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The diophantine approximation deals with the approximation of real numbers (or real vectors) with rational numbers (or rational vectors). See the article Wikipedia article Diophantine_approximation for more information.
A continued fraction is a representation of a real number in terms of a sequence of integers denoted \([a_0; a_1, a_2, \ldots]\). The well known decimal expansion is another way of representing a real number by a sequence of integers. The value of a continued fraction is defined recursively as:
\[
[a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{[a_1; a_2, \ldots]} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots}}}
\]
In this expansion, all coefficients \(a_n\) are integers and only the value \(a_0\) may be non positive. Note that \(a_0\) is nothing else but the floor (this remark provides a way to build the continued fraction expansion from a given real number). As examples

\[
\frac{45}{38} = 1 + \frac{1}{5 + \frac{1}{2 + \frac{1}{3}}}
\]

\[
\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{292 + \frac{1}{\ldots}}}}
\]

It is quite remarkable that

- any real number admits a unique continued fraction expansion
- finite expansions correspond to rationals
- ultimately periodic expansions correspond to quadratic numbers (ie numbers of the form \(a + b\sqrt{D}\) with \(a\) and \(b\) rationals and \(D\) square free positive integer)
- two real numbers \(x\) and \(y\) have the same tail (up to a shift) in their continued fraction expansion if and only if there are integers \(a, b, c, d\) with \(|ad - bc| = 1\) and such that \(y = (ax + b)/(cx + d)\).

Moreover, the rational numbers obtained by truncation of the expansion of a real number gives its so-called best approximations. For more informations on continued fractions, you may have a look at Wikipedia article Continued_fraction.

EXAMPLES:

If you want to create the continued fraction of some real number you may either use its method continued_fraction (if it exists) or call continued_fraction():
It is also possible to create a continued fraction from a list of partial quotients:

```
sage: continued_fraction([-3,1,2,3,4,1,2])
[-3; 1, 2, 3, 4, 1, 2]
```

Even infinite:

```
sage: w = words.ThueMorseWord([1,2])
sage: w
word: 12211212211221122211211211212221...
sage: continued_fraction(w)
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
```

To go back and forth between the value (as a real number) and the partial quotients (seen as a finite or infinite list) you can use the methods `quotients` and `value`:

```
sage: cf = (13/27).continued_fraction()
sage: cf.quotients()
[0, 2, 13]
sage: cf.value()
13/27
```

```
sage: cf = continued_fraction(pi)
sage: cf.quotients()
lazy list [3, 7, 15, ...]
sage: cf.value()
pi
```

```
sage: w = words.FibonacciWord([1,2])
sage: cf = continued_fraction(w)
sage: cf.quotients()
word: 121121121121121121121121121121121121...
sage: v = cf.value()
sage: v
```

(continues on next page)
Recall that quadratic numbers correspond to ultimately periodic continued fractions. For them special methods give access to preperiod and period:

\[
\text{sage: } K.<\sqrt{2}> = \text{QuadraticField}(2)
\]
\[
\text{sage: } cf = \text{continued_fraction}(\sqrt{2}); cf
\]
\[
[1; (2)^*]
\]
\[
\text{sage: } cf.\text{value}()
\]
\[
\sqrt{2}
\]
\[
\text{sage: } cf.\text{preperiod}()
\]
\[
(1,)
\]
\[
\text{sage: } cf.\text{period}()
\]
\[
(2,)
\]

\[
\text{sage: } cf = (3\sqrt{2} + 1/2).\text{continued_fraction}(); cf
\]
\[
[4; (1, 2, 1, 7)^*]
\]

On the following we can remark how the tail may change even in the same quadratic field:

\[
\text{sage: } \text{for } i \text{ in range(20): print(continued_fraction(i*sqrt2))}
\]
\[
[0]
\]
\[
[1; (2)^*]
\]
\[
[2; (1, 4)^*]
\]
\[
[4; (4, 8)^*]
\]
\[
[5; (1, 1, 1, 10)^*]
\]
\[
[7; (14)^*]
\]
\[
...\]
\[
[24; (24, 48)^*]
\]
\[
[25; (2, 5, 6, 5, 2, 50)^*]
\]
\[
[26; (1, 6, 1, 2, 3, 2, 26, 2, 3, 2, 1, 6, 1, 52)^*]
\]

Nevertheless, the tail is preserved under invertible integer homographies:

\[
\text{sage: } \text{apply_homography = lambda } m, z: (m[0, 0]z + m[0, 1]) / (m[1, 0]z + m[1, 1])
\]
\[
\text{sage: } m1 = \text{SL2Z([[60, 13, 83, 18]])}
\]
\[
\text{sage: } m2 = \text{SL2Z([[27, 80, 28, 83]])}
\]
\[
\text{sage: } a = \sqrt{2}/3
\]
\[
\text{sage: } a.\text{continued_fraction}()
\]
\[
[0; 2, (8, 4)^*]
\]
\[
\text{sage: } b = \text{apply_homography}(m1, a)
\]
\[
\text{sage: } b.\text{continued_fraction}()
\]
\[
[0; 1, 2, 1, 1, 1, 1, 6, (8, 4)^*]
\]
Todo:

- Improve numerical approximation (the method _mpfr_() is quite slow compared to the same method for an element of a number field)

- Make a class for generalized continued fractions of the form \(a_0 + b_0/(a_1 + b_1/(a_n))\) (the standard continued fractions are when all \(b_n = 1\) while the Hirzebruch-Jung continued fractions are the one for which \(b_n = -1\) for all \(n\)). See Wikipedia article Generalized_continued_fraction.

- look at the function ContinuedFractionApproximationOfRoot in GAP

AUTHORS:

- Vincent Delecroix (2014): cleaning, refactorisation, documentation from the old implementation in contfrac (trac ticket #14567).

class sage.rings.continued_fraction.ContinuedFraction_base
Bases: sage.structure.sage_object.SageObject

Base class for (standard) continued fractions.

If you want to implement your own continued fraction, simply derived from this class and implement the following methods:

- def quotient(self, n): return the \(n\)-th quotient of self as a Sage integer
- def length(self): the number of partial quotients of self as a Sage integer or Infinity.

and optionally:

- def value(self): return the value of self (an exact real number)

This base class will provide:

- computation of convergents in convergent(), numerator() and denominator()
- comparison with other continued fractions (see __richcmp__())
- elementary arithmetic function floor(), ceil(), sign()
- accurate numerical approximations _mpfr_()

All other methods, in particular the ones involving binary operations like sum or product, rely on the optional method value() (and not on convergents) and may fail at execution if it is not implemented.

additive_order()

Return the additive order of this continued fraction, which we defined to be the additive order of its value.

EXAMPLES:
apply_homography\( (a, b, c, d, \text{forward\_value}=\text{False}) \)

Return the continued fraction of \((ax + b)/(cx + d)\).

This is computed using Gosper's algorithm, see `continued_fraction_gosper`.

**INPUT:**

- \(a, b, c, d\) – integers
- `forward_value` – boolean (default: `False`) whether the returned continued fraction is given the symbolic value of \((ax + b)/(cx + d)\) and not only the list of partial quotients obtained from Gosper's algorithm.

**EXAMPLES:**

```python
sage: (5 * 13/6 - 2) / (3 * 13/6 - 4)
53/15
sage: continued_fraction(13/6).apply_homography(5, -2, 3, -4).value()
53/15
```

We demonstrate now the effect of the optional argument `forward_value`:

```python
sage: cf = continued_fraction(pi)
sage: h1 = cf.apply_homography(35, -27, 12, -5)
sage: h1
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 9, 12, 1, 1, 1, 3...

sage: h1.value()
2.536941776086946?

sage: h2 = cf.apply_homography(35, -27, 12, -5, forward_value=True)
sage: h2
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 9, 12, 1, 1, 1, 3...

sage: h2.value()
(35*\pi - 27)/(12*\pi - 5)
```

**REFERENCES:**

- [Gos1972]
- [Knu1998] Exercise 4.5.3.15
- [LS1998]

`ceil()`

Return the ceil of `self`.

**EXAMPLES:**

```python
sage: cf = continued_fraction([2,1,3,4])
sage: cf.ceil()
3
```

`convergent(n)`

Return the \(n\)-th partial convergent to `self`.
EXAMPLES:

```python
sage: a = continued_fraction(pi); a
[3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: a.convergent(3)
355/113
sage: a.convergent(15)
411557987/131002976
```

**convergents()**

Return the list of partial convergents of `self`.

If `self` is an infinite continued fraction, then the object returned is a `lazy_list_generic` which behave like an infinite list.

EXAMPLES:

```python
sage: a = continued_fraction(23/157); a
[0; 6, 1, 4, 1, 3]
sage: a.convergents()
[0, 1/6, 1/7, 5/34, 6/41, 23/157]
```

Todo: Add an example with infinite list.

**denominator**

Return the denominator of the `n`-th partial convergent of `self`.

EXAMPLES:

```python
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.denominator(0)
1
sage: c.denominator(12)
25510582
sage: c.denominator(152)
1255341492699841451528811722575401081588363886480089431843026103930863337221076748
```

**floor()**

Return the floor of `self`.

EXAMPLES:

```python
sage: cf = continued_fraction([2,1,2,3])
sage: cf.floor()
2
```

**is_minus_one()**

Test whether `self` is minus one.

EXAMPLES:

```python
sage: continued_fraction(-1).is_minus_one()
True
sage: continued_fraction(1).is_minus_one()
```

(continues on next page)
False
\texttt{sage}: \text{continued\_fraction}(0).\text{is\_minus\_one}()
False
\texttt{sage}: \text{continued\_fraction}(-2).\text{is\_minus\_one}()
False
\texttt{sage}: \text{continued\_fraction}([-1,1]).\text{is\_minus\_one}()
False

\textbf{is\_one}()
Test whether \texttt{self} is one.

\textbf{EXAMPLES}:

\texttt{sage}: \text{continued\_fraction}(1).\text{is\_one}()
True
\texttt{sage}: \text{continued\_fraction}(5/4).\text{is\_one}()
False
\texttt{sage}: \text{continued\_fraction}(0).\text{is\_one}()
False
\texttt{sage}: \text{continued\_fraction}(\pi).\text{is\_one}()
False

\textbf{is\_zero}()
Test whether \texttt{self} is zero.

\textbf{EXAMPLES}:

\texttt{sage}: \text{continued\_fraction}(0).\text{is\_zero}()
True
\texttt{sage}: \text{continued\_fraction}((0,1)).\text{is\_zero}()
False
\texttt{sage}: \text{continued\_fraction}(-1/2).\text{is\_zero}()
False
\texttt{sage}: \text{continued\_fraction}(\pi).\text{is\_zero}()
False

\textbf{multiplicative\_order}()
Return the multiplicative order of this continued fraction, which we defined to be the multiplicative order of its value.

\textbf{EXAMPLES}:

\texttt{sage}: \text{continued\_fraction}(-1).\text{multiplicative\_order}()
2
\texttt{sage}: \text{continued\_fraction}(1).\text{multiplicative\_order}()
1
\texttt{sage}: \text{continued\_fraction}(\pi).\text{multiplicative\_order}()
+\infty

\texttt{n}(\text{prec}=None, \text{digits}=None, \text{algorithm}=None)
Return a numerical approximation of this continued fraction with \texttt{prec} bits (or decimal \texttt{digits}) of precision.

\textbf{INPUT}:

\begin{itemize}
  \item \texttt{prec} – precision in bits
\end{itemize}
• digits – precision in decimal digits (only used if prec is not given)
• algorithm – ignored for continued fractions

If neither prec nor digits is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

```python
sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...

sage: cf.numerical_approx(prec=53)
1.28102513329557
```

The method \( n \) is a shortcut to this one:

```python
sage: cf.n(digits=25)
1.281025133295569815552930

sage: cf.n(digits=33)
1.28102513329556981555293038097590
```

**numerator**

Return the numerator of the \( n \)-th partial convergent of self.

EXAMPLES:

```python
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]

sage: c.numerator(0)
3
sage: c.numerator(12)
80143857

sage: c.numerator(152)
394377161121226696274373881260074821315726659658874495172739349744692124535300528
```

**numerical_approx**

Return a numerical approximation of this continued fraction with prec bits (or decimal digits) of precision.

INPUT:

• prec – precision in bits
• digits – precision in decimal digits (only used if prec is not given)
• algorithm – ignored for continued fractions

If neither prec nor digits is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

```python
sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...

sage: cf.numerical_approx(prec=53)
1.28102513329557
```

The method \( n \) is a shortcut to this one:
.. code-block::

    sage: cf.n(digits=25)
    1.281025133295569815552930
    sage: cf.n(digits=33)
    1.2810251332955698155529303809759

.. math:: p(n)

   Return the numerator of the \( n \)-th partial convergent of \( \text{self} \).

   EXAMPLES:

   .. code-block::

      sage: c = continued_fraction(pi); c
      [3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 2, ...]
      sage: c.numerator(0)
      3
      sage: c.numerator(12)
      80143857
      sage: c.numerator(152)
      3943771611212266274373881260007482131572665965887449517273934974469212453530528

.. math:: q(n)

   Return the denominator of the \( n \)-th partial convergent of \( \text{self} \).

   EXAMPLES:

   .. code-block::

      sage: c = continued_fraction(pi); c
      [3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 2, ...]
      sage: c.denominator(0)
      1
      sage: c.denominator(12)
      25510582
      sage: c.denominator(152)
      1255341492699841451528811722575401081588363886480008943184302610393086333722107674

.. math:: \text{quotients}()

   Return the list of partial quotients of \( \text{self} \).

   If \( \text{self} \) is an infinite continued fraction, then the object returned is a \text{lazy_list_generic} which behaves like an infinite list.

   EXAMPLES:

   .. code-block::

      sage: a = continued_fraction(23/157); a
      [0; 6, 1, 4, 1, 3]
      sage: a.quotients()
      [0, 6, 1, 4, 1, 3]

      Todo: Add an example with infinite list.

.. math:: \text{sign}()

   Return the sign of \( \text{self} \) as an Integer.

   The sign is defined to be \( 0 \) if \( \text{self} \) is \( 0 \), \( 1 \) if \( \text{self} \) is positive and \(-1\) if \( \text{self} \) is negative.

   EXAMPLES:
```
sage: continued_fraction(tan(pi/7)).sign()
1
sage: continued_fraction(-34/2115).sign()
-1
sage: continued_fraction([0]).sign()
0
```

```
str(nterms=10, unicode=False, join=True)

Return a string representing this continued fraction.

INPUT:

• nterms – the maximum number of terms to use
• unicode – (default False) whether to use unicode character
• join – (default True) if False instead of returning a string return a list of string, each of them representing a line

EXAMPLES:

```
sage: print(continued_fraction(pi).str())
1
3 + ----------------------------------------------------
  1
  7 + .....................................................
  1
  15 + ...................................................
  1
  1 + .................................................
  1
  292 + ...............................................
  1
  1 + .............................................
  1
  1 + ............................
  1
  1 + ..........................
  1
  2 + ........................
  1
  1 + ...

sage: print(continued_fraction(pi).str(nterms=1))
3 + ...

sage: print(continued_fraction(pi).str(nterms=2))
1
3 + ........................
  7 + ...

sage: print(continued_fraction(243/354).str())
1
-----------------------------------------------
  1
  1 + --------------------------------------
  1
  1 + ----------------------------------
  1
  2 + -------------
  1
  1 + ...
```

(continues on next page)
1 5 + --------
   1
3 + ---
 2

\[ \text{sage: } \text{continued_fraction}(243/354).str(join=\text{False}) \]
\[
[\' 1 \\
 1 \\
 1 \\
 1 \\
 1 \\
 2 \\
 1 \\
 1 \\
 5 \\
 1 \\
 3 + ---
 2 \]
\]

\[ \text{sage: } \text{print}(\text{continued_fraction}(243/354).str(unicode=\text{True})) \]
\[
1
---------
1
1 + ----- 1
2 + ----- 1
5 + --- 1
3 + - 2
\]

\text{class} \ \text{sage.rings.continued_fraction.} \text{ContinuedFraction\_infinite}(w, \text{value=\text{None}}, \text{check=\text{True}})

\text{Bases: sage.rings.continued_fraction.} \text{ContinuedFraction\_base}

A continued fraction defined by an infinite sequence of partial quotients.

\text{EXAMPLES:}

\[ \text{sage: } t = \text{continued_fraction(words.ThueMorseWord([1,2]))}; t \]
\[ [1; 2, 2, 1, 2, 1, 2, 1, 2, 2, 1...] \]

\[ \text{sage: } t.n(\text{digits=100}) \]
\[ 1.
\text{→} 4223887366827854883415471160245658253068791089917118293118924529164567472725658833124554129620726... \]

We check that comparisons work well:

\[ \text{sage: } t > \text{continued_fraction(1)} \text{ and } t < \text{continued_fraction(3/2)} \]
\[ \text{True} \]

\[ \text{sage: } t < \text{continued_fraction(1)} \text{ or } t > \text{continued_fraction(2)} \]
\[ \text{False} \]

Can also be called with a value option:
sage: def f(n):
....:     if n % 3 == 2:
....:         return 2*(n+1)//3
....:     return 1
sage: w = Word(f, alphabet=NN)
sage: w
word: 1,1,2,1,1,4,1,1,1,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,22,1,1,...
˓→24,1,1,26,1,...
sage: cf = continued_fraction(w, value=e-1)
sage: cf
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1,...]

In that case a small check is done on the input:

sage: cf = continued_fraction(w, value=pi)
Traceback (most recent call last):
...  
ValueError: value evaluates to 3.141592653589794? while the continued
fraction evaluates to 1.718281828459046? in Real Interval Field
with 53 bits of precision.

length()
Return infinity.

EXAMPLES:

sage: w = words.FibonacciWord([3,13])
sage: cf = continued_fraction(w)
sage: cf.length()  
+Infinity

quotient(n)
Return the n-th partial quotient of self.

INPUT:

• n – an integer

EXAMPLES:

sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf.quotient(0)  
1
sage: cf.quotient(1)  
3
sage: cf.quotient(2)  
1

quotients()
Return the infinite list from which this continued fraction was built.

EXAMPLES:

sage: w = words.FibonacciWord([1,5])
sage: cf = continued_fraction(w)
(continues on next page)
sage: cf.quotients()
word: 151151151151151151151151151151151151151...

value()
Return the value of self.
If this value was provided on initialization, just return this value otherwise return an element of the real lazy field.

EXAMPLES:

```sage
def f(n):
    if n % 3 == 2:
        return 2*(n+1)//3
    return 1
sage: w = Word(f, alphabet=NN)
sage: w
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,22,
     1,1,24,1,1,26,1,...
sage: cf = continued_fraction(w, value=e-1)
sage: cf
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1,...]
sage: cf.value()
e - 1
```

class sage.rings.continued_fraction.ContinuedFraction_periodic(x1, x2=None, check=True)
Bases: sage.rings.continued_fraction.ContinuedFraction_base

Continued fraction associated with rational or quadratic number.
A rational number has a finite continued fraction expansion (or ultimately 0). The one of a quadratic number, ie a number of the form $a + b\sqrt{D}$ with $a$ and $b$ rational, is ultimately periodic.

Note: This class stores a tuple _x1 for the preperiod and a tuple _x2 for the period. In the purely periodic case _x1 is empty while in the rational case _x2 is the tuple (0,).

length()
Return the number of partial quotients of self.

EXAMPLES:

```sage: continued_fraction(2/5).length()
3
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)^*]
sage: cf.length()
+Infinity
```
period()  
Return the periodic part of self.

EXAMPLES:

```
sage: K.<sqrt3> = QuadraticField(3)
sage: cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
sage: cf.period()
(1, 2)
```

```
sage: for k in xsrange(2,40):
    ....:     if not k.is_square():
    ....:         s = QuadraticField(k).gen()
    ....:         cf = continued_fraction(s)
    ....:         print('%2d %d %s
    ' % (k, len(cf.period()), cf))
```

2 1 [1; (2)*]
3 2 [1; (1, 2)*]
5 1 [2; (4)*]
6 2 [2; (2, 4)*]
7 4 [2; (1, 1, 1, 4)*]
8 2 [2; (1, 4)*]
10 1 [3; (6)*]
11 2 [3; (3, 6)*]
12 2 [3; (2, 6)*]
13 5 [3; (1, 1, 1, 1, 6)*]
14 4 [3; (1, 2, 1, 6)*]
...
35 2 [5; (1, 10)*]
37 1 [6; (12)*]
38 2 [6; (6, 12)*]
39 2 [6; (4, 12)*]
```

period_length()  
Return the number of partial quotients of the preperiodic part of self.

EXAMPLES:

```
sage: continued_fraction(2/5).period_length()
1
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.period_length()
1
```

preperiod()  
Return the preperiodic part of self.

EXAMPLES:

```
sage: K.<sqrt3> = QuadraticField(3)
sage: cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
sage: cf.preperiod()
(1,)
```
**preperiod_length()**

Return the number of partial quotients of the preperiodic part of \( \text{self} \).

**EXAMPLES:**

```python
sage: continued_fraction(2/5).preperiod_length()
3
sage: cf = continued_fraction([[0,1],[2]]); cf
[0; 1, (2)*]
sage: cf.preperiod_length()
2
```

**quotient(\( n \))**

Return the \( n \)-th partial quotient of \( \text{self} \).

**EXAMPLES:**

```python
sage: cf = continued_fraction([[12,5],[1,3]]);
[sage: cf.quotient(i) for i in range(10)]
[12, 5, 1, 3, 1, 3, 1, 3, 1, 3]
```

**value()**

Return the value of \( \text{self} \) as a quadratic number (with square free discriminant).

**EXAMPLES:**

Some purely periodic examples:

```python
sage: cf = continued_fraction([[0],[2]]); cf
[[2]*]
sage: v = cf.value(); v
sqrt2 + 1
sage: v.continued_fraction()
[[2]*]
sage: cf = continued_fraction([[0],[1,2]]); cf
[[1, 2]*]
sage: v = cf.value(); v
1/2*sqrt3 + 1/2
sage: v.continued_fraction()
[[1, 2]*]
```

The number \( \sqrt{3} \) that appear above is actually internal to the continued fraction. In order to be access it from the console:

```python
sage: cf.value().parent().inject_variables()
Defining sqrt3
sage: sqrt3
sqrt3
```
Some ultimately periodic but non periodic examples:

```python
sage: cf = continued_fraction([(1,), (2,)]); cf
[1; (2)*]
sage: v = cf.value(); v
sqrt2
sage: v.continued_fraction()
[1; (2)*]
sage: cf = continued_fraction([(1,3),(1,2)]); cf
[1; 3, (1, 2)*]
sage: v = cf.value(); v
-sqrt3 + 3
sage: v.continued_fraction()
[1; 3, (1, 2)*]
sage: cf = continued_fraction([(-5,18), (1,3,1,5)])
sage: cf.value().continued_fraction() == cf
True
sage: cf = continued_fraction([(-1,),(1,)])
```

```python
sage: cf.value().continued_fraction() == cf
True
```

```python
class sage.rings.continued_fraction.ContinuedFraction_real(x)
Bases: sage.rings.continued_fraction.ContinuedFraction_base
```

Continued fraction of a real (exact) number.

This class simply wraps a real number into an attribute (that can be accessed through the method `value()`). The number is assumed to be irrational.

**EXAMPLES:**

```python
sage: cf = continued_fraction(pi)
sage: cf
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...
```

```python
sage: cf.value()
pi
```

```python
sage: cf = continued_fraction(e)
sage: cf
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]
sage: cf.value()
e
```

**length()**

Return infinity

**EXAMPLES:**

```python
sage: continued_fraction(pi).length()
+Infinity
```
**quotient**(*n*)

Return the *n*-th quotient of *self*.

**EXAMPLES:**

```python
sage: cf = continued_fraction(pi)
sage: cf.quotient(27)
13
sage: cf.quotient(2552)
152
sage: cf.quotient(10000)  # long time
5
```

The algorithm is not efficient with element of the symbolic ring and, if possible, one can always prefer number fields elements. The reason is that, given a symbolic element *x*, there is no automatic way to evaluate in RIF an expression of the form \((a*x+b)/(c*x+d)\) where both the numerator and the denominator are extremely small:

```python
sage: a1 = pi
sage: c1 = continued_fraction(a1)
sage: p0 = c1.numerator(12); q0 = c1.denominator(12)
sage: p1 = c1.numerator(13); q1 = c1.denominator(13)
sage: num = (q0*a1 - p0); num.n()
1.49011611938477e-8
sage: den = (q1*a1 - p1); den.n()
-2.98023223876953e-8
sage: a1 = -num/den
sage: RIF(a1)
[-Infinity .. +Infinity]
```

The same computation with an element of a number field instead of *pi* gives a very satisfactory answer:

```python
sage: K.<a2> = NumberField(x^3 - 2, embedding=1.25)
sage: c2 = continued_fraction(a2)
sage: p0 = c2.numerator(111); q0 = c2.denominator(111)
sage: p1 = c2.numerator(112); q1 = c2.denominator(112)
sage: num = (q0*a2 - p0); num.n()
-4.56719261665907e46
sage: den = (q1*a2 - p1); den.n()
-3.65375409332726e47
sage: a2 = -num/den
sage: b2 = RIF(a2); b2
1.002685823312715?
```

The consequence is that the precision needed with *c1* grows when we compute larger and larger partial quotients:

```python
sage: c1.quotient(100)
2
sage: c1._xa.parent()
Real Interval Field with 353 bits of precision
sage: c1.quotient(200)
3
```

(continues on next page)
sage: c1._xa.parent()
Real Interval Field with 753 bits of precision
sage: c1.quotient(300)
5
sage: c1._xa.parent()
Real Interval Field with 1053 bits of precision
sage: c2.quotient(200)
6
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
sage: c2.quotient(500)
1
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
sage: c2.quotient(1000)
1
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision

\texttt{value()}

Return the value of \texttt{self} (the number from which it was built).

\textbf{EXAMPLES:}

\begin{verbatim}
sage: cf = continued_fraction(e)
sage: cf.value()
e
\end{verbatim}

sage.rings.continued_fraction.check_and_reduce_pair\(x1, x2=\text{None}\)
There are often two ways to represent a given continued fraction. This function makes it canonical.
In the very special case of the number 0 we return the pair ((0,), (0,)).

sage.rings.continued_fraction.continued_fraction\(x, value=\text{None}\)
Return the continued fraction of \texttt{x}.

\textbf{INPUT:}

- \texttt{x} – a number or a list of partial quotients (for finite development) or two list of partial quotients (preperiod and period for ultimately periodic development)

\textbf{EXAMPLES:}

A finite continued fraction may be initialized by a number or by its list of partial quotients:

\begin{verbatim}
sage: continued_fraction(12/571)
[0; 47, 1, 1, 2, 2]
sage: continued_fraction([3, 2, 1, 4])
[3, 2, 1, 4]
\end{verbatim}

It can be called with elements defined from symbolic values, in which case the partial quotients are evaluated in a lazy way:

\begin{verbatim}
sage: c = continued_fraction(golden_ratio); c
[1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...
\end{verbatim}
If you want to play with quadratic numbers (such as golden_ratio and sqrt(2) above), it is much more convenient to use number fields as follows since preperiods and periods are computed:

```sage
sage: K.<sqrt5> = NumberField(x^2-5, embedding=2.23)
sage: my_golden_ratio = (1 + sqrt5)/2
sage: cf = continued_fraction((1+sqrt5)/2); cf
[(1)*]
sage: cf.convergent(12)
377/233
sage: cf.period()
(1,)
sage: cf = continued_fraction(2/3+sqrt5/5); cf
[1; 8, (1, 3, 1, 1, 3, 9)*

sage: L.<sqrt2> = NumberField(x^2-2, embedding=1.41)
sage: cf = continued_fraction(sqrt2); cf
[1; (2)*

sage: cf = continued_fraction(sqrt2/3); cf
[0; 2, (8, 4)*
```

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It is also possible to go the other way around, build a ultimately periodic continued fraction from its preperiod and its period and get its value back:

```
sage: cf = continued_fraction([(1,1),(2,8)]); cf
[1; 1, (2, 8)*]
sage: cf.value()
2/11*sqrt5 + 14/11
```

It is possible to deal with higher degree number fields but in that case the continued fraction expansion is known to be aperiodic:

```
sage: K.<a> = NumberField(x^3-2, embedding=1.25)
sage: cf = continued_fraction(a); cf
[1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, ...]
```

Note that initial rounding can result in incorrect trailing partial quotients:

```
sage: continued_fraction(RealField(39)(e))
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 2]
```

Note the value returned for floating point number is the continued fraction associated to the rational number you obtain with a conversion:

```
sage: for _ in range(10):
....:     x = RR.random_element()
....:     cff = continued_fraction(x)
....:     cfe = QQ(x).continued_fraction()
....:     assert cff == cfe, "%s %s %s"%(x,cff,cfe)
```

```
sage.rings.continued_fraction.continued_fraction_list(x, type='std', partial_convergents=False, bits=None, nterms=None)
```

Return the (finite) continued fraction of \(x\) as a list.

The continued fraction expansion of \(x\) are the coefficients \(a_i\) in

\[
x = a_0 + \frac{1}{a_1 + \frac{1}{(...)}},
\]

with \(a_0\) integer and \(a_1, \ldots\) positive integers. The Hirzebruch-Jung continued fraction is the one for which the + signs are replaced with − signs

\[
x = a_0 - \frac{1}{a_1 - \frac{1}{(...)}}
\]

See also:

`continued_fraction()`

INPUT:

- \(x\) – exact rational or floating-point number. The number to compute the continued fraction of.
- \(type\) – either “std” (default) for standard continued fractions or “hj” for Hirzebruch-Jung ones.
- \(partial\_convergents\) – boolean. Whether to return the partial convergents.
- \(bits\) – an optional integer that specify a precision for the real interval field that is used internally.
- \(nterms\) – integer. The upper bound on the number of terms in the continued fraction expansion to return.
OUTPUT:

A list of integers, the coefficients in the continued fraction expansion of \( x \). If \( \text{partial\_convergents} \) is set to \( \text{True} \), then return a pair containing the coefficient list and the partial convergents list is returned.

EXAMPLES:

```sage
sage: continued_fraction_list(45/19)
[2, 2, 1, 2, 2]
sage: 2 + 1/(2 + 1/(1 + 1/(2 + 1/2)))
45/19
sage: continued_fraction_list(45/19,type="hj")
[3, 2, 3, 2, 3]
sage: 3 - 1/(2 - 1/(3 - 1/(2 - 1/3)))
45/19
```

Specifying \( \text{bits} \) or \( \text{nterms} \) modify the length of the output:

```sage
sage: continued_fraction_list(e, bits=20)
[2, 1, 2, 1, 1, 4, 2]
sage: continued_fraction_list(sqrt(2)+sqrt(3), bits=30)
[3, 6, 1, 5, 7, 2]
sage: continued_fraction_list(pi, bits=53)
[3, 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14]
sage: continued_fraction_list(log(3/2), nterms=15)
[0, 2, 2, 6, 1, 11, 2, 1, 2, 1, 4, 3, 1, 1]
sage: continued_fraction_list(tan(sqrt(pi)), nterms=20)
[-5, 9, 4, 1, 1, 1, 1, 5, 1, 1, 1, 1, 1, 2, 1, 4, 3, 1, 63]
```

When the continued fraction is infinite (i.e. \( x \) is an irrational number) and the parameters \( \text{bits} \) and \( \text{nterms} \) are not specified then a warning is raised:

```sage
sage: continued_fraction_list(sqrt(2))
doctest:...: UserWarning: the continued fraction of sqrt(2) seems infinite, return.
→only the first 20 terms
[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
sage: continued_fraction_list(sqrt(4/19))
doctest:...: UserWarning: the continued fraction of 2*sqrt(1/19) seems infinite,
→return only the first 20 terms
[0, 2, 5, 1, 1, 2, 1, 16, 1, 2, 1, 1, 5, 4, 5, 1, 2, 4, 3, 1]
```

An examples with the list of partial convergents:

```sage
sage: continued_fraction_list(RR(pi), partial_convergents=True)
([3, 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 3],
 [(3, 1),
  (22, 7),
  (333, 106),
  (355, 113),
  (103993, 33102),
  (104348, 33215),
  (208341, 66317),
  (312689, 99532),
  (continues on next page)
```

(continues on next page)
sage.rings.continued_fraction.convergents(x)
Return the (partial) convergents of the number x.

EXAMPLES:

```python
sage: from sage.rings.continued_fraction import convergents
sage: convergents(143/255)
[0, 1, 1/2, 4/7, 5/9, 9/16, 14/25, 23/41, 60/107, 143/255]
```

sage.rings.continued_fraction.last_two_convergents(x)
Given the list x that consists of numbers, return the two last convergents \( p_{n-1}, q_{n-1}, p_n, q_n \).

This function is principally used to compute the value of a ultimately periodic continued fraction.

OUTPUT: a 4-tuple of Sage integers

EXAMPLES:

```python
sage: from sage.rings.continued_fraction import last_two_convergents
sage: last_two_convergents([])
(0, 1, 1, 0)
sage: last_two_convergents([0])
(1, 0, 0, 1)
sage: last_two_convergents([-1,1,3,2])
(-1, 4, -2, 9)
```

sage.rings.continued_fraction.rat_interval_cf_list(r1, r2)
Return the common prefix of the rationals \( r_1 \) and \( r_2 \) seen as continued fractions.

OUTPUT: a list of Sage integers.

EXAMPLES:

```python
sage: from sage.rings.continued_fraction import rat_interval_cf_list
sage: rat_interval_cf_list(257/113, 5224/2297)
[2, 3, 1, 1, 1, 4]
```

```python
for prec in range(10,54):
    R = RealIntervalField(prec)
    for _ in range(100):
        x = R.random_element() * R.random_element() + R.random_element() / 100
        l = x.lower().exact_rational()
        u = x.upper().exact_rational()
        if l.floor() != u.floor():
            continue
        cf = rat_interval_cf_list(l, u)
        a = continued_fraction(cf).value()
        b = continued_fraction(cf+[1]).value()
        if a > b:
            continue
```

(continues on next page)
a, b = b, a
assert a <= l
assert b >= u
CHAPTER TWO

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