – an optional integer to specify the length of subwords

• `element_constructor` – an optional function that will be used to build the subwords

EXAMPLES:

```python
sage: S = Subwords(['a', 'b', 'c']); S
Subwords of ['a', 'b', 'c']
sage: S.first()
[]
sage: S.last()
['a', 'b', 'c']
sage: S.list()
[[''], ['a'], ['b'], ['c'], ['a', 'b'], ['a', 'c'], ['b', 'c'], ['a', 'b', 'c']]
```

The same example using string, tuple or a word:

```python
sage: S = Subwords('abc'); S
Subwords of 'abc'
sage: S.list()
['', 'a', 'b', 'c', 'ab', 'ac', 'bc', 'abc']
```

```python
sage: S = Subwords((1,2,3)); S
Subwords of (1, 2, 3)
sage: S.list()
[(], [1], [2], [3], [1, 2], [1, 3], [2, 3], [1, 2, 3])
```

```python
sage: w = Word([1,2,3])
sage: S = Subwords(w); S
Subwords of word: 123
sage: S.list()
[word: , word: 1, word: 2, word: 3, word: 12, word: 13, word: 23, word: 123]
```

Using word with specified length:

```python
sage: S = Subwords(['a', 'b', 'c'], 2); S
Subwords of ['a', 'b', 'c'] of length 2
sage: S.list()
[['a', 'b'], ['a', 'c'], ['b', 'c']]
```

An example that uses the `element_constructor` argument:
```python
sage: p = Permutation([3,2,1])
sage: Subwords(p, element_constructor=tuple).list()
[(3, 2), (3, 1), (2, 1), (3, 2, 1)]
sage: Subwords(p, 2, element_constructor=tuple).list()
[(3, 2), (3, 1), (2, 1)]
```

class sage.combinat.subword.Subwords_w(w, element_constructor)

Bases: Parent

Subwords of a given word.

cardinality()

EXAMPLES:

```python
sage: Subwords([1,2,3]).cardinality()
8
```

first()

EXAMPLES:

```python
sage: Subwords([1,2,3]).first()
[]
sage: Subwords((1,2,3)).first()
()
sage: Subwords('123').first()
''
```

last()

EXAMPLES:

```python
sage: Subwords([1,2,3]).last()
[1, 2, 3]
sage: Subwords((1,2,3)).last()
(1, 2, 3)
sage: Subwords('123').last()
'123'
```

random_element()

Return a random subword with uniform law.

EXAMPLES:

```python
sage: S1 = Subwords([1,2,3,2,1,3])
sage: S2 = Subwords([4,6,6,7,4,5,5])
sage: for i in range(100):
    ....:     w = S1.random_element()
    ....:     if w in S2:
    ....:         assert(not w)
sage: for i in range(100):
    ....:     w = S2.random_element()
    ....:     if w in S1:
    ....:         assert(not w)
```
class sage.combinat.subword.Subwords_wk(w, k, element_constructor)

    Bases: Subwords_w

    Subwords with fixed length of a given word.

cardinality()

    Return the number of subwords of w of length k.

    EXAMPLES:

    sage: Subwords([1,2,3], 2).cardinality()
    3

first()

    EXAMPLES:

    sage: Subwords([1,2,3],2).first()
    [1, 2]
    sage: Subwords([1,2,3],0).first()
    []
    sage: Subwords((1,2,3),2).first()
    (1, 2)
    sage: Subwords((1,2,3),0).first()
    ()
    sage: Subwords('123',2).first()
    '12'
    sage: Subwords('123',0).first()
    ''

code: last()

    EXAMPLES:

    sage: Subwords([1,2,3],2).last()
    [2, 3]
    sage: Subwords([1,2,3],0).last()
    []
    sage: Subwords((1,2,3),2).last()
    (2, 3)
    sage: Subwords((1,2,3),0).last()
    ()
    sage: Subwords('123',2).last()
    '23'
    sage: Subwords('123',0).last()
    ''

random_element()

    Return a random subword of given length with uniform law.

    EXAMPLES:

    sage: S1 = Subwords([1,2,3,2,1],3)
    sage: S2 = Subwords([4,4,5,4,5,4,4,4],3)
    sage: for i in range(100):
    ....:     w = S1.random_element()
    ....:     if w in S2:

(continues on next page)
....:     assert(not w)

sage: for i in range(100):
    ....:     w = S2.random_element()
    ....:     if w in S1:
        ....:         assert(not w)

sage.combinat.subword.smallest_positions(word, subword, pos=0)

Return the smallest positions for which subword appears as a subword of word.

If pos is specified, then it returns the positions of the first appearance of subword starting at pos.

If subword is not found in word, then return False.

EXAMPLES:

sage: sage.combinat.subword.smallest_positions([1,2,3,4], [2,4])
[1, 3]
sage: sage.combinat.subword.smallest_positions([1,2,3,4,4], [2,4])
[1, 3]
sage: sage.combinat.subword.smallest_positions([1,2,3,3,4,4], [3,4])
[2, 4]
sage: sage.combinat.subword.smallest_positions([1,2,3,3,4,4], [3,4],2)
[2, 4]
sage: sage.combinat.subword.smallest_positions([1,2,3,3,4,4], [3,4],3)
[3, 4]
sage: sage.combinat.subword.smallest_positions([1,2,3,4], [2,3])
[1, 2]
sage: sage.combinat.subword.smallest_positions([1,2,3,4], [5,5])
False
sage: sage.combinat.subword.smallest_positions([1,3,3,4,5],[3,5])
[1, 4]
sage: sage.combinat.subword.smallest_positions([1,3,3,5,4,5,3,5],[3,5,3])
[1, 3, 6]
sage: sage.combinat.subword.smallest_positions([1,3,3,5,4,5,3,5],[3,5,3],2)
[2, 3, 6]
sage: sage.combinat.subword.smallest_positions([1,2,3,4,3,3,4],[2,3,3,1])
False
sage: sage.combinat.subword.smallest_positions([1,3,3,5,4,5,3,5],[3,5,3],3)
False

5.1.337 Subword complex

Fix a Coxeter system \((W, S)\). The subword complex \(SC(Q, w)\) associated to a word \(Q \in S^*\) and an element \(w \in W\) is the simplicial complex whose ground set is the set of positions in \(Q\) and whose facets are complements of sets of positions defining a reduced expression for \(w\).

A subword complex is a shellable sphere if and only if the Demazure product of \(Q\) equals \(w\), otherwise it is a shellable ball.

The code is optimized to be used with ReflectionGroup, it works as well with CoxeterGroup, but many methods fail for WeylGroup.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: W = ReflectionGroup(['A',3]); I = list(W.index_set())
# optional - gap3
sage: Q = I + W.w0.coxeter_sorting_word(I); Q
# optional - gap3
[1, 2, 3, 1, 2, 3, 1, 2, 1]

sage: S = SubwordComplex(Q,W.w0)
# optional - gap3
sage: for F in S: print("\{} \{}\".format(F, F.root_configuration())) # optional - gap3
(0, 1, 2) [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
(0, 1, 8) [(1, 0, 0), (0, 1, 0), (0, 0, -1)]
(0, 2, 6) [(1, 0, 0), (0, 1, 1), (0, -1, 0)]
(0, 6, 7) [(1, 0, 0), (0, 0, 1), (0, -1, -1)]
(0, 7, 8) [(1, 0, 0), (0, -1, 0), (0, 0, -1)]
(1, 2, 3) [(1, 1, 0), (0, 0, 1), (-1, 0, 0)]
(1, 3, 8) [(1, 1, 0), (-1, 0, 0), (0, 0, -1)]
(2, 3, 4) [(1, 1, 1), (0, 1, 0), (-1, -1, 0)]
(2, 4, 6) [(1, 1, 1), (-1, 0, 0), (0, -1, 0)]
(3, 4, 5) [(0, 1, 0), (0, 0, 1), (-1, -1, -1)]
(4, 5, 6) [(0, 1, 1), (-1, -1, -1), (0, -1, 0)]
(5, 6, 7) [(-1, 0, 0), (0, 0, 1), (0, -1, -1)]
(5, 7, 8) [(-1, 0, 0), (0, -1, 0), (0, 0, -1)]

Testing that the implementation also works with CoxeterGroup:

```python
sage: W = CoxeterGroup(['A',3]); I = list(W.index_set())
sage: Q = I + W.w0.coxeter_sorting_word(I); Q
[1, 2, 3, 1, 2, 3, 1, 2, 1]

sage: S = SubwordComplex(Q,W.w0); S
Subword complex of type ['A', 3] for Q = (1, 2, 3, 1, 2, 3, 1, 2, 1) and pi = [1, 2, 3, -1, 2, 1]

sage: P = S.increasing_flip_poset(); P; len(P.cover_relations())
Finite poset containing 14 elements
21

The root configuration works:

```python
sage: for F in S: print("\{} \{}\".format(F, F.root_configuration()))
(0, 1, 2) [(1, 0, 0), (0, 1, 0), (0, 0, 1)]
(0, 1, 8) [(1, 0, 0), (0, 1, 0), (0, 0, -1)]
(0, 2, 6) [(1, 0, 0), (0, 1, 1), (0, -1, 0)]
(0, 6, 7) [(1, 0, 0), (0, 0, 1), (0, -1, -1)]
(0, 7, 8) [(1, 0, 0), (0, -1, 0), (0, 0, -1)]
(1, 2, 3) [(1, 1, 0), (0, 0, 1), (-1, 0, 0)]
(1, 3, 8) [(1, 1, 0), (-1, 0, 0), (0, 0, -1)]
(2, 3, 4) [(1, 1, 1), (0, 1, 0), (-1, -1, 0)]
(2, 4, 6) [(1, 1, 1), (-1, 0, 0), (0, -1, 0)]
(3, 4, 5) [(0, 1, 0), (0, 0, 1), (-1, -1, -1)]
(4, 5, 6) [(0, 1, 1), (-1, -1, -1), (0, -1, 0)]
(5, 6, 7) [(-1, 0, 0), (0, 0, 1), (0, -1, -1)]
(5, 7, 8) [(-1, 0, 0), (0, -1, 0), (0, 0, -1)]

And the weight configuration also works:

```
Combinatorics, Release 10.0

```python
sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1],w)
sage: F = SC([1,2])
sage: F.extended_weight_configuration()
[(4/3, 2/3), (2/3, 4/3), (-2/3, 2/3), (2/3, 4/3), (-2/3, 2/3)]
sage: F.extended_weight_configuration(coefficients=(1,2))
[(4/3, 2/3), (4/3, 8/3), (-2/3, 2/3), (4/3, 8/3), (-2/3, 2/3)]
```

One finally can compute the brick polytope, using all functionality on weight configurations, though it does not realize to live in real space:

```python
sage: W = CoxeterGroup(['A',3]); I = list(W.index_set())
sage: Q = I + W.w0.coxeter_sorting_word(I)
sage: S = SubwordComplex(Q,W.w0)
sage: S.brick_polytope()
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 14 vertices
doctest:...: RuntimeWarning: the polytope is built with rational vertices
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 32 vertices
```

AUTHORS:
- Christian Stump: initial version
- Vincent Pilaud: greedy flip algorithm, minor improvements, documentation

REFERENCES:

class sage.combinat.subword_complex.SubwordComplex(Q, w, algorithm='inductive')

Fix a Coxeter system \((W, S)\). The subword complex \(SC(Q, w)\) associated to a word \(Q \in S^*\) and an element \(w \in W\) is the simplicial complex whose ground set is the set of positions in \(Q\) and whose facets are complements of sets of positions defining a reduced expression for \(w\).

A subword complex is a shellable sphere if and only if the Demazure product of \(Q\) equals \(w\), otherwise it is a shellable ball.

**Warning:** This implementation only works for groups build using `CoxeterGroup`, and does not work with groups build using `WeylGroup`.

**EXAMPLES:**

As an example, dual associahedra are subword complexes in type \(A_{n-1}\) given by the word \([1, \ldots, n, 1, \ldots, n-1, \ldots, 1, 2, 1]\) and the permutation \(w_0\).

```python
sage: W = ReflectionGroup(['A',2]) # optional - gap3
sage: w = W.from_reduced_word([1,2,1]) # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w); SC # optional - gap3
Subword complex of type ['A', 2] for Q = (1, 2, 1, 2, 1) and pi = [1, 2, 1]
```
sage: SC.facets()  # optional - gap3
[(0, 1), (0, 4), (1, 2), (2, 3), (3, 4)]

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([[1,2,1,2,1], w]); SC
Subword complex of type ['A', 2] for Q = (1, 2, 1, 2, 1) and pi = [1, 2, 1]
sage: SC.facets()
[(0, 1), (0, 4), (1, 2), (2, 3), (3, 4)]

REFERENCES: [KnuMil], [PilStu]

Element
alias of SubwordComplexFacet

barycenter()
Return the barycenter of the brick polytope of self.

See also:
brick_polytope()

EXAMPLES:

sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: SC = SubwordComplex([[1,2,1,2,1], W.w0])  # optional - gap3
sage: SC.barycenter()  # optional - gap3
(2/3, 4/3)

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([[1,2,1,2,1], w])
sage: SC.barycenter()
(4/3, 8/3)

brick_fan()
Return the brick fan of self.

It is the normal fan of the brick polytope of self. It is formed by the cones generated by the weight configurations of the facets of self.

See also:
weight_cone

EXAMPLES:

sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([[1,2,1,2,1], w])  # optional - gap3
sage: SC.brick_fan()  # optional - gap3
Rational polyhedral fan in 2-d lattice N

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([[1,2,1,2,1], w])
sage: SC.brick_fan()
Rational polyhedral fan in 2-d lattice N
brick_polytope(coefficients=None)

Return the brick polytope of self.

This polytope is the convex hull of the brick vectors of self.

INPUT:

- coefficients – (optional) a list of coefficients used to scale the fundamental weights

See also:

brick_vectors()

EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])                 # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)       # optional - gap3
sage: X = SC.brick_polytope(); X                   # optional - gap3
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 5 vertices

sage: Y = SC.brick_polytope(coefficients=[1,2]); Y  # optional - gap3
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 5 vertices

sage: X == Y                                        # optional - gap3
False

sage: W = CoxeterGroup(['A',2])

sage: SC = SubwordComplex([1,2,1,2,1], W.w0)

sage: X = SC.brick_polytope(); X                   # optional - gap3
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 5 vertices

sage: W = ReflectionGroup(['H',3])                 # optional - gap3

sage: c = W.index_set(); Q = c + tuple(W.w0.coxeter_sorting_word(c))  # optional - gap3
sage: SC = SubwordComplex(Q, W.w0)                  # optional - gap3
sage: SC.brick_polytope()                           # optional - gap3
doctest:...:
RuntimeWarning: the polytope is built with rational vertices
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 32 vertices
```

brick_vectors(coefficients=None)

Return the list of all brick vectors of facets of self.

INPUT:

- coefficients – (optional) a list of coefficients used to scale the fundamental weights

See also:

brick_vector

EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])                 # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)       # optional - gap3
sage: SC.brick_vectors()                            # optional - gap3
[[5/3, 7/3), (5/3, 1/3), (2/3, 7/3), (-1/3, 4/3), (-1/3, 1/3)]

sage: SC.brick_vectors(coefficients=(1,2))         # optional - gap3
[[7/3, 11/3), (7/3, 2/3), (4/3, 11/3), (-2/3, 5/3), (-2/3, 2/3)]
```

(continues on next page)
```python
sage: W = CoxeterGroup(['A',2])
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)
sage: SC.brick_vectors()
[(10/3, 14/3), (10/3, 2/3), (4/3, 14/3), (-2/3, 8/3), (-2/3, 2/3)]
sage: SC.brick_vectors(coefficients=(1,2))
[(14/3, 22/3), (14/3, 4/3), (8/3, 22/3), (-4/3, 10/3), (-4/3, 4/3)]
```

**cartan_type()**

Return the Cartan type of self.

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)  # optional - gap3
sage: SC.cartan_type()  # optional - gap3
['A', 2]
```

**cover_relations(label=False)**

Return the set of cover relations in the associated poset.

**INPUT:**

- label – boolean (default False) whether or not to label the cover relations by the position of flip

**OUTPUT:**

a list of pairs of facets

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)  # optional - gap3
sage: sorted(SC.cover_relations())  # optional - gap3
[((0, 1), (0, 4)),((0, 1), (1, 2)),((0, 4), (3, 4)),((1, 2), (2, 3)),((2, 3), (3, 4))]
```

```
```

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```
**dimension()**

Return the dimension of `self`.

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,1,2,1], W.w0)  # optional - gap3
sage: SC.dimension()  # optional - gap3
1

sage: W = CoxeterGroup(['A',2])
sage: SC = SubwordComplex([1,2,1,1,2,1], W.w0)
sage: SC.dimension()  # optional - gap3
1
```

**facets()**

Return all facets of `self`.

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)  # optional - gap3
sage: SC.facets()  # optional - gap3
[(0, 1), (0, 4), (1, 2), (2, 3), (3, 4)]

sage: W = CoxeterGroup(['A',2])
```

```python
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1], w)
sage: SC.facets()  # optional - gap3
[(0, 1), (0, 4), (1, 2), (2, 3), (3, 4)]
```

**greedy_facet(side='positive')**

Return the negative (or positive) greedy facet of `self`.

This is the lexicographically last (or first) facet of `self`.

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)  # optional - gap3
sage: SC.greedy_facet(side="positive")  # optional - gap3
(0, 1)
sage: SC.greedy_facet(side="negative")  # optional - gap3
(3, 4)

sage: W = CoxeterGroup(['A',2])
```

```python
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1], w)
sage: SC.greedy_facet(side="positive")  # optional - gap3
(0, 1)
sage: SC.greedy_facet(side="negative")  # optional - gap3
(3, 4)
```
group()

Return the group associated to self.

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])                  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])              # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)           # optional - gap3
sage: SC.group()                                    # optional - gap3
Irreducible real reflection group of rank 2 and type A2
```

increasing_flip_graph(label=True)

Return the increasing flip graph of the subword complex.

OUTPUT:

a directed graph

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])                  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)        # optional - gap3
sage: SC.increasing_flip_graph()                    # optional - gap3
Digraph on 5 vertices
```

increasing_flip_poset()

Return the increasing flip poset of the subword complex.

OUTPUT:

a poset

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])                  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)        # optional - gap3
sage: SC.increasing_flip_poset()                    # optional - gap3
Finite poset containing 5 elements
```
**interval**(\(I, J\))

Return the interval \([IJ]\) in the increasing flip graph subword complex.

**INPUT:**

- \(I, J\) – two facets

**OUTPUT:**

a set of facets

**EXAMPLES:**

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)  # optional - gap3
sage: F = SC([1,2])  # optional - gap3
sage: SC.interval(F, F)  # optional - gap3
{(1, 2)}
```

```
sage: W = CoxeterGroup(['A',2])
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)
sage: F = SC([1,2])
sage: SC.interval(F, F)
{(1, 2)}
```

**is_ball()**

Return True if the subword complex \(\text{self}\) is a ball.

This is the case if and only if it is not a sphere.

**EXAMPLES:**

```
sage: W = ReflectionGroup(['A',3])  # optional - gap3
sage: w = W.from_reduced_word([2,3,2])  # optional - gap3
sage: SC = SubwordComplex([3,2,3,2,3], w)  # optional - gap3
sage: SC.is_ball()  # optional - gap3
False
```

```
sage: SC = SubwordComplex([3,2,1,3,2,3], w)  # optional - gap3
sage: SC.is_ball()  # optional - gap3
True
```

```
sage: W = CoxeterGroup(['A',3])
sage: w = W.from_reduced_word([2,3,2])
sage: SC = SubwordComplex([3,2,3,2,3], w)
sage: SC.is_ball()  # optional - gap3
False
```

**is_double_root_free()**

Return True if \(\text{self}\) is double-root-free.

This means that the root configurations of all facets do not contain a root twice.

**EXAMPLES:**

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)  # optional - gap3
sage: SC.is_double_root_free()  # optional - gap3
```

(continues on next page)
sage: SC.is_double_root_free()    # optional - gap3
True

sage: SC = SubwordComplex([1,1,2,2,1,1], w)    # optional - gap3
sage: SC.is_double_root_free()    # optional - gap3
True

sage: SC = SubwordComplex([1,2,1,2,1,2], w)    # optional - gap3
sage: SC.is_double_root_free()    # optional - gap3
False

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1], w)
sage: SC.is_double_root_free()    # optional - gap3
True

is_pure()

Return True since all subword complexes are pure.

EXAMPLES:

sage: W = ReflectionGroup(['A',3])    # optional - gap3
sage: w = W.from_reduced_word([2,3,2])
sage: SC = SubwordComplex([3,2,3,2,3], w)    # optional - gap3
sage: SC.is_pure()    # optional - gap3
True

sage: W = CoxeterGroup(['A',3])

sage: SC = SubwordComplex([3,2,3,2,3], w)    # optional - gap3
sage: SC.is_pure()    # optional - gap3
True

is_root_independent()

Return True if self is root-independent.

This means that the root configuration of any (or equivalently all) facets is linearly independent.

EXAMPLES:

sage: W = ReflectionGroup(['A',2])    # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)    # optional - gap3
sage: SC.is_root_independent()    # optional - gap3
True

sage: SC = SubwordComplex([1,2,1,2,1,2], W.w0)    # optional - gap3
sage: SC.is_root_independent()    # optional - gap3
False

sage: W = CoxeterGroup(['A',2])

sage: SC = SubwordComplex([1,2,1,2,1], W.w0)

sage: SC.is_root_independent()    # optional - gap3
True
**is_sphere()**

Return True if the subword complex self is a sphere.

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',3]) # optional - gap3
sage: w = W.from_reduced_word([2,3,2]) # optional - gap3
sage: SC = SubwordComplex([3,2,3,2,3], w) # optional - gap3
sage: SC.is_sphere() # optional - gap3
True

sage: SC = SubwordComplex([3,2,1,3,2,3], w) # optional - gap3
sage: SC.is_sphere() # optional - gap3
False

sage: W = CoxeterGroup(['A',3])
sage: w = W.from_reduced_word([2,3,2])
sage: SC = SubwordComplex([3,2,3,2,3], w)
sage: SC.is_sphere() # optional - gap3
True
```

**kappa_preimages()**

Return a dictionary containing facets of self as keys, and list of elements of self.group() as values.

See also:

*kappa_preimage*

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A',2]) # optional - gap3
sage: w = W.from_reduced_word([1,2,1]) # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w) # optional - gap3
sage: kappa = SC.kappa_preimages() # optional - gap3
sage: for F in SC: print("{} {}").format(F, [w.reduced_word() for w in kappa[F]]) # optional - gap3
(0, 1) []
(0, 4) [[2], [2, 1]]
(1, 2) [[1]]
(2, 3) [[1, 2]]
(3, 4) [[1, 2, 1]]

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1], w)
sage: kappa = SC.kappa_preimages()
sage: for F in SC: print("{} {}").format(F, [w.reduced_word() for w in kappa[F]])
(0, 1) []
(0, 4) [[2], [2, 1]]
(1, 2) [[1]]
(2, 3) [[1, 2]]
(3, 4) [[1, 2, 1]]
```

**minkowski_summand(i)**

Return the i th Minkowski summand of self.
INPUT:

\(i\) – an integer defining a position in the word \(Q\)

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)  # optional - gap3
sage: SC.minkowski_summand(1)  # optional - gap3
A 0-dimensional polyhedron in \(\mathbb{Q}^2\) defined as the convex hull of 1 vertex

sage: W = CoxeterGroup(['A',2])
sage: SC = SubwordComplex([1,2,1,2,1], W.w0)
sage: SC.minkowski_summand(1)
A 0-dimensional polyhedron in \(\mathbb{Q}^2\) defined as the convex hull of 1 vertex
```

\(\pi()\)

Return the element in the Coxeter group associated to \(self\).

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)  # optional - gap3
sage: SC.pi().reduced_word()
[1, 2, 1]

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1], w)
sage: SC.pi().reduced_word()
[1, 2, 1]
```

\(\text{word()}\)

Return the word in the simple generators associated to \(self\).

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1], w)  # optional - gap3
sage: SC.word()
(1, 2, 1, 2, 1)

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1], w)
sage: SC.word()
(1, 2, 1, 2, 1)
```

class \texttt{sage.combinat.subword_complex.SubwordComplexFacet}(parent, positions, facet_test=True)

Bases: Simplex, Element

A facet of a subword complex.

Facets of the subword complex \(SC(Q,w)\) are complements of sets of positions in \(Q\) defining a reduced expression for \(w\).
EXAMPLES:

```python
sage: W = ReflectionGroup(['A', 2])  # optional - gap3
sage: w = W.from_reduced_word([1, 2, 1])  # optional - gap3
sage: SC = SubwordComplex([1, 2, 1, 2, 1], w)  # optional - gap3
sage: F = SC[0]; F  # optional - gap3
(0, 1)
```

```python
sage: W = CoxeterGroup(['A', 2])
sage: w = W.from_reduced_word([1, 2, 1])
sage: SC = SubwordComplex([1, 2, 1, 2, 1], w)
sage: F = SC[0]; F  # optional - gap3
(0, 1)
```

**brick_vector**

Return the brick vector of `self`.

This is the sum of the weight vectors in the extended weight configuration.

**INPUT:**

- coefficients – (optional) a list of coefficients used to scale the fundamental weights

**See also:**

**extended_weight_configuration()**

**EXAMPLES:**

```python
sage: W = ReflectionGroup(['A', 2])  # optional - gap3
sage: w = W.from_reduced_word([1, 2, 1])  # optional - gap3
sage: SC = SubwordComplex([1, 2, 1, 2, 1], w)  # optional - gap3
sage: F = SC[0]; F  # optional - gap3
(1, 2)
```

```python
sage: F.extended_weight_configuration()  # optional - gap3
[(2/3, 1/3), (1/3, 2/3), (-1/3, 1/3), (1/3, 2/3), (-1/3, 1/3)]
```

```python
sage: F.brick_vector()  # optional - gap3
(2/3, 7/3)
```

```python
sage: F.brick_vector(coefficients=[1, 2])  # optional - gap3
(4/3, 11/3)
```

```python
sage: W = CoxeterGroup(['A', 2])
sage: w = W.from_reduced_word([1, 2, 1])
sage: SC = SubwordComplex([1, 2, 1, 2, 1], w)
sage: F = SC[0]
sage: F.brick_vector()  # optional - gap3
(4/3, 11/3)
```

```python
sage: F.brick_vector(coefficients=[1, 2])  # optional - gap3
(8/3, 22/3)
```

**extended_root_configuration()**

Return the extended root configuration of `self`.

Let $Q = q_1, \ldots, q_m \in S^*$ and $w \in W$. The extended root configuration of a facet $I$ of $SC(Q, w)$ is the sequence $r(I, 1), \ldots, r(I, m)$ of roots defined by $r(I, k) = \Pi Q_{[k-1] \setminus I}(\alpha q_k)$, where $\Pi Q_{[k-1] \setminus I}$ is the product of the simple reflections $q_i$ for $i \in [k-1] \setminus I$ in this order.

The extended root configuration is used to perform flips efficiently.
See also:

flip()

EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])       # optional - gap3
sage: w = W.from_reduced_word([1,2,1])   # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1],w) # optional - gap3
sage: F = SC([1,2]); F                    # optional - gap3
(1, 2)
sage: F.extended_root_configuration()   # optional - gap3
[(1, 0), (1, 1), (-1, 0), (1, 1), (0, 1)]
```

extended_weight_configuration(coefficients=None)

Return the extended weight configuration of self.

Let $Q = q_1 \ldots q_m \in S^*$ and $w \in W$. The extended weight configuration of a facet $I$ of $SC(Q,w)$ is the sequence $w(I,1), \ldots, w(I,m)$ of weights defined by $w(I,k) = \Pi Q_{[k-1]\setminus I}(\varpi_{q_k})$, where $\Pi Q_{[k-1]\setminus I}$ is the product of the simple reflections $q_i$ for $i \in [k-1]\setminus I$ in this order.

The extended weight configuration is used to compute the brick vector.

INPUT:

* coefficients – (optional) a list of coefficients used to scale the fundamental weights

See also:

brick_vector()

EXAMPLES:

```python
sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1],w)
sage: F = SC([1,2])
sage: F.extended_weight_configuration()  # optional - gap3
[(4/3, 2/3), (2/3, 4/3), (-2/3, 2/3), (2/3, 4/3), (-2/3, 2/3)]
sage: F.extended_weight_configuration(coefficients=(1,2))  # optional - gap3
[(4/3, 2/3), (4/3, 8/3), (-2/3, 2/3), (4/3, 8/3), (-2/3, 2/3)]
```

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flip(i, return_position=False)

Return the facet obtained after flipping position i in self.

INPUT:

• i – position in the word Q (integer).
• return_position – boolean (default: False) tells whether the new position should be returned as well.

OUTPUT:

• The new subword complex facet.
• The new position if return_position is True.

EXAMPLES:

```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([1,2,1,1,2,1],w)  # optional - gap3
sage: F = SC([1,2]); F  # optional - gap3
(1, 2)
sage: F.flip(1)  # optional - gap3
(2, 3)
sage: F.flip(1, return_position=True)  # optional - gap3
((2, 3), 3)
sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,1,2,1],w)
sage: F = SC([1,2]); F
(1, 2)
sage: F.flip(1)
(2, 3)
sage: F.flip(1, return_position=True)
((2, 3), 3)
```

is_vertex()

Return True if self is a vertex of the brick polytope of self.parent.

A facet is a vertex of the brick polytope if its root cone is pointed. Note that this property is always satisfied for root-independent subword complexes.

See also:

root_cone()

EXAMPLES:

```
sage: W = ReflectionGroup(['A',1])  # optional - gap3
sage: w = W.from_reduced_word([1])  # optional - gap3
sage: SC = SubwordComplex([1,1,1],w)  # optional - gap3
sage: F = SC([0,1]); F.is_vertex()  # optional - gap3
True
sage: F = SC([0,2]); F.is_vertex()  # optional - gap3
False
sage: W = ReflectionGroup(['A',2])  # optional - gap3
```

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(continued from previous page)

\begin{CodeBlock}
\begin{verbatim}
\textbf{sage}: w = W.from_reduced_word([1,2,1])  # optional - gap3
\textbf{sage}: SC = SubwordComplex([1,2,1,2,1,1,2,1],w)  # optional - gap3
\textbf{sage}: F = SC([0,1,2,3]); F.is_vertex()  # optional - gap3
True
\textbf{sage}: F = SC([0,1,2,6]); F.is_vertex()  # optional - gap3
False
\textbf{sage}: W = CoxeterGroup(['A',2])
\textbf{sage}: w = W.from_reduced_word([1,2,1])
\textbf{sage}: SC = SubwordComplex([1,2,1,2,1,1,2,1],w)
\textbf{sage}: F = SC([0,1,2,3]); F.is_vertex()  # optional - gap3
True
\textbf{sage}: F = SC([0,1,2,6]); F.is_vertex()  # optional - gap3
False
\end{verbatim}
\end{CodeBlock}

\textbf{kappa_preimage()}

Return the fiber of \texttt{self} under the $\kappa$ map.

The $\kappa$ map sends an element $w \in W$ to the unique facet of $I \in SC(Q, w)$ such that the root configuration of $I$ is contained in $w(\Phi^+)$. In other words, $w$ is in the preimage of \texttt{self} under $\kappa$ if and only if $w^{-1}$ sends every root in the root configuration to a positive root.

\textbf{EXAMPLES:}

\begin{CodeBlock}
\begin{verbatim}
\textbf{sage}: W = ReflectionGroup(['A',2])  # optional - gap3
\textbf{sage}: w = W.from_reduced_word([1,2,1])
\textbf{sage}: SC = SubwordComplex([1,2,1,2,1,1,2,1],w)
\textbf{sage}: F = SC([1,2]); F  # optional - gap3
(1, 2)
\textbf{sage}: F.kappa_preimage()  # optional - gap3
[(1,4)(2,3)(5,6)]
\textbf{sage}: F = SC([0,4]); F  # optional - gap3
(0, 4)
\textbf{sage}: F.kappa_preimage()  # optional - gap3
[(1,3)(2,5)(4,6), (1,2,6)(3,4,5)]
\textbf{sage}: W = CoxeterGroup(['A',2])
\textbf{sage}: w = W.from_reduced_word([1,2,1])
\textbf{sage}: SC = SubwordComplex([1,2,1,2,1,1,2,1],w)
\textbf{sage}: F = SC([1,2]); F  # optional - gap3
(1, 2)
\textbf{sage}: F.kappa_preimage()  # optional - gap3
[
[-1  1]
[ 0  1]
]
\textbf{sage}: F = SC([0,4]); F  # optional - gap3
(0, 4)
\textbf{sage}: F.kappa_preimage()  # optional - gap3
\end{verbatim}
\end{CodeBlock}

(continues on next page)
\begin{verbatim}
[ [ 1  0] [-1  1]
 [ 1 -1], [-1  0]
]
\end{verbatim}

\texttt{plot}(list\_colors=\texttt{None}, labels=[], thickness=3, fontsize=14, shift=(0, 0), compact=False, roots=True, **\texttt{args})

In type $A$ or $B$, plot a pseudoline arrangement representing the facet $\texttt{self}$.

Pseudoline arrangements are graphical representations of facets of types $A$ or $B$ subword complexes.

INPUT:

- \texttt{list\_colors} -- list (default: []) to change the colors of the pseudolines.
- \texttt{labels} -- list (default: []) to change the labels of the pseudolines.
- \texttt{thickness} -- integer (default: 3) for the thickness of the pseudolines.
- \texttt{fontsize} -- integer (default: 14) for the size of the font used for labels.
- \texttt{shift} -- couple of coordinates (default: $(0,0)$) to change the origin.
- \texttt{compact} -- boolean (default: False) to require a more compact representation.
- \texttt{roots} -- boolean (default: True) to print the extended root configuration.

EXAMPLES:

\begin{verbatim}
sage: W = ReflectionGroup(['A',2])                      # optional - gap3
sage: w = W.from\_reduced\_word([1,2,1])                  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1],w)                 # optional - gap3
sage: F = SC([1,2]); F.plot()                             # optional - gap3
Graphics object consisting of 26 graphics primitives
sage: W = CoxeterGroup(['A',2])                           # optional - gap3
sage: w = W.from\_reduced\_word([1,2,1])                  # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1],w)                 # optional - gap3
sage: F = SC([1,2]); F.plot()                             # optional - gap3
Graphics object consisting of 26 graphics primitives
sage: W = ReflectionGroup(['B',3])                        # optional - gap3
sage: c = W.from\_reduced\_word([1,2,3])                  # optional - gap3
sage: Q = c.reduced\_word()*2 + W.w0.coxeter\_sorting\_word(c) # optional - gap3
sage: SC = SubwordComplex(Q, W.w0)                       # optional - gap3
sage: F = SC[15]; F.plot()                                # optional - gap3
Graphics object consisting of 53 graphics primitives
\end{verbatim}

REFERENCES: \cite{PilStu}

\texttt{root\_cone}()

Return the polyhedral cone generated by the root configuration of $\texttt{self}$.

See also:

\texttt{root\_configuration}()

EXAMPLES:
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```python
sage: W = ReflectionGroup(['A',1])  # optional - gap3
sage: w = W.from_reduced_word([1])   # optional - gap3
sage: SC = SubwordComplex([[1,1,1],w]) # optional - gap3
sage: F = SC([0,2]); F.root_cone()   # optional - gap3
1-d cone in 1-d lattice N

sage: W = CoxeterGroup(['A',1])
```

```
sage: w = W.from_reduced_word([1])
```

```
sage: SC = SubwordComplex([[1,1,1],w])
```

```
sage: F = SC([0,2]); F.root_cone()   # optional - gap3
1-d cone in 1-d lattice N
```

root_configuration()

Return the root configuration of self.

Let $Q = q_1 \ldots q_m \in S^*$ and $w \in W$. The root configuration of a facet $I = [i_1, \ldots, i_n]$ of $\mathcal{SC}(Q, w)$ is the sequence $r(I, i_1), \ldots, r(I, i_n)$ of roots defined by $r(I, k) = \Pi Q[k-1] \setminus I(\alpha_{q_k})$, where $\Pi Q[k-1] \setminus I$ is the product of the simple reflections $q_i$ for $i \in [k-1] \setminus I$ in this order.

EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([[1,2,1,2,1],w])  # optional - gap3
sage: F = SC([1,2]); F.root_cone()   # optional - gap3
(1, 2)
sage: F.root_configuration()       # optional - gap3
[(1, 1), (-1, 0)]
```

```python
sage: W = CoxeterGroup(['A',2])
```

```
sage: w = W.from_reduced_word([1,2,1])
```

```
sage: SC = SubwordComplex([[1,2,1,2,1],w])
```

```
sage: F = SC([1,2]); F.root_cone()   # optional - gap3
(1, 2)
sage: F.root_configuration()       # optional - gap3
[(1, 1), (-1, 0)]
```

show(*kwds, **args)

Show the facet self.

See also:

plot()

EXAMPLES:

```python
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])  # optional - gap3
sage: SC = SubwordComplex([[1,2,1,2,1],w])  # optional - gap3
sage: F = SC([1,2]); F.show()   # optional - gap3
```

upper_root_configuration()

Return the positive roots of the root configuration of self.

EXAMPLES:

```python
```
sage: W = ReflectionGroup(['A',2])  # optional - gap3
sage: w = W.from_reduced_word([1,2,1])   # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1],w)  # optional - gap3
sage: F = SC([1,2]); F  # optional - gap3
(1, 2)

sage: F.root_configuration()  # optional - gap3
[(1, 1), (-1, 0)]

sage: F.upper_root_configuration()  # optional - gap3
[(1, 0)]

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1],w)
sage: F = SC([1,2]); F  # optional - gap3
(1, 2)
sage: F.upper_root_configuration()  # optional - gap3
[(1, 0)]

weight_cone()

Return the polyhedral cone generated by the weight configuration of self.

See also:
weight_configuration()

EXAMPLES:

weight_configuration()

Return the weight configuration of self.

Let \( Q = q_1 \ldots q_m \in S^* \) and \( w \in W \). The weight configuration of a facet \( I = [i_1, \ldots, i_n] \) of \( SC(Q,w) \) is the sequence \( w^I(i_1), \ldots, w^I(i_n) \) of weights defined by \( w^I(k) = \Pi q_{[k-1] \setminus I} \omega_q \), where \( \Pi q_{[k-1] \setminus I} \) is the product of the simple reflections \( q_i \) for \( i \in [k-1] \setminus I \) in this order.

EXAMPLES:
sage: W = ReflectionGroup(['A',2]) # optional - gap3
sage: w = W.from_reduced_word([1,2,1]) # optional - gap3
sage: SC = SubwordComplex([1,2,1,2,1,w]) # optional - gap3
sage: F = SC([1,2]); F # optional - gap3
(1, 2)
sage: F.weight_configuration() # optional - gap3
[(1/3, 2/3), (-1/3, 1/3)]

sage: W = CoxeterGroup(['A',2])
sage: w = W.from_reduced_word([1,2,1])
sage: SC = SubwordComplex([1,2,1,2,1,w])
sage: F = SC([1,2]); F
(1, 2)
sage: F.weight_configuration() # optional - gap3
[(2/3, 4/3), (-2/3, 2/3)]

5.1.338 Super Tableaux

AUTHORS:

• Matthew Lancellotti (2007): initial version

class sage.combinat.super_tableau.SemistandardSuperTableau(parent, t, check=True,
preprocessed=False)

Bases: Tableau

A semistandard super tableau.

A semistandard super tableau is a tableau with primed positive integer entries. As defined in [Muth2019], a semistandard super tableau weakly increases along the rows and down the columns. Also, the letters of even parity (unprimed) strictly increases down the columns, and letters of odd parity (primed) strictly increases along the rows. Note that Sage uses the English convention for partitions and tableaux; the longer rows are displayed on top.

INPUT:

• t – a tableau, a list of iterables, or an empty list

EXAMPLES:

sage: t = SemistandardSuperTableau([['1p',2,'3'],[2,3]]); t
[[1', 2, 3'], [2, 3]]
sage: t.shape()
[3, 2]
sage: t.pp() # pretty printing
1' 2 3'
2 3
sage: t = Tableau([['1p',2],[2]])
sage: s = SemistandardSuperTableau(t); s
[[1', 2], [2]]
sage: SemistandardSuperTableau([]) # The empty tableau
[]
check()

Check that self is a valid semistandard super tableau.

class sage.combinat.super_tableau.SemistandardSuperTableaux(/**kwds)**

Bases: SemistandardTableaux

The set of semistandard super tableaux.

A semistandard super tableau is a tableau with primed positive integer entries. As defined in [Muth2019], a semistandard super tableau weakly increases along the rows and down the columns. Also, the letters of even parity (unprimed) strictly increases down the columns, and letters of odd parity (primed) strictly increases along the rows. Note that Sage uses the English convention for partitions and tableaux; the longer rows are displayed on top.

EXAMPLES:

```sage
SST = SemistandardSuperTableaux(); SST
Semistandard super tableaux
```

**Element**

alias of SemistandardSuperTableau

class sage.combinat.super_tableau.SemistandardSuperTableaux_all

Bases: SemistandardSuperTableaux

All semistandard super tableaux.

class sage.combinat.super_tableau.StandardSuperTableau(parent, t, check=True, preprocessed=False)

Bases: SemistandardSuperTableau

A standard super tableau.

A standard super tableau is a semistandard super tableau whose entries are in bijection with positive primed integers $1', 1'^2, 2', \ldots, n$.

For more information refer [Muth2019].

INPUT:

* t – a Tableau, a list of iterables, or an empty list

EXAMPLES:

```sage
t = StandardSuperTableau([["1'","1","2'","2","3'","3'"],[3,"4'"],[1', 1', 2', 2', 3', 3']]); t
[[1', 1', 2', 2', 3', 3'], [3, 4']]
sage: t.shape()
[5, 2]
sage: t.pp() # pretty printing
1' 1 2' 2 3'
3 4'
sage: t.is_standard()
True
sage: StandardSuperTableau([]) # The empty tableau
[]
```

check()

Check that self is a standard tableau.
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\textbf{is\_standard()}

Return \texttt{True} since \texttt{self} is a standard super tableau.

EXAMPLES:

\begin{verbatim}
  sage: StandardSuperTableau([['1p', 1], ['2p', 2]]).is_standard()
  True
\end{verbatim}

\textbf{class} \texttt{sage.combinat.super_tableau.StandardSuperTableaux(**kwds)}

\textbf{Bases:} \texttt{SemistandardSuperTableaux, Parent}

The set of standard super tableaux.

A standard super tableau is a tableau whose entries are primed positive integers, which are strictly increasing in rows and down columns and contains each letters from 1',1,2'…n exactly once.

For more information refer [Muth2019].

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{n} – a non-negative integer or a partition.
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
  sage: SST = StandardSuperTableaux()
  sage: SST
  Standard super tableaux
  sage: SST([["1'", 1], ["2'", 2], [3,"4'"],[1,'1', 2', 3', [3, 4']]])
  [[1', 1, 2', 3', 3, 4']]
  sage: SST = StandardSuperTableaux(3)
  sage: SST
  Standard super tableaux of size 3
  sage: SST.first()
  [[1', 1, 2']]  
  sage: SST.last()
  [[1'], [1], [2']]  
  sage: SST.cardinality()
  4
  sage: SST.list()
  [[[1', 1'], [1', 2'], [1]], [[1', 1], [2']], [[1'], [1], [2']]]
  sage: SST = StandardSuperTableaux([3,2])
  sage: SST
  Standard super tableaux of shape [3, 2]
\end{verbatim}

\textbf{Element}

alias of \texttt{StandardSuperTableau}

\textbf{class} \texttt{sage.combinat.super_tableau.StandardSuperTableaux\_all}

\textbf{Bases:} \texttt{StandardSuperTableaux, DisjointUnionEnumeratedSets}

All standard super tableaux.

\textbf{class} \texttt{sage.combinat.super_tableau.StandardSuperTableaux\_shape(p)}

\textbf{Bases:} \texttt{StandardSuperTableaux}

Standard super tableaux of a fixed shape \texttt{p}.

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cardinality()

Return the number of standard super tableaux of given shape.

The standard super tableaux of a fixed shape \( p \) are in bijection with the corresponding standard tableaux (under the alphabet relabeling). Refer \texttt{sage.combinat.tableau.StandardTableaux\_shape} for more details.

**EXAMPLES:**

```
sage: StandardSuperTableaux([3,2,1]).cardinality()
16
sage: StandardSuperTableaux([2,2]).cardinality()
2
sage: StandardSuperTableaux([5]).cardinality()
1
sage: StandardSuperTableaux([6,5,3]).cardinality()
6651216
sage: StandardSuperTableaux([]).cardinality()
1
```

class \texttt{sage.combinat.super\_tableau.StandardSuperTableaux\_size}(n)

**Bases:** \texttt{StandardSuperTableaux, DisjointUnionEnumeratedSets}

Standard super tableaux of fixed size \( n \).

**EXAMPLES:**

```
sage: [ t for t in StandardSuperTableaux(1) ]
[[[1']]]
sage: [ t for t in StandardSuperTableaux(2) ]
[[[1', 1]], [[1'], [1]]]
sage: [ t for t in StandardSuperTableaux(3) ]
[[[1', 1, 2']], [[[1', 2'], [1]], [[1', 1], [2']], [[1'], [1], [2']]],
[[1'], 1, 2', 2]],
[[[1', 1, 2'], [2']],
[[1', 1', 2'], [2]],
[[1', 2'], [1, 2]],
[[1', 1], [2', 2]],
[[1', 2], [1], [2']],
[[1', 2'], [1], [2]],
[[1', 1], [2'], [2]],
[[1'], [1], [2'], [2]]]
```

cardinality()

Return the number of all standard super tableaux of size \( n \).

The standard super tableaux of size \( n \) are in bijection with the corresponding standard tableaux (under the alphabet relabeling). Refer \texttt{sage.combinat.tableau.StandardTableaux\_size} for more details.

**EXAMPLES:**

```
sage: StandardSuperTableaux(3).cardinality()
4
sage: ns = [1,2,3,4,5,6]
```

(continues on next page)
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5.1.339 Super Partitions

AUTHORS:

• Mike Zabrocki

A super partition of size $n$ and fermionic sector $m$ is a pair consisting of a strict partition of some integer $r$ of length $m$ (that may end in a 0) and an integer partition of $n - r$.

This module provides tools for manipulating super partitions.

Super partitions are the indexing set for symmetric functions in super space.

Super partitions may be input in two different formats: one as a pair consisting of fermionic (strict partition) and a bosonic (partition) part and the other as a list of integer values where the negative entries come first and are listed in strict order followed by the positive values in weak order.

A super partition is displayed as two partitions separated by a semicolon as a default. Super partitions may also be displayed as a weakly increasing sequence of integers that are strict if the numbers are not positive.

These combinatorial objects index the space of symmetric polynomials in two sets of variables, one commuting and one anti-commuting, and they are known as symmetric functions in super space (hence the origin of the name super partitions).

EXAMPLES:

```python
sage: st = [StandardSuperTableaux(n) for n in ns]
sage: all(st.cardinality() == len(st.list()) for st in st)
True
sage: StandardSuperTableaux(50).cardinality() # long time
2788695605342342839104615869259776
```

---

5.1. Comprehensive Module List
False

```
sage: [-2, -2, 2, 1] in SuperPartitions(7,2)
False

REFERENCES:
• [JL2016]

class sage.combinat.superpartition.SuperPartition(parent, lst, check=True, immutable=True)

   Bases: ClonableArray

   A super partition.

   A super partition of size \( n \) and fermionic sector \( m \) is a pair consisting of a strict partition of some integer \( r \) of length \( m \) (that may end in a 0) and an integer partition of \( n - r \).

   EXAMPLES:

   sage: sp = SuperPartition([[1,0],[2,2,1]]); sp
   [1, 0; 2, 2, 1]
   sage: sp[0]
   (1, 0)
   sage: sp[1]
   (2, 2, 1)
   sage: sp.fermionic_degree()
   2
   sage: sp.bosonic_degree()
   6
   sage: sp.length()
   5
   sage: sp.conjugate()
   [4, 2; ]

   a_part()
   The antisymmetric part as a list of strictly decreasing integers.

   OUTPUT:
   • a list

   EXAMPLES:

   sage: SuperPartition([[3,1],[2,2,1]]).antisymmetric_part()
   [3, 1]
   sage: SuperPartition([[2,1,0],[3,3]]).antisymmetric_part()
   [2, 1, 0]

   add_horizontal_border_strip_star(h)
   Return a list of super partitions that differ from self by a horizontal strip.

   The notion of horizontal strip comes from the Pieri rule for the Schur-star basis of symmetric functions in super space (see Theorem 7 from [JL2016]).

   INPUT:
   • \( h \) – number of cells in the horizontal strip

   OUTPUT:
add_horizontal_border_strip_star_bar(h)

List super partitions that differ from self by a horizontal strip.

The notion of horizontal strip comes from the Pieri rule for the Schur-star-bar basis of symmetric functions in super space (see Theorem 10 from [JL2016]).

INPUT:

• h – number of cells in the horizontal strip

OUTPUT:

• a list of super partitions

EXAMPLES:

```
sage: SuperPartition([[4,1],[3]]).add_horizontal_border_strip_star_bar(3)
[[3, 1; 7],
 [4, 1; 6],
 [3, 0; 6, 2],
 [3, 1; 6, 1],
 [4, 0; 5, 2],
 [4, 1; 5, 1],
 [3, 0; 5, 3],
 [3, 1; 5, 2],
 [4, 0; 4, 3],
 [4, 1; 4, 2],
 [4, 1; 3, 3]]
sage: SuperPartition([[2,1],[3]]).add_horizontal_border_strip_star_bar(2)
[[2, 1; 5], [2, 0; 4, 2], [2, 1; 4, 1], [2, 0; 3, 3], [2, 1; 3, 2]]
```

(continues on next page)
antisymmetric_part()
The antisymmetric part as a list of strictly decreasing integers.

OUTPUT:
  • a list

EXAMPLES:

\[
\text{sage: SuperPartition([[3,1],[2,2,1]]).antisymmetric_part()}
\]
\[ [3, 1] \]

\[
\text{sage: SuperPartition([[2,1,0],[3,3]]).antisymmetric_part()}
\]
\[ [2, 1, 0] \]

bi_degree()
Return the bidegree of \texttt{self}, which is a pair consisting of the bosonic and fermionic degree.

OUTPUT:
  • a tuple of two integers

EXAMPLES:

\[
\text{sage: SuperPartition([[3,1],[2,2,1]]).bi_degree()}
\]
\[ (9, 2) \]

\[
\text{sage: SuperPartition([[2,1,0],[3,3]]).bi_degree()}
\]
\[ (9, 3) \]

bosonic_degree()
Return the bosonic degree of \texttt{self}.

The \textit{bosonic degree} is the sum of the sizes of the antisymmetric and symmetric parts.

OUTPUT:
  • an integer

EXAMPLES:

\[
\text{sage: SuperPartition([[3,1],[2,2,1]]).bosonic_degree()}
\]
\[ 9 \]

\[
\text{sage: SuperPartition([[2,1,0],[3,3]]).bosonic_degree()}
\]
\[ 9 \]

bosonic_length()
Return the length of the partition of the symmetric part.

OUTPUT:
  • an integer

EXAMPLES:
check()
Check that self is a valid super partition.

EXAMPLES:

```
sage: SP = SuperPartition([[1],[1]])
sage: SP.check()
```

conjugate()
Conjugate of a super partition.

The conjugate of a super partition is defined by conjugating the circled diagram.

OUTPUT:
- a SuperPartition

EXAMPLES:

```
sage: SuperPartition([[3, 1, 0], [4, 3, 2, 1]]).conjugate()
[6, 4, 1; 3]
sage: all(sp == sp.conjugate().conjugate() for sp in SuperPartitions(4))
True
sage: all(sp.conjugate() in SuperPartitions(3,2) for sp in SuperPartitions(3,2))
True
```

degree()
Return the bosonic degree of self.

The bosonic degree is the sum of the sizes of the antisymmetric and symmetric parts.

OUTPUT:
- an integer

EXAMPLES:

```
sage: SuperPartition([[3,1],[2,2,1]]).bosonic_degree()
9
sage: SuperPartition([[2,1,0],[3,3]]).bosonic_degree()
9
```

dominates(other)
Return True if and only if self dominates other.

If the symmetric and anti-symmetric parts of self and other are not the same size then the result is False.

EXAMPLES:

```
sage: LA = SuperPartition([[2,1],[2,1,1]])
sage: LA.dominates([[2,1],[3,1]])
False
sage: LA.dominates([[2,1],[1,1,1,1]])
```

(continues on next page)
True
sage: LA.dominates([[3],[2,1,1]])
False
sage: LA.dominates([[1],[1]*6])
False

fermionic_degree()

Return the fermionic degree of self.
The fermionic degree is the length of the antisymmetric part.

OUTPUT:

• an integer

EXAMPLES:

sage: SuperPartition([[3,1],[2,2,1]]).fermionic_degree()
2
sage: SuperPartition([[2,1,0],[3,3]]).fermionic_degree()
3

fermionic_sector()

Return the fermionic degree of self.
The fermionic degree is the length of the antisymmetric part.

OUTPUT:

• an integer

EXAMPLES:

sage: SuperPartition([[3,1],[2,2,1]]).fermionic_degree()
2
sage: SuperPartition([[2,1,0],[3,3]]).fermionic_degree()
3

static from_circled_diagram(shape, corners)

Construct a super partition from a circled diagram.

A circled diagram consists of a partition of the concatenation of the antisymmetric and symmetric parts
and a list of addable cells of the partition which indicate the location of the circled cells.

INPUT:

• shape – a partition or list of integers
• corners – a list of removable cells of shape

OUTPUT:

• a SuperPartition

EXAMPLES:

sage: SuperPartition.from_circled_diagram([3, 2, 2, 1, 1], [(0, 3), (3, 1)])
[3, 2, 2, 1, 1] 3
sage: SuperPartition.from_circled_diagram([3, 3, 2, 1], [(2, 2), (3, 1), (4, 1)])
[3, 3, 2, 1, 1] 3

(continues on next page)
\[0)\])
\[2, 1, 0; 3, 3\]
\sage: from_cd = SuperPartition.from_circled_diagram
\sage: all(sp == from_cd(sp.to_circled_diagram()) for sp in SuperPartitions(4))
\[True\]

**length()**

Return the length of self, which is the sum of the lengths of the antisymmetric and symmetric part.

**OUTPUT:**

• an integer

**EXAMPLES:**

\sage: SuperPartition([\[3,1\],\[2,2,1\]]).length()
\[5\]
\sage: SuperPartition([\[2,1,0\],\[3,3\]]).length()
\[5\]

**s_part()**

The symmetric part as a list of weakly decreasing integers.

**OUTPUT:**

• a list

**EXAMPLES:**

\sage: SuperPartition([\[3,1\],\[2,2,1\]]).symmetric_part()
\[2, 2, 1\]
\sage: SuperPartition([\[2,1,0\],\[3,3\]]).symmetric_part()
\[3, 3\]

**shape_circled_diagram()**

A concatenated partition with an extra cell for each antisymmetric part

**OUTPUT:**

• a partition

**EXAMPLES:**

\sage: SuperPartition([\[3,1\],\[2,2,1\]]).shape_circled_diagram()
\[4, 2, 2, 2, 1\]
\sage: SuperPartition([\[2,1,0\],\[3,3\]]).shape_circled_diagram()
\[3, 3, 3, 2, 1\]

**sign()**

Return the sign of a permutation of cycle type the symmetric part of self.

**OUTPUT:**

• either 1 or -1

**EXAMPLES:**
```python
sage: SuperPartition([[1,0],[3,1,1]]).sign()
-1
sage: SuperPartition([[1,0],[3,2,1]]).sign()
1
sage: sum(sp.sign()/sp.zee() for sp in SuperPartitions(6,0))
0
```

### symmetric_part()

The symmetric part as a list of weakly decreasing integers.

**OUTPUT:**

- a list

**EXAMPLES:**

```python
sage: SuperPartition([[3,1],[2,2,1]]).symmetric_part()
[2, 2, 1]
sage: SuperPartition([[2,1,0],[3,3]]).symmetric_part()
[3, 3]
```

### to_circled_diagram()

The shape of the circled diagram and a list of addable cells

A circled diagram consists of a partition for the outer shape and a list of removable cells of the partition indicating the location of the circled cells

**OUTPUT:**

- a list consisting of a partition and a list of pairs of integers

**EXAMPLES:**

```python
sage: SuperPartition([[3,1],[2,2,1]]).to_circled_diagram()
[[3, 2, 2, 1, 1], [(0, 3), (3, 1)]]
sage: SuperPartition([[2,1,0],[3,3]]).to_circled_diagram()
[[3, 3, 2, 1], [(2, 2), (3, 1), (4, 0)]]
sage: from_cd = SuperPartition.from_circled_diagram
sage: all(sp == from_cd(*sp.to_circled_diagram()) for sp in SuperPartitions(4))
True
```

### to_composition()

Concatenate the antisymmetric and symmetric parts to a composition.

**OUTPUT:**

- a (possibly weak) composition

**EXAMPLES:**

```python
sage: SuperPartition([[3,1],[2,2,1]]).to_composition()
[3, 1, 2, 2, 1]
sage: SuperPartition([[2,1,0],[3,3]]).to_composition()
[2, 1, 0, 3, 3]
sage: SuperPartition([[2,1,0],[3,3]]).to_composition().parent()
Compositions of non-negative integers
```
to_list()  
The list of two lists with the antisymmetric and symmetric parts.  

EXAMPLES:

```
sage: SuperPartition([[1],[1]]).to_list()  
[[1], [1]]
sage: SuperPartition([], [1]).to_list()  
[[], [1]]
```

to_partition()  
Concatenate and sort the antisymmetric and symmetric parts to a partition.  

OUTPUT:  
• a partition  

EXAMPLES:

```
sage: SuperPartition([[3,1],[2,2,1]]).to_partition()  
[3, 2, 2, 1, 1]
sage: SuperPartition([[2,1,0],[3,3]]).to_partition()  
[3, 3, 2, 1]
sage: SuperPartition([[2,1,0],[3,3]]).to_partition().parent()  
Partitions
```

zee()  
Return the centralizer size of a permutation of cycle type symmetric part of self.  

OUTPUT:  
• a positive integer  

EXAMPLES:

```
sage: SuperPartition([[1,0],[3,1,1]]).zee()  
6
sage: SuperPartition([[1],[2,2,1]]).zee()  
8
sage: sum(1/sp.zee() for sp in SuperPartitions(6,0))  
1
```

class sage.combinat.superpartition.SuperPartitions(is_infinite=False)  

Bases: UniqueRepresentation, Parent  

Super partitions.  

A super partition of size $n$ and fermionic sector $m$ is a pair consisting of a strict partition of some integer $r$ of length $m$ (that may end in a 0) and an integer partition of $n - r$.  

INPUT:  
• $n$ – an integer (optional: default None)  
• $m$ – if $n$ is specified, an integer (optional: default None)  

Super partitions are the indexing set for symmetric functions in super space.  

EXAMPLES:
sage: SuperPartitions()
Super Partitions
sage: SuperPartitions(2)
Super Partitions of 2
sage: SuperPartitions(2).cardinality()
8
sage: SuperPartitions(4,2)
Super Partitions of 4 and of fermionic sector 2
sage: [[2,0],[1,1]] in SuperPartitions(4,2)
True
sage: [[1,0],[1,1]] in SuperPartitions(4,2)
False
sage: [[1,0],[2,1]] in SuperPartitions(4)
True
sage: [[1,0],[2,2,1]] in SuperPartitions(4)
False
sage: [[1,0],[2,1]] in SuperPartitions()
True
sage: [[1,1],[2,1]] in SuperPartitions()
False

Element
alias of SuperPartition

options = Current options for SuperPartition - display: default

class sage.combinat.superpartition.SuperPartitions_all
    Bases: SuperPartitions
    Initialize self.

class sage.combinat.superpartition.SuperPartitions_n(n)
    Bases: SuperPartitions
    Initialize self.

class sage.combinat.superpartition.SuperPartitions_n_m(n,m)
    Bases: SuperPartitions
    Initialize self.

5.1.340 Symmetric Group Algebra

sage.combinat.symmetric_group_algebra.HeckeAlgebraSymmetricGroupT(R, n, q=Nothing)
Return the Hecke algebra of the symmetric group S_n on the T-basis with quantum parameter q over the ring R.
If R is a commutative ring and q is an invertible element of R, and if n is a nonnegative integer, then the Hecke algebra of the symmetric group S_n over R with quantum parameter q is defined as the algebra generated by the generators T_1, T_2, ..., T_{n-1} with relations

T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}

for all i < n - 1 (“braid relations”),

T_i T_j = T_j T_i

for all i, j such that |i - j| > 1.
for all $i$ and $j$ such that $|i - j| > 1$ (“locality relations”), and

$$T_i^2 = q + (q - 1)T_i$$

for all $i$ (the “quadratic relations”, also known in the form $(T_i + 1)(T_i - q) = 0$). (This is only one of several existing definitions in literature, not all of which are fully equivalent. We are following the conventions of [Go1993].) For any permutation $w \in S_n$, we can define an element $T_w$ of this Hecke algebra by setting $T_w = T_{i_1}T_{i_2} \cdots T_{i_k}$, where $w = s_{i_1}s_{i_2} \cdots s_{i_k}$ is a reduced word for $w$ (with $s_i$ meaning the transposition $(i, i+1)$, and the product of permutations being evaluated by first applying $s_{i_k}$, then $s_{i_{k-1}}$, etc.). This element is independent of the choice of the reduced decomposition, and can be computed in Sage by calling $H[w]$ where $H$ is the Hecke algebra and $w$ is the permutation.

The Hecke algebra of the symmetric group $S_n$ with quantum parameter $q$ over $R$ can be seen as a deformation of the group algebra $RS_n$; indeed, it becomes $RS_n$ when $q = 1$.

**Warning:** The multiplication on the Hecke algebra of the symmetric group does *not* follow the global option `mult` of the `Permutations` class (see `options()`). It is always as defined above. It does not match the default option (`mult=l2r`) of the symmetric group algebra!

**EXAMPLES:**

```python
sage: HeckeAlgebraSymmetricGroupT(QQ, 3)
Hecke algebra of the symmetric group of order 3 on the T basis over Univariate Polynomial Ring in q over Rational Field
sage: HeckeAlgebraSymmetricGroupT(QQ, 3, 2)
Hecke algebra of the symmetric group of order 3 with q=2 on the T basis over Rational Field
```

The multiplication on the Hecke algebra follows a different convention than the one on the symmetric group algebra does by default:

```python
sage: H3 = HeckeAlgebraSymmetricGroupT(QQ, 3)
sage: H3([1,3,2]) * H3([2,1,3])
T[3, 1, 2]
sage: S3 = SymmetricGroupAlgebra(QQ, 3)
sage: S3([1,3,2]) * S3([2,1,3])
[2, 3, 1]
sage: TestSuite(H3).run()
```

**class** `sage.combinat.symmetric_group_algebra.HeckeAlgebraSymmetricGroup_generic(<R,n>, q=None)`

**Bases:** `CombinatorialFreeModule`

**one_basis()**

Return the identity permutation.

**EXAMPLES:**

```python
sage: HeckeAlgebraSymmetricGroupT(QQ, 3).one()  # indirect doctest
T[1, 2, 3]
```
Return the variable or parameter $q$.

**EXAMPLES:**

```python
sage: HeckeAlgebraSymmetricGroupT(QQ, 3).q()
q
sage: HeckeAlgebraSymmetricGroupT(QQ, 3, 2).q()
2
```

class sage.combinat.symmetric_group_algebra.HeckeAlgebraSymmetricGroup_t(R, n, q=None)

Bases: HeckeAlgebraSymmetricGroup_generic

**algebra_generators()**

Return the generators of the algebra.

**EXAMPLES:**

```python
sage: HeckeAlgebraSymmetricGroupT(QQ,3).algebra_generators()
[T[2, 1, 3], T[1, 3, 2]]
```

**jucys_murphy($k$)**

Return the Jucys-Murphy element $J_k$ of the Hecke algebra.

These Jucys-Murphy elements are defined by

$$J_k = (T_{k-1}T_{k-2} \cdots T_1)(T_1T_2 \cdots T_{k-1}).$$

More explicitly,

$$J_k = q^{k-1} + \sum_{l=1}^{k-1} (q^l - q^{l-1})T_{(l,k)}.$$ 

For generic $q$, the $J_k$ generate a maximal commutative sub-algebra of the Hecke algebra.

**Warning:** The specialization $q = 1$ does not map these elements $J_k$ to the Young-Jucys-Murphy elements of the group algebra $R S_n$. (Instead, it maps the “reduced” Jucys-Murphy elements $(J_k - q^{k-1})/(q - 1)$ to the Young-Jucys-Murphy elements of $R S_n$.)

**EXAMPLES:**

```python
sage: H3 = HeckeAlgebraSymmetricGroupT(QQ,3)
sage: j2 = H3.jucys_murphy(2); j2
q*T[1, 2, 3] + (q-1)*T[2, 1, 3]
sage: j3 = H3.jucys_murphy(3); j3
q^2*T[1, 2, 3] + (q^2-q)*T[1, 3, 2] + (q-1)*T[3, 2, 1]
sage: j2*j3 == j3*j2
True
sage: j0 = H3.jucys_murphy(0); j0 == H3.one()
True
```

```
Traceback (most recent call last):
...
ValueError: k (= 0) must be between 1 and n (= 3)
```
product_on_basis($perm1, perm2$)

EXAMPLES:

```
sage: H3 = HeckeAlgebraSymmetricGroupT(QQ, 3, 1)
sage: a = H3([2,1,3])+2*H3([1,2,3])-H3([3,2,1])
sage: a^2  #indirect doctest
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = QS3([2,1,3])+2*QS3([1,2,3])-QS3([3,2,1])
sage: a^2
6*[1, 2, 3] + 4*[2, 1, 3] - [2, 3, 1] - [3, 1, 2] - 4*[3, 2, 1]
```
sage.combinat.symmetric_group_algebra.SymmetricGroupAlgebra($R$, $W$, category=None)

Return the symmetric group algebra of order $W$ over the ring $R$.

INPUT:

- $W$ – a symmetric group; alternatively an integer $n$ can be provided, as shorthand for Permutations($n$).
- $R$ – a base ring
- $category$ – a category (default: the category of $W$)

This supports several implementations of the symmetric group. At this point this has been tested with $W=$Permutations($n$) and $W=$SymmetricGroup($n$).

**Warning:** Some features are failing in the latter case, in particular if the domain of the symmetric group is not $1, \ldots, n$.

**Note:** The brave can also try setting $W=$WeylGroup(['A',n-1]), but little support for this currently exists.

**EXAMPLES:**

```
sage: QS3 = SymmetricGroupAlgebra(QQ, 3); QS3
Symmetric group algebra of order 3 over Rational Field
sage: QS3(1)
[1, 2, 3]
sage: QS3(2)
2*[1, 2, 3]
sage: basis = [QS3(p) for p in Permutations(3)]
sage: a = sum(basis); a
[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1]
sage: a^2
6*[1, 2, 3] + 6*[1, 3, 2] + 6*[2, 1, 3] + 6*[2, 3, 1] + 6*[3, 1, 2] + 6*[3, 2, 1]
sage: a^2 == 6*a
True
sage: b = QS3([3, 1, 2])
sage: b
[3, 1, 2]
sage: b*a
[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1]
sage: b*a == a
True
```

We now construct the symmetric group algebra by providing explicitly the underlying group:

```
sage: SGA = SymmetricGroupAlgebra(QQ, Permutations(4)); SGA
Symmetric group algebra of order 4 over Rational Field
sage: SGA.group()
Standard permutations of 4
sage: SGA.an_element()
[1, 2, 3, 4] + 2*[1, 2, 4, 3] + 3*[1, 3, 2, 4] + [4, 1, 2, 3]
sage: SGA = SymmetricGroupAlgebra(QQ, SymmetricGroup(4)); SGA
Symmetric group algebra of order 4 over Rational Field
```

(continues on next page)
sage: SGA.group()
Symmetric group of order 4! as a permutation group
sage: SGA.an_element()
() + (2,3,4) + 2*(1,3)(2,4) + 3*(1,4)(2,3)

sage: SGA = SymmetricGroupAlgebra(QQ, WeylGroup(['A', 3], prefix='s')); SGA
Symmetric group algebra of order 4 over Rational Field
sage: SGA.group()
Weyl Group of type ['A', 3] (as a matrix group acting on the ambient space)
sage: SGA.an_element()
s1*s2*s3 + 3*s2*s3*s1*s2 + 2*s3*s1 + 1

The preferred way to construct the symmetric group algebra is to go through the usual algebra method:

sage: SGA = Permutations(3).algebra(QQ); SGA
Symmetric group algebra of order 3 over Rational Field
sage: SGA.group()
Standard permutations of 3
sage: SGA = SymmetricGroup(3).algebra(QQ); SGA
Symmetric group algebra of order 3 over Rational Field
sage: SGA.group()
Symmetric group of order 3! as a permutation group

The canonical embedding from the symmetric group algebra of order \(n\) to the symmetric group algebra of order \(p > n\) is available as a coercion:

sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: QS4 = SymmetricGroupAlgebra(QQ, 4)
sage: QS4.coerce_map_from(QS3)
Generic morphism:
  From: Symmetric group algebra of order 3 over Rational Field
  To:   Symmetric group algebra of order 4 over Rational Field
sage: x3 = QS3([3,1,2]) + 2 * QS3([2,3,1]); x3
2*[2, 3, 1] + [3, 1, 2]
sage: QS4(x3)
2*[2, 3, 1, 4] + [3, 1, 2, 4]

This allows for mixed expressions:

sage: x4 = 3*QS4([3, 1, 4, 2])
sage: x3 + x4
2*[2, 3, 1, 4] + [3, 1, 2, 4] + 3*[3, 1, 4, 2]
sage: QS0 = SymmetricGroupAlgebra(QQ, 0)
sage: QS1 = SymmetricGroupAlgebra(QQ, 1)
sage: x0 = QS0([])
sage: x1 = QS1([1])
sage: x0 * x1
[1]
sage: x3 - (2*x0 + x1) - x4
-3*[1, 2, 3, 4] + 2*[2, 3, 1, 4] + [3, 1, 2, 4] - 3*[3, 1, 4, 2]
Caveat: to achieve this, constructing `SymmetricGroupAlgebra(QQ, 10)` currently triggers the construction of all symmetric group algebras of smaller order. Is this a feature we really want to have?

**Warning:** The semantics of multiplication in symmetric group algebras with index set `Permutations(n)` is determined by the order in which permutations are multiplied, which currently defaults to “in such a way that multiplication is associative with permutations acting on integers from the right”, but can be changed to the opposite order at runtime by setting the global variable `Permutations.options['mult']` (see `sage.combinat.permutation.Permutations.options()`). On the other hand, the semantics of multiplication in symmetric group algebras with index set `SymmetricGroup(n)` does not depend on this global variable. (This has the awkward consequence that the coercions between these two sorts of symmetric group algebras do not respect multiplication when this global variable is set to ‘r2l’.) In view of this, it is recommended that code not rely on the usual multiplication function, but rather use the methods `left_action_product()` and `right_action_product()` for multiplying permutations (these methods don’t depend on the setting). See github issue #14885 for more information.

We conclude by constructing the algebra of the symmetric group as a monoid algebra:

```python
sage: QS3 = SymmetricGroupAlgebra(QQ, 3, category=Monoids())
sage: QS3.category()
Category of finite dimensional cellular monoid algebras over Rational Field
sage: TestSuite(QS3).run(skip=['_test_construction'])
```

```python
class sage.combinat.symmetric_group_algebra.SymmetricGroupAlgebra_n(R, W, category)
    Bases: GroupAlgebra_class

    algebra_generators()
    Return generators of this group algebra (as algebra) as a list of permutations.
    The generators used for the group algebra of $S_n$ are the transposition $(2,1)$ and the $n$-cycle $(1,2,\ldots,n)$, unless $n \leq 1$ (in which case no generators are needed).

    EXAMPLES:
    ```
sage: SymmetricGroupAlgebra(ZZ,5).algebra_generators()
Family ([(2, 1, 3, 4, 5), (2, 3, 4, 5, 1)])
    sage: SymmetricGroupAlgebra(QQ,0).algebra_generators()
Family ()
    sage: SymmetricGroupAlgebra(QQ,1).algebra_generators()
Family ()
```
```

```python
    antipode(x)
    Return the image of the element $x$ of self under the antipode of the Hopf algebra self (where the comultiplication is the usual one on a group algebra).
    Explicitly, this is obtained by replacing each permutation $\sigma$ by $\sigma^{-1}$ in $x$ while keeping all coefficients as they are.

    EXAMPLES:
    ```
sage: QS4 = SymmetricGroupAlgebra(QQ, 4)
sage: QS4.antipode(2 * QS4([1, 3, 4, 2]) - 1/2 * QS4([1, 4, 2, 3]))
-1/2*[1, 3, 4, 2] + 2*[1, 4, 2, 3]
```
```
```
sage: all( QS4.antipode(QS4(p)) == QS4(p.inverse()) for p in Permutations(4) )
True

sage: ZS3 = SymmetricGroupAlgebra(ZZ, 3)
sage: ZS3.antipode(ZS3.zero())
0
sage: ZS3.antipode(-ZS3(Permutation([2, 3, 1])))
-[3, 1, 2]

binary_unshuffle_sum(k)

Return the $k$-th binary unshuffle sum in the group algebra $\text{self}$.  

The $k$-th binary unshuffle sum in the symmetric group algebra $RS_n$ over a ring $R$ is defined as the sum of all permutations $\sigma \in S_n$ satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(n)$.

This element has the property that, if it is denoted by $t_k$, and if the $k$-th semi-RSW element (see semi_rsw_element()) is denoted by $s_k$, then $s_k S(t_k)$ and $t_k S(s_k)$ both equal the $k$-th Reiner-Saliola-Welker shuffling element of $RS_n$ (see rsw_shuffling_element()).

The $k$-th binary unshuffle sum is the image of the complete non-commutative symmetric function $S^{(k,n-k)}$ in the ring of non-commutative symmetric functions under the canonical projection on the symmetric group algebra (through the descent algebra).

EXAMPLES:

The binary unshuffle sums on $QS_3$:

sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: QS3.binary_unshuffle_sum(0)
[1, 2, 3]
sage: QS3.binary_unshuffle_sum(1)
[1, 2, 3] + [2, 1, 3] + [3, 1, 2]
sage: QS3.binary_unshuffle_sum(2)
[1, 2, 3] + [1, 3, 2] + [2, 3, 1]
sage: QS3.binary_unshuffle_sum(3)
[1, 2, 3]
sage: QS3.binary_unshuffle_sum(4)
0

Let us check the relation with the $k$-th Reiner-Saliola-Welker shuffling element stated in the docstring:

sage: def test_rsw(n):
.....:     ZSn = SymmetricGroupAlgebra(ZZ, n)
.....:     for k in range(1, n):
.....:         a = ZSn.semi_rsw_element(k)
.....:         b = ZSn.binary_unshuffle_sum(k)
.....:         c = ZSn.left_action_product(a, ZSn.antipode(b))
.....:         d = ZSn.left_action_product(b, ZSn.antipode(a))
.....:         e = ZSn.rsw_shuffling_element(k)
.....:         if c != e or d != e:
.....:             return False
.....:     return True
sage: test_rsw(3)
True
Let us also check the statement about the complete non-commutative symmetric function:

```
sage: def test_rsw_ncsf(n):
    ....:     ZSn = SymmetricGroupAlgebra(ZZ, n)
    ....:     NSym = NonCommutativeSymmetricFunctions(ZZ)
    ....:     S = NSym.S()
    ....:     for k in range(1, n):
    ....:         a = S(Composition([k, n-k])).to_symmetric_group_algebra()
    ....:         if a != ZSn.binary_unshuffle_sum(k):
    ....:             return False
    ....:     return True
sage: test_rsw_ncsf(3)
True
sage: test_rsw_ncsf(4)
True
sage: test_rsw_ncsf(5)  # long time
True
```

**canonical_embedding**(other)

Return the canonical coercion of self into a symmetric group algebra other.

**INPUT:**

- other – a symmetric group algebra with order \( p \) satisfying \( p \geq n \), where \( n \) is the order of self, over a ground ring into which the ground ring of self coerces.

**EXAMPLES:**

```
sage: QS2 = SymmetricGroupAlgebra(QQ, 2)
sage: QS4 = SymmetricGroupAlgebra(QQ, 4)
sage: phi = QS2.canonical_embedding(QS4); phi
Generic morphism:
    From: Symmetric group algebra of order 2 over Rational Field
    To:   Symmetric group algebra of order 4 over Rational Field
sage: x = QS2([2,1]) + 2 * QS2([1,2])
sage: phi(x)
2*[1, 2, 3, 4] + [2, 1, 3, 4]
sage: loads(dumps(phi))
Generic morphism:
    From: Symmetric group algebra of order 2 over Rational Field
    To:   Symmetric group algebra of order 4 over Rational Field
```

(continues on next page)
sage: phi = ZS2.canonical_embedding(QS2); phi
Generic morphism:
  From: Symmetric group algebra of order 2 over Integer Ring
  To: Symmetric group algebra of order 2 over Rational Field

sage: QS4.canonical_embedding(QS2)
Traceback (most recent call last):
...
ValueError: There is no canonical embedding from Symmetric group
  algebra of order 2 over Rational Field to Symmetric group
  algebra of order 4 over Rational Field

sage: QS4g = SymmetricGroup(4).algebra(QQ)
sage: QS4.canonical_embedding(QS4g)(QS4([1,3,2,4]))
(2,3)
sage: QS4g.canonical_embedding(QS4)(QS4g((2,3)))
[1, 3, 2, 4]
sage: ZS2.canonical_embedding(QS4g)(ZS2([2,1]))
(1,2)
sage: ZS2g = SymmetricGroup(2).algebra(ZZ)
sage: ZS2g.canonical_embedding(QS4)(ZS2g((1,2)))
[2, 1, 3, 4]

cell_module(la, **kwds)
Return the cell module indexed by la.

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 3)
sage: M = S.cell_module(Partition([2,1])); M
Cell module indexed by [2, 1] of Cellular basis of
  Symmetric group algebra of order 3 over Rational Field

We check that the input la is standardized:

sage: N = S.cell_module([2,1])
sage: M is N
True

cell_module_indices(la)
Return the indices of the cell module of self indexed by la.

This is the finite set \( M(\lambda) \).

EXAMPLES:

sage: S = SymmetricGroupAlgebra(QQ, 4)
sage: S.cell_module_indices([3,1])
Standard tableaux of shape [3, 1]

cell_poset()
Return the cell poset of self.

EXAMPLES:
central_orthogonal_idempotent(la, block=True)

Return the central idempotent for the symmetric group of order \( n \) corresponding to the indecomposable block to which the partition \( la \) is associated.

If self.base_ring() contains \( \mathbb{Q} \), this corresponds to the classical central idempotent corresponding to the irreducible representation indexed by \( la \).

Alternatively, if self.base_ring() has characteristic \( p > 0 \), then Theorem 2.8 in [Mur1983] provides that \( la \) is associated to an idempotent \( f_\mu \), where \( \mu \) is the \( p \)-core of \( la \). This \( f_\mu \) is a sum of classical idempotents,

\[
\sum_{c(\lambda) = \mu} e_\lambda,
\]

where the sum ranges over the partitions \( \lambda \) of \( n \) with \( p \)-core equal to \( \mu \).

INPUT:

- \( la \) – a partition of self.n or a self.base_ring().characteristic()-core of such a partition
- block – boolean (default: True); when False, this returns the classical idempotent associated to \( la \) (defined over \( \mathbb{Q} \))

OUTPUT:

If \( \text{block=False} \) and the corresponding coefficients are not defined over self.base_ring(), then return None. Otherwise return an element of self.

EXAMPLES:

Asking for block idempotents in any characteristic, by passing a partition of self.n:

```
sage: S0 = SymmetricGroup(4).algebra(QQ)
sage: S2 = SymmetricGroup(4).algebra(GF(2))
sage: S3 = SymmetricGroup(4).algebra(GF(3))
sage: S0.central_orthogonal_idempotent([2,1,1])
3/8*() - 1/8*(3,4) - 1/8*(2,3) - 1/8*(2,4) - 1/8*(1,2)
- 1/8*(1,2)(3,4) + 1/8*(1,2,3,4) + 1/8*(1,2,4,3)
+ 1/8*(1,3,4,2) - 1/8*(1,3) - 1/8*(1,3)(2,4)
+ 1/8*(1,3,2,4) + 1/8*(1,4,3,2) - 1/8*(1,4)
+ 1/8*(1,4,2,3) - 1/8*(1,4)(2,3)
sage: S2.central_orthogonal_idempotent([2,1,1])
() + (1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3)
sage: idem = S3.central_orthogonal_idempotent([4]); idem
() + (1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3)
sage: idem == S3.central_orthogonal_idempotent([1,1,1,1])
True
sage: S3.central_orthogonal_idempotent([2,2])
() + (1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3)
```

Asking for block idempotents in any characteristic, by passing \( p \)-cores:
```python
sage: S0.central_orthogonal_idempotent([1,1])
Traceback (most recent call last):
  ...  
ValueError: [1, 1] is not a partition of integer 4
sage: S2.central_orthogonal_idempotent([1])
() 
```

```python
sage: S2.central_orthogonal_idempotent([1])
Traceback (most recent call last):
  ...  
ValueError: the 2-core of [1] is not a 2-core of a partition of 4
sage: S3.central_orthogonal_idempotent([1])
() + (1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3)
sage: S3.central_orthogonal_idempotent([7])
() + (1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3)
```

Asking for classical idempotents:

```python
sage: S3.central_orthogonal_idempotent([2,2], block=False) is None
True
sage: S3.central_orthogonal_idempotent([2,1,1], block=False)
(3,4) + (2,3) + (2,4) + (1,2) + (1,2)(3,4) + 2*(1,2,3,4) + 2*(1,2,4,3) + 2*(1,3,4,2) + (1,3)(2,4) + 2*(1,3,2,4) + 2*(1,4,3,2) + (1,4) + 2*(1,4,2,3) + (1,4)(2,3)
```

See also:

- `sage.combinat.partition.Partition.core()`

### central_orthogonal_idempotents()

Return a maximal list of central orthogonal idempotents for `self`.

This method does not require that `self` be semisimple, relying on Nakayama’s Conjecture whenever `self`. `base_ring()` has positive characteristic.

**EXAMPLES:**

```python
sage: QS3 = SymmetricGroupAlgebra(QQ,3)
sage: a = QS3.central_orthogonal_idempotents()
sage: a[0]  # [3]
1/6*[1, 2, 3] + 1/6*[1, 3, 2] + 1/6*[2, 1, 3] + 1/6*[2, 3, 1] + 1/6*[3, 1, 2] + 1/6*[3, 2, 1]
sage: a[1]  # [2, 1]
2/3*[1, 2, 3] - 1/3*[2, 3, 1] - 1/3*[3, 1, 2]
```

See also:

- `central_orthogonal_idempotent()`

### dft(form='seminormal', mult='l2r')

Return the discrete Fourier transform for `self`.

**INPUT:**
**Combinatorics, Release 10.0**

- **mult** – string (default: '\(l2r\)). If set to \(r2l\), this causes the method to use the antipodes \((antipode())\) of the seminormal basis instead of the seminormal basis.

**EXAMPLES:**

```python
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: QS3.dft()
[ 1 1 1 1 1 1]
[ 1 1/2 -1 -1/2 -1/2 1/2]
[ 0 3/4 0 3/4 -3/4 -3/4]
[ 0 1 0 -1 1 -1]
[ 1 -1/2 1 -1/2 -1/2 -1/2]
[ 1 -1 -1 1 1 -1]
```

**epsilon_ik**(itab, ktab, star=0, mult='l2r')

Return the seminormal basis element of **self** corresponding to the pair of tableaux itab and ktab (or restrictions of these tableaux, if the optional variable star is set).

**INPUT:**

- **itab, ktab** – two standard tableaux of size \(n\).
- **star** – integer (default: 0).
- **mult** – string (default: 'l2r'). If set to \(r2l\), this causes the method to return the antipode \((antipode())\) of \(\epsilon(I, K)\) instead of \(\epsilon(I, K)\) itself.

**OUTPUT:**

The element \(\epsilon(I, K)\), where \(I\) and \(K\) are the tableaux obtained by removing all entries higher than \(n - \text{star}\) from itab and ktab, respectively. Here, we are using the notations from **seminormal_basis**().

**EXAMPLES:**

```python
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = QS3.epsilon_ik([[1,2,3]], [[1,2,3]]); a
1/6*[1, 2, 3] + 1/6*[1, 3, 2] + 1/6*[2, 1, 3] + 1/6*[2, 3, 1] + 1/6*[3, 1, 2] + 1/6*[3, 2, 1]
sage: QS3.dft()*vector(a)
(1, 0, 0, 0, 0, 0)
sage: a = QS3.epsilon_ik([[1,2],[3]], [[1,2],[3]]); a
1/3*[1, 2, 3] - 1/6*[1, 3, 2] + 1/3*[2, 1, 3] - 1/6*[2, 3, 1] - 1/6*[3, 1, 2] - 1/6*[3, 2, 1]
sage: QS3.dft()*vector(a)
(0, 0, 0, 1, 0, 0)
```

Let us take some properties of the seminormal basis listed in the docstring of **seminormal_basis**(), and verify them on the situation of \(S_3\).

First, check the formula

\[
\epsilon(T) = \frac{1}{\kappa_{sh(T)}} \epsilon(T) e(T) \epsilon(T).
\]

In fact:

```python
sage: from sage.combinat.symmetric_group_algebra import e
dsage: def test_sn1(n):
....:     QSn = SymmetricGroupAlgebra(QQ, n)
(continues on next page)```
Next, we check the identity
\[ \epsilon(T, S) = \frac{1}{\kappa_{\text{sh}(T)}} \epsilon(S) \pi_{T, S} \epsilon(T) \epsilon(T) \]
which we used to define \( \epsilon(T, S) \). In fact:

```
sage: from sage.combinat.symmetric_group_algebra import e
sage: def test_sn2(n):
    ....:     QSn = SymmetricGroupAlgebra(QQ, n)
    ....:     mul = QSn.left_action_product
    ....:     QSn1 = SymmetricGroupAlgebra(QQ, n - 1)
    ....:     for lam in Partitions(n):
    ....:         k = prod(lam.hooks())
    ....:         for T in StandardTableaux(lam):
    ....:             TT = T.restrict(n-1)
    ....:             SS = S.restrict(n-1)
    ....:             eTT = QSn1.epsilon_ik(TT, TT)
    ....:             eSS = QSn1.epsilon_ik(SS, SS)
    ....:             eTS = QSn.epsilon_ik(T, S)
    ....:             piTS = [0] * n
    ....:             for (i, j) in T.cells():
    ....:                 piTS[T[i][j] - 1] = S[i][j]
    ....:             piTS = QSn(Permutation(piTS))
    ....:             if k * eTS != mul(mul(eSS, piTS), mul(e(T), eTT)):
    ....:                 return False
    ....: return True
sage: test_sn2(3)
True
sage: test_sn2(4)  # long time
True
```

Let us finally check the identity
\[ \epsilon(T, S)\epsilon(U, V) = \delta_{T, V} \epsilon(U, S) \]
In fact:
```python
def test_sn3(lam):
    n = lam.size()
    QSn = SymmetricGroupAlgebra(QQ, n)
    mul = QSn.left_action_product
    for T in StandardTableaux(lam):
        for S in StandardTableaux(lam):
            for U in StandardTableaux(lam):
                for V in StandardTableaux(lam):
                    lhs = mul(QSn.epsilon_ik(T, S), QSn.epsilon_ik(U, V))
                    if T == V:
                        rhs = QSn.epsilon_ik(U, S)
                    else:
                        rhs = QSn.zero()
                    if rhs != lhs:
                        return False
    return True
```

```
sage: all( test_sn3(lam) for lam in Partitions(3) )
True
sage: all( test_sn3(lam) for lam in Partitions(4) )  # long time
True
```

**jucys_murphy**

Return the Jucys-Murphy element $J_k$ (also known as a Young-Jucys-Murphy element) for the symmetric group algebra `self`.

The Jucys-Murphy element $J_k$ in the symmetric group algebra $RS_n$ is defined for every $k \in \{1, 2, \ldots, n\}$ by

$$J_k = (1, k) + (2, k) + \cdots + (k - 1, k) \in RS_n,$$

where the addends are transpositions in $S_n$ (regarded as elements of $RS_n$). We note that there is not a dependence on $n$, so it is often suppressed in the notation.

**EXAMPLES:**

```
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: QS3.jucys_murphy(1)
(1)
sage: QS3.jucys_murphy(2)
(2, 1, 3)
sage: QS3.jucys_murphy(3)
(1, 3, 2) + (3, 2, 1)
sage: QS4 = SymmetricGroupAlgebra(QQ, 4)
sage: j3 = QS4.jucys_murphy(3); j3
(1, 3, 2) + (3, 2, 1)
sage: j4 = QS4.jucys_murphy(4); j4
(1, 2, 4, 3) + (1, 4, 3, 2) + (4, 2, 3, 1)
sage: j3*j4 == j4*j3
True
sage: QS5 = SymmetricGroupAlgebra(QQ, 5)
sage: QS5.jucys_murphy(4)
(1, 2, 4, 3, 5) + (1, 4, 3, 2, 5) + (4, 2, 3, 1, 5)
```
left_action_product(left, right)

Return the product of two elements left and right of self, where multiplication is defined in such a way that for two permutations p and q, the product pq is the permutation obtained by first applying q and then applying p. This definition of multiplication is tailored to make multiplication of permutations associative with their action on numbers if permutations are to act on numbers from the left.

**EXAMPLES:**

```python
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: p1 = Permutation([2, 1, 3])
sage: p2 = Permutation([3, 1, 2])
sage: QS3.left_action_product(QS3(p1), QS3(p2))
[3, 2, 1]
sage: x = QS3([1, 2, 3]) - 2*QS3([1, 3, 2])
sage: y = 1/2 * QS3([3, 1, 2]) + 3*QS3([1, 2, 3])
sage: QS3.left_action_product(x, y)
3*[1, 2, 3] - 6*[1, 3, 2] - [2, 1, 3] + 1/2*[3, 1, 2]
sage: QS3.left_action_product(0, x)
0
```

The method coerces its input into the algebra self:

```python
sage: QS4 = SymmetricGroupAlgebra(QQ, 4)
sage: QS4.left_action_product(QS3([1, 2, 3]), QS3([2, 1, 3]))
[2, 1, 3, 4]
sage: QS4.left_action_product(1, Permutation([4, 1, 2, 3]))
[4, 1, 2, 3]
```

**Warning:** Note that coercion presently works from permutations of n into the n-th symmetric group algebra, and also from all smaller symmetric group algebras into the n-th symmetric group algebra, but not from permutations of integers smaller than n into the n-th symmetric group algebra.

monomial_from_smaller_permutation(permutation)

Convert permutation into a permutation, possibly extending it to the appropriate size, and return the corresponding basis element of self.

**EXAMPLES:**

```python
sage: QS5 = SymmetricGroupAlgebra(QQ, 5)
sage: QS5.monomial_from_smaller_permutation([],)
[1, 2, 3, 4, 5]
sage: QS5.monomial_from_smaller_permutation(Permutation([3,1,2]))
[3, 1, 2, 4, 5]
sage: QS5.monomial_from_smaller_permutation([5,3,4,1,2])
[5, 3, 4, 1, 2]
sage: QS5.monomial_from_smaller_permutation(SymmetricGroup(2)([1,2]))
[2, 1, 3, 4, 5]
sage: QS5g = SymmetricGroup(5).algebra(QQ)
sage: QS5g.monomial_from_smaller_permutation([2,1])
(1,2)
```

retract_direct_product(f, m)
Return the direct-product retract of the element $f \in RS_n$ to $RS_m$, where $m \leq n$ (and where $RS_n$ is self).

If $m$ is a nonnegative integer less or equal to $n$, then the direct-product retract from $S_n$ to $S_m$ is defined as an $R$-linear map $S_n \to S_m$ which sends every permutation $p \in S_n$ to

$$dret(p) \quad \text{if } dret(p) \text{ is defined;}$$

$$0 \quad \text{otherwise}.$$ 

Here $dret(p)$ denotes the direct-product retract of the permutation $p$ to $S_m$, which is defined in `retract_direct_product()`.

**EXAMPLES:**

```sage
sage: SGA3 = SymmetricGroupAlgebra(QQ, 3)
sage: SGA3.retract_direct_product(2*SGA3([1,2,3]) - 4*SGA3([2,1,3]) + 7*SGA3([1, ˓→3,2]), 2)
2*[1, 2] - 4*[2, 1]
sage: SGA3.retract_direct_product(2*SGA3([1,3,2]) - 5*SGA3([2,3,1]), 2)
0

sage: SGA5 = SymmetricGroupAlgebra(QQ, 5)
sage: SGA5.retract_direct_product(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 4)
11*[3, 2, 1, 4, 5]

sage: SGA5.retract_direct_product(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 3)
-6*[1, 3, 2] + 11*[3, 2, 1, 4, 5]

sage: SGA5.retract_direct_product(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 2)
0

sage: SGA5.retract_direct_product(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 1)
2*[1]

sage: SGA5.retract_direct_product(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 3)
8*[1, 2, 3] - 6*[1, 3, 2]

sage: SGA5.retract_direct_product(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 1)
2*[1]

sage: SGA5.retract_direct_product(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 0)
2*[]
```

See also:

`retract_plain()`, `retract_okounkov_vershik()`

**retract_okounkov_vershik($f$, $m$)**

Return the Okounkov-Vershik retract of the element $f \in RS_n$ to $RS_m$, where $m \leq n$ (and where $RS_n$ is self).

If $m$ is a nonnegative integer less or equal to $n$, then the Okounkov-Vershik retract from $S_n$ to $S_m$ is defined as an $R$-linear map $S_n \to S_m$ which sends every permutation $p \in S_n$ to the Okounkov-Vershik retract of the permutation $p$ to $S_m$, which is defined in `retract_okounkov_vershik()`.

**EXAMPLES:**
sage: SGA3 = SymmetricGroupAlgebra(QQ, 3)
sage: SGA3.retract_okounkov_vershik(2*SGA3([1,2,3]) - 4*SGA3([2,1,3]) + 7*SGA3([1,3,2]), 2)
9*[1, 2] - 4*[2, 1]
sage: SGA3.retract_okounkov_vershik(2*SGA3([1,3,2]) - 5*SGA3([2,3,1]), 2)
2*[1, 2] - 5*[2, 1]

sage: SGA5 = SymmetricGroupAlgebra(QQ, 5)
sage: SGA5.retract_okounkov_vershik(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 4)
-6*[1, 3, 2, 4] + 8*[1, 4, 2, 3] + 11*[3, 2, 1, 4]
sage: SGA5.retract_okounkov_vershik(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 3)
2*[1, 3, 2] + 11*[3, 2, 1]
sage: SGA5.retract_okounkov_vershik(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 2)
13*[1, 2]
sage: SGA5.retract_okounkov_vershik(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 1)
13*[1]

sage: SGA5.retract_okounkov_vershik(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 3)
8*[1, 2, 3] - 6*[1, 3, 2]
sage: SGA5.retract_okounkov_vershik(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 1)
2*[1]
sage: SGA5.retract_okounkov_vershik(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 0)
0

See also:

retract_plain(), retract_direct_product()

**retract_plain(f, m)**

Return the plain retract of the element \( f \in RS_n \) to \( RS_m \), where \( m \leq n \) (and where \( RS_n \) is self).

If \( m \) is a nonnegative integer less or equal to \( n \), then the plain retract from \( S_n \) to \( S_m \) is defined as an \( R \)-linear map \( S_n \to S_m \) which sends every permutation \( p \in S_n \) to

\[
pret(p) \quad \text{if pret}(p) \text{ is defined;}
0 \quad \text{otherwise}
\]

Here \( \text{pret}(p) \) denotes the plain retract of the permutation \( p \) to \( S_m \), which is defined in **retract_plain()**.

**EXAMPLES:**

sage: SGA3 = SymmetricGroupAlgebra(QQ, 3)
sage: SGA3.retract_plain(2*SGA3([1,2,3]) - 4*SGA3([2,1,3]) + 7*SGA3([1,3,2]), 2)
2*[1, 2] - 4*[2, 1]
sage: SGA3.retract_plain(2*SGA3([1,3,2]) - 5*SGA3([2,3,1]), 2)
0

sage: SGA5 = SymmetricGroupAlgebra(QQ, 5)
Combinatorics, Release 10.0

sage: SGA5.retract_plain(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 4)
11*[3, 2, 1, 4]
sage: SGA5.retract_plain(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 3)
11*[3, 2, 1]
sage: SGA5.retract_plain(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 2)
0
sage: SGA5.retract_plain(8*SGA5([1,4,2,5,3]) - 6*SGA5([1,3,2,5,4]) + 11*SGA5([3,2,1,4,5]), 1)
0
sage: SGA5.retract_plain(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 3)
8*[1, 2, 3] - 6*[1, 3, 2]
sage: SGA5.retract_plain(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 1)
8*[1]
sage: SGA5.retract_plain(8*SGA5([1,2,3,4,5]) - 6*SGA5([1,3,2,4,5]), 0)
8*[]

See also:

retract_direct_product(), retract_okounkov_vershik()

right_action_product(left, right)

Return the product of two elements left and right of self, where multiplication is defined in such a way that for two permutations p and q, the product pq is the permutation obtained by first applying p and then applying q. This definition of multiplication is tailored to make multiplication of permutations associative with their action on numbers if permutations are to act on numbers from the right.

EXAMPLES:

sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: p1 = Permutation([2, 1, 3])
sage: p2 = Permutation([3, 1, 2])
sage: QS3.right_action_product(QS3(p1), QS3(p2))
[1, 3, 2]
sage: x = QS3([1, 2, 3]) - 2*QS3([1, 3, 2])
sage: y = 1/2 * QS3([3, 1, 2]) + 3*QS3([1, 2, 3])
sage: QS3.right_action_product(x, y)
3*[1, 2, 3] - 6*[1, 3, 2] + 1/2*[3, 1, 2] - [3, 2, 1]
sage: QS3.right_action_product(0, x)
0

The method coerces its input into the algebra self:

sage: QS4 = SymmetricGroupAlgebra(QQ, 4)
sage: QS4.right_action_product(QS3([1, 2, 3]), QS3([2, 1, 3]))
[2, 1, 3, 4]
sage: QS4.right_action_product(1, Permutation([4, 1, 2, 3]))
[4, 1, 2, 3]
**Warning:** Note that coercion presently works from permutations of $n$ into the $n$-th symmetric group algebra, and also from all smaller symmetric group algebras into the $n$-th symmetric group algebra, but not from permutations of integers smaller than $n$ into the $n$-th symmetric group algebra.

**rsw_shuffling_element**(k)

Return the $k$-th Reiner-Saliola-Welker shuffling element in the group algebra `self`.

The $k$-th Reiner-Saliola-Welker shuffling element in the symmetric group algebra $RS_n$ over a ring $R$ is defined as the sum $\sum_{\sigma \in S_n} \text{noninv}_k(\sigma) \cdot \sigma$, where for every permutation $\sigma$, the number $\text{noninv}_k(\sigma)$ is the number of all $k$-noninversions of $\sigma$ (that is, the number of all $k$-element subsets of $\{1, 2, \ldots, n\}$ on which $\sigma$ restricts to a strictly increasing map). See `sage.combinat.permutation.number_of_noninversions()` for the `noninv` map.

This element is more or less the operator $\nu_{k,1^{n-k}}$ introduced in [RSW2011]; more precisely, $\nu_{k,1^{n-k}}$ is the left multiplication by this element.

It is a nontrivial theorem (Theorem 1.1 in [RSW2011]) that the operators $\nu_{k,1^{n-k}}$ (for fixed $n$ and varying $k$) pairwise commute. It is a conjecture (Conjecture 1.2 in [RSW2011]) that all their eigenvalues are integers (which, in light of their commutativity and easily established symmetry, yields that they can be simultaneously diagonalized over $\mathbb{Q}$ with only integer eigenvalues).

**EXAMPLES:**

The Reiner-Saliola-Welker shuffling elements on $\mathbb{Q}S_3$:

```
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: QS3.rsw_shuffling_element(0)
[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1]
sage: QS3.rsw_shuffling_element(1)
3*[1, 2, 3] + 3*[1, 3, 2] + 3*[2, 1, 3] + 3*[2, 3, 1] + 3*[3, 1, 2] + 3*[3, 2, 1]
sage: QS3.rsw_shuffling_element(2)
3*[1, 2, 3] + 2*[1, 3, 2] + 2*[2, 1, 3] + [2, 3, 1] + [3, 1, 2]
sage: QS3.rsw_shuffling_element(3)
[1, 2, 3]
sage: QS3.rsw_shuffling_element(4)
0
```

Checking the commutativity of Reiner-Saliola-Welker shuffling elements (we leave out the ones for which it is trivial):

```
sage: def test_rsw_comm(n):
...    QSn = SymmetricGroupAlgebra(QQ, n)
...    rsws = [QSn.rsw_shuffling_element(k) for k in range(2, n)]
...    return all( all( rsws[i] * rsws[j] == rsws[j] * rsws[i] for j in range(i) )
...                for i in range(len(rsws)) )
sage: test_rsw_comm(3)
True
sage: test_rsw_comm(4)
True
sage: test_rsw_comm(5)  # long time
True
```
**Note:** For large \( k \) (relative to \( n \)), it might be faster to call `QSn.left_action_product(QSn.semi_rsw_element(k), QSn.antipode(binary_unshuffle_sum(k)))` than `QSn.rsw_shuffling_element(n)`.

See also:

`semi_rsw_element()`, `binary_unshuffle_sum()`

**semi_rsw_element\( (k) \)**

Return the \( k \)-th semi-RSW element in the group algebra `self`.

The \( k \)-th semi-RSW element in the symmetric group algebra \( RS_n \) over a ring \( R \) is defined as the sum of all permutations \( \sigma \in S_n \) satisfying \( \sigma(1) < \sigma(2) < \cdots < \sigma(k) \).

This element has the property that, if it is denoted by \( s_k \), then \( s_k S(s_k) \) is \( (n-k)! \) times the \( k \)-th Reiner-Saliola-Welker shuffling element of \( RS_n \) (see `rsw_shuffling_element()`). Here, \( S \) denotes the antipode of the group algebra \( RS_n \).

The \( k \)-th semi-RSW element is the image of the complete non-commutative symmetric function \( S(k,1^{n-k}) \) in the ring of non-commutative symmetric functions under the canonical projection on the symmetric group algebra (through the descent algebra).

**EXAMPLES:**

The semi-RSW elements on \( QS_3 \):

```python
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: QS3.semi_rsw_element(0)
[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1]
sage: QS3.semi_rsw_element(1)
[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + [2, 3, 1] + [3, 1, 2] + [3, 2, 1]
sage: QS3.semi_rsw_element(2)
[1, 2, 3] + [1, 3, 2] + [2, 3, 1]
sage: QS3.semi_rsw_element(3)
[1, 2, 3]
sage: QS3.semi_rsw_element(4)
0
```

Let us check the relation with the \( k \)-th Reiner-Saliola-Welker shuffling element stated in the docstring:

```python
sage: def test_rsw(n):
    ...:     ZSn = SymmetricGroupAlgebra(ZZ, n)
    ...:     for k in range(1, n):
    ...:         a = ZSn.semi_rsw_element(k)
    ...:         b = ZSn.left_action_product(a, ZSn.antipode(a))
    ...:         if factorial(n-k) * ZSn.rsw_shuffling_element(k) != b:
    ...:             return False
    ...:     return True
sage: test_rsw(3)
True
sage: test_rsw(4)
True
sage: test_rsw(5)  # long time
True
```

Let us also check the statement about the complete non-commutative symmetric function:
sage: def test_rsw_ncsf(n):
....:     ZSn = SymmetricGroupAlgebra(ZZ, n)
....:     NSym = NonCommutativeSymmetricFunctions(ZZ)
....:     S = NSym.S()
....:     for k in range(1, n):
....:         a = S(Composition([k] + [1]*(n-k))).to_symmetric_group_algebra()
....:         if a != ZSn.semi_rsw_element(k):
....:             return False
....:     return True

sage: test_rsw_ncsf(3)
True
sage: test_rsw_ncsf(4)
True
sage: test_rsw_ncsf(5) # long time
True

seminormal_basis(mult='l2r')

Return a list of the seminormal basis elements of self.

The seminormal basis of a symmetric group algebra is defined as follows:

Let \( n \) be a nonnegative integer. Let \( R \) be a \( \mathbb{Q} \)-algebra. In the following, we will use the "left action" convention for multiplying permutations. This means that for all permutations \( p \) and \( q \) in \( S_\mathbf{n} \), the product \( pq \) is defined in such a way that \( (pq)(i) = p(q(i)) \) for each \( i \in \{1, 2, \ldots, n\} \) (this is the same convention as in \texttt{left_action_product()}, but not the default semantics of the \( \ast \) operator on permutations in Sage).

Thus, for instance, \( s_2s_1 \) is the permutation obtained by first transposing 1 with 2 and then transposing 2 with 3 (where \( s_i = (i, i+1) \)).

For every partition \( \lambda \) of \( n \), let

\[
\kappa_\lambda = \frac{n!}{f^\lambda}
\]

where \( f^\lambda \) is the number of standard Young tableaux of shape \( \lambda \). Note that \( \kappa_\lambda \) is an integer, namely the product of all hook lengths of \( \lambda \) (by the hook length formula). In Sage, this integer can be computed by using \texttt{sage.combinat.symmetric_group_algebra.kappa()}.

Let \( T \) be a standard tableau of size \( n \).

Let \( a(T) \) denote the formal sum (in \( RS_n \)) of all permutations in \( S_n \) which stabilize the rows of \( T \) (as sets), i. e., which map each entry \( i \) of \( T \) to an entry in the same row as \( i \). (See \texttt{sage.combinat.symmetric_group_algebra.a()} for an implementation of this.)

Let \( b(T) \) denote the signed formal sum (in \( RS_n \)) of all permutations in \( S_n \) which stabilize the columns of \( T \) (as sets). Here, “signed” means that each permutation is multiplied with its sign. (This is implemented in \texttt{sage.combinat.symmetric_group_algebra.b()}.)

Define an element \( e(T) \) of \( RS_n \) to be \( a(T)b(T) \). (This is implemented in \texttt{sage.combinat.symmetric_group_algebra.e()} for \( R = \mathbb{Q} \)).

Let \( \text{sh}(T) \) denote the shape of \( T \). (See \texttt{shape()}.)

Let \( \overline{T} \) denote the standard tableau of size \( n - 1 \) obtained by removing the letter \( n \) (along with its cell) from \( T \) (if \( n \geq 1 \)).

Now, we define an element \( \epsilon(T) \) of \( RS_n \). We define it by induction on the size \( n \) of \( T \), so we set \( \epsilon(\emptyset) = 1 \) and only need to define \( \epsilon(T) \) for \( n \geq 1 \), assuming that \( \epsilon(\overline{T}) \) is already defined. We do this by setting

\[
\epsilon(T) = \frac{1}{\kappa_{\text{sh}(T)}} \epsilon(\overline{T}) e(T) \epsilon(T).
\]
This element $\epsilon(T)$ is implemented as `sage.combinat.symmetric_group_algebra.epsilon()` for $R = \mathbb{Q}$, but it is also a particular case of the elements $\epsilon(T, S)$ defined below.

Now let $S$ be a further tableau of the same shape as $T$ (possibly equal to $T$). Let $\pi_{T, S}$ denote the permutation in $S_n$ such that applying this permutation to the entries of $T$ yields the tableau $S$. Define an element $\epsilon(T, S)$ of $RS_n$ by

$$\epsilon(T, S) = \frac{1}{\kappa_{sh(T)}} \epsilon(S) \pi_{T, S} \epsilon(T) = \frac{1}{\kappa_{sh(T)}} \epsilon(S) a(T) \pi_{T, S} b(T) \epsilon(T).$$

This element $\epsilon(T, S)$ is called *Young’s seminormal unit corresponding to the bitableau $(T, S)$*, and is the return value of `epsilon_ik()` applied to $T$ and $S$. Note that $\epsilon(T, T) = \epsilon(T)$.

If we let $\lambda$ run through all partitions of $n$, and $(T, S)$ run through all pairs of tableaux of shape $\lambda$, then the elements $\epsilon(T, S)$ form a basis of $RS_n$. This basis is called *Young’s seminormal basis* and has the properties that

$$\epsilon(T, S) \epsilon(U, V) = \delta_{T, V} \epsilon(U, S)$$

(where $\delta$ stands for the Kronecker delta).

**Warning:** Because of our convention, we are multiplying our elements in reverse of those given in some papers, for example [Ram1997]. Using the other convention of multiplying permutations, we would instead have $\epsilon(U, V) \epsilon(T, S) = \delta_{T, V} \epsilon(U, S)$.

In other words, Young’s seminormal basis consists of the matrix units in a (particular) Artin-Wedderburn decomposition of $RS_n$ into a direct product of matrix algebras over $\mathbb{Q}$.

The output of `seminormal_basis()` is a list of all elements of the seminormal basis of `self`.

**INPUT:**

- `mult` – string (default: 'l2r'). If set to 'r2l', this causes the method to return the list of the antipodes (`antipode()`) of all $\epsilon(T, S)$ instead of the $\epsilon(T, S)$ themselves.

**EXAMPLES:**

```
sage: QS3 = SymmetricGroupAlgebra(QQ,3)
sage: QS3.seminormal_basis()
[1/6*[1, 2, 3] + 1/6*[1, 3, 2] + 1/6*[2, 1, 3] + 1/6*[2, 3, 1] + 1/6*[3, 1, 2] + 1/6*[3, 2, 1],
1/3*[1, 2, 3] + 1/6*[1, 3, 2] - 1/3*[2, 1, 3] - 1/6*[2, 3, 1] - 1/6*[3, 1, 2] + 1/6*[3, 2, 1],
1/3*[1, 3, 2] + 1/3*[2, 1, 3] - 1/3*[2, 3, 1] - 1/3*[3, 1, 2] + 1/3*[3, 2, 1],
1/4*[1, 3, 2] - 1/4*[2, 1, 3] + 1/4*[2, 3, 1] - 1/4*[3, 1, 2] - 1/4*[3, 2, 1],
1/6*[1, 2, 3] - 1/6*[1, 3, 2] - 1/6*[2, 1, 3] + 1/6*[2, 3, 1] + 1/6*[3, 1, 2] - 1/6*[3, 2, 1]
```

`specht_module(D)`

Return the Specht module of `self` indexed by the diagram $D$.

**EXAMPLES:**
sage: SGA = SymmetricGroupAlgebra(QQ, 5)
sage: SM = SGA.specht_module(Partition([3,1,1]))
sage: SM
Specht module of [(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)] over Rational Field
sage: s = SymmetricFunctions(QQ).s()
sage: s(SM.frobenius_image())
s[3, 1, 1]

sage: SM = SGA.specht_module([(1,1),(1,3),(2,2),(3,1),(3,2)])
sage: SM
Specht module of [(1, 1), (1, 3), (2, 2), (3, 1), (3, 2)] over Rational Field
sage: s(SM.frobenius_image())
s[2, 2, 1] + s[3, 1, 1] + s[3, 2]

\textbf{specht\_module\_dimension}(D)

Return the dimension of the Specht module of self indexed by D.

EXAMPLES:

sage: SGA = SymmetricGroupAlgebra(QQ, 5)
sage: SGA.specht_module_dimension(Partition([3,1,1]))
6
sage: SGA.specht_module_dimension([(1,1),(1,3),(2,2),(3,1),(3,2)])
16

sage.combinat.symmetric_group_algebra.a(tableau, star=0, base\_ring=\texttt{Rational Field})

The row projection operator corresponding to the Young tableau \texttt{tableau} (which is supposed to contain every integer from 1 to its size precisely once, but may and may not be standard).

This is the sum (in the group algebra of the relevant symmetric group over \texttt{Q}) of all the permutations which preserve the rows of \texttt{tableau}. It is called $a_{\text{tableau}}$ in [EGHLSVY], Section 4.2.

INPUT:

- \texttt{tableau} – Young tableau which contains every integer from 1 to its size precisely once.
- \texttt{star} – nonnegative integer (default: 0). When this optional variable is set, the method computes not the row projection operator of \texttt{tableau}, but the row projection operator of the restriction of \texttt{tableau} to the entries 1, 2, ..., \texttt{tableau.size()} - \texttt{star} instead.
- \texttt{base\_ring} – commutative ring (default: \texttt{QQ}). When this optional variable is set, the row projection operator is computed over a user-determined base ring instead of \texttt{Q}. (Note that symmetric group algebras currently don't preserve coercion, so e. g. a symmetric group algebra over \texttt{Z} does not coerce into the corresponding one over \texttt{Q}; so convert manually or choose your base rings wisely!)

EXAMPLES:

sage: from sage.combinat.symmetric_group_algebra import a
sage: a([[1,2]])
[1, 2] + [2, 1]
sage: a([[1],[2]])
[1, 2]
sage: a([[]])
[]
sage: a([[1, 5], [2, 3], [4]])
[1, 2, 3, 4, 5] + [1, 3, 2, 4, 5] + [5, 2, 3, 4, 1] + [5, 3, 2, 4, 1]
Combinatorics, Release 10.0

sage: a([[1,4], [2,3]], base_ring=ZZ)
[1, 2, 3, 4] + [1, 3, 2, 4] + [4, 2, 3, 1] + [4, 3, 2, 1]

sage.combinat.symmetric_group_algebra.b(tableau, star=0, base_ring=Rational Field)

The column projection operator corresponding to the Young tableau `tableau` (which is supposed to contain every integer from 1 to its size precisely once, but may and may not be standard).

This is the signed sum (in the group algebra of the relevant symmetric group over \(\mathbb{Q}\)) of all the permutations which preserve the column of `tableau` (where the signs are the usual signs of the permutations). It is called \(b_{\text{tableau}}\) in [EGHLSVY], Section 4.2.

**INPUT:**
- `tableau` – Young tableau which contains every integer from 1 to its size precisely once.
- `star` – nonnegative integer (default: 0). When this optional variable is set, the method computes not the column projection operator of `tableau`, but the column projection operator of the restriction of `tableau` to the entries 1, 2, ..., `tableau.size()` - `star` instead.
- `base_ring` – commutative ring (default: \(\mathbb{Q}\)). When this optional variable is set, the column projection operator is computed over a user-determined base ring instead of \(\mathbb{Q}\). (Note that symmetric group algebras currently don’t preserve coercion, so e.g. a symmetric group algebra over \(\mathbb{Z}\) does not coerce into the corresponding one over \(\mathbb{Q}\); so convert manually or choose your base rings wisely!)

**EXAMPLES:**

```python
sage: from sage.combinat.symmetric_group_algebra import b
sage: b([[1,2]])
[1, 2]
```

```python
sage: b([[1,2]])
[1, 2] - [2, 1]
```

```python
sage: b([[1]])
[]
```

```python
sage: b([[1, 2, 4], [5, 3]])
[1, 2, 3, 4, 5] - [1, 3, 2, 4, 5] - [5, 2, 3, 4, 1] + [5, 3, 2, 4, 1]
```

```python
sage: b([[1, 2, 3], [2, 3]], base_ring=ZZ)
[1, 2, 3, 4] - [1, 2, 4, 3] - [2, 1, 3, 4] + [2, 1, 4, 3]
```

```python
sage: b([[1, 4], [2, 3]], base_ring=Integers(5))
[1, 2, 3, 4] + 4*[1, 2, 4, 3] + 4*[2, 1, 3, 4] + [2, 1, 4, 3]
```

With the `l2r` setting for multiplication, the unnormalized Young symmetrizer \(e(\text{tableau})\) should be the product \(b(\text{tableau}) \ast a(\text{tableau})\) for every tableau. Let us check this on the standard tableaux of size 5:

```python
sage: from sage.combinat.symmetric_group_algebra import a, b, e
sage: all( e(t) == b(t) * a(t) for t in StandardTableaux(5) )
True
```

sage.combinat.symmetric_group_algebra.e(tableau, star=0)

The unnormalized Young projection operator corresponding to the Young tableau `tableau` (which is supposed to contain every integer from 1 to its size precisely once, but may and may not be standard).

If \(n\) is a nonnegative integer, and \(T\) is a Young tableau containing every integer from 1 to \(n\) exactly once, then the unnormalized Young projection operator \(e(T)\) is defined by

\[
e(T) = a(T)b(T) \in \mathbb{Q}S_n,
\]
where \( a(T) \in QS_n \) is the sum of all permutations in \( S_n \) which fix the rows of \( T \) (as sets), and \( b(T) \in QS_n \) is the signed sum of all permutations in \( S_n \) which fix the columns of \( T \) (as sets). Here, “signed” means that each permutation is multiplied with its sign; and the product on the group \( S_n \) is defined in such a way that \((pq)(i) = p(q(i))\) for any permutations \( p \) and \( q \) and any \( 1 \leq i \leq n \).

Note that the definition of \( e(T) \) is not uniform across literature. Others define it as \( b(T)a(T) \) instead, or include certain scalar factors (we do not, whence “unnormalized”).

**EXAMPLES:**

```
sage: from sage.combinat.symmetric_group_algebra import e
sage: e([[1,2]])
[1, 2] + [2, 1]
sage: e([[1],[2]])
[1, 2] - [2, 1]
sage: e([])
[]
```

There are differing conventions for the order of the symmetrizers and antisymmetrizers. This example illustrates our conventions:

```
sage: e([[1,2],[3]])
[1, 2, 3] + [2, 1, 3] - [3, 1, 2] - [3, 2, 1]
```

To obtain the product \( b(T)a(T) \), one has to take the antipode of this:

```
sage: QS3 = parent(e([[1,2],[3]]))
sage: QS3.antipode(e([[1,2],[3]]))
[1, 2, 3] + [2, 1, 3] - [2, 3, 1] - [3, 2, 1]
```

**See also:**

```
sage.combinat.symmetric_group_algebra.e_hat(tab, star=0)
```

The Young projection operator corresponding to the Young tableau \( \text{tab} \) (which is supposed to contain every integer from 1 to its size precisely once, but may and may not be standard). This is an idempotent in the rational group algebra.

If \( n \) is a nonnegative integer, and \( T \) is a Young tableau containing every integer from 1 to \( n \) exactly once, then the Young projection operator \( \hat{e}(T) \) is defined by

\[
\hat{e}(T) = \frac{1}{\kappa_\lambda} a(T)b(T) \in QS_n,
\]

where \( \lambda \) is the shape of \( T \), where \( \kappa_\lambda \) is \( n! \) divided by the number of standard tableaux of shape \( \lambda \), where \( a(T) \in QS_n \) is the sum of all permutations in \( S_n \) which fix the rows of \( T \) (as sets), and where \( b(T) \in QS_n \) is the signed sum of all permutations in \( S_n \) which fix the columns of \( T \) (as sets). Here, “signed” means that each permutation is multiplied with its sign; and the product on the group \( S_n \) is defined in such a way that \((pq)(i) = p(q(i))\) for any permutations \( p \) and \( q \) and any \( 1 \leq i \leq n \).

Note that the definition of \( \hat{e}(T) \) is not uniform across literature. Others define it as \( \frac{1}{\kappa_\lambda} b(T)a(T) \) instead.

**EXAMPLES:**

```
sage: from sage.combinat.symmetric_group_algebra import e_hat
sage: e_hat([[1,2,3]])
1/6*[1, 2, 3] + 1/6*[1, 3, 2] + 1/6*[2, 1, 3] + 1/6*[2, 3, 1] + 1/6*[3, 1, 2] + 1/
```
There are differing conventions for the order of the symmetrizers and antisymmetrizers. This example illustrates our conventions:

\[ e_\hat{\cdot}([[1,2],[3]]) = \frac{1}{3}[[1,2,3]] + \frac{1}{3}[[2,1,3]] - \frac{1}{3}[[3,1,2]] - \frac{1}{3}[[3,2,1]] \]

See also:

\[ e() \]

\[ e_ik(itab, ktab, star=0) \]

\[ \text{sage.combinat.symmetric_group_algebra.} \text{epsilon}_ik(itab, ktab, star=0) \]

\[ \text{sage.combinat.symmetric_group_algebra.} \text{epsilon}(tab, star=0) \]

\[ \text{sage.combinat.symmetric_group_algebra.} \text{epsilon}_ik(itab, ktab, star=0) \]

\[ \text{sage.combinat.symmetric_group_algebra.} \text{epsilon}(tab, star=0) \]

\[ \text{sage.combinat.symmetric_group_algebra.} \text{epsilon}_ik(itab, ktab, star=0) \]
\[
\frac{1}{4} [1, 3, 2] - \frac{1}{4} [2, 3, 1] + \frac{1}{4} [3, 1, 2] - \frac{1}{4} [3, 2, 1]
\]

```
sage: epsilon_ik([[1,2],[3]], [[1,3],[2]], star=1)
Traceback (most recent call last):
  ... 
ValueError: the two tableaux must be of the same shape
```

\textbf{sage.combinat.symmetric_group_algebra.kappa(\textit{alpha})}

Return $\kappa_\alpha$, which is $n!$ divided by the number of standard tableaux of shape $\alpha$ (where $\alpha$ is a partition of $n$).

**INPUT:**
- \textit{alpha} – integer partition (can be encoded as a list).

**OUTPUT:**
The factorial of the size of \textit{alpha}, divided by the number of standard tableaux of shape \textit{alpha}. Equivalently, the product of all hook lengths of \textit{alpha}.

**EXAMPLES:**
```
sage: from sage.combinat.symmetric_group_algebra import kappa
sage: kappa(Partition([2,1]))
3
sage: kappa([2,1])
3
```

\textbf{sage.combinat.symmetric_group_algebra.pi_ik(\textit{itab}, \textit{ktab})}

Return the permutation $p$ which sends every entry of the tableau \textit{itab} to the respective entry of the tableau \textit{ktab}, as an element of the corresponding symmetric group algebra.

This assumes that \textit{itab} and \textit{ktab} are tableaux (possibly given just as lists of lists) of the same shape.

**EXAMPLES:**
```
sage: from sage.combinat.symmetric_group_algebra import pi_ik
sage: pi_ik([[1,3],[2]], [[1,2],[3]])
[1, 3, 2]
```

\textbf{sage.combinat.symmetric_group_algebra.seminormal_test(\textit{n})}

Run a variety of tests to verify that the construction of the seminormal basis works as desired. The numbers appearing are results in James and Kerber’s ‘Representation Theory of the Symmetric Group’ [JK1981].

**EXAMPLES:**
```
sage: from sage.combinat.symmetric_group_algebra import seminormal_test
sage: seminormal_test(3)
True
```
5.1.341 Representations of the Symmetric Group

Todo:

- construct the product of two irreducible representations.
- implement Induction/Restriction of representations.

Warning: This code uses a different convention than in Sagan’s book “The Symmetric Group”

```python
class sage.combinat.symmetric_group_representations.SpechtRepresentation(parent, partition):
    Bases: SymmetricGroupRepresentation_generic_class

    representation_matrix(permutation)
    Return the matrix representing the permutation in this irreducible representation.

    Note: This method caches the results.

    EXAMPLES:

    sage: spc = SymmetricGroupRepresentation([3,1], 'specht')
    sage: spc.representation_matrix(Permutation([2,1,3,4]))
    [ 0 -1  0]
    [-1  0  0]
    [ 0  0  1]
    sage: spc.representation_matrix(Permutation([3,2,1,4]))
    [0 0 1]
    [0 1 0]
    [1 0 0]

    scalar_product(u, v)
    Return 0 if u+v is not a permutation, and the signature of the permutation otherwise.
    This is the scalar product of a vertex u of the underlying Yang-Baxter graph with the vertex v in the ‘dual’ Yang-Baxter graph.

    EXAMPLES:

    sage: spc = SymmetricGroupRepresentation([3,2], 'specht')
    sage: spc.scalar_product((1,0,2,1,0),(0,3,0,3,0))
    -1
    sage: spc.scalar_product((1,0,2,1,0),(3,0,0,3,0))
    0

    scalar_product_matrix(permutation=None)
    Return the scalar product matrix corresponding to permutation.
    The entries are given by the scalar products of u and permutation.action(v), where u is a vertex in the underlying Yang-Baxter graph and v is a vertex in the dual graph.

    EXAMPLES:
```
```python
sage: spc = SymmetricGroupRepresentation([3,1], 'specht')

sage: spc.scalar_product_matrix()

[ 1 0 0]
[ 0 -1 0]
[ 0 0 1]
```

```python
class sage.combinat.symmetric_group_representations.SpechtRepresentations(n, ring=None, cache_matrices=True):
    ...

The irreducible representation of the symmetric group corresponding to partition.

INPUT:

- partition – a partition of a positive integer
- implementation – string (default: "specht"), one of:
  - "seminormal" - for Young’s seminormal representation
  - "orthogonal" - for Young’s orthogonal representation
  - "specht" - for Specht's representation
- ring – the ring over which the representation is defined
- cache_matrices – boolean (default: True) if True, then any representation matrices that are computed are cached

EXAMPLES:

Young’s orthogonal representation: the matrices are orthogonal.

```python
sage: orth = SymmetricGroupRepresentation([2,1], "orthogonal"); orth

Orthogonal representation of the symmetric group corresponding to [2, 1]
```

```python
sage: all(a*a.transpose() == a.parent().identity_matrix() for a in orth)

True
```

```python
sage: orth = SymmetricGroupRepresentation([3,2], "orthogonal"); orth

Orthogonal representation of the symmetric group corresponding to [3, 2]
```

```python
sage: orth([1,3,2,4,5])

[ 1 0 0 0 0]
[ 0 1 0 0 0]
[ 0 0 -1 0 0]
[ 0 0 0 1 0]
[ 0 0 0 0 -1]
```

```python
sage: orth([1,3,2,4,5])

[ 1 0 0 0 0]
[ 0 -1/2 1/2*sqrt(3) 0 0]
[ 0 1/2*sqrt(3) 1/2 0 0]
```

(continues on next page)
\[
\begin{bmatrix}
0 & 0 & 0 & -1/2 & 1/2\sqrt{3} \\
0 & 0 & 0 & 1/2\sqrt{3} & 1/2
\end{bmatrix}
\]

\textbf{sage: orth([1,2,4,3,5])}
\[
\begin{bmatrix}
-1/3 & 2/3\sqrt{2} & 0 & 0 & 0 \\
2/3\sqrt{2} & 1/3 & 0 & 0 & 0
\end{bmatrix}
\]

\textbf{sage: orth([2/3\sqrt{2}, 1/3])}
\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\textbf{sage: orth([-1/3, 2/3\sqrt{2}, 0, 0, 0]}
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

\textbf{The Specht representation:}

\textbf{sage: spc = SymmetricGroupRepresentation([3,2], "specht")}
\textbf{sage: spc.scalar_product_matrix(Permutation([1,2,3,4,5]))}
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{bmatrix}
\]

\textbf{sage: spc.scalar_product_matrix(Permutation([5,4,3,2,1]))}
\[
\begin{bmatrix}
1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]

\textbf{sage: spc.scalar_product_matrix(Permutation([5,4,3,2,1]))}
\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1
\end{bmatrix}
\]

\textbf{sage: spc.scalar_product_matrix(Permutation([5,4,3,2,1]))}
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

\textbf{By default, any representation matrices that are computed are cached:}

\textbf{sage: spc = SymmetricGroupRepresentation([3,2], "specht")}
\textbf{sage: spc([5,4,3,2,1])}
\[
\begin{bmatrix}
1 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1
\end{bmatrix}
\]

\textbf{sage: spc._cache__representation_matrix}
\[
\{(([5, 4, 3, 2, 1]), ()): \begin{bmatrix}
1 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1
\end{bmatrix}\}
\]

This can be turned off with the keyword cache_matrices:

\textbf{sage: spc = SymmetricGroupRepresentation([3,2], "specht", cache_matrices=False)}
\textbf{sage: spc([5,4,3,2,1])}
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(continued from previous page)

\[
\begin{bmatrix}
1 & -1 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 \\
0 & 1 & -1 & -1 & 1 \\
0 & 1 & 0 & -1 & 1
\end{bmatrix}
\]

\textbf{sage:} \ \texttt{hasattr(spc, \ '_cache\_representation\_matrix\')}
False

**Note:** The implementation is based on the paper [Las].

REFERENCES:

AUTHORS:

- Franco Saliola (2009-04-23)

class \texttt{sage.combinat.symmetric\_group\_representations.SymmetricGroupRepresentation\_generic\_class}(parent, partition)

Bases: \texttt{Element}

Generic methods for a representation of the symmetric group.

**to\_character()**

Return the character of the representation.

**EXAMPLES:**

The trivial character:

\texttt{sage: \rho = SymmetricGroupRepresentation([3])}
\texttt{sage: \chi = \rho.to\_character(); \chi}
Character of Symmetric group of order 3! as a permutation group
\texttt{sage: \chi.values()}
[1, 1, 1]
\texttt{sage: all(\chi(g) == 1 \text{ for } g \text{ in } SymmetricGroup(3))}
True

The sign character:

\texttt{sage: \rho = SymmetricGroupRepresentation([1,1,1])}
\texttt{sage: \chi = \rho.to\_character(); \chi}
Character of Symmetric group of order 3! as a permutation group
\texttt{sage: \chi.values()}
[1, -1, 1]
\texttt{sage: all(\chi(g) == g.sign() \text{ for } g \text{ in } SymmetricGroup(3))}
True

The defining representation:

\texttt{sage: \texttt{triv = SymmetricGroupRepresentation([4])}}
\texttt{sage: hook = SymmetricGroupRepresentation([3,1])}
\texttt{sage: def\_rep = \texttt{lambda} \ p : triv(p).block\_sum(hook(p)).trace()}

(continues on next page)
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```
sage: list(map(def_rep, Permutations(4)))
[4, 2, 2, 1, 1, 2, 2, 0, 1, 0, 0, 1, 1, 0, 2, 1, 0, 0, 1, 1, 2, 0, 0]
sage: [p.to_matrix().trace() for p in Permutations(4)]
[4, 2, 2, 1, 1, 2, 2, 0, 1, 0, 0, 1, 1, 0, 2, 1, 0, 0, 1, 1, 2, 0, 0]
```

**verify_representation()**
Verify the representation.

This tests that the images of the simple transpositions are involutions and tests that the braid relations hold.

**EXAMPLES:**
```
sage: spc = SymmetricGroupRepresentation([1,1,1])
sage: spc.verify_representation()
True
sage: spc = SymmetricGroupRepresentation([4,2,1])
sage: spc.verify_representation()
True
```

sage.combinat.symmetric_group_representations.

**SymmetricGroupRepresentations**(n, implementation='specht', ring=None, cache_matrices=True)

Irreducible representations of the symmetric group.

**INPUT:**

- n – positive integer
- implementation – string (default: "specht"), one of:
  - "seminormal" - for Young’s seminormal representation
  - "orthogonal" - for Young’s orthogonal representation
  - "specht" - for Specht’s representation
- ring – the ring over which the representation is defined
- cache_matrices – boolean (default: True) if True, then any representation matrices that are computed are cached

**EXAMPLES:**

Young’s orthogonal representation: the matrices are orthogonal.
```
sage: orth = SymmetricGroupRepresentations(3, "orthogonal"); orth
Orthogonal representations of the symmetric group of order 3! over Symbolic Ring
sage: orth.list()
[Orthogonal representation of the symmetric group corresponding to [3],
 Orthogonal representation of the symmetric group corresponding to [2, 1],
 Orthogonal representation of the symmetric group corresponding to [1, 1, 1]]
sage: orth([2,1])([1,2,3])
[1 0]
[0 1]
```

Young’s seminormal representation.
The Specht Representation.

```
sage: spc = SymmetricGroupRepresentations(5, "specht"); spc
Specht representations of the symmetric group of order 5! over Integer Ring
sage: spc([3,2])([5,4,3,2,1])
[ 1 -1 0 1 0]
[ 0 0 -1 0 1]
[ 0 0 0 -1 1]
[ 0 1 -1 -1 1]
[ 0 1 0 -1 1]
```

**Note:** The implementation is based on the paper [Las].

**AUTHORS:**

- Franco Saliola (2009-04-23)

```python
class sage.combinat.symmetric_group_representations.SymmetricGroupRepresentations_class(n, ring=None, cache_matrices=True):
    Bases: UniqueRepresentation, Parent
    Generic methods for the CombinatorialClass of irreducible representations of the symmetric group.

cardinality()
    Return the cardinality of self.

    EXAMPLES:
    sage: sp = SymmetricGroupRepresentations(4, "specht")
    sage: sp.cardinality()
    5
```
**representation_matrix(permutation)**

Return the matrix representing permutation.

**EXAMPLES:**

```python
sage: orth = SymmetricGroupRepresentation([2,1], "orthogonal")
sage: orth.representation_matrix(Permutation([2,1,3]))
[ 1 0]
[ 0 -1]
sage: orth.representation_matrix(Permutation([1,3,2]))
[ -1/2 1/2*sqrt(3)]
[1/2*sqrt(3) 1/2]
```

```python
sage: norm = SymmetricGroupRepresentation([2,1], "seminormal")
sage: p = PermutationGroupElement([2,1,3])
sage: norm.representation_matrix(p)
[ 1 0]
[ 0 -1]
sage: p = PermutationGroupElement([1,3,2])
sage: norm.representation_matrix(p)
[ -1/2 3/2]
[ 1/2 1/2]
```

**representation_matrix_for_simple_transposition(i)**

Return the matrix representing the transposition that swaps i and i+1.

**EXAMPLES:**

```python
sage: orth = SymmetricGroupRepresentation([2,1], "orthogonal")
sage: orth.representation_matrix_for_simple_transposition(1)
[ 1 0]
[ 0 -1]
sage: orth.representation_matrix_for_simple_transposition(2)
[ -1/2 1/2*sqrt(3)]
[1/2*sqrt(3) 1/2]
```

```python
sage: norm = SymmetricGroupRepresentation([2,1], "seminormal")
sage: norm.representation_matrix_for_simple_transposition(1)
[ 1 0]
[ 0 -1]
sage: norm.representation_matrix_for_simple_transposition(2)
[ -1/2 3/2]
[ 1/2 1/2]
```

**class** sage.combinat.symmetric_group_representations.YoungRepresentations_Orthogonal(n, ring=None, cache_matrices=True)

**Bases:** SymmetricGroupRepresentations_class

**Element**

alias of YoungRepresentation_Orthogonal

**class** sage.combinat.symmetric_group_representations.YoungRepresentations_Seminormal(n, ring=None, cache_matrices=True)
Bases: SymmetricGroupRepresentations_class

Element
alias of YoungRepresentation_Seminormal

sage.combinat.symmetric_group_representations.partition_to_vector_of_contents(partition, reverse=False)

Return the “vector of contents” associated to partition.

EXAMPLES:

```python
sage: from sage.combinat.symmetric_group_representations import partition_to_vector_of_contents
sage: partition_to_vector_of_contents([3,2])
(0, 1, 2, -1, 0)
```

5.1.342 T-sequences

T-sequences are tuples of four (-1, 0, 1) sequences of length \( t \) where for every \( i \) exactly one sequence has a nonzero entry at index \( i \) and for which the nonperiodic autocorrelation function is equal to zero (i.e. they are complementary). See Definition 7.5 of [Seb2017].

These can be constructed from Turyn sequences. In particular, if Turyn sequences of length \( l \) exists, there will be T-sequences of length \( 4l - 1 \) and \( 2l - 1 \).

Turyn sequences are tuples of four (-1, +1) sequences \( X, U, Y, V \) of length \( l, l, l - 1, l - 1 \) with nonperiodic autocorrelation equal to zero and the additional constraints that:

- the first element of \( X \) is 1
- the last element of \( X \) is -1
- the last element of \( U \) is 1

The nonperiodic autocorrelation of a family of sequences \( X = \{A_1, A_2, ..., A_n\} \) is defined as (see Definition 7.2 of [Seb2017]):

\[
N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + ... + a_{n,i}a_{n,i+j})
\]

AUTHORS:
- Matteo Cati (2022-11-16): initial version

sage.combinat.t_sequences.T_sequences_construction_from_base_sequences(base_sequences, check=True)

Construct T-sequences of length \( 2n + p \) from base sequences of length \( n + p, n + p, n, n \).

Given base sequences \( A, B, C, D \), the T-sequences are constructed as described in [KTR2005]:

\[
T_1 = \frac{1}{2}(A + B); 0_n
\]
\[
T_2 = \frac{1}{2}(A - B); 0_n
\]
\[
T_3 = 0_{n+p} + \frac{1}{2}(C + D)
\]
\[
T_4 = 0_{n+p} + \frac{1}{2}(C - D)
\]
INPUT:

- `base_sequences` – the base sequences that should be used to construct the T-sequences.
- `check` – boolean, if true (default) checks that the sequences created are T-sequences before returning them.

EXAMPLES:

```python
sage: from sage.combinat.t_sequences import turyn_sequences_smallcases, T_sequences_construction_from_base_sequences
sage: seqs = turyn_sequences_smallcases(4)
sage: T_sequences_construction_from_base_sequences(seqs)
[[1, 1, -1, 0, 0, 0, 0],
 [0, 0, 0, -1, 0, 0, 0],
 [0, 0, 0, 0, 1, 0, 1],
 [0, 0, 0, 1, 0, 0, 0]]
```

sage.combinat.t_sequences.T_sequences_construction_from_turyn_sequences(turyn_sequences, check=True)

Construct T-sequences of length $4l - 1$ from Turyn sequences of length $l$.

Given Turyn sequences $X, U, Y, V$, the T-sequences are constructed as described in theorem 7.7 of [Seb2017]:

\[
T_1 = 1; 0_{4l-2}
\]
\[
T_2 = 0; X/Y; 0_{2l-1}
\]
\[
T_3 = 0_{2l}; U/0_{l-2}
\]
\[
T_4 = 0_{2l} + 0_{l}/V
\]

INPUT:

- `turyn_sequences` – the Turyn sequences that should be used to construct the T-sequences.
- `check` – boolean, if true (default) checks that the sequences created are T-sequences before returning them.

EXAMPLES:

```python
sage: from sage.combinat.t_sequences import turyn_sequences_smallcases, T_sequences_construction_from_turyn_sequences, is_T_sequences_set
sage: seqs = turyn_sequences_smallcases(4)
sage: T_sequences_construction_from_turyn_sequences(seqs)
[[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
 [0, 1, 1, 1, 1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0],
 [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, -1, 0, 1],
 [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 1, 0]]
```

sage.combinat.t_sequences.T_sequences_smallcases(t, existence=False, check=True)

Construct T-sequences for some small values of $t$.

This method will try to use the constructions defined in `T_sequences_construction_from_base_sequences()` and `T_sequences_construction_from_turyn_sequences()` together with the Turyn sequences stored in `turyn_sequences_smallcases()`, or base sequences created by `base_sequences_smallcases()`.

This function contains also some T-sequences taken directly from [CRSKKY1989].

INPUT:

- `t` – integer, the length of the T-sequences to construct.
- `existence` – boolean (default false). If true, this method only returns whether a T-sequences of the given size can be constructed.
• check – boolean, if true (default) check that the sequences are T-sequences before returning them.

EXAMPLES:
By default, this method returns the four T-sequences

```
sage: from sage.combinat.t_sequences import T_sequences_smallcases, is_T_sequences_
             → set
sage: T_sequences_smallcases(9)
[[1, 1, 0, 1, 0, 0, 0, 0, 0],
 [0, 0, -1, 0, 1, 0, 0, 0, 0],
 [0, 0, 0, 0, 0, 1, 0, 0, -1],
 [0, 0, 0, 0, 0, 0, 1, -1, 0]]
```

If the existence flag is passed, the method returns a boolean

```
sage: T_sequences_smallcases(9, existence=True)
True
```

```
sage.combinat.t_sequences.base_sequences_construction(turyn_type_seqs, check=True)
```

Construct base sequences of length $2n - 1, 2n - 1, n, n$ from Turyn type sequences of length $n, n, n, n - 1$.

Given Turyn type sequences $X, Y, Z, W$ of length $n, n, n, n - 1$, Theorem 1 of [KTR2005] shows that the following are base sequences of length $2n - 1, 2n - 1, n, n$:

- $A = Z; W$
- $B = Z; -W$
- $C = X$
- $D = Y$

INPUT:
- `turyn_type_seqs` – The list of 4 Turyn type sequences that should be used to construct the base sequences.
- `check` – boolean, if True (default) check that the resulting sequences are base sequences before returning them.

OUTPUT: A list containing the four base sequences.

EXAMPLES:

```
sage: from sage.combinat.t_sequences import base_sequences_construction
sage: X = [1,1,-1,1,-1,1,-1,1]
```

```
sage: Y = [1,-1,-1,-1,-1,-1,-1,1]
```

```
sage: Z = [1,-1,-1,1,1,1,1,-1]
```

```
sage: W = [1,1,1,-1,1,1,-1]
```

```
sage: base_sequences_construction([X, Y, Z, W])
[[1, -1, -1, 1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, -1],
 [1, -1, -1, 1, 1, 1, -1, -1, -1, -1, -1, 1, -1, 1, -1, 1],
 [1, 1, -1, 1, -1, 1, -1, -1, 1, -1, -1, 1, -1, 1, -1, 1]]
```

See also:

```
is_base_sequences_tuple()
```

```
sage.combinat.t_sequences.base_sequences_smallcases(n, p, existence=False, check=True)
```

Construct base sequences of length $n + p, n + p, n, n$ from available data.
The function uses the construction `base_sequences_construction()`, together with Turyn type sequences from `turyn_type_sequences_smallcases()` to construct base sequences with \( p = n - 1 \).

Furthermore, this function uses also Turyn sequences (i.e. base sequences with \( p = 1 \)) from `turyn_sequences_smallcases()`.

**INPUT:**

- \( n \) – integer, the length of the last two base sequences.
- \( p \) – integer, \( n + p \) will be the length of the first two base sequences.
- `existence` – boolean (default False). If True, the function will only check whether the base sequences can be constructed.
- `check` – boolean, if True (default) check that the resulting sequences are base sequences before returning them.

**OUTPUT:**

If `existence` is False, the function returns a list containing the four base sequences, or raises an error if the base sequences cannot be constructed. If `existence` is True, the function returns a boolean, which is True if the base sequences can be constructed and False otherwise.

**EXAMPLES:**

```python
sage: from sage.combinat.t_sequences import base_sequences_smallcases
sage: base_sequences_smallcases(8, 7)
[[1, -1, -1, 1, 1, 1, 1, -1, 1, 1, 1, -1, 1, 1, -1],
 [1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, 1, -1, -1, 1],
 [1, 1, -1, 1, -1, 1, -1, 1],
 [1, -1, -1, -1, -1, -1, -1, 1]]
```

If `existence` is True, the function returns a boolean

```python
sage: base_sequences_smallcases(8, 7, existence=True)
True
sage: base_sequences_smallcases(7, 5, existence=True)
False
```

`sage.combinat.t_sequences.is_T_sequences_set(sequences, verbose=False)`

Check if a family of sequences is composed of T-sequences.

Given 4 (-1, 0, +1) sequences, they will be T-sequences if (Definition 7.4 of [Seb2017]):

- they have all the same length \( t \)
- for each index \( i \), exactly one sequence is nonzero at \( i \)
- the nonperiodic autocorrelation is equal to 0

**INPUT:**

- `sequences` – a list of four sequences.
- `verbose` – a boolean (default False). If true the function will be verbose when the sequences do not satisfy the contraints.

**EXAMPLES:**
```python
sage: from sage.combinat.t_sequences import is_T_sequences_set
sage: seqs = [[1, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, -1], [0, 0, 0, 0, 0]]
```

```python
sage: is_T_sequences_set(seqs)
True
```

```python
sage: seqs = [[1, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, -1], [0, 0, 0, 0, 0]]
```

```python
sage: is_T_sequences_set(seqs, verbose=True)
There should be exactly a nonzero element at every index, found 2 such elements...
```

```
False
```

```
```

```python
sage.combinat.t_sequences.is_base_sequences_tuple(base_sequences, verbose=False)
```

Check if the given sequences are base sequences.

Four (-1, +1) sequences $A, B, C, D$ of length $n + p, n + p, n, n$ are called base sequences if for all $j \geq 1$:

$$N_A(j) + N_B(j) + N_C(j) + N_D(j) = 0$$

where $N_X(j)$ is the nonperiodic autocorrelation (See definition in [KTR2005]).

INPUT:

- `base_sequences` – The list of 4 sequences that should be checked.
- `verbose` – a boolean (default false). If true the function will be verbose when the sequences do not satisfy the constraints.

EXAMPLES:

```python
sage: from sage.combinat.t_sequences import is_base_sequences_tuple
sage: seqs = [[1, -1, -1, 1, 1, 1, 1, -1, 1, 1, 1, -1, 1, 1, -1], [1, -1, -1, 1, 1, -1, 1, -1, -1, -1, -1, 1, -1, -1, 1], [1, 1, -1, 1, -1, 1, -1, 1], [1, -1, -1, -1, -1, -1, -1, 1]]
```

```python
sage: is_base_sequences_tuple(seqs)
True
```

If verbose is true, the function will be verbose

```python
sage: seqs = [[1, -1], [1, 1], [-1], [2]]
```

```python
sage: is_base_sequences_tuple(seqs, verbose=True)
Base sequences should only contain -1, +1, found 2
False
```

See also:

`base_sequences_construction()`

```
```

```python
sage.combinat.t_sequences.is_skew(seq, verbose=False)
```

Check if the given sequence is skew.

A sequence $X = \{x_1, x_2, ..., x_n\}$ is defined skew (according to Definition 7.4 of [Seb2017]) if $n$ is even and $x_i = -x_{n-i+1}$.

INPUT:

- `seq` – the sequence that should be checked.
- `verbose` – a boolean (default false). If true the function will be verbose when the sequences do not satisfy the constraints.

EXAMPLES:
sage: from sage.combinat.t_sequences import is_skew
sage: is_skew([1, -1, 1, -1, 1, -1])
True
sage: is_skew([1, -1, -1, -1], verbose=True)
Constraint not satisfied at index 1
False

sage.combinat.t_sequences.is_symmetric(seq, verbose=False)

Check if the given sequence is symmetric.

A sequence $X = \{x_1, x_2, ..., x_n\}$ is defined symmetric (according to Definition 7.4 of [Seb2017]) if $n$ is odd and $x_i = x_{n-i+1}$.

INPUT:

- seq – the sequence that should be checked.
- verbose – a boolean (default false). If true the function will be verbose when the sequences do not satisfy the constraints.

EXAMPLES:

sage: from sage.combinat.t_sequences import is_symmetric
sage: is_symmetric([1, -1, 1, -1, 1])
True
sage: is_symmetric([1, -1, 1, 1, 1], verbose=True)
Constraint not satisfied at index 1
False

sage.combinat.t_sequences.turyn_sequences_smallcases(l, existence=False)

Construction of Turyn sequences for small values of $l$.

The data is taken from [Seb2017] and [CRSKKY1989].

INPUT:

- l – integer, the length of the Turyn sequences.
- existence – boolean (default False). If true, only return whether the Turyn sequences are available for the given length.

EXAMPLES:

By default, this method returns the four Turyn sequences

sage: from sage.combinat.t_sequences import turyn_sequences_smallcases
sage: turyn_sequences_smallcases(4)
[[1, 1, -1, -1], [1, 1, -1, 1], [1, 1, 1], [1, -1, 1]]

If we pass the existence flag, the method will return a boolean

sage: turyn_sequences_smallcases(4, existence=True)
True

sage.combinat.t_sequences.turyn_type_sequences_smallcases(n, existence=False)

Construction of Turyn type sequences for small values of $n$.

The data is taken from [KTR2005] for $n = 36$, and from [BDKR2013] for $n \leq 32$.

INPUT:
• \( n \) – integer, the length of the Turyn type sequences.

• \texttt{existence} – boolean (default False). If true, only return whether the Turyn type sequences are available for the given length.

**EXAMPLES:**

By default, this method returns the four Turyn type sequences

```
sage: from sage.combinat.t_sequences import turyn_type_sequences_smallcases
sage: turyn_type_sequences_smallcases(4)
[[1, 1, 1, 1], [1, 1, -1, 1], [1, 1, -1, -1], [1, -1, 1]]
```

If we pass the \texttt{existence} flag, the method will return a boolean

```
sage: turyn_type_sequences_smallcases(4, existence=True)
True
```

**ALGORITHM:**

The Turyn type sequences are stored in hexadecimal format. Given \( n \) hexadecimal digits \( h_1, h_2, ..., h_n \), it is possible to get the Turyn type sequences by converting each \( h_i \) (\( 1 \leq i \leq n - 1 \)) into a four digits binary number. Then, the \( j \)-th binary digit is 0 if the \( i \)-th number in the \( j \)-th sequence is 1, and it is 1 if the number in the sequence is -1.

For the \( n \)-th digit, it should be converted to a 3 digits binary number, and then the same mapping as before can be used (see also [BDKR2013]).

### 5.1.343 Tableaux

**AUTHORS:**

• Mike Hansen (2007): initial version
• Jason Bandlow (2011): updated to use Parent/Element model, and many minor fixes
• Andrew Mathas (2012-13): completed the transition to the parent/element model begun by Jason Bandlow
• Travis Scrimshaw (11-22-2012): Added tuple options, changed \texttt{*katabolism*} to \texttt{*catabolism*}. Cleaned up documentation.
• Andrew Mathas (2016-08-11): Row standard tableaux added
• Oliver Pechenik (2018): Added increasing tableaux.

This file consists of the following major classes:

**Element classes:**

• \texttt{Tableau}
• \texttt{SemistandardTableau}
• \texttt{StandardTableau}
• \texttt{RowStandardTableau}
• \texttt{IncreasingTableau}

**Factory classes:**

• \texttt{Tableaux}
• \texttt{SemistandardTableaux}
• **StandardTableaux**
• **RowStandardTableaux**
• **IncreasingTableaux**

Parent classes:
• **Tableaux_all**
• **Tableaux_size**
• **SemistandardTableaux_all** (facade class)
• **SemistandardTableaux_size**
• **SemistandardTableaux_size_inf**
• **SemistandardTableaux_size_weight**
• **SemistandardTableaux_shape**
• **SemistandardTableaux_shape_inf**
• **SemistandardTableaux_shape_weight**
• **StandardTableaux_all** (facade class)
• **StandardTableaux_size**
• **StandardTableaux_shape**
• **IncreasingTableaux_all** (facade class)
• **IncreasingTableaux_size**
• **IncreasingTableaux_size_inf**
• **IncreasingTableaux_size_weight**
• **IncreasingTableaux_shape**
• **IncreasingTableaux_shape_inf**
• **IncreasingTableaux_shape_weight**
• **RowStandardTableaux_all** (facade class)
• **RowStandardTableaux_size**
• **RowStandardTableaux_shape**

For display options, see `Tableaux.options()`.

---

**Todo:**

• Move methods that only apply to semistandard tableaux from tableau to semistandard tableau
• Copy/move functionality to skew tableaux
• Add a class for tableaux of a given shape (eg `Tableaux_shape`)

```python
class sage.combinat.tableau.IncreasingTableau(parent, t, check=True)
    Bases: Tableau
    A class to model an increasing tableau.
```

INPUT:
• \(\tau\) – a tableau, a list of iterables, or an empty list

An increasing tableau is a tableau whose entries are positive integers that are strictly increasing across rows and strictly increasing down columns.

**EXAMPLES:**

```sage
t = IncreasingTableau([[1,2,3],[2,3]]); t
[[1, 2, 3], [2, 3]]
sage: t.shape()
[3, 2]
sage: t.pp() # pretty printing
1 2 3
2 3
t = Tableau([[1,2],[2]])
sage: s = IncreasingTableau(t); s
[[1, 2], [2]]
sage: IncreasingTableau([]) # The empty tableau
[]
```

You can also construct an *IncreasingTableau* from the appropriate *Parent* object:

```sage
IT = IncreasingTableaux()
sage: IT([[1, 2, 3], [4, 5]])
[[1, 2, 3], [4, 5]]
```

See also:

- *Tableaux*
- *Tableau*
- *SemistandardTableaux*
- *SemistandardTableau*
- *StandardTableaux*
- *StandardTableau*
- *IncreasingTableaux*

**K_bender_knuth\(i\)**

Return the \(i\)-th K-Bender-Knuth operator (as defined in [DPS2017]) applied to `self`.

The \(i\)-th K-Bender-Knuth operator swaps the letters \(i\) and \(i+1\) everywhere where doing so would not break increasingness.

**EXAMPLES:**

```sage
T = IncreasingTableau([[1,3,4],[2,4,5]])
sage: T.K_bender_knuth(2)
[[1, 2, 4], [3, 4, 5]]
sage: T.K_bender_knuth(3)
[[1, 3, 4], [2, 4, 5]]
```

**K_evacuation\(ceiling=None\)**

Return the K-evacuation involution from [TY2009] to `self`.

**EXAMPLES:**

```sage
```
sage: T = IncreasingTableau([[1,3,4],[2,4,5]])
sage: T.K_evacuation()
[[1, 2, 4], [2, 3, 5]]
sage: T.K_evacuation(6)
[[2, 3, 5], [3, 4, 6]]
sage: U = IncreasingTableau([[1,3,4],[3,4,5],[5]])
sage: U.K_evacuation()
[[1, 2, 3], [2, 3, 5], [3]]

K_promotion(ceiling=None)

Return the K-promotion operator from [Pec2014] applied to self.

EXAMPLES:

sage: T = IncreasingTableau([[1,3,4],[2,4,5]])
sage: T.K_promotion()
[[1, 2, 3], [3, 4, 5]]
sage: T.K_promotion(6)
[[1, 2, 3], [3, 4, 6]]
sage: U = IncreasingTableau([[1,3,4],[3,4,5],[5]])
sage: U.K_promotion()
[[2, 3, 4], [3, 4, 5], [4]]

K_promotion_inverse(ceiling=None)

Return the inverse of K-promotion operator applied to self.

EXAMPLES:

sage: T = IncreasingTableau([[1,3,4],[2,4,5]])
sage: T.K_promotion_inverse()
[[1, 2, 4], [3, 4, 5]]
sage: T.K_promotion_inverse(6)
[[2, 4, 5], [3, 5, 6]]
sage: U = IncreasingTableau([[1,3,4],[3,4,5],[5]])
sage: U.K_promotion_inverse()
[[1, 2, 4], [2, 4, 5], [4]]

check()

Check that self is a valid increasing tableau.

descent_set()

Compute the descents of the increasing tableau self as defined in [DPS2017].

The number i is a descent of an increasing tableau if some instance of i + 1 appears in a lower row than some instance of i.

Note: This notion is close to the notion of descent for a standard tableau but is unrelated to the notion for semistandard tableaux.

EXAMPLES:

sage: T = IncreasingTableau([[1,2,4],[3,5,6]])
sage: T.descent_set()
Combinatorics, Release 10.0

(continued from previous page)

\[ [2, 4] \]
sage: U = IncreasingTableau([[1,3,4],[2,4,5]])
sage: U.descent_set()
\[ [1, 3, 4] \]
sage: V = IncreasingTableau([[1,3,4],[3,4,5],[4,5]])
sage: V.descent_set()
\[ [3, 4] \]

dual_K_evacuation(ceiling=None)

Return the dual K-evacuation involution applied to self.

EXAMPLES:

sage: T = IncreasingTableau([[1,3,4],[2,4,5]])
sage: T.dual_K_evacuation()
\[ [[1, 2, 4], [2, 3, 5]] \]
sage: T.dual_K_evacuation(6)
\[ [[2, 3, 5], [3, 4, 6]] \]
sage: U = IncreasingTableau([[1,3,4],[3,4,5],[5]])
sage: U.dual_K_evacuation()
\[ [[1, 2, 3], [2, 3, 5], [3]] \]

class sage.combinat.tableau.IncreasingTableaux(**kwds)

Bases: Tableaux

A factory class for the various classes of increasing tableaux.

An increasing tableau is a tableau whose entries are positive integers that are strictly increasing across rows and strictly increasing down columns. Note that Sage uses the English convention for partitions and tableaux; the longer rows are displayed on top.

INPUT:

Keyword arguments:
- size – the size of the tableaux
- shape – the shape of the tableaux
- eval – the weight (also called binary content) of the tableaux, where values can be either 0 or 1 with position \( i \) being 1 if and only if \( i \) can appear in the tableaux
- max_entry – positive integer or infinity (\( \infty \)); the maximum entry for the tableaux; if size or shape are specified, max_entry defaults to be size or the size of shape

Positional arguments:
- the first argument is interpreted as either size or shape according to whether it is an integer or a partition
- the second keyword argument will always be interpreted as eval

**Warning:** The eval is not the usual notion of eval or weight, where the \( i \)-th entry counts how many \( i \)'s appear in the tableau.

EXAMPLES:
sage: IT = IncreasingTableaux([2,1]); IT
Increasing tableaux of shape [2, 1] and maximum entry 3
sage: IT.list()
[[[1, 3], [2]], [[1, 2], [3]], [[1, 2], [2]], [[1, 3], [3]], [[2, 3], [3]]]

sage: IT = IncreasingTableaux(3); IT
Increasing tableaux of size 3 and maximum entry 3
sage: IT.list()
[[[1, 2, 3]],
 [[[1, 3], [2]],
 [[[1, 2], [3]],
 [[[1, 2], [2]],
 [[[1, 3], [3]],
 [[[2, 3], [3]],
 [[[1], [2], [3]]]]

sage: IT = IncreasingTableaux(3, max_entry=2); IT
Increasing tableaux of size 3 and maximum entry 2
sage: IT.list()
[[[1, 2], [2]]]

sage: IT = IncreasingTableaux(3, max_entry=4); IT
Increasing tableaux of size 3 and maximum entry 4
sage: IT.list()
[[[1, 2, 3]],
 [[[1, 2, 4]],
 [[[1, 3, 4]],
 [[[2, 3, 4]],
 [[[1, 3], [2]],
 [[[1, 2], [3]],
 [[[1, 2], [2]],
 [[[1, 4], [3]],
 [[[1, 3], [4]],
 [[[1, 3], [3]],
 [[[1, 4], [4]],
 [[[2, 4], [3]],
 [[[2, 3], [4]],
 [[[2, 3], [3]],
 [[[2, 4], [4]],
 [[[3, 4], [4]],
 [[[1], [2], [3]],
 [[[1], [2], [4]],
 [[[1], [3], [4]],
 [[[2], [3], [4]],
 [[[2], [3], [3]],
 [[[2], [4], [4]],
 [[[3], [4], [4]],
 [[[1], [2], [3]],
 [[[1], [2], [4]],
 [[[1], [3], [4]],
 [[[2], [3], [4]],

sage: IT = IncreasingTableaux(3, max_entry=oo); IT
Increasing tableaux of size 3
sage: IT[123]
[[5, 7], [6]]
sage: list(IT)
[[], [[1]], [[2]], [[1, 2]], [[1], [2]]]
sage: IT[4]
[[1], [2]]
sage: IncreasingTableaux()[0]
[]

See also:

- Tableaux
- Tableau
- SemistandardTableaux
- SemistandardTableau
- StandardTableaux
- StandardTableau
- IncreasingTableau

Element
alias of IncreasingTableau

class sage.combinat.tableau.IncreasingTableaux_all(max_entry=None)
Bases: IncreasingTableaux, DisjointUnionEnumeratedSets

All increasing tableaux.

EXAMPLES:

sage: T = IncreasingTableaux()
sage: T.cardinality()
+Infinity

sage: T = IncreasingTableaux(max_entry=3)
sage: list(T)
[[],
 [1],
 [2],
 [3],
 [1, 2],
 [1, 3],
 [2, 3],
 [1], [2],
 [1], [3],
 [2], [3],
 [1, 2, 3],
 [1, 3], [2],
 [1, 2], [3],
 [1, 2], [2],
 [1, 3], [3],
 [2, 3], [3],
 [1], [2], [3]]
class sage.combinat.tableau.IncreasingTableaux_shape(p, max_entry=None)

Bases: IncreasingTableaux

Increasing tableaux of fixed shape \( p \) with a given max entry.

An increasing tableau with max entry \( i \) is required to have all its entries less or equal to \( i \). It is not required to actually contain an entry \( i \).

INPUT:

- \( p \) – a partition
- \( \text{max\_entry} \) – the max entry; defaults to the size of \( p \)

class sage.combinat.tableau.IncreasingTableaux_shape_inf(p)

Bases: IncreasingTableaux

Increasing tableaux of fixed shape \( p \) and no maximum entry.

class sage.combinat.tableau.IncreasingTableaux_shape_weight(p, wt)

Bases: IncreasingTableaux_shape

Increasing tableaux of fixed shape \( p \) and binary weight \( wt \).

class sage.combinat.tableau.IncreasingTableaux_size(n, max_entry=None)

Bases: IncreasingTableaux

Increasing tableaux of fixed size \( n \).

class sage.combinat.tableau.IncreasingTableaux_size_inf(n)

Bases: IncreasingTableaux

Increasing tableaux of fixed size \( n \) with no maximum entry.

class sage.combinat.tableau.IncreasingTableaux_size_weight(n, wt)

Bases: IncreasingTableaux

Increasing tableaux of fixed size \( n \) and weight \( wt \).

class sage.combinat.tableau.RowStandardTableau(parent, t, check=True)

Bases: Tableau

A class to model a row standard tableau.

A row standard tableau is a tableau whose entries are positive integers from 1 to \( m \) that increase along rows.

INPUT:

- \( t \) – a Tableau, a list of iterables, or an empty list

EXAMPLES:

```
sage: t = RowStandardTableau([[3,4,5],[1,2]]); t
[[3, 4, 5], [1, 2]]
sage: t.shape()
[3, 2]
sage: t.pp()  # pretty printing
3 4 5
1 2
sage: t.is_standard()
False
sage: RowStandardTableau([])  # The empty tableau
sage.combinat.tableau.RowStandardTableau([[]])
```
When using code that will generate a lot of tableaux, it is more efficient to construct a `RowStandardTableau` from the appropriate `Parent` object:

```python
sage: ST = RowStandardTableaux()
sage: ST([[3, 4, 5], [1, 2]])
[[3, 4, 5], [1, 2]]
```

See also:

- `Tableau`
- `StandardTableau`
- `SemistandardTableau`
- `Tableaux`
- `StandardTableaux`
- `RowStandardTableaux`
- `SemistandardTableaux`

```python
class sage.combinat.tableau.RowStandardTableaux
Bases: Tableaux

A factory for the various classes of row standard tableaux.

INPUT:
- either a non-negative integer (possibly specified with the keyword `n`) or a partition

OUTPUT:
- with no argument, the class of all standard tableaux
- with a non-negative integer argument, `n`, the class of all standard tableaux of size `n`
- with a partition argument, the class of all standard tableaux of that shape

A row standard tableau is a tableau that contains each of the entries from 1 to `n` exactly once and is increasing along rows.

All classes of row standard tableaux are iterable.

EXAMPLES:

```python
sage: ST = RowStandardTableaux(3); ST
Row standard tableaux of size 3
sage: ST.first()
[[1, 2, 3]]
```
sage: ST.last()
[[3], [1], [2]]
sage: ST.cardinality()
10
sage: ST.list()
[[[1, 2, 3]],
 [[2, 3], [1]],
 [[1, 2], [3]],
 [[1, 3], [2]],
 [[3], [2], [1]],
 [[2], [3], [1]],
 [[1], [3], [2]],
 [[1], [2], [3]],
 [[2], [1], [3]],
 [[3], [1], [2]]]

See also:

- `Tableaux`
- `Tableau`
- `SemistandardTableaux`
- `SemistandardTableau`
- `RowStandardTableau`
- `StandardSkewTableaux`

**Element**

alias of `RowStandardTableau`

class sage.combinat.tableau.RowStandardTableaux_all

Bases: `RowStandardTableaux`, `DisjointUnionEnumeratedSets`

All row standard tableaux.

class sage.combinat.tableau.RowStandardTableaux_shape(p)

Bases: `RowStandardTableaux`

Row Standard tableaux of a fixed shape p.

cardinality()

Return the number of row standard tableaux of this shape.

This is just the index of the corresponding Young subgroup in the full symmetric group.

EXAMPLES:

sage: RowStandardTableaux([3,2,1]).cardinality()
60
sage: RowStandardTableaux([2,2]).cardinality()
6
sage: RowStandardTableaux([5]).cardinality()
1
sage: RowStandardTableaux([6,5,5,3]).cardinality()
class sage.combinat.tableau.RowStandardTableaux_size(n)

Bases: RowStandardTableaux, DisjointUnionEnumeratedSets

Row standard tableaux of fixed size \(n\).

EXAMPLES:

```python
sage: [ t for t in RowStandardTableaux(1) ]
[[[1]]]
sage: [ t for t in RowStandardTableaux(2) ]
[[[1, 2]], [[2], [1]], [[1], [2]]]
sage: list(RowStandardTableaux(3))
[[[1, 2, 3]], 
[[2, 3], [1]], 
[[1, 2], [3]], 
[[1, 3], [2]], 
[[3], [2], [1]], 
[[2], [3], [1]], 
[[1], [3], [2]], 
[[1], [2], [3]], 
[[2], [1], [3]], 
[[3], [1], [2]]]
```

an_element()

Return a particular element of the class.

EXAMPLES:

```python
sage: RowStandardTableaux(4).an_element()
[[1, 2, 3, 4]]
```

class sage.combinat.tableau.SemistandardTableau(parent, t, check=True)

A class to model a semistandard tableau.

INPUT:

- \(t\) – a tableau, a list of iterables, or an empty list

OUTPUT:

- A SemistandardTableau object constructed from \(t\).

A semistandard tableau is a tableau whose entries are positive integers, which are weakly increasing in rows and strictly increasing down columns.

EXAMPLES:

```python
sage: t = SemistandardTableau([[1,2,3],[2,3]]); t
[[1, 2, 3], [2, 3]]
sage: t.shape()
[3, 2]
```
When using code that will generate a lot of tableaux, it is slightly more efficient to construct a SemistandardTableau from the appropriate Parent object:

```
sage: SST = SemistandardTableaux()
sage: SST([[1, 2, 3], [4, 5]])
[[[1, 2, 3], [4, 5]]]
```

See also:

- Tableaux
- Tableau
- SemistandardTableaux
- StandardTableaux
- StandardTableau

```python
class sage.combinat.tableau.SemistandardTableaux(**kwds):
    Bases: Tableaux
    A factory class for the various classes of semistandard tableaux.
    
    INPUT:
    
    Keyword arguments:
    - `size` – The size of the tableaux
    - `shape` – The shape of the tableaux
    - `eval` – The weight (also called content or evaluation) of the tableaux
    - `max_entry` – A maximum entry for the tableaux. This can be a positive integer or infinity (oo). If `size` or `shape` are specified, `max_entry` defaults to be `size` or the size of `shape`.

    Positional arguments:
    - The first argument is interpreted as either `size` or `shape` according to whether it is an integer or a partition
    - The second keyword argument will always be interpreted as `eval`

    OUTPUT:
    - The appropriate class, after checking basic consistency tests. (For example, specifying `eval` implies a value for `max_entry`).
```
A semistandard tableau is a tableau whose entries are positive integers, which are weakly increasing in rows and strictly increasing down columns. Note that Sage uses the English convention for partitions and tableaux; the longer rows are displayed on top.

Classes of semistandard tableaux can be iterated over if and only if there is some restriction.

EXAMPLES:

```
sage: SST = SemistandardTableaux([2, 1]); SST
Semistandard tableaux of shape [2, 1] and maximum entry 3

sage: SST.list()
[[[1, 1], [2]],
 [1, 1], [3]],
 [[1, 2], [2]],
 [[1, 2], [3]],
 [[1, 3], [2]],
 [[1, 3], [3]],
 [[2, 1], [2]],
 [[2, 1], [3]],
 [[2, 1], [2]],
 [[2, 2], [3]],
 [[2, 3], [3]]]

sage: SST = SemistandardTableaux(3); SST
Semistandard tableaux of size 3 and maximum entry 3

sage: SST.list()
[[[1, 1, 1]],
 [[1, 1, 2]],
 [[1, 1, 3]],
 [[1, 2, 2]],
 [[1, 2, 3]],
 [[1, 3, 3]],
 [[2, 2, 2]],
 [[2, 2, 3]],
 [[2, 3, 3]],
 [[3, 3, 3]],
 [[1, 1], [2]],
 [[1, 2], [2]],
 [[1, 2], [3]],
 [[1, 3], [2]],
 [[1, 3], [3]],
 [[2, 2], [2]],
 [[2, 2], [3]],
 [[2, 3], [2]],
 [[2, 3], [3]],
 [[1], [2], [3]]]

sage: SST = SemistandardTableaux(3, max_entry=2); SST
Semistandard tableaux of size 3 and maximum entry 2

sage: SST.list()
[[[1, 1, 1]],
 [[1, 1, 2]],
 [[1, 2, 2]],
 [[2, 2, 2]],
 [[1, 1], [2]],
 [[1, 2], [2]],
 [[1, 2], [2]]]

sage: SST = SemistandardTableaux(3, max_entry=oo); SST

(continues on next page)
Semistandard tableaux of size 3

sage: SST[123]
[[3, 4], [6]]

sage: SemistandardTableaux(max_entry=2)[11]
[[1, 1], [2]]

sage: SemistandardTableaux()[0]
[]

See also:

- Tableaux
- Tableau
- SemistandardTableau
- StandardTableaux
- StandardTableau

Element

alias of SemistandardTableau

class sage.combinat.tableau.SemistandardTableaux_all(max_entry=None)

Bases: SemistandardTableaux, DisjointUnionEnumeratedSets

All semistandard tableaux.

list()

class sage.combinat.tableau.SemistandardTableaux_shape(p, max_entry=None)

Bases: SemistandardTableaux

Semistandard tableaux of fixed shape \( p \) with a given max entry.

A semistandard tableau with max entry \( i \) is required to have all its entries less or equal to \( i \). It is not required to actually contain an entry \( i \).

INPUT:

- \( p \) – a partition
- max_entry – the max entry; defaults to the size of \( p \)

cardinality(algorithm='hook')

Return the cardinality of self.

INPUT:

- algorithm – (default: 'hook') any one of the following:
  - 'hook' – use Stanley’s hook length formula
  - 'sum' – sum over the compositions of \( \text{max}_\text{entry} \) the number of semistandard tableau with shape and given weight vector
This is computed using Stanley's hook length formula:

\[ f_\lambda = \prod_{u \in \lambda} \frac{n + c(u)}{h(u)}. \]

where \( n \) is the max_entry, \( c(u) \) is the content of \( u \), and \( h(u) \) is the hook length of \( u \). See [Sta-EC2] Corollary 7.21.4.

EXAMPLES:

```python
sage: SemistandardTableaux([2,1]).cardinality()
8
sage: SemistandardTableaux([2,2,1]).cardinality()
75
sage: SymmetricFunctions(QQ).schur()([2,2,1]).expand(5)(1,1,1,1,1) # cross check
75
sage: SemistandardTableaux([5]).cardinality()
126
sage: SemistandardTableaux([3,2,1]).cardinality()
896
sage: SemistandardTableaux([3,2,1], max_entry=7).cardinality()
2352
sage: SemistandardTableaux([6,5,4,3,2,1], max_entry=30).cardinality()
20836107592001331200
sage: ssts = [SemistandardTableaux(p, max_entry=6) for p in Partitions(5)]
sage: all(sst.cardinality() == sst.cardinality(algorithm='sum')
      for sst in ssts)
True
```

random_element() 

Return a uniformly distributed random tableau of the given shape and max_entry.

Uses the algorithm from [Kra1999] based on the Novelli-Pak-Stoyanovskii bijection

EXAMPLES:

```python
sage: S = SemistandardTableaux([2, 2, 1, 1])
sage: S.random_element() in S
True
sage: S = SemistandardTableaux([2, 2, 1, 1], max_entry=7)
sage: S.random_element() in S
True
```

class sage.combinat.tableau.SemistandardTableaux_shape_inf(p) 

Bases: SemistandardTableaux 

Semistandard tableaux of fixed shape \( p \) and no maximum entry.

class sage.combinat.tableau.SemistandardTableaux_shape_weight(p, mu) 

Bases: SemistandardTableaux_shape 

Semistandard tableaux of fixed shape \( p \) and weight \( \mu \).

cardinality() 

Return the number of semistandard tableaux of the given shape and weight, as computed by kostka_number function of symmetrica.

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EXAMPLES:

```python
sage: SemistandardTableaux([2,2], [2, 1, 1]).cardinality()
1
sage: SemistandardTableaux([2,2,2], [2, 2, 1,1]).cardinality()
1
sage: SemistandardTableaux([2,2,2], [2, 2, 2]).cardinality()
1
sage: SemistandardTableaux([3,2,1], [2, 2, 2]).cardinality()
2
```

```
list()
```

Return a list of all semistandard tableaux in self generated by symmetrica.

EXAMPLES:

```python
sage: SemistandardTableaux([2,2], [2, 1, 1]).list()
[[[1, 1], [2, 3]]]
```

```python
sage: SemistandardTableaux([2,2,2], [2, 2, 1,1]).list()
[[[1, 1], [2, 2], [3, 4]]]
```

```python
sage: SemistandardTableaux([2,2,2], [2, 2, 2]).list()
[[[1, 1], [2, 2], [3, 3]]]
```

```python
sage: SemistandardTableaux([3,2,1], [2, 2, 2]).list()
[[[1, 1, 2], [2, 3], [3]], [[1, 1, 3], [2, 2], [3]]]
```

```
class sage.combinat.tableau.SemistandardTableaux_size(n, max_entry=None)
```

Bases: SemistandardTableaux

Semistandard tableaux of fixed size n.

```python
cardinality()
```

Return the cardinality of self.

EXAMPLES:

```python
sage: SemistandardTableaux(3).cardinality()
19
```

```python
sage: SemistandardTableaux(4).cardinality()
116
```

```python
sage: SemistandardTableaux(4, max_entry=2).cardinality()
9
```

```python
sage: SemistandardTableaux(4, max_entry=10).cardinality()
4225
```

```python
sage: ns = list(range(1, 6))
sage: ssts = [ SemistandardTableaux(n) for n in ns ]
sage: all(sst.cardinality() == len(sst.list()) for sst in ssts)
True
```

```
random_element()
```

Generate a random SemistandardTableau with uniform probability.

The RSK algorithm gives a bijection between symmetric $k \times k$ matrices of nonnegative integers that sum to $n$ and semistandard tableaux with size $n$ and maximum entry $k$.

The number of $k \times k$ symmetric matrices of nonnegative integers having sum of elements on the diagonal $i$ and sum of elements above the diagonal $j$ is $\binom{k+i-1}{k-1} \binom{(i+j-1)}{(i-1)}$. We first choose the sum of the elements
on the diagonal randomly weighted by the number of matrices having that trace. We then create random integer vectors of length \( k \) having that sum and use them to generate a \( k \times k \) diagonal matrix. Then we take a random integer vector of length \( \binom{k}{2} \) summing to half the remainder and distribute it symmetrically to the remainder of the matrix.

Applying RSK to the random symmetric matrix gives us a pair of identical \textit{SemistandardTableau} of which we choose the first.

EXAMPLES:

```python
sage: SemistandardTableaux(6).random_element() # random
[[1, 1, 2], [3, 5, 5]]
sage: SemistandardTableaux(6, max_entry=7).random_element() # random
[[2, 4, 4, 6, 6, 6]]
```

**class** \texttt{sage.combinat.tableau.SemistandardTableaux\_size\_inf}(\(n\))

Bases: \texttt{SemistandardTableaux}

Semistandard tableaux of fixed size \( n \) with no maximum entry.

\texttt{list}()

**class** \texttt{sage.combinat.tableau.SemistandardTableaux\_size\_weight}(\(n, \mu\))

Bases: \texttt{SemistandardTableaux}

Semistandard tableaux of fixed size \( n \) and weight \( \mu \).

\texttt{cardinality}()

Return the cardinality of self.

EXAMPLES:

```python
sage: SemistandardTableaux(3, [2,1]).cardinality()
2
sage: SemistandardTableaux(4, [2,2]).cardinality()
3
```

**class** \texttt{sage.combinat.tableau.StandardTableau}(\textit{parent}, \textit{t}, \texttt{check}=\texttt{True})

Bases: \texttt{SemistandardTableau}

A class to model a standard tableau.

\texttt{INPUT}:

- \textit{t} – a Tableau, a list of iterables, or an empty list

A standard tableau is a semistandard tableau whose entries are exactly the positive integers from 1 to \( n \), where \( n \) is the size of the tableau.

EXAMPLES:

```python
sage: t = StandardTableau([[1,2,3],[4,5]]); t
[[1, 2, 3], [4, 5]]
sage: t.shape()
[3, 2]
sage: t.pp() # pretty printing
1 2 3
4 5
sage: t.is_standard()
```

(continues on next page)
True
sage: StandardTableau([])  # The empty tableau
[]
sage: StandardTableau([[1,2,3],[4,5]]) in RowStandardTableaux()
True

When using code that will generate a lot of tableaux, it is more efficient to construct a StandardTableau from the appropriate Parent object:

sage: ST = StandardTableaux()
sage: ST([[1, 2, 3], [4, 5]])
[[1, 2, 3], [4, 5]]

See also:

- Tableaux
- Tableau
- SemistandardTableaux
- SemistandardTableau
- StandardTableaux

check()

Check that self is a standard tableau.

dominates(t)

Return True if self dominates the tableau t.

That is, if the shape of the tableau restricted to \( k \) dominates the shape of t restricted to \( k \), for \( k = 1, 2, \ldots, n \).

When the two tableaux have the same shape, then this ordering coincides with the Bruhat ordering for the corresponding permutations.

INPUT:

- t – a tableau

EXAMPLES:

sage: s = StandardTableau([[1,2,3],[4,5]])
sage: t = StandardTableau([[1,2],[3,5],[4]])
sage: s.dominates(t)
True
sage: t.dominates(s)
False
sage: all(StandardTableau(s).dominates(t) for t in StandardTableaux([3,2]))
True
sage: s.dominates([[1,2,3,4,5]])
False

down()

An iterator for all the standard tableaux that can be obtained from self by removing a cell. Note that this iterates just over a single tableau (or nothing if self is empty).

EXAMPLES:
Combinatorics, Release 10.0

```
sage: t = StandardTableau([[1,2],[3]])
sage: [x for x in t.down()]  
[[[1, 2]]]
sage: t = StandardTableau([])  
sage: [x for x in t.down()]  
[]
```

down_list()

Return a list of all the standard tableaux that can be obtained from `self` by removing a cell. Note that this is just a singleton list if `self` is nonempty, and an empty list otherwise.

EXAMPLES:
```
sage: t = StandardTableau([[1,2],[3]])
sage: t.down_list()  
[[[1, 2]]]
sage: t = StandardTableau([])  
sage: t.down_list()  
[]
```

is_standard()

Return `True` since `self` is a standard tableau.

EXAMPLES:
```
sage: StandardTableau([[1, 3], [2, 4]]).is_standard()  
True
```

promotion(\(n=None\))

Return the image of `self` under the promotion operator.

The promotion operator, applied to a standard tableau \(t\), does the following:

Remove the letter \(n\) from \(t\), thus leaving a hole where it used to be. Apply jeu de taquin to move this hole southwest (in French notation) until it reaches the inner boundary of \(t\). Fill 0 into the hole once jeu de taquin has completed. Finally, add 1 to each letter in the tableau. The resulting standard tableau is the image of \(t\) under the promotion operator.

This definition of promotion is precisely the one given in [Hai1992] (p. 90). It is the inverse of the maps called “promotion” in [Sag2011] (p. 23) and in [Stan2009].

See the promotion() method for a more general operator.

EXAMPLES:
```
sage: ST = StandardTableaux(7)  
sage: all( st.promotion().promotion_inverse() == st for st in ST )  
# long time  
True
sage: all( st.promotion_inverse().promotion() == st for st in ST )  
# long time  
True
sage: st = StandardTableau([[1,2,5],[3,4]])  
sage: parent(st.promotion())  
Standard tableaux
```

promotion_inverse(\(n=None\))

Return the image of `self` under the inverse promotion operator. The optional variable \(m\) should be set to the size of `self` minus 1 for a minimal speedup; otherwise, it defaults to this number.

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The inverse promotion operator, applied to a standard tableau \( t \), does the following:

Remove the letter 1 from \( t \), thus leaving a hole where it used to be. Apply jeu de taquin to move this hole northeast (in French notation) until it reaches the outer boundary of \( t \). Fill \( n + 1 \) into this hole, where \( n \) is the size of \( t \). Finally, subtract 1 from each letter in the tableau. This yields a new standard tableau.

This definition of inverse promotion is the map called “promotion” in [Sag2011] (p. 23) and in [Stan2009], and is the inverse of the map called “promotion” in [Hai1992] (p. 90).

See the \texttt{promotion_inverse()} method for a more general operator.

**EXAMPLES:**

```python
sage: t = StandardTableau([[1,3],[2,4]])
sage: t.promotion_inverse()=[[1, 2], [3, 4]]
```

We check the equivalence of two definitions of inverse promotion on standard tableaux:

```python
sage: ST = StandardTableaux(7)
sage: def bk_promotion_inverse7(st):
....:  st2 = st
....:  for i in range(1, 7):
....:      st2 = st2.bender_knuth_involution(i, check=False)
....:  return st2
sage: all( bk_promotion_inverse7(st) == st.promotion_inverse() for st in ST )  # long time
True
```

**standard_descents()**

Return a list of the integers \( i \) such that \( i \) appears strictly further north than \( i + 1 \) in \( self \) (this is not to say that \( i \) and \( i + 1 \) must be in the same column). The list is sorted in increasing order.

**EXAMPLES:**

```python
sage: StandardTableau( [[1,3,4],[2,5]] ).standard_descents() [1, 4]
sage: StandardTableau( [[1,2],[3,4]] ).standard_descents() [2]
sage: StandardTableau( [[1,2,5],[3,4],[6,7],[8],[9]] ).standard_descents() [2, 5, 7, 8]
sage: StandardTableau( [] ).standard_descents() []
```

**standard_major_index()**

Return the major index of the standard tableau \( self \) in the standard meaning of the word. The major index is defined to be the sum of the descents of \( self \) (see \texttt{standard_descents()} for their definition).

**EXAMPLES:**

```python
sage: StandardTableau( [[1,4,5],[2,6],[3]] ).standard_major_index() 8
sage: StandardTableau( [[1,2],[3,4]] ).standard_major_index() 2
sage: StandardTableau( [[1,2,3],[4,5]] ).standard_major_index() 3
```
**standard_number_of_descents()**

Return the number of all integers \( i \) such that \( i \) appears strictly further north than \( i + 1 \) in \( \text{self} \) (this is not to say that \( i \) and \( i + 1 \) must be in the same column). A list of these integers can be obtained using the `standard_descents()` method.

**EXAMPLES:**

```python
sage: StandardTableau( [[1,2],[3,4],[5]] ).standard_number_of_descents()
2
sage: StandardTableau( [] ).standard_number_of_descents()
0
sage: tabs = StandardTableaux(5)
sage: all( t.standard_number_of_descents() == t.schuetzenberger_involution().schuetzenberger_involution().standard_number_of_descents() for t in tabs )
True
```

**up()**

An iterator for all the standard tableaux that can be obtained from \( \text{self} \) by adding a cell.

**EXAMPLES:**

```python
sage: t = StandardTableau([[1,2]])
sage: [x for x in t.up()]
[[[1, 2, 3]], [[1, 2], [3]]]
```

**up_list()**

Return a list of all the standard tableaux that can be obtained from \( \text{self} \) by adding a cell.

**EXAMPLES:**

```python
sage: t = StandardTableau([[1,2]])
sage: t.up_list()
[[[1, 2, 3]], [[1, 2], [3]]]
```

**class sage.combinat.tableau.StandardTableaux(**kwds)**

Bases: `SemistandardTableaux`

A factory for the various classes of standard tableaux.

**INPUT:**

- Either a non-negative integer (possibly specified with the keyword \( n \)) or a partition.

**OUTPUT:**

- With no argument, the class of all standard tableaux
- With a non-negative integer argument, \( n \), the class of all standard tableaux of size \( n \)
- With a partition argument, the class of all standard tableaux of that shape.

A standard tableau is a semistandard tableau which contains each of the entries from 1 to \( n \) exactly once.

All classes of standard tableaux are iterable.

**EXAMPLES:**

```python
sage: ST = StandardTableaux(3); ST
Standard tableaux of size 3
```
sage: ST.first()
[[1, 2, 3]]
sage: ST.last()
[[1], [2], [3]]
sage: ST.cardinality()
4
tage: ST.list()
[[[1, 2, 3]], [[1, 3], [2]], [[1, 2], [3]], [[1], [2], [3]]]

See also:

- Tableaux
- Tableau
- SemistandardTableaux
- SemistandardTableau
- StandardTableau
- StandardSkewTableaux

Element

alias of StandardTableau
class sage.combinat.tableau.StandardTableaux_all

Bases: StandardTableaux, DisjointUnionEnumeratedSets

All standard tableaux.
class sage.combinat.tableau.StandardTableaux_shape(p)

Bases: StandardTableaux

Semistandard tableaux of a fixed shape $p$.
cardinality()

Return the number of standard Young tableaux of this shape.

This method uses the so-called hook length formula, a formula for the number of Young tableaux associated with a given partition. The formula says the following: Let $\lambda$ be a partition. For each cell $c$ of the Young diagram of $\lambda$, let the hook length of $c$ be defined as 1 plus the number of cells horizontally to the right of $c$ plus the number of cells vertically below $c$. The number of standard Young tableaux of shape $\lambda$ is then $n!$ divided by the product of the hook lengths of the shape of $\lambda$, where $n = |\lambda|$.

For example, consider the partition $[3, 2, 1]$ of 6 with Ferrers diagram:

```
# # #
# #
#
```

When we fill in the cells with their respective hook lengths, we obtain:

```
5 3 1
3 1
1
```
The hook length formula returns

\[
\frac{6!}{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 16.
\]

EXAMPES:

```
sage: StandardTableaux([3,2,1]).cardinality()
sage: StandardTableaux([2,2]).cardinality()
sage: StandardTableaux([5]).cardinality()
sage: StandardTableaux([6,5,5,3]).cardinality()
sage: StandardTableaux([]).cardinality()
```

REFERENCES:


**list()**

Return a list of the standard Young tableaux of the specified shape.

```
sage: StandardTableaux([2,2]).list()
sage: StandardTableaux([5]).list()
sage: StandardTableaux([3,2,1]).list()
```

**random_element()**

Return a random standard tableau of the given shape using the Greene-Nijenhuis-Wilf Algorithm.

```
sage: t = StandardTableaux([2,2]).random_element()
sage: t.shape()
```

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[2, 2]
sage: StandardTableaux([]).random_element()
[]

class sage.combinat.tableau.StandardTableaux_size(n)
    Bases: StandardTableaux, DisjointUnionEnumeratedSets
    Standard tableaux of fixed size n.

    EXAMPLES:

    sage: [ t for t in StandardTableaux(1) ]
    [[[1]]]
    sage: [ t for t in StandardTableaux(2) ]
    [[[1, 2]], [[1], [2]]]
    sage: [ t for t in StandardTableaux(3) ]
    [[[1, 2, 3]], [[1, 3], [2]], [[1, 2], [3]], [[1], [2], [3]]]
    sage: StandardTableaux(4)[:]
    [[[1, 2, 3, 4]],
     [[1, 3, 4], [2]],
     [[1, 2, 4], [3]],
     [[1, 2, 3], [4]],
     [[1, 3], [2, 4]],
     [[1, 2], [3, 4]],
     [[1, 4], [2], [3]],
     [[1, 3], [2], [4]],
     [[1, 2], [3], [4]],
     [[1], [2], [3], [4]]]

cardinality()
    Return the number of all standard tableaux of size n.

    The number of standard tableaux of size n is equal to the number
    of involutions in the symmetric group Sn. This is a consequence
    of the symmetry of the RSK correspondence, that if \( \sigma \mapsto (P, Q) \),
    then \( \sigma^{-1} \mapsto (Q, P) \). For more information, see Wikipedia
    article Robinson-Schensted-Knuth_correspondence#Symmetry.

    ALGORITHM:

    The algorithm uses the fact that standard tableaux of size n are in
    bijection with the involutions of size n, (see page 41 in section 4.1 of [Ful1997]).
    For each number of fixed points, you count the number of ways
    to choose those fixed points multiplied by the number of perfect
    matchings on the remaining values.

    EXAMPLES:

    sage: StandardTableaux(3).cardinality()
    4
    sage: ns = [1,2,3,4,5,6]
    sage: sts = [StandardTableaux(n) for n in ns]
    sage: all(st.cardinality() == len(st.list()) for st in sts)
    False

    The cardinality can be computed without constructing all elements in this set, so this computation is fast
    (see also github issue #28273):
random_element()

Return a random StandardTableau with uniform probability.

This algorithm uses the fact that the Robinson-Schensted correspondence returns a pair of identical standard
Young tableaux (SYTs) if and only if the permutation was an involution. Thus, generating a random SYT
is equivalent to generating a random involution.

To generate an involution, we first need to choose its number of fixed points $k$ (if the size of the involution is
even, the number of fixed points will be even, and if the size is odd, the number of fixed points will be odd).
To do this, we choose a random integer $r$ between 0 and the number $N$ of all involutions of size $n$. We then
decompose the interval $\{1, 2, \ldots, N\}$ into subintervals whose lengths are the numbers of involutions of
size $n$ with respectively $0, 1, \ldots, \lfloor N/2 \rfloor$ fixed points. The interval in which our random integer $r$ lies then
decides how many fixed points our random involution will have. We then place those fixed points randomly
and then compute a perfect matching (an involution without fixed points) on the remaining values.

EXAMPLES:

```
sage: StandardTableaux(10).random_element()  # random
[[1, 3, 6], [2, 5, 7], [4, 8], [9], [10]]
sage: StandardTableaux(0).random_element()
[]
sage: StandardTableaux(1).random_element()
[[1]]
```

class sage.combinat.tableau.Tableau(parent, t, check=True)

Bases: ClonableList

A class to model a tableau.

INPUT:

- $t$ – a Tableau, a list of iterables, or an empty list

OUTPUT:

- A Tableau object constructed from $t$.

A tableau is abstractly a mapping from the cells in a partition to arbitrary objects (called entries). It is often
represented as a finite list of nonempty lists (or, more generally an iterator of iterables) of weakly decreasing
lengths. This list, in particular, can be empty, representing the empty tableau.

Note that Sage uses the English convention for partitions and tableaux; the longer rows are displayed on top.

EXAMPLES:

```
sage: t = Tableau([[1,2,3],[4,5]]); t
[[1, 2, 3], [4, 5]]
sage: t.shape()
[3, 2]
sage: t.pp()  # pretty printing
 1 2 3
 4 5
sage: t.is_standard()
True
```

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When using code that will generate a lot of tableaux, it is slightly more efficient to construct a Tableau from the appropriate Parent object:

```python
sage: T = Tableaux()
sage: T([[1, 2, 3], [4, 5]])
[[1, 2, 3], [4, 5]]
```

See also:
- `Tableaux`
- `SemistandardTableaux`
- `SemistandardTableau`
- `StandardTableaux`
- `StandardTableau`

**add_entry**(`cell`, `m`)

Return the result of setting the entry in cell `cell` equal to `m` in the tableau `self`.

This tableau has larger size than `self` if `cell` does not belong to the shape of `self`; otherwise, the tableau has the same shape as `self` and has the appropriate entry replaced.

**INPUT:**
- `cell` – a pair of nonnegative integers

**OUTPUT:**

The tableau `self` with the entry in cell `cell` set to `m`. This entry overwrites an existing entry if `cell` already belongs to `self`, or is added to the tableau if `cell` is a cocorner of the shape `self`. (Either way, the input is not modified.)

**Note:** Both coordinates of `cell` are interpreted as starting at 0. So, `cell == (0, 0)` corresponds to the northwesternmost cell.

**EXAMPLES:**

```python
sage: s = StandardTableau([[1,2,5],[3,4]]); s.pp()
  1  2  5
  3  4
sage: t = s.add_entry( (1,2), 6); t.pp()
  1  2  5
  3  4  6
sage: t.category()
Category of elements of Standard tableaux
sage: s.add_entry( (2,0), 6).pp()
  1  2  5
  3  4
```
**anti_restrict**($n$)

Return the skew tableau formed by removing all of the cells from `self` that are filled with a number at most $n$.

**EXAMPLES:**

```python
sage: t = Tableau([[1,2,3],[4,5]]); t
[[1, 2, 3], [4, 5]]
sage: t.anti_restrict(1)
[[None, 2, 3], [4, 5]]
sage: t.anti_restrict(2)
[[None, None, 3], [4, 5]]
sage: t.anti_restrict(3)
[[None, None, None], [4, 5]]
sage: t.anti_restrict(4)
[[None, None, None], [None, 5]]
sage: t.anti_restrict(5)
[[None, None, None], [None, None]]
```

**atom()**

**EXAMPLES:**

```python
sage: Tableau([[1,2],[3,4]]).atom()
[2, 2]
sage: Tableau([[1,2,3],[4,5],[6]]).atom()
[3, 2, 1]
```

**bender_knuth_involution**($k$, rows=None, check=True)

Return the image of `self` under the $k$-th Bender–Knuth involution, assuming `self` is a semistandard tableau.

Let $T$ be a tableau, then a *lower free $k$ in $T$* means a cell of $T$ which is filled with the integer $k$ and whose direct lower neighbor is not filled with the integer $k + 1$ (in particular, this lower neighbor might not exist at all). Let an *upper free $k + 1$ in $T$* mean a cell of $T$ which is filled with the integer $k + 1$ and whose direct upper neighbor is not filled with the integer $k$ (in particular, this neighbor might not exist at all). It is clear that for any row $r$ of $T$, the lower free $k$'s and the upper free $k + 1$'s in $r$ together form a contiguous interval or $r$.

The `$k$-th Bender–Knuth switch at row $i$` changes the entries of the cells in this interval in such a way that if it used to have $a$ entries of $k$ and $b$ entries of $k + 1$, it will now have $b$ entries of $k$ and $a$ entries of $k + 1$. For fixed $k$, the $k$-th Bender–Knuth switches for different $i$ commute. The composition of the $k$-th Bender–Knuth switches for all rows is called the `$k$-th Bender-Knuth involution`. This is used to show that the Schur functions defined by semistandard tableaux are symmetric functions.
INPUT:

- k – an integer
- rows – (Default None) When set to None, the method computes the k-th Bender–Knuth involution as defined above. When an iterable, this computes the composition of the k-th Bender–Knuth switches at row i over all i in rows. When set to an integer i, the method computes the k-th Bender–Knuth switch at row i. Note the indexing of the rows starts with 1.
- check – (Default: True) Check to make sure self is semistandard. Set to False to avoid this check.

OUTPUT:

The image of self under either the k-th Bender–Knuth involution, the k-th Bender–Knuth switch at a certain row, or the composition of such switches, as detailed in the INPUT section.

EXAMPLES:

```
sage: t = Tableau([[1,1,3,4,5,6,7],[2,2,4,6,7,7,7],[3,4,5,8,8,9],[6,6,7,10],
    ... [7,8,8,11],[8]])
sage: t.bender_knuth_involution(1) == t
True
sage: t.bender_knuth_involution(2)
[[1, 1, 2, 4, 4, 5, 6, 7], [2, 3, 4, 6, 7, 7, 7], [3, 4, 5, 8, 8, 9], [6, 6, 7, 10], [7, 8, 8, 11], [8]]
sage: t.bender_knuth_involution(3)
[[1, 1, 3, 3, 3, 5, 6, 7], [2, 2, 4, 6, 7, 7, 7], [3, 4, 5, 8, 8, 9], [6, 6, 7, 10], [7, 8, 8, 11], [8]]
sage: t.bender_knuth_involution(4)
[[1, 1, 3, 4, 5, 5, 6, 7], [2, 2, 4, 6, 7, 7, 7], [3, 5, 5, 8, 8, 9], [6, 6, 7, 10], [7, 8, 8, 11], [8]]
sage: t.bender_knuth_involution(5)
[[1, 1, 3, 4, 4, 5, 6, 7], [2, 2, 4, 5, 7, 7, 7], [3, 4, 6, 8, 8, 9], [5, 5, 7, 10], [7, 8, 8, 11], [8]]
sage: t.bender_knuth_involution(666) == t
True
sage: t.bender_knuth_involution(4, 2) == t
True
sage: t.bender_knuth_involution(4, 3)
[[1, 1, 3, 4, 4, 5, 6, 7], [2, 2, 4, 6, 7, 7, 7], [3, 5, 5, 8, 8, 9], [6, 6, 7, 10], [7, 8, 8, 11], [8]]
```

The rows keyword can be an iterator:

```
sage: t.bender_knuth_involution(6, iter([1,2])) == t
False
sage: t.bender_knuth_involution(6, iter([3,4])) == t
True
```

The Bender–Knuth involution is an involution:

```
sage: T = SemistandardTableaux(shape=[3,1,1], max_entry=4)
sage: all(all(t.bender_knuth_involution(k).bender_knuth_involution(k) == t for k in range(1,5)) for t in T)
True
```

The same holds for the single switches:
Locality of the Bender–Knuth involutions:

\[
\text{sage: all(all(t.bender_knuth_involution(k).bender_knuth_involution(l) == t.bender_knuth_involution(l).bender_knuth_involution(k) for k in range(1,5) for l in range(1,5) if abs(k - l) > 1) for t in T)}
\]
True

Berenstein and Kirillov [KB1995] have shown that \((s_1 s_2)^6 = id\) (for tableaux of straight shape):

\[
\text{sage: p = lambda t, k: t.bender_knuth_involution(k).bender_knuth_involution(k + 1)}
\]
\[
\text{sage: all(p(p(p(p(p(p(t,1),1),1),1),1),1),1) == t for t in T)}
\]
True

However, \((s_2 s_3)^6 = id\) is false:

\[
\text{sage: p = lambda t, k: t.bender_knuth_involution(k).bender_knuth_involution(k + 1)}
\]
\[
\text{sage: t = Tableau([[1,2,2],[3,4]])}
\]
\[
\text{sage: x = t}
\]
\[
\text{sage: for i in range(6): x = p(x, 2)}
\]
\[
\text{sage: x}
\]
\[
[[1, 2, 3], [2, 4]]
\]
\[
\text{sage: x == t}
\]
False

\text{bump}(x)

Insert \(x\) into \(\text{self}\) using Schensted’s row-bumping (or row-insertion) algorithm.

\text{EXAMPLES:}

\[
\text{sage: t = Tableau([[1,2],[3]])}
\]
\[
\text{sage: t.bump(1)}
\]
\[
[[1, 1], [2], [3]]
\]
\[
\text{sage: t}
\]
\[
[[1, 2], [3]]
\]
\[
\text{sage: t.bump(2)}
\]
\[
[[1, 2, 2], [3]]
\]
\[
\text{sage: t.bump(3)}
\]
\[
[[1, 2, 3], [3]]
\]
\[
\text{sage: t}
\]
\[
[[1, 2], [3]]
\]
\[
\text{sage: t = Tableau([[1,2,2,3],[2,3,5,5],[4,4,6],[5,6]])}
\]
\[
\text{sage: t.bump(2)}
\]
\[
[[1, 2, 2, 2], [2, 3, 3, 5], [4, 4, 5], [5, 6, 6]]
\]
\[
\text{sage: t.bump(1)}
\]
\[
[[1, 1, 2, 3], [2, 2, 5, 5], [3, 4, 6], [4, 6], [5]]
\]

\text{bump_multiply}(other)

Multiply two tableaux using Schensted’s bump.
This product makes the set of semistandard tableaux into an associative monoid. The empty tableau is the unit in this monoid. See pp. 11-12 of [Ful1997].

The same product operation is implemented in a different way in \texttt{slide\_multiply()}.

**EXAMPLES:**

```python
sage: t = Tableau([[1,2,2,3],[2,3,5,5],[4,4,6],[5,6]])
sage: t2 = Tableau([[1,2],[3]])
sage: t.bump\_multiply(t2)
[[1, 1, 2, 2, 3], [2, 2, 3, 5], [3, 4, 5], [4, 6, 6], [5]]
```

catabolism()

Remove the top row of \texttt{self} and insert it back in using column Schensted insertion (starting with the largest letter).

**EXAMPLES:**

```python
sage: Tableau([]).catabolism()
[]
sage: Tableau([[1,2,3,4,5]]).catabolism()
[[1, 2, 3, 4, 5]]
```

catabolism\_projector\((\texttt{parts})\)

**EXAMPLES:**

```python
sage: t = Tableau([[1,1,3,3],[2,3],[3]])
sage: t.catabolism\_projector([[4,2,1]])
[[1, 1, 3, 3], [2, 3], [3]]
sage: t.catabolism\_projector([[1]])
[]
sage: t.catabolism\_projector([[2,1],[1]])
[]
sage: t.catabolism\_projector([[1],[4,1]])
[[1, 1, 3, 3], [2, 3], [3]]
```

catabolism\_sequence()

Perform \texttt{catabolism()} on \texttt{self} until it returns a tableau consisting of a single row.

**EXAMPLES:**

```python
sage: t = Tableau([[1,2,3,4,5,6,8],[7,9]])
sage: t.catabolism\_sequence()
[[[1, 2, 3, 4, 5, 6, 8], [7, 9]],
[[1, 2, 3, 4, 5, 6, 7, 9], [8]],
[[1, 2, 3, 4, 5, 6, 7, 8], [9]],
[[1, 2, 3, 4, 5, 6, 7, 8, 9]]]
sage: Tableau([]).catabolism\_sequence()
[]]
```

cells()

Return a list of the coordinates of the cells of \texttt{self}. 

Coordinates start at 0, so the northwesternmost cell (in English notation) has coordinates \((0, 0)\).

**EXAMPLES:**

```
sage: Tableau([[1,2],[3,4]]).cells()
[(0, 0), (0, 1), (1, 0), (1, 1)]
```

### cells_containing(i)

Return the list of cells in which the letter \(i\) appears in the tableau \(self\). The list is ordered with cells appearing from left to right.

Cells are given as pairs of coordinates \((a, b)\), where both rows and columns are counted from 0 (so \(a = 0\) means the cell lies in the leftmost column of the tableau, etc.).

**EXAMPLES:**

```
sage: t = Tableau([[1,1,3],[2,3,5],[4,5]])
sage: t.cells_containing(5)
[(2, 1), (1, 2)]
sage: t.cells_containing(4)
[(2, 0)]
sage: t.cells_containing(6)
[]
sage: t = Tableau([[1,1,2,4],[2,4,4],[4]])
sage: t.cells_containing(4)
[(2, 0), (1, 1), (1, 2), (0, 3)]
sage: t = Tableau([[1,1,2,8,9],[2,5,6,11],[3,7,7,13],[4,8,9],[5],[13],[14]])
sage: t.cells_containing(8)
[(3, 1), (0, 3)]
sage: Tableau([]).cells_containing(3)
[]
```

### charge()

Return the charge of the reading word of \(self\). See `charge()` for more information.

**EXAMPLES:**

```
sage: Tableau([[1,1],[2,2],[3]]).charge()
0
sage: Tableau([[1,1,3],[2,2]]).charge()
1
sage: Tableau([[1,1,2],[2],[3]]).charge()
1
sage: Tableau([[1,1,2],[2,3]]).charge()
2
sage: Tableau([[1,1,2,3],[2]]).charge()
2
sage: Tableau([[1,1,2,2],[3]]).charge()
3
sage: Tableau([[1,1,2,2,3]]).charge()
4
```
check()

Check that self is a valid straight-shape tableau.

EXAMPLES:

```
sage: t = Tableau([[1,1],[2]])
sage: t.check()

sage: t = Tableau([[None, None, 1], [2, 4], [3, 4, 5]])  # indirect doctest
Traceback (most recent call last):
... ValueError: a tableau must be a list of iterables of weakly decreasing length
```

cocharge()

Return the cocharge of the reading word of self. See cocharge() for more information.

EXAMPLES:

```
sage: Tableau([[1,1],[2,2],[3]]).cocharge()
4
sage: Tableau([[1,1,3],[2,2]]).cocharge()
3
sage: Tableau([[1,1,2],[2],[3]]).cocharge()
3
sage: Tableau([[1,1,2],[2,3]]).cocharge()
2
sage: Tableau([[1,1,2,3],[2]]).cocharge()
2
sage: Tableau([[1,1,2,2],[3]]).cocharge()
1
sage: Tableau([[1,1,2,2,3]]).cocharge()
0
```

codegree(e, multicharge=(0,))


The codegree of a tableau is an integer that is defined recursively by successively stripping off the number $k$, for $k = n, n-1, \ldots, 1$ and at stage adding the number of addable cell of the same residue minus the number of removable cells of the same residue as $k$ and are above $k$ in the diagram.

The codegree of the tableau $T$ gives the degree of “dual” homogeneous basis element of the Graded Specht module that is indexed by $T$.

INPUT:

- e – the quantum characteristic
- multicharge – (default: [0]) the multicharge

OUTPUT:

The codegree of the tableau self, which is an integer.

EXAMPLES:

```
sage: StandardTableau([[1,3,5],[2,4]]).codegree(3)
0
sage: StandardTableau([[1,2,5],[3,4]]).codegree(3)
```

(continues on next page)
### column_stabilizer()

Return the PermutationGroup corresponding to the column stabilizer of self.

This assumes that every integer from 1 to the size of self appears exactly once in self.

**EXAMPLES:**

```python
sage: cs = Tableau([[1,2,3],[4,5]]).column_stabilizer()
sage: cs.order() == factorial(2)*factorial(2)
True
sage: PermutationGroupElement(((1,3,2),(4,5))) in cs
False
sage: PermutationGroupElement(((1,4))) in cs
True
```

### components()

This function returns a list containing itself. It exists mainly for compatibility with TableauTuple as it allows constructions like the example below.

**EXAMPLES:**

```python
sage: t = Tableau([[1,2,3],[4,5]])
sage: for s in t.components(): print(s.to_list())
[[1, 2, 3], [4, 5]]
```

### conjugate()

Return the conjugate of self.

**EXAMPLES:**

```python
sage: Tableau([[1,2],[3,4]]).conjugate()
[[1, 3], [2, 4]]
sage: c = StandardTableau([[1,2],[3,4]]).conjugate()
sage: c.parent()
Standard tableaux
```

### content($k, multicharge=[0])

Return the content of $k$ in the standard tableau self.

The content of $k$ is $c - r$ if $k$ appears in row $r$ and column $c$ of the tableau.

The multicharge is a list of length 1 which gives an offset for all of the contents. It is included mainly for compatibility with sage.combinat.tableau_tuple.TableauTuple().

**EXAMPLES:**

```python
sage: StandardTableau([[1,2],[3,4]]).content(3)
-1
sage: StandardTableau([[1,2],[3,4]]).content(6)
Traceback (most recent call last):
```

---

**5.1. Comprehensive Module List**

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ValueError: 6 does not appear in tableau

corners()
Return the corners of the tableau self.

EXAMPLES:

```
sage: Tableau([[1, 4, 6], [2, 5], [3]]).corners()
[(0, 2), (1, 1), (2, 0)]
sage: Tableau([[1, 3], [2, 4]]).corners()
[(1, 1)]
```

degree(e, multicharge=(0,))

The degree is an integer that is defined recursively by successively stripping off the number \(k\), for \(k = n, n-1, \ldots, 1\) and at stage adding the number of addable cell of the same residue minus the number of removable cells of the same residue as \(k\) and which are below \(k\) in the diagram.

The degrees of the tableau \(T\) gives the degree of the homogeneous basis element of the graded Specht module that is indexed by \(T\).

INPUT:

- \(e\) – the quantum characteristic
- multicharge – (default: \([0]\)) the multicharge

OUTPUT:

The degree of the tableau self, which is an integer.

EXAMPLES:

```
sage: StandardTableau([[1,2,5],[3,4]]).degree(3)
0
sage: StandardTableau([[1,2,5],[3,4]]).degree(4)
1
```

descents()
Return a list of the cells \((i,j)\) such that \(\text{self}[i][j] > \text{self}[i-1][j]\).

**Warning:** This is not to be confused with the descents of a standard tableau.

EXAMPLES:

```
sage: Tableau([[1,4],[2,3]]).descents()
[(1, 0)]
sage: Tableau([[1,2],[3,4]]).descents()
[(1, 0), (1, 1)]
sage: Tableau([[1,2,3],[4,5]]).descents()
[(1, 0), (1, 1)]
```
entries()

Return the tuple of all entries of self, in the order obtained by reading across the rows from top to bottom (in English notation).

EXAMPLES:

```
sage: t = Tableau([[1,3], [2]])
sage: t.entries()
(1, 3, 2)
```

entry(cell)

Return the entry of cell cell in the tableau self. Here, cell should be given as a tuple (i, j) of zero-based coordinates (so the northwesternmost cell in English notation is (0, 0)).

EXAMPLES:

```
sage: t = Tableau([[1,2],[3,4]])
sage: t.entry( (0,0) )
1
sage: t.entry( (1,1) )
4
```

evacuation(n=None, check=True)

Return the evacuation of the tableau self.

This is an alias for schuetzenberger_involution().

This method relies on the analogous method on words, which reverts the word and then complements all letters within the underlying ordered alphabet. If n is specified, the underlying alphabet is assumed to be [1, 2, ..., n]. If no alphabet is specified, n is the maximal letter appearing in self.

INPUT:

• n – an integer specifying the maximal letter in the alphabet (optional)
• check – (Default: True) Check to make sure self is semistandard. Set to False to avoid this check. (optional)

OUTPUT:

• a tableau, the evacuation of self

EXAMPLES:

```
sage: t = Tableau([[1,1,1],[2,2]])
sage: t.evacuation(3)
[[2, 2, 3], [3, 3]]
sage: t = Tableau([[1,2,3],[4,5]])
sage: t.evacuation()
[[1, 2, 5], [3, 4]]
sage: t = Tableau([[1,3,5,7],[2,4,6],[8,9]])
sage: t.evacuation()
[[1, 2, 6, 8], [3, 4, 9], [5, 7]]
sage: t = Tableau([])
sage: t.evacuation()
```

sage: t = StandardTableau([[1,2,3],[4,5]])
sage: s = t.evacuation()
sage: s.parent()
Standard tableaux
evaluation()
Return the weight of the tableau self. Trailing zeroes are omitted when returning the weight.
The weight of a tableau $T$ is the sequence $(a_1, a_2, a_3, \ldots)$, where $a_k$ is the number of entries of $T$ equal to $k$. This sequence contains only finitely many nonzero entries.
The weight of a tableau $T$ is the same as the weight of the reading word of $T$, for any reading order.
EXAMPLES:
sage: Tableau([[1,2],[3,4]]).weight()
[1, 1, 1, 1]
sage: Tableau([]).weight()
[]
sage: Tableau([[1,3,3,7],[4,2],[2,3]]).weight()
[1, 2, 3, 1, 0, 0, 1]
first_column_descent()
Return the first cell where self is not column standard.
Cells are ordered left to right along the rows and then top to bottom. That is, the cell $(r, c)$ with $r$ and $c$ minimal such that the entry in position $(r, c)$ is bigger than the entry in position $(r, c + 1)$. If there is no such cell then None is returned - in this case the tableau is column strict.
OUTPUT:
The first cell which there is a descent or None if no such cell exists.
EXAMPLES:
sage: Tableau([[1,4,5],[2,3]]).first_column_descent()
(0, 1)
sage: Tableau([[1,2,3],[4]]).first_column_descent() is None
True
first_row_descent()
Return the first cell where the tableau self is not row standard.
Cells are ordered left to right along the rows and then top to bottom. That is, the cell $(r, c)$ with $r$ and $c$ minimal such that the entry in position $(r, c)$ is bigger than the entry in position $(r, c + 1)$. If there is no such cell then None is returned - in this case the tableau is row strict.
OUTPUT:
The first cell which there is a descent or None if no such cell exists.
EXAMPLES:
flush()

Return the number of flush segments in self, as in [Sal2014].

Let \( 1 \leq i < k \leq r + 1 \) and suppose \( \ell \) is the smallest integer greater than \( k \) such that there exists an \( \ell \)-segment in the \((i + 1)\)-st row of \( T \). A \( k \)-segment in the \( i \)-th row of \( T \) is called flush if the leftmost box in the \( k \)-segment and the leftmost box of the \( \ell \)-segment are in the same column of \( T \). If, however, no such \( \ell \) exists, then this \( k \)-segment is said to be flush if the number of boxes in the \( k \)-segment is equal to \( \theta_i \), where \( \theta_i = \lambda_i - \lambda_{i+1} \) and the shape of \( T \) is \( \lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r) \). Denote the number of flush \( k \)-segments in \( T \) by \( \text{flush}(T) \).

EXAMPLES:

```python
sage: t = Tableau([[1,1,2,3,5],[2,3,5,5],[3,4]])
sage: t.flush()
3
```

height()

Return the height of self.

EXAMPLES:

```python
sage: Tableau([[1,2,3],[4,5]]).height()
2
sage: Tableau([[1,2,3]]).height()
1
```

hillman_grassl()

Return the image of the \( \lambda \)-array self under the Hillman-Grassl correspondence (as a \texttt{WeakReversePlanePartition}).

This relies on interpreting self as a \( \lambda \)-array in the sense of \texttt{hillman_grassl}.

Fix a partition \( \lambda \) (see \texttt{Partition}). We draw all partitions and tableaux in English notation.

A \( \lambda \)-array will mean a tableau of shape \( \lambda \) whose entries are nonnegative integers. (No conditions on the order of these entries are made. Note that 0 is allowed.)

A weak reverse plane partition of shape \( \lambda \) (short: \( \lambda \)-rpp) will mean a \( \lambda \)-array whose entries weakly increase along each row and weakly increase along each column.

The Hillman-Grassl correspondence \( H \) is the map that sends a \( \lambda \)-array \( M \) to a \( \lambda \)-rpp \( H(M) \) defined recursively as follows:

- If all entries of \( M \) are 0, then \( H(M) = M \).
Otherwise, let \( s \) be the index of the leftmost column of \( M \) containing a nonzero entry. Let \( r \) be the index of the bottommost nonzero entry in the \( s \)-th column of \( M \). Let \( M' \) be the \( \lambda \)-array obtained from \( M \) by subtracting 1 from the \((r,s)\)-th entry of \( M \). Let \( Q = (q_{i,j}) \) be the image \( H(M') \) (which is already defined by recursion).

Define a sequence \((i_1,j_1), (i_2,j_2), \ldots, (i_n,j_n)\) of boxes in the diagram of \( \lambda \) (actually a lattice path made of southward and westward steps) as follows: Set \((i_1,j_1) = (r, \lambda_r)\) (the rightmost box in the \( r \)-th row of \( \lambda \)). If \((i_k,j_k)\) is defined for some \( k \geq 1 \), then \((i_{k+1},j_{k+1})\) is constructed as follows: If \( q_{i_k+1,j_k} \) is well-defined and equals \( q_{i_k,j_k} \), then we set \((i_{k+1},j_{k+1}) = (i_k + 1, j_k)\). Otherwise, if \( j_k = s \), then the sequence ends here. Otherwise, we set \((i_{k+1},j_{k+1}) = (i_k,j_k - 1)\).

Let \( H(M) \) be the array obtained from \( Q \) by adding 1 to the \((i_k,j_k)\)-th entry of \( Q \) for each \( k \in \{1,2,\ldots,n\} \).

See [Gans1981] (Section 3) for this construction.

See also:

- \( \text{hillman_grassl()} \) for the Hillman-Grassl correspondence as a standalone function.
- \( \text{hillman_grassl_inverse()} \) for the inverse map.

**EXAMPLES:**

```python
sage: a = Tableau([[2, 1, 1], [0, 2, 0], [1, 1]])
```

```python
sage: A = a.hillman_grassl(); A
[[2, 2, 4], [2, 3, 4], [3, 5]]
```

```python
sage: A.parent(), a.parent()
(Weak Reverse Plane Partitions, Tableaux)
```

**insert_word**(\( w, \text{left=False} \))

Insert the word \( w \) into the tableau \( self \) letter by letter using Schensted insertion. By default, the word \( w \) is being processed from left to right, and the insertion used is row insertion. If the optional keyword \( \text{left} \) is set to True, the word \( w \) is being processed from right to left, and column insertion is used instead.

**EXAMPLES:**

```python
sage: t0 = Tableau([])
```

```python
sage: w = [1,1,2,3,3,3,3]
```

```python
sage: t0.insert_word(w)
[[1, 1, 2, 3, 3, 3, 3]]
```

```python
sage: t0.insert_word(w,left=True)
[[1, 1, 2, 3, 3, 3, 3]]
```

```python
sage: w.reverse()
```

```python
sage: t0.insert_word(w)
[[1, 1, 3, 3], [2, 3], [3]]
```

```python
sage: t1 = Tableau([[1,3],[2]])
```

```python
sage: t1.insert_word([4,5])
[[1, 3, 4, 5], [2]]
```

```python
sage: t1.insert_word([4,5], left=True)
[[1, 3], [2, 5], [4]]
```

**inversion_number()**

Return the inversion number of \( self \).

The inversion number is defined to be the number of inversions of \( self \) minus the sum of the arm lengths of the descents of \( self \) (see the \( \text{inversions()} \) and \( \text{descents()} \) methods for the relevant definitions).
**Warning:** This has none of the meanings in which the word “inversion” is used in the theory of standard tableaux.

**EXAMPLES:**

```python
sage: t = Tableau([[1,2,3],[2,5]])
sage: t.inversion_number()
0
sage: t = Tableau([[1,2,4],[3,5]])
sage: t.inversion_number()
0
```

**inversions()**

Return a list of the inversions of `self`.

Let $T$ be a tableau. An inversion is an attacking pair $(c, d)$ of the shape of $T$ (see `attacking_pairs()` for a definition of this) such that the entry of $c$ in $T$ is greater than the entry of $d$.

**Warning:** Do not mistake this for the inversions of a standard tableau.

**EXAMPLES:**

```python
sage: t = Tableau([[1,2,3],[2,5]])
sage: t.inversions()
[((1, 1), (0, 0))]
sage: t = Tableau([[1,4,3],[5,2],[2,6],[3]])
sage: t.inversions()
[((0, 1), (0, 2)), ((1, 0), (1, 1)), ((1, 1), (0, 0)), ((2, 1), (1, 0))]
```

**is_column_increasing**(weak=False)

Return `True` if the entries in each column are in increasing order, and `False` otherwise.

By default, this checks for strictly increasing columns. Set `weak` to `True` to test for weakly increasing columns.

**EXAMPLES:**

```python
sage: T = Tableau([[1, 1, 3], [1, 2]])
sage: T.is_column_increasing(weak=True)
True
sage: T.is_column_increasing()
False
sage: Tableau([[2], [1]]).is_column_increasing(weak=True)
False
```

**is_column_strict()**

Return `True` if `self` is a column strict tableau and `False` otherwise.

A tableau is column strict if the entries in each column are in (strictly) increasing order.

**EXAMPLES:**
Tableau([[1, 3], [2, 4]]).is_column_strict()  # True
Tableau([[1, 2], [2, 4]]).is_column_strict()  # True
Tableau([[2, 3], [2, 4]]).is_column_strict()  # False
Tableau([[5, 3], [2, 4]]).is_column_strict()  # False
Tableau([]).is_column_strict()  # True
Tableau([[1, 4, 2]]).is_column_strict()  # True
Tableau([[1, 4, 2], [2, 5]]).is_column_strict()  # True
Tableau([[1, 4, 2], [2, 3]]).is_column_strict()  # False

is_increasing()  
Return True if self is an increasing tableau and False otherwise.  
A tableau is increasing if it is both row strict and column strict.

EXAMPLES:

Tableau([[1, 3], [2, 4]]).is_increasing()  # True
Tableau([[1, 2], [2, 4]]).is_increasing()  # True
Tableau([[2, 3], [2, 4]]).is_increasing()  # False
Tableau([[5, 3], [2, 4]]).is_increasing()  # False
Tableau([[1, 2, 3], [2, 3], [3]]).is_increasing()  # True

is_k_tableau(k)  
Checks whether self is a valid weak k-tableau.

EXAMPLES:

t = Tableau([[1,2,3],[2,3],[3]])
t.is_k_tableau(3)  # True
t = Tableau([[1,1,3],[2,2],[3]])
t.is_k_tableau(3)  # False

is_key_tableau()  
Return True if self is a key tableau or False otherwise.

A tableau is a key tableau if the set of entries in the j-th column is a subset of the set of entries in the (j-1)-st column.

REFERENCES:
  • [LS1990]
• [Wil2010]

**EXAMPLES:**

```python
sage: t = Tableau([[1,1,1],[2,3],[3]])
sage: t.is_key_tableau()
True

sage: t = Tableau([[1,1,2],[2,3],[3]])
sage: t.is_key_tableau()
False
```

**`is_rectangular()`**

Return True if the tableau `self` is rectangular and False otherwise.

**EXAMPLES:**

```python
sage: Tableau([[1,2],[3,4]]).is_rectangular()
True
sage: Tableau([[1,2,3],[4,5],[6]]).is_rectangular()
False
sage: Tableau([]).is_rectangular()
True
```

**`is_row_increasing(weak=False)`**

Return True if the entries in each row are in increasing order, and False otherwise.

By default, this checks for strictly increasing rows. Set `weak` to True to test for weakly increasing rows.

**EXAMPLES:**

```python
sage: T = Tableau([[1, 1, 3], [1, 2]])
sage: T.is_row_increasing(weak=True)
True
sage: T.is_row_increasing()
False
sage: Tableau([[2, 1]]).is_row_increasing(weak=True)
False
```

**`is_row_strict()`**

Return True if `self` is a row strict tableau and False otherwise.

A tableau is row strict if the entries in each row are in (strictly) increasing order.

**EXAMPLES:**

```python
sage: Tableau([[1, 3], [2, 4]]).is_row_strict()
True
sage: Tableau([[1, 2], [2, 4]]).is_row_strict()
True
sage: Tableau([[2, 3], [2, 4]]).is_row_strict()
True
sage: Tableau([[5, 3], [2, 4]]).is_row_strict()
False
```

**`is_semistandard()`**

Return True if `self` is a semistandard tableau, and False otherwise.
A tableau is semistandard if its rows weakly increase and its columns strictly increase.

```python
sage: Tableau([[1,1],[1,2]]).is_semistandard()
False
sage: Tableau([[1,2],[1,2]]).is_semistandard()
False
sage: Tableau([[1,1],[2,2]]).is_semistandard()
True
sage: Tableau([[1,2],[2,3]]).is_semistandard()
True
sage: Tableau([[4,1],[3,2]]).is_semistandard()
False
```

**is_standard()**

Return True if self is a standard tableau and False otherwise.

```python
sage: Tableau([[1, 3], [2, 4]]).is_standard()
True
sage: Tableau([[1, 2], [2, 4]]).is_standard()
False
sage: Tableau([[2, 3], [2, 4]]).is_standard()
False
sage: Tableau([[5, 3], [2, 4]]).is_standard()
False
```

**k_weight(k)**

Return the $k$-weight of self.

A tableau has $k$-weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ if there are exactly $\alpha_i$ distinct residues for the cells occupied by the letter $i$ for each $i$. The residue of a cell in position $(a, b)$ is $a - b$ modulo $k + 1$.

This definition is the one used in [Ive2012] (p. 12).

```python
sage: Tableau([[1,2],[2,3]]).k_weight(1)
[1, 1, 1]
sage: Tableau([[1,2],[2,3]]).k_weight(2)
[1, 2, 1]
sage: t = Tableau([[1,1,1,2,5],[2,3,6],[3],[4]])
sage: t.k_weight(1)
[2, 1, 1, 1, 1]
sage: t.k_weight(2)
[3, 2, 2, 1, 1]
sage: t.k_weight(3)
[3, 1, 2, 1, 1]
sage: t.k_weight(4)
[3, 2, 1, 1, 1]
sage: t.k_weight(5)
[3, 2, 1, 1, 1]
```

**lambda_catabolism(part)**
Return the part-catabolism of self, where part is a partition (which can be just given as an array).

For a partition \( \lambda \) and a tableau \( T \), the \( \lambda \)-catabolism of \( T \) is defined by performing the following steps.

1. Truncate the parts of \( \lambda \) so that \( \lambda \) is contained in the shape of \( T \). Let \( m \) be the length of this partition.
2. Let \( T_a \) be the first \( m \) rows of \( T \), and \( T_b \) be the remaining rows.
3. Let \( S_a \) be the skew tableau \( T_a/\lambda \).
4. Concatenate the reading words of \( S_a \) and \( T_b \), and insert into a tableau.

**EXAMPLES:**

```python
sage: Tableau([[1,1,3],[2,4,5]]).lambda_catabolism([2,1])
[[3, 5], [4]]
sage: t = Tableau([[1,1,3,3],[2,3],[3]])
sage: t.lambda_catabolism([1])
[[1, 2, 3, 3, 3], [3]]
sage: t.lambda_catabolism([1,1])
[[1, 3, 3, 3], [3]]
sage: t.lambda_catabolism([2,1])
[[3, 3, 3, 3]]
sage: t.lambda_catabolism([4,2,1])
[]
sage: t.lambda_catabolism([5,1])
[[3, 3]]
sage: t.lambda_catabolism([4,1])
[[3, 3]]
```

**last_letter_lequal(tab2)**

Return True if self is less than or equal to tab2 in the last letter ordering.

**EXAMPLES:**

```python
sage: st = StandardTableaux([3,2])
sage: f = lambda b: 1 if b else 0
sage: matrix([ [ f(t1.last_letter_lequal(t2)) for t2 in st] for t1 in st] )
[1 1 1 1 1]
[0 1 1 1 1]
[0 0 1 1 1]
[0 0 0 1 1]
[0 0 0 0 1]
```

**left_key_tableau()**

Return the left key tableau of self.

The left key tableau of a tableau \( T \) is the key tableau whose entries are weakly lesser than the corresponding entries in \( T \), and whose column reading word is subject to certain conditions. See [LS1990] for the full definition.

**ALGORITHM:**

The following algorithm follows [Wil2010]. Note that if \( T \) is a key tableau then the output of the algorithm is \( T \).
To compute the left key tableau \( L \) of a tableau \( T \) we iterate over the columns of \( T \). Let \( T_j \) be the \( j \)-th column of \( T \) and iterate over the entries in \( T_j \) from bottom to top. Initialize the corresponding entry \( k \) in \( L \) as the largest entry in \( T_j \). Scan the columns to the left of \( T_j \) and with each column update \( k \) to be the lowest entry in that column which is weakly less than \( k \). Update \( T_j \) and all columns to the left by removing all scanned entries.

See also:

- \texttt{is_key_tableau()}

**EXAMPLES:**

```python
sage: t = Tableau([[1,2],[2,3]])
sage: t.left_key_tableau()
[[1, 1], [2, 2]]
sage: t = Tableau([[1,1,2,4],[2,3,3],[4],[5]])
sage: t.left_key_tableau()
[[1, 1, 1, 2], [2, 2, 2], [4], [5]]
```

\texttt{leq(secondtab)}

Check whether each entry of \texttt{self} is less-or-equal to the corresponding entry of a further tableau \texttt{secondtab}.

INPUT:

- \texttt{secondtab} – a tableau of the same shape as \texttt{self}

**EXAMPLES:**

```python
sage: T = Tableau([[1, 2], [3]])
sage: S = Tableau([[1, 3], [3]])
sage: G = Tableau([[2, 1], [4]])
sage: H = Tableau([[1, 2], [4]])
sage: T.leq(S)
True
sage: T.leq(T)
True
sage: T.leq(G)
False
sage: T.leq(H)
True
sage: S.leq(T)
False
sage: S.leq(G)
False
sage: S.leq(H)
False
sage: G.leq(H)
False
sage: G.leq(G)
False
```

\texttt{level()}

Return the level of \texttt{self}, which is always 1.

This function exists mainly for compatibility with \texttt{TableauTuple}.
**EXAMPLES:**

```python
sage: Tableau([[1,2,3],[4,5]]).level()
1
```

**major_index()**

Return the major index of `self`.

The major index of a tableau $T$ is defined to be the sum of the number of descents of $T$ (defined in `descents()`) with the sum of their legs' lengths.

**Warning:** This is not to be confused with the major index of a standard tableau.

**EXAMPLES:**

```python
sage: Tableau( [[1,4],[2,3]] ).major_index()
1
sage: Tableau( [[1,2],[3,4]] ).major_index()
2
```

If the major index would be defined in the sense of standard tableaux theory, then the following would give 3 for a result:

```python
sage: Tableau( [[1,2,3],[4,5]] ).major_index()
2
```

**plot(descents=False)**

Return a plot `self`.

If English notation is set then the first row of the tableau is on the top:

```
1 2 3 4
2 3
5
```

Whereas if French notation is set, the first row of the tableau is on the bottom:

```
5
2 3
1 2 3 4
```

**INPUT:**
• descents – boolean (default: False); if True, then the descents are marked in the tableau; only valid if self is a standard tableau

EXAMPLES:

```
sage: t = Tableau([[1,2,4],[3]])
sage: t.plot()
Graphics object consisting of 11 graphics primitives
sage: t.plot(descents=True)
Graphics object consisting of 12 graphics primitives
sage: t = Tableau([[2,2,4],[3]])
sage: t.plot()
Graphics object consisting of 11 graphics primitives
sage: t.plot(descents=True)
Traceback (most recent call last):
...
ValueError: the tableau must be standard for 'descents=True'
```

`pp()`

Pretty print a string of the tableau.

EXAMPLES:

```
sage: T = Tableau([[1,2,3],[3,4],[5]])
sage: T.pp()
  1  2  3
  3  4
  5
sage: Tableaux.options.convention="french"
sage: T.pp()
  5
  3  4
  1  2  3
sage: Tableaux.options._reset()
```

`promotion(n)`

Return the image of self under the promotion operator.

**Warning:** You might know this operator as the inverse promotion operator – literature does not agree on the name. You might also be looking for the Lapointe-Lascoux-Morse promotion operator (`promotion_operator()`).

The promotion operator, applied to a tableau $t$, does the following:

Iterate over all letters $n + 1$ in the tableau $t$, from left to right. For each of these letters, do the following:

• Remove the letter from $t$, thus leaving a hole where it used to be.

• Apply jeu de taquin to move this hole southwest (in French notation) until it reaches the inner boundary of $t$.

• Fill 0 into the hole once jeu de taquin has completed.

Once this all is done, add 1 to each letter in the tableau. This is not always well-defined. Restricted to the class of semistandard tableaux whose entries are all $\leq n + 1$, this is the usual promotion operator defined
When `self` is a standard tableau of size $n + 1$, this definition of promotion is precisely the one given in [Hai1992] (p. 90). It is the inverse of the maps called “promotion” in [Sag2011] (p. 23) and in [Stan2009].

**Warning:** To my (Darij’s) knowledge, the fact that the above promotion operator really is the inverse of the “inverse promotion operator” `promotion_inverse()` for semistandard tableaux has never been proven in literature. Corrections are welcome.

**REFERENCES:**

- [Hai1992]
- [Sag2011]

**EXAMPLES:**

```python
sage: t = Tableau([[1,2],[3,3]])
sage: t.promotion(2)
[[1, 1], [2, 3]]

sage: t = Tableau([[1,1,1],[2,2,3],[3,4,4]])
sage: t.promotion(3)
[[1, 1, 2], [2, 2, 3], [3, 4, 4]]

sage: t = Tableau([[1,2],[2]])
sage: t.promotion(3)
[[2, 3], [3]]

sage: t = Tableau([[1,1,3],[2,3]])
sage: t.promotion(2)
[[1, 1, 2], [2, 3]]

sage: t = Tableau([])
sage: t.promotion(2)
[]
```

`promotion_inverse(n)`

Return the image of `self` under the inverse promotion operator.

**Warning:** You might know this operator as the promotion operator (without “inverse”) – literature does not agree on the name.

The inverse promotion operator, applied to a tableau $t$, does the following:

Iterate over all letters 1 in the tableau $t$, from right to left. For each of these letters, do the following:

- Remove the letter from $t$, thus leaving a hole where it used to be.
- Apply jeu de taquin to move this hole northeast (in French notation) until it reaches the outer boundary of $t$. 

• Fill \( n + 2 \) into the hole once jeu de taquin has completed.

Once this all is done, subtract 1 from each letter in the tableau. This is not always well-defined. Restricted to the class of semistandard tableaux whose entries are all \( \leq n + 1 \), this is the usual inverse promotion operator defined on this class.

When \( \text{self} \) is a standard tableau of size \( n + 1 \), this definition of inverse promotion is the map called “promotion” in [Sag2011] (p. 23) and in [Stan2009], and is the inverse of the map called “promotion” in [Hai1992] (p. 90).

**Warning:** To my (Darij’s) knowledge, the fact that the above “inverse promotion operator” really is the inverse of the promotion operator \( \text{promotion()} \) for semistandard tableaux has never been proven in literature. Corrections are welcome.

**EXAMPLES:**

```sage
t = Tableau([[1,2],[3,3]])
t.promotion_inverse(2)
[[1, 2], [2, 3]]

sage: t = Tableau([[1,2],[2,3]])
t.promotion_inverse(2)
[[1, 1], [2, 3]]

sage: t = Tableau([[1,2,5],[3,3,6],[4,7]])
t.promotion_inverse(8)
[[1, 2, 4], [2, 5, 9], [3, 6]]

sage: t = Tableau([])
t.promotion_inverse(2)
[]
```

**promotion_operator\( (i) \)**

Return a list of semistandard tableaux obtained by the \( i \)-th Lapointe-Lascoux-Morse promotion operator from the semistandard tableau \( \text{self} \).

**Warning:** This is not Schuetzenberger’s jeu de taquin promotion! For the latter, see \text{promotion()} and \text{promotion_inverse()}.

This operator is defined by taking the maximum entry \( m \) of \( T \), then adding a horizontal \( i \)-strip to \( T \) in all possible ways, each time filling this strip with \( m + 1 \)'s, and finally letting the permutation \( \sigma_1 \sigma_2 \cdots \sigma_m = (2,3,\ldots,m+1,1) \) act on each of the resulting tableaux via the Lascoux-Schuetzenberger action (\text{symmetric_group_action_on_values()}). This method returns the list of all resulting tableaux. See [LLM2003] for the purpose of this operator.

**EXAMPLES:**

```sage
t = Tableau([[1,2],[3]])
t.promotion_operator(1)
[[[1, 2, 4], [3]], [[1, 2], [3, 4]], [[1, 2], [3], [4]]]

sage: t = Tableau([[1,2],[3]])
t.promotion_operator(2)
[[[1, 1, 2, 4], [3]],
```

(continues on next page)
[[1, 1, 4], [2, 3]],
[[1, 1, 2], [3], [4]],
[[1, 1], [2, 3], [4]]
\par
\texttt{sage: Tableau([[1]]).promotion_operator(2)}
\begin{verbatim}
[[[1, 1, 2], [[1, 1], [2]]]
\end{verbatim}
\par
\texttt{sage: Tableau([[1,1],[2]]).promotion_operator(3)}
\begin{verbatim}
[[[1, 1, 1, 3], [2, 2]],
[[1, 1, 1, 2], [2], [3]],
[[1, 1, 1], [2, 2], [3]]]
\end{verbatim}
\par
The example from [LLM2003] p. 12:
\begin{verbatim}
\texttt{sage: Tableau([[1,1],[2,2]]).promotion_operator(3)}
\end{verbatim}
\begin{verbatim}
[[[1, 1, 1, 3, 3], [2, 2]],
[[1, 1, 1, 3], [2, 2], [3]],
[[1, 1, 1], [2, 2], [3, 3]]
\end{verbatim}

\texttt{raise_action_from_words(f, *args)}
\par
EXAMPLES:
\begin{verbatim}
\texttt{sage: from sage.combinat.tableau import symmetric_group_action_on_values}
\texttt{sage: import functools}
\texttt{sage: t = Tableau([[1,1,3,3],[2,3],[3]])}
\texttt{sage: f = functools.partial(t.raise_action_from_words, symmetric_group_action_\rightarrow on_values)}
\texttt{sage: f([1,2,3])}
\end{verbatim}
\begin{verbatim}
[[1, 1, 3, 3], [2, 3], [3]]
\end{verbatim}
\begin{verbatim}
\texttt{sage: f([3,2,1])}
\end{verbatim}
\begin{verbatim}
[[1, 1, 1, 1], [2, 3], [3]]
\end{verbatim}
\begin{verbatim}
\texttt{sage: f([1,3,2])}
\end{verbatim}
\begin{verbatim}
[[1, 1, 2, 2], [2, 2], [3]]
\end{verbatim}

\texttt{reading_word_permutation()}
\par
Return the permutation obtained by reading the entries of the standardization of \texttt{self} row by row, starting with the bottommost row (in English notation).
\par
EXAMPLES:
\begin{verbatim}
\texttt{sage: StandardTableau([[1,2],[3,4]]).reading_word_permutation()}
\end{verbatim}
\begin{verbatim}
[3, 4, 1, 2]
\end{verbatim}
\par
Check that \texttt{github issue #14724} is fixed:
\begin{verbatim}
\texttt{sage: SemistandardTableau([[1,1]]).reading_word_permutation()}
\end{verbatim}
\begin{verbatim}
[1, 2]
\end{verbatim}

\texttt{reduced_column_word()}
\par
Return the lexicographically minimal reduced expression for the permutation that maps the conjugate of \texttt{initial_tableau()} to \texttt{self}.
\par
This reduced expression is a minimal length coset representative for the corresponding Young subgroup. In one line notation, the permutation is obtained by concatenating the columns of the tableau in order from top to bottom.
EXAMPLES:

```python
sage: StandardTableau([[1,4,6],[2,5],[3]]).reduced_column_word()
[]
sage: StandardTableau([[1,4,5],[2,6],[3]]).reduced_column_word()
[5]
sage: StandardTableau([[1,3,6],[2,5],[4]]).reduced_column_word()
[3]
sage: StandardTableau([[1,3,5],[2,6],[4]]).reduced_column_word()
[3, 5]
sage: StandardTableau([[1,2,5],[3,6],[4]]).reduced_column_word()
[3, 2, 5]
```

reduced_lambda_catabolism(part)

EXAMPLES:

```python
sage: t = Tableau([[1,1,3,3],[2,3],[3]])
sage: t.reduced_lambda_catabolism([[])
[[1, 1, 3, 3], [2, 3], [3]]
sage: t.reduced_lambda_catabolism([1])
[[1, 2, 3, 3, 3], [3]]
sage: t.reduced_lambda_catabolism([1,1])
[[1, 3, 3, 3], [3]]
sage: t.reduced_lambda_catabolism([2,1])
[[3, 3, 3, 3]]
sage: t.reduced_lambda_catabolism([4,2,1])
[]
sage: t.reduced_lambda_catabolism([5,1])
0
sage: t.reduced_lambda_catabolism([4,1])
0
```

reduced_row_word()

Return the lexicographically minimal reduced expression for the permutation that maps the initial_tableau() to self.

This reduced expression is a minimal length coset representative for the corresponding Young subgroup. In one line notation, the permutation is obtained by concatenating the rows of the tableau in order from top to bottom.

EXAMPLES:

```python
sage: StandardTableau([[1,2,3],[4,5],[6]]).reduced_row_word()
[]
sage: StandardTableau([[1,2,3],[4,6],[5]]).reduced_row_word()
[5]
sage: StandardTableau([[1,2,4],[3,6],[5]]).reduced_row_word()
[3, 5]
sage: StandardTableau([[1,2,5],[3,6],[4]]).reduced_row_word()
[3, 5, 4]
sage: StandardTableau([[1,2,6],[3,5],[4]]).reduced_row_word()
[3, 4, 5, 4]
```

residue(k, e, multicharge=(0,))

Return the residue of the integer $k$ in the tableau self.
The residue of $k$ in a standard tableau is $c - r + m$ in $\mathbb{Z}/e\mathbb{Z}$, where $k$ appears in row $r$ and column $c$ of the tableau with multicharge $m$.

**INPUT:**
- $k$ – an integer in $\{1, 2, \ldots, n\}$
- $e$ – an integer in $\{0, 2, 3, 4, 5, \ldots\}$
- multicharge – (default: [0]) a list of length 1

The multicharge is a list of length 1 which gives an offset for all of the contents. It is included mainly for compatibility with residue().

**OUTPUT:**
The residue in $\mathbb{Z}/e\mathbb{Z}$.

**EXAMPLES:**

```sage
sage: StandardTableau([[1,2,5],[3,4]]).residue(1,3)
0
sage: StandardTableau([[1,2,5],[3,4]]).residue(2,3)
1
sage: StandardTableau([[1,2,5],[3,4]]).residue(3,3)
2
sage: StandardTableau([[1,2,5],[3,4]]).residue(4,3)
0
sage: StandardTableau([[1,2,5],[3,4]]).residue(5,3)
2
sage: StandardTableau([[1,2,5],[3,4]]).residue(6,3)
Traceback (most recent call last):
  ... ValueError: 6 does not appear in the tableau
```

**residue_sequence(e, multicharge=(0,))**

Return the `sage.combinat.tableau_residues.ResidueSequence` of the tableau `self`.

**INPUT:**
- $e$ – an integer in $\{0, 2, 3, 4, 5, \ldots\}$
- multicharge – (default: [0]) a sequence of integers of length 1

The multicharge is a list of length 1 which gives an offset for all of the contents. It is included mainly for compatibility with residue().

**OUTPUT:**
The corresponding residue sequence of the tableau; see ResidueSequence.

**EXAMPLES:**

```sage
sage: StandardTableauTuple([[1,2],[3,4]]).residue_sequence(2)
2-residue sequence (0,1,1,0) with multicharge (0)
sage: StandardTableauTuple([[1,2],[3,4]]).residue_sequence(3)
3-residue sequence (0,1,2,0) with multicharge (0)
sage: StandardTableauTuple([[1,2],[3,4]]).residue_sequence(4)
4-residue sequence (0,1,3,0) with multicharge (0)
```
**restrict(n)**

Return the restriction of the semistandard tableau `self` to `n`. If possible, the restricted tableau will have the same parent as this tableau.

If `T` is a semistandard tableau and `n` is a nonnegative integer, then the restriction of `T` to `n` is defined as the (semistandard) tableau obtained by removing all cells filled with entries greater than `n` from `T`.

**Note:** If only the shape of the restriction, rather than the whole restriction, is needed, then the faster method `restriction_shape()` is preferred.

**EXAMPLES:**

```sage
sage: Tableau([[1,2],[3],[4]]).restrict(3)
[[1, 2], [3]]
sage: StandardTableau([[1,2],[3],[4]]).restrict(2)
[[1], [2]]
sage: Tableau([[1,2,3],[2,4,4],[3]]).restrict(0)
[]
sage: Tableau([[1,2,3],[2,4,4],[3]]).restrict(2)
[[1, 2], [2]]
sage: Tableau([[1,2,3],[2,4,4],[3]]).restrict(3)
[[1, 2, 3], [2], [3]]
sage: Tableau([[1,2,3],[2,4,4],[3]]).restrict(5)
[[1, 2, 3], [2, 4, 4], [3]]
```

If possible the restricted tableau will belong to the same category as the original tableau:

```sage
sage: S = StandardTableau([[1,2,4,7],[3,5],[6]]); S.category()
Category of elements of Standard tableaux
sage: S.restrict(4).category()
Category of elements of Standard tableaux
sage: SS=StandardTableaux([4,2,1])([[1,2,4,7],[3,5],[6]]); SS.category()
Category of elements of Standard tableaux of shape [4, 2, 1]
sage: SS.restrict(4).category()
Category of elements of Standard tableaux
```

**restriction_shape(n)**

Return the shape of the restriction of the semistandard tableau `self` to `n`.

If `T` is a semistandard tableau and `n` is a nonnegative integer, then the restriction of `T` to `n` is defined as the (semistandard) tableau obtained by removing all cells filled with entries greater than `n` from `T`.

This method computes merely the shape of the restriction. For the restriction itself, use `restrict()`.

**EXAMPLES:**
reverse_bump(loc)

Reverse row bump the entry of self at the specified location loc (given as a row index or a corner (r, c) of the tableau).

This is the reverse of Schensted’s row-insertion algorithm. See Section 1.1, page 8, of Fulton’s [Ful1997].

**INPUT:**

- loc – Can be either of the following:
  - The coordinates (r, c) of the square to reverse-bump (which must be a corner of the tableau);
  - The row index r of this square.

  Note that both r and c are 0-based, i.e., the topmost row and the leftmost column are the 0-th row and the 0-th column.

**OUTPUT:**

An ordered pair consisting of:

1. The resulting (smaller) tableau;
2. The entry bumped out at the end of the process.

**See also:**

bump()

**EXAMPLES:**

This is the reverse of Schensted’s bump:

```python
def reverse_bump(loc):
    return ...  # Implementation of reverse_bump
```

```python
sage: Tableau([[1,2,2,4],[2,3,3],[4]]).reverse_bump(2)
[[1,2,2,2,4],[3,3,3],[4]]
```

Some errors caused by wrong input:
Some edge cases:

- `sage: T.reverse_bump(0)`
  `([], 1)`
- `sage: Tableau([[1,1]]).reverse_bump(0)`
  `([[1]], 1)`
- `sage: Tableau([]).reverse_bump(0)`
  Traceback (most recent call last):
  ...  
  `IndexError: list index out of range`

Note: Reverse row bumping is only implemented for tableaux with weakly increasing and strictly increasing columns (though the tableau does not need to be an instance of class `SemistandardTableau`).

**right_key_tableau()**

Return the right key tableau of `self`.

The right key tableau of a tableau $T$ is a key tableau whose entries are weakly greater than the corresponding entries in $T$, and whose column reading word is subject to certain conditions. See [LS1990] for the full definition.

ALGORITHM:

The following algorithm follows [Wil2010]. Note that if $T$ is a key tableau then the output of the algorithm is $T$.

To compute the right key tableau $R$ of a tableau $T$ we iterate over the columns of $T$. Let $T_j$ be the $j$-th column of $T$ and iterate over the entries in $T_j$ from bottom to top. Initialize the corresponding entry $k$ in $R$ to be the largest entry in $T_j$, Scan the bottom of each column of $T$ to the right of $T_j$, updating $k$ to be the scanned entry whenever the scanned entry is weakly greater than $k$. Update $T_j$ and all columns to the right by removing all scanned entries.

See also:

- `is_key_tableau()`

EXAMPLES:

```
sage: t = Tableau([[1,2],[2,3]])
sage: t.right_key_tableau()
[[2, 2], [3, 3]]
```
\begin{verbatim}
sage: t = Tableau([[1,1,2,4],[2,3,3],[4],[5]])
sage: t.right_key_tableau()
[[2, 2, 2, 4], [3, 4, 4], [4], [5]]
\end{verbatim}

\textbf{rotate\_180()}

Return the tableau obtained by rotating \texttt{self} by 180 degrees.

This only works for rectangular tableaux.

**EXAMPLES:**

\begin{verbatim}
sage: Tableau([[1,2],[3,4]]).rotate\_180()
[[4, 3], [2, 1]]
\end{verbatim}

\textbf{row\_stabilizer()}

Return the \texttt{PermutationGroup} corresponding to the row stabilizer of \texttt{self}.

This assumes that every integer from 1 to the size of \texttt{self} appears exactly once in \texttt{self}.

**EXAMPLES:**

\begin{verbatim}
sage: rs = Tableau([[1,2,3],[4,5]]).row\_stabilizer()
sage: rs.order() == factorial(3)*factorial(2)
True
sage: PermutationGroupElement([(1,3,2),(4,5)]) in rs
True
sage: PermutationGroupElement([(1,4)]) in rs
False
sage: rs = Tableau([[1, 2],[3]]).row\_stabilizer()
sage: PermutationGroupElement([[1,2],[3]]) in rs
True
sage: rs = Tableau([[1],[2],[3]]).row\_stabilizer()
sage: rs.order()
1
sage: rs = Tableau([[2,4,5],[1,3]]).row\_stabilizer()
sage: rs.order()
12
sage: rs = Tableau([]).row\_stabilizer()
sage: rs.order()
1
\end{verbatim}

\textbf{schensted\_insert}(i, left=False)

Insert \texttt{i} into \texttt{self} using Schensted’s row-bumping (or row-insertion) algorithm.

**INPUT:**

- \texttt{i} – a number to insert
- \texttt{left} – (default: \texttt{False}) boolean; if set to \texttt{True}, the insertion will be done from the left. That is, if one thinks of the algorithm as appending a letter to the reading word of \texttt{self}, we append the letter to the left instead of the right

**EXAMPLES:**
sage: t = Tableau([[3,5],[7]])
sage: t.schensted_insert(8)
[[3, 5, 8], [7]]
sage: t.schensted_insert(8, left=True)
[[3, 5], [7], [8]]

schuetzenberger_involution(n=None, check=True)

Return the Schuetzenberger involution of the tableau self.

This method relies on the analogous method on words, which reverts the word and then complements all letters within the underlying ordered alphabet. If \( n \) is specified, the underlying alphabet is assumed to be \([1, 2, \ldots, n]\). If no alphabet is specified, \( n \) is the maximal letter appearing in self.

INPUT:
- \( n \) – an integer specifying the maximal letter in the alphabet (optional)
- \( \text{check} \) – (Default: True) Check to make sure self is semistandard. Set to False to avoid this check.

OUTPUT:
- a tableau, the Schuetzenberger involution of self

EXAMPLES:

```sage
t = Tableau([[1,1,1],[2,2]])
sage: t.schuetzenberger_involution(3)
[[2, 2, 3], [3, 3]]
t = Tableau([[1,2,3],[4,5]])
sage: t.schuetzenberger_involution()
[[1, 2, 5], [3, 4]]
t = Tableau([[1,3,5,7],[2,4,6],[8,9]])
sage: t.schuetzenberger_involution()
[[1, 2, 6, 8], [3, 4, 9], [5, 7]]
t = Tableau([])
sage: t.schuetzenberger_involution()
[]
t = StandardTableau([[1,2,3],[4,5]])
sage: s = t.schuetzenberger_involution()
sage: s.parent()
Standard tableaux
```

seg()

Return the total number of segments in self, as in [Sal2014].

Let \( T \) be a tableau. We define a \( k \)-segment of \( T \) (in the \( i \)-th row) to be a maximal consecutive sequence of \( k \)-boxes in the \( i \)-th row for any \( i + 1 \leq k \leq r + 1 \). Denote the total number of \( k \)-segments in \( T \) by \( \text{seg}(T) \).

REFERENCES:
- [Sal2014]

EXAMPLES:
sage: t = Tableau([[1,1,2,3,5],[2,3,5,5],[3,4]])
sage: t.seg()
6

sage: B = crystals.Tableaux("A4",shape=[4,3,2,1])
sage: t = B[31].to_tableau()
sage: t.seg()
3

shape()
Return the shape of a tableau self.

EXAMPLES:

sage: Tableau([[1,2,3],[4,5],[6]]).shape() [3, 2, 1]

sage: Tableau([[1, 4, 6], [2, 5], [3]]).size()
6
sage: Tableau([[1, 3], [2, 4]]).size()
4

slide_multiply(other)
Multiply two tableaux using jeu de taquin.

This product makes the set of semistandard tableaux into an associative monoid. The empty tableau is the
unit in this monoid.

See pp. 15 of [Ful1997].

The same product operation is implemented in a different way in bump_multiply().

EXAMPLES:

sage: t = Tableau([[1,2,2,3],[2,3,5,5],[4,4,6],[5,6]])
sage: t2 = Tableau([[1,2],[3]])
sage: t.slide_multiply(t2)
[[1, 1, 2, 2, 3], [2, 2, 3, 5], [3, 4, 5], [4, 6, 6], [5]]

socle()
EXAMPLES:

sage: Tableau([[1,2],[3,4]]).socle()
2
sage: Tableau([[1,2,3,4]]).socle()
4

standardization(check=True)
Return the standardization of self, assuming self is a semistandard tableau.

The standardization of a semistandard tableau \( T \) is the standard tableau \( st(T) \) of the same shape
as \( T \) whose reversed reading word is the standardization of the reversed reading word of \( T \).
The standardization of a word $w$ can be formed by replacing all 1’s in $w$ by 1, 2, \ldots, $k_1$ from left to right, all 2’s in $w$ by $k_1 + 1, k_1 + 2, \ldots, k_2$, and repeating for all letters which appear in $w$. See also $\text{Word.standard_permutation()}$.

**INPUT:**

- check – (Default: True) Check to make sure self is semistandard. Set to False to avoid this check.

**EXAMPLES:**

```python
sage: t = Tableau([[1,3,3,4],[2,4,4],[5,16]])
sage: t.standardization()
[[1, 3, 4, 7], [2, 5, 6], [8, 9]]
```

Standard tableaux are fixed under standardization:

```python
sage: all((t == t.standardization() for t in StandardTableaux(6)))
True
sage: t = Tableau([])
sage: t.standardization()
[]
```

The reading word of the standardization is the standardization of the reading word:

```python
sage: T = SemistandardTableaux(shape=[5,2,2,1], max_entry=4)
sage: all(t.to_word().standard_permutation() == t.standardization().reading_word_permutation() for t in T) # long time
True
```

**sulzgruber_correspondence()**

Return the image of the $\lambda$-array self under the Sulzgruber correspondence (as a $\text{WeakReversePlanePartition}$).

This relies on interpreting self as a $\lambda$-array in the sense of hillman_grassl. See hillman_grassl for definitions of the objects involved.

The Sulzgruber correspondence is the map $\Phi_{\lambda}$ from [Sulzgr2017] Section 7, and is the map $\xi_{\lambda}^{-1}$ from [Pak2002] Section 5. It is denoted by $\text{RSK}$ in [Hopkins2017]. It is the inverse of the Pak correspondence (pak_correspondence()). The following description of the Sulzgruber correspondence follows [Hopkins2017] (which denotes it by $\text{RSK}$):

Fix a partition $\lambda$ (see $\text{Partition()}$). We draw all partitions and tableaux in English notation.

A $\lambda$-array will mean a tableau of shape $\lambda$ whose entries are nonnegative integers. (No conditions on the order of these entries are made. Note that 0 is allowed.)

A weak reverse plane partition of shape $\lambda$ (short: $\lambda$-rpp) will mean a $\lambda$-array whose entries weakly increase along each row and weakly increase along each column.

We shall also use the following notation: If $(u, v)$ is a cell of $\lambda$, and if $\pi$ is a $\lambda$-rpp, then:

- the lower bound of $\pi$ at $(u, v)$ (denoted by $\pi_{< (u, v)}$) is defined to be $\max\{\pi_{u-1, v}, \pi_{u, v-1}\}$ (where $\pi_{0, v}$ and $\pi_{u, 0}$ are understood to mean 0).
- the upper bound of $\pi$ at $(u, v)$ (denoted by $\pi_{> (u, v)}$) is defined to be $\min\{\pi_{u+1, v}, \pi_{u, v+1}\}$ (where $\pi_{i, j}$ is understood to mean $+\infty$ if $(i, j)$ is not in $\lambda$; thus, the upper bound at a corner cell is $+\infty$).
- toggling $\pi$ at $(u, v)$ means replacing the entry $\pi_{u, v}$ of $\pi$ at $(u, v)$ by $\pi_{< (u, v)} + \pi_{> (u, v)} - \pi_{u, v}$ (this is well-defined as long as $(u, v)$ is not a corner of $\lambda$).
Note that every $\lambda$-rpp $\pi$ and every cell $(u, v)$ of $\lambda$ satisfy $\pi_{< (u,v)} \leq \pi_{u,v} \leq \pi_{>(u,v)}$. Note that toggling a $\lambda$-rpp (at a cell that is not a corner) always results in a $\lambda$-rpp. Also, toggling is an involution).

The Pak correspondence $\xi_{\lambda}$ sends a $\lambda$-rpp $\pi$ to a $\lambda$-array $\xi_{\lambda}(\pi)$. It is defined by recursion on $\lambda$ (that is, we assume that $\xi_{\mu}$ is already defined for every partition $\mu$ smaller than $\lambda$), and its definition proceeds as follows:

- If $\lambda = \emptyset$, then $\xi_{\lambda}$ is the obvious bijection sending the only $\emptyset$-rpp to the only $\emptyset$-array.
- Pick any corner $c = (i, j)$ of $\lambda$, and let $\mu$ be the result of removing this corner $c$ from the partition $\lambda$. (The exact choice of $c$ is immaterial.)
- Let $\pi'$ be what remains of $\pi$ when the corner cell $c$ is removed.
- For each positive integer $k$ such that $(i - k, j - k)$ is a cell of $\lambda$, toggle $\pi'$ at $(i - k, j - k)$. (All these togglings commute, so the order in which they are made is immaterial.)
- Let $M = \xi_{\mu}(\pi')$.
- Extend the $\mu$-array $M$ to a $\lambda$-array $M'$ by adding the cell $c$ and writing the number $\pi_{i,j} - \pi_{< (i,j)}$ into this cell.
- Set $\xi_{\lambda}(\pi) = M'$.

See also:

$sulzgruber_correspondence()$ for the Sulzgruber correspondence as a standalone function.

$pak_correspondence()$ for the inverse map.

EXAMPLES:

```
sage: a = Tableau([[2, 1, 1], [0, 2, 0], [1, 1]])
sage: A = a.sulzgruber_correspondence(); A
[[0, 1, 4], [1, 5, 5], [3, 6]]
sage: A.parent(), a.parent()
(Weak Reverse Plane Partitions, Tableaux)
sage: a = Tableau([[1, 3], [0, 1]])
sage: a.sulzgruber_correspondence()
[[0, 4], [1, 5]]
```

$symmetric_group_action_on_entries(w)$

Return the tableau obtained form this tableau by acting by the permutation $w$.

Let $T$ be a standard tableau of size $n$, then the action of $w \in S_n$ is defined by permuting the entries of $T$ (recall they are $1, 2, \ldots, n$). In particular, suppose the entry at cell $(i, j)$ is $a$, then the entry becomes $w(a)$. In general, the resulting tableau $wT$ may not be standard.

**Note:** This is different than $symmetric_group_action_on_values()$ which is defined on semistandard tableaux and is guaranteed to return a semistandard tableau.

**Input:**

- $w$ – a permutation

**Examples:**
Return the image of the semistandard tableau \( self \) under the action of the permutation \( \text{perm} \) using the Lascoux-Schuetzenberger action of the symmetric group \( S_n \) on the semistandard tableaux with ceiling \( n \).

If \( n \) is a nonnegative integer, then the Lascoux-Schuetzenberger action is a group action of the symmetric group \( S_n \) on the set of semistandard Young tableaux with ceiling \( n \) (that is, with entries taken from the set \( \{1, 2, \ldots, n\} \)). It is defined as follows:

Let \( i \in \{1, 2, \ldots, n-1\} \), and let \( T \) be a semistandard tableau with ceiling \( n \). Let \( w \) be the reading word (\text{to_word()}\) of \( T \). Replace all letters \( i \) in \( w \) by closing parentheses, and all letters \( i+1 \) in \( w \) by opening parentheses. Whenever an opening parenthesis stands left of a closing parenthesis without there being any parentheses in between (it is allowed to have letters in-between as long as they are not parentheses), consider these two parentheses as matched with each other, and replace them back by the letters \( i+1 \) and \( i \). Repeat this procedure until there are no more opening parentheses standing left of closing parentheses. Then, let \( a \) be the number of opening parentheses in the word, and \( b \) the number of closing parentheses (notice that all opening parentheses are right of all closing parentheses). Replace the first \( a \) parentheses by the letters \( i \), and replace the remaining \( b \) parentheses by the letters \( i+1 \). Let \( w' \) be the resulting word. Let \( T' \) be the tableau with the same shape as \( T \) but with reading word \( w' \). This tableau \( T' \) can be shown to be semistandard. We define the image of \( T \) under the action of the simple transposition \( s_i = (i, i+1) \in S_n \) to be this tableau \( T' \). It can be shown that these actions of the transpositions \( s_1, s_2, \ldots, s_{n-1} \) satisfy the Moore-Coxeter relations of \( S_n \), and thus this extends to a unique action of the symmetric group \( S_n \) on the set of semistandard tableaux with ceiling \( n \). This is the Lascoux-Schuetzenberger action.

This action of the symmetric group \( S_n \) on the set of all semistandard tableaux of given shape \( \lambda \) with entries in \( \{1, 2, \ldots, n\} \) is the one defined in [Loth02] Theorem 5.6.3. In particular, the action of \( s_i \) is denoted by \( \sigma_i \) in said source. (Beware of the typo in the definition of \( \sigma_i \): it should say \( \sigma_i(a_i^r a_{i+1}^s) = a_i^r a_{i+1}^s \), not \( \sigma_i(a_i^r a_{i+1}^s) = a_i^s a_{i+1}^r \).)

EXAMPLES:

```python
sage: t = Tableau([[1,1,3,3],[2,3],[3]])
sage: t.symmetric_group_action_on_values([1,2,3])
[[1, 1, 3, 3], [2, 3], [3]]
sage: t.symmetric_group_action_on_values([2,1,3])
[[1, 2, 3, 3], [2, 3], [3]]
sage: t.symmetric_group_action_on_values([3,1,2])
[[1, 2, 2, 3], [2, 3], [3]]
sage: t.symmetric_group_action_on_values([2,3,1])
[[1, 1, 1, 3], [2, 2], [3]]
sage: t.symmetric_group_action_on_values([3,2,1])
[[1, 1, 1, 3], [2, 2], [3]]
sage: t.symmetric_group_action_on_values([1,3,2])
[[1, 1, 2, 2], [2, 2], [3]]
```
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**to_Gelfand_Tsetlin_pattern()**

Return the *Gelfand-Tsetlin pattern* corresponding to `self` when semistandard.

**EXAMPLES:**

```python
sage: T = Tableau([[1,2,3],[2,3],[3]])
sage: G = T.to_Gelfand_Tsetlin_pattern(); G
[[3, 2, 1], [2, 1], [1]]
sage: G.to_tableau() == T
True
sage: T = Tableau([[1,3],[2]])
sage: T.to_Gelfand_Tsetlin_pattern()
[[2, 1, 0], [1, 1], [1]]
```

**to_chain**(max_entry=None)

Return the chain of partitions corresponding to the (semi)standard tableau `self`.

The optional keyword parameter `max_entry` can be used to customize the length of the chain. Specifically, if this parameter is set to a nonnegative integer `n`, then the chain is constructed from the positions of the letters 1, 2, ..., `n` in the tableau.

**EXAMPLES:**

```python
sage: Tableau([[1,2],[3,4]]).to_chain()
[[], [1], [2], [1, 2], [2, 1, 2]]
sage: Tableau([[1,1],[2]]).to_chain()
[[], [2], [2, 1]]
sage: Tableau([[1,1],[3]]).to_chain()
[[], [2], [2], [2, 1]]
sage: Tableau([]).to_chain()
[[]]
sage: Tableau([[1,1],[2],[3]]).to_chain(max_entry=2)
[[], [2], [2, 1]]
sage: Tableau([[1,1],[2],[3]]).to_chain(max_entry=3)
[[], [2], [2, 1], [2, 1, 1]]
sage: Tableau([[1,1],[2],[3]]).to_chain(max_entry=4)
[[], [2], [2, 1], [2, 1, 1], [2, 1, 1, 1]]
sage: Tableau([[1,1,2],[2,3],[4,5]]).to_chain(max_entry=6)
[[], [2], [3, 1], [3, 2], [3, 2, 1], [3, 2, 2], [3, 2, 2]]
```

**to_list()**

Return `self` as a list of lists (not tuples!).

**EXAMPLES:**

```python
sage: t = Tableau([[1,2],[3,4]])
sage: l = t.to_list(); l
[[1, 2], [3, 4]]
sage: l[0][0] = 2
sage: t
[[1, 2], [3, 4]]
```

**to_sign_matrix**(max_entry=None)

Return the sign matrix of `self`.

A sign matrix is an \(m \times n\) matrix of 0’s, 1’s and -1’s such that the partial sums of each column is either 0 or 1 and the partial sums of each row is non-negative. [Ava2007]
INPUT:
- **max_entry** – A non-negative integer, the maximum allowable number in the tableau. Defaults to the largest entry in the tableau if not specified.

EXAMPLES:

```
sage: t = SemistandardTableau([[1,1,1,2,4],[3,3,4],[4,5],[6,6]])
sage: t.to_sign_matrix(6)
[ 0 0 0 1 0 0]
[ 0 1 0 -1 0 0]
[ 1 -1 0 1 0 0]
[ 0 0 1 -1 1 1]
[ 0 0 0 1 -1 0]
sage: t = Tableau([[1,2,4],[3,5]])
sage: t.to_sign_matrix(7)
[ 0 0 0 1 0 0 0]
[ 0 1 0 -1 1 0 0]
[ 1 -1 1 0 -1 0 0]
sage: t = Tableau(((4,5,4,3),(2,1,3)))
sage: t.to_sign_matrix(5)
[ 0 0 1 0 0]
[ 0 0 0 1 0]
[ 1 0 -1 -1 1]
[-1 1 0 1 -1]
sage: s = Tableau(((1,0,-2,4),(3,4,5)))
sage: s.to_sign_matrix(6)
Traceback (most recent call last):
  ... ValueError: the entries must be non-negative integers
```

to_word()
An alias for **to_word_by_row**.

EXAMPLES:

```
sage: Tableau([[1,2],[3,4]]).to_word()
word: 3412
sage: Tableau([[1, 4, 6], [2, 5], [3]]).to_word()
word: 325146
```

**to_word_by_column()**
Return the word obtained from a column reading of the tableau self (starting with the leftmost column, reading every column from bottom to top).

EXAMPLES:

```
sage: Tableau([[1,2],[3,4]]).to_word_by_column()
word: 3142
sage: Tableau([[1, 4, 6], [2, 5], [3]]).to_word_by_column()
word: 321546
```

**to_word_by_row()**
Return the word obtained from a row reading of the tableau self (starting with the lowermost row, reading every row from left to right).

EXAMPLES:
sage: Tableau([[1, 2], [3, 4]]).to_word_by_row()
word: 3412
sage: Tableau([[1, 4, 6], [2, 5], [3]]).to_word_by_row()
word: 325146

vertical_flip()

Return the tableau obtained by vertically flipping the tableau self.
This only works for rectangular tableaux.

EXAMPLES:

sage: Tableau([[1, 2], [3, 4]]).vertical_flip()
[[3, 4], [1, 2]]

weight()

Return the weight of the tableau self. Trailing zeroes are omitted when returning the weight.
The weight of a tableau $T$ is the sequence $(a_1, a_2, a_3, \ldots)$, where $a_k$ is the number of entries of $T$ equal to $k$. This sequence contains only finitely many nonzero entries.
The weight of a tableau $T$ is the same as the weight of the reading word of $T$, for any reading order.

EXAMPLES:

sage: Tableau([[1, 2], [3, 4]]).weight()
[1, 1, 1, 1]
sage: Tableau([]).weight()
[]
sage: Tableau([[1, 3, 3, 7], [4, 2], [2, 3]]).weight()
[1, 2, 3, 1, 0, 0, 1]

class sage.combinat.tableau.Tableau_class(parent, t, check=True)

Bases: Tableau

This exists solely for unpickling Tableau_class objects.

class sage.combinat.tableau.Tableaux

Bases: UniqueRepresentation, Parent

A factory class for the various classes of tableaux.

INPUT:

- n (optional) – a non-negative integer

OUTPUT:

- If n is specified, the class of tableaux of size n. Otherwise, the class of all tableaux.

A tableau in Sage is a finite list of lists, whose lengths are weakly decreasing, or an empty list, representing the empty tableau. The entries of a tableau can be any Sage objects. Because of this, no enumeration through the set of Tableaux is possible.

EXAMPLES:
```
sage: T = Tableaux(); T
Tableaux
sage: T3 = Tableaux(3); T3
Tableaux of size 3
sage: [['a','b']] in T
True
sage: [['a','b']] in T3
False
sage: t = T3([[1,1,1]]); t
[[1, 1, 1]]
sage: t in T
True
sage: t.parent()
Tableaux of size 3
sage: T([]) # the empty tableau
[]
sage: T.category()
Category of sets
```

See also:

- `Tableau`
- `SemistandardTableaux`
- `SemistandardTableau`
- `StandardTableaux`
- `StandardTableau`

```
Element

alias of Tableau

options = Current options for Tableaux - ascii_art: repr - convention: English -
         display: list - latex: diagram
```

class sage.combinat.tableau.Tableaux_all

Bases: `Tableaux`

Initializes the class of all tableaux

```
an_element()

    Return a particular element of the class.
```

class sage.combinat.tableau.Tableaux_size(n)

Bases: `Tableaux`

Tableaux of a fixed size n.

```
an_element()

    Return a particular element of the class.
```

```
sage.combinat.tableau.from_chain(chain)

    Return a semistandard tableau from a chain of partitions.
```

EXAMPLES:
```python
sage: from sage.combinat.tableau import from_chain
sage: from_chain([[], [2], [2, 1], [3, 2, 1]])
[[1, 1, 3], [2, 3], [3]]
```

`sage.combinat.tableau.from_shape_and_word(shape, w, convention='French')`  
Return a tableau from a shape and word.

**INPUT:**
- `shape` – a partition
- `w` – a word whose length equals that of the partition
- `convention` – a string which can take values "French" or "English"; the default is "French"

**OUTPUT:**
A tableau, whose shape is `shape` and whose reading word is `w`. If the convention is specified as "French", the reading word is to be read starting from the top row in French convention (= the bottom row in English convention). If the convention is specified as "English", the reading word is to be read starting with the top row in English convention.

**EXAMPLES:**

```python
sage: from sage.combinat.tableau import from_shape_and_word
sage: t = Tableau([[1, 3], [2], [4]])
sage: shape = t.shape(); shape
[2, 1, 1]
sage: word = t.to_word(); word
word: 4213
sage: from_shape_and_word(shape, word)
[[1, 3], [2], [4]]
sage: word = Word(flatten(t))
sage: from_shape_and_word(shape, word, convention="English")
[[1, 3], [2], [4]]
```

`sage.combinat.tableau.symmetric_group_action_on_values(word, perm)`  
Return the image of the word `word` under the Lascoux-Schuetzenberger action of the permutation `perm`.

See `Tableau.symmetric_group_action_on_values()` for the definition of the Lascoux-Schuetzenberger action on semistandard tableaux. The transformation that the reading word of the tableau undergoes in said definition is precisely the Lascoux-Schuetzenberger action on words.

**EXAMPLES:**

```python
sage: from sage.combinat.tableau import symmetric_group_action_on_values
sage: symmetric_group_action_on_values([1,1,1],[2,1,3])
[1, 1, 1]
sage: symmetric_group_action_on_values([1,1,1],[2,1,3])
[2, 2, 2]
sage: symmetric_group_action_on_values([1,2,1],[2,1,3])
[2, 2, 1]
sage: symmetric_group_action_on_values([2,2,2],[2,1,3])
[1, 1, 1]
sage: symmetric_group_action_on_values([2,2,2],[2,1,3])
[2, 1, 1]
sage: symmetric_group_action_on_values([2,2,3,1,1,2,2,3],[1,3,2])
```

(continues on next page)
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sage.combinat.tableau.unmatched_places(w, open, close)

Given a word $w$ and two letters open and close to be treated as opening and closing parentheses (respectively), return a pair $(xs, ys)$ that encodes the positions of the unmatched parentheses after the standard parenthesis matching procedure is applied to $w$.

More precisely, $xs$ will be the list of all $i$ such that $w[i]$ is an unmatched closing parenthesis, while $ys$ will be the list of all $i$ such that $w[i]$ is an unmatched opening parenthesis. Both lists returned are in increasing order.

EXAMPLES:

sage: from sage.combinat.tableau import unmatched_places
sage: unmatched_places([2,2,2,1,1,1],2,1)
([], [])
sage: unmatched_places([1,1,1,2,2,2],2,1)
([0, 1, 2], [3, 4, 5])
sage: unmatched_places([], 2, 1)
([], [])
sage: unmatched_places([1,2,4,6,2,1,5,3], 2, 1)
([0], [1])
sage: unmatched_places([2,2,1,2,4,6,2,1,5,3], 2, 1)
([], [0, 3])
sage: unmatched_places([3,1,1,1,2,1,2], 2, 1)
([1, 2, 3], [6])

5.1.344 Residue sequences of tableaux

A residue sequence for a StandardTableau, or StandardTableauTuple, of size $n$ is an $n$-tuple $(i_1, i_2, \ldots, i_n)$ of elements of $\mathbb{Z}/e\mathbb{Z}$ for some positive integer $e \geq 1$. Such sequences arise in the representation theory of the symmetric group and the closely related cyclotomic Hecke algebras, and cyclotomic quiver Hecke algebras, where the residue sequences play a similar role to weights in the representations of Lie groups and Lie algebras. These Hecke algebras are semisimple when $e$ is “large enough” and in these cases residue sequences are essentially the same as content sequences (see sage.combinat.partition.Partition.content()) and it is not difficult to see that residue sequences are in bijection with the set of standard tableaux. In the non-semisimple case, when $e$ is “small”, different standard tableaux can have the same residue sequence. In this case the residue sequences describe how to decompose modules into generalized eigenspaces for the Jucys-Murphy elements for these algebras.

By definition, if $t$ is a StandardTableau of size $n$ then the residue sequence of $t$ is the $n$-tuple $(i_1, i_2, \ldots, i_n)$ determined by $e$ and a multicharge $\kappa = (\kappa_1, \ldots, \kappa_l)$ by setting $i_m = \kappa_k + c - r + e\mathbb{Z}$, if $m$ appears in component $k$, row
$r$ and column $c$ of $t$. These sequences arise in the representation theory of the cyclotomic Hecke algebras of type A, which are also known as Ariki-Koike algebras.

The residue classes are constructed from standard tableaux:

```
sage: StandardTableau([[1,2],[3,4]]).residue_sequence(2)
2-residue sequence (0,1,1,0) with multicharge (0)
sage: StandardTableau([[1,2],[3,4]]).residue_sequence(3)
3-residue sequence (0,1,2,0) with multicharge (0)
```

One of the most useful functions of a $ResidueSequence$ is that it can return the $StandardTableaux_residue$ and $StandardTableaux_residue_shape$ that contain all of the tableaux with this residue sequence. Again, these are best accessed via the standard tableaux classes:

```
sage: res = StandardTableau([[1,2],[3,4]]).residue_sequence(2)
sage: res.standard_tableaux()
Standard tableaux with 2-residue sequence (0,1,1,0) and multicharge (0)
sage: res.standard_tableaux()
[[[1, 2, 4], [3]],
[[1, 2], [3, 4]],
[[1, 2], [3], [4]],
[[1, 3, 4], [2]],
[[1, 3], [2, 4]],
[[1, 3], [2], [4]]]
sage: res.standard_tableaux(shape=[4])
Standard (4)-tableaux with 2-residue sequence (0,1,1,0) and multicharge (0)
sage: res.standard_tableaux(shape=[4])
[]
sage: res=StandardTableauTuple([[[5]],[[1,2],[3,4]]]).residue_sequence(3,[0,0])
sage: res.standard_tableaux()
Standard tableaux with 3-residue sequence (0,1,2,0,0) and multicharge (0,0)
sage: res.standard_tableaux(shape=[4],[[1],[2,2]])
[[5], [[1, 2], [3, 4]], [[4], [[1, 2], [3, 5]]]]
```

These residue sequences are particularly useful in the graded representation theory of the cyclotomic KLR algebras and the cyclotomic Hecke algebras of type $A$; see [DJM1998] and [BK2009].

This module implements the following classes:

- $ResidueSequence$
- $ResidueSequences$

See also:

- $Partitions$
- $PartitionTuples$
- $StandardTableaux_residue$

5.1. Comprehensive Module List 3427
Todo: Strictly speaking this module implements residue sequences of type $A_{k}^{(1)}$. Residue sequences of other types also need to be implemented.

AUTHORS:

- Andrew Mathas (2016-07-01): Initial version

class sage.combinat.tableau_residues.ResidueSequence(parent, residues, check)
Bases: ClonableArray

A residue sequence.

The residue sequence of a tableau $t$ (of partition or partition tuple shape) is the sequence $(i_1, i_2, \ldots, i_n)$ where $i_k$ is the residue of $l$ in $t$, for $k = 1, 2, \ldots, n$, where $n$ is the size of $t$. Residue sequences are important in the representation theory of the cyclotomic Hecke algebras of type $G(r, 1, n)$, and of the cyclotomic quiver Hecke algebras, because they determine the eigenvalues of the Jucys-Murphy elements upon all modules. More precisely, they index and completely determine the irreducible representations of the (cyclotomic) Gelfand-Tsetlin algebras.

Rather than being called directly, residue sequences are best accessed via the standard tableaux classes `StandardTableau` and `StandardTableauTuple`.

INPUT:

Can be of the form:

- `ResidueSequence(e, res)`.
- `ResidueSequence(e, multicharge, res)`.

where $e$ is a positive integer not equal to 1 and `res` is a sequence of integers (the residues).

EXAMPLES:

```python
sage: res = StandardTableauTuple([[[1,3],[6]],[[2,7],[4],[5]]]).residue_sequence(3, (0,5))
sage: res
3-residue sequence (0,2,1,1,0,2,0) with multicharge (0,2)
sage: res.quantum_characteristic()
3
sage: res.level()
2
sage: res.size()
7
sage: res.residues()
[0, 2, 1, 1, 0, 2, 0]
sage: res.restrict(2)
```
3-residue sequence (0,2) with multicharge (0,2)

```python
sage: res.standard_tableaux([[2,1],[1],[2,1]])
```
Standard (2,1|1|2,1)-tableaux with 3-residue sequence (0,2,1,1,0,2,0) and _multicharge (0,2)

```python
sage: res.standard_tableaux([[2,2],[3]]).list()
[]
sage: res.standard_tableaux([[2,2],[3]])[:]
[]
sage: res.standard_tableaux()
Standard tableaux with 3-residue sequence (0,2,1,1,0,2,0) and multicharge (0,2)

```python
sage: res.standard_tableaux()[:10]
```

![Tableau Output]

The TestSuite fails _test_pickling because __getitem__ does not support slices, so we skip this.

### base_ring()

Return the base ring for the residue sequence.

If the `quantum_characteristic()` of the residue sequence `self` is \( e \) then the base ring for the sequence is \( \mathbb{Z}/e\mathbb{Z} \), or \( \mathbb{Z} \) if \( e = 0 \).

**EXAMPLES:**

```python
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).base_ring()
Ring of integers modulo 3
```

### block()

Return a dictionary \( \beta \) that determines the block associated to the residue sequence `self`.

Two Specht modules for a cyclotomic Hecke algebra of type \( A \) belong to the same block, in this sense, if and only if the residue sequences of their standard tableaux have the same block in this sense. The blocks of these algebras are actually indexed by positive roots in the root lattice of an affine special linear group. Instead of than constructing the root lattice, this method simply returns a dictionary \( \beta \) where the keys are residues \( i \) and where the value of the key \( i \) is equal to the numbers of nodes in the residue sequence `self` that are equal to \( i \). The dictionary \( \beta \) corresponds to the positive root:

\[
\sum_{i \in I} \beta_i \alpha_i \in \mathbb{Q}^+,
\]

These positive roots also index the blocks of the cyclotomic KLR algebras of type \( A \).

We return a dictionary because when the `quantum_characteristic()` is 0, the Cartan type is \( A_\infty \), in which case the simple roots are indexed by the integers, which is infinite.

**EXAMPLES:**

5.1. Comprehensive Module List 3429
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, [0,0,0], [0,1,2,0,1,2,0,1,2]).block()
{0: 3, 1: 3, 2: 3}

check()

Raise a ValueError if self is not a residue sequence.

EXAMPLES:

sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, [0,0,1], [0,0,1,1,2,2,3,3]).check()
sage: ResidueSequence(3, [0,0,1], [2,0,1,1,2,2,3,3]).check()

level()

Return the level of the residue sequence. That is, the level of the corresponding (tuples of) standard tableaux.

The level of a residue sequence is the length of its multicharge(). This is the same as the level of the standard_tableaux() that belong to the residue class of tableaux determined by self.

EXAMPLES:

sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).level()
3

multicharge()

Return the multicharge for the residue sequence self.

The $\epsilon$-residue sequences are associated with a cyclotomic Hecke algebra with Hecke parameter $q$ of quantum_characteristic() $\epsilon$ and multicharge $(\kappa_1, \ldots, \kappa_l)$. This means that the cyclotomic parameters of the Hecke algebra are $q^{\kappa_1}, \ldots, q^{\kappa_l}$. Equivalently, the Hecke algebra is determined by the dominant weight

$$\sum_{r \in \mathbb{Z}/\epsilon \mathbb{Z}} \kappa_r \Lambda_r \in P^+.$$ 

EXAMPLES:

sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).multicharge()
(0, 0, 1)

negative()

Return the negative of the residue sequence self.

That is, if self is the residue sequence $(i_1, \ldots, i_n)$ then return $(-i_1, \ldots, -i_n)$. Taking the negative residue sequences is a shadow of tensoring with the sign representation from the cyclotomic Hecke algebras of type $A$.

EXAMPLES:

sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).negative()
3-residue sequence (0,0,2,2,1,1,0,0) with multicharge (0,0,1)
quantum_characteristic()  
Return the quantum characteristic of the residue sequence self.

The \(e\)-residue sequences are associated with a cyclotomic Hecke algebra that has a parameter \(q\) of quantum characteristic \(e\). This is the smallest positive integer such that \(1 + q + \cdots + q^{e-1} = 0\), or \(e = 0\) if no such integer exists.

EXAMPLES:

```python
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).quantum_characteristic()
3
```

residues()  
Return a list of the residue sequence.

EXAMPLES:

```python
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3,(0,0,1),[0,0,1,1,2,2,3,3]).residues()
[0, 0, 1, 1, 2, 2, 0, 0]
```

restrict(m)  
Return the subsequence of this sequence of length \(m\).

The residue sequence self is of the form \((r_1, \ldots, r_n)\). The function returns the residue sequence \((r_1, \ldots, r_m)\), with the same quantum_characteristic() and multicharge().

EXAMPLES:

```python
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3,(0,0,1),[0,0,1,1,2,2,3,3]).restrict(7)
3-residue sequence (0,0,1,1,2,2,0) with multicharge (0,0,1)
sage: ResidueSequence(3,(0,0,1),[0,0,1,1,2,2,3,3]).restrict(6)
3-residue sequence (0,0,1,1,2,2) with multicharge (0,0,1)
sage: ResidueSequence(3,(0,0,1),[0,0,1,1,2,2,3,3]).restrict(4)
3-residue sequence (0,0,1,1) with multicharge (0,0,1)
```

restrict_row(cell, row)  
Return a residue sequence for the tableau obtained by swapping the row in ending in cell with the row that is row rows above it and which has the same length.

The residue sequence self is of the form \((r_1, \ldots, r_n)\). The function returns the residue sequence \((r_1, \ldots, r_m)\), with the same quantum_characteristic() and multicharge().

EXAMPLES:

```python
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, [0,1,2,2,0,1]).restrict_row((1,2),1)
3-residue sequence (2,0,1,0,1) with multicharge (0)
sage: ResidueSequence(3, [1,0], [0,1,2,2,0,1]).restrict_row((1,1,2),1)
3-residue sequence (2,0,1,0,1) with multicharge (1,0)
```

row_standard_tableaux(shape=None)  
Return the residue-class of row standard tableaux that have residue sequence self.

INPUT:
• shape – (optional) a partition or partition tuple of the correct level

OUTPUT:
An iterator for the row standard tableaux with this residue sequence. If the shape is given then only tableaux of this shape are returned, otherwise all of the full residue-class of row standard tableaux, or row standard tableaux tuples, is returned. The residue sequence self specifies the multicharge() of the tableaux which, in turn, determines the level() of the tableaux in the residue class.

EXAMPLES:

```sage
donate from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3,(0,0,0),[0,1,2,0,1,2,0,1,2]).row_standard_tableaux()
Row standard tableaux with 3-residue sequence (0,1,2,0,1,2,0,1,2) and
→multicharge (0,0,0)
sage: ResidueSequence(3,(0,0,0),[0,1,2,0,1,2,0,1,2]).row_standard_tableaux([[3],
→[3],[3]])
Row standard (3|3|3)-tableaux with 3-residue sequence (0,1,2,0,1,2,0,1,2) and
→multicharge (0,0,0)
```

size()
Return the size of the residue sequence.
This is the size, or length, of the residue sequence, which is the same as the size of the standard_tableaux() that belong to the residue class of tableaux determined by self.

EXAMPLES:

```sage
donate from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).size()
8
```

standard_tableaux(shape=None)
Return the residue-class of standard tableaux that have residue sequence self.

INPUT:
• shape – (optional) a partition or partition tuple of the correct level

OUTPUT:
An iterator for the standard tableaux with this residue sequence. If the shape is given then only tableaux of this shape are returned, otherwise all of the full residue-class of standard tableaux, or standard tableaux tuples, is returned. The residue sequence self specifies the multicharge() of the tableaux which, in turn, determines the level() of the tableaux in the residue class.

EXAMPLES:

```sage
donate from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3, (0,0,1), [0,0,1,1,2,2,3,3]).standard_tableaux()
Standard tableaux with 3-residue sequence (0,0,1,1,2,2,3,3) and
→multicharge (0,0,0)
sage: ResidueSequence(3, (0,0,0),[0,1,2,0,1,2,0,1,2]).standard_tableaux()
Standard tableaux with 3-residue sequence (0,1,2,0,1,2,0,1,2) and
→multicharge (0,0,0)
```
**swap_residues(i, j)**

Return the new residue sequence obtained by swapping the residues for i and j.

**INPUT:**

- i and j – two integers between 1 and the length of the residue sequence

If residue sequence self is of the form \((r_1, \ldots, r_n)\), and \(i < j\), then the residue sequence \((r_1, \ldots, r_j, \ldots, r_i, \ldots, r_m)\), with the same `quantum_characteristic()` and `multicharge()`, is returned.

**EXAMPLES:**

```python
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: res = ResidueSequence(3,(0,0,1),[0,0,1,2,2,3,3]); res
3-residue sequence (0,0,1,1,2,2,0,0) with multicharge (0,0,1)
sage: ser = res.swap_residues(2,6); ser
3-residue sequence (0,2,1,1,2,0,0,0) with multicharge (0,0,1)
sage: res == ser
False
```

**class** `sage.combinat.tableau_residues.ResidueSequences(e, multicharge=(0,))`

**Bases:** `UniqueRepresentation, Parent`

A parent class for `ResidueSequence`.

This class exists because `ResidueSequence` needs to have a parent. Apart from being a parent the only useful method that it provides is `cell_residue()`, which is a short-hand for computing the residue of a cell using the `ResidueSequence.quantum_characteristic()` and `ResidueSequence.multicharge()` for the residue class.

**EXAMPLES:**

```python
sage: from sage.combinat.tableau_residues import ResidueSequences
sage: ResidueSequences(e=0, multicharge=(0,1,2))
0-residue sequences with multicharge (0, 1, 2)
sage: ResidueSequences(e=0, multicharge=(0,1,2)) == ResidueSequences(e=0, multicharge=(0,1,2))
True
sage: ResidueSequences(e=0, multicharge=(0,1,2)) == ResidueSequences(e=3, multicharge=(0,1,2))
False
sage: ResidueSequences(e=0, multicharge=(0,1,2)).element_class
<class 'sage.combinat.tableau_residues.ResidueSequences_with_category.element_class'>
```

**Element**

alias of `ResidueSequence`

**an_element()**

Return a particular element of `self`.

**EXAMPLES:**

```python
sage: TableauTuples().an_element()
([[1], [[2]], [[3]], [[4]], [[5]], [[6]], [[7]])
```
cell_residue(*args)

Return the residue a cell with respect to the quantum characteristic and the multicharge of the residue sequence.

INPUT:

• r and c – the row and column indices in level one
• k, r and c – the component, row and column indices in higher levels

EXAMPLES:

```
sage: from sage.combinat.tableau_residues import ResidueSequences
sage: ResidueSequences(3).cell_residue(1,1)
0
sage: ResidueSequences(3).cell_residue(2,1)
2
sage: ResidueSequences(3).cell_residue(3,1)
1
sage: ResidueSequences(3).cell_residue(3,2)
2
sage: ResidueSequences(3,(0,1,2)).cell_residue(0,0,0)
0
sage: ResidueSequences(3,(0,1,2)).cell_residue(0,1,0)
2
sage: ResidueSequences(3,(0,1,2)).cell_residue(0,1,2)
1
sage: ResidueSequences(3,(0,1,2)).cell_residue(1,0,0)
1
sage: ResidueSequences(3,(0,1,2)).cell_residue(1,1,0)
0
sage: ResidueSequences(3,(0,1,2)).cell_residue(1,0,1)
2
sage: ResidueSequences(3,(0,1,2)).cell_residue(2,0,0)
2
sage: ResidueSequences(3,(0,1,2)).cell_residue(2,1,0)
1
sage: ResidueSequences(3,(0,1,2)).cell_residue(2,0,1)
0
```

check_element(element)

Check that element is a residue sequence with multicharge self.multicharge().

This is weak criteria in that we only require that element is a tuple of elements in the underlying base ring of self. Such a sequence is always a valid residue sequence, although there may be no tableaux with this residue sequence.

EXAMPLES:

```
sage: from sage.combinat.tableau_residues import ResidueSequence
sage: ResidueSequence(3,(0,0,1),[0,0,1,1,2,2,3,3]) # indirect doctest
3-residue sequence (0,0,1,1,2,2,0,0) with multicharge (0,0,1)
sage: ResidueSequence(3,(0,0,1),[2,0,1,4,2,2,5,3]) # indirect doctest
3-residue sequence (2,0,1,1,2,2,2,0) with multicharge (0,0,1)
sage: ResidueSequence(3,(0,0,1),[2,0,1,1,2,2,3,3]) # indirect doctest
3-residue sequence (2,0,1,1,2,2,0,0) with multicharge (0,0,1)
```
5.1.345 TableauTuples

A TableauTuple is a tuple of tableaux. These objects arise naturally in representation theory of the wreath products of cyclic groups and the symmetric groups where the standard tableau tuples index bases for the ordinary irreducible representations. This generalises the well-known fact the ordinary irreducible representations of the symmetric groups have bases indexed by the standard tableaux of a given shape. More generally, TableauTuples, or multitableaux, appear in the representation theory of the degenerate and non-degenerate cyclotomic Hecke algebras and in the crystal theory of the integral highest weight representations of the affine special linear groups.

A TableauTuple is an ordered tuple \((t^{(1)}, t^{(2)}, \ldots, t^{(l)})\) of tableaux. The length of the tuple is its level and the tableaux \(t^{(1)}, t^{(2)}, \ldots, t^{(l)}\) are the components of the TableauTuple.

A tableaux can be thought of as the labelled diagram of a partition. Analogously, a TableauTuple is the labelled diagram of a PartitionTuple. That is, a TableauTuple is a tableau of PartitionTuple shape. As much as possible, TableauTuples behave in exactly the same way as Tableaux. There are obvious differences in that the cells of a partition are ordered pairs \((r, c)\), where \(r\) is a row index and \(c\) a column index, whereas the cells of a PartitionTuple are ordered triples \((k, r, c)\), with \(r\) and \(c\) as before and \(k\) indexes the component.

Frequently, we will call a TableauTuple a tableau, or a tableau of PartitionTuple shape. If the shape of the tableau is known this should not cause any confusion.

**Warning:** In sage the convention is that the \((k, r, c)\)-th entry of a tableau tuple \(t\) is the entry in row \(r\), column \(c\) and component \(k\) of the tableau. This is because it makes much more sense to let \(t[k]\) be component of the tableau. In particular, we want \(t(k, r, c) = t[k][r][c]\). In the literature, the cells of a tableau tuple are usually written in the form \((r, c, k)\), where \(r\) is the row index, \(c\) is the column index, and \(k\) is the component index.

The same convention applies to the cells of PartitionTuples.

**Note:** As with partitions and tableaux, the cells are 0-based. For example, the (lexicographically) first cell in any non-empty tableau tuple is \([0, 0, 0]\).

**EXAMPLES:**

```sage
sage: TableauTuple([[1,2,3],[4,5]])
[[1, 2, 3], [4, 5]]

sage: t = TableauTuple([[ [6,7], [8,9] ], [[1,2,3],[4,5]] ]); t
([[6, 7], [8, 9]], [[1, 2, 3], [4, 5]])

sage: t.pp()
6 7 1 2 3
8 9 4 5

sage: t[0,0,1] 7

sage: t[1,0,1] 2

sage: t.shape() (([2, 2], [3, 2])

sage: t.size() 9

sage: t.level() 2

sage: t.components()
[[[6, 7], [8, 9]], [[1, 2, 3], [4, 5]]]
```

(continues on next page)
One reason for implementing `TableauTuples` is to be able to consider `StandardTableauTuples`. These objects arise in many areas of algebraic combinatorics. In particular, they index bases for the Specht modules of the cyclotomic Hecke algebras of type $G(r,1,n)$. A `StandardTableauTuple` of tableau whose entries are increasing along rows and down columns in each component and which contain the numbers $1, 2, \ldots, n$, where the shape of the `StandardTableauTuple` is a `PartitionTuple` of $n$.

As tableaux (of partition shape) are in natural bijection with 1-tuples of tableaux all of the `TableauTuple` classes return an ordinary `Tableau` when given `TableauTuple` of level 1.

There is one situation where a 1-tuple of tableau is not actually a `Tableau`; tableaux generated by the `StandardTableauTuples()` iterators must have the correct parents, so in this one case 1-tuples of tableaux are different from `Tableaux`:
sage: StandardTableauTuples()[:10]
[(), ([1]), ([2]), ([1, 2]), ([1], [2]), ([1], []), ([], [1]), ([], [], []), ([1, 2, 3]),
([1, 3], [2])]

AUTHORS:
- Andrew Mathas (2016-08-11): Row standard tableaux added

Element classes:
- TableauTuples
- StandardTableauTuples
- RowStandardTableauTuples

Factory classes:
- TableauTuples
- StandardTableauTuples
- RowStandardTableauTuples

Parent classes:
- TableauTuples_all
- TableauTuples_level
- TableauTuples_size
- TableauTuples_level_size
- StandardTableauTuples_all
- StandardTableauTuples_level
- StandardTableauTuples_size
- StandardTableauTuples_level_size
- StandardTableauTuples_shape
- StandardTableaux_residue
- StandardTableaux_residue_shape
- RowStandardTableauTuples_all
- RowStandardTableauTuples_level
- RowStandardTableauTuples_size
- RowStandardTableauTuples_level_size
Todo: Implement semistandard tableau tuples as defined in [DJM1998].

Much of the combinatorics implemented here is motivated by this and subsequent papers on the representation theory of these algebras.

class sage.combinat.tableau_tuple.RowStandardTableauTuple(
    parent, t, check=True)

    Bases: TableauTuple

    A class for row standard tableau tuples of shape a partition tuple.

    A row standard tableau tuple of size $n$ is an ordered tuple of row standard tableaux (see RowStandardTableau), with entries $1, 2, \ldots, n$ such that, in each component, the entries are in increasing order along each row. If the tableau in component $k$ has shape $\lambda^{(k)}$ then $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ is a PartitionTuple.

    Note: The tableaux appearing in a RowStandardTableauTuple are row strict, but individually they are not standard tableaux because the entries in any single component of a RowStandardTableauTuple will typically not be in bijection with $\{1, 2, \ldots, n\}$.

    INPUT:
    • $t$ – a tableau, a list of (standard) tableau or an equivalent list

    OUTPUT:
    • A RowStandardTableauTuple object constructed from $t$.

    Note: Sage uses the English convention for (tuples of) partitions and tableaux: the longer rows are displayed on top. As with PartitionTuple, in Sage the cells, or nodes, of partition tuples are 0-based. For example, the (lexicographically) first cell in any non-empty partition tuple is $[0, 0, 0]$. Further, the coordinates $[k, r, c]$ in a TableauTuple refer to the component, row and column indices, respectively.

    EXAMPLES:
    
    sage: t = RowStandardTableauTuple([[4,7],[3]], [[2,6,8],[1,5]], [[9]]); t
    ([4, 7], [3], [[2, 6, 8], [1, 5], [9]])
    sage: t.pp()
When using code that will generate a lot of tableaux, it is slightly more efficient to construct a `RowStandardTableauTuple` from the appropriate parent object:

```python
sage: RST = RowStandardTableauTuples()
sage: RST([[4,5],[7]], [[1,2,3],[6,8]], [[9]])
([[4, 5], [7]], [[1, 2, 3], [6, 8]], [[9]])
```

See also:
- `RowTableau`
- `RowTableaux`
- `TableauTuples`
- `TableauTuple`
- `StandardTableauTuples`
- `StandardTableauTuple`
- `RowStandardTableauTuples`

### codegree(e, multicharge)


The codegree of a tableau is an integer that is defined recursively by successively stripping off the number $k$, for $k = n, n-1, \ldots, 1$ and at stage adding the number of addable cell of the same residue minus the number of removable cells of the same residue as $k$ and which are above $k$ in the diagram.

The codegree of the tableau `self` gives the degree of “dual” homogeneous basis element of the graded Specht module which is indexed by `self`.

**INPUT:**
- `e` – the quantum characteristic
- `multicharge` – the multicharge

**OUTPUT:**
The codegree of the tableau self, which is an integer.

EXAMPLES:

```python
sage: StandardTableauTuple([[1]], [], []).codegree(0,(0,0,0))
0
sage: StandardTableauTuple([], [[1]], []).codegree(0,(0,0,0))
1
sage: StandardTableauTuple([], [], [[1]]).codegree(0,(0,0,0))
2
sage: StandardTableauTuple([[1]], [[2]], []).codegree(0,(0,0,0))
-1
sage: StandardTableauTuple([[1]], [], [[2]]).codegree(0,(0,0,0))
0
sage: StandardTableauTuple([], [[1]], [[2]]).codegree(0,(0,0,0))
1
sage: StandardTableauTuple([], [], [[1],[2]]).codegree(0,(0,0,0))
1
sage: StandardTableauTuple([], [[2]], [[1]]).codegree(0,(0,0,0))
2
sage: StandardTableauTuple([], [[2]], [[1]]).codegree(0,(0,0,0))
3
```

```
degree(e, multicharge)
```


The degree of a tableau is an integer that is defined recursively by successively stripping off the number \( k \), for \( k = n, n-1, \ldots, 1 \), and at stage adding the count of the number of addable cell of the same residue minus the number of removable cells of them same residue as \( k \) and that are below \( k \) in the diagram.

Note that even though this degree function was defined by Brundan-Kleshchev-Wang [BKW2011] the underlying combinatorics is much older, going back at least to Misra and Miwa.

The degrees of the tableau \( T \) gives the degree of the homogeneous basis element of the graded Specht module which is indexed by \( T \).

**INPUT:**

- \( e \) – the quantum characteristic \( e \)
- \( \text{multicharge} \) – (default: [0]) the multicharge

**OUTPUT:**

The degree of the tableau self, which is an integer.

**EXAMPLES:**

```python
sage: StandardTableauTuple([[1]], [], []).degree(0,(0,0,0))
2
sage: StandardTableauTuple([], [[1]], []).degree(0,(0,0,0))
1
sage: StandardTableauTuple([], [], [[1]]).degree(0,(0,0,0))
0
sage: StandardTableauTuple([[1]], [[2]], []).degree(0,(0,0,0))
3
sage: StandardTableauTuple([[1]], [], [[2]]).degree(0,(0,0,0))
2
```
Combinatorics, Release 10.0

sage: StandardTableauTuple([][[1]], [[2]]).degree(0,(0,0,0))
1
sage: StandardTableauTuple([[2],[1]], []).degree(0,(0,0,0))
1
sage: StandardTableauTuple([[2], [], [1]]).degree(0,(0,0,0))
0
sage: StandardTableauTuple([], [2], [1]).degree(0,(0,0,0))
-1

inverse(k)
Return the cell containing k in the tableau tuple self.

EXAMPLES:

sage: RowStandardTableauTuple([[3,4],[1,2]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(1)
(0, 1, 0)
sage: RowStandardTableauTuple([[3,4],[1,2]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(2)
(0, 1, 1)
sage: RowStandardTableauTuple([[3,4],[1,2]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(3)
(0, 0, 0)
sage: RowStandardTableauTuple([[3,4],[1,2]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(4)
(0, 0, 1)
sage: RowStandardTableauTuple([[1,2],[3,4]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(1)
(0, 0, 0)
sage: RowStandardTableauTuple([[1,2],[3,4]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(2)
(0, 0, 1)
sage: RowStandardTableauTuple([[1,2],[3,4]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(3)
(0, 1, 0)
sage: RowStandardTableauTuple([[1,2],[3,4]], [[5,6,7],[8]], [[9,10],[11],[12]]).inverse(12)
(2, 2, 0)

residue_sequence(e, multicharge)
Return the sage.combinat.tableau_residues.ResidueSequence of self.

INPUT:

- e – integer in {0, 2, 3, 4, 5, ...}
- multicharge – a sequence of integers of length equal to the level/length of self

OUTPUT:

The residue sequence of the tableau.

EXAMPLES:
sage: RowStandardTableauTuple([[5],[3,4],[1,2]]).residue_sequence(3,[0,0])
3-residue sequence (2,0,1,0) with multicharge (0,0)
sage: StandardTableauTuple([[5],[1,2],[3,4]]).residue_sequence(3,[0,1])
3-residue sequence (1,2,0,1,0) with multicharge (0,1)
sage: StandardTableauTuple([[5],[1,2],[3,4]]).residue_sequence(3,[0,2])
3-residue sequence (2,0,1,2,0) with multicharge (0,2)

class sage.combinat.tableau_tuple.RowStandardTableauTuples

Bases: TableauTuples

A factory class for the various classes of tuples of row standard tableau.

INPUT:

There are three optional arguments:

• level – the level() of the tuples of tableaux
• size – the size() of the tuples of tableaux
• shape – a list or a partition tuple specifying the shape() of the row standard tableau tuples

It is not necessary to use the keywords. If they are not used then the first integer argument specifies the level() and the second the size() of the tableau tuples.

OUTPUT:

The appropriate subclass of RowStandardTableauTuples.

A tuple of row standard tableau is a tableau whose entries are positive integers which increase from left to right along the rows in each component. The entries do NOT need to increase from left to right along the components.

Note: Sage uses the English convention for (tuples of) partitions and tableaux: the longer rows are displayed on top. As with PartitionTuple, in sage the cells, or nodes, of partition tuples are 0-based. For example, the (lexicographically) first cell in any non-empty partition tuple is [0, 0, 0].

EXAMPLES:

sage: tabs = RowStandardTableauTuples([[2],[1,1]]); tabs
Row standard tableau tuples of shape ([2], [1, 1])
sage: tabs.cardinality()
12
sage: tabs[:]
[[[3, 4]], [[2], [1, 1]]],
[[[2, 4]], [[3], [1]]],
[[[1, 4]], [[3], [2]]],
[[[1, 2]], [[4], [3]]],
[[[1, 3]], [[4], [2]]],
[[[2, 3]], [[4], [1]]],
[[[1, 4]], [[2], [3]]],
[[[1, 3]], [[2], [4]]],
[[[1, 2]], [[3], [4]]],
[[[2, 3]], [[1], [4]]],
[[[2, 4]], [[1], [3]]],
[[[3, 4]], [[1], [2]]]
sage: tabs = RowStandardTableauTuples(level=3); tabs
Row standard tableau tuples of level 3
sage: tabs[100]
([], [], [[2, 3], [1]])

sage: RowStandardTableauTuples()[0]
([], [], [], [], [])

See also:

- TableauTuples
- Tableau
- RowStandardTableau
- RowStandardTableauTuples

Element
alias of RowStandardTableauTuple

level_one_parent_class
alias of RowStandardTableaux_all

shape()
Return the shape of the set of RowStandardTableauTuples, or None if it is not defined.

EXAMPLES:

sage: tabs=RowStandardTableauTuples(shape=[[5,2],[3,2],[],[1,1,1],[3]]); tabs
Row standard tableau tuples of shape ([5, 2], [3, 2], [], [1, 1, 1], [3])
sage: tabs.shape()
([5, 2], [3, 2], [], [1, 1, 1], [3])
sage: RowStandardTableauTuples().shape() is None
True

class sage.combinat.tableau_tuple.RowStandardTableauTuples_all
Bases: RowStandardTableauTuples, DisjointUnionEnumeratedSets
Default class of all RowStandardTableauTuples with an arbitrary level() and size().

an_element()
Return a particular element of the class.

EXAMPLES:

sage: RowStandardTableauTuples().an_element()
([[4, 5, 6, 7]], [[2, 3]], [[1]])

class sage.combinat.tableau_tuple.RowStandardTableauTuples_level(level)
Bases: RowStandardTableauTuples, DisjointUnionEnumeratedSets
Class of all RowStandardTableauTuples with a fixed level and arbitrary size.

an_element()
Return a particular element of the class.

EXAMPLES:
```python
sage: RowStandardTableauTuples(2).an_element()
([[1]], [[2, 3]])
sage: RowStandardTableauTuples(3).an_element()
([[1]], [[2, 3]], [[4, 5, 6, 7]])
```

class sage.combinat.tableau_tuple.RowStandardTableauTuples_level_size(level, size)

Bases: `RowStandardTableauTuples`, `DisjointUnionEnumeratedSets`

Class of all `RowStandardTableauTuples` with a fixed level and a fixed size.

**an_element()**

Return a particular element of `self`.

**EXAMPLES:**

```python
sage: RowStandardTableauTuples(5, size=2).an_element()
([], [], [], [], [[1], [2]])
sage: RowStandardTableauTuples(2, size=4).an_element()
([[1]], [[2, 3], [4]])
```

class sage.combinat.tableau_tuple.RowStandardTableauTuples_residue(residue)

Bases: `RowStandardTableauTuples`

Class of all row standard tableau tuples with a fixed residue sequence.

Implicitly, this also specifies the quantum characteristic, multicharge and hence the level and size of the tableaux.

**Note:** This class is not intended to be called directly, but rather, it is accessed through the row standard tableaux.

**EXAMPLES:**

```python
sage: RowStandardTableau([[3,4,5],[1,2]]).residue_sequence(2).row_standard_tableaux()  # Row standard tableaux with 2-residue sequence (1,0,0,1,0) and multicharge (0)
sage: RowStandardTableau([[3,4,5],[1,2]]).residue_sequence(3).row_standard_tableaux()  # Row standard tableaux with 3-residue sequence (2,0,1,2) and multicharge (0)
sage: RowStandardTableauTuple([[[5,6],[7]],[[1,2,3],[4]]]).residue_sequence(2,(0,1)).row_standard_tableaux()  # Row standard tableaux with 2-residue sequence (0,1,0,1,0,1,1) and multicharge (0,0)
sage: RowStandardTableauTuple([[[5,6],[7]],[[1,2,3],[4]]]).residue_sequence(3,(0,1)).row_standard_tableaux()  # Row standard tableaux with 3-residue sequence (1,2,0,0,0,1,2) and multicharge (0,1)
```

**an_element()**

Return a particular element of `self`.

**EXAMPLES:**

```python
sage: RowStandardTableau([[2,3],[1]]).residue_sequence(3).row_standard_tableaux().an_element()  # Continuing on next page
[[2, 3], [1]]
sage: StandardTableau([[1,3],[2]]).residue_sequence(3).row_standard_tableaux().an_element()  # Continuing on next page
```
level()

Return the level of self.

EXAMPLES:

```python
sage: RowStandardTableau([[2,3],[1]]).residue_sequence(3,(0,1)).row_standard_tableaux().level()
2
sage: StandardTableau([[1,2],[3]]).residue_sequence(3,(0,1)).row_standard_tableaux().level()
2
sage: RowStandardTableauTuple([[4],[[2,3],[1]]]).residue_sequence(3,(0,1)).row_standard_tableaux().level()
2
sage: StandardTableauTuple([[4],[[1,3],[2]]]).residue_sequence(3,(0,1)).row_standard_tableaux().level()
2
```

multicharge()

Return the multicharge of self.

EXAMPLES:

```python
sage: RowStandardTableau([[2,3],[1]]).residue_sequence(3,(0,1)).row_standard_tableaux().multicharge()
(0, 1)
sage: StandardTableau([[1,2],[3]]).residue_sequence(3,(0,1)).row_standard_tableaux().multicharge()
(0, 1)
sage: RowStandardTableauTuple([[4],[[2,3],[1]]]).residue_sequence(3,(0,1)).row_standard_tableaux().multicharge()
(0, 1)
sage: StandardTableauTuple([[4],[[1,3],[2]]]).residue_sequence(3,(0,1)).row_standard_tableaux().multicharge()
(0, 1)
```

quantum_characteristic()

Return the quantum characteristic of self.

EXAMPLES:

```python
sage: RowStandardTableau([[2,3],[1]]).residue_sequence(3,(0,1)).row_standard_tableaux().quantum_characteristic()
3
sage: StandardTableau([[1,2],[3]]).residue_sequence(3,(0,1)).row_standard_tableaux().quantum_characteristic()
```
sage: RowStandardTableauTuple([[4],[[2,3],[1]]]).residue_sequence(3,(0,1)).row_standard_tableaux().quantum_characteristic()  

3

sage: StandardTableauTuple([[4],[[1,3],[2]]]).residue_sequence(3,(0,1)).row_standard_tableaux().quantum_characteristic()  

3

residue_sequence()
Return the residue sequence of self.

EXAMPLES:

sage: RowStandardTableau([[2,3],[1]]).residue_sequence(3,(0,1)).row_standard_tableaux().residue_sequence()  

3-residue sequence (2,0,1) with multicharge (0,1)

sage: StandardTableau([[1,2],[3]]).residue_sequence(3,(0,1)).row_standard_tableaux().residue_sequence()  

3-residue sequence (0,1,2) with multicharge (0,1)

sage: RowStandardTableauTuple([[4],[[2,3],[1]]]).residue_sequence(3,(0,1)).row_standard_tableaux().residue_sequence()  

3-residue sequence (0,1,2,0) with multicharge (0,1)

sage: StandardTableauTuple([[4],[[1,3],[2]]]).residue_sequence(3,(0,1)).row_standard_tableaux().residue_sequence()  

3-residue sequence (1,0,2,0) with multicharge (0,1)

size()
Return the size of self.

EXAMPLES:

sage: RowStandardTableau([[2,3],[1]]).residue_sequence(3,(0,1)).row_standard_tableaux().size()  

3

sage: StandardTableau([[1,2],[3]]).residue_sequence(3,(0,1)).row_standard_tableaux().size()  

3

sage: RowStandardTableauTuple([[4],[[2,3],[1]]]).residue_sequence(3,(0,1)).row_standard_tableaux().size()  

4

sage: StandardTableauTuple([[4],[[1,3],[2]]]).residue_sequence(3,(0,1)).row_standard_tableaux().size()  

4

class sage.combinat.tableau_tuple.RowStandardTableauTuples_residue_shape(residue, shape)
Bases: RowStandardTableauTuples_residue
All row standard tableau tuples with a fixed residue and shape.

INPUT:

- shape – the shape of the partitions or partition tuples
- residue – the residue sequence of the label

EXAMPLES:
```
sage: res = RowStandardTableauTuple([[3, 6], [1]], [[5, 7], [4], [2]]).residue_sequence(3, (0, 0))
sage:.tabs = res.row_standard_tableaux([[2, 1], [2, 1, 1]]); tabs
Row standard (2,1|2,1^2)-tableaux with 3-residue sequence (2,1,0,2,0,1,1) and multicharge (0,0)
sage: tabs.shape()
((2, 1), (2, 1, 1))
sage: tabs.level()
2
sage: tabs[:6]
[[[[5, 7], [4]], [[3, 6], [1], [2]]],
 [[[5, 7], [1]], [[3, 6], [4], [2]]],
 [[[3, 7], [4]], [[5, 6], [1], [2]]],
 [[[3, 7], [1]], [[5, 6], [4], [2]]],
 [[[5, 6], [4]], [[3, 7], [1], [2]]],
 [[[5, 6], [1]], [[3, 7], [4], [2]]]]
```

class sage.combinat.tableau_tuple.RowStandardTableauTuples_shape(shape)
Bases: RowStandardTableauTuples

Class of all RowStandardTableauTuples of a fixed shape.

an_element()

Return a particular element of self.

EXAMPLES:
```
sage: RowStandardTableauTuples([[2],[2,1]]).an_element()
([], [[4, 5]], [[1, 3], [2]])
sage: RowStandardTableauTuples([[10],[[],[]]]).an_element()
([], [], [[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]])
```

cardinality()

Return the number of row standard tableau tuples of with the same shape as the partition tuple self.

This is just the index of the corresponding Young subgroup in the full symmetric group.

EXAMPLES:
```
sage: RowStandardTableauTuples([[3,2,1],[[]]]).cardinality()
60
sage: RowStandardTableauTuples([[1],[1],[1]]).cardinality()
6
sage: RowStandardTableauTuples([[2],[1],[1]]).cardinality()
60
```

class sage.combinat.tableau_tuple.RowStandardTableauTuples_size(size)
Bases: RowStandardTableauTuples, DisjointUnionEnumeratedSets

Class of all RowStandardTableauTuples with an arbitrary level and a fixed size.

an_element()

Return a particular element of the class.

EXAMPLES:
class sage.combinat.tableau_tuple.StandardTableauTuple(parent, t, check=True)

Bases: RowStandardTableauTuple

A class to model a standard tableau of shape a partition tuple. This is a tuple of standard tableau with entries 1, 2, ..., n, where n is the size of the underlying partition tuple, such that the entries increase along rows and down columns in each component of the tuple.

sage: s = StandardTableauTuple([[1,2,3],[4,5]])
sage: t = StandardTableauTuple([[1,2],[3,5],[4]])
sage: s.dominates(t) True
sage: t.dominates(s) False
sage: StandardTableauTuple([[1,2,3],[4,5]]) in RowStandardTableauTuples() True

The tableaux appearing in a StandardTableauTuple are both row and column strict, but individually they are not standard tableaux because the entries in any single component of a StandardTableauTuple will typically not be in bijection with \{1,2,...,n\}.

INPUT:
• t – a tableau, a list of (standard) tableau or an equivalent list

OUTPUT:
• A StandardTableauTuple object constructed from t.

Note: Sage uses the English convention for (tuples of) partitions and tableaux: the longer rows are displayed on top. As with PartitionTuple, in sage the cells, or nodes, of partition tuples are 0-based. For example, the (lexicographically) first cell in any non-empty partition tuple is [0,0,0]. Further, the coordinates [k,r,c] in a TableauTuple refer to the component, row and column indices, respectively.

EXAMPLES:

sage: t = TableauTuple([[1,3,4],[7,9]],[[2,8,11],[6]],[[5,10]])
sage: t
[[[1, 3, 4], [7, 9]], [[2, 8, 11], [6]], [[5, 10]]]

sage: t[0][0][0] 1
sage: t[1][1][0] 6
sage: t[2][0][0] 5
sage: t[2][0][1] 10

sage: t = StandardTableauTuple([[4,5],[7],[1,2,3],[6,8]],[[9]])

sage: t.pp()
4 5 1 2 3 9
7 6 8

sage: t.shape()
[[2, 1], [3, 2], [1]]

sage: t[0].pp() # pretty printing
(continues on next page)
When using code that will generate a lot of tableaux, it is slightly more efficient to construct a `StandardTableauTuple` from the appropriate parent object:

```
sage: STT = StandardTableauTuples()
sage: STT([[4,5],[7]], [[1,2,3],[6,8]], [[9]])
([[4, 5], [7]], [[1, 2, 3], [6, 8]], [[9]])
```

See also:

- `Tableau`
- `Tableaux`
- `TableauTuples`
- `TableauTuple`
- `StandardTableauTuples`

`dominates(t)`

Return `True` if the tableau (tuple) `self` dominates the tableau `t`. The two tableaux do not need to be of the same shape.

EXAMPLES:

```
sage: s = StandardTableauTuple([[1,2,3],[4,5]])
sage: t = StandardTableauTuple([[1,2],[3,5],[4]])
sage: s.dominates(t)
True
sage: t.dominates(s)
False
```

`restrict(m=None)`

Return the restriction of the standard tableau `self` to `m`, which defaults to one less than the current `size()`.

EXAMPLES:

```
sage: StandardTableauTuple([[5],[1,2],[3,4]]).restrict(6)
([], [], [1, 2, 3, 4])
sage: StandardTableauTuple([[5],[1,2],[3,4]]).restrict(5)
([], [1, 2], [3, 4])
sage: StandardTableauTuple([[5],[1,2],[3,4]]).restrict(4)
([], [1, 2], [3, 4])
sage: StandardTableauTuple([[5],[1,2],[3,4]]).restrict(3)
([], [1, 2], [3])
sage: StandardTableauTuple([[5],[1,2],[3,4]]).restrict(2)
(continues on next page)
```
(continued from previous page)

$$\left(\emptyset, \left[\begin{array}{l} 1 \end{array}\right]\right)$$

```
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).restrict(1)
(\emptyset, \left[\begin{array}{l} 1 \end{array}\right])
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).restrict(0)
(\emptyset, \emptyset)
```

Where possible the restricted tableau belongs to the same category as the tableau self:

```
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Standard tableau tuples
sage: StandardTableauTuples([[1],[2,2]])([[5],[[1,2],[3,4]]]).restrict(3).
→ category()
Category of elements of Standard tableau tuples
sage: StandardTableauTuples(level=2)([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Standard tableau tuples of level 2
```

**to_chain()**

Return the chain of partitions corresponding to the standard tableau tuple self.

**EXAMPLES:**

```
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).to_chain()
[(\emptyset, \emptyset),
 (\emptyset, [1]),
 (\emptyset, [2]),
 ([1], [2, 2]),
 ([1], [2, 2])]
```

### class sage.combinat.tableau_tuple.StandardTableauTuples

**Bases:** RowStandardTableauTuples

A factory class for the various classes of tuples of standard tableau.

**INPUT:**

There are three optional arguments:

- **level** – the level() of the tuples of tableaux
- **size** – the size() of the tuples of tableaux
- **shape** – a list or a partition tuple specifying the shape() of the standard tableau tuples

It is not necessary to use the keywords. If they are not used then the first integer argument specifies the level() and the second the size() of the tableau tuples.

**OUTPUT:**

The appropriate subclass of StandardTableauTuples.

A tuple of standard tableau is a tableau whose entries are positive integers which increase from left to right along the rows, and from top to bottom down the columns, in each component. The entries do NOT need to increase from left to right along the components.
Note:  Sage uses the English convention for (tuples of) partitions and tableaux: the longer rows are displayed on top. As with PartitionTuple, in sage the cells, or nodes, of partition tuples are 0-based. For example, the (lexicographically) first cell in any non-empty partition tuple is [0, 0, 0].

EXAMPLES:

```
sage: tabs=StandardTableauTuples([[3],[2,2]]); tabs
Standard tableau tuples of shape ([3], [2, 2])
sage: tabs.cardinality()
70
sage: tabs[10:16]
[([[[1, 2, 3]], [[4, 6], [5, 7]]),
  ([[[1, 2, 4]], [[3, 6], [5, 7]]),
  ([[[1, 3, 4]], [[2, 6], [5, 7]]),
  ([[[2, 3, 4]], [[1, 6], [5, 7]]),
  ([[[1, 2, 5]], [[3, 6], [4, 7]]),
  ([[[1, 3, 5]], [[2, 6], [4, 7]]])
```

```
sage: tabs=StandardTableauTuples(level=3); tabs
Standard tableau tuples of level 3
sage: tabs[100]
([[[1, 2], [3]]], [], [[4]])
sage: StandardTableauTuples()[0]
()
```

See also:

- TableauTuples
- Tableau
- StandardTableau
- StandardTableauTuples

**Element**

alias of StandardTableauTuple

**level_one_parent_class**

alias of StandardTableaux_all

**shape()**

Return the shape of the set of StandardTableauTuples, or None if it is not defined.

EXAMPLES:

```
sage: tabs=StandardTableauTuples(shape=[[5,2],[3,2],[],[1,1,1],[3]]); tabs
Standard tableau tuples of shape ([5, 2], [3, 2], [], [1, 1, 1], [3])
sage: tabs.shape()
([5, 2], [3, 2], [], [1, 1, 1], [3])
sage: StandardTableauTuples().shape() is None
True
```
class sage.combinat.tableau_tuple.StandardTableauTuples_all
    Bases: StandardTableauTuples, DisjointUnionEnumeratedSets
    
    Default class of all StandardTableauTuples with an arbitrary level() and size().

class sage.combinat.tableau_tuple.StandardTableauTuples_level(level)
    Bases: StandardTableauTuples, DisjointUnionEnumeratedSets
    
    Class of all StandardTableauTuples with a fixed level and arbitrary size.

    an_element()
        Return a particular element of the class.

        EXAMPLES:
        
        sage: StandardTableauTuples(size=2).an_element()
        ([[1]], [[2]], [], [])
        sage: StandardTableauTuples(size=4).an_element()
        ([[1]], [[2, 3, 4]], [], [])

class sage.combinat.tableau_tuple.StandardTableauTuples_level_size(level, size)
    Bases: StandardTableauTuples, DisjointUnionEnumeratedSets
    
    Class of all StandardTableauTuples with a fixed level and a fixed size.

    an_element()
        Return a particular element of the class.

        EXAMPLES:
        
        sage: StandardTableauTuples(5, size=2).an_element()
        ([], [], [], [], [[1], [2]])
        sage: StandardTableauTuples(2, size=4).an_element()
        ([[1]], [[2, 3]], [4])

cardinality()
    
    Return the number of elements in this set of tableaux.

    EXAMPLES:
    
    sage: StandardTableauTuples(3, 2).cardinality()
    12
    sage: StandardTableauTuples(4, 6).cardinality()
    31936

class sage.combinat.tableau_tuple.StandardTableauTuples_shape(shape)
    Bases: StandardTableauTuples
    
    Class of all StandardTableauTuples of a fixed shape.

    an_element()
        Return a particular element of the class.

        EXAMPLES:
        
        sage: StandardTableauTuples([[2], [2, 1]]).an_element()
        ([[2], [2, 1]], [[1, 3], [5]])
        sage: StandardTableauTuples([[10], [], []]).an_element()
        ([[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]], [], [])
cardinality()

Return the number of standard Young tableau tuples of with the same shape as the partition tuple self.

Let $\mu = (\mu^{(1)}, \ldots, \mu^{(l)})$ be the shape of the tableaux in self and let $m_k = |\mu^{(k)}|$, for $1 \leq k \leq l$. Multiplying by a (unique) coset representative of the Young subgroup $S_{m_1} \times \cdots \times S_{m_l}$ inside the symmetric group $S_n$, we can assume that $t$ is standard and the numbers $1, 2, \ldots, n$ are entered in order from to right along the components of the tableau. Therefore, there are

$$\binom{n}{m_1, \ldots, m_l} \prod_{k=1}^{l} |\text{Std}(\mu^{(k)})|$$

standard tableau tuples of this shape, where $|\text{Std}(\mu^{(k)})|$ is the number of standard tableau of shape $\mu^{(k)}$, for $1 \leq k \leq l$. This is given by the hook length formula.

EXAMPLES:

```python
sage: StandardTableauTuples([[3,2,1],[1]]).cardinality()
16
sage: StandardTableauTuples([[1],[1],[1]]).cardinality()
6
sage: StandardTableauTuples([[2,1],[1],[1]]).cardinality()
40
sage: StandardTableauTuples([[3,2,1],[3,2]]).cardinality()
36960
```

last()

Return the last standard tableau tuple in self, with respect to the order that they are generated by the iterator.

This is just the standard tableau tuple with the numbers $1, 2, \ldots, n$, where $n$ is size(), entered in order down the columns form right to left along the components.

EXAMPLES:

```python
sage: StandardTableauTuples([[2],[2,2]]).last().pp()
5 6 1 3
 2 4
```

random_element()

Return a random standard tableau in self.

We do this by randomly selecting addable nodes to place $1, 2, \ldots, n$. Of course we could do this recursively, but it is more efficient to keep track of the (changing) list of addable nodes as we go.

EXAMPLES:

```python
sage: StandardTableauTuples([[2],[2,1]]).random_element() # random
([[1, 2]], [[3, 4], [5]])
```

class sage.combinat.tableau_tuple.StandardTableauTuples_size(size)

Bases: StandardTableauTuples, DisjointUnionEnumeratedSets

Class of all StandardTableauTuples with an arbitrary level and a fixed size.

an_element()

Return a particular element of the class.

EXAMPLES:
```python
sage: StandardTableauTuples(size=2).an_element()
([[1]], [[2]], [], [])
sage: StandardTableauTuples(size=4).an_element()
([[1], [2, 3, 4]], [], [])
```

class sage.combinat.tableau_tuple.StandardTableaux_residue(residue)

Bases: StandardTableauTuples

Class of all standard tableau tuples with a fixed residue sequence.

Implicitly, this also specifies the quantum characteristic, multicharge and hence the level and size of the tableaux.

Note: This class is not intended to be called directly, but rather, it is accessed through the standard tableaux.

EXAMPLES:

```python
sage: StandardTableau([[1,2,3],[4,5]]).residue_sequence(2).standard_tableaux()
Standard tableaux with 2-residue sequence (0,1,0,1,0) and multicharge (0)
sage: StandardTableau([[1,2,3],[4,5]]).residue_sequence(3).standard_tableaux()
Standard tableaux with 3-residue sequence (0,1,2,2,0) and multicharge (0)
sage: StandardTableauTuple([[5,6],[7]], [[1,2,3],[4]]).residue_sequence(2,(0,0)).standard_tableaux()
Standard tableaux with 2-residue sequence (0,1,0,1,0,1,1) and multicharge (0,0)
sage: StandardTableauTuple([[5,6],[7]], [[1,2,3],[4]]).residue_sequence(3,(0,1)).standard_tableaux()
Standard tableaux with 3-residue sequence (1,2,0,0,0,1,2) and multicharge (0,1)
```

class sage.combinat.tableau_tuple.StandardTableaux_residue_shape(residue, shape)

Bases: StandardTableaux_residue

All standard tableau tuples with a fixed residue and shape.

INPUT:

- shape – the shape of the partitions or partition tuples
- residue – the residue sequence of the label

EXAMPLES:

```python
sage: res = StandardTableauTuple([[1,3],[6]], [[2,7],[4],[5]]).residue_sequence(3, (0,0))
sage: tabs = res.standard_tableaux([[2,1],[2,1,1]]); tabs
Standard (2,1^2)-tableaux with 3-residue sequence (0,0,1,2,1,2,1) and multicharge (0,0)
sage: tabs.shape()
([2, 1], [2, 1, 1])
sage: tabs.level()
2
sage: tabs[:6]

([(2, 7), [6]], [[1, 3], [4], [5]]),
([(1, 7), [6]], [[2, 3], [4], [5]]),
([(2, 3), [6]], [[1, 7], [4], [5]]),
([(1, 3), [6]], [[2, 7], [4], [5]]),
```

(continues on next page)
an_element()

Return a particular element of self.

EXAMPLES:

```
sage: T = StandardTableau([[1,3],[2]]).residue_sequence(3).standard_tableaux([2,1])
sage: T.an_element()
[[1, 3], [2]]
```

class sage.combinat.tableau_tuple.TableauTuple(parent, t, check=True)

Bases: CombinatorialElement

A class to model a tuple of tableaux.

INPUT:

- t – a list or tuple of Tableau, a list or tuple of lists of lists

OUTPUT:

- The Tableau tuple object constructed from t.

A TableauTuple is a tuple of tableau of shape a PartitionTuple. These combinatorial objects are useful in several areas of algebraic combinatorics. In particular, they are important in:

- the representation theory of the complex reflection groups of type $G(l, 1, n)$ and the representation theory of the associated (degenerate and non-degenerate) Hecke algebras. See, for example, [DJM1998]
- the crystal theory of (quantum) affine special linear groups and its integral highest weight modules and their canonical bases. See, for example, [BK2009].

These apparently different and unrelated contexts are, in fact, intimately related as in characteristic zero the cyclotomic Hecke algebras categorify the canonical bases of the integral highest weight modules of the quantum affine special linear groups.

The level() of a tableau tuple is the length of the tuples. This corresponds to the level of the corresponding highest weight module.

In sage a TableauTuple looks an behaves like a real tuple of (level 1) Tableau. Many of the operations which are defined on Tableau extend to TableauTuples. Tableau tuples of level 1 are just ordinary Tableau.

In sage, the entries of Tableaux can be very general, including arbitrarily nested lists, so some lists can be interpreted either as a tuple of tableaux or simply as tableaux. If it is possible to interpret the input to TableauTuple as a tuple of tableaux then TableauTuple returns the corresponding tuple. Given a 1-tuple of tableaux the tableau itself is returned.

EXAMPLES:

```
sage: t = TableauTuple([[6,9,10],[11]], [[1,2,3],[4,5]], [[7],[8]] ); t
([[6, 9, 10], [11]], [[1, 2, 3], [4, 5]], [[7], [8]])
sage: t.level()
3
sage: t.size()
11
sage: t.shape()
(continues on next page)
sage: t.is_standard()
True
sage: t.pp()  # pretty printing
   6 9 10    1 2 3    7
  11        4 5     8
sage: t.category()
Category of elements of Tableau tuples
sage: t.parent()
Tableau tuples

sage: s = TableauTuple([['a', 'c', 'b'], ['d', 'e'], [(2, 1)]]); s
([[a', 'c', 'b'], ['d', 'e'], [[(2, 1)]])

sage: s.shape()
([3, 2], [1])

sage: s.size()
6

sage: TableauTuple([], [], []))  # The empty 3-tuple of tableaux
([], [], []

sage: TableauTuple([[1, 2, 3], [4, 5]])
[[1, 2, 3], [4, 5]]

sage: TableauTuple([[1, 2, 3], [4, 5]]) == Tableau([[1, 2, 3], [4, 5]])
True

See also:

- StandardTableauTuple
- StandardTableauTuples
- StandardTableau
- StandardTableaux
- TableauTuple
- TableauTuples
- Tableau
- Tableaux

Element

alias of Tableau

add_entry(cell, m)

Set the entry in cell equal to m. If the cell does not exist then extend the tableau, otherwise just replace the entry.

EXAMPLES:

sage: s = StandardTableauTuple([[3, 4, 7], [6, 8]], [[9, 13], [12]], [[1, 5], [2, 11], →[10]])); s.pp()
3 4 7 9 13 1 5

(continues on next page)
sage: t = s.add_entry( (0,0,3),14); t.pp(); t.category()
\[
3 4 7 14 9 13 1 5
6 8 12 2 11
\]

Category of elements of Standard tableau tuples

sage: t = s.add_entry( (0,0,3),15); t.pp(); t.category()
\[
3 4 7 15 9 13 1 5
6 8 12 2 11
\]

Category of elements of Tableau tuples

sage: t = s.add_entry( (1,1,1),14); t.pp(); t.category()
\[
3 4 7 9 13 1 5
6 8 12 14 2 11
\]

Category of elements of Standard tableau tuples

sage: t = s.add_entry( (2,1,1),14); t.pp(); t.category()
\[
3 4 7 9 13 1 5
6 8 12 2 14
\]

Category of elements of Tableau tuples

sage: t = s.add_entry( (2,1,2),14); t.pp(); t.category()

Traceback (most recent call last):
...
IndexError: (2, 1, 2) is not an addable cell of the tableau

**cells_containing(m)**

Return the list of cells in which the letter \( m \) appears in the tableau \( \textsf{self} \).

The list is ordered with cells appearing from left to right.

**EXAMPLES:**

```python
sage: t = TableauTuple([[4,5], [[1,1,2,4],[2,4,4], [4]], [[1,3,4],[3,4]]])
sage: t.cells_containing(4)
[[(0, 0, 0), (1, 2, 0), (1, 1, 1), (1, 1, 2), (1, 0, 3), (2, 1, 1), (2, 0, 2)]
sage: t.cells_containing(6)
[]
```

**charge()**

Return the charge of the reading word of \( \textsf{self} \).

See **charge()** for more information.

**EXAMPLES:**
cocharge()

Return the cocharge of the reading word of self.

See cocharge() for more information.

EXAMPLES:

\[
\text{sage: TableauTuple([[4,5]],[[1,1,2,4],[2,4,4],[4]],[[1,3,4],[3,4]])}.charge()
4
\]

column_stabilizer()

Return the PermutationGroup corresponding to self. That is, return subgroup of the symmetric group of degree \( \text{size()} \) which is the column stabilizer of self.

EXAMPLES:

\[
\text{sage: cs = TableauTuple([[1,2,3],[4,5]],[[6,7]],[[8],[9]])}.column_stabilizer()
\]

\[
\text{sage: cs.order()}
8
\]

\[
\text{sage: PermutationGroupElement([(1,3,2),(4,5)]) in cs}
False
\]

\[
\text{sage: PermutationGroupElement([(1,4)]) in cs}
True
\]

components()

Return a list of the components of tableau tuple self.

The components are the individual Tableau which are contained in the tuple self.

For compatibility with TableauTuples of level() 1, components() should be used to iterate over the components of TableauTuples.

EXAMPLES:

\[
\text{sage: for t in TableauTuple([[1,2,3],[4,5]]).components(): t.pp()}
1 2 3
4 5
\]

\[
\text{sage: for t in TableauTuple([[1,2,3],[4,5]],[[6,7],[8,9]]) components(): t.pp()}
1 2 3
4 5
6 7
8 9
\]

conjugate()

Return the conjugate of the tableau tuple self.

The conjugate tableau tuple \( T' \) is the TableauTuple obtained from \( T \) by reversing the order of the components and conjugating each component – that is, swapping the rows and columns of the all of Tableau in \( T \) (see sage.combinat.tableau.Tableau.conjugate()).

EXAMPLES:
content \((k, \text{multicharge})\)

Return the content \(k\) in \(self\).

The content of \(k\) in a standard tableau. That is, if \(k\) appears in row \(r\) and column \(c\) of the tableau, then we return \(c - r + a_k\), where the multicharge is \((a_1, a_2, \ldots, a_l)\) and \(l\) is the level of the tableau.

The multicharge determines the dominant weight

\[
\Lambda = \sum_{i=1}^{l} \Lambda_{a_i}
\]

defined above. In the combinatorics, the multicharge simply offsets the contents in each component so that the cell \((k, r, c)\) has content \(a_k + c - r\).

INPUT:

- \(k\) – an integer in \(\{1, 2, \ldots, n\}\)
- \(\text{multicharge}\) – a sequence of integers of length \(l\)

Here \(l\) is the level() and \(n\) is the size() of \(self\).

EXAMPLES:

```python
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).content(3,[0,0])
-1
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).content(3,[0,1])
0
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).content(3,[0,2])
1
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).content(6,[0,2])
Traceback (most recent call last):
  ...
  ValueError: 6 must be contained in the tableaux
```

entries()

Return a sorted list of all entries of \(self\), in the order obtained by reading across the rows.

EXAMPLES:

```python
sage: TableauTuple([[1,2],[[3,4],[5,6,7],[8],[[9,10],[11],[12]]]]).entries()
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
sage: TableauTuple([[1,2],[[3,4],[9,10],[11],[12]],[[5,6,7],[8]]]).entries()
[1, 2, 3, 4, 9, 10, 11, 12, 5, 6, 7, 8]
```

every \((l, r, c)\)

Return the entry of the cell \((l, r, c)\) in \(self\).

A cell is a tuple \((l, r, c)\) of coordinates, where \(l\) is the component index, \(r\) is the row index, and \(c\) is the column index.

EXAMPLES:
Combinatorics, Release 10.0

```python
sage: t = TableauTuple([[1,2],[3,4],[5,6,7],[8],[9,10],[11],[12]])
sage: t.entry(1, 0, 0)
5
sage: t.entry(1, 1, 1)
Traceback (most recent call last):
  ... IndexError: tuple index out of range
```

**first_column_descent()**

Return the first cell of self is not column standard.

Cells are ordered left to right along the rows and then top to bottom. That is, return the cell \((k, r, c)\) with \((k, r, c)\) minimal such that the entry in position \((k, r, c)\) is bigger than the entry in position \((k, r, c + 1)\). If there is no such cell then None is returned - in this case the tableau is column strict.

**OUTPUT:**

The cell corresponding to the first column descent or None if the tableau is column strict.

**EXAMPLES:**

```python
sage: TableauTuple([[3,5,6],[2,4,5],[1,4,5],[2,3]]).first_column_descent()
(0, 0, 0)
sage: Tableau([[1,2,3],[4],[5,6,7],[8,9]]).first_column_descent()
is None
```

**first_row_descent()**

Return the first cell of self that is not row standard.

Cells are ordered left to right along the rows and then top to bottom. That is, the cell minimal \((k, r, c)\) such that the entry in position \((k, r, c)\) is bigger than the entry in position \((k, r, c + 1)\). If there is no such cell then None is returned - in this case the tableau is row strict.

**OUTPUT:**

The cell corresponding to the first row descent or None if the tableau is row strict.

**EXAMPLES:**

```python
sage: TableauTuple([[5,6,7],[1,2],[1,3,2],[4]]).first_row_descent()
(1, 0, 1)
sage: TableauTuple([[1,2,3],[4],[6,7,8],[1,2,3],[1,11]]).first_row_descent()
is None
```

**is_column_strict()**

Return True if the tableau self is column strict and False otherwise.

A tableau tuple is column strict if the entries in each column of each component are in increasing order, when read from top to bottom.

**EXAMPLES:**

```python
sage: TableauTuple([[5,6,7],[1,2],[1,3,2],[4]]).is_column_strict()
True
sage: TableauTuple([[1,2],[2,4],[4,5,6],[7,8]]).is_column_strict()
True
```

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is_column_strict()

Return True if the tableau self is column strict and False otherwise.

A tableau tuple is column strict if the entries in each column of each component are in increasing order, when read from top to bottom.

EXAMPLES:

sage: TableauTuple([[1],[2,3],[2,4]]).is_column_strict()
False
sage: TableauTuple([[1],[2,2],[4,5]]).is_column_strict()
True
sage: TableauTuple([[1,2],[6,7],[4,8],[6,9],[]]).is_column_strict()
True

is_row_strict()

Return True if the tableau self is row strict and False otherwise.

A tableau tuple is row strict if the entries in each row of each component are in increasing order, when read from left to right.

EXAMPLES:

sage: TableauTuple([[1],[2,3],[2,4]]).is_row_strict()
True
sage: TableauTuple([[2,2],[4,5]]).is_row_strict()
True
sage: TableauTuple([[1,2],[6,7],[4,8],[6,9],[]]).is_row_strict()
True

is_standard()

Return True if the tableau self is a standard tableau and False otherwise.

A tableau tuple is standard if it is row standard, column standard and the entries in the tableaux are 1, 2, ..., n, where n is the size() of the underlying partition tuple of self.

EXAMPLES:

sage: TableauTuple([[5,7],[8],[1,3],[2,4],[6]]).is_standard()
True
sage: TableauTuple([[1,2],[2,4],[4,5,6],[7,8]]).is_standard()
True
sage: TableauTuple([[1],[2,3],[2,4]]).is_standard()
True
sage: TableauTuple([[1],[2,2],[4,5]]).is_row_strict()
False
sage: TableauTuple([[1,2],[6,7],[4,8],[6,9],[]]).is_standard()
False

level()

Return the level of the tableau self.

This is just the number of components in the tableau tuple self.

EXAMPLES:

sage: TableauTuple([[7,8,9],[1,2,3],[4,5],[6]]).level()
3
pp()

Pretty printing for the tableau tuple self.

EXAMPLES:

```python
sage: TableauTuple([[1,2,3],[4,5]], [[1,2,3],[4,5]]).pp()
1 2 3 1 2 3
4 5 4 5
sage: TableauTuple([[1,2],[3],[4]], [[6,7,8],[10,11],[12],[13]]).pp()
1 2 6 7 8
3 10 11
4 12
13
sage: t = TableauTuple([[1,2,3],[4,5],[6],[9]], [[1,2,3],[4,5,8],[11,12,13],[14]])
sage: t.pp()
1 2 3 1 2 3 11 12 13
4 5 4 5 8 14
6 9
sage: TableauTuples.options(convention="french")
sage: t.pp()
9 6 4 5 4 5 8 14
1 2 3 1 2 3 11 12 13
sage: TableauTuples.options._reset()
```

reduced_column_word()

Return the lexicographically minimal reduced expression for the permutation that maps the `initial_column_tableau()` to self.

This reduced expression is a minimal length coset representative for the corresponding Young subgroup. In one line notation, the permutation is obtained by concatenating the rows of the tableau from top to bottom in each component, and then left to right along the components.

EXAMPLES:

```python
sage: StandardTableauTuple([[7,9],[8]], [[1,4,6],[2,5],[3]]).reduced_column_word()
[]
sage: StandardTableauTuple([[7,9],[8]], [[1,3,6],[2,5],[4]]).reduced_column_word()
[3]
sage: StandardTableauTuple([[6,9],[8]], [[1,3,7],[2,5],[4]]).reduced_column_word()
[3, 6]
sage: StandardTableauTuple([[6,8],[9]], [[1,3,7],[2,5],[4]]).reduced_column_word()
[3, 6, 8]
sage: StandardTableauTuple([[5,8],[9]], [[1,3,7],[2,6],[4]]).reduced_column_word()
[3, 6, 5, 8]
```

reduced_row_word()
Return the lexicographically minimal reduced expression for the permutation that maps the
initial_tableau() to self.

This reduced expression is a minimal length coset representative for the corresponding Young subgroup. In
one line notation, the permutation is obtained by concatenating the rows of the tableau from top to bottom
in each component, and then left to right along the components.

EXAMPLES:

```
sage: StandardTableauTuple([[[1,2],[3]],[[4,5,6],[7,8],[9]]]).reduced_row_word()
[]
sage: StandardTableauTuple([[[1,2],[3]],[[4,5,6],[7,9],[8]]]).reduced_row_word()
[8]
sage: StandardTableauTuple([[[1,2],[3]],[[4,5,7],[6,9],[8]]]).reduced_row_word()
[6, 8]
sage: StandardTableauTuple([[[1,2],[3]],[[4,5,8],[6,9],[7]]]).reduced_row_word()
[6, 8, 7]
sage: StandardTableauTuple([[[1,2],[3]],[[4,5,9],[6,8],[7]]]).reduced_row_word()
[6, 8, 7, 8, 7]
sage: StandardTableauTuple([[[7,9],[8]],[[1,3,5],[2,6],[4]]]).reduced_row_word()
[2, 3, 2, 1, 4, 3, 2, 5, 4, 3, 6, 5, 4, 3, 2, 7, 6, 5, 8, 7, 6, 5, 4]
```

residue\( (k, e, \text{multicharge}) \)

Return the residue of the integer \( k \) in the tableau \( \text{self} \).

The residue of \( k \) is \( c - r + a_k \) in \( \mathbb{Z}/e\mathbb{Z} \), where \( k \) appears in row \( r \) and column \( c \) of the tableau and the
multicharge is \( (a_1, a_2, \ldots, a_l) \).

The multicharge determines the dominant weight

\[
\sum_{i=1}^{l} \Lambda_{a_i}
\]

for the affine special linear group. In the combinatorics, it simply offsets the contents in each component
so that the cell \((k,0,0)\) has content \(a_k\).

INPUT:

- \( k \) – an integer in \{1,2,\ldots,n\}
- \( e \) – an integer in \{0,2,3,4,\ldots\}
- \multicharge \( \) – a list of integers of length \( l \)

Here \( l \) is the level() and \( n \) is the size() of \( \text{self} \).

OUTPUT:

The residue of \( k \) in a standard tableau. That is,

EXAMPLES:

```
sage: StandardTableauTuple([[[5]],[[1,2],[3,4]]]).residue(1, 3,[0,0])

0
sage: StandardTableauTuple([[[5]],[[1,2],[3,4]]]).residue(1, 3,[0,1])

1
sage: StandardTableauTuple([[[5]],[[1,2],[3,4]]]).residue(1, 3,[0,2])

2
```

(continues on next page)
sage: StandardTableauTuple([[5],[[1,2],[3,4]]]).residue(6, 3, [0,2])
Traceback (most recent call last):
  ... 
ValueError: 6 must be contained in the tableaux

\texttt{restrict}(m=None)

Return the restriction of the standard tableau \texttt{self} to \texttt{m}.

The restriction is the subtableau of \texttt{self} whose entries are less than or equal to \texttt{m}.

By default, \texttt{m} is one less than the current size.

**EXAMPLES:**

\begin{verbatim}
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict()
([], [[1, 2], [3, 4]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(6)
([[5], [[1, 2], [3, 4]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(5)
([[5], [[1, 2], [3, 4]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(4)
([], [[1, 2], [3, 4]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(3)
([], [[1, 2], [3]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(2)
([], [[1, 2]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(1)
([], [[1]])
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(0)
([], [])
\end{verbatim}

Where possible the restricted tableau belongs to the same category as the original tableaux:

\begin{verbatim}
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples
sage: TableauTuples(level=2)([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples of level 2
\end{verbatim}

\texttt{row_stabilizer}()

Return the \texttt{PermutationGroup} corresponding to \texttt{self}. That is, return subgroup of the symmetric group of degree \texttt{size()} which is the row stabilizer of \texttt{self}.

**EXAMPLES:**

\begin{verbatim}
sage: rs = TableauTuple([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples
sage: TableauTuple([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples
sage: TableauTuples(level=2)([[5],[[1,2],[3,4]]]).restrict(3).category()
Category of elements of Tableau tuples of level 2
\end{verbatim}
shape()

Return the *PartitionTuple* which is the shape of the tableau tuple `self`.

**EXAMPLES:**

```
sage: TableauTuple([[7,8,9],[],[[1,2,3],[4,5],[6]]]).shape()
([3], [], [3, 2, 1])
```

size()

Return the *size* of the tableau tuple `self`.

This is just the number of boxes, or the *size*, of the underlying *PartitionTuple*.

**EXAMPLES:**

```
sage: TableauTuple([[7,8,9],[],[[1,2,3],[4,5],[6]]]).size()
9
```

`symmetric_group_action_on_entries(w)`

Return the action of a permutation `w` on `self`.

Consider a standard tableau tuple `T = (t^{(1)}, t^{(2)}, \ldots, t^{(l)})` of size `n`, then the action of `w \in S_n` is defined by permuting the entries of `T` (recall they are 1, 2, \ldots, `n`). In particular, suppose the entry at cell `(k, i, j)` is `a`, then the entry becomes `w(a)`. In general, the resulting tableau tuple `wT` may not be standard.

**INPUT:**

- `w` – a permutation

**EXAMPLES:**

```
sage: TableauTuple([[1,2],[4]], [[3,5]]).symmetric_group_action_on_entries(Permutation(((4,5))))
([1, 2], [5])
sage: TableauTuple([[1,2],[4]], [[3,5]]).symmetric_group_action_on_entries(Permutation(((1,2))))
([[2, 1], [4]], [[3, 5]])
```

to_list()

Return the list representation of the tableaux tuple `self`.

**EXAMPLES:**

```
sage: TableauTuple([ [[[1,2,3],[4,5]], [[6,7],[8,9]] ]]).to_list()
[[[1, 2, 3], [4, 5]], [[6, 7], [8, 9]]]
```

to_permutation()

Return a permutation with the entries in the tableau tuple `self`.

The permutation is obtained from `self` by reading the entries of the tableau tuple in order from left to right along the rows, and then top to bottom, in each component and then left to right along the components.

**EXAMPLES:**

```
sage: TableauTuple([[[1,2],[3,4]], [[5,6,7],[8]], [[9,10],[11],[12]]]).to_permutation()
[12, 11, 9, 10, 8, 5, 6, 7, 3, 4, 1, 2]
```
to_word()
Return a word obtained from a row reading of the tableau tuple self.

EXAMPLES:

\[
\text{sage: TableauTuple}([[1,2],[3,4]],[[5,6,7],[8]],[[9,10],[11],[12]]) . \text{to_word_by_row()}
\]
word: 12,11,9,10,8,5,6,7,3,4,1,2

to_word_by_column()
Return the word obtained from a column reading of the tableau tuple self.

EXAMPLES:

\[
\text{sage: TableauTuple}([[1,2],[3,4]],[[5,6,7],[8]],[[9,10],[11],[12]]) . \text{to_word_by_column()}
\]
word: 12,11,9,10,8,5,6,7,3,1,4,2

to_word_by_row()
Return a word obtained from a row reading of the tableau tuple self.

EXAMPLES:

\[
\text{sage: TableauTuple}([[1,2],[3,4]],[[5,6,7],[8]],[[9,10],[11],[12]]) . \text{to_word_by_row()}
\]
word: 12,11,9,10,8,5,6,7,3,4,1,2

up(n=None)
An iterator for all the TableauTuple that can be obtained from self by adding a cell with the label n. If n is not specified then a cell with label n will be added to the tableau tuple, where n-1 is the size of the tableau tuple before any cells are added.

EXAMPLES:

\[
\text{sage: list(TableauTuple}([[1,2]],[[3]]) . \text{up()})
\]
([[1, 2, 4], [3]),
 ([[1, 2], [4], [3]),
 ([[1, 2], [3, 4]],
 ([[1, 2], [3], [4]])]

class sage.combinat.tableau_tuple.TableauTuples
Bases: UniqueRepresentation, Parent
A factory class for the various classes of tableau tuples.

INPUT:
There are three optional arguments:

• shape – determines a PartitionTuple which gives the shape of the TableauTuples
• level – the level of the tableau tuples (positive integer)
• size – the size of the tableau tuples (non-negative integer)

It is not necessary to use the keywords. If they are not specified then the first integer argument specifies the level and the second the size of the tableaux.

OUTPUT:
The corresponding class of tableau tuples.

The entries of a tableau can be any sage object. Because of this, no enumeration of the set of TableauTuples is possible.

EXAMPLES:

```sage
t3 = TableauTuples(3); t3
T3 = TableauTuples(3)
sage: [[a',b']] in TableauTuples()
True
sage: [[a',b']] in TableauTuples(level=3)
False
sage: t = TableauTuples(level=3)([[],[[1,1,1]],[]]); t
([], [[1, 1, 1]], [])
sage: t in T3
True
sage: t in TableauTuples()
True
sage: t in TableauTuples(size=3)
True
sage: t in TableauTuples(size=4)
False
sage: t in StandardTableauTuples()
False
sage: t.parent()
Tableau tuples of level 3
sage: t.category()
Category of elements of Tableau tuples of level 3
```

See also:

- Tableau
- StandardTableau
- StandardTableauTuples

Element

alias of TableauTuple

level()

Return the level of a tableau tuple in self, or None if different tableau tuples in self can have different sizes. The level of a tableau tuple is just the level of the underlying PartitionTuple.

EXAMPLES:

```sage:
TableauTuples().level() is None
True
sage: TableauTuples(7).level()
7
```

level_one_parent_class

alias of Tableaux_all
list()

If the set of tableau tuples self is finite then this function returns the list of these tableau tuples. If the class is infinite an error is returned.

EXAMPLES:

```
sage: StandardTableauTuples([[1], [2], [3], [4], [5]]).list()
[[[1], [2], [3], [4], [5]]],
[[[1], [2], [3], [4]], [[5]]],
[[[1], [2], [3]], [[4], [5]]],
[[[1], [2]], [[3], [4], [5]]],
[[[1]], [[2], [3], [4], [5]]],
[[[1]], [[2], [3], [4]]],
[[[1]], [[2], [3]]],
[[[1]], [[2]]],
[[[1]], []],
[[[1], [2], [3], [4], [5]]],
[[[1], [2], [3], [4]], [[5]]],
[[[1], [2], [3]], [[4], [5]]],
[[[1], [2]], [[3], [4], [5]]],
[[[1]], [[2], [3], [4], [5]]],
[[[1]], [[2], [3], [4]]],
[[[1]], [[2], [3]]],
[[[1]], [[2]]],
[[[1]], []],
```

```
options = Current options for Tableaux - ascii_art: repr - convention: English - display: list - latex: diagram
```

size()

Return the size of a tableau tuple in self, or None if different tableau tuples in self can have different sizes. The size of a tableau tuple is just the size of the underlying PartitionTuple.

EXAMPLES:

```
sage: TableauTuples(size=14).size()
14
```

class sage.combinat.tableau_tuple.TableauTuples_all

Bases: TableauTuples

The parent class of all TableauTuples, with arbitrary level and size.

an_element()

Return a particular element of the class.

EXAMPLES:

```
sage: TableauTuples().an_element()
[[[1]], [[2]], [[3]], [[4]], [[5]], [[6]], [[7]]]
```

class sage.combinat.tableau_tuple.TableauTuples_level(level)

Bases: TableauTuples

Class of all TableauTuples with a fixed level and arbitrary size.
**an_element()**

Return a particular element of the class.

**EXAMPLES:**

```
sage: TableauTuples(3).an_element()
([], [], [])
```

```
sage: TableauTuples(5).an_element()
([], [], [], [], []
```

```
sage: T = TableauTuples(0)
Traceback (most recent call last):
...
ValueError: the level must be a positive integer
```

**class** sage.combinat.tableau_tuple.TableauTuples_level_size(level, size)

Bases: TableauTuples

Class of all TableauTuples with a fixed level and a fixed size.

**an_element()**

Return a particular element of the class.

**EXAMPLES:**

```
sage: TableauTuples(3,0).an_element()
([], [], []
```

```
sage: TableauTuples(3,1).an_element()
([[1]], [], []
```

```
sage: TableauTuples(3,2).an_element()
([[1, 2]], [], []
```

**class** sage.combinat.tableau_tuple.TableauTuples_size(size)

Bases: TableauTuples

Class of all TableauTuples with a arbitrary level and fixed size.

**an_element()**

Return a particular element of the class.

**EXAMPLES:**

```
sage: TableauTuples(size=3).an_element()
([], [[1, 2, 3]], []
```

```
sage: TableauTuples(size=0).an_element()
([], [], []
```

**5.1.346 Generalized Tamari lattices**

These lattices depend on three parameters \(a, b\) and \(m\), where \(a\) and \(b\) are coprime positive integers and \(m\) is a nonnegative integer.

The elements are *Dyck paths* in the \((a \times b)\)-rectangle. The order relation depends on \(m\).

To use the provided functionality, you should import Generalized Tamari lattices by typing:

```
sage: from sage.combinat.tamari_lattices import GeneralizedTamariLattice
```
Then,

```
sage: GeneralizedTamariLattice(3,2)
Finite lattice containing 2 elements
sage: GeneralizedTamariLattice(4,3)
Finite lattice containing 5 elements
```

The classical Tamari lattices are special cases of this construction and are also available directly using the catalogue of posets, as follows:

```
sage: posets.TamariLattice(3)
Finite lattice containing 5 elements
```

See also:

For more detailed information see `TamariLattice()`, `GeneralizedTamariLattice()`.

```
sage.combinat.tamari_lattices.DexterSemilattice(n)
```

Return the $n$-th Dexter meet-semilattice.

INPUT:

- $n$ – a nonnegative integer (the index)

OUTPUT:

a finite meet-semilattice

The elements of the semilattice are Dyck paths in the $(n + 1 \times n)$-rectangle.

EXAMPLES:

```
sage: posets.DexterSemilattice(3)
Finite meet-semilattice containing 5 elements
```

```
sage: P = posets.DexterSemilattice(4); P
Finite meet-semilattice containing 14 elements
sage: len(P.maximal_chains())
15
sage: len(P.maximal_elements())
4
sage: P.chain_polynomial()
q^5 + 19*q^4 + 47*q^3 + 42*q^2 + 14*q + 1
```

REFERENCES:

- [Cha18]

```
sage.combinat.tamari_lattices.GeneralizedTamariLattice(a, b, m=1, check=True)
```

Return the $(a, b)$-Tamari lattice of parameter $m$.

INPUT:

- $a$ and $b$ – coprime integers with $a \geq b$
- $m$ – a nonnegative integer such that $a \geq bm$

OUTPUT:

- a finite lattice (the lattice property is only conjectural in general)
The elements of the lattice are *Dyck paths* in the \((a \times b)\)-rectangle.

The parameter \(m\) (slope) is used only to define the covering relations. When the slope \(m\) is 0, two paths are comparable if and only if one is always above the other.

The usual *Tamari lattice* of index \(b\) is the special case \(a = b + 1\) and \(m = 1\).

Other special cases give the \(m\)-Tamari lattices studied in [BMFPR].

**EXAMPLES:**

```python
sage: from sage.combinat.tamari_lattices import GeneralizedTamariLattice
sage: GeneralizedTamariLattice(3, 2)
Finite lattice containing 2 elements
sage: GeneralizedTamariLattice(4, 3)
Finite lattice containing 5 elements
sage: GeneralizedTamariLattice(4, 4)
Traceback (most recent call last):
  ... ValueError: the numbers a and b must be coprime with a\geq b
sage: GeneralizedTamariLattice(7, 5, 2)
Traceback (most recent call last):
  ... ValueError: the condition a\geq b*m does not hold
sage: P = GeneralizedTamariLattice(5, 3); P
Finite lattice containing 7 elements
```

**REFERENCES:**

sage.combinat.tamari_lattices.TamariLattice\((n, m=1)\)

Return the \(n\)-th Tamari lattice.

Using the slope parameter \(m\), one can also get the \(m\)-Tamari lattices.

**INPUT:**

- \(n\) – a nonnegative integer (the index)
- \(m\) – an optional nonnegative integer (the slope, default to 1)

**OUTPUT:**

a finite lattice

In the usual case, the elements of the lattice are *Dyck paths* in the \((n + 1 \times n)\)-rectangle. For a general slope \(n\), the elements are Dyck paths in the \((mn + 1 \times n)\)-rectangle.

See Tamari lattice for mathematical background.

**EXAMPLES:**

```python
sage: posets.TamariLattice(3)
Finite lattice containing 5 elements
sage: posets.TamariLattice(3, 2)
Finite lattice containing 12 elements
```

**REFERENCES:**

- [BMFPR]
sage.combinat.tamari_lattices.paths_in_triangle(i, j, a, b)

Return all Dyck paths from (0, 0) to (i, j) in the (a × b)-rectangle.

This means that at each step of the path, one has \(a y \geq bx\).

A path is represented by a sequence of 0 and 1, where 0 is an horizontal step (1, 0) and 1 is a vertical step (0, 1).

INPUT:

- \(a\) and \(b\) – coprime integers with \(a \geq b\)
- \(i\) and \(j\) – nonnegative integers with \(1 \geq j \geq \frac{b}{a} \geq 0\)

OUTPUT:

- a list of paths

EXAMPLES:

```python
sage: from sage.combinat.tamari_lattices import paths_in_triangle
sage: paths_in_triangle(2,2,2,2)
[[(1, 0, 1, 0), (1, 1, 0, 0)]

sage: paths_in_triangle(2,3,4,4)
[[(1, 0, 1, 0, 1), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 1, 0, 0)]

sage: paths_in_triangle(2,1,4,4)
Traceback (most recent call last):
  ... ValueError: the endpoint is not valid
```

sage.combinat.tamari_lattices.swap(p, i, m=1)

Perform a covering move in the \((a, b)\)-Tamari lattice of parameter \(m\).

The letter at position \(i\) in \(p\) must be a 0, followed by at least one 1.

INPUT:

- \(p\) – a Dyck path in the \((a \times b)\)-rectangle
- \(i\) – an integer between 0 and \(a + b - 1\)

OUTPUT:

- a Dyck path in the \((a \times b)\)-rectangle

EXAMPLES:

```python
sage: from sage.combinat.tamari_lattices import swap
sage: swap((1,0,1,0,0),1)
(1, 1, 0, 0, 0)

sage: swap((1,0,1,0,0,1,1,0,0,0),3)
(1, 1, 0, 1, 0, 0, 0, 0, 0)
```

sage.combinat.tamari_lattices.swap_dexter(p, i)

Perform covering moves in the \((a, b)\)-Dexter posets.

The letter at position \(i\) in \(p\) must be a 0, followed by at least one 1.

INPUT:

- \(p\) – a Dyck path in the \((a \times b)\)-rectangle

EXAMPLES:
• \( i \) – an integer between 0 and \( a + b - 1 \)

**OUTPUT:**

• a list of Dyck paths in the \((a \times b)\)-rectangle

**EXAMPLES:**

```python
sage: from sage.combinat.tamari_lattices import swap_dexter
sage: swap_dexter((1,0,1,0,0),1)
[(1, 1, 0, 0, 0)]
sage: swap_dexter((1,1,0,0,1,1,0,0,0),3)
[(1, 1, 0, 1, 1, 0, 0, 0, 0), (1, 1, 1, 1, 0, 0, 0, 0, 0)]
sage: swap_dexter((1,1,0,1,0,0),2)
[]
```

### 5.1.347 Tiling Solver

Tiling a \( n \)-dimensional polyomino with \( n \)-dimensional polyominoes.

This module defines two classes:

• `sage.combinat.tiling.Polyomino` class, to represent polyominoes in arbitrary dimension. The goal of this class is to return all the rotated, reflected and/or translated copies of a polyomino that are contained in a certain box.

• `sage.combinat.tiling.TilingSolver` class, to solve the problem of tiling a \( n \)-dimensional polyomino with a set of \( n \)-dimensional polyominoes. One can specify if rotations and reflections are allowed or not and if pieces can be reused or not. This class convert the tiling data into rows of a matrix that are passed to the DLX solver. It also allows to compute the number of solutions.

This uses dancing links code which is in Sage. Dancing links were originally introduced by Donald Knuth in 2000 [Knuth1]. Knuth used dancing links to solve tilings of a region by 2d pentaminoes. Here we extend the method to any dimension.

In particular, the `sage.games.quantumino` module is based on the Tiling Solver and allows to solve the 3d Quantumino puzzle.

**AUTHOR:**

• Sébastien Labbé, June 2011, initial version

• Sébastien Labbé, July 2015, count solutions up to rotations

• Sébastien Labbé, April 2017, tiling a polyomino, not only a rectangular box

**EXAMPLES:**

### 2d Easy Example

Here is a 2d example. Let us try to fill the \( 3 \times 2 \) rectangle with a \( 1 \times 2 \) rectangle and a \( 2 \times 2 \) square. Obviously, there are two solutions:

```python
sage: from sage.combinat.tiling import TilingSolver, Polyomino
sage: p = Polyomino([(0,0), (0,1)])
sage: q = Polyomino([(0,0), (0,1), (1,0), (1,1)])
sage: T = TilingSolver([p,q], box=[3,2])
sage: it = T.solve()
(continues on next page)
Scott's pentamino problem

As mentioned in the introduction of [Knuth1], Scott's pentamino problem consists in tiling a chessboard leaving the center four squares vacant with the 12 distinct pentaminoes.

The 12 pentaminoes:

```python
sage: from sage.combinat.tiling import Polyomino
sage: I = Polyomino(((0,0),(1,0),(2,0),(3,0),(4,0)], color='brown')
sage: N = Polyomino(((1,0),(1,1),(1,2),(0,2),(0,3)], color='yellow')
sage: L = Polyomino(((0,0),(1,0),(0,1),(0,2),(0,3)], color='magenta')
sage: U = Polyomino(((0,0),(1,0),(0,1),(0,2),(1,2)], color='violet')
sage: X = Polyomino(((1,0),(0,1),(1,1),(1,2),(2,1)], color='pink')
sage: W = Polyomino(((2,0),(2,1),(1,1),(1,2),(0,2)], color='green')
sage: P = Polyomino(((1,0),(2,0),(0,1),(1,1),(2,1)], color='orange')
sage: F = Polyomino(((1,0),(1,1),(0,1),(2,1),(2,2)], color='gray')
sage: Z = Polyomino(((0,0),(1,0),(1,1),(1,2),(2,2)], color='yellow')
sage: T = Polyomino(((0,0),(0,1),(1,1),(2,1),(0,2)], color='red')
sage: Y = Polyomino(((0,0),(1,0),(2,0),(3,0),(2,1)], color='green')
sage: V = Polyomino(((0,0),(0,1),(0,2),(1,0),(2,0)], color='blue')
```

A 8 × 8 chessboard leaving the center four squares vacant:

```python
sage: import itertools
sage: s = set(itertools.product(range(8), repeat=2))
sage: s.difference_update(((3,3), (3,4), (4,3), (4,4))]
sage: chessboard = Polyomino(s)
sage: len(chessboard)
60
```

This problem is represented by a matrix made of 1568 rows and 72 columns. It has 65 different solutions up to isometries:

```python
sage: from sage.combinat.tiling import TilingSolver
sage: T
Tiling solver of 12 pieces into a box of size 60
Rotation allowed: True
Reflection allowed: True
```
Reusing pieces allowed: False

sage: len(T.rows())  # long time
1568

sage: T.number_of_solutions()  # long time
520

sage: 520 / 8
65

Showing one solution:

sage: solution = next(T.solve())  # long time
sage: G = sum([piece.show2d() for piece in solution], Graphics())  # long time
sage: G.show(aspect_ratio=1, axes=False)  # long time

1d Easy Example

Here is an easy one dimensional example where we try to tile a stick of length 6 with three sticks of length 1, 2 and 3. There are six solutions:

sage: p = Polyomino([[0]])
sage: q = Polyomino([[0],[1]])
sage: r = Polyomino([[0],[1],[2]])
sage: T = TilingSolver([p,q,r], box=[6])
sage: len(T.rows())
15
sage: it = T.solve()
sage: next(it)
[Polyomino: [(0)], Color: gray, Polyomino: [(1), (2)], Color: gray, Polyomino: [(3), (4), (5)], Color: gray]
sage: next(it)
[Polyomino: [(0)], Color: gray, Polyomino: [(1), (2), (3)], Color: gray, Polyomino: [(4), (5)], Color: gray]
sage: T.number_of_solutions()
6

2d Puzzle allowing reflections

The following is a puzzle owned by Florent Hivert:

sage: from sage.combinat.tiling import Polyomino, TilingSolver
sage: L = []
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)], 'yellow'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2)], 'black'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,3)], 'gray'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,0)], 'cyan'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,0),(1,1)], 'red'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,1),(1,2)], 'blue'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(0,3),(1,1),(1,2)], 'green'))
sage: L.append(Polyomino([[0,1),(0,2),(0,3),(1,0),(1,1),(1,3)], 'magenta'))
sage: L.append(Polyomino([[0,1),(0,2),(0,3),(1,0),(1,1),(1,2)], 'orange'))
sage: L.append(Polyomino([[0,0),(0,1),(0,2),(1,0),(1,1),(1,2)], 'pink'))
By default, rotations are allowed and reflections are not. In this case, there are no solution for tiling a $8 \times 8$ rectangular box:

```
sage: T = TilingSolver(L, box=(8,8))
sage: T.number_of_solutions()                # long time (2.5 s)
0
```

If reflections are allowed, there are solutions. Solve the puzzle and show one solution:

```
sage: T = TilingSolver(L, box=(8,8), reflection=True)
sage: solution = next(T.solve())            # long time (7s)
sage: G = sum([piece.show2d() for piece in solution], Graphics()) # long time (<1s)
sage: G.show(aspect_ratio=1, axes=False)    # long time (2s)
```

Compute the number of solutions:

```
sage: T.number_of_solutions()                # long time (2.6s)
328
```

Create a animation of all the solutions:

```
sage: a = T.animate()                        # not tested
sage: a                                        # not tested
Animation with 328 frames
```

### 3d Puzzle

The same thing done in 3d *without* allowing reflections this time:

```
sage: from sage.combinat.tiling import Polyomino, TilingSolver
sage: L = []
sage: L.append(Polyomino([(0,0,0),(0,1,0),(0,2,0),(0,3,0),(1,0,0),(1,1,0),(1,2,0),(1,3,0)]))

Solve the puzzle and show one solution:

```
sage: T = TilingSolver(L, box=(8,8,1))
sage: solution = next(T.solve())            # long time (8s)
sage: G = sum([p.show3d(size=0.85) for p in solution], Graphics()) # long time (<1s)
sage: G.show(aspect_ratio=1, viewer='tachyon') # long time (2s)
```

Let us compute the number of solutions:

```
sage: T.number_of_solutions()                # long time (3s)
328
```
Donald Knuth example: the Y pentamino

Donald Knuth [Knuth1] considered the problem of packing 45 Y pentaminoes into a $15 \times 15$ square:

```python
sage: from sage.combinat.tiling import Polyomino, TilingSolver
sage: y = Polyomino([(0,0),(1,0),(2,0),(3,0),(2,1)])
sage: T = TilingSolver([y], box=(5,10), reusable=True, reflection=True)
sage: T.number_of_solutions()
10
sage: solution = next(T.solve())
sage: G = sum([p.show2d() for p in solution], Graphics())
sage: G.show(aspect_ratio=1)  # long time (2s)
```

Up to the symmetries of the square, there are 212 distinct solutions:

```python
sage: 1696 // 8
212
```

Animation of Donald Knuth’s dancing links

Animation of the solutions:

```python
sage: from sage.combinat.tiling import Polyomino, TilingSolver
sage: Y = Polyomino([(0,0),(1,0),(2,0),(3,0),(2,1)], color='yellow')
sage: T = TilingSolver([Y], box=(15,15), reusable=True, reflection=True)
sage: a = T.animate(stop=40)  # long time # optional -- ImageMagick
sage: a                        # long time # optional -- ImageMagick
```

Incremental animation of the solutions (one piece is removed/added at a time):

```python
sage: a = T.animate('incremental', stop=40)  # long time # optional -- ImageMagick
sage: a                                 # long time # optional -- ImageMagick
```

5d Easy Example

Here is a 5d example. Let us try to fill the $2 \times 2 \times 2 \times 2 \times 2$ rectangle with reusable $1 \times 1 \times 1 \times 1 \times 1$ rectangles. Obviously, there is one solution:

```python
sage: from sage.combinat.tiling import Polyomino, TilingSolver
sage: p = Polyomino([(0,0,0,0,0)])
sage: T = TilingSolver([p], box=(2,2,2,2,2), reusable=True)
sage: rows = T.rows()               # long time (3s)
sage: rows                         # long time (fast)
[[0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]]
```

(continues on next page)
sage: T.number_of_solutions()  # long time (fast)
1

REFERENCES:
class sage.combinat.tiling.Polyomino(coords, color='gray', dimension=None)

Bases: SageObject

A polyomino in $\mathbb{Z}^d$.
The polyomino is the union of the unit square (or cube, or n-cube) centered at those coordinates. Such an object should be connected, but the code does not make this assumption.

INPUT:
- coords – iterable of integer coordinates in $\mathbb{Z}^d$
- color – string (default: 'gray'), color for display
- dimension – integer (default: None), dimension of the space, if None, it is guessed from the coords if coords is non empty

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino
sage: Polyomino([(0,0,0), (0,1,0), (1,1,0), (1,1,1)], color='blue')
Polyomino: [(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1)], Color: blue
```

boundary() Return the boundary of a 2d polyomino.

INPUT:
- self - a 2d polyomino

OUTPUT:
- list of edges (an edge is a pair of adjacent 2d coordinates)

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0), (1,0), (0,1), (1,1)])
sage: sorted(p.boundary())
[((-0.5, -0.5), (-0.5, 0.5)), ((-0.5, -0.5), (0.5, -0.5)), ((-0.5, 0.5), (-0.5,
→1.5)), ((-0.5, 1.5), (0.5, 1.5)), ((0.5, -0.5), (1.5, -0.5)), ((0.5, 1.5), (1.
→1.5)), ((1.5, -0.5), (1.5, 0.5)), ((1.5, 0.5), (1.5, 1.5))]
sage: len(_)
8
```

bounding_box() EXAMPLES:
```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0),(1,0,0),(1,1,0),(1,1,1),(1,2,0)], color='deeppink')
sage: p.bounding_box()
[[0, 0, 0], [1, 2, 1]]
```

**canonical()**

Return the translated copy of `self` having minimal and nonnegative coordinates.

**canonical_isometric_copies**(orientation_preserving=True, mod_box_isometries=False)

Return the list of image of `self` under isometries of the $n$-cube where the coordinates are all nonnegative and minimal.

**INPUT:**

- orientation_preserving – bool (optional, default: True); if True, the group of isometries of the $n$-cube is restricted to those that preserve the orientation, i.e. of determinant 1.
- mod_box_isometries – bool (default: False), whether to quotient the group of isometries of the $n$-cube by the subgroup of isometries of the $a_1 \times a_2 \cdots \times a_n$ rectangular box where are the $a_i$ are assumed to be distinct.

**OUTPUT:**

set of Polyomino

**EXAMPLES:**

```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0), (0,1,0), (1,1,0), (1,1,1)], color='blue')
sage: s = p.canonical_isometric_copies()
sage: len(s)
12
```

With the non orientation-preserving:

```python
sage: s = p.canonical_isometric_copies(orientation_preserving=False)
sage: len(s)
24
```

Modulo rotation by angle 180 degrees:

```python
sage: s = p.canonical_isometric_copies(mod_box_isometries=True)
sage: len(s)
3
```
center()

Return the center of the polyomino.

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0),(0,0,1)])
sage: p.center()
(0, 0, 1/2)
```

In 3d:

```
sage: p = Polyomino([(0,0,0),(1,0,0),(1,1,0),(1,1,1),(1,2,0)], color='deeppink')
sage: p.center()
(4/5, 4/5, 1/5)
```

In 2d:

```
sage: p = Polyomino([(0,0),(1,0),(1,1),(1,2)])
sage: p.center()
(3/4, 3/4)
```

color(color=None)

Return or change the color of the polyomino.

INPUT:

• color – string, RBG tuple or None (default: None), if None, it returns the current color

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0), (0,1,0), (1,1,0), (1,1,1)], color='blue')
sage: p.color()
'blue'
```

frozenset()

Return the elements of $\mathbb{Z}^d$ in the polyomino as a frozenset.

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0), (0,1,0), (1,1,0), (1,1,1)], color='red')
sage: p.frozenset()
frozenset({(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1)})
```

intersection(other)

Return the intersection of self and other.

INPUT:

• other - a polyomino

OUTPUT:

polyomino

EXAMPLES:
sage: from sage.combinat.tiling import Polyomino
sage: a = Polyomino([(0,0)])
sage: b = Polyomino([(0,0), (0,1), (1,1), (2,1)])
sage: a.intersection(b)
Polyomino: [(0, 0)], Color: gray
sage: a.intersection(b+(1,1))
Polyomino: [], Color: gray

\textbf{isometric\_copies}(\texttt{box, orientation\_preserving=True, mod\_box\_isometries=False})

Return the translated and isometric images of \texttt{self} that lies in the box.

\textbf{INPUT}:

- \texttt{box} – Polyomino or tuple of integers (size of a box)
- \texttt{orientation\_preserving} – bool (optional, default: True); If True, the group of isometries of the \(n\)-cube is restricted to those that preserve the orientation, i.e. of determinant 1.
- \texttt{mod\_box\_isometries} – bool (default: False), whether to quotient the group of isometries of the \(n\)-cube by the subgroup of isometries of the \(a_1 \times a_2 \cdots \times a_n\) rectangular box where are the \(a_i\) are assumed to be distinct.

\textbf{EXAMPLES}:

sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0),(1,0,0),(1,1,0),(1,1,1),(1,2,0)], color='deeppink')
sage: L = list(p.isometric_copies(box=(5,8,2)))
sage: len(L)
360

sage: p = Polyomino([(0,0,0),(1,0,0),(1,1,0),(1,2,0),(1,2,1)], color='orange')

sage: L = list(p.isometric_copies(box=(5,8,2)))

sage: len(L)
180

sage: L = list(p.isometric_copies((5,8,2), False))

sage: len(L)
360

sage: L = list(p.isometric_copies((5,8,2), mod_box_isometries=True))

sage: len(L)
45

sage: p = Polyomino([(0,0), (1,0), (0,1)])
sage: b = Polyomino([(0,0), (1,0), (2,0), (0,1), (1,1), (0,2)])

sage: sorted(p.isometric_copies(b), key=lambda p: p.sorted_list())
[[Polyomino: [(0, 0), (0, 1), (1, 0)], Color: gray,
  Polyomino: [(0, 0), (0, 1), (1, 1)], Color: gray,
  Polyomino: [(0, 0), (1, 0), (1, 1)], Color: gray,
  Polyomino: [(0, 1), (0, 2), (1, 1)], Color: gray,
  Polyomino: [(0, 1), (1, 0), (1, 1)], Color: gray,
  Polyomino: [(1, 0), (1, 1), (2, 0)], Color: gray]]

\textbf{isometric\_copies\_intersection}(\texttt{box, orientation\_preserving=True})

Return the set of non empty intersections of isometric images of \texttt{self} with a polyomino.

\textbf{INPUT}:
• box – Polyomino or tuple of integers (size of a box)

• orientation_preserving – bool (optional, default: True); if True, the group of isometries of the $n$-cube is restricted to those that preserve the orientation, i.e. of determinant 1.

EXAMPLES:

```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0),(1,0)], color='deeppink')
```

```python
sage: sorted(sorted(a.frozenset()) for a in p.isometric_copies_
→intersection(box=(2,3)))
```

```python
[(0, 0),
[(0, 0), (0, 1)],
[(0, 0), (1, 0)],
[(0, 1)],
[(0, 1), (0, 2)],
[(0, 1), (1, 1)],
[(0, 2)],
[(0, 2), (1, 2)],
[(1, 0)],
[(1, 0), (1, 1)],
[(1, 1)],
[(1, 1), (1, 2)],
[(1, 2)]
```

neighbor_edges()

Return an iterator over the pairs of neighbor coordinates inside of the polyomino.

Two points $P$ and $Q$ in the polyomino are neighbor if $P - Q$ has one coordinate equal to $+1$ or $-1$ and zero everywhere else.

EXAMPLES:

```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0),(0,0,1)])
```

```python
[sage: sorted(edge) for edge in p.neighbor_edges()]
```

```python
[[[0, 0, 0), (0, 0, 1)]
```

In 3d:

```python
sage: p = Polyomino([(0,0,0),(1,0,0),(1,1,0),(1,1,1),(1,2,0)], color='deeppink')
```

```python
sage: L = sorted(sorted(edge) for edge in p.neighbor_edges())
```

```python
sage: for a in L: a
```

```python
[(0, 0, 0), (1, 0, 0)]
[(1, 0, 0), (1, 1, 0)]
[(1, 1, 0), (1, 1, 1)]
[(1, 1, 0), (1, 2, 0)]
```

In 2d:

```python
sage: p = Polyomino([(0,0),(1,0),(1,1),(1,2)])
```

```python
sage: L = sorted(sorted(edge) for edge in p.neighbor_edges())
```

```python
sage: for a in L: a
```

```python
[(0, 0), (1, 0)]
[(1, 0), (1, 1)]
[(1, 1), (1, 2)]
```
**self_surrounding**(radius, remove_incomplete_copies=True, ncpus=None)

Return a list of isometric copies of `self` surrounding it with an annulus of given radius.

**INPUT:**

- `self` - a polyomino of dimension 2
- `radius` - integer
- `remove_incomplete_copies` – bool (default: True), whether to keep only complete copies of `self` in the output
- `ncpus` – integer (default: None), maximal number of subprocesses to use at the same time. If None, it detects the number of effective CPUs in the system using `sage.parallel.ncpus.ncpus()`. If ncpus=1, the first solution is searched serially.

**OUTPUT:**

list of polyominoes

**EXAMPLES:**

```python
sage: from sage.combinat.tiling import Polyomino
sage: H = Polyomino([(−1, 1), (−1, 4), (−1, 7), (0, 0), (0, 1), (0, 2),
....: (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (1, 1), (1, 2),
....: (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 0), (2, 2),
....: (2, 3), (2, 5), (2, 6), (2, 8)])
sage: solution = H.self_surrounding(8)
sage: G = sum([p.show2d() for p in solution], Graphics())
sage: solution = H.self_surrounding(8, remove_incomplete_copies=False)
sage: G = sum([p.show2d() for p in solution], Graphics())
```

**show2d**(size=0.7, color='black', thickness=1)

Return a 2d Graphic object representing the polyomino.

**INPUT:**

- `self` - a polyomino of dimension 2
- `size` - number (optional, default: 0.7), the size of each square.
- `color` - color (optional, default: 'black'), color of the boundary line.
- `thickness` - number (optional, default: 1), how thick the boundary line is.

**EXAMPLES:**

```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0),(1,0),(1,1),(1,2)], color='deeppink')
sage: p.show2d()  # long time (0.5s)
```

**show3d**(size=1)

Return a 3d Graphic object representing the polyomino.

**INPUT:**

- `self` - a polyomino of dimension 3
- `size` - number (optional, default: 1), the size of each 1 \times 1 \times 1 cube. This does a homothety with respect to the center of the polyomino.
EXAMPLES:

```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0), (0,1,0), (1,1,0), (1,1,1)], color='blue')
sage: p.show3d() # long time (2s)
```

```
Graphics3d Object
```

```python
sage: p.sorted_list()
```

```
[(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1)]
```

```
translated_copies(box)
```

Return an iterator over the translated images of self inside a polyomino.

INPUT:

- box – Polyomino or tuple of integers (size of a box)

OUTPUT:

iterator of 3d polyominoes

EXAMPLES:

```python
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino([(0,0,0),(1,0,0),(1,1,0),(1,1,1),(1,2,0)], color='deeppink')
sage: for t in p.translated_copies(box=(5,8,2)): t
```

```
Polyomino: [(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 2, 0)], Color: →deeppink
Polyomino: [(0, 1, 0), (1, 1, 0), (1, 2, 0), (1, 2, 1), (1, 3, 0)], Color: →deeppink
Polyomino: [(0, 2, 0), (1, 2, 0), (1, 3, 0), (1, 3, 1), (1, 4, 0)], Color: →deeppink
Polyomino: [(0, 3, 0), (1, 3, 0), (1, 4, 0), (1, 4, 1), (1, 5, 0)], Color: →deeppink
Polyomino: [(0, 4, 0), (1, 4, 0), (1, 5, 0), (1, 5, 1), (1, 6, 0)], Color: →deeppink
Polyomino: [(0, 5, 0), (1, 5, 0), (1, 6, 0), (1, 6, 1), (1, 7, 0)], Color: →deeppink
Polyomino: [(1, 0, 0), (2, 0, 0), (2, 1, 0), (2, 1, 1), (2, 2, 0)], Color: →deeppink
Polyomino: [(1, 1, 0), (2, 1, 0), (2, 2, 0), (2, 2, 1), (2, 3, 0)], Color: →deeppink
Polyomino: [(1, 2, 0), (2, 2, 0), (2, 3, 0), (2, 3, 1), (2, 4, 0)], Color: →deeppink
Polyomino: [(1, 3, 0), (2, 3, 0), (2, 4, 0), (2, 4, 1), (2, 5, 0)], Color: →deeppink
Polyomino: [(1, 4, 0), (2, 4, 0), (2, 5, 0), (2, 5, 1), (2, 6, 0)], Color: →deeppink
Polyomino: [(1, 5, 0), (2, 5, 0), (2, 6, 0), (2, 6, 1), (2, 7, 0)], Color: →deeppink
(continues on next page)
This method is independent of the translation of the polyomino:

```
sage: q = Polyomino([(0,0,0), (1,0,0)])
sage: list(q.translated_copies((2,2,1)))
[Polyomino: [(0, 0, 0), (1, 0, 0)], Color: gray, Polyomino: [(0, 1, 0), (1, 1, 0)], Color: gray]
sage: q = Polyomino([(34,7,-9), (35,7,-9)])
sage: list(q.translated_copies((2,2,1)))
[Polyomino: [(0, 0, 0), (1, 0, 0)], Color: gray, Polyomino: [(0, 1, 0), (1, 1, 0)], Color: gray]
```

Inside smaller boxes:

```
sage: list(p.translated_copies(box=(2,2,3)))
[]
sage: list(p.translated_copies(box=(2,3,2)))
[Polyomino: [(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 2, 0)], Color: deeppink]
sage: list(p.translated_copies(box=(3,2,2)))
[]
sage: list(p.translated_copies(box=(1,1,1)))
[]
```

Using a Polyomino as input:

```
sage: b = Polyomino([(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)])
sage: p = Polyomino([(0,0)])
```
translated_copies_intersection(box)

Return the set of non empty intersections of translated images of self with a polyomino.

INPUT:

* box – Polyomino or tuple of integers (size of a box)

OUTPUT:

set of 3d polyominos

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino
sage: p = Polyomino(((0,0), (1,0), (0,1)))
sage: b = Polyomino(((0,0), (1,0), (2,0), (0,1), (1,1), (0,2)))
sage: list(p.translated_copies_intersection(b))
[[(0, 0), (1, 0)], [(0, 1), (1, 0)], [(0, 2), (1, 0)], [(1, 0), (2, 0)], [(1, 1), (2, 0)]]
```

Using a Polyomino as input:

```
sage: b = Polyomino(((0,0), (0,1), (0,2), (1,0), (2,0)))
sage: p = Polyomino(((0,0), (1,0)))
sage: sorted(sorted(a.frozenset()) for a in p.translated_copies_intersection(b))
[[[(0, 0), (0, 1), (0, 2)], [(1, 0), (2, 0)]]]
```

class sage.combinat.tiling.TilingSolver(pieces, box, rotation=True, reflection=False, reusable=False, outside=False)

Bases: SageObject

Tiling solver

Solve the problem of tiling a polyomino with a certain number of polyominos.
INPUT:

- `pieces` – iterable of Polyominoes
- `box` – Polyomino or tuple of integers (size of a box)
- `rotation` – bool (optional, default: `True`), whether to allow rotations
- `reflection` – bool (optional, default: `False`), whether to allow reflections
- `reusable` – bool (optional, default: `False`), whether to allow the pieces to be reused
- `outside` – bool (optional, default: `False`), whether to allow pieces to partially go outside of the box (all non-empty intersection of the pieces with the box are considered)

EXAMPLES:

By default, rotations are allowed and reflections are not allowed:

```
sage: from sage.combinat.tiling import TilingSolver, Polyomino
sage: p = Polyomino([(0,0,0)])
sage: q = Polyomino([(0,0,0), (0,0,1)])
sage: r = Polyomino([(0,0,0), (0,0,1), (0,0,2)])
sage: T = TilingSolver([p,q,r], box=(1,1,6))
sage: T
Tiling solver of 3 pieces into a box of size 6
Rotation allowed: True
Reflection allowed: False
Reusing pieces allowed: False
```

Solutions are given by an iterator:

```
sage: it = T.solve()
sage: for p in next(it): p
Polyomino: [(0, 0, 0)], Color: gray
Polyomino: [(0, 0, 1), (0, 0, 2)], Color: gray
Polyomino: [(0, 0, 3), (0, 0, 4), (0, 0, 5)], Color: gray
```

Another solution:

```
sage: for p in next(it): p
Polyomino: [(0, 0, 0)], Color: gray
Polyomino: [(0, 0, 1), (0, 0, 2), (0, 0, 3)], Color: gray
Polyomino: [(0, 0, 4), (0, 0, 5)], Color: gray
```

Tiling of a polyomino by polyominos:

```
sage: b = Polyomino([(0,0), (1,0), (1,1), (2,1), (1,2), (2,2), (0,3), (1,3)])
sage: p = Polyomino([(0,0), (1,0)])
sage: T = TilingSolver([p], box=b, reusable=True)
sage: T.number_of_solutions()
2
```

```
animate(partial=None, stop=None, size=0.75, axes=False)
```

Return an animation of evolving solutions.

INPUT:

- `partial` - string (optional, default: `None`), whether to include partial (incomplete) solutions. It can be one of the following:
- None - include only complete solutions
- 'common_prefix' - common prefix between two consecutive solutions
- 'incremental' - one piece change at a time

- stop - integer (optional, default: None), number of frames
- size - number (optional, default: 0.75), the size of each 1 \times 1 square. This does a homothety with respect to the center of each polyomino.
- axes - bool (optional, default: False), whether the x and y axes are shown.

EXAMPLES:

```
sage: from sage.combinat.tiling import Polyomino, TilingSolver
sage: y = Polyomino([(0,0),(1,0),(2,0),(3,0),(2,1)], color='cyan')

sage: T = TilingSolver([y], box=(5,10), reusable=True, reflection=True)
sage: a = T.animate()
sage: a
# optional -- ImageMagick # long time
Animation with 10 frames
```

Include partial solutions (common prefix between two consecutive solutions):

```
sage: a = T.animate('common_prefix')
sage: a
# optional -- ImageMagick # long time
Animation with 19 frames
```

Incremental solutions (one piece removed or added at a time):

```
sage: a = T.animate('incremental')
```

```
sage: a.show()
# optional -- ImageMagick # long time
```

The show function takes arguments to specify the delay between frames (measured in hundredths of a second, default value 20) and the number of iterations (default value 0, which means to iterate forever). To iterate 4 times with half a second between each frame:

```
sage: a.show(delay=50, iterations=4)
# optional -- ImageMagick # long time
```

Limit the number of frames:

```
sage: a = T.animate('incremental', stop=13) # not tested
sage: a
Animation with 13 frames
```

```
coord_to_int_dict()
```

Return a dictionary mapping coordinates to integers.

OUTPUT:

```
dict
```

EXAMPLES:
sage: from sage.combinat.tiling import TilingSolver, Polyomino
sage: p = Polyomino(((0,0,0),))
reusable pieces:
sage: q = Polyomino(((0,0,0), (0,0,1)))
reusable pieces:
sage: r = Polyomino(((0,0,0), (0,0,1), (0,0,2)))
reusable pieces:
sage: T = TilingSolver([p,q,r], box=(1,1,6))
sage: A = T.coord_to_int_dict()
sage: sorted(A.items())
[((0, 0, 0), 3), ((0, 0, 1), 4), ((0, 0, 2), 5), ((0, 0, 3), 6), ((0, 0, 4), 7), ((0, 0, 5), 8)]

Reusable pieces:
sage: p = Polyomino(((0,0), (0,1)))
reusable pieces:
sage: q = Polyomino(((0,0), (0,1), (1,0), (1,1)))
reusable pieces:
sage: T = TilingSolver([p,q], box=[3,2], reusable=True)
reusable pieces:
sage: B = T.coord_to_int_dict()
reusable pieces:
sage: sorted(B.items())
[((0, 0), 0), ((0, 1), 1), ((1, 0), 2), ((1, 1), 3), ((2, 0), 4), ((2, 1), 5)]

dlx_solver()
return the sage DLX solver of that tiling problem.
OUTPUT:
DLX Solver
EXAMPLES:

int_to_coord_dict()
return a dictionary mapping integers to coordinates.
EXAMPLES:

(continues on next page)
Continued from previous page:

```python
sage: q = Polyomino([(0,0), (0,1), (1,0), (1,1)])
sage: T = TilingSolver([p,q], box=[3,2], reusable=True)
sage: B = T.int_to_coord_dict()
sage: sorted(B.items())
[(0, (0, 0)), (1, (0, 1)), (2, (1, 0)), (3, (1, 1)), (4, (2, 0)), (5, (2, 1))]
```

**is_suitable()**

Return whether the volume of the box is equal to sum of the volume of the polyominoes and the number of rows sent to the DLX solver is larger than zero.

If these conditions are not verified, then the problem is not suitable in the sense that there are no solution.

**EXAMPLES:**

```python
sage: from sage.combinat.tiling import TilingSolver, Polyomino
sage: p = Polyomino([(0,0)])
sage: q = Polyomino([(0,0), (0,1)])
sage: r = Polyomino([(0,0), (0,1), (0,2)])
sage: T = TilingSolver([p,q,r], box=(1,1,6))
sage: T.is_suitable()
True
sage: T = TilingSolver([p,q,r], box=(1,1,7))
sage: T.is_suitable()
False
```

**nrows_per_piece()**

Return the number of rows necessary by each piece.

**OUTPUT:**

list

**EXAMPLES:**

```python
sage: from sage.games.quantumino import QuantuminoSolver
sage: q = QuantuminoSolver(0)
Sage: T = q.tiling_solver()
Sage: T.nrows_per_piece()
# long time (10s)
[360, 360, 360, 360, 360, 180, 180, 672, 672, 360, 360, 180, 180, 360, 360, 180]
```

**number_of_solutions()**

Return the number of distinct solutions.

**OUTPUT:**

integer

**EXAMPLES:**

```python
sage: from sage.combinat.tiling import TilingSolver, Polyomino
sage: p = Polyomino([(0,0)])
sage: q = Polyomino([(0,0), (0,1)])
sage: r = Polyomino([(0,0), (0,1), (0,2)])
sage: T = TilingSolver([p,q,r], box=(1,6))
sage: T.number_of_solutions()
6
```
```python
sage: T = TilingSolver([p,q,r], box=(1,7))
sage: T.number_of_solutions()
0
```

**pieces()**

Return the list of pieces.

**OUTPUT:**

list of 3d polyominoes

**EXAMPLES:**

```python
from sage.combinat.tiling import TilingSolver, Polyomino
p = Polyomino([[0,0,0]])
q = Polyomino([[0,0,0], (0,0,1)])
r = Polyomino([[0,0,0], (0,0,1), (0,0,2)])
T = TilingSolver([p,q,r], box=(1,1,6))
for p in T._pieces:
polyomino: [(0, 0, 0)], Color: gray
polyomino: [(0, 0, 0), (0, 0, 1)], Color: gray
polyomino: [(0, 0, 0), (0, 0, 1), (0, 0, 2)], Color: gray
```

**row_to_polyomino(row_number)**

Return a polyomino associated to a row.

**INPUT:**

- row_number – integer, the i-th row

**OUTPUT:**

polyomino

**EXAMPLES:**

```python
from sage.combinat.tiling import TilingSolver, Polyomino
a = Polyomino([[0,0,0], (0,0,1), (1,0,0)], color='blue')
b = Polyomino([[0,0,0], (1,0,0), (0,1,0)], color='red')
T = TilingSolver([a,b], box=(2,1,3))
len(T.rows())
16
T.row_to_polyomino(7)
polyomino: [(0, 0, 2), (1, 0, 1), (1, 0, 2)], Color: blue
T.row_to_polyomino(13)
polyomino: [(0, 0, 1), (1, 0, 1), (1, 0, 2)], Color: red
```

**rows()**

Creation of the rows

**EXAMPLES:**

```python
from sage.combinat.tiling import TilingSolver, Polyomino
p = Polyomino([[0,0,0]])
```
sage: q = Polyomino([(0,0,0), (0,0,1)])

sage: r = Polyomino([(0,0,0), (0,0,1), (0,0,2)])

sage: T = TilingSolver([p,q,r], box=(1,1,6))

sage: rows = T.rows()

sage: for row in rows: row

[0, 3]
[0, 4]
[0, 5]
[0, 6]
[0, 7]
[0, 8]
[1, 3, 4]
[1, 4, 5]
[1, 5, 6]
[1, 6, 7]
[1, 7, 8]
[2, 3, 4, 5]
[2, 4, 5, 6]
[2, 5, 6, 7]
[2, 6, 7, 8]

rows_for_piece(i, mod_box_isometries=False)

Return the rows for the i-th piece.

INPUT:

• i – integer, the i-th piece
• mod_box_isometries – bool (default: False), whether to consider only rows for positions up to
  the action of the quotient the group of isometries of the $n$-cube by the subgroup of isometries of the
  $a_1 \times a_2 \cdots \times a_n$ rectangular box where are the $a_i$ are assumed to be distinct.

EXAMPLES:

sage: from sage.combinat.tiling import TilingSolver, Polyomino

sage: p = Polyomino([(0,0,0)])

sage: q = Polyomino([(0,0,0), (0,0,1)])

sage: r = Polyomino([(0,0,0), (0,0,1), (0,0,2)])

sage: T = TilingSolver([p,q,r], box=(1,1,6))

sage: T.rows_for_piece(0)

[[0, 3], [0, 4], [0, 5], [0, 6], [0, 7], [0, 8]]

sage: T.rows_for_piece(1)

[[1, 3, 4], [1, 4, 5], [1, 5, 6], [1, 6, 7], [1, 7, 8]]

sage: T.rows_for_piece(2)

[[2, 3, 4, 5], [2, 4, 5, 6], [2, 5, 6, 7], [2, 6, 7, 8]]

Less rows when using mod_box_isometries=True:

sage: a = Polyomino([(0,0,0), (0,0,1), (1,0,0)])

sage: b = Polyomino([(0,0,0), (1,0,0), (0,1,0)])

sage: T = TilingSolver([a,b], box=(2,1,3))

sage: T.rows_for_piece(0)

[[0, 2, 3, 5],
 [0, 3, 4, 6],
(continues on next page)
solve(\textit{partial} = \texttt{None})

Return an iterator of list of polyominoes that are an exact cover of the box.

INPUT:

- \texttt{partial} - string (optional, default: \texttt{None}), whether to include partial (incomplete) solutions. It can be one of the following:
  - \texttt{None} - include only complete solution
  - \texttt{'common_prefix'} - common prefix between two consecutive solutions
  - \texttt{'incremental'} - one piece change at a time

OUTPUT:

iterator of list of polyominoes

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.tiling import TilingSolver, Polyomino
sage: p = Polyomino([[0, 0, 0]])
sage: q = Polyomino([[0, 0, 0], [0, 0, 1]])
sage: r = Polyomino([[0, 0, 0], [0, 0, 1], [0, 0, 2]])
sage: T = TilingSolver([p, q, r], box=(1, 1, 6))
sage: it = T.solve()  
sage: for p in next(it): p
Polyomino: \[(0, 0), \text{ Color: gray}\]
Polyomino: \[(0, 0), (0, 1), \text{ Color: gray}\]
Polyomino: \[(0, 0), (0, 1), (0, 2), \text{ Color: gray}\]

sage: for p in next(it): p
Polyomino: \[(0, 0), \text{ Color: gray}\]
Polyomino: \[(0, 0), (0, 1), (0, 2), \text{ Color: gray}\]

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Polyomino: [(0, 0, 3), (0, 0, 4), (0, 0, 5)], Color: gray

\[ \text{sage: for } p \text{ in next(it): p} \]
Polyomino: [(0, 0, 0)], Color: gray

\[ \text{sage: for } p \text{ in next(it): p} \]
Polyomino: [(0, 0, 0)], Color: gray

Polyomino: [(0, 0, 1), (0, 0, 2), (0, 0, 3)], Color: gray
Polyomino: [(0, 0, 4), (0, 0, 5)], Color: gray

\[ \text{sage: for } p \text{ in next(it): p} \]
Polyomino: [(0, 0, 0), (0, 0, 1)], Color: gray
Polyomino: [(0, 0, 2), (0, 0, 3), (0, 0, 4)], Color: gray
Polyomino: [(0, 0, 5)], Color: gray

Color preservation when the polyomino can be reused:

\[ \text{sage: p = Polyomino([[0,0,0]], color='yellow')} \]
\[ \text{sage: q = Polyomino([[0,0,0], (0,0,1)], color='yellow'}) \]
\[ \text{sage: r = Polyomino([[0,0,0], (0,0,1), (0,0,2)], color='yellow'}) \]
\[ \text{sage: T = TilingSolver([p,q,r], box=(1,1,6), reusable=True)} \]
\[ \text{sage: it = T.solve()} \]
\[ \text{sage: for } p \text{ in next(it): p} \]

Polyomino: [(0, 0, 0)], Color: yellow
Polyomino: [(0, 0, 1)], Color: yellow
Polyomino: [(0, 0, 2)], Color: yellow
Polyomino: [(0, 0, 3)], Color: yellow
Polyomino: [(0, 0, 4)], Color: yellow
Polyomino: [(0, 0, 5)], Color: yellow

space()
Return an iterator over all the non negative integer coordinates contained in the space to tile.

\[ \text{EXAMPLES:} \]
\[ \text{sage: from sage.combinat.tiling import TilingSolver, Polyomino} \]
\[ \text{sage: p = Polyomino([[0,0,0]])} \]
\[ \text{sage: q = Polyomino([[0,0,0], (0,0,1)])} \]
\[ \text{sage: r = Polyomino([[0,0,0], (0,0,1), (0,0,2)])} \]
\[ \text{sage: T = TilingSolver([p,q,r], box=(1,1,6))} \]
\[ \text{sage: list(T.space())} \]

[(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 0, 4), (0, 0, 5)]

starting_rows()
Return the starting rows for each piece.

\[ \text{EXAMPLES:} \]
\[ \text{sage: from sage.combinat.tiling import TilingSolver, Polyomino} \]
\[ \text{sage: p = Polyomino([[0,0,0]])} \]
\[ \text{sage: q = Polyomino([[0,0,0], (0,0,1)])} \]
\[ \text{sage: r = Polyomino([[0,0,0], (0,0,1), (0,0,2)])} \]
\[ \text{sage: T = TilingSolver([p,q,r], box=(1,1,6))} \]
\[ \text{sage: T.starting_rows()} \]
[0, 6, 11, 15]
sage.combinat.tiling.ncube_isometry_group\(n, \text{orientation\_preserving}=\text{True}\)

Return the isometry group of the \(n\)-cube as a list of matrices.

**INPUT:**
- \(n\) – positive integer, dimension of the space
- \text{orientation\_preserving} – bool (optional, default: True), whether the orientation is preserved

**OUTPUT:**
list of matrices

**EXAMPLES:**

```python
sage: from sage.combinat.tiling import ncube_isometry_group
sage: ncube_isometry_group(2)
[[1 0] [0 1] [-1 0] [0 -1]
 [0 1], [-1 0], [ 0 -1], [ 1 0]]
sage: ncube_isometry_group(2, orientation_preserving=False)
[[1 0] [0 -1] [1 0] [0 1] [0 1] [-1 0] [-1 0]
 [0 1], [-1 0], [ 0 -1], [-1 0], [1 0], [ 0 -1], [ 1 0], [ 0 1]]
```

There are 24 orientation preserving isometries of the 3-cube:

```python
sage: ncube_isometry_group(3)
[[1 0 0] [1 0 0] [1 0 0] [0 1 0] [0 1 0] [0 1 0] [0 0 1]
 [0 1 0], [0 0 1], [0 0 -1], [-1 0 0], [0 0 1], [0 1 0], [0 0 1], [1 0 0], [1 0 0],
 [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0]
 [0 0 1], [0 0 1], [0 0 -1], [-1 0 0], [0 0 -1], [-1 0 0], [0 0 -1], [-1 0 0], [0 0 -1],
 [0 0 1], [0 0 1], [0 0 1], [0 0 1], [0 0 1], [0 0 1], [0 0 1], [0 0 1], [0 0 1], [0 0 1],
 [0 -1 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0]
 [0 0 -1], [0 0 -1], [0 0 -1], [-1 0 0], [0 0 -1], [-1 0 0], [0 0 -1], [-1 0 0], [0 0 -1],
 [0 0 -1] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0] [-1 0 0]
```

sage.combinat.tiling.ncube_isometry_group_cosets\(orientation\_preserving=\text{True}\)

Return the quotient of the isometry group of the \(n\)-cube by the the isometry group of the rectangular parallelepiped.

**INPUT:**
- \(n\) – positive integer, dimension of the space
- \text{orientation\_preserving} – bool (optional, default: True), whether the orientation is preserved

**OUTPUT:**
list of cosets, each coset being a sorted list of matrices

EXAMPLES:

```
sage: from sage.combinat.tiling import ncube_isometry_group_cosets
sage: sorted(ncube_isometry_group_cosets(2))
[[[-1 0]
  [0 -1]
],
 [1 0], [-1 0]]
sage: sorted(ncube_isometry_group_cosets(2, False))
[[[-1 0] [-1 0] [1 0] [1 0]
  [0 -1] [0 1] [0 -1] [0 1]
],
 [0 -1] [0 1] [0 1] [0 1]]
sage: sorted(ncube_isometry_group_cosets(3))
[[[-1 0 0] [-1 0 0] [1 0 0] [1 0 0]
  [0 -1 0] [0 1 0] [0 -1 0] [0 1 0]
  [0 0 1], [0 0 -1], [0 0 -1], [0 0 1]
],
 [1 0 0], [-1 0 0], [-1 0 0], [1 0 0]
]]
```
5.1.348 Transitive ideal closure tool

sage.combinat.tools.transitive_ideal(f, x)

Return a list of all elements reachable from \( x \) in the abstract reduction system whose reduction relation is given by the function \( f \).

In more elementary terms:

If \( S \) is a set, and \( f \) is a function sending every element of \( S \) to a list of elements of \( S \), then we can define a digraph on the vertex set \( S \) by drawing an edge from \( s \) to \( t \) for every \( s \in S \) and every \( t \in f(s) \).

If \( x \in S \), then an element \( y \in S \) is said to be reachable from \( x \) if there is a path \( x \to y \) in this graph.

Given \( f \) and \( x \), this method computes the list of all elements of \( S \) reachable from \( x \).

Note that if there are infinitely many such elements, then this method will never halt.

For more powerful versions, see \texttt{sage.combinat.backtrack}

EXAMPLES:

```python
sage: f = lambda x: [x-1] if x > 0 else []
sage: sage.combinat.tools.transitive_ideal(f, 10)
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

5.1.349 Combinatorial triangles for posets and fans

This provides several classes and methods to convert between them. Elements of the classes are polynomials in two variables \( x \) and \( y \), possibly with other parameters. The conversion methods amount to specific invertible rational change-of-variables involving \( x \) and \( y \).

These polynomial are called triangles because their supports, the sets of exponents where their coefficients can be non-zero, have a triangular shape.

The M-triangle class is motivated by the generating series of Möbius numbers for graded posets. A typical example is:

```python
sage: W = SymmetricGroup(4)
sage: posets.NoncrossingPartitions(W).M_triangle()
M: x^3*y^3 - 6*x^2*y^3 + 6*x^2*y^2 + 10*x*y^3 - 16*x*y^2 - 5*y^3 + 6*x*y + 10*y^2 - 6*y + 1
sage: unicode_art(_)
⎛ –5 10 –6 1 ⎞
⎜ 10 –16 6 0 ⎟
⎜ –6 6 0 0 ⎟
⎝ 1 0 0 0 ⎠
```

The F-triangle class is motivated by the generating series of pure simplicial complexes endowed with a distinguished facet. One can also think about complete fans endowed with a distinguished maximal cone. A typical example is:

```python
sage: C = ClusterComplex(['A',3])
sage: f = C.greedy_facet()
sage: C.F_triangle(f)
F: 5*x^3 + 5*x^2*y + 3*x*y^2 + y^3 + 10*x^2 + 8*x*y + 3*y^2 + 6*x + 3*y + 1
sage: unicode_art(_)
⎛ 1 0 0 0 ⎞
⎝ 0 0 0 0 ⎠
```

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The H-triangles are related to the F-triangles by a relationship similar to the classical link between the f-vector and the h-vector of a simplicial complex.

The Gamma-triangles are related to the H-triangles by an analog of the relationship between gamma-vectors and h-vectors of flag simplicial complexes.

class sage.combinat.triangles_FHM.F_triangle(poly, variables=None)

Bases: Triangle

Class for the F-triangles.

h()

Return the associated H-triangle.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import F_triangle
sage: x,y = polygens(ZZ,'x,y')
sage: ft = F_triangle(1+x*y)
sage: ft.h()
H: x*y + 1
```

m()

Return the associated M-triangle.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import H_triangle
sage: x, y = polygens(ZZ,'x,y')
sage: H_triangle(1+x*y).f()
F: x + y + 1
sage: _.m()
M: x*y - y + 1
```

vector()

Return the f-vector as a polynomial in one variable.

This is obtained by letting $y = x$.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import F_triangle
sage: x, y = polygens(ZZ,'x,y')
sage: ft = 2*x^2 + 2*x*y + y^2 + 3*x + 2*y + 1
sage: F_triangle(ft).vector()
5*x^2 + 5*x + 1
```
class sage.combinat.triangles_FHM.Gamma_triangle(poly, variables=None)

Bases: Triangle

Class for the Gamma-triangles.

h()

Return the associated H-triangle.

The transition between Gamma-triangles and H-triangles is defined by

\[ H(x, y) = (1 + x)^d \sum_{0 \leq i, 0 \leq j \leq d - 2i} \gamma_{i,j} \left( \frac{x}{1 + x} \right)^i \left( \frac{1 + xy}{1 + x} \right)^j \]

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import Gamma_triangle
sage: x, y = polygen(ZZ, 'x,y')
sage: g = y**2 + x
sage: Gamma_triangle(g).h()
H: x^2*y^2 + 2*x*y + x + 1
```

vector()

Return the gamma-vector as a polynomial in one variable.

This is obtained by letting \( y = 1 \).

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import Gamma_triangle
sage: x, y = polygen(ZZ, 'x,y')
sage: gt = y**2 + x
sage: Gamma_triangle(gt).vector()
x + 1
```

class sage.combinat.triangles_FHM.H_triangle(poly, variables=None)

Bases: Triangle

Class for the H-triangles.

f()

Return the associated F-triangle.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import H_triangle
sage: x, y = polygens(ZZ, 'x,y')
sage: H_triangle(1+x*y).f()
F: x + y + 1
```

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sage: flo = H_triangle(1+4*x+2*x**2+x*y*(4+8*x)+
....: x**2*y**2*(6+4*x)+4*(x*y)**3+(x*y)**4).f(); flo
F: 7*x^4 + 12*x^3*y + 10*x^2*y^2 + 4*x*y^3 + y^4 + 20*x^3
+ 28*x^2*y + 16*x*y^2 + 4*y^3 + 20*x^2 + 20*x*y
+ 6*x^2 + 8*x + 4*y + 1
sage: flo(-1-x,-1-y) == flo
True

\texttt{gamma}() \hfill \texttt{m}()

Return the associated Gamma-triangle.

In some cases, this is a more condensed way to encode the same amount of information.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.triangles_FHM import H_triangle
sage: x, y = polygen(ZZ, 'x,y')
sage: ht = x**2*y**2 + 2*x*y + x + 1
sage: H_triangle(ht).gamma()
Γ: y^2 + x
\end{verbatim}

\begin{verbatim}
sage: W = SymmetricGroup(5)
sage: P = posets.NoncrossingPartitions(W)
sage: P.M_triangle().h().gamma()
Γ: y^4 + 3*x*y^2 + 2*x^2 + 2*x*y + x
\end{verbatim}

\texttt{transpose}()

Return the transposed H-triangle.

OUTPUT:

another H-triangle

This operation is an involution. When seen as a matrix, it performs a symmetry with respect to the northwest-southeast diagonal.

EXAMPLES:

\begin{verbatim}
sage: from sage.combinat.triangles_FHM import H_triangle
sage: h = polygen(ZZ, 'h')
sage: x, y = polygens(h.parent(), 'x,y')
sage: ht = H_triangle(x^2*y^2 + 2*x*y + 2*h - 4*x + 1, variables=[x,y])
sage: ht.m()
M: x^2*y^2 + (-2*h + 2)*x*y^2 + (2*h - 2)*x*y + (2*h - 3)*y^2 + (-2*h + 2)*y + 1
\end{verbatim}

(continues on next page)
Combinatorics, Release 10.0

sage: H_triangle(x^2*y^2 + 2*x*y + x + 1).transpose()
H: x^2*y^2 + x^2*y + 2*x*y + 1

vector()
Return the h-vector as a polynomial in one variable.
This is obtained by letting $y = 1$.

EXAMPLES:

sage: from sage.combinat.triangles_FHM import H_triangle
sage: x, y = polygen(ZZ, 'x,y')
sage: ht = x**2*y**2 + 2*x*y + x + 1
sage: H_triangle(ht).vector()
x^2 + 3*x + 1

class sage.combinat.triangles_FHM.M_triangle(poly, variables=None)
Bases: Triangle
Class for the M-triangles.
This is motivated by generating series of Möbius numbers of graded posets.

EXAMPLES:

sage: x, y = polygens(ZZ, 'x,y')
sage: P = Poset({2:[1]})
sage: P.M_triangle()
M: x*y - y + 1

dual()
Return the dual M-triangle.
This is the M-triangle of the dual poset, hence an involution.
When seen as a matrix, this performs a symmetry with respect to the northwest-southeast diagonal.

EXAMPLES:

sage: from sage.combinat.triangles_FHM import M_triangle
sage: x, y = polygens(ZZ, 'x,y')
sage: mt = M_triangle(x*y - y + 1)
sage: mt.dual() == mt
True

f()
Return the associated F-triangle.

EXAMPLES:

sage: from sage.combinat.triangles_FHM import M_triangle
sage: x, y = polygens(ZZ, 'x,y')
sage: M_triangle(1-y+x*y).f()
F: x + y + 1
h()  
Return the associated H-triangle.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import M_triangle
sage: x, y = polygens(ZZ, 'x,y')
sage: M_triangle(1-y+x*y).h()
H: x*y + 1
```

transmute()  
Return the image of self by an involution.

OUTPUT:
another M-triangle

The involution is defined by converting to an H-triangle, transposing the matrix, and then converting back to an M-triangle.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import M_triangle
sage: x, y = polygens(ZZ, 'x,y')
sage: nc3 = x^2*y^2 - 3*x*y^2 + 3*x*y + 2*y^2 - 3*y + 1
sage: m = M_triangle(nc3)
sage: m2 = m.transmute(); m2
M: 2*x^2*y^2 - 3*x*y^2 + 2*x*y + y^2 - 2*y + 1
sage: m2.transmute() == m
True
```

class sage.combinat.triangles_FHM.Triangle(poly, variables=None)

Bases: SageObject

Common class for different kinds of triangles.

This serves as a base class for F-triangles, H-triangles, M-triangles and Gamma-triangles.

The user should use these subclasses directly.

The input is a polynomial in two variables. One can also give a polynomial with more variables and specify two chosen variables.

EXAMPLES:

```
sage: from sage.combinat.triangles_FHM import Triangle
sage: x, y = polygens(ZZ, 'x,y')
sage: ht = Triangle(1+4*x+2*x*y)
sage: unicode_art(ht)
\[
\begin{pmatrix}
\emptyset & 2 \\
1 & 4 \\
\end{pmatrix}
\]
```

matrix()  
Return the associated matrix for display.

EXAMPLES:
sage: from sage.combinat.triangles_FHM import H_triangle
sage: x, y = polygens(ZZ, 'x,y')

polynomial()

Return the triangle as a bare polynomial.

EXAMPLES:

sage: from sage.combinat.triangles_FHM import H_triangle
sage: x, y = polygens(ZZ, 'x,y')

truncate(d)

Return the truncated triangle.

INPUT:

• d – integer

As a polynomial, this means that all monomials with a power of either \( x \) or \( y \) greater than or equal to \( d \) are dismissed.

EXAMPLES:

5.1.350 Tuples

class sage.combinat.tuple.Tuples(S, k)

Bases: Parent, UniqueRepresentation

Return the enumerated set of ordered tuples of \( S \) of length \( k \).

An ordered tuple of length \( k \) of set is an ordered selection with repetition and is represented by a list of length \( k \) containing elements of set.

EXAMPLES:

(continues on next page)
sage: K.<a> = GF(4, 'a')
sage: mset = [x for x in K if x != 0]
sage: Tuples(mset,2).list()
[[a, a], [a + 1, a], [1, a], [a, a + 1], [a + 1, a + 1], [1, a + 1], [a, 1], [a + 1, 1], [1, 1]]

cardinality()

EXAMPLES:

sage: S = [1,2,3,4,5]
sage: Tuples(S,2).cardinality()
25

sage: S = [1,1,2,3,4,5]
sage: Tuples(S,2).cardinality()
25

cardinality()

EXAMPLES:

sage: S = [1,2]
sage: UnorderedTuples(S,3).list()
[[1, 1, 1], [1, 1, 2], [1, 2, 2], [2, 2, 2]]

sage: UnorderedTuples(['a','b','c'],2).list()
[['a', 'a'], ['a', 'b'], ['a', 'c'], ['b', 'b'], ['b', 'c'], ['c', 'c']]

cardinality()

EXAMPLES:

sage: S = [1,2,3,4,5]
sage: UnorderedTuples(S,2).cardinality()
15

list()

EXAMPLES:

sage: S = [1,2]
sage: UnorderedTuples(S,3).list()
sage.combinat.tuple.UnorderedTuples_sk
alias of UnorderedTuples

5.1.351 Introduction to combinatorics in Sage

This thematic tutorial is a translation by Hugh Thomas of the combinatorics chapter written by Nicolas M. Thiéry in the book “Calcul Mathématique avec Sage” [CMS2012]. It covers mainly the treatment in Sage of the following combinatorial problems: enumeration (how many elements are there in a set \( S \)), listing (generate all the elements of \( S \), or iterate through them), and random selection (choosing an element at random from a set \( S \) according to a given distribution, for example the uniform distribution). These questions arise naturally in the calculation of probabilities (what is the probability in poker of obtaining a straight or a four-of-a-kind of aces?), in statistical physics, and also in computer algebra (the number of elements in a finite field), or in the analysis of algorithms. Combinatorics covers a much wider domain (partial orders, representation theory, …) for which we only give a few pointers towards the possibilities offered by Sage.

Todo: Add link to some thematic tutorial on graphs

A characteristic of computational combinatorics is the profusion of types of objects and sets that one wants to manipulate. It would be impossible to describe them all or, a fortiori, to implement them all. After some Initial examples, this chapter illustrates the underlying method: supplying the basic building blocks to describe common combinatorial sets Common enumerated sets, tools for combining them to construct new examples Constructions, and generic algorithms for solving uniformly a large class of problems Generic algorithms.

This is a domain in which Sage has much more extensive capabilities than most computer algebra systems, and it is rapidly expanding; at the same time, it is still quite new, and has many unnecessary limitations and incoherences.

Initial examples

Poker and probability

We begin by solving a classic problem: enumerating certain combinations of cards in the game of poker, in order to deduce their probability.

A card in a poker deck is characterized by a suit (hearts, diamonds, spades, or clubs) and a value (2, 3, …, 10, jack, queen, king, ace). The game is played with a full deck, which consists of the Cartesian product of the set of suits and the set of values:

\[
\text{Cards} = \text{Suits} \times \text{Values} = \{(s, v) \mid s \in \text{Suits} \text{ et } v \in \text{Values}\}.
\]

We construct these examples in Sage:

```
sage: Suits = Set(['Hearts', 'Diamonds', 'Spades', 'Clubs'])
sage: Values = Set([2, 3, 4, 5, 6, 7, 8, 9, 10,
                 .....:
                 "Jack", "Queen", "King", "Ace")]
sage: Cards = cartesian_product([Values, Suits])
```
There are 4 suits and 13 possible values, and therefore $4 \times 13 = 52$ cards in the poker deck:

```python
sage: Suits.cardinality()
4
sage: Values.cardinality()
13
sage: Cards.cardinality()
52
```

Draw a card at random:

```python
sage: Cards.random_element()  # random
(6, 'Clubs')
```

Now we can define a set of cards:

```python
sage: Set([Cards.random_element(), Cards.random_element()])  # random
{(2, 'Hearts'), (4, 'Spades')}
```

This problem should eventually disappear: it is planned to change the implementation of Cartesian products so that their elements are immutable by default.

Returning to our main topic, we will be considering a simplified version of poker, in which each player directly draws five cards, which form his hand. The cards are all distinct and the order in which they are drawn is irrelevant; a hand is therefore a subset of size 5 of the set of cards. To draw a hand at random, we first construct the set of all possible hands, and then we ask for a randomly chosen element:

```python
sage: Hands = Subsets(Cards, 5)
sage: Hands.random_element()  # random
{(4, 'Hearts'), (9, 'Diamonds'), (8, 'Spades'), (9, 'Clubs'), (7, 'Hearts')}
```

The total number of hands is given by the number of subsets of size 5 of a set of size 52, which is given by the binomial coefficient $\binom{52}{5}$:

```python
sage: binomial(52,5)
2598960
```

One can also ignore the method of calculation, and simply ask for the size of the set of hands:

```python
sage: Hands.cardinality()
2598960
```

The strength of a poker hand depends on the particular combination of cards present. One such combination is the flush: this is a hand all of whose cards have the same suit. (In principle, straight flushes should be excluded; this will be the goal of an exercise given below.) Such a hand is therefore characterized by the choice of five values from among the thirteen possibilities, and the choice of one of four suits. We will construct the set of all flushes, so as to determine how many there are:

```python
sage: Flushes = cartesian_product([Subsets(Values, 5), Suits])
sage: Flushes.cardinality()
5148
```

The probability of obtaining a flush when drawing a hand at random is therefore:
or about two in a thousand:

```
sage: 1000.0 * Flashes.cardinality() / Hands.cardinality()
1.98079231692677
```

We will now attempt a little numerical simulation. The following function tests whether a given hand is a flush or not:

```
sage: def is_flush(hand):
    ....:     return len(set(suit for (val, suit) in hand)) == 1
```

We now draw 10000 hands at random, and count the number of flushes obtained (this takes about 10 seconds):

```
sage: n = 10000
sage: nflush = 0
sage: for i in range(n):  # long time
    ....:     hand = Hands.random_element()
    ....:     if is_flush(hand):
    ....:         nflush += 1
sage: n, nflush  # random
(10000, 18)
```

### Exercises

A hand containing four cards of the same value is called a *four of a kind*. Construct the set of four of a kind hands (Hint: use `Arrangements` to choose a pair of distinct values at random, then choose a suit for the first value). Calculate the number of four of a kind hand, list them, and then determine the probability of obtaining a four of a kind when drawing a hand at random.

A hand all of whose cards have the same suit, and whose values are consecutive, is called a *straight flush* rather than a *flush*. Count the number of straight flushes, and then deduce the correct probability of obtaining a flush when drawing a hand at random.

Calculate the probability of each of the poker hands (see Wikipedia article [Poker hands](https://en.wikipedia.org/wiki/Poker_hands)), and compare them with the results of simulations.

### Enumeration of trees using generating functions

In this section, we discuss the example of complete binary trees, and illustrate in this context many techniques of enumeration in which formal power series play a natural role. These techniques are quite general, and can be applied whenever the combinatorial objects in question admit a recursive definition (grammar) (see [Species, decomposable combinatorial classes](https://en.wikipedia.org/wiki/Species_(mathematics)) for an automated treatment). The goal is not a formal presentation of these methods; the calculations are rigorous, but most of the justifications will be skipped.

A *complete binary tree* is either a leaf $L$, or a node to which two complete binary trees are attached (see Figure: The five complete binary trees with four leaves).
Find by hand all the complete binary trees with \( n = 1, 2, 3, 4, 5 \) leaves (see Exercise: complete binary tree iterator to find them using Sage).

Our goal is to determine the number \( c_n \) of complete binary trees with \( n \) leaves (in this section, except when explicitly stated otherwise, “trees” always means complete binary trees). This is a typical situation in which one is not only interested in a single set, but in a family of sets, typically parameterized by \( n \in \mathbb{N} \).

According to the solution of Exercise: enumeration of binary trees, the first terms are given by \( c_1, \ldots, c_5 = 1, 1, 2, 5, 14 \). The simple fact of knowing these few numbers is already very valuable. In fact, this permits research in a gold mine of information: the Online Encyclopedia of Integer Sequences, commonly called “Sloane”, the name of its principal author, which contains more than 190000 sequences of integers:

\[
\text{sage: oeis([1,1,2,5,14])} \quad \# \text{optional -- internet}
\]

\[
0: \text{A000108: Catalan numbers: ...}
\]

\[
1: \ldots
\]

\[
2: \ldots
\]

The result suggests that the trees are counted by one of the most famous sequences, the Catalan numbers. Looking through the references supplied by the Encyclopedia, we see that this is really the case: the few numbers above form a digital fingerprint of our objects, which enable us to find, in a few seconds, a precise result from within an abundant literature.

Our next goal is to recover this result using Sage. Let \( C_n \) be the set of trees with \( n \) leaves, so that \( c_n = |C_n| \); by convention, we will define \( C_0 = \emptyset \) and \( c_0 = 0 \). The set of all trees is then the disjoint union of the sets \( C_n \):

\[
C = \biguplus_{n \in \mathbb{N}} C_n.
\]

Having named the set \( C \) of all trees, we can translate the recursive definition of trees into a set-theoretic equation:

\[
C \approx \{L\} \uplus C \times C.
\]

In words: a tree \( t \) (which is by definition in \( C \)) is either a leaf (so in \( \{L\} \)) or a node to which two trees \( t_1 \) and \( t_2 \) have been attached, and which we can therefore identify with the pair \( (t_1, t_2) \) (in the Cartesian product \( C \times C \)).

The founding idea of algebraic combinatorics, introduced by Euler in a letter to Goldbach of 1751 to treat a similar problem, is to manipulate all the numbers \( c_n \) simultaneously, by encoding them as coefficients in a formal power series, called the generating function of the \( c_n \)’s:

\[
C(z) = \sum_{n \in \mathbb{N}} c_n z^n,
\]

where \( z \) is a formal variable (which means that we do not have to worry about questions of convergence). The beauty of this idea is that set-theoretic operations (\( A \uplus B, A \times B \)) translate naturally into algebraic operations on the corresponding series (\( A(z) + B(z), A(z) \cdot B(z) \)), in such a way that the set-theoretic equation satisfied by \( C \) can be translated directly into an algebraic equation satisfied by \( C(z) \):

\[
C(z) = z + C(z) \cdot C(z).
\]
Now we can solve this equation with Sage. In order to do so, we introduce two variables, $C$ and $z$, and we define the equation:

\[
\text{sage: } C, z = \text{var('C,z')}
\]
\[
\text{sage: } \text{sys = [ } C == z + C*C \text{ ]}
\]

There are two solutions, which happen to have closed forms:

\[
\text{sage: } \text{sol = solve(sys, C, solution_dict=True); sol}
\]
\[
\left[\{C: -1/2*sqrt(-4*z + 1) + 1/2\}, \{C: 1/2*sqrt(-4*z + 1) + 1/2\}\right]
\]
\[
\text{sage: } \text{s0 = sol[0][C]; s1 = sol[1][C]}
\]

and whose Taylor series begin as follows:

\[
\text{sage: } \text{s0.series(z, 6)}
\]
\[
1*z + 1*z^2 + 2*z^3 + 5*z^4 + 14*z^5 + \text{Order}(z^6)
\]
\[
\text{sage: } \text{s1.series(z, 6)}
\]
\[
1 + (-1)*z + (-1)*z^2 + (-2)*z^3 + (-5)*z^4 + (-14)*z^5 + \text{Order}(z^6)
\]

The second solution is clearly aberrant, while the first one gives the expected coefficients. Therefore, we set:

\[
\text{sage: } C = \text{s0}
\]

We can now calculate the next terms:

\[
\text{sage: } \text{C.series(z, 11)}
\]
\[
1*z + 1*z^2 + 2*z^3 + 5*z^4 + 14*z^5 + 42*z^6 + 132*z^7 + 429*z^8 + 1430*z^9 + 4862*z^{10} + \text{Order}(z^{11})
\]

or calculate, more or less instantaneously, the 100-th coefficient:

\[
\text{sage: } \text{C.series(z, 101).coefficient(z,100)}
\]
\[
227508830794229349661819540395688853956041682601541047340
\]

It is unfortunate to have to recalculate everything if at some point we wanted the 101-st coefficient. Lazy power series (see sage.rings.lazy_series_ring) come into their own here, in that one can define them from a system of equations without solving it, and, in particular, without needing a closed form for the answer. We begin by defining the ring of lazy power series:

\[
\text{sage: } \text{L.<z> = LazyPowerSeriesRing(QQ)}
\]

Then we create a “free” power series, which we name, and which we then define by a recursive equation:

\[
\text{sage: } \text{C = L.undefined(valuation=1)}
\]
\[
\text{sage: } \text{C.define( z + C * C )}
\]

\[
\text{sage: } \text{[C.coefficient(i) for i in range(11)]}
\]
\[
[0, 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862]
\]

At any point, one can ask for any coefficient without having to redefine $C$:

\[
\text{sage: } \text{C.coefficient(100)}
\]
\[
227508830794229349661819540395688853956041682601541047340
\]
We now return to the closed form of $C(z)$:

\begin{verbatim}
sage: z = var('z')
sage: C = s0; C
-1/2*sqrt(-4*z + 1) + 1/2
\end{verbatim}

The $n$-th coefficient in the Taylor series for $C(z)$ being given by $\frac{1}{n!}C(z)^{(n)}(0)$, we look at the successive derivatives $C(z)^{(n)}(z)$:

\begin{verbatim}
sage: derivative(C, z, 1)
1/sqrt(-4*z + 1)
sage: derivative(C, z, 2)
2/(-4*z + 1)^(3/2)
sage: derivative(C, z, 3)
12/(-4*z + 1)^(5/2)
\end{verbatim}

This suggests the existence of a simple explicit formula, which we will now seek. The following small function returns $d_n = n!c_n$:

\begin{verbatim}
sage: def d(n):
    return derivative(s0, n).subs(z=0)
\end{verbatim}

Taking successive quotients:

\begin{verbatim}
sage: [ (d(n+1) / d(n)) for n in range(1,17) ]
[2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62]
\end{verbatim}

we observe that $d_n$ satisfies the recurrence relation $d_{n+1} = (4n-2)d_n$, from which we deduce that $c_n$ satisfies the recurrence relation $c_{n+1} = \frac{4n-2}{n+1}c_n$. Simplifying, we find that $c_n$ is the $(n-1)$-th Catalan number:

\[ c_n = \text{Catalan}(n-1) = \frac{1}{n} \binom{2(n-1)}{n-1}. \]

We check this:

\begin{verbatim}
sage: n = var('n')
sage: c = 1/n*binomial(2*(n-1),n-1)
sage: [c.subs(n=k) for k in range(1, 11)]
[1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862]
sage: [catalan_number(k-1) for k in range(1, 11)]
[1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862]
\end{verbatim}

We can now calculate coefficients much further; here we calculate $c_{100000}$ which has more than 60000 digits:

\begin{verbatim}
sage: cc = c(n = 100000)
This takes a couple of seconds:
\end{verbatim}

\begin{verbatim}
sage: %time cc = c(100000) # not tested
CPU times: user 2.34 s, sys: 0.00 s, total: 2.34 s
Wall time: 2.34 s
sage: ZZ(cc).ndigits()
60198
\end{verbatim}
The methods which we have used generalize to all recursively defined objects: the system of set-theoretic equations can be translated into a system of equations on the generating function, which enables the recursive calculation of its coefficients. If the set-theoretic equations are simple enough (for example, if they only involve Cartesian products and disjoint unions), the equation for $C(z)$ is algebraic. This equation has, in general, no closed-form solution. However, using confinement, one can deduce a linear differential equation which $C(z)$ satisfies. This differential equation, in turn, can be translated into a recurrence relation of fixed length on its coefficients $c_n$. In this case, the series is called $D$-finite. After the initial calculation of this recurrence relation, the calculation of coefficients is very fast. All these steps are purely algorithmic, and it is planned to port into Sage the implementations which exist in Maple (the gfun and combstruct packages) or MuPAD-Combinat (the decomposableObjects library).

For the moment, we illustrate this general procedure in the case of complete binary trees. The generating function $C(z)$ is a solution to an algebraic equation $P(z, C(z)) = 0$, where $P = P(x, y)$ is a polynomial with coefficients in $Q$. In the present case, $P = y^2 - y + x$. We formally differentiate this equation with respect to $z$:

```python
sage: x, y, z = var('x, y, z')
sage: P = function('P')(x, y)
sage: C = function('C')(z)
sage: equation = P(x=z, y=C) == 0
sage: diff(equation, z)
\frac{dC(z)}{dz} \frac{\partial P}{\partial y}(z, C(z)) + \frac{\partial P}{\partial x}(z, C(z)) = 0
```

or, in a more readable format,

$$\frac{dC(z)}{dz} = -\frac{\partial P}{\partial y}(z, C(z)).$$

From this we deduce:

$$\frac{dC(z)}{dz} = -\frac{\partial P}{\partial y}(z, C(z)).$$

In the case of complete binary trees, this gives:

```python
sage: P = y^2 - y + x
sage: Px = diff(P, x); Py = diff(P, y)
sage: - Px / Py
-1/(2*y - 1)
```

Recall that $P(z, C(z)) = 0$. Thus, we can calculate this fraction mod $P$ and, in this way, express the derivative of $C(z)$ as a polynomial in $C(z)$ with coefficients in $Q(z)$. In order to achieve this, we construct the quotient ring $R = Q(x)[y]/(P)$:

```python
sage: Qx = QQ['x'].fraction_field()
sage: Qxy = Qx['y']
sage: R = Qxy.quo(P); R
Univariate Quotient Polynomial Ring in ybar
over Fraction Field of Univariate Polynomial Ring in x
over Rational Field with modulus y^2 - y + x
```

Note: $ybar$ is the name of the variable $y$ in the quotient ring.

**Todo:** add link to some tutorial on quotient rings

We continue the calculation of this fraction in $R$: 5.1. Comprehensive Module List
```
sage: fraction = - R(Px) / R(Py); fraction
(1/2/(x - 1/4))*y - 1/4/(x - 1/4)
```

Note: The following variant does not work yet:
```
sage: fraction = R( - Px / Py ); fraction
# todo: not implemented
Traceback (most recent call last):
...
TypeError: denominator must be a unit
```

We lift the result to \( Q(x)[y] \) and then substitute \( z \) and \( C(z) \) to obtain an expression for \( \frac{d}{dz} C(z) \):
```
sage: fraction = fraction.lift(); fraction
(1/2/(x - 1/4))*y - 1/4/(x - 1/4)
sage: fraction(x=z, y=C)
2*C(z)/(4*z - 1) - 1/(4*z - 1)
```
or, more legibly,
\[
\frac{\partial C(z)}{\partial z} = \frac{1}{1 - 4z} - \frac{2}{1 - 4z} C(z).
\]

In this simple case, we can directly deduce from this expression a linear differential equation with coefficients in \( Q[z] \):
```
sage: equadiff = diff(C,z) == fraction(x=z, y=C)
sage: equadiff
diff(C(z), z) == 2*C(z)/(4*z - 1) - 1/(4*z - 1)
sage: equadiff = equadiff.simplify_rational()
sage: equadiff = equadiff.rhs().denominator()
sage: equadiff
(4*z - 1)*diff(C(z), z) - 2*C(z) + 1 == 0
```
or, more legibly,
\[
(1 - 4z) \frac{\partial C(z)}{\partial z} + 2C(z) - 1 = 0.
\]

It is trivial to verify this equation on the closed form:
```
sage: Cf = sage.symbolic.function_factory.function('C')
sage: equadiff = equadiff.substitute_function(C, s0.function(z))
(4*z - 1)/sqrt(-4*z + 1) + sqrt(-4*z + 1) == 0
sage: bool(equadiff.substitute_function(Cf, s0.function(z)))
True
```

In the general case, one continues to calculate successive derivatives of \( C(z) \). These derivatives are **confined** in the quotient ring \( Q(z)[C]/(P) \) which is of finite dimension \( \deg P \) over \( Q(z) \). Therefore, one will eventually find a linear relation among the first \( \deg P \) derivatives of \( C(z) \). Putting it over a single denominator, we obtain a linear differential equation of degree \( \leq \deg P \) with coefficients in \( Q[z] \). By extracting the coefficient of \( z^n \) in the differential equation, we obtain the desired recurrence relation on the coefficients; in this case we recover the relation we had already found, based on the closed form:
\[
c_{n+1} = \frac{(4n - 2)}{n + 1} c_n
\]
After fixing the correct initial conditions, it becomes possible to calculate the coefficients of $C(z)$ recursively:

```python
sage: def C(n):
    return 1 if n <= 1 else (4*n-6)/n * C(n-1)
sage: [ C(i) for i in range(10) ]
[1, 1, 1, 2, 5, 14, 42, 132, 429, 1430]
```

If $n$ is too large for the explicit calculation of $c_n$, a sequence asymptotically equivalent to the sequence of coefficients $c_n$ may be sought. Here again, there are generic techniques. The central tool is complex analysis, specifically, the study of the generating function around its singularities. In the present instance, the singularity is at $z_0 = 1/4$ and one would obtain $c_n \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}$.

**Summary**

We see here a general phenomenon of computer algebra: the best *data structure* to describe a complicated mathematical object (a real number, a sequence, a formal power series, a function, a set) is often an equation defining the object (or a system of equations, typically with some initial conditions). Attempting to find a closed-form solution to this equation is not necessarily of interest: on the one hand, such a closed form rarely exists (e.g., the problem of solving a polynomial by radicals), and on the other hand, the equation, in itself, contains all the necessary information to calculate algorithmically the properties of the object under consideration (e.g., a numerical approximation, the initial terms or elements, an asymptotic equivalent), or to calculate with the object itself (e.g., performing arithmetic on power series). Therefore, instead of solving the equation, we look for the equation describing the object which is best suited to the problem we want to solve.

As we saw in our example, confinement (for example, in a finite dimensional vector space) is a fundamental tool for studying such equations. This notion of confinement is widely applicable in elimination techniques (linear algebra, Gröbner bases, and their algebro-differential generalizations). The same tool is central in algorithms for automatic summation and automatic verification of identities (Gosper’s algorithm, Zeilberger’s algorithm, and their generalizations; see also *Exercise: alternating sign matrices*).

**Todo:** add link to some tutorial on summation

All these techniques and their many generalizations are at the heart of very active topics of research: automatic combinatorics and analytic combinatorics, with major applications in the analysis of algorithms. It is likely, and desirable, that they will be progressively implemented in Sage.

**Common enumerated sets**

**First example: the subsets of a set**

Fix a set $E$ of size $n$ and consider the subsets of $E$ of size $k$. We know that these subsets are counted by the binomial coefficients $\binom{n}{k}$. We can therefore calculate the number of subsets of size $k = 2$ of $E = \{1, 2, 3, 4\}$ with the function `binomial`:

```python
sage: binomial(4, 2)
6
```

Alternatively, we can construct the set $\mathcal{P}_2(E)$ of all the subsets of size 2 of $E$, then ask its cardinality:

```python
sage: S = Subsets([1,2,3,4], 2)
sage: S.cardinality()
6
```
Once $S$ has been constructed, we can also obtain the list of its elements:

```
sage: S.list()
[[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4]]
```

or select an element at random:

```
sage: S.random_element()     # random
{1, 4}
```

More precisely, the object $S$ models the set $\mathcal{P}_2(E)$ equipped with a fixed order (here, lexicographic order). It is therefore possible to ask for its 5-th element, keeping in mind that, as with Python lists, the first element is numbered zero:

```
sage: S.unrank(4)
{2, 4}
```

As a shortcut, in this setting, one can also use the notation:

```
sage: S[4]
{2, 4}
```

but this should be used with care because some sets have a natural indexing other than by $(0, 1, \ldots)$.

Conversely, one can calculate the position of an object in this order:

```
sage: s = S([2,4]); s
{2, 4}
sage: S.rank(s)
4
```

Note that $S$ is not the list of its elements. One can, for example, model the set $\mathcal{P} (\mathcal{P}(\mathcal{P}(E)))$ and calculate its cardinality ($2^{2^4}$):

```
sage: E = Set([1,2,3,4])
sage: S = Subsets(Subsets(Subsets(E))); S
Subsets of Subsets of Subsets of {1, 2, 3, 4}
sage: n = S.cardinality(); n
200352993040684979072351560255750447825475569751419265016973...
```

which is roughly $2 \cdot 10^{19728}$:

```
sage: n.ndigits()
19729
```

or ask for its 237102124-th element:

```
sage: S.unrank(237102123)   # random print output
{{1, 2, 4}, {1, 4}, {}, {1, 3, 4}, {1, 2, 4}, {4}, {2, 3}, {1, 3}, {2}},
{{1, 3}, {2, 4}, {1, 2, 4}, {}, {3, 4}}}
```

It would be physically impossible to construct explicitly all the elements of $S$, as there are many more of them than there are particles in the universe (estimated at $10^{82}$).

Remark: it would be natural in Python to use `len(S)` to ask for the cardinality of $S$. This is not possible because Python requires that the result of `len` be an integer of type `int`; this could cause overflows, and would not permit the return of `{Infinity}` for infinite sets:
Partitions of integers

We now consider another classic problem: given a positive integer \( n \), in how many ways can it be written in the form of a sum \( n = i_1 + i_2 + \cdots + i_\ell \), where \( i_1, \ldots, i_\ell \) are positive integers? There are two cases to distinguish:

- the order of the elements in the sum is not important, in which case we call \((i_1, \ldots, i_\ell)\) a partition of \( n \);
- the order of the elements in the sum is important, in which case we call \((i_1, \ldots, i_\ell)\) a composition of \( n \).

We will begin with the partitions of \( n = 5 \); as before, we begin by constructing the set of these partitions:

```
sage: P5 = Partitions(5); P5
Partitions of the integer 5
```

then we ask for its cardinality:

```
sage: P5.cardinality()
7
```

We look at these 7 partitions; the order being irrelevant, the entries are ordered, by convention, in decreasing order.

```
sage: P5.list()
[[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1],
 [1, 1, 1, 1, 1]]
```

The calculation of the number of partitions uses the Rademacher formula (Wikipedia article Partition_(number_theory)), implemented in C and highly optimized, which makes it very fast:

```
sage: Partitions(100000).cardinality()
27493510569775980054758203125984302147328114964173055050716607366215962...
```

Partitions of integers are combinatorial objects naturally equipped with many operations. They are therefore returned as objects that are richer than simple lists:

```
sage: P7 = Partitions(7)
sage: p = P7.unrank(5); p
[4, 2, 1]
sage: type(p)
<class 'sage.combinat.partition.Partitions_n_with_category.element_class'>
```

For example, they can be represented graphically by a Ferrers diagram:

```
sage: print(p.ferrers_diagram())
 ****
 **
 *
```

We leave it to the user to explore by introspection the available operations.

Note that we can also construct a partition directly by:

```
sage: print(p.ferrers_diagram())
 ****
 **
 *
```

We leave it to the user to explore by introspection the available operations.

Note that we can also construct a partition directly by:
If one wants to restrict the possible values of the parts $i_1, \ldots, i_\ell$ of the partition as, for example, when giving change, one can use `WeightedIntegerVectors`. For example, the following calculation:

```python
sage: WeightedIntegerVectors(8, [2,3,5]).list()
[[0, 1, 1], [1, 2, 0], [4, 0, 0]]
```

shows that to make 8 dollars using 2, 3, and 5 dollar bills, one can use a 3 and a 5 dollar bill, or a 2 and two 3 dollar bills, or four 2 dollar bills.

Compositions of integers are manipulated the same way:

```python
sage: C5 = Compositions(5); C5
Compositions of 5
sage: C5.cardinality()
16
sage: C5.list()
[[1, 1, 1, 1, 1], [1, 1, 1, 2], [1, 1, 2, 1], [1, 1, 3],
 [1, 2, 1, 1], [1, 2, 2], [1, 3, 1], [1, 4], [2, 1, 1, 1],
 [2, 1, 2], [2, 2, 1], [2, 3], [3, 1, 1], [3, 2], [4, 1], [5]]
```

The number 16 above seems significant and suggests the existence of a formula. We look at the number of compositions of $n$ ranging from 0 to 9:

```python
sage: [ Compositions(n).cardinality() for n in range(10) ]
[1, 1, 2, 4, 8, 16, 32, 64, 128, 256]
```

Similarly, if we consider the number of compositions of 5 by length, we find a line of Pascal’s triangle:

```python
sage: x = var('x')
sage: sum( x^len(c) for c in C5 )
x^5 + 4*x^4 + 6*x^3 + 4*x^2 + x
```

The above example uses a functionality which we have not seen yet: C5 being iterable, it can be used like a list in a for loop or a comprehension ([Set comprehension and iterators]).

Prove the formulas suggested by the above examples for the number of compositions of $n$ and the number of compositions of $n$ of length $k$; investigate by introspection whether Sage uses these formulas for calculating cardinalities.
Some other finite enumerated sets

Essentially, the principle is the same for all the finite sets with which one wants to do combinatorics in Sage: begin by constructing an object which models this set, and then supply appropriate methods, following a uniform interface\(^1\). We now give a few more typical examples.

Intervals of integers:

```
sage: C = IntegerRange(3, 21, 2); C
{3, 5, ..., 19}
sage: C.cardinality()
9
sage: C.list()
[3, 5, 7, 9, 11, 13, 15, 17, 19]
```

Permutations:

```
sage: C = Permutations(4); C
Standard permutations of 4
sage: C.cardinality()
24
sage: C.list()
[[1, 2, 3, 4], [1, 2, 4, 3], [1, 3, 2, 4], [1, 3, 4, 2],
 [1, 4, 2, 3], [1, 4, 3, 2], [2, 1, 3, 4], [2, 1, 4, 3],
 [2, 3, 1, 4], [2, 3, 4, 1], [2, 4, 1, 3], [2, 4, 3, 1],
 [3, 1, 2, 4], [3, 1, 4, 2], [3, 2, 1, 4], [3, 2, 4, 1],
 [3, 4, 1, 2], [3, 4, 2, 1], [4, 1, 2, 3], [4, 1, 3, 2],
 [4, 2, 1, 3], [4, 2, 3, 1], [4, 3, 1, 2], [4, 3, 2, 1]]
```

Set partitions:

```
sage: C = SetPartitions(['a', 'b', 'c'])
sage: C # random print output
Set partitions of {'a', 'c', 'b'}
sage: C.cardinality()
5
sage: C.list()
[{'a', 'b', 'c'},
 {'a', 'b'}, {'c'},
 {'a', 'c'}, {'b'},
 {'a'}, {'b'}, {'c'}]
```

Partial orders on a set of 8 elements, up to isomorphism:

```
sage: C = Posets(8); C
Posets containing 8 elements
sage: C.cardinality()
16999
sage: C.unrank(20).plot()
Graphics object consisting of 20 graphics primitives
```

\(^1\) Or at least that should be the case; there are still many corners to clean up.
One can iterate through all graphs up to isomorphism. For example, there are 34 simple graphs with 5 vertices:

```
sage: len(list(graphs(5)))
34
```

Here are those with at most 4 edges:

```
sage: up_to_four_edges = list(graphs(5, lambda G: G.size() <= 4))
sage: pretty_print(*up_to_four_edges)
```
However, the set $C$ of these graphs is not yet available in Sage; as a result, the following commands are not yet implemented:

```
\begin{verbatim}
sage: C = Graphs(5) # todo: not implemented
sage: C.cardinality() # todo: not implemented
34
sage: Graphs(19).cardinality() # todo: not implemented
24637809253125004524383007491432768
sage: Graphs(19).random_element() # todo: not implemented
Graph on 19 vertices
\end{verbatim}
```

What we have seen so far also applies, in principle, to finite algebraic structures like the dihedral groups:

5.1. Comprehensive Module List
sage: G = DihedralGroup(4); G
Dihedral group of order 8 as a permutation group
sage: G.cardinality()
8
sage: G.list()
[((), (1,3)(2,4), (1,4,3,2), (1,2,3,4), (2,4), (1,3), (1,4)(2,3), (1,2)(3,4)]

or the algebra of $2 \times 2$ matrices over the finite field $\mathbb{Z}/2\mathbb{Z}$:

sage: C = MatrixSpace(GF(2), 2)
sage: C.list()
[[0, 0], [1, 0], [0, 1], [0, 0], [1, 1], [1, 0], [0, 1], [0, 1],
[0, 0], [0, 0], [0, 0], [0, 0], [1, 1], [0, 1], [0, 0], [1, 1], [0, 1], [0, 1],
[0, 1], [1, 1], [1, 1], [1, 0], [0, 1], [1, 1],
[1, 1], [1, 0], [0, 1], [1, 1], [1, 1], [1, 1]]
sage: C.cardinality()
16

Exercise

List all the monomials of degree 5 in three variables (see IntegerVectors). Manipulate the ordered set partitions OrderedSetPartitions and standard tableaux (StandardTableaux).

Exercise

List the alternating sign matrices of size 3, 4, and 5 (AlternatingSignMatrices), and try to guess the definition. The discovery and proof of the formula for the enumeration of these matrices (see the method cardinality), motivated by calculations of determinants in physics, is quite a story. In particular, the first proof, given by Zeilberger in 1992 was automatically produced by a computer program. It was 84 pages long, and required nearly a hundred people to verify it.

Exercise

Calculate by hand the number of vectors in $(\mathbb{Z}/2\mathbb{Z})^5$, and the number of matrices in $GL_3(\mathbb{Z}/2\mathbb{Z})$ (that is to say, the number of invertible $3 \times 3$ matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$). Verify your answer with Sage. Generalize to $GL_n(\mathbb{Z}/q\mathbb{Z})$. 
Set comprehension and iterators

We will now show some of the possibilities offered by Python for constructing (and iterating through) sets, with a notation that is flexible and close to usual mathematical usage, and in particular the benefits this yields in combinatorics.

We begin by constructing the finite set \( \{ i^2 \mid i \in \{1, 3, 7\} \} \):

```
sage: [ i^2 for i in [1, 3, 7] ]
[1, 9, 49]
```

and then the same set, but with \( i \) running from 1 to 9:

```
sage: [ i^2 for i in range(1,10) ]
[1, 4, 9, 16, 25, 36, 49, 64, 81]
```

A construction of this form in Python is called set comprehension. A clause can be added to keep only those elements with \( i \) prime:

```
sage: [ i^2 for i in range(1,10) if is_prime(i) ]
[4, 9, 25, 49]
```

Combining more than one set comprehension, it is possible to construct the set \( \{ (i, j) \mid 1 \leq j < i < 5 \} \):

```
sage: [ (i,j) for i in range(1,6) for j in range(1,i) ]
[(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3),
 (5, 1), (5, 2), (5, 3), (5, 4)]
```

or to produce Pascal’s triangle:

```
sage: [[binomial(n,i) for i in range(n+1)] for n in range(10)]
[[1],
 [1, 1],
 [1, 2, 1],
 [1, 3, 3, 1],
 [1, 4, 6, 4, 1],
 [1, 5, 10, 10, 5, 1],
 [1, 6, 15, 20, 15, 6, 1],
 [1, 7, 21, 35, 35, 21, 7, 1],
 [1, 8, 28, 56, 70, 56, 28, 8, 1],
 [1, 9, 36, 84, 126, 126, 84, 36, 9, 1]]
```

The execution of a set comprehension is accomplished in two steps; first an iterator is constructed, and then a list is filled with the elements successively produced by the iterator. Technically, an iterator is an object with a method \texttt{next} which returns a new value each time it is called, until it is exhausted. For example, the following iterator \texttt{it}:

```
sage: it = (binomial(3, i) for i in range(4))
```

returns successively the binomial coefficients \( \binom{3}{i} \) with \( i = 0, 1, 2, 3 \):

```
sage: next(it)
1
sage: next(it)
3
sage: next(it)
3
```

(continues on next page)
When the iterator is finally exhausted, an exception is raised:

```
sage: next(it)
Traceback (most recent call last):
  ...,
StopIteration
```

More generally, an `iterable` is a Python object `L` (a list, a set, ...) over whose elements it is possible to iterate. Technically, the iterator is constructed by `iter(L)`. In practice, the commands `iter` and `next` are used very rarely, since `for` loops and list comprehensions provide a much pleasanter syntax:

```
sage: for s in Subsets(3):
    print(s)
{}
{1}
{2}
{3}
{1, 2}
{1, 3}
{2, 3}
{1, 2, 3}
```

```
sage: [ s.cardinality() for s in Subsets(3) ]
[0, 1, 1, 1, 2, 2, 2, 3]
```

What is the point of an iterator? Consider the following example:

```
sage: sum( [ binomial(8, i) for i in range(9) ] )
256
```

When it is executed, a list of 9 elements is constructed, and then it is passed as an argument to `sum` to add them up. If, on the other hand, the iterator is passed directly to `sum` (note the absence of square brackets):

```
sage: sum( binomial(8, i) for i in range(9) )
256
```

the function `sum` receives the iterator directly, and can short-circuit the construction of the intermediate list. If there are a large number of elements, this avoids allocating a large quantity of memory to fill a list which will be immediately destroyed\(^2\).

Most functions that take a list of elements as input will also accept an iterator (or an iterable) instead. To begin with, one can obtain the list (or the tuple) of elements of an iterator as follows:

```
sage: list(binomial(8, i) for i in range(9))
[1, 8, 28, 56, 70, 56, 28, 8, 1]
sage: tuple(binomial(8, i) for i in range(9))
(1, 8, 28, 56, 70, 56, 28, 8, 1)
```

We now consider the functions `all` and `any` which denote respectively the \(n\)-ary and \(or\):

\(^2\) Technical detail: `range` returns an iterator on \(\{0, \ldots, 8\}\) while `range` returns the corresponding list. Starting in Python 3.0, `range` will behave like `range`, and `range` will no longer be needed.
The following example verifies that all primes from 3 to 99 are odd:

```
sage: all( is_odd(p) for p in range(3,100) if is_prime(p) )
True
```

A Mersenne prime is a prime of the form $2^p - 1$. We verify that, for $p < 1000$, if $2^p - 1$ is prime, then $p$ is also prime:

```
sage: def mersenne(p):
    return 2^p -1
sage: [ is_prime(p)
    ....: for p in range(1000) if is_prime(mersenne(p)) ]
[True, True, True, True, True, True, True, True, True, True, True, True, True, True]
```

Is the converse true?

**Exercise**

Try the two following commands and explain the considerable difference in the length of the calculations:

```
sage: all( is_prime(mersenne(p))
    ....: for p in range(1000) if is_prime(p) )
False
```

```
sage: all( [ is_prime(mersenne(p))
    ....: for p in range(1000) if is_prime(mersenne(p)) ] )
False
```

We now try to find the smallest counter-example. In order to do this, we use the Sage function `exists`:

```
sage: exists( (p for p in range(1000) if is_prime(p)),
    ....: lambda p: not is_prime(mersenne(p)) )
(True, 11)
```

Alternatively, we could construct an iterator on the counter-examples:

```
sage: counter_examples = (p for p in range(1000)
    ....: if is_prime(p) and not is_prime(mersenne(p)))
sage: next(counter_examples)
11
sage: next(counter_examples)
23
```
Exercise

What do the following commands do?

```
sage: cubes = \[t**3 \text{ for } t \text{ in range(-999,1000)}\]
sage: exists((x,y) \text{ for } x \text{ in cubes for } y \text{ in cubes}, \# \text{ long time (3s, 2012)}
....: lambda x_y: x_y[0] + x_y[1] == 218)
(True, (-125, 343))
sage: exists(((x,y) \text{ for } x \text{ in cubes for } y \text{ in cubes}), \# \text{ long time (2s, 2012)}
....: lambda x_y: x_y[0] + x_y[1] == 218)
(True, (-125, 343))
```

Which of the last two is more economical in terms of time? In terms of memory? By how much?

Exercise

Try each of the following commands, and explain its result. If possible, hide the result first and try to guess it before launching the command.

Todo: hide the results by default

**Warning:** it will be necessary to interrupt the execution of some of the commands

```
sage: x = var('x')
sage: sum( x^\text{len(s)} \text{ for } s \text{ in Subsets(8)} )
x^8 + 8*x^7 + 28*x^6 + 56*x^5 + 70*x^4 + 56*x^3 + 28*x^2 + 8*x + 1
sage: sum( x^p.length() \text{ for } p \text{ in Permutations(3)} )
x^3 + 2*x^2 + 2*x + 1
sage: factor(sum( x^p.length() \text{ for } p \text{ in Permutations(3)} ))
(x^2 + x + 1)*(x + 1)
sage: P = Permutations(5)
sage: all( p \text{ in } P \text{ for } p \text{ in } P )
True
sage: for p in GL(2, 2): print(p); print("")
```

```
[1 0]
[0 1]
[0 1]
[1 0]
[0 1]
[1 1]
[1 1]
[0 1]
[1 1]
[1 0]
```
Operations on iterators

Python provides numerous tools for manipulating iterators; most of them are in the itertools library, which can be imported by:

```python
sage: import itertools
```

We will demonstrate some applications, taking as a starting point the permutations of 3:

```python
sage: list(Permutations(3))
[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]
```

We can list the elements of a set by numbering them:
or select only the elements in positions 2, 3, and 4 (analogue of $1[1:4]$):

```python
sage: import itertools
sage: list(itertools.islice(Permutations(3), int(1), int(4)))
[[1, 3, 2], [2, 1, 3], [2, 3, 1]]
```

To apply a function to all the elements, one can do:

```python
sage: [z.cycle_type() for z in Permutations(3)]
[[1, 1, 1], [2, 1], [2, 1], [3], [3], [2, 1]]
```

and similarly to select the elements satisfying a certain condition:

```python
sage: [z for z in Permutations(3) if z.has_pattern([1,2])]
[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2]]
```

**Implementation of new iterators**

It is easy to construct new iterators, using the keyword `yield` instead of `return` in a function:

```python
sage: def f(n):
    ....:     for i in range(n):
    ....:         yield i
```

After the yield, execution is not halted, but only suspended, ready to be continued from the same point. The result of the function is therefore an iterator over the successive values returned by yield:

```python
sage: g = f(4)
sage: next(g)
0
sage: next(g)
1
sage: next(g)
2
sage: next(g)
3
sage: next(g)
Traceback (most recent call last):
  ...
StopIteration
```

The function could be used as follows:

```python
sage: [ x for x in f(5) ]
[0, 1, 2, 3, 4]
```

This model of computation, called *continuation*, is very useful in combinatorics, especially when combined with recursion. Here is how to generate all words of a given length on a given alphabet:
sage: def words(alphabet, l):
    ...:     if l == 0:
    ...:         yield []
    ...:     else:
    ...:         for word in words(alphabet, l-1):
    ...:             for l in alphabet:
    ...:                 yield word + [l]

sage: [ w for w in words(['a', 'b'], 3) ]
[['a', 'a', 'a'], ['a', 'a', 'b'], ['a', 'b', 'a'], ['b', 'a', 'a'], ['b', 'b', 'a']]

These words can then be counted by:

sage: sum(1 for w in words(['a', 'b', 'c', 'd'], 10))
1048576

Counting the words one by one is clearly not an efficient method in this case, since the formula \(n^k\) is also available; note, though, that this is not the stupidest possible approach - it does, at least, avoid constructing the entire list in memory.

We now consider Dyck words, which are well-parenthesized words in the letters "(" and ")". The function below generates all the Dyck words of a given length (where the length is the number of pairs of parentheses), using the recursive definition which says that a Dyck word is either empty or of the form \((w_1)w_2\) where \(w_1\) and \(w_2\) are Dyck words:

```python
sage: def dyck_words(l):
    ...:     if l==0:
    ...:         yield ''
    ...:     else:
    ...:         for k in range(l):
    ...:             for w1 in dyck_words(k):
    ...:                 for w2 in dyck_words(l-k-1):
    ...:                     yield '(()'+w1+')'+w2
```

Here are all the Dyck words of length 4:

```python
sage: list(dyck_words(4))
['(()())', '()(()())', '(()())()', '()(()())', '(()())', '
((()))', '(())()()', '(()()))', '((()))', '
((()))', '((()))', '(()()))', '(()))']
```

Counting them, we recover a well-known sequence:

```python
sage: [ sum(1 for w in dyck_words(l)) for l in range(10) ]
[1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862]
```

**Exercise: complete binary tree iterator**

Construct an iterator on the set \(C_n\) of complete binary trees with \(n\) leaves (see *Enumeration of trees using generating functions*).

Hint: Sage 4.8.2 does not yet have a native data structure to represent complete binary trees. One simple way to represent them is to define a formal variable `Leaf` for the leaves and a formal 2-ary function `Node`.

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```sage
var('Leaf')
Leaf
sage: function('Node', nargs=2)
Node
```

The second tree in *Figure: The five complete binary trees with four leaves* can be represented by the expression:

```sage
tr = Node(Node(Leaf, Node(Leaf, Leaf)), Leaf)
```

### Constructions

We will now see how to construct new sets starting from these building blocks. In fact, we have already begun to do this with the construction of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{1, 2, 3, 4\})))$ in the previous section, and to construct the example of sets of cards in *Initial examples*.

Consider a large Cartesian product:

```sage
C = cartesian_product([Compositions(8), Permutations(20)]); C
```

The Cartesian product of (Compositions of 8, Standard permutations of 20)

```sage
C.cardinality()
```

Clearly, it is impractical to construct the list of all the elements of this Cartesian product! And, in the following example, $H$ is equipped with the usual combinatorial operations and also its structure as a product group:

```sage
G = DihedralGroup(4)
sage: H = cartesian_product([G,G])
sage: H in Groups()
True
sage: H.an_element()
((1,3), (1,3))
sage: t = H([G.gen(0), G.gen(0)])
sage: t
((1,2,3,4), (1,2,3,4))
sage: t*t
((1,3)(2,4), (1,3)(2,4))
```

We now construct the union of two existing disjoint sets:

```sage
C = DisjointUnionEnumeratedSets(
....: [ Compositions(4), Permutations(3)]
)
sage: C
Disjoint union of Family (Compositions of 4,
Standard permutations of 3)
sage: C.cardinality()
14
sage: C.list()
[[1, 1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 3], [2, 1, 1], [2, 2],
 [3, 1], [4], [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1],
 [3, 1, 2], [3, 2, 1]]
```

It is also possible to take the union of more than two disjoint sets, or even an infinite number of them. We will now construct the set of all permutations, viewed as the union of the sets $\mathcal{P}_n$ of permutations of size $n$. We begin by
constructing the infinite family \( F = (P_n)_{n \in \mathbb{N}} \):

```python
sage: F = Family(NonNegativeIntegers(), Permutations); F
Lazy family (\langle \text{sage.combinat.permutation.Permutations}\rangle(i))_{i \in \text{Non negative integers}}
```

```python
sage: F.keys()
Non negative integers
```

```python
sage: F[1000]
Standard permutations of 1000
```

Now we can construct the disjoint union \( \bigcup_{n \in \mathbb{N}} P_n \):

```python
sage: U = DisjointUnionEnumeratedSets(F); U
Disjoint union of
Lazy family (\langle \text{sage.combinat.permutation.Permutations}\rangle(i))_{i \in \text{Non negative integers}}
```

It is an infinite set:

```python
sage: U.cardinality()
+Infinity
```

which doesn’t prohibit iteration through its elements, though it will be necessary to interrupt it at some point:

```python
sage: for p in U: # not tested
....:     print(p)
[]
[1]
[1, 2]
[2, 1]
[1, 2, 3]
[1, 3, 2]
[2, 1, 3]
[2, 3, 1]
[3, 1, 2]
...
```

Note: the above set could also have been constructed directly with:

```python
sage: U = Permutations(); U
Standard permutations
```

**Summary**

*Sage* provides a library of common enumerated sets, which can be combined by standard constructions, giving a toolbox that is flexible (but which could still be expanded). It is also possible to add new building blocks to *Sage* with a few lines (see the code in *FiniteEnumeratedSets().example()*). This is made possible by the uniformity of the interfaces and the fact that *Sage* is based on an object-oriented language. Also, very large or even infinite sets can be manipulated thanks to lazy evaluation strategies (iterators, etc.).

There is no magic to any of this: under the hood, *Sage* applies the usual rules (for example, that the cardinality of \( E \times E \) is \(|E|^2\)); the added value comes from the capacity to manipulate complicated constructions. The situation is comparable to *Sage*’s implementation of differential calculus: *Sage* applies the usual rules for differentiation of functions and their
compositions, where the added value comes from the possibility of manipulating complicated formulas. In this sense, Sage implements a calculus of finite enumerated sets.

**Generic algorithms**

**Lexicographic generation of lists of integers**

Among the classic enumerated sets, especially in algebraic combinatorics, a certain number are composed of lists of integers fixed sum, such as partitions, compositions, or integer vectors. These examples can also have supplementary constraints added to them. Here are some examples. We start with the integer vectors with sum $10$ and length $3$, with parts bounded below by $2$, $4$, and $2$ respectively:

```sage
IntegerVectors(10, 3, min_part=2, max_part=5,
...:    inner=[2, 4, 2]).list()
[[4, 4, 2], [3, 5, 2], [3, 4, 3], [2, 5, 3], [2, 4, 4]]
```

The compositions of $5$ with each part at most $3$, and with length $2$ or $3$:

```sage
Compositions(5, max_part=3,
...:    min_length=2, max_length=3).list()
[[3, 2], [3, 1, 1], [2, 3], [2, 2, 1], [2, 1, 2], [1, 3, 1],
 [1, 2, 2], [1, 1, 3]]
```

The strictly decreasing partitions of $5$:

```sage
Partitions(5, max_slope=-1).list()
[[5], [4, 1], [3, 2]]
```

These sets share the same underlying algorithmic structure, implemented in the more general (and slightly more cumbersome) class `IntegerListsLex`. This class models sets of vectors $(\ell_0, \ldots, \ell_k)$ of non-negative integers, with constraints on the sum and the length, and bounds on the parts and on the consecutive differences between the parts. Here are some more examples:

```sage
IntegerListsLex(10, length=3,
...:    min_part=2, max_part=5,
...:    floor=[2, 4, 2]).list()
[[4, 4, 2], [3, 5, 2], [3, 4, 3], [2, 5, 3], [2, 4, 4]]
```

```sage
IntegerListsLex(5, min_part=1, max_part=3,
...:    min_length=2, max_length=3).list()
[[3, 2], [3, 1, 1], [2, 3], [2, 2, 1], [2, 1, 2], [1, 3, 1],
 [1, 2, 2], [1, 1, 3]]
```

```sage
IntegerListsLex(5, min_part=1, max_slope=-1).list()
[[5], [4, 1], [3, 2]]
```

```sage
list(Compositions(5, max_length=2))
[[5], [4, 1], [3, 2], [2, 3], [1, 4]]
```

```sage
list(IntegerListsLex(5, max_length=2, min_part=1))
[[5], [4, 1], [3, 2], [2, 3], [1, 4]]
```

The point of the model of `IntegerListsLex` is in the compromise between generality and efficiency. The main algorithm permits iteration through the elements of such a set $S$ in reverse lexicographic order with a good complexity.
in most practical use cases. Roughly speaking, the time needed to iterate through all the elements of $S$ is proportional to the number of elements, where the proportion factor is controlled by the length $l$ of the longest element of $S$. In addition, the memory usage is also controlled by $l$, which is to say negligible in practice.

This algorithm is based on a very general principle for traversing a decision tree, called *branch and bound*: at the top level, we run through all the possible choices for $\ell_0$; for each of these choices, we run through all the possible choices for $\ell_1$, and so on. Mathematically speaking, we have put the structure of a prefix tree on the elements of $S$: a node of the tree at depth $k$ corresponds to a prefix $\ell_0, \ldots, \ell_k$ of one (or more) elements of $S$ (see Figure: The prefix tree of the partitions of 5).

![Figure: The prefix tree of the partitions of 5.](image)

The usual problem with this type of approach is to avoid bad decisions which lead to leaving the prefix tree and exploring dead branches; this is particularly problematic because the growth of the number of elements is usually exponential in the depth. It turns out that the constraints listed above are simple enough to be able to reasonably predict when a sequence $\ell_0, \ldots, \ell_k$ is a prefix of some element $S$. Hence, most dead branches can be pruned.

**Integer points in polytopes**

Although the algorithm for iteration in `IntegerListsLex` is efficient, its counting algorithm is naive: it just iterates over all the elements.

There is an alternative approach to treating this problem: modelling the desired lists of integers as the set of integer points of a polytope, that is to say, the set of solutions with integer coordinates of a system of linear inequalities. This is a very general context in which there exist advanced counting algorithms (e.g. Barvinok), which are implemented in libraries like *LattE*. Iteration does not pose a hard problem in principle. However, there are two limitations that justify the existence of `IntegerListsLex`. The first is theoretical: lattice points in a polytope only allow modelling of problems of a fixed dimension (length). The second is practical: at the moment only the library PALP has a *Sage* interface, and though it offers multiple capabilities for the study of polytopes, in the present application it only produces a list of lattice points, without providing either an iterator or non-naive counting:
This polytope can be visualized in 3D with \texttt{L.plot3d()} (see Figure: The polytope L and its integer points, in cross-eyed stereographic perspective.).

![Figure: The polytope L and its integer points, in cross-eyed stereographic perspective.](image)
Species, decomposable combinatorial classes

In *Enumeration of trees using generating functions*, we showed how to use the recursive definition of binary trees to count them efficiently using generating functions. The techniques we used there are very general, and apply whenever the sets involved can be defined recursively (depending on who you ask, such a set is called a *decomposable combinatorial class* or, roughly speaking, a *combinatorial species*). This includes all the types of trees, but also permutations, compositions, functional graphs, etc.

Here, we illustrate just a few examples using the Sage library on combinatorial species:

```sage
from sage.combinat.species.library import *

o = var('o')

BT = CombinatorialSpecies(min=1)
Leaf = SingletonSpecies()
BT.define( Leaf + (BT*BT) )

BT5 = BT.isotypes([o]*5)
BT5.cardinality()
14
BT5.list()
[[o*(o*(o*(o*o))), o*(o*((o*o)*o)), o*((o*o)*(o*o)),
  o*((o*(o*o))*o), o*(((o*o)*o)*o), (o*o)*(o*(o*o)),
  (o*o)*((o*o)*o), (o*(o*o))*(o*o), ((o*o)*o)*(o*o),
  ((o*(o*o))*o)*o, (o*((o*o)*o))*o, ((o*o)*(o*o))*o,
  ((o*(o*o))*o)*o, (((o*o)*o)*o)*o],
[[(o*(o*o))*o]*o, (((o*o)*o)*o)*o]]
```

The trees are constructed using a generic recursive structure; the display is therefore not wonderful. To do better, it would be necessary to provide Sage with a more specialized data structure with the desired display capabilities.

We recover the generating function for the Catalan numbers:

```sage
g = BT.isotype_generating_series(); g
z + z^2 + 2*z^3 + 5*z^4 + 14*z^5 + 42*z^6 + 132*z^7 + O(z^8)
```

which is returned in the form of a lazy power series:

```sage
g[100]

227508830794229349661819540395688853956041682601541047340
```

We finish with the Fibonacci words, which are binary words without two consecutive “1”s. They admit a natural recursive definition:

```sage
Eps = EmptySetSpecies()
Z0 = SingletonSpecies()
Z1 = Eps*SingletonSpecies()
FW = CombinatorialSpecies()
FW.define(Eps + Z0*FW + Z1*Eps + Z1*Z0*FW)
```

The Fibonacci sequence is easily recognized here, hence the name:
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```sage
L = FW.isotype_generating_series()[:15]; L
[1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987]
```

```sage
oeis(L)  # optional -- internet
0: A000045: Fibonacci numbers: F(n) = F(n-1) + F(n-2) with F(0) = 0 and F(1) = 1.
1: ...
2: ...
```

This is an immediate consequence of the recurrence relation. One can also generate immediately all the Fibonacci words of a given length, with the same limitations resulting from the generic display.

```sage
FW3 = FW.isotypes([o]*3)
```

```sage
FW3.list()  
[o*(o*(o*{})), o*(o*(o*{})), o*(o*(o*{})), o*(o*(o*{})), o*(o*(o*{})), o*(o*(o*{}))]
```

Graphs up to isomorphism

We saw in Some other finite enumerated sets that Sage could generate graphs and partial orders up to isomorphism. We will now describe the underlying algorithm, which is the same in both cases, and covers a substantially wider class of problems.

We begin by recalling some notions. A graph \( G = (V, E) \) is a set \( V \) of vertices and a set \( E \) of edges connecting these vertices; an edge is described by a pair \( \{u, v\} \) of distinct vertices of \( V \). Such a graph is called labelled; its vertices are typically numbered by considering \( V = \{1, 2, 3, 4, 5\} \).

In many problems, the labels on the vertices play no role. Typically a chemist wants to study all the possible molecules with a given composition, for example the alkanes with \( n = 8 \) atoms of carbon and \( 2n + 2 = 18 \) atoms of hydrogen. He therefore wants to find all the graphs consisting of \( 8 \) vertices with \( 4 \) neighbours, and \( 18 \) vertices with a single neighbour. The different carbon atoms, however, are all considered to be identical, and the same for the hydrogen atoms. The problem of our chemist is not imaginary; this type of application is actually at the origin of an important part of the research in graph theory on isomorphism problems.

Working by hand on a small graph it is possible, as in the example of Some other finite enumerated sets, to make a drawing, erase the labels, and “forget” the geometrical information about the location of the vertices in the plane. However, to represent a graph in a computer program, it is necessary to introduce labels on the vertices so as to be able to describe how the edges connect them together. To compensate for the extra information which we have introduced, we then say that two labelled graphs \( g_1 \) and \( g_2 \) are isomorphic if there is a bijection from the vertices of \( g_1 \) to those of \( g_2 \), which maps bijectively the edges of \( g_1 \) to those of \( g_2 \); an unlabelled graph is then an equivalence class of labelled graphs.

In general, testing if two labelled graphs are isomorphic is expensive. However, the number of graphs, even unlabelled, grows very rapidly. Nonetheless, it is possible to list unlabelled graphs very efficiently considering their number. For example, the program Nauty can list the 12005168 simple graphs with 10 vertices in 20 seconds.

As in Lexicographic generation of lists of integers, the general principle of the algorithm is to organize the objects to be enumerated into a tree that one traverses.

For this, in each equivalence class of labelled graphs (that is to say, for each unlabelled graph) one fixes a convenient canonical representative. The following are the fundamental operations:

- Testing whether a labelled graph is canonical
- Calculating the canonical representative of a labelled graph

These unavoidable operations remain expensive; one therefore tries to minimize the number of calls to them.
The canonical representatives are chosen in such a way that, for each canonical labelled graph $G$, there is a canonical choice of an edge whose removal produces a canonical graph again, which is called the father of $G$. This property implies that it is possible to organize the set of canonical representatives as a tree: at the root, the graph with no edges; below it, its unique child, the graph with one edge; then the graphs with two edges, and so on. The set of children of a graph $G$ can be constructed by augmentation, adding an edge in all the possible ways to $G$, and then selecting, from among those graphs, the ones that are still canonical\(^3\). Recursively, one obtains all the canonical graphs.

In what sense is this algorithm generic? Consider for example planar graphs (graphs which can be drawn in the plane without edges crossing): by removing an edge from a planar graph, one obtains another planar graph; so planar graphs form a subtree of the previous tree. To generate them, exactly the same algorithm can be used, selecting only the children which are planar:

```
sage: [len(list(graphs(n, property = lambda G: G.is_planar())))
....: for n in range(7)]
[1, 1, 2, 4, 11, 33, 142]
```

In a similar fashion, one can generate any family of graphs closed under deletion of an edge, and in particular any family characterized by a forbidden subgraph. This includes for example forests (graphs without cycles), bipartite graphs (graphs without odd cycles), etc. This can be applied to generate:

- partial orders, via the bijection with Hasse diagrams which are oriented graphs without cycles and without edges implied by the transitivity of the order relation;
- lattices (not implemented in Sage), via the bijection with the meet semi-lattice obtained by deleting the maximal vertex; in this case an augmentation by vertices rather than by edges is used.

REFERENCES:

### 5.1.352 Vector Partitions

AUTHORS:

- Amritanshu Prasad (2013): Initial version
- Shriya M (2022): Added new parameters such as distinct, parts and is_repeateable

```python
def IntegerVectorsIterator(vect, min=None):
    """Return an iterator over the list of integer vectors which are componentwise less than or equal to vect, and lexicographically greater than or equal to min."

    INPUT:

    - vect -- A list of non-negative integers
    - min -- A list of non-negative integers dominated elementwise by vect

    OUTPUT:

    A list in lexicographic order of all integer vectors (as lists) which are dominated elementwise by vect and are greater than or equal to min in lexicographic order.

    EXAMPLES:

```

\(^3\) In practice, an efficient implementation would exploit the symmetries of $G$, i.e., its automorphism group, to reduce the number of children to explore, and to reduce the cost of each test of canonicity.
Fig. 4: Figure: The generation tree of simple graphs with 4 vertices.
```python
sage: from sage.combinat.vector_partition import IntegerVectorsIterator
sage: list(IntegerVectorsIterator([1, 1]))
[[0, 0], [0, 1], [1, 0], [1, 1]]
sage: list(IntegerVectorsIterator([1, 1], min=[1, 0]))
[[1, 0], [1, 1]]
```

```python
class sage.combinat.vector_partition.VectorPartition(parent, vecpar)
Bases: CombinatorialElement

A vector partition is a multiset of integer vectors.

**partition_at_vertex**(i)

Return the partition obtained by sorting the i-th elements of the vectors in the vector partition.

**EXAMPLES:**

```python
sage: V = VectorPartition([[1, 2, 1], [2, 4, 1]])
sage: V.partition_at_vertex(1)
[4, 2]
```

**sum**()

Return the sum vector as a list.

**EXAMPLES:**

```python
sage: V = VectorPartition([[3, 2, 1], [2, 2, 1]])
sage: V.sum()
[5, 4, 2]
```

```python
class sage.combinat.vector_partition.VectorPartitions(vec, min=None, parts=None, distinct=False, is_repeatable=None)
Bases: UniqueRepresentation, Parent

Class of all vector partitions of vec with all parts greater than or equal to min in lexicographic order, with parts from parts.

A vector partition of vec is a list of vectors with non-negative integer entries whose sum is vec.

**INPUT:**

- vec - Integer vector
- min - Integer vector dominated elementwise by vec
- parts - Finite list of possible parts
- distinct - Boolean, set to True if only vector partitions with distinct parts are enumerated
- is_repeatable - Boolean function on parts which gives True in parts that can be repeated

**EXAMPLES:**

If min is not specified, then the class of all vector partitions of vec is created:

```python
sage: VP = VectorPartitions([2, 2])
sage: for vecpar in VP:
    ....:     print(vecpar)
[0, 1], [0, 1], [1, 0], [1, 0]]
```
```
If `distinct` is set to be `True`, then distinct part partitions are created:

```
sage: VP = VectorPartitions([2,2], distinct = True)
sage: list(VP)
[[[0, 1], [1, 0], [1, 1]],
 [[[0, 1], [2, 1]],
 [[[0, 2], [2, 0]],
 [[[1, 0], [1, 2]],
 [[[2, 2]]]]
```

If `min` is specified, then the class consists of only those vector partitions whose parts are all greater than or equal to `min` in lexicographic order:

```
sage: VP = VectorPartitions([2, 2], min = [1, 0])
sage: for vecpar in VP:
....:     print(vecpar)
[[1, 0], [1, 2]]
[[1, 1], [1, 1]]
[[2, 2]]
sage: VP = VectorPartitions([2, 2], min = [1, 0], distinct = True)
sage: for vecpar in VP:
....:     print(vecpar)
[[1, 0], [1, 2]]
[[2, 2]]
```

If `parts` is specified, then the class consists only of those vector partitions whose parts are from `parts`:

```
sage: Vec_Par = VectorPartitions([2,2], parts=[[0,1],[1,0],[1,1]])
sage: list(Vec_Par)
[[[0, 1], [0, 1], [1, 0], [1, 0]], [[0, 1], [1, 0], [1, 1]], [[1, 1], [1, 1]]]
```

If `is_repeatable` is specified, then the parts which satisfy the boolean function `is_repeatable` are allowed to be repeated:

```
sage: Vector_Partitions = VectorPartitions([2,2], parts=[[0,1],[1,0],[1,1]], is_repeatable=lambda vec: sum(vec)%2!=0)
sage: list(Vector_Partitions)
[[[0, 1], [0, 1], [1, 0], [1, 0]], [[0, 1], [1, 0], [1, 1]]]
```

Element

```
alias of VectorPartition
```

```
sage.combinat.vector_partition.find_min(vect)
```

Return a string of `0’s with one `1 at the location where the list `vect` has its last entry which is not equal to `0. 

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INPUT:

- \texttt{vec} – A list of integers

OUTPUT:

A list of the same length with 0's everywhere, except for a 1 at the last position where \texttt{vec} has an entry not equal to 0.

EXAMPLES:

```
sage: from sage.combinat.vector_partition import find_min
sage: find_min([2, 1])
[0, 1]
sage: find_min([2, 1, 0])
[0, 1, 0]
```

### 5.1.353 Abstract word (finite or infinite)

This module gathers functions that works for both finite and infinite words.

AUTHORS:

- Sébastien Labbé
- Franco Saliola

EXAMPLES:

```
sage: a = 0.618
sage: g = words.CodingOfRotationWord(alpha=a, beta=1-a, x=a)
sage: f = words.FibonacciWord()
sage: p = f.longest_common_prefix(g, length='finite')
sage: p
word: 0100101001001010010100100101001001010010...
sage: p.length()
231
```

```python
class sage.combinat.words.abstract_word.Word_class
    Bases: SageObject

    apply_morphism(morphism)
        Returns the word obtained by applying the morphism to self.

        INPUT:

        - \texttt{morphism} - Can be an instance of WordMorphism, or anything that can be used to construct one.

        EXAMPLES:

        ```
sage: w = Word("ab")
sage: d = {'a':'ab', 'b':'ba'}
sage: w.apply_morphism(d)
word: abba
sage: w.apply_morphism(WordMorphism(d))
word: abba
```
```
\begin{verbatim}
sage: w = Word('ababa')
sage: d = dict(a='ab', b='ba')
sage: d
{'a': 'ab', 'b': 'ba'}
sage: w.apply_morphism(d)
word: abbaabbaab

For infinite words:

sage: t = words.ThueMorseWord([0,1]); t
word: 0110100110100101101001011001101001100110110...
sage: t.apply_morphism({0:8,1:9})
word: 89989889988998989898898989988989898998...
\end{verbatim}

**complete_return_words_iterator** *(fact)*

Returns an iterator over all the complete return words of fact in self (without unicity).

A complete return word \( u \) of a factor \( v \) is a factor starting by the given factor \( v \) and ending just after the next occurrence of this factor \( v \). See for instance [1].

**INPUT:**

- fact - a non empty finite word

**OUTPUT:**

iterator

**EXAMPLES:**

\begin{verbatim}
sage: TM = words.ThueMorseWord()
sage: fact = Word([0,1,1,0,1])
sage: it = TM.complete_return_words_iterator(fact)
sage: next(it)
word: 01101001100101101
sage: next(it)
word: 01101001011001101
sage: next(it)
word: 011010011001011001101
sage: next(it)
word: 0110100101101
sage: next(it)
word: 01101001100101101
sage: next(it)
word: 01101001011001101
\end{verbatim}

**REFERENCES:**


**delta()**

Returns the image of self under the delta morphism.

This is the word composed of the length of consecutive runs of the same letter in a given word.

**OUTPUT:**

Word over integers
EXAMPLES:

For finite words:

```python
sage: W = Words('0123456789')
sage: W('22112122').delta()
word: 22112
sage: W('555008').delta()
word: 321
sage: W().delta()
word:
sage: Word('aabbaba').delta()
word: 22112
```

For infinite words:

```python
sage: t = words.ThueMorseWord()
sage: t.delta()
word: 1211222112122112112221122112211222112112112221...
```

**factor_occurrences_iterator**(fact)

Returns an iterator over all occurrences (including overlapping ones) of fact in self in their order of appearance.

**INPUT:**

- fact - a non empty finite word

**OUTPUT:**

iterator

**EXAMPLES:**

```python
sage: TM = words.ThueMorseWord()
sage: fact = Word([0,1,1,0,1])
sage: it = TM.factor_occurrences_iterator(fact)
sage: next(it)
0
sage: next(it)
12
sage: next(it)
24
```

```python
sage: u = Word('121')
sage: w = Word('121213211213')
sage: list(w.factor_occurrences_iterator(u))
[0, 2, 8]
```

**finite_differences**(mod=None)

Return the word obtained by the differences of consecutive letters of self.

**INPUT:**

- self - A word over the integers.
- mod - (default: None) It can be one of the following:
  - None or 0: result is over the integers
integer: result is over the integers modulo \(\text{mod}\).

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: w = Word([x^2 for x in range(10)])</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: w.finite_differences()</code></td>
<td>word: 1,3,5,7,9,11,13,15,17</td>
</tr>
<tr>
<td><code>sage: w.finite_differences(mod=4)</code></td>
<td>word: 1313131311</td>
</tr>
<tr>
<td><code>sage: w.finite_differences(mod=0)</code></td>
<td>word: 1,3,5,7,9,11,13,15,17</td>
</tr>
</tbody>
</table>

**first_occurrence\(\text{other}, \text{start}=0\)**

Return the position of the first occurrence of \text{other} in \text{self}.

If \text{other} is not a factor of \text{self}, it returns None or loops forever when \text{self} is an infinite word.

**INPUT:**

- \text{other} – a finite word
- \text{start} – integer (default: 0), where the search starts

**OUTPUT:**

integer or None

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: w = Word('01234567890123456789')</code></td>
<td>3</td>
</tr>
<tr>
<td><code>sage: w.first_occurrence(Word('3456'))</code></td>
<td>3</td>
</tr>
<tr>
<td><code>sage: w.first_occurrence(Word('3456'), start=7)</code></td>
<td>13</td>
</tr>
</tbody>
</table>

When the factor is not present, None is returned:

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: w.first_occurrence(Word('3456'), start=17)</code> is None</td>
<td>True</td>
</tr>
<tr>
<td><code>sage: w.first_occurrence(Word('3333'))</code> is None</td>
<td>True</td>
</tr>
</tbody>
</table>

Also works for searching a finite word in an infinite word:

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: w = Word('0123456789')^oo</code></td>
<td>3</td>
</tr>
<tr>
<td><code>sage: w.first_occurrence(Word('3456'))</code></td>
<td>1003</td>
</tr>
<tr>
<td><code>sage: w.first_occurrence(Word('3456'), start=1000)</code></td>
<td>1003</td>
</tr>
</tbody>
</table>

But it will loop for ever if the factor is not found:

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: w.first_occurrence(Word('3333'))</code> # not tested -- infinite loop</td>
<td></td>
</tr>
</tbody>
</table>

The empty word occurs in a word:

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: Word('123').first_occurrence(Word(''), 0)</code></td>
<td>0</td>
</tr>
</tbody>
</table>
is_empty()  

Returns True if the length of self is zero, and False otherwise.

EXAMPLES:

```python
sage: it = iter([])
sage: Word(it).is_empty()
True
sage: it = iter([1,2,3])
sage: Word(it).is_empty()
False
sage: from itertools import count
sage: Word(count()).is_empty()
False
```

is_finite()  

Returns whether this word is known to be finite.

**Warning:** A word defined by an iterator such that its end has never been reached will returns False.

EXAMPLES:

```python
sage: Word([]).is_finite()
True
sage: Word('a').is_finite()
True
sage: TM = words.ThueMorseWord()
sage: TM.is_finite()
False
```

iterated_right_palindromic_closure\( (f=None, \text{algorithm}=\text{'recursive'}) \)  

Returns the iterated \((f-)\)-palindromic closure of self.

**INPUT:**

- \(f\) - involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).
- \(\text{algorithm}\) - string (default: 'recursive') specifying which algorithm to be used when computing the iterated palindromic closure. It must be one of the two following values:
  - 'definition' - computed using the definition
  - 'recursive' - computation based on an efficient formula that recursively computes the iterated right palindromic closure without having to recompute the longest \(f\)-palindromic suffix at each iteration [2].
OUTPUT:

word – the iterated \( f \)-palindromic closure of self

EXAMPLES:

```python
sage: W = Word('123').iterated_right_palindromic_closure()
word: 1213121
```

```python
sage: w = Word('abc')
sage: w.iterated_right_palindromic_closure()
word: abacaba
```

```python
sage: w = Word('aaa')
sage: w.iterated_right_palindromic_closure()
word: aaa
```

```python
sage: w = Word('abbab')
sage: w.iterated_right_palindromic_closure()
word: ababaababaababaababaababaababaababaababaababaababaababa
```

A right \( f \)-palindromic closure:

```python
sage: f = WordMorphism('a->b,b->a')
sage: w = Word('abbab')
sage: w.iterated_right_palindromic_closure(f=f)
word: abbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabba
```

An infinite word:

```python
sage: t = words.ThueMorseWord('ab')
sage: t.iterated_right_palindromic_closure()
word: ababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababa
```

There are two implementations computing the iterated right \( f \)-palindromic closure, the latter being much more efficient:

```python
sage: w = Word('abaab')
sage: u = w.iterated_right_palindromic_closure(algorithm='definition')
sage: v = w.iterated_right_palindromic_closure(algorithm='recursive')
sage: u
word: abaabaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababa
sage: u == v
True
```

REFERENCES:


**length()**

Returns the length of self.

**lex_greater(other)**

Returns True if self is lexicographically greater than other.

**EXAMPLES:**

```
sage: w = Word([1,2,3])
sage: u = Word([1,3,2])
sage: v = Word([3,2,1])
sage: w.lex_greater(u)
False
sage: v.lex_greater(w)
True
sage: a = Word("abba")
sage: b = Word("abbb")
sage: a.lex_greater(b)
False
sage: b.lex_greater(a)
True
```

For infinite words:

```
sage: t = words.ThueMorseWord()
sage: t[:10].lex_greater(t)
False
sage: t.lex_greater(t[:10])
True
```

**lex_less(other)**

Returns True if self is lexicographically less than other.

**EXAMPLES:**

```
sage: w = Word([1,2,3])
sage: u = Word([1,3,2])
sage: v = Word([3,2,1])
sage: w.lex_less(u)
True
sage: v.lex_less(w)
False
sage: a = Word("abba")
sage: b = Word("abbb")
sage: a.lex_less(b)
True
sage: b.lex_less(a)
False
```

For infinite words:
```python
sage: t = words.ThueMorseWord()
sage: t.lex_less(t[:10])
False
sage: t[:10].lex_less(t)
True
```

`longest_common_prefix(other, length='unknown')`

Returns the longest common prefix of self and other.

**INPUT:**

- `other` - word
- `length` - string (optional, default: 'unknown') the length type of the resulting word if known. It may be one of the following:
  - 'unknown'
  - 'finite'
  - 'infinite'

**EXAMPLES:**

```python
sage: f = lambda n : add(Integer(n).digits(2)) % 2
sage: t = Word(f)
sage: u = t[:10]
sage: t.longest_common_prefix(u)
word: 0110100110
```

The longest common prefix of two equal infinite words:

```python
sage: t1 = Word(f)
sage: t2 = Word(f)
sage: t1.longest_common_prefix(t2)
word: 0110100110010110100101100110100110010110...
```

Useful to study the approximation of an infinite word:

```python
sage: a = 0.618
sage: g = words.CodingOfRotationWord(alpha=a, beta=1-a, x=a)
sage: f = words.FibonacciWord()
sage: p = f.longest_common_prefix(g, length='finite')
sage: p.length()
231
```

`longest_periodic_prefix(period=1)`

Returns the longest prefix of self having the given period.

**INPUT:**

- `period` - positive integer (optional, default 1)

**OUTPUT:**

word

**EXAMPLES:**
sage: Word([]).longest_periodic_prefix()
word:
sage: Word([1]).longest_periodic_prefix()
word: 1
sage: Word([1,2]).longest_periodic_prefix()
word: 1
sage: Word([1,1,2]).longest_periodic_prefix()
word: 11
sage: Word([1,2,1,2,1,3]).longest_periodic_prefix(2)
word: 12121
sage: type(_)
<class 'sage.combinat.words.word.FiniteWord_iter_with_caching'>
sage: Word(lambda n:0).longest_periodic_prefix()
word: 0000000000000000000000000000000000000000...

palindrome_prefixes_iterator(max_length=None)
Returns an iterator over the palindrome prefixes of self.

INPUT:

• max_length - non negative integer or None (optional, default: None) the maximum length of the prefixes

OUTPUT:

iterator

EXAMPLES:

sage: w = Word('abaaba')
sage: for pp in w.palindrome_prefixes_iterator(): pp
word: a
word: aba
word: abaaba
sage: for pp in w.palindrome_prefixes_iterator(max_length=4): pp
word: a
word: aba

You can iterate over the palindrome prefixes of an infinite word:

sage: f = words.FibonacciWord()
sage: for pp in f.palindrome_prefixes_iterator(max_length=20): pp
word: 0
word: 010
word: 010010
word: 0100100101001001
word: 0100101001001001010010100101001010...

parent()
Returns the parent of self.

partial_sums(start, mod=None)
Returns the word defined by the partial sums of its prefixes.
INPUT:

- `self` - A word over the integers.
- `start` - integer, the first letter of the resulting word.
- `mod` - (default: None) It can be one of the following:
  - None or 0: result is over the integers
  - integer: result is over the integers modulo mod.

EXAMPLES:

```python
sage: w = Word(range(10))
sage: w.partial_sums(0)
word: 0,0,1,3,6,10,15,21,28,36,45
sage: w.partial_sums(1)
word: 1,1,2,4,7,11,16,22,29,37,46
sage: w = Word([1,2,3,1,2,3,2,2,2,2])
sage: w.partial_sums(0, mod=None)
word: 0,1,3,6,7,9,12,14,16,18,20
sage: w.partial_sums(0, mod=0)
word: 0,1,3,6,7,9,12,14,16,18,20
sage: w.partial_sums(0, mod=8)
word: 01367146024
sage: w.partial_sums(0, mod=4)
word: 01323102020
sage: w.partial_sums(0, mod=2)
word: 01101100000
sage: w.partial_sums(0, mod=1)
word: 0000000000
```

`prefixes_iterator(max_length=None)`

Returns an iterator over the prefixes of self.

INPUT:

- `max_length` - non negative integer or None (optional, default: None) the maximum length of the prefixes

OUTPUT:

iterator

EXAMPLES:

```python
sage: w = Word('abaaba')
sage: for p in w.prefixes_iterator(): p
word: a
word: ab
word: aba
word: abaa
word: abaab
word: abaaba
sage: for p in w.prefixes_iterator(max_length=3): p
```

(continues on next page)
word: a
word: ab
word: aba

You can iterate over the prefixes of an infinite word:

```python
sage: f = words.FibonacciWord()
sage: for p in f.prefixes_iterator(max_length=8): p
word: 0
word: 01
word: 010
word: 0100
word: 01001
word: 010010
word: 0100101
word: 01001010
```

`return_words_iterator(fact)`

Returns an iterator over all the return words of `fact` in `self` (without unicity).

**INPUT:**

- `fact` - a non empty finite word

**OUTPUT:**

iterator

**EXAMPLES:**

```python
sage: w = Word('baccabcbacabca')
sage: b = Word('b')
sage: list(w.return_words_iterator(b))
[word: bacca, word: bcc, word: bac]
sage: TM = words.ThueMorseWord()
sage: fact = Word([0, 1, 1, 0, 1])
sage: it = TM.return_words_iterator(fact)
sage: next(it)
word: 011010011001
sage: next(it)
word: 011010010110
sage: next(it)
word: 0110100110010110
sage: next(it)
word: 01101001
sage: next(it)
word: 011010011001
```

`string_rep()`

Returns the (truncated) raw sequence of letters as a string.
EXAMPLES:

```python
sage: Word('abbabaab').string_rep()
'abbabaab'
sage: Word([0, 1, 0, 0, 1]).string_rep()
'01001'
sage: Word([0,10,101]).string_rep()
'0,10,101'
sage: WordOptions(letter_separator='-')

sage: Word([0,1,10,101]).string_rep()
'0-1-10-101'
sage: WordOptions(letter_separator=',')
```

```python
sage: from itertools import count
sage: Word(count()).sum_digits()
word: 01101001101001100101100100010110110100110110...
```

```python
sage: Word(primes(1000)).sum_digits()
word: 1001110100111010111011001011101110011011...
```

```python
sage: Word(primes(1000)).sum_digits(base=3)
word: 2100002020002222122121022210222122111022...
```

```python
sage: Word(primes(1000)).sum_digits(base=3, mod=3)
word: 2100002020002222122121022210222122111022...
```

```python
sage: Word(primes(1000)).sum_digits(base=3, mod=2)
word: 0111111111111111111111111111111111111111...
```

```python
sage: Word(primes(1000)).sum_digits(base=10, mod=7)
word: 2350241354435041006132432241353546006304...
```

sum_digits(base=2, mod=None)

Return the sequence of the sum modulo $mod$ of the digits written in base $base$ of self.

INPUT:

- self - word over natural numbers
- base - integer (default: 2), greater or equal to 2
- mod - modulo (default: None), can take the following values:
  - integer – the modulo
  - None - the value $base$ is considered for the modulo.

EXAMPLES:

The Thue-Morse word:

```python
sage: from itertools import count
sage: Word(count()).sum_digits()
word: 01101001101001100101100100010110110100110110...
```

Sum of digits modulo 2 of the prime numbers written in base 2:

```python
sage: Word(primes(1000)).sum_digits()
word: 1001110100111010111011001011101110011011...
```

Sum of digits modulo 3 of the prime numbers written in base 3:

```python
sage: Word(primes(1000)).sum_digits(base=3)
word: 2100002020002222122121022210222122111022...
```

Sum of digits modulo 2 of the prime numbers written in base 3:

```python
sage: Word(primes(1000)).sum_digits(base=3, mod=3)
word: 2100002020002222122121022210222122111022...
```

Sum of digits modulo 7 of the prime numbers written in base 10:

```python
sage: Word(primes(1000)).sum_digits(base=10, mod=7)
word: 2350241354435041006132432241353546006304...
```

Negative entries:
Combinatorics on words

Main modules and their methods:

- Abstract word (finite or infinite)
- Finite word
- Infinite word
- Alphabet
- Set of words
- Word paths
- Word morphisms/substitutions
- Shuffle product of words
- Suffix Tries and Suffix Trees

Main classes and functions meant to be used by the user:


A list of common words can be accessed through words.<tab> and are listed in the words catalog.

Internal representation of words:

- Word classes
• Fast word datatype using an array of unsigned char
• Datatypes for finite words
• Datatypes for words defined by iterators and callables

Options:
• User-customizable options for words

See WordOptions().

5.1.355 Alphabet

AUTHORS:
• Franco Saliola (2008-12-17) : merged into sage
• Vincent Delecroix and Stepan Starosta (2012): remove classes for alphabet and use other Sage classes otherwise (TotallyOrderedFiniteSet, FiniteEnumeratedSet, ...). More shortcut to standard alphabets.

EXAMPLES:
sage: build_alphabet("ab")
{'a', 'b'}
sage: build_alphabet([0,1,2])
{0, 1, 2}
sage: build_alphabet(name="PP")
Positive integers
sage: build_alphabet(name="NN")
Non negative integers
sage: build_alphabet(name="lower")
{'a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i', 'j', 'k', 'l', 'm', 'n', 'o', 'p', 'q', 'r', 's', 't', 'u', 'v', 'w', 'x', 'y', 'z'}

sage.combinat.words.alphabet.Alphabet(data=None, names=None, name=None)

Return an object representing an ordered alphabet.

INPUT:
• data – the letters of the alphabet; it can be:
  – a list/tuple/iterable of letters; the iterable may be infinite
  – an integer \( n \) to represent \( \{1, \ldots, n\} \), or infinity to represent \( \mathbb{N} \)
• names – (optional) a list for the letters (i.e. variable names) or a string for prefix for all letters; if given a list, it must have the same cardinality as the set represented by data
• name – (optional) if given, then return a named set and can be equal to: 'lower', 'upper', 'space', 'underscore', 'punctuation', 'printable', 'binary', 'octal', 'decimal', 'hexadecimal', 'radix64'.

You can use many of them at once, separated by spaces: 'lower punctuation' represents the union of the two alphabets 'lower' and 'punctuation'.

Alternatively, name can be set to "positive integers" (or "PP") or "natural numbers" (or "NN"). name cannot be combined with data.

EXAMPLES:
If the argument is a Set, it just returns it:
Combinatorics, Release 10.0

```
sage: build_alphabet(ZZ) is ZZ
True
sage: F = FiniteEnumeratedSet('abc')
sage: build_alphabet(F) is F
True
```

If a list, tuple or string is provided, then it builds a proper Sage class (`TotallyOrderedFiniteSet`):

```
sage: build_alphabet([0, 1, 2])
{0, 1, 2}
sage: F = build_alphabet('abc'); F
{'a', 'b', 'c'}
sage: print(type(F).__name__)
TotallyOrderedFiniteSet_with_category
```

If an integer and a set is given, then it constructs a `TotallyOrderedFiniteSet`:

```
sage: build_alphabet(3, ['a', 'b', 'c'])
{'a', 'b', 'c'}
```

If an integer and a string is given, then it considers that string as a prefix:

```
sage: build_alphabet(3, 'x')
{'x0', 'x1', 'x2'}
```

If no data is provided, `name` may be a string which describe an alphabet. The available names decompose into two families. The first one are ‘positive integers’, ‘PP’, ‘natural numbers’ or ‘NN’ which refer to standard set of numbers:

```
sage: build_alphabet(name="positive integers")
Positive integers
sage: build_alphabet(name="PP")
Positive integers
sage: build_alphabet(name="natural numbers")
Non negative integers
sage: build_alphabet(name="NN")
Non negative integers
```

The other families for the option `name` are among ‘lower’, ‘upper’, ‘space’, ‘underscore’, ‘punctuation’, ‘printable’, ‘binary’, ‘octal’, ‘decimal’, ‘hexadecimal’, ‘radix64’ which refer to standard set of characters. Theses names may be combined by separating them by a space:

```
sage: build_alphabet(name="lower")
{'a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i', 'j', 'k', 'l', 'm', 'n', 'o', 'p', 'q →', 'r', 's', 't', 'u', 'v', 'w', 'x', 'y', 'z'}
sage: build_alphabet(name="hexadecimal")
{'0', '1', '2', '3', '4', '5', '6', '7', '8', '9', 'a', 'b', 'c', 'd', 'e', 'f'}
sage: build_alphabet(name="decimal punctuation")
{'0', '1', '2', '3', '4', '5', '6', '7', '8', '9', '.', ',', ';', ':', '? →'}
```

In the case the alphabet is built from a list or a tuple, the order on the alphabet is given by the elements themselves:

```
5.1. Comprehensive Module List 3553
```
sage: A = build_alphabet([0,2,1])
sage: A(0) < A(2)
True
sage: A(2) < A(1)
False

If a different order is needed, you may use `TotallyOrderedFiniteSet` and set the option `facade` to `False`. That way, the comparison fits the order of the input:

sage: A = TotallyOrderedFiniteSet([4,2,6,1], facade=False)
sage: A(4) < A(2)
True
sage: A(1) < A(6)
False

Be careful, the element of the set in the last example are no more integers and do not compare equal with integers:

sage: type(A.an_element())
<class 'sage.sets.totally_ordered_finite_set.TotallyOrderedFiniteSet_with_category.element_class'>
sage: A(1) == 1
False
sage: 1 == A(1)
False

We give an example of an infinite alphabet indexed by the positive integers and the prime numbers:

sage: build_alphabet(oo, 'x')
Lazy family (x(i))_{i in Non negative integers}
sage: build_alphabet(Primes(), 'y')
Lazy family (y(i))_{i in Set of all prime numbers: 2, 3, 5, 7, ...}

`sage.combinat.words.alphabet.build_alphabet(data=None, names=None, name=None)`

Return an object representing an ordered alphabet.

INPUT:

- `data` – the letters of the alphabet; it can be:
  - a list/tuple/iterable of letters; the iterable may be infinite
  - an integer $n$ to represent $\{1, \ldots, n\}$, or infinity to represent $\mathbb{N}$
- `names` – (optional) a list for the letters (i.e. variable names) or a string for prefix for all letters; if given a list, it must have the same cardinality as the set represented by `data`
- `name` – (optional) if given, then return a named set and can be equal to: 'lower', 'upper', 'space', 'underscore', 'punctuation', 'printable', 'binary', 'octal', 'decimal', 'hexadecimal', 'radix64'.

You can use many of them at once, separated by spaces: 'lower punctuation' represents the union of the two alphabets 'lower' and 'punctuation'.

Alternatively, `name` can be set to "positive integers" (or "PP") or "natural numbers" (or "NN").

name cannot be combined with `data`.

EXAMPLES:
If the argument is a Set, it just returns it:

```
sage: build_alphabet(ZZ) is ZZ
True
sage: F = FiniteEnumeratedSet('abc')
sage: build_alphabet(F) is F
True
```

If a list, tuple or string is provided, then it builds a proper Sage class  
(TotallyOrderedFiniteSet):

```
sage: build_alphabet([0,1,2])
{0, 1, 2}
sage: F = build_alphabet('abc'); F
{a', 'b', 'c'}
sage: print(type(F).__name__)
TotallyOrderedFiniteSet_with_category
```

If an integer and a set is given, then it constructs a  
(TotallyOrderedFiniteSet):

```
sage: build_alphabet(3, ['a', 'b', 'c'])
{'a', 'b', 'c'}
```

If an integer and a string is given, then it considers that string as a prefix:

```
sage: build_alphabet(3, 'x')
{'x0', 'x1', 'x2'}
```

If no data is provided, name may be a string which describe an alphabet. The available names decompose into  
two families. The first one are ‘positive integers’, ‘PP’, ‘natural numbers’ or ‘NN’ which refer to standard set of  
numbers:

```
sage: build_alphabet(name="positive integers")
Positive integers
sage: build_alphabet(name="PP")
Positive integers
sage: build_alphabet(name="natural numbers")
Non negative integers
sage: build_alphabet(name="NN")
Non negative integers
```

names may be combined by separating them by a space:

```
sage: build_alphabet(name="lower")
{'a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i', 'j', 'k', 'l', 'm', 'n', 'o', 'p', 'q  
˓→', 'r', 's', 't', 'u', 'v', 'w', 'x', 'y', 'z'}
sage: build_alphabet(name="hexadecimal")
{'0', '1', '2', '3', '4', '5', '6', '7', '8', '9', 'a', 'b', 'c', 'd', 'e', 'f'}
sage: build_alphabet(name="decimal punctuation")
{'0', '1', '2', '3', '4', '5', '6', '7', '8', '9', ' ', '!', '(', ')', '.', ',', ':'}
```

In the case the alphabet is built from a list or a tuple, the order on the alphabet is given by the elements themselves:
If a different order is needed, you may use `TotallyOrderedFiniteSet` and set the option `facade` to `False`. That way, the comparison fits the order of the input:

```python
sage: A = TotallyOrderedFiniteSet([4,2,6,1], facade=False)
sage: A(4) < A(2)
True
sage: A(1) < A(6)
False
```

Be careful, the element of the set in the last example are no more integers and do not compare equal with integers:

```python
sage: type(A.an_element())
<class 'sage.sets.totally_ordered_finite_set.TotallyOrderedFiniteSet_with_category.element_class'>
sage: A(1) == 1
False
sage: 1 == A(1)
False
```

We give an example of an infinite alphabet indexed by the positive integers and the prime numbers:

```python
sage: build_alphabet(oo, 'x')
Lazy family (x(i))_{i in Non negative integers}
sage: build_alphabet(Primes(), 'y')
Lazy family (y(i))_{i in Set of all prime numbers: 2, 3, 5, 7, ...}
```

### 5.1.356 Finite word

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- Franco Saliola
- Julien Leroy (March 2010): `reduced_rauzy_graph`

**EXAMPLES:**
**Creation of a finite word**

Finite words from Python strings, lists and tuples:

```sage
sage: Word("abbabaab")
word: abbabaab
sage: Word([0, 1, 1, 0, 1, 0, 0, 1])
word: 01101001
sage: Word( ('a', 0, 5, 7, 'b', 9, 8) )
word: a057b98
```

Finite words from functions:

```sage
sage: f = lambda n : n%3
sage: Word(f, length=13)
word: 0120120120120
```

Finite words from iterators:

```sage
sage: from itertools import count
sage: Word(count(), length=10)
word: 0123456789
sage: Word( iter('abbccdef') )
word: abbccdef
```

Finite words from words via concatenation:

```sage
sage: u = Word("abcabba")
sage: v = Word([0, 4, 8, 8, 3])
sage: u * v
word: abcabba04883
sage: v * u
word: 04883abcabba
sage: u + v
word: abcabba04883
sage: u^3 * v^(8/5)
word: abcabbaabccabbaabccabba04883048
```

Finite words from infinite words:

```sage
sage: vv = v^Infinity
sage: vv[10000:10015]
word: 048830488304883
```

Finite words in a specific combinatorial class:

```sage
sage: W = Words("ab")
sage: W
Finite and infinite words over {'a', 'b'}
sage: W("abbabaab")
word: abbabaab
sage: W(["a", "b", "b", "a", "b", "a", "a", "b"])
word: abbbababa
```

(continues on next page)
sage: W( iter('ababab') )
word: ababab

Finite word as the image under a morphism:

sage: m = WordMorphism({0:[4,4,5,0], 5:[0,5,5], 4:[4,0,0,0]})
sage: m(0)
word: 4450
sage: m(0, order=2)
word: 400040000554450
sage: m(0, order=3)
word: 4000445044504450400044504450445044500550...

Note: The following two finite words have the same string representation:

sage: w = Word('010120')
sage: z = Word([0, 1, 0, 1, 2, 0])
sage: w
word: 010120
sage: z
word: 010120
but are not equal:

sage: w == z
False

Indeed, w and z are defined on different alphabets:

sage: w[2]
'0'
sage: z[2]
0

Functions and algorithms

There are more than 100 functions defined on a finite word. Here are some of them:

sage: w = Word('abaabba'); w
word: abaabba
sage: w.is_palindrome()
False
sage: w.is_lyndon()
False
sage: w.number_of_factors()
28
sage: w.critical_exponent()
3
```python
sage: print(w.lyndon_factorization())
(ab, aabbb, a)
sage: print(w.crochemore_factorization())
(a, b, a, ab, bb, a)
```

```python
sage: st = w.suffix_tree()
sage: st
Implicit Suffix Tree of the word: abaabbba
sage: st.show(word_labels=True)
```

```python
sage: T = words.FibonacciWord('ab')
sage: T.longest_common_prefix(Word('abaababbbbbbb'))
word: abaababa
```

As matrix and many other sage objects, words have a parent:

```python
sage: u = Word('xyxxyxyyy')
sage: u.parent()
Finite words over Set of Python objects of class 'object'
```

```python
sage: v = Word('xyxxyxyyy', alphabet='xy')
sage: v.parent()
Finite words over {'x', 'y'}
```

### Factors and Rauzy Graphs

Enumeration of factors, the successive values returned by `next(it)` can appear in a different order depending on hardware. Therefore we mark the three first results of the test `random`. The important test is that the iteration stops properly on the fourth call:

```python
sage: w = Word([4,5,6])**7
sage: it = w.factor_iterator(4)
sage: next(it)  # random
word: 6456
sage: next(it)  # random
word: 5645
sage: next(it)  # random
word: 4564
sage: next(it)
Traceback (most recent call last):
... StopIteration
```

The set of factors:

```python
sage: sorted(w.factor_set(3))
[word: 456, word: 564, word: 645]
sage: sorted(w.factor_set(4))
[word: 4564, word: 5645, word: 6456]
sage: w.factor_set().cardinality()
61
```
Rauzy graphs:

```python
sage: f = words.FibonacciWord()[30]
sage: f.rauzy_graph(4)
Looped digraph on 5 vertices
sage: f.reduced_rauzy_graph(4)
Looped multi-digraph on 2 vertices
```

Left-special and bispecial factors:

```python
sage: f.number_of_left_special_factors(7)
1
sage: f.bispecial_factors()
[word: , word: 0, word: 010, word: 010010, word: 01001010010]
```

class `sage.combinat.words.finite_word.CallableFromListOfWords`(`words`)  
Bases: `tuple`  
A class to create a callable from a list of words. The concatenation of a list of words is obtained by creating a word from this callable.

class `sage.combinat.words.finite_word.Factorization`(`iterable=`, `/`)  
Bases: `list`  
A list subclass having a nicer representation for factorization of words.

class `sage.combinat.words.finite_word.FiniteWord_class`  
Bases: `Word_class`  

`BWT()`  
Return the Burrows-Wheeler Transform (BWT) of `self`.

The Burrows-Wheeler transform of a finite word \( w \) is obtained from \( w \) by first listing the conjugates of \( w \) in lexicographic order and then concatenating the final letters of the conjugates in this order. See [BW1994].

EXAMPLES:

```python
sage: Word('abaccaaba').BWT()  
word: cbaabaaca
sage: Word('abaab').BWT()  
word: bbaaa
sage: Word('bbabba').BWT()  
word: cbbbaaa
sage: Word('aabab').BWT()  
word: bbaa
sage: Word('').BWT()  
word: 
```

`LZ_decomposition()`  
Return the Crochemore factorization of `self` as an ordered list of factors.

The Crochemore factorization or the Lempel-Ziv decomposition of a finite word \( w \) is the unique factorization: \((x_1, x_2, \ldots, x_n)\) of \( w \) with each \( x_i \) satisfying either: C1. \( x_i \) is a letter that does not appear in \( u = x_1 \ldots x_{i-1} \); C2. \( x_i \) is the longest prefix of \( v = x_i \ldots x_n \) that also has an occurrence beginning within \( u = x_1 \ldots x_{i-1} \). See [Cro1983].
EXAMPLES:

```python
sage: x = Word('abababb')
sage: x.crochemore_factorization()
(a, b, abab, b)
sage: mul(x.crochemore_factorization()) == x
True
sage: y = Word('abaababacabba')
sage: y.crochemore_factorization()
(a, b, a, aba, ba, c, ab, ba)
sage: mul(y.crochemore_factorization()) == y
True
sage: x = Word([0,1,0,1,0,1,1])
sage: x.crochemore_factorization()
(0, 1, 0101, 1)
sage: mul(x.crochemore_factorization()) == x
True
```

**abelian_complexity**\( (n) \)

Return the number of abelian vectors of factors of length \( n \) of `self`.

**EXAMPLES:**

```python
sage: w = words.FibonacciWord()[100]
sage: [w.abelian_complexity(i) for i in range(20)]
[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
sage: w = words.ThueMorseWord()[100]
sage: [w.abelian_complexity(i) for i in range(20)]
[1, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2]
```

**abelian_vector**

Return the abelian vector of `self` counting the occurrences of each letter.

The vector is defined w.r.t. the order of the alphabet of the parent. See also `evaluation_dict()`.

**INPUT:**

- `self` – word having a parent on a finite alphabet

**OUTPUT:**

a list

**EXAMPLES:**

```python
sage: W = Words('ab')
sage: W('aaabbb').abelian_vector()
[3, 5]
sage: W('a').abelian_vector()
[1, 0]
sage: W().abelian_vector()
[0, 0]
```

The result depends on the alphabet of the parent:
```python
sage: W = Words('abc')
sage: W('aabaa').abelian_vector()
[4, 1, 0]
```

**abelian_vectors(n)**

Return the abelian vectors of factors of length \( n \) of \( \text{self} \).

The vectors are defined w.r.t. the order of the alphabet of the parent.

**OUTPUT:**

a set of tuples

**EXAMPLES:**

```python
sage: W = Words([0,1,2])
sage: w = W([0,1,1,0,1,2,0,2,0,2])
sage: w.abelian_vectors(3)
{(1, 0, 2), (1, 1, 1), (1, 2, 0), (2, 0, 1)}
sage: w[:5].abelian_vectors(3)
{(1, 2, 0)}
sage: w[5:].abelian_vectors(3)
{(1, 0, 2), (2, 0, 1)}
```

The word must be defined with a parent on a finite alphabet:

```python
sage: from itertools import count
sage: w = Word(count(), alphabet=NN)
sage: w[:2].abelian_vectors(2)
Traceback (most recent call last):
...
TypeError: The alphabet of the parent is infinite; define the word with a parent on a finite alphabet
```

**apply_permutation_to_letters(permutation)**

Return the word obtained by applying the permutation \( \text{permutation} \) of the alphabet of \( \text{self} \) to each letter of \( \text{self} \).

**EXAMPLES:**

```python
sage: w = Words('abcd')('abcd')
sage: p = [2,1,4,3]
sage: w.apply_permutation_to_letters(p)
word: badc
sage: u = Words('dabc')('abcd')
sage: u.apply_permutation_to_letters(p)
word: dcba
```
apply_permutation_to_positions(permutation)
Return the word obtained by permuting the positions of the letters in self according to the permutation permutation.

EXAMPLES:

sage: w = Words('abcd')('abcd')
sage: w.apply_permutation_to_positions([2,1,4,3])
word: badc
sage: u = Words('dabc')('abcd')
sage: u.apply_permutation_to_positions([2,1,4,3])
word: badc
sage: w.apply_permutation_to_positions(Permutation([2,1,4,3]))
word: badc
sage: w.apply_permutation_to_positions(PermutationGroupElement([2,1,4,3]))
word: badc
sage: Word([1,2,3,4]).apply_permutation_to_positions([3,4,2,1])
word: 3421

balance()
Return the balance of self.

The balance of a word is the smallest number $q$ such that self is $q$-balanced [FV2002].

A finite or infinite word $w$ is said to be $q$-balanced if for any two factors $u, v$ of $w$ of the same length, the difference between the number of $x$'s in each of $u$ and $v$ is at most $q$ for all letters $x$ in the alphabet of $w$. A 1-balanced word is simply said to be balanced. See Chapter 2 of [Lot2002].

OUTPUT:
integer

EXAMPLES:

sage: Word('1111111').balance()
0
sage: Word('00101010111').balance()
2
sage: Word('0101010101').balance()
1

sage: w = Word('11112222')
sage: w.is_balanced(2)
False
sage: w.is_balanced(3)
False
sage: w.is_balanced(4)
True
sage: w.is_balanced(5)
True
sage: w.balance()
4

**bispecial_factors**(\(n=None\))

Return the bispecial factors (of length \(n\)).

A factor \(u\) of a word \(w\) is **bispecial** if it is right special and left special.

**INPUT:**

* \(n\) – integer (optional, default: None). If None, it returns all bispecial factors.

**OUTPUT:**

a list of words

**EXAMPLES:**

```
sage: w = words.FibonacciWord()[:30]
sage: w.bispecial_factors()
[word: , word: 0, word: 010, word: 010010, word: 01001010010]
```

```
sage: w = words.ThueMorseWord()[:30]
sage: for i in range(10):
    ....:     print("{} {}") .format(i, sorted(w.bispecial_factors(i))))
0 [word: ]
1 [word: 0, word: 1]
2 [word: 01, word: 10]
3 [word: 010, word: 101]
4 [word: 0110, word: 1001]
5 []
6 [word: 011001, word: 100110]
7 []
8 [word: 10010110]
9 []
```

**bispecial_factors_iterator**(\(n=None\))

Return an iterator over the bispecial factors (of length \(n\)).

A factor \(u\) of a word \(w\) is **bispecial** if it is right special and left special.

**INPUT:**

* \(n\) – integer (optional, default: None). If None, it returns an iterator over all bispecial factors.

**EXAMPLES:**

```
sage: w = words.ThueMorseWord()[:30]
sage: for i in range(10):
    ....:     for u in sorted(w.bispecial_factors_iterator(i)):
    ....:         print("{} {}") .format(i,u))
0
1 0
1 1
2 01
2 10
3 010
```

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3 101
4 0110
4 1001
6 011001
6 100110
8 10010110

```
sage: key = lambda u : (len(u), u)
sage: for u in sorted(w.bispecial_factors_iterator(), key=key): u
word: 0
word: 1
word: 01
word: 10
word: 010
word: 101
word: 0110
word: 1001
word: 011001
word: 100110
word: 10010110
```

border()
Return the longest word that is both a proper prefix and a proper suffix of self.

EXAMPLES:

```
sage: Word('121212').border()
word: 1212
sage: Word('12321').border()
word: 1
sage: Word().border() is None
True
```

charge(check=True)
Return the charge of self. This is defined as follows.

If \(w\) is a permutation of length \(n\), (in other words, the evaluation of \(w\) is \((1, 1, \ldots, 1)\)), the statistic \(\text{charge}(w)\) is given by \(\sum_{i=1}^{n} c_i(w)\) where \(c_i(w) = 0\) and \(c_i(w)\) is defined recursively by setting \(p_i\) equal to 1 if \(i\) appears to the right of \(i - 1\) in \(w\) and 0 otherwise. Then we set \(c_i(w) = c_i-1(w) + p_i\).

EXAMPLES:

```
sage: Word([1, 2, 3]).charge()
3
sage: Word([3, 5, 1, 4, 2]).charge() == 0 + 1 + 1 + 2 + 2
True
```

If \(w\) is not a permutation, but the evaluation of \(w\) is a partition, the charge of \(w\) is defined to be the sum of its charge subwords (each of which will be a permutation). The first charge subword is found by starting at the end of \(w\) and moving left until the first 1 is found. This is marked, and we continue to move to the left until the first 2 is found, wrapping around from the beginning of the word back to the end, if necessary. We mark this 2, and continue on until we have marked the largest letter in \(w\). The marked letters, with relative order preserved, form the first charge subword of \(w\). This subword is removed, and the next charge
subword is found in the same manner from the remaining letters. In the following example, \( w_1, w_2, w_3 \) are the charge subwords of \( w \).

**EXAMPLES:**

```
sage: w = Word([5,2,3,4,4,1,1,2,2,3])
sage: w1 = Word([5, 2, 4, 1, 3])
sage: w2 = Word([3, 4, 1, 2])
sage: w3 = Word([1, 2])
sage: w.charge() == w1.charge() + w2.charge() + w3.charge()
True
```

Finally, if \( w \) does not have partition content, we apply the Lascoux-Schützenberger standardization operators \( s_i \) in such a manner as to obtain a word with partition content. (The word we obtain is independent of the choice of operators.) The charge is then defined to be the charge of this word:

```
sage: Word([3,3,2,1,1]).charge()
0
sage: Word([1,2,3,1,2]).charge()
2
```

Note that this differs from the definition of charge given in Macdonald’s book. The difference amounts to a choice of reading a word from left-to-right or right-to-left. The choice in Sage was made to agree with the definition of a reading word of a tableau in Sage, and seems to be the more common convention in the literature.

See [Mac1995], [LLM2003], and [LLT].

**cocharge()**

Return the cocharge of \( \text{self} \). For a word \( w \), this can be defined as \( n_{ev} - ch(w) \), where \( ch(w) \) is the charge of \( w \) and \( ev \) is the evaluation of \( w \), and \( n_{ev} \) is \( \sum_{i<j} \min(ev_i, ev_j) \).

**EXAMPLES:**

```
sage: Word([1,2,3]).cocharge()
0
sage: Word([3,2,1]).cocharge()
3
sage: Word([1,1,2]).cocharge()
0
sage: Word([2,1,2]).cocharge()
1
```

**coerce(other)**

Try to return a pair of words with a common parent; raise an exception if this is not possible.

This function begins by checking if both words have the same parent. If this is the case, then no work is done and both words are returned as-is.

Otherwise it will attempt to convert \( \text{other} \) to the domain of \( \text{self} \). If that fails, it will attempt to convert \( \text{self} \) to the domain of \( \text{other} \). If both attempts fail, it raises a \( \text{TypeError} \) to signal failure.

**EXAMPLES:**

```
sage: W1 = Words('abc'); W2 = Words('ab')
sage: w1 = W1('abc'); w2 = W2('abba'); w3 = W1('baab')
sage: w1.parent() is w2.parent()
(continues on next page)```
False
sage: a, b = w1.coerce(w2)
sage: a.parent() is b.parent()
True
sage: w1.parent() is w2.parent()
False

\texttt{colored\_vector}(x=0, y=0, width='default', height=1, cmap='hsv', thickness=1, label=None)

Return a vector (Graphics object) illustrating self. Each letter is represented by a coloured rectangle.

If the parent of self is a class of words over a finite alphabet, then each letter in the alphabet is assigned a unique colour, and this colour will be the same every time this method is called. This is especially useful when plotting and comparing words defined on the same alphabet.

If the alphabet is infinite, then the letters appearing in the word are used as the alphabet.

INPUT:

- \( x \) – (default: 0) bottom left x-coordinate of the vector
- \( y \) – (default: 0) bottom left y-coordinate of the vector
- \( \text{width} \) – (default: 'default') width of the vector. By default, the width is the length of self.
- \( \text{height} \) – (default: 1) height of the vector
- \( \text{thickness} \) – (default: 1) thickness of the contour
- \( \text{cmap} \) – (default: 'hsv') color map; for available color map names type: \text{import matplotlib.cm; list(matplotlib.cm.datad)}
- \( \text{label} \) – string (default: None) a label to add on the colored vector

OUTPUT:

Graphics

EXAMPLES:

sage: Word(range(20)).colored_vector()
Graphics object consisting of 21 graphics primitives
sage: Word(range(100)).colored_vector(0,0,10,1)
Graphics object consisting of 101 graphics primitives
sage: Words(range(100))(range(10)).colored_vector()
Graphics object consisting of 11 graphics primitives
sage: w = Word('abbabaab')
sage: w.colored_vector()
Graphics object consisting of 9 graphics primitives
sage: w.colored_vector(cmap='autumn')
Graphics object consisting of 9 graphics primitives
sage: Word(range(20)).colored_vector(label='Rainbow')
Graphics object consisting of 23 graphics primitives

When two words are defined under the same parent, same letters are mapped to same colors:

sage: W = Words(range(20))
sage: w = W(range(20))
sage: y = W(range(10,20))

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sage: y.colored_vector(y=1, x=10) + w.colored_vector()
Graphics object consisting of 32 graphics primitives

commutes_with(other)
Return True if self commutes with other, and False otherwise.

EXAMPLES:

sage: Word('12').commutes_with(Word('12'))
True
sage: Word('12').commutes_with(Word('11'))
False
sage: Word().commutes_with(Word('21'))
True

complete_return_words(fact)
Return the set of complete return words of fact in self.
This is the set of all factors starting by the given factor and ending just after the next occurrence of this
factor. See for instance [JV2000].

INPUT:
• fact – a non-empty finite word

OUTPUT:
a Python set of finite words

EXAMPLES:

sage: s = Word('2133123213231').complete_return_words(Word('2'))
sage: sorted(s)
[word: 2132, word: 213312, word: 2332]
sage: Word('').complete_return_words(Word('213'))
set()
sage: Word('1212').complete_return_words(Word('1212'))
{word: 121212}

concatenate(other)
Return the concatenation of self and other.

INPUT:
• other – a word over the same alphabet as self

EXAMPLES:
Concatenation may be made using + or * operations:

sage: w = Word('abadafd')
sage: y = Word([5,3,5,8,7])
sage: w * y
word: abadafd53587
sage: w + y
word: abadafd53587
sage: w.concatenate(y)
word: abadafd53587
Both words must be defined over the same alphabet:

```sage
z = Word('1223', alphabet = '123')
sage: z + y
Traceback (most recent call last):
  ... ValueError: 5 not in alphabet
```

Eventually, it should work:

```sage
z = Word('1223', alphabet = '123')
sage: z + y
#todo: not implemented
```

**conjugate(pos)**

Return the conjugate at pos of self.

pos can be any integer, the distance used is the modulo by the length of self.

EXAMPLES:

```sage
Word('12112').conjugate(1)
word: 21121
sage: Word().conjugate(2)
word:
sage: Word('12112').conjugate(8)
word: 12112
sage: Word('12112').conjugate(-1)
word: 21211
```

**conjugate_position(other)**

Return the position where self is conjugate with other. Return None if there is no such position.

EXAMPLES:

```sage
Word('12113').conjugate_position(Word('31211'))
1
sage: Word('12131').conjugate_position(Word('12113')) is None
True
sage: Word().conjugate_position(Word('123')) is None
True
```

**conjugates()**

Return the list of unique conjugates of self.

EXAMPLES:

```sage
Word(range(6)).conjugates()
[word: 012345,
 word: 123450,
 word: 234501,
 word: 345012,
 word: 450123,
 word: 501234]
sage: Word('cbbca').conjugates()
[word: cbbca, word: bbcac, word: bcacb, word: cacbb, word: acbcb]
```
The result contains each conjugate only once:

```
sage: Word('abcabc').conjugates()
[word: abcabc, word: bcabca, word: cabcab]
```

**conjugates_iterator()**

Return an iterator over the conjugates of `self`.

**EXAMPLES:**

```
sage: it = Word(range(4)).conjugates_iterator()
sage: for w in it: w
word: 0123
word: 1230
word: 2301
word: 3012
```

**content(n=None)**

Return content of `self`.

**INPUT:**

- `n` – (optional) an integer specifying the maximal letter in the alphabet

**OUTPUT:**

- a list where the $i$-th entry indicates the multiplicity of the $i$-th letter in the alphabet in `self`

**EXAMPLES:**

```
sage: w = Word([1,2,4,3,2,2,2])
sage: w.content()
[1, 4, 1, 1]
sage: w = Word([3,1])
sage: w.content()
[1, 1]
sage: w.content(n=3)
[1, 0, 1]
sage: w = Word([2,4],alphabet=[1,2,3,4])
sage: w.content(n=3)
[0, 1, 0]
sage: w.content()
[0, 1, 0, 1]
```

**count(letter)**

Return the number of occurrences of `letter` in `self`.

**INPUT:**

- `letter` - a letter

**OUTPUT:**

- integer

**EXAMPLES:**
sage: w = Word('abbabaab')
sage: w.number_of_letter_occurrences('a')
4
sage: w.number_of_letter_occurrences('ab')
0

This method is equivalent to list(w).count(letter) and tuple(w).count(letter), thus count is an alias for the method number_of_letter_occurrences:

sage: list(w).count('a')
4
sage: w.count('a')
4

But notice that if s and w are strings, Word(s).count(w) counts the number occurrences of w as a letter in Word(s) which is not the same as s.count(w) which counts the number of occurrences of the string w inside s:

sage: s = 'abbabaab'
sage: s.count('ab')
3
sage: Word(s).count('ab')
0

See also:

sage.combinat.words.finite_word.FiniteWord_class.number_of_factor_occurrences()
critical_exponent()

Return the critical exponent of self.

The critical exponent of a word is the supremum of the order of all its (finite) factors. See [Dej1972].

Note: The implementation here uses the suffix tree to enumerate all the factors. It should be improved (especially when the critical exponent is larger than 2).

EXAMPLES:

sage: Word('aaba').critical_exponent()
2
sage: Word('aaba').critical_exponent()
2
sage: Word('aabaaba').critical_exponent()
7/3
sage: Word('ab').critical_exponent()
1
sage: Word('aba').critical_exponent()
3/2
sage: words.ThueMorseWord()[:20].critical_exponent()
2

For the Fibonacci word, the critical exponent is known to be $\left(5 + \sqrt{5}\right)/2$. With a prefix of length 500, we obtain a lower bound:
sage: words.FibonacciWord()[:500].critical_exponent()
320/89

It is an error to compute the critical exponent of the empty word:

```
sage: Word('').critical_exponent()
Traceback (most recent call last):
  ...  
ValueError: no critical exponent for empty word
```

crochemore_factorization()
Return the Crochemore factorization of self as an ordered list of factors.

The Crochemore factorization or the Lempel-Ziv decomposition of a finite word \( w \) is the unique factorization: \( (x_1, x_2, \ldots, x_n) \) of \( w \) with each \( x_i \) satisfying either: C1. \( x_i \) is a letter that does not appear in \( u = x_1 \cdots x_{i-1} \); C2. \( x_i \) is the longest prefix of \( v = x_i \cdots x_n \) that also has an occurrence beginning within \( u = x_1 \cdots x_{i-1} \). See [Cro1983].

**EXAMPLES:**

```
sage: x = Word('abababb')
sage: x.crochemore_factorization()
(a, b, abab, b)
sage: mul(x.crochemore_factorization()) == x
True
sage: y = Word('abaababacabba')
sage: y.crochemore_factorization()
(a, b, a, aba, ba, c, ab, ba)
sage: mul(y.crochemore_factorization()) == y
True
sage: x = Word([0,1,0,1,0,1,1])
sage: x.crochemore_factorization()
(0, 1, 0101, 1)
sage: mul(x.crochemore_factorization()) == x
True
```

defect(f=None)
Return the defect of self.

The defect of a finite word \( w \) is given by the difference between the maximum number of possible palindromic factors in a word of length \( |w| \) and the actual number of palindromic factors contained in \( w \). It is well known that the maximum number of palindromic factors in \( w \) is \( |w| + 1 \) (see [DJP2001]).

An optional involution on letters \( f \) can be given. In that case, the \( f \)-palindromic defect (or pseudopalin-
dromic defect, or theta-palindromic defect) of \( w \) is returned. It is a generalization of defect to \( f \)-palindromes. More precisely, the defect is \( D(w) = |w| + 1 - g_f(w) - |P\ A\ L_f(w)| \), where \( P\ A\ L_f(w) \) denotes the set of \( f \)-palindromic factors of \( w \) (including the empty word) and \( g_f(w) \) is the number of pairs \( \{a, f(a)\} \) such that \( a \) is a letter, \( a \) is not equal to \( f(a) \), and \( a \) or \( f(a) \) occurs in \( w \). In the case of usual palindromes (i.e., for \( f \) not given or equal to the identity), \( g_f(w) = 0 \) for all \( w \). See [BHNR2004] for usual palindromes and [Star2011] for \( f \)-palindromes.

**INPUT:**

- \( f \) – involution (default: None) on the alphabet of \( self \). It must be callable on letters as well as words (e.g. \texttt{WordMorphism}). The default value corresponds to usual palindromes, i.e., \( f \) equal to the identity.

**OUTPUT:**
an integer – If $f$ is None, the palindromic defect of self; otherwise, the $f$-palindromic defect of self.

**EXAMPLES:**

```python
sage: Word('ara').defect()
0
sage: Word('abcacba').defect()
1
```

It is known that Sturmian words (see [DJP2001]) have zero defect:

```python
sage: words.FibonacciWord()[:100].defect()
0
```

It is even conjectured that the defect of an aperiodic word which is a fixed point of a primitive morphism is either 0 or infinite (see [BBGL2008]):

```python
sage: w = words.ThueMorseWord()
sage: w[:50].defect()
12
sage: w[:100].defect()
16
sage: w[:300].defect()
52
```

For generalized defect with an involution different from the identity, there is always a letter which is not a palindrome! This is the reason for the modification of the definition:

```python
sage: f = WordMorphism('a->b,b->a')
sage: Word('a').defect(f)
0
sage: Word('ab').defect(f)
0
sage: Word('aa').defect(f)
1
sage: Word('abbabaabbaababba').defect(f)
3
```

```python
sage: f = WordMorphism('a->b,b->a,c->c')
sage: Word('cabc').defect(f)
0
sage: Word('abcaab').defect(f)
2
```

Other examples:
sage: Word('000000000000').defect()
0
sage: Word('011010011001').defect()
2
sage: Word('0101001010001').defect()
0
sage: Word().defect()
0
sage: Word('abbabaabbaababba').defect()
2

deg_inv_lex_less(other, weights=None)

Return True if the word self is degree inverse lexicographically less than other.

EXAMPLES:

sage: Word([1,2,4]).deg_inv_lex_less(Word([1,3,2]))
False
sage: Word([3,2,1]).deg_inv_lex_less(Word([1,2,3]))
True

deg_lex_less(other, weights=None)

Return True if self is degree lexicographically less than other, and False otherwise. The weight of each letter in the ordered alphabet is given by weights, which defaults to [1, 2, 3, ...].

EXAMPLES:

sage: Word([1,2,3]).deg_lex_less(Word([1,3,2]))
True
sage: Word([3,2,1]).deg_lex_less(Word([1,2,3]))
False
sage: W = Words(range(5))
sage: W([1,2,4]).deg_lex_less(W([1,3,2]))
False
sage: Word("abba").deg_lex_less(Word("abbb"), dict(a=1,b=2))
True
sage: Word("abba").deg_lex_less(Word("baba"), dict(a=1,b=2))
True
sage: Word("abba").deg_lex_less(Word("aaba"), dict(a=1,b=2))
False
sage: Word("abba").deg_lex_less(Word("aaba"), dict(a=1,b=0))
True

deg_rev_lex_less(other, weights=None)

Return True if self is degree reverse lexicographically less than other.

EXAMPLES:

sage: Word([3,2,1]).deg_rev_lex_less(Word([1,2,3]))
False
sage: Word([1,2,4]).deg_rev_lex_less(Word([1,3,2]))
False
sage: Word([1,2,3]).deg_rev_lex_less(Word([1,2,4]))
True
degree(weights=None)
Return the weighted degree of self, where the weighted degree of each letter in the ordered alphabet is
given by weights, which defaults to [1, 2, 3, ...].

INPUT:
• weights – a list or a tuple, or a dictionary keyed by the letters occurring in self.

EXAMPLES:

```sage
sage: Word([1,2,3]).degree()
6
sage: Word([3,2,1]).degree()
6
sage: Words("ab")("abba").degree()
6
sage: Words("ab")("abba").degree([0,2])
4
sage: Words("ab")("abba").degree([-1,-1])
-4
sage: Words("ab")("aabba").degree([1,1])
5
sage: Words([1,2,4])([1,2,4]).degree()
6
sage: Word([1,2,4]).degree()
7
sage: Word("aabba").degree({'a':1,'b':2})
7
sage: Word([0,1,0]).degree({0:17,1:0})
34
```

delta()
Return the image of self under the delta morphism.

The delta morphism, also known as the run-length encoding, is the word composed of the length of con-
ssecutive runs of the same letter in a given word.

EXAMPLES:

```sage
sage: W = Words('0123456789')
sage: W('221121212').delta()
word: 22112
sage: W('555008').delta()
word: 321
sage: W().delta()
word:
sage: Word('aababaa').delta()
word: 22112
```

delta_derivate(W=None)
Return the derivative under delta for self.

EXAMPLES:

```sage
sage: W = Words('1234')
sage: W('121212121').delta_derivate()
(continues on next page)
```
**delta_derivative_left**(W=None)

Return the derivative under delta for `self`.

**EXAMPLES:**

```python
sage: W = Words('12')
sage: W('12211').delta_derivative_left()
word: 22
sage: W('1').delta_derivative_left(Words([1]))
word: 1
sage: W('2112').delta_derivative_left()
word: 21
sage: W('2211').delta_derivative_left()
word: 22
sage: W('112').delta_derivative_left()
word: 21
sage: W('11222').delta_derivative_left(Words([1, 2, 3]))
word: 3
```

**delta_derivative_right**(W=None)

Return the right derivative under delta for `self`.

**EXAMPLES:**

```python
sage: W = Words('12')
sage: W('12211').delta_derivative_right()
word: 122
sage: W('1').delta_derivative_right(Words([1]))
word: 1
sage: W('2112').delta_derivative_right()
word: 12
sage: W('2211').delta_derivative_right()
word: 22
sage: W('112').delta_derivative_right()
word: 2
sage: W('11222').delta_derivative_right(Words([1, 2, 3]))
word: 23
```

**delta_inv**(W=None, s=None)

Lift `self` via the delta operator to obtain a word containing the letters in alphabet (default is [0, 1]). The letters used in the construction start with `s` (default is alphabet[0]) and cycle through alphabet.
INPUT:

- alphabet – an iterable
- s – an object in the iterable

EXAMPLES:

```
sage: W = Words([1, 2])
sage: W([2, 2, 1, 1]).delta_inv()
word: 112212
sage: W([1, 1, 1, 1]).delta_inv(Words('123'))
word: 1231
sage: W([2, 2, 1, 1, 2]).delta_inv(s=2)
word: 22112122
```

**evaluation()**

Return the abelian vector of self counting the occurrences of each letter. The vector is defined w.r.t. the order of the alphabet of the parent. See also `evaluation_dict()`.

INPUT:

- self – word having a parent on a finite alphabet

OUTPUT:

a list

EXAMPLES:

```
sage: W = Words('ab')
sage: W('aaabbb').abelian_vector()
[3, 5]
sage: W('a').abelian_vector()
[1, 0]
sage: W().abelian_vector()
[0, 0]
```

The result depends on the alphabet of the parent:

```
sage: W = Words('abc')
sage: W('aabaa').abelian_vector()
[4, 1, 0]
```

**evaluation_dict()**

Return a dictionary keyed by the letters occurring in self with values the number of occurrences of the letter.

EXAMPLES:

```
sage: Word([2,1,4,2,3,4,2]).evaluation_dict()
{1: 1, 2: 3, 3: 1, 4: 2}
sage: Word('badbcdb').evaluation_dict()
{'a': 1, 'b': 3, 'c': 1, 'd': 2}
sage: Word().evaluation_dict()
{}
```
Combinatorics, Release 10.0

```python
sage: f = Word('1213121').evaluation_dict() # keys appear in random order
{'1': 4, '2': 2, '3': 1}
```

**evaluation_partition()**

Return the evaluation of the word w as a partition.

**EXAMPLES:**

```python
sage: Word("acdabda").evaluation_partition()
[3, 2, 1, 1]
sage: Word([2,1,4,2,3,4,2]).evaluation_partition()
[3, 2, 1, 1]
```

**evaluation_sparse()**

Return a list representing the evaluation of self. The entries of the list are two-element lists [a, n], where a is a letter occurring in self and n is the number of occurrences of a in self.

**EXAMPLES:**

```python
sage: sorted(Word([4,4,2,5,2,1,4,1]).evaluation_sparse())
[(1, 2), (2, 2), (4, 3), (5, 1)]
sage: sorted(Word("abcaccab").evaluation_sparse())
[('a', 3), ('b', 2), ('c', 3)]
```

**exponent()**

Return the exponent of self.

**OUTPUT:**

integer – the exponent

**EXAMPLES:**

```python
sage: Word('1231').exponent()
1
sage: Word('121212').exponent()
3
sage: Word().exponent()
0
```

**factor_complexity(n)**

Return the number of distinct factors of length n of self.

**INPUT:**

- n – the length of the factors.

**EXAMPLES:**

```python
sage: w = words.FibonacciWord()[:100]
sage: [w.factor_complexity(i) for i in range(20)]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]
sage: w = words.ThueMorseWord()[:1000]
sage: [w.factor_complexity(i) for i in range(20)]
[1, 2, 4, 6, 10, 12, 16, 20, 22, 24, 28, 32, 36, 40, 42, 44, 46, 48, 52, 56]
```
factor_iterator\((n=None)\)
Generate distinct factors of \texttt{self}.

INPUT:

- \(n\) – an integer, or \texttt{None}.

OUTPUT:

If \(n\) is an integer, returns an iterator over all distinct factors of length \(n\). If \(n\) is \texttt{None}, returns an iterator generating all distinct factors.

EXAMPLES:

\begin{verbatim}
sage: w = Word('1213121')
sage: sorted( w.factor_iterator(0) )
[word: ]
sage: sorted( w.factor_iterator(10) )
[]
sage: sorted( w.factor_iterator(1) )
[word: 1, word: 2, word: 3]
sage: sorted( w.factor_iterator(4) )
[word: 1213, word: 1312, word: 2131, word: 3121]
sage: sorted( w.factor_iterator() )
[word: , word: 1, word: 12, word: 121, word: 1213, word: 12131, word: 121312,  
word: 1213121, word: 13, word: 131, word: 1312, word: 13121, word: 2, word:  
21, word: 213, word: 2131, word: 21312, word: 213121, word: 3, word: 31,  
word: 312, word: 3121]
sage: u = Word([1,2,1,2,3])
sage: sorted( u.factor_iterator(0) )
[word: ]
sage: sorted( u.factor_iterator(10) )
[]
sage: sorted( u.factor_iterator(1) )
[word: 1, word: 2, word: 3]
sage: sorted( u.factor_iterator(5) )
[word: 12123]
sage: sorted( u.factor_iterator() )
[word: , word: 1, word: 12, word: 121, word: 1212, word: 12123, word: 123,  
word: 2, word: 21, word: 212, word: 2123, word: 23, word: 3]
sage: xxx = Word("xxx")
sage: sorted( xxx.factor_iterator(0) )
[word: ]
sage: sorted( xxx.factor_iterator(4) )
[]
sage: sorted( xxx.factor_iterator(2) )
[word: xx]
sage: sorted( xxx.factor_iterator() )
[word: , word: x, word: xx, word: xxx]
sage: e = Word()
sage: sorted( e.factor_iterator(0) )
[word: ]
\end{verbatim}
factor_occurrences_in\texttt{(}\texttt{other}\texttt{)}

Return an iterator over all occurrences (including overlapping ones) of \texttt{self} in \texttt{other} in their order of appearance.

\textbf{Warning:} This method is deprecated since 2020 and will be removed in a later version of SageMath. Use \texttt{factor_occurrences_iterator()} instead.

EXAMPLES:

\begin{verbatim}
sage: u = Word(['121'])
sage: w = Word(['1213211213'])
sage: list(u.factor_occurrences_in(w))
doctest:warning...
DeprecationWarning: f.factor_occurrences_in(w) is deprecated.
Use w.factor_occurrences_iterator(f) instead.
See https://github.com/sagemath/sage/issues/30187 for details.
[0, 2, 8]
\end{verbatim}

factor_set\texttt{(}n=\texttt{None, algorithm='suffix tree'}}

Return the set of factors (of length \texttt{n}) of \texttt{self}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{n} – an integer or \texttt{None} (default: \texttt{None}).
  \item \texttt{algorithm} – string (default: \texttt{'suffix tree'}), takes the following values:
    \begin{itemize}
      \item \texttt{'suffix tree'} – construct and use the suffix tree of the word
      \item \texttt{'naive'} – algorithm uses a sliding window
    \end{itemize}
\end{itemize}

\textbf{OUTPUT:}

If \texttt{n} is an integer, returns the set of all distinct factors of length \texttt{n}. If \texttt{n} is \texttt{None}, returns the set of all distinct factors.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: w = Word(['121'])
sage: sorted(w.factor_set())
[\texttt{word: 1, word: 2, word: 3}]
sage: sorted(w.factor_set(algorithm='naive'))
[\texttt{word: 1, word: 2, word: 3}]
\end{verbatim}

\begin{verbatim}
sage: w = Word(['1213121'])
sage: for i in range(w.length()): sorted(w.factor_set(i))
[\texttt{word: 1, word: 2, word: 3}]
\end{verbatim}

(continues on next page)
sage: w = Word([1,2,1,2,3])
sage: s = w.factor_set()
sage: sorted(s)
[word: 1, word: 12, word: 121, word: 1212, word: 12123, word: 123, word: 2, word: 21, word: 212, word: 2123, word: 23, word: 3]

find(sub, start=0, end=None)

Return the index of the first occurrence of sub in self, such that sub is contained within self[start:end]. Return -1 on failure.

INPUT:
• sub – string, list, tuple or word to search for.
• start – non-negative integer (default: 0) specifying the position from which to start the search.
• end – non-negative integer (default: None) specifying the position at which the search must stop. If None, then the search is performed up to the end of the string.

OUTPUT:
a non-negative integer or -1

EXAMPLES:

sage: w = Word([0,1,0,0,1])
sage: w.find(Word([1,0]))
1

The sub argument can also be a tuple or a list:

sage: w.find([1,0])
1
sage: w.find((1,0))
1

Examples using start and end:

sage: w.find(Word([0,1]), start=1)
3
sage: w.find(Word([0,1]), start=1, end=5)
3
sage: w.find(Word([0,1]), start=1, end=4) == -1
True
sage: w.find(Word([1,1])) == -1
True
sage: w.find("aa")
-1

Instances of Word_str handle string inputs as well:
```
sage: w = Word('abac')
sage: w.find('a')
0
sage: w.find('ba')
1
```

**first_pos_in(other)**

Return the position of the first occurrence of `self` in `other`, or `None` if `self` is not a factor of `other`.

**Warning:** This method is deprecated since 2020 and will be removed in a later version of SageMath. Use `first_occurrence()` instead.

**EXAMPLES:**
```
sage: Word('12').first_pos_in(Word('131231'))
doctest:warning...
DeprecationWarning: f.first_pos_in(w) is deprecated.
Use w.first_occurrence(f) instead.
See https://github.com/sagemath/sage/issues/30187 for details.
2
sage: Word('32').first_pos_in(Word('131231'))
is None
True
```

**foata_bijection()**

Return word `self` under the Foata bijection.

The Foata bijection \( \phi \) is a bijection on the set of words of given content (by a slight generalization of Section 2 in [FS1978]). It can be defined by induction on the size of the word: Given a word \( w_1 w_2 \cdots w_n \), start with \( \phi(w_1) = w_1 \). At the \( i \)-th step, if \( \phi(w_1 w_2 \cdots w_i) = v_1 v_2 \cdots v_i \), we define \( \phi(w_1 w_2 \cdots w_i w_{i+1}) \) by placing \( w_{i+1} \) on the end of the word \( v_1 v_2 \cdots v_i \) and breaking the word up into blocks as follows. If \( w_{i+1} \geq v_i \), place a vertical line to the right of each \( v_k \) for which \( w_{i+1} \geq v_k \). Otherwise, if \( w_{i+1} < v_i \), place a vertical line to the right of each \( v_k \) for which \( w_{i+1} < v_k \). In either case, place a vertical line at the start of the word as well. Now, within each block between vertical lines, cyclically shift the entries one place to the right.

For instance, to compute \( \phi([4, 1, 5, 4, 2, 2, 3]) \), the sequence of words is

- 4,
- \([4|1 \rightarrow 41,\]
- \([4|1|5 \rightarrow 415,\]
- \([415|4 \rightarrow 5414,\]
- \([5|4|14|2 \rightarrow 54412,\]
- \([5441|2|2 \rightarrow 154422,\]
- \([1|5442|2|3 \rightarrow 1254423.\)

So \( \phi([4, 1, 5, 4, 2, 2, 3]) = [1, 2, 5, 4, 2, 3]. \)

**See also:**

*Foata bijection on Permutations.*

**EXAMPLES:**
```python
sage: w = Word([2,2,2,1,1,1])
sage: w.foata_bijection()
word: 112221
sage: w = Word([2,2,1,2,2,2,1,1,2,1])
sage: w.foata_bijection()
word: 2122212211
sage: w = Word([4,1,5,4,2,2,3])
sage: w.foata_bijection()
word: 1254423
```

**good_suffix_table()**

Return a table of the maximum skip you can do in order not to miss a possible occurrence of `self` in a word.

This is a part of the Boyer-Moore algorithm to find factors. See [BM1977].

**EXAMPLES:**

```python
sage: Word('121321').good_suffix_table()
[5, 5, 5, 5, 3, 3, 1]
sage: Word('12412').good_suffix_table()
[3, 3, 3, 3, 3, 1]
```

**has_period(p)**

Return True if `self` has the period p, False otherwise.

**Note:** By convention, integers greater than the length of `self` are periods of `self`.

**INPUT:**

- `p` – an integer to check if it is a period of `self`.

**EXAMPLES:**

```python
sage: w = Word('ababa')
sage: w.has_period(2)
True
sage: w.has_period(3)
False
sage: w.has_period(4)
True
sage: w.has_period(-1)
False
sage: w.has_period(5)
True
sage: w.has_period(6)
True
```

**has_prefix(other)**

Test whether `self` has `other` as a prefix.

**INPUT:**

- `other` – a word, or data describing a word
OUTPUT:

boolean

EXAMPLES:

```
sage: w = Word("abbaabababa")
sage: u = Word("abbab")
sage: w.has_prefix(u)
True
sage: u.has_prefix(w)
False
sage: u.has_prefix("abbab")
True
```

```
sage: w = Word([0,1,1,0,1,0,0,1,0,1,0,1,0])
sage: u = Word([0,1,0,1,0])
sage: w.has_prefix(u)
True
sage: u.has_prefix(w)
False
sage: u.has_prefix([0,1,1,0,1])
True
```

`has_suffix(other)`

Test whether `self` has `other` as a suffix.

**Note:** Some word datatype classes, like `WordDatatype_str`, override this method.

INPUT:

• `other` – a word, or data describing a word

OUTPUT:

boolean

EXAMPLES:

```
sage: w = Word("abbaabababa")
sage: u = Word("ababa")
sage: w.has_suffix(u)
True
sage: u.has_suffix(w)
False
sage: u.has_suffix("ababa")
True
```

```
sage: w = Word([0,1,1,0,1,0,0,1,0,1,0,1,0])
sage: u = Word([0,1,0,1,0])
sage: w.has_suffix(u)
True
sage: u.has_suffix(w)
False
```
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\begin{verbatim}
sage: u.has_suffix([0,1,0,1,0])
True

\textbf{implicit suffix tree()}

Return the implicit suffix tree of \texttt{self}.

The suffix tree of a word \( w \) is a compactification of the suffix trie for \( w \). The compactification removes all nodes that have exactly one incoming edge and exactly one outgoing edge. It consists of two components: a tree and a word. Thus, instead of labelling the edges by factors of \( w \), we can label them by indices of the occurrence of the factors in \( w \).

Type \texttt{sage.combinat.words.suffix_trees.ImplicitSuffixTree?} for more information.

\textbf{Examples:}

\begin{verbatim}
sage: w = Word("cacao")
\end{verbatim}
\begin{verbatim}
sage: w.implicit_suffix_tree()
Implicit Suffix Tree of the word: cacao
\end{verbatim}

\begin{verbatim}
sage: w = Word([0,1,0,1,1])
\end{verbatim}
\begin{verbatim}
sage: w.implicit_suffix_tree()
Implicit Suffix Tree of the word: 01011
\end{verbatim}

\textbf{inv lex less(other)}

Return True if \texttt{self} is inverse lexicographically less than \texttt{other}.

\textbf{Examples:}

\begin{verbatim}
sage: Word([1,2,4]).inv_lex_less(Word([1,3,2]))
False
\end{verbatim}
\begin{verbatim}
sage: Word([3,2,1]).inv_lex_less(Word([1,2,3]))
True
\end{verbatim}

\textbf{inversions()}

Return a list of the inversions of \texttt{self}. An inversion is a pair \((i, j)\) of non-negative integers \( i < j \) such that \texttt{self}[i] > \texttt{self}[j].

\textbf{Examples:}

\begin{verbatim}
sage: Word([1,2,3,2,2,1]).inversions()
[[0, 1], [0, 2], [0, 3], [0, 4], [1, 2], [1, 3], [1, 5], [2, 3], [2, 4], [2, 5], [3, 5], [4, 5]]
\end{verbatim}
\begin{verbatim}
sage: Words([3,2,1])([1,2,3,2,2,1]).inversions()
[[0, 1], [0, 2], [0, 3], [0, 4], [1, 2], [1, 3], [1, 5], [2, 3], [2, 4], [2, 5], [3, 5], [4, 5]]
\end{verbatim}
\begin{verbatim}
sage: Word('abbaba').inversions()
[[0, 1], [0, 2], [0, 3], [0, 4], [1, 2], [1, 3], [1, 5], [2, 3], [2, 4], [2, 5], [3, 5], [4, 5]]
\end{verbatim}
\begin{verbatim}
sage: Words('ba')('abbaba').inversions()
[[0, 1], [0, 2], [0, 3], [0, 4], [1, 2], [1, 3], [1, 5], [2, 3], [2, 4], [2, 5], [3, 5], [4, 5]]
\end{verbatim}

\textbf{is balanced}(q=1)

Return True if \texttt{self} is \( q \)-balanced, and False otherwise.

A finite or infinite word \( w \) is said to be \( q \)-balanced if for any two factors \( u, v \) of \( w \) of the same length, the difference between the number of \( x \)'s in each of \( u \) and \( v \) is at most \( q \) for all letters \( x \) in the alphabet of \( w \). A 1-balanced word is simply said to be balanced. See for instance [CFZ2000] and Chapter 2 of [Lot2002].

\textbf{INPUT:}

5.1. comprehensive module list 3585
• q – integer (default: 1), the balance level

OUTPUT:

boolean – the result

EXAMPLES:

```
sage: Word('1213121').is_balanced()
True
sage: Word('1122').is_balanced()
False
sage: Word('121333121').is_balanced()
False
sage: Word('121333121').is_balanced(2)
False
sage: Word('121333121').is_balanced(3)
True
sage: Word('121122121').is_balanced()
False
sage: Word('121122121').is_balanced(2)
True
```

is_cadence(seq)

Return True if seq is a cadence of self, and False otherwise.

A cadence is an increasing sequence of indexes that all map to the same letter.

EXAMPLES:

```
sage: Word('121132123').is_cadence([0, 2, 6])
True
sage: Word('121132123').is_cadence([0, 1, 2])
False
sage: Word('121132123').is_cadence([])
True
```

is_christoffel()

Return True if self is a Christoffel word, and False otherwise.

The Christoffel word of slope \( \frac{p}{q} \) is obtained from the Cayley graph of \( \mathbb{Z}/(p + q)\mathbb{Z} \) with generator \( q \) as follows. If \( u \rightarrow v \) is an edge in the Cayley graph, then, \( v = u + p \mod p + q \). Let \( a, b \) be the alphabet of \( w \). Label the edge \( u \rightarrow v \) by \( a \) if \( u < v \) and \( b \) otherwise. The Christoffel word is the word obtained by reading the edge labels along the cycle beginning from 0.

Equivalently, \( w \) is a Christoffel word iff \( w \) is a symmetric non-empty word and \( w[1 : n - 1] \) is a palindrome.

See for instance [Ber2007] and [BLRS2009].

INPUT:

• self – word

OUTPUT:

boolean – True if self is a Christoffel word, False otherwise.

EXAMPLES:
is_christoffel()

Return True if self is a Christoffel word, and False otherwise.

INPUT:

• self – a finite word

OUTPUT:

bool

EXAMPLES:

sage: Word('00100101').is_christoffel()
True
sage: Word('aab').is_christoffel()
True
sage: Word('').is_christoffel()
False
sage: Word('123123123').is_christoffel()
False
sage: Word('00100').is_christoffel()
False
sage: Word('0').is_christoffel()
True

is_conjugate_with(other)

Return True if self is a conjugate of other, and False otherwise.

INPUT:

• other – a finite word

OUTPUT:

bool

EXAMPLES:

sage: w = Word([0..20])
sage: z = Word([7..20] + [0..6])
sage: w
word: 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20
sage: z
word: 7,8,9,10,11,12,13,14,15,16,17,18,19,20,0,1,2,3,4,5,6
sage: w.is_conjugate_with(z)
True
sage: z.is_conjugate_with(w)
True
sage: u = Word([4]*21)
sage: u
word: 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
sage: u.is_conjugate_with(w)
False
sage: u.is_conjugate_with(z)
False

Both words must be finite:

sage: w = Word(iter([2]*100),length='unknown')
sage: z = Word([2]*100)
sage: z.is_conjugate_with(w)  #TODO: Not implemented for word of unknown length
True
sage: wf = Word(iter([2]*100),length='finite')
sage: z.is_conjugate_with(wf)
True
sage: wf.is_conjugate_with(z)
True

is_cube()

Return True if self is a cube, and False otherwise.
EXAMPLES:

```python
sage: Word('012012012').is_cube()
True
sage: Word('01010101').is_cube()
False
sage: Word().is_cube()
True
sage: Word('012012').is_cube()
False
```

**is_cube_free()**

Return True if *self* does not contain cubes, and False otherwise.

EXAMPLES:

```python
sage: Word('12312').is_cube_free()
True
sage: Word('32221').is_cube_free()
False
sage: Word().is_cube_free()
True
```

**is_empty()**

Return True if the length of *self* is zero, and False otherwise.

EXAMPLES:

```python
sage: Word([]).is_empty()
True
sage: Word('a').is_empty()
False
```

**is_factor(*other*)**

Return True if *self* is a factor of *other*, and False otherwise.

A finite word \(u \in A^*\) is a factor of a finite word \(v \in A^*\) if there exists \(p, s \in A^*\) such that \(v = pus\).

EXAMPLES:

```python
sage: u = Word('2113')
sage: w = Word('12312332131233121132123')
sage: u.is_factor(w)
True
sage: u = Word('321')
sage: w = Word('1231241231312312312')
sage: u.is_factor(w)
False
```

The empty word is factor of another word:

```python
sage: Word().is_factor(Word())
True
sage: Word().is_factor(Word('a'))
True
```
sage: Word().is_factor(Word([1,2,3]))
True
sage: Word().is_factor(Word(lambda n:n, length=5))
True

isfinite()

Return True.

EXAMPLES:

sage: Word([]).isFinite()
True
sage: Word('a').isFinite()
True

isfull(f=None)

Return True if self has defect 0, and False otherwise.

A word is full (or rich) if its defect is zero (see [BHNR2004]).

If f is given, then the f-palindromic defect is used (see [PeSt2011]).

INPUT:

• f – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPUT:

boolean – If f is None, whether self is full; otherwise, whether self is full of f-palindromes.

EXAMPLES:

sage: words.ThueMorseWord()[:100].isFull()
False
sage: words.FibonacciWord()[:100].isFull()
True
sage: Word('000000000000000').isFull()
True
sage: Word('011010011001').isFull()
False
sage: Word('2194').isFull()
True
sage: Word().isFull()
True

sage: f = WordMorphism('a->b,b->a')
sage: Word().is_full(f)
True
sage: w = Word('ab')
sage: w.is_full()
True
sage: w.is_full(f)
True
sage: f = WordMorphism('a->b,b->a')
sage: Word('abab').is_full(f)
True
sage: Word('abba').is_full(f)
False

A simple example of an infinite word full of f-palindromes:

sage: p = WordMorphism({0:'abc',1:'ab'})
sage: f = WordMorphism('a->b,b->a,c->c')
sage: p(words.FibonacciWord()[:50]).is_full(f)
True
sage: p(words.FibonacciWord()[:150]).is_full(f)
True

is_lyndon()

Return True if self is a Lyndon word, and False otherwise.

A Lyndon word is a non-empty word that is lexicographically smaller than each of its proper suffixes (for the given order on its alphabet). That is, \( w \) is a Lyndon word if \( w \) is non-empty and for each factorization \( w = uv \) (with \( u, v \) both non-empty), we have \( w < v \).

Equivalently, \( w \) is a Lyndon word if \( w \) is a non-empty word that is lexicographically smaller than each of its proper conjugates for the given order on its alphabet.

See for instance [Lot1983].

EXAMPLES:

sage: Word('123132133').is_lyndon()
True
sage: Word().is_lyndon()
False
sage: Word('122112').is_lyndon()
False

is_overlap()

Return True if self is an overlap, and False otherwise.

EXAMPLES:

sage: Word('12121').is_overlap()
True
sage: Word('123').is_overlap()
False
sage: Word('1231').is_overlap()
False
sage: Word('123123').is_overlap()
False
sage: Word('1231231').is_overlap()
True
sage: Word().is_overlap()
False

is_palindrome(f=None)

Return True if self is a palindrome (or a \( f \)-palindrome), and False otherwise.
Let \( f : \Sigma \to \Sigma \) be an involution that extends to a morphism on \( \Sigma^* \). We say that \( w \in \Sigma^* \) is a \( f \)-\textit{palindrome} if \( w = f(\tilde{w}) \) [Lab2008]. Also called \( f \)-\textit{pseudo-palindrome} [AZZ2005].

**INPUT:**

- \( f \) – involution (default: None) on the alphabet of \( \text{self} \). It must be callable on letters as well as words (e.g. \texttt{WordMorphism}). The default value corresponds to usual palindromes, i.e., \( f \) equal to the identity.

**EXAMPLES:**

```python
sage: Word('esope reste ici et se repose').is_palindrome()
False
sage: Word('esoperesteicietserepose').is_palindrome()
True
sage: Word('I saw I was I').is_palindrome()
True
sage: Word('abcbba').is_palindrome()
True
sage: Word('abcdbba').is_palindrome()
False
```

Some \( f \)-palindromes:

```python
sage: f = WordMorphism('a->b,b->a')
sage: Word('aababb').is_palindrome(f)
True

sage: f = WordMorphism('a->b,b->a,c->c')
sage: Word('abacbacbab').is_palindrome(f)
True

sage: f = WordMorphism({'a':'b','b':'a'})
sage: Word('aababb').is_palindrome(f)
True

sage: f = WordMorphism({0:[1],1:[0]})
sage: w = words.ThueMorseWord()[:8]; w
word: 01101001
sage: w.is_palindrome(f)
True
```

The word must be in the domain of the involution:

```python
sage: f = WordMorphism('a->a')
sage: Word('aababb').is_palindrome(f)
Traceback (most recent call last):
  ...
KeyError: 'b'
```

**is_prefix**(*other*)

Return True if \( \text{self} \) is a prefix of \( \text{other} \), and False otherwise.

**EXAMPLES:**
```python
sage: w = Word('0123456789')
sage: y = Word('012345')
sage: y.is_prefix(w)
True
sage: w.is_prefix(y)
False
sage: w.is_prefix(Word())
False
sage: Word().is_prefix(w)
True
sage: Word().is_prefix(Word())
True
```

**is_primitive()**

Return True if self is primitive, and False otherwise.

A finite word \( w \) is primitive if it is not a positive integer power of a shorter word.

**EXAMPLES:**

```python
sage: Word('1231').is_primitive()
True
sage: Word('111').is_primitive()
False
```

**is_proper_prefix(other)**

Return True if self is a proper prefix of other, and False otherwise.

**EXAMPLES:**

```python
sage: Word('12').is_proper_prefix(Word('123'))
True
sage: Word('12').is_proper_prefix(Word('12'))
False
sage: Word().is_proper_prefix(Word('123'))
True
sage: Word('123').is_proper_prefix(Word('12'))
False
sage: Word().is_proper_prefix(Word())
False
```

**is_proper_suffix(other)**

Return True if self is a proper suffix of other, and False otherwise.

**EXAMPLES:**

```python
sage: Word('23').is_proper_suffix(Word('123'))
True
sage: Word('12').is_proper_suffix(Word('12'))
False
sage: Word().is_proper_suffix(Word('123'))
True
sage: Word('123').is_proper_suffix(Word('12'))
False
```
**is_quasiperiodic()**

Return True if `self` is quasiperiodic, and False otherwise.

A finite or infinite word \(w\) is *quasiperiodic* if it can be constructed by concatenations and superpositions of one of its proper factors \(u\), which is called a *quasiperiod* of \(w\). See for instance [AE1993], [Mar2004], and [GLR2008].

**EXAMPLES:**

```python
sage: Word('abaababaabaababaaba').is_quasiperiodic()
True
sage: Word('abacaba').is_quasiperiodic()
False
sage: Word('a').is_quasiperiodic()
False
sage: Word().is_quasiperiodic()
False
sage: Word('abaaba').is_quasiperiodic()
True
```

**is_rich**

Return True if `self` has defect 0, and False otherwise.

A word is *full* (or *rich*) if its defect is zero (see [BHNR2004]).

If \(f\) is given, then the \(f\)-palindromic defect is used (see [PeSt2011]).

**INPUT:**

- \(f\) – involution (default: None) on the alphabet of `self`. It must be callable on letters as well as words (e.g. `WordMorphism`).

**OUTPUT:**

boolean – If \(f\) is None, whether `self` is full; otherwise, whether `self` is full of \(f\)-palindromes.

**EXAMPLES:**

```python
sage: words.ThueMorseWord()[:100].is_full()
False
sage: words.FibonacciWord()[:100].is_full()
True
sage: Word('000000000000000').is_full()
True
sage: Word('011010011001').is_full()
False
sage: Word('2194').is_full()
True
sage: Word().is_full()
True
```
sage: w.is_full(f)
True

sage: f = WordMorphism('a->b,b->a')
sage: Word('abab').is_full(f)
True
sage: Word('abba').is_full(f)
False

A simple example of an infinite word full of f-palindromes:

sage: p = WordMorphism({0:'abc',1:'ab'})
sage: f = WordMorphism('a->b,b->a,c->c')
sage: p(words.FibonacciWord()[:50]).is_full(f)
True
sage: p(words.FibonacciWord()[:150]).is_full(f)
True

is_smooth_prefix()

Return True if self is the prefix of a smooth word, and False otherwise.

Let $A_k = \{1, \ldots, k\}, k \geq 2$. An infinite word $w$ in $A_k^\omega$ is said to be smooth if and only if for all positive integers $m$, $\Delta^m(w)$ is in $A_k^\omega$, where $\Delta(w)$ is the word obtained from $w$ by composing the length of consecutive runs of the same letter in $w$. See for instance [BL2003] and [BDLV2006].

INPUT:

- self – must be a word over the integers to get something other than False

OUTPUT:

boolean – whether self is a smooth prefix or not

EXAMPLES:

sage: W = Words([1, 2])
sage: W([1, 1, 2, 2, 1, 2, 1, 1]).is_smooth_prefix()
True
sage: W([1, 2, 1, 2, 1, 2, 1, 2, 1, 1, 2]).is_smooth_prefix()
False

is_square()

Return True if self is a square, and False otherwise.

EXAMPLES:

sage: Word([1,0,0,1]).is_square()
False
sage: Word('1212').is_square()
True
sage: Word('1213').is_square()
False
sage: Word('12123').is_square()
False
sage: Word().is_square()
True
is_square_free()

Return True if self does not contain squares, and False otherwise.

EXAMPLES:

```python
sage: Word('12312').is_square_free()
True
sage: Word('31212').is_square_free()
False
sage: Word().is_square_free()
True
```

is_sturmian_factor()

Tell whether self is a factor of a Sturmian word.

The finite word self must be defined on a two-letter alphabet.

Equivalently, tells whether self is balanced. The advantage over the is_balanced method is that this one runs in linear time whereas is_balanced runs in quadratic time.

OUTPUT:

boolean – the result

EXAMPLES:

```python
sage: w = Word('011101101101101101',alphabet='01')
sage: w.is_sturmian_factor()
True
sage: words.LowerMechanicalWord(random(),alphabet='01')[:100].is_sturmian_factor()
True
sage: words.CharacteristicSturmianWord(random())[:100].is_sturmian_factor()
True
sage: w = Word('aabb',alphabet='ab')
sage: w.is_sturmian_factor()
False
sage: s1 = WordMorphism('a->ab,b->b')
sage: s2 = WordMorphism('a->ba,b->b')
sage: s3 = WordMorphism('a->a,b->ba')
sage: s4 = WordMorphism('a->a,b->ab')
sage: W = Words('ab')
sage: w = W('ab')
sage: for i in range(8): w = choice([s1,s2,s3,s4])(w)
sage: w.is_sturmian_factor()
True
```

Famous words:

```python
sage: words.FibonacciWord()[:100].is_sturmian_factor()
True
sage: words.ThueMorseWord()[:1000].is_sturmian_factor()
False
```
sage: words.KolakoskiWord()[1000].is_sturmian_factor()
False

See [Arn2002], [Ser1985], and [SU2009].

AUTHOR:
• Thierry Monteil

is_subword_of(other)

Return True if self is a subword of other, and False otherwise.

A finite word u is a subword of a finite word v if u is a subsequence of v. See Chapter 6 on Subwords in [Lot1997].

Some references define subword as a consecutive subsequence. Use is_factor() if this is what you need.

INPUT:
other – a finite word

EXAMPLES:

sage: Word('bb').is_subword_of(Word('ababa'))
True
sage: Word('bbb').is_subword_of(Word('ababa'))
False

sage: Word().is_subword_of(Word('123'))
True
sage: Word('123').is_subword_of(Word('32113321323321'))
True
sage: Word('321').is_subword_of(Word('111222121122133111222332'))
False

See also:

longest_common_subword() number_of_subword_occurrences() is_factor()

is_suffix(other)

Return True if self is a suffix of other, and False otherwise.

EXAMPLES:

sage: w = Word('0123456789')
sage: y = Word('56789')
sage: y.is_suffix(w)
True
sage: w.is_suffix(y)
False
sage: Word('579').is_suffix(w)
False
sage: Word().is_suffix(y)
True
sage: w.is_suffix(Word())
False
sage: Word().is_suffix(Word())
True
is_symmetric\( (f=None) \)

Return True if self is symmetric (or \( f \)-symmetric), and False otherwise.

A word is symmetric (resp. \( f \)-symmetric) if it is the product of two palindromes (resp. \( f \)-palindromes). See [BHNR2004] and [DeLuca2006].

INPUT:

- \( f \) – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

EXAMPLES:

```
sage: Word('ababab').is_symmetric()
True
sage: Word('ababa').is_symmetric()
True
sage: Word('aababaabba').is_symmetric()
False
sage: Word('aabbaababba').is_symmetric()
False
sage: f = WordMorphism('a->b,b->a')
sage: Word('aabbaababba').is_symmetric(f)
True
```

is_tangent()

Tell whether self is a tangent word.

The finite word self must be defined on a two-letter alphabet.

A binary word is said to be tangent if it can appear in infinitely many cutting sequences of a smooth curve, where each cutting sequence is observed on a progressively smaller grid.

This class of words strictly contains the class of 1-balanced words, and is strictly contained in the class of 2-balanced words.

This method runs in linear time.

OUTPUT:

boolean – the result

EXAMPLES:

```
sage: w = Word('0110110110111011101',alphabet='01')
sage: w.is_tangent()
True
```

Some tangent words may not be balanced:

```
sage: Word('aabb',alphabet='ab').is_balanced()
False
sage: Word('aabb',alphabet='ab').is_tangent()
True
```

Some 2-balanced words may not be tangent:

```
sage: Word('aaabb',alphabet='ab').is_tangent()
False
```
sage: Word('aaabb',alphabet='ab').is_balanced(2)
True

Famous words:

sage: words.FibonacciWord()[:100].is_tangent()
True
sage: words.ThueMorseWord()[:1000].is_tangent()
True
sage: words.KolakoskiWord()[:1000].is_tangent()
False

See [Mon2010].

AUTHOR:
• Thierry Monteil

is_yamanouchi(n=None)

Return whether self is Yamanouchi.

A word $w$ is Yamanouchi if, when read from right to left, it always has weakly more $i$'s than $i+1$'s for all $i$ that appear in $w$.

INPUT:
• n – (optional) an integer specifying the maximal letter in the alphabet

EXAMPLES:

sage: w = Word([1,2,4,3,2,2,2])
sage: w.is_yamanouchi()
False
sage: w = Word([2,3,4,3,1,2,1,1,2,1])
sage: w.is_yamanouchi()
True
sage: w = Word([3,1])
sage: w.is_yamanouchi(n=3)
False
sage: w.is_yamanouchi()
True
sage: w = Word([3,1],alphabet=[1,2,3])
sage: w.is_yamanouchi()
False
sage: w = Word([2,1,1,2])
sage: w.is_yamanouchi()
False

iterated_left_palindromic_closure(f=None)

Return the iterated left ($f$-)palindromic closure of self.

INPUT:
• $f$ – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPUT:
word – the left iterated \( f \)-palindromic closure of \( \text{self} \).

**EXAMPLES:**

```
sage: Word('123').iterated_left_palindromic_closure()
word: 3231323
sage: f = WordMorphism('a->b,b->a')
sage: Word('ab').iterated_left_palindromic_closure(f=f)
word: abbaab
sage: Word('aab').iterated_left_palindromic_closure(f=f)
word: abbaabbaab
```

**lacunas** \((f=None)\)

Return the list of all the lacunas of \( \text{self} \).

A *lacuna* is a position in a word where the longest \((f-)\)palindromic suffix is not unioccurrent (see [BMBL2008]).

**INPUT:**

* \( f \) – involution (default: None) on the alphabet of \( \text{self} \). It must be callable on letters as well as words (e.g. WordMorphism). The default value corresponds to usual palindromes, i.e., \( f \) equal to the identity.

**OUTPUT:**

a list – list of all the lacunas of \( \text{self} \)

**EXAMPLES:**

```
sage: w = Word([0,1,1,2,3,4,5,1,13,3])
sage: w.lacunas()
[7, 9]
sage: words.ThueMorseWord()[:100].lacunas()
[8, 9, 24, 25, 32, 33, 34, 35, 36, 37, 38, 39, 96, 97, 98, 99]
sage: f = WordMorphism({0:[1],1:[0]})
sage: words.ThueMorseWord()[:50].lacunas(f)
[0, 2, 4, 12, 16, 17, 18, 19, 48, 49]
```

**last_position_dict()**

Return a dictionary that contains the last position of each letter in \( \text{self} \).

**EXAMPLES:**

```
sage: Word('1231232').last_position_dict()
{1: 3, 2: 6, 3: 5}
```

**left_special_factors** \((n=None)\)

Return the left special factors (of length \( n \)).

A factor \( u \) of a word \( w \) is **left special** if there are two distinct letters \( a \) and \( b \) such that \( au \) and \( bu \) are factors of \( w \).

**INPUT:**

* \( n \) – integer (optional, default: None). If None, it returns all left special factors.

**OUTPUT:**

a list of words

**EXAMPLES:**
\begin{Verbatim}
sage: alpha, beta, x = 0.54, 0.294, 0.1415
sage: w = words.CodingOfRotationWord(alpha, beta, x)[:40]
sage: for i in range(5):
....:     print("{} {}").format(i, sorted(w.left_special_factors(i)))
\end{Verbatim}

**left_special_factors_iterator**\(n=None\)

Return an iterator over the left special factors (of length n).

A factor \(u\) of a word \(w\) is left special if there are two distinct letters \(a\) and \(b\) such that \(au\) and \(bu\) are factors of \(w\).

**INPUT:**

* n – integer (optional, default: None). If None, it returns an iterator over all left special factors.

**EXAMPLES:**

\begin{Verbatim}
sage: alpha, beta, x = 0.54, 0.294, 0.1415
sage: w = words.CodingOfRotationWord(alpha, beta, x)[:40]
sage: sorted(w.left_special_factors_iterator(3))
[\text{word: 000, word: 01}]
sage: sorted(w.left_special_factors_iterator(4))
[\text{word: 0000, word: 0101}]
sage: sorted(w.left_special_factors_iterator(5))
[\text{word: 00000, word: 01010}]
\end{Verbatim}

**length**

Return the length of self.

**length_border**

Return the length of the border of self.

The border of a word is the longest word that is both a proper prefix and a proper suffix of self.

**EXAMPLES:**

\begin{Verbatim}
sage: Word('121').length_border()
1
sage: Word('1').length_border()
0
sage: Word('1212').length_border()
2
sage: Word('111').length_border()
2
sage: Word().length_border() is None
True
\end{Verbatim}

**length_maximal_palindrome**\(j, m=None, f=None\)

Return the length of the longest palindrome centered at position j.

**INPUT:**
• **j** – rational, position of the symmetry axis of the palindrome. Must return an integer when doubled. It is an integer when the center of the palindrome is a letter.

• **m** – integer (default: None), minimal length of palindrome, if known. The parity of m can’t be the same as the parity of 2j.

• **f** – involution (default: None), on the alphabet. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPut:

length of the longest \( f \)-palindrome centered at position \( j \)

EXAMPLES:

```
sage: Word('01001010').length_maximal_palindrome(3/2)
0
sage: Word('01101001').length_maximal_palindrome(3/2)
4
sage: Word('01010').length_maximal_palindrome(j=3, f='0->1,1->0')
0
sage: Word('01010').length_maximal_palindrome(j=2.5, f='0->1,1->0')
4
sage: Word('0222220').length_maximal_palindrome(3, f='0->1,1->0,2->2')
5
```

```
sage: w = Word('abcdcbaxyzyyx')
sage: w.length_maximal_palindrome(3)
7
sage: w.length_maximal_palindrome(3, 3)
7
sage: w.length_maximal_palindrome(3.5)
0
sage: w.length_maximal_palindrome(9.5)
6
sage: w.length_maximal_palindrome(9.5, 2)
6
```

**lengths_maximal_palindromes** \( f=None \)

Return the length of maximal palindromes centered at each position.

INPUT:

• **f** – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPut:

a list – The length of the maximal palindrome (or \( f \)-palindrome) with a given symmetry axis (letter or space between two letters).

EXAMPLES:

```
sage: Word('01010').lengths_maximal_palindromes()
[0, 1, 3, 0, 3, 0, 3, 1, 4, 1, 0, 1, 0]
sage: Word('00000').lengths_maximal_palindromes()
[0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0]
sage: Word('a').lengths_maximal_palindromes()
(continues on next page)
```

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[0, 1, 0]  
sage: Word('').lengths_maximal_palindromes()  
[0]  
sage: Word().lengths_maximal_palindromes()  
[0]  
sage: f = WordMorphism('a->b,b->a')  
sage: Word('abbabaab').lengths_maximal_palindromes(f)  
[0, 0, 2, 0, 0, 0, 2, 0, 8, 0, 2, 0, 0, 0, 2, 0, 0]  

lengths_unioccurrent_lps(f=None)

Return the list of the lengths of the unioccurrent longest (f)-palindromic suffixes (lps) for each non-empty prefix of self. No unioccurrent lps are indicated by None.

It corresponds to the function $H_w$ defined in [BMBL2008] and [BMBFLR2008].

INPUT:

• f – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism). The default value corresponds to usual palindromes, i.e., f equal to the identity.

OUTPUT:

a list – list of the length of the unioccurrent longest palindromic suffix (lps) for each non-empty prefix of self. No unioccurrent lps are indicated by None.

EXAMPLES:

sage: w = Word([0,1,1,2,3,4,5,1,13,3])  
sage: w.lengths_unioccurrent_lps()  
[1, 1, 2, 1, 1, 1, None, 1, None]  
sage: f = words.FibonacciWord()[:20]  
sage: f.lengths_unioccurrent_lps() == f.lps_lengths()[1:]  
True  
sage: t = words.ThueMorseWord()  
sage: t[:20].lengths_unioccurrent_lps()  
[1, 1, 2, 4, 3, 3, 2, 4, None, None, 6, 8, 10, 12, 14, 16, 6, 8, 10, 12]  
sage: f = WordMorphism({1:[0],0:[1]})  
sage: t[:15].lengths_unioccurrent_lps(f)  
[None, 2, None, 2, None, 4, 6, 8, 4, 6, 4, 6, None, 4, 6]

letters()

Return the list of letters that appear in this word, listed in the order of first appearance.

EXAMPLES:

sage: Word([0,1,1,0,1,0,0,1]).letters()  
[0, 1]  
sage: Word("cacao").letters()  
['c', 'a', 'o']

longest_backward_extension(x, y)

Compute the length of the longest factor of self that ends at x and that matches a factor that ends at y.

INPUT:

• x, y – positions in self
EXAMPLES:

```python
sage: w = Word('0011001')
sage: w.longest_backward_extension(6, 2)
3
sage: w.longest_backward_extension(1, 4)
1
sage: w.longest_backward_extension(1, 3)
0
```

The method also accepts negative positions indicating the distance from the end of the word (in order to be consistent with how negative indices work with lists). For instance, for a word of length 7, using positions 6 and −5 is the same as using positions 6 and 2:

```python
sage: w.longest_backward_extension(6, -5)
3
sage: w.longest_backward_extension(-6, 4)
1
```

**longest_common_subword(other)**

Return a longest subword of `self` and `other`.

A subword of a word is a subset of the word’s letters, read in the order in which they appear in the word.

For more information, see Wikipedia article Longest_common_subsequence_problem.

INPUT:

- `other` – a word

ALGORITHM:

For any indices `i, j`, we compute the longest common subword `lcs[i,j]` of `self[:i]` and `other[:j]`. This can be easily obtained as the longest of

- `lcs[i-1,j]`
- `lcs[i,j-1]`
- `lcs[i-1,j-1]+self[i] if self[i]==other[j]`

EXAMPLES:

```python
sage: v1 = Word("abc")
sage: v2 = Word("ace")
sage: v1.longest_common_subword(v2)
word: ac
sage: w1 = Word("1010101010101010101010101010101010101010")

sage: w2 = Word("0011001100110011001100110011001100110011")
sage: w1.longest_common_subword(w2)
word: 00110011001100110011010101010
```

See also:

- `is_subword_of()`

**longest_common_suffix(other)**

Return the longest common suffix of `self` and `other`.

EXAMPLES:
longest_forward_extension(x, y)

Compute the length of the longest factor of self that starts at x and that matches a factor that starts at y.

INPUT:

* x, y – positions in self

EXAMPLES:

```python
sage: w = Word('0011001')
sage: w.longest_forward_extension(0, 4)
3
sage: w.longest_forward_extension(0, 2)
0
```

The method also accepts negative positions indicating the distance from the end of the word (in order to be consistent with how negative indices work with lists). For instance, for a word of length 7, using positions −3 and 2 is the same as using positions 4 and 2:

```python
sage: w.longest_forward_extension(1, -2)
2
sage: w.longest_forward_extension(4, -3)
3
```

lps(f=None, l=None)

Return the longest palindromic (or f-palindromic) suffix of self.

INPUT:

* f – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).
* l – integer (default: None) the length of the longest palindrome suffix of `self[:-1]`, if known.

OUTPUT:

word – If f is None, the longest palindromic suffix of self; otherwise, the longest f-palindromic suffix of self.

EXAMPLES:

```python
sage: Word('0111').lps()
word: 111
```
sage: Word('011101').lps()
word: 101
sage: Word('6667').lps()
word: 7
sage: Word('abbabaab').lps()
word: baab
sage: Word().lps()
word:
sage: f = WordMorphism('a->b,b->a')
sage: Word('abbabaab').lps(f=f)
word: abbabaab
sage: w = Word('33412321')
sage: w.lps(l=3)
word: 12321
sage: Y = Word
sage: w = Y('01101001')
sage: w.lps(l=2)
word: 1001
sage: w.lps()
word: 1001
sage: w.lps(l=None)
word: 1001
sage: Y().lps(l=2)
Traceback (most recent call last):
... IndexError: list index out of range
sage: v = Word('abbabaab')
sage: pal = v[:0]
sage: for i in range(1, v.length()+1):
......: pal = v[:i].lps(l=pal.length())
......: pal
word: a
word: b
word: bb
word: abba
word: bab
word: aba
word: aa
word: baab
sage: f = WordMorphism('a->b,b->a')
sage: v = Word('abbabaab')
sage: pal = v[:0]
sage: for i in range(1, v.length()+1):
......: pal = v[:i].lps(f=f, l=pal.length())
......: pal
word: ab
word: ba
word: ab
word: baba
word: bbabaa

(continues on next page)
lps_lengths(f=None)

Return the length of the longest palindromic suffix of each prefix.

INPUT:

• f – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPUT:

a list – The length of the longest palindromic (or f-palindromic) suffix of each prefix of self.

EXAMPLES:

sage: Word('01101001').lps_lengths()
[0, 1, 1, 2, 4, 3, 3, 2, 4]
sage: Word('00000').lps_lengths()
[0, 1, 2, 3, 4, 5]
sage: Word('0').lps_lengths()
[0, 1]
sage: Word('').lps_lengths()
[0]
sage: Word().lps_lengths()
[0]
sage: f = WordMorphism('a->b,b->a')
sage: Word('abbabaab').lps_lengths(f)
[0, 0, 2, 0, 2, 2, 4, 6, 8]

lyndon_factorization()

Return the Lyndon factorization of self.

The Lyndon factorization of a finite word $w$ is the unique factorization of $w$ as a non-increasing product of Lyndon words, i.e., $w = l_1 \cdots l_n$ where each $l_i$ is a Lyndon word and $l_1 \geq \cdots \geq l_n$. See for instance [Duv1983].

OUTPUT:

the list $[l_1, \ldots, l_n]$ of factors obtained

EXAMPLES:

sage: Word('010010010001000').lyndon_factorization()
(01, 001, 001, 001, 0, 0, 0)
sage: Words('010010010001000').lyndon_factorization()
(0, 10010010001000)
sage: Word('abbababaabbaab').lyndon_factorization()
(abb, ababb, aababb, a)
sage: Words('ba')('abbababaabbaab').lyndon_factorization()
(a, babababaaba, bba)
sage: Word([1,2,1,3,1,2,1]).lyndon_factorization()
(1213, 12, 1)

major_index(final_descent=False)

Return the major index of self.
The major index of a word $w$ is the sum of the descents of $w$.

With the `final_descent` option, the last position of a non-empty word is also considered as a descent.

See also:

*major index on Permutations.*

**EXAMPLES:**

```
sage: w = Word([2,1,3,3,2])
sage: w.major_index()
5
sage: w = Word([2,1,3,3,2])
sage: w.major_index(final_descent=True)
10
```

**minimal_conjugate()**

Return the lexicographically minimal conjugate of this word (see Wikipedia article Lexicographically_minimal_string_rotation).

**EXAMPLES:**

```
sage: Word('213').minimal_conjugate()
word: 132
sage: Word('11').minimal_conjugate()
word: 11
sage: Word('12112').minimal_conjugate()
word: 11212
sage: Word('211').minimal_conjugate()
word: 112
sage: Word('2112112111').minimal_conjugate()
word: 112112112
```

**minimal_period()**

Return the period of self.

Let $A$ be an alphabet. An integer $p \geq 1$ is a period of a word $w = a_1a_2\cdots a_n$ where $a_i \in A$ if $a_i = a_{i+p}$ for $i = 1, \ldots, n - p$. The smallest period of $w$ is called the period of $w$. See Chapter 1 of [Lot2002].

**EXAMPLES:**

```
sage: Word('aba').minimal_period()
2
sage: Word('abab').minimal_period()
2
sage: Word('ababa').minimal_period()
2
sage: Word('ababaa').minimal_period()
2
sage: Word('ababac').minimal_period()
5
sage: Word('aaabaa').minimal_period()
6
sage: Word('aaaaaa').minimal_period()
1
sage: Word('a').minimal_period()
1
```

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### nb_factor_occurrences_in(other)

Return the number of times `self` appears as a factor in `other`.

**Warning:** This method is deprecated since 2020 and will be removed in a later version of SageMath. Use `number_of_factor_occurrences()` instead.

**EXAMPLES:**

```python
sage: Word('123').nb_factor_occurrences_in(Word('11233212313112332121123'))
doctest:warning...
DeprecationWarning: f.nb_factor_occurrences_in(w) is deprecated.
Use w.number_of_factor_occurrences(f) instead.
4
sage: Word('321').nb_factor_occurrences_in(Word('1123321231311233221123'))
0
```

An error is raised for the empty word:

```python
sage: Word('').nb_factor_occurrences_in(Word('123'))
Traceback (most recent call last):
...
NotImplementedError: The factor must be non empty
```

### nb_subword_occurrences_in(other)

Return the number of times `self` appears in `other` as a subword.

This corresponds to the notion of binomial coefficients of two finite words whose properties are presented in the chapter of Lothaire’s book written by Sakarovitch and Simon [Lot1997].

**Warning:** This method is deprecated since 2020 and will be removed in a later version of SageMath. Use `number_of_subword_occurrences()` instead.

**INPUT:**

- `other` – finite word

**EXAMPLES:**

```python
sage: tm = words.ThueMorseWord()
sage: u = Word([0,1,0,1])
sage: u.nb_subword_occurrences_in(tm[:1000])
doctest:warning...
DeprecationWarning: f.nb_subword_occurrences_in(w) is deprecated.
Use w.number_of_subword_occurrences(f) instead.
```
See https://github.com/sagemath/sage/issues/30187 for details.

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```python
sage: u = Word([0,1,0,1,1,0])
sage: u.nb_subword_occurrences_in(tm[:100])
20370432
```

**Note:** This code, based on [MSSY2001], actually computes the number of occurrences of all prefixes of `self` as subwords in all prefixes of `other`. In particular, its complexity is bounded by \( \text{len}(self) \times \text{len}(other) \).

---

**number_of_factor_occurrences(other)**

Return the number of times `other` appears as a factor in `self`.

**INPUT:**

- `other` – a non empty word

**EXAMPLES:**

```python
sage: w = Word('112332312313112332121123')
sage: w.number_of_factor_occurrences(Word('123'))
4
sage: w = Word('11233231231311233221123')
sage: w.number_of_factor_occurrences(Word('321'))
0
```

```python
sage: Word().number_of_factor_occurrences(Word('123'))
0
```

An error is raised for the empty word:

```python
sage: Word('123').number_of_factor_occurrences(Word())
Traceback (most recent call last):
  ...
NotImplementedError: The factor must be non empty
```

---

**number_of_factors(n=None, algorithm='suffix tree')**

Count the number of distinct factors of `self`.

**INPUT:**

- `n` – an integer, or `None`.
- `algorithm` – string (default: 'suffix tree'), takes the following values:
  - 'suffix tree' – construct and use the suffix tree of the word
  - 'naive' – algorithm uses a sliding window

**OUTPUT:**

If `n` is an integer, returns the number of distinct factors of length `n`. If `n` is `None`, returns the total number of distinct factors.

**EXAMPLES:**
sage: w = Word([1,2,1,2,3])
sage: w.number_of_factors()
13
sage: [w.number_of_factors(i) for i in range(6)]
[1, 3, 3, 2, 1]

sage: w = words.ThueMorseWord()[:100]
sage: [w.number_of_factors(i) for i in range(10)]
[1, 2, 4, 6, 10, 12, 16, 20, 22, 24]

sage: Word('1213121').number_of_factors()
22
sage: Word('1213121').number_of_factors(1)
3

sage: Word('a'*100).number_of_factors()
101
sage: Word('a'*100).number_of_factors(77)
1

sage: Word().number_of_factors()
1
sage: Word().number_of_factors(17)
0

sage: blueberry = Word("blueberry")
sage: blueberry.number_of_factors()
43
sage: [blueberry.number_of_factors(i) for i in range(10)]
[1, 6, 8, 7, 6, 5, 4, 3, 2, 1]


number_of_inversions()

Return the number of inversions in self.

An inversion of a word \(w = w_1 \ldots w_n\) is a pair of indices \((i, j)\) with \(i < j\) and \(w_i > w_j\).

See also:
number of inversions on Permutations.

EXAMPLES:

sage: w = Word([2,1,3,3,2])
sage: w.number_of_inversions()
3

number_of_left_special_factors(n)

Return the number of left special factors of length \(n\).

A factor \(u\) of a word \(w\) is left special if there are two distinct letters \(a\) and \(b\) such that \(au\) and \(bu\) are factors of \(w\).

INPUT:

* \(n\) – integer
OUTPUT:

a non-negative integer

EXAMPLES:

```python
sage: w = words.FibonacciWord()[:100]
sage: [w.number_of_left_special_factors(i) for i in range(10)]
[1, 1, 1, 1, 1, 1, 1, 1, 1]
```

```python
sage: w = words.ThueMorseWord()[:100]
sage: [w.number_of_left_special_factors(i) for i in range(10)]
[1, 2, 2, 4, 2, 4, 4, 2, 2, 4]
```

**number_of_letter_occurrences**(letter)

Return the number of occurrences of letter in self.

INPUT:

• letter - a letter

OUTPUT:

• integer

EXAMPLES:

```python
sage: w = Word('abbabaab')
sage: w.number_of_letter_occurrences('a')
4
sage: w.number_of_letter_occurrences('ab')
0
```

This method is equivalent to `list(w).count(letter)` and `tuple(w).count(letter)`, thus `count` is an alias for the method `number_of_letter_occurrences`:

```python
sage: list(w).count('a')
4
sage: w.count('a')
4
```

But notice that if s and w are strings, `Word(s).count(w)` counts the number occurrences of w as a letter in Word(s) which is not the same as `s.count(w)` which counts the number of occurrences of the string w inside s:

```python
sage: s = 'abbabaab'
sage: s.count('ab')
3
sage: Word(s).count('ab')
0
```

See also:

`sage.combinat.words.finite_word.FiniteWord_class.number_of_factor_occurrences()`

**number_of_right_special_factors**(n)

Return the number of right special factors of length n.
A factor $u$ of a word $w$ is right special if there are two distinct letters $a$ and $b$ such that $ua$ and $ub$ are factors of $w$.

INPUT:
- $n$ – integer

OUTPUT:
a non-negative integer

EXAMPLES:

```python
sage: w = words.FibonacciWord()[:100]
sage: [w.number_of_right_special_factors(i) for i in range(10)]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

```python
sage: w = words.ThueMorseWord()[:100]
sage: [w.number_of_right_special_factors(i) for i in range(10)]
[1, 2, 2, 4, 2, 4, 4, 2, 2, 4]
```

`number_of_subword_occurrences(other)`

Return the number of times `other` appears in `self` as a subword.

This corresponds to the notion of `binomial coefficient` of two finite words whose properties are presented in the chapter of Lothaire’s book written by Sakarovitch and Simon [Lot1997].

INPUT:
- `other` – finite word

EXAMPLES:

```python
sage: tm = words.ThueMorseWord()
sage: u = Word([0,1,0,1])
sage: tm[:1000].number_of_subword_occurrences(u)
2604124996
```

```python
sage: u = Word([0,1,0,1,1,0])
sage: tm[:100].number_of_subword_occurrences(u)
20370432
```

Note: This code, based on [MSSY2001], actually compute the number of occurrences of all prefixes of `self` as subwords in all prefixes of `other`. In particular, its complexity is bounded by $\text{len}(\text{self}) \ast \text{len}(\text{other})$.

`order()`

Return the order of `self`.

Let $p(w)$ be the period of a word $w$. The positive rational number $|w|/p(w)$ is the order of $w$. See Chapter 8 of [Lot2002].

OUTPUT:
rational – the order

EXAMPLES:
sage: Word('abaaba').order()
2
sage: Word('ababaaba').order()
8/5
sage: Word('a').order()
1
sage: Word('aa').order()
2
sage: Word().order()
0

overlap_partition(other, delay=0, p=None, involution=None)

Return the partition of the alphabet induced by the overlap of self and other with the given delay.

The partition of the alphabet is given by the equivalence relation obtained from the symmetric, reflexive and transitive closure of the set of pairs of letters \( R_{u,v,d} = \{(u_k, v_{k-d}) : 0 \leq k < n, 0 \leq k - d < m\} \) where \( u = u_0u_1 \cdots u_{n-1}, v = v_0v_1 \cdots v_{m-1} \) are two words on the alphabet \( A \) and \( d \) is an integer.

The equivalence relation defined by \( R \) is inspired from [Lab2008].

INPUT:

- other – word on the same alphabet as self
- delay – integer (default: 0)
- p – disjoint sets data structure (optional, default: None), a partition of the alphabet into disjoint sets to start with. If None, each letter start in distinct equivalence classes.
- involution – callable (optional, default: None), an involution on the alphabet. If involution is not None, the relation \( R_{u,v,d} \cup R_{\text{involution}(u),\text{involution}(v),d} \) is considered.

OUTPUT:

a disjoint set data structure

EXAMPLES:

sage: W = Words(list('abc012345'))
sage: u = W('abc')
sage: v = W('01234')
sage: u.overlap_partition(v)
{{'0'}, {'a'}}, {{'1'}, {'b'}}, {{'2'}, {'c'}}, {{'3'}}, {{'4'}}, {{'5'}}
sage: u.overlap_partition(v, 2)
{{'0'}, {'c'}}, {{'1'}}, {{'2'}}, {{'3'}}, {{'4'}}, {{'5'}}, {'a'}, {'b'}
sage: u.overlap_partition(v, -1)
{{'0'}}, {'1'}, {'a'}, {'2'}, {'b'}, {'3'}, {'c'}, {'4'}, {'5'}}

You can re-use the same disjoint set and do more than one overlap:

sage: p = u.overlap_partition(v, 2)
sage: p
{{'0'}, {'c'}}, {{'1'}}, {{'2'}}, {{'3'}}, {{'4'}}, {{'5'}}, {'a'}, {'b'}

sage: u.overlap_partition(v, 1, p)
{{'0'}, {'1'}, {'b'}, {'c'}}, {{'2'}}, {{'3'}}, {{'4'}}, {{'5'}}, {'a'}

The function overlap_partition can be used to study equations on words. For example, if a word \( w \) overlaps itself with delay \( d \), then \( d \) is a period of \( w \):
sage: W = Words(range(20))
sage: w = W(range(14)); w
word: 0,1,2,3,4,5,6,7,8,9,10,11,12,13
sage: d = 5
sage: p = w.overlap_partition(w, d)
sage: m = WordMorphism(p.element_to_root_dict())
sage: w2 = m(w); w2
word: 56789567895678
sage: w2.minimal_period() == d
True

If a word is equal to its reversal, then it is a palindrome:

sage: W = Words(range(20))
sage: w = W(range(17)); w
word: 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16
sage: p = w.overlap_partition(w.reversal(), 0)
sage: m = WordMorphism(p.element_to_root_dict())
sage: w2 = m(w); w2
word: 01234567876543210
sage: w2.parent()
Finite words over {0, 1, 2, 3, 4, 5, 6, 7, 8, 17, 18, 19}
sage: w2.is_palindrome()
True

If the reversal of a word $w$ is factor of its square $w^2$, then $w$ is symmetric, i.e. the product of two palindromes:

sage: W = Words(range(10))
sage: w = W(range(10)); w
word: 0123456789
sage: p = (w*w).overlap_partition(w.reversal(), 4)
sage: m = WordMorphism(p.element_to_root_dict())
sage: w2 = m(w); w2
word: 0110456654
sage: w2.is_symmetric()
True

If the image of the reversal of a word $w$ under an involution $f$ is factor of its square $w^2$, then $w$ is $f$-symmetric:

sage: W = Words([-11,-9,...,11])
sage: w = W([1,3,...,11])
sage: w
word: 1,3,5,7,9,11
sage: inv = lambda x:-x
sage: f = WordMorphism(dict( (a, inv(a)) for a in W.alphabet()))
sage: p = (w*w).overlap_partition(f(w).reversal(), 2, involution=f)
sage: m = WordMorphism(p.element_to_root_dict())
sage: m(w)
word: 1,-1,5,7,-7,-5
sage: m(w).is_symmetric(f)
True
palindrome_prefixes()
Return a list of all palindrome prefixes of self.

OUTPUT:
a list – A list of all palindrome prefixes of self.

EXAMPLES:

```
sage: w = Word('abaaba')
sage: w.palindrome_prefixes()  
[word: , word: a, word: aba, word: abaaba]
sage: w = Word('abbbbbbbbb')
sage: w.palindrome_prefixes()  
[word: , word: a]
```

palindromes(f=None)
Return the set of all palindromic (or \(f\)-palindromic) factors of self.

INPUT:

* \(f\) – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPUT:
a set – If \(f\) is None, the set of all palindromic factors of self; otherwise, the set of all \(f\)-palindromic factors of self.

EXAMPLES:

```
sage: sorted(Word('01101001').palindromes())
[word: , word: 0, word: 00, word: 010, word: 0110, word: 1, word: 1001, word: ... ˓→101, word: 11]
sage: sorted(Word('00000').palindromes())
[word: , word: 0, word: 00, word: 000, word: 0000, word: 00000]
sage: sorted(Word('0').palindromes())
[word: , word: 0]
sage: sorted(Word('').palindromes())
[word: ]
sage: sorted(Word('').palindromes(f))
[word: ]
sage: f = WordMorphism('a->b,b->a')
sage: sorted(Word('abbabaab').palindromes(f))
[word: , word: ab, word: abbabaab, word: ba, word: baba, word: bbabaa]
```

palindromic_closure(side='right', f=None)
Return the shortest palindrome having self as a prefix (or as a suffix if side is 'left').

See [DeLuca2006].

INPUT:

* side – 'right' or 'left' (default: 'right') the direction of the closure
* \(f\) – involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

OUTPUT:
a word – If \( f \) is None, the right palindromic closure of \( \text{self} \); otherwise, the right \( f \)-palindromic closure of \( \text{self} \). If \( \text{side} \) is 'left', the left palindromic closure.

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: Word('1233').palindromic_closure()</code></td>
<td>word: 123321</td>
</tr>
<tr>
<td><code>sage: Word('12332').palindromic_closure()</code></td>
<td>word: 123321</td>
</tr>
<tr>
<td><code>sage: Word('0110343').palindromic_closure()</code></td>
<td>word: 01103430110</td>
</tr>
<tr>
<td><code>sage: Word('0110343').palindromic_closure(side='left')</code></td>
<td>word: 3430110343</td>
</tr>
<tr>
<td><code>sage: Word('01105678').palindromic_closure(side='left')</code></td>
<td>word: 876501105678</td>
</tr>
<tr>
<td><code>sage: w = Word('abbaba')</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: w.palindromic_closure()</code></td>
<td>word: abbababba</td>
</tr>
<tr>
<td><code>sage: f = WordMorphism('a-&gt;b,b-&gt;a')</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: w.palindromic_closure(f=f)</code></td>
<td>word: abbababab</td>
</tr>
<tr>
<td><code>sage: w.palindromic_closure(f=f, side='left')</code></td>
<td>word: babaabbaba</td>
</tr>
</tbody>
</table>

**palindromic_complexity\((n)\)**

Return the number of distinct palindromic factors of length \( n \) of \( \text{self} \).

**INPUT:**

- \( n \) – the length of the factors.

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: w = words.FibonacciWord()[:100]</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: [w.palindromic_complexity(i) for i in range(20)]</code></td>
<td>[1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]</td>
</tr>
<tr>
<td><code>sage: w = words.ThueMorseWord()[:1000]</code></td>
<td></td>
</tr>
<tr>
<td><code>sage: [w.palindromic_complexity(i) for i in range(20)]</code></td>
<td>[1, 2, 2, 2, 2, 0, 4, 0, 4, 0, 4, 0, 4, 0, 2, 0, 2, 0, 4, 0]</td>
</tr>
</tbody>
</table>

**palindromic_lacunas_study\((f=None)\)**

Return interesting statistics about longest \((f-)\)palindromic suffixes and lacunas of \( \text{self} \) (see [BMBL2008] and [BMBFLR2008]).

Note that a word \( w \) has at most \(|w| + 1 \) different palindromic factors (see [DJP2001]). For \( f \)-palindromes (or pseudopalindromes or theta-palindromes), the maximum number of \( f \)-palindromic factors is \(|w| + 1 - g_f(w)\), where \( g_f(w) \) is the number of pairs \( \{a, f(a)\} \) such that \( a \) is a letter, \( a \) is not equal to \( f(a) \), and \( a \) or \( f(a) \) occurs in \( w \), see [Star2011].

**INPUT:**

- \( f \) – involution (default: None) on the alphabet of \( \text{self} \). It must be callable on letters as well as words (e.g. \( \text{WordMorphism} \)). The default value corresponds to usual palindromes, i.e., \( f \) equal to the identity.

**OUTPUT:**
• list – list of the length of the longest palindromic suffix (lps) for each non-empty prefix of self

• list – list of all the lacunas, i.e. positions where there is no unioccurrent lps

• set – set of palindromic factors of self

EXAMPLES:

```
sage: a,b,c = Word('abbabaabbaab').palindromic_lacunas_study()
sage: a
[1, 1, 2, 4, 3, 3, 2, 4, 2, 4, 6, 8]
sage: b
[8, 9]
sage: c               # random order
set([word: , word: b, word: bab, word: abba, word: bb, word: aa, word: baabbaab,
     word: baab, word: aba, word: aabbaa, word: a])

sage: f = WordMorphism('a->b,b->a')
sage: a,b,c = Word('abbabaab').palindromic_lacunas_study(f=f)
sage: a
[0, 2, 0, 2, 2, 4, 6, 8]
sage: b
[0, 2, 4]
sage: c               # random order
set([word: , word: ba, word: baba, word: ab, word: bbabaa, word: abbabaab])
sage: c == set([Word(), Word('ba'), Word('baba'), Word('ab'), Word('bbabaa'),
              Word('abbabaab'))]
True
```

periods(divide_length=False)

Return a list containing the periods of self between 1 and \( n - 1 \), where \( n \) is the length of self.

INPUT:

• divide_length – boolean (default: False). When set to True, then only periods that divide the length of self are considered.

OUTPUT:

a list of positive integers

EXAMPLES:

```
sage: w = Word('ababab')
sage: w.periods()
[2, 4]
sage: w.periods(divide_length=True)
[2]
sage: w = Word('ababa')
sage: w.periods()
[2, 4]
sage: w.periods(divide_length=True)
[]
```

phi()

Apply the phi function to self and return the result. This is the word obtained by taking the first letter of the words obtained by iterating delta on self.
OUTPUT:

a word – the result of the phi function

EXAMPLES:

```python
sage: W = Words([1, 2])
sage: W([2, 2, 1, 1, 2, 2, 1, 2, 2, 1, 1, 2]).phi()
word: 222222
sage: W([2, 1, 2, 2, 1, 2, 2, 1, 2, 1]).phi()
word: 212113
sage: W().phi()
word:
sage: Word([2, 1, 2, 2, 1, 2, 2, 1, 1, 2]).phi()
word: 212113
sage: Word([2, 3, 1, 1, 2, 3, 2, 3, 1, 2]).phi()
word: 21215
sage: Word("aabbabaabaabba").phi()
word: a22222
sage: w = Word([2, 3, 1, 1, 2, 1, 2, 3, 1, 2, 2, 3, 1, 2])
```

See [BL2003] and [BDLV2006].

**phi_inv** (*W=None*)

Apply the inverse of the phi function to self.

INPUT:

- **self** – a word over the integers
- **W** – a parent object of words defined over integers

OUTPUT:

a word – the inverse of the phi function

EXAMPLES:

```python
sage: W = Words([1, 2])
sage: W([2, 2, 2, 2, 1, 2]).phi_inv()
word: 22112122
sage: W([2, 2, 2]).phi_inv(Words([2, 3]))
word: 2233
```

**prefix_function_table()**

Return a vector containing the length of the proper prefix-suffixes for all the non-empty prefixes of self.

EXAMPLES:

```python
sage: Word('121321').prefix_function_table()
[0, 0, 1, 0, 0, 1]
sage: Word('1241245').prefix_function_table()
[0, 0, 0, 1, 2, 3, 0]
sage: Word().prefix_function_table()
[]
```

**primitive()**

Return the primitive of self.

EXAMPLES:
primitive_length()

Return the length of the primitive of self.

EXAMPLES:

```sage
sage: Word('12312').primitive_length()
4
sage: Word('121212').primitive_length()
2
```

quasiperiods()

Return the quasiperiods of self as a list ordered from shortest to longest.

Let \( w \) be a finite or infinite word. A quasiperiod of \( w \) is a proper factor \( u \) of \( w \) such that the occurrences of \( u \) in \( w \) entirely cover \( w \), i.e., every position of \( w \) falls within some occurrence of \( u \) in \( w \). See for instance [AE1993], [Mar2004], and [GLR2008].

EXAMPLES:

```sage
sage: Word('abaababaabaababaaba').quasiperiods()
[word: aba, word: abaaba, word: abaababaaba]
sage: Word('abaaba').quasiperiods()
[word: aba]
sage: Word('abacaba').quasiperiods()
[]
```

rauzy_graph(n)

Return the Rauzy graph of the factors of length \( n \) of self.

The vertices are the factors of length \( n \) and there is an edge from \( u \) to \( v \) if \( ua = bv \) is a factor of length \( n + 1 \) for some letters \( a \) and \( b \).

INPUT:

* \( n \) – integer

EXAMPLES:

```sage
sage: w = Word(range(10)); w
word: 0123456789
sage: g = w.rauzy_graph(3); g
Looped digraph on 8 vertices
sage: WordOptions(identifier='')
sage: g.vertices(sort=True)
[012, 123, 234, 345, 456, 567, 678, 789]
sage: g.edges(sort=True)
[(012, 123, 3),
 (123, 234, 4),
 (234, 345, 5),
 (345, 456, 6),
 (456, 567, 7),
 (continues on next page)
```
sage: WordOptions(identifier='word: ')

sage: f = words.FibonacciWord()[:100]
sage: f.rauzy_graph(8)
Looped digraph on 9 vertices

sage: w = Word('1111111')
sage: g = w.rauzy_graph(3)
sage: g.edges(sort=True)
[(word: 111, word: 111, word: 1)]

sage: w = Word('111')
sage: for i in range(5) : w.rauzy_graph(i)
Looped multi-digraph on 1 vertex
Looped digraph on 1 vertex
Looped digraph on 1 vertex
Looped digraph on 0 vertices

Multi-edges are allowed for the empty word:

sage: W = Words('abcde')
sage: w = W('abc')
sage: w.rauzy_graph(0)
Looped multi-digraph on 1 vertex

sage: _.edges(sort=True)
[(word: , word: , word: a),
 (word: , word: , word: b),
 (word: , word: , word: c)]

reduced_rauzy_graph(n)
Return the reduced Rauzy graph of order n of self.

INPUT:

• n – a non-negative integer. Every vertex of a reduced Rauzy graph of order n is a factor of length n of self.

OUTPUT:

a looped multi-digraph

DEFINITION:

For infinite periodic words (resp. for finite words of type $u'u[0 : j]$), the reduced Rauzy graph of order n (resp. for n smaller or equal to $(i - 1)|u| + j$) is the directed graph whose unique vertex is the prefix p of length n of self and which has an only edge which is a loop on p labelled by $w[n + 1 : |w|]p$ where w is the unique return word to p.

In other cases, it is the directed graph defined as followed. Let $G_n$ be the Rauzy graph of order n of self. The vertices are the vertices of $G_n$ that are either special or not prolongable to the right or to the left. For each couple $(u, v)$ of such vertices and each directed path in $G_n$ from u to v that contains no other vertices
that are special, there is an edge from \( u \) to \( v \) in the reduced Rauzy graph of order \( n \) whose label is the label of the path in \( G_n \).

**Note:** In the case of infinite recurrent non-periodic words, this definition corresponds to the following one that can be found in [BDLGZ2009] and [BPS2008] where a simple path is a path that begins with a special factor, ends with a special factor and contains no other vertices that are special:

The reduced Rauzy graph of factors of length \( n \) is obtained from \( G_n \) by replacing each simple path \( P = v_1v_2...v_\ell \) with an edge \( v_1v_\ell \) whose label is the concatenation of the labels of the edges of \( P \).

**EXAMPLES:**

```
sage: w = Word(range(10)); w
word: 0123456789
sage: g = w.reduced_rauzy_graph(3); g
Looped multi-digraph on 2 vertices
sage: g.vertices(sort=True)
[word: 012, word: 789]
sage: g.edges(sort=True)
[(word: 012, word: 789, word: 3456789)]
```

For the Fibonacci word:

```
sage: f = words.FibonacciWord()[:100]
sage: g = f.reduced_rauzy_graph(8); g
Looped multi-digraph on 2 vertices
sage: g.vertices(sort=True)
[word: 01001010, word: 01010010]
sage: g.edges(sort=True)
[(word: 01001010, word: 01010010, word: 010), (word: 01010010, word: 01001010, word: 10)]
```

For periodic words:

```
sage: from itertools import cycle
sage: w = Word(cycle('abcd'))[:100]
sage: g = w.reduced_rauzy_graph(3)
sage: g.vertices(sort=True)
[(word: abc, word: abc, word: dabc)]
```

```
sage: w = Word('111')
sage: for i in range(5) : w.reduced_rauzy_graph(i)
Looped digraph on 1 vertex
Looped digraph on 1 vertex
Looped digraph on 1 vertex
Looped multi-digraph on 1 vertex
Looped multi-digraph on 0 vertices
```

For ultimately periodic words:

```
sage: sigma = WordMorphism('a->abcd,b->cd,c->cd,d->cd')
sage: w = sigma.fixed_point('a')[:100]; w
word: abcdcdcdcdccdcddcddcdccdcddcdccdcddcdccdcddcdccdcddcdccdcddcdccdcddcdcdccdcddcd... (continues on next page)```
sage: g = w.reduced_rauzy_graph(5)
sage: g.vertices(sort=True)
[word: abcdc, word: cdcdd]
sage: g.edges(sort=True)
[(word: abcdc, word: cdcdd, word: dc), (word: cdcdd, word: cdcdd, word: dc)]

AUTHOR:
Julien Leroy (March 2010): initial version

return_words(fact)
Return the set of return words of fact in self.
This is the set of all factors starting by the given factor and ending just before the next occurrence of this
factor. See [Dur1998] and [HZ1999].

INPUT:
• fact – a non-empty finite word

OUTPUT:
a Python set of finite words

EXAMPLES:

sage: Word('12331233213231').return_words(Word('2'))
{word: 213, word: 21331, word: 233}
sage: Word().return_words(Word('213'))
set()
sage: Word('121212').return_words(Word('1212'))
{word: 12}

sage: TM = words.ThueMorseWord()[:1000]
sage: sorted(TM.return_words(Word([0])))
[word: 0, word: 01, word: 011]

return_words_derivate(fact)
Return the word generated by mapping a letter to each occurrence of the return words for the given factor
dropping any dangling prefix and suffix. See for instance [Dur1998].

EXAMPLES:

sage: Word('12131221312313122').return_words_derivate(Word('1'))
word: 123242

rev_lex_less(other)
Return True if the word self is reverse lexicographically less than other.

EXAMPLES:

sage: Word([1,2,4]).rev_lex_less(Word([1,3,2]))
True
sage: Word([3,2,1]).rev_lex_less(Word([1,2,3]))
False
reversal()

Return the reversal of self.

EXAMPLES:

```python
sage: Word('124563').reversal()
word: 365421
```

rfind(sub, start=0, end=None)

Return the index of the last occurrence of sub in self, such that sub is contained within self[start:end]. Return -1 on failure.

INPUT:

- sub – string, list, tuple or word to search for.
- start – non-negative integer (default: 0) specifying the position at which the search must stop.
- end – non-negative integer (default: None) specifying the position from which to start the search. If None, then the search is performed up to the end of the string.

OUTPUT:

a non-negative integer or -1

EXAMPLES:

```python
sage: w = Word([0,1,0,0,1])
sage: w.rfind(Word([0,1]))
3
```

The sub parameter can also be a list or a tuple:

```python
sage: w.rfind([0,1])
3
sage: w.rfind((0,1))
3
```

Examples using the argument start and end:

```python
sage: w.rfind(Word([0,1]), end=4)
0
sage: w.rfind(Word([0,1]), end=5)
3
sage: w.rfind(Word([0,0]), start=2, end=5)
2
sage: w.rfind(Word([0,0]), start=3, end=5)
-1
```

Instances of Word_str handle string inputs as well:

```python
sage: w = Word('abac')
sage: w.rfind('a')
2
```
right_special_factors\( (n=None) \)

Return the right special factors (of length \( n \)).

A factor \( u \) of a word \( w \) is right special if there are two distinct letters \( a \) and \( b \) such that \( ua \) and \( ub \) are factors of \( w \).

INPUT:

\* \( n \) – integer (optional, default: \( None \)). If \( None \), it returns all right special factors.

OUTPUT:

a list of words

EXAMPLES:

```sage
w = words.ThueMorseWord()[:30]
sage: for i in range(5):
    ....:     print("{} {}\n".format(i, sorted(w.right_special_factors(i))))
0 [word: ]
1 [word: 0, word: 1]
2 [word: 01, word: 10]
3 [word: 001, word: 010, word: 101, word: 110]
4 [word: 0110, word: 1001]
```

right_special_factors_iterator\( (n=None) \)

Return an iterator over the right special factors (of length \( n \)).

A factor \( u \) of a word \( w \) is right special if there are two distinct letters \( a \) and \( b \) such that \( ua \) and \( ub \) are factors of \( w \).

INPUT:

\* \( n \) – integer (optional, default: \( None \)). If \( None \), it returns an iterator over all right special factors.

EXAMPLES:

```sage
alpha, beta, x = 0.61, 0.54, 0.3
w = words.CodingOfRotationWord(alpha, beta, x)[:40]
sage: sorted(w.right_special_factors_iterator(3))
[\text{word: 010, word: 101}]
sage: sorted(w.right_special_factors_iterator(4))
[\text{word: 0101, word: 1010}]
sage: sorted(w.right_special_factors_iterator(5))
[\text{word: 00101, word: 11010}]
```

robinson_schensted()

Return the semistandard tableau and standard tableau pair obtained by running the Robinson-Schensted algorithm on \( \text{self} \).

This can also be done by running \( RSK() \) on \( \text{self} \).

EXAMPLES:

```sage
Word([1,1,3,1,2,3,1]).robinson_schensted()
[[[1, 1, 1, 1, 3], [2], [3]], [[1, 2, 3, 5, 6], [4], [7]]]
```

schuetzenberger_involution\( (n=None) \)

Return the Schützenberger involution of the word \( \text{self} \), which is obtained by reverting the word and then
complementing all letters within the underlying ordered alphabet. If \( n \) is specified, the underlying alphabet is assumed to be \([1, 2, \ldots, n]\). If no alphabet is specified, \( n \) is the maximal letter appearing in \( \text{self} \).

**INPUT:**
- \( \text{self} \) – a word
- \( n \) – an integer specifying the maximal letter in the alphabet (optional)

**OUTPUT:**
A word, the Schützenberger involution of \( \text{self} \)

**EXAMPLES:**

```python
sage: w = Word([9,7,4,1,6,2,3])
sage: v = w.schuetzenberger_involution(); v
word: 7849631
```

```python
sage: v.parent()
Finite words over Set of Python objects of class 'object'
```

```python
sage: w = Word([1,2,3],alphabet=[1,2,3,4,5])
sage: v = w.schuetzenberger_involution();v
word: 345
```

```python
sage: v.parent()
Finite words over \{1, 2, 3, 4, 5\}
```

```python
sage: w = Word([1,2,3])
sage: v = w.schuetzenberger_involution(n=5);v
word: 345
```

```python
sage: v.parent()
Finite words over Set of Python objects of class 'object'
```

```python
sage: w = Word([11,32,69,2,53,1,2,3,18,41])
sage: w.schuetzenberger_involution()
word: 29,52,67,68,69,17,68,1,38,59
```

```python
sage: w = Word([],alphabet=[1,2,3,4,5])
sage: w.schuetzenberger_involution()
word:
```

```python
sage: w = Word([])
sage: w.schuetzenberger_involution()
word:
```

**shifted_shuffle** (**other**, **shift=None**)

Return the combinatorial class representing the shifted shuffle product between words \( \text{self} \) and \( \text{other} \). This is the same as the shuffle product of \( \text{self} \) with the word obtained from \( \text{other} \) by incrementing its values (i.e. its letters) by the given \( \text{shift} \).

**INPUT:**
- \( \text{other} \) – finite word over the integers
- \( \text{shift} \) – integer or None (default: None) added to each letter of \( \text{other} \). When \( \text{shift} \) is None, it is replaced by \( \text{self}.\text{length}() \)

**OUTPUT:**
combinatorial class of shifted shuffle products of self and other

**EXAMPLES:**

```python
sage: w = Word([0,1,1])
sage: sp = w.shifted_shuffle(w); sp
Shuffle product of word: 011 and word: 344
sage: sp = w.shifted_shuffle(w, 2); sp
Shuffle product of word: 011 and word: 233
sage: sp.cardinality()
20
sage: WordOptions(identifier='')
sage: sp.list()
[01233, 012133, 012313, 012331, 021133, 021313, 023113, 023131, 023311,
  → 201133, 201313, 201331, 203113, 203131, 203311, 230113, 230131, 230311, 233011]
sage: WordOptions(identifier='word: ')
sage: y = Word('aba')
sage: y.shifted_shuffle(w,2)
Traceback (most recent call last):
  ... ValueError: for shifted shuffle, words must only contain integers as letters
```

**shuffle**(other, overlap=0)

Return the combinatorial class representing the shuffle product between words self and other. This consists of all words of length `self.length() + other.length()` that have both self and other as subwords.

If overlap is non-zero, then the combinatorial class representing the shuffle product with overlaps is returned. The calculation of the shift in each overlap is done relative to the order of the alphabet. For example, a shifted by a is b in the alphabet [a, b, c] and 0 shifted by 1 in [0, 1, 2, 3] is 2.

**INPUT:**

- other – finite word
- overlap – (default: 0) integer or True

**OUTPUT:**

combinatorial class of shuffle product of self and other

**EXAMPLES:**

```python
sage: ab = Word("ab")
sage: cd = Word("cd")
sage: sp = ab.shuffle(cd); sp
Shuffle product of word: ab and word: cd
sage: sp.cardinality()
6
sage: sp.list()
[word: abcd, word: acbd, word: acdb, word: cabd, word: cadb, word: cdab]
sage: w = Word([0,1])
sage: u = Word([2,3])
sage: w.shuffle(w)
Shuffle product of word: 01 and word: 01
sage: u.shuffle(u)
(continues on next page)
```
Shuffle product of word: 23 and word: 23
\texttt{sage: w.shuffle(u)}

Shuffle product of word: 01 and word: 23
\texttt{sage: sp2 = w.shuffle(u,2); sp2}
Overlapping shuffle product of word: 01 and word: 23 with 2 overlaps
\texttt{sage: list(sp2)}
[\text{word: 24}]

\textbf{squares()}

Returns a set of all distinct squares of \texttt{self}.

\textbf{EXAMPLES:}

\texttt{sage: sorted(Word('cacao').squares())}
[\text{word: , word: caca}]
\texttt{sage: sorted(Word('1111').squares())}
[\text{word: , word: 11, word: 1111}]
\texttt{sage: w = Word('00110011010')}
\texttt{sage: sorted(w.squares())}
[\text{word: , word: 00, word: 00110011, word: 01100110, word: 1010, word: 11}]

\textbf{standard_factorization()}

Return the standard factorization of \texttt{self}.

The \textit{standard factorization} of a word \(w\) of length greater than 1 is the factorization \(w = uv\) where \(v\) is the longest proper suffix of \(w\) that is a Lyndon word.

Note that if \(w\) is a Lyndon word of length greater than 1 with standard factorization \(w = uv\), then \(u\) and \(v\) are also Lyndon words and \(u < v\).

See for instance [CFL1958], [Duv1983] and [Lot2002].

\textbf{INPUT:}

- \texttt{self} – finite word of length greater than 1

\textbf{OUTPUT:}

2-tuple \((u, v)\)

\textbf{EXAMPLES:}

\texttt{sage: Words('01')('0010110011').standard_factorization()}
(\text{word: 001011, word: 0011})
\texttt{sage: Words('123')('1223312').standard_factorization()}
(\text{word: 12233, word: 12})
\texttt{sage: Word([3,2,1]).standard_factorization()}
(\text{word: 32, word: 1})

\texttt{sage: w = Word('0010110011',alphabet='01')}
\texttt{sage: w.standard_factorization()}
(\text{word: 001011, word: 0011})
\texttt{sage: w = Word('0010110011',alphabet='10')}
\texttt{sage: w.standard_factorization()}
(\text{word: 001011001, word: 1})
\texttt{sage: w = Word('1223312',alphabet='123')}

(continues on next page)
**standard_permutation()**

Return the standard permutation of the word `self` on the ordered alphabet. It is defined as the permutation with exactly the same inversions as `self`. Equivalently, it is the permutation of minimal length whose inverse sorts `self`.

**EXAMPLES:**

```python
sage: w = Word([1,2,3,2,2,1]); w
word: 123221
sage: p = w.standard_permutation(); p
[1, 3, 6, 4, 5, 2]
sage: v = Word(p.inverse().action(w)); v
word: 112223
sage: [q for q in Permutations(w.length())
    ....: if q.length() <= p.length() and
    ....: q.inverse().action(w) == list(v)]
[[[1, 3, 6, 4, 5, 2]]]
sage: w = Words([1,2,3])([1,2,3,2,2,1,2,1]); w
word: 12322121
sage: p = w.standard_permutation(); p
[1, 4, 8, 5, 6, 2, 7, 3]
sage: Word(p.inverse().action(w))
word: 11122223
sage: w = Words([3,2,1])([1,2,3,2,2,1,2,1]); w
word: 12322121
sage: p = w.standard_permutation(); p
[6, 2, 1, 3, 4, 7, 5, 8]
sage: Word(p.inverse().action(w))
word: 32222111
sage: w = Words('ab')('abbaba'); w
word: abbaba
sage: p = w.standard_permutation(); p
[1, 4, 5, 2, 6, 3]
sage: Word(p.inverse().action(w))
word: aaabbb
sage: w = Words('ba')('abbaba'); w
word: abbaba
sage: p = w.standard_permutation(); p
[4, 1, 2, 5, 3, 6]
sage: Word(p.inverse().action(w))
word: bbbaaa
```

**sturmian_desubstitute_as_possible()**

Sturmian-desubstitute the word `self` as much as possible.

The finite word `self` must be defined on a two-letter alphabet or use at most two letters.
It can be Sturmian desubstituted if one letter appears isolated: the Sturmian desubstitution consists in removing one letter per run of the non-isolated letter. The accelerated Sturmian desubstitution consists in removing a run equal to the length of the shortest inner run from any run of the non-isolated letter (including possible leading and trailing runs even if they have shorter length). The (accelerated) Sturmian desubstitution is done as much as possible. A word is a factor of a Sturmian word if, and only if, the result is the empty word.

OUTPUT:

a finite word defined on a two-letter alphabet

EXAMPLES:

```
sage: u = Word('1011101110111011',alphabet='01') ; u
word: 1011101101110111
sage: v = u.sturmian_desubstitute_as_possible() ; v
word: 01100101
sage: v == v.sturmian_desubstitute_as_possible()
True
sage: Word('azaazaaazaazaaazaz', alphabet='az').sturmian_desubstitute_as_possible()
word:
```

AUTHOR:

• Thierry Monteil

```
subword_complementaries(other)
```

Return the possible complementaries other minus self if self is a subword of other (empty list otherwise). The complementary is made of all the letters that are in other once we removed the letters of self. There can be more than one.

To check whether self is a subword of other (without knowing its complementaries), use self.
is_subword_of(other), and to count the number of occurrences of self in other, use other.
number_of_subword_occurrences(self).

INPUT:

• other – finite word

OUTPUT:

• list of all the complementary subwords of self in other.

EXAMPLES:

```
sage: Word('tamtam').subword_complementaries(Word('ta'))
[]
sage: Word('mta').subword_complementaries(Word('tamtam'))
[word: tam]
sage: Word('ta').subword_complementaries(Word('tamtam'))
[word: mtam, word: amtm, word: tamm]
sage: Word('a').subword_complementaries(Word('a'))
[word: ]
```
**suffix_tree()**

Alias for `implicit_suffix_tree()`.

**EXAMPLES:**

```
sage: Word('ababaab').suffix_tree()
Implicit Suffix Tree of the word: ababaab
```

**suffix_trie()**

Return the suffix trie of self.

The *suffix trie* of a finite word $w$ is a data structure representing the factors of $w$. It is a tree whose edges are labelled with letters of $w$, and whose leafs correspond to suffixes of $w$.

Type `sage.combinat.words.suffix_trees.SuffixTrie?` for more information.

**EXAMPLES:**

```
sage: w = Word("cacao")
sage: w.suffix_trie()
Suffix Trie of the word: cacao
```

```
sage: w = Word([0,1,0,1,1])
sage: w.suffix_trie()
Suffix Trie of the word: 01011
```

**swap**(i, j=None)

Return the word $w$ with entries at positions $i$ and $j$ swapped. By default, $j = i+1$.

**EXAMPLES:**

```
sage: Word([1,2,3]).swap(0,2)
word: 321
sage: Word([1,2,3]).swap(1)
word: 132
sage: Word("abba").swap(1,-1)
word: aabb
```

**swap_decrease**(i)

Return the word with positions $i$ and $i+1$ exchanged if $\text{self}[i] < \text{self}[i+1]$. Otherwise, it returns self.

**EXAMPLES:**

```
sage: w = Word([1,3,2])
sage: w.swap_decrease(0)
word: 312
sage: w.swap_decrease(1)
word: 132
sage: w.swap_decrease(1) is w
True
sage: Words("ab")("abba").swap_decrease(0)
word: baba
sage: Words("ba")("abba").swap_decrease(0)
word: abba
```
**swap_increase(i)**

Return the word with positions \(i\) and \(i+1\) exchanged if \(\text{self}[i] > \text{self}[i+1]\). Otherwise, it returns \(\text{self}\).

**EXAMPLES:**

```
sage: w = Word([1,3,2])
sage: w.swap_increase(1)
word: 123
sage: w.swap_increase(0)
word: 132
sage: w.swap_increase(0) is w
True
sage: Words("ab")("abba").swap_increase(0)
word: abba
sage: Words("ba")("abba").swap_increase(0)
word: baba
```

**to_integer_list()**

Return a list of integers from \([0,1,...,\text{self}.\text{length()-1}]\) in the same relative order as the letters in \(\text{self}\) in the parent.

**EXAMPLES:**

```
sage: from itertools import count
sage: w = Word('abbabaab')
sage: w.to_integer_list()
[0, 1, 1, 0, 1, 0, 0, 1]
sage: w = Word(iter("cacao"), length="finite")
sage: w.to_integer_list()
[1, 0, 1, 0, 2]
sage: w = Words([3,2,1])([2,3,3,1])
sage: w.to_integer_list()
[1, 0, 0, 2]
```

**to_integer_word()**

Return a word over the alphabet \([0,1,...,\text{self}.\text{length()-1}]\) whose letters are in the same relative order as the letters of \(\text{self}\) in the parent.

**EXAMPLES:**

```
sage: from itertools import count
sage: w = Word('abbabaab')
sage: w.to_integer_word()
word: 01101001
sage: w = Word(iter("cacao"), length="finite")
sage: w.to_integer_word()
word: 10102
sage: w = Words([3,2,1])([2,3,3,1])
sage: w.to_integer_word()
word: 1002
```

**to_monoid_element()**

Return \(\text{self}\) as an element of the free monoid with the same alphabet as \(\text{self}\).
Combinatorics, Release 10.0

EXAMPLES:

```
sage: w = Word('aabb')
sage: w.to_monoid_element()  
a^2*b^2  
sage: W = Words('abc')
sage: w = W(w)
sage: w.to_monoid_element()  
a^2*b^2  
```

to_ordered_set_partition()

Return the ordered set partition correspond to self.

If $w$ is a finite word of length $n$, then the corresponding ordered set partition is an ordered set partition $(P_1, P_2, \ldots, P_k)$ of \{1, 2, \ldots, $n$\}, where each block $P_i$ is the set of positions at which the $i$-th smallest letter occurring in $w$ occurs in $w$.

EXAMPLES:

```
sage: w = Word('abbabaab')
sage: w.to_ordered_set_partition()  
[[1, 4, 6, 7], [2, 3, 5, 8]]  
sage: Word([-10, 3, -10, 2]).to_ordered_set_partition()  
[[1, 3], [4], [2]]  
sage: Word([]).to_ordered_set_partition()  
[[]]  
sage: Word('aaaaa').to_ordered_set_partition()  
[[1, 2, 3, 4, 5]]
```

topological_entropy($n$)

Return the topological entropy for the factors of length $n$.

The topological entropy of a sequence $u$ is defined as the exponential growth rate of the complexity of $u$ as the length increases: $H_{\text{top}}(u) = \lim_{n \to \infty} \frac{\log(p_a(n))}{n}$ where $d$ denotes the cardinality of the alphabet and $p_a(n)$ is the complexity function, i.e. the number of factors of length $n$ in the sequence $u$ [Fog2002].

INPUT:

• self – a word defined over a finite alphabet
• $n$ – positive integer

OUTPUT:

real number (a symbolic expression)

EXAMPLES:

```
sage: W = Words([0, 1])
sage: w = W([0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1])
sage: t = w.topological_entropy(3); t  
1/3*log(7)/log(2)  
sage: n(t)  
0.935784974019201
```

```
sage: w = words.ThueMorseWord()[:100]
sage: topo = w.topological_entropy
```

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Combinatorics, Release 10.0

sage: for i in range(0, 41, 5):
    ....:     print("{0} {1:.5f}".format(i, n(topo(i), digits=5)))
0 1.0000
5 0.71699
10 0.48074
15 0.36396
20 0.28774
25 0.23628
30 0.20075
35 0.17270
40 0.14827

If no alphabet is specified, an error is raised:

sage: w = Word(range(20))
sage: w.topological_entropy(3)
Traceback (most recent call last):
...
TypeError: The word must be defined over a finite alphabet

The following is ok:

sage: W = Words(range(20))
sage: w = W(range(20))
sage: w.topological_entropy(3)
1/3*log(18)/log(20)

sage.combinat.words.finite_word.evaluation_dict(w)

Return a dictionary keyed by the letters occurring in \( w \) with values the number of occurrences of the letter.

INPUT:

• \( w \) – a word

sage.combinat.words.finite_word.word_to_ordered_set_partition(w)

Return the ordered set partition corresponding to a finite word \( w \).

If \( w \) is a finite word of length \( n \), then the corresponding ordered set partition is an ordered set partition \( (P_1, P_2, \ldots, P_k) \) of \( \{1, 2, \ldots, n\} \), where each block \( P_i \) is the set of positions at which the \( i \)-th smallest letter occurring in \( w \) occurs in \( w \). (Positions are 1-based.)

This is the same functionality that \texttt{to\_ordered\_set\_partition()} provides, but without the wrapping: The input \( w \) can be given as a list or tuple, not necessarily as a word; and the output is returned as a list of lists (which are the blocks of the ordered set partition in increasing order), not as an ordered set partition.

EXAMPLES:

sage: from sage.combinat.words.finite_word import word_to_ordered_set_partition
sage: word_to_ordered_set_partition([3, 6, 3, 1])
[[4], [1, 3], [2]]
sage: word_to_ordered_set_partition([1, 3, 3, 7])
[[1], [2, 3], [4]]
sage: word_to_ordered_set_partition("noob")
[[4], [1], [2, 3]]
sage: word_to_ordered_set_partition(Word("hell"))
(continues on next page)
5.1.357 Infinite word

AUTHORS:

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• Franco Saliola

EXAMPLES:

Creation of an infinite word

Periodic infinite words:

```python
sage: v = Word([0, 4, 8, 8, 3])
sage: vv = v^Infinity
sage: vv
word: 0488304883048830488304883048830488304883...
```

Infinite words from a function \( f : \mathbb{N} \rightarrow A \) over an alphabet \( A \):

```python
sage: Word(lambda n: n%3)  
word: 0120120120120120120120120120120120120120...
```

```python
sage: def t(n):
....:     return add(Integer(n).digits(base=2)) % 2
sage: Word(t, alphabet = [0, 1])
word: 0110100110010110100101100110100110010110...
```

or as a one-liner:

```python
sage: Word(lambda n : add(Integer(n).digits(base=2)) % 2, alphabet = [0, 1])
word: 0110100110010110100101100110100110010110...
```

Infinite words from iterators:

```python
sage: from itertools import count,repeat
sage: Word( repeat(4) )
word: 4444444444444444444444444444444444444444...
```

```python
sage: Word( count() )
word: 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,
...31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,...
```

Infinite words from morphism

For example, let \( A = \{a, b\} \) and \( \mu : A^* \rightarrow A^* \) be the morphism defined by \( a \mapsto ab, b \mapsto ba \):

```python
sage: from itertools import count,repeat
sage: Word( count() )
word: 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,
...31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,...
```
**5.1.358 Lyndon words**

sage.combinat.words.lyndon_word.LyndonWord(data, check=True)

Construction of a Lyndon word.

INPUT:

- `data` – list
- `check` – bool (optional, default: True) if True, check that the input data represents a Lyndon word.

OUTPUT:

A Lyndon word.

EXAMPLES:

```python
sage: LyndonWord([1,2,2])
word: 122
sage: LyndonWord([1,2,3])
word: 123
sage: LyndonWord([2,1,2,3])
Traceback (most recent call last):
  ... ValueError: not a Lyndon word
```

If `check` is False, then no verification is done:
sage: LyndonWord([2,1,2,3], check=False)
word: 2123

sage.combinat.words.lyndon_word.LyndonWords(e=None, k=None)
Return the combinatorial class of Lyndon words.

A Lyndon word \( w \) is a word that is lexicographically less than all of its rotations. Equivalently, whenever \( w \) is split into two non-empty substrings, \( w \) is lexicographically less than the right substring.

See Wikipedia article Lyndon_word

INPUT:

• no input at all

or

• \( e \) – integer, size of alphabet

• \( k \) – integer, length of the words

or

• \( e \) – a composition

OUTPUT:

A combinatorial class of Lyndon words.

EXAMPLES:

sage: LyndonWords()
Lyndon words

If \( e \) is an integer, then \( e \) specifies the length of the alphabet; \( k \) must also be specified in this case:

sage: LW = LyndonWords(3, 4); LW
Lyndon words from an alphabet of size 3 of length 4
sage: LW.first()
word: 1112
sage: LW.last()
word: 2333
sage: LW.random_element() # random
word: 1232
sage: LW.cardinality()
18

If \( e \) is a (weak) composition, then it returns the class of Lyndon words that have evaluation \( e \):

sage: LyndonWords([2, 0, 1]).list()
[word: 113]
sage: LyndonWords([2, 0, 1, 0, 1]).list()
[word: 1135, word: 1153, word: 1315]
sage: LyndonWords([2, 1, 1]).list()
[word: 1123, word: 1132, word: 1213]

class sage.combinat.words.lyndon_word.LyndonWords_class(alphabet=None)
Bases: UniqueRepresentation, Parent
The set of all Lyndon words.
class sage.combinat.words.lyndon_word.LyndonWords_evaluation(e):
    Bases: UniqueRepresentation, Parent
    The set of Lyndon words on a fixed multiset of letters.
    EXAMPLES:
    sage: L = LyndonWords([1,2,1])
    sage: L
    Lyndon words with evaluation [1, 2, 1]
    sage: L.list()
    [word: 1223, word: 1232, word: 1322]

cardinality()
    Return the number of Lyndon words with the evaluation e.
    EXAMPLES:
    sage: LyndonWords([]).cardinality()
    0
    sage: LyndonWords([2,2]).cardinality()
    1
    sage: LyndonWords([2,3,2]).cardinality()
    30
    Check to make sure that the count matches up with the number of Lyndon words generated:
    sage: comps = [[], [2,2], [3,2,7], [4,2]] + Compositions(4).list()
    sage: lws = [LyndonWords(comp) for comp in comps]
    sage: all(lw.cardinality() == len(lw.list()) for lw in lws)
    True

class sage.combinat.words.lyndon_word.LyndonWords_nk(n, k):
    Bases: UniqueRepresentation, Parent
    Lyndon words of fixed length k over the alphabet \{1, 2, \ldots, n\}.
    INPUT:
    • n – the size of the alphabet
    • k – the length of the words
    EXAMPLES:
    sage: L = LyndonWords(3, 4)
    sage: L.list()
    [word: 1112, word: 1113, word: 1122, word: 1123, ...
     word: 1333, word: 2223, word: 2233, word: 2333]
cardinality()

sage.combinat.words.lyndon_word.StandardBracketedLyndonWords(n, k)

Return the combinatorial class of standard bracketed Lyndon words from [1, ..., n] of length k.

These are in one to one correspondence with the Lyndon words and form a basis for the subspace of degree k of the free Lie algebra of rank n.

EXAMPLES:

```python
sage: SBLW33 = StandardBracketedLyndonWords(3,3); SBLW33
Standard bracketed Lyndon words from an alphabet of size 3 of length 3
sage: SBLW33.first()
[1, [1, 2]]
sage: SBLW33.last()
[[2, 3], 3]
sage: SBLW33.cardinality()
8
sage: SBLW33.random_element() in SBLW33
True
```

class sage.combinat.words.lyndon_word.StandardBracketedLyndonWords_nk(n, k)

Bases: UniqueRepresentation, Parent

cardinality()

EXAMPLES:

```python
sage: StandardBracketedLyndonWords(3, 3).cardinality()
8
sage: StandardBracketedLyndonWords(3, 4).cardinality()
18
```

sage.combinat.words.lyndon_word.standard_bracketing(lw)

Return the standard bracketing of a Lyndon word lw.

EXAMPLES:

```python
sage: import sage.combinat.words.lyndon_word as lyndon_word
sage: [lyndon_word.standard_bracketing(u) for u in LyndonWords(3,3)]
[[1, [1, 2]],
 [1, [1, 3]],
 [[1, 2], 2],
 [1, [2, 3]],
 [[1, 3], 2],
 [1, [3, 3]],
 [2, [2, 3]],
 [[2, 3], 3]]
```

sage.combinat.words.lyndon_word.standard_unbracketing(sblw)

Return flattened sblw if it is a standard bracketing of a Lyndon word, otherwise raise an error.

EXAMPLES:

```python
sage: from sage.combinat.words.lyndon_word import standard_unbracketing
sage: standard_unbracketing([1, [2, 3]])
word: 123
```
(continues on next page)
5.1.359 Word morphisms/substitutions

This module implements morphisms over finite and infinite words.

AUTHORS:

• Sébastien Labbé (2007-06-01): initial version
• Sébastien Labbé (2008-07-01): merged into sage-words
• Sébastien Labbé (2008-12-17): merged into sage
• Sébastien Labbé (2009-02-03): words next generation
• Sébastien Labbé (2009-11-20): allowing the choice of the datatype of the image. Doc improvements.
• Stepan Starosta (2012-11-09): growing letters

EXAMPLES:

Creation of a morphism from a dictionary or a string:

\begin{verbatim}
sage: n = WordMorphism({0:[0,2,2,1],1:[0,2],2:[2,2,1]})
sage: m = WordMorphism('x->xyxsxss,s->xyss,y->ys')
\end{verbatim}

\begin{verbatim}
sage: n
WordMorphism: 0->0221, 1->02, 2->221
sage: m
WordMorphism: s->xyss, x->xyxsxss, y->ys
\end{verbatim}

The codomain may be specified:

\begin{verbatim}
sage: WordMorphism({0:[0,2,2,1],1:[0,2],2:[2,2,1]}, codomain=Words([0,1,2,3,4]))
WordMorphism: 0->0221, 1->02, 2->221
\end{verbatim}

Power of a morphism:

\begin{verbatim}
sage: n^2
WordMorphism: 0->022122122102, 1->0221221, 2->22122102
\end{verbatim}

Image under a morphism:

\begin{verbatim}
sage: m('y')
word: ys
sage: m('xxxy')
word: xyxsxssxyxsxssxyxsxssxyssys
\end{verbatim}

Iterated image under a morphism:
Combinatorics, Release 10.0

```python
sage: m('y', 3)
word: ysyxssyxsxssyssxssyss
```

See more examples in the documentation of the call method (m.__call__?).

Infinite fixed point of morphism:

```python
sage: fix = m.fixed_point('x')
sage: fix
word: xyxsxssyxsxssyssxxyssxssxxyssxssyssyssxyssyssxssxyssxyss
sage: fix.length()
+Infinity
```

Incidence matrix:

```python
sage: matrix(m)
[ 2  3  1]
[ 1  3  0]
[ 1  1  1]
```

Many other functionalities...:

```python
sage: m.is_identity()
False
sage: m.is_endomorphism()
True
```

```python
class sage.combinat.words.morphism.PeriodicPointIterator(m, cycle)
    Bases: object

    (Lazy) constructor of the periodic points of a word morphism.

    This class is mainly used in WordMorphism.periodic_point and WordMorphism.periodic_points.

    EXAMPLES:

    ```python
    sage: from sage.combinat.words.morphism import PeriodicPointIterator
    sage: s = WordMorphism('a->bacca,b->cba,c->aab')
    sage: p = PeriodicPointIterator(s, ['a','b','c'])
    sage: p._cache[0]
    lazy list ['a', 'a', 'b', ...]
    sage: p._cache[1]
    lazy list ['b', 'a', 'c', ...]
    sage: p._cache[2]
    lazy list ['c', 'b', 'a', ...]
    
    get_iterator(i)
    Internal method.

    EXAMPLES:
    ```python
    sage: from sage.combinat.words.morphism import PeriodicPointIterator
    sage: s = WordMorphism('a->bacca,b->cba,c->aab')
    sage: p = PeriodicPointIterator(s, ['a','b','c'])
    sage: p.get_iterator(0)
    <generator object ...get_iterator at ...>
    ```

```
class sage.combinat.words.morphism.WordMorphism(data, domain=None, codomain=None)

WordMorphism class

INPUT:

- data – dict or str or an instance of WordMorphism, the map giving the image of letters
- domain – (optional: None) set of words over a given alphabet. If None, the domain alphabet is computed from data and is sorted.
- codomain – (optional: None) set of words over a given alphabet. If None, the codomain alphabet is computed from data and is sorted.

Note: When the domain or the codomain are not explicitly given, it is expected that the letters are comparable because the alphabets of the domain and of the codomain are sorted.

EXAMPLES:

From a dictionary:

```python
sage: n = WordMorphism({0:[0,2,2,1], 1:[0,2], 2:[2,2,1]})
sage: n
WordMorphism: 0->0221, 1->02, 2->221
```

From a string with '->' as separation:

```python
sage: m = WordMorphism('x->xyxsxss, s->xyss, y->ys')
sage: m
WordMorphism: s->xyss, x->xyxsxss, y->ys
sage: m.domain()
Finite words over {'s', 'x', 'y'}
sage: m.codomain()
Finite words over {'s', 'x', 'y'}
```

Specifying the domain and codomain:

```python
sage: W = FiniteWords([0,1,2])
sage: d = {0:[0,1,2], 1:[0,1,0], 2:[0]}
sage: m = WordMorphism(d, domain=W, codomain=W)
sage: m([0]).parent()
Finite words over {0, 1, 2}
```

When the alphabet is non-sortable, the domain and/or codomain must be explicitly given:

```python
sage: W = FiniteWords(['a',6])
sage: d = {'a':[a,6,'a'], 6:[6,6,6, 'a']}
sage: WordMorphism(d, domain=W, codomain=W)
WordMorphism: 6->666a, a->a6a
```

abelian_rotation_subspace()  
Return the subspace on which the incidence matrix of self acts by roots of unity.

EXAMPLES:
sage: WordMorphism('0->1,1->0').abelian_rotation_subspace()
Vector space of degree 2 and dimension 2 over Rational Field
Basis matrix:
[1 0]
[0 1]
sage: WordMorphism('0->01,1->10').abelian_rotation_subspace()
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]
sage: WordMorphism('0->01,1->1').abelian_rotation_subspace()
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[0 1]
sage: WordMorphism('1->122,2->211').abelian_rotation_subspace()
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[ 1 -1]
sage: WordMorphism('0->1,1->102,2->3,3->4,4->2').abelian_rotation_subspace()
Vector space of degree 5 and dimension 3 over Rational Field
Basis matrix:
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]

The domain needs to be equal to the codomain:

sage: WordMorphism('0->1,1->', codomain=Words('01')).abelian_rotation_subspace()
Vector space of degree 2 and dimension 0 over Rational Field
Basis matrix:
[]

codomain()

Return the codomain of self.

EXAMPLES:

sage: WordMorphism('a->ab,b->a').codomain()
Finite words over {'a', 'b'}
sage: WordMorphism('6->ab,y->5,0->asd').codomain()
Finite words over {'5', 'a', 'b', 'd', 's'}

conjugate(pos)

Return the morphism where the image of the letter by self is conjugated of parameter pos.

INPUT:

• pos - integer

EXAMPLES:

sage: m = WordMorphism('a->abcde')
sage: m.conjugate(0) == m
True
sage: m.conjugate(1)
WordMorphism: a->bcdea

(continues on next page)
\begin{Verbatim}
\texttt{sage: m.conjugate(3)}
WordMorphism: a->deabc
\texttt{sage: WordMorphism('').conjugate(4)}
WordMorphism:
\texttt{sage: m = WordMorphism('a->abcde,b->xyz')}
\texttt{sage: m.conjugate(2)}
WordMorphism: a->cdeab, b->zxy
\end{Verbatim}

domain()

Return domain of self.

EXAMPLES:

\begin{Verbatim}
\texttt{sage: WordMorphism('a->ab,b->a').domain()}
Finite words over \{'a', 'b'\}
\texttt{sage: WordMorphism('b->ba,a->ab').domain()}
Finite words over \{'a', 'b'\}
\texttt{sage: WordMorphism('6->ab,y->5,0->asd').domain()}
Finite words over \{'0', '6', 'y'\}
\end{Verbatim}

dual_map\(k=1\)

Return the dual map \(E^*_k\) of self (see [1]).

\begin{itemize}
\item \textbf{Note: } It is actually implemented only for \(k = 1\).
\end{itemize}

INPUT:

\begin{itemize}
\item \texttt{self} - unimodular endomorphism defined on integers 1, 2, \ldots, d
\item \texttt{k} - integer (optional, default: 1)
\end{itemize}

OUTPUT:

an instance of E1Star - the dual map

EXAMPLES:

\begin{Verbatim}
\texttt{sage: sigma = WordMorphism({1:[2],2:[3],3:[1,2]})}
\texttt{sage: sigma.dual_map()}
E_1^*(1->2, 2->3, 3->12)
\end{Verbatim}

\texttt{sage: sigma.dual_map(k=2)}
Traceback (most recent call last):
...
NotImplementedError: the dual map E_k^* is implemented only for k = 1 (not 2)

REFERENCES:

\begin{itemize}
\end{itemize}

extend_by\(\texttt{other}\)

Return self extended by other.

Let \(\varphi_1 : A^* \to B^*\) and \(\varphi_2 : C^* \to D^*\) be two morphisms. A morphism \(\mu : (A \cup C)^* \to (B \cup D)^*\) corresponds to \(\varphi_1\) extended by \(\varphi_2\) if \(\mu(a) = \varphi_1(a)\) if \(a \in A\) and \(\mu(a) = \varphi_2(a)\) otherwise.

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INPUT:
• other - a WordMorphism.

OUTPUT:
WordMorphism

EXAMPLES:
```
sage: m = WordMorphism('a->ab,b->ba')
sage: n = WordMorphism({0:'1','1':0,'a':5})
sage: m.extend_by(n)
WordMorphism: 0->1, 1->0, a->ab, b->ba
sage: n.extend_by(m)
WordMorphism: 0->1, 1->0, a->5, b->ba
sage: m.extend_by(m)
WordMorphism: a->ab, b->ba
```

```
def fixed_point(letter):
    Return the fixed point of self beginning by the given letter.

A fixed point of morphism \( \varphi \) is a word \( w \) such that \( \varphi(w) = w \).

INPUT:
• self - an endomorphism (or more generally a self-composable morphism), must be prolongable on letter
• letter - in the domain of self, the first letter of the fixed point.

OUTPUT:
• word - the fixed point of self beginning with letter.

EXAMPLES:
```
sage: W = FiniteWords('abc')
```

1. Infinite fixed point:
```
sage: WordMorphism('a->ab,b->ba').fixed_point(letter='a')
word: abbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbb...
```

2. Infinite fixed point of an erasing morphism:
```
sage: WordMorphism('a->ab,b->c->ba', codomain=W).fixed_point(letter='a')
word: ab
```

3. Finite fixed point:
```
sage: WordMorphism('a->ab,b->c->ba', codomain=W).fixed_point(letter='b')
word: b
```
```
sage: _.parent()
Finite words over {'a', 'b', 'c'}
```

(continues on next page)
5. Fixed point of a power of a morphism:

```python
sage: m = WordMorphism('a->ba,b->ab')
sage: (m^2).fixed_point(letter='a')
word: abbabaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabba..."
This shows that issue github issue #13668 has been resolved:

```python
sage: s = "1->321331332133133,2->133321331332133133,3->2133133133321331332133133→"
sage: s = WordMorphism(s)
sage: (s^2).fixed_points()
[]
```

```python
sage: s7r.periodic_point(2)
word: 2,1,1,10,9,8,7,6,5,4,3,2,1,10,9,8,7,6,5,4,3,2,10,9,8,7,6,5,4,3,2,9,8,7,6,
→5,4,3,2,8,...
```

```
has_conjugate_in_classP(f=None)
Return True if self has a conjugate in class f-P.
DEFINITION: Let A be an alphabet. We say that a primitive substitution S is in the class P if there exists a palindrome p and for each b ∈ A a palindrome q_b such that S(b) = pq_b for all b ∈ A. [1]
Let f be an involution on A. We say that a morphism φ is in class f-P if there exists an f-palindrome p and for each α ∈ A there exists an f-palindrome q_α such that φ(α) = pq_α. [2]
INPUT:
• f - involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).
REFERENCES:
EXAMPLES:
```
sage: fibo = WordMorphism('a->ab,b->a')
sage: fibo.has_conjugate_in_classP()
```
True
sage: (fibo^2).is_in_classP()
False
sage: (fibo^2).has_conjugate_in_classP()
True

**has_left_conjugate()**

Return True if all the non empty images of `self` begins with the same letter.

**EXAMPLES:**

```python
sage: m = WordMorphism('a->abcde,b->xyz')
sage: m.has_left_conjugate()
False
sage: WordMorphism('b->xyz').has_left_conjugate()
True
sage: WordMorphism('').has_left_conjugate()
True
sage: WordMorphism('a->,b->xyz').has_left_conjugate()
True
sage: WordMorphism('a->abbab,b->abb').has_left_conjugate()
True
sage: WordMorphism('a->abbab,b->abb,c->').has_left_conjugate()
True
```

**has_right_conjugate()**

Return True if all the non empty images of `self` ends with the same letter.

**EXAMPLES:**

```python
sage: m = WordMorphism('a->abcde,b->xyz')
sage: m.has_right_conjugate()
False
sage: WordMorphism('b->xyz').has_right_conjugate()
True
sage: WordMorphism('').has_right_conjugate()
True
sage: WordMorphism('a->,b->xyz').has_right_conjugate()
True
sage: WordMorphism('a->abbab,b->abb').has_right_conjugate()
True
sage: WordMorphism('a->abbab,b->abb,c->').has_right_conjugate()
True
```

**image(letter)**

Return the image of a letter.

**INPUT:**

- letter – a letter in the domain alphabet

**OUTPUT:**

word
Note: The letter is assumed to be in the domain alphabet (no check done). Hence, this method is faster than the __call__ method suitable for words input.

EXAMPLES:

```python
sage: m = WordMorphism('a->ab,b->ac,c->a')
sage: m.image('b')
word: ac
```

```python
sage: s = WordMorphism({('a', 1): [('a', 1), ('a', 2)], ('a', 2): [('a', 1)])}
sage: s.image(('a', 1))
word: ('a', 1), ('a', 2)
```

```python
sage: s = WordMorphism({'b':[1,2], 'a':[2,3,4], 'z':[9,8,7]})
sage: s.image('b')
word: 12
```

```python
sage: s = WordMorphism('a->abcd,b->cd,c->dd,d->', domain=Words('a'), codomain=Words('a')).immortal_letters()
[]
```

images()  
Return the list of all the images of the letters of the alphabet under self.

EXAMPLES:

```python
sage: sorted(WordMorphism('a->ab,b->a').images())
[word: a, word: ab]
sage: sorted(WordMorphism('6->ab,y->5,0->asd').images())
[word: 5, word: ab, word: asd]
```

immortal_letters()  
Return the list of immortal letters.

A letter a is immortal for the morphism s if the length of the iterates of $|s^n(a)|$ is larger than zero as n goes to infinity.

Requires this morphism to be self-composable.

EXAMPLES:

```python
sage: WordMorphism('a->a').immortal_letters()
['a']
sage: WordMorphism('a->b,b->a').immortal_letters()
['a', 'b']
sage: WordMorphism('a->abcd,b->cd,c->dd,d->').immortal_letters()
['a']
sage: WordMorphism('a->bc,b->cac,c->de,d->,e->').immortal_letters()
['a', 'b']
sage: WordMorphism('a->', domain=Words('a'), codomain=Words('a')).immortal_letters()
[]
```
incidence_matrix()

Return the incidence matrix of the morphism. The order of the rows and columns are given by the order defined on the alphabet of the domain and the codomain.

The matrix returned is over the integers. If a different ring is desired, use either the change_ring function or the matrix function.

EXAMPLES:

```sage
sage: m = WordMorphism('a->abc,b->a,c->c')
sage: m.incidence_matrix()

[1 1 0]
[1 0 0]
[1 0 1]
sage: m = WordMorphism('a->abc,b->a,c->c,d->abbccccabca,e->abc')
sage: m.incidence_matrix()

[1 1 0 3 1]
[1 0 0 3 1]
[1 0 1 5 1]
```

infinite_repetitions_primitive_roots(w=None, allow_growing=None)

Return the set of primitive roots (up to conjugacy) of infinite repetitions from the language \{m^n(w)|n \geq 0\}, where \(m\) is this morphism and \(w\) is a word inputted as a parameter.

Requires this morphism to be an endomorphism.

The word \(v^w\) is an infinite repetition (in other words, an infinite periodic factor) of a language, if \(v\) is a non-empty word and for each positive integer \(k\) the word \(v^k\) is a factor of some word from the language. It turns out that a language created by iterating a morphism has a finite number of primitive roots of infinite repetitions.

If \(v\) is a primitive root of an infinite repetition, then all its conjugations are also primitive roots of an infinite repetition. For simplicity’s sake this method returns only the lexicographically minimal one from each conjugacy class.

INPUT:

- \(w\) – finite iterable (default: self.domain().alphabet()). Represents a word used to start the language.
- allow_growing – boolean or None (default: None). If False, return only the primitive roots that contain no growing letters. If True, return only the primitive roots that contain at least one growing letter. If None, return both.

ALGORITHM:

The algorithm used is described in detail in [KS2015].

EXAMPLES:

```sage
sage: m = WordMorphism('a->aba,b->aba,c->cd,d->e,e->d')
sage: inf_reps = m.infinite_repetitions_primitive_roots('ac')
sage: sorted(inf_reps)
[word: aab, word: de]
```
allow_growing parameter:

```python
sage: sorted(m.infinite_repetitions_primitive_roots('ac', True))
[word: aab]
sage: sorted(m.infinite_repetitions_primitive_roots('ac', False))
[word: de]
```

Incomplete check that these words are indeed the primitive roots of infinite repetitions:

```python
sage: SL = m._language_naive(10, Word('ac'))
sage: all(x in SL for x in inf_reps)  # True
sage: all(x^2 in SL for x in inf_reps)  # True
sage: all(x^3 in SL for x in inf_reps)  # True
```

Large example:

```python
sage: m = WordMorphism('a->1b5,b->fcg,c->dae,d->432,e->678,f->f,g->g,1->2,2->3,˓
    →3->4,4->1,5->6,6->7,7->8,8->5')
sage: sorted(m.infinite_repetitions_primitive_roots('a'))
[word: 1432f2143f3214f4321f, word: 5678g8567g7856g6785g]
```

is_empty()

Return True if the cardinality of the domain is zero and False otherwise.

```python
sage: WordMorphism('').is_empty()  # True
sage: WordMorphism('a->a').is_empty()  # False
```

is_endomorphism()

Return whether self is an endomorphism, that is if the domain coincide with the codomain.

```python
sage: WordMorphism('a->ab,b->a').is_endomorphism()  # True
sage: WordMorphism('6->ab,y->5,0->asd').is_endomorphism()  # False
sage: WordMorphism('a->a,b->aa,c->aaa').is_endomorphism()  # False
sage: Wabc = Words('abc')
sage: m = WordMorphism('a->a,b->aa,c->aaa', codomain=Wabc)
sage: m.is_endomorphism()  # True
```

We check that github issue #8674 is fixed:

```python
sage: P = WordPaths('abcd')
sage: m = WordMorphism('a->adab,b->ab,c->cbcd,d->cd', domain=P, codomain=P)
sage: m.is_endomorphism()  # True
```
is_erasing()

Return True if self is an erasing morphism, i.e. the image of a letter is the empty word.

EXAMPLES:

```
sage: WordMorphism('a->ab,b->a').is_erasing()
False
sage: WordMorphism('6->ab,y->5,0->asd').is_erasing()
False
sage: WordMorphism('6->ab,y->5,0->asd,7->').is_erasing()
True
sage: WordMorphism('').is_erasing()
False
```

is_growing(letter=None)

Return True if letter is a growing letter.

A letter $a$ is growing for the morphism $s$ if the length of the iterates of $|s^n(a)|$ tend to infinity as $n$ goes to infinity.

INPUT:

* letter – None or a letter in the domain of self

**Note**: If letter is None, this returns True if self is everywhere growing, i.e., all letters are growing letters (see [CassNic10]), and that self must be an endomorphism.

EXAMPLES:

```
sage: WordMorphism('0->01,1->1').is_growing('0')
True
sage: WordMorphism('0->01,1->1').is_growing('1')
False
sage: WordMorphism('0->01,1->10').is_growing()
True
sage: WordMorphism('0->1,1->2,2->01').is_growing()
True
sage: WordMorphism('0->01,1->1').is_growing()
False
```

The domain needs to be equal to the codomain:

```
sage: WordMorphism('0->01,1->0,2->1',codomain=Words('012')).is_growing()
True
```

Test of erasing morphisms:

```
sage: WordMorphism('0->01,1->').is_growing('0')
False
sage: m = WordMorphism('a->bc,b->bcc,c->',codomain=Words('abc'))
sage: m.is_growing('a')
False
sage: m.is_growing('b')
False
```

(continues on next page)
is\_identity()

Return True if self is the identity morphism.

EXAMPLES:

```
sage: m = WordMorphism('a->a,b->b,c->c,d->e')
sage: m.is_identity()
False
sage: WordMorphism('a->a,b->b,c->c').is_identity()
True
sage: WordMorphism('a->a,b->b,c->cb').is_identity()
False
sage: m = WordMorphism('a->b,b->c,c->a')
sage: (m^2).is_identity()
False
sage: (m^3).is_identity()
True
sage: (m^4).is_identity()
False
sage: WordMorphism('').is_identity()
True
sage: WordMorphism({0:[0],1:[1]}).is_identity()
True
```

We check that github issue #8618 is fixed:

```
sage: t = WordMorphism({'a1': ['a2'], 'a2': ['a1']})
sage: (t*t).is_identity()
True
```

is\_in\_classP(f=None)

Return True if self is in class P (or f-P).

DEFINITION: Let A be an alphabet. We say that a primitive substitution S is in the class P if there exists a palindrome p and for each b ∈ A a palindrome q_b such that S(b) = pq_b for all b ∈ A. [1]

Let f be an involution on A. “We say that a morphism ϕ is in class f-P if there exists an f-palindrome p and for each α ∈ A there exists an f-palindrome q_α such that ϕ(α) = pq_α. [2]

INPUT:

• f - involution (default: None) on the alphabet of self. It must be callable on letters as well as words (e.g. WordMorphism).

REFERENCES:


EXAMPLES:
sage: WordMorphism('a->baba,b->bba').is_in_classP()
True
sage: tm = WordMorphism('a->ab,b->ba')
sage: tm.is_in_classP()
False
sage: f = WordMorphism('a->b,b->a')
sage: tm.is_in_classP(f=f)
True
sage: (tm^2).is_in_classP()
True
sage: (tm^2).is_in_classP(f=f)
False
sage: fibo = WordMorphism('a->ab,b->a')
sage: fibo.is_in_classP()
True
sage: fibo.is_in_classP(f=f)
False
sage: (fibo^2).is_in_classP()
False
sage: f = WordMorphism('a->b,b->a,c->c')
sage: WordMorphism('a->acbcc,b->acbab,c->acbba').is_in_classP(f)
True

is_injective()
Return whether this morphism is injective.

ALGORITHM:
Uses a version of Wikipedia article Sardinas–Patterson_algorithm. Time complexity is on average quadratic with regards to the size of the morphism.

EXAMPLES:

sage: WordMorphism('a->0,b->10,c->110,d->111').is_injective()
True
sage: WordMorphism('a->00,b->01,c->012,d->20001').is_injective()
False

is_involution()
Return True if self is an involution, i.e. its square is the identity.

INPUT:
• self - an endomorphism

EXAMPLES:

sage: WordMorphism('a->b,b->a').is_involution()
True
sage: WordMorphism('a->b,b->ba').is_involution()
False
sage: WordMorphism({0:[1],1:[0]}).is_involution()
True

is_primitive()
Return True if self is primitive.
A morphism $\varphi$ is primitive if there exists a positive integer $k$ such that for all $\alpha \in \Sigma$, $\varphi^k(\alpha)$ contains all the letters of $\Sigma$.

INPUT:
- self - an endomorphism

ALGORITHM:

Exercises 8.7.8, p.281 in [1]: (c) Let $y(M)$ be the least integer $e$ such that $M^e$ has all positive entries. Prove that, for all primitive matrices $M$, we have $y(M) \leq (d - 1)^2 + 1$. (d) Prove that the bound $y(M) \leq (d - 1)^2 + 1$ is best possible.

EXAMPLES:

```sage
sage: tm = WordMorphism('a->ab,b->ba')
sage: tm.is_primitive()
True
sage: fibo = WordMorphism('a->ab,b->a')
sage: fibo.is_primitive()
True
sage: m = WordMorphism('a->bb,b->aa')
sage: m.is_primitive()
False
sage: f = WordMorphism({0:[1],1:[0]})
sage: f.is_primitive()
False

sage: s = WordMorphism('a->b,b->c,c->ab')
sage: s.is_primitive()
True
sage: s = WordMorphism('a->b,b->c,c->d,d->e,e->f,f->g,g->h,h->ab')
sage: s.is_primitive()
True
```

REFERENCES:

is_prolongable(letter)

Return True if self is prolongable on letter.

A morphism $\varphi$ is prolongable on a letter $a$ if $a$ is a prefix of $\varphi(a)$.

INPUT:
- self - its codomain must be an instance of Words
- letter - a letter in the domain alphabet

OUTPUT:
Boolean

EXAMPLES:

```sage
sage: WordMorphism('a->ab,b->a').is_prolongable(letter='a')
True
sage: WordMorphism('a->ab,b->a').is_prolongable(letter='b')
```

We check that github issue #8595 is fixed:

```
sage: s = WordMorphism({('a', 1) : [('a', 1), ('a', 2)], ('a', 2) : [('a', 1)]})
sage: s.is_prolongable(('a',1))
True
```

**is_pushy**(w=None)

Return whether the language \( \{m^n(w) | n \geq 0 \} \) is pushy, where \( m \) is this morphism and \( w \) is a word inputted as a parameter.

Requires this morphism to be an endomorphism.

A language created by iterating a morphism is pushy, if its words contain an infinite number of factors containing no growing letters. It turns out that this is equivalent to having at least one infinite repetition containing no growing letters.

See infinite_repetitions_primitive_roots() and is_growing().

**INPUT:**

- \( w \) – finite iterable (default: self.domain().alphabet()). Represents a word used to start the language.

**EXAMPLES:**

```
sage: WordMorphism('a->abca,b->bc,c->bcb').is_pushy()
False
sage: WordMorphism('a->abc,b->,c->bcb').is_pushy()
True
```

**is_repetitive**(w=None)

Return whether the language \( \{m^n(w) | n \geq 0 \} \) is repetitive, where \( m \) is this morphism and \( w \) is a word inputted as a parameter.

Requires this morphism to be an endomorphism.

A language is repetitive, if for each positive integer \( k \) there exists a word \( u \) such that \( u^k \) is a factor of some word of the language.

It turns out that for languages created by iterating a morphism this is equivalent to having at least one infinite repetition (this property is also known as strong repetitiveness).

See infinite_repetitions_primitive_roots().

**INPUT:**

- \( w \) – finite iterable (default: self.domain().alphabet()). Represents a word used to start the language.
EXAMPLES:

This method can be used to check whether a purely morphic word is not k-power free for all positive integers k. For example, the language containing just the Thue-Morse word and its prefixes is not repetitive, since the Thue-Morse word is cube-free:

```
sage: WordMorphism('a->ab,b->ba').is_repetitive('a')
False
```

Similarly, the Hanoi word is square-free:

```
sage: WordMorphism('a->aC,A->ac,b->cb,c->bA,C->ba').is_repetitive('a')
False
```

However, this method solves a more general problem, as it can be called on any morphism m and with any word w:

```
sage: WordMorphism('a->c,b->cda,c->a,d->abc').is_repetitive('bd')
True
```

**is_self_composable()**

Return whether the codomain of self is contained in the domain.

EXAMPLES:

```
sage: f = WordMorphism('a->a,b->a')
sage: f.is_endomorphism()
False
sage: f.is_self_composable()
True
```

**is_unboundedly_repetitive(w=None)**

Return whether the language \( \{m^n(w)|n \geq 0\} \) is unboundedly repetitive, where m is this morphism and w is a word inputted as a parameter.

Requires this morphism to be an endomorphism.

A language created by iterating a morphism is unboundedly repetitive, if it has at least one infinite repetition containing at least one growing letter.

See **infinite_repetitions_primitive_roots()** and **is_growing()**.

INPUT:

* w – finite iterable (default: self.domain().alphabet()). Represents a word used to start the language.

EXAMPLES:

```
sage: WordMorphism('a->abca,b->bc,c->').is_unboundedly_repetitive()
True
sage: WordMorphism('a->abc,b->,c->bcb').is_unboundedly_repetitive()
False
```

**is_uniform(k=None)**

Return True if self is a k-uniform morphism.
Let \( k \) be a positive integer. A morphism \( \phi \) is called \( k \)-uniform if for every letter \( \alpha \), we have \( |\phi(\alpha)| = k \). In other words, all images have length \( k \). A morphism is called uniform if it is \( k \)-uniform for some positive integer \( k \).

**INPUT:**
- \( k \) - a positive integer or None. If set to a positive integer, then the function return True if self is \( k \)-uniform. If set to None, then the function return True if self is uniform.

**EXAMPLES:**

```python
sage: phi = WordMorphism('a->ab,b->a')
sage: phi.is_uniform()
False
sage: phi.is_uniform(k=1)
False
sage: tau = WordMorphism('a->ab,b->ba')
sage: tau.is_uniform()
True
sage: tau.is_uniform(k=1)
False
sage: tau.is_uniform(k=2)
True
```

**language**\((n, u=None)\)

Return the words of length \( n \) in the language generated by this substitution.

Given a non-erasing substitution \( s \) and a word \( u \) the DOL-language generated by \( s \) and \( u \) is the union of the factors of \( s^n(u) \) where \( n \) is a non-negative integer.

**INPUT:**
- \( n \) – non-negative integer - length of the words in the language
- \( u \) – a word or None (optional, default None) - if set to None some letter of the alphabet is used

**OUTPUT:** a Python set

**EXAMPLES:**

The fibonacci morphism:

```python
sage: s = WordMorphism({0: [0,1], 1:[0]})
sage: sorted(s.language(3))
[word: 001, word: 010, word: 100, word: 101]
sage: len(s.language(1000))
1001
sage: all(len(s.language(n)) == n+1 for n in range(100))
True
```

A growing but non-primitive example. The DOL-languages generated by 0 and 2 are different:

```python
sage: s = WordMorphism({0: [0,1], 1:[0], 2:[2,0,2]})
sage: u = s.fixed_point(0)
sage: A0 = u[:200].factor_set(5)
sage: B0 = s.language(5, [0])
sage: set(A0) == B0
```

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True

\[v = s\.fixed\.point(2)\]
\[A2 = v[:200].factor\.set(5)\]
\[B2 = s\.language(5, [2])\]
\[set(A2) == B2\]
\[True\]

\[\text{len}(A0), \text{len}(A2)\]
\[(6, 20)\]

The Chacon transformation (non-primitive):

\[s = \text{WordMorphism}([\theta: [0,0,1,0], 1:[1]])\]
\[\text{sorted}(s\.language(10))\]
\[[\text{word}: 0001000101,\]
\[\text{word}: 0001010010,\]
\[\ldots\]
\[\text{word}: 1010010001,\]
\[\text{word}: 1010010100]\]

`latex_layout(layout=None)`

Get or set the actual latex layout (oneliner vs array).

**INPUT:**

- `layout` - string (default: None), can take one of the following values:
  - None - Returns the actual latex layout. By default, the layout is 'array'
  - 'oneliner' - Set the layout to 'oneliner'
  - 'array' - Set the layout to 'array'

**EXAMPLES:**

\[s = \text{WordMorphism}('a->ab,b->ba')\]
\[s\.latex\.layout()\]

'array'
\[s\.latex\.layout('oneliner')\]
\[s\.latex\.layout()\]

'oneliner'

`letter_growth_types()`

Return the mortal, polynomial and exponential growing letters.

The growth of \(|s^n(\alpha)|\) as \(n\) goes to \(\infty\) is always of the form \(\alpha^n n^\beta\) (where \(\alpha\) is a Perron number and \(\beta\) an integer).

Without doing any linear algebra three cases can be differentiated: mortal (ultimately empty or \(\alpha = 0\)); polynomial (\(\alpha = 1\)); exponential (\(\alpha > 1\)). This is what is done in this method.

It requires this morphism to be an endomorphism.

**OUTPUT:**

The output is a 3-tuple of lists (mortal, polynomial, exponential) where:

- **mortal**: list of mortal letters
• polynomial: a list of lists where polynomial[i] is the list of letters with growth \( n^i \).

• exponential: list of at least exponentially growing letters

EXAMPLES:

```python
sage: s = WordMorphism('a->abc,b->bc,c->c')

sage: mortal, poly, expo = s.letter_growth_types()

sage: mortal
[]

sage: poly
[['c'], ['b'], ['a']]

sage: expo
[]
```

When three mortal letters (c, d, and e), and two letters (a, b) are not growing:

```python
sage: s = WordMorphism('a->bc,b->cac,c->de,d->,e->')

sage: s^20
WordMorphism: a->cacde, b->debcde, c->, d->, e->

sage: mortal, poly, expo = s.letter_growth_types()

sage: mortal
['c', 'd', 'e']

sage: poly
[['a', 'b']]

sage: expo
[]
```

```python
sage: s = WordMorphism('a->abcd,b->bc,c->c,d->a')

sage: mortal, poly, expo = s.letter_growth_types()

sage: mortal
[]

sage: poly
[['c'], ['b']]

sage: expo
['a', 'd']
```

list_of_conjugates()

Return the list of all the conjugate morphisms of self.

DEFINITION:

Recall from Lothaire [1] (Section 2.3.4) that \( \varphi \) is right conjugate of \( \varphi' \), noted \( \varphi \bowtie \varphi' \), if there exists \( u \in \Sigma^* \) such that

\[
\varphi(\alpha)u = u\varphi'(\alpha),
\]

for all \( \alpha \in \Sigma \), or equivalently that \( \varphi(x)u = u\varphi'(x) \), for all words \( x \in \Sigma^* \). Clearly, this relation is not symmetric so that we say that two morphisms \( \varphi \) and \( \varphi' \) are conjugate, noted \( \varphi \bowtie \varphi' \), if \( \varphi \bowtie \varphi' \) or \( \varphi' \bowtie \varphi \). It is easy to see that conjugacy of morphisms is an equivalence relation.

REFERENCES:


EXAMPLES:
```python
sage: m = WordMorphism('a->abbab,b->abb')
sage: m.list_of_conjugates()
[WordMorphism: a->babba, b->bab,
  WordMorphism: a->abbab, b->abb,
  WordMorphism: a->bbaba, b->bba,
  WordMorphism: a->ababb, b->abb,
  WordMorphism: a->babba, b->bba,
  WordMorphism: a->abbab, b->bab]
sage: m = WordMorphism('a->aaa,b->aa')
sage: m.list_of_conjugates()
[WordMorphism: a->aaa, b->aa]
sage: WordMorphism('').list_of_conjugates()
[WordMorphism: ]
sage: m = WordMorphism('a->aba,b->aba')
sage: m.list_of_conjugates()
[WordMorphism: a->baa, b->baa,
  WordMorphism: a->aab, b->aab,
  WordMorphism: a->aba, b->aba]
sage: m = WordMorphism('a->abb,b->abbab,c->')
sage: m.list_of_conjugates()
[WordMorphism: a->bab, b->babba, c->,
  WordMorphism: a->abb, b->abbab, c->,
  WordMorphism: a->bba, b->bbaba, c->,
  WordMorphism: a->bab, b->babab, c->,
  WordMorphism: a->bba, b->babba, c->,
  WordMorphism: a->bab, b->abbab, c->]
```

**partition_of_domain_alphabet()**

Return a partition of the domain alphabet.

Let \( \varphi : \Sigma^* \to \Sigma^* \) be an involution. There exists a triple of sets \((A, B, C)\) such that

- \( A \cup B \cup C = \Sigma; \)
- \( A, B \) and \( C \) are mutually disjoint and
- \( \varphi(A) = B, \varphi(B) = A, \varphi(C) = C. \)

These sets are not unique.

**INPUT:**

- `self` - An involution.

**OUTPUT:**

A tuple of three sets

**EXAMPLES:**

```python
sage: m = WordMorphism('a->b,b->a')
sage: m.partition_of_domain_alphabet()  # random ordering
({'a'}, {'b'}, { })
sage: m = WordMorphism('a->b,b->a,c->c')
sage: m.partition_of_domain_alphabet()  # random ordering
({'a'}, {'b'}, {'c'})
```

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sage: m = WordMorphism('a->a,b->b,c->c')
sage: m.partition_of_domain_alphabet() # random ordering
({}, {}, {'a', 'c', 'b'})
sage: m = WordMorphism('A->T,T->A,C->G,G->C')
sage: m.partition_of_domain_alphabet() # random ordering
({'A', 'C'}, {'T', 'G'}, {})
Make sure that github issue #31454 is fixed:

```
sage: WordMorphism('a->a,b->bb').periodic_points()
[[word: bbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbb...]]
```

## pisot_eigenvector_left()

Return the left eigenvector of the incidence matrix associated to the largest eigenvalue (in absolute value).

Unicity of the result is guaranteed when the multiplicity of the largest eigenvalue is one, for example when 
self is a Pisot irreducible substitution.

A substitution is Pisot irreducible if the characteristic polynomial of its incidence matrix is irreducible over 
\( \mathbb{Q} \) and has all roots, except one, of modulus strictly smaller than 1.

**INPUT:**

- **self** - a Pisot irreducible substitution.

**EXAMPLES:**

```
sage: m = WordMorphism('a->aaaabbc,b->aaabbc,c->aabc')
sage: matrix(m)
[4 3 2]
[2 2 1]
[1 1 1]
sage: m.pisot_eigenvector_left()
(1, 0.8392867552141611?, 0.5436890126920763?)
```

## pisot_eigenvector_right()

Return the right eigenvector of the incidence matrix associated to the largest eigenvalue (in absolute value).

Unicity of the result is guaranteed when the multiplicity of the largest eigenvalue is one, for example when 
self is a Pisot irreducible substitution.

A substitution is Pisot irreducible if the characteristic polynomial of its incidence matrix is irreducible over 
\( \mathbb{Q} \) and has all roots, except one, of modulus strictly smaller than 1.

**INPUT:**

- **self** - a Pisot irreducible substitution.

**EXAMPLES:**

```
sage: m = WordMorphism('a->aaaabbc,b->aaabbc,c->aabc')
sage: matrix(m)
[4 3 2]
```

rauzy_fractal_plot(n=None, exchange=False, eig=None, translate=None, prec=53, colormap='hsv', opacity=None, plot_origin=None, plot_basis=False, point_size=None)

Return a plot of the Rauzy fractal associated with a substitution.

The substitution does not have to be irreducible. The usual definition of a Rauzy fractal requires that its dominant eigenvalue is a Pisot number but the present method doesn’t require this, allowing to plot some interesting pictures in the non-Pisot case (see the examples below).

For more details about the definition of the fractal and the projection which is used, see Section 3.1 of [1].

Plots with less than 100,000 points take a few seconds, and several millions of points can be plotted in reasonable time.

Other ways to draw Rauzy fractals (and more generally projections of paths) can be found in sage.combinat.words.paths.FiniteWordPath_all.plot_projection() or in sage.combinat.e_one_star().

OUTPUT:
A Graphics object.

INPUT:

• n - integer (default: None) The number of points used to plot the fractal. Default values: 1000 for a 1D fractal, 50000 for a 2D fractal, 10000 for a 3D fractal.
• exchange - boolean (default: False). Plot the Rauzy fractal with domain exchange.
• eig - a real element of QQbar of degree >= 2 (default: None). The eigenvalue used to plot the fractal. It must be an eigenvalue of self.incidence_matrix(). The one used by default is the maximal eigenvalue of self.incidence_matrix() (usually a Pisot number), but for substitutions with more than 3 letters other interesting choices are sometimes possible.
• translate - a list of vectors of RR^{size_alphabet}, or a dictionary from the alphabet to lists of vectors (default: None). Plot translated copies of the fractal. This option allows to plot tilings easily. The projection used for these vectors is the same as the projection used for the canonical basis to plot the fractal. If the input is a list, all the pieces will be translated and plotted. If the input is a dictionary, each piece will be translated and plotted accordingly to the vectors associated with each letter in the dictionary. Note: by default, the Rauzy fractal placed at the origin is not plotted with the translate option; the vector (0,0,...,0) has to be added manually.
• prec - integer (default: 53). The number of bits used in the floating point representations of the points of the fractal.
• colormap - color map or dictionary (default: 'hsv'). It can be one of the following:
  – string - a coloring map. For available coloring map names type: sorted(colormaps)
  – dict - a dictionary of the alphabet mapped to colors.
• opacity - a dictionary from the alphabet to the real interval [0,1] (default: None). If none is specified, all letters are plotted with opacity 1.
• plot_origin - a couple (k,c) (default: None). If specified, mark the origin by a point of size k and color c.
**plot_basis** - boolean (default: False). Plot the projection of the canonical basis with the fractal.

**point_size** - float (default: None). The size of the points used to plot the fractal.

EXAMPLES:

1. The Rauzy fractal of the Tribonacci substitution:

```python
sage: s = WordMorphism('1->12,2->13,3->1')
sage: s.rauzy_fractal_plot()  # long time
Graphics object consisting of 3 graphics primitives
```

2. The “Hokkaido” fractal. We tweak the plot using the plotting options to get a nice reusable picture, in which we mark the origin by a black dot:

```python
sage: s = WordMorphism('a->ab,b->c,c->d,d->e,e->a')
sage: G = s.rauzy_fractal_plot(n=100000, point_size=3, plot_origin=(50, ˓→"black"))  # not tested
sage: G.show(figsize=10, axes=false)  # not tested
```

3. Another “Hokkaido” fractal and its domain exchange:

```python
sage: s = WordMorphism({1:[2], 2:[4,3], 3:[4], 4:[5,3], 5:[6], 6:[1]})
sage: s.rauzy_fractal_plot()  # not tested takes > 1 second
sage: s.rauzy_fractal_plot(exchange=True)  # not tested takes > 1 second
```

4. A three-dimensional Rauzy fractal:

```python
sage: s = WordMorphism('1->12,2->13,3->14,4->1')
sage: s.rauzy_fractal_plot()  # not tested takes > 1 second
```

5. A one-dimensional Rauzy fractal (very scattered):

```python
sage: s = WordMorphism('1->2122,2->1')
sage: s.rauzy_fractal_plot().show(figsize=20)  # not tested takes > 1 ˓→second
```

6. A high resolution plot of a complicated fractal:

```python
sage: s = WordMorphism('1->23,2->123,3->112233')
sage: G = s.rauzy_fractal_plot(n=300000)  # not tested takes > 1 second
sage: G.show(axes=false, figsize=20)  # not tested takes > 1 second
```

7. A nice colorful animation of a domain exchange:

```python
sage: s = WordMorphism('1->21,2->3,3->4,4->25,5->6,6->7,7->1')
sage: L = [s.rauzy_fractal_plot(), s.rauzy_fractal_plot(exchange=True)]
  # not tested takes > 1 second
sage: animate(L, axes=false).show(delay=100)  # not tested takes > 1 ˓→second
```

8. Plotting with only one color:

```python
sage: s = WordMorphism('1->12,2->31,3->1')
sage: s.rauzy_fractal_plot(colormap={'1':'black', '2':'black', '3':'black'})  # not tested takes > 1 second
```
9. Different fractals can be obtained by choosing another (non-Pisot) eigenvalue:

```
sage: s = WordMorphism('1->12,2->3,3->45,4->5,5->6,6->7,7->8,8->1')
sage: E = s.incidence_matrix().eigenvalues()
sage: x = [x for x in E if -0.8 < x < -0.7][0]
sage: s.rauzy_fractal_plot()  # not tested takes > 1 second
sage: s.rauzy_fractal_plot(eig=x)  # not tested takes > 1 second
```

10. A Pisot reducible substitution with seemingly overlapping tiles:

```
sage: s = WordMorphism({1:[1,2], 2:[2,3], 3:[4], 4:[5], 5:[6], 6:[7], 7:[8],
                    8:[9], 9:[10], 10:[1]})
sage: s.rauzy_fractal_plot()  # not tested takes > 1 second
```

11. A non-Pisot reducible substitution with a strange Rauzy fractal:

```
sage: s = WordMorphism({1:[3,2], 2:[3,3], 3:[4], 4:[1]})
sage: s.rauzy_fractal_plot()  # not tested takes > 1 second
```

12. A substitution with overlapping tiles. We use the options colormap and opacity to study how the tiles overlap:

```
sage: s = WordMorphism('1->213,2->4,3->5,4->1,5->21')
sage: s.rauzy_fractal_plot()  # not tested, takes > 1 second
sage: s.rauzy_fractal_plot(colormap={'1': 'red', '4': 'purple'})  # not tested, takes > 1 second
sage: s.rauzy_fractal_plot(opacity={'1': 0.1, '2': 1, '3': 0.1, '4': 0.1, '5': 0.1}, n=150000)  # not tested takes > 1 second
```

13. Funny experiments by playing with the precision of the float numbers used to plot the fractal:

```
sage: s = WordMorphism('1->12,2->13,3->1')
sage: s.rauzy_fractal_plot(prec=6)  # not tested
sage: s.rauzy_fractal_plot(prec=9)  # not tested
sage: s.rauzy_fractal_plot(prec=15)  # not tested
sage: s.rauzy_fractal_plot(prec=19)  # not tested
sage: s.rauzy_fractal_plot(prec=25)  # not tested
```

14. Using the translate option to plot periodic tilings:

```
sage: s = WordMorphism('1->12,2->13,3->1')
sage: s.rauzy_fractal_plot(n=10000, translate=[(0,0,0),(-1,0,1),(0,-1,1),(1,-1,0),(1,0,-1),(0,1,-1),(-1,1,0)])  # not tested takes > 1 second
```
Combinatorics, Release 10.0

```python
sage: s = WordMorphism({1:[1,2], 2:[3], 3:[4,3], 4:[5], 5:[6], 6:[1]})
sage: s.rauzy_fractal_plot()  # not tested takes > 1 second
sage: D = {1:[(0,0,0,0,0,0), (0,1,0,0,0,0)], 3:[(0,0,0,0,0,0), (0,1,0,0,0,0), (0,0,0,0,0,0)], 6:[(0,1,0,0,0,0)]}
sage: s.rauzy_fractal_plot(n=30000, translate=D)  # not tested takes > 1 second
```

16. Plot the projection of the canonical basis with the fractal:

```python
sage: s = WordMorphism({1:[2,1], 2:[3], 3:[6,4], 4:[5,1], 5:[6], 6:[7], 7:[8], 8:[9], 9:[1]})
sage: s.rauzy_fractal_plot(plot_basis=True)  # not tested takes > 1 second
```

REFERENCES:


AUTHOR:

Timo Jolivet (2012-06-16)

**rauzy_fractal_points**

```
rauzy_fractal_points(n=None, exchange=False, eig=None, translate=None, prec=53)
```

Return a dictionary of list of points associated with the pieces of the Rauzy fractal of `self`.

INPUT:

See the method `rauzy_fractal_plot()` for a description of the options and more examples.

OUTPUT:

dictionary of list of points

EXAMPLES:

The Rauzy fractal of the Tribonacci substitution and the number of points in the piece of the fractal associated with '1', '2' and '3' are respectively:

```python
sage: s = WordMorphism('1->12,2->13,3->1')
sage: D = s.rauzy_fractal_points(n=100)
sage: len(D['1'])
54
sage: len(D['2'])
30
sage: len(D['3'])
16
```

AUTHOR:

Timo Jolivet (2012-06-16)

**rauzy_fractal_projection**

```
rauzy_fractal_projection(eig=None, prec=53)
```

Return a dictionary giving the projection of the canonical basis.

See the method `rauzy_fractal_plot()` for more details about the projection.

INPUT:

- **eig** - a real element of QQbar of degree >= 2 (default: None). The eigenvalue used for the projection. It must be an eigenvalue of `self.incidence_matrix()`. The one used by default is the maximal
eigenvalue of \texttt{self.incidence_matrix()} (usually a Pisot number), but for substitutions with more than 3 letters other interesting choices are sometimes possible.

- \texttt{prec} - integer (default: 53). The number of bits used in the floating point representations of the coordinates.

**OUTPUT:**

dictionary, letter -> vector, giving the projection

**EXAMPLES:**

The projection for the Rauzy fractal of the Tribonacci substitution is:

```
sage: s = WordMorphism('1->12,2->13,3->1')
sage: s.rauzy_fractal_projection()
{'1': (1.00000000000000, 0.000000000000000),
 '2': (-1.41964337760708, -0.606290729207199),
 '3': (-0.771844506346038, 1.11514250803994)}
```

**AUTHOR:**

Timo Jolivet (2012-06-16)

\texttt{restrict\_domain}(alphabet)

Return a restriction of \texttt{self} to the given alphabet.

**INPUT:**

- \texttt{alphabet} - an iterable

**OUTPUT:**

WordMorphism

**EXAMPLES:**

```
sage: m = WordMorphism('a->b,b->a')
sage: m.restrict_domain('a')
WordMorphism: a->b
sage: m.restrict_domain('')
WordMorphism: 
sage: m.restrict_domain('A')
WordMorphism: 
sage: m.restrict_domain('Aa')
WordMorphism: a->b
```

The input alphabet must be iterable:

```
sage: m.restrict_domain(66)
Traceback (most recent call last):
  ...
TypeError: 'sage.rings.integer.Integer' object is not iterable
```

\texttt{reversal()} 

Return the reversal of \texttt{self}.

**EXAMPLES:**
**simplify_alphabet_size**(Z=None)

If this morphism is simplifiable, return morphisms $h$ and $k$ such that this morphism is simplifiable with respect to $h$ and $k$, otherwise raise `ValueError`.

This method is quite fast if this morphism is non-injective, but very slow if it is injective.

Let $f : X^* \to Y^*$ be a morphism. Then $f$ is simplifiable with respect to morphisms $h : X^* \to Z^*$ and $k : Z^* \to Y^*$, if $f = k \circ h$ and $|Z| < |X|$. If also $Y \subseteq X$, then the morphism $g : Z^* \to Z^* = h \circ k$ is a simplification of $f$ (with respect to $h$ and $k$).

Loosely speaking, a morphism is simplifiable if it contains “more letters than is needed”. Non-injectivity implies simplifiability. Simplification preserves some properties of the original morphism (e.g. repetitiveness).

For more information see Section 3 in [KO2000].

**INPUT:**

- `Z` – iterable (default: `self.domain().alphabet()`), whose elements are used as an alphabet for the simplification.

**EXAMPLES:**

Example of a simplifiable (non-injective) morphism:

```python
sage: f = WordMorphism('a->aca,b->badc,c->acab,d->adc')
sage: h, k = f.simplify_alphabet_size('xyz'); h, k
(WordMorphism: a->x, b->zy, c->xz, d->y, WordMorphism: x->aca, y->adc, z->b)
```

```python
sage: k * h == f
True
```

```python
sage: g = h * k; g
WordMorphism: x->xxzx, y->xyxz, z->zy
```

Example of a simplifiable (injective) morphism:

```python
sage: f = WordMorphism('a->abcc,b->abcd,c->abdc,d->abdd')
sage: h, k = f.simplify_alphabet_size('xyz'); h, k
(WordMorphism: a->xyy, b->xyz, c->xzy, d->xzz, WordMorphism: x->ab, y->c, z->d)
```

```python
sage: k * h == f
True
```

```python
sage: g = h * k; g
WordMorphism: x->xyyxyz, y->xzy, z->xzz
```

Example of a non-simplifiable morphism:

```python
sage: WordMorphism('a->aa').simplify_alphabet_size()
Traceback (most recent call last):
...
ValueError: self (a->aa) is not simplifiable
```

Example of an erasing morphism:
Example of a morphism, that is not an endomorphism:

```python
sage: f = WordMorphism('a->xx,b->xy,c->yx,d->yy')
sage: h, k = f.simplify_alphabet_size(NN); h, k
(WordMorphism: a->00, b->01, c->10, d->11, WordMorphism: 0->x, 1->y)
sage: k * h == f
True
sage: len(k.domain().alphabet()) < len(f.domain().alphabet())
True
```

sage.combinat.words.morphism.simplify_until_injective()

Return a quadruplet \((g, h, k, i)\), where \(g\) is an injective simplification of this morphism with respect to \(h\), \(k\) and \(i\).

Requires this morphism to be an endomorphism.

This methods basically calls `simplify_alphabet_size()` until the returned simplification is injective. If this morphism is already injective, a quadruplet \((g, h, k, i)\) is still returned, where \(g\) is this morphism, \(h\) and \(k\) are the identity morphisms and \(i\) is 0.

Let \(f : X^* \to Y^*\) be a morphism and \(Y \subseteq X\). Then \(g : Z^* \to Z^*\) is an injective simplification of \(f\) with respect to morphisms \(h : X^* \to Z^*\) and \(k : Z^* \to Y^*\) and a positive integer \(i\), if \(g\) is injective, \(|Z| < |X|\), \(g^i = h \circ k\) and \(f^i = k \circ h\).

For more information see Section 4 in [KO2000].

EXAMPLES:

```python
sage: f = WordMorphism('a->abc,b->a,c->bc')
sage: g, h, k, i = f.simplify_until_injective(); g, h, k, i
(WordMorphism: a->aa, b->a, c->bc, WordMorphism: a->abc, b->a, c->bc, WordMorphism: a->abc, 2)
sage: g.is_injective()
True
sage: g**i == h * k
True
sage: f**i == k * h
True
```

sage.combinat.words.morphism.get_cycles(f, domain)

Return the list of cycles of the function \(f\) contained in domain.

INPUT:

- \(f\) - function.
- \(domain\) - iterable, a subdomain of the domain of definition of \(f\).

EXAMPLES:
5.1.360 Word paths

This module implements word paths, which is an application of Combinatorics on Words to Discrete Geometry. A word path is the representation of a word as a discrete path in a vector space using a one-to-one correspondence between the alphabet and a set of vectors called steps. Many problems surrounding 2d lattice polygons (such as questions of self-intersection, area, inertia moment, etc.) can be solved in linear time (linear in the length of the perimeter) using theory from Combinatorics on Words.

On the square grid, the encoding of a path using a four-letter alphabet (for East, North, West and South directions) is also known as the Freeman chain code [1,2] (see [3] for further reading).

AUTHORS:

- Arnaud Bergeron (2008): Initial version, path on the square grid

EXAMPLES:

The combinatorial class of all paths defined over three given steps:

```
sage: P = WordPaths('abc', steps=[(1,2), (-3,4), (0,-3)]); P
```
Word Paths over 3 steps

This defines a one-to-one correspondence between alphabet and steps:

```
sage: d = P.letters_to_steps()
sage: sorted(d.items())
[('a', (1, 2)), ('b', (-3, 4)), ('c', (0, -3))]
```

Creation of a path from the combinatorial class P defined above:

```
sage: p = P('abaccba'); p
Path: abaccba
```

Many functions can be used on p: the coordinates of its trajectory, ask whether p is a closed path, plot it and many other:

```
sage: list(p.points())
[(0, 0), (1, 2), (-2, 6), (-1, 8), (-1, 5), (-1, 2), (-4, 6), (-3, 8)]
sage: p.is_closed()
False
```

(continues on next page)
To obtain a list of all the available word path specific functions, use help(p):

```
sage: help(p)
Help on FiniteWordPath_2d_str in module sage.combinat.words.paths object:
...
Methods inherited from FiniteWordPath_2d:
...
Methods inherited from FiniteWordPath_all:
...
```

Since p is a finite word, many functions from the word library are available:

```
sage: p.crochemore_factorization()
(a, b, a, c, c, ba)
sage: p.is_palindrome()
False
sage: p[:3]
Path: aba
sage: len(p)
7
```

P also herits many functions from Words:

```
sage: P = WordPaths('rs', steps=[(1,2), (-1,4)]); P
Word Paths over 2 steps
sage: P.alphabet()
{'r', 's'}
sage: list(P.iterate_by_length(3))
[Path: rrr,
 Path: rrs,
 Path: rsr,
 Path: rss,
 Path: srr,
 Path: srs,
 Path: ssr,
 Path: sss]
```

When the number of given steps is half the size of alphabet, the opposite of vectors are used:

```
sage: P = WordPaths('abcd', [(1,0), (0,1)]);
P
Word Paths over 2 steps
sage: sorted(P.letters_to_steps().items())
[('a', (1, 0)), ('b', (0, 1)), ('c', (-1, 0)), ('d', (0, -1))]
```

Some built-in combinatorial classes of paths:

```
sage: P = WordPaths('abAB', steps='square_grid'); P
Word Paths on the square grid
sage: D = WordPaths('()', steps='dyck'); D
Finite Dyck paths
```

(continues on next page)
sage: d = D('()', ''); d
Path: ()

sage: d.plot()
Graphics object consisting of 3 graphics primitives

sage: P = WordPaths('abcdef', steps='triangle_grid')

sage: p = P('babadefadabcdefafafabacdefa')

sage: p.plot()
Graphics object consisting of 3 graphics primitives

Vector steps may be in more than 2 dimensions:

sage: d = [(1,0,0), (0,1,0), (0,0,1)]

sage: P = WordPaths(alphabet='abc', steps=d); P
Word Paths over 3 steps

sage: p = P('abcabcabcaababcababcacabcacabcacacbacbcac')

sage: p.plot()
Graphics3d Object

sage: d = [(1,3,5,1), (-5,1,-6,0), (0,0,1,9), (4,2,-1,0)]

sage: P = WordPaths(alphabet='rstu', steps=d); P
Word Paths over 4 steps

sage: p = P('rtusuusususuturrsust'); p
Path: rtusuusususuturrsust

sage: p.end_point()
(5, 31, -26, 30)

sage: CubePaths = WordPaths('abcABC', steps='cube_grid'); CubePaths
Word Paths on the cube grid

sage: CubePaths('abcabaabcabAAAAA').plot()
Graphics3d Object

The input data may be a str, a list, a tuple, a callable or a finite iterator:

sage: P = WordPaths([0, 1, 2, 3])

sage: P([0,1,2,3,2,1,2,3,2])
Path: 01232132

sage: P([0,1,2,3,2,1,2,3,2])
Path: 01232132

sage: P(lambda n:n%4, length=10)
Path: 0123012301

sage: P(iter([0,3,2,1]), length='finite')
Path: 0321

REFERENCES:

Combinatorics, Release 10.0

- [5] Wikipedia article Dyck_word

```python
class sage.combinat.words.paths.FiniteWordPath_2d
    Bases: FiniteWordPath_all

    animate()

    Returns an animation object illustrating the path growing step by step.

    EXAMPLES:

        sage: P = WordPaths('abAB')
        sage: p = P('aababbb')
        sage: a = p.animate(); print(a)
        Animation with 9 frames
        sage: show(a) # long time # optional -- ImageMagick
        sage: show(a, delay=35, iterations=3) # long time # optional -- ImageMagick

        sage: P = WordPaths('abcdef', steps='triangle')
        sage: p = P('abcdef')
        sage: a = p.animate(); print(a)
        Animation with 8 frames
        sage: show(a) # long time # optional -- ImageMagick

    If the path is closed, the plain polygon is added at the end of the animation:

        sage: P = WordPaths('abAB')
        sage: p = P('ababAbABABaB')
        sage: a = p.animate(); print(a)
        Animation with 14 frames
        sage: show(a) # long time # optional -- ImageMagick

    Another example illustrating a Fibonacci tile:

        sage: w = words.fibonacci_tile(2)
        sage: a = w.animate(); print(a)
        Animation with 54 frames
        sage: show(a) # long time # optional -- ImageMagick

    The first 4 Fibonacci tiles in an animation:

        sage: a = words.fibonacci_tile(0).animate()
        sage: b = words.fibonacci_tile(1).animate()
        sage: c = words.fibonacci_tile(2).animate()
        sage: d = words.fibonacci_tile(3).animate()
        sage: print(a*b*c*d)
        Animation with 296 frames
        sage: show(a*b*c*d) # long time # optional -- ImageMagick

    Note: If ImageMagick is not installed, you will get an error message like this:

        convert: not found
```

(continues on next page)
Error: ImageMagick does not appear to be installed. Saving an animation to a GIF file or displaying an animation requires ImageMagick, so please install it and try again.

See www.imagemagick.org, for example.

**area()**

Returns the area of a closed path.

**INPUT:**

- `self` - a closed path

**EXAMPLES:**

```sage
sage: P = WordPaths('abcd', steps=[(1,1),(-1,1),(-1,-1),(1,-1)])
sage: p = P('abcd')
sage: p.area()
#todo: not implemented
2
```

**height()**

Returns the height of `self`.

The height of a $2d$-path is merely the difference between the highest and the lowest $y$-coordinate of each points traced by it.

**OUTPUT:**

- non negative real number

**EXAMPLES:**

```sage
sage: Freeman = WordPaths('abAB')
sage: Freeman('aababaabbAA').height()
5
```

The function is well-defined if `self` is not simple or close:

```sage
sage: Freeman('aabAB').height()
1
sage: Freeman('abbABA').height()
2
```

This works for any $2d$-paths:

```sage
sage: Paths = WordPaths('ab', steps=[(1,0),(1,1)])
sage: p = Paths('abbaa')
sage: p.height()
2
sage: DyckPaths = WordPaths('ab', steps='dyck')
sage: p = DyckPaths('abaabb')
sage: p.height()
2
sage: w = WordPaths('abcABC', steps='triangle')('ababcaABC')
sage: w.height()
2.59807621135332
```
height_vector()

Return the height at each point.

EXAMPLES:

```python
sage: Paths = WordPaths('ab', steps=[(1,0),(0,1)])
sage: p = Paths('abba')
sage: p.height_vector()
[0, 0, 1, 2, 3, 3]
```

plot(pathoptions={'rgbcolor': 'red', 'thickness': 3}, fill=True, filloptions={'rgbcolor': 'red', 'alpha': 0.2}, startpoint=True, startoptions={'rgbcolor': 'red', 'pointsize': 100}, endarrow=True, arrowoptions={'rgbcolor': 'red', 'arrowsize': 20, 'width': 3}, gridlines=False, gridoptions={})

Returns a 2d Graphics illustrating the path.

INPUT:

- pathoptions - (dict, default:dict(rgbcolor=‘red’,thickness=3)), options for the path drawing
- fill - (boolean, default: True), if fill is True and if the path is closed, the inside is colored
- filloptions - (dict, default:dict(rgbcolor=‘red’,alpha=0.2)), options for the inside filling
- startpoint - (boolean, default: True), draw the start point?
- startoptions - (dict, default:dict(rgbcolor=‘red’,pointsize=100)) options for the start point drawing
- endarrow - (boolean, default: True), draw an arrow end at the end?
- arrowoptions - (dict, default:dict(rgbcolor=‘red’,arrowsize=20, width=3)) options for the end point arrow
- gridlines- (boolean, default: False), show gridlines?
- gridoptions - (dict, default: {}), options for the gridlines

EXAMPLES:

A non closed path on the square grid:

```python
sage: P = WordPaths('abAB')
sage: P('abababAABAB').plot()
Graphics object consisting of 3 graphics primitives
```

A closed path on the square grid:

```python
sage: P('ababABAB').plot()
Graphics object consisting of 4 graphics primitives
```

A Dyck path:

```python
sage: P = WordPaths('()', steps='dyck')
sage: P('()()((())').plot()
Graphics object consisting of 3 graphics primitives
```

A path in the triangle grid:

```python
sage: P = WordPaths('abcdef', steps='triangle_grid')
sage: P('abcdedededefab').plot()
Graphics object consisting of 3 graphics primitives
```
A polygon of length 220 that tiles the plane in two ways:

```python
sage: P = WordPaths('abAB')
sage: P(→'aBabAbabaBaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaBabaB```n

With gridlines:

```python
sage: P('ababababab').plot(gridlines=True)
```

**plot_directive_vector** *(options={’rgbcolor’: ’blue’})*

Returns an arrow 2d graphics that goes from the start of the path to the end.

**INPUT:**

- options - dictionary, default: {’rgbcolor’: ‘blue’} graphic options for the arrow

If the start is the same as the end, a single point is returned.

**EXAMPLES:**

```python
sage: P = WordPaths('abcd'); P
Word Paths on the square grid
sage: p = P('aaaccacacacacacaccccbdd'); p
Path: aaaccacacacacacaccccbdd
sage: R = p.plot() + p.plot_directive_vector()
sage: R.axes(False)
sage: R.set_aspect_ratio(1)
sage: R.plot()
Graphics object consisting of 4 graphics primitives
```

**width()**

Returns the width of self.

The height of a $2d$-path is merely the difference between the rightmost and the leftmost $x$-coordinate of each points traced by it.

**OUTPUT:**

- non negative real number

**EXAMPLES:**

```python
sage: Freeman = WordPaths('abAB')
sage: Freeman('aababaabbAA').width()
5
```

The function is well-defined if self is not simple or close:

```python
sage: Freeman('aabAAB').width()
2
sage: Freeman('ABBa').width()
1
```

This works for any $2d$-paths:
```python
sage: Paths = WordPaths('ab', steps=[(1,0),(1,1)])
sage: p = Paths('abbaa')
sage: p.width()
5
sage: DyckPaths = WordPaths('ab', steps='dyck')
sage: p = DyckPaths('abaabb')
sage: p.width()
6
sage: w = WordPaths('abcABC', steps='triangle')('ababcaaaBC')
sage: w.width()
4.5
```

**width_vector()**

Return the width at each point.

EXAMPLES:

```python
sage: Paths = WordPaths('ab', steps=[(1,0),(0,1)])
sage: p = Paths('abbba')
sage: p.width_vector()
[0, 1, 1, 1, 1, 2]
```

**xmax()**

Returns the maximum of the x-coordinates of the path.

EXAMPLES:

```python
sage: P = WordPaths('0123')
sage: p = P('0101013332')
sage: p.xmax()
3
```

This works for any 2d-paths:

```python
sage: Paths = WordPaths('ab', steps=[(1,-1),(-1,1)])
sage: p = Paths('ababa')
sage: p.xmax()
1
sage: DyckPaths = WordPaths('ab', steps='dyck')
sage: p = DyckPaths('abaabb')
sage: p.xmax()
6
sage: w = WordPaths('abcABC', steps='triangle')('ababcaaaBC')
sage: w.xmax()
4.5
```

**xmin()**

Returns the minimum of the x-coordinates of the path.

EXAMPLES:

```python
sage: P = WordPaths('0123')
sage: p = P('0101013332')
sage: p.xmin()
0
```
This works for any $2d$-paths:

```python
sage: Paths = WordPaths('ab', steps=[(1,0),(-1,1)])
sage: p = Paths('abbaa')
sage: p.xmin()
-2
sage: DyckPaths = WordPaths('ab', steps='dyck')
sage: p = DyckPaths('abaabb')
sage: p.xmin()
0
sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
sage: w.xmin()
0.000000000000000
```

**ymin()**

Returns the minimum of the y-coordinates of the path.

**EXAMPLES:**

```python
sage: P = WordPaths('0123')
sage: p = P('0101013332')
sage: p.ymin()
0
```

This works for any $2d$-paths:

```python
sage: Paths = WordPaths('ab', steps=[(1,-1),(-1,1)])
sage: p = Paths('ababa')
sage: p.ymin()
0
sage: DyckPaths = WordPaths('ab', steps='dyck')
sage: p = DyckPaths('abaabb')
sage: p.ymin()
2
sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
sage: w.ymin()
2.59807621135332
```

**ymax()**

Returns the maximum of the y-coordinates of the path.

**EXAMPLES:**

```python
sage: P = WordPaths('0123')
sage: p = P('0101013332')
sage: p.ymax()
3
```

This works for any $2d$-paths:

```python
sage: Paths = WordPaths('ab', steps=[(1,-1),(-1,1)])
sage: p = Paths('ababa')
sage: p.ymax()
0
sage: DyckPaths = WordPaths('ab', steps='dyck')
sage: p = DyckPaths('abaabb')
sage: p.ymax()
2
sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
sage: w.ymax()
2.59807621135332
```
sage: p = DyckPaths('abaabb')
sage: p.ymin()
0
sage: w = WordPaths('abcABC', steps='triangle')('ababcaabC')
sage: w.ymin()
0.000000000000000

5.1. Comprehensive Module List
class sage.combinat.words.paths.FiniteWordPath_3d_callable(parent, callable, length=None)
   Bases: WordDatatype_callable, FiniteWordPath_3d_callable

class sage.combinat.words.paths.FiniteWordPath_3d_callable_with_caching(parent, callable, length=None)
   Bases: WordDatatype_callable_with_caching, FiniteWordPath_3d_callable

class sage.combinat.words.paths.FiniteWordPath_3d_iter(parent, iter, length=None)
   Bases: WordDatatype_iter, FiniteWordPath_3d_iter

class sage.combinat.words.paths.FiniteWordPath_3d_iter_with_caching(parent, iter, length=None)
   Bases: WordDatatype_iter_with_caching, FiniteWordPath_3d_iter

class sage.combinat.words.paths.FiniteWordPath_3d_list
   Bases: WordDatatype_list, FiniteWordPath_3d

class sage.combinat.words.paths.FiniteWordPath_3d_str
   Bases: WordDatatype_str, FiniteWordPath_3d

class sage.combinat.words.paths.FiniteWordPath_3d_tuple
   Bases: WordDatatype_tuple, FiniteWordPath_3d

class sage.combinat.words.paths.FiniteWordPath_all
   Bases: SageObject

directive_vector()  
   Returns the directive vector of self.
   
   The directive vector is the vector starting at the start point and ending at the end point of the path self.
   
   EXAMPLES:

   \[
   \begin{align*}
   \text{sage: } & \text{WordPaths('abcdef')('ababab').directive_vector()} \\
   & (6, 2*sqrt3) \\
   \text{sage: } & \text{WordPaths('abAB')('ababab').directive_vector()} \\
   & (4, 4) \\
   \text{sage: } & \text{P = WordPaths('abcABC', steps='cube_grid')} \\
   \text{sage: } & \text{P('abababCC').directive_vector()} \\
   & (4, 4, -2) \\
   \text{sage: } & \text{WordPaths('abcdef')('abcdef').directive_vector()} \\
   & (0, 0) \\
   \text{sage: } & \text{P = WordPaths('abc', steps=[(1,3,7,9),(-4,1,0,0),(0,32,1,8)])} \\
   \text{sage: } & \text{P('abcabababacaacccbbcac').directive_vector()} \\
   & (-16, 254, 63, 128)
   \end{align*}
   \]

end_point()  
   Returns the end point of the path.
   
   EXAMPLES:

   \[
   \begin{align*}
   \text{sage: } & \text{WordPaths('abcdef')('ababab').end_point()} \\
   & (6, 2*sqrt3)
   \end{align*}
   \]
sage: WordPaths('abAB')('abababab').end_point()
(4, 4)
sage: P = WordPaths('abcABC', steps='cube_grid')
sage: P('ababababCC').end_point()
(4, 4, -2)
sage: WordPaths('abcdef')('abcdef').end_point()
(0, 0)
sage: P = WordPaths('abc', steps=[(1,3,7,9),(-4,1,0,0),(0,32,1,8)])
sage: P('abcabababacaacccbbcac').end_point()
(-16, 254, 63, 128)

\textbf{is\_closed()}

Returns True if the path is closed, i.e. if the origin and the end of the path are equal.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P = WordPaths('abcd', steps=[[1,0],[0,1],[-1,0],[0,-1]])
sage: P('abcd').is_closed()
True
sage: P('abc').is_closed()
False
sage: P('').is_closed()
True
sage: P('aacacc').is_closed()
True
\end{verbatim}

\textbf{is\_simple()}

Returns True if the path is simple, i.e. if all its points are distincts.

If the path is closed, the last point is not considered.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P = WordPaths('abcdef', steps='triangle_grid');P
Word Paths on the triangle grid
sage: P('abc').is_simple()
True
sage: P('abcde').is_simple()
True
sage: P('abcdef').is_simple()
True
sage: P('ad').is_simple()
True
sage: P('aabdee').is_simple()
False
\end{verbatim}

\textbf{is\_tangent()}

The \texttt{is\_tangent()} method, which is implemented for words, has an extended meaning for word paths, which is not implemented yet.

\textbf{AUTHOR:}

- Thierry Monteil
\texttt{plot\_projection}(v=None, letters=None, color=None, ring=None, size=12, kind='right')

Return an image of the projection of the successive points of the path into the space orthogonal to the given vector.

INPUT:

- \texttt{self} - a word path in a 3 or 4 dimension vector space
- \texttt{v} - vector (optional, default: None) If None, the directive vector (i.e. the end point minus starting point) of the path is considered.
- \texttt{letters} - iterable (optional, default: None) of the letters to be projected. If None, then all the letters are considered.
- \texttt{color} - dictionary (optional, default: None) of the letters mapped to colors. If None, automatic colors are chosen.
- \texttt{ring} - ring (optional, default: None) where to do the computations. If None, RealField(53) is used.
- \texttt{size} - number (optional, default: 12) size of the points.
- \texttt{kind} - string (optional, default 'right') either 'right' or 'left'. The color of a letter is given to the projected prefix to the right or the left of the letter.

OUTPUT:

2d or 3d Graphic object.

EXAMPLES:

The Rauzy fractal:

\begin{Verbatim}
\texttt{sage: s = WordMorphism('1->12,2->13,3->1')}
\texttt{sage: D = s.fixed_point('1')}
\texttt{sage: v = s.pisot_eigenvector_right()}
\texttt{sage: P = WordPaths('123',[(1,0,0),(0,1,0),(0,0,1)])}
\texttt{sage: w = P(D[:200])}
\texttt{sage: w.plot\_projection(v)}  \# long time (2s)
\end{Verbatim}

Graphics object consisting of 200 graphics primitives

In this case, the abelianized vector doesn't give a good projection:

\begin{Verbatim}
\texttt{sage: w.plot\_projection()}  \# long time (2s)
\end{Verbatim}

Graphics object consisting of 200 graphics primitives

You can project only the letters you want:

\begin{Verbatim}
\texttt{sage: w.plot\_projection(v, letters='12')}  \# long time (2s)
\end{Verbatim}

Graphics object consisting of 168 graphics primitives

You can increase or decrease the precision of the computations by changing the ring of the projection matrix:

\begin{Verbatim}
\texttt{sage: w.plot\_projection(v, ring=RealField(20))}  \# long time (2s)
\end{Verbatim}

Graphics object consisting of 200 graphics primitives

You can change the size of the points:

\begin{Verbatim}
\texttt{sage: w.plot\_projection(v, size=30)}  \# long time (2s)
\end{Verbatim}

Graphics object consisting of 200 graphics primitives
You can assign the color of a letter to the projected prefix to the right or the left of the letter:

```
sage: w.plot_projection(v, kind='left')  # long time (2s)
Graphics object consisting of 200 graphics primitives
```

To remove the axis, do like this:

```
sage: r = w.plot_projection(v)
sage: r.axes(False)
sage: r  # long time (2s)
Graphics object consisting of 200 graphics primitives
```

You can assign different colors to each letter:

```
sage: color = {'1':'purple', '2':(.2,.3,.4), '3': 'magenta'}
sage: w.plot_projection(v, color=color)  # long time (2s)
Graphics object consisting of 200 graphics primitives
```

The 3d-Rauzy fractal:

```
sage: s = WordMorphism('1->12,2->13,3->14,4->1')
sage: D = s.fixed_point('1')
sage: v = s.pisot_eigenvector_right()
sage: P = WordPaths('1234',[(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)])
sage: w = P(D[:200])
sage: w.plot_projection(v)
Graphics3d Object
```

The dimension of vector space of the parent must be 3 or 4:

```
sage: P = WordPaths('ab', [(1, 0), (0, 1)])
sage: p = P('aabbabbab')
sage: p.plot_projection()
Traceback (most recent call last):
...  
TypeError: The dimension of the vector space (=2) must be 3 or 4
```

```
points(include_last=True)
```

Returns an iterator yielding a list of points used to draw the path represented by this word.

**INPUT:**

- `include_last` - bool (default: True) whether to include the last point

**EXAMPLES:**

A simple closed square:

```
sage: P = WordPaths('abAB')
sage: list(P('abAB').points())
[[(0, 0), (1, 0), (1, 1), (0, 1), (0, 0)]
```

A simple closed square without the last point:

```
sage: list(P('abAB').points(include_last=False))
[[(0, 0), (1, 0), (1, 1), (0, 1)]
```
Combinatorics, Release 10.0

\begin{verbatim}
sage: list(P('abaB').points())
[(0, 0), (1, 0), (1, 1), (2, 1), (2, 0)]
\end{verbatim}

\textbf{projected\_path}(v=None, ring=None)

Return the path projected into the space orthogonal to the given vector.

**INPUT:**

- \textbf{v} - vector (optional, default: None) If None, the directive vector (i.e. the end point minus starting point) of the path is considered.
- \textbf{ring} - ring (optional, default: None) where to do the computations. If None, RealField(53) is used.

**OUTPUT:**

word path

**EXAMPLES:**

The projected path of the tribonacci word:

\begin{verbatim}
sage: s = WordMorphism('1->12,2->13,3->1')
sage: D = s.fixed_point('1')
sage: v = s.pisot_eigenvector_right()
sage: P = WordPaths('123',[(1,0,0),(0,1,0),(0,0,1)])
sage: w = P(D[:1000])
sage: p = w.projected_path(v)
sage: p
Path: 1213121121312121312112131213121121312121...
sage: p[:20].plot()
Graphics object consisting of 3 graphics primitives
\end{verbatim}

The \textbf{ring} argument allows to change the precision of the projected steps:

\begin{verbatim}
sage: p = w.projected_path(v, RealField(10))
sage: p
Path: 1213121121312121312112131213121121312121...
sage: p.parent().letters_to_steps()
{'1': (-0.53, 0.00), '2': (0.75, -0.48), '3': (0.41, 0.88)}
\end{verbatim}

\textbf{projected\_point\_iterator}(v=None, ring=None)

Return an iterator of the projection of the orbit points of the path into the space orthogonal to the given vector.

**INPUT:**

- \textbf{v} - vector (optional, default: None) If None, the directive vector (i.e. the end point minus starting point) of the path is considered.
- \textbf{ring} - ring (optional, default: None) where to do the computations. If None, RealField(53) is used.

**OUTPUT:**

iterator of points

**EXAMPLES:**

Projected points of the Rauzy fractal:
sage: s = WordMorphism('1->12,2->13,3->1')
sage: D = s.fixed_point('1')
sage: v = s.pisot_eigenvector_right()
sage: P = WordPaths('123',[[1,0,0],[0,1,0],[0,0,1]])
sage: w = P(D[:200])
sage: it = w.projected_point_iterator(v)

sage: for i in range(6): next(it)
(0.000000000000000, 0.000000000000000)
(-0.526233343362516, 0.000000000000000)
(0.220830337618112, -0.477656250512816)
(-0.305403005744404, -0.477656250512816)
(0.100767309386062, 0.400890564600664)
(-0.425466033976454, 0.400890564600664)

Projected points of a 2d path:

sage: P = WordPaths('ab','ne')
sage: p = P('aabbabab')
sage: it = p.projected_point_iterator(ring=RealField(20))

sage: for i in range(8): next(it)
(0.00000)
(0.78087)
(1.5617)
(0.93704)
(0.31235)
(1.0932)
(0.46852)
(-0.15617)

start_point()

Return the starting point of self.

OUTPUT:

vector

EXAMPLES:

sage: WordPaths('abcdef')(abcdef).start_point()
(0, 0)
sage: WordPaths('abcdef', steps='cube_grid')(abcdef).start_point()
(0, 0, 0)
sage: P = WordPaths('ab', steps=[(1,0,0,0),(0,1,0,0)])
sage: P('abbba').start_point()
(0, 0, 0, 0)

tikz_trajectory()

Returns the trajectory of self as a tikz str.

EXAMPLES:

sage: P = WordPaths('abcdef')
sage: p = P('abcde')
sage: p.tikz_trajectory()
class sage.combinat.words.paths.FiniteWordPath_all_callable(parent, callable, length=None)
    Bases: WordDatatype_callable, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_all_callable_with_caching(parent, callable, length=None)
    Bases: WordDatatype_callable_with_caching, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_all_iter(parent, iter, length=None)
    Bases: WordDatatype_iter, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_all_iter_with_caching(parent, iter, length=None)
    Bases: WordDatatype_iter_with_caching, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_all_list
    Bases: WordDatatype_list, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_all_str
    Bases: WordDatatype_str, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_all_tuple
    Bases: WordDatatype_tuple, FiniteWordPath_all, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid
    Bases: FiniteWordPath_3d

class sage.combinat.words.paths.FiniteWordPath_cube_grid_callable(parent, callable, length=None)
    Bases: WordDatatype_callable, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_callable_with_caching(parent, callable, length=None)
    Bases: WordDatatype_callable_with_caching, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_iter(parent, iter, length=None)
    Bases: WordDatatype_iter, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_iter_with_caching(parent, iter, length=None)
    Bases: WordDatatype_iter_with_caching, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_list
    Bases: WordDatatype_list, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_str
    Bases: WordDatatype_str, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_tuple
    Bases: WordDatatype_tuple, FiniteWordPath_cube_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_cube_grid_dyck
    Bases: FiniteWordPath_2d
class sage.combinat.words.paths.FiniteWordPath_dyck_callable
    Bases: WordDatatype_callable, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_dyck_callable_with_caching
    Bases: WordDatatype_callable_with_caching, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_dyck_iter
    Bases: WordDatatype_iter, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_dyck_iter_with_caching
    Bases: WordDatatype_iter_with_caching, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_dyck_list
    Bases: WordDatatype_list, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_dyck_str
    Bases: WordDatatype_str, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_dyck_tuple
    Bases: WordDatatype_tuple, FiniteWordPath_dyck, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_hexagonal_grid
    Bases: FiniteWordPath_triangle_grid

INPUT:

• parent - a parent object inheriting from Words_all that has the alphabet attribute defined
• *args, **kwds - arguments accepted by AbstractWord

EXAMPLES:

sage: F = WordPaths('abcdef', steps='hexagon'); F
Word Paths on the hexagonal grid
sage: f = F('aaabbbccdef'); f
Path: aaabbbccdef

sage: f == loads(dumps(f))
True
class sage.combinat.words.paths.FiniteWordPath_hexagonal_grid_list
    Bases: WordDatatype_list, FiniteWordPath_hexagonal_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_hexagonal_grid_str
    Bases: WordDatatype_str, FiniteWordPath_hexagonal_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_hexagonal_grid_tuple
    Bases: WordDatatype_tuple, FiniteWordPath_hexagonal_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east
    Bases: FiniteWordPath_2d

class sage.combinat.words.paths.FiniteWordPath_north_east_callable
    (parent, callable, length=None)
    Bases: WordDatatype_callable, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east_callable_with_caching
    (parent, callable, length=None)
    Bases: WordDatatype_callable_with_caching, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east_iter
    (parent, iter, length=None)
    Bases: WordDatatype_iter, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east_iter_with_caching
    (parent, iter, length=None)
    Bases: WordDatatype_iter_with_caching, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east_list
    Bases: WordDatatype_list, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east_str
    Bases: WordDatatype_str, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_north_east_tuple
    Bases: WordDatatype_tuple, FiniteWordPath_north_east, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_square_grid
    Bases: FiniteWordPath_2d

area()
    Returns the area of a closed path.

    INPUT:
     - self - a closed path

    EXAMPLES:

    sage: P = WordPaths('abAB', steps='square_grid')
    sage: P('abAB').area()
    1
    sage: P('aabbAABB').area()
    4
    sage: P('aabbABAB').area()
    3

    The area of the Fibonacci tiles:
### is_closed()

Returns True if self represents a closed path and False otherwise.

**EXAMPLES:**

```python
sage: P = WordPaths('abAB', steps='square_grid')
sage: P('aA').is_closed()
True
sage: P('abAB').is_closed()
True
sage: P('ababAABB').is_closed()
True
sage: P('aaabbbAABB').is_closed()
False
sage: P('ab').is_closed()
False
```

### is_simple()

Returns True if the path is simple, i.e. if all its points are distincts.

If the path is closed, the last point is not considered.
Note: The linear algorithm described in the thesis of Xavier Provençal should be implemented here.

EXAMPLES:

```
sage: P = WordPaths('abAB', steps='square_grid')
sage: P('abab').is_simple()
True
sage: P('abAB').is_simple()
True
sage: P('abA').is_simple()
True
sage: P('aabABB').is_simple()
False
sage: P().is_simple()
True
sage: P('A').is_simple()
True
sage: P('aA').is_simple()
True
sage: P('aaA').is_simple()
False
```

REFERENCES:


```
tikz_trajectory()
Returns the trajectory of self as a tikz str.
```

EXAMPLES:

```
sage: f = words.fibonacci_tile(1)
sage: f.tikz_trajectory()
'(0, 0) -- (0, -1) -- (-1, -1) -- (-1, -2) -- (0, -2) -- (0, -3) -- (1, -3) -- (1, -2) -- (2, -2) -- (2, -1) -- (1, -1) -- (1, 0) -- (0, 0)'
```

```python
class sage.combinat.words.paths.FiniteWordPath_square_grid_callable(parent, callable, length=None)

Bases: WordDatatype_callable, FiniteWordPath_square_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_square_grid_callable_with_caching(parent, callable, length=None)

Bases: WordDatatype_callable_with_caching, FiniteWordPath_square_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_square_grid_iter(parent, iter, length=None)

Bases: WordDatatype_iter, FiniteWordPath_square_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_square_grid_iter_with_caching(parent, iter, length=None)

Bases: WordDatatype_iter_with_caching, FiniteWordPath_square_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_square_grid_list

Bases: WordDatatype_list, FiniteWordPath_square_grid, FiniteWord_class
```
class sage.combinat.words.paths.FiniteWordPath_square_grid_str
    Bases: WordDatatype_str, FiniteWordPath_square_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_square_grid_tuple
    Bases: WordDatatype_tuple, FiniteWordPath_square_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid
    Bases: FiniteWordPath_2d

    xmax()
        Returns the maximum of the x-coordinates of the path.
        EXAMPLES:

        sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
        sage: w.xmax()
        4.50000000000000
        sage: w = WordPaths('abcABC', steps='triangle')('ABAcacacabababcacbAC')
        sage: w.xmax()
        4.00000000000000

    xmin()
        Returns the minimum of the x-coordinates of the path.
        EXAMPLES:

        sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
        sage: w.xmin()
        0.000000000000000
        sage: w = WordPaths('abcABC', steps='triangle')('ABAcacacabababcacbAC')
        sage: w.xmin()
        -3.00000000000000

    ymax()
        Returns the maximum of the y-coordinates of the path.
        EXAMPLES:

        sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
        sage: w.ymax()
        2.59807621135332
        sage: w = WordPaths('abcABC', steps='triangle')('ABAcacacabababcacbAC')
        sage: w.ymax()
        8.66025403784439

    ymin()
        Returns the minimum of the y-coordinates of the path.
        EXAMPLES:

        sage: w = WordPaths('abcABC', steps='triangle')('ababcaaBC')
        sage: w.ymin()
        0.000000000000000
        sage: w = WordPaths('abcABC', steps='triangle')('ABAcacacabababcacbAC')
        sage: w.ymin()
        -0.866025403784439
class sage.combinat.words.paths.FiniteWordPath_triangle_grid_callable(parent, callable, length=None)

Bases: WordDatatype_callable, FiniteWordPath_triangle_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid_callable_with_caching(parent, callable, length=None)

Bases: WordDatatype_callable_with_caching, FiniteWordPath_triangle_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid_iter(parent, iter, length=None)

Bases: WordDatatype_iter, FiniteWordPath_triangle_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid_iter_with_caching(parent, iter, length=None)

Bases: WordDatatype_iter_with_caching, FiniteWordPath_triangle_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid_list

Bases: WordDatatype_list, FiniteWordPath_triangle_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid_str

Bases: WordDatatype_str, FiniteWordPath_triangle_grid, FiniteWord_class

class sage.combinat.words.paths.FiniteWordPath_triangle_grid_tuple

Bases: WordDatatype_tuple, FiniteWordPath_triangle_grid, FiniteWord_class

sage.combinat.words.paths.WordPaths(alphabet, steps=None)

Returns the combinatorial class of paths of the given type of steps.

INPUT:

- alphabet - ordered alphabet
- steps - (default is None). It can be one of the following:
  - an iterable ordered container of as many vectors as there are letters in the alphabet. The vectors are associated to the letters according to their order in steps. The vectors can be a tuple or anything that can be passed to vector function.
  - an iterable ordered container of k vectors where k is half the size of alphabet. The vectors and their opposites are associated to the letters according to their order in steps (given vectors first, opposite vectors after).
  - None: In this case, the type of steps are guessed from the length of alphabet.
  - ‘square_grid’ or ‘square’: (default when size of alphabet is 4) The order is : East, North, West, South.
  - ‘triangle_grid’ or ‘triangle’:
  - ‘hexagonal_grid’ or ‘hexagon’: (default when size of alphabet is 6)
  - ‘cube_grid’ or ‘cube’:
  - ‘north_east’, ‘ne’ or ‘NE’: (the default when size of alphabet is 2)
  - ‘dyck’:

OUTPUT:

- The combinatorial class of all paths of the given type.

EXAMPLES:

The steps can be given explicitly:
<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>sage: WordPaths('abc', steps=[(1,2), (-1,4), (0,-3)])</code></td>
<td>Word Paths over 3 steps</td>
</tr>
<tr>
<td>Different type of input alphabet:</td>
<td></td>
</tr>
<tr>
<td><code>sage: WordPaths(range(3), steps=[(1,2), (-1,4), (0,-3)])</code></td>
<td>Word Paths over 3 steps</td>
</tr>
<tr>
<td><code>sage: WordPaths(['cric', 'crac', 'croc'], steps=[(1,2), (1,4), (0,3)])</code></td>
<td>Word Paths over 3 steps</td>
</tr>
<tr>
<td>Directions can be in three dimensions as well:</td>
<td></td>
</tr>
<tr>
<td><code>sage: WordPaths('ab', steps=[(1,2,2),(-1,4,2)])</code></td>
<td>Word Paths over 2 steps</td>
</tr>
<tr>
<td>When the number of given steps is half the size of alphabet, the opposite of vectors are used:</td>
<td></td>
</tr>
<tr>
<td><code>sage: P = WordPaths('abcd', [(1,0), (0,1)])</code></td>
<td>Word Paths over 4 steps</td>
</tr>
<tr>
<td><code>sage: sorted(P.letters_to_steps().items())</code></td>
<td></td>
</tr>
<tr>
<td><code>[('a', (1, 0)), ('b', (0, 1)), ('c', (-1, 0)), ('d', (0, -1))]</code></td>
<td></td>
</tr>
<tr>
<td>When no steps are given, default classes are returned:</td>
<td></td>
</tr>
<tr>
<td><code>sage: WordPaths('ab')</code></td>
<td>Word Paths in North and East steps</td>
</tr>
<tr>
<td><code>sage: WordPaths(range(4))</code></td>
<td>Word Paths on the square grid</td>
</tr>
<tr>
<td><code>sage: WordPaths(range(6))</code></td>
<td>Word Paths on the hexagonal grid</td>
</tr>
<tr>
<td>There are many type of built-in steps…</td>
<td></td>
</tr>
<tr>
<td>On a two letters alphabet:</td>
<td></td>
</tr>
<tr>
<td><code>sage: WordPaths('ab', steps='north_east')</code></td>
<td>Word Paths in North and East steps</td>
</tr>
<tr>
<td><code>sage: WordPaths('ab', steps='dyck')</code></td>
<td>Finite Dyck paths</td>
</tr>
<tr>
<td>On a four letters alphabet:</td>
<td></td>
</tr>
<tr>
<td><code>sage: WordPaths('ruld', steps='square_grid')</code></td>
<td>Word Paths on the square grid</td>
</tr>
<tr>
<td>On a six letters alphabet:</td>
<td></td>
</tr>
<tr>
<td><code>sage: WordPaths('abcdef', steps='hexagonal_grid')</code></td>
<td>Word Paths on the hexagonal grid</td>
</tr>
<tr>
<td><code>sage: WordPaths('abcdef', steps='triangle_grid')</code></td>
<td>Word Paths on the triangle grid</td>
</tr>
<tr>
<td><code>sage: WordPaths('abcdef', steps='cube_grid')</code></td>
<td>Word Paths on the cube grid</td>
</tr>
</tbody>
</table>

5.1. Comprehensive Module List
class sage.combinat.words.paths.WordPaths_all(alphabet, steps)

Bases: FiniteWords

The combinatorial class of all paths, i.e of all words over an alphabet where each letter is mapped to a step (a vector).

letters_to_steps()

Returns the dictionary mapping letters to vectors (steps).

EXAMPLES:

```python
sage: d = WordPaths('ab').letters_to_steps()
sage: sorted(d.items())
[('a', (0, 1)), ('b', (1, 0))]

sage: d = WordPaths('abcd').letters_to_steps()
sage: sorted(d.items())
[('a', (1, 0)), ('b', (0, 1)), ('c', (-1, 0)), ('d', (0, -1))]

sage: d = WordPaths('abcdef').letters_to_steps()
sage: sorted(d.items())
[('a', (1, 0)), ('b', (1/2, 1/2*sqrt3)), ('c', (-1/2, 1/2*sqrt3)), ('d', (-1, 0)), ('e', (-1/2, -1/2*sqrt3)), ('f', (1/2, -1/2*sqrt3))]
```

vector_space()

Return the vector space over which the steps of the paths are defined.

EXAMPLES:

```python
sage: WordPaths('ab', steps='dyck').vector_space()
Ambient free module of rank 2 over the principal ideal domain Integer Ring

sage: WordPaths('ab', steps='north_east').vector_space()
Ambient free module of rank 2 over the principal ideal domain Integer Ring

sage: WordPaths('abcd', steps='square_grid').vector_space()
Ambient free module of rank 2 over the principal ideal domain Integer Ring

sage: WordPaths('abcdef', steps='hexagonal_grid').vector_space()
Vector space of dimension 2 over Number Field in sqrt3 with defining polynomial
˓→x^2 - 3 with sqrt3 = 1.732050807568878?

sage: WordPaths('abcdef', steps='cube_grid').vector_space()
Ambient free module of rank 3 over the principal ideal domain Integer Ring

sage: WordPaths('abcdef', steps='triangle_grid').vector_space()
Vector space of dimension 2 over Number Field in sqrt3 with defining polynomial
˓→x^2 - 3 with sqrt3 = 1.732050807568878?
```

class sage.combinat.words.paths.WordPaths_cube_grid(alphabet)

Bases: WordPaths_all

The combinatorial class of all paths on the cube grid.

class sage.combinat.words.paths.WordPaths_dyck(alphabet)

Bases: WordPaths_all

The combinatorial class of all Dyck paths.
class sage.combinat.words.paths.WordPaths_hexagonal_grid(alphabet)
    Bases: WordPaths_triangle_grid
    The combinatorial class of all paths on the hexagonal grid.

class sage.combinat.words.paths.WordPaths_north_east(alphabet)
    Bases: WordPaths_all
    The combinatorial class of all paths using North and East directions.

class sage.combinat.words.paths.WordPaths_square_grid(alphabet)
    Bases: WordPaths_all
    The combinatorial class of all paths on the square grid.

class sage.combinat.words.paths.WordPaths_triangle_grid(alphabet)
    Bases: WordPaths_all
    The combinatorial class of all paths on the triangle grid.

5.1.361 Shuffle product of words

See also:
The module `sage.combinat.shuffle` contains a more general implementation of shuffle product.

class sage.combinat.words.shuffle_product.ShuffleProduct_shifted(w1, w2, check=True)
    Bases: ShuffleProduct_w1w2
    Shifted shuffle product of \( w_1 \) with \( w_2 \).
    This is the shuffle product of \( w_1 \) with the word obtained by adding the length of \( w_1 \) to every letter of \( w_2 \).
    Note that this class is meant to be used for words; it misbehaves when \( w_1 \) is a permutation or composition.
    INPUT:
    - check – boolean (default True) whether to check that all words in the shuffle product belong to the correct parent
    EXAMPLES:

sage: from sage.combinat.words.shuffle_product import ShuffleProduct_shifted
sage: w, u = Word([1,2]), Word([3,4])
sage: S = ShuffleProduct_shifted(w,u)
sage: S == loads(dumps(S))
True

class sage.combinat.words.shuffle_product.ShuffleProduct_w1w2(w1, w2, check=True)
    Bases: Parent, UniqueRepresentation
    The shuffle product of the two words \( w_1 \) and \( w_2 \).
    If \( u \) and \( v \) are two words, then the shuffle product of \( u \) and \( v \) is a certain multiset of words defined as follows:
    Let \( a \) and \( b \) be the lengths of \( u \) and \( v \), respectively. For every \( a \)-element subset \( I \) of \( \{1, 2, \ldots, a + b\} \), let \( w(I) \) be the length-\( a + b \) word such that:
    - for every \( 1 \leq k \leq a \), the \( i_k \)-th letter of \( w(I) \) is the \( k \)-th letter of \( u \), where \( i_k \) is the \( k \)-th smallest element of \( I \);
    - for every \( 1 \leq l \leq b \), the \( j_l \)-th letter of \( w(I) \) is the \( l \)-th letter of \( v \), where \( j_l \) is the \( l \)-th smallest element of \( \{1, 2, \ldots, a + b\} \setminus I \).
The shuffle product of $u$ and $v$ is then the multiset of all $w(I)$ with $I$ ranging over the $a$-element subsets of \{1,2,\ldots,a+b\}.

**INPUT:**

- check – boolean (default True) whether to check that all words in the shuffle product belong to the correct parent

**EXAMPLES:**

```python
sage: from sage.combinat.words.shuffle_product import ShuffleProduct_w1w2
sage: W = Words([1,2,3,4])
sage: s = ShuffleProduct_w1w2(W([1,2]),W([3,4]))
sage: sorted(s)
[word: 1234, word: 1324, word: 1342, word: 3124, word: 3142, word: 3412]
sage: s == loads(dumps(s))
True
sage: TestSuite(s).run()

sage: s = ShuffleProduct_w1w2(W([1,4,3]),W([2]))
sage: sorted(s)
[word: 1243, word: 1423, word: 1432, word: 2143]

sage: s = ShuffleProduct_w1w2(W([1,4,3]),W([]))
sage: sorted(s)
[word: 143]
```

cardinality()

Return the number of words in the shuffle product of $w_1$ and $w_2$.

This is understood as a multiset cardinality, not as a set cardinality; it does not count the distinct words only.

It is given by \(\binom{l_1+l_2}{l_1}\), where $l_1$ is the length of $w_1$ and where $l_2$ is the length of $w_2$.

**EXAMPLES:**

```python
sage: from sage.combinat.words.shuffle_product import ShuffleProduct_w1w2
sage: w, u = map(Words("abcd"), ["ab", "cd"])
sage: S = ShuffleProduct_w1w2(w,u)
sage: S.cardinality()
6

sage: w, u = map(Words("ab"), ["ab", "ab"])
sage: S = ShuffleProduct_w1w2(w,u)
sage: S.cardinality()
6
```
5.1.362 Suffix Tries and Suffix Trees

**class** sage.combinat.words.suffix_trees.DecoratedSuffixTree(w)

   Bases: ImplicitSuffixTree

   The decorated suffix tree of a word.

   A **decorated suffix tree** of a word \( w \) is the suffix tree of \( w \) marked with the end point of all squares in the \( w \).

   The symbol \$ \ is appended to \( w \) to ensure that each final state is a leaf of the suffix tree.

   **INPUT:**
   - \( w \) – a finite word

   **EXAMPLES:**

   ```
sage: from sage.combinat.words.suffix_trees import DecoratedSuffixTree
sage: w = Word('0011001')
sage: DecoratedSuffixTree(w)
Decorated suffix tree of : 0011001$
```

**ALGORITHM:**

When using 'pair' as output, the squares are retrieved in linear time. The algorithm is an implementation of the one proposed in [DS2004].

**square_vocabulary**(output='pair')

   Return the list of distinct squares of \( self.word \).

   Two types of outputs are available *pair* and *word*. The algorithm is only truly linear if *output* is set to *pair*. A pair is a tuple \((i, l)\) that indicates the factor \( self.word()[i:i+l] \). The option 'word' return word objects.

   **INPUT:**
   - *output* – (default: "pair") either "pair" or "word"

   **EXAMPLES:**

   ```
sage: from sage.combinat.words.suffix_trees import DecoratedSuffixTree
sage: w = Word('aabb')
sage: sorted(DecoratedSuffixTree(w).square_vocabulary())
[(0, 0), (0, 2), (2, 2)]
sage: w = Word('00110011010')
sage: sorted(DecoratedSuffixTree(w).square_vocabulary(output="word"))
[word: , word: 00, word: 00110011, word: 01100110, word: 1010, word: 11]
```

**class** sage.combinat.words.suffix_trees.ImplicitSuffixTree(word)

   Bases: SageObject

   Construct the implicit suffix tree of a word \( w \).

   The suffix tree of a word \( w \) is a compactification of the suffix trie for \( w \). The compactification removes all nodes that have exactly one incoming edge and exactly one outgoing edge. It consists of two components: a tree and a word. Thus, instead of labelling the edges by factors of \( w \), we can labelled them by indices of the occurrence of the factors in \( w \).
The following is a straightforward implementation of Ukkonen’s on-line algorithm for constructing the implicit suffix tree [Ukko1995]. It constructs the suffix tree for w[:i] from that of w[:i-1].

GENERAL IDEA. The suffix tree of w[:i+1] can be obtained from that of w[:i] by visiting each node corresponding to a suffix of w[:i] and modifying the tree by applying one of two rules (either append a new node to the tree, or split an edge into two). The “active state” is the node where the algorithm begins and the “suffix link” carries us to the next node that needs to be dealt with.

TREE. The tree is modelled as an automaton, which is stored as a dictionary of dictionaries: it is keyed by the nodes of the tree, and the corresponding dictionary is keyed by pairs \((i, j)\) of integers representing the word \(w[i-1:j]\). This makes it faster to look up a particular transition beginning at a specific node.

STATES/NODES. The states will always be -1, 0, 1, ..., n. The state -1 is special and is only used for the purposes of the algorithm. All transitions map -1 to 0, so this information is not explicitly stored in the transition function.

EXPLICIT/IMPLICIT NODES. By definition, some of the nodes will not be states, but merely locations along an edge; these are called implicit nodes. A node \(r\) (implicit or explicit) is referenced as a pair \((s, (k, p))\) where \(s\) is an ancestor of \(r\) and \(w[k-1:p]\) is the word read by transitioning from \(s\) to \(r\) in the suffix trie. A reference pair is canonical if \(s\) is the closest ancestor of \(r\).

SUFFIX LINK. The algorithm makes use of a map from (some) nodes to other nodes, called the suffix link. This is stored as a dictionary.

ACTIVE STATE. We store as \(\_\_\_active\_state\) the active state of the tree, the state where the algorithm will begin when processing the next letter.

RUNNING TIME. The running time and storage space of the algorithm is linear in the length of the word \(w\) (whereas for a suffix tree it is quadratic).

REFERENCES:

- [Ukko1995]

EXAMPLES:

```
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: w = Words("aco")("cacao")
sage: t = ImplicitSuffixTree(w);
Implicit Suffix Tree of the word: cacao
sage: ababb = Words([0,1])([0,1,0,1,1])
sage: s = ImplicitSuffixTree(ababb);
Implicit Suffix Tree of the word: 01011
```

LZ_decomposition()  

Return a list of index of the beginning of the block of the Lempel-Ziv decomposition of \(self-word\).  

The Lempel-Ziv decomposition is the factorisation \(u_1...u_k\) of a word \(w = x_1...x_n\) such that \(u_i\) is the longest prefix of \(u_i...u_k\) that has an occurrence starting before \(u_i\) or a letter if this prefix is empty.

OUTPUT:

Return a list \(iB\) of index such that the blocks of the decomposition are \(self-word()[iB[k]:iB[k+1]]\)

EXAMPLES:

```
sage: w = Word('abababb')
sage: T = w.suffix_tree()
sage: T.LZ_decomposition()
[0, 1, 2, 6, 7]
```

(continues on next page)
**active_state()**

Returns the active state of the suffix tree.

**EXAMPLES:**

```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2]))
sage: t.active_state()
(0, (6, 6))
```

**edge_iterator()**

Returns an iterator over the edges of the suffix tree. The edge from $u$ to $v$ labelled by $(i, j)$ is returned as the tuple $(u, v, (i, j))$.

**EXAMPLES:**

```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: sorted( ImplicitSuffixTree(Word("aaaaa")).edge_iterator() )
[(0, 1, (0, None))]
sage: sorted( ImplicitSuffixTree(Word([0,1,0,1])).edge_iterator() )
[(0, 1, (0, None)), (0, 2, (1, None))]
sage: sorted( ImplicitSuffixTree(Word()).edge_iterator() )
[]
```

**factor_iterator**(n=None)

Generate distinct factors of self.

**INPUT:**

* n - an integer, or None.

**OUTPUT:**

* If n is an integer, returns an iterator over all distinct factors of length n. If n is None, returns an iterator generating all distinct factors.

**EXAMPLES:**

```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: sorted( ImplicitSuffixTree(Word("cacao")).factor_iterator() )
[word: , word: a, word: ac, word: aca, word: acao, word: ao, word: c, word: ca, ...
```

(continues on next page)
leftmost_covering_set()

Compute the leftmost covering set of square pairs in self.word(). Return a square as a pair \((i, l)\) designating factor \(self.word()[i:i+l]\).

A leftmost covering set is a set such that the leftmost occurrence \((j, l)\) of a square in \(self.word()\) is covered by a pair \((i, l)\) in the set for all types of squares. We say that \((j, l)\) is covered by \((i, l)\) if \((i, l)\) \((i+1,l), \ldots, (j, l)\) are all squares.

The set is returned in the form of a list \(P\) such that \(P[i]\) contains all the lengths of squares starting at \(i\) in the set. The lists \(P[i]\) are sorted in decreasing order.

The algorithm used is described in [DS2004].

EXAMPLES:

```python
sage: w = Word('abaabaabbaabaaba')
sage: T = w.suffix_tree()
sage: T.leftmost_covering_set()
[[6], [6], [2], [], [], [], [], [2], [], [], [], [], [], [], [], []]
sage: w = Word('abaca')
sage: T = w.suffix_tree()
sage: T.leftmost_covering_set()
[[], [], [], [], []]
sage: T = Word('aaaaa').suffix_tree()
sage: T.leftmost_covering_set()
[[4, 2], [], [], [], []]
```

number_of_factors\((n=None)\)

Count the number of distinct factors of self.word().

INPUT:

• \(n\) - an integer, or None.

OUTPUT:

• If \(n\) is an integer, returns the number of distinct factors of length \(n\). If \(n\) is None, returns the total number of distinct factors.

EXAMPLES:
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: t = ImplicitSuffixTree(Word([1,2,1,3,1,2,1]))
sage: t.number_of_factors()
22
sage: t.number_of_factors(1)
3
sage: t.number_of_factors(9)
0
sage: t.number_of_factors(0)
1

sage: t = ImplicitSuffixTree(Word("cacao"))
sage: t.number_of_factors()
13
sage: list(map(t.number_of_factors, range(10)))
[1, 3, 3, 3, 2, 1, 0, 0, 0, 0]

sage: t = ImplicitSuffixTree(Word("c"*1000))
sage: t.number_of_factors()
1001
sage: t.number_of_factors(17)
1
sage: t.number_of_factors(0)
1

sage: ImplicitSuffixTree(Word()).number_of_factors()
1

sage: blueberry = ImplicitSuffixTree(Word("blueberry"))
sage: blueberry.number_of_factors()
43
sage: list(map(blueberry.number_of_factors, range(10)))
[1, 6, 8, 7, 6, 5, 4, 3, 2, 1]

plot(word_labels=False, layout='tree', tree_root=0, tree_orientation='up', vertex_colors=None, edge_labels=True, *args, **kwds)

Returns a Graphics object corresponding to the transition graph of the suffix tree.

INPUT:

* word_labels - boolean (default: False) if False, labels the edges by pairs \(i, j\); if True, labels the edges by word[\(i:j\)].
* layout - (default: 'tree')
* tree_root - (default: 0)
* tree_orientation - (default: 'up')
* vertex_colors - (default: None)
* edge_labels - (default: True)

EXAMPLES:
Combinatorics, Release 10.0

 sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
 sage: ImplicitSuffixTree(Word('cacao')).plot(word_labels=True)
 Graphics object consisting of 23 graphics primitives
 sage: ImplicitSuffixTree(Word('cacao')).plot(word_labels=False)
 Graphics object consisting of 23 graphics primitives

process_letter(letter)

Modifies the current implicit suffix tree producing the implicit suffix tree for self.word() + letter.

EXAMPLES:

 sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
 sage: w = Words('aco')('cacao')
sage: t = ImplicitSuffixTree(w[:-1]); t
Implicit Suffix Tree of the word: caca
 sage: t.process_letter(w[-1]); t
Implicit Suffix Tree of the word: cacao
 sage: W = Words([0,1])
sage: s = ImplicitSuffixTree(W([0,1,0,1])); s
Implicit Suffix Tree of the word: 0101
 sage: s.process_letter(W([1])[0]); s
Implicit Suffix Tree of the word: 01011

show(word_labels=None, *args, **kwds)

Displays the output of self.plot().

INPUT:

• word_labels - (default: None) if False, labels the edges by pairs (i,j); if True, labels the edges by word[i:j].

EXAMPLES:

 sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
 sage: w = Words([0,1])
sage: s = ImplicitSuffixTree(W([0,1,0,1])); s
Implicit Suffix Tree of the word: 0101
 sage: s.process_letter(W([1])[0]); s
Implicit Suffix Tree of the word: 01011

states()

Returns the states (explicit nodes) of the suffix tree.

EXAMPLES:

 sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
 sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2])); t
Implicit Suffix Tree of the word: 01010
 sage: t.states()
[0, 1, 2, 3, 4, 5, 6, 7]

suffix_link(state)

Evaluates the suffix link map of the implicit suffix tree on state. Note that the suffix link is not defined for all states.
The suffix link of a state $x'$ that corresponds to the suffix $x$ is defined to be -1 is $x'$ is the root (0) and $y'$ is the state corresponding to the suffix $x[1:]$.

INPUT:

• state - a state

EXAMPLES:

```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2]))

sage: t.suffix_link(3)
5
sage: t.suffix_link(5)
0
sage: t.suffix_link(0)
-1
sage: t.suffix_link(-1)
Traceback (most recent call last):
  ...TypeError: there is no suffix link from -1
```

**suffix_walk***(edge, l)*

Return the state of “w” if the input state is “aw”.

If the input state (edge, 1) is path labeled “aw” with “a” a letter, the output is the state which is path labeled “w”.

INPUT:

• edge – the edge containing the state

• l – the string-depth of the state on edge (l>0)

OUTPUT:

Return ("explicit", end_node) if the state of w is an explicit state and ("implicit", edge, d) is obtained by reading d letters on edge.

EXAMPLES:

```python
sage: T = Word('00110111011').suffix_tree()

sage: T.suffix_walk((0, 5), 1)
('explicit', 0)

sage: T.suffix_walk((7, 3), 1)
('implicit', (9, 4), 1)
```

**to_digraph***(word_labels=False)*

Returns a DiGraph object of the transition graph of the suffix tree.

INPUT:

• word_labels - boolean (default: False) if False, labels the edges by pairs (i,j); if True, labels the edges by word[i:j].

EXAMPLES:
```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2]))
sage: t.to_digraph()
Digraph on 8 vertices
to_explicit_suffix_tree()
Converts self to an explicit suffix tree. It is obtained by processing an end of string letter as if it were a regular letter, except that no new leaf nodes are created (thus, the only thing that happens is that some implicit nodes become explicit).

EXAMPLES:
```
```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2]))
sage: t.to_explicit_suffix_tree()
```
```python
transition_function(word, node=0)
Returns the node obtained by starting from node and following the edges labelled by the letters of word. Returns ("explicit", end_node) if we end at end_node, or ("implicit", edge, d) if we end d spots along an edge.

INPUT:
• word - a word
• node - (default: 0) starting node

EXAMPLES:
```
```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2]))
sage: t.transition_function(W([0,1,0]))
('implicit', (3, 1), 1)
sage: t.transition_function(W([0,1,2]))
('explicit', 4)
sage: t.transition_function(W([0,1,2]), 5)
('explicit', 2)
sage: t.transition_function(W([0,1]), 5)
('implicit', (5, 2), 2)
```
```python
transition_function_dictionary()
Returns the transition function as a dictionary of dictionaries. The format is consistent with the input format for DiGraph.

EXAMPLES:
```
```python
sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree
sage: W = Words([0,1,2])
sage: t = ImplicitSuffixTree(W([0,1,0,1,2]))
sage: t.transition_function_dictionary()
```
```python
```
sage: t = ImplicitSuffixTree(W("cac"))
sage: t.transition_function_dictionary()
{0: {1: (0, None), 2: (1, None)}}

sage: W = Words([0,1])
sage: t = ImplicitSuffixTree(W([0,1,0]))
sage: t.transition_function_dictionary()
{0: {1: (0, None), 2: (1, None)}}

trie_type_dict()

Returns a dictionary in a format compatible with that of the suffix trie transition function.

EXAMPLES:

sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree, SuffixTrie

sage: W = Words("ab")
sage: t = ImplicitSuffixTree(W("aba"))
sage: d = t.trie_type_dict()
sage: len(d)
5
sage: d  # random

uncompactify()

Returns the tree obtained from self by splitting edges so that they are labelled by exactly one letter. The resulting tree is isomorphic to the suffix trie.

EXAMPLES:

sage: from sage.combinat.words.suffix_trees import ImplicitSuffixTree, SuffixTrie

sage: abbab = Words("ab")("abbab")
sage: s = SuffixTrie(abbab)
sage: t = ImplicitSuffixTree(abbab)
sage: t.uncompactify().is_isomorphic(s.to_digraph())
True

word()

Returns the word whose implicit suffix tree this is.

class sage.combinat.words.suffix_trees.SuffixTrie(word)

Bases: SageObject

Construct the suffix trie of the word w.

The suffix trie of a finite word w is a data structure representing the factors of w. It is a tree whose edges are labelled with letters of w, and whose leafs correspond to suffixes of w.

This is a straightforward implementation of Algorithm 1 from [Ukko1995]. It constructs the suffix trie of w[:i] from that of w[:i-1].

A suffix trie is modelled as a deterministic finite-state automaton together with the suffix_link map. The set of states corresponds to factors of the word (below we write x' for the state corresponding to x); these are always
0, 1, ..., The state 0 is the initial state, and it corresponds to the empty word. For the purposes of the algorithm, there is also an auxiliary state -1. The transition function $t$ is defined as:

$t(-1,a) = 0$ for all letters $a$; and
$t(x',a) = y'$ for all $x',y' \in Q$ such that $y = xa$,

and the suffix link function is defined as:

$suffix\_link(0) = -1;
 suffix\_link(x') = y'$, if $x = ay$ for some letter $a$.

REFERENCES:
* [Ukko1995]

EXAMPLES:

```python
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("cao")("cacao")
sage: t = SuffixTrie(w); t
Suffix Trie of the word: cacao
```

```python
sage: e = Words("ab")()
sage: t = SuffixTrie(e); t
Suffix Trie of the word: 
```

```python
sage: t.process_letter("a"); t
Suffix Trie of the word: a
```

```python
sage: t.process_letter("b"); t
Suffix Trie of the word: ab
```

```python
sage: t.process_letter("a"); t
Suffix Trie of the word: aba
```

**active_state()**

Returns the active state of the suffix trie. This is the state corresponding to the word as a suffix of itself.

EXAMPLES:

```python
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("cao")("cacao")
sage: t = SuffixTrie(w)
sage: t.active_state()
8
```

```python
sage: u = Words([0,1])([0,1,1,0,1,0,0,1])
sage: s = SuffixTrie(u)
sage: s.active_state()
22
```

**final_states()**

Returns the set of final states of the suffix trie. These are the states corresponding to the suffixes of `self.word()`. They are obtained by repeatedly following the suffix link from the active state until we reach 0.

EXAMPLES:
```python
from sage.combinat.words.suffix_trees import SuffixTrie
w = Words("cao")("cacao")
t = SuffixTrie(w)
t.final_states() == Set([8, 9, 10, 11, 12, 0])
```

**has_suffix(word)**
Return True if and only if word is a suffix of self.word().

**Examples:**
```python
from sage.combinat.words.suffix_trees import SuffixTrie
w = Words("cao")("cacao")
t = SuffixTrie(w)
[t.has_suffix(w[i:]) for i in range(w.length()+1)]
[True, True, True, True, True, True]
```

**node_to_word(state=0)**
Returns the word obtained by reading the edge labels from 0 to state.

**Input:**
- state - (default: 0) a state

**Examples:**
```python
from sage.combinat.words.suffix_trees import SuffixTrie
w = Words("abc")("abcba")
t = SuffixTrie(w)
t.node_to_word(10)
word: abcba
t.node_to_word(7)
word: abcb
```

**plot(layout='tree', tree_root=0, tree_orientation='up', vertex_colors=None, edge_labels=True, *args, **kwds)**
Returns a Graphics object corresponding to the transition graph of the suffix trie.

**Examples:**
```python
from sage.combinat.words.suffix_trees import SuffixTrie
SuffixTrie(Word("cacao")).plot()
```

**process_letter(letter)**
Modify self to produce the suffix trie for self.word() + letter.

**Note:** letter must occur within the alphabet of the word.

**Examples:**
```python
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("ab")("ababba")
sage: t = SuffixTrie(w); t
Suffix Trie of the word: ababba
sage: t.process_letter("a"); t
Suffix Trie of the word: ababbaa
```

`show(*args, **kwds)`

Displays the output of `self.plot()`.

**EXAMPLES:**

```python
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("cao")("cac")
sage: t = SuffixTrie(w)
sage: t.show()
```

`states()`

Return the states of the automaton defined by the suffix trie.

**EXAMPLES:**

```python
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words([0,1])([0,1,1])
sage: t = SuffixTrie(w)
sage: t.states()
[0, 1, 2, 3, 4]
```

```python
sage: u = Words("aco")("cacao")
sage: s = SuffixTrie(u)
sage: s.states()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]
```

`suffix_link(state)`

Evaluates the suffix link map of the suffix trie on `state`. Note that the suffix link map is not defined on -1.

**INPUT:**

- `state` - a state

**EXAMPLES:**

```python
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("cao")("cacao")
sage: t = SuffixTrie(w)
sage: list(map(t.suffix_link, range(13)))
[-1, 0, 3, 0, 5, 1, 7, 2, 9, 10, 11, 12, 0]
sage: t.suffix_link(0)
-1
```

`to_digraph()`

Returns a `DiGraph` object of the transition graph of the suffix trie.

**EXAMPLES:**
sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("cao")("cac")

sage: t = SuffixTrie(w)

sage: d = t.to_digraph(); d
Digraph on 6 vertices

sage: d.adjacency_matrix()

\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}

transition_function\(\text{(node, word)}\)

Returns the state reached by beginning at \text{node} and following the arrows in the transition graph labelled by the letters of \text{word}.

INPUT:

\begin{itemize}
\item node - a node
\item word - a word
\end{itemize}

EXAMPLES:

sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words([0,1])([0,1,0,1,1])

sage: t = SuffixTrie(w)

sage: all(t.transition_function(u, letter) == v
.....: for (u, letter), v) in t._transition_function.items())
True

word()

Returns the word whose suffix tree this is.

EXAMPLES:

sage: from sage.combinat.words.suffix_trees import SuffixTrie
sage: w = Words("abc")("abcba")

sage: t = SuffixTrie(w)

sage: t.word()
word: abcba

sage: t.word() == w
True

5.1.363 Word classes

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class sage.combinat.words.word.FiniteWord_callable(parent, callable, length=None)

Bases: WordDatatype_callable, FiniteWord_class

Finite word represented by a callable.

For such word \( w \), type \( w \). and hit Tab key to see the list of functions defined on \( w \).

EXAMPLES:

```
sage: f = lambda n : 3 if n > 8 else 6
sage: w = Word(f, length=30, caching=False)
sage: w
word: 666666663333333333333333333333
sage: w.is_symmetric()
True
```

class sage.combinat.words.word.FiniteWord_callable_with_caching(parent, callable, length=None)

Bases: WordDatatype_callable_with_caching, FiniteWord_class

Finite word represented by a callable (with caching).

For such word \( w \), type \( w \). and hit Tab key to see the list of functions defined on \( w \).

EXAMPLES:

```
sage: f = lambda n : n % 3
sage: w = Word(f, length=32)
sage: w
word: 01201201201201201201201201201201
sage: w.border()
word: 01201201201201201201201201201
```

class sage.combinat.words.word.FiniteWord_char

Bases: WordDatatype_char, FiniteWord_class

Finite word represented by an array unsigned char *(i.e. integers between 0 and 255).

For any word \( w \), type \( w <\text{TAB}> \) to see the functions that can be applied to \( w \).

EXAMPLES:

```
sage: W = Words(range(20))
sage: w = W(list(range(1, 10)) * 2)
sage: type(w)
<class 'sage.combinat.words.word.FiniteWord_char'>
sage: w
word: 123456789123456789
sage: w.is_palindrome()
False
sage: (w*w[::-1]).is_palindrome()
True
sage: (w[:-1:]*w[::-1]).is_palindrome()
True
sage: w.is_lyndon()
False
```

(continues on next page)
sage: W(list(range(10)) + [10, 10]).is_lyndon()
    True
sage: w.is_square_free()
    False
sage: w[:-1].is_square_free()
    True
sage: u = W([randint(0,10) for i in range(10)])
sage: (u*u).is_square()
    True
sage: (u*u*u).is_cube()
    True
sage: len(w.factor_set())
    127
sage: w.rauzy_graph(5)
    Looped digraph on 9 vertices
sage: u = W([1,2,3])
sage: w.first_occurrence(u)
    0
sage: w.first_occurrence(u, start=1)
    9

class sage.combinat.words.word.FiniteWord_iter(parent, iter, length=None)
    Bases: WordDatatype_iter, FiniteWord_class
    Finite word represented by an iterator.
    For such word w, type w. and hit Tab key to see the list of functions defined on w.
    EXAMPLES:

    sage: w = Word(iter(range(10)), caching=False)
    sage: w
     word: 0123456789
    sage: w.finite_differences()
     word: 111111111

class sage.combinat.words.word.FiniteWord_iter_with_caching(parent, iter, length=None)
    Bases: WordDatatype_iter_with_caching, FiniteWord_class
    Finite word represented by an iterator (with caching).
    For such word w, type w. and hit Tab key to see the list of functions defined on w.
    EXAMPLES:

    sage: w = Word(iter('abcdef'))
    sage: w
     word: cdefab

class sage.combinat.words.word.FiniteWord_list
    Bases: WordDatatype_list, FiniteWord_class

5.1. Comprehensive Module List
Finite word represented by a Python list.

For any word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

**EXAMPLES:**
```
sage: w = Word(range(10))
sage: w.iterated_right_palindromic_closure()
word: 0102010301020104010201030102010501020103...
```

**class** `sage.combinat.words.word.FiniteWord_morphic`(

\( \text{parent, morphism, letter, coding=}\)\( \text{None, length=}\)\( \text{+Infinity} \)

Bases: `WordDatatype_morphic`, `FiniteWord_class`

Finite morphic word.

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

**EXAMPLES:**
```
sage: m = WordMorphism("a->ab,b->")
sage: w = m.fixed_point("a")
sage: w
word: ab
```

**class** `sage.combinat.words.word.FiniteWord_str`

Bases: `WordDatatype_str`, `FiniteWord_class`

Finite word represented by a Python str.

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

**EXAMPLES:**
```
sage: w = Word('abcdef')
sage: w.is_square()
False
```

**class** `sage.combinat.words.word.FiniteWord_tuple`

Bases: `WordDatatype_tuple`, `FiniteWord_class`

Finite word represented by a Python tuple.

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

**EXAMPLES:**
```
sage: w = Word(())
sage: w.is_empty()
True
```

**class** `sage.combinat.words.word.InfiniteWord_callable`

\( \text{parent, callable, length=}\)\( \text{None} \)

Bases: `WordDatatype_callable`, `InfiniteWord_class`

Infinite word represented by a callable.

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

Infinite words behave like a Python list : they can be sliced using square brackets to define for example a prefix or a factor.

**EXAMPLES:**

class sage.combinat.words.word.InfiniteWord_callable_with_caching(parent, callable, length=None)

Bases: WordDatatype_callable_with_caching, InfiniteWord_class

Infinite word represented by a callable (with caching).

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

Infinite words behave like a Python list: they can be sliced using square brackets to define for example a prefix or a factor.

EXAMPLES:

```python
sage: w = Word(lambda n:n)
sage: factor = w[4:13]
sage: factor
word: 4,5,6,7,8,9,10,11,12
```

class sage.combinat.words.word.InfiniteWord_iter(parent, iter, length=None)

Bases: WordDatatype_iter, InfiniteWord_class

Infinite word represented by an iterable.

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

Infinite words behave like a Python list: they can be sliced using square brackets to define for example a prefix or a factor.

EXAMPLES:

```python
sage: from itertools import chain, cycle
sage: w = Word(chain('letsgo', cycle('forever')), caching=False)
sage: w
word: letsgoforeverforeverforeverforeverforeve...
sage: prefix = w[:100]
sage: prefix
word: letsgoforeverforeverforeverforever...
sage: prefix.is_lyndon()
False
```

class sage.combinat.words.word.InfiniteWord_iter_with_caching(parent, iter, length=None)

Bases: WordDatatype_iter_with_caching, InfiniteWord_class

Infinite word represented by an iterable (with caching).

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

Infinite words behave like a Python list: they can be sliced using square brackets to define for example a prefix or a factor.

EXAMPLES:
sage: from itertools import cycle
sage: w = Word(cycle([9,8,4]))

sage: w
word: 9849849849849849849849849849849849849849...

sage: prefix = w[:23]

sage: prefix
word: 98498498498498498498498

sage: prefix.minimal_period()
3

class sage.combinat.words.word.InfiniteWord_morphic(parent, morphism, letter, coding=None, length=+Infinity)

Bases: WordDatatype_morphic, InfiniteWord_class

Morphic word of infinite length.

For such word \( w \), type \( w \). and hit Tab key to see the list of functions defined on \( w \).

Infinite words behave like a Python list: they can be sliced using square brackets to define for example a prefix or a factor.

EXAMPLES:

sage: m = WordMorphism('a->ab,b->a')

sage: w = m.fixed_point('a')

sage: w
word: abaababaababaababaababaababaababaababaabaababaababaababaababaababaababaababaababaab...

sage.combinat.words.word.Word(data=None, alphabet=None, length=None, datatype=None, caching=True, RSK_data=None)

Construct a word.

INPUT:

- **data** – (default: None) list, string, tuple, iterator, free monoid element, None (shorthand for []), or a callable defined on \([0,1,...,\text{length}]\).

- **alphabet** – any argument accepted by Words

- **length** – (default: None) This is dependent on the type of data. It is ignored for words defined by lists, strings, tuples, etc., because they have a naturally defined length. For callables, this defines the domain of definition, which is assumed to be \([0, 1, 2, ..., \text{length}-1]\). For iterators: Infinity if you know the iterator will not terminate (default); "unknown" if you do not know whether the iterator terminates; "finite" if you know that the iterator terminates, but do not know the length.

- **datatype** – (default: None) None, "list", "str", "tuple", "iter", "callable". If None, then the function tries to guess this from the data.

- **caching** – (default: True) True or False. Whether to keep a cache of the letters computed by an iterator or callable.

- **RSK_data** – (Optional) Default: None A semistandard and a standard Young tableau to run the inverse RSK bijection on.

**Note:** Be careful when defining words using callables and iterators. It appears that islice does not pickle correctly causing various errors when reloading. Also, most iterators do not support copying and should not support pickling by extension.
EXAMPLES:

Empty word:

```
sage: Word()
word: 
```

Word with string:

```
sage: Word("abbabaab")
word: abbabaab
```

Word with string constructed from other types:

```
sage: Word([0,1,1,0,1,0,0,1], datatype="str")
word: 01101001
sage: Word((0,1,1,0,1,0,0,1), datatype="str")
word: 01101001
```

Word with list:

```
sage: Word([0,1,1,0,1,0,0,1])
word: 01101001
```

Word with list constructed from other types:

```
sage: Word("01101001", datatype="list")
word: 01101001
sage: Word((0,1,1,0,1,0,0,1), datatype="list")
word: 01101001
```

Word with tuple:

```
sage: Word((0,1,1,0,1,0,0,1))
word: 01101001
```

Word with tuple constructed from other types:

```
sage: Word([0,1,1,0,1,0,0,1], datatype="tuple")
word: 01101001
sage: Word("01101001", datatype="str")
word: 01101001
```

Word with iterator:

```
sage: from itertools import count
sage: Word(count())
word: 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,
 →29,30,31,32,33,34,35,36,37,38,39,...
sage: Word(iter("abbabaab")) # iterators default to infinite words
word: abbabaab
sage: Word(iter("abbabaab"), length="unknown")
word: abbabaab
sage: Word(iter("abbabaab"), length="finite")
word: abbabaab
```

5.1. Comprehensive Module List
Word with function (a 'callable'):

```
sage: f = lambda n : add(Integer(n).digits(2)) % 2
sage: Word(f)
word: 0110100110110011011001101100110110011010110...
sage: Word(f, length=8)
word: 01101001
```

Word over a string with a parent:

```
sage: w = Word("abbabaab", alphabet="abc"); w
word: abbabaab
sage: w.parent()
Finite words over {'a', 'b', 'c'}
```

Word from a free monoid element:

```
sage: M.<x,y,z> = FreeMonoid(3)
sage: Word(x^3*y*x*z^2*x)
word: xxxyxzxz
```

The default parent is the combinatorial class of all words:

```
sage: w = Word("abbabaab"); w
word: abbabaab
sage: w.parent()
Finite words over Set of Python objects of class 'object'
```

We can also input a semistandard tableau and a standard tableau to obtain a word from the inverse RSK algorithm using the RSK_data option:

```
sage: p = Tableau([[1,2,2],[3]]); q = Tableau([[1,2,4],[3]])
sage: Word(RSK_data=[p, q])
word: 1322
```

```
class sage.combinat.words.word.Word_iter(parent, iter, length=None)
Bases: WordDatatype_iter, Word_class

Word of unknown length (finite or infinite) represented by an iterable.

For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

Words behave like a Python list : they can be sliced using square braquets to define for example a prefix or a factor.

EXAMPLES:

```
sage: w = Word(iter([1,1,4,9]*1000), length='unknown', caching=False)
sage: w
word: 1149114911491149114911491149114911491149...
sage: w.delta()
word: 2112112112112112112112112112112112112112...
```

```
class sage.combinat.words.word.Word_iter_with_caching(parent, iter, length=None)
Bases: WordDatatype_iter_with_caching, Word_class

Word of unknown length (finite or infinite) represented by an iterable (with caching).
```
For such word \( w \), type \( w \) and hit Tab key to see the list of functions defined on \( w \).

Words behave like a Python list: they can be sliced using square brackets to define for example a prefix or a factor.

EXAMPLES:

```
sage: w = Word(iter([1,2,3]*1000), length='unknown')
sage: w
word: 1231231231231231231231231231231231231231...
sage: w.finite_differences(mod=2)
word: 1101101101101101101101101101101101101101...
```

### 5.1.364 Fast word datatype using an array of unsigned char

**class** `sage.combinat.words.word_char.WordDatatype_char`

Bases: `WordDatatype`

A Fast class for words represented by an array `unsigned char *`.

Currently, only handles letters in \([0,255]\).

**concatenate**(other)

Concatenation of `self` and `other`.

EXAMPLES:

```
sage: W = Words([0,1,2])
sage: W([0,2,1]).concatenate([0,0,0])
word: 021000
```

**has_prefix**(other)

Test whether `other` is a prefix of `self`.

**INPUT:**

- `other` – a word or a sequence (e.g. tuple, list)

**EXAMPLES:**

```
sage: W = Words([0,1,2])
sage: w = W([0,1,1,0,1,2,0])
sage: w.has_prefix([0,1,1])
True
sage: w.has_prefix([0,1,2])
False
sage: w.has_prefix(w)
True
sage: w.has_prefix(w[:-1])
True
sage: w.has_prefix(w[1:])
False
```

**is_empty**

Return whether the word is empty.

**EXAMPLES:**
Combinatorics, Release 10.0

```python
sage: W = Words([0,1,2])
sage: W([0,1,2,2]).is_empty()
False
sage: W([]).is_empty()
True
```

**is_square()**

Return True if self is a square, and False otherwise.

**EXAMPLES:**

```python
sage: w = Word([n % 4 for n in range(48)], alphabet=[0,1,2,3])
sage: w.is_square()
True
sage: w = Word([n % 4 for n in range(49)], alphabet=[0,1,2,3])
sage: w.is_square()
False
sage: (w*w).is_square()
True
```

**length()**

Return the length of the word as a Sage integer.

**EXAMPLES:**

```python
sage: W = Words([0,1,2,3,4])
sage: w = W([0,1,2,0,3,2,1])
sage: w.length()
7
sage: type(w.length())
<class 'sage.rings.integer.Integer'>
```

**letters()**

Return the list of letters that appear in this word, listed in the order of first appearance.

**EXAMPLES:**

```python
sage: W = Words(5)
sage: W([1,3,1,2,2,3,1]).letters()
[1, 3, 2]
```

**longest_common_prefix(other)**

Return the longest common prefix of this word and other.

**EXAMPLES:**

```python
sage: W = Words([0,1,2])
sage: W([0,1,0,2]).longest_common_prefix([0,1])
word: 01
sage: u = W([0,1,0,0,1])
sage: v = W([0,1,0,2])
```

(continues on next page)
Using infinite words is also possible (and the return type is also a of the same type as self):

```
sage: W([0,1,0,0]).longest_common_prefix(words.FibonacciWord())
word: 0100
sage: type(_)
<class 'sage.combinat.words.word.FiniteWord_char'>
```

An example of an intensive usage:

```
sage: W = Words([0,1])
sage: w = words.FibonacciWord()
sage: w = W(list(w[:5000]))
sage: L = [[len(w[n:].longest_common_prefix(w[n+fibonacci(i):]))
       ....: for i in range(5,15)] for n in range(1,1000)]
sage: for n,l in enumerate(L):
    ....:     if l.count(0) > 4:
    ....:         print("{} {}".format(n+1,l))
375 [0, 13, 0, 34, 0, 89, 0, 233, 0, 233]
376 [0, 12, 0, 33, 0, 88, 0, 232, 0, 232]
608 [8, 0, 21, 0, 55, 0, 144, 0, 377, 0]
609 [7, 0, 20, 0, 54, 0, 143, 0, 376, 0]
985 [0, 13, 0, 34, 0, 89, 0, 233, 0, 610]
986 [0, 12, 0, 33, 0, 88, 0, 232, 0, 609]
```

### longest_common_suffix(other)

Return the longest common suffix between this word and other.

**EXAMPLES:**

```
sage: W = Words([0,1,2])
sage: W([0,1,0,0,1,2]).longest_common_suffix([2,0,2])
word: 02
sage: u = W([0,1,0,0,1])
sage: v = W([1,2,0,0,1])
sage: u.longest_common_suffix(v)
word: 001
sage: v.longest_common_suffix(u)
word: 001
```

### sage.combinat.words.word_char.reversed_word_iterator(w)

This function exists only because it is not possible to use yield in the special method __reversed__.

**EXAMPLES:**

```
sage: W = Words([0,1,2])
sage: w = W([0,1,0,0,1,2])
sage: list(reversed(w)) # indirect doctest
[2, 1, 0, 0, 1, 0]
```
5.1.365 Datatypes for finite words

```python
class sage.combinat.words.word_datatypes.WordDatatype
    Bases: object
    The generic WordDatatype class.
    Any word datatype must contain two attributes (at least):
    - _parent
    - _hash

    They are automatically defined here and it’s not necessary (and forbidden) to define them anywhere else.

class sage.combinat.words.word_datatypes.WordDatatype_list
    Bases: WordDatatype
    Datatype class for words defined by lists.

    length()
    Return the length of the word.

    EXAMPLES:

    sage: w = Word([0,1,1,0])
    sage: w.length()
    4

    number_of_letter_occurrences(a)
    Returns the number of occurrences of the letter a in the word self.

    INPUT:
    • a - a letter

    OUTPUT:
    • integer

    EXAMPLES:

    sage: w = Word([0,1,1,0,1])
    sage: w.number_of_letter_occurrences(0)
    2
    sage: w.number_of_letter_occurrences(1)
    3
    sage: w.number_of_letter_occurrences(2)
    0

    See also:
    sage.combinat.words.finite_word.FiniteWord_class.number_of_factor_occurrences()

class sage.combinat.words.word_datatypes.WordDatatype_str
    Bases: WordDatatype
    Datatype for words defined by strings.
```
\texttt{find}(\textit{sub}, \textit{start}=0, \textit{end}=\text{None})

Returns the index of the first occurrence of \textit{sub} in \textit{self}, such that \textit{sub} is contained within \textit{self}[\textit{start}:\textit{end}]. Returns -1 on failure.

**INPUT:**

- \textit{sub} - string or word to search for.
- \textit{start} - non negative integer (default: 0) specifying the position from which to start the search.
- \textit{end} - non negative integer (default: None) specifying the position at which the search must stop. If None, then the search is performed up to the end of the string.

**OUTPUT:**

non negative integer or -1

**EXAMPLES:**

\begin{verbatim}
sage: w = Word("abbabaabababa")
sage: w.find("a")
0
sage: w.find("a", 4)
5
sage: w.find("a", 4, 5)
-1
\end{verbatim}

\texttt{has\_prefix}(\textit{other})

Test whether \textit{self} has \textit{other} as a prefix.

**INPUT:**

- \textit{other} - a word (an instance of \texttt{Word\_class}) or a \texttt{str}.

**OUTPUT:**

- boolean

**EXAMPLES:**

\begin{verbatim}
sage: w = Word("abbabaabababa")
sage: u = Word("abbab")
sage: w.has\_prefix(u)
True
sage: u.has\_prefix(w)
False
sage: u.has\_prefix("abbab")
True
\end{verbatim}

\texttt{has\_suffix}(\textit{other})

Test whether \textit{self} has \textit{other} as a suffix.

**INPUT:**

- \textit{other} - a word (an instance of \texttt{Word\_class}) or a \texttt{str}.

**OUTPUT:**

- boolean

**EXAMPLES:**

\begin{verbatim}
sage: w = Word("abbabaabababa")
sage: u = Word("ababa")
sage: w.has_suffix(u)
True
sage: u.has_suffix(w)
False
sage: u.has_suffix("ababa")
True

is_prefix(other)
Test whether self is a prefix of other.

INPUT:

• other - a word (an instance of Word_class) or a str.

OUTPUT:

• boolean

EXAMPLES:

sage: w = Word("abbabaabababa")
sage: u = Word("abbab")
sage: w.is_prefix(u)
False
sage: u.is_prefix(w)
True
sage: u.is_prefix("abbabaabababa")
True

is_suffix(other)
Test whether self is a suffix of other.

INPUT:

• other - a word (an instance of Word_class) or a str.

OUTPUT:

• boolean

EXAMPLES:

sage: w = Word("abbabaabababa")
sage: u = Word("ababa")
sage: w.is_suffix(u)
False
sage: u.is_suffix(w)
True
sage: u.is_suffix("abbabaabababa")
True

length()
Return the length of the word.

EXAMPLES:
\texttt{sage: w = Word("abbaabababa")}
\texttt{sage: w.length()}
13

\texttt{sage: w = Word("abbaabababa")}
\texttt{sage: w.number_of_letter_occurrences('a')}
7
\texttt{sage: w.number_of_letter_occurrences('b')}
6
\texttt{sage: w.number_of_letter_occurrences('c')}
0

\texttt{sage: w.number_of_letter_occurrences('abb')}
0

See also:
\texttt{sage.combinat.words.finite_word.FiniteWord_class.number_of_factor_occurrences()}

\texttt{partition(sep)}

Search for the separator \texttt{sep} in \texttt{S}, and return the part before it, the separator itself, and the part after it. The concatenation of the terms in the list gives back the initial word.

See also the \texttt{split} method.

\textbf{Note:} This just wraps Python's builtin \texttt{str::partition()} for \texttt{str}.

\texttt{sage: w = Word("MyTailorIsPoor")}
\texttt{sage: w.partition("Tailor")}
[word: My, word: Tailor, word: IsPoor]

\texttt{sage: w = Word("3230301030323212323032321210121232121010")}
\texttt{sage: l = w.partition("323")}
\texttt{[word: , word: 323, word: 0301030323212323032321210121232121010]}
\texttt{sage: print(l)}
[\texttt{word: , word: 323, word: 0301030323212323032321210121232121010}]
\texttt{sage: sum(l, Word("")) == w}
\texttt{True}
If the separator is not a string an error is raised:

```python
sage: w = Word("le papa du papa du papa etait un petit pioupiou")
sage: w.partition(Word(['p','a','p','a']))
Traceback (most recent call last):
  ...
ValueError: the separator must be a string
```

**rfind**(sub, start=0, end=None)

Returns the index of the last occurrence of sub in self, such that sub is contained within self[start:end]. Returns -1 on failure.

**INPUT:**
- sub - string or word to search for.
- start - non negative integer (default: 0) specifying the position at which the search must stop.
- end - non negative integer (default: None) specifying the position from which to start the search. If None, then the search is performed up to the end of the string.

**OUTPUT:**
- non negative integer or -1

**EXAMPLES:**

```python
sage: w = Word("abbabaabababa")
sage: w.rfind("a")
12
sage: w.rfind("a", 4, 8)
6
sage: w.rfind("a", 4, 5)
-1
```

**split**(sep=None, maxsplit=None)

Returns a list of words, using sep as a delimiter string. If maxsplit is given, at most maxsplit splits are done. See also the partition method.

**Note:** This just wraps Python’s builtin `str::split()` for `str`.

**INPUT:**
- sep - string or word (optional, default: None)
- maxsplit - positive integer (optional, default: None)

**OUTPUT:**
- a list of words

**EXAMPLES:**

You can split along white space to find words in a sentence:

```python
sage: w = Word("My tailor is poor")
sage: w.split(" ")
[word: My, word: tailor, word: is, word: poor]
```
Combinatorics, Release 10.0

The python behavior is kept when no argument is given:

```python
sage: w.split()
[word: My, word: tailor, word: is, word: poor]
```

You can split in two words letters to get the length of blocks in the other letter:

```python
sage: w = Word("ababbabaaba")
sage: w.split('a')
[word: , word: b, word: bb, word: b, word: , word: b, word: ]
sage: w.split('b')
[word: a, word: a, word: , word: a, word: aa, word: a]
```

You can split along words:

```python
sage: w = Word("3230301030323212323032321")
sage: w.split("32")
[word: , word: 30301030, word: , word: 12, word: 30, word: , word: 1]
```

If the separator is not a string a ValueError is raised:

```python
sage: w = Word("le papa du papa du papa etait un petit pioupiou")
sage: w.split(Word([\'p',\'a','p','a'])))
Traceback (most recent call last):
 ... 
ValueError: the separator must be a string
```

```python
class sage.combinat.words.word_datatypes.WordDatatype_tuple
    Bases: WordDatatype
    Datatype class for words defined by tuples.

    length()
    Return the length of the word.

    EXAMPLES:

    sage: w = Word((0,1,1,0))
sage: w.length()
4
```

5.1.366 Common words

AUTHORS:

- Franco Saliola (2008-12-17): merged into sage
- Sébastien Labbé (2008-12-17): merged into sage
- Arnaud Bergeron (2008-12-17): merged into sage
- Amy Glen (2008-12-17): merged into sage
- Sébastien Labbé (2009-12-19): Added S-adic words (github issue #7543)

USE:
To see a list of all word constructors, type \texttt{words.} and then press the Tab key. The documentation for each constructor includes information about each word, which provides a useful reference.

REFERENCES:

EXAMPLES:

\begin{verbatim}
sage: t = words.ThueMorseWord(); t
t: 0110100110011010100101100110100110010110...
\end{verbatim}

\begin{verbatim}
class sage.combinat.words.word_generators.LowerChristoffelWord(p, q, alphabet=(0, 1),
algorithm='cf')

Bases: FiniteWord_list

Returns the lower Christoffel word of slope \( p/q \), where \( p \) and \( q \) are relatively prime non-negative integers, over the given two-letter alphabet.

The Christoffel word of slope \( p/q \) is obtained from the Cayley graph of \( \mathbb{Z}/(p+q)\mathbb{Z} \) with generator \( q \) as follows. If \( u \to v \) is an edge in the Cayley graph, then \( v = u + p \mod p + q \). Label the edge \( u \to v \) by \( \text{alphabet}[1] \) if \( u < v \) and \( \text{alphabet}[0] \) otherwise. The Christoffel word is the word obtained by reading the edge labels along the cycle beginning from 0.

EXAMPLES:

\begin{verbatim}
sage: words.LowerChristoffelWord(4,7)
word: 00100100100101
sage: words.LowerChristoffelWord(4,7,alphabet='ab')
word: aabaabaabab
\end{verbatim}

\textbf{markoff_number()}

Return the Markoff number associated to the Christoffel word \texttt{self}.

The Markoff number of a Christoffel word \( w \) is \( \text{trace}(M(w))/3 \), where \( M(w) \) is the 2 \times 2 matrix obtained by applying the morphism: 0 \to \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad 1 \to \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}

EXAMPLES:

\begin{verbatim}
sage: w0 = words.LowerChristoffelWord(4,7)
sage: w1, w2 = w0.standard_factorization()
sage: (m0,m1,m2) = (w.markoff_number() for w in (w0,w1,w2))
sage: (m0,m1,m2)
(294685, 13, 7561)
sage: m0**2 + m1**2 + m2**2 == 3*m0*m1*m2
True
\end{verbatim}

\textbf{standard_factorization()}

Returns the standard factorization of the Christoffel word \texttt{self}.

The standard factorization of a Christoffel word \( w \) is the unique factorization of \( w \) into two Christoffel words.

EXAMPLES:

\begin{verbatim}
sage: w = words.LowerChristoffelWord(5,9)
sage: w
word: 00100100100101
\end{verbatim}

(continues on next page)
sage: w1, w2 = w.standard_factorization()
sage: w1
word: 001
sage: w2
word: 00100100101

sage: w = words.LowerChristoffelWord(51, 37)
sage: w1, w2 = w.standard_factorization()
sage: w1
word: 0101010101
sage: w2
word: 0101010101010101010101010101010101010101...
sage: w1 * w2 == w
True

class sage.combinat.words.word_generators.WordGenerator
Bases: object

Constructor of several famous words.

EXAMPLES:

sage: words.ThueMorseWord()
word: 0110100110010110100101100110100110010110...

sage: words.FibonacciWord()
word: 0100101001001010010100100101001001010010...

sage: words.ChristoffelWord(5, 8)
word: 0010010100101

sage: words.RandomWord(10, 4)  # not tested random
word: 1311131221

sage: words.CodingOfRotationWord(alpha=0.618, beta=0.618)
word: 10101010110101101011011010110110101101101...

sage: tm = WordMorphism('a->ab,b->ba')
sage: fib = WordMorphism('a->ab,b->a')
sage: tmword = words.ThueMorseWord([0, 1])
sage: from itertools import repeat
sage: words.s_adic(tmword, repeat('a'), {0: tm, 1: fib})
word: abbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabba...

Note: To see a list of all word constructors, type words. and then hit the Tab key. The documentation for each constructor includes information about each word, which provides a useful reference.

BaumSweetWord()

Returns the Baum-Sweet Word.
The Baum-Sweet Sequence is an infinite word over the alphabet \{0, 1\} defined by the following string substitution rules:

\[
\begin{align*}
00 & \to 0000 \\
01 & \to 1001 \\
10 & \to 0100 \\
11 & \to 1101
\end{align*}
\]

The substitution rule above can be considered as a morphism on the submonoid of \{0, 1\} generated by \{00, 01, 10, 11\} (which is a free monoid on these generators).

It is also defined as the concatenation of the terms from the Baum-Sweet Sequence:

\[
b_n = \begin{cases} 
0, & \text{if } n = 0 \\
1, & \text{if } m \text{ is even} \\
b_{\frac{m-1}{2}}, & \text{if } m \text{ is odd} 
\end{cases}
\]

where \( n = m4^k \) and \( m \) is not divisible by 4 if \( m \neq 0 \).

The individual terms of the Baum-Sweet Sequence are also given by:

\[
b_n = \begin{cases} 
1, & \text{if the binary representation of } n \text{ contains no block of consecutive 0's of odd length} \\
0, & \text{otherwise}
\end{cases}
\]

for \( n > 0 \) with \( b_0 = 1 \).

For more information see: Wikipedia article Baum-Sweet_sequence.

EXAMPLES:

Baum-Sweet Word:

```python
sage: w = words.BaumSweetWord(); w
word: 1101100101001001100100000100100101001001...
```

Block Definition:

```python
sage: w = words.BaumSweetWord()
sage: f = lambda n: '1' if all(len(x)%2==0 for x in bin(n)[2:].split('1')) else '0'
sage: all(f(i) == w[i] for i in range(1,100))
True
```

CharacteristicSturmianWord(slope, alphabet=(0, 1), bits=None)

Returns the characteristic Sturmian word (also called standard Sturmian word) of given slope.

Over a binary alphabet \( \{a, b\} \), the characteristic Sturmian word \( c_\alpha \) of irrational slope \( \alpha \) is the infinite word satisfying \( s_{\alpha,0} = ac_\alpha \) and \( s'_{\alpha,0} = bc_\alpha \), where \( s_{\alpha,0} \) and \( s'_{\alpha,0} \) are respectively the lower and upper mechanical words with slope \( \alpha \) and intercept 0. Equivalently, for irrational \( \alpha \), \( c_\alpha = s_{\alpha,\alpha} = s'_{\alpha,\alpha} \).

Let \( \alpha = [0, d_1 + 1, d_2, d_3, \ldots] \) be the continued fraction expansion of \( \alpha \). It has been shown that the characteristic Sturmian word of slope \( \alpha \) is also the limit of the sequence: \( s_0 = b, s_1 = a, \ldots, s_{n+1} = s_n d_n s_{n-1} \) for \( n > 0 \).

See Section 2.1 of [Loth02] for more details.

INPUT:

- \( \text{slope} \) - the slope of the word. It can be one of the following:
– real number in ]0, 1[
– iterable over the continued fraction expansion of a real number in ]0, 1[
• alphabet - any container of length two that is suitable to build an instance of OrderedAlphabet (list, tuple, str, ...)
• bits - integer (optional and considered only if slope is a real number) the number of bits to consider when computing the continued fraction.

OUTPUT:
word

ALGORITHM:
Let \([0, d_1 + 1, d_2, d_3, \ldots]\) be the continued fraction expansion of \(\alpha\). Then, the characteristic Sturmian word of slope \(\alpha\) is the limit of the sequence: \(s_0 = b, s_1 = a\) and \(s_{n+1} = s_{d_n} s_{n-1}\) for \(n > 0\).

EXAMPLES:
From real slope:

```sage
words.CharacteristicSturmianWord(1/golden_ratio^2)
```
word: 0100101001001010010100100101001001010010...

```sage
words.CharacteristicSturmianWord(4/5)
```
word: 11110

```sage
words.CharacteristicSturmianWord(5/14)
```
word: 01001001001001

```sage
words.CharacteristicSturmianWord(pi-3)
```
word: 0000001000000100000010000001000000100000...

From an iterator of the continued fraction expansion of a real:

```sage
def cf():
    ....: yield 0
    ....: yield 2
    ....: while True: yield 1
sage: F = words.CharacteristicSturmianWord(cf()); F
```
word: 010010010010010010100100100101001001010010...

```sage:Fib = words.FibonacciWord(); Fib
```
word: 010010010010010010100100100101001001010010...

```sage:F[[:10000]] == Fib[:10000]
```
True

The alphabet may be specified:

```sage:
words.CharacteristicSturmianWord(cf(), 'rs')
```
word: rsrrrsrrrrssrrrsrrrsrrrrrrrsrrrrrrrsrssr...

The characteristic sturmian word of slope \((\sqrt{3} - 1)/2\):

```sage:
words.CharacteristicSturmianWord((sqrt(3)-1)/2)
```
word: 010010010010010010100100100101001001010010...

The same word defined from the continued fraction expansion of \((\sqrt{3} - 1)/2\):
```python
sage: from itertools import cycle, chain
sage: it = chain([0], cycle([2, 1]))
sage: words.CharacteristicSturmianWord(it)
word: 01001001001001001001010010010010100100101...
```

The first terms of the standard sequence of the characteristic sturmian word of slope \((\sqrt{3} - 1)/2\):

```python
sage: words.CharacteristicSturmianWord([0, 2])
word: 01
sage: words.CharacteristicSturmianWord([0, 2, 1])
word: 010
sage: words.CharacteristicSturmianWord([0, 2, 1, 2])
word: 01001001
sage: words.CharacteristicSturmianWord([0, 2, 1, 2, 1])
word: 01001001010
sage: words.CharacteristicSturmianWord([0, 2, 1, 2, 1, 2])
word: 010010010100100100101001001001
```

ChristoffelWord

alias of LowerChristoffelWord

CodingOfRotationWord(alpha, beta, x=0, alphabet=(0, 1))

Returns the infinite word obtained from the coding of rotation of parameters \((\alpha, \beta, x)\) over the given two-letter alphabet.

The coding of rotation corresponding to the parameters \((\alpha, \beta, x)\) is the symbolic sequence \(u = (u_n)_{n \geq 0}\) defined over the binary alphabet \(\{0, 1\}\) by \(u_n = 1\) if \(x + n\alpha \in [0, \beta]\) and \(u_n = 0\) otherwise. See [AC03].

EXAMPLES:

```python
sage: alpha = 0.45
sage: beta = 0.48
sage: words.CodingOfRotationWord(0.45, 0.48)
word: 1101010101001010101011010101010010101010...
```

```python
sage: words.CodingOfRotationWord(0.45, 0.48, alphabet='xy')
word: yyxyxyxyxyxyxyxyxyxyxyxyxyxyxyxyxyxyxyxyxy...
```

FibonacciWord(alphabet=(0, 1), construction_method='recursive')

Returns the Fibonacci word on the given two-letter alphabet.

INPUT:

- alphabet – any container of length two that is suitable to build an instance of OrderedAlphabet (list, tuple, str, ...)

- construction_method – can be any of the following: “recursive”, “fixed point”, “function” (see below for definitions).

Recursive construction: the Fibonacci word is the limit of the following sequence of words: \(S_0 = 0, S_1 = 01, S_n = S_{n-1}S_{n-2}\) for \(n \geq 2\).

Fixed point construction: the Fibonacci word is the fixed point of the morphism: \(0 \mapsto 01\) and \(1 \mapsto 0\). Hence, it can be constructed by the following read-write process:
1. beginning at the first letter of 01,
2. if the next letter is 0, append 01 to the word;
3. if the next letter is 1, append 1 to the word;
4. move to the next letter of the word.

Function: Over the alphabet \{1, 2\}, the \(n\)-th letter of the Fibonacci word is \(\lfloor (n + 2)\phi \rfloor - \lfloor (n + 1)\phi \rfloor\) where \(\phi = (1 + \sqrt{5})/2\) is the golden ratio.

\[
\text{EXAMPLES:}
\]

```python
sage: w = words.FibonacciWord(construction_method="recursive"); w
word: 0100101001001010010100100101001001010010...
```

```python
sage: v = words.FibonacciWord(construction_method="recursive", alphabet='ab'); v
word: ababababababaabababaabababaabababaabababa...
```

```python
sage: u = words.FibonacciWord(construction_method="fixed point"); u
word: 0100101001001010010100100101001001010010...
```

```python
sage: words.FibonacciWord(construction_method="fixed point", alphabet=[4, 1])
word: 4144144144144144144144144144144144144144...
```

```python
sage: words.FibonacciWord([0,1], 'function')
word: 0100101001001010010100100101001001010010...
```

```python
sage: words.FibonacciWord('ab', 'function')
word: ababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababababab...```

FixedPointOfMorphism\((\text{morphism}, \text{first_letter})\)

Returns the fixed point of the morphism beginning with \text{first_letter}.

A fixed point of a morphism \(\varphi\) is a word \(w\) such that \(\varphi(w) = w\).

INPUT:

- \text{morphism} – endomorphism prolongable on \text{first_letter}. It must be something that WordMorphism’s constructor understands (dict, str, ...).
- \text{first_letter} – the first letter of the fixed point

OUTPUT:

The fixed point of the morphism beginning with \text{first_letter}

\[
\text{EXAMPLES:}
\]

```python
sage: mu = {0:[0,1], 1:[1,0]}
sage: tm = words.FixedPointOfMorphism(mu, 0); tm
word: 01101001101001101001101001101001101001101001100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110100110...
```

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```

```python
sage: TM = words.ThueMorseWord()
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```
KolakoskiWord\((alphabet=(1, 2))\)

Returns the Kolakoski word over the given alphabet and starting with the first letter of the alphabet.

Let \( A = \{a, b\} \) be an alphabet, where \( a \) and \( b \) are two distinct positive integers. The Kolakoski word \( K_{a,b} \) over \( A \) and starting with \( a \) is the unique infinite word \( w \) such that \( w = \Delta(w) \), where \( \Delta(w) \) is the word encoding the runs of \( w \) (see \texttt{delta()} method on words for more details).

Note that \( K_{a,b} \neq K_{b,a} \). On the other hand, the words \( K_{a,b} \) and \( K_{b,a} \) are the unique two words over \( A \) that are fixed by \( \Delta \).

Also note that the Kolakoski word is also known as the Oldenburger word.

INPUT:

- \texttt{alphabet} - (default: (1,2)) an iterable of two positive integers

OUTPUT:

infinite word

EXAMPLES:

The usual Kolakoski word:

\[
sage: w = \text{words.KolakoskiWord()}\\
sage: w\\
word: 12211212212211211212211211221221221...\\
sage: w.delta()\\
word: 12211212212211211212211211221221221221...
\]

The other Kolakoski word on the same alphabet:

\[
sage: w = \text{words.KolakoskiWord(alphabet = (2,1))}\\
sage: w\\
word: 22112212212211211212211212212212211...\\
sage: w.delta()\\
word: 22112212212211211212211212212212212211...
\]

It is naturally generalized to any two integers alphabet:

\[
sage: w = \text{words.KolakoskiWord(alphabet = (2,5))}\\
sage: w\\
word: 22552222555552255225522555552255555225555522555552255555225555522555552...\\
sage: w.delta()\\
word: 22552222555552255225522555552255555225555522555552255555225555522555552...
\]

REFERENCES:
**LowerChristoffelWord**

alias of **LowerChristoffelWord**

**LowerMechanicalWord**(alpha, rho=0, alphabet=None)

Returns the lower mechanical word with slope $\alpha$ and intercept $\rho$.

The lower mechanical word $s_{\alpha, \rho}$ with slope $\alpha$ and intercept $\rho$ is defined by $s_{\alpha, \rho}(n) = \lfloor \alpha(n + 1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor$. [Loth02]

**INPUT:**

- alpha – real number such that $0 \leq \alpha \leq 1$
- rho – real number (optional, default: 0)
- alphabet – iterable of two elements or None (optional, default: None)

**OUTPUT:**

infinite word

**EXAMPLES:**

```sage```
lower_mechanical_word = words.LowerMechanicalWord(1/golden_ratio**2)
word: 001001001001001001001001001001001001001001...```

```sage```
lower_mechanical_word = words.LowerMechanicalWord(1/5)
word: 00010001000100010001000100010001000100010001...```

```sage```
lower_mechanical_word = words.LowerMechanicalWord(1/pi)
word: 0001001001001001001001001001001001001001001...```

**MinimalSmoothPrefix**(n)

This function finds and returns the minimal smooth prefix of length $n$.

See [BMP2007] for a definition.

**INPUT:**

- n – the desired length of the prefix

**OUTPUT:**

word – the prefix

**Note:** Be patient, this function can take a really long time if asked for a large prefix.

**EXAMPLES:**

```sage```
minimal_smooth_prefix = words.MinimalSmoothPrefix(10)
word: 1212212112```

**PalindromicDefectWord**(k=1, alphabet='ab')

Return the finite word $w = ab^k ab^{k-1} aab^{k-1} ab^{k-1} a$.

As described by Brlek, Hamel, Nivat and Reutenauer in [BHN04], this finite word $w$ is such that the infinite periodic word $w^\omega$ has palindromic defect $k$.

**INPUT:**

- k – positive integer (optional, default: 1)
- alphabet – iterable (optional, default: 'ab') of size two
OUTPUT:
finite word

EXAMPLES:

```python
sage: words.PalindromicDefectWord(10)
word: abbbbbbbbbabbbbbbabbbbbbabbbbbbabbbbb
```

```python
sage: w = words.PalindromicDefectWord(3)
sage: w
word: abbbabbaabbbba
sage: w.defect()
0
sage: (w^2).defect()
3
sage: (w^3).defect()
3
```

On other alphabets:

```python
sage: words.PalindromicDefectWord(3, alphabet='cd')
word: cdddcddcdcdcdcc
sage: words.PalindromicDefectWord(3, alphabet=['c', 3])
word: c333c33cc33c333c
```

**RandomWord** *(n, m=2, alphabet=None)*

Return a random word of length *n* over the given *m*-letter alphabet.

**INPUT:**

- *n* - integer, the length of the word
- *m* - integer (default 2), the size of the output alphabet
- *alphabet* - (default is \{0, 1, ..., *m* − 1\}) any container of length *m* that is suitable to build an instance of OrderedAlphabet (list, tuple, str, ...)

**EXAMPLES:**

```python
sage: words.RandomWord(10)  # random results
word: 0110100101
sage: words.RandomWord(10, 4)  # random results
word: 0322313320
sage: words.RandomWord(100, 7)  # random results
word: 2630644023642516442650025611300034413310...
```

```python
sage: words.RandomWord(100, 7, range(-3,4))  # random results
word: 1,3,-1,-1,3,2,2,0,1,-2,1,-1,-3,-2,2,0,3,0,-3,0,3,0,-2,-2,2,0,1,-3,2,-2,-2,
-2,0,2,1,-2,-3,-2,-1,0,...
```

```python
sage: words.RandomWord(100, 5, "abcde")  # random results
word: acebaaccdbdbbdeadeebdeebbebaaacbadaac...
```

```python
sage: words.RandomWord(17, 5, "abcde")  # random results
word: dcacbbecbddebaadd
```

**StandardEpisturmianWord** *(directive_word)*

Returns the standard episturmian word (or epistandard word) directed by directive_word. Over a 2-letter alphabet, this function gives characteristic Sturmian words.
An infinite word $w$ over a finite alphabet $A$ is said to be \emph{standard episturmian} (or \emph{epistandard}) iff there exists an infinite word $x_1 x_2 x_3 \cdots$ over $A$ (called the \emph{directive word} of $w$) such that $w$ is the limit as $n$ goes to infinity of $Pal(x_1 \cdots x_n)$, where $Pal$ is the iterated palindromic closure function.

Note that an infinite word is \emph{episturmian} if it has the same set of factors as some epistandard word.

See for instance [DJP2001], [JP2002], and [GJ2007].

INPUT:

- \texttt{directive\_word} - an infinite word or a period of a periodic infinite word

EXAMPLES:

```python
sage: Fibonacci = words.StandardEpisturmianWord(Words('ab')('ab')); Fibonacci
word: abaababaababaababaababaababaabbaabaabaaba...
sage: Tribonacci = words.StandardEpisturmianWord(Words('abc')('abc'));
word: abacabacabacabacabacabacabacabababa...
sage: S = words.StandardEpisturmianWord(Words('abcd')('aabcabada')); S
word: aabaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababa...
sage: S[:25]
word: aabaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaab
Thue-Morse word on other alphabets:

```
sage: t = words.ThueMorseWord('ab'); t
word: abbabaabbaababaabbaabbaabbaabbaabbaabba...
```

```
sage: t = words.ThueMorseWord(['L1', 'L2'])
sage: t[:8]
word: L1,L2,L2,L1,L2,L1,L1,L2
```

Generalized Thue Morse word:

```
sage: words.ThueMorseWord(alphabet=(0,1,2), base=2)
word: 0112120122012001220120012001220...
```

```
sage: t = words.ThueMorseWord(alphabet=(0,1,2), base=5); t
word: 0120112012201200120112012120122012001201...
```

```
sage: t[100:130].critical_exponent()
10/3
```

REFERENCES:

**UpperChristoffelWord** \((p, q, alphabet=(0, 1))\)

Returns the upper Christoffel word of slope \(p/q\), where \(p\) and \(q\) are relatively prime non-negative integers, over the given alphabet.

The upper Christoffel word of slope \(p/q\) is equal to the reversal of the lower Christoffel word of slope \(p/q\). Equivalently, if \(xuy\) is the lower Christoffel word of slope \(p/q\), where \(x\) and \(y\) are letters, then \(yux\) is the upper Christoffel word of slope \(p/q\) (because \(u\) is a palindrome).

**INPUT:**

- **alphabet** - any container of length two that is suitable to build an instance of OrderedAlphabet (list, tuple, str, ...)

**EXAMPLES:**

```
sage: words.UpperChristoffelWord(1,0)
word: 1
```

```
sage: words.UpperChristoffelWord(0,1)
word: 0
```

```
sage: words.UpperChristoffelWord(1,1)
word: 10
```

```
sage: words.UpperChristoffelWord(4,7)
word: 10100100100
```

**UpperMechanicalWord** \((alpha, rho=0, alphabet=None)\)

Returns the upper mechanical word with slope \(\alpha\) and intercept \(\rho\)

The upper mechanical word \(s'_{\alpha, \rho}\) with slope \(\alpha\) and intercept \(\rho\) is defined by \(s'_{\alpha, \rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil\). [Loth02]

**INPUT:**

- **alpha** – real number such that \(0 \leq \alpha \leq 1\)
• rho – real number (optional, default: 0)
• alphabet – iterable of two elements or None (optional, default: None)

OUTPUT:
infinite word

EXAMPLES:

```
sage: words.UpperMechanicalWord(1/golden_ratio^2)
word: 1010010100101001001010010100101001...
sage: words.UpperMechanicalWord(1/5)
word: 10000100001000010000100001000010000...
sage: words.UpperMechanicalWord(1/pi)
word: 1001001001001001001001000100100100100100...
```

dual_fibonacci_tile(n)

Returns the \( n \)-th dual Fibonacci Tile \[\text{[BmBGL09]}\].

EXAMPLES:

```
sage: for i in range(4): words.dual_fibonacci_tile(i)
Path: 3210
Path: 32123032301032303212321012321010121232101232101010...
```

fibonacci_tile(n)

Returns the \( n \)-th Fibonacci Tile \[\text{[BmBGL09]}\].

EXAMPLES:

```
sage: for i in range(3): words.fibonacci_tile(i)
Path: 3210
Path: 323030101212
Path: 323030103032321232123210123212321010121232101232101010...
```

s_adic(sequence, letters, morphisms=None)

Returns the \( s \)-adic infinite word obtained from a sequence of morphisms applied on a letter.

DEFINITION (from \[\text{Fogg}\]):

Let \( w \) be a infinite word over an alphabet \( A = A_0 \). A standard representation of \( w \) is obtained from a sequence of substitutions \( \sigma_k : A_{k+1} \to A_k \) and a sequence of letters \( a_k \in A_k \) such that:

\[
\lim_{k \to \infty} \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_k (a_k).
\]

Given a set of substitutions \( S \), we say that the representation is \( S \)-adic standard if the substitutions are chosen in \( S \).

INPUT:

• sequence - An iterable sequence of indices or of morphisms. It may be finite or infinite. If sequence is infinite, the image of the \((i+1)\)-th letter under the \((i+1)\)-th morphism must start with the \(i\)-th letter.
• letters - A letter or a sequence of letters.
• morphisms - dict, list, callable or None (optional, default None) an object that maps indices to morphisms. If None, then sequence must consist of morphisms.
OUTPUT:

A word.

EXAMPLES:

Let us define three morphisms and compute the first nested successive prefixes of the $s$-adic word:

```python
sage: m1 = WordMorphism('e->gh,f->hg')
sage: m2 = WordMorphism('c->ef,d->e')
sage: m3 = WordMorphism('a->cd,b->dc')
sage: words.s_adic([m1], 'e')
word: gh
sage: words.s_adic([m1, m2], 'ec')
word: ghhg
sage: words.s_adic([m1, m2, m3], 'eca')
word: ghhggh
```

When the given sequence of morphism is finite, one may simply give the last letter, i.e. 'a', instead of giving all of them, i.e. 'eca':

```python
sage: words.s_adic([m1, m2, m3], 'a')
word: ghhggh
sage: words.s_adic([m1, m2, m3], 'b')
word: ghghhg
```

If the letters don't satisfy the hypothesis of the algorithm (nested prefixes), an error is raised:

```python
sage: words.s_adic([m1, m2, m3], 'ecb')
Traceback (most recent call last):
  ... ValueError: the hypothesis of the algorithm used is not satisfied; the image of
  the 3-th letter (=b) under the 3-th morphism (=a->cd, b->dc) should start
  with the 2-th letter (=c)
```

Let's define the Thue-Morse morphism and the Fibonacci morphism which will be used below to illustrate more examples and let's import the `repeat` tool from the `itertools`:

```python
sage: tm = WordMorphism('a->ab,b->ba')
sage: fib = WordMorphism('a->ab,b->a')
sage: from itertools import repeat
```

Two trivial examples of infinite $s$-adic words:

```python
sage: words.s_adic(repeat(tm), repeat('a'))
word: abbabaabaababababaaabbaabbaababbaababbaababbaababbaab...  
```

```python
sage: words.s_adic(repeat(fib), repeat('a'))
word: ababaabbaababaababaababaababaababaababaababaababaababaab...  
```

A less trivial infinite $s$-adic word:

```python
sage: D = {4:tm, 5:fib}
sage: tmword = words.ThueMorseWord([4, 5])
sage: it = (D[a] for a in tmword)
```

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(continued from previous page)

sage: words.s_adic(it, repeat('a'))
word: abbaababbaabbaabbaababbaababbaabbaababba...
The same thing using a sequence of indices:
sage: tmword = words.ThueMorseWord([0,1])
sage: words.s_adic(tmword, repeat('a'), [tm,fib])
word: abbaababbaabbaabbaababbaababbaabbaababba...
The correspondence of the indices may be given as a dict:
sage: words.s_adic(tmword, repeat('a'), {0:tm,1:fib})
word: abbaababbaabbaabbaababbaababbaabbaababba...
because dict are more versatile for indices:
sage: tmwordTF = words.ThueMorseWord('TF')
sage: words.s_adic(tmwordTF, repeat('a'), {'T':tm,'F':fib})
word: abbaababbaabbaabbaababbaababbaabbaababba...
or by a callable:
sage: f = lambda n: tm if n == 0 else fib
sage: words.s_adic(words.ThueMorseWord(), repeat('a'), f)
word: abbaababbaabbaabbaababbaababbaabbaababba...
Random infinite 𝑠-adic words:
sage:
sage:
....:
sage:
word:
sage:
word:
sage:
word:

from sage.misc.prandom import randint
def it():
while True: yield randint(0,1)
words.s_adic(it(), repeat('a'), [tm,fib]) # random
abbaabababbaababbaabbaababbaabababbaabba...
words.s_adic(it(), repeat('a'), [tm,fib]) # random
abbaababbaabbaababbaababbaabbaababbaabba...
words.s_adic(it(), repeat('a'), [tm,fib]) # random
abaaababaabaabaaababaabaaababaaababaabaa...

An example where the sequences cycle on two morphisms and two letters:
sage:
sage:
sage:
sage:
word:
sage:
word:

G = WordMorphism('a->cd,b->dc')
H = WordMorphism('c->ab,d->ba')
from itertools import cycle
words.s_adic([G,H],'ac')
cddc
words.s_adic(cycle([G,H]),cycle('ac'))
cddcdccddccdcddcdccdcddccddcdccddccdcddc...

The morphism 𝜎 : 𝑎 ↦→ 𝑏𝑎, 𝑏 ↦→ 𝑏 can’t satisfy the hypothesis of the nested prefixes, but one may compute
arbitrarily long finite words having the limit 𝜎 𝜔 (𝑎):
sage: sigma = WordMorphism('a->ba,b->b')
sage: words.s_adic(repeat(sigma),repeat('a'))
(continues on next page)

5.1. Comprehensive Module List

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The following examples illustrates an $S$-adic word defined over an infinite set $S$ of morphisms $x_h$:

```
sage: x = lambda h:WordMorphism({1:[2],2:[3]+[1]*(h+1),3:[3]+[1]*h})
sage: for h in [0,1,2,3]:
    ....:     print("{} {}").format(h, x(h))
0 1->2, 2->3
1 1->2, 2->31, 3->3
2 1->2, 2->311, 3->31
3 1->2, 2->3111, 3->311
sage: w = Word(lambda n : valuation(n+1, 2) ); w
word: 0102010301020104010201030102010501020103...
sage: s = words.s_adic(w, repeat(3), x); s
word: 32322322232223222322232223222322232223222322...
sage: prefixe = s[:10000]
sage: list(map(prefixe.number_of_factors, range(15)))
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]
sage: [_[i+1] - _[i] for i in range(len(_)-1)]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

AUTHORS:

- Sébastien Labbé (2009-12-18): initial version
5.1.367 Datatypes for words defined by iterators and callables

```python
class sage.combinat.words.word_infinite_datatypes.WordDatatype_callable(parent, callable, length=None):

    Bases: WordDatatype

    Datatype for a word defined by a callable.

class sage.combinat.words.word_infinite_datatypes.WordDatatype_callable_with_caching(parent, callable, length=None):

    Bases: WordDatatype_callable

    Datatype for a word defined by a callable.

flush()

    Empty the associated cache of letters.

    EXAMPLES:

    The first 40 (by default) values are always cached:

    sage: w = words.ThueMorseWord()
    sage: w._letter_cache
    {0: 0, 1: 1, 2: 1, 3: 0, 4: 1, 5: 0, 6: 0, 7: 1, 8: 1, 9: 0, 10: 0, 11: 1, 12: 1,
     13: 0, 14: 1, 15: 0, 16: 1, 17: 0, 18: 0, 19: 1, 20: 0, 21: 1, 22: 1, 23: 0,
     24: 0, 25: 1, 26: 1, 27: 0, 28: 1, 29: 0, 30: 0, 31: 1, 32: 1, 33: 0, 34: 0,
     35: 1, 36: 0, 37: 1, 38: 1, 39: 0}
    sage: w[100]
    1
    sage: w._letter_cache
    {0: 0, 1: 1, 2: 1, 3: 0, 4: 1, 5: 0, 6: 0, 7: 1, 8: 1, 9: 0, 10: 0, 11: 1, 12: 1,
     13: 0, 14: 1, 15: 0, 16: 1, 17: 0, 18: 0, 19: 1, 20: 0, 21: 1, 22: 1, 23: 0,
     24: 0, 25: 1, 26: 1, 27: 0, 28: 1, 29: 0, 30: 0, 31: 1, 32: 1, 33: 0, 34: 0,
     35: 1, 36: 0, 37: 1, 38: 1, 39: 0, 100: 1}
    sage: w.flush()
    sage: w._letter_cache
    {}
```

```python
class sage.combinat.words.word_infinite_datatypes.WordDatatype_iter(parent, iter, length=None):

    Bases: WordDatatype

    INPUT:

    • parent - a parent
    • iter - an iterator
    • length - (default: None) the length of the word

    EXAMPLES:

    sage: w = Word(iter("abbabaab"), length="unknown", caching=False); w
    word: abbabaab
    sage: isinstance(w, sage.combinat.words.word_infinite_datatypes.WordDatatype_iter)
    True
    sage: w.length() is None
    False
```

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sage: w.length()
8
sage: s = "abbabaabbaababbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaa...
flush()
Delete the cached values.

EXAMPLES:

```
sage: from itertools import count
sage: w = Word(count())
sage: w._last_index, len(w._list)
(39, 40)
sage: w[43]
43
sage: w._last_index, len(w._list)
(43, 44)
sage: w.flush()
sage: w._last_index, w._list
(-1, [])
```

5.1.368 User-customizable options for words

```
sage.combinat.words.word_options.WordOptions(**kwargs)
```
Sets the global options for elements of the word class. The defaults are for words to be displayed in list notation.

**INPUT:**
- display - 'string' (default), or 'list', words are displayed in string or list notation.
- truncate - boolean (default: True), whether to truncate the string output of long words (see truncate_length below).
- truncate_length - integer (default: 40), if the length of the word is greater than this integer, then the word is truncated.
- letter_separator - (string, default: ",") if the string representation of letters have length greater than 1, then the letters are separated by this string in the string representation of the word.

If no parameters are set, then the function returns a copy of the options dictionary.

**EXAMPLES:**

```
sage: w = Word([2,1,3,12])
sage: u = Word("abba")
sage: WordOptions(display='list')
sage: w
word: [2, 1, 3, 12]
sage: u
word: ['a', 'b', 'b', 'a']
sage: WordOptions(display='string')
sage: w
word: 2,1,3,12
sage: u
word: abba
```
5.1.369 Set of words

To define a new class of words, please refer to the documentation file: sage/combinat/words/notes/word_inheritance_howto.rst

AUTHORS:

• Franco Saliola (2008-12-17): merged into sage
• Sebastien Labbe (2008-12-17): merged into sage
• Arnaud Bergeron (2008-12-17): merged into sage
• Sebastien Labbe (2009-07-21): Improved morphism iterator (github issue #6571).
• Vincent Delecroix (2015): classes simplifications (github issue #19619)

EXAMPLES:

```
sage: Words()
Finite and infinite words over Set of Python objects of class 'object'
sage: Words(4)
Finite and infinite words over {1, 2, 3, 4}
sage: Words(4,5)
Words of length 5 over {1, 2, 3, 4}
sage: FiniteWords('ab')
Finite words over {'a', 'b'}
sage: InfiniteWords('natural numbers')
Infinite words over Non negative integers
```

```python
class sage.combinat.words.words.AbstractLanguage(alphabet=None, category=None):
    Bases: Parent
    
    Abstract base class
    
    This is not to be used by any means. This class gather previous features of set of words (prior to github issue #19619). In the future that class might simply disappear or become a common base class for all languages. In the latter case, its name would possibly change to Language.

    alphabet()

    EXAMPLES:

    ```
sage: Words(NN).alphabet()
Non negative integer semiring
sage: InfiniteWords([1,2,3]).alphabet()
{1, 2, 3}
sage: InfiniteWords('ab').alphabet()
{'a', 'b'}
sage: FiniteWords([1,2,3]).alphabet()
{1, 2, 3}
sage: FiniteWords().alphabet()
Set of Python objects of class 'object'
```

    identity_morphism()

    Returns the identity morphism from self to itself.
```
EXAMPLES:

```python
sage: W = Words('ab')
sage: W.identity_morphism()
WordMorphism: a->a, b->b
```

```python
sage: W = Words(range(3))
sage: W.identity_morphism()
WordMorphism: 0->0, 1->1, 2->2
```

There is no support yet for infinite alphabet:

```python
sage: W = Words(alphabet=Alphabet(name='NN'))
sage: W
Finite and infinite words over Non negative integers
sage: W.identity_morphism()
Traceback (most recent call last):
  ... NotImplementedError: size of alphabet must be finite
```

```python
class sage.combinat.words.words.FiniteOrInfiniteWords(alphabet)
    Bases: AbstractLanguage

INPUT:
    • alphabet – the underlying alphabet

cardinality()
    Return the cardinality of this set of words.

EXAMPLES:

```python
sage: Words('abcd').cardinality()
+Infinity
sage: Words('a').cardinality()
+Infinity
sage: Words('').cardinality()
1
```

factors()
    Return the set of finite words.

EXAMPLES:

```python
sage: Words('ab').finite_words()
Finite words over {'a', 'b'}
```

finite_words()
    Return the set of finite words.

EXAMPLES:

```python
sage: Words('ab').finite_words()
Finite words over {'a', 'b'}
```
infinite_words()

Return the set of infinite words.

EXAMPLES:

```
sage: Words('ab').infinite_words()
Infinite words over {'a', 'b'}
```

iterate_by_length(length)

Return an iterator over the words of given length.

EXAMPLES:

```
sage: [w.string_rep() for w in Words('ab').iterate_by_length(3)]
['aaa', 'aab', 'aba', 'abb', 'baa', 'bab', 'bba', 'bbb']
```

shift()

Return the set of infinite words.

EXAMPLES:

```
sage: Words('ab').infinite_words()
Infinite words over {'a', 'b'}
```

class sage.combinat.words.words.FiniteWords(alphabet=None, category=None)

Bases: AbstractLanguage

The set of finite words over a fixed alphabet.

EXAMPLES:

```
sage: W = FiniteWords('ab')
sage: W
Finite words over {'a', 'b'}
```

cardinality()

Return the cardinality of this set.

EXAMPLES:

```
sage: FiniteWords('').cardinality()
1
sage: FiniteWords('a').cardinality()
+Infinity
```

factors()

Return itself.

EXAMPLES:

```
sage: FiniteWords('ab').factors()
Finite words over {'a', 'b'}
```

iter_morphisms(arg=None, codomain=None, min_length=1)

Iterate over all morphisms with domain self and the given codomain.

INPUT:
• **arg** - (optional, default: None) It can be one of the following:
  
  – *None* - then the method iterates through all morphisms.
  
  – tuple \((a, b)\) of two integers - It specifies the range \(\text{range}(a, b)\) of values to consider for the sum of the length of the image of each letter in the alphabet.
  
  – list of nonnegative integers - The length of the list must be equal to the size of the alphabet, and the \(i\)-th integer of **arg** determines the length of the word mapped to by the \(i\)-th letter of the (ordered) alphabet.

• **codomain** - (default: None) a combinatorial class of words. By default, **codomain** is **self**.

• **min_length** - (default: 1) nonnegative integer. If **arg** is not specified, then iterate through all the morphisms where the length of the images of each letter in the alphabet is at least **min_length**. This is ignored if **arg** is a list.

**OUTPUT:**

iterator

**EXAMPLES:**

Iterator over all non-erasing morphisms:

```python
sage: W = FiniteWords('ab')
sage: it = W.iter_morphisms()
sage: for _ in range(7): next(it)
  WordMorphism: a->a, b->a
  WordMorphism: a->a, b->b
  WordMorphism: a->b, b->a
  WordMorphism: a->b, b->b
  WordMorphism: a->aa, b->a
  WordMorphism: a->aa, b->b
  WordMorphism: a->ab, b->a
  WordMorphism: a->ab, b->b
```

Iterator over all morphisms including erasing morphisms:

```python
sage: W = FiniteWords('ab')
sage: it = W.iter_morphisms(min_length=0)
sage: for _ in range(7): next(it)
  WordMorphism: a->, b->
  WordMorphism: a->a, b->
  WordMorphism: a->a, b->b
  WordMorphism: a->b, b->
  WordMorphism: a->b, b->a
  WordMorphism: a->aa, b->
  WordMorphism: a->aa, b->b
  WordMorphism: a->ab, b->
  WordMorphism: a->ab, b->b
```

Iterator over morphisms where the sum of the lengths of the images of the letters is in a specific range:

```python
sage: for m in W.iter_morphisms((0, 3), min_length=0): m
  WordMorphism: a->aa, b->
  WordMorphism: a->ab, b->
  WordMorphism: a->ba, b->
  WordMorphism: a->bb, b->
  WordMorphism: a->a, b->a
  WordMorphism: a->a, b->b
```
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<table>
<thead>
<tr>
<th>WordMorphism: a-&gt;b, b-&gt;a</th>
</tr>
</thead>
<tbody>
<tr>
<td>WordMorphism: a-&gt;b, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;b, b-&gt;</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;, b-&gt;aa</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;, b-&gt;ab</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;, b-&gt;ba</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;, b-&gt;bb</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;, b-&gt;b</td>
</tr>
</tbody>
</table>

```
sage: for m in W.iter_morphisms((2, 4)): m
```

<table>
<thead>
<tr>
<th>WordMorphism: a-&gt;aa, b-&gt;a</th>
</tr>
</thead>
<tbody>
<tr>
<td>WordMorphism: a-&gt;aa, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ab, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ab, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ba, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ba, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;bb, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;bb, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;aa</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;ab</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;ba</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;bb</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;b, b-&gt;aa</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;b, b-&gt;ab</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;b, b-&gt;ba</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;b, b-&gt;bb</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;a, b-&gt;b</td>
</tr>
</tbody>
</table>

Iterator over morphisms with specific image lengths:

```
sage: for m in W.iter_morphisms([0, 0]): m
```

| WordMorphism: a->, b-> |

```
sage: for m in W.iter_morphisms([0, 1]): m
```

<table>
<thead>
<tr>
<th>WordMorphism: a-&gt;aa, b-&gt;a</th>
</tr>
</thead>
<tbody>
<tr>
<td>WordMorphism: a-&gt;aa, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ab, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ab, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ba, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ba, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;bb, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;bb, b-&gt;b</td>
</tr>
</tbody>
</table>

```
sage: for m in W.iter_morphisms([2, 1]): m
```

<table>
<thead>
<tr>
<th>WordMorphism: a-&gt;aa, b-&gt;a</th>
</tr>
</thead>
<tbody>
<tr>
<td>WordMorphism: a-&gt;aa, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ab, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ab, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ba, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;ba, b-&gt;b</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;bb, b-&gt;a</td>
</tr>
<tr>
<td>WordMorphism: a-&gt;bb, b-&gt;b</td>
</tr>
</tbody>
</table>

```
sage: for m in W.iter_morphisms([2, 2]): m
```

| WordMorphism: a->aa, b->aa |

(continues on next page)
WordMorphism: a->aa, b->ab
WordMorphism: a->aa, b->ba
WordMorphism: a->aa, b->bb
WordMorphism: a->ab, b->aa
WordMorphism: a->ab, b->ab
WordMorphism: a->ab, b->ba
WordMorphism: a->ab, b->bb
WordMorphism: a->ba, b->aa
WordMorphism: a->ba, b->ab
WordMorphism: a->ba, b->ba
WordMorphism: a->ba, b->bb
WordMorphism: a->bb, b->aa
WordMorphism: a->bb, b->ab
WordMorphism: a->bb, b->ba
WordMorphism: a->bb, b->bb

The codomain may be specified as well:

```
sage: Y = FiniteWords('xyz')
sage: for m in W.iter_morphisms([0, 2], codomain=Y): m
WordMorphism: a->, b->xx
WordMorphism: a->, b->xy
WordMorphism: a->, b->xz
WordMorphism: a->, b->yx
WordMorphism: a->, b->yy
WordMorphism: a->, b->yz
WordMorphism: a->, b->zx
WordMorphism: a->, b->zy
WordMorphism: a->, b->zz
```

```
sage: for m in Y.iter_morphisms([0,2,1], codomain=W): m
WordMorphism: x->, y->aa, z->a
WordMorphism: x->, y->aa, z->b
WordMorphism: x->, y->ab, z->a
WordMorphism: x->, y->ab, z->b
WordMorphism: x->, y->ba, z->a
WordMorphism: x->, y->ba, z->b
WordMorphism: x->, y->bb, z->a
WordMorphism: x->, y->bb, z->b
```

```
sage: it = W.iter_morphisms(codomain=Y)
sage: for _ in range(10): next(it)
WordMorphism: a->x, b->x
WordMorphism: a->x, b->y
WordMorphism: a->x, b->z
WordMorphism: a->y, b->x
WordMorphism: a->y, b->y
WordMorphism: a->y, b->z
WordMorphism: a->z, b->x
WordMorphism: a->z, b->y
WordMorphism: a->z, b->z
WordMorphism: a->xx, b->x
```

iterate_by_length(l=1)
Returns an iterator over all the words of self of length l.
INPUT:

- \( l \) - integer (default: 1), the length of the desired words

EXAMPLES:

```python
sage: W = FiniteWords('ab')
sage: list(W.iterate_by_length(1))
[word: a, word: b]
sage: list(W.iterate_by_length(2))
[word: aa, word: ab, word: ba, word: bb]
sage: list(W.iterate_by_length(3))
[word: aaa,
 word: aab,
 word: aba,
 word: abb,
 word: baa,
 word: bab,
 word: bba,
 word: bbb]
sage: list(W.iterate_by_length('a'))
Traceback (most recent call last):
 ... TypeError: the parameter l (= 'a') must be an integer
```

```python
random_element(length=None, *args, **kwds)
```

Returns a random finite word on the given alphabet.

INPUT:

- \( length \) – (optional) the length of the word. If not set, will use a uniformly random number between 0 and 10.
- all other argument are transmitted to the random generator of the alphabet

EXAMPLES:

```python
sage: W = FiniteWords(5)
sage: W.random_element()
# random word: 5114325445423521544531411434451152142155...
sage: W = FiniteWords(ZZ)
sage: W.random_element()  # random
word: 5,-1,-1,-1,0,0,0,0,-3,-11
sage: W.random_element(length=4, x=0, y=4)  # random
word: 1003
```

```
shift()
```

Return the set of infinite words on the same alphabet.

EXAMPLES:

```python
sage: FiniteWords('ab').shift()
Infinite words over {'a', 'b'}
```

```
class sage.combinat.words.words.InfiniteWords(alphabet=None, category=None)
Bases: AbstractLanguage
```
cardinality()

Return the cardinality of this set

EXAMPLES:

```
sage: InfiniteWords('ab').cardinality()
+Infinity
sage: InfiniteWords('a').cardinality()
1
sage: InfiniteWords('').cardinality()
0
```

factors()

Return the set of finite words on the same alphabet.

EXAMPLES:

```
sage: InfiniteWords('ab').factors()
Finite words over {'a', 'b'}
```

random_element(*args, **kwds)

Return a random infinite word.

EXAMPLES:

```
sage: W = InfiniteWords('ab')
sage: W.random_element()  # random
word: abbbabbaabbabbaabbaabbbbbaabbbbb...
sage: W = InfiniteWords(ZZ)
sage: W.random_element(x=2, y=4)  # random
word: 333322322232233332233223222323232323...
```

shift()

Return itself.

EXAMPLES:

```
sage: InfiniteWords('ab').shift()
Infinite words over {'a', 'b'}
```

```

sage.combinat.words.words.Words(alphabet=None, length=None, finite=True, infinite=True)

Returns the combinatorial class of words of length k over an alphabet.

EXAMPLES:

```
sage: Words()
Finite and infinite words over Set of Python objects of class 'object'
sage: Words(length=7)
Words of length 7 over Set of Python objects of class 'object'
sage: Words(5)
Finite and infinite words over {1, 2, 3, 4, 5}
sage: Words(5, 3)
Words of length 3 over {1, 2, 3, 4, 5}
sage: Words(5, infinite=False)
```

(continues on next page)
Finite words over \{1, 2, 3, 4, 5\}
sage: Words(5, finite=False)
Infinite words over \{1, 2, 3, 4, 5\}
sage: Words('ab')
Finite and infinite words over \{'a', 'b'}\)
sage: Words('ab', 2)
Words of length 2 over \{'a', 'b'}\)
sage: Words('ab', infinite=False)
Finite words over \{'a', 'b'}\)
sage: Words('ab', finite=False)
Infinite words over \{'a', 'b'}\)
sage: Words('positive integers', infinite=False)
Infinite words over Positive integers
sage: Words('natural numbers')
Finite and infinite words over Non negative integers

```python
class sage.combinat.words.words.Words_n(words, n):
    Bases: Parent
    The set of words of fixed length on a given alphabet.

    alphabet()
    Return the underlying alphabet.

    EXAMPLES:
    sage: Words([0,1], 4).alphabet()
    {0, 1}

cardinality()
    Returns the number of words of length \(n\) from alphabet.

    EXAMPLES:
    sage: Words(['a','b','c'], 4).cardinality()
    81
    sage: Words(3, 4).cardinality()
    81
    sage: Words(0,0).cardinality()
    1
    sage: Words(5,0).cardinality()
    1
    sage: Words(['a','b','c'],0).cardinality()
    1
    sage: Words(0,1).cardinality()
    0
    sage: Words(5,1).cardinality()
    5
    sage: Words(['a','b','c'],1).cardinality()
    3
    sage: Words(7,13).cardinality()
    96889010407
    sage: Words(['a','b','c','d','e','f','g'],13).cardinality()
    96889010407
```
iterate_by_length(length)

All words in this class are of the same length, so use iterator instead.

list()

Returns a list of all the words contained in self.

EXAMPLES:

```python
sage: Words(0,0).list()
[ ]
sage: Words(5,0).list()
[ ]
sage: Words(['a','b','c'],0).list()
[ ]
sage: Words(5,1).list()
[word: 1, word: 2, word: 3, word: 4, word: 5]
sage: Words(['a','b','c'],2).list()
[word: aa, word: ab, word: ac, word: ba, word: bb, word: bc, word: ca, word: cb, word: cc]
```

random_element(*args, **kwds)

Return a random word in this set.

EXAMPLES:

```python
sage: W = Words('ab', 4)
sage: W.random_element() # random
word: bbab
sage: W.random_element() in W
True

sage: W = Words(ZZ, 5)
sage: W.random_element() # random
word: 1,2,2,-1,12
sage: W.random_element() in W
True
```

5.1.370 Yang-Baxter Graphs

class sage.combinat.yang_baxter_graph.SwapIncreasingOperator(i)

Bases: SwapOperator

class sage.combinat.yang_baxter_graph.SwapOperator(i)

Bases: SageObject

The operator that swaps the items in positions i and i+1.

EXAMPLES:

```python
sage: from sage.combinat.yang_baxter_graph import SwapOperator
sage: s3 = SwapOperator(3)
sage: s3 == loads(dumps(s3))
True
```
position()

self is the operator that swaps positions i and i+1. This method returns i.

EXAMPLES:

```python
sage: from sage.combinat.yang_baxter_graph import SwapOperator
sage: s3 = SwapOperator(3)
sage: s3.position()
3
```

sage.combinat.yang_baxter_graph.YangBaxterGraph(partition=None, root=None, operators=None)

Construct the Yang-Baxter graph from root by repeated application of operators, or the Yang-Baxter graph associated to partition.

INPUT:

The user needs to provide either partition or both root and operators, where

- partition – a partition of a positive integer
- root – the root vertex
- operator - a function that maps vertices u to a list of tuples of the form (v, l) where v is a successor of u and l is the label of the edge from u to v.

OUTPUT:

- Either:
  - YangBaxterGraph_partition - if partition is defined
  - YangBaxterGraph_generic - if partition is None

EXAMPLES:

The Yang-Baxter graph defined by a partition \([p_1,\ldots, p_k]\) is the labelled directed graph with vertex set obtained by bubble-sorting \((p_k-1, p_k-2, \ldots, 0, \ldots, p_1-1, p_1-2, \ldots, 0)\); there is an arrow from \(u\) to \(v\) labelled by \(i\) if \(v\) is obtained by swapping the \(i\)-th and \((i+1)\)-th elements of \(u\). For example, if the partition is \([3, 1]\), then we begin with \((0, 2, 1, 0)\) and generate all tuples obtained from it by swapping two adjacent entries if they are increasing:

```python
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: bubbleswaps = [SwapIncreasingOperator(i) for i in range(3)]
sage: Y = YangBaxterGraph(root=(0,2,1,0), operators=bubbleswaps); Y
Yang-Baxter graph with root vertex (0, 2, 1, 0)
sage: Y.vertices(sort=True)
[(0, 2, 1, 0), (2, 0, 1, 0), (2, 1, 0, 0)]
```

The partition keyword is a shorthand for the above construction:

```python
sage: Y = YangBaxterGraph(partition=[3,1]); Y
Yang-Baxter graph of [3, 1], with top vertex (0, 2, 1, 0)
sage: Y.vertices(sort=True)
[(0, 2, 1, 0), (2, 0, 1, 0), (2, 1, 0, 0)]
```

The permutahedron can be realized as a Yang-Baxter graph:

```python
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: swappers = [SwapIncreasingOperator(i) for i in range(3)]
```
The Cayley graph of a finite group can be realized as a Yang-Baxter graph:

```python
def left_multiplication_by(g):
    return lambda h: h * g
G = CyclicPermutationGroup(4)
operators = [left_multiplication_by(gen) for gen in G.gens()]
Y = YangBaxterGraph(root=G.identity(), operators=operators); Y
Yang-Baxter graph with root vertex ()
```
EXAMPLES:

```
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: ops = [SwapIncreasingOperator(i) for i in range(3)]
sage: Y = YangBaxterGraph(root=(0,2,1,0), operators=ops)
sage: Y.edges()
[((0, 2, 1, 0), (2, 0, 1, 0), Swap-if-increasing at position 0), ((2, 0, 1, 0),
  (2, 1, 0, 0), Swap-if-increasing at position 1)]
```

```
plot(*args, **kwds)

Plot self as a digraph.

EXAMPLES:

```
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: ops = [SwapIncreasingOperator(i) for i in range(4)]
sage: Y = YangBaxterGraph(root=(1,0,2,1,0), operators=ops)
sage: Y.plot()
Graphics object consisting of 16 graphics primitives
sage: Y.plot(edge_labels=False)
Graphics object consisting of 11 graphics primitives
```

```
relabel_edges(edge_dict, inplace=True)

Relabel the edges of self.

INPUT:

- edge_dict -- a dictionary keyed by the (unlabelled) edges.

EXAMPLES:

```
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: ops = [SwapIncreasingOperator(i) for i in range(3)]
sage: Y = YangBaxterGraph(root=(0,2,1,0), operators=ops)
sage: def relabel_op(op, u):
    ...:     i = op.position()
    ...:     return u[:i] + u[i:i+2][::-1] + u[i+2:]
sage: Y.edges()
[((0, 2, 1, 0), (2, 0, 1, 0), Swap-if-increasing at position 0), ((2, 0, 1, 0),
  (2, 1, 0, 0), Swap-if-increasing at position 1)]
sage: d = {((0,2,1,0),(2,0,1,0)):17, ((2,0,1,0),(2,1,0,0)):27}
sage: Y.relabel_edges(d, inplace=False).edges()
[((0, 2, 1, 0), (2, 0, 1, 0), 17), ((2, 0, 1, 0), (2, 1, 0, 0), 27)]
sage: Y.edges()
[((0, 2, 1, 0), (2, 0, 1, 0), Swap-if-increasing at position 0), ((2, 0, 1, 0),
  (2, 1, 0, 0), Swap-if-increasing at position 1)]
sage: Y.relabel_edges(d, inplace=True)
sage: Y.edges()
[((0, 2, 1, 0), (2, 0, 1, 0), 17), ((2, 0, 1, 0), (2, 1, 0, 0), 27)]
```

```
relabel_vertices(v, relabel_operator, inplace=True)

Relabel the vertices u of self by the object obtained from u by applying the
relabel_operator to v along a path from self.root() to u.

Note that the self.root() is paired with v.

INPUT:

```
• \(v\) – tuple, Permutation, ...

• inplace – if True, modifies self; otherwise returns a modified copy of self.

EXAMPLES:

```python
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: ops = [SwapIncreasingOperator(i) for i in range(3)]
sage: Y = YangBaxterGraph(root=(0,2,1,0), operators=ops)
sage: def relabel_op(op, u):
...:     i = op.position()
...:     return u[:i] + u[i:i+2][::-1] + u[i+2:]
sage: d = Y.relabel_vertices((1,2,3,4), relabel_op, inplace=False); d

Yang-Baxter graph with root vertex (1, 2, 3, 4)
```

```python
sage: Y.vertices(sort=True)
[(0, 2, 1, 0), (2, 0, 1, 0), (2, 1, 0, 0)]
```

```python
e = Y.relabel_vertices((1,2,3,4), relabel_op); e
```

```python
sage: Y.vertices(sort=True)
[(1, 2, 3, 4), (2, 1, 3, 4), (2, 3, 1, 4)]
```

```
root()

Return the root vertex of self.

If self is the Yang-Baxter graph of the partition \([p_1, p_2, \ldots, p_k]\), then this is the vertex \((p_k - 1, p_k - 2, \ldots, 0, \ldots, p_1 - 1, p_1 - 2, \ldots, 0)\).

EXAMPLES:

```python
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: ops = [SwapIncreasingOperator(i) for i in range(4)]
sage: Y = YangBaxterGraph(root=(1,0,2,1,0), operators=ops)
sage: Y.root()
(1, 0, 2, 1, 0)
```

```python
sage: Y = YangBaxterGraph(root=(0,1,0,2,1,0), operators=ops)
sage: Y.root()
(0, 1, 0, 2, 1, 0)
```

```python
sage: Y = YangBaxterGraph(root=(1,0,3,2,1,0), operators=ops)
sage: Y.root()
(1, 0, 3, 2, 1, 0)
```

```python
sage: Y = YangBaxterGraph(partition=[3,2])
sage: Y.root()
(1, 0, 2, 1, 0)
```

```
successors(v)

Return the successors of the vertex v.

EXAMPLES:

```python
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
sage: ops = [SwapIncreasingOperator(i) for i in range(4)]
sage: Y = YangBaxterGraph(root=(1,0,2,1,0), operators=ops)
sage: Y.successors(Y.root())
[(1, 2, 0, 1, 0)]
```

```python
sage: sorted(Y.successors((1, 2, 0, 1, 0)))
[(1, 2, 1, 0, 0), (2, 1, 0, 1, 0)]
```
vertex_relabelling_dict($v$, relabel_operator)

Return a dictionary pairing vertices $u$ of self with the object obtained from $v$ by applying the relabel_operator along a path from the root to $u$.

Note that the root is paired with $v$.

INPUT:
- $v$ – an object
- relabel_operator – function mapping a vertex and a label to the image of the vertex

OUTPUT:
- dictionary pairing vertices with the corresponding image of $v$

EXAMPLES:

```
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
dsage: ops = [SwapIncreasingOperator(i) for i in range(3)]
dsage: Y = YangBaxterGraph(root=(0,2,1,0), operators=ops)
dsage: def relabel_operator(op, u):
    ....:     i = op.position()
    ....:     return u[:i] + u[i:i+2][::-1] + u[i+2:]
dsage: Y.vertex_relabelling_dict((1,2,3,4), relabel_operator)
{(0, 1, 2, 3, 4): (1, 2, 3, 4),
 (2, 1, 0, 0): (2, 1, 3, 4),
 (2, 1, 0, 0): (2, 3, 1, 4)}
```

vertices(sort=False)

Return the vertices of self.

INPUT:
- sort – boolean (default False) whether to sort the vertices

EXAMPLES:

```
sage: from sage.combinat.yang_baxter_graph import SwapIncreasingOperator
dsage: ops = [SwapIncreasingOperator(i) for i in range(3)]
dsage: Y = YangBaxterGraph(root=(0,2,1,0), operators=ops)
dsage: Y.vertices(sort=True)
[(0, 2, 1, 0), (2, 0, 1, 0), (2, 1, 0, 0)]
```

class sage.combinat.yang_baxter_graph.YangBaxterGraph_partition($partition$)

Bases: YangBaxterGraph_generic

A class to model the Yang-Baxter graph of a partition.

The Yang-Baxter graph defined by a partition $[p_1, \ldots, p_k]$ is the labelled directed graph with vertex set obtained by bubble-sorting $(p_k - 1, p_k - 2, \ldots, 0, \ldots, p_1 - 1, p_1 - 2, \ldots, 0)$; there is an arrow from $u$ to $v$ labelled by $i$ if $v$ is obtained by swapping the $i$-th and $(i + 1)$-th elements of $u$.

**Note:** This is a lazy implementation: the digraph is only computed when it is needed.

EXAMPLES:
sage: Y = YangBaxterGraph(partition=[3,2,1]); Y
Yang-Baxter graph of [3, 2, 1], with top vertex (0, 1, 0, 2, 1, 0)
sage: loads(dumps(Y)) == Y
True

AUTHORS:

• Franco Saliola (2009-04-23)

relabel_vertices(v, inplace=True)

Relabel the vertices of self with the object obtained from v by applying the transpositions corresponding to the edge labels along some path from the root to the vertex.

INPUT:

• v – tuple, Permutation, ...

• inplace – if True, modifies self; otherwise returns a modified copy of self.

EXAMPLES:

sage: Y = YangBaxterGraph(partition=[3,1]); Y
Yang-Baxter graph of [3, 1], with top vertex (0, 2, 1, 0)
sage: d = Y.relabel_vertices((1,2,3,4), inplace=False); d
Digraph on 3 vertices
sage: Y.vertices(sort=True)
[(0, 2, 1, 0), (2, 0, 1, 0), (2, 1, 0, 0)]
sage: e = Y.relabel_vertices((1,2,3,4)); e
sage: Y.vertices(sort=True)
[(1, 2, 3, 4), (2, 1, 3, 4), (2, 3, 1, 4)]

vertex_relabelling_dict(v)

Return a dictionary pairing vertices u of self with the object obtained from v by applying transpositions corresponding to the edges labels along a path from the root to u.

Note that the root is paired with v.

INPUT:

• v – an object

OUTPUT:

• dictionary pairing vertices with the corresponding image of v

EXAMPLES:

sage: Y = YangBaxterGraph(partition=[3,1])
sage: Y.vertex_relabelling_dict((1,2,3,4))
{(0, 2, 1, 0): (1, 2, 3, 4),
 (2, 0, 1, 0): (2, 1, 3, 4),
 (2, 1, 0, 0): (2, 3, 1, 4)}
sage: Y.vertex_relabelling_dict((4,3,2,1))
{(0, 2, 1, 0): (4, 3, 2, 1),
 (2, 0, 1, 0): (3, 4, 2, 1),
 (2, 1, 0, 0): (3, 2, 4, 1)}
5.1.371 C-Finite Sequences

C-finite infinite sequences satisfy homogeneous linear recurrences with constant coefficients:

\[ a_{n+d} = c_0a_n + c_1a_{n+1} + \cdots + c_{d-1}a_{n+d-1}, \quad d > 0. \]

CFiniteSequences are completely defined by their ordinary generating function (o.g.f., which is always a fraction of polynomials over Z or Q).

EXAMPLES:

```
sage: fibo = CFiniteSequence(x/(1-x-x^2))  # the Fibonacci sequence
sage: fibo
C-finite sequence, generated by -x/(x^2 + x - 1)
sage: fibo.parent()
The ring of C-Finite sequences in x over Rational Field
sage: fibo.parent().category()
Category of commutative rings
sage: C.<x> = CFiniteSequences(QQ)
sage: fibo.parent() == C
True
sage: C(x/(1-x-x^2))
C-finite sequence, generated by -x/(x^2 + x - 1)
sage: C(x/(1-x-x^2)) == fibo
True
sage: var('y')
y
sage: CFiniteSequence(y/(1-y-y^2))
C-finite sequence, generated by -y/(y^2 + y - 1)
sage: CFiniteSequence(y/(1-y-y^2)) == fibo
False
```

Finite subsets of the sequence are accessible via python slices:

```
sage: fibo[137]  # the 137th term of the Fibonacci sequence
19134702400093278081449423917
sage: fibo[137] == fibonacci(137)
True
sage: fibo[0:12]
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
sage: fibo[14:4:-2]
[377, 144, 55, 21, 8]
```

They can be created also from the coefficients and start values of a recurrence:

```
sage: r = C.from_recurrence([1,1],[0,1])
sage: r == fibo
True
```

Given enough values, the o.g.f. of a C-finite sequence can be guessed:

```
sage: r = C.guess([0,1,1,2,3,5,8])
sage: r == fibo
True
```
Combinatorics, Release 10.0

See also:

\texttt{fibonacci()}, \texttt{BinaryRecurrenceSequence}

AUTHORS:

\begin{itemize}
\item Ralf Stephan (2014): initial version
\end{itemize}

REFERENCES:

\begin{itemize}
\item [GK1982]
\item [KP2011]
\item [SZ1994]
\item [Zei2011]
\end{itemize}

class \texttt{sage.rings.cfinite_sequence.CFiniteSequence(parent, ogf)}

Bases: \texttt{FieldElement}

Create a C-finite sequence given its ordinary generating function.

INPUT:

\begin{itemize}
\item \texttt{ogf} – a rational function, the ordinary generating function (can be an element from the symbolic ring, fraction field or polynomial ring)
\end{itemize}

OUTPUT:

\begin{itemize}
\item A \texttt{CFiniteSequence} object
\end{itemize}

EXAMPLES:

\begin{verbatim}
sage: CFiniteSequence((2-x)/(1-x-x^2)) # the Lucas sequence
C-finite sequence, generated by (x - 2)/(x^2 + x - 1)
sage: CFiniteSequence(x/(1-x)^3) # triangular numbers
C-finite sequence, generated by -x/(x^3 - 3*x^2 + 3*x - 1)
\end{verbatim}

Polynomials are interpreted as finite sequences, or recurrences of degree 0:

\begin{verbatim}
sage: CFiniteSequence(x^2-4*x^5) Finite sequence [1, 0, 0, -4], offset = 2 sage: CFiniteSequence(1) Finite sequence [1], offset = 0
\end{verbatim}

This implementation allows any polynomial fraction as o.g.f. by interpreting any power of \(x\) dividing the o.g.f. numerator or denominator as a right or left shift of the sequence offset:

\begin{verbatim}
sage: CFiniteSequence(x^2+3/x) Finite sequence [3, 0, 0, 1], offset = -1 sage: CFiniteSequence(1/(x+4/x^3)) Finite sequence [4, 0, 1], offset = -3 sage: P = LaurentPolynomialRing(QQ.fraction_field(), 'X') sage: X=P.gen() sage: CFiniteSequence(1/(1-X)) C-finite sequence, generated by -1/(X - 1)
\end{verbatim}

The o.g.f. is always normalized to get a denominator constant coefficient of +1:

\begin{verbatim}
sage: CFiniteSequence(1/(x-2)) C-finite sequence, generated by 1/(x - 2)
\end{verbatim}
The given ogf is used to create an appropriate parent: it can be a symbolic expression, a polynomial, or a fraction field element as long as it can be coerced into a proper fraction field over the rationals:

```python
sage: var('x')
x
sage: f1 = CFiniteSequence((2-x)/(1-x-x^2))
sage: P.<x> = QQ[]
sage: f2 = CFiniteSequence((2-x)/(1-x-x^2))
sage: f1 == f2
True
sage: f1.parent()
The ring of C-Finite sequences in x over Rational Field
sage: f1.ogf().parent()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
sage: CFiniteSequence(log(x))
Traceback (most recent call last):
  ...TypeError: unable to convert log(x) to a rational
```

closed_form(n=n')

Return a symbolic expression in \( n \), which equals the \( n \)-th term of the sequence.

It is a well-known property of C-finite sequences \( a_n \) that they have a closed form of the type:

\[
a_n = \sum_{i=1}^{d} c_i(n) \cdot r_i^n,
\]

where \( r_i \) are the roots of the characteristic equation and \( c_i(n) \) is a polynomial (whose degree equals the multiplicity of \( r_i \) minus one). This is a natural generalization of Binet’s formula for Fibonacci numbers. See, for instance, [KP2011, Theorem 4.1].

Note that if the o.g.f. has a polynomial part, that is, if the numerator degree is not strictly less than the denominator degree, then this closed form holds only when \( n \) exceeds the degree of that polynomial part. In that case, the returned expression will differ from the sequence for small \( n \).

EXAMPLES:

```python
sage: CFiniteSequence(1/(1-x)).closed_form()
1
sage: CFiniteSequence(x^2/(1-x)).closed_form()
1
sage: CFiniteSequence(1/(1-x^2)).closed_form()
1/2*(-1)^n + 1/2
sage: CFiniteSequence(1/(1+x^3)).closed_form()
1/3*(-1)^n + 1/3*(1/2*I*sqrt(3) + 1/2)^n + 1/3*(-1/2*I*sqrt(3) + 1/2)^n
sage: CFiniteSequence(1/(1-x)/(1-2*x)/(1-3*x)).closed_form()
9/2*3^n - 4*2^n + 1/2
```

Binet’s formula for the Fibonacci numbers:

```python
sage: CFiniteSequence(x/(1-x-x^2)).closed_form()
sqrt(1/5)*(1/2*sqrt(5) + 1/2)^n - sqrt(1/5)*(-1/2*sqrt(5) + 1/2)^n
sage: [__.subs(n=k).full_simplify() for k in range(6)]
[0, 1, 1, 2, 3, 5]
```

(continues on next page)
Examples with multiple roots:

\[
sage: \text{CFiniteSequence}(x^3(x^2+4x+1)/(1-x)^5).\text{closed_form()}
1/4*n^4 + 1/2*n^3 + 1/4*n^2
\]

\[
sage: \text{CFiniteSequence}((1+2*x-x^2)/(1-x)^4/(1+x)^2).\text{closed_form()}
1/12*n^3 - 1/8*(-1)^n*(n + 1) + 3/4*n^2 + 43/24*n + 9/8
\]

\[
sage: \text{CFiniteSequence}(1/(1-x)^3/(1-2*x)^4).\text{closed_form()}
4/3*(n^3 - 3*n^2 + 20*n - 36)*2^n + 1/2*n^2 + 19/2*n + 49
\]

\[
sage: \text{CFiniteSequence}((x/(1-x-x^2))^2).\text{closed_form()}
1/5*(n - sqrt(1/5))*(1/2*sqrt(5) + 1/2)^n + 1/5*(n + sqrt(1/5))*(-1/2*sqrt(5) + 1/2)^n
\]

\[\text{coefficients()}
\]
Return the coefficients of the recurrence representation of the C-finite sequence.

\[\text{OUTPUT:}
• A list of values
\]

\[\text{EXAMPLES:}
\]

\[
sage: \text{C.<x> = CFiniteSequences(QQ)}
sage: \text{lucas = C((2-x)/(1-x-x^2))} \quad \# \text{the Lucas sequence}
sage: \text{lucas.coefficients()}
[1, 1]
\]

\[\text{denominator()}
\]
Return the numerator of the o.g.f of self.

\[\text{EXAMPLES:}
\]

\[
sage: f = \text{CFiniteSequence}((2-x)/(1-x-x^2)); f
\text{C-finite sequence, generated by (x - 2)/(x^2 + x - 1)}
sage: f.denominator()
x^2 + x - 1
\]

\[\text{numerator()}
\]
Return the numerator of the o.g.f of self.

\[\text{EXAMPLES:}
\]

\[
sage: f = \text{CFiniteSequence}((2-x)/(1-x-x^2)); f
\text{C-finite sequence, generated by (x - 2)/(x^2 + x - 1)}
sage: f.numerator()
x - 2
\]

\[\text{ogf()}
\]
Return the ordinary generating function associated with the CFiniteSequence.

\[\text{This is always a fraction of polynomials in the base ring.}
\]

\[\text{EXAMPLES:}
\]
sage: C.<x> = CFiniteSequences(QQ)
sage: r = C.from_recurrence([2],[1])
sage: r.ogf()
-1/2/(x - 1/2)
sage: C(0).ogf()
0

recurrence_repr()

Return a string with the recurrence representation of the C-finite sequence.

OUTPUT:

• A string

EXAMPLES:

sage: C.<x> = CFiniteSequences(QQ)
sage: C((-2*x^3 + x^2 - x + 1)/(2*x^2 - 3*x + 1)).recurrence_repr()
'homogeneous linear recurrence with constant coefficients of degree 2: a(n+2) = a(n+1) - 2*a(n), starting a(0...) = [1, 2, 5, 9]'
sage: C(x^3/(1-x-x^2)).recurrence_repr()
'homogeneous linear recurrence with constant coefficients of degree 2: a(n+2) = a(n+1) + a(n), starting a(3...) = [1, 1, 2, 3]'

series(n)

Return the Laurent power series associated with the CFiniteSequence, with precision n.

INPUT:

• n – a nonnegative integer

EXAMPLES:

sage: C.<x> = CFiniteSequences(QQ)
sage: r = C.from_recurrence([-1,2],[0,1])
sage: s = r.series(4); s
x + 2*x^2 + 3*x^3 + 4*x^4 + O(x^5)
sage: type(s)
<class 'sage.rings.laurent_series_ring_element.LaurentSeries'>

sage.rings.cfinite_sequence.CFiniteSequences(base_ring=None, names=None, category=None)

Return the ring of C-Finite sequences.

The ring is defined over a base ring (Z or Q) and each element is represented by its ordinary generating function (ogf) which is a rational function over the base ring.

INPUT:
• base_ring – the base ring to construct the fraction field representing the C-Finite sequences
• names – (optional) the list of variables.

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C
The ring of C-Finite sequences in x over Rational Field
sage: C.an_element()
C-finite sequence, generated by (x - 2)/(x^2 + x - 1)
sage: C.category()
Category of commutative rings
sage: C.one()
Finite sequence [1], offset = 0
sage: C.zero()
Constant infinite sequence 0.
sage: C(x)
Finite sequence [1], offset = 1
sage: C(1/x)
Finite sequence [1], offset = -1
sage: C((-x + 2)/(-x^2 - x + 1))
C-finite sequence, generated by (x - 2)/(x^2 + x - 1)
```

```python
class sage.rings.cfinite_sequence.CFiniteSequences_generic(polynomial_ring, category)
Bases: CommutativeRing, UniqueRepresentation

The class representing the ring of C-Finite Sequences

Element

alias of CFiniteSequence

an_element()

Return an element of C-Finite Sequences.

OUTPUT:
The Lucas sequence.

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C.an_element()
C-finite sequence, generated by (x - 2)/(x^2 + x - 1)
```

fraction_field()

Return the fraction field used to represent the elements of self.

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C.fraction_field()
Fraction Field of Univariate Polynomial Ring in x over Rational Field
```

from_recurrence(coefficients, values)

Create a C-finite sequence given the coefficients c and starting values a of a homogeneous linear recurrence.

\[ a_{n+d} = c_0a_n + c_1a_{n+1} + \cdots + c_{d-1}a_{n+d-1}, \quad d \geq 0. \]
INPUT:
• coefficients – a list of rationals
• values – start values, a list of rationals

OUTPUT:
• A CFiniteSequence object

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C.from_recurrence([1,1],[0,1])  # Fibonacci numbers
C-finite sequence, generated by -x/(x^2 + x - 1)
sage: C.from_recurrence([-1,2],[0,1])  # natural numbers
C-finite sequence, generated by x/(x^2 - 2x + 1)
sage: r = C.from_recurrence([-1],[1])
sage: s = C.from_recurrence([-1],[1,-1])
sage: r == s
True
sage: r = C(x^3/(1-x-x^2))
sage: s = C.from_recurrence([1,1],[0,0,0,1,1])
sage: r == s
True
sage: C.from_recurrence(1,1)
Traceback (most recent call last):
  ... ValueError: Wrong type for recurrence coefficient list.
```

```
gen(i=0)

Return the i-th generator of self.

INPUT:
• i – an integer (default:0)

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C.gen()
x
sage: x == C.gen()
True
```

```
guess(sequence, algorithm='sage')

Return the minimal CFiniteSequence that generates the sequence.
Assume the first value has index 0.

INPUT:
• sequence – list of integers
• algorithm – string
  – ‘sage’ - the default is to use Sage’s matrix kernel function
  – ‘pari’ - use Pari’s implementation of LLL
  – ‘bm’ - use Sage’s Berlekamp-Massey algorithm
```
OUTPUT:

- a CFiniteSequence, or 0 if none could be found

With the default kernel method, trailing zeroes are chopped off before a guessing attempt. This may reduce the data below the accepted length of six values.

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C.guess([1,2,4,8,16,32])
C-finite sequence, generated by -1/2/(x - 1/2)
sage: r = C.guess([1,2,3,4,5])
Traceback (most recent call last):
...  
ValueError: Sequence too short for guessing.
```

With Berlekamp-Massey, if an odd number of values is given, the last one is dropped. So with an odd number of values the result may not generate the last value:

```python
sage: r = C.guess([1,2,4,8,9], algorithm='bm'); r
C-finite sequence, generated by -1/2/(x - 1/2)
sage: r[0:5]
[1, 2, 4, 8, 16]
```

`ngens()`

Return the number of generators of `self`

EXAMPLES:

```python
sage: from sage.rings.cfinite_sequence import CFiniteSequences
sage: C.<x> = CFiniteSequences(QQ)
sage: C.ngens()
1
```

`polynomial_ring()`

Return the polynomial ring used to represent the elements of `self`.

EXAMPLES:

```python
sage: C.<x> = CFiniteSequences(QQ)
sage: C.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```
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[HT95] W. Huffman and V. Tonchev, The existence of extremal self-dual \([50, 25, 10]\) codes and quasi-symmetric \(2-(49, 9, 6)\) designs, Designs, Codes and Cryptography September 1995, Volume 6, Issue 2, pp 97-106


[Buch00] A. Buch, A Littlewood-Richardson rule for the K-theory of Grassmannians, arXiv math.AG/0004137


[Haz2004] Michiel Hazewinkel, Explicit polynomial generators for the ring of quasisymmetric functions over the integers. arXiv math/0410366v1


[AHM2018] Edward Allen, Joshua Hallam, Sarah Mason, *Dual Immaculate Quasisymmetric Functions Expand Positively into Young Quasisymmetric Schur Functions*. arXiv 1606.03519


[GXZ] A. M. Garsia, G. Xin, M. Zabrocki, A three shuffle case of the compositional parking function conjecture, arXiv 1208.5796v1


Combinatorics, Release 10.0


[Bour] Bourbaki, *Lie Groups and Lie Algebras* IV.2


[SL000081] Sloane’s OEIS sequence A000081


[EilLan53] On the groups $H(\pi, n)$, I, Samuel Eilenberg and Saunders Mac Lane, 1953.


[PR22] Prasad, A., Ram, S., Splitting subspaces and a finite field interpretation of the Touchard-Riordan formula. arXiv 2205.11076


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