CONTENTS

1 Hyperplane arrangements 3
2 Polyhedral computations 65
3 Triangulations 665
4 Miscellaneous 701
5 Helper functions 747
6 Indices and Tables 755
Python Module Index 757
Index 759
Sage includes classes for hyperplane arrangements, polyhedra, toric varieties (including polyhedral cones and fans), triangulations and some other helper classes and functions.
1.1 Hyperplane Arrangements

Before talking about hyperplane arrangements, let us start with individual hyperplanes. This package uses certain linear expressions to represent hyperplanes, that is, a linear expression \(3x + 3y - 5z - 7\) stands for the hyperplane with the equation \(3x + 3y - 5z = 7\). To create it in Sage, you first have to create a \(\text{HyperplaneArrangements}\) object to define the variables \(x, y, z\):

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>sage: H.&lt;x,y,z&gt; = HyperplaneArrangements(QQ)</td>
<td></td>
</tr>
<tr>
<td>sage: h = 3<em>x + 2</em>y - 5*z - 7; h</td>
<td></td>
</tr>
<tr>
<td>sage: h.normal()</td>
<td></td>
</tr>
<tr>
<td>(3, 2, -5)</td>
<td></td>
</tr>
<tr>
<td>sage: h.constant_term()</td>
<td></td>
</tr>
<tr>
<td>-7</td>
<td></td>
</tr>
</tbody>
</table>

The individual hyperplanes behave like the linear expression with regard to addition and scalar multiplication, which is why you can do linear combinations of the coordinates:

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>sage: -2*h</td>
<td></td>
</tr>
<tr>
<td>Hyperplane (-6x - 4y + 10z + 14)</td>
<td></td>
</tr>
<tr>
<td>sage: x, y, z</td>
<td></td>
</tr>
<tr>
<td>(Hyperplane (x + 0y + 0z + 0),</td>
<td></td>
</tr>
<tr>
<td>Hyperplane (0x + y + 0z + 0),</td>
<td></td>
</tr>
<tr>
<td>Hyperplane (0x + 0y + z + 0))</td>
<td></td>
</tr>
</tbody>
</table>

See \(\text{sage.geometry.hyperplane_arrangement.hyperplane}\) for more functionality of the individual hyperplanes.

1.1.1 Arrangements

There are several ways to create hyperplane arrangements:

Notation (i): by passing individual hyperplanes to the \(\text{HyperplaneArrangements}\) object:

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>sage: H.&lt;x,y&gt; = HyperplaneArrangements(QQ)</td>
<td></td>
</tr>
<tr>
<td>sage: box = x</td>
<td>y</td>
</tr>
<tr>
<td>Arrangement (&lt;y - 1</td>
<td>y</td>
</tr>
<tr>
<td>sage: box == H(x, y, x-1, y-1) # alternative syntax</td>
<td></td>
</tr>
<tr>
<td>True</td>
<td></td>
</tr>
</tbody>
</table>

Notation (ii): by passing anything that defines a hyperplane, for example a coefficient vector and constant term:
sage: H = HyperplaneArrangements(QQ, ('x', 'y'))
sage: triangle = H([(1, 0), 0], [(0, 1), 0], [(1, 1), -1]); triangle
Arrangement <y | x | x + y - 1>
sage: H.inject_variables()
Defining x, y
sage: triangle == x | y | x+y-1
True

The default base field is $\mathbb{Q}$, the rational numbers. Finite fields are also supported:

sage: H.<x,y,z> = HyperplaneArrangements(GF(5))
sage: a = H([(1,2,3), 4], [(5,6,7), 8]); a
Arrangement <y + 2*z + 3 | x + 2*y + 3*z + 4>

Number fields are also possible:

sage: # needs sage.rings.number_field
sage: x = polygen(QQ, 'x')
sage: NF.<a> = NumberField(x**4 - 5*x**2 + 5, embedding=1.90)
sage: H.<y,z> = HyperplaneArrangements(NF)
sage: A = H([[(-a**3 + 3*a, -a**2 + 4), 1],
    ...:    [(0, 2*a**2 - 6), 1],
    ...:    [(a**3 - 3*a, -a**2 + 4), 1]])

Arrangement of 5 hyperplanes of dimension 2 and rank 2
sage: A.base_ring()
Number Field in a with defining polynomial x^4 - 5*x^2 + 5
with a = 1.902113032590308?

Notation (iii): a list or tuple of hyperplanes:

sage: H.<x,y,z> = HyperplaneArrangements(GF(5))
sage: k = [x+i for i in range(4)]; k
[Hyperplane x + 0*y + 0*z + 0, Hyperplane x + 0*y + 0*z + 1,
  Hyperplane x + 0*y + 0*z + 2, Hyperplane x + 0*y + 0*z + 3]
sage: H(k)
Arrangement <x | x + 1 | x + 2 | x + 3>

Notation (iv): using the library of arrangements:

sage: # needs sage.graphs
sage: hyperplane_arrangements.braid(4)
Arrangement of 6 hyperplanes of dimension 4 and rank 3
sage: hyperplane_arrangements.semiorder(3)
Arrangement of 6 hyperplanes of dimension 3 and rank 2
sage: # needs sage.graphs
sage: hyperplane_arrangements.graphical(graphs.PetersenGraph())
Arrangement of 15 hyperplanes of dimension 10 and rank 9
sage: hyperplane_arrangements.Ish(5)
Arrangement of 20 hyperplanes of dimension 5 and rank 4

Notation (v): from the bounding hyperplanes of a polyhedron:

sage: a = polytopes.cube().hyperplane_arrangement(); a
Arrangement of 6 hyperplanes of dimension 3 and rank 3
sage: a.n_regions()
New arrangements from old:

```python
sage: # needs sage.graphs
sage: a = hyperplane_arrangements.braid(3)
sage: b = a.add_hyperplane([4, 1, 2, 3])
sage: b
Arrangement <t1 - t2 | t0 - t1 | t0 - t2 | t0 + 2*t1 + 3*t2 + 4>
sage: c = b.deletion([4, 1, 2, 3])
sage: a == c
True
```

```python
sage: # needs sage.combinat sage.graphs
sage: a = hyperplane_arrangements.braid(3)
sage: b = a.union(hyperplane_arrangements.semiorder(3))
sage: b == a | hyperplane_arrangements.semiorder(3)       # alternate syntax
True
sage: b == hyperplane_arrangements.Catalan(3)
True
sage: a
Arrangement <t1 - t2 | t0 - t1 | t0 - t2>
```

```python
sage: a = hyperplane_arrangements.coordinate(4)
sage: h = a.hyperplanes()[0]
sage: b = a.restriction(h)
sage: b == hyperplane_arrangements.coordinate(3)
True
```

1.1.2 Properties of Arrangements

A hyperplane arrangement is *essential* if the normals to its hyperplanes span the ambient space. Otherwise, it is *inessential*. The essentialization is formed by intersecting the hyperplanes by this normal space (actually, it is a bit more complicated over finite fields):

```python
sage: # needs sage.graphs
sage: a = hyperplane_arrangements.braid(4); a
Arrangement of 6 hyperplanes of dimension 4 and rank 3
sage: a.is_essential()
False
sage: a.rank() < a.dimension()       # double-check
True
sage: a.essentialization()
Arrangement of 6 hyperplanes of dimension 3 and rank 3
```

The connected components of the complement of the hyperplanes of an arrangement in \( \mathbb{R}^n \) are called the *regions* of the arrangement:

```python
sage: a = hyperplane_arrangements.semiorder(3)
sage: b = a.essentialization(); b
Arrangement of 6 hyperplanes of dimension 2 and rank 2
sage: b.n_regions()
19
sage: b.regions()
(A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 6 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices and 1...)
```

(continues on next page)
→ray,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices and 1...

→ray,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices and 1...

→ray,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices and 1...

→ray,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices and 1...

→ray,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices and 1...

sage: b.bounded_regions()
(A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 6 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices)
sage: b.n_bounded_regions()
7
sage: a.unbounded_regions()
(A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1...
→line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices, 1 ray, ...
→1 line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1...
→line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices, 1 ray, ...
→1 line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1...
→line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices, 1 ray, ...
→1 line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1...
→line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices, 1 ray, ...
→1 line,
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1...
→line)
The distance between regions is defined as the number of hyperplanes separating them. For example:

```
sage: # needs sage.combinat
sage: r1 = b.regions()[0]
sage: r2 = b.regions()[1]
sage: b.distance_between_regions(r1, r2)
1
sage: [hyp for hyp in b if b.is_separating_hyperplane(r1, r2, hyp)]
[Hyperplane 2*t1 + t2 + 1]
sage: b.distance Enumerator(r1)  # generating function for distances from r1
6*x^3 + 6*x^2 + 6*x + 1
```

**Note:** *bounded region* really mean *relatively bounded* here. A region is relatively bounded if its intersection with space spanned by the normals of the hyperplanes in the arrangement is bounded.

The intersection poset of a hyperplane arrangement is the collection of all nonempty intersections of hyperplanes in the arrangement, ordered by reverse inclusion. It includes the ambient space of the arrangement (as the intersection over the empty set):

```
sage: # needs sage.graphs
sage: a = hyperplane_arrangements.braid(3)
sage: p = a.intersection_poset()
sage: p.is_ranked()
True
sage: p.order_polytope()
A 5-dimensional polyhedron in ZZ^5 defined as the convex hull of 10 vertices
```

The characteristic polynomial is a basic invariant of a hyperplane arrangement. It is defined as

\[ \chi(x) := \sum_{w \in P} \mu(w)x^{\dim(w)} \]

where the sum is \( P \) is the *intersection poset*() of the arrangement and \( \mu \) is the Möbius function of \( P \):

```
sage: # long time
sage: a = hyperplane_arrangements.semiorder(5)
sage: a.characteristic_polynomial()  # about a second on Core i7
x^5 - 20*x^4 + 180*x^3 - 790*x^2 + 1380*x
sage: a.poincare_polynomial()
1380*x^4 + 790*x^3 + 180*x^2 + 20*x + 1
sage: a.n_regions()
2371
sage: charpoly = a.characteristic_polynomial()
sage: charpoly(-1)
-2371
sage: a.n_bounded_regions()
751
sage: charpoly(1)
751
```

For finer invariants derived from the intersection poset, see *whitney_number()* and *doubly_indexed_whitney_number()*.

Miscellaneous methods (see documentation for an explanation):

### 1.1. Hyperplane Arrangements
sage: a = hyperplane_arrangements.semiorder(3)
sage: a.has_good_reduction(5) # needs sage.rings.finite_rings
True
sage: b = a.change_ring(GF(5))
sage: pa = a.intersection_poset() # needs sage.graphs
sage: pb = b.intersection_poset() # needs sage.rings.finite_rings
sage: pa.is_isomorphic(pb) # needs sage.graphs sage.rings.finite_rings
True
sage: a.face_vector() # needs sage.graphs
(0, 12, 30, 19)
sage: a.face_vector() # needs sage.graphs
(0, 12, 30, 19)
sage: a.is_central() False
sage: a.is_linear() False
sage: a.sign_vector((1,1,1)) (-1, 1, -1, 1, -1, 1)
sage: a.varchenko_matrix()[:6, :6]
[ 1 h2 h2*h4 h2*h3 h2*h3*h4 h2*h3*h4*h5
 [ h2 1 h4 h3 h3*h4 h3*h4*h5
 [ h2*h4 h4 1 h3*h4 h3 h3*h5
 [ h2*h3 h3 h3*h4 1 h4 h4*h5
 [ h2*h3*h4 h3*h4 h3 h4 1 h5
 [h2*h3*h4*h5 h3*h4*h5 h3*h5 h4*h5 h5 1]

There are extensive methods for visualizing hyperplane arrangements in low dimensions. See `plot()` for details.

AUTHORS:

- David Perkinson (2013-06): initial version
- Qiaoyu Yang (2013-07)
- Kuai Yu (2013-07)

This module implements hyperplane arrangements defined over the rationals or over finite fields. The original motivation was to make a companion to Richard Stanley’s notes [Sta2007] on hyperplane arrangements.

class sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement (parent, hyperplanes, check=True, backend=None)

Bases: Element

A hyperplane arrangement.
Warning: You should never create `HyperplaneArrangementElement` instances directly, always use the parent.

`add_hyperplane(other)`
The union of `self` with `other`.

INPUT:

- `other` – a hyperplane arrangement or something that can be converted into a hyperplane arrangement

OUTPUT:
A new hyperplane arrangement.

EXAMPLES:

```sage
H.<x,y> = HyperplaneArrangements(QQ)
A = H([1,2,3], [0,1,1], [0,1,-1], [1,-1,0], [1,1,0])
B = H([1,1,1], [1,-1,1], [1,0,-1])
A.union(B)
```
Arrangement of 8 hyperplanes of dimension 2 and rank 2

```sage
A | B  # syntactic sugar
```
Arrangement of 8 hyperplanes of dimension 2 and rank 2

A single hyperplane is coerced into a hyperplane arrangement if necessary:

```sage
A.union(x+y-1)
```
Arrangement of 6 hyperplanes of dimension 2 and rank 2

```sage
A.add_hyperplane(x+y-1)  # alias
```
Arrangement of 6 hyperplanes of dimension 2 and rank 2

```sage
P.<x,y> = HyperplaneArrangements(RR)
C = P(2*x + 4*y + 5)
C.union(A)
```
Arrangement of 6 hyperplanes of dimension 2 and rank 2

`backend()`

Return the backend used for polyhedral objects

OUTPUT:
A string giving the backend or None if none is specified.

EXAMPLES:

By default, no backend is specified:

```sage
H = HyperplaneArrangements(QQ)
A = H()
A.backend()
```

Otherwise, one may specify a polyhedral backend:

```sage
A = H(backend='ppl')
A.backend()
'ppl'
```

```sage
A = H(backend='normaliz')
A.backend()
'normaliz'
```
bounded_regions()

Return the relatively bounded regions of the arrangement.

A region is relatively bounded if its intersection with the space spanned by the normals to the hyperplanes is bounded. This is the same as being bounded in the case that the hyperplane arrangement is essential. It is assumed that the arrangement is defined over the rationals.

OUTPUT:

Tuple of polyhedra. The relatively bounded regions of the arrangement.

See also:

unbounded_regions()

EXAMPLES:

```sage
# needs sage.combinat
sage: A = hyperplane_arrangements.semiorder(3)
sage: A.bounded_regions()
(A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 6 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line,
 A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices and 1 line)
sage: A.bounded_regions()[0].is_compact()  # the regions are only relatively bounded
False
sage: A.is_essential()
False
```

center()

Return the center of the hyperplane arrangement.

The polyhedron defined to be the set of all points in the ambient space of the arrangement that lie on all of the hyperplanes.

OUTPUT:

A polyhedron.

EXAMPLES:

The empty hyperplane arrangement has the entire ambient space as its center:

```sage
H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H()
sage: A.center()
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 lines
```

The Shi arrangement in dimension 3 has an empty center:
The Braid arrangement in dimension 3 has a center that is neither empty nor full-dimensional:

```
sage: A = hyperplane_arrangements.braid(3)  #→needs sage.combinat
sage: A.center()  #→needs sage.combinat
A 1-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex and
1 line
```

**change_ring**(base_ring)

Return hyperplane arrangement over the new base ring.

**INPUT:**

- base_ring – the new base ring; must be a field for hyperplane arrangements

**OUTPUT:**

The hyperplane arrangement obtained by changing the base field, as a new hyperplane arrangement.

**Warning:** While there is often a one-to-one correspondence between the hyperplanes of self and those of self.change_ring(base_ring), there is no guarantee that the order in which they appear in self.hyperplanes() will match the order in which their counterparts in self.cone() will appear in self.change_ring(base_ring).hyperplanes()!

**EXAMPLES:**

```
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H([(1,1), 0], [(2,3), -1])
sage: A.change_ring(FiniteField(2))
Arrangement <y + 1 | x + y>
```

**characteristic_polynomial**()

Return the characteristic polynomial of the hyperplane arrangement.

**OUTPUT:**

The characteristic polynomial in \(\mathbb{Q}[x]\).

**EXAMPLES:**

```
sage: a = hyperplane_arrangements.coordinate(2)
sage: a.characteristic_polynomial()
x^2 - 2*x + 1
```

**closed_faces**(labelled=True)

Return the closed faces of the hyperplane arrangement self (provided that self is defined over a totally ordered field).

Let \(A\) be a hyperplane arrangement in the vector space \(K^n\), whose hyperplanes are the zero sets of the affine-linear functions \(u_1, u_2, \ldots, u_N\). (We consider these functions \(u_1, u_2, \ldots, u_N\), and not just the hyperplanes, as given. We also assume the field \(K\) to be totally ordered.) For any point \(x \in K^n\), we define the **sign vector** of \(x\) to be the vector \((v_1, v_2, \ldots, v_N) \in \{-1, 0, 1\}^N\) such that (for each \(i\)) the number \(v_i\) is the
sign of $u_i(x)$. For any $v \in \{-1, 0, 1\}^N$, we let $F_v$ be the set of all $x \in K^n$ which have sign vector $v$. The nonempty ones among all these subsets $F_v$ are called the open faces of $A$. They form a partition of the set $K^n$.

Furthermore, for any $v = (v_1, v_2, \ldots, v_N) \in \{-1, 0, 1\}^N$, we let $G_v$ be the set of all $x \in K^n$ such that, for every $i$, the sign of $u_i(x)$ is either 0 or $v_i$. Then, $G_v$ is a polyhedron. The nonempty ones among all these polyhedra $G_v$ are called the closed faces of $A$. While several sign vectors $v$ can lead to one and the same closed face $G_v$, we can assign to every closed face a canonical choice of a sign vector: Namely, if $G$ is a closed face of $A$, then the sign vector of $G$ is defined to be the vector $(v_1, v_2, \ldots, v_N) \in \{-1, 0, 1\}^N$ where $x$ is any point in the relative interior of $G$ and where, for each $i$, the number $v_i$ is the sign of $u_i(x)$. (This does not depend on the choice of $x$.)

There is a one-to-one correspondence between the closed faces and the open faces of $A$. It sends a closed face $G$ to the open face $F_v$, where $v$ is the sign vector of $G$; this $F_v$ is also the relative interior of $G$. The inverse map sends any open face $O$ to the closure of $O$.

INPUT:

- labelled - boolean (default: True); if True, then this method returns not the faces itself but rather pairs $(v, F)$ where $F$ is a closed face and $v$ is its sign vector (here, the order and the orientation of the $u_1, u_2, \ldots, u_N$ is as given by self.hyperplanes()).

OUTPUT:

A tuple containing the closed faces as polyhedra, or (if labelled is set to True) the pairs of sign vectors and corresponding closed faces.

Todo: Should the output rather be a dictionary where the keys are the sign vectors and the values are the faces?

EXAMPLES:

```sage
sage: # needs sage.graphs
sage: a = hyperplane_arrangements.braid(2)
sage: a.hyperplanes()
(Hyperplane t0 - t1 + 0,)
sage: a.closed_faces()
(((0,), A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line),
 ((1,), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line),
((-1,), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line))
sage: a.closed_faces(labelled=False)
(A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line,
 A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line,
 A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line)
sage: [(v, F, F.representative_point()) for v, F in a.closed_faces()]
[((0,), A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line, (0, 0)),
 ((1,), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line, (0, -1)),
((-1,), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line, (-1, 0))]
```
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: a = H(x, y+1)
sage: a.hyperplanes()
(Hyperplane 0*x + y + 1, Hyperplane x + 0*y + 0)
sage: [(v, F, F.representative_point()) for v, F in a.closed_faces()]
[((0, 0), A 0-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, (0, -1)),
 (0, 1), A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 ray, (1, -1)),
 (0, -1), A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 ray, (-1, -1)),
 (1, 0), A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 ray, (0, 0)),
 (1, 1), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays, (1, 0)),
 (1, -1), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays, (-1, 0)),
 (-1, 1), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 ray, (0, -2)),
 (-1, -1), A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays, (-1, -2))]
sage: a = hyperplane_arrangements.braid(3)
# needs sage.graphs
sage: a.hyperplanes()
# needs sage.graphs
(Hyperplane 0*t0 + t1 - t2 + 0,
 Hyperplane t0 - t1 + 0*t2 + 0,
 Hyperplane t0 + 0*t1 - t2 + 0)
sage: [(v, F, F.representative_point()) for v, F in a.closed_faces()]
# needs sage.graphs
[((0, 0, 0), A 1-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex and 1 line, (0, 0, 0)),
 (0, 1, 1), A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 1 ray, 1 line, (0, -1, -1)),
 (0, -1, -1), A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 1 ray, 1 line, (-1, 0, 0)),
 (1, 0, 1), A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 1 ray, 1 line, (1, 1, 0)),
 (1, 1, 1), A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (1, 1, 0)),
 (1, -1, 0), A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 1 ray, 1 line, (0, -1, -2)),
 (1, -1, 1), A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (1, 2, 0)),
 (1, -1, -1), A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (1, 2, 0)),
 (-1, 0, -1), A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (1, -2, -1)),
 (-1, 1, 1), A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (0, 1, 0)),
 (-1, -1, 1), A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (0, -2, -1)),
 (-1, 1, -1), A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line, (0, -2, -1)),
 (continues on next page)
Let us check that the number of closed faces with a given dimension computed using `self.closed_faces()` equals the one computed using `face_vector()`:

```python
def test_number(a):
    Qx = PolynomialRing(QQ, 'x'); x = Qx.gen()
    RHS = Qx.sum(vi * x ** i for i, vi in enumerate(a.face_vector()))
    LHS = Qx.sum(x ** F[1].dim() for F in a.closed_faces())
    return LHS == RHS

sage: def test_number(a):
    Qx = PolynomialRing(QQ, 'x'); x = Qx.gen()
    RHS = Qx.sum(vi * x ** i for i, vi in enumerate(a.face_vector()))
    LHS = Qx.sum(x ** F[1].dim() for F in a.closed_faces())
    return LHS == RHS

sage: def test_number(a):
    Qx = PolynomialRing(QQ, 'x'); x = Qx.gen()
    RHS = Qx.sum(vi * x ** i for i, vi in enumerate(a.face_vector()))
    LHS = Qx.sum(x ** F[1].dim() for F in a.closed_faces())
    return LHS == RHS

sage: a = hyperplane_arrangements.Catalan(2)
sage: test_number(a)  # needs sage.combinat
True
sage: a = hyperplane_arrangements.Catalan(2)
sage: test_number(a)  # needs sage.combinat
True
```

### cocharacteristic_polynomial()

Return the cocharacteristic polynomial of `self`.

The cocharacteristic polynomial of a hyperplane arrangement $A$ is defined by

$$
\Psi_A(z) := \sum_{X \in L} |\mu(B, X)| z^{\dim X},
$$

where $L$ is the intersection poset of $A$, $B$ is the minimal element of $L$ (here, the 0 dimensional subspace), and $\mu$ is the Möbius function of $L$.

**OUTPUT:**

The cocharacteristic polynomial in $\mathbb{Z}[z]$.

**EXAMPLES:**

```python
sage: A = hyperplane_arrangements.coordinate(2)
sage: A.cocharacteristic_polynomial()
1 + 2*z + z^2
sage: B = hyperplane_arrangements.braid(3)
sage: B.cocharacteristic_polynomial()
3*z^2 + 3*z + 1
```

### cone(variable='t')

Return the cone over the hyperplane arrangement.

**INPUT:**

- `variable` – string; the name of the additional variable

**OUTPUT:**

A new hyperplane arrangement. Its equations consist of $[0, -d, a_1, \ldots, a_n]$ for each $[d, a_1, \ldots, a_n]$ in the original arrangement and the equation $[0, 1, 0, \ldots, 0]$. 

---
**Warning:** While there is an almost-one-to-one correspondence between the hyperplanes of `self` and those of `self.cone()`, there is no guarantee that the order in which they appear in `self.hyperplanes()` will match the order in which their counterparts in `self.cone().hyperplanes()` will appear in `self.cone().hyperplanes()`!

**EXAMPLES:**

```sage
sage: # needs sage.combinat
sage: a.<x,y,z> = hyperplane_arrangements.semiorder(3)
sage: b = a.cone()
sage: a.characteristic_polynomial().factor()  
(x^2 - 6*x + 12)

sage: b.characteristic_polynomial().factor()  
((x - 1) * x * (x^2 - 6*x + 12))
sage: a.hyperplanes()  
(Hyperplane 0*x + y - z - 1,
 Hyperplane 0*x + y - z + 1,
 Hyperplane x - y + 0*z - 1,
 Hyperplane x - y + 0*z + 1,
 Hyperplane x + 0*y - z - 1,
 Hyperplane x + 0*y - z + 1)

sage: b.hyperplanes()  
(Hyperplane -t + 0*x + y - z + 0,
 Hyperplane -t + x - y + 0*z + 0,
 Hyperplane -t + x + 0*y - z + 0,
 Hyperplane t + 0*x + 0*y + 0*z + 0,
 Hyperplane t + 0*x + y - z + 0,
 Hyperplane t + x - y + 0*z + 0,
 Hyperplane t + x + 0*y - z + 0)
```

**defining_polynomial()**

Return the defining polynomial of `A`.

Let \( A = (H_i)_i \) be a hyperplane arrangement in a vector space \( V \) corresponding to the null spaces of \( \alpha_{H_i} \in V^* \). Then the **defining polynomial** of \( A \) is given by

\[
Q(A) = \prod_i \alpha_{H_i} \in S(V^*).
\]

**EXAMPLES:**

```sage
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H([2*x + y - z, -x - 2*y + z])
sage: p = A.defining_polynomial(); p
-2*x^2 - 5*x*y - 2*y^2 + 3*x*z + 3*y*z - z^2

sage: p.factor()  
(-1) * (x + 2*y - z) * (2*x + y - z)
```

**deletion(hyperplanes)**

Return the hyperplane arrangement obtained by removing \( h \).

**INPUT:**

- \( h \) – a hyperplane or hyperplane arrangement

**OUTPUT:**

A new hyperplane arrangement with the given hyperplane(s) \( h \) removed.
See also:

\( \text{restriction()} \)

EXAMPLES:

```python
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H([0,1,0], [1,0,1], [-1,0,1], [0,1,-1], [0,1,1]); A
Arrangement of 5 hyperplanes of dimension 2 and rank 2
sage: A.deletion(x)
Arrangement <y - 1 | y + 1 | x - y | x + y>
sage: h = H([0,1,0], [0,1,1])
sage: A.deletion(h)
Arrangement <y - 1 | y + 1 | x - y>
```

\textbf{derivation\_module\_basis} (\textit{algorithm}=\textit{\textquote{\textquote{singular}}} )

Return a basis for the derivation module of \textit{self} if one exists, otherwise return \text{None}.

See also:

\text{derivation}\_module\_free\_chain(), is_free()

INPUT:

- \text{algorithm} = (default: \text{\textquote{singular}}) can be one of the following:
  - \text{\textquote{singular}} – use Singular's minimal free resolution
  - \text{\textquote{BC}} – use the algorithm given by Barakat and Cuntz in [BC2012] (much slower than using Singular)

OUTPUT:

A basis for the derivation module (over \( S \), the symmetric space) as vectors of a free module over \( S \).

ALGORITHM:

\textbf{Singular}

This gets the reduced syzygy module of the Jacobian ideal of the defining polynomial \( f \) of \textit{self}. It then checks Saito's criterion that the determinant of the basis matrix is a scalar multiple of \( f \). If the basis matrix is not square or it fails Saito's criterion, then we check if the arrangement is free. If it is free, then we fall back to the Barakat-Cuntz algorithm.

\textbf{BC}

Return the product of the derivation module free chain matrices. See Section 6 of [BC2012].

EXAMPLES:

```python
sage: W = WeylGroup(['A', 2], prefix='s')  # needs sage.combinat sage.groups
sage: A = W.long_element().inversion_arrangement()  # needs sage.combinat sage.groups
sage: A.derivation_module_basis()  # needs sage.combinat sage.groups
[(a1, a2), (0, a1*a2 + a2^2)]
```
**derivation_module_free_chain()**

Return a free chain for the derivation module if one exists, otherwise return `None`.

See also:

* `is_free()`

**EXAMPLES:**

```python
sage: W = WeylGroup(['A',3], prefix='s')
# needs sage.combinat sage.groups
sage: A = W.long_element().inversion_arrangement()
# needs sage.combinat sage.groups
sage: for M in A.derivation_module_free_chain(): print("\n\n\n" + M)  # needs sage.combinat sage.groups
[ 1 0 0]
[ 0 1 0]
[ 0 0 a3]

[ 1 0 0]
[ 0 0 1]
[ 0 a2 0]

[ 1 0 0]
[ 0 -1 -1]
[ 0 a2 -a3]

[ 0 1 0]
[ 0 0 1]
[a1 0 0]

[ 1 0 -1]
[a3 -1 0]
[a1 0 a2]

[ 1 0 0]
[a3 -1 -1]
[ 0 a1 -a2 - a3]
```

**dimension()**

Return the ambient space dimension of the arrangement.

**OUTPUT:**
An integer.

**EXAMPLES:**

```python
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: (x | x-1 | x+1).dimension() 2
sage: H(x).dimension() 2
```

**distance_between_regions(region1, region2)**

Return the number of hyperplanes separating the two regions.

**INPUT:**

- `region1, region2` – regions of the arrangement or representative points of regions
OUTPUT:

An integer. The number of hyperplanes separating the two regions.

EXAMPLES:

```
sage: c = hyperplane_arrangements.coordinate(2)
sage: r = c.region_containing_point([-1, -1])
sage: s = c.region_containing_point([1, 1])
sage: c.distance_between_regions(r, s)
2
sage: c.distance_between_regions(s, s)
0
```

distance_enumerator (base_region)

Return the generating function for the number of hyperplanes at given distance.

INPUT:

• base_region – region of arrangement or point in region

OUTPUT:

A polynomial $f(x)$ for which the coefficient of $x^i$ is the number of hyperplanes of distance $i$ from base_region, i.e., the number of hyperplanes separated by $i$ hyperplanes from base_region.

EXAMPLES:

```
sage: c = hyperplane_arrangements.coordinate(3)
sage: c.distance_enumerator(c.region_containing_point([1,1,1]))
x^3 + 3*x^2 + 3*x + 1
```

doubly_indexed_whitney_number (i, j, kind=1)

Return the $i, j$-th doubly-indexed Whitney number.

If kind=1, this number is obtained by adding the Möbius function values $m(x, y)$ over all $x, y$ in the intersection poset with rank($x$) = $i$ and rank($y$) = $j$.

If kind = 2, this number is the number of elements $x, y$ in the intersection poset such that $x \leq y$ with ranks $i$ and $j$, respectively.

INPUT:

• $i, j$ – integers

• kind – (default: 1) 1 or 2

OUTPUT:

Integer. The $(i,j)$-th entry of the kind Whitney number.

See also:

whitney_number(), whitney_data()

EXAMPLES:

```
sage: # needs sage.combinat
sage: A = hyperplane_arrangements.Shi(3)
sage: A.doubly_indexed_whitney_number(0, 2)
9
sage: A.whitney_number(2)
9
```

(continues on next page)
REFERENCES:

- [GZ1983]

**essentialization()**

Return the essentialization of the hyperplane arrangement.

The essentialization of a hyperplane arrangement whose base field has characteristic 0 is obtained by intersecting the hyperplanes by the space spanned by their normal vectors.

**OUTPUT:**

The essentialization as a new hyperplane arrangement.

**EXAMPLES:**

```sage
sage: a = hyperplane_arrangements.braid(3)
# needs sage.graphs
sage: a.is_essential()  # needs sage.graphs
False
sage: a.essentialization()  # needs sage.graphs
Arrangement <t1 - t2 | t1 + 2*t2 | 2*t1 + t2>
```

```sage
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: B = H([(1,0),1], [(1,0),-1])
sage: B.is_essential()  # needs sage.graphs
False
sage: B.essentialization()
Arrangement <-x + 1 | x + 1>
```

```sage
sage: H.<x,y> = HyperplaneArrangements(GF(2))
sage: C = H([(1,1),1], [(1,1),0])
sage: C.essentialization()
Arrangement <y | y + 1>
```

```sage
sage: h = hyperplane_arrangements.semiorder(4)
sage: h.essentialization()
Arrangement of 12 hyperplanes of dimension 3 and rank 3
```

**face_product** ($F, G, normalize=True$)

Return the product $FG$ in the face semigroup of $\text{self}$, where $F$ and $G$ are two closed faces of $\text{self}$.

The face semigroup of a hyperplane arrangement $\mathcal{A}$ is defined as follows: As a set, it is the set of all open faces of $\text{self}$ (see `closed_faces()`). Its product is defined by the following rule: If $F$ and $G$ are two open faces of $\mathcal{A}$, then $FG$ is an open face of $\mathcal{A}$, and for every hyperplane $H \in \mathcal{A}$, the open face $FG$ lies on the same side of $H$ as $F$ unless $F \subseteq H$, in which case $FG$ lies on the same side of $H$ as $G$. Alternatively, $FG$ can be defined as follows: If $f$ and $g$ are two points in $F$ and $G$, respectively, then $FG$ is the face that contains the point $(f + \varepsilon g)/(1 + \varepsilon)$ for any sufficiently small positive $\varepsilon$.

In our implementation, the face semigroup consists of closed faces rather than open faces (thanks to the 1-to-1 correspondence between open faces and closed faces, this is not really a different semigroup); these closed
faces are given as polyhedra.

The face semigroup of a hyperplane arrangement is always a left-regular band (i.e., a semigroup satisfying the identities \(x^2 = x\) and \(xyx = xy\)). When the arrangement is central, then this semigroup is a monoid. See [Br2000] (Appendix A in particular) for further properties.

**INPUT:**

- \(F, G\) – two faces of \(self\) (as polyhedra)
- normalize – Boolean (default: True); if True, then this method returns the precise instance of \(FG\) in the list returned by \(self\).\textunderscore closed\_faces()\), rather than creating a new instance

**EXAMPLES:**

```python
sage: # needs sage.graphs
sage: a = hyperplane_arrangements.braid(3)
sage: a.hyperplanes()
(Hyperplane 0*t0 + t1 - t2 + 0,
 Hyperplane t0 - t1 + 0*t2 + 0,
 Hyperplane t0 + 0*t1 - t2 + 0)
sage: faces = {F0: F1 for F0, F1 in a.closed_faces()}
sage: xGyEz = faces[(0, 1, 1)] # closed face x >= y = z
sage: xGyEz.representative_point()
(0, -1, -1)
sage: xGyEz = faces[(0, 1, 1)] # closed face x >= y = z
sage: xGyEz.representative_point()
(0, -1, -1)
sage: yGxGz = faces[(1, -1, 1)] # closed face y >= x >= z
sage: yGxGz = faces[(1, 1, 1)] # closed face y >= y >= z
sage: a.face_product(xGyEz, yGxGz) == xGyGz
True
sage: a.face_product(yGxGz, xGyEz) == yGxGz
True
sage: xEzGy = faces[(-1, 1, 0)] # closed face x = z >= y
sage: xGzGy = faces[(-1, 1, 1)] # closed face x >= z >= y
sage: a.face_product(xEzGy, yGxGz) == xGzGy
True
```

**face\_semigroup\_algebra** *(field=None, names='e')*

Return the face semigroup algebra of \(self\).

This is the semigroup algebra of the face semigroup of \(self\) (see \(face\_product\) for the definition of the semigroup).

Due to limitations of the current Sage codebase (e.g., semigroup algebras do not profit from the functionality of the \texttt{FiniteDimensionalAlgebra} class), this is implemented not as a semigroup algebra, but as a \texttt{FiniteDimensionalAlgebra}. The closed faces of \(self\) (in the order in which the \texttt{closed\_faces()} method outputs them) are identified with the vectors \((0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)\) (with the 1 moving from left to right).

**INPUT:**

- field – a field (default: \(\mathbb{Q}\)), to be used as the base ring for the algebra (can also be a commutative ring, but then certain representation-theoretical methods might misbehave)
- names – (default: 'e') string; names for the basis elements of the algebra

**Todo:** Also implement it as an actual semigroup algebra?
EXAMPLES:

```python
sage: # needs sage.graphs
sage: a = hyperplane_arrangements.braid(3)
sage: [(i, F[0]) for i, F in enumerate(a.closed_faces())]
[(0, (0, 0, 0)),
 (1, (0, 1, 1)),
 (2, (0, -1, -1)),
 (3, (1, 0, 1)),
 (4, (1, 1, 1)),
 (5, (1, -1, 0)),
 (6, (1, -1, 1)),
 (7, (1, -1, -1)),
 (8, (-1, 0, -1)),
 (9, (-1, 1, 0)),
 (10, (-1, 1, 1)),
 (11, (-1, 1, -1)),
 (12, (-1, -1, -1))]
sage: U = a.face_semigroup_algebra(); U
Finite-dimensional algebra of degree 13 over Rational Field
sage: e0, e1, e2, e3, e4, e5, e6, e7, e8, e9, e10, e11, e12 = U.basis()
sage: e0 * e1
e1
e0
sage: e0 * e5
e5
sage: e5 * e0
e5
sage: e3 * e2
e6
sage: e7 * e12
sage: e3 * e12
e7
sage: e4 * e8
sage: e8 * e4
sage: e8 * e1
sage: e5 * e12
sage: (e3 + 2*e4) * (e1 - e7)
e4 - e6
sage: U3 = a.face_semigroup_algebra(field=GF(3)); U3
Finite-dimensional algebra of degree 13 over Finite Field of size 3
```

`face_vector()`

Return the face vector.

OUTPUT:

A vector of integers.

The `d`-th entry is the number of faces of dimension `d`. A face is the intersection of a region with a hyperplane of the arrangement.

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```python
sage: A = hyperplane_arrangements.Shi(3)
sage: A.face_vector() # needs sage.combinat
(0, 6, 21, 16)
```

**has_good_reduction** \((p)\)

Return whether the hyperplane arrangement has good reduction mod \(p\).

Let \(A\) be a hyperplane arrangement with equations defined over the integers, and let \(B\) be the hyperplane arrangement defined by reducing these equations modulo a prime \(p\). Then \(A\) has good reduction modulo \(p\) if the intersection posets of \(A\) and \(B\) are isomorphic.

**INPUT:**

- \(p\) – prime number

**OUTPUT:**

A boolean.

**EXAMPLES:**

```python
sage: # needs sage.combinat
sage: a = hyperplane_arrangements.semiorder(3)
sage: a.has_good_reduction(5)
True
sage: a.has_good_reduction(3)
False
sage: b = a.change_ring(GF(3))
sage: a.characteristic_polynomial()
x^3 - 6*x^2 + 12*x
sage: b.characteristic_polynomial() # not equal to that for a
x^3 - 6*x^2 + 10*x
```

**hyperplanes** ()

Return the hyperplanes in the arrangement as a tuple.

**OUTPUT:**

An integer.

**EXAMPLES:**

```python
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H([1,1,0], [2,3,-1], [4,5,3])
sage: A.hyperplanes()
(Hyperplane x + 0*y + 1, Hyperplane 3*x - y + 2, Hyperplane 5*x + 3*y + 4)
```

Note that the hyperplanes can be indexed as if they were a list:

```python
sage: A[0]
Hyperplane x + 0*y + 1
```

**intersection_poset** \((element_label='int')\)

Return the intersection poset of the hyperplane arrangement.

**INPUT:**

- \(element_label\) – (default: "int") specify how an intersection should be represented; must be one of the following:

```
The poset of non-empty intersections of hyperplanes, with intersections represented by integers, subsets of integers or subspaces (see the examples for more details).

**EXAMPLES:**

By default, the elements of the poset are the integers from 0 through the cardinality of the poset minus one. The element labelled 0 always corresponds to the ambient vector space, and the hyperplanes themselves are labelled 1, 2, ..., \( n \), where \( n \) is the number of hyperplanes of the arrangement.

```sage
A = hyperplane_arrangements.coordinate(2)
L = A.intersection_poset(); L  # needs sage.combinat
Finite poset containing 4 elements
sage: sorted(L)  # needs sage.combinat
[0, 1, 2, 3]
sage: L.level_sets()  # needs sage.combinat
[[0], [1, 2], [3]]
```

```sage
# needs sage.combinat
A = hyperplane_arrangements.semiorder(3)
L = A.intersection_poset(); L  # needs sage.combinat
Finite poset containing 19 elements
sage: sorted(L)  # needs sage.combinat
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]
sage: [sorted(level_set) for level_set in L.level_sets()]  # needs sage.combinat
[[{}], [0], [0, 2, 3, 4, 5, 6], [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]]
```

By passing the argument `element_label="subset"`, each element of the intersection poset is labelled by the set of indices of the hyperplanes whose intersection is said element. The index of a hyperplane is its index in `self.hyperplanes()`.

```sage
A = hyperplane_arrangements.semiorder(3)
L = A.intersection_poset(element_label='subset')  # needs sage.combinat
sage: [sorted(level, key=sorted) for level in L.level_sets()]  # needs sage.combinat
[[{}], [0], [0, 3], [0, 4], [1], [1, 3, 4], [2], [2, 3], [2, 4], [3], [4]]
```

One can instead use affine subspaces as elements, which is what is used to compute the poset in the first place:

```sage
H.<x,y> = HyperplaneArrangements(QQ)
A = H((y, y-1, y+1, x-y, x+y))
L = A.intersection_poset(element_label='subset')  # needs sage.combinat
sage: sorted(L, key=sorted)  # needs sage.combinat
[{}, {0}, {0, 3}, {0, 4}, {1}, {1, 3, 4}, {2}, {2, 3}, {2, 4}, {3}]```
sage: A = hyperplane_arrangements.coordinate(2)
sage: L = A.intersection_poset(element_label='subspace'); L
    #← needs sage.combinat
Finite poset containing 4 elements
sage: sorted(L, key=lambda S: (S.dimension(),
    #← needs sage.combinat
....:
[Affine space p + W where:
  p = (0, 0)
  W = Vector space of degree 2 and dimension 0 over Rational Field
     Basis matrix: [],
Affine space p + W where:
  p = (0, 0)
  W = Vector space of degree 2 and dimension 1 over Rational Field
     Basis matrix: [0 1],
Affine space p + W where:
  p = (0, 0)
  W = Vector space of degree 2 and dimension 1 over Rational Field
     Basis matrix: [1 0],
Affine space p + W where:
  p = (0, 0)
  W = Vector space of dimension 2 over Rational Field]

\textbf{is\_central} \ (\textit{certificate}=False)

Test whether the intersection of all the hyperplanes is nonempty.

A hyperplane arrangement is central if the intersection of all the hyperplanes in the arrangement is nonempty.

INPUT:

\begin{itemize}
  \item certificate – boolean (default: False); specifies whether to return the center as a polyhedron (possibly empty) as part of the output
\end{itemize}

OUTPUT:

If \textit{certificate} is \text{True}, returns a tuple containing:

1. A boolean
2. The polyhedron defined to be the intersection of all the hyperplanes

If \textit{certificate} is \text{False}, returns a boolean.

EXAMPLES:

\begin{verbatim}
sage: a = hyperplane_arrangements.braid(2)  #← needs sage.graphs
sage: a.is_central()  #← needs sage.graphs
True
\end{verbatim}

The Catalan arrangement in dimension 3 is not central:

\begin{verbatim}
sage: b = hyperplane_arrangements.Catalan(3)
sage: b.is_central(certificate=True)
(False, The empty polyhedron in \text{QQ}^3)
\end{verbatim}

The empty arrangement in dimension 5 is central:
Combinatorial and Discrete Geometry, Release 10.3

\begin{verbatim}
sage: H = HyperplaneArrangements(QQ, names=tuple(['x'+str(i) for i in range(7)]))
sage: c = H()
sage: c.is_central(certificate=True)
(True, A 7-dimensional polyhedron in QQ^7 defined as the convex hull of 1 vertex and 7 lines)
\end{verbatim}

**is_essential()**

Test whether the hyperplane arrangement is essential.

A hyperplane arrangement is essential if the span of the normals of its hyperplanes spans the ambient space.

See also:

**essentialization()**

OUTPUT:

A boolean indicating whether the hyperplane arrangement is essential.

EXAMPLES:

\begin{verbatim}
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: H(x, x+1).is_essential()
False
sage: H(x, y).is_essential()
True
\end{verbatim}

**is_formal()**

Return if self is formal.

A hyperplane arrangement is formal if it is 3-generated [Yuz1993], where $k$-generated is defined in minimal_generated_number().

EXAMPLES:

\begin{verbatim}
sage: P.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 3*x+5*z, 3*x+4*y+5*z)
sage: B = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 2*x+3*y+4*z, x+3*z, x+2*y+3*z)
sage: A.is_formal()
True
sage: B.is_formal()
False
\end{verbatim}

**is_free(algorithm='singular')**

Return if self is free.

A hyperplane arrangement $A$ is free if the module of derivations $\text{Der}(A)$ is a free $S$-module, where $S$ is the corresponding symmetric space.

INPUT:

- **algorithm** – (default: "singular") can be one of the following:
  - "singular" – use Singular's minimal free resolution
  - "BC" – use the algorithm given by Barakat and Cuntz in [BC2012] (much slower than using Singular)

ALGORITHM:
singular

Check that the minimal free resolution has length at most 2 by using Singular.

BC

This implementation follows [BC2012] by constructing a chain of free modules

\[ D(A) = D(A_n) < D(A_{n-1}) < \cdots < D(A_1) < D(A_0) \]

corresponding to some ordering of the arrangements \( A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset A_n = A \). Such a chain is found by using a backtracking algorithm.

EXAMPLES:

For type \( A \) arrangements, chordality is equivalent to freeness. We verify that in type \( A_3 \):

```python
sage: W = WeylGroup(['A', 3], prefix='s')  # needs sage.combinat sage.groups
sage: for x in W:
    # needs sage.combinat sage.groups
    ....: A = x.inversion_arrangement()
    ....: assert A.matroid().is_chordal() == A.is_free()
```

is_linear()

Test whether all hyperplanes pass through the origin.

OUTPUT:

A boolean. Whether all the hyperplanes pass through the origin.

EXAMPLES:

```python
sage: a = hyperplane_arrangements.semiorder(3)
sage: a.is_linear()
False
sage: b = hyperplane_arrangements.braid(3)  # needs sage.graphs
sage: b.is_linear()  # needs sage.graphs
True
```

is_separating_hyperplane (region1, region2, hyperplane)

Test whether the hyperplane separates the given regions.

INPUT:

- region1, region2 – polyhedra or list/tuple/iterable of coordinates which are regions of the arrangement or an interior point of a region
- hyperplane – a hyperplane
OUTPUT:
A boolean. Whether the hyperplane `hyperplane` separate the given regions.

EXAMPLES:

```python
sage: A.<x,y> = hyperplane_arrangements.coordinate(2)
sage: A.is_separating_hyperplane([1,1], [2,1], y)
False
sage: A.is_separating_hyperplane([1,1], [-1,1], x)
True
```

```python
sage: r = A.region_containing_point([1,1])
sage: s = A.region_containing_point([-1,1])
sage: A.is_separating_hyperplane(r, s, x)
True
```

`is_simplicial()`
Test whether the arrangement is simplicial.

A region is simplicial if the normal vectors of its bounding hyperplanes are linearly independent. A hyperplane arrangement is said to be simplicial if every region is simplicial.

OUTPUT:
A boolean whether the hyperplane arrangement is simplicial.

EXAMPLES:

```python
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H([[0,1,1,1], [0,1,2,3]])
sage: A.is_simplicial()
True
```

```python
sage: A = H([[0,1,1,1], [0,1,2,3], [0,1,3,2]])
sage: A.is_simplicial()
True
```

```python
sage: A = H([[0,1,1,1], [0,1,2,3], [0,1,3,2], [0,2,1,3]])
sage: A.is_simplicial()
False
```

`matroid()`
Return the matroid associated to `self`.

Let $A$ denote a central hyperplane arrangement and $n_H$ the normal vector of some hyperplane $H \in A$. We define a matroid $M_A$ as the linear matroid spanned by $\{n_H | H \in A\}$. The matroid $M_A$ is such that the lattice of flats of $M$ is isomorphic to the intersection lattice of $A$ (Proposition 3.6 in [Sta2007]).

EXAMPLES:

```python
sage: P.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 2*x+3*y+4*z)
sage: M = A.matroid(); M
Linear matroid of rank 3 on 7 elements represented over the Rational Field
```

We check the lattice of flats is isomorphic to the intersection lattice:

```python
sage: f = sum([list(M.flats(i)) for i in range(M.rank() + 1)], [])
sage: PF = Poset([f, lambda x, y: x < y])
```

(continues on next page)
\texttt{minimal\_generated\_number}() \\
Return the minimum $k$ such that \texttt{self} is $k$-generated.

Let $A$ be a central hyperplane arrangement. Let $W_k$ denote the solution space of the linear system corresponding to the linear dependencies among the hyperplanes of $A$ of length at most $k$. We say $A$ is $k$-generated if $\dim W_k = \rank A$.

Equivalently this says all dependencies forming the Orlik-Terao ideal are generated by at most $k$ hyperplanes.

EXAMPLES:

We construct Example 2.2 from [Yuz1993]:

\begin{verbatim}
sage: P.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 2*x+3*y+4*z, 3*x+5*z, 3*x+4*y+5*z)
sage: B = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 2*x+3*y+4*z, x+3*z, x+2*y+3*z)
sage: A.minimal_generated_number()
3
sage: B.minimal_generated_number()
4
\end{verbatim}

\texttt{n\_bounded\_regions}() \\
Return the number of (relatively) bounded regions.

OUTPUT:

An integer. The number of relatively bounded regions of the hyperplane arrangement.

EXAMPLES:

\begin{verbatim}
sage: A = hyperplane_arrangements.semiorder(3)
sage: A.n_bounded_regions()
7
\end{verbatim}

\texttt{n\_hyperplanes}() \\
Return the number of hyperplanes in the arrangement.

OUTPUT:

An integer.

EXAMPLES:

\begin{verbatim}
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H([1,1,0], [2,3,-1], [4,5,3])
sage: A.n_hyperplanes()
3
sage: len(A)  # equivalent
3
\end{verbatim}

\texttt{n\_regions}() \\
The number of regions of the hyperplane arrangement.

OUTPUT:
An integer.

EXAMPLES:

```python
sage: A = hyperplane_arrangements.semiorder(3)
sage: A.n_regions()
19
```

**orlik_solomon_algebra** *(base_ring=None, ordering=None, **kwds)*

Return the Orlik-Solomon algebra of *self*.

**INPUT:**

- `base_ring` – (default: the base field of *self*) the ring over which the Orlik-Solomon algebra will be defined
- `ordering` – (optional) an ordering of the ground set

**EXAMPLES:**

```python
sage: P.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 2*x+3*y+4*z)
sage: A.orlik_solomon_algebra()
Orlik-Solomon algebra of Linear matroid of rank 3 on 7 elements represented over the Rational Field
sage: A.orlik_solomon_algebra(base_ring=ZZ)
Orlik-Solomon algebra of Linear matroid of rank 3 on 7 elements represented over the Rational Field
```

**orlik_terao_algebra** *(base_ring=None, ordering=None, **kwds)*

Return the Orlik-Terao algebra of *self*.

**INPUT:**

- `base_ring` – (default: the base field of *self*) the ring over which the Orlik-Terao algebra will be defined
- `ordering` – (optional) an ordering of the ground set

**EXAMPLES:**

```python
sage: P.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = P(x, y, z, x+y+z, 2*x+y+z, 2*x+3*y+z, 2*x+3*y+4*z)
sage: A.orlik_terao_algebra()
Orlik-Terao algebra of Linear matroid of rank 3 on 7 elements represented over the Rational Field represented over the Rational Field
sage: A.orlik_terao_algebra(base_ring=QQ['t'])
Orlik-Terao algebra of Linear matroid of rank 3 on 7 elements represented over the Rational Field over Univariate Polynomial Ring in t over Rational Field
```

**plot** *(**kwds)***

Plot the hyperplane arrangement.

**OUTPUT:**

A graphics object.

**EXAMPLES:**
poincare_polynomial()

Return the Poincaré polynomial of the hyperplane arrangement.

OUTPUT:

The Poincaré polynomial in \( \mathbb{Q}[x] \).

EXAMPLES:

```python
sage: a = hyperplane_arrangements.coordinate(2)
sage: a.poincare_polynomial()
x^2 + 2*x + 1
```

poset_of_regions(B=None, numbered_labels=True)

Return the poset of regions for a central hyperplane arrangement.

The poset of regions is a partial order on the set of regions where the regions are ordered by \( R \leq R' \) if and only if \( S(R) \subseteq S(R') \) where \( S(R) \) is the set of hyperplanes which separate the region \( R \) from the base region \( B \).

INPUT:

- \( B \) – a region (optional; default: None); if None, then an arbitrary region is chosen as the base region.
- \( numbered_labels \) – bool (optional; default: True); if True, then the elements of the poset are numbered. Else they are labelled with the regions themselves.

OUTPUT:

A Poset object containing the poset of regions.

EXAMPLES:

```python
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H([[0,1,1,1], [0,1,2,3]])
sage: A.poset_of_regions()  # needs sage.combinat
Finite poset containing 4 elements
sage: A.poset_of_regions(numbered_labels=False)  # needs sage.combinat sage.graphs
Finite poset containing 6 elements
sage: A = hyperplane_arrangements.braid(3)
sage: A.poset_of_regions()  # needs sage.combinat
Finite poset containing 6 elements
sage: A.poset_of_regions(numbered_labels=False)  # needs sage.combinat sage.graphs
Finite poset containing 6 elements
sage: A = hyperplane_arrangements.braid(4)
sage: A.poset_of_regions()  # needs sage.combinat
Finite poset containing 24 elements
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H([[0,1,1,1], [0,1,2,3], [0,1,3,2], [0,2,1,3]])
sage: R = A.regions()
sage: base_region = R[3]
sage: A.poset_of_regions(B=base_region)  # needs sage.combinat sage.graphs
```

(continues on next page)
primitive_eulerian_polynomial()

Return the primitive Eulerian polynomial of self.

The primitive Eulerian polynomial of a hyperplane arrangement \(A\) is defined [BHS2023] by

\[
P_A(z) := \sum_{X \in L} |\mu(B, X)|(z - 1)^{\text{codim} X},
\]

where \(L\) is the intersection poset of \(A\), \(B\) is the minimal element of \(L\) (here, the 0 dimensional subspace), and \(\mu\) is the Möbius function of \(L\).

OUTPUT:

The primitive Eulerian polynomial in \(\mathbb{Z}[z]\).

EXAMPLES:

```python
sage: A = hyperplane_arrangements.coordinate(2)
sage: A.primitive_eulerian_polynomial()
z^2
sage: B = hyperplane_arrangements.braid(3)
sage: B.primitive_eulerian_polynomial()
z^2 + z
sage: H = hyperplane_arrangements.Shi(['B',2]).cone()
sage: H.primitive_eulerian_polynomial()
z^3 + 3*z^2 + 4*z
sage: H = hyperplane_arrangements.graphical(graphs.CycleGraph(4))
sage: H.primitive_eulerian_polynomial()
z^3 + 3*z^2 - z
```

We verify Example 2.4 in [BHS2023] for \(k = 2, 3, 4, 5\):

```python
sage: R.<x,y> = HyperplaneArrangements(QQ)
sage: for k in range(2,6):
    ....:     H = R([x+j*y for j in range(k)])
    ....:     H.primitive_eulerian_polynomial()
z^2
z^2 + z
z^2 + 2*z
z^2 + 3*z
```

We verify Equation (4) in [BHS2023] on some examples:

```python
sage: R.<x> = ZZ[]
sage: Arr = [hyperplane_arrangements.braid(n) for n in range(2,6)]
sage: all(R(A.cocharacteristic_polynomial()(1/(x-1)) * (x-1)^A.dimension()) == R(A.primitive_eulerian_polynomial()) for A in Arr)
True
```

We compute types \(H_3\) and \(F_4\) in Table 1 of [BHS2023]:

1.1. Hyperplane Arrangements
We verify Proposition 2.5 in [BHS2023] on the braid arrangement $B_k$ for $k = 2, 3, 4, 5$:

```
sage: B = [hyperplane_arrangements.braid(k) for k in range(2,6)]
sage: all(H.is_simplicial() for H in B)
True
sage: all(c > 0 for H in B for c in H.primitive_eulerian_polynomial().coefficients())
True
```

We verify Example 9.4 in [BHS2023] showing a hyperplane arrangement whose primitive Eulerian polynomial does not have real roots (in general, the graphical arrangement of a cycle graph corresponds to the arrangements in Example 9.4):

```
sage: H = hyperplane_arrangements.graphical(graphs.CycleGraph(5))
sage: pep = H.primitive_eulerian_polynomial(); pep
z^4 + 6*z^3 - 4*z^2 + z
sage: pep.roots(QQbar)
[(-6.626418492719221?, 1),
 (0, 1),
 (0.3132092463596102? - 0.2298065541510677?*I, 1),
 (0.3132092463596102? + 0.2298065541510677?*I, 1)]
sage: pep.roots(AA)
[(-6.626418492719221?, 1), (0, 1)]
```

**rank ()**

Return the rank.

**OUTPUT:**

The dimension of the span of the normals to the hyperplanes in the arrangement.

**EXAMPLES:**

```
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H(\[[0, 1, 2, 3],[\-3, 4, 5, 6]\])
sage: A.dimension()
3
sage: A.rank()
2
```

(continues on next page)
(Hyperplane $0^*t_0 + t_1 - t_2 + 0$,
Hyperplane $t_0 - t_1 + 0^*t_2 + 0$,
Hyperplane $t_0 + 0^*t_1 - t_2 + 0$)
sage: B.dimension()
3
sage: B.rank()
2

sage: p = polytopes.simplex(5, project=True)
sage: H = p.hyperplane_arrangement()
sage: H.rank()
5

region_containing_point($p$)
The region in the hyperplane arrangement containing a given point.
The base field must have characteristic zero.

INPUT:

• $p$ – point

OUTPUT:

A polyhedron. A ValueError is raised if the point is not interior to a region, that is, sits on a hyperplane.

EXAMPLES:

```
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H([(1,0), 0], [(0,1), 1], [(0,1), -1], [(1,-1), 0], [(1,1), 0])
sage: A.region_containing_point([1,2])
```

regions()
Return the regions of the hyperplane arrangement.
The base field must have characteristic zero.

OUTPUT:

A tuple containing the regions as polyhedra.
The regions are the connected components of the complement of the union of the hyperplanes as a subset of $\mathbb{R}^n$.

EXAMPLES:

```
sage: a = hyperplane_arrangements.braid(2)  # needs sage.graphs
sage: a.regions()                           # needs sage.graphs
(A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line,
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex, 1 ray, 1 line)
```
(A 2-dimensional polyhedron in \( \mathbb{Q}^2 \) defined as the convex hull of 1 vertex and 2 rays, A 2-dimensional polyhedron in \( \mathbb{Q}^2 \) defined as the convex hull of 1 vertex and 2 rays, A 2-dimensional polyhedron in \( \mathbb{Q}^2 \) defined as the convex hull of 1 vertex and 2 rays, A 2-dimensional polyhedron in \( \mathbb{Q}^2 \) defined as the convex hull of 1 vertex and 2 rays)

```python
sage: chessboard = []
sage: N = 8
sage: for x0 in range(N + 1):
    ...:     for y0 in range(N + 1):
    ...:         chessboard.extend([(x-x0, y-y0)])
```  
```
sage: chessboard = H(chessboard)
```  
```
sage: len(chessboard.bounded_regions())
# long time, 359 ms on a Core i7
64
```  
Example 6 of [KP2020]:

```python
sage: from itertools import product
sage: def zero_one(d):
    ...:     for x in product([0,1], repeat=d):
    ...:         if any(x):
    ...:             yield [0] + list(x)
```  
```
sage: K.<x,y> = HyperplaneArrangements(QQ)
sage: A = K(*zero_one(2))
sage: len(A.regions())
6
```  
```
sage: K.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = K(*zero_one(3))
sage: len(A.regions())
32
```  
```
sage: K.<x,y,z,w> = HyperplaneArrangements(QQ)
sage: A = K(*zero_one(4))
sage: len(A.regions())
370
```  
```
sage: K.<x,y,z,w,r> = HyperplaneArrangements(QQ)
sage: A = K(*zero_one(5))
sage: len(A.regions())
# not tested (~25s)
11292
```  
It is possible to specify the backend:

```python
sage: # needs sage.rings.number_field
sage: K.<q> = CyclotomicField(9)
sage: L.<r9> = NumberField((q + q**(-1)).minpoly(),
    ...:     embedding=AA(q + q**-1))
```  
```
sage: norms = [[1, 1/3*(-2*r9**2-r9+1), 0],
    ...:          [1, -r9**2 - r9, 0],
    ...:          [1, -r9**2 + 1, 0],
    ...:          [1, -r9**2, 0],
    ...:          [1, r9**2 - 4, -r9**2+3]]
sage: H.<x,y,z> = HyperplaneArrangements(L)
sage: A = H(backend='normaliz')
```  
```
sage: for v in norms:
```
....: a, b, c = v
....: A = A.add_hyperplane(a*x + b*y + c*z)
sage: R = A.regions()  # optional - pynormaliz
˓→pynormaliz
sage: R[0].backend()  # optional - pynormaliz
˓→pynormaliz
normaliz

restriction(\textit{hyperplane})

Return the restriction to a hyperplane.

INPUT:

- hyperplane – a hyperplane of the hyperplane arrangement

OUTPUT:

The restriction of the hyperplane arrangement to the given hyperplane.

EXAMPLES:

\begin{verbatim}
\sage: # needs sage.graphs
\sage: A.<u,x,y,z> = hyperplane_arrangements.braid(4); A
Arrangement of 6 hyperplanes of dimension 4 and rank 3
\sage: H = A[0]; H
Hyperplane 0*u + 0*x + y - z + 0
\sage: R = A.restriction(H); R
Arrangement <x - z | u - x | u - z>
\sage: D = A.deletion(H); D
Arrangement of 5 hyperplanes of dimension 4 and rank 3
\sage: ca = A.characteristic_polynomial();
\sage: cr = R.characteristic_polynomial();
\sage: cd = D.characteristic_polynomial();
\sage: ca
x^4 - 6*x^3 + 11*x^2 - 6*x
\sage: cd - cr
x^4 - 6*x^3 + 11*x^2 - 6*x
\end{verbatim}

See also:

\texttt{deletion()}

\texttt{sign_vector(\textit{p})}

Indicates on which side of each hyperplane the given point \textit{p} lies.

The base field must have characteristic zero.

INPUT:

- \textit{p} – point as a list/tuple/iterable

OUTPUT:

A vector whose entries are in \([-1, 0, +1]\).

EXAMPLES:

\begin{verbatim}
\sage: H.<x,y> = HyperplaneArrangements(QQ)
\sage: A = H([(1,0), [0,1]), [0,1]); A
Arrangement <y + 1 | x>
\sage: A.sign_vector([2, -2])
\end{verbatim}
unbounded_regions()

Return the relatively bounded regions of the arrangement.

OUTPUT:

tuple of polyhedra. The regions of the arrangement that are not relatively bounded. It is assumed that the
arrangement is defined over the rationals.

See also:

bounded_regions()

EXAMPLES:

sage: # needs sage.combinat
sage: A = hyperplane_arrangements.semiorder(3)
sage: B = A.essentialization()
sage: B.n_regions() - B.n_bounded_regions()
12
sage: B.unbounded_regions()
(A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 3 vertices and 1 ray,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 3 vertices and 1 ray,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 1 vertex and 2 rays,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 3 vertices and 1 ray,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 1 vertex and 2 rays,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 3 vertices and 1 ray,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 1 vertex and 2 rays,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 3 vertices and 1 ray,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 1 vertex and 2 rays,
 A 2-dimensional polyhedron in QQ^2 defined
 as the convex hull of 1 vertex and 2 rays)

union(other)

The union of self with other.

INPUT:

• other — a hyperplane arrangement or something that can be converted into a hyperplane arrangement

OUTPUT:

A new hyperplane arrangement.
EXAMPLES:

```python
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: A = H([1,2,3], [0,1,1], [0,1,-1], [1,-1,0], [1,1,0])
sage: B = H([1,1,1], [1,-1,1], [1,0,-1])
sage: A.union(B)
Arrangement of 8 hyperplanes of dimension 2 and rank 2
sage: A | B  # syntactic sugar
Arrangement of 8 hyperplanes of dimension 2 and rank 2
```

A single hyperplane is coerced into a hyperplane arrangement if necessary:

```python
sage: A.union(x+y-1)
Arrangement of 6 hyperplanes of dimension 2 and rank 2
sage: A.add_hyperplane(x+y-1)  # alias
Arrangement of 6 hyperplanes of dimension 2 and rank 2
```

**varchenko_matrix** *(names='h')*

Return the Varchenko matrix of the arrangement.

Let $H_1, \ldots, H_s$ and $R_1, \ldots, R_t$ denote the hyperplanes and regions, respectively, of the arrangement. Let $S = Q[h_1, \ldots, h_s]$, a polynomial ring with indeterminate $h_i$ corresponding to hyperplane $H_i$. The Varchenko matrix is the $t \times t$ matrix with $i, j$-th entry the product of those $h_k$ such that $H_k$ separates $R_i$ and $R_j$.

**INPUT:**

- `names` – string or list/tuple/iterable of strings. The variable names for the polynomial ring $S$.

**OUTPUT:**

The Varchenko matrix.

**EXAMPLES:**

```python
sage: a = hyperplane_arrangements.coordinate(3)
sage: v = a.varchenko_matrix(); v
[ 1  h2  h1  h1*h2 h0*h1*h2 h0*h1  h0*h2  h0]
[ h2  1  h1*h2 h1 h0*h1 h0*h1*h2 h0  h0*h2]
[ h1 h1*h2  1 h2 h0*h2 h0 h0*h1*h2 h0*h1]
[ h1*h2 h1 h2  1 h0 h0*h2 h0*h1 h0*h1*h2]
[ h0*h1*h2 h0*h1 h0*h2 h0  1 h2 h1 h1*h2]
[ h0*h1 h0*h1*h2 h0 h0*h2 h2  1 h1*h2 h1]
[ h0*h2 h0 h0*h1*h2 h0*h1 h1 h1*h2  1 h2]
[ h0 h0*h2 h0*h1 h0*h1*h2 h1*h2 h1 h2  1]
sage: factor(det(v))
(h2 - 1)^4 * (h2 + 1)^4 * (h1 - 1)^4 * (h1 + 1)^4 * (h0 - 1)^4 * (h0 + 1)^4
```

**vertices** *(exclude_sandwiched=False)*

Return the vertices.

The vertices are the zero-dimensional faces, see `face_vector()`.

**INPUT:**

- `exclude_sandwiched` – boolean (default: False). Whether to exclude hyperplanes that are sandwiched between parallel hyperplanes. Useful if you only need the convex hull.
OUTPut:
The vertices in a sorted tuple. Each vertex is returned as a vector in the ambient vector space.

EXAMPLES:

```python
sage: # needs sage.combinat
sage: A = hyperplane_arrangements.Shi(3).essentialization()
sage: A.dimension()
2
sage: A.face_vector()
(6, 21, 16)
sage: A.vertices()
((-2/3, 1/3), (-1/3, -1/3), (0, -1), (0, 0), (1/3, -2/3), (2/3, -1/3))
sage: point2d(A.vertices(), size=20) + A.plot()  # needs sage.plot
Graphics object consisting of 7 graphics primitives
```

```python
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: chessboard = []
sage: N = 8
sage: for x0 in range(N + 1):
....:     for y0 in range(N + 1):
....:         chessboard.extend([x-x0, y-y0])
sage: chessboard = H(chessboard)
sage: len(chessboard.vertices(exclude_sandwiched=True))
81
sage: chessboard.vertices(exclude_sandwiched=True)
((0, 0), (0, 8), (8, 0), (8, 8))
```

whitney_data()
Return the Whitney numbers.

See also:

whitney_number(), doubly_indexed_whitney_number()

OUTPUT:

A pair of integer matrices. The two matrices are the doubly-indexed Whitney numbers of the first or second kind, respectively. The $i, j$-th entry is the $i, j$-th doubly-indexed Whitney number.

EXAMPLES:

```python
sage: A = hyperplane_arrangements.Shi(3)
sage: A.whitney_data()  # needs sage.combinat
([ 1 -6  9] [ 1  6  6]
[ 0  6 -15] [ 0  6 15]
[ 0  0  6], [ 0  0  6])
```

whitney_number($k$, kind=1)
Return the $k$-th Whitney number.

If kind=1, this number is obtained by summing the Möbius function values $\mu(x, x)$ over all $x$ in the intersection poset with rank($x$) = $k$.

If kind=2, this number is the number of elements $x, y$ in the intersection poset such that $x \leq y$ with ranks $i$ and $j$, respectively.
See [GZ1983] for more details.

**INPUT:**
- \( k \) – integer
- \( \text{kind} \) – 1 or 2 (default: 1)

**OUTPUT:**
Integer. The \( k \)-th Whitney number.

**See also:**
- `doubly_indexed_whitney_number()`
- `whitney_data()`

**EXAMPLES:**

```
sage: # needs sage.combinat
sage: A = hyperplane_arrangements.Shi(3)
sage: A.whitney_number(0)
1
sage: A.whitney_number(1)
-6
sage: A.whitney_number(2)
9
sage: A.characteristic_polynomial()
x^3 - 6*x^2 + 9*x
sage: A.whitney_number(1, kind=2)
6
sage: p = A.intersection_poset()
sage: r = p.rank_function()
sage: len([i for i in p if r(i) == 1])
6
```

```python
class sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangements(base_ring, names=())
```

**Bases:** `Parent, UniqueRepresentation`

Hyperplane arrangements.

For more information on hyperplane arrangements, see `sage.geometry.hyperplane_arrangement.arrangement`.

**INPUT:**
- `base_ring` – ring; the base ring
- `names` – tuple of strings; the variable names

**EXAMPLES:**

```
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: x
Hyperplane x + 0*y + 0
sage: x + y
Hyperplane x + y + 0
sage: H(x, y, x-1, y-1)
Arrangement <y - 1 | y | x - 1 | x>
```

**Element**
- alias of `HyperplaneArrangementElement`
ambient_space()

Return the ambient space.

The ambient space is the parent of hyperplanes. That is, new hyperplanes are always constructed internally from the ambient space instance.

EXAMPLES:

```
sage: L.<x, y> = HyperplaneArrangements(QQ)
sage: L.ambient_space()([[1,0], 0])
Hyperplane x + 0*y + 0
sage: L.ambient_space()([[1,0], 0]) == x
True
```

base_ring()

Return the base ring.

OUTPUT:

The base ring of the hyperplane arrangement.

EXAMPLES:

```
sage: L.<x,y> = HyperplaneArrangements(QQ)
sage: L.base_ring()
Rational Field
```

change_ring(base_ring)

Return hyperplane arrangements over a different base ring.

INPUT:

- base_ring -- a ring; the new base ring.

OUTPUT:

A new `HyperplaneArrangements` instance over the new base ring.

EXAMPLES:

```
sage: L.<x,y> = HyperplaneArrangements(QQ)
sage: L.gen(0)
Hyperplane x + 0*y + 0
sage: L.change_ring(RR).gen(0)
Hyperplane 1.000000000000000*x + 0.000000000000000*y + 0.000000000000000
```

gen(i)

Return the i-th coordinate hyperplane.

INPUT:

- i -- integer

OUTPUT:

A linear expression.

EXAMPLES:

```
sage: L.<x, y, z> = HyperplaneArrangements(QQ); L
Hyperplane arrangements in
3-dimensional linear space over Rational Field with coordinates x, y, z
```

(continues on next page)
**sage:** L.gen(0)
Hyperplane $x + 0*y + 0*z + 0$

**gens()**

Return the coordinate hyperplanes.

**OUTPUT:**

A tuple of linear expressions, one for each linear variable.

**EXAMPLES:**

```sage
def L = HyperplaneArrangements(QQ, ('x', 'y', 'z'))
sage: L.gens()
(Hyperplane $x + 0*y + 0*z + 0$,
 Hyperplane $0*x + y + 0*z + 0$,
 Hyperplane $0*x + 0*y + z + 0$)
```

**ngens()**

Return the number of linear variables.

**OUTPUT:**

An integer.

**EXAMPLES:**

```sage
def L.<x, y, z> = HyperplaneArrangements(QQ); L
Hyperplane arrangements in 3-dimensional linear space
 over Rational Field with coordinates x, y, z
sage: L.ngens()
3
```

### 1.2 Library of Hyperplane Arrangements

A collection of useful or interesting hyperplane arrangements. See [sage.geometry.hyperplane_arrangement.arrangement](#) for details about how to construct your own hyperplane arrangements.

**class**

`sage.geometry.hyperplane_arrangement.library.HyperplaneArrangementLibrary`

**Bases:** object

The library of hyperplane arrangements.

**Catalan** ($n, K=Rational~Field, names=None$)

Return the Catalan arrangement.

**INPUT:**

- $n$ – integer
- $K$ – field (default: $Q$)
- $names$ – tuple of strings or $None$ (default); the variable names for the ambient space
The arrangement of $3n(n-1)/2$ hyperplanes $\{x_i - x_j = -1, 0, 1 : 1 \leq i \leq j \leq n\}$.

**Examples:**

```python
sage: hyperplane_arrangements.Catalan(5)
Arrangement of 30 hyperplanes of dimension 5 and rank 4
```

**Coxeter** ($\text{data}, K=\text{Rational Field}, \text{names}=None$)

Return the Coxeter arrangement.

This generalizes the braid arrangements to crystallographic root systems.

**Input:**

- **data** – either an integer or a Cartan type (or coercible into; see “CartanType”)
- **$K$** – field (default: $\mathbb{Q}$)
- **names** – tuple of strings or None (default); the variable names for the ambient space

**Output:**

- If **data** is an integer $n$, return the braid arrangement in dimension $n$, i.e. the set of $n(n-1)$ hyperplanes: $\{x_i - x_j = 0, 1: 1 \leq i \leq j \leq n\}$. This corresponds to the Coxeter arrangement of Cartan type $A_{n-1}$.
- If **data** is a Cartan type, return the Coxeter arrangement of given type.

The Coxeter arrangement of a given crystallographic Cartan type is defined by the inner products $\langle a, x \rangle = 0$ where $a \in \Phi^+$ runs over positive roots of the root system $\Phi$.

**Examples:**

```python
sage: hyperplane_arrangements.Coxeter(4)
Arrangement of 6 hyperplanes of dimension 4 and rank 3
sage: hyperplane_arrangements.Coxeter("B4")
Arrangement of 16 hyperplanes of dimension 4 and rank 4
sage: hyperplane_arrangements.Coxeter("A3")
Arrangement of 6 hyperplanes of dimension 4 and rank 3
```

If the Cartan type is not crystallographic, the Coxeter arrangement is not implemented yet:

```python
sage: hyperplane_arrangements.Coxeter("H3")
Traceback (most recent call last):
  ...
NotImplementedError: Coxeter arrangements are not implemented for non-
→ crystallographic Cartan types
```

The characteristic polynomial is pre-computed using the results of Terao, see [Ath2000]:

```python
sage: # needs sage.combinat
sage: hyperplane_arrangements.Coxeter("A3").characteristic_polynomial()
n^3 - 6*n^2 + 11*n - 6
```

**G_Shi** ($G, K=\text{Rational Field}, \text{names}=None$)

Return the Shi hyperplane arrangement of a graph $G$.

**Input:**

- **$G$** – graph

---

**Output:**

The arrangement of $3n(n-1)/2$ hyperplanes $\{x_i - x_j = -1, 0, 1 : 1 \leq i \leq j \leq n\}$.

**Examples:**

```python
sage: hyperplane_arrangements.Catalan(5)
Arrangement of 30 hyperplanes of dimension 5 and rank 4
```
• $K$ – field (default: $\mathbb{Q}$)
• $\text{name}$ – tuple of strings or $\text{None}$ (default); the variable names for the ambient space

OUTPUT:
The Shi hyperplane arrangement of the given graph $G$.

EXAMPLES:

```
sage: # needs sage.graphs
sage: G = graphs.CompleteGraph(5)
sage: hyperplane_arrangements.G_Shi(G)
Arrangement of 20 hyperplanes of dimension 5 and rank 4
sage: g = graphs.HouseGraph()
sage: hyperplane_arrangements.G_Shi(g)
Arrangement of 12 hyperplanes of dimension 5 and rank 4
sage: a = hyperplane_arrangements.G_Shi(graphs.WheelGraph(4)); a
Arrangement of 12 hyperplanes of dimension 4 and rank 3
```

$\text{G\_semiorder}(G, K=\text{Rational Field}, \text{names}=\text{None})$

Return the semiorder hyperplane arrangement of a graph.

INPUT:
• $G$ – graph
• $K$ – field (default: $\mathbb{Q}$)
• $\text{name}$ – tuple of strings or $\text{None}$ (default); the variable names for the ambient space

OUTPUT:
The semiorder hyperplane arrangement of a graph $G$ is the arrangement $\{x_i - x_j = -1, 1\}$ where $ij$ is an edge of $G$.

EXAMPLES:

```
sage: # needs sage.graphs
sage: G = graphs.CompleteGraph(5)
sage: hyperplane_arrangements.G_semiorder(G)
Arrangement of 20 hyperplanes of dimension 5 and rank 4
sage: g = graphs.HouseGraph()
sage: hyperplane_arrangements.G_semiorder(g)
Arrangement of 12 hyperplanes of dimension 5 and rank 4
```

$Ish(n, K=\text{Rational Field}, \text{names}=\text{None})$

Return the Ish arrangement.

INPUT:
• $n$ – integer
• $K$ – field (default: $\mathbb{Q}$)
• $\text{name}$ – tuple of strings or $\text{None}$ (default); the variable names for the ambient space

OUTPUT:
The Ish arrangement, which is the set of $n(n - 1)$ hyperplanes.

$$\{x_i - x_j = 0 : 1 \leq i \leq j \leq n\} \cup \{x_1 - x_j = i : 1 \leq i \leq j \leq n\}.$$
sage: # needs sage.combinat
sage: a = hyperplane_arrangements.Ish(3); a
Arrangement of 6 hyperplanes of dimension 3 and rank 2
sage: a.characteristic_polynomial()
x^3 - 6*x^2 + 9*x
sage: b = hyperplane_arrangements.Shi(3)
sage: b.characteristic_polynomial()
x^3 - 6*x^2 + 9*x

REFERENCES:

• [AR2012]

IshB \((n, K=\text{Rational Field}, \text{names=\text{None}})\)

Return the type B Ish arrangement.

INPUT:

• \(n\) – integer
• \(K\) – field (default: \(\text{QQ}\))
• \(\text{names}\) – tuple of strings or \text{None} (default); the variable names for the ambient space

OUTPUT:

The type \(B\) Ish arrangement, which is the set of \(2n^2\) hyperplanes
\[\{x_i \pm x_j = 0 : 1 \leq i < j \leq n\} \cup \{x_i = a : 1 \leq i \leq n, \quad i - n \leq a \leq n - i + 1\}.

EXAMPLES:

sage: a = hyperplane_arrangements.IshB(2)
sage: a
Arrangement of 8 hyperplanes of dimension 2 and rank 2
sage: a.hyperplanes()
(Hyperplane 0*t0 + t1 - 1,
 Hyperplane 0*t0 + t1 + 0,
 Hyperplane t0 - t1 + 0,
 Hyperplane t0 + 0*t1 - 2,
 Hyperplane t0 + 0*t1 - 1,
 Hyperplane t0 + 0*t1 + 0,
 Hyperplane t0 + 0*t1 + 1,
 Hyperplane t0 + t1 + 0)
sage: a.cone().is_free()
True

sage: a = hyperplane_arrangements.IshB(3); a
Arrangement of 18 hyperplanes of dimension 3 and rank 3
sage: a.characteristic_polynomial()
x^3 - 18*x^2 + 108*x - 216
sage: b = hyperplane_arrangements.Shi(['B', 3])
sage: b.characteristic_polynomial()
x^3 - 18*x^2 + 108*x - 216

REFERENCES:

• [TT2023]
Shi (data, K=Rational Field, names=None, m=1)

Return the Shi arrangement.

INPUT:
- data – either an integer or a Cartan type (or coercible into; see “CartanType”)
- K – field (default: QQ)
- names – tuple of strings or None (default); the variable names for the ambient space
- m – integer (default: 1)

OUTPUT:
- If data is an integer \( n \), return the Shi arrangement in dimension \( n \), i.e. the set of \( n(n-1) \) hyperplanes: \( \{ x_i - x_j = 0, 1 : 1 \leq i \leq j \leq n \} \). This corresponds to the Shi arrangement of Cartan type \( A_{n-1} \).
- If data is a Cartan type, return the Shi arrangement of given type.
- If \( m > 1 \), return the \( m \)-extended Shi arrangement of given type.

The \( m \)-extended Shi arrangement of a given crystallographic Cartan type is defined by the inner product \( \langle a, x \rangle = k \) for \(-m < k \leq m \) and \( a \in \Phi^+ \) is a positive root of the root system \( \Phi \).

EXAMPLES:

\[
\text{sage: } \text{# needs sage.combinat}
\text{sage: } \text{hyperplane_arrangements.Shi(4)}
\text{Arrangement of 12 hyperplanes of dimension 4 and rank 3}
\text{sage: } \text{hyperplane_arrangements.Shi("A3")}
\text{Arrangement of 12 hyperplanes of dimension 4 and rank 3}
\text{sage: } \text{hyperplane_arrangements.Shi("A3", m=2)}
\text{Arrangement of 24 hyperplanes of dimension 4 and rank 3}
\]

(continues on next page)
sage: hyperplane_arrangements.Shi("B4")
Arrangement of 32 hyperplanes of dimension 4 and rank 4
sage: hyperplane_arrangements.Shi("B4", m=3)
Arrangement of 96 hyperplanes of dimension 4 and rank 4
sage: hyperplane_arrangements.Shi("C3")
Arrangement of 18 hyperplanes of dimension 3 and rank 3
sage: hyperplane_arrangements.Shi("D4", m=3)
Arrangement of 72 hyperplanes of dimension 4 and rank 4
sage: hyperplane_arrangements.Shi("E6")
Arrangement of 72 hyperplanes of dimension 8 and rank 6
sage: hyperplane_arrangements.Shi("E6", m=2)
Arrangement of 144 hyperplanes of dimension 8 and rank 6

If the Cartan type is not crystallographic, the Shi arrangement is not defined:

sage: hyperplane_arrangements.Shi("H4")
Traceback (most recent call last):
...  
NotImplementedError: Shi arrangements are not defined for non-
→ crystallographic Cartan types

The characteristic polynomial is pre-computed using the results of [Ath1996]:

sage: # needs sage.combinat
sage: hyperplane_arrangements.Shi("A3").characteristic_polynomial()
x^4 - 12*x^3 + 48*x^2 - 64*x
sage: hyperplane_arrangements.Shi("A3", m=2).characteristic_polynomial()
x^4 - 24*x^3 + 192*x^2 - 512*x
sage: hyperplane_arrangements.Shi("C3").characteristic_polynomial()
x^3 - 18*x^2 + 108*x - 216
sage: hyperplane_arrangements.Shi("E6").characteristic_polynomial()
x^8 - 72*x^7 + 2160*x^6 - 34560*x^5 + 311040*x^4 - 1492992*x^3 + 2985984*x^2
sage: hyperplane_arrangements.Shi("B4", m=3).characteristic_polynomial()
x^4 - 96*x^3 + 3456*x^2 - 55296*x + 331776

bigraphical \((G, A=None, K=\text{Rational Field}, names=None)\)

Return a bigraphical hyperplane arrangement.

INPUT:

• \(G\) – graph
• \(A\) – list, matrix, dictionary (default: None gives semiorder), or the string ‘generic’
• \(K\) – field (default: \(\mathbb{Q}\))
• \(names\) – tuple of strings or None (default); the variable names for the ambient space

OUTPUT:

The hyperplane arrangement with hyperplanes \(x_i - x_j = A[i, j]\) and \(x_j - x_i = A[j, i]\) for each edge \(v_i, v_j\) of \(G\). The indices \(i, j\) are the indices of elements of \(G\).vertices().

EXAMPLES:

sage: # needs sage.graphs
sage: G = graphs.CycleGraph(4)
sage: G.edges(sort=True)
[(0, 1, None), (0, 3, None), (1, 2, None), (2, 3, None)]
sage: G.edges(sort=True, labels=False)
[(0, 1), (0, 3), (1, 2), (2, 3)]
sage: A = {0:{1:1, 3:2}, 1:{0:3, 2:0}, 2:{1:2, 3:1}, 3:{2:0, 0:2}}
sage: HA = hyperplane_arrangements.bigraphical(G, A)
sage: HA.n_regions()
63
sage: hyperplane_arrangements.bigraphical(G, 'generic').n_regions()
65
sage: hyperplane_arrangements.bigraphical(G).n_regions()
59

REFERENCES:

• [HP2016]

braid\((n, K=\text{Rational Field}, \text{name}=\text{None})\)
The braid arrangement.

INPUT:

• \(n\) – integer

• \(K\) – field (default: \(\mathbb{Q}\))

• \(\text{name}\) – tuple of strings or \(\text{None}\) (default); the variable names for the ambient space

OUTPUT:

The hyperplane arrangement consisting of the \(n(n-1)/2\) hyperplanes \(x_i - x_j = 0 : 1 \leq i \leq j \leq n\).

EXAMPLES:

sage: hyperplane_arrangements.braid(4) # needs sage.graphs
Arrangement of 6 hyperplanes of dimension 4 and rank 3

coordinate\((n, K=\text{Rational Field}, \text{name}=\text{None})\)
Return the coordinate hyperplane arrangement.

INPUT:

• \(n\) – integer

• \(K\) – field (default: \(\mathbb{Q}\))

• \(\text{name}\) – tuple of strings or \(\text{None}\) (default); the variable names for the ambient space

OUTPUT:

The coordinate hyperplane arrangement, which is the central hyperplane arrangement consisting of the coor-
dinate hyperplanes \(x_i = 0\).

EXAMPLES:

sage: hyperplane_arrangements.coordinate(5)
Arrangement of 5 hyperplanes of dimension 5 and rank 5

graphical\((G, K=\text{Rational Field}, \text{name}=\text{None})\)
Return the graphical hyperplane arrangement of a graph \(G\).

INPUT:

• \(G\) – graph
• \( K \) – field (default: \( \mathbb{Q} \))
• \( \text{names} \) – tuple of strings or \text{None} (default); the variable names for the ambient space

OUTPUT:
The graphical hyperplane arrangement of a graph \( G \), which is the arrangement \( \{ x_i - x_j = 0 \} \) for all edges \( ij \) of the graph \( G \).

EXAMPLES:

```sage
# needs sage.graphs
sage: G = graphs.CompleteGraph(5)
sage: hyperplane_arrangements.graphical(G)
Arrangement of 10 hyperplanes of dimension 5 and rank 4
sage: g = graphs.HouseGraph()
sage: hyperplane_arrangements.graphical(g)
Arrangement of 6 hyperplanes of dimension 5 and rank 4
```

\textbf{linial} \((n, K=\text{Rational Field}, \text{names}=\text{None})\)

Return the linial hyperplane arrangement.

INPUT:
• \( n \) – integer
• \( K \) – field (default: \( \mathbb{Q} \))
• \( \text{names} \) – tuple of strings or \text{None} (default); the variable names for the ambient space

OUTPUT:
The linial hyperplane arrangement is the set of hyperplanes \( \{ x_i - x_j = 1 : 1 \leq i < j \leq n \} \).

EXAMPLES:

```sage
sage: a = hyperplane_arrangements.linial(4); a
Arrangement of 6 hyperplanes of dimension 4 and rank 3
sage: a.characteristic_polynomial()
\text{x}^4 - 6\text{x}^3 + 15\text{x}^2 - 14\text{x}
```

\textbf{semiorder} \((n, K=\text{Rational Field}, \text{names}=\text{None})\)

Return the semiorder arrangement.

INPUT:
• \( n \) – integer
• \( K \) – field (default: \( \mathbb{Q} \))
• \( \text{names} \) – tuple of strings or \text{None} (default); the variable names for the ambient space

OUTPUT:
The semiorder arrangement, which is the set of \( n(n - 1) \) hyperplanes \( \{ x_i - x_j = -1, 1 : 1 \leq i < j \leq n \} \).

EXAMPLES:

```sage
sage: hyperplane_arrangements.semiorder(4)
Arrangement of 12 hyperplanes of dimension 4 and rank 3
```

\texttt{sage.geometry.hyperplane_arrangement.library.make_parent} \((\text{base_ring}, \text{dimension}, \text{names}=\text{None})\)
Construct the parent for the hyperplane arrangements.
For internal use only.

INPUT:
- `base_ring` – a ring
- `dimension` – integer
- `names` – `None` (default) or a list/tuple/iterable of strings

OUTPUT:
A new `HyperplaneArrangements` instance.

EXAMPLES:
```
sage: from sage.geometry.hyperplane_arrangement.library import make_parent
sage: make_parent(QQ, 3)
Hyperplane arrangements in 3-dimensional linear space over
Rational Field with coordinates t0, t1, t2
```

1.3 Hyperplanes

**Note:** If you want to learn about Sage’s hyperplane arrangements then you should start with `sage.geometry.hyperplane_arrangement.arrangement`. This module is used to represent the individual hyperplanes, but you should never construct the classes from this module directly (but only via the `HyperplaneArrangements`.

A linear expression, for example, $3x + 3y - 5z - 7$ stands for the hyperplane with the equation $x + 3y - 5z = 7$. To create it in Sage, you first have to create a `HyperplaneArrangements` object to define the variables $x, y, z$:

```
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = 3*x + 2*y - 5*z - 7; h
Hyperplane 3*x + 2*y - 5*z - 7
sage: h.coefficients()
[-7, 3, 2, -5]
sage: h.normal()
(3, 2, -5)
sage: h.constant_term()
-7
sage: h.change_ring(GF(3))
Hyperplane 0*x + 2*y + z + 2
sage: h.point()
(21/38, 7/19, -35/38)
sage: h.linear_part()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 3/5]
[ 0 1 2/5]
```

Another syntax to create hyperplanes is to specify coefficients and a constant term:

```sage
V = H.ambient_space(); V
3-dimensional linear space over Rational Field with coordinates x, y, z
```
Combinatorial and Discrete Geometry, Release 10.3

sage: h in V
True
sage: V([3, 2, -5], -7)
Hyperplane 3*x + 2*y - 5*z - 7

Or constant term and coefficients together in one list/tuple/iterable:

sage: V([-7, 3, 2, -5])
Hyperplane 3*x + 2*y - 5*z - 7
sage: v = vector([-7, 3, 2, -5]); v
(-7, 3, 2, -5)
sage: V(v)
Hyperplane 3*x + 2*y - 5*z - 7

Note that the constant term comes first, which matches the notation for Sage's `Polyhedron()`

sage: Polyhedron(ieqs=[(4,1,2,3)]).Hrepresentation()
(An inequality (1, 2, 3) x + 4 >= 0,)

The difference between hyperplanes as implemented in this module and hyperplane arrangements is that:

- hyperplane arrangements contain multiple hyperplanes (of course),
- linear expressions are a module over the base ring, and these module structure is inherited by the hyperplanes.

The latter means that you can add and multiply by a scalar:

sage: h = 3*x + 2*y - 5*z - 7; h
Hyperplane 3*x + 2*y - 5*z - 7
sage: -h
Hyperplane -3*x - 2*y + 5*z + 7
sage: h + x
Hyperplane 4*x + 2*y - 5*z - 7
sage: h + 7
Hyperplane 3*x + 2*y - 5*z + 0
sage: 3*h
Hyperplane 9*x + 6*y - 15*z - 21
sage: h * RDF(3)
Hyperplane 9.0*x + 6.0*y - 15.0*z - 21.0

Which you can't do with hyperplane arrangements:

sage: arrangement = H(h, x, y, x+y-1); arrangement
Arrangement <y | x | x + y - 1 | 3*x + 2*y - 5*z - 7>
sage: arrangement + x
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +:
'Hyperplane arrangements in 3-dimensional linear space over Rational Field with coordinates x, y, z' and
'Hyperplane arrangements in 3-dimensional linear space over Rational Field with coordinates x, y, z'

class sage.geometry.hyperplane_arrangement.hyperplane.AmbientVectorSpace(base_ring, names=())

Bases: LinearExpressionModule

The ambient space for hyperplanes.
This class is the parent for the Hyperplane instances.

Element
alias of Hyperplane

dimension()
Return the ambient space dimension.

symmetric_space()
Construct the symmetric space of self.

class sage.geometry.hyperplane_arrangement.hyperplane.Hyperplane
Bases: LinearExpression
A hyperplane.
You should always use AmbientVectorSpace to construct instances of this class.
INPUT:

- parent – the parent AmbientVectorSpace
- coefficients – a vector of coefficients of the linear variables
- constant – the constant term for the linear expression

EXAMPLES:

```sage
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: x+y-1
Hyperplane x + y - 1
sage: ambient = H.ambient_space()
sage: ambient._element_constructor_(x+y-1)
Hyperplane x + y - 1
```

For technical reasons, we must allow the degenerate cases of an empty space and of a full space:

```sage
sage: 0*x
Hyperplane 0*x + 0*y + 0
sage: 0*x + 1
Hyperplane 0*x + 0*y + 1
sage: x + 0 == x + ambient(0)  # because coercion requires them
True
```

**dimension()**

The dimension of the hyperplane.

OUTPUT:

An integer.

EXAMPLES:

```sage
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = x + y + z - 1
sage: h.dimension()
2
```

**intersection(other)**

The intersection of self with other.

INPUT:

- other – a hyperplane, a polyhedron, or something that defines a polyhedron

OUTPUT:

A polyhedron.

EXAMPLES:

```sage
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = x + y + z - 1
sage: h.intersection(x - y)
A 1-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex and 1 line
sage: h.intersection(polytopes.cube())
A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 3 vertices
```
linear_part()

The linear part of the affine space.

OUTPUT:

Vector subspace of the ambient vector space, parallel to the hyperplane.

EXAMPLES:

```
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = x + 2*y + 3*z - 1
sage: h.linear_part()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1/3]
[ 0 1 -2/3]
```

linear_part_projection(point)

Orthogonal projection onto the linear part.

INPUT:

• point – vector of the ambient space, or anything that can be converted into one; not necessarily on the hyperplane

OUTPUT:

Coordinate vector of the projection of point with respect to the basis of linear_part(). In particular, the length of this vector is one less than the ambient space dimension.

EXAMPLES:

```
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = x + 2*y + 3*z - 4
sage: h.linear_part()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1/3]
[ 0 1 -2/3]
sage: p1 = h.linear_part_projection(0); p1
(0, 0)
sage: p2 = h.linear_part_projection([3,4,5]); p2
(8/7, 2/7)
sage: p3 = h.linear_part().basis()
[(1, 0, -1/3),
 (0, 1, -2/3)]
sage: p3 = h.linear_part_projection([1,1,1]); p3
(4/7, 1/7)
```

normal()

Return the normal vector.

OUTPUT:

A vector over the base ring.

EXAMPLES:
orthogonal_projection (point)

Return the orthogonal projection of a point.

INPUT:

- point – vector of the ambient space, or anything that can be converted into one; not necessarily on the hyperplane

OUTPUT:

A vector in the ambient vector space that lies on the hyperplane.

In finite characteristic, a ValueError is raised if the the norm of the hyperplane normal is zero.

EXAMPLES:

```sage
sage: H.<x, y, z> = HyperplaneArrangements(QQ)
sage: h = x + 2*y + 3*z - 4
sage: p1 = h.orthogonal_projection(0); p1
(2/7, 4/7, 6/7)
sage: p1 in h
True
sage: p2 = h.orthogonal_projection([3,4,5]); p2
(10/7, 6/7, 2/7)
sage: p1 in h
True
sage: p3 = h.orthogonal_projection([1,1,1]); p3
(6/7, 5/7, 4/7)
sage: p3 in h
True
```

plot (**kwds)

Plot the hyperplane.

OUTPUT:

A graphics object.

EXAMPLES:

```sage
sage: L.<x, y> = HyperplaneArrangements(QQ)
sage: (x + y - 2).plot() # needs sage.plot
Graphics object consisting of 2 graphics primitives
```

point ()

Return the point closest to the origin.

OUTPUT:

A vector of the ambient vector space. The closest point to the origin in the $L^2$-norm.

In finite characteristic a random point will be returned if the norm of the hyperplane normal vector is zero.
EXAMPLES:

```sage
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = x + 2*y + 3*z - 4
sage: h.point()
(2/7, 4/7, 6/7)
sage: h.point() in h
True
sage: # needs sage.rings.finite_rings
sage: H.<x,y,z> = HyperplaneArrangements(GF(3))
sage: h = 2*x + y + z + 1
sage: h.point()
(1, 0, 0)
sage: h.point().base_ring()
Finite Field of size 3
sage: H.<x,y,z> = HyperplaneArrangements(GF(3))
sage: h = x + y + z + 1
sage: h.point()
(2, 0, 0)
```

**polyhedron(****kwds***)

Return the hyperplane as a polyhedron.

OUTPUT:

A `Polyhedron()` instance.

EXAMPLES:

```sage
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: h = x + 2*y + 3*z - 4
sage: P = h.polyhedron(); P
A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex and 2 lines
sage: P.Hrepresentation()
(An equation (1, 2, 3) x - 4 == 0,)
sage: P.Vrepresentation()
(A line in the direction (0, 3, -2),
A line in the direction (3, 0, -1),
A vertex at (0, 0, 4/3))
```

**primitive**(signed=True)

Return hyperplane defined by primitive equation.

INPUT:

- `signed` – boolean (optional, default: True); whether to preserve the overall sign

OUTPUT:

Hyperplane whose linear expression has common factors and denominators cleared. That is, the same hyperplane (with the same sign) but defined by a rescaled equation. Note that different linear expressions must define different hyperplanes as comparison is used in caching.

If `signed`, the overall rescaling is by a positive constant only.

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: h = -1/3*x + 1/2*y - 1; h
Hyperplane -1/3*x + 1/2*y - 1
sage: h.primitive()
Hyperplane -2*x + 3*y - 6
sage: h == h.primitive()
False
sage: (4*x + 8).primitive()
Hyperplane x + 0*y + 2
sage: (4*x - y - 8).primitive(signed=True) # default
Hyperplane 4*x - y - 8
sage: (4*x - y - 8).primitive(signed=False)
Hyperplane -4*x + y + 8
```

```to_symmetric_space()```

Return self considered as an element in the corresponding symmetric space.

**EXAMPLES:**

```
sage: L.<x, y> = HyperplaneArrangements(QQ)
sage: h = -1/3*x + 1/2*y
sage: h.to_symmetric_space()
-1/3*x + 1/2*y
sage: hp = -1/3*x + 1/2*y - 1
sage: hp.to_symmetric_space()
Traceback (most recent call last):
  ... ValueError: the hyperplane must pass through the origin
```

### 1.4 Affine Subspaces of a Vector Space

An affine subspace of a vector space is a translation of a linear subspace. The affine subspaces here are only used internally in hyperplane arrangements. You should not use them for interactive work or return them to the user.

**EXAMPLES:**

```
sage: from sage.geometry.hyperplane_arrangement.affine_subspace import AffineSubspace
sage: a = AffineSubspace([1,0,0,0], QQ^4)
sage: a.dimension()
4
sage: a.point()
(1, 0, 0, 0)
sage: a.linear_part()
Vector space of dimension 4 over Rational Field
sage: a
Affine space p + W where:
  p = (1, 0, 0, 0)
  W = Vector space of dimension 4 over Rational Field
sage: b = AffineSubspace((1,0,0,0), matrix(QQ, [[1,2,3,4]]).right_kernel())
sage: c = AffineSubspace((0,2,0,0), matrix(QQ, [[0,0,1,2]]).right_kernel())
sage: b.intersection(c)
Affine space p + W where:
  p = (-3, 2, 0, 0)
```

(continues on next page)
\[ W = \text{Vector space of degree 4 and dimension 2 over Rational Field} \]

Basis matrix:
\[
\begin{bmatrix}
1 & 0 & -1 & 1/2 \\
0 & 1 & -2 & 1
\end{bmatrix}
\]

\texttt{sage: b < a}
True

\texttt{sage: c < b}
False

\texttt{sage: A = AffineSubspace([8,38,21,250], VectorSpace(GF(19),4))}

\texttt{sage: A}
Affine space \( p + W \) where:
\[
p = (8, 0, 2, 3) \\
W = \text{Vector space of dimension 4 over Finite Field of size 19}
\]

\textbf{class} \texttt{sage.geometry.hyperplane_arrangement.affine_subspace.AffineSubspace(p, V)}

\texttt{Bases: SageObject}

An affine subspace.

\textbf{INPUT:}

- \( p \) – list/tuple/iterable representing a point on the affine space
- \( V \) – vector subspace

\textbf{OUTPUT:}

Affine subspace parallel to \( V \) and passing through \( p \).

\textbf{EXAMPLES:}

\texttt{sage: from sage.geometry.hyperplane_arrangement.affine_subspace import…
  \_\_\_AffineSubspace
\texttt{sage: a = AffineSubspace([1,0,0,0], VectorSpace(QQ,4))}
\texttt{sage: a}
Affine space \( p + W \) where:
\[
p = (1, 0, 0, 0) \\
W = \text{Vector space of dimension 4 over Rational Field}
\]

\textbf{dimension()}

Return the dimension of the affine space.

\textbf{OUTPUT:}

An integer.

\textbf{EXAMPLES:}

\texttt{sage: from sage.geometry.hyperplane_arrangement.affine_subspace import…
  \_\_\_AffineSubspace
\texttt{sage: a = AffineSubspace([1,0,0,0], VectorSpace(QQ,4))
\texttt{sage: a.dimension()}
4

\textbf{intersection(other)}

Return the intersection of \texttt{self} with \texttt{other}.

\textbf{INPUT:}

- \( \texttt{other} \) – an \texttt{AffineSubspace
A new affine subspace, (or None if the intersection is empty).

**EXAMPLES:**

```python
sage: from sage.geometry.hyperplane_arrangement.affine_subspace import AffineSubspace
sage: V = VectorSpace(QQ,3)
sage: U = V.subspace([[1,0,0], [0,1,0]])
sage: W = V.subspace([[0,1,0], [0,0,1]])
sage: A = AffineSubspace((0,0,0), U)
sage: B = AffineSubspace((1,1,1), W)
sage: A.intersection(B)
Affine space p + W where:
  p = (1, 1, 0)
  W = Vector space of degree 3 and dimension 1 over Rational Field
Basis matrix:
[0 1 0]
sage: C = AffineSubspace((0,0,1), U)
sage: A.intersection(C)
sage: C = AffineSubspace((7,8,9), U.complement())
sage: A.intersection(C)
Affine space p + W where:
  p = (7, 8, 0)
  W = Vector space of degree 3 and dimension 0 over Rational Field
Basis matrix:
[]
sage: A.intersection(C).intersection(B)
```

**linear_part()**

Return the linear part of the affine space.

**OUTPUT:**

A vector subspace of the ambient space.

**EXAMPLES:**

```python
sage: from sage.geometry.hyperplane_arrangement.affine_subspace import AffineSubspace
sage: A = AffineSubspace([2,3,1], matrix(QQ, [[1,2,3]]).right_kernel())
sage: A.linear_part()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[ 1 0 -1/3]
[ 0 1 -2/3]
sage: A.linear_part().ambient_vector_space()
Vector space of dimension 3 over Rational Field
```

**point()**

Return a point \( p \) in the affine space.
OUTPUT:

A point of the affine space as a vector in the ambient space.

EXAMPLES:

```
sage: from sage.geometry.hyperplane_arrangement.affine_subspace import...
    ...
    AffineSubspace
sage: A = AffineSubspace([[2,3,1], VectorSpace(QQ,3))
sage: A.point()
(2, 3, 1)
```

### 1.5 Plotting of Hyperplane Arrangements

**Plot Options:**

Besides the usual plot options (enter `plot?`), the plot command for hyperplane arrangements includes the following:

- `hyperplane_colors` – Color or list of colors, one for each hyperplane (default: equally spread range of hues).
- `hyperplane_labels` – Boolean, 'short', 'long' (default: False). If False, no labels are shown; if ‘short’ or ‘long’, the hyperplanes are given short or long labels, respectively. If True, the hyperplanes are given long labels.
- `label_colors` – Color or list of colors, one for each hyperplane (default: black).
- `label_fontsize` – Size for hyperplane_label font (default: 14). This does not work for 3d plots.
- `label_offsets` – Amount by which labels are offset from `h.point()` for each hyperplane `h`. The format is different for each dimension: if the hyperplanes have dimension 0, the offset can be a single number or a list of numbers, one for each hyperplane; if the hyperplanes have dimension 1, the offset can be a single 2-tuple, or a list of 2-tuples, one for each hyperplane; if the hyperplanes have dimension 2, the offset can be a single 3-tuple or a list of 3-tuples, one for each hyperplane. (Defaults: 0-dim: 0.1, 1-dim: (0,1), 2-dim: (0,0,0.2)).
- `hyperplane_legend` – Boolean, 'short', 'long' (default: 'long'; in 3-d: False). If False, no legend is shown; if True, 'short', or 'long', the legend is shown with the default, long, or short labeling, respectively. (For arrangements of lines or planes, only.)
- `hyperplane_opacities` – A number or list of numbers, one for each hyperplane, between 0 and 1. Only applies to 3d plots.
- `point_sizes` – Number or list of numbers, one for each hyperplane giving the sizes of points in a zero-dimensional arrangement (default: 50).
- `ranges` – Range for the parameters or a list of ranges of parameters, one for each hyperplane, for the parametric plots of the hyperplanes. If a single positive integer `r` is given for `ranges`, then all parameters run from `-r` to `r`. Otherwise, for a line in the plane, the range has the form `[a,b]` (default: `[-3,3]`), and for a plane in 3-space, the range has the form `[[a,b],[c,d]]` (default: `[[[-3,3],[3,3]]`). The ranges are centered around `hyperplane_arrangement.point()`.

**Examples:**

```
sage: H3.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H3([(1,0,0), 0], [(0,0,1), 5])
sage: A.plot(hyperplane_opacities=0.5, hyperplane_labels=True,....:    hyperplane_legend=False)
Graphics3d Object
```

(continues on next page)
sage: c = H3([(1,0,0),0], [(0,0,1),5])
sage: c.plot(ranges=10)
#→ needs sage.plot
Graphics3d Object
sage: c.plot(ranges=[[9.5,10], [-3,3]])
#→ needs sage.plot
Graphics3d Object
sage: c.plot(ranges=[[9.5,10], [-3,3]], [[-6,6], [-5,5]])
#→ needs sage.plot
Graphics3d Object

sage: H2.<s,t> = HyperplaneArrangements(QQ)
sage: h = H2([(1,1),0], [(1,-1),0], [(0,1),2])
sage: h.plot(ranges=20)
#→ needs sage.plot
Graphics object consisting of 3 graphics primitives
sage: h.plot(ranges=[[-1, 10]])
#→ needs sage.plot
Graphics object consisting of 3 graphics primitives
sage: h.plot(ranges=[[-1, 1], [-5, 5], [-1, 10]])
#→ needs sage.plot
Graphics object consisting of 3 graphics primitives

sage: a = hyperplane_arrangements.coordinate(3)
sage: opts = {'hyperplane_colors':['yellow', 'green', 'blue']}
sage: opts['hyperplane_labels'] = True
sage: opts['label_offsets'] = [(0,2,2), (2,0,2), (2,2,0)]
sage: opts['hyperplane_legend'] = False
sage: opts['hyperplane_opacities'] = 0.7
sage: a.plot(**opts)
#→ needs sage.plot
Graphics3d Object
sage: opts['hyperplane_labels'] = 'short'
sage: a.plot(**opts)
#→ needs sage.plot
Graphics3d Object

sage: H.<u> = HyperplaneArrangements(QQ)
sage: pts = H(3*u+4, 2*u+5, 7*u+1)
sage: pts.plot(hyperplane_colors=[yellow,black,blue])
#→ needs sage.plot
Graphics object consisting of 3 graphics primitives
sage: pts.plot(point_sizes=[50,100,200], hyperplane_colors=blue)
#→ needs sage.plot
Graphics object consisting of 3 graphics primitives

sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: a = H(x, y+1, y+2)
sage: a.plot(hyperplane_labels=True, label_colors='blue', label_fontsize=18)
#→ needs sage.plot
Graphics3d Object
sage: a.plot(hyperplane_labels=True, label_colors=['red','green','black'])
#→ needs sage.plot
Graphics3d Object

sage.geometry.hyperplane_arrangement.plot.legend_3d(hyperplane_arrangement, hyperplane_colors, length)
Create plot of a 3d legend for an arrangement of planes in 3-space. The length parameter determines whether short or long labels are used in the legend.

**INPUT:**
- `hyperplane_arrangement` – a hyperplane arrangement
- `hyperplane_colors` – list of colors
- `length` – either 'short' or 'long'

**OUTPUT:**
- A graphics object.

**EXAMPLES:**
```
sage: a = hyperplane_arrangements.semiorder(3)
sage: from sage.geometry.hyperplane_arrangement.plot import legend_3d
sage: legend_3d(a, list(colors.values())[:6], length='long')
# needs sage.combinat sage.plot
Graphics object consisting of 6 graphics primitives
sage: b = hyperplane_arrangements.semiorder(4)
sage: c = b.essentialization()
sage: legend_3d(c, list(colors.values())[:12], length='long')
# needs sage.combinat sage.plot
Graphics object consisting of 12 graphics primitives
sage: legend_3d(c, list(colors.values())[:12], length='short')
# needs sage.combinat sage.plot
Graphics object consisting of 12 graphics primitives
sage: p = legend_3d(c, list(colors.values())[:12], length='short')
# needs sage.combinat sage.plot
sage: p.set_legend_options(ncol=4)
# needs sage.combinat sage.plot
sage: type(p) # needs sage.combinat sage.plot
<class 'sage.plot.graphics.Graphics'>
```

`sage.geometry.hyperplane_arrangement.plot.plot(hyperplane_arrangement, **kwds)`

Return a plot of the hyperplane arrangement.

If the arrangement is in 4 dimensions but inessential, a plot of the essentialization is returned.

**Note:** This function is available as the `plot()` method of hyperplane arrangements. You should not call this function directly, only through the method.

**INPUT:**
- `hyperplane_arrangement` – the hyperplane arrangement to plot
- `**kwds` – plot options: see `sage.geometry.hyperplane_arrangement.plot`.

**OUTPUT:**
A graphics object of the plot.

**EXAMPLES:**
sage: B = hyperplane_arrangements.semiorder(4)
sage: B.plot()  # needs sage.combinat sage.plot
Displaying the essentialization.
Graphics3d Object

sage.geometry.hyperplane_arrangement.plot.plot_hyperplane(hyperplane, **kwds)
Return the plot of a single hyperplane.

INPUT:

• **kwds – plot options: see below

OUTPUT:

A graphics object of the plot.

Plot Options

Besides the usual plot options (enter plot?), the plot command for hyperplanes includes the following:

• hyperplane_label – Boolean value or string (default: True). If True, the hyperplane is labeled with its equation, if a string, it is labeled by that string, otherwise it is not labeled.

• label_color – (Default: 'black') Color for hyperplane_label.

• label_fontsize – Size for hyperplane_label font (default: 14) (does not work in 3d, yet).

• label_offset – (Default: 0-dim: 0.1, 1-dim: (0,1), 2-dim: (0,0,0.2)) Amount by which label is offset from hyperplane.point().

• point_size – (Default: 50) Size of points in a zero-dimensional arrangement or of an arrangement over a finite field.

• ranges – Range for the parameters for the parametric plot of the hyperplane. If a single positive number \( r \) is given for the value of ranges, then the ranges for all parameters are set to \([-r, r] \). Otherwise, for a line in the plane, ranges has the form \([a, b]\) (default: \([-3,3]\)), and for a plane in 3-space, the ranges has the form \([[a, b], [c, d]]\) (default: \([-3,3],[-3,3]\)). (The ranges are centered around hyperplane.point().)

EXAMPLES:

sage: H1.<x> = HyperplaneArrangements(QQ)
sage: a = 3*x + 4
sage: a.plot()  # indirect doctest  # needs sage.plot
Graphics object consisting of 3 graphics primitives

sage: a.plot(point_size=100, hyperplane_label='hello')  # needs sage.plot
Graphics object consisting of 3 graphics primitives

sage: H2.<x,y> = HyperplaneArrangements(QQ)
sage: b = 3*x + 4*y + 5
sage: b.plot()  # needs sage.plot
Graphics object consisting of 2 graphics primitives

sage: b.plot(ranges=(1,5), label_offset=(2,-1))  # needs sage.plot
Graphics object consisting of 2 graphics primitives

(continues on next page)
sage: opts = {'hyperplane_label': True, 'label_color': 'green', 'label_fontsize': 24, 'label_offset': (0,1.5)}
sage: b.plot(**opts)  # needs sage.plot
Graphics object consisting of 2 graphics primitives

sage: # needs sage.plot
sage: H3.<x,y,z> = HyperplaneArrangements(QQ)
sage: c = 2*x + 3*y + 4*z + 5
sage: c.plot()
Graphics3d Object
sage: c.plot(label_offset=(1,0,1), color='green', label_color='red', frame=False)
Graphics3d Object
sage: d = -3*x + 2*y + 2*z + 3
sage: d.plot(opacity=0.8)
Graphics3d Object
sage: e = 4*x + 2*z + 3
sage: e.plot(ranges=[[-1,1],[0,8]], label_offset=(2,2,1), aspect_ratio=1)
Graphics3d Object
2.1 Polyhedra

2.1.1 Library of commonly used, famous, or interesting polytopes

This module gathers several constructors of polytopes that can be reached through \texttt{polytopes}. For example, here is the hypercube in dimension 5:

```
sage: polytopes.hypercube(5)
A 5-dimensional polyhedron in ZZ^5 defined as the convex hull of 32 vertices
```

The following constructions are available:

- \texttt{Birkhoff_polytope()}
- \texttt{associahedron()}
- \texttt{bitruncated_six_hundred_cell()}
- \texttt{buckyball()}
- \texttt{cantellated_one_hundred_twenty_cell()}
- \texttt{cantellated_six_hundred_cell()}
- \texttt{cantitruncated_one_hundred_twenty_cell()}
- \texttt{cantitruncated_six_hundred_cell()}
- \texttt{cross_polytope()}
- \texttt{cube()}
- \texttt{cuboctahedron()}
- \texttt{cyclic_polytope()}
- \texttt{dodecahedron()}
- \texttt{flow_polytope()}
- \texttt{Gosset_3_21()}
- \texttt{grand_antiprism()}
- \texttt{great_rhombicuboctahedron()}
- \texttt{hypercube()}
- \texttt{hypersimplex()}
- \texttt{icosahedron()}
- \texttt{icosidodecahedron()}
- \texttt{Kirkman_icosahedron()}
- \texttt{octahedron()}
- \texttt{omnitruncated_one_hundred_twenty_cell()}
- \texttt{omnitruncated_six_hundred_cell()}
- \texttt{one_hundred_twenty_cell()}
- \texttt{parallelotope()}

continues on next page
class sage.geometry.polyhedron.library.Polytopes
    Bases: object

A class of constructors for commonly used, famous, or interesting polytopes.

**Birkhoff_polytope** (*n*, *backend=None*)

Return the Birkhoff polytope with *n*! vertices.

The vertices of this polyhedron are the (flattened) *n* by *n* permutation matrices. So the ambient vector space has dimension *n*² but the dimension of the polyhedron is (*n* − 1)².

**INPUT:**

- *n* – a positive integer giving the size of the permutation matrices.
- *backend* – the backend to use to create the polytope.

**See also:**

`sage.matrix.matrix2.Matrix.as_sum_of_permutations()` – return the current matrix as a sum of permutation matrices

**EXAMPLES:**

```python
sage: b3 = polytopes.Birkhoff_polytope(3)
sage: b3.f_vector()
(1, 6, 15, 18, 9, 1)
sage: b3.ambient_dim(), b3.dim()
(9, 4)
sage: b3.is_lattice_polytope()
(continues on next page)
```
Combinatorial and Discrete Geometry, Release 10.3

Gosset_3_21 (backend=None)

Return the Gosset 3_21 polytope.

The Gosset 3_21 polytope is a uniform 7-polytope. It has 56 vertices, and 702 facets: 126 3_11 and 576 6-simplex. For more information, see the Wikipedia article 3_21_polytope.

INPUT:

• backend – the backend to use to create the polytope.

EXAMPLES:

```python
sage: g = polytopes.Gosset_3_21(); g
A 7-dimensional polyhedron in ZZ^8 defined as the convex hull of 56 vertices
sage: g.f_vector() # not tested (~16s)
(1, 56, 756, 4032, 10080, 12096, 6048, 702, 1)
```

Kirkman_icosahedron (backend=None)

Return the Kirkman icosahedron.

The Kirkman icosahedron is a 3-polytope with integer coordinates: (±9, ±6, ±6), (±12, ±4, 0), (0, ±12, ±8), (±6, 0, ±12). See [Fe2012] for more information.

INPUT:

• backend – the backend to use to create the polytope.

EXAMPLES:

```python
sage: ki = polytopes.Kirkman_icosahedron()
sage: ki.f_vector()
(1, 20, 38, 20, 1)
sage: ki.volume()
6528
sage: vertices = ki.vertices()
sage: edges = [[vector(edge[0]),vector(edge[1])] for edge in ki.bounded_edges()]
sage: edge_lengths = [norm(edge[0]-edge[1]) for edge in edges]
sage: sorted(set(edge_lengths))
[7, 8, 9, 11, 12, 14, 16]
```

bitruncated_six_hundred_cell (exact=True, backend=None)

Return the bitruncated 600-cell.

2.1. Polyhedra 67
The bitruncated 600-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 3600 vertices. For more information see Wikipedia article Bitruncated 600-cell.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

**INPUT:**
- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` – the backend to use to create the polytope.

**EXAMPLES:**

```sage
sage: polytopes.runcinated_six_hundred_cell(exact=True, backend='normaliz')
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 3600 vertices
```

**buckyball** `(exact=True, base_ring=None, backend=None)`

Return the bucky ball.

The bucky ball, also known as the truncated icosahedron is an Archimedean solid. It has 32 faces and 60 vertices.

**See also:**

`icosahedron()`

**INPUT:**
- `exact` – (boolean, default True) If False use an approximate ring for the coordinates.
- `base_ring` – the ring in which the coordinates will belong to. If it is not provided and exact=True it will be the number field $\mathbb{Q}[\phi]$ where $\phi$ is the golden ratio and if exact=False it will be the real double field.
- `backend` – the backend to use to create the polytope.

**EXAMPLES:**

```sage
sage: bb = polytopes.buckyball()  # long time
sage: bb.f_vector()  # long time
(1, 60, 90, 32, 1)
sage: bb.base_ring()  # long time
Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?
```

A much faster implementation using floating point approximations:

```sage
sage: bb = polytopes.buckyball(exact=False)
```
sage: bb.base_ring()  
# needs sage.groups
Real Double Field

Its facets are 5 regular pentagons and 6 regular hexagons:

```sage
sage: sum(1 for f in bb.facets() if len(f.vertices()) == 5)  
# needs sage.groups
12
```

```sage
sage: sum(1 for f in bb.facets() if len(f.vertices()) == 6)  
# needs sage.groups
20
```

cantellated_one_hundred_twenty_cell (exact=True, backend=None)

Return the cantellated 120-cell.

The cantellated 120-cell is a 4-dimensional 4-uniform polytope in the \( H_4 \) family. It has 3600 vertices. For more information see Wikipedia article Cantellated 120-cell.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend ‘cdd’) returns a numerical inconsistency error, and thus cannot be computed.

**INPUT:**

- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` – the backend to use to create the polytope.

**EXAMPLES:**

```sage
sage: polytopes.cantellated_one_hundred_twenty_cell(backend='normaliz')  
# not tested - long time
```

A 4-dimensional polyhedron in \( \mathbb{A}^4 \) defined as the convex hull of 3600 vertices

cantellated_six_hundred_cell (exact=False, backend=None)

Return the cantellated 600-cell.

The cantellated 600-cell is a 4-dimensional 4-uniform polytope in the \( H_4 \) family. It has 3600 vertices. For more information see Wikipedia article Cantellated 600-cell.

**Warning:** The coordinates are inexact by default. The computation with inexact coordinates (using the backend ‘cdd’) issues a UserWarning on inconsistencies.

**INPUT:**

- `exact` - (boolean, default False) if True use exact coordinates instead of floating point approximations.
- `backend` – the backend to use to create the polytope.

**EXAMPLES:**
It is possible to use the backend 'normaliz' to get an exact representation:

```
sage: polytopes.cantitruncated_six_hundred_cell(exact=True, backend='normaliz')
```

A 4-dimensional polyhedron in AA^4 defined as the convex hull of 3600 vertices.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend `cdd`) returns a numerical inconsistency error, and thus cannot be computed.

**INPUT:**
- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.

**EXAMPLES:**

```
sage: polytopes.cantitruncated_six_hundred_cell(exact=True, backend='normaliz')
```

A 4-dimensional polyhedron in AA^4 defined as the convex hull of 3600 vertices.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend `cdd`) returns a numerical inconsistency error, and thus cannot be computed.

**INPUT:**
- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.
EXAMPLES:

```
sage: polytopes.cantitruncated_six_hundred_cell(exact=True, # not tested - very long time
.....: backend='normaliz')
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 7200 vertices
```

cross_polytope(dim, backend=None)

Return a cross-polytope in dimension dim.

A cross-polytope is a higher dimensional generalization of the octahedron. It is the convex hull of the $2d$ points $(\pm 1, 0, \ldots, 0), (0, \pm 1, \ldots, 0), \ldots, (0, 0, \ldots, \pm 1)$. See the Wikipedia article Cross-polytope for more information.

INPUT:

- dim – integer. The dimension of the cross-polytope.
- backend – the backend to use to create the polytope.

EXAMPLES:

```
sage: four_cross = polytopes.cross_polytope(4)
sage: four_cross.f_vector()
(1, 8, 24, 32, 16, 1)
sage: four_cross.is_simple()
False
```

cube(intervals=None, backend=None)

Return the cube.

The cube is the Platonic solid that is obtained as the convex hull of the eight $\pm 1$ vectors of length 3 (by default). Alternatively, the cube is the product of three intervals from intervals.

See also:

hypercube()

INPUT:

- intervals – list (default=None). It takes the following possible inputs:
  - If the input is None (the default), returns the convex hull of the eight $\pm 1$ vectors of length three.
  - 'zero_one' – (string). Return the 0/1-cube.
  - a list of 3 lists of length 2. The cube will be a product of these three intervals.
- backend – the backend to use to create the polytope.

OUTPUT:

A cube as a polyhedron object.

EXAMPLES:

Return the $\pm 1$-cube:

```
sage: c = polytopes.cube()
sage: c
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: c.f_vector()
(1, 8, 12, 6, 1)
```

(continues on next page)
Return the 0/1-cube:

```python
sage: cc = polytopes.cube(intervals = 'zero_one')
sage: cc.vertices_list()
[[1, 0, 0],
 [1, 1, 0],
 [1, 1, 1],
 [1, 0, 1],
 [0, 0, 1],
 [0, 0, 0],
 [0, 1, 0],
 [0, 1, 1]]
```

cuboctahedron (backend=None)

Return the cuboctahedron.

The cuboctahedron is an Archimedean solid with 12 vertices and 14 faces dual to the rhombic dodecahedron. It can be defined as the convex hull of the twelve vertices \((0, \pm 1, \pm 1), (\pm 1, 0, \pm 1)\) and \((\pm 1, \pm 1, 0)\). For more information, see the Wikipedia article Cuboctahedron.

**INPUT:**

- `backend` – the backend to use to create the polytope.

**See also:**

`rhombic_dodecahedron()`

**EXAMPLES:**

```python
sage: co = polytopes.cuboctahedron()
sage: co.f_vector()
(1, 12, 24, 14, 1)
```

Its facets are 8 triangles and 6 squares:

```python
sage: sum(1 for f in co.facets() if len(f.vertices()) == 3)
8
sage: sum(1 for f in co.facets() if len(f.vertices()) == 4)
6
```

Some more computation:

```python
sage: co.volume()
20/3
sage: co.ehrhart_polynomial()  # optional - latte_int
20/3*t^3 + 8*t^2 + 10/3*t + 1
```

cyclic_polytope (dim, n, base_ring=Rational Field, backend=None)

Return a cyclic polytope.

A cyclic polytope of dimension `dim` with `n` vertices is the convex hull of the points \((t, t^2, \ldots, t^{dim})\) with \(t \in \{0, 1, \ldots, n - 1\}\). For more information, see the Wikipedia article Cyclic_polytope.
INPUT:

• \texttt{dim} – positive integer. the dimension of the polytope.
• \texttt{n} – positive integer. the number of vertices.
• \texttt{base\_ring} – either \texttt{QQ} (default) or \texttt{RDF}.
• \texttt{backend} – the backend to use to create the polytope.

EXAMPLES:

```python
sage: c = polytopes.cyclic_polytope(4,10)
sage: c.f_vector()
(1, 10, 45, 70, 35, 1)
```

\textbf{dodecahedron} \texttt{(exact=True, base\_ring=None, backend=None)}

Return a dodecahedron.

The dodecahedron is the Platonic solid dual to the \texttt{icosahedron()}.  

INPUT:

• \texttt{exact} – (boolean, default \texttt{True}) If \texttt{False} use an approximate ring for the coordinates.
• \texttt{base\_ring} – (optional) the ring in which the coordinates will belong to. Note that this ring must contain \(\sqrt{5}\). If it is not provided and \texttt{exact=True} it will be the number field \(\mathbb{Q}[\sqrt{5}]\) and if \texttt{exact=False} it will be the real double field.
• \texttt{backend} – the backend to use to create the polytope.

EXAMPLES:

```python
sage: # needs sage.groups sage.rings.number_field
sage: d12 = polytopes.dodecahedron()
sage: d12.f_vector()
(1, 20, 30, 12, 1)
sage: d12.volume()
-176*sqrt5 + 400
sage: numerical_approx(_)
6.45203596003699
sage: d12 = polytopes.dodecahedron(exact=False)  #...
# needs sage.groups
sage: d12.base_ring()
# needs sage.groups
Real Double Field
```

Here is an error with a field that does not contain \(\sqrt{5}\):

```python
sage: polytopes.dodecahedron(base\_ring=QQ)  #...
# needs sage.groups sage.symbolic
Traceback (most recent call last):
...
TypeError: unable to convert 1/4*sqrt(5) + 1/4 to a rational
```

\textbf{static edge\_polytope} \texttt{(backend=None)}

Return the edge polytope of \texttt{self}.

The edge polytope (EP) of a Graph on \(n\) vertices is the polytope in \(Z^n\) defined as the convex hull of \(e_i + e_j\) for each edge \((i,j)\). Here \(e_1, \ldots, e_n\) denotes the standard basis.

INPUT:

2.1. Polyhedra
Combinatorial and Discrete Geometry, Release 10.3

- **backend** – string or None (default); the backend to use; see `sage.geometry.polyhedron.constructor.Polyhedron()

**EXAMPLES:**

The EP of a 4-cycle is a square:

```
sage: G = graphs.CycleGraph(4)
sage: P = G.edge_polytope(); P  # needs sage.geometry.polyhedron
A 2-dimensional polyhedron in Z^4 defined as the convex hull of 4 vertices
```

The EP of a complete graph on 4 vertices is cross polytope:

```
sage: G = graphs.CompleteGraph(4)
sage: P = G.edge_polytope(); P  # needs sage.geometry.polyhedron
A 3-dimensional polyhedron in Z^4 defined as the convex hull of 6 vertices
```

sage: P.is_combinatorially_isomorphic(polytopes.cross_polytope(3))  # needs sage.geometry.polyhedron
```
```true```

The EP of a graph is isomorphic to the subdirect sum of its connected components EPs:

```
sage: n = randint(3, 6)
sage: G1 = graphs.RandomGNP(n, 0.2)  # needs networkx
sage: n = randint(3, 6)
sage: G2 = graphs.RandomGNP(n, 0.2)  # needs networkx
sage: G = G1.disjoint_union(G2)  # needs networkx
sage: P = G.edge_polytope()  # needs networkx sage.geometry.polyhedron
```

sage: P1 = G1.edge_polytope()  # needs networkx sage.geometry.polyhedron
sage: P2 = G2.edge_polytope()  # needs networkx sage.geometry.polyhedron
sage: P1.is_combinatorially_isomorphic(P2)  # needs networkx sage.geometry.polyhedron
```
```true```

All trees on $n$ vertices have isomorphic EPs:

```
sage: n = randint(4, 10)
sage: G1 = graphs.RandomTree(n)
sage: G2 = graphs.RandomTree(n)
sage: P1 = G1.edge_polytope()  # needs sage.geometry.polyhedron
sage: P2 = G2.edge_polytope()  # needs sage.geometry.polyhedron
```

sage: P1.is_combinatorially_isomorphic(P2)  # needs sage.geometry.polyhedron
```
```true```

However, there are still many different EPs:

```
sage: len(list(graphs(5)))
34
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```python
sage: polys = []
sage: for G in graphs(5):
    # needs sage.geometry.polyhedron
    ....:     P = G.edge_polytope()
    ....:     for P1 in polys:
    ....:         if P.is_combinatorially_isomorphic(P1):
    ....:             break
    ....:     else:
    ....:         polys.append(P)
sage: len(polys)
19
```

**static flow_polytope** *(edges=None, ends=None, backend=None)*

Return the flow polytope of a digraph.

The flow polytope of a directed graph is the polytope consisting of all nonnegative flows on the graph with a given set \( S \) of sources and a given set \( T \) of sinks.

A flow on a directed graph \( G \) with a given set \( S \) of sources and a given set \( T \) of sinks means an assignment of a nonnegative real to each edge of \( G \) such that the flow is conserved in each vertex outside of \( S \) and \( T \), and there is a unit of flow entering each vertex in \( S \) and a unit of flow leaving each vertex in \( T \). These flows clearly form a polytope in the space of all assignments of reals to the edges of \( G \).

The polytope is empty unless the sets \( S \) and \( T \) are equinumerous.

By default, \( S \) is taken to be the set of all sources (i.e., vertices of indegree 0) of \( G \), and \( T \) is taken to be the set of all sinks (i.e., vertices of outdegree 0) of \( G \). If a different choice of \( S \) and \( T \) is desired, it can be specified using the optional `ends` parameter.

The polytope is returned as a polytope in \( \mathbb{R}^m \), where \( m \) is the number of edges of the digraph \( self \). The \( k \)-th coordinate of a point in the polytope is the real assigned to the \( k \)-th edge of \( self \). The order of the edges is the one returned by `self.edges(sort=True)`. If a different order is desired, it can be specified using the optional `edges` parameter.

The faces and volume of these polytopes are of interest. Examples of these polytopes are the Chan-Robbins-Yuen polytope and the Pitman-Stanley polytope [PS2002].

**INPUT:**

- `edges` – list (default: None); a list of edges of `self`. If not specified, the list of all edges of `self` is used with the default ordering of `self.edges(sort=True)`. This determines which coordinate of a point in the polytope will correspond to which edge of `self`. It is also possible to specify a list which contains not all edges of `self`; this results in a polytope corresponding to the flows which are 0 on all remaining edges. Notice that the edges entered here must be in the precisely same format as outputted by `self.edges()`; so, if `self.edges()` outputs an edge in the form `(1, 3, None)`, then `(1, 3)` will not do!

- `ends` – (optional, default: `(self.sources(), self.sinks())`) a pair \((S, T)\) of an iterable \( S \) and an iterable \( T \).

- `backend` – string or None (default); the backend to use; see `sage.geometry.polyhedron.constructor.Polyhedron()`

**Note:** Flow polytopes can also be built through the `polytopes.<tab>` object:

2.1. Polyhedra 75
EXAMPLES:

A commutative square:

```python
sage: G = DiGraph({1: [2, 3], 2: [4], 3: [4]})
sage: fl = G.flow_polytope(); fl
# needs sage.geometry.polyhedron
A 1-dimensional polyhedron in QQ^4 defined as the convex hull of 2 vertices
sage: fl.vertices()
# needs sage.geometry.polyhedron
(A vertex at (0, 1, 1, 0), A vertex at (1, 0, 0, 1))
```

Using a different order for the edges of the graph:

```python
sage: ordered_edges = G.edges(sort=True, key=lambda x: x[0] - x[1])
sage: fl = G.flow_polytope(edges=ordered_edges); fl
# needs sage.geometry.polyhedron
A 1-dimensional polyhedron in QQ^4 defined as the convex hull of 2 vertices
sage: fl.vertices()
# needs sage.geometry.polyhedron
(A vertex at (0, 1, 1, 0), A vertex at (1, 0, 0, 1))
```

A tournament on 4 vertices:

```python
sage: H = digraphs.TransitiveTournament(4)
sage: fl = H.flow_polytope(); fl
# needs sage.geometry.polyhedron
A 3-dimensional polyhedron in QQ^6 defined as the convex hull of 4 vertices
sage: fl.vertices()
# needs sage.geometry.polyhedron
(A vertex at (0, 0, 1, 0, 0, 0),
 A vertex at (0, 1, 0, 0, 0, 1),
 A vertex at (1, 0, 0, 0, 1, 0),
 A vertex at (1, 0, 0, 1, 0, 1))
```

Restricting to a subset of the edges:

```python
sage: fl = H.flow_polytope(edges=[(0, 1, None), (1, 2, None),
 ....: (2, 3, None), (0, 3, None)]); fl
# needs sage.geometry.polyhedron
A 1-dimensional polyhedron in QQ^4 defined as the convex hull of 2 vertices
sage: fl.vertices()
# needs sage.geometry.polyhedron
(A vertex at (0, 0, 0, 1), A vertex at (1, 1, 1, 0))
```

Using a different choice of sources and sinks:

```python
sage: # needs sage.geometry.polyhedron
sage: fl = H.flow_polytope(ends=[[1], [3]]); fl
```

(continues on next page)
A 1-dimensional polyhedron in $\mathbb{Q}^6$ defined as the convex hull of 2 vertices

```
sage: fl.vertices()
(A vertex at (0, 0, 0, 1, 0, 1), A vertex at (0, 0, 0, 0, 1, 0))
```

```
sage: fl = H.flow_polytope(ends=[(0, 1), [3]]); fl
The empty polyhedron in $\mathbb{Q}^6$
```

```
sage: fl = H.flow_polytope(ends=[(3), [0]]); fl
The empty polyhedron in $\mathbb{Q}^6$
```

```
sage: fl = H.flow_polytope(ends=[(0, 1), (2, 3)]); fl
A 3-dimensional polyhedron in $\mathbb{Q}^6$ defined as the convex hull of 5 vertices
```

```
sage: fl.vertices()
(A vertex at (0, 0, 1, 1, 0, 0),
 A vertex at (0, 1, 0, 0, 1, 0),
 A vertex at (1, 0, 0, 2, 0, 1),
 A vertex at (1, 0, 0, 1, 1, 0),
 A vertex at (0, 1, 0, 1, 0, 1))
```

```
sage: fl = H.flow_polytope(edges=[(0, 1, None), (1, 2, None),
....: (2, 3, None), (0, 2, None),
....: (1, 3, None)],
....: ends=[(0, 1), (2, 3)]); fl
A 2-dimensional polyhedron in $\mathbb{Q}^5$ defined as the convex hull of 4 vertices
```

```
sage: fl.vertices()
(A vertex at (0, 0, 0, 1, 1),
 A vertex at (1, 2, 1, 0, 0),
 A vertex at (1, 1, 0, 0, 1),
 A vertex at (0, 1, 1, 1, 0))
```

A digraph with one source and two sinks:

```
sage: Y = DiGraph({1: [2], 2: [3, 4]})
sage: Y.flow_polytope()
needs sage.geometry.polyhedron
The empty polyhedron in $\mathbb{Q}^3$
```

A digraph with one vertex and no edge:

```
sage: Z = DiGraph({1: []})
sage: Z.flow_polytope()
needs sage.geometry.polyhedron
A 0-dimensional polyhedron in $\mathbb{Q}^0$ defined as the convex hull of 1 vertex
```

A digraph with multiple edges (github issue #28837):

```
sage: G = DiGraph([(0, 1), (0, 1)], multiedges=True); G
Multi-digraph on 2 vertices
sage: P = G.flow_polytope(); P
needs sage.geometry.polyhedron
A 1-dimensional polyhedron in $\mathbb{Q}^2$ defined as the convex hull of 2 vertices
```

```
sage: P.vertices()
needs sage.geometry.polyhedron
(A vertex at (1, 0), A vertex at (0, 1))
sage: P.lines()
needs sage.geometry.polyhedron
()`
generalized_permutahedron(coxeter_type, point=None, exact=True, regular=False, backend=None)

Return the generalized permutahedron of type `coxeter_type` as the convex hull of the orbit of `point`
in the fundamental cone.

This generalized permutahedron lies in the vector space used in the geometric representation, that is, in the
default case, the dimension of generalized permutahedron equals the dimension of the space.

**INPUT:**

- `coxeter_type` – a Coxeter type; given as a pair `[type,rank]`, where type is a letter and rank is the
  number of generators.
- `point` – a list (default: None); a point given by its coordinates in the weight basis. If None is given,
  the point `(1,1,1,...)` is used.
- `exact` - (boolean, default True) if False use floating point approximations instead of exact coordi-
nates
- `regular` – boolean (default: False); whether to apply a linear transformation making the vertex
  figures isometric.
- `backend` – backend to use to create the polytope; (default: None)

**EXAMPLES:**

```python
sage: perm_a3 = polytopes.generalized_permutahedron(['A',3]); perm_a3
# needs sage.combinat
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 24 vertices
```

You can put the starting point along the hyperplane of the first generator:

```python
sage: perm_a3_011 = polytopes.generalized_permutahedron(['A',3], [0,1,1])
sage: perm_a3_011
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 12 vertices
sage: perm_a3_110 = polytopes.generalized_permutahedron(['A',3], [1,1,0])
sage: perm_a3_110
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 12 vertices
sage: perm_a3_110.is_combinatorially_isomorphic(perm_a3_011)
True
sage: perm_a3_101 = polytopes.generalized_permutahedron(['A',3], [1,0,1])
sage: perm_a3_101
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 12 vertices
sage: perm_a3_110.is_combinatorially_isomorphic(perm_a3_101)
False
sage: perm_a3_011.f_vector()
(1, 12, 18, 8, 1)
sage: perm_a3_101.f_vector()
(1, 12, 24, 14, 1)
```

The usual output does not necessarily give a polyhedron with isometric vertex figures:

```python
sage: perm_a2 = polytopes.generalized_permutahedron(['A',2]); perm_a2
# needs sage.combinat
```

(continues on next page)
A vertex at (1, 0),
A vertex at (1, 1))

It works also with Coxeter types that lead to non-rational coordinates:

```
sage: perm_b3 = polytopes.generalized_permutahedron(['B',3])  # long...
˓→time, needs sage.combinat sage.rings.number_field
sage: perm_b3  # long...
˓→time, needs sage.combinat sage.rings.number_field
A 3-dimensional polyhedron in
(Number Field in a with defining polynomial x^2 - 2 with a = 1.
˓→414213562373095?)^3
defined as the convex hull of 48 vertices
```

Setting regular=True applies a linear transformation to get isometric vertex figures and the result is inscribed. This cannot be done using rational coordinates. We first do the computations using floating point approximations (RDF):

```
sage: perm_a2_inexact = polytopes.generalized_permutahedron(
˓→[A,2], exact=False)
sage: sorted(perm_a2_inexact.vertices())  #...
˓→needs sage.combinat
[A vertex at (-1.0, -1.0),
A vertex at (-1.0, 0.0),
A vertex at (0.0, -1.0),
A vertex at (0.0, 1.0),
A vertex at (1.0, 0.0),
A vertex at (1.0, 1.0)]
```

```
sage: perm_a2_inexact_reg = polytopes.generalized_permutahedron(
˓→[A,2], exact=False, regular=True)
sage: sorted(perm_a2_inexact_reg.vertices())  #...
˓→needs sage.combinat
[A vertex at (-1.0, 0.0),
A vertex at (-0.5, -0.8660254038),
A vertex at (-0.5, 0.8660254038),
A vertex at (0.5, -0.8660254038),
A vertex at (0.5, 0.8660254038),
A vertex at (1.0, 0.0)]
```

We can do the same computation using exact arithmetic with the field AA:

```
sage: perm_a2_reg = polytopes.generalized_permutahedron(
˓→[A,2], regular=True)
sage: V = sorted(perm_a2_reg.vertices()); V  # random
˓→needs sage.combinat sage.rings.number_field
[A vertex at (-1, 0),
A vertex at (-1/2, -0.866025403784439?),
A vertex at (-1/2, 0.866025403784439?),
A vertex at (1/2, -0.866025403784439?),
A vertex at (1/2, 0.866025403784439?),
A vertex at (1.000000000000000?, 0.?e-18)]
```

Even though the numbers look like floating point approximations, the computation is actually exact. We can

2.1. Polyhedra
cleanup the display a bit using `exactify`:

```python
clean up the display a bit using `exactify`:
```sage: for v in V:
    # needs sage.combinat sage.rings.number_field
    ....: for x in v:
    ....:     x.exactify()
sage: V
    # needs sage.combinat sage.rings.number_field
    [A vertex at (-1, 0),
     A vertex at (-1/2, -0.866025403784439?),
     A vertex at (-1/2, 0.866025403784439?),
     A vertex at (1/2, -0.866025403784439?),
     A vertex at (1/2, 0.866025403784439?),
     A vertex at (1, 0)]
sage: perm_a2_reg.is_inscribed()
    # needs sage.combinat sage.rings.number_field
    True
```

Larger examples take longer:

```python
Larger examples take longer:
```sage: # needs sage.combinat sage.rings.number_field
sage: perm_a3_reg = polytopes.generalized_permutahedron( # long time
    ....: ['A', 3], regular=True); perm_a3_reg
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 24 vertices
sage: perm_a3_reg.is_inscribed() # long time
    True
sage: perm_b3_reg = polytopes.generalized_permutahedron( # long time
    ....: ['B', 3], regular=True); perm_b3_reg
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 48 vertices
```

It is faster with the backend `'number_field'`, which internally uses an embedded number field instead of doing the computations directly with the base ring (AA):

```python
It is faster with the backend `'number_field'`, which internally uses an embedded number field instead of doing the computations directly with the base ring (AA):
```sage: # needs sage.combinat sage.rings.number_field
sage: perm_a3_reg_nf = polytopes.generalized_permutahedron( # long time
    ....: ['A', 3], regular=True, backend='number_field'); perm_a3_reg_nf
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 24 vertices
sage: perm_a3_reg_nf.is_inscribed() # long time
    True
sage: perm_b3_reg_nf = polytopes.generalized_permutahedron( # long time
    ....: ['B', 3], regular=True, backend='number_field'); perm_b3_reg_nf
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 48 vertices
```

It is even faster with the backend `'normaliz'`:

```python
It is even faster with the backend `'normaliz'`:
```sage: # optional - pynormaliz, needs sage.combinat sage.rings.number_field
sage: perm_a3_reg_norm = polytopes.generalized_permutahedron( # long time
    ....: ['A', 3], regular=True, backend='normaliz'); perm_a3_reg_norm
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 24 vertices
sage: perm_a3_reg_norm.is_inscribed() # long time
    True
sage: perm_b3_reg_norm = polytopes.generalized_permutahedron( # long time
    ....: ['B', 3], regular=True, backend='normaliz'); perm_b3_reg_norm
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 48 vertices
```

The speedups from using backend `'normaliz'` allow us to go even further:
sage: perm_h3 = polytopes.generalized_permutahedron(["H",3], backend='normaliz'); perm_h3
A 3-dimensional polyhedron defined as the convex hull of 120 vertices

sage: perm_f4 = polytopes.generalized_permutahedron(["F",4], backend='normaliz'); perm_f4
A 4-dimensional polyhedron defined as the convex hull of 1152 vertices

See also:

- permutahedron()
- permutahedron()

grand_antiprism(exact=True, backend=None, verbose=False)

Return the grand antiprism.

The grand antiprism is a 4-dimensional non-Wythoffian uniform polytope. The coordinates were taken from http://eusebeia.dyndns.org/4d/gap. For more information, see the Wikipedia article Grand_antiprism.

Warning: The coordinates are exact by default. The computation with exact coordinates is not as fast as with floating point approximations. If you find this method to be too slow, consider using floating point approximations.

INPUT:

- exact - (boolean, default True) if False use floating point approximations instead of exact coordinates
- backend – the backend to use to create the polytope.

EXAMPLES:

sage: gap = polytopes.grand_antiprism() # not tested - very long time
sage: gap
A 4-dimensional polyhedron in (Number Field in sqrt5 with defining polynomial x^2 - 5 with sqrt5 = 2.236067977499790?)^4 defined as the convex hull of 100 vertices

Computation with the backend 'normaliz' is instantaneous:

sage: gap_norm = polytopes.grand_antiprism(backend='normaliz') # optional - pynormaliz, needs sage.rings.number_field
sage: gap_norm
A 4-dimensional polyhedron in (Number Field in sqrt5 with defining polynomial x^2 - 5 with sqrt5 = 2.236067977499790?)^4 defined as the convex hull of 100 vertices

Computation with approximated coordinates is also faster, but inexact:
sage: gap = polytopes.grand_antiprism(exact=False)  # random
sage: gap
A 4-dimensional polyhedron in RDF^4 defined as the convex hull of 100 vertices
sage: gap.f_vector()
(1, 100, 500, 720, 320, 1)
sage: len(list(gap.bounded_edges()))
500

**great_rhombicuboctahedron** (*exact*=*True*, *base_ring*=*None*, *backend*=*None*)

Return the great rhombicuboctahedron.

The great rhombicuboctahedron (or truncated cuboctahedron) is an Archimedean solid with 48 vertices and 26 faces. For more information see the [Wikipedia article](https://en.wikipedia.org/wiki/Truncated_cuboctahedron).

**INPUT:**

- **exact** – (boolean, default True) If False use an approximate ring for the coordinates.
- **base_ring** – the ring in which the coordinates will belong to. If it is not provided and exact=True it will be a the number field \(\mathbb{Q}[\phi]\) where \(\phi\) is the golden ratio and if exact=False it will be the real double field.
- **backend** – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: gr = polytopes.great_rhombicuboctahedron()  # long time  #...
needs sage.rings.number_field
sage: gr.f_vector()  # long time  #...
needs sage.rings.number_field
(1, 48, 72, 26, 1)
```

A faster implementation is obtained by setting exact=False:

```
sage: gr = polytopes.great_rhombicuboctahedron(exact=False)
sage: gr.f_vector()
(1, 48, 72, 26, 1)
```

Its facets are 4 squares, 8 regular hexagons and 6 regular octagons:

```
sage: sum(1 for f in gr.facets() if len(f.vertices()) == 4)
12
sage: sum(1 for f in gr.facets() if len(f.vertices()) == 6)
8
sage: sum(1 for f in gr.facets() if len(f.vertices()) == 8)
6
```

**hypercube** (*dim*, *intervals*=*None*, *backend*=*None*)

Return a hypercube of the given dimension.

The *dim*-dimensional hypercube is by default the convex hull of the \(2^{dim} \pm 1\) vectors of length *dim*. Alternatively, it is the product of *dim* line segments given in the *intervals*. For more information see the [Wikipedia article](https://en.wikipedia.org/wiki/Hypercube).

**INPUT:**

- **dim** – integer. The dimension of the hypercube.
- **intervals** – (default = None). It takes the following possible inputs:
  - If None (the default), it returns the \(\pm 1\)-cube of dimension *dim*. 

Combinatorial and Discrete Geometry, Release 10.3

- 'zero_one' – (string). Return the 0/1-cube.
- a list of length dim. Its elements are pairs of numbers \((a, b)\) with \(a < b\). The cube will be the product of these intervals.

- backend – the backend to use to create the polytope.

EXAMPLES:

Create the ±1-hypercube of dimension 4:

```sage
sage: four_cube = polytopes.hypercube(4)
sage: four_cube.is_simple()
True
sage: four_cube.base_ring()
Integer Ring
sage: four_cube.volume()
16
sage: four_cube.ehrhart_polynomial()  # optional - latte_int
16*t^4 + 32*t^3 + 24*t^2 + 8*t + 1
```

Return the 0/1-hypercube of dimension 4:

```sage
sage: z_cube = polytopes.hypercube(4, intervals='zero_one')
sage: z_cube.vertices()[0]
A vertex at (1, 0, 1, 1)
sage: z_cube.is_simple()
True
sage: z_cube.base_ring()
Integer Ring
sage: z_cube.volume()
1
sage: z_cube.ehrhart_polynomial()  # optional - latte_int
t^4 + 4*t^3 + 6*t^2 + 4*t + 1
```

Return the 4-dimensional combinatorial cube that is the product of \([0,3]^4\):

```sage
t_cube = polytopes.hypercube(4, intervals=[[0,3]]*4)
```

Checking that t_cube is three times the previous 0/1-cube:

```sage
t_cube == 3 * z_cube
True
```

**hypersimplex** \((\text{dim}, k, \text{project}=\text{False}, \text{backend}=\text{None})\)

Return the hypersimplex in dimension \(\text{dim}\) and parameter \(k\).

The hypersimplex \(\Delta_{d,k}\) is the convex hull of the vertices made of \(k\) ones and \(d - k\) zeros. It lies in the \(d - 1\) hyperplane of vectors of sum \(k\). If you want a projected version to \(\mathbb{R}^{d-1}\) (with floating point coordinates) then set \text{project}=True in the options.

See also:

**simplex()**

INPUT:

- \text{dim} – the dimension
- \text{n} – the numbers \((1, \ldots, n)\) are permuted
• project – (boolean, default False) if True, the polytope is (isometrically) projected to a vector space of dimension dim-1. This operation turns the coordinates into floating point approximations and corresponds to the projection given by the matrix from zero_sum_projection().

• backend – the backend to use to create the polytope.

EXAMPLES:

```python
sage: # needs sage.combinat
sage: h_4_2 = polytopes.hypersimplex(4, 2)
sage: h_4_2
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 6 vertices
sage: h_4_2.f_vector()
(1, 6, 12, 8, 1)
sage: h_4_2.ehrhart_polynomial()  # optional ~-
→latte_int
2/3*t^3 + 2*t^2 + 7/3*t + 1
sage: TestSuite(h_4_2).run()

sage: # needs sage.combinat
sage: h_7_3 = polytopes.hypersimplex(7, 3, project=True)
sage: h_7_3
A 6-dimensional polyhedron in RDF^6 defined as the convex hull of 35 vertices
sage: h_7_3.f_vector()
(1, 35, 210, 350, 245, 84, 14, 1)
sage: TestSuite(h_7_3).run(skip=["_test_pyramid", "_test_lawrence"])
```

icosahedron (exact=True, base_ring=None, backend=None)

Return an icosahedron with edge length 1.

The icosahedron is one of the Platonic solids. It has 20 faces and is dual to the dodecahedron().

INPUT:

• exact – (boolean, default True) If False use an approximate ring for the coordinates.

• base_ring – (optional) the ring in which the coordinates will belong to. Note that this ring must contain \(\sqrt{5}\). If it is not provided and exact=True it will be the number field \(\mathbb{Q}[\sqrt{5}]\) and if exact=False it will be the real double field.

• backend – the backend to use to create the polytope.

EXAMPLES:

```python
sage: ico = polytopes.icosahedron()  #...
→needs sage.rings.number_field
sage: ico.f_vector()  #...
→needs sage.rings.number_field
(1, 12, 30, 20, 1)
sage: ico.volume()  #...
→needs sage.rings.number_field
5/12*sqrt5 + 5/4

Its non exact version:

```python
sage: ico = polytopes.icosahedron(exact=False)  #...
→needs sage.groups
sage: ico.base_ring()  #...
→needs sage.groups
Real Double Field
```

(continues on next page)
A version using \( AA < \text{sage.rings.qqbar.AlgebraicRealField} \): 

\begin{verbatim}
sage: ico = polytopes.icosahedron(base_ring=AA)  # long time
˓→ needs sage.groups, sage.rings.number_field
sage: ico.base_ring()
˓→ needs sage.groups, sage.rings.number_field
Algebraic Real Field
sage: ico.volume()
˓→ needs sage.groups, sage.rings.number_field
2.181694990624913?
\end{verbatim}

Note that if base ring is provided it must contain the square root of 5. Otherwise you will get an error:

\begin{verbatim}
sage: polytopes.icosahedron(base_ring=QQ)  #...
˓→ needs sage.symbolic
Traceback (most recent call last):
...
TypeError: unable to convert 1/4*sqrt(5) + 1/4 to a rational
\end{verbatim}

\textbf{icosidodecahedron (exact=True, backend=None)}

Return the icosidodecahedron.

The Icosidodecahedron is a polyhedron with twenty triangular faces and twelve pentagonal faces. For more information see the Wikipedia article Icosidodecahedron.

INPUT:

- \texttt{exact} – (boolean, default True) If False use an approximate ring for the coordinates.
- \texttt{backend} – the backend to use to create the polytope.

EXAMPLES:

\begin{verbatim}
sage: id = polytopes.icosidodecahedron()  #...
˓→ needs sage.groups, sage.rings.number_field
sage: id.f_vector()
˓→ needs sage.groups, sage.rings.number_field
(1, 30, 60, 32, 1)
\end{verbatim}

\textbf{icosidodecahedron\textunderscore v2 (exact=True, base\_ring=None, backend=None)}

Return the icosidodecahedron.

The icosidodecahedron is an Archimedean solid. It has 32 faces and 30 vertices. For more information, see the Wikipedia article Icosidodecahedron.

INPUT:

- \texttt{exact} – (boolean, default True) If False use an approximate ring for the coordinates.
- \texttt{base\_ring} – the ring in which the coordinates will belong to. If it is not provided and \texttt{exact=True} it will be a the number field \( \mathbb{Q}[\phi] \) where \( \phi \) is the golden ratio and if \texttt{exact=False} it will be the real double field.
- \texttt{backend} – the backend to use to create the polytope.

EXAMPLES:
sage: id = polytopes.icosidodecahedron_V2() # long time - 6secs
sage: id.f_vector() # long time
(1, 30, 60, 32, 1)
sage: id.base_ring() # long time
Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?

A much faster implementation using floating point approximations:

sage: id = polytopes.icosidodecahedron_V2(exact=False)
sage: id.f_vector()
(1, 30, 60, 32, 1)
sage: id.base_ring()
Real Double Field

Its facets are 20 triangles and 12 regular pentagons:

sage: sum(1 for f in id.facets() if len(f.vertices()) == 3)
20
sage: sum(1 for f in id.facets() if len(f.vertices()) == 5)
12

octahedron (backend=None)

Return the octahedron.

The octahedron is a Platonic solid with 6 vertices and 8 faces dual to the cube. It can be defined as the convex hull of the six vertices (0, 0, ±1), (±1, 0, 0) and (0, ±1, 0). For more information, see the Wikipedia article Octahedron.

INPUT:

- backend – the backend to use to create the polytope.

EXAMPLES:

sage: co = polytopes.octahedron()
sage: co.f_vector()
(1, 6, 12, 8, 1)

Its facets are 8 triangles:

sage: sum(1 for f in co.facets() if len(f.vertices()) == 3)
8

Some more computation:

sage: co.volume()
4/3
sage: co.ehrhart_polynomial() # optional - latte_int
4/3*t^3 + 2*t^2 + 8/3*t + 1

omnitruncated_one_hundred_twenty_cell (exact=True, backend=None)

Return the omnitruncated 120-cell.

The omnitruncated 120-cell is a 4-dimensional 4-uniform polytope in the \( H_4 \) family. It has 14400 vertices. For more information see Wikipedia article Omnitruncated 120-cell.
Warning: The coordinates are exact by default. The computation with inexact coordinates (using the
backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

INPUT:

- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.

EXAMPLES:

```sage
polytopes.omnitruncated_one_hundred_twenty_cell(backend='normaliz') # not tested - very long time ~10min
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 14400...
```

`omnitruncated_six_hundred_cell (exact=True, backend=None)`

Return the omnitruncated 120-cell.

The omnitruncated 120-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 14400 vertices. For more information see Wikipedia article Omnitruncated 120-cell.

Warning: The coordinates are exact by default. The computation with inexact coordinates (using the
backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

INPUT:

- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.

EXAMPLES:

```sage
polytopes.omnitruncated_one_hundred_twenty_cell(backend='normaliz') # not tested - very long time ~10min
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 14400...
```

`one_hundred_twenty_cell (exact=True, backend=None, construction='coxeter')`

Return the 120-cell.

The 120-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 600 vertices and 120 facets. For more information see Wikipedia article 120-cell.

Warning: The coordinates are exact by default. The computation with inexact coordinates (using the
backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

INPUT:

- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.
• construction – the construction to use (string, default ‘coxeter’); the other possibility is ‘as_permutahedron’.

EXAMPLES:
The classical construction given by Coxeter in [Cox1969] is given by:

```
sage: polytopes.one_hundred_twenty_cell() # not tested ~-
   → long time ~15 sec.
A 4-dimensional polyhedron in (Number Field in sqrt5 with defining
polynomial x^2 - 5 with sqrt5 = 2.236067977499790?)^4 defined as
the convex hull of 600 vertices
```

The 'normaliz' is faster:

```
sage: P = polytopes.one_hundred_twenty_cell(backend='normaliz'); P
   → optional - pynormaliz
A 4-dimensional polyhedron in (Number Field in sqrt5 with defining
polynomial x^2 - 5 with sqrt5 = 2.236067977499790?)^4 defined as
the convex hull of 600 vertices
```

It is also possible to realize it using the generalized permutahedron of type $H_4$:

```
sage: polytopes.one_hundred_twenty_cell(backend='normaliz', # not tested ~
   → long time
   ....: construction='as_permutahedron')
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 600 vertices
```

**parallelotope** *(generators, backend=None)*

Return the zonotope, or parallelotope, spanned by the generators.

The parallelotope is the multi-dimensional generalization of a parallelogram (2 generators) and a parallelepiped (3 generators).

**INPUT:**

• generators – a list of vectors of same dimension

• backend – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: polytopes.parallelotope([(1, 0), (0, 1)])
   → needs sage.rings.number_field
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: polytopes.parallelotope([[1, 2, 3, 4], [0, 1, 0, 7], [3, 1, 0, 2], [0, 0, 1, 0]])
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 16 vertices
```

```
sage: K = QuadraticField(2, 'sqrt2') #...
   → needs sage.rings.number_field
sage: sqrt2 = K.gen() #...
   → needs sage.rings.number_field
sage: P = polytopes.parallelotope([[1, sqrt2, (1, 0)], (1, -1)]); P #...
   → needs sage.rings.number_field
A 2-dimensional polyhedron in (Number Field in sqrt2 with defining
polynomial x^2 - 2 with sqrt2 = 1.414213562373095?)^2 defined as
the convex hull of 4 vertices
```

**pentakis_dodecahedron** *(exact=True, base_ring=None, backend=None)*

Return the pentakis dodecahedron.
The pentakis dodecahedron (or kisdodecahedron) is a face-regular, vertex-uniform polytope dual to the truncated icosahedron. It has 60 facets and 32 vertices. See the Wikipedia article Pentakis_dodecahedron for more information.

**INPUT:**

- **exact** – (boolean, default True) If False use an approximate ring for the coordinates.
- **base_ring** – the ring in which the coordinates will belong to. If it is not provided and exact=True it will be the number field \( \mathbb{Q}[\phi] \) where \( \phi \) is the golden ratio and if exact=False it will be the real double field.
- **backend** – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: pd = polytopes.pentakis_dodecahedron()  # long time - ~10 sec
sage: pd.n_vertices()  # long time
32
sage: pd.n_inequalities()  # long time
60
```

A much faster implementation is obtained when setting exact=False:

```
sage: pd = polytopes.pentakis_dodecahedron(exact=False)  #...
  ←needs sage.groups
sage: pd.n_vertices()  #...
  ←needs sage.groups
32
sage: pd.n_inequalities()  #...
  ←needs sage.groups
60
```

The 60 are triangles:

```
sage: all(len(f.vertices()) == 3 for f in pd.facets())  #...
  ←needs sage.groups
```

**permutahedron** \((n, \text{project=False, backend=None})\)

Return the standard permutahedron of \((1,...,n)\).

The permutahedron (or permutohedron) is the convex hull of the permutations of \{1,...,n\} seen as vectors. The edges between the permutations correspond to multiplication on the right by an elementary transposition in the SymmetricGroup.

If we take the graph in which the vertices correspond to vertices of the polyhedron, and edges to edges, we get the BubbleSortGraph().

**INPUT:**

- **n** – integer
- **project** – (boolean, default False) if True, the polytope is (isometrically) projected to a vector space of dimension \( \dim-1 \). This operation turns the coordinates into floating point approximations and corresponds to the projection given by the matrix from zero_sum_projection().
- **backend** – the backend to use to create the polytope.

**EXAMPLES:**
As both Hrepresentation and Vrepresentation are known, the permutahedron can be set up with both using the backend field. The following takes very very long time to recompute, e.g. with backend ppl:

```
    sage: polytopes.permutahedron(8, backend='field')  # (~1s)
    A 7-dimensional polyhedron in QQ^8 defined as the convex hull of 40320 vertices
```

```sage
    sage: polytopes.permutahedron(9, backend='field')  # not tested (memory consumption)  # (~5s)
    A 8-dimensional polyhedron in QQ^9 defined as the convex hull of 362880 vertices
```

See also:

- `BubbleSortGraph()`

```
rectified_one_hundred_twenty_cell (exact=True, backend=None)
```

Return the rectified 120-cell.

The rectified 120-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 1200 vertices. For more information see Wikipedia article Rectified 120-cell.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

**INPUT:**

- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.

**EXAMPLES:**

```
    sage: polytopes.rectified_one_hundred_twenty_cell(backend='normaliz')  # not tested - long time
    A 4-dimensional polyhedron in AA^4 defined as the convex hull of 1200 vertices
```
rectified_six_hundred_cell (exact=True, backend=None)
Return the rectified 600-cell.

The rectified 600-cell is a 4-dimensional 4-uniform polytope in the \( H_4 \) family. It has 720 vertices. For more information see Wikipedia article Rectified 600-cell.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

**INPUT:**
- exact - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- backend – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: polytopes.rectified_six_hundred_cell(backend='normaliz')  # not tested ~ long time ~14sec
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 720 vertices
```

regular_polygon (n, exact=True, base_ring=None, backend=None)
Return a regular polygon with \( n \) vertices.

**INPUT:**
- \( n \) – a positive integer, the number of vertices.
- exact – (boolean, default True) if False floating point numbers are used for coordinates.
- base_ring – a ring in which the coordinates will lie. It is None by default. If it is not provided and exact is True then it will be the field of real algebraic number, if exact is False it will be the real double field.
- backend – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: # needs sage.rings.number_field
sage: octagon = polytopes.regular_polygon(8)
sage: octagon
A 2-dimensional polyhedron in AA^2 defined as the convex hull of 8 vertices
sage: octagon.n_vertices() 8
sage: v = octagon.volume()
sage: v
2.828427124746190?
sage: v == 2*QQbar(2).sqrt()
True
```

Its non exact version:

```
sage: polytopes.regular_polygon(3, exact=False).vertices()
(A vertex at (0.0, 1.0),
 A vertex at (0.8660254038, -0.5),
 A vertex at (-0.8660254038, -0.5))
sage: polytopes.regular_polygon(25, exact=False).n_vertices()
25
```
**rhombic_dodecahedron** *(backend=None)*

Return the rhombic dodecahedron.

The rhombic dodecahedron is a polytope dual to the cuboctahedron. It has 14 vertices and 12 faces. For more information see the Wikipedia article Rhombic_dodecahedron.

**INPUT:**

- **backend** – the backend to use to create the polytope.

**See also:**

cuboctahedron()

**EXAMPLES:**

```python
sage: rd = polytopes.rhombic_dodecahedron()
sage: rd.f_vector()
(1, 14, 24, 12, 1)
```

Its facets are 12 quadrilaterals (not all identical):

```python
sage: sum(1 for f in rd.facets() if len(f.vertices()) == 4)
12
```

Some more computations:

```python
sage: p = rd.ehrhart_polynomial()  # optional - latte_int
sage: p
16*t^3 + 12*t^2 + 4*t + 1
sage: [p(i) for i in [1,2,3,4]]  # optional - latte_int
[33, 185, 553, 1233]
sage: [len((i*rd).integral_points()) for i in [1,2,3,4]]
[33, 185, 553, 1233]
```

**rhomboicosidodecahedron** *(exact=True, base_ring=None, backend=None)*

Return the rhomboicosidodecahedron.

The rhomboicosidodecahedron is an Archimedean solid. It has 62 faces and 60 vertices. For more information, see the Wikipedia article Rhombicosidodecahedron.

**INPUT:**

- **exact** – (boolean, default True) If False use an approximate ring for the coordinates.

- **base_ring** – the ring in which the coordinates will belong to. If it is not provided and exact=True it will be a the number field \( \mathbb{Q}[\phi] \) where \( \phi \) is the golden ratio and if exact=False it will be the real double field.

- **backend** – the backend to use to create the polytope.

**EXAMPLES:**

```python
sage: rid = polytopes.rhomboicosidodecahedron()  # long time (6secs)
sage: rid.f_vector()  # long time
(1, 60, 120, 62, 1)
sage: rid.base_ring()  # long time
Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?
```

A much faster implementation using floating point approximations:
sage: rid = polytopes.rhombicosidodecahedron(exact=False)
sage: rid.f_vector()
(1, 60, 120, 62, 1)
sage: rid.base_ring()
Real Double Field

Its facets are 20 triangles, 30 squares and 12 pentagons:

```
sage: sum(1 for f in rid.facets() if len(f.vertices()) == 3)
20
sage: sum(1 for f in rid.facets() if len(f.vertices()) == 4)
30
sage: sum(1 for f in rid.facets() if len(f.vertices()) == 5)
12
```

runcinated_one_hundred_twenty_cell (exact=False, backend=None)

Return the runcinated 120-cell.

The runcinated 120-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 2400 vertices. For more information see Wikipedia article Runcinated 120-cell.

**Warning:** The coordinates are inexact by default. The computation with inexact coordinates (using the backend `cdd`) issues a UserWarning on inconsistencies.

**INPUT:**

- **exact** - (boolean, default False) if True use exact coordinates instead of floating point approximations.
- **backend** – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: polytopes.runcinated_one_hundred_twenty_cell(exact=False) # not tested
--- very long time
```

Doctest: warning ... UserWarning: This polyhedron data is numerically complicated; cdd could not convert between the inexact V and H representation without loss of data. The resulting object might show inconsistencies.

A 4-dimensional polyhedron in RDF^4 defined as the convex hull of 2400 vertices

It is possible to use the backend 'normaliz' to get an exact representation:

```
sage: polytopes.runcinated_one_hundred_twenty_cell(exact=True, # not tested
...
 backend='normaliz')
```

A 4-dimensional polyhedron in AA^4 defined as the convex hull of 2400 vertices

runcitruncated_one_hundred_twenty_cell (exact=False, backend=None)

Return the runcitruncated 120-cell.

The runcitruncated 120-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 7200 vertices. For more information see Wikipedia article Runcitruncated 120-cell.
Warning: The coordinates are inexact by default. The computation with inexact coordinates (using the backend 'cdd') issues a UserWarning on inconsistencies.

INPUT:

- `exact` - (boolean, default False) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.

EXAMPLES:

```
sage: polytopes.runcitruncated_one_hundred_twenty_cell(exact=False) # not tested - very long time
doctest:warning...

UserWarning: This polyhedron data is numerically complicated; cdd could not convert between the inexact V and H representation without loss of data. The resulting object might show inconsistencies.
```

It is possible to use the backend 'normaliz' to get an exact representation:

```
sage: polytopes.runcitruncated_one_hundred_twenty_cell(exact=True, backend='normaliz') # not tested - very long time

A 4-dimensional polyhedron in AA^4 defined as the convex hull of 7200 vertices
```

runcitruncated_six_hundred_cell (exact=True, backend=None)

Return the runcitruncated 600-cell.

The runcitruncated 600-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 7200 vertices. For more information see Wikipedia article Runcitruncated 600-cell.

Warning: The coordinates are exact by default. The computation with inexact coordinates (using the backend 'cdd') returns a numerical inconsistency error, and thus cannot be computed.

INPUT:

- `exact` - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- `backend` - the backend to use to create the polytope.

EXAMPLES:

```
sage: polytopes.runcitruncated_six_hundred_cell(backend='normaliz') # not tested - very long time

A 4-dimensional polyhedron in AA^4 defined as the convex hull of 7200 vertices
```

simplex (dim=3, project=False, base_ring=None, backend=None)

Return the dim dimensional simplex.

The $d$-simplex is the convex hull in $\mathbb{R}^{d+1}$ of the standard basis $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. For more information, see the Wikipedia article Simplex.
INPUT:

- **dim** – The dimension of the simplex, a positive integer.
- **project** – (boolean, default False) if True, the polytope is (isometrically) projected to a vector space of dimension \( dim-1 \). This corresponds to the projection given by the matrix from `zero_sum_projection()`. By default, this operation turns the coordinates into floating point approximations (see `base_ring`).
- **base_ring** – the base ring to use to create the polytope. If `project` is False, this defaults to \( \mathbb{Z} \). Otherwise, it defaults to RDF.
- **backend** – the backend to use to create the polytope.

See also:

`tetrahedron()`

EXAMPLES:

```python
sage: s5 = polytopes.simplex(5)
sage: s5
A 5-dimensional polyhedron in ZZ^6 defined as the convex hull of 6 vertices
sage: s5.f_vector()
(1, 6, 15, 20, 15, 6, 1)

sage: s5 = polytopes.simplex(5, project=True)
sage: s5
A 5-dimensional polyhedron in RDF^5 defined as the convex hull of 6 vertices
```

Its volume is \( \sqrt{d+1}/d! \):

```python
sage: s5 = polytopes.simplex(5, project=True)
sage: s5.volume() # abs tol 1e-10
0.0204124145231931
sage: sqrt(6.) / factorial(5)
0.0204124145231931

sage: s6 = polytopes.simplex(6, project=True)
sage: s6.volume() # abs tol 1e-10
0.00367465459870082
sage: sqrt(7.) / factorial(6)
0.00367465459870082
```

Computation in algebraic reals:

```python
sage: s3 = polytopes.simplex(3, project=True, base_ring=AA)  # needs sage.rings.number_field
sage: s3.volume() == sqrt(3+1) / factorial(3)  # needs sage.rings.number_field
True
```

**six_hundred_cell** (*exact=False, backend=None*)

Return the standard 600-cell polytope.

The 600-cell is a 4-dimensional regular polytope. In many ways this is an analogue of the icosahedron.

**Warning:** The coordinates are not exact by default. The computation with exact coordinates takes a huge amount of time.
INPUT:
• exact - (boolean, default False) if True use exact coordinates instead of floating point approximations
• backend – the backend to use to create the polytope.

EXAMPLES:

```sage
p600 = polytopes.six_hundred_cell(); p600
# needs sage.groups
A 4-dimensional polyhedron in RDF^4 defined as the convex hull of 120 vertices
p600.f_vector()   # long time (~2sec)
# needs sage.groups
(1, 120, 720, 1200, 600, 1)
```

Computation with exact coordinates is currently too long to be useful:

```sage
p600 = polytopes.six_hundred_cell(exact=True)
# tested, needs sage.groups
len(list(p600.bounded_edges()))
# tested, needs sage.groups
720
```

**small_rhombicuboctahedron (exact=True, base_ring=None, backend=None)**

Return the (small) rhombicuboctahedron.

The rhombicuboctahedron is an Archimedean solid with 24 vertices and 26 faces. See the Wikipedia article Rhombicuboctahedron for more information.

INPUT:
• exact – (boolean, default True) If False use an approximate ring for the coordinates.
• base_ring – the ring in which the coordinates will belong to. If it is not provided and exact=True it will be a the number field $\mathbb{Q}[\phi]$ where $\phi$ is the golden ratio and if exact=False it will be the real double field.
• backend – the backend to use to create the polytope.

EXAMPLES:

```sage
sr = polytopes.small_rhombicuboctahedron()
# needs sage.rings.number_field
sr.f_vector()  # needs sage.rings.number_field
(1, 24, 48, 26, 1)
sr.volume()    # needs sage.rings.number_field
80/3*sqrt2 + 32
```

The faces are 8 equilateral triangles and 18 squares:

```sage
sum(1 for f in sr.facets() if len(f.vertices()) == 3)  # needs sage.rings.number_field
8
```

Its non exact version:

```sage
sum(1 for f in sr.facets() if len(f.vertices()) == 4)  # needs sage.rings.number_field
18
```
sage: sr = polytopes.small_rhombicuboctahedron(False)
sage: sr
A 3-dimensional polyhedron in RDF^3 defined as the convex hull of
24 vertices
sage: sr.f_vector()
(1, 24, 48, 26, 1)

snub_cube (exact=False, base_ring=None, backend=None, verbose=False)

Return a snub cube.

The snub cube is an Archimedean solid. It has 24 vertices and 38 faces. For more information see the
Wikipedia article Snub_cube.

The constant $z$ used in constructing this polytope is the reciprocal of the tribonacci constant, that is, the
solution of the equation $x^3 + x^2 + x - 1 = 0$. See Wikipedia article Generalizations_of_Fibonacci_numbers#Tribonacci_numbers.

INPUT:

- exact – (boolean, default False) if True use exact coordinates instead of floating point approxi-
mations
- base_ring – the field to use. If None (the default), construct the exact number field needed (if exact is True) or default to RDF (if exact is True).
- backend – the backend to use to create the polytope. If None (the default), the backend will be
selected automatically.

EXAMPLES:

sage: # needs sage.groups
sage: sc_inexact = polytopes.snub_cube(exact=False); sc_inexact
A 3-dimensional polyhedron in RDF^3 defined as the convex hull of 24 vertices
sage: sc_inexact.f_vector()
(1, 24, 60, 38, 1)

sage: # long time, needs sage.groups sage.rings.number_field
sage: sc_exact = polytopes.snub_cube(exact=True)
sage: sc_exact.f_vector()
(1, 24, 60, 38, 1)
sage: sorted(sc_exact.vertices())
[A vertex at (-1, -z, -z^2),
  A vertex at (-1, -z^2, z),
  A vertex at (-1, z^2, -z),
  A vertex at (-1, z, z^2),
  A vertex at (-z, -1, z^2),
  A vertex at (-z, -z^2, -1),
  A vertex at (-z, z^2, 1),
  A vertex at (-z, 1, -z^2),
  A vertex at (-z^2, -1, -z),
  A vertex at (-z^2, -z, 1),
  A vertex at (-z^2, z, -1),
  A vertex at (-z^2, 1, z),
  A vertex at (z^2, -1, z),
  A vertex at (z^2, -z, -1),
  A vertex at (z^2, z, 1),
  A vertex at (z^2, 1, -z),
  A vertex at (z, -1, -z^2),
  A vertex at (z, -z^2, 1),
  A vertex at (z, z^2, -1),
  A vertex at (z, 1, z^2),
(continues on next page)
A vertex at \((z, z^2, -1)\),
A vertex at \((z, 1, z^2)\),
A vertex at \((1, -z, z^2)\),
A vertex at \((1, -z^2, -z)\),
A vertex at \((1, z^2, z)\),
A vertex at \((1, z, -z^2)\)\]

```
sage: sc_exact.is_combinatorially_isomorphic(sc_inexact)
True
```

**snub_dodecahedron** \((\text{base\_ring}=\text{None}, \text{backend}=\text{None}, \text{verbose}=\text{False})\)

Return the snub dodecahedron.

The snub dodecahedron is an Archimedean solid. It has 92 faces and 60 vertices. For more information, see the Wikipedia article Snub_dodecahedron.

**INPUT:**

- **base\_ring** – the ring in which the coordinates will belong to. If it is not provided it will be the real double field.
- **backend** – the backend to use to create the polytope.

**EXAMPLES:**

Only the backend using the optional normaliz package can construct the snub dodecahedron in reasonable time:

```
sage: sd = polytopes.snub_dodecahedron(base_ring=AA, backend=normaliz)  # optional - pynormaliz, long time

sage: sd.f_vector()  # optional - pynormaliz, long time
(1, 60, 150, 92, 1)

sage: sd.base_ring()  # optional - pynormaliz, long time
Algebraic Real Field
```

Its facets are 80 triangles and 12 pentagons:

```
sage: sum(1 for f in sd.facets())  # optional - pynormaliz, long time
80

sage: sum(1 for f in sd.facets())  # optional - pynormaliz, long time
12
```

**static symmetric_edge_polytope** \((\text{backend}=\text{None})\)

Return the symmetric edge polytope of **self**.

The symmetric edge polytope (SEP) of a Graph on \(n\) vertices is the polytope in \(\mathbb{Z}^n\) defined as the convex hull of \(e_i - e_j\) and \(e_j - e_i\) for each edge \((i,j)\). Here \(e_1, \ldots, e_n\) denotes the standard basis.

**INPUT:**

- **backend** – string or \text{None} (default); the backend to use; see \textit{sage.geometry.polyhedron.constructor.Polyhedron}()}
EXAMPLES:

The SEP of a 4-cycle is a cube:

```
sage: G = graphs.CycleGraph(4)
sage: P = G.symmetric_edge_polytope(); P
#−→ needs sage.geometry.polyhedron
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 8 vertices
sage: P.is_combinatorially_isomorphic(polytopes.cube())
#−→ needs sage.geometry.polyhedron
True
```

The SEP of a complete graph on 4 vertices is a cuboctahedron:

```
sage: G = graphs.CompleteGraph(4)
sage: P = G.symmetric_edge_polytope(); P
#−→ needs sage.geometry.polyhedron
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 12 vertices
sage: P.is_combinatorially_isomorphic(polytopes.cuboctahedron())
#−→ needs sage.geometry.polyhedron
True
```

The SEP of a graph with edges on \(n\) vertices has dimension \(n\) minus the number of connected components:

```
sage: n = randint(5, 12)
sage: G = Graph()
sage: while not G.num_edges():
 #−→ needs networkx
 ....: G = graphs.RandomGNP(n, 0.2)
sage: P = G.symmetric_edge_polytope()
 #−→ needs networkx sage.geometry.polyhedron
sage: P.ambient_dim() == n
 #−→ needs networkx sage.geometry.polyhedron
True
sage: P.dim() == n - G.connected_components_number()
 #−→ needs networkx sage.geometry.polyhedron
True
```

The SEP of a graph is isomorphic to the subdirect sum of its connected components SEP’s:

```
sage: n = randint(3, 6)
sage: G1 = graphs.RandomGNP(n, 0.2)
 #−→ needs networkx
sage: n = randint(3, 6)
sage: G2 = graphs.RandomGNP(n, 0.2)
 #−→ needs networkx
sage: G = G1.disjoint_union(G2)
 #−→ needs networkx
sage: P = G.symmetric_edge_polytope()
 #−→ needs networkx sage.geometry.polyhedron
sage: P1 = G1.symmetric_edge_polytope()
 #−→ needs networkx sage.geometry.polyhedron
sage: P2 = G2.symmetric_edge_polytope()
 #−→ needs networkx sage.geometry.polyhedron
sage: P.is_combinatorially_isomorphic(P1.subdirect_sum(P2))
 #−→ needs networkx sage.geometry.polyhedron
True
```

All trees on \(n\) vertices have isomorphic SEPs:
However, there are still many different SEPs:

```
    sage: len(list(graphs(5)))
    34
    sage: polys = []
    sage: for G in graphs(5):
    ....:     P = G.symmetric_edge_polytope()
    ....:     for P1 in polys:
    ....:         if P.is_combinatorially_isomorphic(P1):
    ....:             break
    ....:     else:
    ....:         polys.append(P)
    sage: len(polys)
    25
```

A non-trivial example of two graphs with isomorphic SEPs:

```
    sage: G1 = graphs.CycleGraph(4)
    sage: G1.add_edges([[0, 5], [5, 2], [1, 6], [6, 2]])
    sage: G2 = copy(G1)
    sage: G2.add_edges([[2, 7], [7, 3]])
    sage: G1.is_isomorphic(G2)
    False
    sage: P1 = G1.symmetric_edge_polytope()  # needs sage.geometry.polyhedron
    sage: P2 = G2.symmetric_edge_polytope()  # needs sage.geometry.polyhedron
    sage: P1.is_combinatorially_isomorphic(P2)  # needs sage.geometry.polyhedron
    True
```

Apparently, glueing two graphs together on a vertex gives isomorphic SEPs:

```
    sage: n = randint(3, 7)
    sage: g1 = graphs.RandomGNP(n, 0.2)  # needs networkx
    sage: g2 = graphs.RandomGNP(n, 0.2)  # needs networkx
    sage: G = g1.disjoint_union(g2)  # needs networkx
    sage: H = copy(G)  # needs networkx
    sage: G.merge_vertices(((0, randrange(n)), (1, randrange(n))))  # (continues on next page)
```
tetrahedron (backend=None)

Return the tetrahedron.

The tetrahedron is a Platonic solid with 4 vertices and 4 faces dual to itself. It can be defined as the convex hull of the 4 vertices (0,0,0), (1,1,0), (1,0,1) and (0,1,1). For more information, see the Wikipedia article Tetrahedron.

INPUT:

• backend – the backend to use to create the polytope.

See also:

simplex()

EXAMPLES:

sage: co = polytopes.tetrahedron()
sage: co.f_vector()
(1, 4, 6, 4, 1)

Its facets are 4 triangles:

sage: sum(1 for f in co.facets() if len(f.vertices()) == 3)
4

Some more computation:

sage: co.volume()
1/3
sage: co.ehrhart_polynomial() # optional – latte_int
1/3*t^3 + t^2 + 5/3*t + 1

truncated_cube (exact=True, base_ring=None, backend=None)

Return the truncated cube.

The truncated cube is an Archimedean solid with 24 vertices and 14 faces. It can be defined as the convex hull of the 24 vertices \((\pm x, \pm 1, \pm 1), (\pm 1, \pm x, \pm 1), (\pm 1, \pm 1, \pm x)\) where \(x = \sqrt{2} - 1\). For more information, see the Wikipedia article Truncated_cube.

INPUT:

• exact – (boolean, default True) If False use an approximate ring for the coordinates.

• base_ring – the ring in which the coordinates will belong to. If it is not provided and exact=True it will be a the number field \(\mathbb{Q}[\sqrt{2}]\) and if exact=False it will be the real double field.

• backend – the backend to use to create the polytope.
EXAMPLES:

```python
sage: co = polytopes.truncated_cube()  # needs sage.rings.number_field
```

Its facets are 8 triangles and 6 octagons:

```python
sage: sum(1 for f in co.facets() if len(f.vertices()) == 3)  # needs sage.rings.number_field
8
sage: sum(1 for f in co.facets() if len(f.vertices()) == 8)  # needs sage.rings.number_field
6
```

Some more computation:

```python
sage: co.volume()  # needs sage.rings.number_field
56/3*sqrt2 - 56/3
```

**truncated_dodecahedron** (*exact=True, base_ring=None, backend=None*)

Return the truncated dodecahedron.

The truncated dodecahedron is an Archimedean solid. It has 32 faces and 60 vertices. For more information, see the Wikipedia article Truncated dodecahedron.

**INPUT:**

- `exact` – (boolean, default True) If False use an approximate ring for the coordinates.
- `base_ring` – the ring in which the coordinates will belong to. If it is not provided and `exact=True` it will be a the number field $\mathbb{Q}[\phi]$ where $\phi$ is the golden ratio and if `exact=False` it will be the real double field.
- `backend` – the backend to use to create the polytope.

**EXAMPLES:**

```python
sage: td = polytopes.truncated_dodecahedron()  # needs sage.rings.number_field
sage: td.f_vector()  # needs sage.rings.number_field
(1, 60, 90, 32, 1)
sage: td.base_ring()  # needs sage.rings.number_field
Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?
```

Its facets are 20 triangles and 12 regular decagons:

```python
sage: sum(1 for f in td.facets() if len(f.vertices()) == 3)  # needs sage.rings.number_field
20
sage: sum(1 for f in td.facets() if len(f.vertices()) == 10)  # needs sage.rings.number_field
12
```
The faster implementation using floating point approximations does not fully work unfortunately, see https://github.com/cddlib/cddlib/pull/7 for a detailed discussion of this case:

```python
sage: td = polytopes.truncated_dodecahedron(exact=False)  # random
doctest:warning
... UserWarning: This polyhedron data is numerically complicated; cdd
could not convert between the inexact V and H representation
without loss of data. The resulting object might show
inconsistencies.
sage: td.f_vector()
Traceback (most recent call last):
... ValueError: not all vertices are intersections of facets
```

```
truncated_icosidodecahedron (exact=True, base_ring=None, backend=None)

Return the truncated icosidodecahedron.

The truncated icosidodecahedron is an Archimedean solid. It has 62 faces and 120 vertices. For more information, see the Wikipedia article Truncated_icosidodecahedron.

INPUT:

• `exact` – (boolean, default True) If False use an approximate ring for the coordinates.

• `base_ring` – the ring in which the coordinates will belong to. If it is not provided and `exact=True` it will be the number field \( \mathbb{Q}[\phi] \) where \( \phi \) is the golden ratio and if `exact=False` it will be the real double field.

• `backend` – the backend to use to create the polytope.

EXAMPLES:

```python
sage: ti = polytopes.truncated_icosidodecahedron()  # long time
sage: ti.f_vector()                                  # long time
(1, 120, 180, 62, 1)
sage: ti.base_ring()                                # long time
Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?
```

The implementation using floating point approximations is much faster:

```
sage: ti = polytopes.truncated_icosidodecahedron(exact=False)  # random
sage: ti.f_vector()
(1, 120, 180, 62, 1)
sage: ti.base_ring()
Real Double Field
```

Its facets are 30 squares, 20 hexagons and 12 decagons:

```
sage: sum(1 for f in ti.facets() if len(f.vertices()) == 4)
30
sage: sum(1 for f in ti.facets() if len(f.vertices()) == 6)
20
sage: sum(1 for f in ti.facets() if len(f.vertices()) == 10)
12
```
truncated_octahedron (backend=None)

Return the truncated octahedron.

The truncated octahedron is an Archimedean solid with 24 vertices and 14 faces. It can be defined as the convex hull of all the permutations of \((0, \pm 1, \pm 2)\). For more information, see the Wikipedia article Truncated_octahedron.

This is also known as the permutahedron of dimension 3.

INPUT:

- \texttt{backend} – the backend to use to create the polytope.

EXAMPLES:

```python
sage: co = polytopes.truncated_octahedron()
sage: co.f_vector()
(1, 24, 36, 14, 1)
```

Its facets are 6 squares and 8 hexagons:

```python
sage: sum(1 for f in co.facets() if len(f.vertices()) == 4)
6
sage: sum(1 for f in co.facets() if len(f.vertices()) == 6)
8
```

Some more computation:

```python
sage: co.volume()
32
sage: co.ehrhart_polynomial()  # optional - latte_int
32*t^3 + 18*t^2 + 6*t + 1
```

truncated_one_hundred_twenty_cell (exact=True, backend=None)

Return the truncated 120-cell.

The truncated 120-cell is a 4-dimensional 4-uniform polytope in the \(H_4\) family. It has 2400 vertices. For more information see Wikipedia article Truncated 120-cell.

**Warning:** The coordinates are exact by default. The computation with inexact coordinates (using the backend `cdd`) returns a numerical inconsistency error, and thus cannot be computed.

INPUT:

- \texttt{exact} - (boolean, default True) if True use exact coordinates instead of floating point approximations.
- \texttt{backend} – the backend to use to create the polytope.

EXAMPLES:

```python
sage: polytopes.truncated_one_hundred_twenty_cell(backend='normaliz')  # not tested - long time
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 2400 vertices
```
truncated_six_hundred_cell (exact=False, backend=None)

Return the truncated 600-cell.

The truncated 600-cell is a 4-dimensional 4-uniform polytope in the $H_4$ family. It has 1440 vertices. For more information see Wikipedia article Truncated 600-cell.

**Warning:** The coordinates are not exact by default. The computation with exact coordinates takes a huge amount of time.

**INPUT:**

- `exact` - (boolean, default `False`) if `True` use exact coordinates instead of floating point approximations
- `backend` – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: polytopes.truncated_six_hundred_cell()  # not tested - long time
A 4-dimensional polyhedron in RDF^4 defined as the convex hull of 1440 vertices
```

It is possible to use the backend 'normaliz' to get an exact representation:

```
sage: polytopes.truncated_six_hundred_cell(exact=True, backend='normaliz')  # not tested - long time ~16sec
A 4-dimensional polyhedron in AA^4 defined as the convex hull of 1440 vertices
```

truncated_tetrahedron (backend=None)

Return the truncated tetrahedron.

The truncated tetrahedron is an Archimedean solid with 12 vertices and 8 faces. It can be defined as the convex hull off all the permutations of $(\pm 1, \pm 1, \pm 3)$ with an even number of minus signs. For more information, see the Wikipedia article Truncated_tetrahedron.

**INPUT:**

- `backend` – the backend to use to create the polytope.

**EXAMPLES:**

```
sage: co = polytopes.truncated_tetrahedron()
sage: co.f_vector()
(1, 12, 18, 8, 1)
```

It's facets are 4 triangles and 4 hexagons:

```
sage: sum(1 for f in co.facets() if len(f.vertices()) == 3)
4
sage: sum(1 for f in co.facets() if len(f.vertices()) == 6)
4
```

Some more computation:

```
sage: co.volume()
184/3
sage: co.ehrhart_polynomial()  # optional - latte_int
184/3*t^3 + 28*t^2 + 26/3*t + 1
```
twenty_four_cell (backend=None)

Return the standard 24-cell polytope.

The 24-cell polyhedron (also called icositetrachoron or octaplex) is a regular polyhedron in 4-dimension. For more information see the Wikipedia article 24-cell.

INPUT:

- backend – the backend to use to create the polytope.

EXAMPLES:

```python
sage: p24 = polytopes.twenty_four_cell()
sage: p24.f_vector()
(1, 24, 96, 96, 24, 1)
sage: v = next(p24.vertex_generator())
sage: for adj in v.neighbors(): print(adj)
A vertex at (-1/2, -1/2, -1/2, 1/2)
A vertex at (-1/2, -1/2, 1/2, -1/2)
A vertex at (-1, 0, 0, 0)
A vertex at (-1/2, 1/2, -1/2, -1/2)
A vertex at (0, -1, 0, 0)
A vertex at (0, 0, -1, 0)
A vertex at (0, 0, 0, -1)
A vertex at (1/2, -1/2, -1/2, -1/2)
sage: p24.volume()
2
```

zonotope (generators, backend=None)

Return the zonotope, or parallelotope, spanned by the generators.

The parallelotope is the multi-dimensional generalization of a parallelogram (2 generators) and a parallelepiped (3 generators).

INPUT:

- generators – a list of vectors of same dimension
- backend – the backend to use to create the polytope.

EXAMPLES:

```python
sage: polytopes.parallelotope([ (1,0), (0,1) ])  # needs sage.rings.number_field
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: polytopes.parallelotope([[1,2,3,4], [0,1,0,7], [3,1,0,2], [0,0,1,0]])
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 16 vertices
sage: K = QuadraticField(2, 'sqrt2') # needs sage.rings.number_field
sage: sqrt2 = K.gen() # needs sage.rings.number_field
sage: P = polytopes.parallelotope([[1, sqrt2], (1, -1)]); P  # needs sage.rings.number_field
A 2-dimensional polyhedron in (Number Field in sqrt2 with defining polynomial x^2 - 2 with sqrt2 = 1.414213562373095?)^2 defined as the convex hull of 4 vertices
```

sage.geometry.polyhedron.library.gale_transform_to_polytope (vectors, base_ring=None, backend=None)

Return the polytope associated to the list of vectors forming a Gale transform.
This function is the inverse of `gale_transform()` up to projective transformation.

**INPUT:**

- `vectors` – the vectors of the Gale transform
- `base_ring` – string (default: `None`); the base ring to be used for the construction
- `backend` – string (default: `None`); the backend to use to create the polytope

**Note:** The order of the input vectors will not be preserved.

If the center of the (input) vectors is the origin, the function is much faster and might give a nicer representation of the polytope.

If this is not the case, the vectors will be scaled (each by a positive scalar) accordingly to obtain the polytope.

**See also:**

:`func`~`sage.geometry.polyhedron.library.gale_transform_to_primal`.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.library import gale_transform_to_polytope/nsage: points = polytopes.octahedron().gale_transform()
sage: points
((0, -1), (-1, 0), (1, 1), (1, 1), (-1, 0), (0, -1))
sage: P = gale_transform_to_polytope(points); P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
sage: P.vertices()
(A vertex at (-1, 0, 0),
 A vertex at (0, -1, 0),
 A vertex at (0, 0, -1),
 A vertex at (0, 0, 1),
 A vertex at (0, 1, 0),
 A vertex at (1, 0, 0))
```

One can specify the base ring:

```python
sage: gale_transform_to_polytope(  ....: [(1,1), (-1,-1), (1,0),
 ....:  (-1,0), (1,-1), (-2,1)],
 ....:  base_ring=RDF).vertices()
(A vertex at (-0.64, 1.4, -2.16),
 A vertex at (-0.96, -0.4, -1.24),
 A vertex at (0.6, -2.0, 2.4),
 A vertex at (1.0, 0.0, 0.0),
 A vertex at (0.0, 1.0, 0.0),
 A vertex at (0.0, 0.0, 1.0))
```

One can also specify the backend:
A gale transform corresponds to a polytope if and only if every oriented (linear) hyperplane has at least two vectors on each side. See Theorem 6.19 of [Zie2007]. If this is not the case, one of two errors is raised.

If there is such a hyperplane with no vector on one side, the vectors are not totally cyclic:

```python
sage: gale_transform_to_polytope([(0,1), (1,1), (1,0), (-1,0)])
Traceback (most recent call last):
  ... ValueError: input vectors not totally cyclic
```

If every hyperplane has at least one vector on each side, then the gale transform corresponds to a point configuration. It corresponds to a polytope if and only if this point configuration is convex and if and only if every hyperplane contains at least two vectors of the gale transform on each side.

If this is not the case, an error is raised:

```python
sage: gale_transform_to_polytope([(0,1), (1,1), (1,0), (-1,-1)])
Traceback (most recent call last):
  ... ValueError: the gale transform does not correspond to a polytope
```

Return a point configuration dual to a totally cyclic vector configuration.

This is the dehomogenized vector configuration dual to the input. The dual vector configuration is acyclic and can therefore be dehomogenized as the input is totally cyclic.

**INPUT:**

- `vectors` – the ordered vectors of the Gale transform
- `base_ring` – string (default: `None`); the base ring to be used for the construction
- `backend` – string (default: `None`); the backend to be use to construct a polyhedral, used internally in case the center is not the origin, see `Polyhedron()`

**OUTPUT:** An ordered point configuration as list of vectors.

**Note:** If the center of the (input) vectors is the origin, the function is much faster and might give a nicer representation of the point configuration.

If this is not the case, the vectors will be scaled (each by a positive scalar) accordingly.

**ALGORITHM:**

Step 1: If the center of the (input) vectors is not the origin, we do an appropriate transformation to make it so.
Step 2: We add a row of ones on top of \texttt{Matrix(vectors)}. The right kernel of this larger matrix is the dual configuration space, and a basis of this space provides the dual point configuration.

More concretely, the dual vector configuration (inhomogeneous) is obtained by taking a basis of the right kernel of \texttt{Matrix(vectors)}. If the center of the (input) vectors is the origin, there exists a basis of the right kernel of the form \([[[1], [V]]], \) where \([1]\) represents a row of ones. Then, \([V]\) is a dehomogenization and thus the dual point configuration.

To extend \([1]\) to a basis of \texttt{Matrix(vectors)}, we add a row of ones to \texttt{Matrix(vectors)} and calculate a basis of the right kernel of the obtained matrix.

REFERENCES:

For more information, see Section 6.4 of [Zie2007] or Definition 2.5.1 and Definition 4.1.35 of [DLRS2010].

See also:

\texttt{func} `\texttt{sage.geometry.polyhedron.library.gale_transform_to_polytope}`.

EXAMPLES:

\begin{verbatim}
from sage.geometry.polyhedron.library import gale_transform_to_primal

points = ((0, -1), (-1, 0), (1, 1), (1, 1), (-1, 0), (0, -1))

matrix(RDF, gale_transform_to_primal(points)).norm() < 1e-15
True
\end{verbatim}

One can specify the base ring:

\begin{verbatim}
p = [(1,1), (-1,-1), (1,0), (-1,0), (1,-1), (-2,1)]
gtpp = gale_transform_to_primal(p); gtpp

[[16, -35, 54),
 (24, 10, 31),
 (-15, 50, -60),
 (-25, 0, 0),
 (0, -25, 0),
 (0, 0, -25)]

(matrix(RDF, gtpp)/25 +
 ....: matrix(gale_transform_to_primal(p, base_ring=RDF))).norm() < 1e-15
True
\end{verbatim}

One can also specify the backend to be used internally:

\begin{verbatim}
gale_transform_to_primal(p, backend='field')

[(48, -71, 88),
 (84, -28, 99),
 (-77, 154, -132),
 (-55, 0, 0),
 (0, -55, 0),
 (0, 0, -55)]

gale_transform_to_primal(p, backend='normaliz') # optional →

→pynormaliz

[(16, -35, 54),
 (24, 10, 31),
 (-15, 50, -60),
 (-25, 0, 0),
 (0, -25, 0),
 (0, 0, -25)]
\end{verbatim}

The input vectors should be totally cyclic:
sage: gale_transform_to_primal([(0,1), (1,0), (1,1), (-1,0)])
Traceback (most recent call last):
...
ValueError: input vectors not totally cyclic

sage: gale_transform_to_primal(
....:   [(1,1,0), (-1,-1,0), (1,0,0),
....:    (-1,0,0), (1,-1,0), (-2,1,0)], backend='field')
Traceback (most recent call last):
...
ValueError: input vectors not totally cyclic

sage.geometry.polyhedron.library.project_points(*points, **kwds)
Projects a set of points into a vector space of dimension one less.

**INPUT:**
- points... – the points to project.
- base_ring – (default to RDF if keyword is None or not provided in kwds) the base ring to use.

The projection is isometric to the orthogonal projection on the hyperplane made of zero sum vector. Hence, if the set of points have all equal sums, then their projection is isometric (as a set of points).

The projection used is the matrix given by `zero_sum_projection()`.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.library import project_points
sage: project_points([2,-1,3,2])  # abs tol 1e-15
[(2.1213203435596424, -2.041241452319315, -0.577350269189626)]
sage: project_points([1,2,3], [3,3,5])  # abs tol 1e-15
[(-0.7071067811865475, -1.2247448713915892), (0.0, -1.6329931618554523)]
```

These projections are compatible with the restriction. More precisely, given a vector \( v \), the projection of \( v \) restricted to the first \( i \) coordinates will be equal to the projection of the first \( i + 1 \) coordinates of \( v \):

```python
sage: project_points([1,2])  # abs tol 1e-15
[(-0.7071067811865475)]
sage: project_points([1,2,3])  # abs tol 1e-15
[(-0.7071067811865475, -1.2247448713915892)]
sage: project_points([1,2,3,4])  # abs tol 1e-15
[(-0.7071067811865475, -1.2247448713915892, -1.7320508075688776)]
sage: project_points([1,2,3,4,0])  # abs tol 1e-15
[(-0.7071067811865475, -1.2247448713915892, -1.7320508075688776, 2.0000000000000000e-2360797749979)]
```

Check that it is (almost) an isometry:

```python
sage: V = list(map(vector, IntegerVectors(n=5, length=3)))
sage: P = project_points(*V)
sage: for i in range(21):
    #... needs sage.combinat
        for j in range(21):
            assert abs((V[i]-V[j]).norm() - (P[i]-P[j]).norm()) < 0.00001
```

Example with exact computation:
sage: V = [ vector(v) for v in IntegerVectors(n=4, length=2) ]
sage: P = project_points(*V, base_ring=AA)  # needs sage.combinat sage.rings.number_field
sage: for i in range(len(V)):
    for j in range(len(V)):
        assert (V[i]-V[j]).norm() == (P[i]-P[j]).norm()

sage.geometry.polyhedron.library.zero_sum_projection(d, base_ring=None)

Return a matrix corresponding to the projection on the orthogonal of \((1,1,\ldots,1)\) in dimension \(d\).

The projection maps the orthonormal basis \((1,-1,0,\ldots,0)/\sqrt{2})\), \((1,1,-1,0,\ldots,0)/\sqrt{3})\), \ldots, \((1,1,\ldots,1,-1)/\sqrt{d})\) to the canonical basis in \(\mathbb{R}^{d-1}\).

OUTPUT:

A matrix of dimensions \((d-1) \times d\) defined over base_ring (default: RDF).

EXAMPLES:

\[
\begin{bmatrix}
0.7071067811865475 & -0.7071067811865475 \\
0.4082482904638631 & 0.4082482904638631 & -0.8164965809277261
\end{bmatrix}
\]

Exact computation in AA:

\[
\begin{bmatrix}
0.7071067811865475? & -0.7071067811865475? \\
0.4082482904638630? & 0.4082482904638630? & -0.8164965809277260?
\end{bmatrix}
\]

2.1.2 Polyhedra

In this module, a polyhedron is a convex (possibly unbounded) set in Euclidean space cut out by a finite set of linear inequalities and linear equations. Note that the dimension of the polyhedron can be less than the dimension of the ambient space. There are two complementary representations of the same data:

H(alf-space/Hyperplane)-representation

This describes a polyhedron as the common solution set of a finite number of

- linear inequalities \(A\mathbf{x} + b \geq 0\), and
- linear equations \(C\mathbf{x} + d = 0\).

V(ertex)-representation

The other representation is as the convex hull of vertices (and rays and lines to all for unbounded polyhedra) as generators. The polyhedron is then the Minkowski sum

\[
P = \text{conv}\{v_1, \ldots, v_k\} + \sum_{i=1}^{m} \mathbb{R} r_i + \sum_{j=1}^{n} \mathbb{R} \ell_j
\]

where

- vertices \(v_1, \ldots, v_k\) are a finite number of points. Each vertex is specified by an arbitrary vector, and two points are equal if and only if the vector is the same.
• **rays** $r_1, \ldots, r_m$ are a finite number of directions (directions of infinity). Each ray is specified by a non-zero vector, and two rays are equal if and only if the vectors are the same up to rescaling with a positive constant.

• **lines** $\ell_1, \ldots, \ell_n$ are a finite number of unoriented directions. In other words, a line is equivalent to the set \{$r, -r$\} for a ray $r$. Each line is specified by a non-zero vector, and two lines are equivalent if and only if the vectors are the same up to rescaling with a non-zero (possibly negative) constant.

When specifying a polyhedron, you can input a non-minimal set of inequalities/equations or generating vertices/rays/lines. The non-minimal generators are usually called points, non-extremal rays, and non-extremal lines, but for our purposes it is more convenient to always talk about vertices/rays/lines. Sage will remove any superfluous representation objects and always return a minimal representation. For example, $(0, 0)$ is a superfluous vertex here:

```python
sage: triangle = Polyhedron(vertices=[(0,2), (-1,0), (1,0), (0,0)])
sage: triangle.vertices()
(A vertex at (-1, 0), A vertex at (1, 0), A vertex at (0, 2))
```

See also:

If one only needs to keep track of a system of linear system of inequalities, one should also consider the class for mixed integer linear programming.

• **Mixed Integer Linear Programming**

**Unbounded Polyhedra**

A polytope is defined as a bounded polyhedron. In this case, the minimal representation is unique and a vertex of the minimal representation is equivalent to a 0-dimensional face of the polytope. This is why one generally does not distinguish vertices and 0-dimensional faces. But for non-bounded polyhedra we have to allow for a more general notion of “vertex” in order to make sense of the Minkowski sum presentation:

```python
sage: half_plane = Polyhedron(ieqs=[(0,1,0)])
sage: half_plane.Hrepresentation()
(An inequality (1, 0) x + 0 >= 0,)
sage: half_plane.Vrepresentation()
(A line in the direction (0, 1), A ray in the direction (1, 0), A vertex at (0, 0))
```

Note how we need a point in the above example to anchor the ray and line. But any point on the boundary of the half-plane would serve the purpose just as well. Sage picked the origin here, but this choice is not unique. Similarly, the choice of ray is arbitrary but necessary to generate the half-plane.

Finally, note that while rays and lines generate unbounded edges of the polyhedron they are not in a one-to-one correspondence with them. For example, the infinite strip has two infinite edges (1-faces) but only one generating line:

```python
sage: strip = Polyhedron(vertices=[(1,0),(-1,0)], lines=[(0,1)])
sage: strip.Hrepresentation()
(An inequality (1, 0) x + 1 >= 0, An inequality (-1, 0) x + 1 >= 0)
sage: strip.lines()
(A line in the direction (0, 1),)
sage: [f.ambient_V_indices() for f in strip.faces(1)]
[(0, 2), (0, 1)]
sage: for f in strip.faces(1):
    ....:     print(f.ambient_V_indices())
(0, 2)
(0, 1)
sage: for f in strip.faces(1):
    ....:     print("{} = {}").format(f.ambient_V_indices(), f.as_polyhedron().Hrepresentation())
```

(continues on next page)
Inequalities $A\vec{v} + b \geq 0$ (and, similarly, equations) are specified by a list $[b, A]$:

```
sage: Polyhedron(ieqs=[[0,1,0),(0,0,1),(1,-1,-1)]).Hrepresentation()
(An inequality (-1, -1) x + 1 >= 0,
 An inequality (1, 0) x + 0 >= 0,
 An inequality (0, 1) x + 0 >= 0)
```

See `Polyhedron()` for a detailed description of all possible ways to construct a polyhedron.

### Base Rings

The base ring of the polyhedron can be specified by the `base_ring` optional keyword argument. If not specified, a suitable common base ring for all coordinates and coefficients will be chosen automatically. Important cases are:

- `base_ring=QQ` uses a fast implementation for exact rational numbers.
- `base_ring=ZZ` is similar to `QQ`, but the resulting polyhedron object will have extra methods for lattice polyhedra.
- `base_ring=RDF` uses floating point numbers, this is fast but susceptible to numerical errors.
Polyhedra with symmetries often are defined over some algebraic field extension of the rationals. As a simple example, consider the equilateral triangle whose vertex coordinates involve $\sqrt{3}$. An exact way to work with roots in Sage is the Algebraic Real Field:

```
sage: triangle = Polyhedron([(0,0), (1,0), (1/2, sqrt(3)/2)], base_ring=AA)  # needs sage.rings.number_field sage.symbolic
```

```
sage: triangle.Hrepresentation()  # needs sage.rings.number_field sage.symbolic
(An inequality (-1, -0.5773502691896258?) x + 1 >= 0,
 An inequality (1, -0.5773502691896258?) x + 0 >= 0,
 An inequality (0, 1.154700538379252?) x + 0 >= 0)
```

Without specifying the `base_ring`, the $\sqrt{3}$ would be a symbolic ring element and, therefore, the polyhedron defined over the symbolic ring. This is currently not supported as SR is not exact:

```
sage: Polyhedron([(0,0), (1,0), (1/2, sqrt(3)/2)])
Traceback (most recent call last):
... ValueError: no default backend for computations with Symbolic Ring
```

Even faster than all algebraic real numbers (the field AA) is to take the smallest extension field. For the equilateral triangle, that would be:

```
sage: x = polygen(ZZ, 'x')
sage: K.<sqrt3> = NumberField(x^2 - 3, embedding=AA(3)**(1/2))  # needs sage.rings.number_field
sage: Polyhedron([(0,0), (1,0), (1/2, sqrt3/2)])
A 2-dimensional polyhedron in (Number Field in sqrt3 with defining polynomial x^2 - 3 with sqrt3 = 1.732050807568878?)^2 defined as the convex hull of 3 vertices
```

**Warning:** Be careful when you construct polyhedra with floating point numbers. The only available backend for such computation is cdd which uses machine floating point numbers which have limited precision. If the input consists of floating point numbers and the `base_ring` is not specified, the base ring is set to be the `RealField` with the precision given by the minimal bit precision of the input. Then, if the obtained minimum is 53 bits of precision, the constructor converts automatically the base ring to RDF. Otherwise, it returns an error:

```
sage: Polyhedron(vertices = [[1.12345678901234, 2.12345678901234]])
A 0-dimensional polyhedron in RDF^2 defined as the convex hull of 1 vertex
```

```
sage: Polyhedron(vertices = [[1.12345678901234, 2.12345678901234]], base_ring=RDF)
A 0-dimensional polyhedron in RDF^2 defined as the convex hull of 1 vertex
```

The strongly suggested method to input floating point numbers is to specify the `base_ring` to be RDF:
See also:

* Parents for polyhedra

**Base classes**

Depending on the chosen base ring, a specific class is used to represent the polyhedron object.

See also:

- Base class for polyhedra
- Base class for polyhedra over integers
- Base class for polyhedra over rationals
- Base class for polyhedra over RDF

The most important base class is **Base class for polyhedra** from which other base classes and backends inherit.

**Backends**

There are different backends available to deal with polyhedron objects.

See also:

- cdd backend for polyhedra
- field backend for polyhedra
- normaliz backend for polyhedra
- ppl backend for polyhedra

**Note:** Depending on the backend used, it may occur that different methods are available or not.

**Appendix**

**REFERENCES:**

- Komei Fukuda’s FAQ in Polyhedral Computation

**AUTHORS:**

- Arnaud Bergeron: improvements to triangulation and rendering, 2008
- Sebastien Barthelemy: documentation improvements, 2008
- Volker Braun: refactoring, handle non-compact case, 2009 and 2010
- Andrey Novoseltsev: added lattice_from_incidence, 2010
- Volker Braun: rewrite to use PPL instead of cddlib, 2011
- Volker Braun: Add support for arbitrary subfields of the reals
Construct a polyhedron object.

You may either define it with vertex/ray/line or inequalities/equations data, but not both. Redundant data will automatically be removed (unless minimize=False), and the complementary representation will be computed.

**INPUT:**

- **vertices** – iterable of points. Each point can be specified as any iterable container of base_ring elements. If rays or lines are specified but no vertices, the origin is taken to be the single vertex.

  Instead of vertices, the first argument can also be an object that can be converted to a Polyhedron() via an as_polyhedron() or polyhedron() method. In this case, the following 5 arguments cannot be provided.

- **rays** – list of rays. Each ray can be specified as any iterable container of base_ring elements.

- **lines** – list of lines. Each line can be specified as any iterable container of base_ring elements.

- **ieqs** – list of inequalities. Each line can be specified as any iterable container of base_ring elements. An entry equal to \([-1, 7, 3, 4]\) represents the inequality \(7x_1 + 3x_2 + 4x_3 \geq 1\).

- **eqns** – list of equalities. Each line can be specified as any iterable container of base_ring elements. An entry equal to \([-1, 7, 3, 4]\) represents the equality \(7x_1 + 3x_2 + 4x_3 = 1\).

- **ambient_dim** – integer. The ambient space dimension. Usually can be figured out automatically from the H/V-representation dimensions.

- **base_ring** – a sub-field of the reals implemented in Sage. The field over which the polyhedron will be defined. For QQ and algebraic extensions, exact arithmetic will be used. For RDF, floating point numbers will be used. Floating point arithmetic is faster but might give the wrong result for degenerate input.

- **backend** – string or None (default). The backend to use. Valid choices are
  - 'cdd': use cdd (backend_cdd) with Q or R coefficients depending on base_ring
  - 'normaliz': use normaliz (backend_normaliz) with Z or Q coefficients depending on base_ring
  - 'polymake': use polymake (backend_polymake) with Q, R or QuadraticField coefficients depending on base_ring
  - 'ppl': use ppl (backend_ppl) with Z or Q coefficients depending on base_ring
  - 'field': use python implementation (backend_field) for any field

Some backends support further optional arguments:

- **minimize** – boolean (default: True); whether to immediately remove redundant H/V-representation data; currently not used.

- **verbose** – boolean (default: False); whether to print verbose output for debugging purposes; only supported by the cdd and normaliz backends

- **mutable** – boolean (default: False); whether the polyhedron is mutable

**OUTPUT:**

The polyhedron defined by the input data.

**EXAMPLES:**
Construct some polyhedra:

```python
sage: square_from_vertices = Polyhedron(vertices = [[1, 1], [1, -1], [-1, 1], [-1, -1]])
sage: square_from_ieqs = Polyhedron(ieqs = [[1, 0, 1], [1, 1, 0], [1, 0, -1], [1, -1, 0]])
sage: list(square_from_ieqs.vertex_generator())
[A vertex at (1, -1),
 A vertex at (1, 1),
 A vertex at (-1, 1),
 A vertex at (-1, -1)]
sage: list(square_from_vertices.inequality_generator())
[An inequality (1, 0) x + 1 >= 0,
 An inequality (0, 1) x + 1 >= 0,
 An inequality (-1, 0) x + 1 >= 0,
 An inequality (0, -1) x + 1 >= 0]
sage: p = Polyhedron(vertices = [[1.1, 2.2], [3.3, 4.4]], base_ring=RDF)
sage: p.n_inequalities()
2

The same polyhedron given in two ways:

```python
sage: p = Polyhedron(ieqs = [[0,1,0,0],[0,0,1,0]])
sage: p.Vrepresentation()
(A line in the direction (0, 0, 1),
 A ray in the direction (1, 0, 0),
 A ray in the direction (0, 1, 0),
 A vertex at (0, 0, 0))
sage: q = Polyhedron(vertices=[[0,0,0]], rays=[[1,0,0],[0,1,0]], lines=[[0,0,1]])
sage: q.Hrepresentation()
(An inequality (1, 0, 0) x + 0 >= 0,
 An inequality (0, 1, 0) x + 0 >= 0)
```

Finally, a more complicated example. Take $\mathbb{R}_{\geq 0}^6$ with coordinates $a, b, \ldots, f$ and

- The inequality $e + b \geq c + d$
- The inequality $e + c \geq b + d$
- The equation $a + b + c + d + e + f = 31$

```python
sage: positive_coords = Polyhedron(ieqs=[
 ....: [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0],
 ....: [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]
])
sage: P = Polyhedron(ieqs=positive_coords.inequalities() + (
 ....: [0,0,1,-1,-1,1,0],
 ....: [0,0,-1,1,-1,1,0]),
 ....: eqns=[[-31,1,1,1,1,1,1]])
sage: P
A 5-dimensional polyhedron in QQ^6 defined as the convex hull of 7 vertices
sage: P.dim()
5
sage: P.Vrepresentation()
(A vertex at (31, 0, 0, 0, 0, 0),
 A vertex at (0, 0, 0, 0, 0, 31),
 A vertex at (0, 0, 0, 31/2, 31/2, 0),
 A vertex at (0, 0, 31/2, 31/2, 0, 0),
 A vertex at (0, 0, 0, 31/2, 0, 31/2),
 A vertex at (0, 0, 31/2, 0, 0, 31/2),
 A vertex at (0, 0, 0, 0, 0, 31/2))
```

Regular icosahedron, centered at 0 with edge length 2, with vertices given by the cyclic shifts of $(0, \pm 1, \pm (1 + \sqrt{5})/2)$, cf. Wikipedia article Regular_icosahedron. It needs a number field:
sage: # needs sage.rings.number_field
sage: R0.<r0> = QQ[]
sage: R1.<r1> = NumberField(r0^2-5, embedding=AA(5)**(1/2))
sage: gold = (1+r1)/2
sage: v = [[0, 1, gold], [0, 1, -gold], [0, -1, gold], [0, -1, -gold]]
sage: pp = Permutation((1, 2, 3))

sage: icosah = Polyhedron(  
....:     [pp^2).action(w) for w in v] + [pp.action(w) for w in v] + v,
....:     base_ring=R1)

sage: len(icosah.faces(2))

When the input contains elements of a Number Field, they require an embedding:

sage: # needs sage.rings.number_field
sage: x = polygen(ZZ, x)
sage: K = NumberField(x^2 - 2,s)
sage: s = K.0
sage: L = NumberField(x^3 - 2,t)
sage: t = L.0
sage: P = Polyhedron(vertices=[[0,s], [t,0]])

Traceback (most recent call last):
...
ValueError: invalid base ring

Converting from a given polyhedron:

sage: cb = polytopes.cube(); cb
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: Polyhedron(cb, base_ring=QQ)
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 8 vertices

Converting from other objects to a polyhedron:

sage: quadrant = Cone([(1,0), (0,1)])
sage: Polyhedron(quadrant)
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and 2 rays
sage: Polyhedron(quadrant, base_ring=QQ)
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 rays
sage: o = lattice_polytope.cross_polytope(2)
sage: Polyhedron(o)
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: Polyhedron(o, base_ring=QQ)
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices

sage: p = MixedIntegerLinearProgram(solver='PPL')
sage: x, y = p['x'], p['y']
sage: p.add_constraint(x <= 1)
sage: p.add_constraint(x >= -1)
sage: p.add_constraint(y <= 1)
sage: p.add_constraint(y >= -1)
sage: Polyhedron(o)
A 2-dimensional polyhedron in $\mathbb{Z}^2$ defined as the convex hull of 4 vertices

```
sage: Polyhedron(o, base_ring=QQ)
```

A 2-dimensional polyhedron in $\mathbb{Q}^2$ defined as the convex hull of 4 vertices

```
sage: # needs sage.combinat
sage: H.<x,y> = HyperplaneArrangements(QQ)
sage: h = x + y - 1; h
Hyperplane x + y - 1
sage: Polyhedron(h, base_ring=ZZ)
```

A 1-dimensional polyhedron in $\mathbb{Z}^2$ defined as the convex hull of 1 vertex and 1 line

```
sage: Polyhedron(h)
```

A 1-dimensional polyhedron in $\mathbb{Q}^2$ defined as the convex hull of 1 vertex and 1 line

```
```

Note:

- Once constructed, a `Polyhedron` object is immutable.
- Although the option `base_ring=RDF` allows numerical data to be used, it might not give the right answer for degenerate input data - the results can depend upon the tolerance setting of cdd.

See also:

*Library of polytopes*

### 2.1.3 Parents for Polyhedra

```
sage.geometry.polyhedron.parent.Polyhedra(ambient_space_or_base_ring, ambient_dim,
backend=None, ambient_space=None,
base_ring=None)
```

Construct a suitable parent class for polyhedra

**INPUT:**

- `base_ring` - A ring. Currently there are backends for $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
- `ambient_dim` - integer. The ambient space dimension.
- `ambient_space` - A free module.
- `backend` - string. The name of the backend for computations. There are several backends implemented:
  - `backend="ppl"` uses the Parma Polyhedra Library
  - `backend="cdd"` uses CDD
  - `backend="normaliz"` uses normaliz
  - `backend="polymake"` uses polymake
  - `backend="field"` a generic Sage implementation

**OUTPUT:**

A parent class for polyhedra over the given base ring if the backend supports it. If not, the parent base ring can be larger (for example, $\mathbb{Q}$ instead of $\mathbb{Z}$). If there is no implementation at all, a `ValueError` is raised.
EXAMPLES:

```python
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(AA, 3)  # needs sage.rings.number_field
Polyhedra in AA^3
sage: Polyhedra(ZZ, 3)
Polyhedra in ZZ^3
sage: type(_)
<class 'sage.geometry.polyhedron.parent.Polyhedra_ZZ_ppl_with_category'>
sage: Polyhedra(QQ, 3, backend='cdd')
Polyhedra in QQ^3
sage: type(_)
<class 'sage.geometry.polyhedron.parent.Polyhedra_QQ_cdd_with_category'>
```

CDD does not support integer polytopes directly:

```python
sage: Polyhedra(ZZ, 3, backend='cdd')
Polyhedra in QQ^3
```

Using a more general form of the constructor:

```python
sage: V = VectorSpace(QQ, 3)
sage: Polyhedra(V) is Polyhedra(QQ, 3)
True
sage: Polyhedra(V, backend='field') is Polyhedra(QQ, 3, 'field')
True
sage: Polyhedra(backend='field', ambient_space=V) is Polyhedra(QQ, 3, 'field')
True
sage: M = FreeModule(ZZ, 2)
sage: Polyhedra(M, backend='ppl') is Polyhedra(ZZ, 2, 'ppl')
True
```

```python
class sage.geometry.polyhedron.parent.Polyhedra_QQ_cdd(base_ring, ambient_dim, backend)
    Bases: Polyhedra_base
    Element
        alias of Polyhedron_QQ_cdd

class sage.geometry.polyhedron.parent.Polyhedra_QQ_normaliz(base_ring, ambient_dim, backend)
    Bases: Polyhedra_base
    Element
        alias of Polyhedron_QQ_normaliz

class sage.geometry.polyhedron.parent.Polyhedra_QQ_ppl(base_ring, ambient_dim, backend)
    Bases: Polyhedra_base
    Element
        alias of Polyhedron_QQ_ppl

class sage.geometry.polyhedron.parent.Polyhedra_RDF_cdd(base_ring, ambient_dim, backend)
    Bases: Polyhedra_base
```
Element

alias of Polyhedron_RDF_cdd

class sage.geometry.polyhedron.parent.Polyhedra_ZZ_normaliz(base_ring, ambient_dim, backend)

Bases: Polyhedra_base

Element

alias of Polyhedron_ZZ_normaliz

class sage.geometry.polyhedron.parent.Polyhedra_ZZ_ppl(base_ring, ambient_dim, backend)

Bases: Polyhedra_base

Element

alias of Polyhedron_ZZ_ppl

class sage.geometry.polyhedron.parent.Polyhedra_base(base_ring, ambient_dim, backend)

Bases: UniqueRepresentation, Parent

Polyhedra in a fixed ambient space.

INPUT:

- base_ring – either ZZ, QQ, or RDF. The base ring of the ambient module/vector space.
- ambient_dim – integer. The ambient space dimension.
- backend – string. The name of the backend for computations. There are several backends implemented:
  - backend="ppl" uses the Parma Polyhedra Library
  - backend="cdd" uses CDD
  - backend="normaliz" uses normaliz
  - backend="polymake" uses polymake
  - backend="field" a generic Sage implementation

EXAMPLES:

sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(ZZ, 3)
Polyhedra in ZZ^3

Hrepresentation_space()  
Return the linear space containing the H-representation vectors.

OUTPUT:

A free module over the base ring of dimension ambient_dim() + 1.

EXAMPLES:

sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(ZZ, 2).Hrepresentation_space()
Ambient free module of rank 3 over the principal ideal domain Integer Ring

2.1. Polyhedra
**Vrepresentation_space()**

Return the ambient vector space.

This is the vector space or module containing the Vrepresentation vectors.

**OUTPUT:**

A free module over the base ring of dimension \( \text{ambient_dim}() \).

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(QQ, 4).Vrepresentation_space()
Vector space of dimension 4 over Rational Field
sage: Polyhedra(QQ, 4).ambient_space()
Vector space of dimension 4 over Rational Field
```

**ambient_dim()**

Return the dimension of the ambient space.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(QQ, 3).ambient_dim()
3
```

**ambient_space()**

Return the ambient vector space.

This is the vector space or module containing the Vrepresentation vectors.

**OUTPUT:**

A free module over the base ring of dimension \( \text{ambient_dim}() \).

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(QQ, 4).Vrepresentation_space()
Vector space of dimension 4 over Rational Field
sage: Polyhedra(QQ, 4).ambient_space()
Vector space of dimension 4 over Rational Field
```

**an_element()**

Return a Polyhedron.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(QQ, 4).an_element()
A 4-dimensional polyhedron in QQ^4 defined as the convex hull of 5 vertices
```

**backend()**

Return the backend.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(QQ, 3).backend()
'ppl'
```
**base_extend** *(base_ring, backend=None, ambient_dim=None)*

Return the base extended parent.

**INPUT:**
- base_ring, backend – see `Polyhedron()`.
- ambient_dim – if not None change ambient dimension accordingly.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(ZZ, 3).base_extend(QQ)
Polyhedra in QQ^3
sage: Polyhedra(ZZ, 3).an_element().base_extend(QQ)
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 4 vertices
sage: Polyhedra(QQ, 2).base_extend(ZZ)
Polyhedra in QQ^2
```

**change_ring** *(base_ring, backend=None, ambient_dim=None)*

Return the parent with the new base ring.

**INPUT:**
- base_ring, backend – see `Polyhedron()`.
- ambient_dim – if not None change ambient dimension accordingly.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(ZZ, 3).change_ring(QQ)
Polyhedra in QQ^3
sage: Polyhedra(ZZ, 3).an_element().change_ring(QQ)
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 4 vertices
sage: Polyhedra(RDF, 3).change_ring(QQ).backend()
'cdd'
sage: Polyhedra(QQ, 3).change_ring(ZZ, ambient_dim=4)
Polyhedra in ZZ^4
sage: Polyhedra(QQ, 3, backend='cdd').change_ring(QQ, ambient_dim=4).backend()
'cdd'
```

**empty** *

Return the empty polyhedron.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: P = Polyhedra(QQ, 4)
sage: P.empty()
The empty polyhedron in QQ^4
sage: P.empty().is_empty()
True
```

**list** *

Return the two polyhedra in ambient dimension 0, raise an error otherwise

**EXAMPLES:**
sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: P = Polyhedra(QQ, 3)
sage: P.cardinality()
+Infinity

sage: # needs sage.rings.number_field
sage: P = Polyhedra(AA, 0)
sage: P.category()
Category of finite enumerated polyhedral sets over Algebraic Real Field
sage: P.list()
[The empty polyhedron in AA^0,
A 0-dimensional polyhedron in AA^0 defined as the convex hull of 1 vertex]
sage: P.cardinality()
2

recycle (polyhedron)
Recycle the H/V-representation objects of a polyhedron.

This speeds up creation of new polyhedra by reusing objects. After recycling a polyhedron object, it is not
in a consistent state any more and neither the polyhedron nor its H/V-representation objects may be used any
more.

INPUT:
• polyhedron – a polyhedron whose parent is self.

EXAMPLES:

sage: p = Polyhedron([(0,0),(1,0),(0,1)])
sage: p.parent().recycle(p)

some_elements ()
Return a list of some elements of the semigroup.

EXAMPLES:

sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: Polyhedra(QQ, 4).some_elements()
[A 3-dimensional polyhedron in QQ^4
defined as the convex hull of 4 vertices,
A 4-dimensional polyhedron in QQ^4
defined as the convex hull of 1 vertex and 4 rays,
A 2-dimensional polyhedron in QQ^4
defined as the convex hull of 2 vertices and 1 ray,
The empty polyhedron in QQ^4]
sage: Polyhedra(ZZ, 0).some_elements()
[The empty polyhedron in ZZ^0,
A 0-dimensional polyhedron in ZZ^0 defined as the convex hull of 1 vertex]

universe ()
Return the entire ambient space as polyhedron.

EXAMPLES:

sage: from sage.geometry.polyhedron.parent import Polyhedra
sage: P = Polyhedra(QQ, 4)
sage: P.universe()
A 4-dimensional polyhedron in QQ^4 defined as
the convex hull of 1 vertex and 4 lines

```sage
P.universe().is_universe()
```

```
True
```

**zero()**

Return the polyhedron consisting of the origin, which is the neutral element for Minkowski addition.

**EXAMPLES:**

```sage
from sage.geometry.polyhedron.parent import Polyhedra
p = Polyhedra(QQ, 4).zero(); p
```

```
A 0-dimensional polyhedron in QQ^4 defined as the convex hull of 1 vertex
```

```sage
p + p == p
```

```
True
```

**class** `sage.geometry.polyhedron.parent.Polyhedra_field(base_ring, ambient_dim, backend)`

**Bases:** `Polyhedra_base`

**Element**

alias of `Polyhedron_field`

**class** `sage.geometry.polyhedron.parent.Polyhedra_normaliz(base_ring, ambient_dim, backend)`

**Bases:** `Polyhedra_base`

**Element**

alias of `Polyhedron_normaliz`

**class** `sage.geometry.polyhedron.parent.Polyhedra_number_field(base_ring, ambient_dim, backend)`

**Bases:** `Polyhedra_base`

**Element**

alias of `Polyhedron_number_field`

**class** `sage.geometry.polyhedron.parent.Polyhedra_polymake(base_ring, ambient_dim, backend)`

**Bases:** `Polyhedra_base`

**Element**

alias of `Polyhedron_polymake`

**sage.geometry.polyhedron.parent.does_backend_handle_base_ring(backend)**

Return true, if backend can handle `base_ring`.

**EXAMPLES:**

```sage
from sage.geometry.polyhedron.parent import does_backend_handle_base_ring
does_backend_handle_base_ring(QQ, 'ppl')
```

```
True
```

```sage
does_backend_handle_base_ring(QQ[sqrt(5)], 'ppl')
```

```
# needs sage.rings.number_field sage.symbolic
False
```

```sage
does_backend_handle_base_ring(QQ[sqrt(5)], 'field')
```

```
# needs sage.rings.number_field sage.symbolic
True
```

2.1. Polyhedra 125
2.1.4 Hyperplane and Vertex representation objects for polyhedra

class sage.geometry.polyhedron.representation.Equation(polyhedron_parent)
    
Bases: Hrepresentation

A linear equation of the polyhedron. That is, the polyhedron is strictly smaller-dimensional than the ambient space, and contained in this hyperplane. Inherits from Hrepresentation.

contains(Vobj)

Tests whether the hyperplane defined by the equation contains the given vertex/ray/line.

EXAMPLES:

```sage
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: v = next(p.vertex_generator())
sage: v
A vertex at (0, 0, 0)
sage: a = next(p.equation_generator())
sage: a
An equation (0, 0, 1) x + 0 == 0
sage: a.contains(v)
True
```

interior_contains(Vobj)

Tests whether the interior of the halfspace (excluding its boundary) defined by the inequality contains the given vertex/ray/line.

Note: Return False for any equation.

EXAMPLES:

```sage
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: v = next(p.vertex_generator())
sage: v
A vertex at (0, 0, 0)
sage: a = next(p.equation_generator())
sage: a
An equation (0, 0, 1) x + 0 == 0
sage: a.interior_contains(v)
False
```

is_equation()

Tests if this object is an equation. By construction, it must be.

type()

Return the type (equation/inequality/vertex/ray/line) as an integer.

OUTPUT:

Integer. One of PolyhedronRepresentation.INEQUALITY, .EQUATION, .VERTEX, .RAY, or .LINE.

EXAMPLES:

```sage
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: repr_obj = next(p.equation_generator())
sage: repr_obj.type()
```
class sage.geometry.polyhedron.representation.Hrepresentation(polyhedron_parent)

Bases: PolyhedronRepresentation

The internal base class for H-representation objects of a polyhedron. Inherits from PolyhedronRepresentation.

A()

Return the coefficient vector \( \mathbf{A} \) in \( \mathbf{A}x + \mathbf{b} \).

EXAMPLES:

```python
sage: p = Polyhedron(ieqs = [[0,1,0],[0,0,1],[1,-1,0],[1,0,-1]])
sage: pH = p.Hrepresentation(2)
sage: pH.A()
(1, 0)
```

adjacent()

Alias for neighbors().

b()

Return the constant \( b \) in \( \mathbf{A}x + \mathbf{b} \).

EXAMPLES:

```python
sage: p = Polyhedron(ieqs = [[0,1,0],[0,0,1],[1,-1,0],[1,0,-1]])
sage: pH = p.Hrepresentation(2)
sage: pH.b()
0
```

eval(Vobj)

Evaluate the left hand side \( \mathbf{A}x + \mathbf{b} \) on the given vertex/ray/line.

EXAMPLES:

```python
sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[-1,-1]])
sage: ineq = next(triangle.inequality_generator())
sage: ineq
An inequality (2, -1) x + 1 >= 0
sage: [ ineq.eval(v) for v in triangle.vertex_generator() ]
[0, 0, 3]
sage: [ ineq * v for v in triangle.vertex_generator() ]
[0, 0, 3]
```

If you pass a vector, it is assumed to be the coordinate vector of a point:
Combinatorial and Discrete Geometry, Release 10.3

```
sage: ineq.eval(vector(ZZ, [3,2]))
5
```

**incident()**

Return a generator for the incident H-representation objects, that is, the vertices/rays/lines satisfying the (in)equality.

**EXAMPLES:**

```
sage: triangle = Polyhedron(ieqs=[[1,0],[0,1],[1,-1]])
sage: ineq = next(triangle.inequality_generator())
sage: ineq
An inequality (2, -1) x + 1 >= 0
```

```
sage: [v for v in ineq.incident()]
[A vertex at (-1, -1), A vertex at (0, 1)]
```

```
sage: p = Polyhedron(ieqs=[[0,0,0],[0,1,0],[0,0,1]], rays=[[1,-1,-1]])
sage: ineq = p.Hrepresentation(2)
sage: ineq
An inequality (1, 0, 1) x + 0 >= 0
```

```
sage: [x for x in ineq.incident()]
[A vertex at (0, 0, 0), A vertex at (0, 1, 0), A ray in the direction (1, -1, -1)]
```

**is_H()**

Return True if the object is part of a H-representation (inequality or equation).

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs=[[0,1,0],[0,0,1],[1,-1,0],[1,0,-1]])
sage: pH = p.Hrepresentation(0)
sage: pH.is_H()
True
```

**is_equation()**

Return True if the object is an equation of the H-representation.

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs=[[0,1,0],[0,0,1],[1,-1,0],[1,0,-1]], eqns=[[1,-1,1]])
sage: pH = p.Hrepresentation(0)
sage: pH.is_equation()
True
```

**is_incident(Vobj)**

Return whether the incidence matrix element (Vobj,self) == 1

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs=[[0,0,0,1],[0,0,1,0],[0,1,0,0],
...: [1,-1,0,0],[1,0,-1,0],[1,0,0,-1]])
sage: pH = p.Hrepresentation(0)
sage: pH.is_incident(p.Vrepresentation(1))
True
sage: pH.is_incident(p.Vrepresentation(5))
False
```
**is_inequality()**

Return True if the object is an inequality of the H-representation.

**EXAMPLES:**

```python
sage: p = Polyhedron(ieqs = [[0,1,0],[0,0,1],[1,-1,0],[1,0,-1]])
sage: pH = p.Hrepresentation(0)
sage: pH.is_inequality()
True
```

**neighbors()**

Iterate over the adjacent facets (i.e. inequalities).

Only defined for inequalities.

**EXAMPLES:**

```python
sage: p = Polyhedron(ieqs = [[0,0,0,1],[0,0,1,0],[0,1,0,0],
....: [1,-1,0,0],[1,0,-1,0],[1,0,0,-1]])
sage: pH = p.Hrepresentation(0)
sage: a = list(pH.neighbors())
sage: a[0]
An inequality (0, -1, 0) x + 1 >= 0
sage: list(a[0])
[1, 0, -1, 0]
```

**repr_pretty(**kwds**)**

Return a pretty representation of this equality/inequality.

**INPUT:**

- prefix—a string
- indices—a tuple or other iterable
- latex—a boolean

**OUTPUT:**

A string

**EXAMPLES:**

```python
sage: P = Polyhedron(ieqs=[[0, 1, 0, 0], (1, 2, 1, 0)],
....: eqns=[(1, -1, -1, 1)])
sage: for h in P.Hrepresentation():
....:     print(h.repr_pretty())
x0 + x1 - x2 == 1
x0 >= 0
2*x0 + x1 >= -1
```

**class** `sage.geometry.polyhedron.representation.Inequality(polyhedron_parent)`

**Bases:** `Hrepresentation`

A linear inequality (supporting hyperplane) of the polyhedron. Inherits from `Hrepresentation`.

**contains(Vobj)**

Tests whether the halfspace (including its boundary) defined by the inequality contains the given vertex/ray/line.

**EXAMPLES:**

---

2.1. Polyhedra
sage: p = polytopes.cross_polytope(3)
sage: i1 = next(p.inequality_generator())

sage: [i1.contains(q) for q in p.vertex_generator()]
[True, True, True, True, True, True]

sage: p2 = 3*polytopes.hypercube(3)

sage: [i1.contains(q) for q in p2.vertex_generator()]
[True, True, False, True, False, True, False, False]

interior_contains (Vobj)

Tests whether the interior of the halfspace (excluding its boundary) defined by the inequality contains the
given vertex/ray/line.

EXAMPLES:

sage: p = polytopes.cross_polytope(3)
sage: i1 = next(p.inequality_generator())

sage: [i1.interior_contains(q) for q in p.vertex_generator()]
[False, True, True, False, False, True]

sage: p2 = 3*polytopes.hypercube(3)

sage: [i1.interior_contains(q) for q in p2.vertex_generator()]
[True, True, False, True, False, True, False, False]

If you pass a vector, it is assumed to be the coordinate vector of a point:

sage: P = Polyhedron(vertices=[[1,1],[1,-1],[-1,1],[-1,-1]])
sage: p = vector(ZZ, [1,0])

sage: [ ieq.interior_contains(p) for ieq in P.inequality_generator() ]
[True, True, False, True]

is_facet_defining_inequality (other)

Check if self defines a facet of other.

INPUT:

• other – a polyhedron

See also:

slack_matrix() incidence_matrix()

EXAMPLES:

sage: P = Polyhedron(vertices=[[0,0,0],[0,1,0]], rays=[[1,0,0]])
sage: P.inequalities()
(An inequality (1, 0, 0) x + 0 >= 0,  
An inequality (0, 1, 0) x + 0 >= 0,  
An inequality (0, -1, 0) x + 1 >= 0)

sage: Q = Polyhedron(ieqs=[[0,1,0,0]])

sage: Q.inequalities()[0].is_facet_defining_inequality(P)
True

sage: Q = Polyhedron(ieqs=[[0,2,0,3]])

sage: Q.inequalities()[0].is_facet_defining_inequality(P)
True

sage: Q = Polyhedron(ieqs=[[0,AA(2).sqrt(),0,3]])

# needs sage.rings.number_field

sage: Q.inequalities()[0].is_facet_defining_inequality(P)
True

sage: Q = Polyhedron(ieqs=[[1,1,0,0]])

(continues on next page)
is_inequality()

Return True since this is, by construction, an inequality.

EXAMPLES:

```python
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: a = next(p.inequality_generator())
sage: a.is_inequality()
True
```

outer_normal()

Return the outer normal vector of self.

OUTPUT: The normal vector directed away from the interior of the polyhedron.

EXAMPLES:

```python
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: a = next(p.inequality_generator())
sage: a.outer_normal()
(1, -1, 0)
```

type()

Return the type (equation/inequality/vertex/ray/line) as an integer.

OUTPUT: Integer. One of PolyhedronRepresentation.INEQUALITY, .EQUATION, .VERTEX, .RAY, or .LINE.

EXAMPLES:

```python
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: repr_obj = next(p.inequality_generator())
sage: repr_obj.type()
0
sage: repr_obj.type() == repr_obj.INEQUALITY
True
sage: repr_obj.type() == repr_obj.EQUATION
False
```
class sage.geometry.polyhedron.representation.Line(polyhedron_parent)

Bases: Vrepresentation

A line (Minkowski summand \(\cong \mathbb{R}\)) of the polyhedron. Inherits from Vrepresentation.

evaluated_on(Hobj)

Return \(\mathbf{A} \ell\)^T

EXAMPLES:

```python
sage: p = Polyhedron(ieqs = [[1, 0, 0, 1], [1,1,0,0]])
sage: a = next(p.line_generator())
sage: h = next(p.inequality_generator())
sage: a.evaluated_on(h)
0
```

homogeneous_vector(base_ring=None)

Return homogeneous coordinates for this line.

Since a line is given by a direction, this is the vector with a 0 appended.

INPUT:

- base_ring – the base ring of the vector.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[(2,0)], rays=[(1,0)], lines=[(3,2)])
sage: P.lines()[0].homogeneous_vector()
(3, 2, 0)
sage: P.lines()[0].homogeneous_vector(RDF)
(3.0, 2.0, 0.0)
```

is_line()

Tests if the object is a line. By construction it must be.

type()

Return the type (equation/inequality/vertex/ray/line) as an integer.

OUTPUT:

Integer. One of PolyhedronRepresentation.INEQUALITY, .EQUATION, .VERTEX, .RAY, or .LINE.

EXAMPLES:

```python
sage: p = Polyhedron(ieqs = [[1, 0, 0, 1], [1,1,0,0]])
sage: repr_obj = next(p.line_generator())
sage: repr_obj.type()
4
sage: repr_obj.type() == repr_obj.INEQUALITY
```
class sage.geometry.polyhedron.representation.PolyhedronRepresentation

Bases: sageobject

The internal base class for all representation objects of Polyhedron (vertices/rays/lines and inequalities/equations)

Note: You should not (and cannot) instantiate it yourself. You can only obtain them from a Polyhedron() class.

EQUATION = 1
INEQUALITY = 0
LINE = 4
RAY = 3
VERTEX = 2

count (i)
Count the number of occurrences of i in the coordinates.

INPUT:
• i – Anything.

OUTPUT:
Integer. The number of occurrences of i in the coordinates.

EXAMPLES:

```python
sage: p = Polyhedron(vertices=[(0,1,1,2,1)])
sage: v = p.Vrepresentation(0); v
A vertex at (0, 1, 1, 2, 1)
sage: v.count(1)
3
```

index ()
Return an arbitrary but fixed number according to the internal storage order.

Note: H-representation and V-representation objects are enumerated independently. That is, amongst all vertices/rays/lines there will be one with index()==0, and amongst all inequalities/equations there will be one with index()==0, unless the polyhedron is empty or spans the whole space.

EXAMPLES:
```python
sage: s = Polyhedron(vertices=[[1],[-1]])
sage: first_vertex = next(s.vertex_generator())
sage: first_vertex.index()
0
sage: first_vertex == s.Vrepresentation(0)
True
```

**polyhedron()**

Return the underlying polyhedron.

**vector**(base_ring=None)

Return the vector representation of the H/V-representation object.

**INPUT:**

- base_ring – the base ring of the vector.

**OUTPUT:**

For a V-representation object, a vector of length ambient_dim(). For a H-representation object, a vector of length ambient_dim() + 1.

**EXAMPLES:**

```python
sage: s = polytopes.cuboctahedron()
sage: v = next(s.vertex_generator())
sage: v
A vertex at (-1, -1, 0)
sage: v.vector()
(-1, -1, 0)
sage: v()
(-1, -1, 0)
sage: type(v())
<class 'sage.modules.vector_integer_dense.Vector_integer_dense'>
```

Conversion to a different base ring can be forced with the optional argument:

```python
sage: v.vector(RDF)
(-1.0, -1.0, 0.0)
sage: vector(RDF, v)
(-1.0, -1.0, 0.0)
```

**class** `sage.geometry.polyhedron.representation.Ray(polyhedron_parent)`

**Bases:** `Vrepresentation`

A ray of the polyhedron. Inherits from `Vrepresentation`.

**evaluated_on**(Hobj)

Return $A r^n$

**EXAMPLES:**

```python
sage: p = Polyhedron(ieqs=[[0,0,1],[0,1,0],[1,-1,0]])
sage: a = next(p.ray_generator())
sage: h = next(p.inequality_generator())
sage: a.evaluated_on(h)
0
```
homogeneous_vector (base_ring=None)

Return homogeneous coordinates for this ray.
Since a ray is given by a direction, this is the vector with a 0 appended.

INPUT:

• base_ring – the base ring of the vector.

EXAMPLES:

```sage
P = Polyhedron(vertices=[[2,0]], rays=[[1,0]], lines=[[3,2]])
sage: P.rays()[0].homogeneous_vector()
(1, 0, 0)
sage: P.rays()[0].homogeneous_vector(RDF)
(1.0, 0.0, 0.0)
```

is_ray()

Tests if this object is a ray. Always True by construction.

EXAMPLES:

```sage
p = Polyhedron(ieqs = [[0,0,1],[0,1,0],[1,-1,0]])
sage: a = next(p.ray_generator())
sage: a.is_ray()
True
```

type()

Return the type (equation/inequality/vertex/ray/line) as an integer.

OUTPUT:

Integer. One of PolyhedronRepresentation.INEQUALITY, .EQUATION, .VERTEX, .RAY, or .LINE.

EXAMPLES:

```sage
p = Polyhedron(ieqs = [[0,0,1],[0,1,0],[1,-1,0]])
sage: repr_obj = next(p.ray_generator())
sage: repr_obj.type()
3
sage: repr_obj.type() == repr_obj.INEQUALITY
False
sage: repr_obj.type() == repr_obj.EQUATION
False
sage: repr_obj.type() == repr_obj.VERTEX
False
sage: repr_obj.type() == repr_obj.RAY
True
sage: repr_obj.type() == repr_obj.LINE
False
```

```class` sage.geometry.polyhedron.representation.Vertex (polyhedron_parent)

Bases: Vrepresentation

A vertex of the polyhedron. Inherits from Vrepresentation.

evaluated_on (Hobj)

Return \( \mathbf{A} \mathbf{x} + \mathbf{b} \)

EXAMPLES:
sage: p = polytopes.hypercube(3)
sage: v = next(p.vertex_generator())
sage: h = next(p.inequality_generator())
sage: v
A vertex at (1, -1, -1)
sage: h
An inequality (-1, 0, 0) x + 1 >= 0
sage: v.evaluated_on(h)
0

homogeneous_vector (base_ring=None)

Return homogeneous coordinates for this vertex.

Since a vertex is given by an affine point, this is the vector with a 1 appended.

INPUT:

- base_ring – the base ring of the vector.

EXAMPLES:

sage: P = Polyhedron(vertices=[(2,0)], rays=[(1,0)], lines=[(3,2)])
sage: P.vertices()[0].homogeneous_vector()
(2, 0, 1)
sage: P.vertices()[0].homogeneous_vector(RDF)
(2.0, 0.0, 1.0)

is_integral()

Return whether the coordinates of the vertex are all integral.

OUTPUT:

Boolean.

EXAMPLES:

sage: p = Polyhedron([(1/2,3,5), (0,0,0), (2,3,7)])
sage: [ v.is_integral() for v in p.vertex_generator() ]
[True, False, True]

is_vertex()

Tests if this object is a vertex. By construction it always is.

EXAMPLES:

sage: p = Polyhedron(ieqs = [[0,0,1],[0,1,0],[1,-1,0]])
sage: a = next(p.vertex_generator())
sage: a.is_vertex()
True

type()

Return the type (equation/inequality/vertex/ray/line) as an integer.

OUTPUT:

Integer. One of PolyhedronRepresentation.INEQUALITY, .EQUATION, .VERTEX, .RAY, or .LINE.

EXAMPLES:
sage: p = Polyhedron(vertices = [[0,0,0],[1,1,0],[1,2,0]])
sage: repr_obj = next(p.vertex_generator())
sage: repr_obj.type()
2
sage: repr_obj.type() == repr_obj.INEQUALITY
False
sage: repr_obj.type() == repr_obj.EQUATION
False
sage: repr_obj.type() == repr_obj.VERTEX
True
sage: repr_obj.type() == repr_obj.RAY
False
sage: repr_obj.type() == repr_obj.LINE
False

```python
class sage.geometry.polyhedron.representation.Vrepresentation(polyhedron_parent)
Bases: PolyhedronRepresentation

The base class for V-representation objects of a polyhedron. Inherits from PolyhedronRepresentation.

adjacent()

Alias for neighbors().

incident()

Return a generator for the equations/inequalities that are satisfied on the given vertex/ray/line.

EXAMPLES:

sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[-1,-1]])
sage: ineq = next(triangle.inequality_generator())
sage: ineq
An inequality (2, -1) x + 1 >= 0
sage: [ v for v in ineq.incident() ]
[A vertex at (-1, -1), A vertex at (0, 1)]
sage: p = Polyhedron(vertices=[[0,0,0],[0,1,0],[0,0,1]], rays=[[1,-1,-1]])
sage: ineq = p.Hrepresentation(2)
sage: ineq
An inequality (1, 0, 1) x + 0 >= 0
sage: [ x for x in ineq.incident() ]
[A vertex at (0, 0, 0),
 A vertex at (0, 1, 0),
 A ray in the direction (1, -1, -1)]
```

is_V()

Return True if the object is part of a V-representation (a vertex, ray, or line).

EXAMPLES:

```python
sage: p = Polyhedron(vertices = [[0,0],[1,0],[0,3],[1,3]])
sage: v = next(p.vertex_generator())
sage: v.is_V()
True
```

is_incident(Hobj)

Return whether the incidence matrix element (self,Hobj) == 1

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```
sage: p = polytopes.hypercube(3)
sage: h1 = next(p.inequality_generator())
sage: h1
An inequality (-1, 0, 0) x + 1 >= 0
sage: v1 = next(p.vertex_generator())
sage: v1
A vertex at (1, -1, -1)
sage: v1.is_incident(h1)
True
```

**is_line()**

Return True if the object is a line of the V-representation. This method is over-ridden by the corresponding method in the derived class Line.

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs = [[1, 0, 0, 0, 1], [1, 1, 0, 0, 0], [1, 0, 1, 0, 0], [1, 0, 0, 1, 0], [0, 0, 0, 0, 0]])
sage: line1 = next(p.line_generator())
sage: line1.is_line()
True
sage: v1 = next(p.vertex_generator())
sage: v1.is_line()
False
```

**is_ray()**

Return True if the object is a ray of the V-representation. This method is over-ridden by the corresponding method in the derived class Ray.

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs = [[1, 0, 0, 0, 1], [1, 1, 0, 0, 0], [1, 0, 1, 0, 0], [1, 0, 0, 1, 0], [0, 0, 0, 0, 0]])
sage: r1 = next(p.ray_generator())
sage: r1.is_ray()
True
sage: v1 = next(p.vertex_generator())
sage: v1.is_ray()
False
```

**is_vertex()**

Return True if the object is a vertex of the V-representation. This method is over-ridden by the corresponding method in the derived class Vertex.

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs = [[1, 0, 0, 0, 1], [1, 1, 0, 0, 0], [1, 0, 1, 0, 0], [1, 0, 0, 1, 0], [0, 0, 0, 0, 0]])
sage: r1 = next(p.ray_generator())
sage: r1.is_vertex()
False
```
neighbors()

Return a generator for the adjacent vertices/rays/lines.

EXAMPLES:

```python
sage: p = Polyhedron(vertices = [[0,0],[1,0],[0,3],[1,4]])
sage: v = next(p.vertex_generator())
sage: next(v.neighbors())

A vertex at (0, 3)
```

sage.geometry.polyhedron.representation.repr_pretty(coefficients, type, prefix='x', indices=None, latex=False, style='>=', split=False)

Return a pretty representation of equation/inequality represented by the coefficients.

INPUT:

- coefficients – a tuple or other iterable
- type – either 0 (PolyhedronRepresentation.INEQUALITY) or 1 (PolyhedronRepresentation.EQUATION)
- prefix – a string (default: x)
- indices – a tuple or other iterable
- latex – a boolean
- split – a boolean; (Default: False). If set to True, the output is split into a 3-tuple containing the left-hand side, the relation, and the right-hand side of the object.
- style – either "positive" (making all coefficients positive), or "<=" or ">=".

OUTPUT:

A string or 3-tuple of strings (depending on split).

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.representation import repr_pretty
sage: from sage.geometry.polyhedron.representation import PolyhedronRepresentation
sage: print(repr_pretty((0, 1, 0, 0), PolyhedronRepresentation.INEQUALITY))
x0 >= 0
sage: print(repr_pretty((1, 2, 1, 0), PolyhedronRepresentation.INEQUALITY))
2*x0 + x1 >= -1
sage: print(repr_pretty((1, -1, -1, 1), PolyhedronRepresentation.EQUATION))
-x0 - x1 + x2 == -1
```

2.1.5 Functions for plotting polyhedra

class sage.geometry.polyhedron.plot.Projection(polygon, proj=<function projection_func_identity>)

Bases: SageObject

The projection of a Polyhedron.

This class keeps track of the necessary data to plot the input polyhedron.
coord_index_of(v)

Convert a coordinate vector to its internal index.

EXAMPLES:

```sage
sage: p = polytopes.hypercube(3)
sage: proj = p.projection()
sage: proj.coord_index_of(vector((1,1,1)))
2
```

coord_indices_of(v_list)

Convert list of coordinate vectors to the corresponding list of internal indices.

EXAMPLES:

```sage
sage: p = polytopes.hypercube(3)
sage: proj = p.projection()
sage: proj.coord_indices_of([vector((1,1,1)), vector((1,-1,1))])
[2, 3]
```

coordinates_of(coord_index_list)

Given a list of indices, return the projected coordinates.

EXAMPLES:

```sage
sage: p = polytopes.simplex(4, project=True).projection()
sage: p.coordinates_of([1])
[[-0.7071067812, 0.4082482905, 0.2886751346, 0.2236067977]]
```

identity()

Return the identity projection of the polyhedron.

EXAMPLES:

```sage
sage: # needs sage.groups
sage: p = polytopes.icosahedron(exact=False)
sage: from sage.geometry.polyhedron.plot import Projection
sage: pproj = Projection(p)
sage: ppid = pproj.identity()
sage: ppid.dimension
3
```

render_0d(point_opts=None, line_opts=None, polygon_opts=None)

Return 0d rendering of the projection of a polyhedron into 2-dimensional ambient space.

INPUT:

See plot().

OUTPUT:

A 2-d graphics object.

EXAMPLES:

```sage
sage: print(Polyhedron([]).projection().render_0d().description())  # needs sage.plot
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```python
sage: print(P.projection().render_0d().description())
needs sage.plot
Point set defined by 1 point(s): [0.0, 0.0]
```

**render_1d** *(point_opts=None, line_opts=None, polygon_opts=None)*

Return 1d rendering of the projection of a polyhedron into 2-dimensional ambient space.

**INPUT:**

See plot().

**OUTPUT:**

A 2-d graphics object.

**EXAMPLES:**

```python
sage: Polyhedron([[0],[1]]).projection().render_1d() # needs sage.plot
Graphics object consisting of 2 graphics primitives
```

**render_2d** *(point_opts=None, line_opts=None, polygon_opts=None)*

Return 2d rendering of the projection of a polyhedron into 2-dimensional ambient space.

**EXAMPLES:**

```python
sage: p1 = Polyhedron(vertices=[[1,1]], rays=[[1,1]])
sage: q1 = p1.projection()
sage: p2 = Polyhedron(vertices=[[1,0],[0,1],[0,0]])
sage: q2 = p2.projection()
sage: p3 = Polyhedron(vertices=[[1,2]])
sage: q3 = p3.projection()
sage: p4 = Polyhedron(vertices=[[2,0]], rays=[[1,-1]], lines=[[1,1]])
sage: q4 = p4.projection()
sage: q1.plot() + q2.plot() + q3.plot() + q4.plot() # needs sage.plot
Graphics object consisting of 18 graphics primitives
```

**render_3d** *(point_opts=None, line_opts=None, polygon_opts=None)*

Return 3d rendering of a polyhedron projected into 3-dimensional ambient space.

**EXAMPLES:**

```python
sage: p1 = Polyhedron(vertices=[[1,1,1]], rays=[[1,1,1]])
sage: p2 = Polyhedron(vertices=[[2,0,0],[0,2,0],[0,0,2]])
sage: p3 = Polyhedron(vertices=[[1,0,0],[0,1,0],[0,0,1]],
....:         rays=[[1,-1,-1]])
sage: (p1.projection().plot() + p2.projection().plot())
....: + p3.projection().plot()) # needs sage.plot
Graphics3d Object
```

It correctly handles various degenerate cases:

```python
sage: # needs sage.plot
sage: Polyhedron(lines=[[1,0,0],[0,1,0],[0,0,1]]).plot() # whole space
Graphics3d Object
sage: Polyhedron(vertices=[[1,1,1]], rays=[[1,0,0]],
```

(continues on next page)
The origin is not included, if it is not in the polyhedron (github issue #23555):

```
sage: Q = Polyhedron([[100],[101]])
sage: P = Q*Q*Q; P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: p = P.plot()
# needs sage.plot
sage: p.bounding_box()
# needs sage.plot
((100.0, 100.0, 100.0), (101.0, 101.0, 101.0))
```

Plot 3d polytope with rainbow colors:

```
sage: polytopes.hypercube(3).plot(polygon='rainbow', alpha=0.4)  # needs sage.plot
```

**render_fill_2d(**kwds**)

Return the filled interior (a polygon) of a polyhedron in 2d.

**EXAMPLES:**

```
sage: cps = [i^3 for i in srange(-2, 2, 1/5)]
sage: p = Polyhedron(vertices=[[t^2-1]/(t^2+1), 2*t/(t^2+1)] for t in cps)
sage: proj = p.projection()
sage: filled_poly = proj.render_fill_2d()  # needs sage.plot
sage: filled_poly.axes_width()  # needs sage.plot
0.8
```

**render_line_1d(**kwds**)

Return the line of a polyhedron in 1d.

**INPUT:**

- **kwds** – options passed through to line2d().

**OUTPUT:**
A 2-d graphics object.

**EXAMPLES:**

```python
sage: outline = polytopes.hypercube(1).projection().render_line_1d()  # needs sage.plot
sage: outline._objects[0]  # needs sage.plot
Line defined by 2 points
```

### render_outline_2d(**kwds)**

Return the outline (edges) of a polyhedron in 2d.

**EXAMPLES:**

```python
sage: penta = polytopes.regular_polygon(5)  # needs sage.rings.number_field
sage: outline = penta.projection().render_outline_2d()  # needs sage.plot sage.rings.number_field
sage: outline._objects[0]  # needs sage.plot sage.rings.number_field
Line defined by 2 points
```

### render_points_1d(**kwds)**

Return the points of a polyhedron in 1d.

**INPUT:**

- **kwds** – options passed through to `point2d()`.

**OUTPUT:**

A 2-d graphics object.

**EXAMPLES:**

```python
sage: cube1 = polytopes.hypercube(1)
sage: proj = cube1.projection()
sage: points = proj.render_points_1d()  # needs sage.plot
sage: points._objects  # needs sage.plot
[Point set defined by 2 point(s)]
```

### render_points_2d(**kwds)**

Return the points of a polyhedron in 2d.

**EXAMPLES:**

```python
sage: # needs sage.rings.number_field
sage: hex = polytopes.regular_polygon(6)
sage: proj = hex.projection()
sage: hex_points = proj.render_points_2d()  # needs sage.plot
sage: hex_points._objects  # needs sage.plot
[Point set defined by 6 point(s)]
```

### render_solid_3d(**kwds)**

Return solid 3d rendering of a 3d polytope.

2.1. Polyhedra
EXAMPLES:

```python
sage: p = polytopes.hypercube(3).projection()
sage: p_solid = p.render_solid_3d(opacity=.7)  # needs sage.plot
sage: type(p_solid)  # needs sage.plot
<class 'sage.plot.plot3d.index_face_set.IndexFaceSet'>
```

`render_vertices_3d(**kwds)`

Return the 3d rendering of the vertices.

EXAMPLES:

```python
sage: p = polytopes.cross_polytope(3)
sage: proj = p.projection()
sage: verts = proj.render_vertices_3d()  # needs sage.plot
sage: verts.bounding_box()  # needs sage.plot
((-1.0, -1.0, -1.0), (1.0, 1.0, 1.0))
```

`render_wireframe_3d(**kwds)`

Return the 3d wireframe rendering.

EXAMPLES:

```python
sage: cube = polytopes.hypercube(3)
sage: cube_proj = cube.projection()
sage: wire = cube_proj.render_wireframe_3d()  # needs sage.plot
sage: print(wire.tachyon().split('
')[77])  # for testing  # needs sage.plot
FCylinder base 1.0 1.0 -1.0 apex -1.0 1.0 -1.0 rad 0.005 texture...
```

`schlegel(facet=None, position=None)`

Return the Schlegel projection.

- The facet is orthonormally transformed into its affine hull.
- The position specifies a point coming out of the barycenter of the facet from which the other vertices will be projected into the facet.

INPUT:

- `facet` – a PolyhedronFace. The facet into which the Schlegel diagram is created. The default is the first facet.
- `position` – a positive number. Determines a relative distance from the barycenter of `facet`. A value close to 0 will place the projection point close to the facet and a large value further away. If the given value is too large, an error is returned. If no position is given, it takes the midpoint of the possible point of views along a line spanned by the barycenter of the facet and a valid point outside the facet.

EXAMPLES:

```python
sage: cube4 = polytopes.hypercube(4)
sage: Projection(cube4).schlegel()  # needs sage.plot
```
The 4-cube with a truncated vertex seen into the resulting tetrahedron facet:

```python
sage: tcube4 = cube4.face_truncation(cube4.faces(0)[0])
sage: tcube4.facets()[4]
A 3-dimensional face of a Polyhedron in QQ^4 defined as the convex hull of 4...
˓→vertices
sage: into_tetra = Projection(tcube4).schlegel(tcube4.facets()[4])  #...
˓→needs sage.symbolic
sage: into_tetra.plot()  #...
˓→needs sage.plot sage.symbolic
Graphics3d Object
```

Taking a larger value for the position changes the image:

```python
sage: into_tetra_far = Projection(tcube4).schlegel(tcube4.facets()[4], 4)  #...
˓→needs sage.symbolic
sage: into_tetra_far.plot()  #...
˓→needs sage.plot sage.symbolic
Graphics3d Object
```

A value which is too large or negative give a projection point that sees more than one facet resulting in an error:

```python
sage: Projection(tcube4).schlegel(tcube4.facets()[4], 5)
Traceback (most recent call last):
...
ValueError: the chosen position is too large
```

```python
sage: Projection(tcube4).schlegel(tcube4.facets()[4], -1)
Traceback (most recent call last):
...
ValueError: 'position' should be a positive number
```

**stereographic** (**projection_point**=None)

Return the stereographic projection.

**INPUT:**

- **projection_point** - The projection point. This must be distinct from the polyhedron’s vertices. Default is (1, 0, ..., 0)

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.plot import Projection
sage: proj = Projection(polytopes.buckyball()); proj  # long time
The projection of a polyhedron into 3 dimensions
sage: proj.stereographic([5,2,3]).plot()  # long time  #...
˓→needs sage.plot
Graphics object consisting of 123 graphics primitives
sage: Projection(polytopes.twenty_four_cell()).stereographic([2,0,0,0])
The projection of a polyhedron into 3 dimensions
```

**tikz** (**view**=[0, 0, 1], **angle**=0, **scale**=1, **edge_color**='blue!95!black', **facet_color**='blue!95!black', **opacity**=0.8, **vertex_color**='green', **axis**=False, **output_type**=None)

Return a tikz picture of self as a string or as a TikzPicture according to a projection view and an angle angle obtained via the threejs viewer.
INPUT:

• **view** – list (default: [0,0,1]) representing the rotation axis (see note below).
• **angle** – integer (default: 0) angle of rotation in degree from 0 to 360 (see note below).
• **scale** – integer (default: 1) specifying the scaling of the tikz picture.
• **edge_color** – string (default: ‘blue!95!black’) representing colors which tikz recognize.
• **facet_color** – string (default: ‘blue!95!black’) representing colors which tikz recognize.
• **vertex_color** – string (default: ‘green’) representing colors which tikz recognize.
• **opacity** – real number (default: 0.8) between 0 and 1 giving the opacity of the front facets.
• **axis** – Boolean (default: False) draw the axes at the origin or not.
• **output_type** – string (default: None), valid values are None (deprecated), 'LatexExpr' and 'TikzPicture', whether to return a LatexExpr object (which inherits from Python str) or a TikzPicture object from module sage.misc.latex_standalone

OUTPUT:

*LatexExpr* object or *TikzPicture* object

**Note:** The inputs **view** and **angle** can be obtained by visualizing it using `.show(aspect_ratio=1)`. This will open an interactive view in your default browser, where you can rotate the polytope. Once the desired view angle is found, click on the information icon in the lower right-hand corner and select *Get Viewpoint*. This will copy a string of the form '[x,y,z],angle' to your local clipboard. Go back to Sage and type

\[ \text{Img} = \text{P}.\text{projection().tikz([x,y,z],angle)} \]

The inputs **view** and **angle** can also be obtained from the viewer Jmol:

1) Right click on the image
2) Select ‘’Console’’
3) Select the tab ‘’State’’
4) Scroll to the line ‘’moveto’’

It reads something like:

```
moveto 0.0 {x y z angle} Scale
```

The **view** is then [x,y,z] and **angle** is angle. The following number is the scale.

Jmol performs a rotation of **angle** degrees along the vector [x,y,z] and show the result from the z-axis.

EXAMPLES:

```
sage: # needs sage.plot sage.rings.number_field
sage: P1 = polytopes.small_rhombicuboctahedron()
sage: Image1 = P1.projection().tikz([1,3,5], 175, scale=4,
.....: output_type='TikzPicture')
sage: type(Image1)
<class 'sage.misc.latex_standalone.TikzPicture'>
sage: Image1
\documentclass[tikz]{standalone}
\begin{document}
\begin{tikzpicture}
    \x={(-0.939161cm, 0.244762cm)},
    \y={(0.097442cm, -0.482887cm)},
```

(continues on next page)
A second example:

```sage
P2 = Polyhedron(vertices=[[1, 1], [1, 2], [2, 1]])
Image2 = P2.projection().tikz(scale=3, edge_color='blue!95!black',
   facet_color='orange!95!black', opacity=0.
   vertex_color='yellow', axis=True,
   output_type='TikzPicture')
```

The second example using a LatexExpr as output type:

```sage
# needs sage.plot
Image2 = P2.projection().tikz(scale=3, edge_color='blue!95!black',
   facet_color='orange!95!black', opacity=0.
   vertex_color='yellow', axis=True,
   output_type='LatexExpr')
type(Image2)
<class 'sage.misc.latex.LatexExpr'>
print(','.join(Image2.splitlines()[:4]))
```

(continues on next page)
A third example:

```
sage: with open('polytope-tikz2.tex', 'w') as f:
    # not tested
    _ = f.write(Image2)
```

A fourth example:

```
sage: P = Polyhedron(verts=[1,1,0,0], [1,2,0,0],
    ....:                  [2,1,0,0], [0,0,1,0], [0,0,0,1]); P
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 5 vertices
sage: P.projection().tikz(output_type='TikzPicture')
Traceback (most recent call last):
  ... Not ImplementedError: The polytope has to live in 2 or 3 dimensions.
```

Todo: Make it possible to draw Schlegel diagram for 4-polytopes.
Make it possible to draw 3-polytopes living in higher dimension.

```python
class sage.geometry.polyhedron.plot.ProjectionFuncSchlegel(facet, projection_point)
Bases: object
The Schlegel projection from the given input point.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.plot import ProjectionFuncSchlegel
sage: fcube = polytopes.hypercube(4)
sage: facet = fcube.facets()[0]
sage: proj = ProjectionFuncSchlegel(facet, [0, -1.5, 0, 0])
sage: proj([0,0,0,0])[0]
1.0
```

```python
class sage.geometry.polyhedron.plot.ProjectionFuncStereographic(projection_point)
Bases: object
The stereographic (or perspective) projection onto a codimension-1 linear subspace with respect to a sphere centered at the origin.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.plot import ProjectionFuncStereographic
sage: cube = polytopes.hypercube(3).vertices()
sage: proj = ProjectionFuncStereographic([1.2, 3.4, 5.6])
sage: ppoints = [proj(vector(x)) for x in cube]
sage: ppoints[5]
(-0.0918273..., -0.036375...)
```

```python
sage.geometry.polyhedron.plot.cyclic_sort_vertices_2d(Vlist)
Return the vertices/rays in cyclic order if possible.

Note: This works if and only if each vertex/ray is adjacent to exactly two others. For example, any 2-dimensional polyhedron satisfies this.

See vertex_adjacency_matrix() for a discussion of “adjacent”.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.plot import cyclic_sort_vertices_2d
sage: square = Polyhedron([[1,0],[-1,0],[0,1],[0,-1]])
sage: vertices = [v for v in square.vertex_generator()]
sage: vertices
[A vertex at (-1, 0),
 A vertex at (0, -1),
 A vertex at (0, 1),
 A vertex at (1, 0)]
sage: cyclic_sort_vertices_2d(vertices)
[A vertex at (1, 0),
 A vertex at (0, -1),
 A vertex at (-1, 0),
 A vertex at (0, 1)]
```

Rays are allowed, too:
sage: P = Polyhedron(vertices=[(0, 1), (1, 0), (2, 0), (3, 0), (4, 1)], rays=[(0, → 1)])
sage: P.adjacency_matrix()
[0 1 0 1 0]  
[1 0 1 0 0]  
[0 1 0 1 0]  
[1 0 0 1 0]  
[0 0 1 1 0]
sage: cyclic_sort_vertices_2d(P.Vrepresentation())  
[A vertex at (3, 0),  
A vertex at (1, 0),  
A vertex at (0, 1),  
A ray in the direction (0, 1),  
A vertex at (4, 1)]
sage: P = Polyhedron(vertices=[(0, 1), (1, 0), (2, 0), (3, 0), (4, 1)], rays=[(0, → 1), (1,1)])
sage: P.adjacency_matrix()
[0 1 0 0 0]  
[1 0 1 0 0]  
[0 1 0 1 0]  
[0 0 0 1 0]  
[0 0 1 1 0]
sage: cyclic_sort_vertices_2d(P.Vrepresentation())  
[A ray in the direction (1, 1),  
A vertex at (3, 0),  
A vertex at (1, 0),  
A vertex at (0, 1),  
A ray in the direction (0, 1)]

sage: P = Polyhedron(vertices=[(1,2)], rays=[(0,1)], lines=[(1,0)])
sage: P.adjacency_matrix()
[0 0 1]  
[0 0 0]  
[1 0 0]
sage: cyclic_sort_vertices_2d(P.Vrepresentation())  
[A vertex at (0, 2),  
A line in the direction (1, 0),  
A ray in the direction (0, 1)]
	sage.geometry.polyhedron.plot.projection_func_identity(x)

The identity projection.

EXAMPLES:

sage: from sage.geometry.polyhedron.plot import projection_func_identity
sage: projection_func_identity((1,2,3))
[1, 2, 3]
2.1.6 A class to keep information about faces of a polyhedron

This module gives you a tool to work with the faces of a polyhedron and their relative position. First, you need to find the faces. To get the faces in a particular dimension, use the `face()` method:

```python
sage: P = polytopes.cross_polytope(3)
sage: P.faces(3)
(A 3-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 6...
˓vertices,)
sage: [f.ambient_V_indices() for f in P.facets()]
[(3, 4, 5),
 (2, 4, 5),
 (1, 3, 5),
 (1, 2, 5),
 (0, 3, 4),
 (0, 2, 4),
 (0, 1, 3),
 (0, 1, 2)]
sage: [f.ambient_V_indices() for f in P.faces(1)]
[(4, 5),
 (3, 5),
 (2, 5),
 (1, 5),
 (3, 4),
 (2, 4),
 (0, 4),
 (1, 3),
 (0, 3),
 (1, 2),
 (0, 2),
 (0, 1)]
```

or `face_lattice()` to get the whole face lattice as a poset:

```python
sage: P.face_lattice() #...
˓needs sage.combinat
Finite lattice containing 28 elements
```

The faces are printed in shorthand notation where each integer is the index of a vertex/ray/line in the same order as the containing Polyhedron’s `Vrepresentation()`

```python
sage: face = P.faces(1)[8]; face
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices
sage: face.ambient_V_indices()
(0, 3)
sage: P.Vrepresentation(0)
A vertex at (-1, 0, 0)
sage: P.Vrepresentation(3)
A vertex at (0, 0, 1)
sage: face.vertices()
(A vertex at (-1, 0, 0), A vertex at (0, 0, 1))
```

The face itself is not represented by Sage’s `sage.geometry.polyhedron.constructor.Polyhedron()` class, but by an auxiliary class to keep the information. You can get the face as a polyhedron with the `PolyhedronFace.as_polyhedron()` method:
```python
sage: face.as_polyhedron()
A 1-dimensional polyhedron in ZZ^3 defined as the convex hull of 2 vertices
sage: _.equations()
(An equation (0, 1, 0) x + 0 == 0,
An equation (1, 0, -1) x + 1 == 0)
```

```python
class sage.geometry.polyhedron.face.PolyhedronFace(polynomial, V_indices, H_indices)
Bases: ConvexSet_closed

A face of a polyhedron.

This class is for use in face_lattice().

INPUT:

No checking is performed whether the H/V-representation indices actually determine a face of the polyhedron. You should not manually create PolyhedronFace objects unless you know what you are doing.

OUTPUT:

A PolyhedronFace.

EXAMPLES:

```python
sage: octahedron = polytopes.cross_polytope(3)
sage: inequality = octahedron.Hrepresentation(2)
sage: face_h = tuple([ inequality ])
sage: face_v = tuple( inequality.incident() )
sage: face_h_indices = [ h.index() for h in face_h ]
sage: face_v_indices = [ v.index() for v in face_v ]
sage: from sage.geometry.polyhedron.face import PolyhedronFace
sage: face = PolyhedronFace(octahedron, face_v_indices, face_h_indices)
sage: face
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: face.dim()
2
sage: face.ambient_V_indices()
(0, 1, 2)
sage: face.ambient_Hrepresentation()
(An inequality (1, 1, 1) x + 1 >= 0,)
sage: face.ambient_Vrepresentation()
(A vertex at (-1, 0, 0), A vertex at (0, -1, 0), A vertex at (0, 0, -1))
```

```python
affine_tangent_cone()

Return the affine tangent cone of self as a polyhedron.

It is equal to the sum of self and the cone of feasible directions at any point of the relative interior of self.

OUTPUT:

A polyhedron.

EXAMPLES:

```python
sage: half_plane_in_space = Polyhedron(ieqs=[(0,1,0,0)], eqns=[(0,0,0,1)])
sage: line = half_plane_in_space.faces(1)[0]; line
A 1-dimensional face of a Polyhedron in QQ^3 defined as the convex hull of 1 vertex and 1 line
sage: T_line = line.affine_tangent_cone()
(continues on next page)
```
sage: T_line == half_plane_in_space
True

sage: c = polytopes.cube()
sage: edge = min(c.faces(1))
sage: edge.vertices()
(A vertex at (1, -1, -1), A vertex at (1, 1, -1))
sage: T_edge = edge.affine_tangent_cone()
sage: T_edge.Vrepresentation()
(A line in the direction (0, 1, 0),
A ray in the direction (0, 0, 1),
A vertex at (1, 0, -1),
A ray in the direction (-1, 0, 0))

ambient()

Return the containing polyhedron.

EXAMPLES:

sage: P = polytopes.cross_polytope(3); P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
sage: face = P.facets()[3]; face
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 3
˓→vertices
sage: face.polyhedron()
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices

ambient_H_indices()

Return the indices of the H-representation objects of the ambient polyhedron that make up the
H-representation of self.

See also ambient_Hrepresentation().

OUTPUT:

Tuple of indices

EXAMPLES:

sage: Q = polytopes.cross_polytope(3)
sage: F = Q.faces(1)
sage: [f.ambient_H_indices() for f in F]
[(4, 5),
 (5, 6),
 (4, 7),
 (6, 7),
 (0, 5),
 (3, 4),
 (0, 3),
 (1, 6),
 (0, 1),
 (2, 7),
 (2, 3),
 (1, 2)]

ambient_Hrepresentation(index=None)

Return the H-representation objects of the ambient polytope defining the face.

INPUT:
\begin{itemize}
  \item \texttt{index} – optional. Either an integer or \texttt{None} (default).
\end{itemize}

\textbf{OUTPUT:}

If the optional argument is not present, a tuple of \texttt{H}-representation objects. Each entry is either an inequality or an equation.

If the optional integer \texttt{index} is specified, the \texttt{index}-th element of the tuple is returned.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: square = polytopes.hypercube(2)
sage: for face in square.face_lattice():
    print(face.ambient_Hrepresentation())
(An inequality (-1, 0) x + 1 >= 0, An inequality (0, -1) x + 1 >= 0, An inequality (1, 0) x + 1 >= 0, An inequality (0, 1) x + 1 >= 0)
(An inequality (-1, 0) x + 1 >= 0, An inequality (0, 1) x + 1 >= 0)
(An inequality (-1, 0) x + 1 >= 0, An inequality (0, -1) x + 1 >= 0)
(An inequality (-1, 0) x + 1 >= 0,)
(An inequality (0, -1) x + 1 >= 0, An inequality (1, 0) x + 1 >= 0)
(An inequality (0, -1) x + 1 >= 0,)
(An inequality (1, 0) x + 1 >= 0, An inequality (0, 1) x + 1 >= 0)
(An inequality (1, 0) x + 1 >= 0,)
()
\end{verbatim}

\texttt{ambient_V_indices()}

Return the indices of the \texttt{V}-representation objects of the ambient polyhedron that make up the \texttt{V}-representation of \texttt{self}.

See also \texttt{ambient_Vrepresentation()}.  

\textbf{OUTPUT:}

Tuple of indices

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P = polytopes.cube()
sage: F = P.faces(2)
sage: [f.ambient_V_indices() for f in F]
[(0, 3, 4, 5),
 (0, 1, 5, 6),
 (4, 5, 6, 7),
 (2, 3, 4, 7),
 (1, 2, 6, 7),
 (0, 1, 2, 3)]
\end{verbatim}

\texttt{ambient_Vrepresentation(index=None)}

Return the \texttt{V}-representation objects of the ambient polytope defining the face.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{index} – optional. Either an integer or \texttt{None} (default).
\end{itemize}

\textbf{OUTPUT:}

If the optional argument is not present, a tuple of \texttt{V}-representation objects. Each entry is either a vertex, a ray, or a line.

If the optional integer \texttt{index} is specified, the \texttt{index}-th element of the tuple is returned.
EXAMPLES:

```python
sage: square = polytopes.hypercube(2)
sage: for fl in square.face_lattice():
    print(fl.ambient_Vrepresentation())
(A vertex at (1, -1),)
(A vertex at (1, 1),)
(A vertex at (1, -1), A vertex at (1, 1))
(A vertex at (-1, 1),)
(A vertex at (1, 1), A vertex at (-1, 1))
(A vertex at (-1, -1),)
(A vertex at (1, -1), A vertex at (-1, -1))
(A vertex at (-1, 1), A vertex at (-1, -1),)
(A vertex at (1, -1), A vertex at (1, 1),
A vertex at (-1, 1), A vertex at (-1, -1))
```

ambient_dim()

Return the dimension of the containing polyhedron.

EXAMPLES:

```python
sage: P = Polyhedron(vertices = 
    [[1,0,0,0],[0,1,0,0]])

sage: face = P.faces(1)[0]
sage: face.ambient_dim()
4
```

ambient_vector_space(base_field=None)

Return the ambient vector space. It is the ambient free module of the containing polyhedron tensored with a field.

INPUT:

- base_field – (default: the fraction field of the base ring) a field.

EXAMPLES:

```python
sage: half_plane = Polyhedron(ieqs=[[0,1,0]])

sage: line = half_plane.faces(1)[0]; line
A 1-dimensional face of a Polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line

sage: line.ambient_vector_space()  
Vector space of dimension 2 over Rational Field

sage: line.ambient_vector_space(AA)  
Vector space of dimension 2 over Algebraic Real Field
```

as_polyhedron(**kwds)

Return the face as an independent polyhedron.

OUTPUT:

A polyhedron.

EXAMPLES:

```python
sage: P = polytopes.cross_polytope(3); P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
```

(continues on next page)
sage: face = P.faces(2)[3]; face
A 2-dimensional face of a
Polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: face.as_polyhedron()
A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: P.intersection(face.as_polyhedron()) == face.as_polyhedron()
True

contains (point)
Test whether the polyhedron contains the given point.

INPUT:

• point – a point or its coordinates

EXAMPLES:

sage: half_plane = Polyhedron(ieqs=[(0,1,0)])
sage: line = half_plane.faces(1)[0]; line
A 1-dimensional face of a
Polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line
sage: line.contains([0, 1])
True

As a shorthand, one may use the usual in operator:

sage: [5, 7] in line
False

dim()
Return the dimension of the face.

OUTPUT:

Integer.

EXAMPLES:

sage: fl = polytopes.dodecahedron().face_lattice()  # needs sage.combinat sage.rings.number_field
sage: sorted(x.dim() for x in fl)  # needs sage.combinat sage.rings.number_field
[-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3]

is_compact()
Return whether self is compact.

OUTPUT: Boolean.

EXAMPLES:

sage: half_plane = Polyhedron(ieqs=[(0,1,0)])
sage: line = half_plane.faces(1)[0]; line
A 1-dimensional face of a
Polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line
is_relatively_open()

Return whether self is relatively open.

OUTPUT: Boolean.

EXAMPLES:

```
sage: half_plane = Polyhedron(ieqs=[(0,1,0)])
sage: line = half_plane.faces(1)[0]; line
A 1-dimensional face of a
Polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line
sage: line.is_relatively_open()
True
```

line_generator()

Return a generator for the lines of the face.

EXAMPLES:

```
sage: pr = Polyhedron(rays = [[1,0],[-1,0],[0,1]], vertices = [[-2,-2],
                          [2,3]])
sage: face = pr.faces(1)[0]
sage: next(face.line_generator())
A line in the direction (1, 0)
```

lines()

Return all lines of the face.

OUTPUT:

A tuple of lines.

EXAMPLES:

```
sage: p = Polyhedron(rays = [[1,0],[-1,0],[0,1],[1,1]], vertices = [[-2,-2],
                          [2,3]])
sage: p.lines()
(A line in the direction (1, 0),)
```

n_ambient_Hrepresentation()

Return the number of objects that make up the ambient H-representation of the polyhedron.

See also ambient_Hrepresentation().

OUTPUT:

Integer.

EXAMPLES:

```
sage: p = polytopes.cross_polytope(4)
sage: face = p.face_lattice()[5]; face
A 1-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 2...

```

(continues on next page)
(An inequality \((1, -1, 1, -1) x + 1 \geq 0\),
An inequality \((1, 1, 1, 1) x + 1 \geq 0\),
An inequality \((1, 1, -1, 1) x + 1 \geq 0\),
An inequality \((1, -1, 1, 1) x + 1 \geq 0\))

\[
\text{sage: face.n_ambient_Hrepresentation()}
\]

\[
\text{# needs sage.combinat}
\]

\[
\text{n_ambient_Vrepresentation()}
\]

Return the number of objects that make up the ambient V-representation of the polyhedron.

See also \texttt{ambient_Vrepresentation()}.  

OUTPUT:  

Integer.

EXAMPLES:

\[
\text{sage: p = polytopes.cross_polytope(4)}
\]
\[
\text{sage: face = p.face_lattice()[5]; face}
\]

\[
\text{# needs sage.combinat}
\]

A 1-dimensional face of a Polyhedron in \(\mathbb{Z}^4\) defined as the convex hull of 2 vertices

\[
\text{sage: face.ambient_Vrepresentation()}
\]

\[
\text{# needs sage.combinat}
\]

(A vertex at \((-1, 0, 0, 0)\), A vertex at \((0, 0, -1, 0)\))

\[
\text{sage: face.n_ambient_Vrepresentation()}
\]

\[
\text{# needs sage.combinat}
\]

\[
\text{n_lines()}
\]

Return the number of lines of the face.

OUTPUT:  

Integer.

EXAMPLES:

\[
\text{sage: p = Polyhedron(rays = [[1,0],[-1,0],[0,1],[1,1]], vertices = [[-2,-2],[2,3]])}
\]

\[
\text{sage: p.n_lines()}
\]

\[
1
\]

\[
\text{n_rays()}
\]

Return the number of rays of the face.

OUTPUT:  

Integer.

EXAMPLES:

\[
\text{sage: p = Polyhedron(ieqs = [[0,0,0,1],[0,0,1,0],[1,1,0,0]])}
\]

\[
\text{sage: face = p.faces(2)[0]}
\]

\[
\text{sage: face.n_rays()}
\]

\[
2
\]
**n_vertices()**

Return the number of vertices of the face.

OUTPUT:

Integer.

EXAMPLES:

```
sage: Q = polytopes.cross_polytope(3)
sage: face = Q.faces(2)[0]
sage: face.n_vertices()
3
```

**normal_cone(direction='outer')**

Return the polyhedral cone consisting of normal vectors to hyperplanes supporting self.

INPUT:

- direction – string (default: 'outer'), the direction in which to consider the normals. The other allowed option is 'inner'.

OUTPUT:

A polyhedron.

EXAMPLES:

```
sage: p = Polyhedron(vertices=[[1,2], [2,1], [-2,2], [-2,-2], [2,-2]])
sage: for v in p.face_generator(0):
    ....:    vect = v.vertices()[0].vector()
    ....:    nc = v.normal_cone().rays_list()
    ....:    print("{0} has outer normal cone spanned by {1}").format(vect,nc)
    ....:    (2, 1) has outer normal cone spanned by [1, 0], [1, 1]
    (1, 2) has outer normal cone spanned by [0, 1], [1, 1]
    (2, -2) has outer normal cone spanned by [0, -1], [1, 0]
    (-2, -2) has outer normal cone spanned by [-1, 0], [0, -1]
    (-2, 2) has outer normal cone spanned by [-1, 0], [0, 1]

sage: for v in p.face_generator(0):
    ....:    vect = v.vertices()[0].vector()
    ....:    nc = v.normal_cone(direction='inner').rays_list()
    ....:    print("{0} has inner normal cone spanned by {1}").format(vect,nc)
    ....:    (2, 1) has inner normal cone spanned by [-1, -1], [-1, 0]
    (1, 2) has inner normal cone spanned by [-1, -1], [0, -1]
    (2, -2) has inner normal cone spanned by [-1, 0], [0, 1]
    (-2, -2) has inner normal cone spanned by [0, 1], [1, 0]
    (-2, 2) has inner normal cone spanned by [0, -1], [1, 0]
```

The function works for polytopes that are not full-dimensional:

```
sage: p = polytopes.permutahedron(3)
sage: f1 = p.faces(0)[0]
sage: f2 = p.faces(1)[0]
sage: f3 = p.faces(2)[0]
sage: f1.normal_cone()
```

A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 1 vertex, 2 rays, 1 line

(continues on next page)
Normal cones are only defined for non-empty faces:

```sage
sage: f0 = p.faces(-1)[0]
sage: f0.normal_cone()
Traceback (most recent call last):
  ... ValueError: the empty face does not have a normal cone
```

polyhedron()
Return the containing polyhedron.

EXAMPLES:

```sage
sage: P = polytopes.cross_polytope(3); P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
sage: face = P.facets()[3]; face
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 3 → vertices
sage: face.polyhedron()
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
```

ray_generator()
Return a generator for the rays of the face.

EXAMPLES:

```sage
sage: pi = Polyhedron(ieqs = [[1,1,0],[1,0,1]])
sage: face = pi.faces(1)[1]
sage: next(face.ray_generator())
A ray in the direction (1, 0)
```

rays()
Return the rays of the face.

OUTPUT:
A tuple of rays.

EXAMPLES:

```sage
sage: p = Polyhedron(ieqs = [[0,0,0,1],[0,0,1,0],[1,1,0,0]])
sage: face = p.faces(2)[2]
sage: face.rays()
(A ray in the direction (1, 0, 0), A ray in the direction (0, 1, 0))
```

stacking_locus()
Return the polyhedron containing the points that sees every facet containing self.

OUTPUT:
A polyhedron.
EXAMPLES:

```python
sage: cp = polytopes.cross_polytope(4)
sage: facet = cp.facets()[0]
sage: facet.stacking_locus().vertices()
(A vertex at (1/2, 1/2, 1/2, 1/2),
 A vertex at (1, 0, 0, 0),
 A vertex at (0, 0, 0, 1),
 A vertex at (0, 0, 1, 0),
 A vertex at (0, 1, 0, 0))
sage: face = cp.faces(2)[0]
sage: face.stacking_locus().vertices()
(A vertex at (0, 1, 0, 0),
 A vertex at (0, 0, 1, 0),
 A vertex at (1, 0, 0, 0),
 A vertex at (1, 1, 1, 0),
 A vertex at (1/2, 1/2, 1/2, 1/2),
 A vertex at (1/2, 1/2, 1/2, -1/2))
```

**vertex_generator()**

Return a generator for the vertices of the face.

EXAMPLES:

```python
sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[1,1]])
sage: face = triangle.facets()[0]
sage: for v in face.vertex_generator(): print(v)
A vertex at (1, 0)
A vertex at (1, 1)
sage: type(face.vertex_generator())
<... 'generator'>
```

**vertices()**

Return all vertices of the face.

OUTPUT:

A tuple of vertices.

EXAMPLES:

```python
sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[1,1]])
sage: face = triangle.faces(1)[2]
sage: face.vertices()
(A vertex at (0, 1), A vertex at (1, 0))
```

`sage.geometry.polyhedron.face.combinatorial_face_to_polyhedral_face` *(polyhedron, combinatorial_face)*

Convert a combinatorial face to a face of a polyhedron.

INPUT:

- polyhedron – a polyhedron containing combinatorial_face
- combinatorial_face – a CombinatorialFace

OUTPUT: a `PolyhedronFace`.

EXAMPLES:
```python
sage: from sage.geometry.polyhedron.face import combinatorial_face_to_polyhedral_face
sage: P = polytopes.simplex()
sage: C = P.combinatorial_polyhedron()
sage: it = C.face_iter()
sage: comb_face = next(it)
sage: combinatorial_face_to_polyhedral_face(P, comb_face)
A 2-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 3 vertices
```

## 2.1.7 Generate cdd .ext / .ine file format

`sage.geometry.polyhedron.cdd_file_format.cdd_Hrepresentation(cdd_type, ieqs, eqns, file_output=None)`

Return a string containing the H-representation in cddlib’s ine format.

**INPUT:**

- `file_output` (string; optional) – a filename to which the representation should be written. If set to None (default), representation is returned as a string.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.cdd_file_format import cdd_Hrepresentation
sage: cdd_Hrepresentation('rational', None, [[0,1]])
'H-representation
linearity 1 1
begin
  1 2 rational
  0 1
end'
```

`sage.geometry.polyhedron.cdd_file_format.cdd_Vrepresentation(cdd_type, vertices, rays, lines, file_output=None)`

Return a string containing the V-representation in cddlib’s ext format.

**INPUT:**

- `file_output` (string; optional) – a filename to which the representation should be written. If set to None (default), representation is returned as a string.

**Note:** If there is no vertex given, then the origin will be implicitly added. You cannot write the empty V-representation (which cdd would refuse to process).

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.cdd_file_format import cdd_Vrepresentation
sage: print(cdd_Vrepresentation('rational', [[0,0]], [[1,0]], [[0,1]]))
V-representation
linearity 1 1
begin
  3 3 rational
  0 0 1
  0 1 0
  1 0 0
end
```
2.1.8 Formal modules generated by polyhedra

Class `sage.geometry.polyhedron.modules.formal_polyhedra_module.FormalPolyhedraModule` is for formal modules generated by polyhedra.

It is formal because it is free – it does not know about linear relations of polyhedra.

A formal polyhedral module is graded by dimension.

**INPUT:**
- `base_ring` – base ring of the module; unrelated to the base ring of the polyhedra
- `dimension` – the ambient dimension of the polyhedra
- `basis` – the basis

**EXAMPLES:**

```python
def closed_interval(a, b):
    return Polyhedron(vertices=[[a], [b]])
```

A three-dimensional vector space of polyhedra:

```python
sage: I01 = closed_interval(0, 1); I01.rename("conv([0], [1])")
sage: I11 = closed_interval(1, 1); I11.rename("{[1]}")
sage: I12 = closed_interval(1, 2); I12.rename("conv([1], [2])")
sage: basis = [I01, I11, I12]
sage: M = FormalPolyhedraModule(QQ, 1, basis=basis); M
Free module generated by {conv([0], [1]), {[1]}, conv([1], [2])} over Rational Field
```

A one-dimensional subspace; bases of subspaces just use the indexing set 0, . . . , \(d - 1\), where \(d\) is the dimension:

```python
sage: M_lower = M.submodule([M(I11)]); M_lower
Free module generated by {0} over Rational Field
sage: M_lower.print_options(prefix='S')
sage: M_lower.is_submodule(M)
True
sage: x = M(I01) - 2*M(I11) + M(I12)
sage: M_lower.reduce(x)
S[0] + S[2]
sage: M_lower.retract.domain() is M
True
sage: y = M_lower.retract(M(I11)); y
```

(continues on next page)
sage: M_lower.lift(y)
[{{[1]}},]

Quotient space; bases of quotient space are families indexed by elements of the ambient space:

sage: M_mod_lower = M.quotient_module(M_lower); M_mod_lower
Free module generated by {conv([0], [1]), conv([1], [2])} over Rational Field
sage: M_mod_lower.print_options(prefix='Q')
sage: M_mod_lower.retract(x)
Q[conv([0], [1])] + Q[conv([1], [2])]
sage: M_mod_lower.retract(M(I01) - 2*M(I11) + M(I12)) == M_mod_lower.retract(M(I01) + M(I12))
True

\texttt{degree\_on\_basis}\ (m)

The degree of an element of the basis is defined as the dimension of the polyhedron.

\textbf{INPUT:}

- \texttt{m} – an element of the basis (a polyhedron)

\textbf{EXAMPLES:}

```
sage: from sage.geometry.polyhedron.modules.formal_polyhedra_module import FormPolyhedraModule

sage: def closed_interval(a,b):
    return Polyhedron(\text{vertices}=[a, b])

sage: I01 = closed_interval(0, 1); I01.rename("conv([0], [1])")

sage: I11 = closed_interval(1, 1); I11.rename("{[1]}")

sage: I12 = closed_interval(1, 2); I12.rename("conv([1], [2])")

sage: I02 = closed_interval(0, 2); I02.rename("conv([0], [2])")

sage: M = FormalPolyhedraModule(QQ, 1, basis=[I01, I11, I12, I02])
```

We can extract homogeneous components:

```
sage: O = M(I01) + M(I11) + M(I12)
sage: O.homogeneous_component(0)
[{{[1]}},]
sage: O.homogeneous_component(1)
[\text{conv([0], [1])} + \text{conv([1], [2])}]
```

We note that modulo the linear relations of polyhedra, this would only be a filtration, not a grading, as the following example shows:

```
sage: X = M(I01) + M(I12) - M(I02)
sage: X.degree()
1
sage: Y = M(I11)
sage: Y.degree()
0
```
2.2 Lattice polyhedra

2.2.1 Lattice and reflexive polytopes

This module provides tools for work with lattice and reflexive polytopes. A convex polytope is the convex hull of finitely many points in $\mathbb{R}^n$. The dimension $n$ of a polytope is the smallest $n$ such that the polytope can be embedded in $\mathbb{R}^n$.

A lattice polytope is a polytope whose vertices all have integer coordinates.

If $L$ is a lattice polytope, the dual polytope of $L$ is

$$\{y \in \mathbb{Z}^n : x \cdot y \geq -1 \text{ all } x \in L\}$$

A reflexive polytope is a lattice polytope, such that its polar is also a lattice polytope, i.e. it is bounded and has vertices with integer coordinates.

This Sage module uses Package for Analyzing Lattice Polytopes (PALP), which is a program written in C by Maximilian Kreuzer and Harald Skarke, which is freely available under the GNU license terms at http://hep.itp.tuwien.ac.at/~kreuzer/CY/. Moreover, PALP is included standard with Sage.

PALP is described in the paper arXiv math.SC/0204356. Its distribution also contains the application nef.x, which was created by Erwin Riegler and computes nef-partitions and Hodge data for toric complete intersections.

ACKNOWLEDGMENT: polytope.py module written by William Stein was used as an example of organizing an interface between an external program and Sage. William Stein also helped Andrey Novoseltsev with debugging and tuning of this module.

Robert Bradshaw helped Andrey Novoseltsev to realize plot3d function.

Note: IMPORTANT: PALP requires some parameters to be determined during compilation time, i.e., the maximum dimension of polytopes, the maximum number of points, etc. These limitations may lead to errors during calls to different functions of this module. Currently, a ValueError exception will be raised if the output of poly.x or nef.x is empty or contains the exclamation mark. The error message will contain the exact command that caused an error, the description and vertices of the polytope, and the obtained output.

Data obtained from PALP and some other data is cached and most returned values are immutable. In particular, you cannot change the vertices of the polytope or their order after creation of the polytope.

If you are going to work with large sets of data, take a look at all_* functions in this module. They precompute different data for sequences of polynomials with a few runs of external programs. This can significantly affect the time of future computations. You can also use dump/load, but not all data will be stored (currently only faces and the number of their internal and boundary points are stored, in addition to polytope vertices and its polar).

AUTHORS:

- Andrey Novoseltsev (2007-01-15): all_* functions
- Andrey Novoseltsev (2008-04-01): second version, including:
  - dual nef-partitions and necessary convex_hull and minkowski_sum
  - built-in sequences of 2- and 3-dimensional reflexive polytopes
  - plot3d, skeleton_show
- Andrey Novoseltsev (2009-08-26): dropped maximal dimension requirement
- Andrey Novoseltsev (2010-12-15): new version of nef-partitions
sage.geometry.lattice_polytope.LatticePolytope(data, compute_vertices=True, n=0, lattice=None)

Construct a lattice polytope.

**INPUT:**

- **data** – points spanning the lattice polytope, specified as one of:
  - a *point collection* (this is the preferred input and it is the quickest and the most memory efficient one);
  - an iterable of iterables (for example, a list of vectors) defining the point coordinates;
  - a file with matrix data, opened for reading, or
  - a filename of such a file, see `read_palp_point_collection()` for the file format;
- **compute_vertices** – boolean (default: True). If True, the convex hull of the given points will be computed for determining vertices. Otherwise, the given points must be vertices;
- **n** – an integer (default: 0) if data is a name of a file, that contains data blocks for several polytopes, the n-th block will be used;
- **lattice** – the ambient lattice of the polytope. If not given, a suitable lattice will be determined automatically, most likely the toric lattice $M$ of the appropriate dimension.

**OUTPUT:**

- a *lattice polytope*.

**EXAMPLES:**

```sage
points = [(1,0,0), (0,1,0), (0,0,1), (-1,0,0), (0,-1,0), (0,0,-1)]
sage: p = LatticePolytope(points)
sage: p
3-d reflexive polytope in 3-d lattice M
sage: p.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
```

We draw a pretty picture of the polytope in 3-dimensional space:

```sage
sage: p.plot3d().show() #...
←needs palp sage.plot
```

Now we add an extra point, which is in the interior of the polytope...
You can suppress vertex computation for speed but this can lead to mistakes:

```python
sage: p = LatticePolytope(points, compute_vertices=False)
```

Given points must be in the lattice:

```python
sage: LatticePolytope([[1/2], [3/2]])
Traceback (most recent call last):
... ValueError: points [[1/2], [3/2]] are not in 1-d lattice M!
```

But it is OK to create polytopes of non-maximal dimension:

```python
sage: p = LatticePolytope([(1,0,0), (0,1,0), (0,0,0),
...: (-1,0,0), (0,-1,0), (0,0,0), (0,0,0)])
sage: p
2-d lattice polytope in 3-d lattice M
sage: p.vertices()
M(-1, 0, 0), M( 0, -1, 0), M( 1, 0, 0), M( 0, 1, 0)
in 3-d lattice M
```

An empty lattice polytope can be considered as well:

```python
sage: p = LatticePolytope([], lattice=ToricLattice(3).dual()); p
-1-d lattice polytope in 3-d lattice M
sage: p.lattice_dim()
3
sage: p.npoints()
0
sage: p.nfacets()
0
sage: p.points()
Empty collection in 3-d lattice M
sage: p.faces()  # needs sage.graphs
Empty collection
```

```python
class sage.geometry.lattice_polytope.LatticePolytopeClass
    (points=None, compute_vertices=None, ambient=None,
    ambient_vertex_indices=None, ambient_facet_indices=None)

Bases: ConvexSet_compact, Hashable, LatticePolytope

Create a lattice polytope.
```

2.2. Lattice polyhedra 167
Warning: This class does not perform any checks of correctness of input nor does it convert input into the standard representation. Use LatticePolytope() to construct lattice polytopes.

Lattice polytopes are immutable, but they cache most of the returned values.

INPUT:
The input can be either:

• points – PointCollection;
• compute_vertices – boolean.

or (these parameters must be given as keywords):

• ambient – ambient structure, this polytope must be a face of ambient;
• ambient_vertex_indices – increasing list or tuple of integers, indices of vertices of ambient generating this polytope;
• ambient_facet_indices – increasing list or tuple of integers, indices of facets of ambient generating this polytope.

OUTPUT:

• lattice polytope.

Note: Every polytope has an ambient structure. If it was not specified, it is this polytope itself.

adjacent ()
Return faces adjacent to self in the ambient face lattice.

Two distinct faces $F_1$ and $F_2$ of the same face lattice are adjacent if all of the following conditions hold:

• $F_1$ and $F_2$ have the same dimension $d$;
• $F_1$ and $F_2$ share a facet of dimension $d - 1$;
• $F_1$ and $F_2$ are facets of some face of dimension $d + 1$, unless $d$ is the dimension of the ambient structure.

OUTPUT:

• tuple of lattice polytopes.

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o.adjacent()  # needs sage.graphs

(1-d face of 3-d reflexive polytope in 3-d lattice M,
 1-d face of 3-d reflexive polytope in 3-d lattice M,
 1-d face of 3-d reflexive polytope in 3-d lattice M,
 1-d face of 3-d reflexive polytope in 3-d lattice M)
```
**affine_transform**\((a=1, b=0)\)
Return \(a \cdot P + b\), where \(P\) is this lattice polytope.

**Note:**

1. While \(a\) and \(b\) may be rational, the final result must be a lattice polytope, i.e. all vertices must be integral.
2. If the transform (restricted to this polytope) is bijective, facial structure will be preserved, e.g. the first facet of the image will be spanned by the images of vertices which span the first facet of the original polytope.

**INPUT:**

- \(a\) - (default: 1) rational scalar or matrix
- \(b\) - (default: 0) rational scalar or vector, scalars are interpreted as vectors with the same components

**EXAMPLES:**

```python
sage: o = lattice_polytope.cross_polytope(2)
sage: o.vertices()
M( 1, 0),
M( 0, 1),
M(-1, 0),
M( 0, -1)
in 2-d lattice M
sage: o.affine_transform(2).vertices()
M( 2, 0),
M( 0, 2),
M(-2, 0),
M( 0, -2)
in 2-d lattice M
sage: o.affine_transform(1,1).vertices()
M(2, 1),
M(1, 2),
M(0, 1),
M(1, 0)
in 2-d lattice M
sage: o.affine_transform(b=1).vertices()
M(2, 1),
M(1, 2),
M(0, 1),
M(1, 0)
in 2-d lattice M
sage: o.affine_transform(b=(1, 0)).vertices()
M(2, 0),
M(1, 1),
M(0, 0),
M(1, -1)
in 2-d lattice M
sage: a = matrix(QQ, 2, [1/2, 0, 0, 3/2])
sage: o.polar().vertices()
N( 1, 1),
N( 1, -1),
N(-1, -1),
N(-1, 1)
in 2-d lattice N
sage: o.polar().affine_transform(a, (1/2, -1/2)).vertices()
```

(continues on next page)
M(1, 1),
M(1, -2),
M(0, -2),
M(0, 1)
in 2-d lattice M

While you can use rational transformation, the result must be integer:

```
sage: o.affine_transform(a)
Traceback (most recent call last):
...
ValueError: points
[(1/2, 0), (0, 3/2), (-1/2, 0), (0, -3/2)]
are not in 2-d lattice M!
```

ambient()

Return the ambient structure of self.

OUTPUT:

• lattice polytope containing self as a face.

EXAMPLES:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.ambient()
3-d reflexive polytope in 3-d lattice M
sage: o.ambient() is o
True
sage: # needs sage.graphs
sage: face = o.faces(1)[0]
sage: face
1-d face of 3-d reflexive polytope in 3-d lattice M
sage: face.ambient()
3-d reflexive polytope in 3-d lattice M
sage: face.ambient() is o
True
```

ambient_dim()

Return the dimension of the ambient lattice of self.

An alias is ambient_dim().

OUTPUT:

• integer.

EXAMPLES:

```
sage: p = LatticePolytope([(1,0)])
sage: p.lattice_dim()
2
sage: p.dim()
0
```

ambient_facet_indices()

Return indices of facets of the ambient polytope containing self.
OUTPUT:

- increasing tuple of integers.

EXAMPLES:

The polytope itself is not contained in any of its facets:

```sage
sage: o = lattice_polytope.cross_polytope(3)
sage: o.ambient_facet_indices()
()  # needs sage.graphs
```

But each of its other faces is contained in one or more facets:

```sage
sage: face = o.faces(1)[0]
sage: face.ambient_facet_indices()
(4, 5)  # needs sage.graphs
sage: face.vertices()
M(1, 0, 0),
M(0, 1, 0)
in 3-d lattice M
sage: o.facets()[face.ambient_facet_indices()[0]].vertices()
M(1, 0, 0),
M(0, 1, 0),
M(0, 0, -1)
in 3-d lattice M
```

**ambient_ordered_point_indices()**

Return indices of points of the ambient polytope contained in this one.

OUTPUT:

- tuple of integers such that ambient points in this order are geometrically ordered, e.g. for an edge points will appear from one end point to the other.

EXAMPLES:

```sage
sage: cube = lattice_polytope.cross_polytope(3).polar()
sage: face = cube.facets()[0]  # needs sage.graphs
sage: face.ambient_ordered_point_indices()  # needs palp sage.graphs
(5, 8, 4, 9, 10, 11, 6, 12, 7)
sage: cube.points(face.ambient_ordered_point_indices())  # needs palp sage.graphs
N(-1, -1, -1),
N(-1, -1, 0),
N(-1, -1, 1),
N(-1, 0, -1),
N(-1, 0, 0),
N(-1, 0, 1),
N(-1, 1, -1),
N(-1, 1, 0),
N(-1, 1, 1)
in 3-d lattice N
```

**ambient_point_indices()**

Return indices of points of the ambient polytope contained in this one.

OUTPUT:
• tuple of integers, the order corresponds to the order of points of this polytope.

EXAMPLES:

```sage
cube = lattice_polytope.cross_polytope(3).polar()
sage: face = cube.facets()[0]  # needs sage.graphs
sage: face.ambient_point_indices()  # needs palp sage.graphs
(4, 5, 6, 7, 8, 9, 10, 11, 12)
sage: cube.points(face.ambient_point_indices()) == face.points()  # needs palp sage.graphs
True
```

`ambient_vector_space (base_field=None)`

Return the ambient vector space.

It is the ambient lattice (`lattice()`) tensored with a field.

INPUT:

• base_field – (default: the rationals) a field.

EXAMPLES:

```sage
p = LatticePolytope([(1,0)])
sage: p.ambient_vector_space()  # needs sage.rings.number_field
Vector space of dimension 2 over Rational Field
sage: p.ambient_vector_space(AA)  # needs sage.rings.number_field
Vector space of dimension 2 over Algebraic Real Field
```

`ambient_vertex_indices ()`

Return indices of vertices of the ambient structure generating self.

OUTPUT:

• increasing tuple of integers.

EXAMPLES:

```sage
o = lattice_polytope.cross_polytope(3)
sage: o.ambient_vertex_indices()  # needs sage.graphs
(0, 1, 2, 3, 4, 5)
sage: face = o.faces(1)[0]  # needs sage.graphs
sage: face.ambient_vertex_indices()  # needs sage.graphs
(0, 1)
```

`boundary_point_indices ()`

Return indices of (relative) boundary lattice points of this polytope.

OUTPUT:

• increasing tuple of integers.

EXAMPLES:

All points but the origin are on the boundary of this square:
sage: square = lattice_polytope.cross_polytope(2).polar()
sage: square.points()  # ...
needs palp
N( 1, 1),
N( 1, -1),
N(-1, -1),
N(-1, 1),
N(-1, 0),
N( 0, -1),
N( 0, 0),
N( 0, 1),
N( 1, 0)
in 2-d lattice N
sage: square.boundary_point_indices()  # ...
needs palp
(0, 1, 2, 3, 4, 5, 7, 8)

For an edge the boundary is formed by the end points:

sage: face = square.edges()[0]  # ...
needs sage.graphs
sage: face.points()  # ...
needs sage.graphs
N(-1, -1),
N(-1, 1),
N(-1, 0)
in 2-d lattice N
sage: face.boundary_point_indices()  # ...
needs sage.graphs
(0, 1)

boundary_points()

Return (relative) boundary lattice points of this polytope.

OUTPUT:

• a point collection.

EXAMPLES:

All points but the origin are on the boundary of this square:

sage: square = lattice_polytope.cross_polytope(2).polar()
sage: square.boundary_points()  # ...
needs palp
N( 1, 1),
N( 1, -1),
N(-1, -1),
N(-1, 1),
N(-1, 0),
N( 0, -1),
N( 0, 0),
N( 0, 1),
N( 1, 0)
in 2-d lattice N

For an edge the boundary is formed by the end points:
contains (*args)
Check if a given point is contained in self.

INPUT:
• an attempt will be made to convert all arguments into a single element of the ambient space of self; if it fails, False will be returned

OUTPUT:
• True if the given point is contained in self, False otherwise

EXAMPLES:

```
sage: p = lattice_polytope.cross_polytope(2)
sage: p.contains(p.lattice()(1,0))
True
sage: p.contains((1,0))
True
sage: p.contains(1,0)
True
sage: p.contains((2,0))
False
```

dim()
Return the dimension of this polytope.

EXAMPLES:
We create a 3-dimensional octahedron and check its dimension:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.dim()
3
```
Now we create a 2-dimensional diamond in a 3-dimensional space:

```
sage: p = LatticePolytope([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.dim()
2
sage: p.lattice_dim()
3
```

distances (point=None)
Return the matrix of distances for this polytope or distances for the given point.
The matrix of distances m gives distances m[i,j] between the i-th facet (which is also the i-th vertex of the polar polytope in the reflexive case) and j-th point of this polytope.
If point is specified, integral distances from the point to all facets of this polytope will be computed.

EXAMPLES: The matrix of distances for a 3-dimensional octahedron:
sage: o = lattice_polytope.cross_polytope(3)
sage: o.distances()
\#...  
\[
\begin{array}{cccccc}
  2 & 0 & 0 & 2 & 2 & 1 \\
  2 & 2 & 0 & 0 & 2 & 1 \\
  2 & 2 & 0 & 0 & 0 & 1 \\
  2 & 0 & 2 & 0 & 2 & 0 \\
  0 & 0 & 2 & 2 & 0 & 1 \\
  0 & 0 & 2 & 2 & 2 & 1 \\
  0 & 2 & 0 & 2 & 0 & 1 \\
  0 & 2 & 2 & 2 & 0 & 0 \\
\end{array}
\]

Distances from facets to the point (1,2,3):

sage: o.distances([1,2,3])
(-3, 1, 7, 3, 1, -5, -1, 5)

It is OK to use RATIONAL coordinates:

sage: o.distances([1,2,3/2])
(-3/2, 5/2, 11/2, 3/2, -1/2, -7/2, 1/2, 7/2)

sage: o.distances([1,2,sqrt(2)])
\#...
 \[
\text{Traceback (most recent call last):}
...
\text{TypeError: unable to convert sqrt(2) to an element of Rational Field}
\]

Now we create a non-spanning polytope:

sage: p = LatticePolytope([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.distances()
\#...
\[
\begin{array}{cccccc}
  2 & 2 & 0 & 0 & 1 \\
  2 & 0 & 0 & 2 & 1 \\
  0 & 0 & 2 & 2 & 1 \\
  0 & 2 & 2 & 0 & 1 \\
\end{array}
\]

sage: p.distances((1/2, 3, 0))
\#...
\[
(9/2, -3/2, -5/2, 7/2)
\]

This point is not even in the affine subspace of the polytope:

sage: p.distances((1, 1, 1))
\#...
\[
(3, 1, -1, 1)
\]

dual ()

Return the dual face under face duality of polar reflexive polytopes.

This duality extends the correspondence between vertices and facets.

OUTPUT:

• a lattice polytope.

EXAMPLES:
```sage
# needs sage.graphs
sage: o = lattice_polytope.cross_polytope(4)
sage: e = o.edges()[0]; e
1-d face of 4-d reflexive polytope in 4-d lattice M
sage: ed = e.dual(); ed
2-d face of 4-d reflexive polytope in 4-d lattice N
sage: ed.ambient() is e.ambient().polar()
True
sage: e.ambient_vertex_indices() == ed.ambient_facet_indices()
True
sage: e.ambient_facet_indices() == ed.ambient_vertex_indices()
True

dual_lattice()

Return the dual of the ambient lattice of self.

OUTPUT:

- a lattice. If possible (that is, if lattice() has a dual() method), the dual lattice is returned. Otherwise, \( \mathbb{Z}^n \) is returned, where \( n \) is the dimension of self.

EXAMPLES:

```sage
sage: LatticePolytope([(1,0)]).dual_lattice()
2-d lattice N
sage: LatticePolytope([], lattice=ZZ^3).dual_lattice()
Ambient free module of rank 3
over the principal ideal domain Integer Ring
```

edges()

Return edges (faces of dimension 1) of self.

OUTPUT:

- tuple of lattice polytopes.

EXAMPLES:

```sage
sage: o = lattice_polytope.cross_polytope(3)
sage: o.edges()  # needs sage.graphs
(1-d face of 3-d reflexive polytope in 3-d lattice M, ...
  1-d face of 3-d reflexive polytope in 3-d lattice M)
sage: len(o.edges())  # needs sage.graphs
12
```

face_lattice()

Return the face lattice of self.

This lattice will have the empty polytope as the bottom and this polytope itself as the top.

OUTPUT:

- finite poset of lattice polytopes.

EXAMPLES:

Let’s take a look at the face lattice of a square:
sage: square = LatticePolytope([(0,0), (1,0), (1,1), (0,1)])
sage: L = square.face_lattice(); L

```
Finite lattice containing 10 elements with distinguished linear extension
```

To see all faces arranged by dimension, you can do this:

```
sage: for level in L.level_sets(): print(level)

[-1-d face of 2-d lattice polytope in 2-d lattice M]
[0-d face of 2-d lattice polytope in 2-d lattice M, 0-d face of 2-d lattice polytope in 2-d lattice M, 0-d face of 2-d lattice polytope in 2-d lattice M, 0-d face of 2-d lattice polytope in 2-d lattice M]
[1-d face of 2-d lattice polytope in 2-d lattice M, 1-d face of 2-d lattice polytope in 2-d lattice M, 1-d face of 2-d lattice polytope in 2-d lattice M, 1-d face of 2-d lattice polytope in 2-d lattice M]
[2-d lattice polytope in 2-d lattice M]
```

For a particular face you can look at its actual vertices…

```
sage: face = L.level_sets()[1][0]
sage: face.vertices()
M(0, 0)
in 2-d lattice M
```

… or you can see the index of the vertex of the original polytope that corresponds to the above one:

```
sage: face.ambient_vertex_indices()
(0,)
sage: square.vertex(0)
M(0, 0)
```

An alternative to extracting faces from the face lattice is to use `faces()` method:

```
sage: face is square.faces(dim=0)[0]
True
```

The advantage of working with the face lattice directly is that you can (relatively easily) get faces that are related to the given one:

```
sage: face = L.level_sets()[1][0]
sage: D = L.hasse_diagram()
sage: sorted(D.neighbors(face))
[-1-d face of 2-d lattice polytope in 2-d lattice M, 1-d face of 2-d lattice polytope in 2-d lattice M, 1-d face of 2-d lattice polytope in 2-d lattice M]
```

However, you can achieve some of this functionality using `facets()`, `facet_of()`, and `adjacent()` methods.
Note that if \( p \) is a face of \( \text{superp} \), then the face lattice of \( p \) consists of (appropriate) faces of \( \text{superp} \):

```
sage: # needs sage.graphs
sage: superp = LatticePolytope([(1,2,3,4), (5,6,7,8),
.....: (1,2,4,8), (1,3,9,7)])
sage: superp.face_lattice()
Finite lattice containing 16 elements with distinguished linear extension
sage: superp.face_lattice().top()
3-d lattice polytope in 4-d lattice M
sage: p = superp.facets()[0]
sage: p
2-d face of 3-d lattice polytope in 4-d lattice M
sage: p.face_lattice()
Finite poset containing 8 elements with distinguished linear extension
sage: p.face_lattice().bottom()
-1-d face of 3-d lattice polytope in 4-d lattice M
sage: p.face_lattice().top()
2-d face of 3-d lattice polytope in 4-d lattice M
sage: p.face_lattice().top() is p
True
```

**faces** *(dim=None, codim=None)*

Return faces of \( \text{self} \) of specified (co)dimension.

**INPUT:**

- \( \text{dim} \) – integer, dimension of the requested faces;
- \( \text{codim} \) – integer, codimension of the requested faces.

**Note:** You can specify at most one parameter. If you don’t give any, then all faces will be returned.

**OUTPUT:**

- if either \( \text{dim} \) or \( \text{codim} \) is given, the output will be a tuple of lattice polytopes;
- if neither \( \text{dim} \) nor \( \text{codim} \) is given, the output will be the tuple of tuples as above, giving faces of all existing dimensions. If you care about inclusion relations between faces, consider using \( \text{face_lattice()} \) or \( \text{adjacent()}, \text{facet_of()}, \text{and facets()} \).
EXAMPLES:

Let’s take a look at the faces of a square:

```python
sage: square = LatticePolytope([(0,0), (1,0), (1,1), (0,1)])
sage: square.faces()  # needs sage.graphs
((-1-d face of 2-d lattice polytope in 2-d lattice M,),
 (0-d face of 2-d lattice polytope in 2-d lattice M,)
 (0-d face of 2-d lattice polytope in 2-d lattice M,)
 (0-d face of 2-d lattice polytope in 2-d lattice M,)
 (1-d face of 2-d lattice polytope in 2-d lattice M,)
 (1-d face of 2-d lattice polytope in 2-d lattice M,)
 (1-d face of 2-d lattice polytope in 2-d lattice M,),
 (2-d lattice polytope in 2-d lattice M,))
```

Its faces of dimension one (i.e., edges):

```python
sage: square.faces(dim=1)  # needs sage.graphs
(1-d face of 2-d lattice polytope in 2-d lattice M,
 1-d face of 2-d lattice polytope in 2-d lattice M,
 1-d face of 2-d lattice polytope in 2-d lattice M,
 1-d face of 2-d lattice polytope in 2-d lattice M)
```

Its faces of codimension one are the same (also edges):

```python
sage: square.faces(codim=1) is square.faces(dim=1)  # needs sage.graphs
True
```

Let’s pick a particular face:

```python
sage: face = square.faces(dim=1)[0]  # needs sage.graphs
```

Now you can look at the actual vertices of this face…

```python
sage: face.vertices()  # needs sage.graphs
M(0, 0),
M(0, 1)
in 2-d lattice M
```

… or you can see indices of the vertices of the original polytope that correspond to the above ones:

```python
sage: face.ambient_vertex_indices()  # needs sage.graphs
(0, 3)
sage: square.vertices(face.ambient_vertex_indices())  # needs sage.graphs
M(0, 0),
M(0, 1)
in 2-d lattice M
```

**facet_constant** \(i\)

Return the constant in the \(i\)-th facet inequality of this polytope.
This is equivalent to `facet_constants()[i]`.

**INPUT:**
- \( i \) – integer; the index of the facet

**OUTPUT:**
- integer – the constant in the \( i \)-th facet inequality.

**See also:**

`facet_constants()`, `facet_normal()`, `facet_normals()`, `facets()`.

**EXAMPLES:**

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.facet_constant(0)
1
sage: o.facet_constant(0) == o.facet_constants()[0]
True
```

### `facet_constants()`

Return facet constants of `self`.

Facet inequalities have form \( n \cdot x + c \geq 0 \) where \( n \) is the inner normal and \( c \) is a constant.

**OUTPUT:**
- an integer vector

**See also:**

`facet_constant()`, `facet_normal()`, `facet_normals()`, `facets()`.

**EXAMPLES:**

For reflexive polytopes all constants are 1:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
sage: o.facet_constants()
(1, 1, 1, 1, 1, 1, 1, 1)
```

Here is an example of a 3-dimensional polytope in a 4-dimensional space with 3 facets containing the origin:

```
sage: p = LatticePolytope([[0,0,0,0), (1,1,1,3), ....: (1,-1,1,3)], (-1,-1,1,3)])
sage: p.vertices()
M( 1, 0, 0, 0),
M( 1, 1, 1, 3),
M( 1, -1, 1, 3),
M(-1, -1, 1, 3)
in 4-d lattice M
sage: p.facet_constants()
(0, 0, 3, 0)
```
facet_normal(i)
Return the inner normal to the i-th facet of this polytope.
This is equivalent to facet_normals()[i].

INPUT:
• i – integer; the index of the facet

OUTPUT:
• a vector

See also:
facet_constant(), facet_constants(), facet_normals(), facets().

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o.facet_normal(0)
N(1, -1, -1)
sage: o.facet_normal(0) == o.facet_normals()[0]
True
```

facet_normals()
Return inner normals to the facets of self.
If this polytope is not full-dimensional, facet normals will define this polytope in the affine subspace spanned by it.

OUTPUT:
• a point collection in the dual_lattice() of self.

See also:
facet_constant(), facet_constants(), facet_normal(), facets().

EXAMPLES:

Normals to facets of an octahedron are vertices of a cube:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
sage: o.facet_normals()
N( 1, -1, -1),
N( 1, 1, -1),
N( 1, 1, 1),
N(-1, -1, 1),
N(-1, -1, -1),
N(-1, 1, -1),
N(-1, 1, 1)
in 3-d lattice N
```
Here is an example of a 3-dimensional polytope in a 4-dimensional space:

```
sage: p = LatticePolytope([(0,0,0,0), (1,1,1,3),
                        (1,-1,1,3), (-1,-1,1,3)])

sage: p.vertices()
M( 0, 0, 0, 0),
M( 1, 1, 1, 3),
M( 1, -1, 1, 3),
M(-1, -1, 1, 3)
in 4-d lattice M

sage: p.facet_normals()
N( 0, 3, 0, 1),
N( 1, -1, 0, 0),
N( 0, 0, 0, -1),
N(-3, 0, 0, 1)
in 4-d lattice N

sage: p.facet_constants()
(0, 0, 3, 0)
```

Now we manually compute the distance matrix of this polytope. Since it is a simplex, each line (corresponding to a facet) should consist of zeros (indicating generating vertices of the corresponding facet) and a single positive number (since our normals are inner):

```
sage: matrix([[n * v + c for v in p.vertices()]
            for n, c in zip(p.facet_normals(), p.facet_constants())])

[0 6 0 0]
[0 0 2 0]
[3 0 0 0]
[0 0 0 6]
```

---

**facet_of()**

Return elements of the ambient face lattice having `self` as a facet.

**OUTPUT:**

- tuple of `lattice polytopes`.

**EXAMPLES:**

```
sage: square = LatticePolytope([(0,0), (1,0), (1,1), (0,1)])
sage: square.facet_of()
()  
sage: face = square.faces(0)[0]  
sage: len(face.facet_of())
2
sage: face.facet_of()[1]
1-d face of 2-d lattice polytope in 2-d lattice M
```

---

**facets()**

Return facets (faces of codimension 1) of `self`.

**OUTPUT:**

- tuple of `lattice polytopes`.

**EXAMPLES:**

---
sage: o = lattice_polytope.cross_polytope(3)
sage: o.facets()  # needs sage.graphs
(2-d face of 3-d reflexive polytope in 3-d lattice M, ...
2-d face of 3-d reflexive polytope in 3-d lattice M)
sage: len(o.facets())  # needs sage.graphs
8

incidence_matrix()
Return the incidence matrix.

Note: The columns correspond to facets/facet normals in the order of `facet_normals()`, the rows correspond to the vertices in the order of `vertices()`.

EXAMPLES:

```
sage: o = lattice_polytope.cross_polytope(2)
sage: o.incidence_matrix()
[0 0 1 1]
[0 1 1 0]
[1 1 0 0]
[1 0 0 1]
sage: o.faces(1)[0].incidence_matrix()  # needs sage.graphs
[1 0]
[0 1]

sage: o = lattice_polytope.cross_polytope(4)
sage: o.incidence_matrix().column(3).nonzero_positions()  # needs sage.graphs
[3, 4, 5, 6]
sage: o.facets()[3].ambient_vertex_indices()  # needs sage.graphs
(3, 4, 5, 6)
```

index()
Return the index of this polytope in the internal database of 2- or 3-dimensional reflexive polytopes. Databases are stored in the directory of the package.

Note: The first call to this function for each dimension can take a few seconds while the dictionary of all polytopes is constructed, but after that it is cached and fast.

Return type
integer

EXAMPLES: We check what is the index of the “diamond” in the database:

```
sage: d = lattice_polytope.cross_polytope(2)
sage: d.index()  # needs paip
3
```
Note that polytopes with the same index are not necessarily the same:

```sage
d.vertices()
M( 1, 0),
M( 0, 1),
M(-1, 0),
M( 0, -1)
in 2-d lattice M
```

```sage
lattice_polytope.ReflexivePolytope(2,3).vertices()
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

But they are in the same $GL(\mathbb{Z}^n)$ orbit and have the same normal form:

```sage
d.normal_form()  #...
→ needs sage.groups
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

```sage
lattice_polytope.ReflexivePolytope(2,3).normal_form()  #...
→ needs sage.groups
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

**interior_point_indices()**

Return indices of (relative) interior lattice points of this polytope.

**OUTPUT:**

- increasing tuple of integers.

**EXAMPLES:**

The origin is the only interior point of this square:

```sage
square = lattice_polytope.cross_polytope(2).polar()
sage: square.points()  #...
→ needs palp
N( 1, 1),
N( 1, -1),
N(-1, -1),
N(-1, 1),
N(-1, 0),
N( 0, -1),
N( 0, 0),
N( 0, 1),
N( 1, 0)
in 2-d lattice N
```

```sage
square.interior_point_indices()  #...
→ needs palp
(6,)
```
Its edges also have a single interior point each:

```python
sage: face = square.edges()[0]  # needs sage.graphs
sage: face.interior_point_indices()  # needs sage.graphs
N(-1, 0) in 2-d lattice N
```

**interior_points()**

Return (relative) boundary lattice points of this polytope.

**OUTPUT:**

- a *point collection*.

**EXAMPLES:**

The origin is the only interior point of this square:

```python
sage: square = lattice_polytope.cross_polytope(2).polar()
sage: square.interior_points()  # needs palp
N(0, 0) in 2-d lattice N
```

Its edges also have a single interior point each:

```python
sage: face = square.edges()[0]  # needs sage.graphs
sage: face.interior_point_indices()  # needs sage.graphs
N(-1, 0) in 2-d lattice N
```

**is_reflexive()**

Return True if this polytope is reflexive.

**EXAMPLES:** The 3-dimensional octahedron is reflexive (and 4319 other 3-polytopes):

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o.is_reflexive()
True
```

But not all polytopes are reflexive:

```python
sage: p = LatticePolytope([[(1,0,0), (0,1,17), (-1,0,0), (0,-1,0)]])
sage: p.is_reflexive()
False
```

Only full-dimensional polytopes can be reflexive (otherwise the polar set is not a polytope at all, since it is unbounded):
sage: p = LatticePolytope([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.is_reflexive()
False

**lattice**
Return the ambient lattice of `self`.

**OUTPUT:**
- a lattice.

**EXAMPLES:**

```
sage: lattice_polytope.cross_polytope(3).lattice()
3-d lattice M
```

**lattice_dim**
Return the dimension of the ambient lattice of `self`.
An alias is `ambient_dim()`.

**OUTPUT:**
- integer.

**EXAMPLES:**

```
sage: p = LatticePolytope([(1,0)])
sage: p.lattice_dim()
2
```

**linearly_independent_vertices**
Return a maximal set of linearly independent vertices.

**OUTPUT:**
A tuple of vertex indices.

**EXAMPLES:**

```
sage: L = LatticePolytope([[0, 0], [-1, 1], [-1, -1]])
sage: L.linearly_independent_vertices()
(1, 2)
sage: L = LatticePolytope([[0, 0, 0]])
sage: L.linearly_independent_vertices()
()
sage: L = LatticePolytope([[0, 1, 0]])
sage: L.linearly_independent_vertices()
(0,)
```

**nef_partitions** *(keep_symmetric=False, keep_products=True, keep_projections=True, hodge_numbers=False)*
Return 2-part nef-partitions of `self`.

**INPUT:**
- `keep_symmetric` – (default: `False`) if `True`, “-s” option will be passed to `nef.x` in order to keep symmetric partitions, i.e. partitions related by lattice automorphisms preserving `self`;
• keep_products – (default: True) if True, “-D” option will be passed to nef.x in order to keep product partitions, with corresponding complete intersections being direct products;

• keep_projections – (default: True) if True, “-P” option will be passed to nef.x in order to keep projection partitions, i.e. partitions with one of the parts consisting of a single vertex;

• hodge_numbers – (default: False) if False, “-p” option will be passed to nef.x in order to skip Hodge numbers computation, which takes a lot of time.

OUTPUT:

• a sequence of nef-partitions.

Type NefPartition? for definitions and notation.

EXAMPLES:

Nef-partitions of the 4-dimensional cross-polytope:

```
sage: p = lattice_polytope.cross_polytope(4)
sage: p.nef_partitions()
#...
[ Nef-partition {0, 1, 4, 5} ⊔ {2, 3, 6, 7} (direct product),
  Nef-partition {0, 1, 2, 4} ⊔ {3, 5, 6, 7},
  Nef-partition {0, 1, 2, 4, 5} ⊔ {3, 6, 7},
  Nef-partition {0, 1, 2, 4, 5, 6} ⊔ {3, 7} (direct product),
  Nef-partition {0, 1, 2, 3} ⊔ {4, 5, 6, 7},
  Nef-partition {0, 1, 2, 3, 4} ⊔ {5, 6, 7},
  Nef-partition {0, 1, 2, 3, 4, 5} ⊔ {6, 7},
  Nef-partition {0, 1, 2, 3, 4, 5, 6} ⊔ {7} (projection)
]
```

Now we omit projections:

```
sage: p.nef_partitions(keep_projections=False)
#...
[ Nef-partition {0, 1, 4, 5} ⊔ {2, 3, 6, 7} (direct product),
  Nef-partition {0, 1, 2, 4} ⊔ {3, 5, 6, 7},
  Nef-partition {0, 1, 2, 4, 5} ⊔ {3, 6, 7},
  Nef-partition {0, 1, 2, 4, 5, 6} ⊔ {3, 7} (direct product),
  Nef-partition {0, 1, 2, 3} ⊔ {4, 5, 6, 7},
  Nef-partition {0, 1, 2, 3, 4} ⊔ {5, 6, 7},
  Nef-partition {0, 1, 2, 3, 4, 5} ⊔ {6, 7}
]
```

Currently Hodge numbers cannot be computed for a given nef-partition:

```
sage: p.nef_partitions()[1].hodge_numbers()
#...
Traceback (most recent call last):
... Not Implemented Error: use nef_partitions (hodge_numbers=True)!
```

But they can be obtained from nef.x for all nef-partitions at once. Partitions will be exactly the same:

```
sage: p.nef_partitions(hodge_numbers=True)  # long time (2s on sage.math, ...
#...
[ (continues on next page) ]
```
Nef-partition \{0, 1, 4, 5\} \sqcup \{2, 3, 6, 7\} (direct product),
Nef-partition \{0, 1, 2, 4\} \sqcup \{3, 5, 6, 7\},
Nef-partition \{0, 1, 2, 4, 5\} \sqcup \{3, 6, 7\},
Nef-partition \{0, 1, 2, 4, 5, 6\} \sqcup \{3, 7\} (direct product),
Nef-partition \{0, 1, 2, 3\} \sqcup \{4, 5, 6, 7\},
Nef-partition \{0, 1, 2, 3, 4\} \sqcup \{5, 6, 7\},
Nef-partition \{0, 1, 2, 3, 4, 5\} \sqcup \{6, 7\},
Nef-partition \{0, 1, 2, 3, 4, 5, 6\} \sqcup \{7\} (projection)

Now it is possible to get Hodge numbers:

```python
sage: p.nef_partitions(hodge_numbers=True)[1].hodge_numbers()  #...
needs palp
(20,)
```

Since nef-partitions are cached, their Hodge numbers are accessible after the first request, even if you do not specify `hodge_numbers=True` anymore:

```python
sage: p.nef_partitions()[1].hodge_numbers()  #...
needs palp
(20,)
```

We illustrate removal of symmetric partitions on a diamond:

```python
sage: p = lattice_polytope.cross_polytope(2)
sage: p.nef_partitions()  #...
needs palp

[  Nef-partition \{0, 2\} \sqcup \{1, 3\} (direct product),
   Nef-partition \{0, 1\} \sqcup \{2, 3\},
   Nef-partition \{0, 1, 2\} \sqcup \{3\} (projection)
]
sage: p.nef_partitions(keep_symmetric=True)  #...
needs palp

[  Nef-partition \{0, 1, 3\} \sqcup \{2\} (projection),
   Nef-partition \{0, 2, 3\} \sqcup \{1\} (projection),
   Nef-partition \{0, 3\} \sqcup \{1, 2\},
   Nef-partition \{1, 2, 3\} \sqcup \{0\} (projection),
   Nef-partition \{1, 3\} \sqcup \{0, 2\} (direct product),
   Nef-partition \{2, 3\} \sqcup \{0, 1\},
   Nef-partition \{0, 1, 2\} \sqcup \{3\} (projection)
]
```

Nef-partitions can be computed only for reflexive polytopes:

```python
sage: p = LatticePolytope([(1,0,0), (0,1,0), (0,0,2),
....:                      (-1,0,0), (0,-1,0), (0,0,-1)])
sage: p.nef_partitions()  #...
needs palp
Traceback (most recent call last):
...
ValueError: The given polytope is not reflexive!
Polytope: 3-d lattice polytope in 3-d lattice M
```

```
Run nef.x with given keys on vertices of this polytope.

INPUT:
- keys - a string of options passed to nef.x. The key “-f” is added automatically.

OUTPUT: the output of nef.x as a string.

EXAMPLES: This call is used internally for computing nef-partitions:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: s = o.nef_x("-N -V -p")    #→needs palp
sage: s                          # output contains random time    #→needs palp
M:27 8 N:7 6 codim=2 #part=5
3 6 Vertices of P:
  1  0  0  -1  0  0
  0  1  0  -1  0  0
  0  0  1  0  0  -1
P:0 V:2 4 5  0sec 0cpu
P:2 V:3 4 5  0sec 0cpu
P:3 V:4 5  0sec 0cpu
np=3 d:1 p:1  0sec 0cpu
```

```
nfacets ()
Return the number of facets of this polytope.

EXAMPLES: The number of facets of the 3-dimensional octahedron:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o.nfacets()
8
```

The number of facets of an interval is 2:

```python
sage: LatticePolytope([([1], [2]]).nfacets()
2
```

Now consider a 2-dimensional diamond in a 3-dimensional space:

```python
sage: p = LatticePolytope(([1, 0, 0], (0, 1, 0), (-1, 0, 0), (0, -1, 0)))
sage: p.nfacets()
4
```

```
normal_form (algorithm="palp_native", permutation=False)
Return the normal form of vertices of self.

Two full-dimensional lattice polytopes are in the same $GL(\mathbb{Z}^n)$-orbit if and only if their normal forms are the same. Normal form is not defined and thus cannot be used for polytopes whose dimension is smaller than the dimension of the ambient space.

The original algorithm was presented in [KS1998] and implemented in PALP. A modified version of the PALP algorithm is discussed in [GK2013] and available here as "palp_modified".

INPUT:
- algorithm=(default: "palp_native") The algorithm which is used to compute the normal form. Options are:
- "palp" – Run external PALP code, usually the fastest option when it works; but reproducible crashes have been observed in dimension 5 and higher.

- "palp_native" – The original PALP algorithm implemented in sage. Currently competitive with PALP in many cases.

- "palp_modified" – A modified version of the PALP algorithm which determines the maximal vertex-facet pairing matrix first and then computes its automorphisms, while the PALP algorithm does both things concurrently.

- permutation – boolean (default: False); if True, the permutation applied to vertices to obtain the normal form is returned as well. Note that the different algorithms may return different results that nevertheless lead to the same normal form.

OUTPUT:

- a point collection in the lattice() of self or a tuple of it and a permutation.

EXAMPLES:

We compute the normal form of the “diamond”:

```python
sage: d = LatticePolytope([(1,0), (0,1), (-1,0), (0,-1)])
sage: d.vertices()
M( 1, 0),
M( 0, 1),
M(-1, 0),
M( 0, -1)
in 2-d lattice M
sage: d.normal_form() #...

← needs sage.groups
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

The diamond is the 3rd polytope in the internal database:

```python
sage: d.index() #...
3
sage: d #...
← needs palp
2-d reflexive polytope #3 in 2-d lattice M
```

You can get it in its normal form (in the default lattice) as

```python
sage: lattice_polytope.ReflexivePolytope(2, 3).vertices()
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

It is not possible to compute normal forms for polytopes which do not span the space:

```python
sage: p = LatticePolytope([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.normal_form() Traceback (most recent call last):
```

(continues on next page)
We can perform the same examples using other algorithms:

```python
sage: o = lattice_polytope.cross_polytope(2)
sage: o.normal_form(algorithm="palp_native")  # needs sage.groups
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

The following examples demonstrate the speed of the available algorithms. In low dimensions, the default algorithm, "palp_native", is the fastest. As the dimension increases, "palp" is relatively faster than "palp_native". "palp_native" is usually much faster than "palp_modified". In some cases when the polytope has high symmetry, however, "palp_native" is slower:

```python
sage: # not tested
sage: o = lattice_polytope.cross_polytope(2)
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp")
625 loops, best of 3: 3.07 ms per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_native")
625 loops, best of 3: 0.445 ms per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_modified")
625 loops, best of 3: 5.01 ms per loop
sage: o = lattice_polytope.cross_polytope(3)
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp")
625 loops, best of 3: 3.22 ms per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_native")
625 loops, best of 3: 2.73 ms per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_modified")
625 loops, best of 3: 20.7 ms per loop
sage: o = lattice_polytope.cross_polytope(4)
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp")
625 loops, best of 3: 4.84 ms per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_native")
625 loops, best of 3: 55.6 ms per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_modified")
625 loops, best of 3: 129 ms per loop
sage: o = lattice_polytope.cross_polytope(5)
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp")
10 loops, best of 3: 0.0364 s per loop
sage: %timeit o.normal_form.clear_cache(); o.normal_form("palp_native")
10 loops, best of 3: 1.68 s per loop
```
Note that the algorithm "palp" may crash for higher dimensions because of the overflow errors as mentioned in github issue #13525#comment:9. Then use "palp_native" instead, which is usually faster than "palp_modified". Below is an example where "palp" fails and "palp_native" is much faster than "palp_modified":

```
sage: P = LatticePolytope([[-3, -3, -6, -6, -1], [3, 3, 6, 6, 1], [-3, -3, -6, -6, 1],
   ...:[-3, 0, -6, -6, 0], [-3, 0, -3, -6, 0], [-3, 0, -3, -6, 0], [-3, 0, 0, 0, 1]],
   ...:[0, -3, -6, -6, 0], [0, -3, -3, -6, 0], [0, -3, -3, -6, 0], [0, -3, -3, -6, 0],
   ...:[0, 0, -3, -6, 0], [0, 0, -3, -3, -6, 0], [0, 0, -3, -3, -6, 0], [0, 0, -3, -3, -6, 0],
   ...:[0, 0, 0, 0, -1], [3, 0, 0, 0, 0], [3, 0, 0, 0, 0], [3, 0, 0, 0, 0],
   ...:[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1]])
sage: P.normal_form(algorithm="palp")  # not tested
Traceback (most recent call last):
  ...RuntimeError: Error executing ... for a polytope sequence!
Output:
b'*** stack smashing detected ***: terminated\nAborted\n'
sage: P.normal_form(algorithm="palp_native")  #...
```

(continues on next page)
\[ M(0, 1, -3, 0, 0), \]
\[ M(0, 0, -3, -3, 3), \]
\[ M(0, 1, 0, 3, -3), \]
\[ M(0, -1, 0, -3, 3), \]
\[ M(0, 0, 3, 3, -3), \]
\[ M(0, -1, 3, 0, 0), \]
\[ M(12, 0, -6, -3, 3), \]
\[ M(12, -1, -6, -6, 6), \]
\[ M(0, 0, 0, 3, 0), \]
\[ M(12, 0, -9, -3, 6), \]
\[ M(12, -1, -6, -3, 6), \]
\[ M(-6, 0, 6, 3, -6), \]
\[ M(6, 0, -3, -3, 0), \]
\[ M(6, -1, 0, -3, 0), \]
\[ M(-12, 1, 9, 6, -6), \]
\[ M(6, 0, -6, 0, 3), \]
\[ M(0, 0, 0, 0, -3), \]
\[ M(0, -1, 0, -3, 0), \]
\[ M(0, 0, -3, 0, 0), \]
\[ M(0, -1, 0, 0, 0), \]
\[ M(12, -1, -9, -6, 6), \]
\[ M(12, -1, -6, -3, 3) \]

in 5-d lattice \( M \)

\texttt{ sage: P.normal_form(algorithm="palp_modified") } # not tested (22s;...}
\texttt{ MemoryError on 32 bit), needs sage.groups}

\( M(6, 0, 0, 0, 0), \)
\( M(-6, 0, 0, 0, 0), \)
\( M(0, 1, 0, 0, 0), \)
\( M(0, 0, 3, 0, 0), \)
\( M(0, 1, 0, 3, 0), \)
\( M(0, 0, 0, 0, 3), \)
\( M(-6, 1, 6, 3, -6), \)
\( M(-6, 0, 6, 0, -3), \)
\( M(-12, 1, 6, 3, -6), \)
\( M(-6, 1, 6, 3, 0), \)
\( M(-6, 0, 6, 3, 0), \)
\( M(6, 0, -6, -3, 6), \)
\( M(-12, 1, 6, 3, -6), \)
\( M(-12, 0, 9, 3, -6), \)
\( M(0, 0, 0, -3, 0), \)
\( M(-12, 1, 6, 6, -6), \)
\( M(-12, 0, 6, 3, -3), \)
\( M(0, 1, -3, 0, 0), \)
\( M(0, 0, -3, -3, 3), \)
\( M(0, 1, 0, 3, -3), \)
\( M(0, -1, 0, -3, 3), \)
\( M(0, 0, 3, 3, -3), \)
\( M(0, -1, 3, 0, 0), \)
\( M(12, 0, -6, -3, 3), \)
\( M(12, -1, -6, -6, 6), \)
\( M(0, 0, 0, 3, 0), \)
\( M(12, 0, -9, -3, 6), \)
\( M(12, -1, -6, -3, 6), \)
\( M(-6, 0, 6, 3, -6), \)
\( M(6, 0, -3, -3, 0), \)
\( M(6, -1, 0, -3, 0), \)
M(-12, 1, 9, 6, -6),
M( 6, 0, -6, 0, 3),
M( 6, -1, -6, -3, 6),
M( 0, 0, 0, 0, -3),
M( 0, -1, 0, -3, 0),
M( 0, -1, 0, -3, 0),
M( 0, 0, -3, 0, 0),
M( 12, -1, -9, -6, 6),
M(-12, -1, -9, -6, 6),

in 5-d lattice M

sage: %timeit P.normal_form.clear_cache(); P.normal_form("palp_native")  # not tested
10 loops, best of 3: 0.137 s per loop

sage: %timeit P.normal_form.clear_cache(); P.normal_form("palp_modified")  # not tested
10 loops, best of 3: 22.2 s per loop

npoints()

Return the number of lattice points of this polytope.

EXAMPLES: The number of lattice points of the 3-dimensional octahedron and its polar cube:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.npoints()  # needs palp
7
```

```
sage: cube = o.polar()
sage: cube.npoints()  # needs palp
27
```

nvertices()

Return the number of vertices of this polytope.

EXAMPLES: The number of vertices of the 3-dimensional octahedron and its polar cube:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.nvertices()  
6
```

```
sage: cube = o.polar()
sage: cube.nvertices()  
8
```

origin()

Return the index of the origin in the list of points of self.

OUTPUT:

• integer if the origin belongs to this polytope, None otherwise.

EXAMPLES:

```
sage: p = lattice_polytope.cross_polytope(2)
sage: p.origin()  # needs palp
4
```

```
sage: p.point(p.origin())  # needs palp
```

(continues on next page)
Now we make sure that the origin of non-full-dimensional polytopes can be identified correctly (github issue #10661):

```python
sage: LatticePolytope(((1,0,0), (-1,0,0))).origin()
2
```

parent()

Return the set of all lattice polytopes.

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o.parent()
Set of all Lattice Polytopes
```

plot3d(show_facets=True, facet_opacity=0.5, facet_color=(0, 1, 0), facet_colors=None, show_edges=True, edge_thickness=3, edge_color=(0.5, 0.5, 0.5), show_vertices=True, vertex_size=10, vertex_color=(1, 0, 0), show_points=True, point_size=10, point_color=(0, 0, 1), show_vindices=None, vindex_color=(0, 0, 0), vlabels=None, show_pindices=None, pindex_color=(0, 0, 0), index_shift=1.1)

Return a 3d-plot of this polytope.

Polytopes with ambient dimension 1 and 2 will be plotted along x-axis or in xy-plane respectively. Polytopes of dimension 3 and less with ambient dimension 4 and greater will be plotted in some basis of the spanned space.

By default, everything is shown with more or less pretty combination of size and color parameters.

INPUT:

Most of the parameters are self-explanatory:

- `show_facets` - (default: True)
- `facet_opacity` - (default: 0.5)
- `facet_color` - (default: (0,1,0))
- `facet_colors` - (default: None) if specified, must be a list of colors for each facet separately, used instead of `facet_color`
- `show_edges` - (default: True) whether to draw edges as lines
- `edge_thickness` - (default: 3)
- `edge_color` - (default: (0.5,0.5,0.5))
- `show_vertices` - (default: True) whether to draw vertices as balls
- `vertex_size` - (default: 10)


- vertex_color - (default:(1,0,0))
- show_points - (default:True) whether to draw other points as balls
- point_size - (default:10)
- point_color - (default:(0,0,1))
- show_vindices - (default:as same as show_vertices) whether to show indices of vertices
- vindex_color - (default:(0,0,0)) color for vertex labels
- vlabels - (default:None) if specified, must be a list of labels for each vertex, default labels are vertex indices
- show_pindices - (default:as same as show_points) whether to show indices of other points
- pindex_color - (default:(0,0,0)) color for point labels
- index_shift - (default:1.1)) if 1, labels are placed exactly at the corresponding points. Otherwise the label position is computed as a multiple of the point position vector.

EXAMPLES: The default plot of a cube:

```
sage: c = lattice_polytope.cross_polytope(3).polar()
sage: c.plot3d()  # needs palp sage.plot
Graphics3d Object
```

Plot without facets and points, shown without the frame:

```
sage: c.plot3d(show_facets=false,  # needs palp sage.plot
....:  show_points=false).show(frame=False)
```

Plot with facets of different colors:

```
sage: c.plot3d(facet_colors=rainbow(c.nfacets(), 'rgbtuple'))  # needs palp sage.plot
Graphics3d Object
```

It is also possible to plot lower dimensional polytopes in 3D (let's also change labels of vertices):

```
sage: c2 = lattice_polytope.cross_polytope(2)
sage: c2.plot3d(vlabels=['A', 'B', 'C', 'D'])  # needs palp sage.plot
Graphics3d Object
```

point (i)

Return the i-th point of this polytope, i.e. the i-th column of the matrix returned by points().

EXAMPLES: First few points are actually vertices:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
```

(continues on next page)
The only other point in the octahedron is the origin:

```
sage: o.point(6)
```

```
M(0, 0, 0)
sage: o.points()
```

```
M(1, 0, 0),
M(0, 1, 0),
M(0, 0, 1),
M(-1, 0, 0),
M(0, -1, 0),
M(0, 0, -1),
M(0, 0, 0)
in 3-d lattice M
```

`points(*args, **kwds)`

Return all lattice points of `self`.

**INPUT:**

- any arguments given will be passed on to the returned object.

**OUTPUT:**

- a `point collection`.

**EXAMPLES:**

Lattice points of the octahedron and its polar cube:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.points()
```

```
M(1, 0, 0),
M(0, 1, 0),
M(0, 0, 1),
M(-1, 0, 0),
M(0, -1, 0),
M(0, 0, -1),
M(0, 0, 0)
in 3-d lattice M
```

```
sage: cube = o.polar()
sage: cube.points()
```

```
N(1, -1, -1),
N(1, 1, -1),
N(1, 1, 1),
N(1, -1, 1),
N(-1, -1, 1),
N(-1, -1, -1),
N(-1, 1, -1),
N(-1, 1, 1),
N(-1, 1, 1),
```

(continues on next page)
Lattice points of a 2-dimensional diamond in a 3-dimensional space:

```python
sage: p = LatticePolytope([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.points()  #← needs palp
M( 1, 0, 0),
M( 0, 1, 0),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, 0)
in 3-d lattice M
```

Only two of the above points:

```python
sage: p.points(1, 3)  #← needs palp
M(0, 1, 0),
M(0, -1, 0)
in 3-d lattice M
```

We check that points of a zero-dimensional polytope can be computed:

```python
sage: p = LatticePolytope([[1]])
sage: p.points()
M(1)
in 1-d lattice M
```

**polar()**

Return the polar polytope, if this polytope is reflexive.

EXAMPLES: The polar polytope to the 3-dimensional octahedron:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: cube = o.polar()
sage: cube
3-d reflexive polytope in 3-d lattice N
The polar polytope “remembers” the original one:

```
sage: cube.polar()
3-d reflexive polytope in 3-d lattice M
sage: cube.polar().polar() is cube
True
```

Only reflexive polytopes have polars:

```
sage: p = LatticePolytope([(1,0,0), (0,1,0), (0,0,2), ....: (-1,0,0), (0,-1,0), (0,0,-1)])
sage: p.polar()
Traceback (most recent call last):
...
ValueError: The given polytope is not reflexive!
Polytope: 3-d lattice polytope in 3-d lattice M
```

**poly_x** *(keys, reduce_dimension=False)*

Run poly.x with given keys on vertices of this polytope.

**INPUT:**

- keys - a string of options passed to poly.x. The key “f” is added automatically.
- reduce_dimension - (default: False) if True and this polytope is not full-dimensional, poly.x will be called for the vertices of this polytope in some basis of the spanned affine space.

**OUTPUT:** the output of poly.x as a string.

**EXAMPLES:** This call is used for determining if a polytope is reflexive or not:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: print(o.poly_x("e"))
#...
8 3 Vertices of P-dual <-> Equations of P
-1 -1 1
 1 -1 1
-1 1 1
 1 1 1
-1 -1 -1
 1 -1 -1
-1 1 -1
 1 1 -1
```

Since PALP has limits on different parameters determined during compilation, the following code is likely to fail, unless you change default settings of PALP:

```
sage: BIG = lattice_polytope.cross_polytope(7)
sage: BIG
7-d reflexive polytope in 7-d lattice M
sage: BIG.poly_x("e")  #...
←needs palp
Traceback (most recent call last):
...
ValueError: Error executing 'poly.x -fe' for the given polytope!
Output:
Please increase POLY_Dmax to at least 7
```

You cannot call poly.x for polytopes that don’t span the space (if you could, it would crush anyway):
Combinatorial and Discrete Geometry, Release 10.3

```
sage: p = LatticePolytope([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.poly_x("e")  # needs palp
Traceback (most recent call last):
  ...:
ValueError: Cannot run PALP for a 2-dimensional polytope in a 3-dimensional space!

But if you know what you are doing, you can call it for the polytope in some basis of the spanned space:
```
sage: print(p.poly_x("e", reduce_dimension=True))  # needs palp
4 2  Equations of P
-1  1  0
 1  1  2
-1 -1  0
 1 -1  2
```

**polyhedron(****kwds**)**

Return the Polyhedron object determined by this polytope's vertices.

EXAMPLES:
```
sage: o = lattice_polytope.cross_polytope(2)
sage: o.polyhedron()
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
```

**show3d()**

Show a 3d picture of the polytope with default settings and without axes or frame.

See self.plot3d? for more details.

EXAMPLES:
```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.show3d()  # needs palp sage.plot
```

**skeleton()**

Return the graph of the one-skeleton of this polytope.

EXAMPLES:
```
sage: d = lattice_polytope.cross_polytope(2)
sage: g = d.skeleton(); g  # needs palp sage.graphs
Graph on 4 vertices
sage: g.edges(sort=True)  # needs palp sage.graphs
[(0, 1, None), (0, 3, None), (1, 2, None), (2, 3, None)]
```

**skeleton_points**(k=1)

Return the increasing list of indices of lattice points in k-skeleton of the polytope (k is 1 by default).

EXAMPLES: We compute all skeleton points for the cube:
sage: o = lattice_polytope.cross_polytope(3)
sage: c = o.polar()
sage: c.skeleton_points()  # needs palp sage.graphs
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 19, 21, 22, 23, 25, 26]

The default was 1-skeleton:

sage: c.skeleton_points(k=1)  # needs palp sage.graphs
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 19, 21, 22, 23, 25, 26]

0-skeleton just lists all vertices:

sage: c.skeleton_points(k=0)  # needs palp sage.graphs
[0, 1, 2, 3, 4, 5, 6, 7]

2-skeleton lists all points except for the origin (point #17):

sage: c.skeleton_points(k=2)  # needs palp sage.graphs
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26]

3-skeleton includes all points:

sage: c.skeleton_points(k=3)  # needs palp
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]

It is OK to compute higher dimensional skeletons - you will get the list of all points:

sage: c.skeleton_points(k=100)  # needs palp
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]

skeleton_show(normal=None)
Show the graph of one-skeleton of this polytope. Works only for polytopes in a 3-dimensional space.

INPUT:

• normal - a 3-dimensional vector (can be given as a list), which should be perpendicular to the screen.
  If not given, will be selected randomly (new each time and it may be far from “nice”).

EXAMPLES: Show a pretty picture of the octahedron:

sage: o = lattice_polytope.cross_polytope(3)
sage: o.skeleton_show([1,2,4])  # needs palp sage.plot

Does not work for a diamond at the moment:

sage: d = lattice_polytope.cross_polytope(2)
sage: d.skeleton_show()  
Traceback (most recent call last):

(continues on next page)
traverse_boundary()

Return a list of indices of vertices of a 2-dimensional polytope in their boundary order.

Needed for plot3d function of polytopes.

EXAMPLES:

```sage
p = lattice_polytope.cross_polytope(2).polar()
sage: p.traverse_boundary()
[3, 0, 1, 2]
```

vertex(i)

Return the i-th vertex of this polytope, i.e. the i-th column of the matrix returned by vertices().

EXAMPLES: Note that numeration starts with zero:

```sage
o = lattice_polytope.cross_polytope(3)
sage: o.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
sage: o.vertex(3)
M(-1, 0, 0)
```

vertex_facet_pairing_matrix()

Return the vertex facet pairing matrix $PM$.

Return a matrix whose the $i, j$th entry is the height of the $j$th vertex over the $i$th facet. The ordering of the vertices and facets is as in vertices() and facets().

EXAMPLES:

```sage
L = lattice_polytope.cross_polytope(3)
sage: L.vertex_facet_pairing_matrix()
[2 0 0 0 2 2]
[2 2 0 0 0 2]
[2 2 2 0 0 0]
[2 0 2 0 2 0]
[0 0 2 2 0 0]
[0 2 0 2 0 0]
```

vertices(*args, **kwds)

Return vertices of self.

INPUT:

- any arguments given will be passed on to the returned object.

OUTPUT:
• a point collection.

EXAMPLES:

Vertices of the octahedron and its polar cube are in dual lattices:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: o.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
sage: cube = o.polar()
sage: cube.vertices()
N( 1, -1, -1),
N( 1, 1, -1),
N( 1, 1, 1),
N(-1, -1, 1),
N(-1, -1, -1),
N(-1, 1, -1),
N(-1, 1, 1)
in 3-d lattice N
```

class sage.geometry.lattice_polytope.NefPartition(data, Delta_polar, check=True)
Bases: SageObject, Hashable

Create a nef-partition.

INPUT:

• data – a list of integers, the \(i\)-th element of this list must be the part of the \(i\)-th vertex of Delta_polar in this nef-partition;

• Delta_polar – a lattice polytope;

• check – by default the input will be checked for correctness, i.e. that data indeed specify a nef-partition.

If you are sure that the input is correct, you can speed up construction via check=False option.

OUTPUT:

• a nef-partition of Delta_polar.

Let \(M\) and \(N\) be dual lattices. Let \(\Delta \subset M^\mathbb{R}\) be a reflexive polytope with polar \(\Delta^\circ \subset N^\mathbb{R}\). Let \(X_\Delta\) be the toric variety associated to the normal fan of \(\Delta\). A nef-partition is a decomposition of the vertex set \(V\) of \(\Delta^\circ\) into a disjoint union \(V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_{k-1}\) such that divisors \(E_i = \sum_{v \in V_i} D_v\) are Cartier (here \(D_v\) are prime torus-invariant Weil divisors corresponding to vertices of \(\Delta^\circ\)). Equivalently, let \(\nabla_i \subset N^\mathbb{R}\) be the convex hull of vertices from \(V_i\) and the origin. These polytopes form a nef-partition if their Minkowski sum \(\nabla \subset N^\mathbb{R}\) is a reflexive polytope.

The dual nef-partition is formed by polytopes \(\Delta_i \subset M^\mathbb{R}\) of \(E_i\), which give a decomposition of the vertex set of \(\nabla^\circ \subset M^\mathbb{R}\) and their Minkowski sum is \(\Delta\), i.e. the polar duality of reflexive polytopes switches convex hull and
Minkowski sum for dual nef-partitions:

\[ \Delta^\circ = \text{Conv}(\nabla_0, \nabla_1, \ldots, \nabla_{k-1}), \]
\[ \nabla = \nabla_0 + \nabla_1 + \cdots + \nabla_{k-1}, \]
\[ \Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_{k-1}, \]
\[ \nabla^\circ = \text{Conv}(\Delta_0, \Delta_1, \ldots, \Delta_{k-1}). \]

One can also interpret the duality of nef-partitions as the duality of the associated cones. Below \( \overline{M} = M \times \mathbb{Z}^k \) and \( \overline{N} = N \times \mathbb{Z}^k \) are dual lattices.

The Cayley polytope \( P \subset \overline{M}_\mathbb{R} \) of a nef-partition is given by \( P = \text{Conv}(\Delta_0 \times e_0, \Delta_1 \times e_1, \ldots, \Delta_{k-1} \times e_{k-1}) \), where \( \{e_i\}_{i=0}^{k-1} \) is the standard basis of \( \mathbb{Z}^k \). The dual Cayley polytope \( P^* \subset \overline{N}_\mathbb{R} \) is the Cayley polytope of the dual nef-partition.

The Cayley cone \( C \subset \overline{M}_\mathbb{R} \) of a nef-partition is the cone spanned by its Cayley polytope. The dual Cayley cone \( C^\vee \subset \overline{M}_\mathbb{R} \) is the usual dual cone of \( C \). It turns out, that \( C^\vee \) is spanned by \( P^* \).

It is also possible to go back from the Cayley cone to the Cayley polytope, since \( C \) is a reflexive Gorenstein cone supported by \( P \); primitive integral ray generators of \( C \) are contained in an affine hyperplane and coincide with vertices of \( P \).

See Section 4.3.1 in [CK1999] and references therein for further details, or [BN2008] for a purely combinatorial approach.

**EXAMPLES:**

It is very easy to create a nef-partition for the octahedron, since for this polytope any decomposition of vertices is a nef-partition. We create a 3-part nef-partition with the 0-th and 1-st vertices belonging to the 0-th part (recall that numeration in Sage starts with 0), the 2-nd and 5-th vertices belonging to the 1-st part, and 3-rd and 4-th vertices belonging to the 2-nd part:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0,0,1,2,2,1], o)
sage: np
Nef-partition {0, 1} ⊔ {2, 5} ⊔ {3, 4}
```

The octahedron plays the role of \( \Delta^\circ \) in the above description:

```
sage: np.Delta_polar() is o
True
```

The dual nef-partition (corresponding to the “mirror complete intersection”) gives decomposition of the vertex set of \( \nabla^\circ \):

```
sage: np.dual()
Nef-partition {0, 1, 2} ⊔ {3, 4} ⊔ {5, 6, 7}
sage: np.nabla_polar().vertices()
N(-1, -1, 0),
N(-1, 0, 0),
N( 0, -1, 0),
N( 0, 0, -1),
N( 0, 0, 1),
N( 1, 0, 0),
N( 0, 1, 0),
N( 1, 1, 0)
in 3-d lattice N
```

Of course, \( \nabla^\circ \) is \( \Delta^\circ \) from the point of view of the dual nef-partition:
Instead of constructing nef-partitions directly, you can request all 2-part nef-partitions of a given reflexive polytope (they will be computed using nef.x program from PALP):

```
sage: o.nef_partitions()  # needs palp
[ Nef-partition {0, 1, 3} ⊔ {2, 4, 5},
  Nef-partition {0, 1, 3, 4} ⊔ {2, 5} (direct product),
  Nef-partition {0, 1, 2} ⊔ {3, 4, 5},
  Nef-partition {0, 1, 2, 3} ⊔ {4, 5},
  Nef-partition {0, 1, 2, 3, 4} ⊔ {5} (projection) ]
```

**Delta (i=None)**

Return the polytope $\Delta$ or $\Delta_i$ corresponding to self.

**INPUT:**

- i – an integer. If not given, $\Delta$ will be returned.

**OUTPUT:**

- a lattice polytope.

See nef-partition class documentation for definitions and notation.

**EXAMPLES:**

```
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition {0, 1, 3} ⊔ {2, 4, 5}
sage: mp = np.Delta().polar() is o
True
sage: np.Delta().vertices()
N( 1, -1, -1),
N( 1,  1, -1),
N( 1,  1,  1),
N(-1, -1,  1),
N(-1,  1, -1),
N(-1,  1,  1),
in 3-d lattice N
sage: np.Delta(0).vertices()
N(-1,  0,  0),
N( 1,  0,  0),
```

(continues on next page)
Delta_polar()

Return the polytope $\Delta^o$ corresponding to self.

OUTPUT:

- a lattice polytope.

See nef-partition class documentation for definitions and notation.

EXAMPLES:

```sage
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition {0, 1, 3} ⊔ {2, 4, 5}
sage: np.Delta_polar() is o
True
```

Deltas()

Return the polytopes $\Delta_i$ corresponding to self.

OUTPUT:

- a tuple of lattice polytopes.

See nef-partition class documentation for definitions and notation.

EXAMPLES:

```sage
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition {0, 1, 3} ⊔ {2, 4, 5}
sage: np.Delta().vertices()
N(-1, -1, 0),
N(-1, 0, -1),
N( 1, 0, 0),
N( 1, -1, 0)
in 3-d lattice N
sage: [Delta_i.vertices() for Delta_i in np.Deltas()]
[N(-1, -1, 0),
 N( 1, 0, 0),
 N( 1, -1, 0)
in 3-d lattice N,
 N(0, 0, -1),
 N(0, 1, 1),
 N(0, 0, 1),
 N(0, 1, -1)
in 3-d lattice N]
sage: np.nabla_polar().vertices()
N(-1, -1, 0),
N( 1, -1, 0),
N( 1, 0, 0),
N(-1, 0, 0),
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

\[ N(-1, 0, 0), \]
\[ N(0, 1, -1), \]
\[ N(0, 1, 1), \]
\[ N(0, 0, 1), \]
\[ N(0, 0, -1) \]
in 3-d lattice \( N \)

\[ \text{dual}() \]

Return the dual nef-partition.

OUTPUT:

- a nef-partition.

See the class documentation for the definition.

ALGORITHM:

See Proposition 3.19 in [BN2008].

**Note:** Automatically constructed dual nef-partitions will be ordered, i.e. vertex partition of \( \nabla \) will look like \( \{0, 1, 2\} \cup \{3, 4, 5, 6\} \cup \{7, 8\} \).

**EXAMPLES:**

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition {0, 1, 3} \cup {2, 4, 5}
sage: np.dual()
Nef-partition {0, 1, 2, 3} \cup {4, 5, 6, 7}
sage: np.dual().Delta() is np.nabla()
True
sage: np.dual().nabla(0) is np.Delta(0)
True
```

\[ \text{hodge_numbers}() \]

Return Hodge numbers corresponding to self.

OUTPUT:

- a tuple of integers (produced by nef.x program from PALP).

**EXAMPLES:**

Currently, you need to request Hodge numbers when you compute nef-partitions:

```python
sage: # long time, needs palp
sage: p = lattice_polytope.cross_polytope(5)
sage: np = p.nef_partitions()[0] # 4s on sage.math, 2011
sage: np.hodge_numbers()
Traceback (most recent call last):
...  
NotImplementedError: use nef_partitions(hodge_numbers=True)!
sage: np = p.nef_partitions(hodge_numbers=True)[0]  # 13s on sage.math, 2011
sage: np.hodge_numbers()
(19, 19)
```

2.2. Lattice polyhedra 207
nabla\( (i=None) \)

Return the polytope \( \nabla \) or \( \nabla_i \), corresponding to self.

**INPUT:**

- \( i \) – an integer. If not given, \( \nabla \) will be returned.

**OUTPUT:**

- a *lattice polytope*.

See *nef-partition* class documentation for definitions and notation.

**EXAMPLES:**

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition \{0, 1, 3\} \cup \{2, 4, 5\}
sage: np.Delta_polar().vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
sage: np.nabla(0).vertices()
M(-1, 0, 0),
M( 1, 0, 0),
M( 0, 1, 0)
in 3-d lattice M
sage: np.nabla().vertices()
M(-1, 0, 1),
M(-1, 0, -1),
M( 1, 0, 1),
M( 1, 0, -1),
M( 0, 1, 1),
M( 0, 1, -1),
M( 1, -1, 0),
M(-1, -1, 0)
in 3-d lattice M
```

nabla_polar()

Return the polytope \( \nabla^o \) corresponding to self.

**OUTPUT:**

- a *lattice polytope*.

See *nef-partition* class documentation for definitions and notation.

**EXAMPLES:**

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition \{0, 1, 3\} \cup \{2, 4, 5\}
sage: np.nabla_polar().vertices()
N(-1, -1, 0),
N( 1, -1, 0),
N( 1, 0, 0),
N(-1, -1, 0),
N(-1, -1, 0),
N(-1, -1, 0),
N( 1, -1, 0),
N( 1, 0, 0),
N(-1, -1, 0),
N(-1, -1, 0),
N(-1, -1, 0),
```

(continues on next page)
\begin{verbatim}
N(0, 1, -1),
N(0, 1, 1),
N(0, 0, 1),
N(0, 0, -1)
in 3-d lattice N
sage: np.nabla_polar() is np.dual().Delta_polar()
True
\end{verbatim}

\textbf{nablas()}

Return the polytopes $\nabla_i$ corresponding to \texttt{self}.

\textbf{OUTPUT:}

- a tuple of \textbf{lattice polytopes}.

See \texttt{nef-partition} class documentation for definitions and notation.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition {0, 1, 3} ⊔ {2, 4, 5}
sage: np.Delta_polar().vertices()
M(1, 0, 0),
M(0, 1, 0),
M(0, 0, 1),
M(-1, 0, 0),
M(0, -1, 0),
M(0, 0, -1)
in 3-d lattice M
sage: [nabla_i.vertices() for nabla_i in np.nablas()]
[[M(-1, 0, 0),
  M(1, 0, 0),
  M(0, 1, 0)
in 3-d lattice M,
  M(0, -1, 0),
  M(0, 0, -1),
  M(0, 0, 1)
in 3-d lattice M]
\end{verbatim}

\textbf{nparts()}

Return the number of parts in \texttt{self}.

\textbf{OUTPUT:}

- an integer.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition {0, 1, 3} ⊔ {2, 4, 5}
sage: np.nparts()
2
\end{verbatim}

\textbf{part (i, all_points=False)}

Return the $i$-th part of \texttt{self}.

\textbf{INPUT:}
• \( i \) – an integer

• \( \text{all_points} \) – (default: False) whether to list all lattice points or just vertices

**OUTPUT:**

• a tuple of integers, indices of vertices (or all lattice points) of \( \Delta^o \) belonging to \( V_i \).

See \texttt{nef-partition} class documentation for definitions and notation.

**EXAMPLES:**

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition \{0, 1, 3\} \cup \{2, 4, 5\}
sage: np.part(0)
(0, 1, 3)
sage: np.part(0, all_points=True)  # needs palp
(0, 1, 3)
sage: np.dual().part(0)
(0, 1, 2, 3)
sage: np.dual().part(0, all_points=True)  # needs palp
(0, 1, 2, 3, 8)
```

\textbf{part_of} \((i)\)

Return the index of the part containing the \( i \)-th vertex.

**INPUT:**

• \( i \) – an integer.

**OUTPUT:**

• an integer \( j \) such that the \( i \)-th vertex of \( \Delta^o \) belongs to \( V_j \).

See \texttt{nef-partition} class documentation for definitions and notation.

**EXAMPLES:**

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition \{0, 1, 3\} \cup \{2, 4, 5\}
sage: np.part_of(3)  # needs palp
0
sage: np.part_of(2)
1
```

\textbf{part_of_point} \((i)\)

Return the index of the part containing the \( i \)-th point.

**INPUT:**

• \( i \) – an integer.

**OUTPUT:**

• an integer \( j \) such that the \( i \)-th point of \( \Delta^o \) belongs to \( \nabla_j \).

**Note:** Since a nef-partition induces a partition on the set of boundary lattice points of \( \Delta^o \), the value of \( j \) is well-defined for all \( i \) but the one that corresponds to the origin, in which case this method will raise a
ValueError exception. (The origin always belongs to all $\nabla_j$.)

See nef-partition class documentation for definitions and notation.

**EXAMPLES:**

We consider a relatively complicated reflexive polytope #2252 (easily accessible in Sage as ReflexivePolytope(3, 2252), we create it here explicitly to avoid loading the whole database):

```python
sage: p = LatticePolytope([(1,0,0), (0,1,0), (0,0,1), (0,1,-1),
....: (0,-1,1), (-1,1,0), (0,-1,-1), (-1,-1,0), (-1,-1,2)])
sage: np = p.nef_partitions()[0]; np
Nef-partition \{1, 2, 5, 7, 8\} ⊔ \{0, 3, 4, 6\}
```

We see that the polytope has 6 more points in addition to vertices. One of them is the origin:

```python
sage: p.origin()
14
```

But the remaining 5 are partitioned by `np`:

```python
sage: [n for n in range(p.npoints()) if p.origin() != n and np.part_of_point(n) == 0] [1, 2, 5, 7, 8]
sage: [n for n in range(p.npoints()) if p.origin() != n and np.part_of_point(n) == 1] [0, 3, 4, 6, 10, 12]
```

**parts** *(all_points=False)*

Return all parts of `self`.

**INPUT:**

- `all_points` – (default: False) whether to list all lattice points or just vertices

**OUTPUT:**

- a tuple of tuples of integers. The $i$-th tuple contains indices of vertices (or all lattice points) of $\Delta^\circ$ belonging to $V_i$

See nef-partition class documentation for definitions and notation.

**EXAMPLES:**
sage: o = lattice_polytope.cross_polytope(3)
sage: np = NefPartition([0, 0, 1, 0, 1, 1], o); np
Nef-partition (0, 1, 3) \cup (2, 4, 5)
sage: np.parts()
((0, 1, 3), (2, 4, 5))
sage: np.parts(all_points=True)  # needs palp
((0, 1, 3), (2, 4, 5))
sage: np.dual().parts()
((0, 1, 2, 3), (4, 5, 6, 7))
sage: np.dual().parts(all_points=True)  # needs palp
((0, 1, 2, 3, 8), (4, 5, 6, 7, 10))

sage.geometry.lattice_polytope.ReflexivePolytope(dim, n)
Return the \( n \)-th 2- or 3-dimensional reflexive polytope.

**Note:**
1. Numeration starts with zero: \( 0 \leq n \leq 15 \) for \( \dim = 2 \) and \( 0 \leq n \leq 4318 \) for \( \dim = 3 \).
2. During the first call, all reflexive polytopes of requested dimension are loaded and cached for future use, so the first call for 3-dimensional polytopes can take several seconds, but all consecutive calls are fast.
3. Equivalent to ReflexivePolytopes(dim)[n] but checks bounds first.

**EXAMPLES:**
The 3rd 2-dimensional polytope is “the diamond”:

sage: ReflexivePolytope(2, 3)
2-d reflexive polytope #3 in 2-d lattice M
sage: lattice_polytope.ReflexivePolytope(2,3).vertices()
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M

There are 16 reflexive polygons and numeration starts with 0:

sage: ReflexivePolytope(2,16)
Traceback (most recent call last):
...  
ValueError: there are only 16 reflexive polygons!

It is not possible to load a 4-dimensional polytope in this way:

sage: ReflexivePolytope(4,16)
Traceback (most recent call last):
...  
NotImplementedError: only 2- and 3-dimensional reflexive polytopes are available!

sage.geometry.lattice_polytope.ReflexivePolytopes(dim)
Return the sequence of all 2- or 3-dimensional reflexive polytopes.
Note: During the first call the database is loaded and cached for future use, so repetitive calls will return the same object in memory.

**Parameters**

- **dim (2 or 3)** – dimension of required reflexive polytopes

**Return type**

list of lattice polytopes

**EXAMPLES:**

There are 16 reflexive polygons:

```
sage: len(ReflexivePolytopes(2))
16
```

It is not possible to load 4-dimensional polytopes in this way:

```
sage: ReflexivePolytopes(4)
Traceback (most recent call last):
...
NotImplementedError: only 2- and 3-dimensional reflexive polytopes are available!
```

class sage.geometry.lattice_polytope.SetOfAllLatticePolytopesClass

Bases: Set_generic

sage.geometry.lattice_polytope.all_cached_data(polytopes)

Compute all cached data for all given polytopes and their polars.

This function does it MUCH faster than member functions of LatticePolytope during the first run. So it is recommended to use this function if you work with big sets of data. None of the polytopes in the given sequence should be constructed as the polar polytope to another one.

**INPUT:** a sequence of lattice polytopes.

**EXAMPLES:** This function has no output, it is just a fast way to work with long sequences of polytopes. Of course, you can use short sequences as well:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: lattice_polytope.all_cached_data([o])  # ...
→ needs palp
```

sage.geometry.lattice_polytope.all_facet_equations(polytopes)

Compute polar polytopes for all reflexive and equations of facets for all non-reflexive polytopes.

```
all_facet_equations and all_polars are synonyms.
```

This function does it MUCH faster than member functions of LatticePolytope during the first run. So it is recommended to use this function if you work with big sets of data.

**INPUT:** a sequence of lattice polytopes.

**EXAMPLES:** This function has no output, it is just a fast way to work with long sequences of polytopes. Of course, you can use short sequences as well:
sage: o = lattice_polytope.cross_polytope(3)
sage: lattice_polytope.all_polars([o])  #...
→ needs palp
sage: o.polar()  #...
→ needs palp
3-d reflexive polytope in 3-d lattice N

sage.geometry.lattice_polytope.all_nef_partitions(polytopes, keep_symmetric=False)
Compute nef-partitions for all given polytopes.

This function does it MUCH faster than member functions of LatticePolytope during the first run. So it is recommended to use this function if you work with big sets of data.

Note: member function is_reflexive will be called separately for each polytope. It is strictly recommended to call all_polars on the sequence of polytopes before using this function.

INPUT: a sequence of lattice polytopes.

EXAMPLES: This function has no output, it is just a fast way to work with long sequences of polytopes. Of course, you can use short sequences as well:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: lattice_polytope.all_nef_partitions([o])  #...
→ needs palp
sage: o.nef_partitions()  #...
→ needs palp
[ Nef-partition {0, 1, 3} ⊔ {2, 4, 5},
  Nef-partition {0, 1, 3, 4} ⊔ {2, 5} (direct product),
  Nef-partition {0, 1, 2} ⊔ {3, 4, 5},
  Nef-partition {0, 1, 2, 3} ⊔ {4, 5},
  Nef-partition {0, 1, 2, 3, 4} ⊔ {5} (projection) ]
```

You cannot use this function for non-reflexive polytopes:

```
sage: p = LatticePolytope([(1,0,0), (0,1,0), (0,0,2),
  ....: (1,0,0), (0,-1,0), (0,0,-1)])
sage: lattice_polytope.all_nef_partitions([o, p])  #...
→ needs palp
Traceback (most recent call last):
... ...
ValueError: nef-partitions can be computed for reflexive polytopes only
```
needs palp
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1),
M( 0, 0, 0)
in 3-d lattice M

sage.geometry.lattice_polytope.all_polars(polytopes)
Compute polar polytopes for all reflexive and equations of facets for all non-reflexive polytopes.
allexfacet_equations and all_polars are synonyms.
This function does it MUCH faster than member functions of LatticePolytope during the first run. So it is recommended to use this functions if you work with big sets of data.
INPUT: a sequence of lattice polytopes.
EXAMPLES: This function has no output, it is just a fast way to work with long sequences of polytopes. Of course, you can use short sequences as well:

```
sage: o = lattice_polytope.cross_polytope(3)
sage: lattice_polytope.all_polars([o])  #...
needs palp
sage: o.polar()  #...
needs palp
3-d reflexive polytope in 3-d lattice N
```

sage.geometry.lattice_polytope.convex_hull(points)
Compute the convex hull of the given points.

**Note:** points might not span the space. Also, it fails for large numbers of vertices in dimensions 4 or greater

**INPUT:**
• points - a list that can be converted into vectors of the same dimension over ZZ.

**OUTPUT:**
list of vertices of the convex hull of the given points (as vectors).

**EXAMPLES:** Let’s compute the convex hull of several points on a line in the plane:

```
sage: lattice_polytope.convex_hull([[1,2],[3,4],[5,6],[7,8]])
[(1, 2), (7, 8)]
```

sage.geometry.lattice_polytope.cross_polytope(dim)
Return a cross-polytope of the given dimension.

**INPUT:**
• dim – an integer.

**OUTPUT:**
• a lattice polytope.
EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: o
3-d reflexive polytope in 3-d lattice M
sage: o.vertices()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
```

```
sage.geometry.lattice_polytope.is_LatticePolytope(x)
Check if x is a lattice polytope.

INPUT:

- x – anything.

OUTPUT:

- True if x is a lattice polytope, False otherwise.

EXAMPLES:

```python
sage: from sage.geometry.lattice_polytope import is_LatticePolytope
sage: is_LatticePolytope(1)
False
sage: p = LatticePolytope([(1,0), (0,1), (-1,-1)])
sage: p
# needs palp
2-d reflexive polytope #0 in 2-d lattice M
sage: is_LatticePolytope(p)
True
```

```
sage.geometry.lattice_polytope.is_NefPartition(x)
Check if x is a nef-partition.

INPUT:

- x – anything.

OUTPUT:

- True if x is a nef-partition and False otherwise.

EXAMPLES:

```python
sage: from sage.geometry.lattice_polytope import is_NefPartition
sage: is_NefPartition(1)
False
sage: o = lattice_polytope.cross_polytope(3)
sage: np = o.nef_partitions()[0]; np
# needs palp
Nef-partition {0, 1, 3} ⊔ {2, 4, 5}
sage: is_NefPartition(np)
# needs palp
True
```
sage.geometry.lattice_polytope.minkowski_sum(points1, points2)

Compute the Minkowski sum of two convex polytopes.

Note: Polytopes might not be of maximal dimension.

INPUT:

- points1, points2 - lists of objects that can be converted into vectors of the same dimension, treated as vertices of two polytopes.

OUTPUT: list of vertices of the Minkowski sum, given as vectors.

EXAMPLES: Let's compute the Minkowski sum of two line segments:

```
sage: lattice_polytope.minkowski_sum([[1,0],[-1,0]],[[0,1],[0,-1]])
[(1, 1), (1, -1), (-1, 1), (-1, -1)]
```

sage.geometry.lattice_polytope.positive_integer_relations(points)

Return relations between given points.

INPUT:

- points - lattice points given as columns of a matrix

OUTPUT:

matrix of relations between given points with non-negative integer coefficients

EXAMPLES: This is a 3-dimensional reflexive polytope:

```
sage: p = LatticePolytope([[1,0,0], [0,1,0], 
                        ....:   (-1,-1,0), (0,0,1), (-1,0,-1)])
sage: p.points()  # needs palp
M( 1, 0, 0),
M( 0, 1, 0),
M(-1, -1, 0),
M( 0, 0, 1),
M(-1, 0, -1),
M( 0, 0, 0)
in 3-d lattice M
```

We can compute linear relations between its points in the following way:

```
sage: p.points().matrix().kernel().echelonized_basis_matrix()  # needs palp
[ 1 0 0 1 1 0]
[ 0 1 1 -1 -1 0]
[ 0 0 0 0 0 1]
```

However, the above relations may contain negative and rational numbers. This function transforms them in such a way, that all coefficients are non-negative integers:

```
sage: points = p.points().column_matrix()
sage: lattice_polytope.positive_integer_relations(points)  # needs palp
[1 0 0 1 1 0]
[1 1 0 0 0]
```

(continues on next page)
sage.geometry.lattice_polytope.read_all_polytopes(\texttt{file\_name})

Read all polytopes from the given file.

INPUT:

- \texttt{file\_name} – a string with the name of a file with VERTICES of polytopes.

OUTPUT:

- a sequence of polytopes.

EXAMPLES:

We use poly.x to compute two polar polytopes and read them:

\begin{verbatim}
sage: # needs palp
ds = lattice_polytope.cross_polytope(2)
do = lattice_polytope.cross_polytope(3)
sage: result_name = lattice_polytope._palp("poly.x -fe", \[d, o\])
sage: with open(result_name) as f:
....:
    print(f.read())
4 2 Vertices of P-dual <-> Equations of P
  -1  1
  1  1
-1 -1
  1 -1
8 3 Vertices of P-dual <-> Equations of P
  -1 -1  1
  1 -1  1
-1  1  1
  1  1  1
-1 -1 -1
  1 -1 -1
-1  1 -1
  1  1 -1
sage: lattice_polytope.read_all_polytopes(result_name)
[2-d reflexive polytope #14 in 2-d lattice M,
  3-d reflexive polytope in 3-d lattice M]
sage: os.remove(result_name)
\end{verbatim}

sage.geometry.lattice_polytope.read_palp_matrix(\texttt{data}, \texttt{permutation=False})

Read and return an integer matrix from a string or an opened file.

First input line must start with two integers \(m\) and \(n\), the number of rows and columns of the matrix. The rest of
the first line is ignored. The next \(m\) lines must contain \(n\) numbers each.

If \(m>\text{n}\), returns the transposed matrix. If the string is empty or EOF is reached, returns the empty matrix, con-
structed by \texttt{matrix()}

INPUT:

- \texttt{data} – Either a string containing the filename or the file itself
  containing the output by PALP.
• permutation – (default: False) If True, try to retrieve the permutation output by PALP. This parameter makes sense only when PALP computed the normal form of a lattice polytope.

OUTPUT:
A matrix or a tuple of a matrix and a permutation.

EXAMPLES:

```
sage: lattice_polytope.read_palp_matrix("2 3 comment \n 1 2 3 \n 4 5 6")
[1 2 3]
[4 5 6]
sage: lattice_polytope.read_palp_matrix("3 2 Will be transposed \n 1 2 \n 3 4 \n 5 6")
[1 3 5]
[2 4 6]
```

sage.geometry.lattice_polytope.set_palp_dimension(d)
Set the dimension for PALP calls to d.

INPUT:
• d – an integer from the list [4,5,6,11] or None.

OUTPUT:
• none.

PALP has many hard-coded limits, which must be specified before compilation, one of them is dimension. Sage includes several versions with different dimension settings (which may also affect other limits and enable certain features of PALP). You can change the version which will be used by calling this function. Such a change is not done automatically for each polytope based on its dimension, since depending on what you are doing it may be necessary to use dimensions higher than that of the input polytope.

EXAMPLES:
Let's try to work with a 7-dimensional polytope:

```
sage: p = lattice_polytope.cross_polytope(7)
sage: p._palp("poly.x -fv")
# needs palp
Traceback (most recent call last):
...
ValueError: Error executing 'poly.x -fv' for the given polytope!
Output:
Please increase POLY_Dmax to at least 7
```

However, we can work with this polytope by changing PALP dimension to 11:

```
sage: lattice_polytope.set_palp_dimension(11)
sage: p._palp("poly.x -fv")
# needs palp
'7 14 Vertices of P...'
```

Let's go back to default settings:

```
sage: lattice_polytope.set_palp_dimension(None)
```

sage.geometry.lattice_polytope.skip_palp_matrix(data, n=1)
Skip matrix data in a file.

INPUT:
• **data**: opened file with blocks of matrix data in the following format: A block consisting of \( m+1 \) lines has the number \( m \) as the first element of its first line.

• **\( n \)** -(default: 1) integer, specifies how many blocks should be skipped

If EOF is reached during the process, raises ValueError exception.

**EXAMPLES:** We create a file with vertices of the square and the cube, but read only the second set:

```python
sage: # needs palp
sage: d = lattice_polytope.cross_polytope(2)
sage: o = lattice_polytope.cross_polytope(3)
sage: result_name = lattice_polytope._palp("poly.x -fe", [d, o])
sage: with open(result_name) as f:
....:
    print(f.read())
4 2 Vertices of P-dual <-> Equations of P
 -1 1
 1 1
-1 -1
 1 -1
8 3 Vertices of P-dual <-> Equations of P
 -1 -1 1
 1 -1 1
-1 1 1
 1 1 1
-1 -1 -1
 1 -1 -1
-1 1 -1
 1 1 -1
```

```python
sage: f = open(result_name)
sage: lattice_polytope.skip_palp_matrix(f)
sage: lattice_polytope.read_palp_matrix(f)
[-1 1 -1 1 -1 1 -1 1]
[-1 -1 1 1 -1 -1 1 1]
[ 1 1 1 1 -1 -1 -1 -1]
sage: f.close()
sage: os.remove(result_name)
sage.geometry.lattice_polytope.write_palp_matrix(m, ofile=None, comment='', format=None)
```

Write \( m \) into \( ofile \) in PALP format.

**INPUT:**

• \( m \) – a matrix over integers or a **point collection**.

• \( ofile \) – a file opened for writing (default: stdout)

• **comment** – a string (default: empty) see output description

• **format** – a format string used to print matrix entries.

**OUTPUT:**

• nothing is returned, output written to \( ofile \) has the format

  – First line: number_of_rows number_of_columns comment

  – Next number_of_rows lines: rows of the matrix.

**EXAMPLES:**
2.2.2 Lattice Euclidean Group Elements

The classes here are used to return particular isomorphisms of PPL lattice polytopes.

```
class sage.geometry.polyhedron.lattice_euclidean_group_element.LatticeEuclideanGroupElement
```

Bases: `SageObject`

An element of the lattice Euclidean group.

Note that this is just intended as a container for results from LatticePolytope_PPL. There is no group-theoretic functionality to speak of.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL, C_Polyhedron
sage: from sage.geometry.polyhedron.lattice_euclidean_group_element import LatticeEuclideanGroupElement
sage: M = LatticeEuclideanGroupElement([[1,2],[2,3],[-1,2]], [1,2,3])
sage: M
The map A*x+b with A=
[ 1  2]
[ 2  3]
[-1  2]  
b =
(1, 2, 3)
sage: M._A
[ 1  2]
[ 2  3]
[-1  2]
sage: M._b
(1, 2, 3)
sage: M(vector([0,0]))
(1, 2, 3)
sage: M(LatticePolytope_PPL((0,0),(1,0),(0,1)))
A 2-dimensional lattice polytope in ZZ^3 with 3 vertices
sage: _.vertices()
{(1, 2, 3), (2, 4, 2), (3, 5, 5)}
```

codomain_dim()  

Return the dimension of the codomain lattice

**EXAMPLES:**
```python
sage: from sage.geometry.polyhedron.lattice_euclidean_group_element import _LatticeEuclideanGroupElement
sage: M = _LatticeEuclideanGroupElement([[1,2],[2,3],[-1,2]], [1,2,3])
sage: M
The map \( A \cdot x + b \) with \( A = \\
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
-1 & 2 \\
\end{bmatrix} \\
b = \\
(1, 2, 3)
sage: M.codomain_dim()
3
```

Note that this is not the same as the rank. In fact, the codomain dimension depends only on the matrix shape, and not on the rank of the linear mapping:

```python
sage: zero_map = _LatticeEuclideanGroupElement([[0,0],[0,0],[0,0]], [0,0,0])
sage: zero_map.codomain_dim()
3
```

```python
def domain_dim()
    Return the dimension of the domain lattice
    EXAMPLES:
```
```python
sage: from sage.geometry.polyhedron.lattice_euclidean_group_element import _LatticeEuclideanGroupElement
sage: M = _LatticeEuclideanGroupElement([[1,2],[2,3],[-1,2]], [1,2,3])
sage: M
The map \( A \cdot x + b \) with \( A = \\
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
-1 & 2 \\
\end{bmatrix} \\
b = \\
(1, 2, 3)
sage: M.domain_dim()
2
```

```python
exception
sage.geometry.polyhedron.lattice_euclidean_group_element.LatticePolytopeError
    Bases: Exception
    Base class for errors from lattice polytopes
```
```python
exception sage.geometry.polyhedron.lattice_euclidean_group_element.LatticePolytopeNoEmbeddingError
    Bases: LatticePolytopeError
    Raised when no embedding of the desired kind can be found.
```
```python
exception sage.geometry.polyhedron.lattice_euclidean_group_element.LatticePolytopesNotIsomorphicError
    Bases: LatticePolytopeError
    Raised when two lattice polytopes are not isomorphic.
```
2.2.3 Access the PALP database(s) of reflexive lattice polytopes

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.palp_database import PALPreader
sage: for lp in PALPreader(2):
    # needs sage.graphs
    ....:    cone = Cone([(1,r[0],r[1]) for r in lp.vertices()])
    ....:    fan = Fan([cone])
    ....:    X = ToricVariety(fan)
    ....:    ideal = X.affine_algebraic_patch(cone).defining_ideal()
    ....:    print("{} {}\n".format(lp.n_vertices(), ideal.hilbert_series()))
3 (t^2 + 7*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
3 (t^2 + 6*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
3 (t^2 + 2*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
3 (t^2 + 4*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 5*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 3*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 2*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 6*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 6*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 2*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
4 (t^2 + 4*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
5 (t^2 + 5*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
5 (t^2 + 3*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
5 (t^2 + 4*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
6 (t^2 + 4*t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
```

**class** `sage.geometry.polyhedron.palp_database.PALPreader(dim, data_basename=None, output='Polyhedron')`

Bases: `SageObject`

Read PALP database of polytopes.

**INPUT:**

- `dim` – integer. The dimension of the polyhedra
- `data_basename` – string or `None` (default). The directory and database base filename (PALP usually uses 'zzdb') name containing the PALP database to read. Defaults to the built-in database location.
- `output` – string. How to return the reflexive polyhedron data. Allowed values = 'list', 'Polyhedron' (default), 'pointcollection', and 'PPL'. Case is ignored.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.palp_database import PALPreader
sage: polygons = PALPreader(2)
```

(continues on next page)
A 2-dimensional polyhedron in \( \mathbb{Z}^2 \) defined as the convex hull of 3 vertices

```
sage: type(_)
<class 'sage.geometry.polyhedron.parent.Polyhedra_ZZ_ppl_with_category.element_class'>
```

```
sage: next(iter(PALPreader(2, output='PPL')))
```

A 2-dimensional lattice polytope in \( \mathbb{Z}^2 \) with 3 vertices

```
sage: type(_)
<class 'sage.geometry.polyhedron.ppl_lattice_polygon.LatticePolygon_PPL_class'>
```

```
sage: next(iter(PALPreader(2, output='PointCollection')))
[[1, 0], [0, 1], [-1, -1]]
in Ambient free module of rank 2 over the principal ideal domain Integer Ring
```

```
sage: type(_)
<class 'sage.geometry.point_collection.PointCollection'>
```

```
class sage.geometry.polyhedron.palp_database.Reflexive4dHodge(h11, h21, data_basename=None, **kwds)

Bases: PALPreader

Read the PALP database for Hodge numbers of 4d polytopes.

The database is very large and not installed by default. You can install it with the shell command `sage -i polytopes_db_4d`.

INPUT:

- \( h11, h21 \) – Integers. The Hodge numbers of the reflexive polytopes to list.

Any additional keyword arguments are passed to `PALPreader`.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.palp_database import Reflexive4dHodge
sage: ref = Reflexive4dHodge(1,101)  # optional - polytopes_db_4d
sage: next(iter(ref)).Vrepresentation()  # optional - polytopes_db_4d
(A vertex at (-1, -1, -1, -1), A vertex at (0, 0, 0, 1),
 A vertex at (0, 0, 1, 0), A vertex at (0, 1, 0, 0), A vertex at (1, 0, 0, 0))
```

2.2.4 Fast Lattice Polygons using PPL

See `ppl_lattice_polytope` for the implementation of arbitrary-dimensional lattice polytopes. This module is about the specialization to 2 dimensions. To be more precise, the `LatticePolygon_PPL_class` is used if the ambient space is of dimension 2 or less. These all allow you to cyclically order (see `LatticePolygon_PPL_class.ordered_vertices()`) the vertices, which is in general not possible in higher dimensions.

```
class sage.geometry.polyhedron.ppl_lattice_polygon.LatticePolygon_PPL_class

Bases: LatticePolytope_PPL_class

A lattice polygon

This includes 2-dimensional polytopes as well as degenerate (0 and 1-dimensional) lattice polygons. Any polytope in 2d is a polygon.
```
find_isomorphism(polytope)

Return a lattice isomorphism with polytope.

INPUT:

• polytope – a polytope, potentially higher-dimensional.

OUTPUT:

A LatticeEuclideanGroupElement. It is not necessarily invertible if the affine dimension of self or polytope is not two. A LatticePolytopesNotIsomorphicError is raised if no such isomorphism exists.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import *
   →LatticePolytope_PPL
sage: L1 = LatticePolytope_PPL((1,0),(0,1),(0,0))
sage: L2 = LatticePolytope_PPL((1,0,3),(0,1,0),(0,0,1))
sage: iso = L1.find_isomorphism(L2)
```
```python
sage: iso(L1) == L2
True
```
```python
sage: L1 = LatticePolytope_PPL((0, 1), (3, 0), (0, 3), (1, 0))
```
```python
sage: L2 = LatticePolytope_PPL((0,0,2,1),(0,1,2,0),(2,0,0,3),(2,3,0,0))
```
```python
sage: iso = L1.find_isomorphism(L2)
```
```python
sage: iso(L1) == L2
True
```

The following polygons are isomorphic over \( \mathbb{Q} \), but not as lattice polytopes:

```python
sage: L1 = LatticePolytope_PPL((1,0),(0,1),(-1,-1))
```
```python
sage: L2 = LatticePolytope_PPL((0, 0), (0, 1), (1, 0))
```
```python
sage: L1.find_isomorphism(L2)
Traceback (most recent call last):
  ... LatticePolytopesNotIsomorphicError: different number of integral points
```
```python
sage: L2.find_isomorphism(L1)
Traceback (most recent call last):
  ... LatticePolytopesNotIsomorphicError: different number of integral points
```

is_isomorphic(polytope)

Test if self and polytope are isomorphic.

INPUT:

• polytope – a lattice polytope.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import *
   →LatticePolytope_PPL
sage: L1 = LatticePolytope_PPL((1,0),(0,1),(0,0))
```
```python
sage: L2 = LatticePolytope_PPL((1,0,3),(0,1,0),(0,0,1))
```
```python
sage: L1.is_isomorphic(L2)
True
```
ordered_vertices()
Return the vertices of a lattice polygon in cyclic order.

OUTPUT:
A tuple of vertices ordered along the perimeter of the polygon. The first point is arbitrary.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: square = LatticePolytope_PPL((0,0), (1,1), (0,1), (1,0))
sage: square.vertices()
((0, 0), (0, 1), (1, 0), (1, 1))
sage: square.ordered_vertices()
((0, 0), (1, 0), (1, 1), (0, 1))
```

plot()
Plot the lattice polygon.

OUTPUT:
A graphics object.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: P = LatticePolytope_PPL((1,0), (0,1), (0,0), (2,2))
sage: P.plot()
# Graphics object consisting of 6 graphics primitives

sage: LatticePolytope_PPL([0], [1]).plot()
# Graphics object consisting of 3 graphics primitives

sage: LatticePolytope_PPL([0]).plot()
# Graphics object consisting of 2 graphics primitives
```

sub_polytopes()
Return a list of all lattice sub-polygons up to isomorphism.

OUTPUT:
All non-empty sub-lattice polytopes up to isomorphism. This includes self as improper sub-polytope, but excludes the empty polytope. Isomorphic sub-polytopes that can be embedded in different places are only returned once.

EXAMPLES:

```
sage: P1xP1 = LatticePolytope_PPL((1,0), (0,1), (-1,0), (0,-1))
sage: P1xP1.sub_polytopes()
(A 2-dimensional lattice polytope in ZZ^2 with 4 vertices,
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices,
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices,
A 1-dimensional lattice polytope in ZZ^2 with 2 vertices,
A 1-dimensional lattice polytope in ZZ^2 with 2 vertices,
A 0-dimensional lattice polytope in ZZ^2 with 1 vertex)
```
sage.geometry.polyhedron.ppl_lattice_polygon.polar_P1xP1_polytope()

The polar of the $P^1 \times P^1$ polytope

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polygon import polar_P1xP1_
    →polytope
sage: polar_P1xP1_polytope()
A 2-dimensional lattice polytope in ZZ^2 with 4 vertices
sage: _.vertices()
((0, 0), (0, 2), (2, 0), (2, 2))
```

sage.geometry.polyhedron.ppl_lattice_polygon.polar_P2_112_polytope()

The polar of the $P^2[1,1,2]$ polytope

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polygon import polar_P2_112_
    →polytope
sage: polar_P2_112_polytope()
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices
sage: _.vertices()
((0, 0), (0, 2), (4, 0))
```

sage.geometry.polyhedron.ppl_lattice_polygon.polar_P2_polytope()

The polar of the $P^2$ polytope

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polygon import polar_P2_polytope
sage: polar_P2_polytope()
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices
sage: _.vertices()
((0, 0), (0, 3), (3, 0))
```

sage.geometry.polyhedron.ppl_lattice_polygon.sub_reflexive_polygons()

Return all lattice sub-polygons of reflexive polygons.

OUTPUT:
A tuple of all lattice sub-polygons. Each sub-polygon is returned as a pair sub-polygon, containing reflexive polygon.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polygon import sub_reflexive_
    →polygons
sage: l = sub_reflexive_polygons(); l[5]
(A 2-dimensional lattice polytope in ZZ^2 with 6 vertices, A 2-dimensional lattice polytope in ZZ^2 with 3 vertices)
sage: len(l)
33
```

sage.geometry.polyhedron.ppl_lattice_polygon.subpolygons_of_polar_P1xP1()

The lattice sub-polygons of the polar $P^1 \times P^1$ polytope

OUTPUT:
A tuple of lattice polytopes.

EXAMPLES:
sage: from sage.geometry.polyhedron.ppl_lattice_polygon import subpolygons_of_polar_P1xP1
sage: len(subpolygons_of_polar_P1xP1())
20

sage.geometry.polyhedron.ppl_lattice_polygon.subpolygons_of_polar_P2()
The lattice sub-polygons of the polar $P^2$ polytope
OUTPUT:
A tuple of lattice polytopes.
EXAMPLES:

sage: from sage.geometry.polyhedron.ppl_lattice_polygon import subpolygons_of_polar_P2
sage: len(subpolygons_of_polar_P2())
27

sage.geometry.polyhedron.ppl_lattice_polygon.subpolygons_of_polar_P2_112()
The lattice sub-polygons of the polar $P^2[1,1,2]$ polytope
OUTPUT:
A tuple of lattice polytopes.
EXAMPLES:

sage: from sage.geometry.polyhedron.ppl_lattice_polygon import subpolygons_of_polar_P2_112
sage: len(subpolygons_of_polar_P2_112())
28

2.2.5 Fast Lattice Polytopes using PPL.

The LatticePolytope_PPL() class is a thin wrapper around PPL polyhedra. Its main purpose is to be fast to construct, at the cost of being much less full-featured than the usual polyhedra. This makes it possible to iterate with it over the list of all 473800776 reflexive polytopes in 4 dimensions.

Note: For general lattice polyhedra you should use Polyhedron() with base_ring=ZZ.

The class derives from the PPL ppl.polyhedron.C_Polyhedron class, so you can work with the underlying generator and constraint objects. However, integral points are generally represented by Z-vectors. In the following, we always use generator to refer to the PPL generator objects and vertex (or integral point) for the corresponding Z-vector.

EXAMPLES:

sage: vertices = [(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (-9, -6, -1, -1)]
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: P = LatticePolytope_PPL(vertices); P
A 4-dimensional lattice polytope in ZZ^4 with 5 vertices
sage: P.integral_points()
((-9, -6, -1, -1), (-3, -2, 0, 0), (-2, -1, 0, 0), (-1, -1, 0, 0),
 (-1, 0, 0, 0), (0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0))

(continues on next page)
Fibrations of the lattice polytopes are defined as lattice sub-polytopes and give rise to fibrations of toric varieties for suitable fan refinements. We can compute them using `fibration_generator()`

```python
sage: F = next(P.fibration_generator(2))
sage: F.vertices()
((1, 0, 0, 0), (0, 1, 0, 0), (-3, -2, 0, 0))
```

Finally, we can compute automorphisms and identify fibrations that only differ by a lattice automorphism:

```python
sage: square = LatticePolytope_PPL((-1,-1), (-1,1), (1,-1), (1,1))
sage: fibers = [ f.vertices() for f in square.fibration_generator(1) ]; fibers
[[(1, 0), (-1, 0)], [(0, 1), (0, -1)], [(-1, -1), (1, 1)], [(-1, 1), (1, -1)]]
```

AUTHORS:

- Volker Braun: initial version, 2012

`sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL(*args)`

Construct a new instance of the PPL-based lattice polytope class.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: LatticePolytope_PPL((0,0), (1,0), (0,1))
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices
sage: Lattice_polytope_PPL((0,0), (1,0), (0,1))
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices
```

A `TypeError` is raised if the arguments do not specify a lattice polytope:
... TypeError: unable to convert rational 1/2 to an integer

```python
sage: from ppl import point, Generator_System, C_Polyhedron, Linear_Expression
sage: p = point(Linear_Expression([2, 3], 0), 5); p
#...
needs pplpy
```

```python
sage: p = point(2/5, 3/5)
#...
needs pplpy
```

```python
sage: LatticePolytope_PPL(p)
#...
needs pplpy
```

```python
Traceback (most recent call last):
...
TypeError: generator is not a lattice polytope generator
```

```python
sage: P = C_Polyhedron(Generator_System(p)); P
#...
needs pplpy
```

```python
A 0-dimensional polyhedron in QQ^2 defined as the convex hull of 1 point
```

```python
sage: LatticePolytope_PPL(P)
#...
needs pplpy
```

```python
Traceback (most recent call last):
...
TypeError: polyhedron has non-integral generators
```

```python
class sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class
Bases: C_Polyhedron

The lattice polytope class.

You should use `LatticePolytope_PPL()` to construct instances.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: poly_4d = LatticePolytope_PPL((-9,-6,0,0), (0,1,0,0), (1,0,0,0)); poly_4d
A 2-dimensional lattice polytope in ZZ^4 with 3 vertices
```

```python
affine_lattice_polytope()
```

Return the lattice polytope restricted to `affine_space()`.

**OUTPUT:**

A new, full-dimensional lattice polytope.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: poly_4d = LatticePolytope_PPL((-9,-6,0,0), (0,1,0,0), (1,0,0,0)); poly_4d
A 2-dimensional lattice polytope in ZZ^4 with 3 vertices
sage: poly_4d.space_dimension()
4
sage: poly_2d = poly_4d.affine_lattice_polytope(); poly_2d
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices
sage: poly_2d.space_dimension()
2
```
affine_space()
Return the affine space spanned by the polytope.

OUTPUT:
The free module $\mathbb{Z}^n$, where $n$ is the dimension of the affine space spanned by the points of the polytope.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
  -> LatticePolytope_PPL
sage: point = LatticePolytope_PPL((1,2,3))
sage: point.affine_space()
Free module of degree 3 and rank 0 over Integer Ring
Echelon basis matrix:
[]
sage: line = LatticePolytope_PPL((1,1,1), (1,2,3))
sage: line.affine_space()
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[0 1 2]
```

ambient_space()
Return the ambient space.

OUTPUT:
The free module $\mathbb{Z}^d$, where $d$ is the ambient space dimension.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
  -> LatticePolytope_PPL
sage: point = LatticePolytope_PPL((1,2,3))
sage: point.ambient_space()
Ambient free module of rank 3 over the principal ideal domain Integer Ring
```

base_projection(fiber)
The projection that maps the sub-polytope fiber to a single point.

OUTPUT:
The quotient module of the ambient space modulo the affine_space() spanned by the fiber.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
  -> LatticePolytope_PPL
sage: poly = LatticePolytope_PPL((-9,-6,-1,-1),
  ....: (0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0))
sage: fiber = next(poly.fibration_generator(2))
sage: poly.base_projection(fiber)
Finitely generated module V/W over Integer Ring with invariants (0, 0)
```

base_projection_matrix(fiber)
The projection that maps the sub-polytope fiber to a single point.

OUTPUT:
An integer matrix that represents the projection to the base.
See also:

The `base_projection()` yields equivalent information, and is easier to use. However, just returning the
matrix has lower overhead.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: poly = LatticePolytope_PPL((-9,-6,-1,-1),
.....: (0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0))
sage: fiber = next(poly.fibration_generator(2))
sage: poly.base_projection_matrix(fiber)
[ 0 0 -1 0]
[ 0 0 0 -1]
```

Note that the basis choice in `base_projection()` for the quotient is usually different:

```python
sage: proj = poly.base_projection(fiber)
sage: proj_matrix = poly.base_projection_matrix(fiber)
sage: proj_matrix*p for p in poly.integral_points()
[(-1, -1), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 1), (1, 0)]
[proj(p) for p in poly.integral_points()]
[(1, 1), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, -1), (-1, 0)]
```

**base_rays** *(fiber, points)*

Return the primitive lattice vectors that generate the direction given by the base projection of points.

**INPUT:**

- fiber – a sub-lattice polytope defining the `base_projection()`.
- points – the points to project to the base.

**OUTPUT:**

A tuple of primitive \(\mathbb{Z}\)-vectors.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: poly = LatticePolytope_PPL((-9,-6,-1,-1),
.....: (0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0))
sage: fiber = next(poly.fibration_generator(2))
sage: poly.base_rays(fiber, poly.integral_points_not_interior_to_facets())
((-1, -1), (0, 1), (1, 0))
```

**bounding_box** ()

Return the coordinates of a rectangular box containing the non-empty polytope.

**OUTPUT:**

A pair of tuples `(box_min, box_max)` where `box_min` are the coordinates of a point bounding the coordinates of the polytope from below and `box_max` bounds the coordinates from above.
contains \((\text{point\_coordinates})\)
Test whether point is contained in the polytope.

**INPUT:**

- \text{point\_coordinates} – a list/tuple/iterable of rational numbers. The coordinates of the point.

**OUTPUT:** Boolean.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
    LatticePolytope_PPL
sage: LatticePolytope_PPL((0,0), (1,0), (0,1)).bounding_box()
((0, 0), (1, 1))
```

contains\_origin()  
Test whether the polytope contains the origin

**OUTPUT:** Boolean.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
    LatticePolytope_PPL
sage: line = LatticePolytope_PPL((1,2,3), (-1,-2,-3))
sage: line.contains([0,0,0])
True
sage: line.contains([1,0,0])
False
```

embed\_in\_reflexive\_polytope\((\text{output='hom'})\)
Find an embedding as a sub-polytope of a maximal reflexive polytope.

**INPUT:**

- \text{hom} – string. One of 'hom' (default), 'polytope', or points. How the embedding is returned. See the output section for details.

**OUTPUT:**

An embedding into a reflexive polytope. Depending on the \text{output} option slightly different data is returned.

- If \text{output}='hom', a map from a reflexive polytope onto \text{self} is returned.
- If \text{output}='polytope', a reflexive polytope that contains \text{self} (up to a lattice linear transformation) is returned. That is, the domain of the \text{output}='hom' map is returned. If the affine span of \text{self} is less or equal 2-dimensional, the output is one of the following three possibilities:  
  \text{polar\_P2\_polytope()}, \text{polar\_P1xP1\_polytope()}, or \text{polar\_P2\_112\_polytope()}. 

2.2. Lattice polyhedra 233
• If `output='points'`, a dictionary containing the integral points of `self` as keys and the corresponding integral point of the reflexive polytope as value.

If there is no such embedding, a `LatticePolytopeNoEmbeddingError` is raised. Even if it exists, the ambient reflexive polytope is usually not uniquely determined and a random but fixed choice will be returned.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
    →LatticePolytope_PPL
sage: polygon = LatticePolytope_PPL((0,0,2,1), (0,1,2,0), (2,3,0,0), (2,0,0,3))
sage: polygon.embed_in_reflexive_polytope()
The map $A\times+b$ with
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$$
b = (-1, 0, 3, 0)
sage: polygon.embed_in_reflexive_polytope('polytope')
A 2-dimensional lattice polytope in ZZ^2 with 3 vertices
sage: polygon.embed_in_reflexive_polytope('points')
{(0, 0, 2, 1): (1, 0),
 (0, 1, 2, 0): (0, 1),
 (1, 0, 1, 2): (2, 0),
 (1, 1, 1, 1): (1, 1),
 (1, 2, 1, 0): (0, 2),
 (2, 0, 0, 3): (3, 0),
 (2, 1, 0, 2): (2, 1),
 (2, 2, 0, 1): (1, 2),
 (2, 3, 0, 0): (0, 3)}
sage: LatticePolytope_PPL((0,0), (4,0), (0,4)).embed_in_reflexive_polytope()
Traceback (most recent call last):
...
LatticePolytopeNoEmbeddingError: not a sub-polytope of a reflexive polygon
```

**fibration_generator** (*dim*)

Generate the lattice polytope fibrations.

For the purposes of this function, a lattice polytope fiber is a sub-lattice polytope. Projecting the plane spanned by the subpolytope to a point yields another lattice polytope, the base of the fiber.

INPUT:

• *dim* – integer. The dimension of the lattice polytope fiber.

OUTPUT:

A generator yielding the distinct lattice polytope fibers of given dimension.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
    →LatticePolytope_PPL
sage: p = LatticePolytope_PPL((-9,-6,-1,-1),
    ....:     (0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0))
sage: list(p.fibration_generator(2))
[234 Chapter 2. Polyhedral computations

[A 2-dimensional lattice polytope in ZZ^4 with 3 vertices]
**has_IP_property()**

Whether the lattice polytope has the IP property.

That is, the polytope is full-dimensional and the origin is an interior point not on the boundary.

**OUTPUT:** Boolean.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
    →LatticePolytope_PPL
sage: LatticePolytope_PPL((-1,-1), (0,1), (1,0)).has_IP_property()
True
sage: LatticePolytope_PPL((-1,-1), (1,1)).has_IP_property()
False
```

**integral_points()**

Return the integral points in the polyhedron.

Uses the naive algorithm (iterate over a rectangular bounding box).

**OUTPUT:**

The list of integral points in the polyhedron. If the polyhedron is not compact, a `ValueError` is raised.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import...
    →LatticePolytope_PPL
sage: LatticePolytope_PPL((-1,-1), (1,0), (1,1), (0,1)).integral_points()
((-1, -1), (0, 0), (0, 1), (1, 0), (1, 1))

sage: simplex = LatticePolytope_PPL((1,2,3), (2,3,7), (-2,-3,-11))

sage: simplex.integral_points()
((-2, -3, -11), (0, 0, -2), (1, 2, 3), (2, 3, 7))

The polyhedron need not be full-dimensional:

```python
sage: simplex = LatticePolytope_PPL((1,2,3,5), (2,3,7,5), (-2,-3,-11,5))

sage: simplex.integral_points()
((-2, -3, -11, 5), (0, 0, -2, 5), (1, 2, 3, 5), (2, 3, 7, 5))

sage: point = LatticePolytope_PPL((2,3,7))

sage: point.integral_points()
((2, 3, 7),)

sage: empty = LatticePolytope_PPL()

sage: empty.integral_points()
()
```

Here is a simplex where the naive algorithm of running over all points in a rectangular bounding box no longer works fast enough:

```python
sage: v = [(1,0,7,-1), (-2,-2,4,-3), (-1,-1,-1,4), (2,9,0,-5), (-2,-1,5,1)]

sage: simplex = LatticePolytope_PPL(v); simplex
A 4-dimensional lattice polytope in ZZ^4 with 5 vertices

sage: len(simplex.integral_points())
49
```

Finally, the 3-d reflexive polytope number 4078:
```python
sage: v = [(1,0,0), (0,1,0), (0,0,1), (0,0,-1), (0,-2,1),
....:  (1,2,-1), (-1,2,-2), (-1,1,-2), (-1,-1,2), (-1,-3,2)]
sage: P = LatticePolytope_PPL(*v)
sage: pts1 = P.integral_points() # Sage's own code
sage: pts2 = LatticePolytope(v).points() #
˓→needs palp
sage: for p in pts1: p.set_immutable()
sage: set(pts1) == set(pts2) #
˓→needs palp
True
sage: len(Polyhedron(v).integral_points()) # takes about 1 ms
23
sage: len(LatticePolytope(v).points()) # takes about 13 ms #
˓→needs palp
23
sage: len(LatticePolytope_PPL(*v).integral_points()) # takes about 0.5 ms
23
```

**integral_points_not_interior_to_facets()**

Return the integral points not interior to facets.

**OUTPUT:**

A tuple whose entries are the coordinate vectors of integral points not interior to facets (codimension one faces) of the lattice polytope.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: square = LatticePolytope_PPL((-1,-1), (-1,1), (1,-1), (1,1))
sage: square.n_integral_points()
9
sage: square.integral_points_not_interior_to_facets()
((-1, -1), (-1, 1), (0, 0), (1, -1), (1, 1))
```

**is_bounded()**

Return whether the lattice polytope is compact.

**OUTPUT:**

Always True, since polytopes are by definition compact.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: LatticePolytope_PPL((0,0), (1,0), (0,1)).is_bounded()
True
```

**is_full_dimensional()**

Return whether the lattice polytope is full dimensional.

**OUTPUT:**

Boolean. Whether the affine_dimension() equals the ambient space dimension.

**EXAMPLES:**
is_simplex()

Return whether the polyhedron is a simplex.

OUTPUT:

Boolean, whether the polyhedron is a simplex (possibly of strictly smaller dimension than the ambient space).

EXAMPLES:

```sage
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import... →LatticePolytope_PPL
sage: p = LatticePolytope_PPL((0,0), (0,1))
sage: p.is_full_dimensional()
False
sage: q = LatticePolytope_PPL((0,0), (0,1), (1,0))
sage: q.is_full_dimensional()
True
```

lattice_automorphism_group(points=None, point_labels=None)

The integral subgroup of the restricted automorphism group.

INPUT:

- points – A tuple of coordinate vectors or None (default). If specified, the points must form complete orbits under the lattice automorphism group. If None all vertices are used.
- point_labels – A tuple of labels for the points or None (default). These will be used as labels for the do permutation group. If None, the points will be used themselves.

OUTPUT:

The integral subgroup of the restricted automorphism group acting on the given points, or all vertices if not specified.

EXAMPLES:

```sage
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import... →LatticePolytope_PPL
sage: Z3square = LatticePolytope_PPL((0,0), (1,2), (2,1), (3,3))
sage: Z3square.lattice_automorphism_group()  # needs sage.graphs sage.groups
Permutation Group with generators [(), ((1,2),(2,1)), ((0,0),(3,3)), ((0,0),(3,3))((1,2),(2,1))]
sage: G1 = Z3square.lattice_automorphism_group(point_labels=(1,2,3,4))  # needs sage.graphs sage.groups
sage: G1  # needs sage.graphs sage.groups
Permutation Group with generators [(), (2,3), (1,4), (1,4)(2,3)]
sage: G1.cardinality()  # needs sage.graphs sage.groups
4
sage: G2 = Z3square.restricted_automorphism_group(vertex_labels=(1,2,3,4))  # needs sage.graphs sage.groups
```

(continues on next page)
 sage: G2 == PermutationGroup([[(2,3)], [(1,2),(3,4)], [(1,4)]])  
# needs sage.graphs sage.groups
True

 sage: G2.cardinality()  
# needs sage.graphs sage.groups
8

 sage: points = Z3square.integral_points(); points
((0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3))

 sage: Z3square.lattice_automorphism_group(points,  
# needs sage.graphs sage.groups
.....: point_labels=(1,2,3,4,5,6))
Permutation Group with generators [(), (3,4), (1,6)(2,5), (1,6)(2,5)(3,4)]

Point labels also work for lattice polytopes that are not full-dimensional, see github issue #16669:

 sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
 sage: lp = LatticePolytope_PPL((1,0,0), (0,1,0), (-1,-1,0))
 sage: lp.lattice_automorphism_group(point_labels=(0,1,2))
# needs sage.graphs sage.groups
Permutation Group with generators [(), (1,2), (0,1), (0,1,2), (0,2,1), (0,2)]

n_integral_points()  
Return the number of integral points.

 OUTPUT:

 Integer. The number of integral points contained in the lattice polytope.

 EXAMPLES:

 sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
 sage: LatticePolytope_PPL((0,0), (1,0), (0,1)).n_integral_points()
3

n_vertices()  
Return the number of vertices.

 OUTPUT:

 An integer, the number of vertices.

 EXAMPLES:

 sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
 sage: LatticePolytope_PPL((0,0), (1,0), (0,1)).n_vertices()
3

pointsets_mod_automorphism(pointsets)  
Return pointsets modulo the automorphisms of self.

 INPUT:

 • polytopes – a tuple/list/iterable of subsets of the integral points of self.

 OUTPUT:
Representatives of the point sets modulo the `lattice_automorphism_group()` of `self`.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: square = LatticePolytope_PPL((-1,-1), (-1,1), (1,-1), (1,1))
sage: fibers = [f.vertices() for f in square.fibration_generator(1)]
sage: square.pointsets_mod_automorphism(fibers)
# needs sage.graphs sage.groups
(frozenset({(-1, -1), (1, 1)}), frozenset({(-1, 0), (1, 0)}))
sage: cell24 = LatticePolytope_PPL(...:
......:  (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,-1,-1,1), (0,0,-1,1),
......:  (0,-1,0,1), (1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (-1,1,1,0),
......:  (-1,0,0,1), (0,0,0,1),
......:  (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,-1,0,0), (1,0,-1),
......:  (-1,1,0,0), (-1,0,1,0), (0,0,1,-1), (0,0,0,-1))
sage: fibers = [f.vertices() for f in cell24.fibration_generator(2)]
sage: cell24.pointsets_mod_automorphism(fibers)  # long time
# needs sage.graphs sage.groups
(frozenset({(-1, 0, 0, 0), (-1, 0, 0, 1), (0, 0, 0, -1),
......:  (1, 0, 0, 1),
......:  (0, 0, 0, 1), (1, 0, 0, -1), (1, 0, 0, 0)}),
......:  (1, 0, 0, 0), (1, 1, 1, 0), (1, -1, -1, 0),)
```

`restricted_automorphism_group(vertex_labels=None)`

Return the restricted automorphism group.

First, let the linear automorphism group be the subgroup of the Euclidean group $E(d) = GL(d, \mathbb{R}) \ltimes \mathbb{R}^d$ preserving the $d$-dimensional polyhedron. The Euclidean group acts in the usual way $\vec{x} \mapsto A\vec{x} + b$ on the ambient space. The restricted automorphism group is the subgroup of the linear automorphism group generated by permutations of vertices. If the polytope is full-dimensional, it is equal to the full (unrestricted) automorphism group.

**INPUT:**

- `vertex_labels` - a tuple or None (default). The labels of the vertices that will be used in the output permutation group. By default, the vertices are used themselves.

**OUTPUT:**

A `PermutationGroup` acting on the vertices (or the `vertex_labels`, if specified).

**REFERENCES:**

[BSS2009]

**EXAMPLES:**

```python
sage: # needs sage.graphs sage.groups
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: Z3square = LatticePolytope_PPL((0,0), (1,2), (2,1), (3,3))
sage: G1234 = Z3square.restricted_automorphism_group(
```

(continues on next page)
....: vertex_labels=(1,2,3,4)
sage: G1234 == PermutationGroup([[2, 3], [1, 2], [3, 4]])
True
sage: G = Z3square.restricted_automorphism_group()
sage: G == PermutationGroup([[1, 2], [2, 1]], [[0, 0], [1, 2]],
....: [[0, 0], [3, 3]], [[0, 0], [3, 3]])
True
sage: set(G.domain()) == set(Z3square.vertices())
True
sage: set(tuple(x)
....: for x in G.orbit(Z3square.vertices()[0]))
....: == set([(0, 0), (1, 2), (3, 3), (2, 1)])
True
sage: cell24 = LatticePolytope_PPL(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,-1,-1,1), (0,0,-1,1),
....: (0,-1,0,1), (-1,0,0,1), (1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (-1,-1,1,-1),
....: (1,1,1,-1), (-1,1,1,0), (-1,1,0,0), (1,1,0,0), (1,0,-
....: 1,0),
....: (0,1,1,-1), (-1,1,1,0), (-1,1,0,0), (-1,0,1,0), (0,-1,-1,1), (0,0,0,
....: -1))

sage: cell24.restricted_automorphism_group().cardinality()
1152

**sub_polytope_generator()**

Generate the maximal lattice sub-polytopes.

**OUTPUT:**

A generator yielding the maximal (with respect to inclusion) lattice sub polytopes. That is, each can be gotten as the convex hull of the integral points of self with one vertex removed.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: P = LatticePolytope_PPL((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,-1,-1,1), (0,0,-1,1),
....: (0,-1,0,1), (-1,0,0,1), (1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (-1,-1,1,-1),
....: (1,1,1,-1), (-1,1,1,0), (-1,1,0,0), (1,1,0,0), (1,0,-
....: 1,0),
....: (0,1,1,-1), (-1,1,1,0), (-1,1,0,0), (-1,0,1,0), (0,-1,-1,1), (0,0,0,
....: -1))
sage: for p in P.sub_polytope_generator():
....: print(p.vertices())
((0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0))
((-1, -1, -1), (0, 0, 0), (0, 1, 0), (1, 0, 0))
((-1, -1, -1), (0, 0, 0), (0, 0, 1), (1, 0, 0))
((-1, -1, -1), (0, 0, 0), (0, 0, 1), (0, 1, 0))
```

**vertices()**

Return the vertices as a tuple of Z-vectors.

**OUTPUT:**

A tuple of Z-vectors. Each entry is the coordinate vector of an integral points of the lattice polytope.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: p = LatticePolytope_PPL((-9,-6,-1,-1),
....: (0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0))
sage: p.vertices()
((-9, -6, -1, -1), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0))
```
vertices_saturating(constraint)

Return the vertices saturating the constraint.

INPUT:

- constraint – a constraint (inequality or equation) of the polytope.

OUTPUT:

The tuple of vertices saturating the constraint. The vertices are returned as \( \mathbb{Z} \)-vectors, as in vertices().

EXAMPLES:

```sage
sage: from sage.geometry.polyhedron.ppl_lattice_polytope import LatticePolytope_PPL
sage: p = LatticePolytope_PPL((0,0), (0,1), (1,0))
sage: ieq = next(iter(p.constraints())); ieq
x0>=0
sage: p.vertices_saturating(ieq)
((0, 0), (0, 1))
```

2.2.6 Generating Function of Polyhedron’s Integral Points

This module provides `generating_function_of_integral_points()` which computes the generating function of the integral points of a polyhedron.

The main function is accessible via `sage.geometry.polyhedron.base.Polyhedron_base.generating_function_of_integral_points()` as well.

Various

AUTHORS:

- Daniel Krenn (2016, 2021)

ACKNOWLEDGEMENT:

- Daniel Krenn is supported by the Austrian Science Fund (FWF): P 24644-N26 and by the Austrian Science Fund (FWF): P 28466-N35.

Functions
Return the multivariate generating function of the integral points of the polyhedron. To be precise, this returns

\[ \sum_{(r_0, \ldots, r_{d-1}) \in \text{polyhedron} \cap \mathbb{Z}^d} y_0^{r_0} \cdots y_{d-1}^{r_{d-1}}. \]

**INPUT:**

- **polyhedron** – an instance of `Polyhedron_base` (see also `sage.geometry.polyhedron.constructor`)
- **split** – (default: `False`) a boolean or list
  - `split=False` computes the generating function directly, without any splitting.
  - When `split` is a list of disjoint polyhedra, then for each of these polyhedra, `polyhedron` is intersected with it, its generating function computed and all these generating functions are summed up.
  - `split=True` splits into \( d! \) disjoint polyhedra.
- **result_as_tuple** – (default: `None`) a boolean or `None`
  This specifies whether the output is a (partial) factorization (`result_as_tuple=False`) or a sum of such (partial) factorizations (`result_as_tuple=True`). By default (`result_as_tuple=None`), this is automatically determined. If the output is a sum, it is represented as a tuple whose entries are the summands.
- **indices** – (default: `None`) a list or tuple
  If this is `None`, this is automatically determined.
- **name** – (default: `'y'`) a string
  The variable names of the Laurent polynomial ring of the output are this string followed by an integer.
- **names** – a list or tuple of names (strings), or a comma separated string
  `name` is extracted from `names`, therefore `names` has to contain exactly one variable name, and `name` and `names` cannot be specified both at the same time.
- **Factorization_sort** (default: `False`) and **Factorization_simplify** (default: `True`) – booleans
  These are passed on to `sage.structure.factorization.Factorization` when creating the result.
- **sort_factors** – (default: `False`) a boolean
  If set, then the factors of the output are sorted such that the numerator is first and only then all factors of the denominator. It is ensured that the sorting is always the same; use this for doctesting.

**OUTPUT:**

The generating function as a (partial) `Factorization` of the result whose factors are Laurent polynomials.
The result might be a tuple of such factorizations (depending on the parameter `result_as_tuple`) as well.

**Note:** At the moment, only polyhedra with nonnegative coordinates (i.e. a polyhedron in the nonnegative orthant) are handled.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.generating_function import generating_function_of_integral_points

sage: P2 = (....:  Polyhedron(ieqs=[(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, -1)]),
....:  Polyhedron(ieqs=[(0, -1, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0)]))
sage: generating_function_of_integral_points(P2[0], sort_factors=True)
1 * (-y0 + 1)^-1 * (-y1 + 1)^-1 * (-y0*y2 + 1)^-1
sage: generating_function_of_integral_points(P2[1], sort_factors=True)
1 * (-y1 + 1)^-1 * (-y2 + 1)^-1 * (-y0*y2 + 1)^-1
sage: (P2[0] & P2[1]).Hrepresentation()
(An equation (1, 0, -1) x + 0 == 0,  
An inequality (1, 0, 0) x + 0 >= 0,  
An inequality (0, 1, 0) x + 0 >= 0)
sage: generating_function_of_integral_points(P2[0] & P2[1], sort_factors=True)
1 * (-y1 + 1)^-1 * (-y0*y2 + 1)^-1

sage: P3 = (....:  Polyhedron(....:    ieqs=[(0, 0, 0, 0, 1), (0, 0, 0, 1, 0),
....:    (0, 0, 1, 0, -1), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0)]),
....:  Polyhedron(....:    ieqs=[(0, 0, 0, 0, 1), (0, 0, 0, 1, 0),
....:    (1, -1, 0, 1, 1), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)]),
....:  Polyhedron(....:    ieqs=[(0, 0, 0, 0, 1), (0, 0, 0, 1, 0),
....:    (0, 0, 1, 0, 0), (0, 1, 0, 1, 1), (1, 0, 0, 1, 0)]),
....:  Polyhedron(....:    ieqs=[(0, 0, 0, 0, 1), (0, 0, 0, 1, 0),
....:    (0, 0, 1, 0, 0), (0, 1, 0, -1), (1, 0, 0, 1, 0)]),
....:  Polyhedron(....:    ieqs=[(0, 0, 0, 0, 1), (0, 0, 0, 1, 0),
....:    (0, 0, 1, 0, 0), (0, 1, 0, 1, 1), (1, 0, 0, 1, 0)]))
sage: def intersect(I):
....:    I = iter(I)
....:    result = next(I)
....:    for i in I:
....:        result &= i
....:    return result
sage: for J in subsets(range(len(P3))):
....:    if not J:
....:        continue
....:    P = intersect(\[P3[j] for j in J\])
....:    print(\'{}(An inequality (0, 0, 0, 1) x + 0 >= 0, \nAn inequality (0, 0, 1, 0) x + 0 >= 0, \nAn inequality (0, 1, 0, -1) x + 0 >= 0,\'.format(J, P.Hrepresentation()))
....:    print(generating_function_of_integral_points(P, sort_factors=True))
[0]: (An inequality (0, 0, 0, 1) x + 0 >= 0,  
An inequality (0, 0, 1, 0) x + 0 >= 0,  
An inequality (0, 1, 0, -1) x + 0 >= 0, 
(continues on next page)
An inequality $(1, 0, -1, -1) \times -1 \geq 0$.

$\begin{align*}
&\text{y0} \times (-y0 + 1)^{-1} \times (-y1 + 1)^{-1} \times (-y0 \times y2 + 1)^{-1} \times (-y0 \times y1 \times y3 + 1)^{-1} \\
&\text{(1): (An inequality (0, -1, 0, 1) x + 0 >= 0,}
&\text{An inequality (0, 0, 1, 0) x + 0 >= 0,}
&\text{An inequality (0, 1, 0, 0) x + 0 >= 0,}
&\text{An inequality (1, -1, -1, 0) x - 1 >= 0,}
&\text{An inequality (1, 0, 0, -1) x + 0 >= 0)}
\end{align*}$

$(-y0 \times 2 \times y2 \times y3 - y0 \times 2 \times y3 + y0 \times y3 + y0) \times$

$\begin{align*}
&\text{(-y0 + 1)^{-1} \times (-y0 \times y2 + 1)^{-1} \times (-y0 \times y3 + 1)^{-1}} \\
&\text{(-y0 \times y1 \times y3 + 1)^{-1} \times (-y0 \times y2 \times y3 + 1)^{-1}} \\
&\text{(0, 1): (An equation (0, 1, 0, -1) x + 0 == 0,}
&\text{An inequality (1, -1, -1, 0) x - 1 >= 0,}
&\text{An inequality (0, 1, 0, 0) x + 0 >= 0,}
&\text{An inequality (0, 0, 1, 0) x + 0 >= 0)}
\end{align*}$

$\begin{align*}
&\text{y0} \times (-y1 + 1)^{-1} \times (-y0 \times y2 + 1)^{-1} \times (-y0 \times y1 \times y3 + 1)^{-1} \\
&\text{(2): (An inequality (-1, 0, 1, 1) x + 1 >= 0,}
&\text{An inequality (-1, 1, 0, 0) x + 1 >= 0,}
&\text{An inequality (0, 0, 0, 1) x + 0 >= 0,}
&\text{An inequality (0, 0, 1, 0) x + 0 >= 0,}
&\text{An inequality (1, 0, 0, -1) x + 1 >= 0,}
&\text{An inequality (1, 1, 0, 0) x + 0 >= 0)}
\end{align*}$

$\begin{align*}
&\text{(-y0 \times 2 \times y1 \times y2 \times y3^2 + y0 \times 2 \times y2 \times 2 \times y3 + y0 \times y1^2 \times y3^2 - y0 \times 2 \times y2 \times y3 + y0 \times y1 \times y2 \times y3 - y0 \times 2 \times y1 \times y3 - 2 \times y0 \times y2 \times y3 - y0 \times y2 + y0 \times y3 - y1 \times y3 + y0 + y3 + 1) \times}$

$\begin{align*}
&\text{(-y1 + 1)^{-1} \times (-y2 + 1)^{-1} \times (-y0 \times y2 + 1)^{-1} \times (-y0 \times y1 \times y3 + 1)^{-1} \times (-y0 \times y2 \times y3 + 1)^{-1}} \\
&\text{(0, 2): (An equation (1, 0, -1, -1) x - 1 == 0,}
&\text{An inequality (-1, 1, 1, 0) x + 1 >= 0,}
&\text{An inequality (1, 0, -1, 0) x - 1 >= 0,}
&\text{An inequality (0, 0, 1, 0) x + 0 >= 0)}
\end{align*}$

$\begin{align*}
&\text{y0} \times (-y1 + 1)^{-1} \times (-y0 \times y2 + 1)^{-1} \times (-y0 \times y1 \times y3 + 1)^{-1} \\
&\text{(1, 2): (An equation (1, -1, -1, 0) x - 1 == 0,}
&\text{An inequality (0, -1, 0, 1) x + 0 >= 0,}
&\text{An inequality (0, 0, 0, 1) x + 0 >= 0,}
&\text{An inequality (1, 0, 0, -1) x + 0 >= 0,}
&\text{An inequality (1, -1, 0, 0) x - 1 >= 0)}
\end{align*}$

$\begin{align*}
&\text{(-y0 \times 2 \times y2 \times y3 + y0 \times y3 + y0) \times}$

$\begin{align*}
&\text{(-y0 \times y2 + 1)^{-1} \times (-y0 \times y1 \times y3 + 1)^{-1} \times (-y0 \times y2 \times y3 + 1)^{-1}} \\
&\text{(0, 1, 2): (An equation (0, 1, 0, -1) x + 0 == 0,}
&\text{An equation (1, -1, -1, 0) x - 1 == 0,}
&\text{An inequality (0, 0, 1, 0) x + 0 >= 0,}
&\text{An inequality (1, -1, 0, 0) x - 1 >= 0)}
\end{align*}$

$\begin{align*}
&\text{y0} \times (-y0 \times y2 + 1)^{-1} \times (-y0 \times y1 \times y3 + 1)^{-1} \\
&\text{(3): (An inequality (-1, 0, 0, 1) x + 0 >= 0,}
&\text{An inequality (0, -1, -1, 1) x - 1 >= 0,}
&\text{An inequality (0, 0, 1, 0) x + 0 >= 0,}
&\text{An inequality (0, 1, 0, 0) x + 0 >= 0,}
&\text{An inequality (1, 0, -1, 0) x + 0 >= 0)}
\end{align*}$

$\begin{align*}
&\text{(-y0 \times y1 \times y3^2 - y0 \times y3^2 + y0 \times y3 + y3) \times}$

$\begin{align*}
&\text{(-y3 + 1)^{-1} \times (-y0 \times y3 + 1)^{-1} \times (-y1 \times y3 + 1)^{-1} \times (-y0 \times y2 \times y3 + 1)^{-1}} \\
&\text{(0, 3): (An equation -1 == 0,)}
\end{align*}$

$\begin{align*}
&\text{0} \\
&\text{(1, 3): (An equation (1, 0, 0, -1) x + 0 == 0,}
&\text{An inequality (1, -1, -1, 0) x - 1 >= 0)}
\end{align*}$

(continues on next page)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
An inequality \((0, 0, 1, 0) x + 0 >= 0,\)
y_0 y_3 \cdot (-y_0 y_1 y_3 + 1)^{-1} \cdot (-y_0 y_2 y_3 + 1)^{-1}

\([0, 1, 3]:\) (An equation \(-1 == 0,\))
0

\([2, 3]:\) (An equation \((0, 1, 1, -1) x + 1 == 0,\)
An inequality \((-1, 1, 1, 0) x + 1 >= 0,\)
An inequality \((0, 0, 1, 0) x + 0 >= 0,\)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
(y_0 y_1 y_3^2 + y_0 y_3 + y_3) \cdot (-y_1 y_3 + 1)^{-1} \cdot (-y_0 y_1 y_3 + 1)^{-1} \cdot (-y_0 y_2 y_3 + 1)^{-1}

\([0, 2, 3]:\) (An equation \(-1 == 0,\))
0

\([1, 2, 3]:\) (An equation \((1, 0, 0, -1) x + 0 == 0,\)
An equation \((1, -1, -1, 0) x + 1 == 0,\)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
An inequality \((1, -1, 0, 0) x - 1 >= 0,\)
y_0 y_3 \cdot (-y_0 y_1 y_3^2 + y_0 y_3 + y_3) \cdot (-y_1 y_3 + 1)^{-1} \cdot (-y_0 y_1 y_3 + 1)^{-1} \cdot (-y_0 y_2 y_3 + 1)^{-1}

\([0, 1, 2, 3]:\) (An equation \(-1 == 0,\))
0

\([4]:\) (An inequality \((-1, -1, 0, 1) x - 1 >= 0,\)
An inequality \((-1, 0, 1, 0) x + 0 >= 0,\)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
An inequality \((1, 0, 0, 0) x + 0 >= 0,\)
y_3 \cdot (-y_2 + 1)^{-1} \cdot (-y_3 + 1)^{-1} \cdot (-y_1 y_3 + 1)^{-1} \cdot (-y_0 y_2 y_3 + 1)^{-1}

\([0, 4]:\) (An equation \(-1 == 0,\))
0

\([1, 4]:\) (An equation \(-1 == 0,\))
0

\([2, 4]:\) (An equation \((1, 1, 0, -1) x + 1 == 0,\)
An inequality \((-1, 0, 1, 0) x + 0 >= 0,\)
An inequality \((1, 0, 0, 0) x + 0 >= 0,\)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
y_3 \cdot (-y_2 + 1)^{-1} \cdot (-y_1 y_3 + 1)^{-1} \cdot (-y_0 y_2 y_3 + 1)^{-1}

\([0, 2, 4]:\) (An equation \(-1 == 0,\))
0

\([3, 4]:\) (An equation \((1, 0, -1, 0) x + 0 == 0,\)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
An inequality \((-1, -1, 0, 1) x - 1 >= 0,\)
An inequality \((1, 0, 0, 0) x + 0 >= 0,\)
y_3 \cdot (-y_3 + 1)^{-1} \cdot (-y_1 y_3 + 1)^{-1} \cdot (-y_0 y_2 y_3 + 1)^{-1}

\([0, 3, 4]:\) (An equation \(-1 == 0,\))
0

\([1, 3, 4]:\) (An equation \(-1 == 0,\))
0

\([2, 3, 4]:\) (An equation \((1, 1, 0, -1) x + 1 == 0,\)
An equation \((1, 0, -1, 0) x + 0 == 0,\)
An inequality \((0, 1, 0, 0) x + 0 >= 0,\)
An inequality \((1, 0, 0, 0) x + 0 >= 0\)

\[
y_3 \cdot (-y_1 \cdot y_3 + 1)^{-1} \cdot (-y_0 \cdot y_2 \cdot y_3 + 1)^{-1}
\]

\[[0, 2, 3, 4]: (An\ equation\ -1 == 0,)
0
\[1, 2, 3, 4]: (An\ equation\ -1 == 0,)
0
\[0, 1, 2, 3, 4]: (An\ equation\ -1 == 0,)
0

```
sage: P = Polyhedron(vertices=[[1], [5]])
sage: P.generating_function_of_integral_points()
y0^5 + y0^4 + y0^3 + y0^2 + y0
```

See also:
This function is accessible via `sage.geometry.polyhedron.base.Polyhedron_base.generating_function_of_integral_points()` as well. More examples can be found there.

## 2.3 Combinatorial Polyhedra

### 2.3.1 Combinatorial polyhedron

This module gathers algorithms for polyhedra that only depend on the vertex-facet incidences and that are called combinatorial polyhedron. The main class is `CombinatorialPolyhedron`. Most importantly, this class allows to iterate quickly through the faces (possibly of given dimension) via the `FaceIterator` object. The `CombinatorialPolyhedron` uses this iterator to quickly generate the f-vector, the edges, the ridges and the face lattice.

Terminology used in this module:

- **Vrep** – [vertices, rays, lines] of the polyhedron.
- **Hrep** – inequalities and equations of the polyhedron.
- **Facets** – facets of the polyhedron.
- **Vrepresentation** – represents a face by the list of Vrep it contains.
- **Hrepresentation** – represents a face by a list of Hrep it is contained in.
- **bit representation** – represents incidences as bitset, where each bit represents one incidence. There might be trailing zeros, to fit alignment requirements. In most instances, faces are represented by the bit representation, where each bit corresponds to a Vrep or facet. Thus a bit representation can either be a Vrep or facet representation depending on context.

**EXAMPLES:**

Construction:

```
sage: P = polytopes.hypercube(4)
sage: C = CombinatorialPolyhedron(P); C
A 4-dimensional combinatorial polyhedron with 8 facets
```

Obtaining edges and ridges:

```
sage: C.edges()[:2]
((A vertex at (1, -1, -1, -1), A vertex at (-1, -1, -1, -1),)
```

(continues on next page)
(A vertex at (-1, -1, -1, 1), A vertex at (-1, -1, -1, -1)))

\[
\begin{align*}
\text{sage: } & \text{C.edges(names=False)[2]} \\
& ((6, 15), (14, 15))
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{C.ridges()[:2]} \\
& ((\text{An inequality } (0, 0, 1, 0) \times + 1 \geq 0, \\
\text{An inequality } (0, 1, 0, 0) \times + 1 \geq 0), \\
\text{An inequality } (0, 0, 0, 1) x + 1 \geq 0, \\
\text{An inequality } (0, 1, 0, 0) x + 1 \geq 0))
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{C.ridges(names=False)[:2]} \\
& ((6, 7), (5, 7))
\end{align*}
\]

Vertex-graph and facet-graph:
\[
\begin{align*}
\text{sage: } & \text{C.vertex_graph()} \\
\text{Graph on 16 vertices}
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{C.facet_graph()} \\
\text{Graph on 8 vertices}
\end{align*}
\]

Face lattice:
\[
\begin{align*}
\text{sage: } & \text{C.face_lattice()} \\
\text{Finite lattice containing 82 elements}
\end{align*}
\]

Face iterator:
\[
\begin{align*}
\text{sage: } & \text{C.face_generator()} \\
\text{Iterator over the proper faces of a 4-dimensional combinatorial polyhedron}
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{C.face_generator(2)} \\
\text{Iterator over the 2-faces of a 4-dimensional combinatorial polyhedron}
\end{align*}
\]

AUTHOR:

- Jonathan Kliem (2019-04)

class sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron

    Bases: SageObject

    The class of the Combinatorial Type of a Polyhedron, a Polytope.

    INPUT:

    - data – an instance of
      - Polyhedron_base
      - or a LatticePolytopeClass
      - or a ConvexRationalPolyhedralCone
      - or an incidence_matrix as in incidence_matrix() In this case you should also specify
        the \text{Vrep} and \text{facets} arguments
      - or list of facets, each facet given as a list of [vertices, rays, lines] if the polyhedron is
        unbounded, then rays and lines and the extra argument nr_lines are required if the polyhedron
contains no lines, the rays can be thought of as the vertices of the facets deleted from a bounded
polyhedron see Polyhedron_base on how to use rays and lines
– or an integer, representing the dimension of a polyhedron equal to its affine hull
– or a tuple consisting of facets and vertices as two ListOfFaces.

• Vrep – (optional) when data is an incidence matrix, it should be the list of [vertices, rays, lines], if the rows in the incidence_matrix should correspond to names

• facets – (optional) when data is an incidence matrix or a list of facets, it should be a list of facets that
would be used instead of indices (of the columns of the incidence matrix).

• unbounded – value will be overwritten if data is a polyhedron; if unbounded and data is incidence matrix or a list of facets, need to specify far_face

• far_face – (semi-optional); if the polyhedron is unbounded this needs to be set to the list of indices of the
rays and line unless data is an instance of Polyhedron_base.

EXAMPLES:

We illustrate all possible input: a polyhedron:
sage: P = polytopes.cube() sage: CombinatorialPolyhedron(P) A 3-dimensional combinatorial polyhedron with 6 facets

a lattice polytope:

```
sage: points = [(1,0,0), (0,1,0), (0,0,1),
........ (-1,0,0), (0,-1,0), (0,0,-1)]
sage: L = LatticePolytope(points)
sage: CombinatorialPolyhedron(L)
A 3-dimensional combinatorial polyhedron with 8 facets
```

a cone:

```
sage: M = Cone([(1,0), (0,1)])
sage: CombinatorialPolyhedron(M)
A 2-dimensional combinatorial polyhedron with 2 facets
```

an incidence matrix:

```
sage: P = Polyhedron(rays=[[0,1]])
sage: data = P.incidence_matrix()
sage: far_face = [i for i in range(2) if not P.Vrepresentation()[i].is_vertex()]
sage: CombinatorialPolyhedron(data, unbounded=True, far_face=far_face)
A 1-dimensional combinatorial polyhedron with 1 facet
```

a list of facets:

```
sage: C = CombinatorialPolyhedron(((1,2,3),(1,2,4),(1,3,4),(2,3,4)))
A 3-dimensional combinatorial polyhedron with 4 facets
```
(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```sage
sage: C.Vrepresentation()
(1, 2, 3, 4)
sage: C.Hrepresentation()
('facet0', 'facet1', 'facet2', 'myfacet3')
```

an integer:

```sage
sage: CombinatorialPolyhedron(-1).f_vector()
(1)
sage: CombinatorialPolyhedron(0).f_vector()
(1, 1)
sage: CombinatorialPolyhedron(5).f_vector()
(1, 0, 0, 0, 0, 1)
```

tuple of ListOfFaces:

```sage
from sage.geometry.polyhedron.combinatorial_polyhedron.
conversions import facets_tuple_to_bit_rep_of_facets,
facets_tuple_to_bit_rep_of_Vrep
sage: bi_pyr = ((0,1,4), (1,2,4), (2,3,4), (3,0,4), (0,1,5),
(1,2,5), (2,3,5), (3,0,5))
sage: facets = facets_tuple_to_bit_rep_of_facets(bi_pyr, 6)
sage: Vrep = facets_tuple_to_bit_rep_of_Vrep(bi_pyr, 6)
sage: C = CombinatorialPolyhedron((facets, Vrep)); C
A 3-dimensional combinatorial polyhedron with 8 facets
sage: C.f_vector()
(1, 6, 12, 8, 1)
```

Specifying that a polyhedron is unbounded is important. The following with a polyhedron works fine:

```sage
P = Polyhedron(ieqs=[[1,-1,0],[1,1,0]])
sage: C = CombinatorialPolyhedron(P)  # this works fine
sage: C
A 2-dimensional combinatorial polyhedron with 2 facets
```

The following is incorrect, as unbounded is implicitly set to False:

```sage
data = P.incidence_matrix()
sage: vert = P.Vrepresentation()
sage: C = CombinatorialPolyhedron(data, Vrep=vert)
sage: C
A 2-dimensional combinatorial polyhedron with 2 facets
sage: C.f_vector()
Traceback (most recent call last):
... ValueError: not all vertices are intersections of facets
```

The correct usage is:

```sage
far_face = [i for i in range(3) if not P.Vrepresentation()[i].is_vertex()]
sage: C = CombinatorialPolyhedron(data, Vrep=vert, unbounded=True,
far_face=far_face)
sage: C
A 2-dimensional combinatorial polyhedron with 2 facets
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

sage: C.f_vector()
(1, 0, 2, 1)
sage: C.vertices()
()

Hrepresentation()

Return a list of names of facets and possibly some equations.

EXAMPLES:

sage: P = polytopes.permutahedron(3)
sage: C = CombinatorialPolyhedron(P)
sage: C.Hrepresentation()
(An inequality (1, 1, 0) x - 3 >= 0,
An inequality (-1, -1, 0) x + 5 >= 0,
An inequality (0, 1, 0) x - 1 >= 0,
An inequality (-1, 0, 0) x + 3 >= 0,
An inequality (1, 0, 0) x - 1 >= 0,
An inequality (0, -1, 0) x + 3 >= 0,
An equation (1, 1, 1) x - 6 == 0)
sage: points = [(1,0,0), (0,1,0), (0,0,1),
....: (-1,0,0), (0,-1,0), (0,0,-1)]
sage: L = LatticePolytope(points)
sage: C = CombinatorialPolyhedron(L)
sage: C.Hrepresentation()
(N(1, -1, -1),
N(1, 1, -1),
N(1, 1, 1),
N(1, -1, 1),
N(-1, -1, 1),
N(-1, 1, -1),
N(-1, 1, 1))
sage: M = Cone([[1,0], [0,1]])
sage: CombinatorialPolyhedron(M).Hrepresentation()
(M(0, 1), M(1, 0))

Vrepresentation()

Return a list of names of [vertices, rays, lines].

EXAMPLES:

sage: P = Polyhedron(rays=[[1,0,0], [0,1,0],
....: [0,0,1], [0,0,-1]])
sage: C = CombinatorialPolyhedron(P)
sage: C.Vrepresentation()
(A line in the direction (0, 0, 1),
A ray in the direction (1, 0, 0),
A vertex at (0, 0, 0),
A ray in the direction (0, 1, 0))
sage: points = [(1,0,0), (0,1,0), (0,0,1),
....: (-1,0,0), (0,-1,0), (0,0,-1)]
sage: L = LatticePolytope(points)
sage: C = CombinatorialPolyhedron(L)

(continues on next page)
a_maximal_chain (Vindex=None, Hindex=None)

Return a maximal chain of the face lattice in increasing order without empty face and whole polyhedron/maximal face.

INPUT:

- Vindex – integer (default: None); prescribe the index of the vertex in the chain
- Hindex – integer (default: None); prescribe the index of the facet in the chain

Each face is given as CombinatorialFace.

EXAMPLES:

```python
sage: P = polytopes.cross_polytope(4)
sage: C = P.combinatorial_polyhedron()
sage: chain = C.a_maximal_chain(); chain
[A 0-dimensional face of a 4-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 4-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 4-dimensional combinatorial polyhedron,
 A 3-dimensional face of a 4-dimensional combinatorial polyhedron]
sage: [face.ambient_V_indices() for face in chain]
[(7,), (6, 7), (5, 6, 7), (4, 5, 6, 7)]

sage: P = polytopes.hypercube(4)
sage: C = P.combinatorial_polyhedron()
sage: chain = C.a_maximal_chain(); chain
[A 0-dimensional face of a 4-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 4-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 4-dimensional combinatorial polyhedron,
 A 3-dimensional face of a 4-dimensional combinatorial polyhedron]
sage: [face.ambient_V_indices() for face in chain]
[(15,), (6, 15), (5, 6, 14, 15), (0, 5, 6, 7, 8, 9, 14, 15)]

sage: # needs sage.combinat
sage: P = polytopes.permutahedron(4)
sage: C = P.combinatorial_polyhedron()
sage: chain = C.a_maximal_chain(); chain
[A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron]
sage: [face.ambient_V_indices() for face in chain]
[(16,), (15, 16), (8, 9, 14, 15, 16, 17)]

sage: P = Polyhedron(rays=[[1,0]], lines=[[0,1]])
sage: C = P.combinatorial_polyhedron()
sage: chain = C.a_maximal_chain()
sage: [face.ambient_V_indices() for face in chain]
[(0, 1)]

sage: P = Polyhedron(rays=[[1,0,0],[0,0,1]], lines=[[0,1,0]])
```

(continues on next page)
Specify an index for the vertex of the chain:

```python
sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: [face.ambient_V_indices() for face in C.a_maximal_chain()]
[(5,), (0, 5), (0, 3, 4, 5)]
sage: [face.ambient_V_indices() for face in C.a_maximal_chain(Vindex=2)]
[(2,), (2, 7), (2, 3, 4, 7)]
```

If the specified vertex is not contained in the specified facet an error is raised:

```python
sage: C.a_maximal_chain(Vindex=0, Hindex=3)
Traceback (most recent call last):
  ... ValueError: the given Vindex is not compatible with the given Hindex
```

An error is raised, if the specified index does not correspond to a facet:

```python
sage: C.a_maximal_chain(Hindex=40)
Traceback (most recent call last):
  ... ValueError: the given Hindex does not correspond to a facet
```

An error is raised, if the specified index does not correspond to a vertex:

```python
sage: C.a_maximal_chain(Vindex=40)
Traceback (most recent call last):
  ... ValueError: the given Vindex does not correspond to a vertex
```

```python
sage: P = Polyhedron(rays=[[1,0,0],[0,0,1]], lines=[[0,1,0]])
sage: C = P.combinatorial_polyhedron()
sage: C.a_maximal_chain(Vindex=0)
Traceback (most recent call last):
  ... ValueError: the given Vindex does not correspond to a vertex
```
sage: P = Polyhedron(rays=[[1,0,0],[0,0,1]])
sage: C = P.combinatorial_polyhedron()
sage: C.a_maximal_chain(Vindex=0)
[A 0-dimensional face of a 2-dimensional combinatorial polyhedron, 
A 1-dimensional face of a 2-dimensional combinatorial polyhedron]
sage: C.a_maximal_chain(Vindex=1)
Traceback (most recent call last):
... 
ValueError: the given Vindex does not correspond to a vertex

choose_algorithm_to_compute_edges_or_ridges (edges_or_ridges)

Use some heuristics to pick primal or dual algorithm for computation of edges resp. ridges.

We estimate how long it takes to compute a face using the primal and the dual algorithm. This may differ 
significantly, so that e.g. visiting all faces with the primal algorithm is faster than using the dual algorithm to 
just visit vertices and edges.

We guess the number of edges and ridges and do a wild estimate on the total number of faces.

INPUT:

• edges_or_ridges – string; one of: * 'edges' * 'ridges'

OUTPUT:

Either 'primal' or 'dual'.

EXAMPLES:

sage: C = polytopes.permutahedron(5).combinatorial_polyhedron()
sage: C.choose_algorithm_to_compute_edges_or_ridges("edges")
'primal'
sage: C.choose_algorithm_to_compute_edges_or_ridges("ridges")
'primal'

sage: C = polytopes.cross_polytope(5).combinatorial_polyhedron()
sage: C.choose_algorithm_to_compute_edges_or_ridges("edges")
'dual'
sage: C.choose_algorithm_to_compute_edges_or_ridges("ridges")
'dual'

sage: C = polytopes.Birkhoff_polytope(5).combinatorial_polyhedron()
sage: C.choose_algorithm_to_compute_edges_or_ridges("edges")
'dual'
sage: C.choose_algorithm_to_compute_edges_or_ridges("ridges")
'primal'
sage: C.choose_algorithm_to_compute_edges_or_ridges("something_else")
Traceback (most recent call last):
... 
ValueError: unknown computation goal something_else

dim()

Return the dimension of the polyhedron.

EXAMPLES:

sage: C = CombinatorialPolyhedron([[1,2,3], (1,2,4), 
....: (1,3,4), (2,3,4)])

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

sage: C.dimension()
3

sage: P = Polyhedron(rays=[[1,0,0],[0,1,0],[0,0,1],[0,0,-1]])
sage: CombinatorialPolyhedron(P).dimension()
3

.. dim is an alias:

sage: CombinatorialPolyhedron(P).dim()
3

dimension()

Return the dimension of the polyhedron.

EXAMPLES:

sage: C = CombinatorialPolyhedron([(1,2,3), (1,2,4),
....: (1,3,4), (2,3,4)])
sage: C.dimension()
3

sage: P = Polyhedron(rays=[[1,0,0],[0,1,0],[0,0,1],[0,0,-1]])
sage: CombinatorialPolyhedron(P).dimension()
3

.. dim is an alias:

sage: CombinatorialPolyhedron(P).dim()
3

dual()

Return the dual/polar of self.

Only defined for bounded polyhedra.

See also:
polar().

EXAMPLES:

sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: D = C.dual()
sage: D.f_vector()
(1, 6, 12, 8, 1)
sage: D1 = P.polar().combinatorial_polyhedron() # Polar is an alias to be consistent with Polyhedron_base:
sage: D1.face_lattice().is_isomorphic(D.face_lattice())
True

Polar is an alias to be consistent with Polyhedron_base:

sage: C.polar().f_vector()
(1, 6, 12, 8, 1)

For unbounded polyhedra, an error is raised:
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],␣
˓→unbounded=True)
sage: C.dual()
Traceback (most recent call last):
...
ValueError: self must be bounded

edges (names=True, algorithm=None)
Return the edges of the polyhedron, i.e. the rank 1 faces.

INPUT:

• names – boolean (default: True); if False, then the Vrepresentatives in the edges are given by their
indices in the Vrepresentation

• algorithm – string (optional); specify whether the face generator starts with facets or vertices: *
'primal' – start with the facets *
'dual' – start with the vertices *
None – choose automatically

Note: To compute edges and f_vector, first compute the edges. This might be faster.

EXAMPLES:

sage: P = polytopes.cyclic_polytope(3,5)
sage: C = CombinatorialPolyhedron(P)
sage: C.edges()
((A vertex at (3, 9, 27), A vertex at (4, 16, 64)),
 (A vertex at (2, 4, 8), A vertex at (4, 16, 64)),
 (A vertex at (1, 1, 1), A vertex at (4, 16, 64)),
 (A vertex at (0, 0, 0), A vertex at (4, 16, 64)),
 (A vertex at (2, 4, 8), A vertex at (3, 9, 27)),
 (A vertex at (0, 0, 0), A vertex at (3, 9, 27)),
 (A vertex at (1, 1, 1), A vertex at (2, 4, 8)),
 (A vertex at (0, 0, 0), A vertex at (2, 4, 8)),
 (A vertex at (0, 0, 0), A vertex at (1, 1, 1)))

sage: C.edges(names=False)
((3, 4), (2, 4), (1, 4), (0, 4), (2, 3), (0, 3), (1, 2), (0, 2), (0, 1))

sage: P = Polyhedron(rays=[[1,0],[0,0]])
sage: C = CombinatorialPolyhedron(P)
sage: C.edges()
((A line in the direction (1, 0), A vertex at (0, 0))),

sage: P = Polyhedron(rockets=[[0,0],[1,0]])
sage: C = CombinatorialPolyhedron(P)
sage: C.edges()
((A vertex at (0, 0), A vertex at (1, 0))),

sage: from itertools import combinations
sage: N = combinations(["a","b","c","d","e"], 4)
sage: C = CombinatorialPolyhedron(N)
sage: C.edges()
(('d', 'e'),
 ('c', 'e'),
 ('b', 'e'),
 ('a', 'e'),
(continues on next page)
Compute the f_vector of the polyhedron.

The f_vector contains the number of faces of dimension $k$ for each $k$ in \( \text{range}(-1, \text{self. dimension}() + 1) \).

**INPUT:**

- `num_threads` – integer (optional); specify the number of threads
- `parallelization_depth` – integer (optional); specify how deep in the lattice the parallelization is done
- `algorithm` – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

**Note:** To obtain edges and/or ridges as well, first do so. This might already compute the f_vector.

**EXAMPLES:**

```sage
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: C.f_vector()
(1, 120, 240, 150, 30, 1)

sage: P = polytopes.cyclic_polytope(6,10)
sage: C = CombinatorialPolyhedron(P)
sage: C.f_vector()
(1, 10, 45, 120, 185, 150, 50, 1)
```

Using two threads:

```sage
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: C.f_vector(num_threads=2)
(1, 120, 240, 150, 30, 1)
```

**face_by_face_lattice_index** `(index)`

Return the element of `CombinatorialPolyhedron.face_lattice()` with corresponding index.

The element will be returned as `CombinatorialFace`.

**EXAMPLES:**

```sage
```
# needs sage.combinat
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: F = C.face_lattice()
sage: F
Finite lattice containing 28 elements
sage: G = F.relabel(C.face_by_face_lattice_index)
sage: G.level_sets()[0]
[A -1-dimensional face of a 3-dimensional combinatorial polyhedron]
sage: G.level_sets()[3]
[A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron]
sage: P = Polyhedron(rays=[[0,1], [1,0]])
sage: C = CombinatorialPolyhedron(P)
sage: F = C.face_lattice()  # needs sage.combinat
sage: G = F.relabel(C.face_by_face_lattice_index)  # needs sage.combinat
sage: G._elements  # needs sage.combinat
(A -1-dimensional face of a 2-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 2-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 2-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 2-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 2-dimensional combinatorial polyhedron)
sage: def f(i): return C.face_by_face_lattice_index(i).ambient_V_indices()
sage: G = F.relabel(f)  # needs sage.combinat
sage: G._elements  # needs sage.combinat
(((), (0,), (0, 1), (0, 2), (0, 1, 2)))

**face_generator** *(dimension=None, algorithm=None, **kwds)*

Iterator over all proper faces of specified dimension.

**INPUT:**

- **dimension** – if specified, then iterate over only this dimension
- **algorithm** – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

**OUTPUT:**

- **FaceIterator**

**Note:** **FaceIterator** can ignore subfaces or supfaces of the current face.

### 2.3. Combinatorial Polyhedra
EXAMPLES:

```python
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(dimension=2)
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_Vrepresentation()
(A vertex at (1, 3, 2, 5, 4),
 A vertex at (2, 3, 1, 5, 4),
 A vertex at (3, 1, 2, 5, 4),
 A vertex at (3, 2, 1, 5, 4),
 A vertex at (2, 1, 3, 5, 4),
 A vertex at (1, 2, 3, 5, 4))
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_Vrepresentation()
(A vertex at (2, 1, 4, 5, 3),
 A vertex at (3, 2, 4, 5, 1),
 A vertex at (3, 1, 4, 5, 2),
 A vertex at (1, 3, 4, 5, 2),
 A vertex at (1, 2, 4, 5, 3),
 A vertex at (2, 3, 4, 5, 1))
sage: face.ambient_Hrepresentation()
(An inequality (0, 0, -1, -1, 0) x + 9 >= 0,
 An inequality (0, 0, 0, -1, 0) x + 5 >= 0,
 An equation (1, 1, 1, 1, 1) x - 15 == 0)
sage: face.ambient_H_indices()
(25, 29, 30)
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_Hindices()
(24, 29, 30)
sage: face.ambient_Vindices()
(32, 89, 90, 94)
sage: C = CombinatorialPolyhedron([[0,1,2],[0,1,3],[0,2,3],[1,2,3]])
sage: it = C.face_generator()
sage: for face in it: face.ambient_Vrepresentation()
(1, 2, 3)
(0, 2, 3)
(0, 1, 2)
(2, 3)
(1, 3)
(1, 2)
(3,)
(2,)
(1,)
(0, 3)
(0, 2)
(0,)
(0, 1)
sage: P = Polyhedron(rays=[[1,0],[0,1]], vertices=[[1,0],[0,1]])
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(1)
```

(continues on next page)
sage: for face in it: face.ambient_Vrepresentation()
(A vertex at (0, 1), A vertex at (1, 0))
(A ray in the direction (1, 0), A vertex at (1, 0))
(A ray in the direction (0, 1), A vertex at (0, 1))

See also:

FaceIterator, CombinatorialFace.

face_iter (dimension=None, algorithm=None, **kwds)

Iterator over all proper faces of specified dimension.

INPUT:

- dimension – if specified, then iterate over only this dimension
- algorithm – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

OUTPUT:

- FaceIterator

Note: FaceIterator can ignore subfaces or supfaces of the current face.

EXAMPLES:

sage: # needs sage.combinat
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(dimension=2)
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_Vrepresentation()
(A vertex at (1, 3, 2, 5, 4),
 A vertex at (2, 3, 1, 5, 4),
 A vertex at (3, 1, 2, 5, 4),
 A vertex at (3, 2, 1, 5, 4),
 A vertex at (2, 1, 3, 5, 4),
 A vertex at (1, 2, 3, 5, 4))
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_Vrepresentation()
(A vertex at (2, 1, 4, 5, 3),
 A vertex at (3, 2, 4, 5, 1),
 A vertex at (3, 1, 4, 5, 2),
 A vertex at (1, 2, 4, 5, 3),
 A vertex at (2, 3, 4, 5, 1))
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_Hrepresentation()
(An inequality (0, 0, -1, -1, 0) x + 9 >= 0,
 An inequality (0, 0, 0, -1, 0) x + 5 >= 0,
 An equation (1, 1, 1, 1, 1) x - 15 == 0)
sage: face.ambient_H_indices()
(25, 29, 30)
sage: face = next(it); face
A 2-dimensional face of a 4-dimensional combinatorial polyhedron
sage: face.ambient_H_indices()
(24, 29, 30)
sage: face.ambient_V_indices()
(32, 89, 90, 94)

sage: C = CombinatorialPolyhedron([[0,1,2],[0,1,3],[0,2,3],[1,2,3]])
sage: it = C.face_generator()
sage: for face in it: face.ambient_Vrepresentation()
(1, 2, 3)
(0, 2, 3)
(0, 1, 3)
(0, 1, 2)
(2, 3)
(1, 3)
(1, 2)
(3,)
(2,)
(1,)
(0, 3)
(0, 2)
(0,)
(0, 1)

sage: P = Polyhedron(rays=[[1,0],[0,1]], vertices=[[1,0],[0,1]])
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(1)
sage: for face in it: face.ambient_Vrepresentation()
(A vertex at (0, 1), A vertex at (1, 0))
(A ray in the direction (1, 0), A vertex at (1, 0))
(A ray in the direction (0, 1), A vertex at (0, 1))

See also:

FaceIterator, CombinatorialFace.

face_lattice()
Generate the face-lattice.

OUTPUT:

• FiniteLatticePoset

Note: Use CombinatorialPolyhedron.face_by_face_lattice_index() to get the face for each index.

Warning: The labeling of the face lattice might depend on architecture and implementation. Relabeling the face lattice with CombinatorialPolyhedron.face_by_face_lattice_index() or the properties obtained from this face will be platform independent.

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```python
sage: P = Polyhedron(rays=[[1,0],[0,1]])
sage: C = CombinatorialPolyhedron(P)
sage: C.face_lattice()
needs sage.combinat
Finite lattice containing 5 elements

sage: P = Polyhedron(rays=[[1,0,0], [-1,0,0], [0,-1,0], [0,1,0]])
sage: C = CombinatorialPolyhedron(P)
sage: P1 = Polyhedron(rays=[[1,0], [-1,0]])
sage: C1 = CombinatorialPolyhedron(P1)
sage: C.is_isomorphic(C1)

sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: C.face_lattice()
needs sage.combinat
Finite lattice containing 542 elements
```

**facet_adjacency_matrix** *(algorithm=None)*

Return the binary matrix of facet adjacencies.

**INPUT:**

- **algorithm** – string (optional); specify whether the face generator starts with facets or vertices: *
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

**See also:**

vertex_adjacency_matrix().

**EXAMPLES:**

```python
sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: C.facet_adjacency_matrix()
[0 1 1 0 1 1]
[1 0 1 1 0]
[1 1 0 1 0 1]
[1 1 0 1 1 0]
[1 0 1 1 0]
```

**facet_graph** *(names=True, algorithm=None)*

Return the facet graph.

The facet graph of a polyhedron consists of ridges as edges and facets as vertices.

**INPUT:**

- **algorithm** – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

If names is False, the vertices of the graph will be the indices of the facets in the Hrepresentation.

**EXAMPLES:**

```python
```
```python
sage: P = polytopes.cyclic_polytope(4,6)
sage: C = CombinatorialPolyhedron(P)
sage: C.facet_graph()  # needs sage.graphs
Graph on 9 vertices
```

**facets** *(names=True)*

Return the facets as lists of [vertices, rays, lines].

If `names` is `False`, then the V representatives in the facets are given by their indices in the V representation.

The facets are the maximal nontrivial faces.

**EXAMPLES:**

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: C.facets()
((A vertex at (1, -1, -1),
  A vertex at (1, 1, -1),
  A vertex at (1, 1, 1),
  A vertex at (1, -1, 1)),
 (A vertex at (1, 1, -1),
  A vertex at (1, 1, 1),
  A vertex at (-1, 1, -1),
  A vertex at (-1, 1, 1)),
 (A vertex at (1, 1, 1),
  A vertex at (1, -1, 1),
  A vertex at (-1, -1, 1),
  A vertex at (-1, 1, 1)),
 (A vertex at (-1, -1, 1),
  A vertex at (-1, -1, -1),
  A vertex at (-1, 1, -1),
  A vertex at (-1, 1, 1)),
 (A vertex at (1, -1, -1),
  A vertex at (1, 1, -1),
  A vertex at (-1, -1, -1),
  A vertex at (-1, 1, -1)),
 (A vertex at (1, -1, 1),
  A vertex at (1, 1, 1),
  A vertex at (-1, -1, 1),
  A vertex at (-1, -1, -1)))
sage: C.facets(names=False)
((0, 1, 2, 3),
 (1, 2, 6, 7),
 (2, 3, 4, 7),
 (4, 5, 6, 7),
 (0, 1, 5, 6),
 (0, 3, 4, 5))
```

The empty face is trivial and hence the 0-dimensional polyhedron does not have facets:

```python
sage: C = CombinatorialPolyhedron(0)
sage: C.facets()
()```

**flag_f_vector** *(args)*

Return the flag f-vector.
For each $-1 < i_0 < \cdots < i_n < d$ the flag f-vector counts the number of flags $F_0 \subset \cdots \subset F_n$ with $F_j$ of dimension $i_j$ for each $0 \leq j \leq n$, where $d$ is the dimension of the polyhedron.

INPUT:

- `args` – integers (optional); specify an entry of the flag-f-vector; must be an increasing sequence of integers

OUTPUT:

- a dictionary, if no arguments were given
- an Integer, if arguments were given

EXAMPLES:

Obtain the entire flag-f-vector:

```
sage: C = polytopes.hypercube(4).combinatorial_polyhedron()
sage: C.flag_f_vector()
```

```
needs sage.combinat
{(-1,): 1,
 (0,): 16,
 (0, 1): 64,
 (0, 1, 2): 192,
 (0, 1, 2, 3): 384,
 (0, 1, 3): 192,
 (0, 2): 96,
 (0, 2, 3): 192,
 (0, 3): 64,
 (1,): 32,
 (1, 2): 96,
 (1, 2, 3): 192,
 (1, 3): 96,
 (2,): 24,
 (2, 3): 48,
 (3,): 8,
 (4,): 1}
```

Specify an entry:

```
sage: C.flag_f_vector(0,3)
```

```
needs sage.combinat
64
```

```
sage: C.flag_f_vector(2)
```

```
needs sage.combinat
24
```

Leading $-1$ and trailing entry of dimension are allowed:

```
sage: C.flag_f_vector(-1,0,3)
```

```
needs sage.combinat
64
```

```
sage: C.flag_f_vector(-1,0,3,4)
```

```
needs sage.combinat
64
```

One can get the number of trivial faces:
Combinatorial and Discrete Geometry, Release 10.3

```
sage: C.flag_f_vector(-1)  # needs sage.combinat
1
sage: C.flag_f_vector(4)  # needs sage.combinat
1
```

Polyhedra with lines, have 0 entries accordingly:

```
sage: C = (Polyhedron(lines=[[1]]) * polytopes.hypercube(2)).combinatorial_polyhedron()
sage: C.flag_f_vector()  # needs sage.combinat
{(-1,): 1, (0, 1): 0, (0, 2): 0, (0,): 0, (1, 2): 8, (1,): 4, (2,): 4, 3: 1}
```

If the arguments are not strictly increasing or out of range, a key error is raised:

```
sage: C.flag_f_vector(-1,0,3,5)  # needs sage.combinat
Traceback (most recent call last):
...  
KeyError: (0, 3, 5)
sage: C.flag_f_vector(-1,3,0)  # needs sage.combinat
Traceback (most recent call last):
...  
KeyError: (3, 0)
```

**graph** *(names=True, algorithm=None)*

Return a graph in which the vertices correspond to vertices of the polyhedron, and edges to bounded rank 1 faces.

**INPUT:**

- **names** – boolean (default: True); if False, then the nodes of the graph are labelled by the indices of the Vrepresentation

- **algorithm** – string (optional); specify whether the face generator starts with facets or vertices: *'primal' – start with the facets* *'dual' – start with the vertices* *None – choose automatically*

**EXAMPLES:**

```
sage: P = polytopes.cyclic_polytope(3,5)
sage: C = CombinatorialPolyhedron(P)
sage: G = C.vertex_graph(); G  # needs sage.graphs
Graph on 5 vertices
sage: sorted(G.degree())  # needs sage.graphs
[3, 3, 4, 4, 4]
```

```
sage: P = Polyhedron(rays=[[1]])
sage: C = CombinatorialPolyhedron(P)
sage: C.graph()  # needs sage.graphs
Graph on 1 vertex
```

**hasse_diagram()**

Return the Hasse diagram of self.
This is the Hasse diagram of the poset of the faces of \texttt{self}: A directed graph consisting of a vertex for each face and an edge for each minimal inclusion of faces.

**Note:** The vertices of the Hasse diagram are given by indices. Use \texttt{CombinatorialPolyhedron.face_by_face_lattice_index()} to relabel.

**Warning:** The indices of the Hasse diagram might depend on architecture and implementation. Relabeling the face lattice with \texttt{CombinatorialPolyhedron.face_by_face_lattice_index()} or the properties obtained from this face will be platform independent.

**EXAMPLES:**

```sage
# needs sage.graphs sage.rings.number_field
sage: P = polytopes.regular_polygon(4).pyramid()
sage: C = CombinatorialPolyhedron(P)
sage: D = C.hasse_diagram(); D
Digraph on 20 vertices
sage: D.average_degree()
21/5
sage: D.relabel(C.face_by_face_lattice_index)
sage: dim_0_vert = D.vertices(sort=True)[1:6]; dim_0_vert
[A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron]
sage: sorted(D.out_degree(vertices=dim_0_vert))
[3, 3, 3, 3, 4]
```

### incidence_matrix()

Return the incidence matrix.

**Note:** The columns correspond to inequalities/equations in the order \texttt{Hrepresentation()}, the rows correspond to vertices/rays/lines in the order \texttt{Vrepresentation()}.

**See also:**

\texttt{incidence_matrix()}.

**EXAMPLES:**

```sage
sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: C.incidence_matrix()
[1 0 0 0 1 1]
[1 1 0 0 1 0]
[1 1 1 0 0 0]
[1 0 1 0 0 1]
[0 0 1 1 0 1]
[0 0 0 1 1 1]
[0 1 0 1 1 0]
[0 1 1 1 0 0]
```

In this case the incidence matrix is only computed once:
The incidence matrix is consistent with incidence_matrix():

```
sage: P = Polyhedron([[0,0]])
sage: P.incidence_matrix()
[1 1]
sage: C = P.combinatorial_polyhedron()
sage: C.incidence_matrix().clear_cache()
sage: P.combinatorial_polyhedron().incidence_matrix()
[1 1]
```

is_bipyramid (certificate=False)

Test whether the polytope is a bipyramid over some other polytope.

INPUT:

- certificate – boolean (default: False); specifies whether to return a vertex of the polytope which is the apex of a pyramid, if found

INPUT:

- certificate – boolean (default: False); specifies whether to return two vertices of the polytope which are the apices of a bipyramid, if found

OUTPUT:

If certificate is True, returns a tuple containing:

1. Boolean.

2. None or a tuple containing:
   a. The first apex.
   b. The second apex.

If certificate is False returns a boolean.

EXAMPLES:

```
sage: C = polytopes.hypercube(4).combinatorial_polyhedron()
sage: C.is_bipyramid()
False
sage: C.is_bipyramid(certificate=True)
(False, None)
sage: C = polytopes.cross_polytope(4).combinatorial_polyhedron()
sage: C.is_bipyramid()
```
sage: C.is_bipyramid(certificate=True)
(True, [A vertex at (1, 0, 0, 0), A vertex at (-1, 0, 0, 0)])

For unbounded polyhedra, an error is raised:

```
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2], unbounded=True)
sage: C.is_pyramid()
Traceback (most recent call last):
...  
ValueError: polyhedron has to be compact
```

ALGORITHM:
Assume all faces of a polyhedron to be given as lists of vertices.

A polytope is a bipyramid with apexes \( v \), \( w \) if and only if for each proper face \( v \in F \) there exists a face \( G \) with \( G \setminus \{w\} = F \setminus \{v\} \) and vice versa (for each proper face \( w \in F \) there exists ...).

To check this property it suffices to check for all facets of the polyhedron.

```
is_compact()  
Return whether the polyhedron is compact

EXAMPLES:
```
```
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2], unbounded=True)
sage: C.is_compact()
False
sage: C = CombinatorialPolyhedron([[0,1], [0,2], [1,2]])
sage: C.is_compact()
True
sage: P = polytopes.simplex()
sage: P.combinatorial_polyhedron().is_compact()
True
sage: P = Polyhedron(rays=P.vertices())
sage: P.combinatorial_polyhedron().is_compact()
False
```

```
is_lawrence_polytope()  
Return True if self is a Lawrence polytope.

A polytope is called a Lawrence polytope if it has a centrally symmetric (normalized) Gale diagram.

Equivalently, there exists a partition \( P_1, \ldots, P_k \) of the vertices \( V \) such that each part \( P_i \) has size 2 or 1 and for each part there exists a facet with vertices exactly \( V \setminus P_i \).

EXAMPLES:
```
```
sage: C = polytopes.simplex(5).combinatorial_polyhedron()
sage: C.is_lawrence_polytope()
True
sage: P = polytopes.hypercube(4).lawrence_polytope()
sage: C = P.combinatorial_polyhedron()
sage: C.is_lawrence_polytope()
True
sage: P = polytopes.hypercube(4)
```

(continues on next page)
sage: C = P.combinatorial_polyhedron()
sage: C.is_lawrence_polytope()
False

For unbounded polyhedra, an error is raised:

sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],
unbounded=True)
sage: C.is_lawrence_polytope()
Traceback (most recent call last):
...
NotImplementedError: this function is implemented for polytopes only

AUTHORS:

• Laith Rastanawi
• Jonathan Kliem

REFERENCES:

For more information, see [BaSt1990].

is_neighborly \((k=None)\)

Return whether the polyhedron is neighborly.

If the input \(k\) is provided, then return whether the polyhedron is \(k\)-neighborly.

A polyhedron is neighborly if every set of \(n\) vertices forms a face for \(n\) up to floor of half the dimension of the polyhedron. It is \(k\)-neighborly if this is true for \(n\) up to \(k\).

INPUT:

• \(k\) – the dimension up to which to check if every set of \(k\) vertices forms a face. If no \(k\) is provided, check up to floor of half the dimension of the polyhedron.

OUTPUT:

• True if the every set of up to \(k\) vertices forms a face,

• False otherwise

See also:

neighborliness()

EXAMPLES:

sage: P = polytopes.cyclic_polytope(8,12)
sage: C = P.combinatorial_polyhedron()
sage: C.is_neighborly()
True
sage: P = polytopes.simplex(6)
sage: C = P.combinatorial_polyhedron()
sage: C.is_neighborly()
True
sage: P = polytopes.cyclic_polytope(4,10)
sage: P = P.join(P)
sage: C = P.combinatorial_polyhedron()
sage: C.is_neighborly()
False
**is_prism** *(certificate=False)*

Test whether the polytope is a prism of some polytope.

**INPUT:**

- certificate – boolean (default: False); specifies whether to return two facets of the polytope which are the bases of a prism, if found

**OUTPUT:**

If certificate is True, returns a tuple containing:

1. Boolean.
2. None or a tuple containing:
   a. List of the vertices of the first base facet.
   b. List of the vertices of the second base facet.

If certificate is False returns a boolean.

**is_pyramid** *(certificate=False)*

Test whether the polytope is a pyramid over one of its facets.

**INPUT:**

- certificate – boolean (default: False); specifies whether to return a vertex of the polytope which is the apex of a pyramid, if found

**OUTPUT:**

If certificate is True, returns a tuple containing:

1. Boolean.
2. The apex of the pyramid or None.

If certificate is False returns a boolean.

**AUTHORS:**

- Laith Rastanawi
- Jonathan Kliem

**EXAMPLES:**

```python
sage: C = polytopes.cross_polytope(4).combinatorial_polyhedron()
sage: C.is_pyramid()
False
sage: C.is_pyramid(certificate=True)
(True, None)
sage: C = polytopes.cross_polytope(4).pyramid().combinatorial_polyhedron()
sage: C.is_pyramid()
True
sage: C.is_pyramid(certificate=True)
(True, A vertex at (1, 0, 0, 0, 0))
sage: C = polytopes.simplex(5).combinatorial_polyhedron()
sage: C.is_pyramid(certificate=True)
(True, A vertex at (1, 0, 0, 0, 0, 0))
```
For unbounded polyhedra, an error is raised:

```
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],
   →unbounded=True)
sage: C.is_pyramid()
Traceback (most recent call last):
...
ValueError: polyhedron has to be compact
```

**is_simple()**

Test whether the polytope is simple.

If the polyhedron is unbounded, return False.

A polytope is simple, if each vertex is contained in exactly $d$ facets, where $d$ is the dimension of the polytope.

**EXAMPLES:**

```
sage: P = polytopes.cyclic_polytope(4,10)
sage: C = P.combinatorial_polyhedron()
sage: C.is_simple()  # False
sage: P = polytopes.hypercube(4)
sage: C = P.combinatorial_polyhedron()
sage: C.is_simple()  # True
```

Return False for unbounded polyhedra:

```
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],
   →unbounded=True)
sage: C.is_simple()  # False
```

**is_simplex()**

Return whether the polyhedron is a simplex.

A simplex is a bounded polyhedron with $d+1$ vertices, where $d$ is the dimension.

**EXAMPLES:**

```
sage: C = CombinatorialPolyhedron(2).is_simplex()
False
sage: C = CombinatorialPolyhedron([[0,1],[0,2],[1,2]]).is_simplex()
True
```

**is_simplicial()**

Test whether the polytope is simplicial.

This method is not implemented for unbounded polyhedra.

A polytope is simplicial, if each facet contains exactly $d$ vertices, where $d$ is the dimension of the polytope.

**EXAMPLES:**

```
sage: P = polytopes.cyclic_polytope(4,10)
sage: C = P.combinatorial_polyhedron()
sage: C.is_simplicial()  # True
sage: P = polytopes.hypercube(4)
```
```
sage: C = P.combinatorial_polyhedron()
sage: C.is_simplicial()
False
```

For unbounded polyhedra, an error is raised:

```
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],
˓→unbounded=True)
sage: C.is_simplicial()
Traceback (most recent call last):
  ... 
NotImplementedError: this function is implemented for polytopes only
```

**join_of_Vrep(**indices**)

Return the smallest face containing all Vrepresentatives indicated by the indices.

**See also:**

join_of_Vrep().

**EXAMPLES:**

```
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(4)
sage: C = CombinatorialPolyhedron(P)
sage: C.join_of_Vrep(0,1)
A 1-dimensional face of a 3-dimensional combinatorial polyhedron
sage: C.join_of_Vrep(0,11).ambient_V_indices()
(0, 1, 10, 11, 12, 13)
sage: C.join_of_Vrep(8).ambient_V_indices()
(8,)
sage: C.join_of_Vrep().ambient_V_indices()
()
```

**meet_of_Hrep(**indices**)

Return the largest face contained in all facets indicated by the indices.

**See also:**

meet_of_Hrep().

**EXAMPLES:**

```
sage: # needs sage.rings.number_field
sage: P = polytopes.dodecahedron()
sage: C = CombinatorialPolyhedron(P)
sage: C.meet_of_Hrep(0)
A 2-dimensional face of a 3-dimensional combinatorial polyhedron
sage: C.meet_of_Hrep(0,11).ambient_H_indices()
(0,)
sage: C.meet_of_Hrep(0,1).ambient_H_indices()
(0, 1)
sage: C.meet_of_Hrep(0,2).ambient_H_indices()
(0, 2)
sage: C.meet_of_Hrep(0,2,3).ambient_H_indices()
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
sage: C.meet_of_Hrep().ambient_H_indices()
()
```

2.3. Combinatorial Polyhedra
**n_facets()**

Return the number of facets.

Is equivalent to `len(self.facets())`.

**EXAMPLES:**

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: C.n_facets()
6

sage: P = polytopes.cyclic_polytope(4, 20)
sage: C = CombinatorialPolyhedron(P)
sage: C.n_facets()
170

sage: P = Polyhedron(lines=[[0,1]], vertices=[[1,0], [-1,0]])
```

Facets are defined to be the maximal nontrivial faces. The 0-dimensional polyhedron does not have nontrivial faces:

```python
sage: C = CombinatorialPolyhedron(0)
sage: C.f_vector()
(1)
sage: C.n_facets()
0
```

**n_vertices()**

Return the number of vertices.

Is equivalent to `len(self.vertices())`.

**EXAMPLES:**

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: C.n_vertices()
8

sage: P = polytopes.cyclic_polytope(4, 20)
sage: C = CombinatorialPolyhedron(P)
sage: C.n_vertices()
20
```

(continues on next page)
neighborliness()

Return the largest \( k \), such that the polyhedron is \( k \)-neighborly.

A polyhedron is \( k \)-neighborly if every set of \( n \) vertices forms a face for \( n \) up to \( k \).

In case of the \( d \)-dimensional simplex, it returns \( d + 1 \).

See also:

is_neighborly()

EXAMPLES:

```
sage: P = polytopes.cyclic_polytope(8,12)
sage: C = P.combinatorial_polyhedron()
sage: C.neighborliness()
4
sage: P = polytopes.simplex(6)
sage: C = P.combinatorial_polyhedron()
sage: C.neighborliness()
7
sage: P = polytopes.cyclic_polytope(4,10)
sage: P = P.join(P)
sage: C = P.combinatorial_polyhedron()
sage: C.neighborliness()
2
```

polar()

Return the dual/polar of self.

Only defined for bounded polyhedra.

See also:
polar().

EXAMPLES:
```python
sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: D = C.dual()
sage: D.f_vector()
(1, 6, 12, 8, 1)
sage: D1 = P.polar().combinatorial_polyhedron()
sage: D1.face_lattice().is_isomorphic(D.face_lattice())  # Needs sage.combinat
True
```

Polar is an alias to be consistent with `Polyhedron_base`:

```python
sage: C.polar().f_vector()
(1, 6, 12, 8, 1)
```

For unbounded polyhedra, an error is raised:

```python
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],
                                   unbounded=True)
sage: C.dual()
Traceback (most recent call last):
  ...
ValueError: self must be bounded
```

`pyramid(new_vertex=None, new_facet=None)`

Return the pyramid of `self`.

**INPUT:**

- `new_vertex` – (optional); specify a new vertex name to set up the pyramid with vertex names
- `new_facet` – (optional); specify a new facet name to set up the pyramid with facet names

**EXAMPLES:**

```python
sage: C = CombinatorialPolyhedron(((1,2,3),(1,2,4),(1,3,4),(2,3,4)))
sage: C1 = C.pyramid()
sage: C1.facets()
((0, 1, 2, 4), (0, 1, 3, 4), (0, 2, 3, 4), (1, 2, 3, 4), (0, 1, 2, 3))
```

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: C1 = C.pyramid()
sage: P1 = P.pyramid()
sage: C2 = P1.combinatorial_polyhedron()
sage: C2.vertex_facet_graph().is_isomorphic(C1.vertex_facet_graph())  # Needs sage.combinat
True
```

One can specify a name for the new vertex:

```python
sage: P = polytopes.cyclic_polytope(4,10)
sage: C = P.combinatorial_polyhedron()
sage: C1 = C.pyramid(new_vertex='apex')
sage: C1.is_pyramid(certificate=True)
(True, 'apex')
sage: C1.facets()[0]
(A vertex at (0, 0, 0, 0),
```
A vertex at (1, 1, 1, 1),
A vertex at (2, 4, 8, 16),
A vertex at (3, 9, 27, 81),
'apex')

One can specify a name for the new facets:

```python
sage: # needs sage.rings.number_field
sage: P = polytopes.regular_polygon(4)
sage: C = P.combinatorial_polyhedron()
sage: C1 = C.pyramid(new_facet='base')
sage: C1.Hrepresentation()
(An inequality (-1/2, 1/2) x + 1/2 >= 0,
  An inequality (-1/2, -1/2) x + 1/2 >= 0,
  An inequality (1/2, 0.50000000000000000?) x + 1/2 >= 0, 
  'base')
```

For unbounded polyhedra, an error is raised:

```python
sage: C = CombinatorialPolyhedron([[0,1], [0,2]], far_face=[1,2],
                                unbounded=True)
sage: C.pyramid()
Traceback (most recent call last):
  ... ValueError: self must be bounded
```

```python
ridges (add_equations=False, names=True, add_equalities=False, algorithm=None)
```

Return the ridges.

The ridges of a polyhedron are the faces contained in exactly two facets.

To obtain all faces of codimension 1 use `CombinatorialPolyhedron.face_generator()` instead.

The ridges will be given by the facets, they are contained in.

INPUT:

* add_equations – if True, then equations of the polyhedron will be added (only applicable when names is True)
* names – boolean (default: True); if False, then the facets are given by their indices
* algorithm – string (optional); specify whether the face generator starts with facets or vertices: *'primal' – start with the facets *'dual' – start with the vertices *None – choose automatically

**Note:** To compute ridges and f_vector, compute the ridges first. This might be faster.

**EXAMPLES:**

```python
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(2)
sage: C = CombinatorialPolyhedron(P)
sage: C.ridges()
((An inequality (1, 0) x - 1 >= 0, 
  An inequality (-1, 0) x + 2 >= 0),)
sage: C.ridges(add_equations=True)
```
Combinatorial and Discrete Geometry, Release 10.3

((An inequality (1, 0) x - 1 >= 0, An equation (1, 1) x - 3 == 0),
  (An inequality (-1, 0) x + 2 >= 0, An equation (1, 1) x - 3 == 0)),)

sage: P = polytopes.cyclic_polytope(4,5)
sage: C = CombinatorialPolyhedron(P)
sage: C.ridges()
((An inequality (24, -26, 9, -1) x + 0 >= 0,
    An inequality (-50, 35, -10, 1) x + 24 >= 0),
  (An inequality (-12, 19, -8, 1) x + 0 >= 0,
    An inequality (-50, 35, -10, 1) x + 24 >= 0),
  (An inequality (8, -14, 7, -1) x + 0 >= 0,
    An inequality (-50, 35, -10, 1) x + 24 >= 0),
  (An inequality (-6, 11, -6, 1) x + 0 >= 0,
    An inequality (-50, 35, -10, 1) x + 24 >= 0),
  (An inequality (-12, 19, -8, 1) x + 0 >= 0,
    An inequality (24, -26, 9, -1) x + 0 >= 0),
  (An inequality (8, -14, 7, -1) x + 0 >= 0,
    An inequality (24, -26, 9, -1) x + 0 >= 0),
  (An inequality (-6, 11, -6, 1) x + 0 >= 0,
    An inequality (-50, 35, -10, 1) x + 24 >= 0),
  (An inequality (-12, 19, -8, 1) x + 0 >= 0,
    An inequality (-12, 19, -8, 1) x + 0 >= 0),
  (An inequality (-6, 11, -6, 1) x + 0 >= 0,
    An inequality (-6, 11, -6, 1) x + 0 >= 0),
  (An inequality (8, -14, 7, -1) x + 0 >= 0))

sage: C.ridges(names=False)
((3, 4),
 (2, 4),
 (1, 4),
 (0, 4),
 (2, 3),
 (1, 3),
 (0, 3),
 (1, 2),
 (0, 2),
 (0, 1))

sage: P = Polyhedron(rays=[[1,0]])
sage: C = CombinatorialPolyhedron(P)
sage: C
A 1-dimensional combinatorial polyhedron with 1 facet
sage: C.ridges()
()
sage: it = C.face_generator(0)
sage: for face in it: face.ambient_Hrepresentation()
(An inequality (1, 0) x + 0 >= 0, An equation (0, 1) x + 0 == 0)

simpliciality()

Return the largest $k$ such that the polytope is $k$-simplicial.

Return the dimension in case of a simplex.

A polytope is $k$-simplicial, if every $k$-face is a simplex.

EXAMPLES:
simplicity()  
Return the largest $k$ such that the polytope is $k$-simple.  

Return the dimension in case of a simplex.

A polytope $P$ is $k$-simple, if every $(d−1−k)$-face is contained in exactly $k+1$ facets of $P$ for $1 \leq k \leq d−1$. Equivalently it is $k$-simple if the polar/dual polytope is $k$-simplicial.

EXAMPLES:

```python
sage: hyper4 = polytopes.hypersimplex(4,2)
sage: CombinatorialPolyhedron(hyper4).simplicity()
1
sage: hyper5 = polytopes.hypersimplex(5,2)
sage: CombinatorialPolyhedron(hyper5).simplicity()
2
sage: hyper6 = polytopes.hypersimplex(6,2)
sage: CombinatorialPolyhedron(hyper6).simplicity()
3
sage: P = polytopes.simplex(3)
sage: CombinatorialPolyhedron(P).simplicity()
3
sage: P = polytopes.simplex(1)
sage: CombinatorialPolyhedron(P).simplicity()
1
```

vertex_adjacency_matrix(algorithm=None)  
Return the binary matrix of vertex adjacencies.

INPUT:

- `algorithm` – string (optional): specify whether the face generator starts with facets or vertices: *
  - `'primal'` – start with the facets *
  - `'dual'` – start with the vertices *
  - `None` – choose automatically
See also:

vertex_adjacency_matrix().

EXAMPLES:

```
sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: C.vertex_adjacency_matrix()
[0 1 0 1 0 1 0 0]
[1 0 1 0 0 1 0]
[0 1 0 1 0 0 1]
[1 0 1 0 1 0 0]
[0 0 1 0 1 0 1]
[1 0 0 0 1 0 1]
[0 1 0 0 1 0 1]
[0 0 1 0 1 0 1]
```

vertex_facet_graph(names=True)

Return the vertex-facet graph.

This method constructs a directed bipartite graph. The nodes of the graph correspond to elements of the Vrepresentation and facets. There is a directed edge from Vrepresentation to facets for each incidence.

If names is set to False, then the vertices (of the graph) are given by integers.

INPUT:

- names – boolean (default: True); if True label the vertices of the graph by the corresponding names of the Vrepresentation resp. Hrepresentation; if False label the vertices of the graph by integers.

EXAMPLES:

```
sage: P = polytopes.hypercube(2).pyramid()
sage: C = CombinatorialPolyhedron(P)
sage: G = C.vertex_facet_graph(); G
Digraph on 10 vertices
sage: C.Vrepresentation()
(A vertex at (0, -1, -1),
 A vertex at (0, -1, 1),
 A vertex at (0, 1, -1),
 A vertex at (0, 1, 1),
 A vertex at (1, 0, 0))
sage: sorted(G.neighbors_out(C.Vrepresentation()[4]))
[An inequality (-1, -1, 0) x + 1 >= 0,
 An inequality (-1, 0, -1) x + 1 >= 0,
 An inequality (-1, 0, 1) x + 1 >= 0,
 An inequality (-1, 1, 0) x + 1 >= 0]
```

If names is True (the default) but the combinatorial polyhedron has been initialized without specifying names to Vrepresentation and Hrepresentation, then indices of the Vrepresentation and the facets will be used along with a string ‘H’ or ‘V’:

```
sage: C = CombinatorialPolyhedron(P.incidence_matrix())
sage: C.vertex_facet_graph().vertices(sort=True)
[('H', 0),
 ('H', 1),
 (continues on next page)
```
If names is False then the vertices of the graph are given by integers:

```
(sage: C.vertex_facet_graph(names=False).vertices(sort=True)  # needs sage.graphs
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9])
```

**vertex_graph** *(names=True, algorithm=None)*

Return a graph in which the vertices correspond to vertices of the polyhedron, and edges to bounded rank 1 faces.

**INPUT:**

- names – boolean (default: True); if False, then the nodes of the graph are labeled by the indices of the V-representation
- algorithm – string (optional); specify whether the face generator starts with facets or vertices: *'primal' – start with the facets* *'dual' – start with the vertices* *None – choose automatically*

**EXAMPLES:**

```
(sage: P = polytopes.cyclic_polytope(3,5)
sage: C = CombinatorialPolyhedron(P)
sage: G = C.vertex_graph(); G  # needs sage.graphs
Graph on 5 vertices
(sage: sorted(G.degree())  # needs sage.graphs
[3, 3, 4, 4, 4])
```

```
(sage: P = Polyhedron(rays=[[1]])
sage: C = CombinatorialPolyhedron(P)
sage: C.graph()  # needs sage.graphs
Graph on 1 vertex
```

**vertices** *(names=True)*

Return the elements in the V-representation that are vertices.

In case of an unbounded polyhedron, there might be lines and rays in the V-representation.

If names is set to False, then the vertices are given by their indices in the V-representation.

**EXAMPLES:**

```
(sage: P = Polyhedron(rays=[[1,0,0],[0,1,0],[0,0,1]])
sage: C = CombinatorialPolyhedron(P)
sage: C.vertices()
(A vertex at (0, 0, 0),)  # needs sage.graphs
(sage: C.Vrepresentation()
```

(continues on next page)
(A vertex at (0, 0, 0),
A ray in the direction (0, 0, 1),
A ray in the direction (0, 1, 0),
A ray in the direction (1, 0, 0))
sage: P = polytopes.cross_polytope(3)
sage: C = CombinatorialPolyhedron(P)
sage: C.vertices()
(A vertex at (-1, 0, 0),
A vertex at (0, -1, 0),
A vertex at (0, 0, -1),
A vertex at (0, 0, 1),
A vertex at (0, 1, 0),
A vertex at (1, 0, 0))
sage: C.vertices(names=False)
(0, 1, 2, 3, 4, 5)
sage: points = [(1,0,0), (0,1,0), (0,0,1),
....: (-1,0,0), (0,-1,0), (0,0,-1)]
sage: L = LatticePolytope(points)
sage: C = CombinatorialPolyhedron(L)
sage: C.vertices()
(M(1, 0, 0), M(0, 1, 0), M(0, 0, 1), M(-1, 0, 0), M(0, -1, 0), M(0, 0, -1))
sage: C.vertices(names=False)
(0, 1, 2, 3, 4, 5)
sage: P = Polyhedron(vertices=[[0,0]])
sage: C = CombinatorialPolyhedron(P)
sage: C.vertices()
(A vertex at (0, 0),)

2.3.2 Combinatorial face of a polyhedron

This module provides the combinatorial type of a polyhedral face.

See also:
sage.geometry.polyhedron.combinatorial_polyhedron.base,
sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.

EXAMPLES:

Obtain a face from a face iterator:

sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: face = next(it); face
A 2-dimensional face of a 3-dimensional combinatorial polyhedron

Obtain a face from a face lattice index:

sage: P = polytopes.simplex(2)
sage: C = CombinatorialPolyhedron(P)
sage: sorted(C.face_lattice()._elements)
#...

needs sage.combinat
[0, 1, 2, 3, 4, 5, 6, 7]
Obtain further information regarding a face:

```sage
sage: P = polytopes.octahedron()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(2)
sage: face = next(it); face
A 2-dimensional face of a 3-dimensional combinatorial polyhedron

sage: face.ambient_Vrepresentation()
(A vertex at (0, 0, 1), A vertex at (0, 1, 0), A vertex at (1, 0, 0))
sage: face.n_ambient_Vrepresentation()
3
sage: face.ambient_H_indices()
(5,)
sage: face.dimension()
2
sage: face.ambient_dimension()
3
```

See also:

`sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face.CombinatorialFace`

AUTHOR:

• Jonathan Kliem (2019-05)
The dimension of the polyhedron:

```sage```
```
sage: face.ambient_dimension()
5
```

The Vrepresentation:

```sage```
```
sage: face.ambient_Vrepresentation()
(A vertex at (6, 36, 216, 1296, 7776),)
sage: face.ambient_V_indices()
(6,)
sage: face.n_ambient_Vrepresentation()
1
```

The Hrepresentation:

```sage```
```
sage: face.ambient_Hrepresentation()
(An inequality (60, -112, 65, -14, 1) x + 0 >= 0,
An inequality (180, -216, 91, -16, 1) x + 0 >= 0,
An inequality (360, -342, 119, -18, 1) x + 0 >= 0,
An inequality (840, -638, 179, -22, 1) x + 0 >= 0,
An inequality (-2754, 1175, -245, 25, -1) x + 2520 >= 0,
An inequality (504, -450, 145, -20, 1) x + 0 >= 0,
An inequality (-1692, 853, -203, 23, -1) x + 1260 >= 0,
An inequality (252, -288, 113, -18, 1) x + 0 >= 0,
An inequality (-844, 567, -163, 19, -1) x + 0 >= 0)
sage: face.ambient_H_indices()
(3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 19)
sage: face.n_ambient_Hrepresentation()
11
```

**ambient_H_indices** *(add_equations=True)*

Return the indices of the Hrepresentation objects of the ambient polyhedron defining the face.

**INPUT:**

- `add_equations` – boolean (default: True); whether or not to include the equations

**EXAMPLES:**

```sage```
```
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(2)
sage: face = next(it)
sage: face.ambient_H_indices(add_equations=False)
(28, 29)
sage: face2 = next(it)
sage: face2.ambient_H_indices(add_equations=False)
(25, 29)
```

Add the indices of the equation:
Another example:

```python
sage: P = polytopes.cyclic_polytope(4,6)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: _ = next(it); _ = next(it)
sage: next(it).ambient_H_indices()
(0, 1, 2, 4, 5, 7)
sage: next(it).ambient_H_indices()
(0, 1, 5, 6, 7, 8)
sage: next(it).ambient_H_indices()
(0, 1, 2, 3, 6, 8)
sage: [next(it).dimension() for _ in range(2)]
[0, 1]
sage: face = next(it)
sage: face.ambient_H_indices()
(4, 5, 7)
```

See also:

`ambient_Hrepresentation()`.

`ambient_Hrepresentation()`

Return the Hrepresentation objects of the ambient polyhedron defining the face.

It consists of the facets/inequalities that contain the face and the equations defining the ambient polyhedron.

EXAMPLES:

```python
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(2)
sage: next(it).ambient_Hrepresentation()
(An inequality (1, 1, 1, 0, 0) x - 6 >= 0,
 An inequality (0, 0, 0, -1, 0) x + 5 >= 0,
 An equation (1, 1, 1, 1, 1) x - 15 == 0)
sage: next(it).ambient_Hrepresentation()
(An inequality (0, 0, -1, -1, 0) x + 9 >= 0,
 An inequality (0, 0, 0, -1, 0) x + 5 >= 0,
 An equation (1, 1, 1, 1, 1) x - 15 == 0)
sage: P = polytopes.cyclic_polytope(4,6)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: next(it).ambient_Hrepresentation()
(An inequality (-20, 29, -10, 1) x + 0 >= 0,
 An inequality (60, -47, 12, -1) x + 0 >= 0,
 An inequality (30, -31, 10, -1) x + 0 >= 0,
 An inequality (10, -17, 8, -1) x + 0 >= 0,
 An inequality (-154, 71, -14, 1) x + 120 >= 0,
(continues on next page)
```
An inequality \((-78, 49, -12, 1) x + 40 \geq 0\)

\textbf{sage:} \texttt{next(it).ambient_Hrepresentation()}

(An inequality \((-50, 35, -10, 1) x + 24 \geq 0,\)
An inequality \((-12, 19, -8, 1) x + 0 \geq 0,\)
An inequality \((-20, 29, -10, 1) x + 0 \geq 0,\)
An inequality \((60, -47, 12, -1) x + 0 \geq 0,\)
An inequality \((-154, 71, -14, 1) x + 120 \geq 0,\)
An inequality \((-78, 49, -12, 1) x + 40 \geq 0)\)

See also:

\texttt{ambient_H_indices()}. \texttt{ambient_V_indices()}

\textbf{ambient_V_indices()}

Return the indices of the Vrepresentation objects of the ambient polyhedron defining the face.

\textbf{EXAMPLES:}

\texttt{sage: \# needs sage.combinat}
\texttt{sage: P = polytopes.permutahedron(5)}
\texttt{sage: C = CombinatorialPolyhedron(P)}
\texttt{sage: it = C.face_generator(dimension=2)}
\texttt{sage: face = next(it)}
\texttt{sage: next(it).ambient_V_indices()}
(32, 91, 92, 93, 94, 95)
\texttt{sage: next(it).ambient_V_indices()}
(32, 89, 90, 94)

\texttt{sage: C = CombinatorialPolyhedron([[0,1,2],[0,1,3],[0,2,3],[1,2,3]])}
\texttt{sage: it = C.face_generator()}\texttt{ for face in it: (face.dimension(), face.ambient_V_indices())}
(2, (1, 2, 3))
(2, (0, 2, 3))
(2, (0, 1, 3))
(2, (0, 1, 2))
(1, (2, 3))
(1, (1, 3))
(1, (1, 2))
(0, (3,))
(0, (2,))
(0, (1,))
(1, (0, 3))
(1, (0, 2))
(0, (0,))
(1, (0, 1))

See also:

\texttt{ambient_Vrepresentation()}. \texttt{ambient_Vrepresentation()}

\textbf{ambient_Vrepresentation()}

Return the Vrepresentation objects of the ambient polyhedron defining the face.

It consists of the vertices/rays/lines that face contains.

\textbf{EXAMPLES:}
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(dimension=2)
sage: face = next(it)
sage: face.ambient_Vrepresentation()
(A vertex at (1, 3, 2, 5, 4),
A vertex at (2, 3, 1, 5, 4),
A vertex at (3, 1, 2, 5, 4),
A vertex at (3, 2, 1, 5, 4),
A vertex at (2, 1, 3, 5, 4),
A vertex at (1, 2, 3, 5, 4))
sage: face = next(it)
sage: face.ambient_Vrepresentation()
(A vertex at (2, 1, 4, 5, 3),
A vertex at (3, 2, 4, 5, 1),
A vertex at (3, 1, 4, 5, 2),
A vertex at (1, 3, 4, 5, 2),
A vertex at (1, 2, 4, 5, 3),
A vertex at (2, 3, 4, 5, 1))
sage: C = CombinatorialPolyhedron(
    [
        [0, 1, 2],
        [0, 1, 3],
        [0, 2, 3],
        [1, 2, 3]
    ]
)
sage: it = C.face_generator()
sage: for face in it:
    (face.dimension(), face.ambient_Vrepresentation())
(2, (1, 2, 3))
(2, (0, 2, 3))
(2, (0, 1, 3))
(2, (0, 1, 2))
(1, (2, 3))
(1, (1, 3))
(1, (1, 2))
(0, (3,))
(0, (2,))
(0, (1,))
(1, (0, 3))
(1, (0, 2))
(0, (0,))
(1, (0, 1))

See also:

ambient_V_indices().
ambient_dimension()

Return the dimension of the polyhedron.

EXAMPLES:

sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: face = next(it)
sage: face.ambient_dimension()
3

as_combinatorial_polyhedron(quotient=False)

Return self as combinatorial polyhedron.

If quotient is True, return the quotient of the polyhedron by self. Let G be the face corresponding to
self in the dual/polar polytope. The quotient is the dual/polar of G.

Let \([\hat{0}, \hat{1}]\) be the face lattice of the ambient polyhedron and \(F\) be self as element of the face lattice. The face lattice of self as polyhedron corresponds to \([\hat{0}, F]\) and the face lattice of the quotient by self corresponds to \([F, \hat{1}]\).

**EXAMPLES:**

```
sage: P = polytopes.cyclic_polytope(7,11)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(4)
sage: f = next(it); f
A 4-dimensional face of a 7-dimensional combinatorial polyhedron
sage: F = f.as_combinatorial_polyhedron(); F
A 4-dimensional combinatorial polyhedron with 5 facets
sage: F.f_vector()
(1, 5, 10, 10, 5, 1)
sage: F_alt = polytopes.cyclic_polytope(4,5).combinatorial_polyhedron()
sage: F_alt.vertex_facet_graph().is_isomorphic(F.vertex_facet_graph())  # needs sage.graphs
True
```

Obtaining the quotient:

```
sage: Q = f.as_combinatorial_polyhedron(quotient=True); Q
A 2-dimensional combinatorial polyhedron with 6 facets
sage: Q
A 2-dimensional combinatorial polyhedron with 6 facets
sage: Q.f_vector()
(1, 6, 6, 1)
```

The Vrepresentation of the face as polyhedron is given by the ambient Vrepresentation of the face in that order:

```
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(2)
sage: f = next(it)
sage: F = f.as_combinatorial_polyhedron()
sage: C.Vrepresentation()
(A vertex at (1, -1, -1),
A vertex at (1, 1, -1),
A vertex at (1, 1, 1),
A vertex at (1, -1, 1),
A vertex at (-1, -1, 1),
A vertex at (-1, -1, -1),
A vertex at (-1, 1, -1),
A vertex at (-1, 1, 1))
sage: f.ambient_Vrepresentation()
(A vertex at (1, -1, -1),
A vertex at (1, -1, 1),
A vertex at (-1, -1, 1),
A vertex at (-1, -1, -1))
sage: F.Vrepresentation()
(0, 1, 2, 3)
```

To obtain the facets of the face as polyhedron, we compute the meet of each facet with the face. The first representative of each element strictly contained in the face is kept:
The Hrepresentation of the quotient by the face is given by the ambient Hrepresentation of the face in that order:

```
sage: it = C.face_generator(1)
sage: f = next(it)
sage: Q = f.as_combinatorial_polyhedron(quotient=True)
sage: C.Hrepresentation()
((An inequality (-1, 0, 0) x + 1 >= 0, An inequality (0, -1, 0) x + 1 >= 0, An inequality (0, 0, -1) x + 1 >= 0, An inequality (1, 0, 0) x + 1 >= 0, An inequality (0, 0, 1) x + 1 >= 0, An inequality (0, 1, 0) x + 1 >= 0))
sage: f.ambient_Hrepresentation()
((An inequality (0, 0, 1) x + 1 >= 0, An inequality (0, 1, 0) x + 1 >= 0))
sage: Q.Hrepresentation()
((1,))
```

To obtain the vertices of the face as polyhedron, we compute the join of each vertex with the face. The first representative of each element strictly containing the face is kept:

```
sage: [g.ambient_H_indices() for g in C.face_generator(0)]
[(3, 4, 5), (0, 4, 5), (2, 3, 5), (0, 2, 5), (1, 3, 4), (0, 1, 4), (1, 2, 3), (0, 1, 2)]
sage: [g.ambient_H_indices() for g in Q.face_generator(0)]
[(1,), (0,)]
```

The method is not implemented for unbounded polyhedra:

```
sage: P = Polyhedron(rays=[[0,1]])*polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(2)
sage: f = next(it)
sage: f.as_combinatorial_polyhedron()
Traceback (most recent call last):
... NotImplementedError: only implemented for bounded polyhedra
```

REFERENCES:

For more information, see Exercise 2.9 of [Zie2007].
Note: This method is tested in `_test_combinatorial_face_as_combinatorial_polyhedron()`.

**dim()**

Return the dimension of the face.

**EXAMPLES:**

```
sage: # needs sage.combinat
sage: P = polytopes.associahedron(['A', 3])
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: face = next(it)
sage: face.dimension()  # dim is an alias:
sage: face.dim()  # needs sage.combinat
2
```

**dimension()**

Return the dimension of the face.

**EXAMPLES:**

```
sage: # needs sage.combinat
sage: P = polytopes.associahedron(['A', 3])
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: face = next(it)
sage: face.dimension()  # dim is an alias:
sage: face.dim()  # needs sage.combinat
2
```

**is_subface**(other)

Return whether self is contained in other.

**EXAMPLES:**

```
sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: it = C.face_generator()
sage: face = next(it)
sage: face.ambient_V_indices()
(0, 3, 4, 5)
sage: face2 = next(it)
sage: face2.ambient_V_indices()
(0, 1, 5, 6)
sage: face.is_subface(face2)  # (continues on next page)
```
Works for faces of the same combinatorial polyhedron; also from different iterators:

```plaintext
sage: it = C.face_generator(algorithm='dual')
sage: v7 = next(it); v7.ambient_V_indices()
(7,)
sage: v6 = next(it); v6.ambient_V_indices()
(6,)
sage: v5 = next(it); v5.ambient_V_indices()
(5,)
sage: face.ambient_V_indices()
(0, 3, 4, 5)
sage: face.is_subface(v7)
False
sage: v7.is_subface(face)
False
sage: v6.is_subface(face)
False
sage: v5.is_subface(face)
True
sage: face2.ambient_V_indices()
(0, 1, 5, 6)
sage: face2.is_subface(v7)
False
sage: v7.is_subface(face2)
False
sage: v6.is_subface(face2)
True
sage: v5.is_subface(face2)
True
```

Only implemented for faces of the same combinatorial polyhedron:

```plaintext
sage: P1 = polytopes.cube()
sage: C1 = P1.combinatorial_polyhedron()
sage: it = C1.face_generator()
sage: other_face = next(it)
sage: other_face.ambient_V_indices()
(0, 3, 4, 5)
sage: face.ambient_V_indices()
(0, 3, 4, 5)
sage: C is C1
False
sage: face.is_subface(other_face)
Traceback (most recent call last):
```

(continues on next page)
n_ambient_Hrepresentation (add_equations=True)

Return the length of the CombinatorialFace.ambient_H_indices().

Might be faster than then using len.

INPUT:

• add_equations – boolean (default: True); whether or not to count the equations

EXAMPLES:

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: all(face.n_ambient_Hrepresentation() == len(face.ambient_Hrepresentation()) for face in it)
True
```

Specifying whether to count the equations or not:

```python
sage: # needs sage.combinat
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(2)
sage: f = next(it)
sage: f.n_ambient_Hrepresentation(add_equations=True)
3
sage: f.n_ambient_Hrepresentation(add_equations=False)
2
```

n_ambient_Vrepresentation()

Return the length of the CombinatorialFace.ambient_V_indices().

Might be faster than using len.

EXAMPLES:

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: all(face.n_ambient_Vrepresentation() == len(face.ambient_Vrepresentation()) for face in it)
True
```

2.3.3 PolyhedronFaceLattice

This module provides a class that stores and sorts all faces of the polyhedron.

CombinatorialPolyhedron implicitly uses this class to generate the face lattice of a polyhedron.

Terminology in this module:

• Vrep – [vertices, rays, lines] of the polyhedron.
• Hrep – inequalities and equations of the polyhedron.
• Facets – facets of the polyhedron.
• Coatoms – the faces from which all others are constructed in the face iterator. This will be facets or Vrep. In non-dual mode, faces are constructed as intersections of the facets. In dual mode, the are constructed theoretically as joins of vertices. The coatoms are represented as incidences with the atoms they contain.
• Atoms – facets or Vrep depending on application of algorithm. Atoms are represented as incidences of coatoms they are contained in.
• Vrepresentation – represents a face by a list of Vrep it contains.
• Hrepresentation – represents a face by a list of Hrep it is contained in.
• bit representation – represents incidences as uint64_t-array, where each bit represents one incidence. There might be trailing zeros, to fit alignment requirements. In most instances, faces are represented by the bit representation, where each bit corresponds to an atom.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.polyhedron_face_lattice 
....: import PolyhedronFaceLattice
sage: P = polytopes.octahedron()
sage: C = CombinatorialPolyhedron(P)
sage: all_faces = PolyhedronFaceLattice(C)
```

See also:

`base, PolyhedronFaceLattice`

AUTHOR:

• Jonathan Kliem (2019-04)

class sage.geometry.polyhedron.combinatorial_polyhedron.
polyhedron_face_lattice.PolyhedronFaceLattice

Bases: object

A class to generate incidences of `CombinatorialPolyhedron`

On initialization all faces of the given `CombinatorialPolyhedron` are added and sorted (except coatoms). The incidences can be used to generate the `face_lattice`

Might generate the faces of the dual polyhedron for speed.

INPUT:

• baseCombinatorialPolyhedron

See also:

`_record_all_faces(), _record_all_faces_helper(), face_lattice(), _compute_face_lattice_incidences()`.

EXAMPLES:

```python
sage: P = polytopes.Birkhoff_polytope(3)
sage: C = CombinatorialPolyhedron(P)
sage: C._record_all_faces()  # indirect doctest
sage: C.face_lattice()  # needs sage.combinat
Finite lattice containing 50 elements
```

ALGORITHM:
The faces are recorded with `FaceIterator` in Bit-representation. Once created, all level-sets but the coatoms are sorted with merge sort. Non-trivial incidences of elements whose rank differs by 1 are determined by intersecting with all coatoms. Then each intersection is looked up in the sorted level sets.

**dual**

```python
get_face(dimension, index)
```

Return the face of dimension `dimension` and index `index`.

**INPUT:**

- `dimension` – dimension of the face
- `index` – index of the face
- `names` – if True returns the names of the `[vertices, rays, lines]` as given on initialization of `CombinatorialPolyhedron`

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.polyhedron_face_\tensor{\rightarrow}{lattice} \\
....: import PolyhedronFaceLattice
sage: P = polytopes.permutahedron(4)
```
2.3.4 Face iterator for polyhedra

This iterator in principle works on every graded lattice, where every interval of length two has exactly 4 elements (diamond property).

It also works on unbounded polyhedra, as those satisfy the diamond property, except for intervals including the empty face. A (slightly generalized) description of the algorithm can be found in [KS2019].

Terminology in this module:

• Coatoms – the faces from which all others are constructed in the face iterator. This will be facets or Vrep. In non-dual mode, faces are constructed as intersections of the facets. In dual mode, they are constructed theoretically as joins of vertices. The coatoms are represented as incidences with the atoms they contain.

• Atoms – facets or Vrep depending on application of algorithm. Atoms are represented as incidences of coatoms they are contained in.

See also:
sage.geometry.polyhedron.combinatorial_polyhedron.base.

EXAMPLES:

Construct a face iterator:

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator
....: import FaceIterator
sage:
P = polytopes.octahedron()
sage:
C = CombinatorialPolyhedron(P)
sage:
FaceIterator(C, False)
```

Iterator over the proper faces of a 3-dimensional combinatorial polyhedron

```python
sage: FaceIterator(C, False, output_dimension=2)
```

Iterator in the non-dual mode starts with facets:

```python
sage: it = FaceIterator(C, False)
sage: [next(it) for _ in range(9)]
```

Iterator in the dual-mode starts with vertices:

```python
sage: it = FaceIterator(C, True)
sage: [next(it) for _ in range(7)]
```
Obtain the V-representation:

```
sage: it = FaceIterator(C, False)
sage: face = next(it)
sage: face.ambient_Vrepresentation()
(A vertex at (0, -1, 0), A vertex at (0, 0, -1), A vertex at (1, 0, 0))
sage: face.n_ambient_Vrepresentation()
3
```

Obtain the facet-representation:

```
sage: it = FaceIterator(C, True)
sage: face = next(it)
sage: face.ambient_Hrepresentation()
(An inequality (-1, -1, 1) x + 1 >= 0,
  An inequality (-1, -1, -1) x + 1 >= 0,
  An inequality (-1, 1, -1) x + 1 >= 0,
  An inequality (-1, 1, 1) x + 1 >= 0)
sage: face.ambient_H_indices()
(4, 5, 6, 7)
sage: face.n_ambient_Hrepresentation()
4
```

In non-dual mode one can ignore all faces contained in the current face:

```
sage: it = FaceIterator(C, False)
sage: face = next(it)
sage: face.ambient_H_indices()
(7,)
sage: it.ignore_subfaces()
sage: [face.ambient_H_indices() for face in it]
[(6,),
 (5,),
 (4,),
 (3,),
 (2,),
 (1,),
 (0,),
 (5, 6),
 (1, 6),
 (0, 1, 5, 6),
 (4, 5),
 (0, 5),
 (0, 3, 4, 5),
 (3, 4),
 (2, 3),
 (0, 3),
 (0, 1, 2, 3),
 (1, 2),
 (0, 1)]
```

In dual mode one can ignore all faces that contain the current face:

```
sage: it = FaceIterator(C, True)
sage: face = next(it)
sage: face.ambient_V_indices()
(5,)
sage: it.ignore_supfaces()
```

(continues on next page)
sage: [face.ambient_V_indices() for face in it]
[(4,),
 (3,),
 (2,),
 (1,),
 (0,),
 (3, 4),
 (2, 4),
 (0, 4),
 (0, 3, 4),
 (0, 2, 4),
 (1, 3),
 (0, 3),
 (0, 1, 3),
 (1, 2),
 (0, 2),
 (0, 1, 2),
 (0, 1)]

There is a special face iterator class for geometric polyhedra. It yields (geometric) polyhedral faces and it also yields trivial faces. Otherwise, it works exactly the same:

sage: from sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator import FaceIterator_geom
sage: P = polytopes.cube()
sage: it = FaceIterator_geom(P)
sage: [next(it) for _ in range(5)]
[A 3-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 8 vertices,
 A -1-dimensional face of a Polyhedron in ZZ^3,
 A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
 A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
 A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices]

AUTHOR:

• Jonathan Kliem (2019-04)

class sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.FaceIterator

A class to iterate over all combinatorial faces of a polyhedron.

Construct all proper faces from the facets. In dual mode, construct all proper faces from the vertices. Dual will be faster for less vertices than facets.

INPUT:

• C—a CombinatorialPolyhedron
• dual—if True, then dual polyhedron is used for iteration (only possible for bounded Polyhedra)
• output_dimension—if not None, then the face iterator will only yield faces of this dimension
See also:

FaceIterator, FaceIterator_geom, CombinatorialPolyhedron.

EXAMPLES:

Construct a face iterator:

\begin{verbatim}
sage: P = polytopes.cuboctahedron()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()
sage: next(it)
A 0-dimensional face of a 3-dimensional combinatorial polyhedron
\end{verbatim}

Construct faces by the dual or not:

\begin{verbatim}
sage: it = C.face_generator(algorithm='primal')
sage: next(it).dimension()
2
sage: it = C.face_generator(algorithm='dual')
sage: next(it).dimension()
0
\end{verbatim}

For unbounded polyhedra only non-dual iteration is possible:

\begin{verbatim}
sage: P = Polyhedron(rays=[[0,0,1], [0,1,0], [1,0,0]])
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator()

(face.ambient_Vrepresentation() for face in it)
[(A vertex at (0, 0, 0),
  A ray in the direction (0, 1, 0),
  A ray in the direction (1, 0, 0)),
 (A vertex at (0, 0, 0),
  A ray in the direction (0, 0, 1),
  A ray in the direction (1, 0, 0)),
 (A vertex at (0, 0, 0),
  A ray in the direction (0, 0, 1),
  A ray in the direction (0, 1, 0)),
 (A vertex at (0, 0, 0),
  A ray in the direction (1, 0, 0),
  A ray in the direction (0, 0, 0)),
 (A vertex at (0, 0, 0),
  A ray in the direction (0, 0, 1))]
sage: it = C.face_generator(algorithm='dual')
Traceback (most recent call last):
...
ValueError: dual algorithm only available for bounded polyhedra
\end{verbatim}

Construct a face iterator only yielding dimension 2 faces:

\begin{verbatim}
sage: P = polytopes.permutahedron(5)
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(dimension=2)
sage: counter = 0
sage: for _ in it: counter += 1
sage: print ('permutahedron(5) has', counter, 'faces of dimension 2')
permutahedron(5) has 150 faces of dimension 2
\end{verbatim}

(continues on next page)
In non-dual mode one can ignore all faces contained in the current face:

```python
sage: P = polytopes.cube()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(algorithm='primal')
sage: face = next(it)
sage: face.ambient_H_indices()
(5,)
sage: it.ignore_subfaces()
sage: [face.ambient_H_indices() for face in it]
[(4,),
 (3,),
 (2,),
 (1,),
 (0,),
 (3, 4),
 (1, 4),
 (0, 4),
 (1, 3, 4),
 (0, 1, 4),
 (2, 3),
 (1, 3),
 (1, 2, 3),
 (1, 2),
 (0, 2),
 (0, 1, 2),
 (0, 1)]
```

```python
sage: it = C.face_generator(algorithm='dual')
sage: next(it)
A 0-dimensional face of a 3-dimensional combinatorial polyhedron
sage: it.ignore_subfaces()
Traceback (most recent call last):
 ... ValueError: only possible when not in dual mode
```

In dual mode one can ignore all faces that contain the current face:

```python
sage: it = C.face_generator(algorithm='dual')
sage: next(it)
A 0-dimensional face of a 3-dimensional combinatorial polyhedron
sage: face = next(it)
sage: face.ambient_V_indices()
(6,)
sage: [face.ambient_V_indices() for face in it]
[(5,),
 (4,),
 (3,),
 (2,),
 (1,),
 (0,),
 (6, 7),
 (4, 7),
 (2, 7),
```
sage: it = C.face_generator(algorithm='primal')
sage: next(it)
A 2-dimensional face of a 3-dimensional combinatorial polyhedron
sage: it.ignore_supfaces()
Traceback (most recent call last):
... 
ValueError: only possible when in dual mode

**ALGORITHM:**

The algorithm to visit all proper faces exactly once is roughly equivalent to the following. A (slightly generalized) description of the algorithm can be found in [KS2019].

**Initialization:**

```python
faces = [set(facet) for facet in P.facets()]
face_iterator(faces, [])
```

The function `face_iterator` is defined recursively. It visits all faces of the polyhedron \(P\), except those contained in any of `visited_all`. It assumes `faces` to be exactly those facets of \(P\) that are not contained in any of the `visited_all`. It assumes `visited_all` to be some list of faces of a polyhedron \(P_2\), which contains \(P\) as one of its faces:

```python
def face_iterator(faces, visited_all):
    while facets:
        one_face = faces.pop()
        maybe_new_faces = [one_face.intersection(face) for face in faces]
        ...
```

At this point we claim that `maybe_new_faces` contains all facets of `one_face`, which we have not visited before.

**Proof:** Let \(F\) be a facet of `one_face`. We have a chain: \(P \supset one_face \supset F\). By the diamond property, there exists a `second_face` with \(P \supset second_face \supset F\).

Now either `second_face` is not an element of `faces`: Hence `second_face` is contained in one of `visited_all`. In particular, \(F\) is contained in `visited_all`.

Or `second_face` is an element of `faces`: Then, intersecting `one_face` with `second_face` gives \(F\).

This concludes the proof.
Moreover, if an element in `maybe_new_faces` is inclusion-maximal and not contained in any of the `visited_all`, it is a facet of `one_face`. Any facet in `maybe_new_faces` of `one_face` is inclusion-maximal.

Hence, in the following loop, an element `face1` in `maybe_new_faces` is a facet of `one_face` if and only if it is not contained in another facet:

```python
... 
maybe_new_faces2 = [] 
for i, face1 in enumerate(maybe_new_faces):
    if (all(not face1 < face2 for face2 in maybe_new_faces[:i])
        and all(not face1 <= face2 for face2 in maybe_new_faces[i+1:])): 
        maybe_new_faces2.append(face1)
... 
```

Now `maybe_new_faces2` contains only facets of `one_face` and some faces contained in any of `visited_all`. It also contains all the facets not contained in any of `visited_all`.

We construct `new_faces` as the list of all facets of `one_face` not contained in any of `visited_all`:

```python
... 
new_faces = [] 
for face1 in maybe_new_faces2: 
    if all(not face1 < face2 for face2 in visited_all): 
        new_faces.append(face1)
... 
```

By induction we can apply the algorithm, to visit all faces of `one_face` not contained in `visited_all`:

```python
... 
face_iterator(new_faces, visited_all)
... 
```

Finally we visit `one_face` and add it to `visited_all`:

```python
... 
visit(one_face)
visited_all.append(one_face)
... 
```

Note: At this point, we have visited exactly those faces, contained in any of the `visited_all`. The function ends here.

**ALGORITHM** for the special case that all intervals of the lattice not containing zero are boolean (e.g. when the polyhedron is simple):

We do not assume any other properties of our lattice in this case. Note that intervals of length 2 not containing zero, have exactly 2 elements now. But the atom-representation of faces might not be unique.

We do the following modifications:

- To check whether an intersection of faces is zero, we check whether the atom-representation is zero. Although not unique, it works to distinct from zero.
- The intersection of two (relative) facets has always codimension 1 unless empty.
- To intersect we now additionally unite the coatom representation. This gives the correct representation of the new face unless the intersection is zero.
- To mark a face as visited, we save its coatom representation.
- To check whether we have seen a face already, we check containment of the coatom representation.
class sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.
FaceIterator_base

Bases: SageObject

A base class to iterate over all faces of a polyhedron.

Construct all proper faces from the facets. In dual mode, construct all proper faces from the vertices. Dual will be faster for less vertices than facets.

See FaceIterator.

current()

Retrieve the last value of next().

EXAMPLES:

```python
sage: P = polytopes.octahedron()
sage: it = P.combinatorial_polyhedron().face_generator()
sage: next(it)
A 0-dimensional face of a 3-dimensional combinatorial polyhedron
sage: it.current()
A 0-dimensional face of a 3-dimensional combinatorial polyhedron
sage: next(it).ambient_V_indices() == it.current().ambient_V_indices()
True
```

dual

dual_subfaces()

The iterator will not visit any faces of the current face.

Only possible when not in dual mode.

EXAMPLES:

```python
sage: P = polytopes.Gosset_3_21()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(algorithm='primal')
sage: n_non_simplex_faces = 1
sage: for face in it:
....:     if face.n_ambient_Vrepresentation() > face.dimension() + 1:
....:         n_non_simplex_faces += 1
....:     else:
....:         it.ignore_subfaces()
....:
sage: n_non_simplex_faces
127
```

Face iterator must not be in dual mode:

```python
sage: it = C.face_generator(algorithm='dual')
sage: _ = next(it)
sage: it.ignore_subfaces()  # Traceback (most recent call last):
... ValueError: only possible when not in dual mode
```

Ignoring the same face as was requested to visit only consumes the iterator:
Face iterator must be set to a face first:

```python
sage: it = C.face_generator(algorithm='primal')
sage: _ = next(it)
sage: it.only_subfaces()
sage: it.ignore_subfaces()
sage: list(it)
[]
sage: it = C.face_generator(algorithm='primal')
sage: it.ignore_subfaces()
Traceback (most recent call last):
... ValueError: iterator not set to a face yet
```

**ignore_subfaces()**

The iterator will not visit any faces containing the current face.

Only possible when in dual mode.

**EXAMPLES:**

```python
sage: P = polytopes.Gosset_3_21()
sage: C = CombinatorialPolyhedron(P)
sage: it = C.face_generator(algorithm='dual')
sage: n_faces_with_non_simplex_quotient = 1
sage: for face in it:
    ....:     n_facets = face.n_ambient_Hrepresentation(add_equations=False)
    ....:     if n_facets > C.dimension() - face.dimension() + 1:
    ....:         n_faces_with_non_simplex_quotient += 1
    ....:     else:
    ....:         it.ignore_subfaces()
....
sage: n_faces_with_non_simplex_quotient
4845
```

Face iterator must be in dual mode:

```python
sage: it = C.face_generator(algorithm='primal')
sage: _ = next(it)
sage: it.ignore_subfaces()
Traceback (most recent call last):
... ValueError: only possible when in dual mode
```

**join_of_Vrep(**indices**)**

Construct the join of the Vrepresentatives indicated by the indices.

This is the smallest face containing all Vrepresentatives with the given indices.

The iterator must be reset if not newly initialized.

**Note:** In the case of unbounded polyhedra, the smallest face containing given Vrepresentatives may not be well defined.

**EXAMPLES:**
sage: P = polytopes.cube()
sage: it = P.face_generator()
sage: it.join_of_Vrep(1)
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1→vertex
sage: it.join_of_Vrep(1,2).ambient_V_indices()
(1, 2)
sage: it.join_of_Vrep(1,3).ambient_V_indices()
(0, 1, 2, 3)
sage: it.join_of_Vrep(1,5).ambient_V_indices()
(0, 1, 5, 6)
sage: P = polytopes.cross_polytope(4)
sage: it = P.face_generator()
sage: it.join_of_Vrep().ambient_V_indices()
()
sage: it.join_of_Vrep(1,3).ambient_V_indices()
(1, 3)
sage: it.join_of_Vrep(1,2).ambient_V_indices()
(1, 2)
sage: it.join_of_Vrep(1,6).ambient_V_indices()
(0, 1, 2, 3, 4, 5, 6, 7)
sage: it.join_of_Vrep(8)
Traceback (most recent call last):
  ... IndexError: coatoms out of range

If the iterator has already been used, it must be reset before:

sage: # needs sage.rings.number_field
sage: P = polytopes.dodecahedron()
sage: it = P.face_generator()
sage: _, next(it), next(it)
sage: next(it).ambient_V_indices()
(15, 16, 17, 18, 19)
sage: it.join_of_Vrep(1,10)
Traceback (most recent call last):
  ... ValueError: please reset the face iterator
sage: it.reset()
sage: it.join_of_Vrep(1,10).ambient_V_indices()
(1, 10)

In the case of an unbounded polyhedron, we try to make sense of the input:

sage: P = polytopes.cube() * Polyhedron(lines=[[1]])
sage: it = P.face_generator()
sage: it.join_of_Vrep(1)
A 1-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 1→vertex and 1 line
sage: it.join_of_Vrep(0, 1)
A 1-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 1→vertex and 1 line
sage: it.join_of_Vrep(0)
Traceback (most recent call last):
  ... ValueError: the join is not well-defined
sage: P = Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,1]])
sage: it = P.face_generator()
sage: it.join_of_Vrep(0)
A 0-dimensional face of a Polyhedron in QQ^2 defined as the convex hull of 1 vertex
sage: it.join_of_Vrep(1)
A 0-dimensional face of a Polyhedron in QQ^2 defined as the convex hull of 1 vertex
sage: it.join_of_Vrep(2)
Traceback (most recent call last):
  ... ValueError: the join is not well-defined
sage: it.join_of_Vrep(0,2)
A 1-dimensional face of a Polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 ray
sage: P = Polyhedron(rays=[[1,0], [0,1]])
sage: it = P.face_generator()
sage: it.join_of_Vrep(0)
A 0-dimensional face of a Polyhedron in ZZ^2 defined as the convex hull of 1 vertex
sage: it.join_of_Vrep(1,2)
A 2-dimensional face of a Polyhedron in ZZ^2 defined as the convex hull of 1 vertex and 2 rays

meet_of_Hrep(*indices)

Construct the meet of the facets indicated by the indices.

This is the largest face contained in all facets with the given indices.

The iterator must be reset if not newly initialized.

EXAMPLES:

sage: P = polytopes.cube()
sage: it = P.face_generator()
sage: it.meet_of_Hrep(1,2)
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices
sage: it.meet_of_Hrep(1,2).ambient_H_indices()
(1, 2)
sage: it.meet_of_Hrep(1,3).ambient_H_indices()
(1, 3)
sage: it.meet_of_Hrep(1,5).ambient_H_indices()
(0, 1, 2, 3, 4, 5)
sage: P = polytopes.cross_polytope(4)
sage: it = P.face_generator()
sage: it.meet_of_Hrep().ambient_H_indices()
()sage: it.meet_of_Hrep(1,3).ambient_H_indices()
(1, 2, 3, 4)
sage: it.meet_of_Hrep(1,2).ambient_H_indices()
(1, 2)
sage: it.meet_of_Hrep(1,6).ambient_H_indices()
(1, 6)
sage: it.meet_of_Hrep(1,2,6).ambient_H_indices()
(1, 2, 6, 7)
sage: it.meet_of_Hrep(1,2,5,6).ambient_H_indices()
(0, 1, 2, 3, 4, 5, 6, 7)
sage: s = cones.schur(4)
sage: C = CombinatorialPolyhedron(s)
sage: it = C.face_generator()
sage: it.meet_of_Hrep(1,2).ambient_H_indices()
(1, 2)
sage: it.meet_of_Hrep(1,2,3).ambient_H_indices()
Traceback (most recent call last):
... IndexError: coatoms out of range

If the iterator has already been used, it must be reset before:

sage: # needs sage.rings.number_field
sage: P = polytopes.dodecahedron()
sage: it = P.face_generator()
sage: _ = next(it), next(it)
sage: next(it).ambient_V_indices()
(15, 16, 17, 18, 19)
sage: it.meet_of_Hrep(9,11)
Traceback (most recent call last):
... ValueError: please reset the face iterator
sage: it.reset()
sage: it.meet_of_Hrep(9,11).ambient_H_indices()
(9, 11)

next()

Must be implemented by a derived class.

only_subfaces()

The iterator will visit all (remaining) subfaces of the current face and then terminate.

EXAMPLES:

sage: P = polytopes.cube()
sage: it = P.face_generator()
sage: next(it).ambient_H_indices()
()
sage: next(it).ambient_H_indices()
(0, 1, 2, 3, 4, 5)
sage: next(it).ambient_H_indices()
(5,)
sage: next(it).ambient_H_indices()
(4,)
sage: it.only_subfaces()
sage: list(f.ambient_H_indices() for f in it)
[(4, 5), (3, 4), (1, 4), (0, 4), (3, 4, 5), (0, 4, 5), (1, 3, 4), (0, 1, 4)]

sage: P = polytopes.Birkhoff_polytope(4)
sage: C = P.combinatorial_polyhedron()
sage: it = C.face_generator()
Face iterator needs to be set to a face first:

```
sage: it = C.face_generator()
sage: it.only_subfaces()
Traceback (most recent call last):
  ... ValueError: iterator not set to a face yet
```

Face iterator must not be in dual mode:

```
sage: it = C.face_generator(algorithm='dual')
sage: _ = next(it)
sage: it.only_subfaces()
Traceback (most recent call last):
  ... ValueError: only possible when not in dual mode
```

```
only_subfaces()
```

The iterator will visit all (remaining) faces containing the current face and then terminate.

**EXAMPLES:**

```
sage: P = polytopes.cross_polytope(3)
sage: it = P.face_generator()
sage: next(it).ambient_V_indices()
(0, 1, 2, 3, 4, 5)
sage: next(it).ambient_V_indices()
()  
```

```
sage: it.only_subfaces()
sage: list(f.ambient_V_indices() for f in it)
[(4, 5), (3, 4), (2, 4), (0, 4), (3, 4, 5), (2, 4, 5), (0, 3, 4), (0, 2, 4)]
```

```
sage: P = polytopes.Birkhoff_polytope(4)
sage: C = P.combinatorial_polyhedron()
```
sage: it = C.face_generator(algorithm='dual')
sage: next(it).ambient_V_indices()
(23,)
sage: next(it).ambient_V_indices()
(22,)
sage: it.only_supfaces()
sage: all(22 in f.ambient_V_indices() for f in it)
True

reset()

Reset the iterator.

The iterator will start with the first face again.

EXAMPLES:

sage: P = polytopes.cube()
sage: C = P.combinatorial_polyhedron()
sage: it = C.face_generator()
sage: next(it).ambient_V_indices()
(0, 3, 4, 5)
sage: it.reset()
sage: next(it).ambient_V_indices()
(0, 3, 4, 5)

class sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.
FaceIterator_geom

Bases: FaceIterator_base

A class to iterate over all geometric faces of a polyhedron.

Construct all faces from the facets. In dual mode, construct all faces from the vertices. Dual will be faster for less vertices than facets.

INPUT:

- P – an instance of Polyhedron_base
- dual – if True, then dual polyhedron is used for iteration (only possible for bounded Polyhedra)
- output_dimension – if not None, then the FaceIterator will only yield faces of this dimension

EXAMPLES:

Construct a geometric face iterator:

sage: P = polytopes.cuboctahedron()
sage: it = P.face_generator()
sage: next(it)
A 3-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 12...
˓→vertices

Construct faces by the dual or not:

sage: it = P.face_generator(algorithm='primal')
sage: _ = next(it), next(it)
sage: next(it).dim()
2
For unbounded polyhedra only non-dual iteration is possible:

```sage
P = Polyhedron(rays=[[0,0,1], [0,1,0], [1,0,0]])

it = P.face_generator()

[(A vertex at (0, 0, 0),
  A ray in the direction (0, 0, 1),
  A ray in the direction (0, 1, 0),
  A ray in the direction (1, 0, 0)),
 ()],
(A vertex at (0, 0, 0),
  A ray in the direction (0, 1, 0),
  A ray in the direction (1, 0, 0)),
(A vertex at (0, 0, 0),
  A ray in the direction (0, 0, 1),
  A ray in the direction (1, 0, 0)),
(A vertex at (0, 0, 0),
  A ray in the direction (0, 0, 1),
  A ray in the direction (0, 1, 0)),
(A vertex at (0, 0, 0), A ray in the direction (0, 1, 0)),
(A vertex at (0, 0, 0), A ray in the direction (1, 0, 0)),
(A vertex at (0, 0, 0), A ray in the direction (1, 0, 0)),
(A vertex at (0, 0, 0), A ray in the direction (0, 0, 1))]
```

```
sage: it = P.face_generator(algorithm='dual')
Traceback (most recent call last):
  ...:
ValueError: cannot iterate over dual of unbounded Polyhedron
```

Construct a Faceliterator only yielding dimension 2 faces:

```sage
P = polytopes.permutahedron(5)

it = P.face_generator(face_dimension=2)

counter = 0

for _ in it: counter += 1

print ('permutahedron(5) has', counter, 'faces of dimension 2')
```

```
permutahedron(5) has 150 faces of dimension 2
```

In non-dual mode one can ignore all faces contained in the current face:

```sage
P = polytopes.cube()

it = P.face_generator(algorithm='primal')

face = next(it)

face.ambient_H_indices()

(5,)

it.ignore_subfaces()

[(4,),
 (3,),
 (continues on next page)
\begin{verbatim}
(2,),
(1,),
(0,),
(3, 4),
(1, 4),
(0, 4),
(1, 3, 4),
(0, 1, 4),
(2, 3),
(1, 3),
(1, 2, 3),
(1, 2),
(0, 2),
(0, 1, 2),
(0, 1]

sage: it = P.face_generator(algorithm='dual')
sage: _ = next(it), next(it)
sage: next(it)
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1␣
→vertex
sage: it.ignore_subfaces()
Traceback (most recent call last):
...
ValueError: only possible when not in dual mode

In dual mode one can ignore all faces that contain the current face:
\end{verbatim}
sage: it = P.face_generator(algorithm='primal')
sage: _, next(it), next(it)
sage: next(it)
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4
→vertices
sage: it.ignore_supfaces()
Traceback (most recent call last):
... ValueError: only possible when in dual mode

See also:

FaceIterator_base.

P

`current()`

Retrieve the last value of `__next__()`.

EXAMPLES:

sage: P = polytopes.octahedron()
sage: it = P.face_generator()
sage: _, next(it), next(it)
sage: next(it)
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1
→vertex
sage: it.current()
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1
→vertex
sage: next(it).ambient_V_indices() == it.current().ambient_V_indices()
True

`reset()`

Reset the iterator.

The iterator will start with the first face again.

EXAMPLES:

sage: P = polytopes.cube()
sage: it = P.face_generator()

(continues on next page)
2.3.5 List of faces

This module provides a class to store faces of a polyhedron in Bit-representation.

This class allocates memory to store the faces in. A face will be stored as vertex-incidences, where each Bit represents an incidence. In conversions there a methods to actually convert facets of a polyhedron to bit-representations of vertices stored in ListOfFaces.

Moreover, ListOfFaces calculates the dimension of a polyhedron, assuming the faces are the facets of this polyhedron.

Each face is stored over-aligned according to the chunktype.

See also:
sage.geometry.polyhedron.combinatorial_polyhedron.base.

EXAMPLES:

Provide enough space to store 20 faces as incidences to 60 vertices:

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces import ListOfFaces
sage: face_list = ListOfFaces(20, 60, 20)
sage: face_list.matrix().is_zero()
True
```

Obtain the facets of a polyhedron:

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions import incidence_matrix_to_bit_rep_of_facets
sage: P = polytopes.cube()
sage: face_list = incidence_matrix_to_bit_rep_of_facets(P.incidence_matrix())
sage: face_list.compute_dimension()
3
```

Obtain the Vrepresentation of a polyhedron as facet-incidences:

```python
sage: # needs sage.combinat
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions import incidence_matrix_to_bit_rep_of_Vrep
sage: P = polytopes.associahedron([3])
sage: face_list = incidence_matrix_to_bit_rep_of_Vrep(P.incidence_matrix())
sage: face_list.compute_dimension()
3
```

Obtain the facets of a polyhedron as ListOfFaces from a facet list:

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions import facets_tuple_to_bit_rep_of_facets
sage: facets = ((0,1,2), (0,1,3), (0,2,3), (1,2,3))
sage: face_list = facets_tuple_to_bit_rep_of_facets(facets, 4)
```

Likewise for the Vrepresentatives as facet-incidences:
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \
......: import facets_tuple_to_bit_rep_of_Vrep
sage: facets = ((0,1,2), (0,1,3), (0,2,3), (1,2,3))

Obtain the matrix of a list of faces:

sage: face_list = facets_tuple_to_bit_rep_of_Vrep(facets, 4)

sage: face_list.matrix()
[1 1 1 0]
[1 1 0 1]
[1 0 1 1]
[0 1 1 1]

See also:

base, face_iterator, conversions, polyhedron_faces_lattice.

AUTHOR:

• Jonathan Kliem (2019-04)

class

sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces.ListOfFaces

Bases: object

A class to store the Bit-representation of faces in.

This class will allocate the memory for the faces.

INPUT:

• n_faces – the number of faces to be stored
• n_atoms – the total number of atoms the faces contain
• n_coatoms – the total number of coatoms of the polyhedron

See also:


EXAMPLES:

sage: from sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces \ 
......: import ListOfFaces
sage: facets = ListOfFaces(5, 13, 5)
sage: facets.matrix().dimensions()
(5, 13)

compute_dimension()

Compute the dimension of a polyhedron by its facets.

This assumes that self is the list of facets of a polyhedron.

EXAMPLES:

sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
......: import facets_tuple_to_bit_rep_of_facets, \ 
......: facets_tuple_to_bit_rep_of_Vrep
sage: bi_pyr = ((0,1,4), (1,2,4), (2,3,4), (3,0,4),
ALGORITHM:
This is done by iteration:

Computes the facets of one of the facets (i.e. the ridges contained in one of the facets). Then computes the dimension of the facet, by considering its facets.

Repeats until a face has only one facet. Usually this is a vertex.

However, in the unbounded case, this might be different. The face with only one facet might be a ray or a line. So the correct dimension of a polyhedron with one facet is the number of \([\text{lines, rays, vertices}]\) that the facet contains.

Hence, we know the dimension of a face, which has only one facet and iteratively we know the dimension of entire polyhedron we started from.

**matrix()**

Obtain the matrix of self.

Each row represents a face and each column an atom.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \
.....: import facets_tuple_to_bit_rep_of_facets, \n.....: facets_tuple_to_bit_rep_of_Vrep
sage: bi_pyr = ((0,1,4), (1,2,4), (2,3,4), (3,0,4), (0,1,5), (1,2,5), (2,3,5), 
 démarche (3,0,5))
\sage: facets = facets_tuple_to_bit_rep_of_facets(bi_pyr, 6)
sage: Vrep = facets_tuple_to_bit_rep_of_Vrep(bi_pyr, 6)
sage: facets.matrix()
\[
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
\sage: facets.matrix().transpose() == Vrep.matrix()
True
```

**pyramid()**

Return the list of faces of the pyramid.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \
.....: import facets_tuple_to_bit_rep_of_facets
sage: facets = ((0,1,2), (0,1,3), (0,2,3), (1,2,3))
sage: face_list = facets_tuple_to_bit_rep_of_facets(facets, 4)
```
 Incorrect facets that illustrate how this method works:

```plaintext
sage: facets = ((0,1,2,3), (0,1,2,3), (0,1,2,3), (0,1,2,3))
sage: face_list = facets_tuple_to_bit_rep_of_facets(facets, 4)
sage: face_list.matrix()
[1 1 1 1]
[1 1 1 1]
[1 1 1 1]
[1 1 1 1]
sage: face_list.pyramid().matrix()
[1 1 1 1 1]
[1 1 1 1 1]
[1 1 1 1 1]
[1 1 1 1 1]
[1 1 1 1 0]
```

2.3.6 Conversions

This module provides conversions to `ListOfFaces` from - an incidence matrix of a polyhedron or - a tuple of facets (as tuple of vertices each).

Also this module provides a conversion from the data of `ListOfFaces`, which is a Bit-vector representing incidences of a face, to a list of entries which are incident.

See also:

`list_of_faces`, `face_iterator`, `base`.

EXAMPLES:

Obtain the facets of a polyhedron as `ListofFaces`:
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
....:     import incidence_matrix_to_bit_rep_of_facets
sage: P = polytopes.simplex(4)

sage: inc = P.incidence_matrix()

sage: mod_inc = inc.delete_columns([i for i, V in enumerate(P.Hrepresentation()) if V._is_equation()])

sage: face_list = incidence_matrix_to_bit_rep_of_facets(mod_inc)

sage: face_list.compute_dimension()

Obtain the Vrepresentation of a polyhedron as facet-incidences stored in *ListOfFaces*:

sage: # needs sage.combinat
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
....:     import incidence_matrix_to_bit_rep_of_Vrep
sage: P = polytopes.associahedron(['A',4])

sage: face_list = incidence_matrix_to_bit_rep_of_Vrep(P.incidence_matrix())

sage: face_list.compute_dimension()

Obtain the facets of a polyhedron as *ListOfFaces* from a facet list:

sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
....:     import facets_tuple_to_bit_rep_of_facets
sage: facets = ((0,1,2), (0,1,3), (0,2,3), (1,2,3))

sage: face_list = facets_tuple_to_bit_rep_of_facets(facets, 4)

Likewise for the Vrep as facet-incidences:

sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
....:     import facets_tuple_to_bit_rep_of_Vrep
sage: facets = ((0,1,2), (0,1,3), (0,2,3), (1,2,3))

sage: face_list = facets_tuple_to_bit_rep_of_Vrep(facets, 4)

AUTHOR:

- Jonathan Kliem (2019-04)

sage.geometry.polyhedron.combinatorial_polyhedron.conversions.facets_tuple_to_bit_rep_of_Vrep

Initialize Vrepresentatives in Bit-representation as *ListOfFaces*.

Each Vrepresentative is represented as the facets it is contained in. Those are the facets of the polar polyhedron, if it exists.

INPUT:

- facets_input – tuple of facets, each facet a tuple of Vrep, Vrep must be exactly range(n_Vrep)

- n_Vrep

OUTPUT:

- *ListOfFaces*

EXAMPLES:
sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
.....:      import facets_tuple_to_bit_rep_of_Vrep, \
.....:      _bit_rep_to_Vrep_list_wrapper
sage: bi_pyr = ((0,1,4), (1,2,4), (2,3,4), (3,0,4), 
.....:         (0,1,5), (1,2,5), (2,3,5), (3,0,5))
.....:   vertices = facets_tuple_to_bit_rep_of_Vrep(bi_pyr, 6)
sage: for i in range(6):
.....:     print(_bit_rep_to_Vrep_list_wrapper(vertices, i))
(0, 3, 4, 7)
(0, 1, 4, 5)
(1, 2, 5, 6)
(2, 3, 6, 7)
(0, 1, 2, 3)
(4, 5, 6, 7)

sage.geometry.polyhedron.combinatorial_polyhedron.conversions.facets_tuple_to_bit_rep_of_facets.

Initializes facets in Bit-representation as ListOfFaces.

INPUT:

  • facets_input – tuple of facets, each facet a tuple of Vrep, Vrep must be exactly range(n_Vrep)
  • n_Vrep

OUTPUT:

  • ListOfFaces

EXAMPLES:

sage: from sage.geometry.polyhedron.combinatorial_polyhedron.conversions \ 
.....:      import facets_tuple_to_bit_rep_of_facets, \
.....:      _bit_rep_to_Vrep_list_wrapper
sage: bi_pyr = ((0,1,4), (1,2,4), (2,3,4), (3,0,4), 
.....:         (0,1,5), (1,2,5), (2,3,5), (3,0,5))
.....:   facets = facets_tuple_to_bit_rep_of_facets(bi_pyr, 6)
sage: for i in range(8):
.....:     print(_bit_rep_to_Vrep_list_wrapper(facets, i))
(0, 1, 4)
(1, 2, 4)
(2, 3, 4)
(0, 3, 4)
(0, 1, 5)
(1, 2, 5)
(2, 3, 5)
(0, 3, 5)


Initialize Vrepresentatives in Bit-representation as ListOfFaces.

Each Vrepresentative is represented as the facets it is contained in. Those are the facets of the polar polyhedron, if it exists.

INPUT:

  • matrix – an incidence matrix as in sage.geometry.polyhedron.base.Polyhedron_base.
    incidence_matrix() with columns corresponding to equations deleted of type sage.matrix.
Initialize facets in Bit-representation as \texttt{ListOfFaces}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{matrix} – an incidence matrix as in \texttt{sage.geometry.polyhedron.base.Polyhedron_base.incidence_matrix()} with columns corresponding to equations deleted of type \texttt{sage.matrix.matrix_dense.Matrix_dense}
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item \texttt{ListOfFaces}
\end{itemize}

\textbf{EXAMPLES:}

```sage
from sage.geometry.polyhedron.combinatorial_polyhedron.conversions import incidence_matrix_to_bit_rep_of_Vrep, _bit_rep_to_Vrep_list_wrapper
P = polytopes.permutahedron(4)
inc = P.incidence_matrix()
mod_inc = inc.delete_columns([i for i, V in enumerate(P.Hrepresentation()) if V.is_equation()])
vertices = incidence_matrix_to_bit_rep_of_Vrep(mod_inc)
vertices.matrix().dimensions()
(24, 14)
for row in vertices.matrix():
    row.nonzero_positions()
```

```
[8, 9, 11]
[8, 10, 11]
[2, 3, 7]
[1, 5, 7]
[4, 5, 7]
[1, 3, 7]
[4, 6, 7]
[2, 6, 7]
[1, 5, 13]
[8, 9, 13]
[1, 9, 11]
[2, 10, 11]
[1, 3, 11]
[2, 3, 11]
[4, 5, 13]
[4, 12, 13]
[8, 12, 13]
[1, 9, 13]
[0, 8, 12]
[0, 4, 12]
[0, 2, 10]
[0, 2, 6]
[0, 8, 10]
[0, 4, 6]
```
2.4 Polyhedral complexes

2.4.1 Finite polyhedral complexes

This module implements the basic structure of finite polyhedral complexes. For more information, see Polyhedral-Complex.

AUTHORS:

• Yuan Zhou (2021-05): initial implementation
List of PolyhedralComplex methods

Maximal cells and cells

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximal_cells()</td>
<td>Return the dictionary of the maximal cells in this polyhedral complex.</td>
</tr>
<tr>
<td>maximal_cell_iterator()</td>
<td>Return an iterator over maximal cells in this polyhedral complex.</td>
</tr>
<tr>
<td>maximal_cells_sorted()</td>
<td>Return the sorted list of all maximal cells in this polyhedral complex.</td>
</tr>
<tr>
<td>n_maximal_cells()</td>
<td>List the maximal cells of dimension $n$ in this polyhedral complex.</td>
</tr>
<tr>
<td>_n_maximal_cells_sorted()</td>
<td>Return the sorted list of maximal cells of dim $n$ in this complex.</td>
</tr>
<tr>
<td>is_maximal_cell()</td>
<td>Return True if the given cell is a maximal cell in this complex.</td>
</tr>
<tr>
<td>cells()</td>
<td>Return the dictionary of the cells in this polyhedral complex.</td>
</tr>
<tr>
<td>cell_iterator()</td>
<td>Return an iterator over cells in this polyhedral complex.</td>
</tr>
<tr>
<td>cells_sorted()</td>
<td>Return the sorted list of all cells in this polyhedral complex.</td>
</tr>
<tr>
<td>n_cells()</td>
<td>List the cells of dimension $n$ in this polyhedral complex.</td>
</tr>
<tr>
<td>_n_cells_sorted()</td>
<td>Return the sorted list of $n$-cells in this polyhedral complex.</td>
</tr>
<tr>
<td>is_cell()</td>
<td>Return True if the given cell is in this polyhedral complex.</td>
</tr>
<tr>
<td>face_poset()</td>
<td>Return the poset of nonempty cells in the polyhedral complex.</td>
</tr>
<tr>
<td>relative_boundary_cells()</td>
<td>List the maximal cells on the boundary of the polyhedral complex.</td>
</tr>
</tbody>
</table>

Properties of the polyhedral complex

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>dimension()</td>
<td>Return the dimension of the polyhedral complex.</td>
</tr>
<tr>
<td>ambient_dimension()</td>
<td>Return the ambient dimension of the polyhedral complex.</td>
</tr>
<tr>
<td>is_pure()</td>
<td>Return True if the polyhedral complex is pure.</td>
</tr>
<tr>
<td>is_full_dimensional()</td>
<td>Return True if the polyhedral complex is full dimensional.</td>
</tr>
<tr>
<td>is_compact()</td>
<td>Return True if the polyhedral complex is bounded.</td>
</tr>
<tr>
<td>is_connected()</td>
<td>Return True if the polyhedral complex is connected.</td>
</tr>
<tr>
<td>is_subcomplex()</td>
<td>Return True if this complex is a subcomplex of the other.</td>
</tr>
<tr>
<td>is_convex()</td>
<td>Return True if the polyhedral complex is convex.</td>
</tr>
<tr>
<td>is_mutable()</td>
<td>Return True if the polyhedral complex is mutable.</td>
</tr>
<tr>
<td>is_immutable()</td>
<td>Return True if the polyhedral complex is not mutable.</td>
</tr>
<tr>
<td>is_simplicial_complex()</td>
<td>Return True if the polyhedral complex is a simplicial complex.</td>
</tr>
<tr>
<td>is_polyhedral_fan()</td>
<td>Return True if the polyhedral complex is a fan.</td>
</tr>
<tr>
<td>is_simplicial_fan()</td>
<td>Return True if the polyhedral complex is a simplicial fan.</td>
</tr>
</tbody>
</table>

New polyhedral complexes from old ones
<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>connected_component()</code></td>
<td>Return the connected component containing a cell as a subcomplex.</td>
</tr>
<tr>
<td><code>connected_components()</code></td>
<td>Return the connected components of this polyhedral complex.</td>
</tr>
<tr>
<td><code>n_skeleton()</code></td>
<td>Return the $n$-skeleton of this polyhedral complex.</td>
</tr>
<tr>
<td><code>stratify()</code></td>
<td>Return the (pure) subcomplex formed by the maximal cells of dim $n$ in this complex.</td>
</tr>
<tr>
<td><code>boundary_subcomplex()</code></td>
<td>Return the boundary subcomplex of this polyhedral complex.</td>
</tr>
<tr>
<td><code>product()</code></td>
<td>Return the (Cartesian) product of this polyhedral complex with another one.</td>
</tr>
<tr>
<td><code>disjoint_union()</code></td>
<td>Return the disjoint union of this polyhedral complex with another one.</td>
</tr>
<tr>
<td><code>union()</code></td>
<td>Return the union of this polyhedral complex with another one.</td>
</tr>
<tr>
<td><code>join()</code></td>
<td>Return the join of this polyhedral complex with another one.</td>
</tr>
<tr>
<td><code>subdivide()</code></td>
<td>Return a new polyhedral complex (with option <code>make_simplicial</code>) subdividing this one.</td>
</tr>
</tbody>
</table>

**Update polyhedral complex**

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>set_immutable()</code></td>
<td>Make this polyhedral complex immutable.</td>
</tr>
<tr>
<td><code>add_cell()</code></td>
<td>Add a cell to this polyhedral complex.</td>
</tr>
<tr>
<td><code>remove_cell()</code></td>
<td>Remove a cell from this polyhedral complex.</td>
</tr>
</tbody>
</table>

**Miscellaneous**

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>plot()</code></td>
<td>Return a Graphic object showing the plot of polyhedral complex.</td>
</tr>
<tr>
<td><code>graph()</code></td>
<td>Return a directed graph corresponding to the 1-skeleton of this polyhedral complex, given that it is bounded.</td>
</tr>
<tr>
<td><code>union_as_polyhedron()</code></td>
<td>Return a <code>Polyhedron</code> which is the union of cells in this polyhedral complex, given that it is convex.</td>
</tr>
</tbody>
</table>

**Classes and functions**

```python
class sage.geometry.polyhedral_complex.PolyhedralComplex:maximal_cells=None,
backend=None,
maximality_check=True,
face_to_face_check=False,
is Mutable=True,
is Immutable=False,
ambient_dim=None)
```

Bases: `GenericCellComplex`

A polyhedral complex.

A **polyhedral complex** $PC$ is a collection of polyhedra in a certain ambient space $\mathbb{R}^n$ such that the following hold.

- If a polyhedron $P$ is in $PC$, then all the faces of $P$ are in $PC$.
- If polyhedra $P$ and $Q$ are in $PC$, then $P \cap Q$ is either empty or a face of both $P$ and $Q$.

In this context, a “polyhedron” means the geometric realization of a polyhedron. This is in contrast to **simplicial complex**, whose cells are abstract simplices. The concept of a polyhedral complex generalizes that of a **geometric** simplicial complex.
Note: This class derives from `GenericCellComplex`, and so inherits its methods. Some of those methods are not listed here; see the `Generic Cell Complex` page instead.

INPUT:

- `maximal_cells` – a list, a tuple, or a dictionary (indexed by dimension) of cells of the Complex. Each cell is of class `Polyhedron` of the same ambient dimension. To set up a `PolyhedralComplex`, it is sufficient to provide the maximal faces. Use keyword argument `partial=True` to set up a partial polyhedral complex, which is a subset of the faces (viewed as relatively open) of a polyhedral complex that is not necessarily closed under taking intersection.

- `maximality_check` – boolean (default: `True`); if `True`, then the constructor checks that each given maximal cell is indeed maximal, and ignores those that are not

- `face_to_face_check` – boolean (default: `False`); if `True`, then the constructor checks whether the cells are face-to-face, and it raises a `ValueError` if they are not

- `is_mutable` and `is_immutable` – boolean (default: `True` and `False` respectively); set `is_mutable=False` or `is_immutable=True` to make this polyhedral complex immutable

- `backend` – string (optional); the name of the backend used for computations on Sage polyhedra; if it is not given, then each cell has its own backend; otherwise it must be one of the following:
  - `'ppl'` - the Parma Polyhedra Library
  - `'cdd'` - CDD
  - `'normaliz'` - normaliz
  - `'polymake'` - polymake
  - `'field'` - a generic Sage implementation

- `ambient_dim` – integer (optional); used to set up an empty complex in the intended ambient space

EXAMPLES:

```python
sage: pc = PolyhedralComplex([  
....:   Polyhedron(vertices=[(1/3, 1/3), (0, 0), (1/7, 2/7)]),  
....:   Polyhedron(vertices=[(1/7, 2/7), (0, 0), (0, 1/4)])])

sage: [p.Vrepresentation() for p in pc.cells_sorted()]
[(A vertex at (0, 0), A vertex at (0, 1/4), A vertex at (1/7, 2/7)),
 (A vertex at (0, 0), A vertex at (1/3, 1/3), A vertex at (1/7, 2/7)),
 (A vertex at (0, 0), A vertex at (0, 1/4)),
 (A vertex at (0, 0), A vertex at (1/7, 2/7)),
 (A vertex at (0, 0), A vertex at (1/3, 1/3)),
 (A vertex at (0, 1/4), A vertex at (1/7, 2/7)),
 (A vertex at (1/3, 1/3), A vertex at (1/7, 2/7)),
 (A vertex at (0, 0)),
 (A vertex at (0, 1/4)),
 (A vertex at (1/7, 2/7)),
 (A vertex at (1/3, 1/3))]  
sage: pc.plot()  
# needs sage.plot
Graphics object consisting of 10 graphics primitives

sage: pc.is_pure()
True
sage: pc.is_full_dimensional()
True
sage: pc.is_compact()
```

(continues on next page)
add_cell (cell)

Add a cell to this polyhedral complex.

INPUT:

• cell – a polyhedron

This changes the polyhedral complex, by adding a new cell and all of its subfaces.

EXAMPLES:

Set up an empty complex in the intended ambient space, then add a cell:

```
sage: pc = PolyhedralComplex(ambient_dim=2)
sage: pc.add_cell(Polyhedron(vertices=[(1, 2), (0, 2)]))
sage: pc
Polyhedral complex with 1 maximal cell
```

If you add a cell which is already present, there is no effect:

```
sage: pc.add_cell(Polyhedron(vertices=[(1, 2)]))
sage: pc
Polyhedral complex with 1 maximal cell
```

Add a cell and check that dimension is correctly updated:

```
sage: pc.add_cell(Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)]))
sage: pc.dimension()
2
```

Add another cell and check that the properties are correctly updated:

```
sage: pc.add_cell(Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]))
sage: pc
Polyhedral complex with 2 maximal cells
```
sage: len(pc._cells[1])
5
sage: pc._face_poset
Finite poset containing 11 elements
sage: pc._is_convex
True
sage: pc._polyhedron.vertices_list()
[[0, 0], [0, 2], [1, 1], [1, 2]]

Add a ray which makes the complex non convex:

sage: pc.add_cell(Polyhedron(rays=[(1, 0)]))
sage: pc
Polyhedral complex with 3 maximal cells
sage: len(pc._cells[1])
6
sage: (pc._is_convex is False) and (pc._polyhedron is None)
True

alexander_whitney(cell, dim_left)
The decomposition of cell in this complex into left and right factors, suitable for computing cup products.

Todo: Implement alexander_whitney() of a polyhedral complex.

EXAMPLES:

sage: pc = PolyhedralComplex([Polyhedron(vertices=[[0], [1]])])

sage: pc.alexander_whitney(None, 1)
Traceback (most recent call last):
  ...  
NotImplementedError: alexander_whitney is not implemented for polyhedral_complex

ambient_dimension()
The ambient dimension of this cell complex: the ambient dimension of each of its cells.

EXAMPLES:

sage: pc = PolyhedralComplex([Polyhedron(ambient_dim=2)])

sage: pc.ambient_dimension()
2

boundary_subcomplex()
Return the sub-polyhedral complex that is the boundary of self.

A point $P$ is on the boundary of a set $S$ if $P$ is in the closure of $S$ but not in the interior of $S$.

EXAMPLES:
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
sage: bd = PolyhedralComplex([p1, p2]).boundary_subcomplex()
0
sage: len(bd.n_maximal_cells(2))
0
sage: bd.n_maximal_cells(1)
4
sage: pt = PolyhedralComplex([p3])
sage: pt.boundary_subcomplex() == pt
True
Test on polyhedral complex which is not pure:

sage: pc_non_pure = PolyhedralComplex([p1, p3])
sage: pc_non_pure.boundary_subcomplex() == pc_non_pure.n_skeleton(1)
True
Test with maximality_check == False:

sage: pc_invalid = PolyhedralComplex([p2, p3],
.....: maximality_check=False)
sage: pc_invalid.boundary_subcomplex() == pc_invalid.n_skeleton(1)
True
Test unbounded cases:

sage: pc1 = PolyhedralComplex([
.....: Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,0], [0,1]])])
sage: pc1.boundary_subcomplex() == pc1.n_skeleton(1)
True
sage: pc1b = PolyhedralComplex([Polyhedron(
.....: vertices=[[1,0,0], [0,1,0]], rays=[[1,0,0],[0,1,0]])])
sage: pc1b.boundary_subcomplex() == pc1b
True
sage: pc2 = PolyhedralComplex([Polyhedron(vertices=[[-1,0], [1,0]], lines=[[0,1]])])
sage: pc2.boundary_subcomplex() == pc2.n_skeleton(1)
True
sage: pc3 = PolyhedralComplex([Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,0], [0,1]]),
.....: Polyhedron(vertices=[[1,0], [0,-1]], rays=[[1,0], [0,-1]])])
sage: pc3.boundary_subcomplex() == pc3.n_skeleton(1)
False

**cell_iterator** (increasing=True)

An iterator for the cells in this polyhedral complex.

**INPUT:**

- increasing – (default True) if True, return cells in increasing order of dimension, thus starting with the zero-dimensional cells; otherwise it returns cells in decreasing order of dimension

**Note:** Among the cells of a fixed dimension, there is no sorting.

**EXAMPLES:**
```python
sage: pc = PolyhedralComplex([  
    ....: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),
    ....: Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
  ])
sage: len(list(pc.cell_iterator()))
11
```

### cells (subcomplex=None)

The cells of this polyhedral complex, in the form of a dictionary: the keys are integers, representing dimension, and the value associated to an integer \( d \) is the set of \( d \)-cells.

**INPUT:**

- `subcomplex` – (optional) if a subcomplex is given then return the cells which are **not** in this subcomplex

**EXAMPLES:**

```python
sage: pc = PolyhedralComplex([  
    ....: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),
    ....: Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
  ])
sage: list(pc.cells().keys())
[2, 1, 0]
```

### cells_sorted (subcomplex=None)

The sorted list of the cells of this polyhedral complex in non-increasing dimensions.

**INPUT:**

- `subcomplex` – (optional) if a subcomplex is given then return the cells which are **not** in this subcomplex

**EXAMPLES:**

```python
sage: pc = PolyhedralComplex([  
    ....: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),
    ....: Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
  ])
sage: len(pc.cells_sorted())
11
sage: pc.cells_sorted()[0].Vrepresentation()
(A vertex at (0, 0), A vertex at (0, 2), A vertex at (1, 2))
```

### chain_complex (subcomplex=None, augmented=False, verbose=False, check=True, dimensions=None, base_ring=Integer Ring, cochain=False)

The chain complex associated to this polyhedral complex.

**Todo:** Implement chain complexes of a polyhedral complex.

**EXAMPLES:**

```python
sage: pc = PolyhedralComplex([  
    ....: Polyhedron(vertices=[(0), (1)])
  ])
sage: pc.chain_complex()
Traceback (most recent call last):
  ...
NotImplementedError: chain_complex is not implemented for polyhedral complex
```
**connected_component** *(cell=None)*

Return the connected component of this polyhedral complex containing a given cell.

**INPUT:**

- `cell` *(default: self.an_element())* a cell of self

**OUTPUT:**

The connected component containing cell. If the polyhedral complex is empty or if it does not contain the given cell, raise an error.

**EXAMPLES:**

```python
sage: t1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: t2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: v1 = Polyhedron(vertices=[(1, 1)])
sage: v2 = Polyhedron(vertices=[(0, 2)])
sage: v3 = Polyhedron(vertices=[(-1, 0)])
sage: o = Polyhedron(vertices=[(0, 0)])
sage: r = Polyhedron(rays=[(1, 0)])
sage: l = Polyhedron(vertices=[(-1, 0)], lines=[(1, -1)])
sage: pc1 = PolyhedralComplex([t1, t2])
sage: pc1.connected_component() == pc1
True
sage: pc1.connected_component(v1) == pc1
True
sage: pc2 = PolyhedralComplex([t1, v2])
sage: pc2.connected_component(t1) == PolyhedralComplex([t1])
True
sage: pc2.connected_component(o) == PolyhedralComplex([t1])
True
sage: pc2.connected_component(v3)
Traceback (most recent call last):
  ... ValueError: the polyhedral complex does not contain the given cell
sage: pc2.connected_component(r)
Traceback (most recent call last):
  ... ValueError: the polyhedral complex does not contain the given cell
sage: pc3 = PolyhedralComplex([t1, t2, r])
sage: pc3.connected_component(v2) == pc3
True
sage: pc4 = PolyhedralComplex([t1, t2, r, l])
sage: pc4.connected_component(o) == pc3
True
sage: pc4.connected_component(v3)
Traceback (most recent call last):
  ... ValueError: the polyhedral complex does not contain the given cell
sage: pc5 = PolyhedralComplex([t1, t2, r, l, v3])
sage: pc5.connected_component(v3) == PolyhedralComplex([v3])
True
sage: PolyhedralComplex([]).connected_component()
Traceback (most recent call last):
  ... ValueError: the empty polyhedral complex has no connected components
```

**connected_components** ()

Return the connected components of this polyhedral complex, as list of (sub-)PolyhedralComplexes.
EXAMPLES:

```python
sage: t1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: t2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: v1 = Polyhedron(vertices=[(1, 1)])
sage: v2 = Polyhedron(vertices=[(0, 2)])
sage: v3 = Polyhedron(vertices=[(-1, 0)])
sage: o = Polyhedron(vertices=[(0, 0)])
sage: r = Polyhedron(rays=[(1, 0)])
sage: l = Polyhedron(vertices=[(-1, 0)], lines=[(-1, -1)])
sage: pc1 = PolyhedralComplex([t1, t2])
sage: len(pc1.connected_components())
1
sage: pc2 = PolyhedralComplex([t1, v2])
sage: len(pc2.connected_components())
2
sage: pc3 = PolyhedralComplex([t1, t2, r])
sage: len(pc3.connected_components())
1
sage: pc4 = PolyhedralComplex([t1, t2, r, l])
```

```python
sage: pc5 = PolyhedralComplex([t1, t2, r, l, v3])
sage: len(pc5.connected_components())
3
```

```python
sage: PolyhedralComplex([]).connected_components()
Traceback (most recent call last):
...
ValueError: the empty polyhedral complex has no connected components
```

dimension()

The dimension of this cell complex: the maximum dimension of its cells.

EXAMPLES:

```python
sage: pc = PolyhedralComplex([...
       Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),
       Polyhedron(vertices=[(1, 2), (0, 2)]))
sage: pc.dimension()
2
sage: empty_pc = PolyhedralComplex([])
sage: empty_pc.dimension()
-1
```

disjoint_union(right)

The disjoint union of this polyhedral complex with another one.

INPUT:

- `right` — the other polyhedral complex (the right-hand factor)

EXAMPLES:

```python
sage: p1 = Polyhedron(vertices=[(-1, 0), (0, 0), (0, 1)])
sage: p2 = Polyhedron(vertices=[(0, -1), (0, 0), (1, 0)])
sage: p3 = Polyhedron(vertices=[(0, -1), (1, -1), (1, 0)])
sage: pc = PolyhedralComplex([p1]).disjoint_union(PolyhedronComplex([p3]))
sage: set(pc.maximal_cell_iterator()) == set([p1, p3])
True
```

(continues on next page)
sage: pc.disjoint_union(PolyhedralComplex([p2]))
Traceback (most recent call last):
  ...  
ValueError: the two complexes are not disjoint

face_poset()

The face poset of this polyhedral complex, the poset of nonempty cells, ordered by inclusion.

EXAMPLES:

sage: pc = PolyhedralComplex([  
....:     Polyhedron(vertices=[(1/3, 1/3), (0, 0), (1, 2)]),
....:     Polyhedron(vertices=[(1, 2), (0, 0), (0, 1/2)]),

sage: poset = pc.face_poset()
sage: poset
Finite poset containing 11 elements
sage: d = {i: i.vertices_matrix() for i in poset}
sage: poset.plot(element_labels=d)  # needs sage.plot
Graphics object consisting of 28 graphics primitives

For a nonbounded polyhedral complex:

sage: pc = PolyhedralComplex([  
....:     Polyhedron(vertices=[(1/3, 1/3), (0, 0), (1, 2)]),
....:     Polyhedron(vertices=[(1, 2), (0, 0), (0, 1/2)]),
....:     Polyhedron(vertices=[(-1/2, -1/2)], lines=[(1, -1)]),
....:     Polyhedron(rays=[(1, 0)])

sage: poset = pc.face_poset()
sage: poset
Finite poset containing 13 elements
sage: d = {i: ''.join([str(v) + 'n' for v in i.Vrepresentation()]) for i in poset}
sage: poset.show(element_labels=d, figsize=15)  # not tested
sage: pc = PolyhedralComplex([  
....:     Polyhedron(rays=[(1,0),(0,1)]),
....:     Polyhedron(rays=[(-1,0),(0,1)]),
....:     Polyhedron(rays=[(-1,0),(0,-1)]),
....:     Polyhedron(rays=[(1,0),(0,-1)])

sage: pc.face_poset()
Finite poset containing 9 elements

graph()

Return the 1-skeleton of this polyhedral complex, as a graph.

The vertices of the graph are of type vector. This raises a Not Implemented Error if the polyhedral complex is unbounded.

Warning: This may give the wrong answer if the polyhedral complex was constructed with maximality_check set to False.

EXAMPLES:

sage: pc = PolyhedralComplex([  
....:     Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),

(continues on next page)
is_cell(c)

Return whether the given cell c is a cell of self.

EXAMPLES:

```
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
sage: pc = PolyhedralComplex([p1, p2])
sage: pc.is_cell(p3)
True
sage: pc.is_cell(Polyhedron(vertices=[(0, 0)]))
True
```

is_compact()

Test for boundedness of the polyhedral complex

EXAMPLES:

```
sage: p1 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 1/2)])
sage: p2 = Polyhedron(rays=[(1, 0)])
sage: PolyhedralComplex([p1]).is_compact()
True
sage: PolyhedralComplex([p1, p2]).is_compact()
False
```

is_connected()

Return whether self is connected.

EXAMPLES:
sage: pc1 = PolyhedralComplex([  ....: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),  ....: Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)]))
sage: pc1.is_connected()
True
sage: pc2 = PolyhedralComplex([  ....: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),  ....: Polyhedron(vertices=[(0, 2)]))
sage: pc2.is_connected()
False
sage: pc3 = PolyhedralComplex([  ....: Polyhedron(vertices=[(1/3, 1/3), (0, 0), (1, 2)]),  ....: Polyhedron(vertices=[(1, 2), (0, 0), (0, 1/2)]),  ....: Polyhedron(vertices=[(-1/2, -1/2)], lines=[(1, -1)]),  ....: Polyhedron(rays=[(1, 0)])])
sage: pc3.is_connected()
False
sage: pc4 = PolyhedralComplex([  ....: Polyhedron(vertices=[(1/3, 1/3), (0, 0), (1, 2)]),  ....: Polyhedron(rays=[(1, 0)])])
sage: pc4.is_connected()
True

is_convex()

Return whether the set of points in self is a convex set.

When self is convex, the union of its cells is a Polyhedron.

See also:

union_as_polyhedron()

EXAMPLES:

sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(0, 0), (1, 1), (2, 0)])
sage: p4 = Polyhedron(vertices=[(2, 2)])
sage: PolyhedralComplex([p1, p2]).is_convex()
True
sage: PolyhedralComplex([p1, p3]).is_convex()
False
sage: PolyhedralComplex([p1, p4]).is_convex()
False

Test unbounded cases:

sage: pc1 = PolyhedralComplex([  ....: Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,0], [0,1]])])
sage: pc1.is_convex()
True
sage: pc2 = PolyhedralComplex([  ....: Polyhedron(vertices=[[-1,0], [1,0]], lines=[[0,1]])])
sage: pc2.is_convex()
True
sage: pc3 = PolyhedralComplex([  ....: Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,0], [0,1]]),  ....: Polyhedron(vertices=[[1,0], [0,-1]], rays=[[1,0], [0,-1]])])

(continues on next page)
The whole 3d space minus the first orthant is not convex:

```
sage: pc5 = PolyhedralComplex([ Polyhedron(rays=[[1,0,0], [0,1,0], [0,0,-1]]), Polyhedron(rays=[[1,0,0], [0,-1,0], [0,0,1]]), Polyhedron(rays=[[1,0,0], [0,1,0], [0,0,1]]), Polyhedron(rays=[[1,0,0], [0,-1,0], [0,0,-1]]), Polyhedron(rays=[[1,0,0], [0,1,0], [0,0,-1]]), Polyhedron(rays=[[1,0,0], [0,1,0], [0,0,1]])])
sage: pc5.is_convex()
False
```

Tests some non-full-dimensional examples:

```
sage: l = PolyhedralComplex([Polyhedron(vertices=[(1, 2), (0, 2)])])
sage: l.is_convex()
True
sage: pc1b = PolyhedralComplex([Polyhedron(vertices=[[0], [1]])])
sage: pc1b.is_immutable()
False
```

**is_full_dimensional()**

Return whether this polyhedral complex is full-dimensional: its dimension is equal to its ambient dimension.

**EXAMPLES:**

```
sage: p1 = Polyhedron(vertices=[[1, 1], (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[[1, 2], (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[[1, 2], (0, 2)])
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: pc.is_full_dimensional()
True
sage: PolyhedronComplex([p3]).is_full_dimensional()
False
```

**is_immutable()**

Return whether self is immutable.

**EXAMPLES:**

```
sage: pc1 = PolyhedralComplex([Polyhedron(vertices=[[0], [1]])])
sage: pc1.is_immutable()
False
```
\begin{code}
\begin{verbatim}
sage: pc2 = PolyhedralComplex([Polyhedron(\text{vertices}=[[0], [1]])],
                           \hspace{1cm} \text{is\_mutable=True})
sage: pc2.is_immutable()
True
sage: pc3 = PolyhedralComplex([Polyhedron(\text{vertices}=[[0], [1]])],
                           \hspace{1cm} \text{is\_immutable=True})
sage: pc3.is_immutable()
True
\end{verbatim}
\end{code}

\textbf{is\_maximal\_cell}(c)

Return whether the given cell $c$ is a maximal cell of \textit{self}.

\textbf{Warning:} This may give the wrong answer if the polyhedral complex was constructed with \texttt{maximality\_check} set to \texttt{False}.

\textbf{EXAMPLES:}

\begin{code}
\begin{verbatim}
sage: p1 = Polyhedron(\text{vertices}=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(\text{vertices}=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(\text{vertices}=[(1, 2), (0, 2)])
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: pc.is_maximal_cell(p1)
True
sage: pc.is_maximal_cell(p3)
False
Wrong answer due to maximality\_check=False:
\end{verbatim}
\end{code}

\begin{code}
\begin{verbatim}
sage: pc_invalid = PolyhedralComplex([p1, p2, p3],
                                  \hspace{1cm} \text{maximality\_check=True})
sage: pc_invalid.is_maximal_cell(p3)
True
\end{verbatim}
\end{code}

\textbf{is\_mutable}()

Return whether \textit{self} is mutable.

\textbf{EXAMPLES:}

\begin{code}
\begin{verbatim}
sage: pc1 = PolyhedralComplex([Polyhedron(\text{vertices}=[[0], [1]])])
sage: pc1.is\_mutable()
True
sage: pc2 = PolyhedralComplex([Polyhedron(\text{vertices}=[[0], [1]])],
                           \hspace{1cm} \text{is\_mutable=True})
sage: pc2.is\_mutable()
False
sage: pc1 == pc2
True
sage: pc3 = PolyhedralComplex([Polyhedron(\text{vertices}=[[0], [1]])],
                           \hspace{1cm} \text{is\_immutable=True})
sage: pc3.is\_mutable()
False
sage: pc2 == pc3
True
\end{verbatim}
\end{code}
is_polyhedral_fan()
Test if this polyhedral complex is a polyhedral fan.

A polyhedral complex is a fan if all of its (maximal) cells are cones.

EXAMPLES:

```python
sage: p1 = Polyhedron(vertices=[(0, 0), (1, 1), (1, 2)])
sage: p2 = Polyhedron(rays=[(1, 0)])
sage: PolyhedralComplex([p1]).is_polyhedral_fan()
False
sage: PolyhedralComplex([p2]).is_polyhedral_fan()
True
sage: halfplane = Polyhedron(rays=[(1, 0), (-1, 0), (0, 1)])
sage: PolyhedralComplex([halfplane]).is_polyhedral_fan()
True
```

is_pure()
Test if this polyhedral complex is pure.

A polyhedral complex is pure if and only if all of its maximal cells have the same dimension.

**Warning:** This may give the wrong answer if the polyhedral complex was constructed with `maximality_check` set to False.

EXAMPLES:

```python
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: pc.is_pure()
True

Wrong answer due to `maximality_check=False`:

```python
sage: pc_invalid = PolyhedralComplex([p1, p2, p3],
....: maximality_check=False)
sage: pc_invalid.is_pure()
False
```

is_simplicial_complex()
Test if this polyhedral complex is a simplicial complex.

A polyhedral complex is simplicial if all of its (maximal) cells are simplices, i.e., every cell is a bounded polytope with \(d+1\) vertices, where \(d\) is the dimension of the polytope.

EXAMPLES:

```python
sage: p1 = Polyhedron(vertices=[(0, 0), (1, 1), (1, 2)])
sage: p2 = Polyhedron(rays=[(1, 0)])
sage: PolyhedralComplex([p1]).is_simplicial_complex()
True
sage: PolyhedralComplex([p2]).is_simplicial_complex()
False
```
**is_simplicial_fan()**
Test if this polyhedral complex is a simplicial fan.

A polyhedral complex is a **simplicial fan** if all of its (maximal) cells are simplical cones, i.e., every cell is a pointed cone (with vertex being the origin) generated by \( d \) linearly independent rays, where \( d \) is the dimension of the cone.

**EXAMPLES:**

```sage
sage: p1 = Polyhedron(vertices=[(0, 0), (1, 1), (1, 2)])
sage: p2 = Polyhedron(rays=[(1, 0)])
sage: PolyhedralComplex([p1]).is_simplicial_fan()  # False
sage: PolyhedralComplex([p2]).is_simplicial_fan()  # True
sage: halfplane = Polyhedron(rays=[(1, 0), (-1, 0), (0, 1)])
sage: PolyhedralComplex([halfplane]).is_simplicial_fan()  # False
```

**is_subcomplex** (other)
Return whether **self** is a subcomplex of **other**.

**INPUT:**

- **other** – a polyhedral complex

Each maximal cell of **self** must be a cell of **other** for this to be **True**.

**EXAMPLES:**

```sage
sage: p1 = Polyhedron(vertices=[(1/3, 1/3), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 1/2)])
sage: p3 = Polyhedron(vertices=[(0, 0), (1, 0)])
sage: pc = PolyhedralComplex([p1, Polyhedron(vertices=[(1, 0)])])
sage: pc.is_subcomplex(PolyhedralComplex([p1, p2, p3]))  # True
sage: pc.is_subcomplex(PolyhedralComplex([p1, p2]))  # False
```

**join** (right)
The join of this polyhedral complex with another one.

**INPUT:**

- **right** – the other polyhedral complex (the right-hand factor)

**EXAMPLES:**

```sage
sage: pc = PolyhedralComplex([Polyhedron(vertices=[[0], [1]])])
sage: pc_join = pc.join(pc)
sage: pc_join
Polyhedral complex with 1 maximal cell
sage: next(pc_join.maximal_cell_iterator()).vertices()
(A vertex at (0, 0, 0),
 A vertex at (0, 0, 1),
 A vertex at (0, 1, 1),
 A vertex at (1, 0, 0))
```

**maximal_cell_iterator** (increasing=False)
An iterator for the maximal cells in this polyhedral complex.
INPUT:

- increasing – (optional, default False) if True, return maximal cells in increasing order of dimension. Otherwise it returns cells in decreasing order of dimension.

**Note:** Among the cells of a fixed dimension, there is no sorting.

**Warning:** This may give the wrong answer if the polyhedral complex was constructed with `maximality_check` set to False.

**EXAMPLES:**

```python
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: len(list(pc.maximal_cell_iterator())) 2
```

Wrong answer due to `maximality_check=False`:

```python
sage: pc_invalid = PolyhedralComplex([p1, p2, p3],
.....: maximaity_check=False)
sage: len(list(pc_invalid.maximal_cell_iterator())) 3
```

**maximal_cells()**

The maximal cells of this polyhedral complex, in the form of a dictionary: the keys are integers, representing dimension, and the value associated to an integer $d$ is the set of $d$-maximal cells.

**Warning:** This may give the wrong answer if the polyhedral complex was constructed with `maximality_check` set to False.

**EXAMPLES:**

```python
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: len(pc.maximal_cells()[2]) 2
sage: 1 in pc.maximal_cells() False
```

Wrong answer due to `maximality_check=False`:

```python
sage: pc_invalid = PolyhedralComplex([p1, p2, p3],
.....: maximaity_check=False)
sage: len(pc_invalid.maximal_cells()[1]) 1
```
maximal_cells_sorted()

Return the sorted list of the maximal cells of this polyhedral complex by non-increasing dimensions.

EXAMPLES:

```python
sage: pc = PolyhedralComplex([
......: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),
......: Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])])
sage: [p.vertices_list() for p in pc.maximal_cells_sorted()]
[[[0, 0], [0, 2], [1, 2]], [[0, 0], [1, 1], [1, 2]]]
```

n_maximal_cells(n)

List of maximal cells of dimension n of this polyhedral complex.

INPUT:

- n – non-negative integer; the dimension

Note: The resulting list need not be sorted. If you want a sorted list of n-cells, use _n_maximal_cells_sorted().

Warning: This may give the wrong answer if the polyhedral complex was constructed with maximality_check set to False.

EXAMPLES:

```python
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: len(pc.n_maximal_cells(2))
2
sage: len(pc.n_maximal_cells(1))
0
```

Wrong answer due to maximality_check=False:

```python
sage: pc_invalid = PolyhedralComplex([p1, p2, p3],
......: maximality_check=False)
sage: len(pc_invalid.n_maximal_cells(1))
1
```

n_skeleton(n)

The n-skeleton of this polyhedral complex.

The n-skeleton of a polyhedral complex is obtained by discarding all of the cells in dimensions larger than n.

INPUT:

- n – non-negative integer; the dimension

See also:

stratify()

EXAMPLES:
```python
sage: pc = PolyhedralComplex([  
....: Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)]),  
....: Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])])
sage: pc.n_skeleton(2)
Polyhedral complex with 2 maximal cells
sage: pc.n_skeleton(1)
Polyhedral complex with 5 maximal cells
sage: pc.n_skeleton(0)
Polyhedral complex with 4 maximal cells
```

```
plot(**kwds)
```

Return a plot of the polyhedral complex, if it is of dim at most 3.

**INPUT:**

- `explosion_factor` – (default: 0) if positive, separate the cells of the complex by extra space. In this case, the following keyword arguments can be passed to `exploded_plot()`:
  - `center` – (default: None, denoting the origin) the center of explosion
  - `sticky_vertices` – (default: False) boolean or dict. Whether to draw line segments between shared vertices of the given polyhedra. A dict gives options for `sage.plot.line()`.
  - `sticky_center` – (default: True) boolean or dict. When `center` is a vertex of some of the polyhedra, whether to draw line segments connecting the `center` to the shifted copies of these vertices. A dict gives options for `sage.plot.line()`.
- `color` – (default: None) if "rainbow", assign a different color to every maximal cell; otherwise, passed on to `plot()`.
- other keyword arguments are passed on to `plot()`.

**EXAMPLES:**

```
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(0, 0), (0, 2), (-1, 1)])
sage: pc1 = PolyhedralComplex([p1, p2, p3, -p1, -p2, -p3])
sage: bb = dict(xmin=-2, xmax=2, ymin=-3, ymax=3, axes=False)
sage: g0 = pc1.plot(color='rainbow', **bb)  # needs sage.plot
sage: g1 = pc1.plot(explosion_factor=0.5, **bb)  # needs sage.plot
sage: g2 = pc1.plot(explosion_factor=1, color='rainbow', alpha=0.5, **bb)  # needs sage.plot
sage: graphics_array([g0, g1, g2]).show(axes=False)  # not tested
```

```
sage: pc2 = PolyhedralComplex([polytopes.hypercube(3)])
sage: pc3 = pc2.subdivide(new_vertices=[(0, 0, 0)])
sage: g3 = pc3.plot(explosion_factor=1, color='rainbow', alpha=0.5, **bb)  # needs sage.plot
sage: g4 = pc4.plot(explosion_factor=1, center=(1, -1, 1), fill='blue', wireframe='white', point={'color':'red', 'size':10}, alpha=0.6, online=True)  # needs sage.plot
sage: pc5 = PolyhedralComplex([  
....: Polyhedron(inequalities=[(1, 0), (0, 1), (0, 0), (1, 1)]),  
....: Polyhedron(inequalities=[(1, 0), (0, 1), (0, 0), (1, 1)])])
```

(continues on next page)
Polyhedron(rays=[[1,0,0], [0,1,0], [0,0,-1]]),
Polyhedron(rays=[[1,0,0], [0,-1,0], [0,0,-1]]),
Polyhedron(rays=[[1,0,0], [0,-1,0], [0,0,1]]),
Polyhedron(rays=[[-1,0,0], [0,-1,0], [0,0,-1]]),
Polyhedron(rays=[[-1,0,0], [0,1,0], [0,0,-1]]),
Polyhedron(rays=[[-1,0,0], [0,1,0], [0,0,1]])

sage: g5 = pc5.plot(explosion_factor=0.3, color='rainbow', alpha=0.8, #
point={'size': 20}, axes=False, online=True)

product (right)
The (Cartesian) product of this polyhedral complex with another one.

INPUT:
• right – the other polyhedral complex (the right-hand factor)

OUTPUT:
• the product self x right

EXAMPLES:

sage: pc = PolyhedralComplex([Polyhedron(vertices=[[0], [1]])])
sage: pc_square = pc.product(pc)
sage: pc_square
Polyhedral complex with 1 maximal cell

sage: next(pc_square.maximal_cell_iterator()).vertices()
(A vertex at (0, 0),
A vertex at (0, 1),
A vertex at (1, 0),
A vertex at (1, 1))

relative_boundary_cells()
Return the maximal cells of the relative-boundary sub-complex.

A point \( P \) is in the relative boundary of a set \( S \) if \( P \) is in the closure of \( S \) but not in the relative interior of \( S \).

Warning: This may give the wrong answer if the polyhedral complex was constructed with maximality_check set to False.

EXAMPLES:

sage: p1 = Polyhedron(vertices=[[1, 1, 0, 0, (1, 2)]])
sage: p2 = Polyhedron(vertices=[[1, 2, 0, 0, (0, 2)]])
sage: p3 = Polyhedron(vertices=[[1, 2, (0, 2)]])
sage: p4 = Polyhedron(vertices=[[2, 2]])
sage: pc = PolyhedralComplex([p1, p2])
sage: rbd_cells = pc.relative_boundary_cells()
sage: len(rbd_cells)
4
sage: all(p.dimension() == 1 for p in rbd_cells)
True
sage: pc_lower_dim = PolyhedralComplex([p3])
sage: sorted((p.vertices() for p in pc_lower_dim.relative_boundary_cells()))
[(A vertex at (0, 2),), (A vertex at (1, 2),)]
Test on polyhedral complex which is not pure:

```python
sage: pc_non_pure = PolyhedralComplex([p1, p3, p4])
sage: (set(pc_non_pure.relative_boundary_cells())
.....: == set([f.as_polyhedron() for f in p1.faces(1)] + [p3, p4]))
True
```

Test with `maximality_check == False`:

```python
sage: pc_invalid = PolyhedralComplex([p2, p3],
.....: maximality_check=False)
sage: (set(pc_invalid.relative_boundary_cells())
.....: == set([f.as_polyhedron() for f in p2.faces(1)]))
True
```

Test unbounded case:

```python
sage: pc3 = PolyhedralComplex([
.....: Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,0], [0,1]]),
.....: Polyhedron(vertices=[[1,0], [0,-1]], rays=[[1,0], [0,-1]])
]
sage: len(pc3.relative_boundary_cells())
4
```

**remove_cell** (`cell`, `check=False`)

Remove cell from self and all the cells that contain cell as a subface.

**INPUT:**

- `cell` — a cell of the polyhedral complex
- `check` — boolean (default: `False`); if True, raise an error if cell is not a cell of this complex

This does not return anything; instead, it changes the polyhedral complex.

**EXAMPLES:**

If you add a cell which is already present, there is no effect:

```python
sage: p1 = Polyhedron(vertices=[[1, 1], (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[[1, 2], (0, 0), (0, 2)])
sage: r = Polyhedron(rays=[[1, 0]])
sage: pc = PolyhedralComplex([p1, p2, r])
sage: pc.dimension()
2
sage: pc.remove_cell(Polyhedron(vertices=[[0, 0], (1, 2)]))
sage: pc.dimension()
1
```

**set_immutable()**

Make this polyhedral complex immutable.
EXAMPLES:

```python
sage: pc = PolyhedralComplex([Polyhedron(vertices=[[0], [1]])])
sage: pc.is_mutable()
True
sage: pc.set_immutable()
sage: pc.is_mutable()
False
```

`stratify(n)`

Return the pure sub-polyhedral complex which is constructed from the $n$-dimensional maximal cells of this polyhedral complex.

See also:

`n_skeleton()`

**Warning:** This may give the wrong answer if the polyhedral complex was constructed with `maximality_check` set to False.

EXAMPLES:

```python
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(1, 2), (0, 2)])
```

```python
sage: pc = PolyhedralComplex([p1, p2, p3])
sage: pc.stratify(2) == pc
True
sage: pc.stratify(1)
Polyhedral complex with 0 maximal cells
```

Wrong answer due to `maximality_check=False`:

```python
sage: pc_invalid = PolyhedralComplex([p1, p2, p3],
....:                                   maximality_check=False)
sage: pc_invalid.stratify(1)
Polyhedral complex with 1 maximal cell
```

`subdivide(make_simplicial=False, new_vertices=None, new_rays=None)`

Construct a new polyhedral complex by iterative stellar subdivision of `self` for each new vertex/ray given.

Currently, subdivision is only supported for bounded polyhedral complex or polyhedral fan.

**INPUT:**

- `make_simplicial` – boolean (default: False); if True, the returned polyhedral complex is simplicial
- `new_vertices, new_rays` – list (optional); new generators to be added during subdivision

**EXAMPLES:**

```python
sage: square_vertices = [(1, 1, 1), (-1, 1, 1), (-1, -1, 1), (1, -1, 1)]
sage: pc = PolyhedralComplex([
....:     Polyhedron(vertices=[(0, 0, 0)] + square_vertices),
....:     Polyhedron(vertices=[(0, 0, 2)] + square_vertices)])
```

```python
sage: pc.is_compact() and not pc.is_simplicial_complex()
True
```

(continues on next page)
Subdivide a polyhedral complex:

```python
sage: subdivided_pc = pc.subdivide(new_vertices=[(0, 0, 1)])
sage: subdivided_pc
Polyhedral complex with 8 maximal cells
sage: subdivided_pc.is_simplicial_complex()
True
sage: simplicial_pc = pc.subdivide(make_simplicial=True)
sage: simplicial_pc
Polyhedral complex with 4 maximal cells
sage: simplicial_pc.is_simplicial_complex()
True
```

Subdivide a polyhedral fan:

```python
sage: fan = PolyhedralComplex([Polyhedron(rays=square_vertices)])
sage: fan
Polyhedral complex with 4 maximal cells
sage: fan.is_polyhedral_fan() and not fan.is_simplicial_fan()
True
sage: subdivided_fan = fan.subdivide(new_vertices=[(0, 0, 1)])
Traceback (most recent call last):
  ... ValueError: new vertices cannot be used for subdivision
sage: subdivided_fan = fan.subdivide(new_rays=[(0, 0, 1)])
sage: subdivided_fan
Polyhedral complex with 4 maximal cells
sage: subdivided_fan.is_simplicial_fan()
True
sage: simplicial_fan = fan.subdivide(make_simplicial=True)
sage: simplicial_fan
Polyhedral complex with 2 maximal cells
sage: simplicial_fan.is_simplicial_fan()
True
```

Subdivide a polyhedral halfspace:

```python
sage: halfspace = PolyhedralComplex([Polyhedron(rays=[(0, 0, 1)],
....: lines=[(1, 0, 0), (0, 1, 0)])])
sage: halfspace
Polyhedral complex with 2 maximal cells
sage: halfspace.is_simplicial_fan()
False
sage: subdiv_halfspace = halfspace.subdivide(make_simplicial=True)
sage: subdiv_halfspace
Polyhedral complex with 4 maximal cells
sage: subdiv_halfspace.is_simplicial_fan()
True
```

**union** (*right*)

The union of this polyhedral complex with another one.

**INPUT:**

- *right* – the other polyhedral complex (the right-hand factor)

**EXAMPLES:**

```python
sage: p1 = Polyhedron(vertices=[(-1, 0), (0, 0), (0, 1)])
sage: p2 = Polyhedron(vertices=[(0, -1), (0, 0), (1, 0)])
sage: p3 = Polyhedron(verticess=[(0, -1), (1, -1), (1, 0)])
sage: pc = PolyhedralComplex([p1]).union(PolyhedralComplex([p3]))
sage: set(pc.maximal_cell_iterator()) == set([p1, p3])
True
sage: pc.union(PolyhedralComplex([p2]))
Polyhedral complex with 3 maximal cells
sage: p4 = Polyhedron(verticess=[(0, -1), (0, 0), (1, 0), (1, -1)])
```
sage: pc.union(PolyhedralComplex([p4]))
Traceback (most recent call last):
...
ValueError: the given cells are not face-to-face

union_as_polyhedron()

Return self as a Polyhedron if self is convex.

EXAMPLES:

sage: p1 = Polyhedron( vertices=[[1, 1], (0, 0), (1, 2)])
sage: p2 = Polyhedron( vertices=[[1, 2], (0, 0), (0, 2)])
sage: p3 = Polyhedron( vertices=[[0, 0], (1, 1), (2, 0)])
sage: P = PolyhedralComplex([p1, p2]).union_as_polyhedron()
sage: P.vertices_list()
[[0, 0], [0, 2], [1, 1], [1, 2]]
sage: PolyhedralComplex([p1, p3]).union_as_polyhedron()  # the polyhedral complex is not convex

wedge (right)

The wedge (one-point union) of self with right.

Todo: Implement the wedge product of two polyhedral complexes.

EXAMPLES:

sage: pc = PolyhedralComplex([Polyhedron( vertices=[[0], [1]])])
sage: pc.wedge(pc)
Traceback (most recent call last):
...
NotImplementedError: wedge is not implemented for polyhedral complex

sage.geometry.polyhedral_complex.cells_list_to_cells_dict(cells_list)

Helper function that returns the dictionary whose keys are the dimensions, and the value associated to an integer \(d\) is the set of \(d\)-dimensional polyhedra in the given list.

EXAMPLES:

sage: p1 = Polyhedron( vertices=[[1, 1], (0, 0), (1, 2)])
sage: p2 = Polyhedron( vertices=[[1, 1], (0, 0)])
sage: p3 = Polyhedron( vertices=[[0, 0]])
sage: p4 = Polyhedron( vertices=[[1, 1]])
sage: sage.geometry.polyhedral_complex.cells_list_to_cells_dict([p1, p2, p3, p4])
{0: {A 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex, a 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex), 1: {A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices), 2: {A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices}}}

sage.geometry.polyhedral_complex.exploded_plot(polyhedra, center, explosion_factor, sticky_vertices, sticky_center, point, **kwds)

Return a plot of several polyhedra in one figure with extra space between them.

INPUT:
• polyhedra – an iterable of Polyhedron_base objects
• center – (default: None, denoting the origin) the center of explosion
• explosion_factor – (default: 1) a nonnegative number; translate polyhedra by this factor of the distance from center to their center
• sticky_vertices – (default: False) boolean or dict. Whether to draw line segments between shared vertices of the given polyhedra. A dict gives options for sage.plot.line().
• sticky_center – (default: True) boolean or dict. When center is a vertex of some of the polyhedra, whether to draw line segments connecting the center to the shifted copies of these vertices. A dict gives options for sage.plot.line().
• color – (default: None) if "rainbow", assign a different color to every maximal cell and every vertex; otherwise, passed on to plot().
• other keyword arguments are passed on to plot().

EXAMPLES:

```python
sage: from sage.geometry.polyhedral_complex import exploded_plot
sage: p1 = Polyhedron(vertices=[(1, 1), (0, 0), (1, 2)])
sage: p2 = Polyhedron(vertices=[(1, 2), (0, 0), (0, 2)])
sage: p3 = Polyhedron(vertices=[(0, 0), (1, 1), (2, 0)])
sage: exploded_plot([p1, p2, p3])  #...
needs sage.plot
Graphics object consisting of 20 graphics primitives
sage: exploded_plot([p1, p2, p3], center=(1, 1))  #...
needs sage.plot
Graphics object consisting of 19 graphics primitives
sage: exploded_plot([p1, p2, p3], center=(1, 1), sticky_vertices=True)  #...
needs sage.plot
Graphics object consisting of 23 graphics primitives
```

2.5 Toric geometry

2.5.1 Toric lattices

This module was designed as a part of the framework for toric varieties (variety, fano_variety).

All toric lattices are isomorphic to \( \mathbb{Z}^n \) for some \( n \), but will prevent you from doing “wrong” operations with objects from different lattices.

AUTHORS:

• Andrey Novoseltsev (2010-07-30): sublattices and quotients.

EXAMPLES:

The simplest way to create a toric lattice is to specify its dimension only:

```python
sage: N = ToricLattice(3)
sage: N
3-d lattice N
```

While our lattice \( N \) is called exactly “N” it is a coincidence: all lattices are called “N” by default:
sage: another_name = ToricLattice(3)
sage: another_name
3-d lattice N

If fact, the above lattice is exactly the same as before as an object in memory:

sage: N is another_name
True

There are actually four names associated to a toric lattice and they all must be the same for two lattices to coincide:

sage: N, N.dual(), latex(N), latex(N.dual())
\( (3\text{-d lattice } N, 3\text{-d lattice } M, N, M) \)

Notice that the lattice dual to \( N \) is called “M” which is standard in toric geometry. This happens only if you allow completely automatic handling of names:

sage: another_N = ToricLattice(3, "N")
sage: another_N.dual()
3-d lattice N*
sage: N is another_N
False

What can you do with toric lattices? Well, their main purpose is to allow creation of elements of toric lattices:

sage: n = N([1,2,3])
sage: n
N(1, 2, 3)
sage: M = N.dual()
sage: m = M(1,2,3)
sage: m
M(1, 2, 3)

Dual lattices can act on each other:

sage: n * m
14
sage: m * n
14

You can also add elements of the same lattice or scale them:

sage: 2 * n
N(2, 4, 6)
sage: n * 2
N(2, 4, 6)
sage: n + n
N(2, 4, 6)

However, you cannot “mix wrong lattices” in your expressions:

sage: n + m
Traceback (most recent call last):
  ...
TypeError: unsupported operand parent(s) for +:
'3-d lattice N' and '3-d lattice M'
sage: n * n

(continues on next page)
Note that \( n \) and \( m \) are not equal to each other even though they are both “just (1, 2, 3).” Moreover, you cannot easily convert elements between toric lattices:

If you really need to consider elements of one lattice as elements of another, you can either use intermediate conversion to “just a vector”:

Or you can create a homomorphism from one lattice to any other:

\textbf{Warning:} While integer vectors (elements of \( \mathbb{Z}^n \)) are printed as (1, 2, 3), in the code (1, 2, 3) is a tuple, which has nothing to do neither with vectors, nor with toric lattices, so the following is probably not what you want while working with toric geometry objects:

Instead, use syntax like

\textbf{class} \ sage.geometry.toric_lattice.ToricLatticeFactory

\textbf{Warning:} While integer vectors (elements of \( \mathbb{Z}^n \)) are printed as (1, 2, 3), in the code (1, 2, 3) is a tuple, which has nothing to do neither with vectors, nor with toric lattices, so the following is probably not what you want while working with toric geometry objects:
• \texttt{latex\_name} – \texttt{string};
• \texttt{latex\_dual\_name} – \texttt{string}.

OUTPUT:
• lattice.

A toric lattice is uniquely determined by its rank and associated names. There are four such “associated names” whose meaning should be clear from the names of the corresponding parameters, but the choice of default values is a little bit involved. So here is the full description of the “naming algorithm”:

1. If no names were given at all, then this lattice will be called “N” and the dual one “M”. These are the standard choices in toric geometry.
2. If \texttt{name} was given and \texttt{dual\_name} was not, then \texttt{dual\_name} will be \texttt{name} followed by “*”.
3. If LaTeX names were not given, they will coincide with the “usual” names, but if \texttt{dual\_name} was constructed automatically, the trailing star will be typeset as a superscript.

EXAMPIES:
Let’s start with no names at all and see how automatic names are given:

```
sage: L1 = ToricLattice(3)
sage: L1
3-d lattice N
sage: L1.dual()
3-d lattice M
```

If we give the name “N” explicitly, the dual lattice will be called “N*”:

```
sage: L2 = ToricLattice(3, "N")
sage: L2
3-d lattice N
sage: L2.dual()
3-d lattice N*
```

However, we can give an explicit name for it too:

```
sage: L3 = ToricLattice(3, "N", "M")
sage: L3
3-d lattice N
sage: L3.dual()
3-d lattice M
```

If you want, you may also give explicit LaTeX names:

```
sage: L4 = ToricLattice(3, "N", "M", r"\mathbb{N}\), r"\mathbb{M}\)"

\mathbb{N}
\mathbb{M}
```

While all four lattices above are called “N”, only two of them are equal (and are actually the same):

```
sage: L1 == L2
False
sage: L1 == L3
True
sage: L1 is L3
```

(continues on next page)
True

```
sage: L1 == L4
False
```

The reason for this is that \( L_2 \) and \( L_4 \) have different names either for dual lattices or for \( \LaTeX \) typesetting.

```
create_key (rank, name=None, dual_name=None, latex_name=None, latex_dual_name=None)
```

Create a key that uniquely identifies this toric lattice.

See `ToricLattice` for documentation.

**Warning:** You probably should not use this function directly.

```
create_object (version, key)
```

Create the toric lattice described by `key`.

See `ToricLattice` for documentation.

**Warning:** You probably should not use this function directly.

```
class sage.geometry.toric_lattice.ToricLattice_ambient (rank, name, dual_name,
        latex_name, latex_dual_name)
```

Bases: `ToricLattice_generic`, `FreeModule_ambient_pid`

Create a toric lattice.

See `ToricLattice` for documentation.

**Warning:** There should be only one toric lattice with the given rank and associated names. Using this class directly to create toric lattices may lead to unexpected results. Please, use `ToricLattice` to create toric lattices.

```
Element
    alias of ToricLatticeElement
ambient_module ()
    Return the ambient module of `self`.

    OUTPUT:
    • toric lattice.
```

**Note:** For any ambient toric lattice its ambient module is the lattice itself.

**EXAMPLES:**

```
sage: N = ToricLattice(3)
sage: N.ambient_module()
3-d lattice N
sage: N.ambient_module() is N
True
```
dual()

Return the lattice dual to self.

OUTPUT:

• toric lattice.

EXAMPLES:

```python
sage: N = ToricLattice(3)
sage: N
3-d lattice N
sage: M = N.dual()
sage: M
3-d lattice M
sage: M.dual() is N
True
```

Elements of dual lattices can act on each other:

```python
sage: n = N(1,2,3)
sage: m = M(4,5,6)
sage: n * m
32
sage: m * n
32
```

plot(**options)

Plot self.

INPUT:

• any options for toric plots (see toric_plotter.options), none are mandatory.

OUTPUT:

• a plot.

EXAMPLES:

```python
sage: N = ToricLattice(3)
sage: N.plot()  # needs sage.plot
Graphics3d Object
```

class sage.geometry.toric_lattice.ToricLattice_generic(base_ring, rank, degree, sparse=False, coordinate_ring=None, category=None)

Bases: FreeModule_generic_pid

Abstract base class for toric lattices.

Element

alias of ToricLatticeElement

construction()

Return the functorial construction of self.

OUTPUT:
• None, we do not think of toric lattices as constructed from simpler objects since we do not want to perform arithmetic involving different lattices.

**direct_sum** *(other)*

Return the direct sum with other.

**INPUT:**

• other — a toric lattice or more general module.

**OUTPUT:**

The direct sum of self and other as $\mathbb{Z}$-modules. If other is a *ToricLattice*, another toric lattice will be returned.

**EXAMPLES:**

```sage
K = ToricLattice(3, 'K')
L = ToricLattice(3, 'L')
N = K.direct_sum(L); N

6-d lattice K+L

N, N.dual(), latex(N), latex(N.dual())

(6-d lattice K+L, 6-d lattice K^*+L^*, K \oplus L, K^* \oplus L^*)
```

With default names:

```sage
N = ToricLattice(3).direct_sum(ToricLattice(2))
N, N.dual(), latex(N), latex(N.dual())

(5-d lattice N+N, 5-d lattice M+M, N \oplus N, M \oplus M)
```

If other is not a *ToricLattice*, fall back to sum of modules:

```sage
ToricLattice(3).direct_sum(ZZ^2)
Free module of degree 5 and rank 5 over Integer Ring
Echelon basis matrix:
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
```

**intersection** *(other)*

Return the intersection of self and other.

**INPUT:**

• other — a toric (sub)lattice duel.

**OUTPUT:**

• a toric (sub)lattice.

**EXAMPLES:**

```sage
N = ToricLattice(3)
Ns1 = N.submodule([N(2,4,0), N(9,12,0)])
Ns2 = N.submodule([N(1,4,9), N(9,2,0)])
N.intersection(Ns2)
Sublattice <N(54, 12, 0)>
```

Note that if one of the intersecting sublattices is a sublattice of another, no new lattices will be constructed:
quotient (sub, check=True, positive_point=None, positive_dual_point=None, **kwds)

Return the quotient of self by the given sublattice sub.

INPUT:

- sub – sublattice of self;
- check – (default: True) whether or not to check that sub is a valid sublattice.

If the quotient is one-dimensional and torsion free, the following two mutually exclusive keyword arguments are also allowed. They decide the sign choice for the (single) generator of the quotient lattice:

- positive_point – a lattice point of self not in the sublattice sub (that is, not zero in the quotient lattice). The quotient generator will be in the same direction as positive_point.
- positive_dual_point – a dual lattice point. The quotient generator will be chosen such that its lift has a positive product with positive_dual_point. Note: if positive_dual_point is not zero on the sublattice sub, then the notion of positivity will depend on the choice of lift!

Further named arguments are passed to the constructor of a toric lattice quotient.

EXAMPLES:

```python
sage: N = ToricLattice(3)
sage: Ns = N.submodule([N(2,4,0), N(9,12,0)])
sage: Q = N/Ns
sage: Q
Quotient with torsion of 3-d lattice N by Sublattice <N(1, 8, 0), N(0, 12, 0)>
```

Attempting to quotient one lattice by a sublattice of another will result in a ValueError:

```python
sage: N = ToricLattice(3)
sage: M = ToricLattice(3, name='M')
sage: Ms = M.submodule([M(2,4,0), M(9,12,0)])
sage: N.quotient(Ms)
Traceback (most recent call last):
... ValueError: M(1, 8, 0) cannot generate a sublattice of 3-d lattice N
```

However, if we forget the sublattice structure, then it is possible to quotient by vector spaces or modules constructed from any sublattice:

```python
sage: N = ToricLattice(3)
sage: M = ToricLattice(3, name='M')
sage: Ms = M.submodule([M(2,4,0), M(9,12,0)])
sage: N.quotient(Ms.vector_space())
Quotient with torsion of 3-d lattice N by Sublattice <N(1, 8, 0), N(0, 12, 0)>
sage: N.quotient(Ms.sparse_module())
Quotient with torsion of 3-d lattice N by Sublattice <N(1, 8, 0), N(0, 12, 0)>
```
See \texttt{ToricLattice\_quotient} for more examples.

\textbf{saturation()}

Return the saturation of \texttt{self}.

\textbf{OUTPUT:}

\begin{itemize}
  \item a toric lattice.
\end{itemize}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: N = ToricLattice(3)
sage: Ns = N.submodule([[1,2,3], (4,5,6)])
sage: Ns
Sublattice <N(1, 2, 3), N(0, 3, 6)>
sage: Ns_sat = Ns.saturation()
sage: Ns_sat
Sublattice <N(1, 0, -1), N(0, 1, 2)>
sage: Ns_sat is Ns_sat.saturation()
True
\end{verbatim}

\textbf{span}(gens, base\_ring=Integer Ring, *args, **kwds)

Return the span of the given generators.

\textbf{INPUT:}

\begin{itemize}
  \item gens – list of elements of the ambient vector space of \texttt{self}.
  \item base\_ring – (default: \texttt{Z}) base ring for the generated module.
\end{itemize}

\textbf{OUTPUT:}

\begin{itemize}
  \item submodule spanned by gens.
\end{itemize}

\textbf{Note:} The output need not be a submodule of \texttt{self}, nor even of the ambient space. It must, however, be contained in the ambient vector space.

See also \texttt{span\_of\_basis()}, \texttt{submodule()}, and \texttt{submodule\_with\_basis()},

\textbf{EXAMPLES:}

\begin{verbatim}
sage: N = ToricLattice(3)
sage: Ns = N.submodule([N.gen(0)])
sage: Ns.span([N.gen(1)])
Sublattice <N(0, 1, 0)>
sage: Ns.submodule([N.gen(1)])
Traceback (most recent call last):
  ...
ArithmeticError: argument gens (= [N(0, 1, 0)]) does not generate a submodule...
\end{verbatim}

\textbf{span\_of\_basis}(basis, base\_ring=Integer Ring, *args, **kwds)

Return the submodule with the given \texttt{basis}.

\textbf{INPUT:}

\begin{itemize}
  \item basis – list of elements of the ambient vector space of \texttt{self}.
  \item base\_ring – (default: \texttt{Z}) base ring for the generated module.
\end{itemize}

\textbf{OUTPUT:}
• submodule spanned by basis.

Note: The output need not be a submodule of self, nor even of the ambient space. It must, however, be contained in the ambient vector space.

See also span(), submodule(), and submodule_with_basis().

EXAMPLES:

```
sage: N = ToricLattice(3)
sage: Ns = N.span_of_basis([1,2,3])
sage: Ns.span_of_basis([2,4,0])
Sublattice <N(2, 4, 0)>
```

```
sage: Ns.span_of_basis([1/5,2/5,0], (1/7,1/7,0))
Free module of degree 3 and rank 2 over Integer Ring
User basis matrix:
[1/5 2/5 0]
[1/7 1/7 0]
```

Of course the input basis vectors must be linearly independent:

```
sage: Ns.span_of_basis([1,2,0], (2,4,0))
Traceback (most recent call last):
...
ValueError: The given basis vectors must be linearly independent.
```

class sage.geometry.toric_lattice.ToricLattice_quotient(V, W, check=True, positive_point=None, positive_dual_point=None, **kwds)

Bases: FGP_Module_class

Construct the quotient of a toric lattice V by its sublattice W.

INPUT:

• V – ambient toric lattice;
• W – sublattice of V;
• check – (default: True) whether to check correctness of input or not.

If the quotient is one-dimensional and torsion free, the following two mutually exclusive keyword arguments are also allowed. They decide the sign choice for the (single) generator of the quotient lattice:

• positive_point – a lattice point of self not in the sublattice sub (that is, not zero in the quotient lattice). The quotient generator will be in the same direction as positive_point.

• positive_dual_point – a dual lattice point. The quotient generator will be chosen such that its lift has a positive product with positive_dual_point. Note: if positive_dual_point is not zero on the sublattice sub, then the notion of positivity will depend on the choice of lift!

Further given named arguments are passed to the constructor of an FGP module.

OUTPUT:

• quotient of V by W.

EXAMPLES:

The intended way to get objects of this class is to use quotient() method of toric lattices:
Here, `sublattice` happens to be of codimension one in `N`. If you want to prescribe the sign of the quotient generator, you can do either:

```python
sage: Q = N.quotient(sublattice, positive_point=N(0,0,-1)); Q
1-d lattice, quotient of 3-d lattice N by Sublattice <N(1, 0, 1), N(0, 1, -1)>
sage: Q.gens()
(N[1, 0, 0],)
```

or:

```python
sage: M = N.dual()
sage: Q = N.quotient(sublattice, positive_dual_point=M(1,0,0)); Q
1-d lattice, quotient of 3-d lattice N by Sublattice <N(1, 0, 1), N(0, 1, -1)>
sage: Q.gens()
(N[1, 0, 0],)
```

**Element**

alias of `ToricLattice_quotient_element`

**base_extend** (*R*)

Return the base change of `self` to the ring `R`.

**INPUT:**

• `R` – either `ZZ` or `QQ`.

**OUTPUT:**

• `self` if `R = ZZ`, quotient of the base extension of the ambient lattice by the base extension of the sublattice if `R = QQ`.

**EXAMPLES:**

```python
sage: N = ToricLattice(3)
sage: Ns = N.submodule([[2,4,0], [9,12,0]])
sage: Q = N/Ns
sage: Q.base_extend(ZZ) is Q
True
sage: Q.base_extend(QQ)
Vector space quotient V/W of dimension 1 over Rational Field where V: Vector space of dimension 3 over Rational Field W: Vector space of degree 3 and dimension 2 over Rational Field Basis matrix:
[1 0 0]
[0 1 0]
```

**coordinate_vector** (*x*, `reduce=False`)

Return coordinates of `x` with respect to the optimized representation of `self`.

**INPUT:**

• `x` – element of `self` or convertible to `self`
• reduce – (default: False); if True, reduce coefficients modulo invariants

**OUTPUT:**

The coordinates as a vector.

**EXAMPLES:**

```sage
sage: N = ToricLattice(3)
sage: Q = N.quotient(N.span([N(1,2,3), N(0,2,1)]), positive_point=N(0,-1,0))
sage: q = Q.gen(0); q
N[0, -1, 0]
sage: q.vector()  # indirect test
(1)
sage: Q.coordinate_vector(q)
(1)
```

dimension()

Return the rank of self.

**OUTPUT:**

Integer. The dimension of the free part of the quotient.

**EXAMPLES:**

```sage
sage: N = ToricLattice(3)
sage: Ns = N.submodule([N(2,4,0), N(9,12,0)])
sage: Q = N/Ns
sage: Q.ngens()  # direct test
2
sage: Q.rank()  # direct test
1
sage: Ns = N.submodule([N(1,4,0)])
sage: Q = N/Ns
sage: Q.ngens()  # direct test
2
sage: Q.rank()  # direct test
2
```

dual()

Return the lattice dual to self.

**OUTPUT:**

• a toric lattice quotient.

**EXAMPLES:**

```sage
sage: N = ToricLattice(3)
sage: Ns = N.submodule([N(1, -1, -1)])
sage: Q = N / Ns
sage: Q.dual()
Sublattice <M(1, 0, 1), M(0, 1, -1)>
```

gens()

Return the generators of the quotient.

**OUTPUT:**

A tuple of ToricLattice_quotient_element generating the quotient.
EXAMPLES:

```sage
N = ToricLattice(3)
sage: Q = N.quotient(N.span([N(1, 2, 3), N(0, 2, 1)]), positive_point=N(0, -1, 0))
sage: Q.gens()
(N[0, -1, 0],)
```

```python
is_torsion_free()
```

Check if self is torsion-free.

OUTPUT:

- True if self has no torsion and False otherwise.

EXAMPLES:

```sage
N = ToricLattice(3)
sage: Ns = N.submodule([N(2, 4, 0), N(9, 12, 0)])
sage: Q = N/Ns
sage: Q.is_torsion_free()
False
sage: Ns = N.submodule([N(1, 4, 0)])
sage: Q = N/Ns
sage: Q.is_torsion_free()
True
```

```python
rank()
```

Return the rank of self.

OUTPUT:

Integer. The dimension of the free part of the quotient.

EXAMPLES:

```sage
N = ToricLattice(3)
sage: Ns = N.submodule([N(2, 4, 0), N(9, 12, 0)])
sage: Q = N/Ns
sage: Q.ngens()
2
sage: Q.rank()
1
sage: Ns = N.submodule([N(1, 4, 0)])
sage: Q = N/Ns
sage: Q.ngens()
2
sage: Q.rank()
2
```

```python
class sage.geometry.toric_lattice.ToricLattice_quotient_element (parent, x, check=True)
```

Bases: FGP_Element

Create an element of a toric lattice quotient.

**Warning:** You probably should not construct such elements explicitly.
• same as for FGP_Element.

OUTPUT:
• element of a toric lattice quotient.

```
set Immutable()
```

Make self immutable.

OUTPUT:
• none.

Note: Elements of toric lattice quotients are always immutable, so this method does nothing, it is introduced for compatibility purposes only.

EXAMPLES:

```
sage: N = ToricLattice(3)
sage: Ns = N.submodule([N(2,4,0), N(9,12,0)])
sage: Q = N/Ns
sage: Q.0.set Immutable()
```

class sage.geometry.toric_lattice.ToricLattice_sublattice

Construct the sublattice of ambient toric lattice generated by gens.

INPUT (same as for FreeModule_submodule_pid):
• ambient – ambient toric lattice for this sublattice;
• gens – list of elements of ambient generating the constructed sublattice;
• see the base class for other available options.

OUTPUT:
• sublattice of a toric lattice with an automatically chosen basis.

See also ToricLattice_sublattice_with_basis if you want to specify an explicit basis.

EXAMPLES:
The intended way to get objects of this class is to use submodule() method of toric lattices:

```
sage: N = ToricLattice(3)
sage: sublattice = N.submodule([(1,1,0), (3,2,1)])
sage: sublattice.has_user_basis()
False
sage: sublattice.basis()
[ N(1, 0, 1),
  N(0, 1, -1)
]
```

For sublattices without user-specified basis, the basis obtained above is the same as the “standard” one:
sage: sublattice.echelonized_basis()
[  
N(1, 0, 1),
N(0, 1, -1)
]

class sage.geometry.toric_lattice.ToricLattice_sublattice_with_basis

Construct the sublattice of ambient toric lattice with given basis.

INPUT (same as for FreeModule_submodule_with_basis_pid):

• ambient – ambient toric lattice for this sublattice;

• basis – list of linearly independent elements of ambient, these elements will be used as the default basis of the constructed sublattice;

• see the base class for other available options.

OUTPUT:

• sublattice of a toric lattice with a user-specified basis.

See also ToricLattice_sublattice if you do not want to specify an explicit basis.

EXAMPLES:
The intended way to get objects of this class is to use submodule_with_basis() method of toric lattices:

sage: N = ToricLattice(3)
sage: sublattice = N.submodule_with_basis([(1,1,0), (3,2,1)])
sage: sublattice.has_user_basis()
True
sage: sublattice.basis()
[  
N(1, 1, 0),
N(3, 2, 1)
]

Even if you have provided your own basis, you still can access the “standard” one:

sage: sublattice.echelonized_basis()
[  
N(1, 0, 1),
N(0, 1, -1)
]
**dual()**

Return the lattice dual to `self`.

**OUTPUT:**

- a **toric lattice quotient**.

**EXAMPLES:**

```
sage: N = ToricLattice(3)
sage: Ns = N.submodule([ (1,1,0), (3,2,1) ])
sage: Ns.dual()
2-d lattice, quotient of 3-d lattice M by Sublattice <M(1, -1, -1)>
```

**plot(**`**options**)**

Plot `self`.

**INPUT:**

- any options for toric plots (see `toric_plotter.options`), none are mandatory.

**OUTPUT:**

- a plot.

**EXAMPLES:**

```
sage: N = ToricLattice(3)
sage: sublattice = N.submodule_with_basis([ (1,1,0), (3,2,1) ])
sage: sublattice.plot()  # needs sage.plot
Graphics3d Object
```

Now we plot both the ambient lattice and its sublattice:

```
sage: N.plot() + sublattice.plot(point_color="red")  # needs sage.plot
Graphics3d Object
```

**sage.geometry.toric_lattice.is_ToricLattice(x)**

Check if `x` is a toric lattice.

**INPUT:**

- `x` – anything.

**OUTPUT:**

- True if `x` is a toric lattice and False otherwise.

**EXAMPLES:**

```
sage: from sage.geometry.toric_lattice import (sage.geometry.toric_lattice.is_ToricLattice)
......: is_ToricLattice)
sage: is_ToricLattice(1)
False
sage: N = ToricLattice(3)
sage: N
3-d lattice N
sage: is_ToricLattice(N)
True
```
Combinatorial and Discrete Geometry, Release 10.3

sage.geometry.toric_lattice.is_ToricLatticeQuotient(x)

Check if x is a toric lattice quotient.

INPUT:

• x – anything.

OUTPUT:

• True if x is a toric lattice quotient and False otherwise.

EXAMPLES:

sage: from sage.geometry.toric_lattice import (....: is_ToricLatticeQuotient)
sage: is_ToricLatticeQuotient(1)
False
sage: N = ToricLattice(3)
sage: N
3-d lattice N
sage: is_ToricLatticeQuotient(N)
False
sage: Q = N / N.submodule([(1,2,3), (3,2,1)])
sage: Q
Quotient with torsion of 3-d lattice N
by Sublattice <N(1, 2, 3), N(0, 4, 8)>
sage: is_ToricLatticeQuotient(Q)
True

2.5.2 Convex rational polyhedral cones

This module was designed as a part of framework for toric varieties (variety, fano_variety). While the emphasis is on strictly convex cones, non-strictly convex cones are supported as well. Work with distinct lattices (in the sense of discrete subgroups spanning vector spaces) is supported. The default lattice is ToricLattice \( N \) of the appropriate dimension. The only case when you must specify lattice explicitly is creation of a 0-dimensional cone, where dimension of the ambient space cannot be guessed.

AUTHORS:

• Andrey Novoseltsev (2010-06-17): substantial improvement during review by Volker Braun.
• Volker Braun (2010-06-21): various spanned/quotient/dual lattice computations added.
• Volker Braun (2010-12-28): Hilbert basis for cones.
• Andrey Novoseltsev (2012-02-23): switch to PointCollection container.

EXAMPLES:

Use \( \text{Cone()} \) to construct cones:

sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
sage: halfspace = Cone([(1,0,0), (0,1,0), (-1,-1,0), (0,0,1)])
sage: positive_xy = Cone([(1,0,0), (0,1,0)])
sage: four_rays = Cone([(1,1,1), (1,-1,1), (-1,-1,1), (-1,1,1)])

For all of the cones above we have provided primitive generating rays, but in fact this is not necessary - a cone can be constructed from any collection of rays (from the same space, of course). If there are non-primitive (or even non-integral) rays, they will be replaced with primitive ones. If there are extra rays, they will be discarded. Of course, this means that
Combinatorial and Discrete Geometry, Release 10.3

Cone() has to do some work before actually constructing the cone and sometimes it is not desirable, if you know for sure that your input is already “good”. In this case you can use options check=False to force Cone() to use exactly the directions that you have specified and normalize=False to force it to use exactly the rays that you have specified. However, it is better not to use these possibilities without necessity, since cones are assumed to be represented by a minimal set of primitive generating rays. See Cone() for further documentation on construction.

Once you have a cone, you can perform numerous operations on it. The most important ones are, probably, ray accessing methods:

```sage
sage: rays = halfspace.rays()
sage: rays
N( 0, 0, 1),
N( 0, 1, 0),
N( 0, -1, 0),
N( 1, 0, 0),
N(-1, 0, 0)
in 3-d lattice N
sage: rays.set()
frozenset({N(-1, 0, 0), N(0, -1, 0), N(0, 0, 1), N(0, 1, 0), N(1, 0, 0)})
sage: rays.matrix()
[ 0 0 1]
[ 0 1 0]
[ 0 -1 0]
[ 1 0 0]
[ 1 0 0]
sage: rays.column_matrix()
[ 0 0 0 1 -1]
[ 0 1 -1 0 0]
[ 1 0 0 0 0]
sage: rays(3)
N(1, 0, 0)
in 3-d lattice N
sage: rays[3]
N(1, 0, 0)
sage: halfspace.ray(3)
N(1, 0, 0)
```

The method rays() returns a PointCollection with the i-th element being the primitive integral generator of the i-th ray. It is possible to convert this collection to a matrix with either rows or columns corresponding to these generators. You may also change the default output_format() of all point collections to be such a matrix.

If you want to do something with each ray of a cone, you can write

```sage
sage: for ray in positive_xy: print(ray)
N(1, 0, 0)
N(0, 1, 0)
```

There are two dimensions associated to each cone - the dimension of the subspace spanned by the cone and the dimension of the space where it lives:

```sage
sage: positive_xy.dim()
2
sage: positive_xy.lattice_dim()
3
```

You also may be interested in this dimension:

2.5. Toric geometry
sage: dim(positive_xy.linear_subspace())
0
sage: dim(halfspace.linear_subspace())
2

Or, perhaps, all you care about is whether it is zero or not:

sage: positive_xy.is_strictly_convex()
True
sage: halfspace.is_strictly_convex()
False

You can also perform these checks:

sage: positive_xy.is_simplicial()
True
sage: four_rays.is_simplicial()
False
sage: positive_xy.is_smooth()
True

You can work with subcones that form faces of other cones:

sage: # needs sage.graphs
sage: face = four_rays.faces(dim=2)[0]
sage: face
2-d face of 3-d cone in 3-d lattice N
sage: face.rays()
N(-1, -1, 1),
N(-1, 1, 1)
in 3-d lattice N
sage: face.ambient_ray_indices()
(2, 3)
sage: four_rays.rays(face.ambient_ray_indices())
N(-1, -1, 1),
N(-1, 1, 1)
in 3-d lattice N

If you need to know inclusion relations between faces, you can use

sage: # needs sage.graphs
sage: L = four_rays.face_lattice()
sage: [len(s) for s in L.level_sets()]
[1, 4, 4, 1]
sage: face = L.level_sets()[2][0]
sage: face.rays()
N(1, 1, 1),
N(1, -1, 1)
in 3-d lattice N
sage: L.hasse_diagram().neighbors_in(face)
[1-d face of 3-d cone in 3-d lattice N,
1-d face of 3-d cone in 3-d lattice N]

Warning: The order of faces in level sets of the face lattice may differ from the order of faces returned by faces(). While the first order is random, the latter one ensures that one-dimensional faces are listed in the same order as generating rays.
When all the functionality provided by cones is not enough, you may want to check if you can do necessary things using polyhedra corresponding to cones:

```sage
sage: four_rays.polyhedron()
A 3-dimensional polyhedron in ZZ^3 defined as
the convex hull of 1 vertex and 4 rays
```

And of course you are always welcome to suggest new features that should be added to cones!

REFERENCES:

- [Ful1993]

`sage.geometry.cone.Cone` *(rays, lattice=None, check=True, normalize=True)*

Construct a (not necessarily strictly) convex rational polyhedral cone.

INPUT:

- **rays** – a list of rays. Each ray should be given as a list or a vector convertible to the rational extension of the given lattice. May also be specified by a `Polyhedron_base` object;
- **lattice** – `ToricLattice`, \( \mathbb{Z}^n \), or any other object that behaves like these. If not specified, an attempt will be made to determine an appropriate toric lattice automatically;
- **check** – by default the input data will be checked for correctness (e.g. that all rays have the same number of components) and generating rays will be constructed from `rays`. If you know that the input is a minimal set of generators of a valid cone, you may significantly decrease construction time using `check=False` option;
- **normalize** – you can further speed up construction using `normalize=False` option. In this case `rays` must be a list of immutable primitive rays in `lattice`. In general, you should not use this option, it is designed for code optimization and does not give as drastic improvement in speed as the previous one.

OUTPUT:

- convex rational polyhedral cone determined by `rays`.

EXAMPLES:

Let’s define a cone corresponding to the first quadrant of the plane (note, you can even mix objects of different types to represent rays, as long as you let this function to perform all the checks and necessary conversions!):

```sage
sage: quadrant = Cone([(1,0), [0,1]])
sage: quadrant
2-d cone in 2-d lattice N
sage: quadrant.rays()
N(1, 0),
N(0, 1)
in 2-d lattice N
```

If you give more rays than necessary, the extra ones will be discarded:

```sage
sage: Cone([(1,0), (0,1), (1,1), (0,1)]).rays()
N(0, 1),
N(1, 0)
in 2-d lattice N
```

However, this work is not done with `check=False` option, so use it carefully!

```sage
sage: Cone([(1,0), (0,1), (1,1), (0,1)], check=False).rays()
N(1, 0),
N(0, 1),
```

(continues on next page)
N(1, 1),
N(0, 1)
in 2-d lattice N

Even worse things can happen with normalize=False option:

```
sage: Cone([[1,0], (0,1)], check=False, normalize=False)
Traceback (most recent call last):
...
AttributeError: 'tuple' object has no attribute 'parent'
```

You can construct different “not” cones: not full-dimensional, not strictly convex, not containing any rays:

```
sage: one_dimensional_cone = Cone([[1,0]])
sage: one_dimensional_cone.dim()
1
sage: half_plane = Cone([[1,0], (0,1), (-1,0)])
sage: half_plane.rays()
N( 0, 1),
N( 1, 0),
N(-1, 0)
in 2-d lattice N
sage: half_plane.is_strictly_convex()
False
sage: origin = Cone([[0,0]])
sage: origin.rays()
Empty collection
in 2-d lattice N
sage: origin.dim()
0
sage: origin.lattice_dim()
2
```

You may construct the cone above without giving any rays, but in this case you must provide `lattice` explicitly:

```
sage: origin = Cone([])
Traceback (most recent call last):
...
ValueError: lattice must be given explicitly if there are no rays!
sage: origin = Cone([], lattice=ToricLattice(2))
sage: origin.dim()
0
sage: origin.lattice_dim()
2
sage: origin.lattice()
2-d lattice N
```

However, the trivial cone in $n$ dimensions has a predefined constructor for you to use:

```
sage: origin = cones.trivial(2)
sage: origin.rays()
Empty collection
in 2-d lattice N
```

Of course, you can also provide `lattice` in other cases:
sage: L = ToricLattice(3, "L")
sage: c1 = Cone([[1,0,0],[1,1,1]], lattice=L)
sage: c1.rays()
L(1, 0, 0),
L(1, 1, 1)
in 3-d lattice L

Or you can construct cones from rays of a particular lattice:

sage: ray1 = L(1,0,0)
sage: ray2 = L(1,1,1)
sage: c2 = Cone([ray1, ray2])
sage: c2.rays()
L(1, 0, 0),
L(1, 1, 1)
in 3-d lattice L
sage: c1 == c2
True

When the cone in question is not strictly convex, the standard form for the “generating rays” of the linear subspace is “basis vectors and their negatives”, as in the following example:

sage: plane = Cone([[1,0], [0,1], [-1,-1]])
sage: plane.rays()
N( 0, 1),
N( 0, -1),
N( 1, 0),
N(-1, 0)
in 2-d lattice N

The cone can also be specified by a `Polyhedron_base`:

sage: p = plane.polyhedron()
sage: Cone(p)
2-d cone in 2-d lattice N
sage: Cone(p) == plane
True

class sage.geometry.cone.ConvexRationalPolyhedralCone(rays=None, lattice=None, ambient=None, ambient_ray_indices=None, PPL=None)

Bases: IntegralRayCollection, Container, ConvexSet_closed, ConvexRationalPolyhedralCone

Create a convex rational polyhedral cone.

**Warning:** This class does not perform any checks of correctness of input nor does it convert input into the standard representation. Use `Cone()` to construct cones.

Cones are immutable, but they cache most of the returned values.

**INPUT:**

The input can be either:

- `rays` – list of immutable primitive vectors in `lattice`;

2.5. Toric geometry
• \texttt{lattice=ToricLattice, \mathbb{Z}^n}, or any other object that behaves like these. If \texttt{None}, it will be determined as \texttt{parent()} of the first ray. Of course, this cannot be done if there are no rays, so in this case you must give an appropriate lattice directly.

or (these parameters must be given as keywords):

• \texttt{ambient} – ambient structure of this cone, a bigger \texttt{cone} or a \texttt{fan}, this cone \textit{must be a face of} ambient;

• \texttt{ambient\_ray\_indices} – increasing list or tuple of integers, indices of rays of ambient generating this cone.

In both cases, the following keyword parameter may be specified in addition:

• \texttt{PPL} – either \texttt{None} (default) or a \texttt{C\_Polyhedron} representing the cone. This serves only to cache the polyhedral data if you know it already. The constructor does not make a copy so the \texttt{PPL} object should not be modified afterwards.

\textbf{OUTPUT:}

• convex rational polyhedral cone.

\textbf{Note:} Every cone has its ambient structure. If it was not specified, it is this cone itself.

\texttt{Hilbert\_basis()}

Return the Hilbert basis of the cone.

Given a strictly convex cone \( C \subset \mathbb{R}^d \), the Hilbert basis of \( C \) is the set of all irreducible elements in the semigroup \( C \cap \mathbb{Z}^d \). It is the unique minimal generating set over \( \mathbb{Z} \) for the integral points \( C \cap \mathbb{Z}^d \).

If the cone \( C \) is not strictly convex, this method finds the (unique) minimal set of lattice points that need to be added to the defining rays of the cone to generate the whole semigroup \( C \cap \mathbb{Z}^d \). But because the rays of the cone are not unique nor necessarily minimal in this case, neither is the returned generating set (consisting of the rays plus additional generators).

See also \texttt{semigroup\_generators()} if you are not interested in a minimal set of generators.

\textbf{OUTPUT:}

• a \texttt{PointCollection}. The rays of \texttt{self} are the first \texttt{self.nrays()} entries.

\textbf{EXAMPLES:}

The following command ensures that the output ordering in the examples below is independent of TOPCOM, you don’t have to use it:

\begin{verbatim}sage: PointConfiguration.set_engine('internal')\end{verbatim}

We start with a simple case of a non-smooth 2-dimensional cone:

\begin{verbatim}sage: Cone([[1,0], (1,2))].Hilbert_basis() N(1, 0), N(1, 2), N(1, 1) in 2-d lattice N\end{verbatim}

Two more complicated example from GAP/toric:

\begin{verbatim}sage: Cone([[1,0], [3,4])].dual().Hilbert_basis() M(0, 1), M(4, -3),\end{verbatim}

(continues on next page)
in 2-d lattice $M$

```python
sage: cone = Cone([[1,2,3,4], [0,1,0,7], [3,1,0,2], [0,0,1,0]]).dual()
sage: cone.Hilbert_basis()
# long time
```

in 4-d lattice $M$

```python
Notastrictly convex cone:

sage: wedge = Cone([(1,0,0), (1,2,0), (0,0,1), (0,0,-1)])
sage: sorted(wedge.semigroup_generators())
[N(0, 0, -1), N(0, 0, 1), N(1, 0, 0), N(1, 1, 0), N(1, 2, 0)]
sage: wedge.Hilbert_basis()
N(1, 2, 0),
N(1, 0, 0),
N(0, 0, 1),
N(0, 0, -1),
N(1, 1, 0)
in 3-d lattice $N$
```

Not full-dimensional cones are ok, too (see github issue #11312):

```python
sage: Cone([[(1,1,0), (-1,1,0)]].Hilbert_basis()
N(1, 1, 0),
N(-1, 1, 0),
N( 0, 1, 0)
in 3-d lattice $N$
```
ALGORITHM:
The primal Normaliz algorithm, see [Normaliz].

Hilbert_coefficients(point, solver, verbose=None, integrality_tolerance=0)
Return the expansion coefficients of point with respect to Hilbert_basis().

INPUT:
- **point** – a lattice() point in the cone, or something that can be converted to a point. For example, a list or tuple of integers.
- **solver** – (default: None) Specify a Mixed Integer Linear Programming (MILP) solver to be used. If set to None, the default one is used. For more information on MILP solvers and which default solver is used, see the method solve of the class MixedIntegerLinearProgram.
- **verbose** – integer (default: 0). Sets the level of verbosity of the LP solver. Set to 0 by default, which means quiet.
- **integrality_tolerance** – parameter for use with MILP solvers over an inexact base ring; see MixedIntegerLinearProgram.get_values().

OUTPUT:
A ZZ-vector of length len(self.Hilbert_basis()) with nonnegative components.

Note: Since the Hilbert basis elements are not necessarily linearly independent, the expansion coefficients are not unique. However, this method will always return the same expansion coefficients when invoked with the same argument.

EXAMPLES:

```sage
sage: cone = Cone([(1,0), (0,1)])
sage: cone.rays()
N(1, 0),
N(0, 1)
in 2-d lattice N
sage: cone.Hilbert_coefficients([3,2])
(3, 2)
```

A more complicated example:

```sage
sage: N = ToricLattice(2)
sage: cone = Cone([N(1,0), N(1,2)])
sage: cone.Hilbert_basis()
N(1, 0),
N(1, 2),
N(1, 1)
in 2-d lattice N
sage: cone.Hilbert_coefficients(N(1,1))
(0, 0, 1)
```

The cone need not be strictly convex:

```sage
sage: N = ToricLattice(3)
sage: cone = Cone([N(1,0,0), N(1,2,0), N(0,0,1), N(0,0,-1)])
sage: cone.Hilbert_basis()
N(1, 2, 0),
N(1, 0, 0),
```

(continues on next page)
Z_operators_gens()

Compute minimal generators of the Z-operators on this cone.

The Z-operators on a cone generalize the Z-matrices over the nonnegative orthant. They are simply negations of the cross_positive_operators_gens().

OUTPUT:

A list of \( n \)-by-\( n \) matrices where \( n \) is the ambient dimension of this cone. Each matrix \( L \) in the list has the property that \( s(L(x)) \leq 0 \) whenever \((x,s)\) is an element of this cone’s discrete_complementarity_set().

The returned matrices generate the cone of Z-operators on this cone; that is,

- Any nonnegative linear combination of the returned matrices is a Z-operator on this cone.
- Every Z-operator on this cone is some nonnegative linear combination of the returned matrices.

See also:

cross_positive_operators_gens(), lyapunov_like_basis(), positive_operators_gens()

REFERENCES:

- [BP1994]
- [Or2018b]

adjacent()

Return faces adjacent to self in the ambient face lattice.

Two distinct faces \( F_1 \) and \( F_2 \) of the same face lattice are adjacent if all of the following conditions hold:

- \( F_1 \) and \( F_2 \) have the same dimension \( d \);
- \( F_1 \) and \( F_2 \) share a facet of dimension \( d - 1 \);
- \( F_1 \) and \( F_2 \) are facets of some face of dimension \( d + 1 \), unless \( d \) is the dimension of the ambient structure.

OUTPUT:

- tuple of cones.

EXAMPLES:

sage: # needs sage.graphs
sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
sage: octant.adjacent() ()
sage: one_face = octant.faces(1)[0]
sage: len(one_face.adjacent())
2
sage: one_face.adjacent()[1]
1-d face of 3-d cone in 3-d lattice N
Things are a little bit subtle with fans, as we illustrate below.

First, we create a fan from two cones in the plane:

```sage
fan = Fan(cones=[[0,1], [1,2]],
....:   rays=[[1,0], (0,1), (-1,0)])
fan = fan.generating_cone(0)
len(fan.adjacent())
```

The second generating cone is adjacent to this one. Now we create the same fan, but embedded into the 3-dimensional space:

```sage
fan = Fan(cones=[[0,1], [1,2]],
....:   rays=[[1,0,0], (0,1,0), (-1,0,0)])
fan = fan.generating_cone(0)
len(fan.adjacent())
```

The result is as before, since we still have:

```sage
fan.dim()
2
```

Now we add another cone to make the fan 3-dimensional:

```sagean = Fan(cones=[[0,1], [1,2], [3]],
....:   rays=[[1,0,0], (0,1,0), (-1,0,0), (0,0,1)])
fan = fan.generating_cone(0)
len(fan.adjacent())
```

Since now `cone` has smaller dimension than `fan`, it and its adjacent cones must be facets of a bigger one, but since `cone` in this example is generating, it is not contained in any other.

```sage
def ambient(self):
    """Return the ambient structure of self."
    OUTPUT:
    • cone or fan containing self as a face.
    EXAMPLES:
    sage: cone = Cone([(1,2,3), (4,6,5), (9,8,7)])
    sage: cone.ambient()
    3-d cone in 3-d lattice N
    """
ambient_ray_indices()

Return indices of rays of the ambient structure generating self.

OUTPUT:

• increasing tuple of integers.

EXAMPLES:

```sage
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.ambient_ray_indices()
(0, 1)
sage: quadrant.facets()[1].ambient_ray_indices()  # needs sage.graphs
(1,)
```

an_affine_basis()

Return points in self that form a basis for the affine hull.

EXAMPLES:

```sage
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.an_affine_basis()  # needs sage.graphs
[(0, 0), (1, 0), (0, 1)]
sage: ray = Cone([(1, 1)])
sage: ray.an_affine_basis()  # needs sage.graphs
[(0, 0), (1, 1)]
sage: line = Cone([(1,0), (-1,0)])
sage: line.an_affine_basis()  # needs sage.graphs
[(1, 0), (0, 0)]
```

cartesian_product (other, lattice=None)

Return the Cartesian product of self with other.

INPUT:

• other – a cone;

• lattice – (optional) the ambient lattice for the Cartesian product cone. By default, the direct sum of the ambient lattices of self and other is constructed.

OUTPUT:

• a cone.

EXAMPLES:

```sage
c = Cone([(1,)])
sage: c = Cone([(1,)])
sage: c.cartesian_product(c)
2-d cone in 2-d lattice N+N
sage: _.rays()
N+N(1, 0),
N+N(0, 1)
in 2-d lattice N+N
```
contains(*args)

Check if a given point is contained in self.

INPUT:

• anything. An attempt will be made to convert all arguments into a single element of the ambient space of self. If it fails, False will be returned.

OUTPUT:

• True if the given point is contained in self, False otherwise.

EXAMPLES:

```python
sage: c = Cone([(1,0), (0,1)])
sage: c.contains(c.lattice()(1,0))
True
sage: c.contains((1,0))
True
sage: c.contains((1,1))
True
sage: c.contains(1,1)
True
sage: c.contains((-1,0))
False
sage: c.contains(c.dual_lattice()(1,0))  # random output (warning)
False
sage: c.contains(c.dual_lattice()(1,0))
False
sage: c.contains(1)
False
sage: c.contains(1/2, sqrt(3))  # needs sage.symbolic
False
```

cross_positive_operators_gens()

Compute minimal generators of the cross-positive operators on this cone.

Any positive operator $P$ on this cone will have $s(P(x)) \geq 0$ whenever $x$ is an element of this cone and $s$ is an element of its dual. By contrast, the cross-positive operators need only satisfy that property on the discrete_complementarity_set(); that is, when $x$ and $s$ are “cross” (orthogonal).

The cross-positive operators (on some fixed cone) themselves form a closed convex cone. This method computes and returns the generators of that cone as a list of matrices.

Cross-positive operators are also called exponentially-positive, since they become positive operators when exponentiated. Other equivalent names are resolvent-positive, essentially-positive, and quasimonotone.

OUTPUT:

A list of $n$-by-$n$ matrices where $n$ is the ambient dimension of this cone. Each matrix $L$ in the list has the property that $s(L(x)) \geq 0$ whenever $(x,s)$ is an element of this cone’s discrete_complementarity_set().

The returned matrices generate the cone of cross-positive operators on this cone; that is,

• Any nonnegative linear combination of the returned matrices is cross-positive on this cone.
• Every cross-positive operator on this cone is some nonnegative linear combination of the returned matrices.

See also:

lyapunov_like_basis(), positive_operators_gens(), Z_operators_gens()

REFERENCES:

• [SV1970]
• [Or2018b]

EXAMPLES:

Cross-positive operators on the nonnegative orthant are negations of Z-matrices; that is, matrices whose off-diagonal elements are nonnegative:

```sage
K = cones.nonnegative_orthant(2)
sage: K.cross_positive_operators_gens()
[[ 0  1]
 [ 0  0],
 [ 0  0],
 [ 0  0]]
```

```sage
K = Cone([(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)])
sage: all(c[i][j] >= 0 for c in K.cross_positive_operators_gens()
.....:     for i in range(c.nrows())
.....:         for j in range(c.ncols())
.....:             if i != j)
True
```

The trivial cone in a trivial space has no cross-positive operators:

```sage
K = cones.trivial(0)
sage: K.cross_positive_operators_gens()
[]
```

Every operator is a cross-positive operator on the ambient vector space:

```sage
K = Cone([(1,), (-1,)])
sage: K.is_full_space()
True
```

```sage
K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space()
True
```

```sage
K = Cone([(1,0), (0,1), (0,-1)])
sage: K.cross_positive_operators_gens()
[[1 0]
 [-1 0]
 [0 0]]
```

A non-obvious application is to find the cross-positive operators on the right half-plane [Or2018b]:

```sage
K = Cone([(1,0), (0,1), (0,-1)])
sage: K.cross_positive_operators_gens()
[[1 0]
 [-1 0]
 [0 0]]
```

(continues on next page)
Cross-positive operators on a subspace are Lyapunov-like and vice-versa:

```python
sage: K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space()
True
sage: lls = span(vector(l.list())
....:     for l in K.lyapunov_like_basis())
sage: cs = span(vector(c.list())
....:     for c in K.cross_positive_operators_gens())
sage: cs == lls
True
```

**discrete_complementarity_set()**

Compute a discrete complementarity set of this cone.

A discrete complementarity set of a cone is the set of all orthogonal pairs \((x, s)\) where \(x\) is in some fixed generating set of the cone, and \(s\) is in some fixed generating set of its dual. The generators chosen for this cone and its dual are simply their \(rays()\).

**OUTPUT:**

A tuple of pairs \((x, s)\) such that,

- \(x\) and \(s\) are nonzero.
- \(s(x)\) is zero.
- \(x\) is one of this cone’s \(rays()\).
- \(s\) is one of the \(rays()\) of this cone’s \(dual()\).

**REFERENCES:**

- [Or2017]

**EXAMPLES:**

Pairs of standard basis elements form a discrete complementarity set for the nonnegative orthant:

```python
sage: K = cones.nonnegative_orthant(2)
sage: K.discrete_complementarity_set()
((N(1, 0), M(0, 1)), (N(0, 1), M(1, 0)))
```

If a cone consists of a single ray, then the second components of a discrete complementarity set for that cone should generate the orthogonal complement of the ray:

```python
sage: K = Cone([(1,0)])
sage: K.discrete_complementarity_set()
((N(1, 0), M(0, 1)), (N(1, 0), M(0, -1)))
```

When a cone is the entire space, its dual is the trivial cone, so the only discrete complementarity set for it is empty:
Likewise for trivial cones, whose duals are the entire space:

```python
sage: cones.trivial(0).discrete_complementarity_set()
()```

dual()

Return the dual cone of self.

OUTPUT:

• cone.

EXAMPLES:

```python
sage: cone = Cone([(1,0), (-1,3)])
sage: cone.dual().rays()
M(0, 1),
M(3, 1)
in 2-d lattice M
```

Now let's look at a more complicated case:

```python
sage: cone = Cone([(-2,-1,2), (4,1,0), (-4,-1,-5), (4,1,5)])
sage: cone.is_strictly_convex()
False
sage: cone.dim()
3
sage: cone.dual().rays()
M(7, -18, -2),
M(1, -4, 0)
in 3-d lattice M
sage: cone.dual().dual() is cone
True
```

We correctly handle the degenerate cases:

```python
sage: N = ToricLattice(2)
sage: Cone([], lattice=N).dual().rays()  # empty cone
M( 1, 0),
M(-1, 0),
M( 0, 1),
M( 0, -1)
in 2-d lattice M
sage: Cone([(1,0)], lattice=N).dual().rays()  # ray in 2d
M(1, 0),
M(0, 1),
M(0, -1)
in 2-d lattice M
sage: Cone([(1,0),(-1,0)], lattice=N).dual().rays()  # line in 2d
M(0, 1),
M(0, -1)
in 2-d lattice M
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

\[
sage: \text{Cone}([\langle 1,0 \rangle, \langle 0,1 \rangle], \text{lattice}=N).\text{dual()}\text{.rays()} \quad \# \text{strictly convex cone}
\]
\[
M(0, 1),
M(1, 0)
in 2-d lattice M
\]
\[
sage: \text{Cone}([\langle 1,0 \rangle, \langle -1,0 \rangle, \langle 0,1 \rangle], \text{lattice}=N).\text{dual()}\text{.rays()} \quad \# \text{half space}
\]
\[
M(0, 1)
in 2-d lattice M
\]
\[
sage: \text{Cone}([\langle 1,0 \rangle, \langle 0,1 \rangle, \langle -1,-1 \rangle], \text{lattice}=N).\text{dual()}\text{.rays()} \quad \# \text{whole space}
\]
Empty collection
in 2-d lattice M

\textbf{embed}(cone)

Return the cone equivalent to the given one, but sitting in \textbf{self} as a face.

You may need to use this method before calling methods of \textit{cone} that depend on the ambient structure, such as \texttt{ambient\_ray\_indices()} or \texttt{facet\_of()}. The cone returned by this method will have \textbf{self} as ambient. If \textit{cone} does not represent a valid cone of \textbf{self}, \texttt{ValueError} exception is raised.

\textbf{Note}: This method is very quick if \textbf{self} is already the ambient structure of \textit{cone}, so you can use without extra checks and performance hit even if \textit{cone} is likely to sit in \textbf{self} but in principle may not.

\textbf{INPUT}:

\begin{itemize}
  \item \textit{cone} – a \textit{cone}.
\end{itemize}

\textbf{OUTPUT}:

\begin{itemize}
  \item a \textit{cone}, equivalent to \textit{cone} but sitting inside \textbf{self}.
\end{itemize}

\textbf{EXAMPLES}:

Let's take a 3-d cone on 4 rays:

\[
sage: c = \text{Cone}([\langle 1,0,1 \rangle, \langle 0,1,1 \rangle, \langle -1,0,1 \rangle, \langle 0,-1,1 \rangle])
\]

Then any ray generates a 1-d face of this cone, but if you construct such a face directly, it will not “sit” inside the cone:

\[
sage: ray = \text{Cone}([\langle 0,-1,1 \rangle])
\]
\[
sage: ray
1-d cone in 3-d lattice N
\]
\[
sage: ray.ambient\_ray\_indices()
(0,)
\]
\[
sage: ray.\text{adjacent}()
\rightarrow \text{needs sage.graphs}
()
\]
\[
sage: ray.\text{ambient}()
1-d cone in 3-d lattice N
\]

If we want to operate with this ray as a face of the cone, we need to embed it first:

\[
sage: \# \text{needs sage.graphs}
\]
\[
sage: \text{e}\_ray = c.\text{embed}(ray)
\]
\[
sage: \text{e}\_ray
1-d face of 3-d cone in 3-d lattice N
\]
\[
sage: \text{e}\_ray.\text{rays}()
N(0, -1, 1)
\]

(continues on next page)
Not every cone can be embedded into a fixed ambient cone:

```
sage: c.embed(Cone([(0,0,1)]))
Traceback (most recent call last):
  ...  
ValueError: 1-d cone in 3-d lattice N is not a face of 3-d cone in 3-d lattice N!
sage: c.embed(Cone([(1,0,1), (-1,0,1)]))
  -> needs sage.graphs
Traceback (most recent call last):
  ...  
ValueError: 2-d cone in 3-d lattice N is not a face of 3-d cone in 3-d lattice N!
```

**face_lattice()**

Return the face lattice of self.

This lattice will have the origin as the bottom (we do not include the empty set as a face) and this cone itself as the top.

**OUTPUT:**

- a finite poset of cones.

**EXAMPLES:**

Let’s take a look at the face lattice of the first quadrant:

```
sage: quadrant = Cone([(1,0), (0,1)])
sage: L = quadrant.face_lattice()  # -- needs sage.combinat sage.graphs
sage: L  # -- needs sage.combinat sage.graphs
Finite lattice containing 4 elements with distinguished linear extension
```

To see all faces arranged by dimension, you can do this:

```
sage: for level in L.level_sets(): print(level)  # -- needs sage.combinat sage.graphs
[0-d face of 2-d cone in 2-d lattice N]
[1-d face of 2-d cone in 2-d lattice N, 1-d face of 2-d cone in 2-d lattice N]
[2-d cone in 2-d lattice N]
```

For a particular face you can look at its actual rays…
Combinatorial and Discrete Geometry, Release 10.3

```
sage: face = L.level_sets()[1][0]  # needs sage.combinat sage.graphs
sage: face.rays()  # needs sage.combinat sage.graphs
N(1, 0)
in 2-d lattice N

... or you can see the index of the ray of the original cone that corresponds to the above one:

sage: face.ambient_ray_indices()  # needs sage.combinat sage.graphs
(0,)
sage: quadrant.ray(0)
N(1, 0)

An alternative to extracting faces from the face lattice is to use \texttt{faces()} method:

sage: face is quadrant.faces(dim=1)[0]  # needs sage.combinat sage.graphs
True

The advantage of working with the face lattice directly is that you can (relatively easily) get faces that are related to the given one:

sage: face = L.level_sets()[1][0]  # needs sage.combinat sage.graphs
sage: D = L.hasse_diagram()  # needs sage.combinat sage.graphs
sage: sorted(D.neighbors(face))  # needs sage.combinat sage.graphs
[0-d face of 2-d cone in 2-d lattice N,
 2-d cone in 2-d lattice N]

However, you can achieve some of this functionality using \texttt{facets()}, \texttt{facet_of()}, and \texttt{adjacent()} methods:

sage: # needs sage.graphs
sage: face = quadrant.faces(1)[0]
sage: face
1-d face of 2-d cone in 2-d lattice N
sage: face.rays()
N(1, 0)
in 2-d lattice N
sage: face.facets()
(0-d face of 2-d cone in 2-d lattice N,)
sage: face.facet_of()
(2-d cone in 2-d lattice N,)
sage: face.adjacent()
(1-d face of 2-d cone in 2-d lattice N,)
sage: face.adjacent()[0].rays()
N(0, 1)
in 2-d lattice N

Note that if \texttt{cone} is a face of \texttt{supercone}, then the face lattice of \texttt{cone} consists of (appropriate) faces of \texttt{supercone}:
```
sage: # needs sage.combinat sage.graphs
sage: supercone = Cone([(1,2,3,4), (5,6,7,8),
                     (1,2,4,8), (1,3,9,7)]
sage: supercone.face_lattice()
Finite lattice containing 16 elements with distinguished linear extension
sage: supercone.face_lattice().top()
4-d cone in 4-d lattice N
sage: cone = supercone.facets()[0]
sage: cone
3-d face of 4-d cone in 4-d lattice N
sage: cone.face_lattice()
Finite poset containing 8 elements with distinguished linear extension
sage: cone.face_lattice().bottom()
0-d face of 4-d cone in 4-d lattice N
sage: cone.face_lattice().top()
3-d face of 4-d cone in 4-d lattice N
sage: cone.face_lattice().top() == cone
True

faces (dim=None, codim=None)

Return faces of self of specified (co)dimension.

INPUT:

- dim – integer, dimension of the requested faces;
- codim – integer, codimension of the requested faces.

Note: You can specify at most one parameter. If you don’t give any, then all faces will be returned.

OUTPUT:

- if either dim or codim is given, the output will be a tuple of cones;
- if neither dim nor codim is given, the output will be the tuple of tuples as above, giving faces of all existing dimensions. If you care about inclusion relations between faces, consider using face_lattice() or adjacent(), facet_of(), and facets().

EXAMPLES:

Let’s take a look at the faces of the first quadrant:

sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.faces()
# needs sage.graphs
((0-d face of 2-d cone in 2-d lattice N,),
  (1-d face of 2-d cone in 2-d lattice N,
  1-d face of 2-d cone in 2-d lattice N),
  (2-d cone in 2-d lattice N,))
sage: quadrant.faces(dim=1)
# needs sage.graphs
(1-d face of 2-d cone in 2-d lattice N,
  1-d face of 2-d cone in 2-d lattice N)
sage: face = quadrant.faces(dim=1)[0]
# needs sage.graphs
Now you can look at the actual rays of this face…

2.5. Toric geometry
... or you can see indices of the rays of the original cone that correspond to the above ray:

```python
sage: face.ambient_ray_indices()
# needs sage.graphs
(0,)
sage: quadrant.ray(0)
N(1, 0)
```

Note that it is OK to ask for faces of too small or high dimension:

```python
sage: quadrant.faces(-1)   # needs sage.graphs
()  
sage: quadrant.faces(3)   # needs sage.graphs
()
```

In the case of non-strictly convex cones even faces of small non-negative dimension may be missing:

```python
sage: halfplane = Cone([(1,0), (0,1), (-1,0)])
sage: halfplane.faces(0)  
sage: halfplane.faces()  
sage: plane = Cone([(1,0), (0,1), (-1,-1)])
sage: plane.faces(1)  
sage: plane.faces()
```

**facet_normals()**

Return inward normals to facets of `self`.

**Note:**

1. For a not full-dimensional cone facet normals will specify hyperplanes whose intersections with the space spanned by `self` give facets of `self`.

2. For a not strictly convex cone facet normals will be orthogonal to the linear subspace of `self`, i.e. they always will be elements of the dual cone of `self`.

3. The order of normals is random, but consistent with `facets()`.

**OUTPUT:**

- a `PointCollection`.

If the ambient `lattice()` of `self` is a toric lattice, the facet normals will be elements of the dual lattice. If it is a general lattice (like \(\mathbb{Z}^n\)) that does not have a `dual()` method, the facet normals will be returned as integral vectors.
EXAMPLES:

```
sage: cone = Cone([(1,0), (-1,3)])
sage: cone.facet_normals()
M(0, 1),
M(3, 1)
in 2-d lattice M
```

Now let's look at a more complicated case:

```
sage: cone = Cone([(-2,-1,2), (4,1,0), (-4,-1,-5), (4,1,5)])
sage: cone.is_strictly_convex()
False
sage: cone.dim()
3
sage: cone.linear_subspace().dimension()
1
sage: lsg = (QQ^3)(cone.linear_subspace().gen(0)); lsg
(1, 1/4, 5/4)
sage: cone.facet_normals()
M(7, -18, -2),
M(1, -4, 0)
in 3-d lattice M
sage: [lsg*normal for normal in cone.facet_normals()]
[0, 0]
```

A lattice that does not have a `dual()` method:

```
sage: Cone([(1,1),(0,1)], lattice=ZZ^2).facet_normals()
(-1, 1),
( 1, 0)
in Ambient free module of rank 2
over the principal ideal domain Integer Ring
```

We correctly handle the degenerate cases:

```
sage: N = ToricLattice(2)
sage: Cone([], lattice=N).facet_normals() # empty cone
Empty collection
in 2-d lattice M
sage: Cone([(1,0)], lattice=N).facet_normals() # ray in 2d
M(1, 0)
in 2-d lattice M
sage: Cone([(1,0),(-1,0)], lattice=N).facet_normals() # line in 2d
Empty collection
in 2-d lattice M
sage: Cone([(1,0),(0,1)], lattice=N).facet_normals() # strictly convex cone
M(0, 1),
M(1, 0)
in 2-d lattice M
sage: Cone([(1,0),(-1,0),(0,1)], lattice=N).facet_normals() # half space
M(0, 1)
in 2-d lattice M
sage: Cone([(1,0),(0,1),(-1,-1)], lattice=N).facet_normals() # whole space
Empty collection
in 2-d lattice M
```

```
facet_of()
```

Return cones of the ambient face lattice having `self` as a facet.

---

2.5. Toric geometry 379
OUTPUT:

• tuple of cones.

EXAMPLES:

```python
sage: # needs sage.graphs
sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
sage: octant.facet_of()

sage: one_face = octant.faces(1)[0]
sage: len(one_face.facet_of())
2
sage: one_face.facet_of()[1]
2-d face of 3-d cone in 3-d lattice N
```

While fan is the top element of its own cone lattice, which is a variant of a face lattice, we do not refer to cones as its facets:

```python
sage: fan = Fan([octant])  # needs sage.graphs
sage: fan.generating_cone(0).facet_of()  # needs sage.graphs

sage: one_cone = fan(1)[0]  # needs sage.graphs
sage: len(one_cone.facet_of())  # needs sage.graphs
2
```

facets()

Return facets (faces of codimension 1) of self.

OUTPUT:

• tuple of cones.

EXAMPLES:

```python
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.facets()
(1-d face of 2-d cone in 2-d lattice N, 1-d face of 2-d cone in 2-d lattice N)
```

incidence_matrix()

Return the incidence matrix.

Note: The columns correspond to facets/facet normals in the order of facet_normals(), the rows correspond to the rays in the order of rays().

EXAMPLES:
interior()  
Return the interior of self.

OUTPUT:

- either self, an empty polyhedron, or an instance of `RelativeInterior`.

EXAMPLES:

```python
sage: c = Cone([(1,0,0), (0,1,0)]); c
2-d cone in 3-d lattice N
sage: c.interior()
The empty polyhedron in ZZ^3
sage: origin = cones.trivial(2); origin
0-d cone in 2-d lattice N
sage: origin.interior()
The empty polyhedron in ZZ^2
sage: K = cones.nonnegative_orthant(2); K
2-d cone in 2-d lattice N
sage: K.interior()
Relative interior of 2-d cone in 2-d lattice N
sage: K2 = Cone([(1,0),(-1,0),(0,1),(0,-1)]); K2
2-d cone in 2-d lattice N
sage: K2.interior()  # K2
True
```

interior_contains(*args)
Check if a given point is contained in the interior of self.

For a cone of strictly lower-dimension than the ambient space, the interior is always empty. You probably want to use `relative_interior_contains()` in this case.

INPUT:

- anything. An attempt will be made to convert all arguments into a single element of the ambient space of self. If it fails, False will be returned.

OUTPUT:

- True if the given point is contained in the interior of self, False otherwise.

EXAMPLES:
sage: c = Cone([[1,0], (0,1)])
sage: c.contains((1,1))
True
sage: c.interior_contains((1,1))
True
sage: c.contains((1,0))
True
sage: c.interior_contains((1,0))
False

intersection \texttt{(other)}

Compute the intersection of two cones.

INPUT:

\begin{itemize}
  \item other - \texttt{cone}.
\end{itemize}

OUTPUT:

\begin{itemize}
  \item \texttt{cone}.
\end{itemize}

This raises \texttt{ValueError} if the ambient space dimensions are not compatible.

EXAMPLES:

\begin{verbatim}
sage: cone1 = Cone([[1,0], (-1, 3)])
sage: cone2 = Cone([(-1,0), (2, 5)])
sage: cone1.intersection(cone2).rays()
  N(-1, 3),
  N( 2, 5)
in 2-d lattice N
\end{verbatim}

The intersection can also be expressed using the operator \&:

\begin{verbatim}
sage: (cone1 \& cone2).rays()
  N(-1, 3),
  N( 2, 5)
in 2-d lattice N
\end{verbatim}

It is OK to intersect cones living in sublattices of the same ambient lattice:

\begin{verbatim}
sage: N = cone1.lattice()
sage: Ns = N.submodule([[(1,1)]])
sage: cone3 = Cone([[(1,1)]], lattice=Ns)
sage: I = cone1.intersection(cone3)
sage: I.rays()
  N(1, 1)
in Sublattice <N(1, 1)>
sage: I.lattice()
Sublattice <N(1, 1)>
\end{verbatim}

But you cannot intersect cones from incompatible lattices without explicit conversion:

\begin{verbatim}
sage: cone1.intersection(cone1.dual())
Traceback (most recent call last):
  ...
ValueError: 2-d lattice N and 2-d lattice M have different ambient lattices!
sage: cone1.intersection(Cone(cone1.dual().rays(), N)).rays()
\end{verbatim}
(continues on next page)
An intersection with a polyhedron returns a polyhedron:

```python
sage: cone = Cone([(1,0), (-1,0), (0,1)])
sage: p = polytopes.hypercube(2)
sage: cone & p
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: sorted(_.vertices_list())
[[-1, 0], [-1, 1], [1, 0], [1, 1]]
```

`is_compact()`

Checks if the cone has no rays.

**OUTPUT:**

- True if the cone has no rays, False otherwise.

**EXAMPLES:**

```python
sage: c0 = cones.trivial(3)
sage: c0.is_trivial() # True
sage: c0.nrays() # 0
```

`is_empty()`

Return whether `self` is the empty set.

Because a cone always contains the origin, this method returns False.

**EXAMPLES:**

```python
sage: trivial_cone = cones.trivial(3)
sage: trivial_cone.is_empty() # False
```

`is_equivalent(other)`

Check if `self` is “mathematically” the same as `other`.

**INPUT:**

- other - cone.

**OUTPUT:**

- True if `self` and `other` define the same cones as sets of points in the same lattice, False otherwise.

There are three different equivalences between cones $C_1$ and $C_2$ in the same lattice:

1. They have the same generating rays in the same order. This is tested by $C_1 == C_2$.
2. They describe the same sets of points. This is tested by $C1.is_equivalent(C2)$.
3. They are in the same orbit of $GL(n, \mathbb{Z})$ (and, therefore, correspond to isomorphic affine toric varieties). This is tested by $C1.is_isomorphic(C2)$.

**EXAMPLES:**
```
sage: cone1 = Cone([(1,0), (-1, 3)])
sage: cone2 = Cone([-1,3), (1, 0)])
sage: cone1.rays()
N( 1, 0),
N(-1, 3)
in 2-d lattice N
sage: cone2.rays()
N(-1, 3),
N( 1, 0)
in 2-d lattice N
sage: cone1 == cone2
False
sage: cone1.is_equivalent(cone2)
True
```

**is_face_of**

Check if self forms a face of another cone.

**INPUT:**

- `cone` – cone.

**OUTPUT:**

- `True` if self is a face of `cone`, `False` otherwise.

**EXAMPLES:**

```
sage: quadrant = Cone([(1,0), (0,1)])
sage: cone1 = Cone([(1,0)])
sage: cone2 = Cone([(1,2)])
sage: quadrant.is_face_of(quadrant)
True
sage: cone1.is_face_of(quadrant)
True
sage: cone2.is_face_of(quadrant)
False
```

Being a face means more than just saturating a facet inequality:

```
sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
sage: cone = Cone([(2,1,0),(1,2,0)])
sage: cone.is_face_of(octant)
False
```

**is_full_dimensional**

Check if this cone is solid.

A cone is said to be solid if it has nonempty interior. That is, if its extreme rays span the entire ambient space.

An alias is `is_full_dimensional()`.

**OUTPUT:**

`True` if this cone is solid, and `False` otherwise.

**See also:**

`is_proper()`

**EXAMPLES:**
The nonnegative orthant is always solid:

```
sage: quadrant = cones.nonnegative_orthant(2)
sage: quadrant.is_solid()
True
sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
sage: octant.is_solid()
True
```

However, if we embed the two-dimensional nonnegative quadrant into three-dimensional space, then the resulting cone no longer has interior, so it is not solid:

```
sage: quadrant = Cone([(1,0,0), (0,1,0)])
sage: quadrant.is_solid()
False
```

**is_full_space()**

Check if this cone is equal to its ambient vector space.

An alias is `is_universe()`.

**OUTPUT:**

True if this cone equals its entire ambient vector space and `False` otherwise.

**EXAMPLES:**

A single ray in two dimensions is not equal to the entire space:

```
sage: K = Cone([(1,0)])
sage: K.is_full_space()
False
```

Neither is the nonnegative orthant:

```
sage: K = cones.nonnegative_orthant(2)
sage: K.is_full_space()
False
```

The right half-space contains a vector subspace, but it is still not equal to the entire space:

```
sage: K = Cone([(1,0), (-1,0), (0,1)])
sage: K.is_full_space()
False
```

However, if we allow conic combinations of both axes, then the resulting cone is the entire two-dimensional space:

```
sage: K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space()
True
```

**is_isomorphic(other)**

Check if `self` is in the same $GL(n,\mathbb{Z})$-orbit as `other`.

**INPUT:**

- `other` - cone.

**OUTPUT:**
• True if self and other are in the same $GL(n, \mathbb{Z})$-orbit, False otherwise.

There are three different equivalences between cones $C_1$ and $C_2$ in the same lattice:

1. They have the same generating rays in the same order. This is tested by $C_1 == C_2$.
2. They describe the same sets of points. This is tested by $C_1.is_equivalent(C_2)$.
3. They are in the same orbit of $GL(n, \mathbb{Z})$ (and, therefore, correspond to isomorphic affine toric varieties). This is tested by $C_1.is_isomorphic(C_2)$.

EXAMPLES:

```python
sage: cone1 = Cone([(1,0), (0, 3)])
sage: m = matrix(ZZ, [(1, -5), (-1, 4)]) # a GL(2,ZZ)-matrix
sage: cone2 = Cone( m*r for r in cone1.rays() )
sage: cone1.is_isomorphic(cone2)
True

sage: cone1 = Cone([(1,0), (0, 3)])
```

• Check if this cone is proper.

A cone is said to be proper if it is closed, convex, solid, and contains no lines. This cone is assumed to be closed and convex; therefore it is proper if it is solid and contains no lines.

OUTPUT:

True if this cone is proper, and False otherwise.

See also:

is_strictly_convex(), is_solid()

EXAMPLES:

The nonnegative orthant is always proper:

```python
sage: quadrant = cones.nonnegative_orthant(2)
sage: quadrant.is_proper()
True
```

However, if we embed the two-dimensional nonnegative quadrant into three-dimensional space, then the resulting cone no longer has interior, so it is not solid, and thus not proper:

```python
sage: quadrant = Cone([(1,0,0), (0,1,0)])
sage: quadrant.is_proper()
False
```

Likewise, a half-space contains at least one line, so it is not proper:

```python
sage: halfspace = Cone([(1,0), (0,1), (-1,0)])
sage: halfspace.is_proper()
False
```
is_relatively_open()

Return whether self is relatively open.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: K = cones.nonnegative_orthant(3)
sage: K.is_relatively_open()
False

sage: K1 = Cone([(1,0), (-1,0)]); K1
1-d cone in 2-d lattice N
sage: K1.is_relatively_open()
True
```

is_simplicial()

Check if self is simplicial.

A cone is called simplicial if primitive vectors along its generating rays form a part of a rational basis of the ambient space.

OUTPUT:

- True if self is simplicial, False otherwise.

EXAMPLES:

```python
sage: cone1 = Cone([(1,0), (0, 3)])
sage: cone2 = Cone([(1,0), (0, 3), (-1,-1)])
sage: cone1.is_simplicial()
True
sage: cone2.is_simplicial()
False
```

is_smooth()

Check if self is smooth.

A cone is called smooth if primitive vectors along its generating rays form a part of an integral basis of the ambient space. Equivalently, they generate the whole lattice on the linear subspace spanned by the rays.

OUTPUT:

- True if self is smooth, False otherwise.

EXAMPLES:

```python
sage: cone1 = Cone([(1,0), (0, 1)])
sage: cone2 = Cone([(1,0), (-1, 3)])
sage: cone1.is_smooth()
True
sage: cone2.is_smooth()
False
```

The following cones are the same up to a $SL(2,\mathbb{Z})$ coordinate transformation:

```python
sage: Cone([(1,0,0), (2,1,-1)]).is_smooth()
True
sage: Cone([(1,0,0), (2,1,1)]).is_smooth()
```

(continues on next page)
is_solid()

Check if this cone is solid.

A cone is said to be solid if it has nonempty interior. That is, if its extreme rays span the entire ambient space.

An alias is is_full_dimensional().

OUTPUT:

True if this cone is solid, and False otherwise.

See also:

is_proper()

EXAMPLES:

The nonnegative orthant is always solid:

```
sage: quadrant = cones.nonnegative_orthant(2)
sage: quadrant.is_solid()
True
```

However, if we embed the two-dimensional nonnegative quadrant into three-dimensional space, then the resulting cone no longer has interior, so it is not solid:

```
sage: quadrant = Cone([(1,0,0), (0,1,0)])
sage: quadrant.is_solid()
False
```

is_strictly_convex()

Check if self is strictly convex.

A cone is called strictly convex if it does not contain any lines.

OUTPUT:

- True if self is strictly convex, False otherwise.

EXAMPLES:

```
sage: cone1 = Cone([(1,0), (0, 1)])
sage: cone1.is_strictly_convex()
True
```
EXAMPLES:

```
sage: c0 = cones.trivial(3)
sage: c0.is_trivial()
True
sage: c0.nrays()
0
```

**is_universe()**

Check if this cone is equal to its ambient vector space.

An alias is *is_universe()*.

**OUTPUT:**

True if this cone equals its entire ambient vector space and False otherwise.

**EXAMPLES:**

A single ray in two dimensions is not equal to the entire space:

```
sage: K = Cone([(1,0)])
sage: K.is_full_space()
False
```

Neither is the nonnegative orthant:

```
sage: K = cones.nonnegative_orthant(2)
sage: K.is_full_space()
False
```

The right half-space contains a vector subspace, but it is still not equal to the entire space:

```
sage: K = Cone([(1,0), (-1,0), (0,1)])
sage: K.is_full_space()
False
```

However, if we allow conic combinations of both axes, then the resulting cone is the entire two-dimensional space:

```
sage: K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space()
True
```

**lineality()**

Return the lineality of this cone.

The lineality of a cone is the dimension of the largest linear subspace contained in that cone.

**OUTPUT:**

A nonnegative integer; the dimension of the largest subspace contained within this cone.

**REFERENCES:**

- [Roc1970]

**EXAMPLES:**

The lineality of the nonnegative orthant is zero, since it clearly contains no lines:
However, if we add another ray so that the entire $x$-axis belongs to the cone, then the resulting cone will have lineality one:

```python
sage: K = Cone([(1,0,0), (-1,0,0), (0,1,0), (0,0,1)])
sage: K.lineality()
1
```

If our cone is all of $\mathbb{R}^2$, then its lineality is equal to the dimension of the ambient space (i.e. two):

```python
sage: K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space()
True
sage: K.lineality()
2
sage: K.lattice_dim()
2
```

Per the definition, the lineality of the trivial cone in a trivial space is zero:

```python
sage: K = cones.trivial(0)
sage: K.lineality()
0
```

**linear_subspace()**

Return the largest linear subspace contained inside of `self`.

**OUTPUT:**

- subspace of the ambient space of `self`.

**EXAMPLES:**

```python
sage: halfplane = Cone([(1,0), (0,1), (-1,0)])
sage: halfplane.linear_subspace()
Vector space of degree 2 and dimension 1 over Rational Field
Basis matrix:
[1 0]
```

**lines()**

Return lines generating the linear subspace of `self`.

**OUTPUT:**

- tuple of primitive vectors in the lattice of `self` giving directions of lines that span the linear subspace of `self`. These lines are arbitrary, but fixed. If you do not care about the order, see also `line_set()`.

**EXAMPLES:**

```python
sage: halfplane = Cone([(1,0), (0,1), (-1,0)])
sage: halfplane.lines()
N(1, 0)
in 2-d lattice N
sage: fullplane = Cone([(1,0), (0,1), (-1,-1)])
sage: fullplane.lines()
(continues on next page)```
Compute a basis of Lyapunov-like transformations on this cone.

A linear transformation $L$ is said to be Lyapunov-like on this cone if $L(x)$ and $s$ are orthogonal for every pair $(x, s)$ in its discrete_complementarity_set(). The set of all such transformations forms a vector space, namely the Lie algebra of the automorphism group of this cone.

OUTPUT:
A list of matrices forming a basis for the space of all Lyapunov-like transformations on this cone.

See also:
cross_positive_operators_gens(), positive_operators_gens(), Z_operators_gens()

REFERENCES:
• [Or2017]
• [RNPA2011]

EXAMPLES:
Every transformation is Lyapunov-like on the trivial cone:

```
sage: K = cones.trivial(2)
sage: M = MatrixSpace(K.lattice().base_field(), K.lattice_dim())
sage: list(M.basis()) == K.lyapunov_like_basis()
True
```

And by duality, every transformation is Lyapunov-like on the ambient space:

```
sage: K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space()  
True
sage: M = MatrixSpace(K.lattice().base_field(), K.lattice_dim())
sage: list(M.basis()) == K.lyapunov_like_basis()
True
```

However, in a trivial space, there are no non-trivial linear maps, so there can be no Lyapunov-like basis:

```
sage: K = cones.trivial(0)
sage: K.lyapunov_like_basis()
[]
```

The Lyapunov-like transformations on the nonnegative orthant are diagonal matrices:

```
sage: K = cones.nonnegative_orthant(1)
sage: K.lyapunov_like_basis()
[[1]]
sage: K = cones.nonnegative_orthant(2)
sage: K.lyapunov_like_basis()
[]
```
Only the identity matrix is Lyapunov-like on the pyramids defined by the one- and infinity-norms [RNPA2011]:

```
sage: l31 = Cone([(1,0,1), (0,-1,1), (-1,0,1), (0,1,1)])
sage: l31.lyapunov_like_basis()
[ [1 0 0], [0 1 0], [0 0 1] ]
sage: l3infty = Cone([(0,1,1), (1,0,1), (0,-1,1), (-1,0,1)])
sage: l3infty.lyapunov_like_basis()
[ [1 0 0], [0 1 0], [0 0 1] ]
```

**lyapunov_rank()**

Compute the Lyapunov rank of this cone.

The Lyapunov rank of a cone is the dimension of the space of its Lyapunov-like transformations — that is, the length of a `lyapunov_like_basis()`. Equivalently, the Lyapunov rank is the dimension of the Lie algebra of the automorphism group of the cone.

**OUTPUT:**

A nonnegative integer representing the Lyapunov rank of this cone.

If the ambient space is trivial, then the Lyapunov rank will be zero. On the other hand, if the dimension of the ambient vector space is \( n > 0 \), then the resulting Lyapunov rank will be between 1 and \( n^2 \) inclusive. If this cone is `is_proper()`, then that upper bound reduces from \( n^2 \) to \( n \). A Lyapunov rank of \( n - 1 \) is not possible (by Lemma 6 [Or2017]) in either case.

**ALGORITHM:**

Algorithm 3 [Or2017] is used. Every closed convex cone is isomorphic to a Cartesian product of a proper cone, a subspace, and a trivial cone. The Lyapunov ranks of the subspace and trivial cone are easy to compute. Essentially, we “peel off” those easy parts of the cone and compute their Lyapunov ranks separately. We then compute the rank of the proper cone by counting a `lyapunov_like_basis()` for it. Summing the individual ranks gives the Lyapunov rank of the original cone.

**REFERENCES:**

- [GT2014]
• [Or2017]
• [RNPA2011]

EXAMPLES:
The Lyapunov rank of the nonnegative orthant is the same as the dimension of the ambient space [RNPA2011]:

```
sage: positives = cones.nonnegative_orhant(1)
sage: positives.lyapunov_rank()
1
sage: quadrant = cones.nonnegative_orhant(2)
sage: quadrant.lyapunov_rank()
2
sage: octant = cones.nonnegative_orhant(3)
sage: octant.lyapunov_rank()
3
```

A vector space of dimension $n$ has Lyapunov rank $n^2$ [Or2017]:

```
sage: Q5 = VectorSpace(QQ, 5)
sage: gs = Q5.basis() + [-r for r in Q5.basis()]
sage: K = Cone(gs)
sage: K.lyapunov_rank()
25
```

A pyramid in three dimensions has Lyapunov rank one [RNPA2011]:

```
sage: l31 = Cone([(1,0,1), (0,-1,1), (-1,0,1), (0,1,1)])
sage: l31.lyapunov_rank()
1
sage: l3infty = Cone([(0,1,1), (1,0,1), (0,-1,1), (-1,0,1)])
sage: l3infty.lyapunov_rank()
1
```

A ray in $n$ dimensions has Lyapunov rank $n^2 - n + 1$ [Or2017]:

```
sage: K = Cone([(1,0,0,0,0)])
sage: K.lyapunov_rank()
21
sage: K.lattice_dim()**2 - K.lattice_dim() + 1
21
```

A subspace of dimension $m$ in an $n$-dimensional ambient space has Lyapunov rank $n^2 - m(n-m)$ [Or2017]:

```
sage: e1 = vector(QQ, [1,0,0,0,0])
sage: e2 = vector(QQ, [0,1,0,0,0])
sage: z = (0,0,0,0,0)
sage: K = Cone([e1, -e1, e2, -e2, z, z, z])
sage: K.lyapunov_rank()
19
sage: K.lattice_dim()**2 - K.dim()*K.codim()
19
```

Lyapunov rank is additive on a product of proper cones [RNPA2011]:

```
sage: l31 = Cone([(1,0,1), (0,-1,1), (-1,0,1), (0,1,1)])
sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
```

(continues on next page)
Two linearly-isomorphic cones have the same Lyapunov rank [RNPA2011]. A cone linearly-isomorphic to the nonnegative octant will have Lyapunov rank 3:

```sage
sage: K = Cone([[1,2,3], [-1,1,0], (1,0,6)])
sage: K.orthogonal_sublattice()
Sublattice <N(1, 1, 1), N(0, -1, 0)>
```

Lyapunov rank is invariant under `dual()` [RNPA2011]:

```sage
sage: K = Cone([[2,2,4], (-1,9,0), (2,0,6)])
sage: K.orthogonal_sublattice() == K.dual().lyapunov_rank()
True
```

**orthogonal_sublattice** (*args, **kwds*)

The sublattice (in the dual lattice) orthogonal to the sublattice spanned by the cone.

Let $M = self.dual_lattice()$ be the lattice dual to the ambient lattice of the given cone $\sigma$. Then, in the notation of [Ful1993], this method returns the sublattice

$$M(\sigma) \overset{\text{def}}{=} \sigma^\perp \cap M \subset M$$

**INPUT:**

- either nothing or something that can be turned into an element of this lattice.

**OUTPUT:**

- if no arguments were given, a **toric sublattice**, otherwise the corresponding element of it.

**EXAMPLES:**

```sage
sage: c = Cone([[1,1,1], (1,-1,1), (-1,-1,1), (-1,1,1)])
sage: c.orthogonal_sublattice()
Sublattice <N(1, 1, 1), N(0, -1, 0)>
```

**plot(** **options**)

Plot self.

**INPUT:**

- any options for toric plots (see `toric_plotter.options`), none are mandatory.

**OUTPUT:**

- a plot.

**EXAMPLES:**
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.plot()

# needs sage.plot

Graphics object consisting of 9 graphics primitives

polyhedron(**kwds)

Return the polyhedron associated to self.

Mathematically this polyhedron is the same as self.

OUTPUT:

• *Polyhedron_base*.

EXAMPLES:

sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.polyhedron()
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and 2 rays
sage: line = Cone([(1,0), (-1,0)])
sage: line.polyhedron()
A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and 1 line

Here is an example of a trivial cone (see github issue #10237):

sage: origin = Cone([], lattice=ZZ^2)
sage: origin.polyhedron()
A 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex

positive_operators_gens(K2=None)

Compute minimal generators of the positive operators on this cone.

A linear operator on a cone is positive if the image of the cone under the operator is a subset of the cone. This concept can be extended to two cones: the image of the first cone under a positive operator is a subset of the second cone, which may live in a different space.

The positive operators (on one or two fixed cones) themselves form a closed convex cone. This method computes and returns the generators of that cone as a list of matrices.

INPUT:

• *K2* – (default: *self*) the codomain cone; the image of this cone under the returned generators is a subset of *K2*.

OUTPUT:

A list of *m*-by-*n* matrices where *m* is the ambient dimension of *K2* and *n* is the ambient dimension of this cone. Each matrix *P* in the list has the property that *P(x)* is an element of *K2* whenever *x* is an element of this cone.

The returned matrices generate the cone of positive operators from this cone to *K2*; that is,

• Any nonnegative linear combination of the returned matrices sends elements of this cone to *K2*.
• Every positive operator on this cone (with respect to *K2*) is some nonnegative linear combination of the returned matrices.

ALGORITHM:
Computing positive operators directly is difficult, but computing their dual is straightforward using the generators of Berman and Gaiha. We construct the dual of the positive operators, and then return the dual of that, which is guaranteed to be the desired positive operators because everything is closed, convex, and polyhedral.

See also:
cross_positive_operators_gens(), lyapunov_like_basis(), Z_operators_gens()

REFERENCES:
- [BG1972]
- [BP1994]
- [Or2018b]

EXAMPLES:

Positive operators on the nonnegative orthant are nonnegative matrices:

```sage
K = Cone([(1,)])
sage: K.positive_operators_gens()

[[1]]

K = Cone([(1,0), (0,1)])
sage: K.positive_operators_gens()

[[1 0]
  [0 1]
  [0 0]
  [0 0]]
```

The trivial cone in a trivial space has no positive operators:

```sage
K = cones.trivial(0)
sage: K.positive_operators_gens()

[]
```

Every operator is positive on the trivial cone:

```sage
K = cones.trivial(1)
sage: K.positive_operators_gens()

[[1], [-1]]
```

Every operator is positive on the ambient vector space:

```sage
K = Cone([(1,), (-1,)])
sage: K.is_full_space()
True
sage: K.positive_operators_gens()

[[1], [-1]]
```

(continues on next page)
A non-obvious application is to find the positive operators on the right half-plane [Or2018b]:

```
sage: K = Cone([(1,0), (0,1), (0,-1)])
sage: K.positive_operators_gens()
[ [1 0] [0 0] [0 0] [0 0], [1 0], [-1 0], [0 1], [ 0 -1]
 [0 0], [0 0], [0 0], [0 0], [1 0], [-1 0], [0 1], [ 0 -1]
]
```

\section*{random_element} \texttt{(ring=Integer Ring)}

Return a random element of this cone.

All elements of a convex cone can be represented as a nonnegative linear combination of its generators. A random element is thus constructed by assigning random nonnegative weights to the generators of this cone. By default, these weights are integral and the resulting random element will live in the same lattice as the cone.

The random nonnegative weights are chosen from \texttt{ring} which defaults to \texttt{ZZ}. When \texttt{ring} is not \texttt{ZZ}, the random element returned will be a vector. Only the rings \texttt{ZZ} and \texttt{QQ} are currently supported.

INPUT:

- \texttt{ring} – (default: \texttt{ZZ}) the ring from which the random generator weights are chosen; either \texttt{ZZ} or \texttt{QQ}.

OUTPUT:

Either a lattice element or vector contained in both this cone and its ambient vector space. If \texttt{ring} is \texttt{ZZ}, a lattice element is returned; otherwise a vector is returned. If \texttt{ring} is neither \texttt{ZZ} nor \texttt{QQ}, then a \texttt{NotImplementedError} is raised.

EXAMPLES:

The trivial element () is always returned in a trivial space:

```
sage: K = cones.trivial(0)
sage: K.random_element()
N()
sage: K.random_element(ring=QQ)
()
```

A random element of the trivial cone in a nontrivial space is zero:

```
sage: K = cones.trivial(3)
sage: K.random_element()
N(0, 0, 0)
sage: K.random_element(ring=QQ)
(0, 0, 0)
```

A random element of the nonnegative orthant should have all components nonnegative:
sage: K = cones.nonnegative_orthant(3)
sage: all(x >= 0 for x in K.random_element())
True
sage: all(x >= 0 for x in K.random_element(ring=QQ))
True

If ring is not ZZ or QQ, an error is raised:

sage: K = Cone([(1,0), (0,1)])
sage: K.random_element(ring=RR)
Traceback (most recent call last):
  ...  
NotImplementedError: ring must be either ZZ or QQ.

relative_interior()

Return the relative interior of self.

OUTPUT:

• either self or an instance of RelativeInterior.

EXAMPLES:

sage: c = Cone([(1,0,0), (0,1,0)]); c
2-d cone in 3-d lattice N
sage: c.relative_interior()
Relative interior of 2-d cone in 3-d lattice N
sage: origin = cones.trivial(2); origin
0-d cone in 2-d lattice N
sage: origin.relative_interior() is origin
True
sage: K1 = Cone([(1,0), (-1,0)]); K1
1-d cone in 2-d lattice N
sage: K1.relative_interior() is K1
True
sage: K2 = Cone([(1,0),(-1,0),(0,1),(0,-1)]); K2
2-d cone in 2-d lattice N
sage: K2.relative_interior() is K2
True

relative_interior_contains(*args)

Check if a given point is contained in the relative interior of self.

For a full-dimensional cone the relative interior is simply the interior, so this method will do the same check as interior_contains(). For a strictly lower-dimensional cone, the relative interior is the cone without its facets.

INPUT:

• anything. An attempt will be made to convert all arguments into a single element of the ambient space of self. If it fails, False will be returned.

OUTPUT:

• True if the given point is contained in the relative interior of self, False otherwise.

EXAMPLES:
**relative_orthogonal_quotient** (supercone)

The quotient of the dual spanned lattice by the dual of the supercone’s spanned lattice.

In the notation of [Ful1993], if supercone = ρ > σ = self is a cone that contains σ as a face, then $M(\rho) = \text{supercone.orthogonal_sublattice()}$ is a saturated sublattice of $M(\sigma) = \text{self.orthogonal_sublattice()}$. This method returns the quotient lattice. The lifts of the quotient generators are $\dim(\rho) - \dim(\sigma)$ linearly independent $M$-lattice lattice points that, together with $M(\rho)$, generate $M(\sigma)$.

**OUTPUT:**

- toric lattice quotient.

If we call the output Mrho, then

- Mrho.cover() == self.orthogonal_sublattice(), and
- Mrho.relations() == supercone.orthogonal_sublattice().

**Note:**

- $M(\sigma)/M(\rho)$ has no torsion since the sublattice $M(\rho)$ is saturated.
- In the codimension one case, (a lift of) the generator of $M(\sigma)/M(\rho)$ is chosen to be positive on $\sigma$.

**EXAMPLES:**

```python
sage: # needs sage.graphs
sage: rho = Cone([(1,1,1,3), (1,-1,1,3), (-1,-1,1,3), (-1,1,1,3)])
sage: rho.orthogonal_sublattice()
Sublattice <M(0, 0, 3, -1)>

sage: sigma = rho.facets()[1]
sage: sigma.orthogonal_sublattice()
Sublattice <M(0, 1, 1, 0), M(0, 0, 3, -1)>

sage: sigma.is_face_of(rho)
True

sage: Q = sigma.relative_orthogonal_quotient(rho); Q
1-d lattice, quotient
of Sublattice <M(0, 1, 1, 0), M(0, 0, 3, -1)>
by Sublattice <M(0, 0, 3, -1)>

sage: Q.gens()
(M[0, 1, 1, 0],)
```

Different codimension:
sage: # needs sage.graphs
sage: rho = Cone([[1,-1,1,3],[-1,-1,1,3]])
sage: sigma = rho.facets()[0]
sage: sigma.orthogonal_sublattice()
Sublattice <M(1, 0, 2, -1), M(0, 1, 1, 0), M(0, 0, 3, -1)>
sage: rho.orthogonal_sublattice()
Sublattice <M(0, 1, 1, 0), M(0, 0, 3, -1)>
sage: sigma.relative_orthogonal_quotient(rho).gens()
(M[-1, 0, -2, 1],)

Sign choice in the codimension one case:

sage: sigma1 = Cone([(1, 2, 3), (1, -1, 1), (-1, 1, 1), (-1, -1, 1)])
# 3d
sage: sigma2 = Cone([(1, 1, -1), (1, 2, 3), (1, -1, 1), (1, -1, -1)])
# 3d
sage: rho = sigma1.intersection(sigma2)
sage: rho.relative_orthogonal_quotient(sigma1).gens()
(M[-5, -2, 3],)
sage: rho.relative_orthogonal_quotient(sigma2).gens()
(M[5, 2, -3],)

relative_orthogonal_quotient (subcone)

The quotient of the spanned lattice by the lattice spanned by a subcone.

In the notation of [Ful1993], let $\mathcal{N}$ be the ambient lattice and $\mathcal{N}_\sigma$ the sublattice spanned by the given cone $\sigma$. If $\rho < \sigma$ is a subcone, then $\mathcal{N}_\rho = \rho.sublattice()$ is a saturated sublattice of $\mathcal{N}_\sigma = self.sublattice()$. This method returns the quotient lattice. The lifts of the quotient generators are $\dim(\sigma) - \dim(\rho)$ linearly independent primitive lattice points that, together with $\mathcal{N}_\rho$, generate $\mathcal{N}_\sigma$.

OUTPUT:

- toric lattice quotient.

Note:

- The quotient $\mathcal{N}_\sigma / \mathcal{N}_\rho$ of spanned sublattices has no torsion since the sublattice $\mathcal{N}_\rho$ is saturated.
- In the codimension one case, the generator of $\mathcal{N}_\sigma / \mathcal{N}_\rho$ is chosen to be in the same direction as the image $\sigma / \mathcal{N}_\rho$.

EXAMPLES:

sage: sigma = Cone([(1,1,1,3),(1,-1,1,3),(-1,-1,1,3),(-1,1,1,3)])
sage: rho = Cone([(-1, -1, 1, 3), (-1, 1, 1, 3)])
sage: sigma.sublattice()
Sublattice <N(1, 1, 1, 3), N(0, -1, 0, 0), N(-1, -1, 0, 0)>
sage: rho.sublattice()
Sublattice <N(-1, -1, 1, 3), N(0, 1, 0, 0)>
sage: sigma.relative_orthogonal_quotient(rho)
1-d lattice, quotient of Sublattice <N(1, 1, 1, 3), N(0, -1, 0, 0), N(-1, -1, 0, 0)>
by Sublattice <N(1, 0, -1, -3), N(0, 1, 0, 0)>
sage: sigma.relative_orthogonal_quotient(rho).gens()
(N[1, 0, 0, 0],)

More complicated example:
sage: rho = Cone([(1, 2, 3), (1, -1, 1)])
sage: sigma = Cone([(1, 2, 3), (1, -1, 1), (-1, 1, 1), (-1, -1, 1)])

sage: N_sigma = sigma.sublattice()
sage: N_rho = rho.sublattice()

sage: N_sigma
Sublattice <N(1, 2, 3), N(1, -1, 1), N(-1, -1, -2)>
sage: N_rho
Sublattice <N(1, -1, 1), N(1, 2, 3)>

sage: sigma.relative_quotient(rho).gens()
(N[-1, -1, -2],)
sage: N = rho.lattice()
sage: N_sigma == N.span(N_rho.gens() + tuple(q.lift() for q in sigma.relative_quotient(rho).gens()))
True

Sign choice in the codimension one case:

sage: sigma1 = Cone([(1, 2, 3), (1, -1, 1), (-1, 1, 1), (-1, -1, 1)])  # 3d
sage: sigma2 = Cone([(1, 1, -1), (1, 2, 3), (1, -1, 1), (1, -1, -1)])  # 3d
sage: rho = sigma1.intersection(sigma2)

sage: rho.sublattice()
Sublattice <N(1, -1, 1), N(1, 2, 3)>
sage: sigma1.relative_quotient(rho)
1-d lattice, quotient of Sublattice <N(1, 2, 3), N(1, -1, 1), N(-1, -1, -2)>
by Sublattice <N(1, 2, 3), N(0, 3, 2)>

sage: sigma1.relative_quotient(rho).gens()
(N[-1, -1, -2],)
sage: sigma2.relative_quotient(rho).gens()
(N[0, 2, 1],)

semigroup_generators()

Return generators for the semigroup of lattice points of self.

OUTPUT:

• a PointCollection of lattice points generating the semigroup of lattice points contained in self.

Note: No attempt is made to return a minimal set of generators, see Hilbert_basis() for that.

EXAMPLES:

The following command ensures that the output ordering in the examples below is independent of TOPCOM, you don’t have to use it:

sage: PointConfiguration.set_engine('internal')

We start with a simple case of a non-smooth 2-dimensional cone:

sage: Cone([(1,0), (1,2)]).semigroup_generators()
N(1, 1),
N(1, 0),
N(1, 2)
in 2-d lattice N

A non-simplicial cone works, too:
sage: cone = Cone([(3,0,-1), (1,-1,0), (0,1,0), (0,0,1)])
sage: sorted(cone.semigroup_generators())
[N(0, 0, 1), N(0, 1, 0), N(1, -1, 0), N(1, 0, 0), N(3, 0, -1)]

GAP’s toric package thinks this is challenging:

sage: cone = Cone([[1,2,3,4], [0,1,0,7], [3,1,0,2], [0,0,1,0]]).dual()
sage: len(cone.semigroup_generators())
2806

The cone need not be strictly convex:

sage: halfplane = Cone([(1,0), (2,1), (-1,0)])
sage: sorted(halfplane.semigroup_generators())
[N(-1, 0), N(0, 1), N(1, 0)]
sage: line = Cone([(1,1,1), (-1,-1,-1)])
sage: sorted(line.semigroup_generators())
[N(-1, -1, -1), N(1, 1, 1)]
sage: wedge = Cone([(1,0,0), (1,2,0), (0,0,1), (0,0,-1)])
sage: sorted(wedge.semigroup_generators())
[N(0, 0, -1), N(0, 0, 1), N(1, 0, 0), N(1, 1, 0), N(1, 2, 0)]

Nor does it have to be full-dimensional (see github issue #11312):

sage: Cone([[1,1,0], (-1,1,0)]).semigroup_generators()
N(0, 0, 0),
N(1, 1, 0),
N(-1, 1, 0)
in 3-d lattice N

Neither full-dimensional nor simplicial:

sage: A = matrix([[1, 3, 0, -1, 0, 1, 1, -2, 15, -2, 0]])

sage: A.elementary_divisors()
[1, 1, 1, 0]
sage: cone3d = Cone([(3,0,-1), (1,-1,0), (0,1,0), (0,0,1)])
sage: rays = (A*vector(v) for v in cone3d.rays())
sage: gens = Cone(rays).semigroup_generators(); sorted(gens)
[N(-2, -1, 0, 17),
N(0, 1, -2, 0),
N(1, -1, 1, 15),
N(3, -4, 5, 45),
N(3, 0, 1, -2)]
sage: set(map(tuple,gens)) == set(tuple(A*r) for r in cone3d.semigroup_generators())
True

ALGORITHM:

If the cone is not simplicial, it is first triangulated. Each simplicial subcone has the integral points of the spanned parallelotope as generators. This is the first step of the primal Normaliz algorithm, see [Normaliz]. For each simplicial cone (of dimension $d$), the integral points of the open parallelotope

$$par(x_1, \ldots, x_d) = \mathbb{Z}^n \cap \{q_1x_1 + \cdots + q_dx_d : 0 \leq q_i < 1\}$$

are then computed [BK2001].

Finally, the union of the generators of all simplicial subcones is returned.
solid_restriction()

Return a solid representation of this cone in terms of a basis of its sublattice().

We define the solid restriction of a cone to be a representation of that cone in a basis of its own sublattice. Since a cone's sublattice is just large enough to hold the cone (by definition), the resulting solid restriction is_solid(). For convenience, the solid restriction lives in a new lattice (of the appropriate dimension) and not actually in the sublattice object returned by sublattice().

OUTPUT:

A solid cone in a new lattice having the same dimension as this cone’s sublattice().

EXAMPLES:

The nonnegative quadrant in the plane is left after we take its solid restriction in space:

```
sage: K = Cone([((1,0,0), (0,1,0))]
sage: K.solid_restriction().rays()
N(0, 1),
N(1, 0)
in 2-d lattice N
```

The solid restriction of a single ray has the same representation regardless of the ambient space:

```
sage: K = Cone([((1,0))]
sage: K.solid_restriction().rays()
N(1)
in 1-d lattice N
```

```
sage: K = Cone([((1,1,1))]
sage: K.solid_restriction().rays()
N(1)
in 1-d lattice N
```

The solid restriction of the trivial cone lives in a trivial space:

```
sage: K = cones.trivial(0)
sage: K.solid_restriction()
0-d cone in 0-d lattice N
```

```
sage: K = cones.trivial(4)
sage: K.solid_restriction()
0-d cone in 0-d lattice N
```

The solid restriction of a solid cone is itself:

```
sage: K = Cone([((1,1),(1,2))]
sage: K.solid_restriction() is K
True
```

strict_quotient()

Return the quotient of self by the linear subspace.

We define the strict quotient of a cone to be the image of this cone in the quotient of the ambient space by the linear subspace of the cone, i.e. it is the “complementary part” to the linear subspace.

OUTPUT:

- cone.

EXAMPLES:
Sublattice \((\ast args, **kwds)\)

The sublattice spanned by the cone.

Let \(\sigma\) be the given cone and \(N = \text{self.lattice()\)} the ambient lattice. Then, in the notation of [Ful1993],
this method returns the sublattice

\[
N_\sigma \equiv \text{span}(N \cap \sigma)
\]

INPUT:

- either nothing or something that can be turned into an element of this lattice.

OUTPUT:

- if no arguments were given, a toric sublattice, otherwise the corresponding element of it.

Note:

- The sublattice spanned by the cone is the saturation of the sublattice generated by the rays of the cone.
- If you only need a \(Q\)-basis, you may want to try the \(\text{basis()}\) method on the result of \(\text{rays()}\).
- The returned lattice points are usually not rays of the cone. In fact, for a non-smooth cone the rays do not generate the sublattice \(N_\sigma\), but only a finite index sublattice.

EXAMPLES:

```python
sage: cone = Cone([(1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, 1, 1)])
sage: cone.rays().basis()
N( 1, 1, 1),
N( 1, -1, 1),
N(-1, -1, 1)
in 3-d lattice N
sage: cone.rays().basis().matrix().det()
```

(continues on next page)
Another example:

```
sage: c = Cone([(1,2,3), (4,-5,1)])
sage: c
2-d cone in 3-d lattice N
sage: c.rays()
N(1, 2, 3),
N(4, -5, 1)
in 3-d lattice N
sage: c.sublattice()
Sublattice <N(4, -5, 1), N(1, 2, 3)>
sage: c.sublattice(5, -3, 4)
N(5, -3, 4)
sage: c.sublattice(1, 0, 0)
Traceback (most recent call last):
...TypeError: element [1, 0, 0] is not in free module
```

```
sublattice_complement(*args, **kwds)
A complement of the sublattice spanned by the cone.
In other words, sublattice() and sublattice_complement() together form a \(\mathbb{Z}\)-basis for the ambient lattice().
In the notation of [Ful1993], let \(\sigma\) be the given cone and \(N = self.lattice()\) the ambient lattice. Then this method returns

\[
N(\sigma) \overset{\text{def}}{=} N / N_{\sigma}
\]

lifted (non-canonically) to a sublattice of \(N\).

INPUT:

- either nothing or something that can be turned into an element of this lattice.

OUTPUT:

- if no arguments were given, a toric sublattice, otherwise the corresponding element of it.

EXAMPLES:

```
sage: C2_Z2 = Cone([(1,0), (1,2)])  # C^2/\mathbb{Z}_2
sage: c1, c2 = C2_Z2.facets()        # needs sage.graphs
sage: c2.sublattice()               # needs sage.graphs
Sublattice <N(1, 2)>
sage: c2.sublattice_complement()   # needs sage.graphs
Sublattice <N(0, 1)>
```

A more complicated example:
sage: c = Cone([(1,2,3), (4,-5,1)])
sage: c.sublattice()
Sublattice \langle N(4, -5, 1), N(1, 2, 3) \rangle
sage: c.sublattice_complement()
Sublattice \langle N(2, -3, 0) \rangle
sage: m = matrix( c.sublattice().gens() + c.sublattice_complement().gens() )
sage: m
\begin{bmatrix}
  4 & -5 & 1 \\
  1 & 2 & 3 \\
  2 & -3 & 0
\end{bmatrix}
sage: m.det()
-1

sublattice_quotient (*args, **kwds)
The quotient of the ambient lattice by the sublattice spanned by the cone.

INPUT:

• either nothing or something that can be turned into an element of this lattice.

OUTPUT:

• if no arguments were given, a quotient of a toric lattice, otherwise the corresponding element of it.

EXAMPLES:

sage: # needs sage.graphs
sage: C2_Z2 = Cone([(1,0), (1,2)])  # C^2/Z_2
sage: c1, c2 = C2_Z2.facets()
sage: c2.sublattice_quotient()
1-d lattice, quotient of 2-d lattice \( N \) by Sublattice \langle N(1, 2) \rangle
sage: N = C2_Z2.lattice()
sage: n = N(1,1)
sage: n_bar = c2.sublattice_quotient(n); n_bar
N[1, 1]
sage: n_bar.lift()
N(1, 1)
sage: vector(n_bar)
(-1)

class sage.geometry.cone.IntegralRayCollection(rays, lattice)

Bases: SageObject, Hashable, Iterable

Create a collection of integral rays.

Warning: No correctness check or normalization is performed on the input data. This class is designed for internal operations and you probably should not use it directly.

This is a base class for convex rational polyhedral cones and fans.

Ray collections are immutable, but they cache most of the returned values.

INPUT:

• rays – list of immutable vectors in lattice;

• lattice – ToricLattice, \( \mathbb{Z}^n \), or any other object that behaves like these. If None, it will be determined as parent() of the first ray. Of course, this cannot be done if there are no rays, so in this case you must
give an appropriate lattice directly. Note that None is not the default value - you always must give this argument explicitly, even if it is None.

OUTPUT:

• collection of given integral rays.

ambient_dim()

Return the dimension of the ambient lattice of self.

An alias is ambient_dim().

OUTPUT:

• integer.

EXAMPLES:

sage: c = Cone([(1,0)])
sage: c.lattice_dim() 2
sage: c.dim() 1

ambient_vector_space(base_field=None)

Return the ambient vector space.

It is the ambient lattice (lattice()) tensored with a field.

INPUT:

• base_field – (default: the rationals) a field.

EXAMPLES:

sage: c = Cone([(1,0)])
Vector space of dimension 2 over Rational Field
sage: c.ambient_vector_space(AA) # needs sage.rings.number_field
Vector space of dimension 2 over Algebraic Real Field

cartesian_product(other, lattice=None)

Return the Cartesian product of self with other.

INPUT:

• other – an IntegralRayCollection;

• lattice – (optional) the ambient lattice for the result. By default, the direct sum of the ambient lattices of self and other is constructed.

OUTPUT:

• an IntegralRayCollection.

By the Cartesian product of ray collections \((r_0, \ldots, r_{n-1})\) and \((s_0, \ldots, s_{m-1})\) we understand the ray collection of the form \(((r_0, 0), \ldots, (r_{n-1}, 0), (0, s_0), \ldots, (0, s_{m-1}))\), which is suitable for Cartesian products of cones and fans. The ray order is guaranteed to be as described.

EXAMPLES:
sage: c = Cone([[(1,)]])
sage: c.cartesian_product(c)  # indirect doctest
2-d cone in 2-d lattice N+N
sage: _.rays()
N+N(1, 0),
N+N(0, 1)
in 2-d lattice N+N

codim()

Return the codimension of self.

The codimension of a collection of rays (of a cone/fan) is the difference between the dimension of the ambient space and the dimension of the subspace spanned by those rays (of the cone/fan).

OUTPUT:

A nonnegative integer representing the codimension of self.

See also:

dim(), lattice_dim()

EXAMPLES:

The codimension of the nonnegative orthant is zero, since the span of its generators equals the entire ambient space:

sage: K = cones.nonnegative_orthant(3)
sage: K.codim()
0

However, if we remove a ray so that the entire cone is contained within the $x$-$y$ plane, then the resulting cone will have codimension one, because the $z$-axis is perpendicular to every element of the cone:

sage: K = Cone([[(1,0,0), (0,1,0)]]
sage: K.codim()
1

If our cone is all of $\mathbb{R}^2$, then its codimension is zero:

sage: K = Cone([[(1,0), (-1,0), (0,1), (0,-1)]]
sage: K.is_full_space()
True
sage: K.lattice_dim()
0
sage: K.codim()
0

And if the cone is trivial in any space, then its codimension is equal to the dimension of the ambient space:

sage: K = cones.trivial(0)
sage: K.lattice_dim()
0
sage: K.codim()
0

sage: K = cones.trivial(1)
sage: K.lattice_dim()
1
sage: K.codim()
1

(continues on next page)
**codimension()**

Return the codimension of self.

The codimension of a collection of rays (of a cone/fan) is the difference between the dimension of the ambient space and the dimension of the subspace spanned by those rays (of the cone/fan).

**OUTPUT:**

A nonnegative integer representing the codimension of self.

**See also:**

`dim()`, `lattice_dim()`

**EXAMPLES:**

The codimension of the nonnegative orthant is zero, since the span of its generators equals the entire ambient space:

```
sage: K = cones.nonnegative_orthant(3)
sage: K.codim() 0
```

However, if we remove a ray so that the entire cone is contained within the \(x-y\) plane, then the resulting cone will have codimension one, because the \(z\)-axis is perpendicular to every element of the cone:

```
sage: K = Cone([(1,0,0), (0,1,0)])
sage: K.codim() 1
```

If our cone is all of \(\mathbb{R}^2\), then its codimension is zero:

```
sage: K = Cone([(1,0), (-1,0), (0,1), (0,-1)])
sage: K.is_full_space() True
sage: K.codim() 0
```

And if the cone is trivial in any space, then its codimension is equal to the dimension of the ambient space:

```
sage: K = cones.trivial(0)
sage: K.lattice_dim() 0
sage: K.codim() 0
sage: K = cones.trivial(1)
sage: K.lattice_dim() 1
sage: K.codim() 1
```
sage: K = cones.trivial(2)
sage: K.lattice_dim()
2
sage: K.codim()
2

dim()

Return the dimension of the subspace spanned by rays of self.

OUTPUT:

• integer.

EXAMPLES:

sage: c = Cone([(1,0)])
sage: c.lattice_dim()
2
sage: c.dim()
1

dual_lattice()

Return the dual of the ambient lattice of self.

OUTPUT:

• lattice. If possible (that is, if lattice() has a dual() method), the dual lattice is returned. Otherwise, \( \mathbb{Z}^n \) is returned, where \( n \) is the dimension of lattice().

EXAMPLES:

sage: c = Cone([(1,0)])
sage: c.dual_lattice()
2-d lattice M
sage: Cone([], ZZ^3).dual_lattice()
Ambient free module of rank 3
over the principal ideal domain Integer Ring

lattice()

Return the ambient lattice of self.

OUTPUT:

• lattice.

EXAMPLES:

sage: c = Cone([(1,0)])
sage: c.lattice()
2-d lattice N
sage: Cone([], ZZ^3).lattice()
Ambient free module of rank 3
over the principal ideal domain Integer Ring

lattice_dim()

Return the dimension of the ambient lattice of self.

An alias is \texttt{ambient_dim()}. 

OUTPUT:
• integer.

EXAMPLES:
```
sage: c = Cone([(1,0)])
sage: c.lattice_dim()
2
sage: c.dim()
1
```

\texttt{nrays()} 

Return the number of rays of \texttt{self}.

OUTPUT:
• integer.

EXAMPLES:
```
sage: c = Cone([(1,0), (0,1)])
sage: c.nrays()
2
```

\texttt{plot(**options)} 

Plot \texttt{self}.

INPUT:
• any options for toric plots (see \texttt{toric_plotter.options}), none are mandatory.

OUTPUT:
• a plot.

EXAMPLES:
```
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant.plot()  # needs \texttt{sage.plot}
Graphics object consisting of 9 graphics primitives
```

\texttt{ray(n)} 

Return the \texttt{n}-th ray of \texttt{self}.

INPUT:
• \texttt{n} – integer, an index of a ray of \texttt{self}. Enumeration of rays starts with zero.

OUTPUT:
• ray, an element of the lattice of \texttt{self}.

EXAMPLES:
```
sage: c = Cone([(1,0), (0,1)])
sage: c.ray(0)
N(1, 0)
```

2.5. Toric geometry
rays(*args)

Return (some of the) rays of self.

INPUT:

- ray_list – a list of integers, the indices of the requested rays. If not specified, all rays of self will be returned.

OUTPUT:

- a PointCollection of primitive integral ray generators.

EXAMPLES:

```python
sage: c = Cone([(1,0), (0,1), (-1, 0)])
sage: c.rays()
N( 0, 1),
N( 1, 0),
N(-1, 0)
in 2-d lattice N
sage: c.rays([0, 2])
N( 0, 1),
N(-1, 0)
in 2-d lattice N
```

You can also give ray indices directly, without packing them into a list:

```python
sage: c.rays(0, 2)
N( 0, 1),
N(-1, 0)
in 2-d lattice N
```

span(base_ring=None)

Return the span of self.

INPUT:

- base_ring – (default: from lattice) the base ring to use for the generated module.

OUTPUT:

A module spanned by the generators of self.

EXAMPLES:

The span of a single ray is a one-dimensional sublattice:

```python
sage: K1 = Cone([(1,1)])
sage: K1.span()
Sublattice <N(1)>
sage: K2 = Cone([(1,0)])
sage: K2.span()
Sublattice <N(1, 0)>
```

The span of the nonnegative orthant is the entire ambient lattice:

```python
sage: K = cones.nonnegative_orthant(3)
sage: K.span() == K.lattice()
True
```

By specifying a base_ring, we can obtain a vector space:
sage: K = Cone([(1,0,0),(0,1,0),(0,0,1)])
sage: K.span(base_ring=QQ)
Vector space of degree 3 and dimension 3 over Rational Field
Basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]

sage.geometry.cone.classify_cone_2d(ray0, ray1, check=True)

Return \((d, k)\) classifying the lattice cone spanned by the two rays.

INPUT:

- ray0, ray1 – two primitive integer vectors. The generators of the two rays generating the two-dimensional cone.
- check – boolean (default: True). Whether to check the input rays for consistency.

OUTPUT:

A pair \((d, k)\) of integers classifying the cone up to \(GL(2, \mathbb{Z})\) equivalence. See Proposition 10.1.1 of [CLS2011] for the definition. We return the unique \((d, k)\) with minimal \(k\), see Proposition 10.1.3 of [CLS2011].

EXAMPLES:

sage: from sage.geometry.cone import classify_cone_2d
sage: classify_cone_2d(ray0, ray1)
(3, 2)
sage: classify_cone_2d(ray0, ray1)
(3, 1)

sage: m = matrix(ZZ, [(19, -14, -115), (-2, 5, 25), (43, -42, -298)])
sage: m.det()  # check that it is in GL(3, ZZ)
-1
sage: classify_cone_2d(m*ray0, m*ray1)
(3, 1)

sage.geometry.cone.integral_length(v)

Compute the integral length of a given rational vector.

INPUT:

- v – any object which can be converted to a list of rationals

OUTPUT:

Rational number \(r\) such that \(v = r \cdot u\), where \(u\) is the primitive integral vector in the direction of \(v\).

EXAMPLES:

sage: from sage.geometry.cone import integral_length
sage: integral_length([1, 2, 4])
1
sage: integral_length([2, 2, 4])
2

(continues on next page)
sage: integral_length([2/3, 2, 4])
2/3

sage.geometry.cone.is_Cone(x)
Check if x is a cone.

INPUT:
• x – anything.

OUTPUT:
• True if x is a cone and False otherwise.

EXAMPLES:

sage: from sage.geometry.cone import is_Cone
sage: is_Cone(1)
False
sage: quadrant = Cone([(1,0), (0,1)])
2-d cone in 2-d lattice N
sage: is_Cone(quadrant)
True

sage.geometry.cone.normalize_rays(rays, lattice)
Normalize a list of rational rays: make them primitive and immutable.

INPUT:
• rays – list of rays which can be converted to the rational extension of lattice;
• lattice – ToricLattice, \(\mathbb{Z}^n\), or any other object that behaves like these. If None, an attempt will be made to determine an appropriate toric lattice automatically.

OUTPUT:
• list of immutable primitive vectors of the lattice in the same directions as original rays.

EXAMPLES:

sage: from sage.geometry.cone import normalize_rays
sage: normalize_rays([(0, 1), (0, 2), (3, 2), (5/7, 10/3)], None)
[N(0, 1), N(0, 1), N(3, 2), N(3, 14)]

sage: L = ToricLattice(2, "L")
sage: normalize_rays([(0, 1), (0, 2), (3, 2), (5/7, 10/3)], L.dual())
[L*(0, 1), L*(0, 1), L*(3, 2), L*(3, 14)]

sage: ray_in_L = L(0,1)
sage: normalize_rays([ray_in_L, (0, 2), (3, 2), (5/7, 10/3)], None)
[L(0, 1), L(0, 1), L(3, 2), L(3, 14)]

sage: normalize_rays([(0, 1), (0, 2), (3, 2), (5/7, 10/3)], ZZ^2)
[(0, 1), (0, 1), (3, 2), (3, 14)]

sage: normalize_rays([(0, 1), (0, 2), (3, 2), (5/7, 10/3)], ZZ^3)
Traceback (most recent call last):
  ... TypeError: cannot convert (0, 1) to Vector space of dimension 3 over Rational Field!
sage: normalize_rays([], ZZ^3)
[]
sage.geometry.cone.random_cone (lattice=None, min_ambient_dim=0, max_ambient_dim=None, min_rays=0, max_rays=None, strictly_convex=None, solid=None)

Generate a random convex rational polyhedral cone.

Lower and upper bounds may be provided for both the dimension of the ambient space and the number of generating rays of the cone. If a lower bound is left unspecified, it defaults to zero. Unspecified upper bounds will be chosen randomly, unless you set solid, in which case they are chosen a little more wisely.

You may specify the ambient lattice for the returned cone. In that case, the min_ambient_dim and max_ambient_dim parameters are ignored.

You may also request that the returned cone be strictly convex (or not). Likewise you may request that it be (non-)solid.

**Warning:** If you request a large number of rays in a low-dimensional space, you might be waiting for a while. For example, in three dimensions, it is possible to obtain an octagon raised up to height one (all z-coordinates equal to one). But in practice, we usually generate the entire three-dimensional space with six rays before we get to the eight rays needed for an octagon. We therefore have to throw the cone out and start over from scratch. This process repeats until we get lucky.

We also refrain from “adjusting” the min/max parameters given to us when a (non-)strictly convex or (non-)solid cone is requested. This means that it may take a long time to generate such a cone if the parameters are chosen unwisely.

For example, you may want to set min_rays close to min_ambient_dim if you desire a solid cone. Or, if you desire a non-strictly-convex cone, then they all contain at least two generating rays. So that might be a good candidate for min_rays.

**INPUT:**

- **lattice** (default: random) – A ToricLattice object in which the returned cone will live. By default a new lattice will be constructed with a randomly-chosen rank (subject to min_ambient_dim and max_ambient_dim).
- **min_ambient_dim** (default: zero) – A nonnegative integer representing the minimum dimension of the ambient lattice.
- **max_ambient_dim** (default: random) – A nonnegative integer representing the maximum dimension of the ambient lattice.
- **min_rays** (default: zero) – A nonnegative integer representing the minimum number of generating rays of the cone.
- **max_rays** (default: random) – A nonnegative integer representing the maximum number of generating rays of the cone.
- **strictly_convex** (default: random) – Whether or not to make the returned cone strictly convex. Specify True for a strictly convex cone, False for a non-strictly-convex cone, or None if you don’t care.
- **solid** (default: random) – Whether or not to make the returned cone solid. Specify True for a solid cone, False for a non-solid cone, or None if you don’t care.

**OUTPUT:**

A new, randomly generated cone.

A **ValueError** will be thrown under the following conditions:

- Any of min_ambient_dim, max_ambient_dim, min_rays, or max_rays are negative.
- max_ambient_dim is less than min_ambient_dim.
• \text{max\_rays} \text{ is less than } \text{min\_rays}.
• Both \text{max\_ambient\_dim} \text{ and } \text{lattice} \text{ are specified.}
• \text{min\_rays} \text{ is greater than four but } \text{max\_ambient\_dim} \text{ is less than three.}
• \text{min\_rays} \text{ is greater than four but } \text{lattice} \text{ has dimension less than three.}
• \text{min\_rays} \text{ is greater than two but } \text{max\_ambient\_dim} \text{ is less than two.}
• \text{min\_rays} \text{ is greater than two but } \text{lattice} \text{ has dimension less than two.}
• \text{min\_rays} \text{ is positive but } \text{max\_ambient\_dim} \text{ is zero.}
• \text{min\_rays} \text{ is positive but } \text{lattice} \text{ has dimension zero.}
• A trivial lattice is supplied and a non-strictly-convex cone is requested.
• A non-strictly-convex cone is requested but \text{max\_rays} \text{ is less than two.}
• A solid cone is requested but \text{max\_rays} \text{ is less than } \text{min\_ambient\_dim}.
• A solid cone is requested but \text{max\_rays} \text{ is less than the dimension of } \text{lattice}.
• A non-solid cone is requested but \text{max\_ambient\_dim} \text{ is zero.}
• A non-solid cone is requested but \text{lattice} \text{ has dimension zero.}
• A non-solid cone is requested but \text{min\_rays} \text{ is so large that it guarantees a solid cone.}

ALGORITHM:

First, a lattice is determined from \text{min\_ambient\_dim} \text{ and } \text{max\_ambient\_dim} \text{ (or from the supplied lattice).}

Then, lattice elements are generated one at a time and added to a cone. This continues until either the cone meets the user’s requirements, or the cone is equal to the entire space (at which point it is futile to generate more).

We check whether or not the resulting cone meets the user’s requirements; if it does, it is returned. If not, we throw it away and start over. This process repeats indefinitely until an appropriate cone is generated.

EXAMPLES:

Generate a trivial cone in a trivial space:

\begin{verbatim}
sage: random_cone(max_ambient_dim=0, max_rays=0)
0-d cone in 0-d lattice N
\end{verbatim}

We can predict the ambient dimension when \text{min\_ambient\_dim} == \text{max\_ambient\_dim}:

\begin{verbatim}
sage: K = random_cone(min_ambient_dim=4, max_ambient_dim=4)
sage: K.lattice_dim()
4
\end{verbatim}

Likewise for the number of rays when \text{min\_rays} == \text{max\_rays}:

\begin{verbatim}
sage: K = random_cone(min_rays=3, max_rays=3)
sage: K.nrays()
3
\end{verbatim}

If we specify a lattice, then the returned cone will live in it:
We can also request a strictly convex cone:

```
sage: K = random_cone(max_ambient_dim=8, min_rays=10, strictly_convex=True)
sage: K.is_strictly_convex()
True
```

Or one that isn’t strictly convex:

```
sage: K = random_cone(min_ambient_dim=5, min_rays=2, strictly_convex=False)
sage: K.is_strictly_convex()
False
```

An example with all parameters set:

```
sage: K = random_cone(min_ambient_dim=4, max_ambient_dim=7, min_rays=2, max_rays=10, strictly_convex=False, solid=True)
sage: 4 <= K.lattice_dim() and K.lattice_dim() <= 7
True
sage: 2 <= K.nrays() and K.nrays() <= 10
True
sage: K.is_strictly_convex()
False
sage: K.is_solid()
True
```

### 2.5.3 Catalog of common polyhedral convex cones

This module provides shortcut functions, grouped under the globally-available `cones` prefix, to create some common cones:

- The nonnegative orthant,
- The rearrangement cone of order \( p \),
- The Schur cone,
- The trivial cone.

At the moment, only convex rational polyhedral cones are supported—specifically, those cones that can be built using the `Cone()` constructor. As a result, each shortcut method can be passed either an ambient dimension `ambient_dim`, or a toric `lattice` (from which the dimension can be inferred) to determine the ambient space.

Here are some typical usage examples:

```
sage: cones.nonnegative_orthant(2).rays()
N(1, 0),
N(0, 1)
in 2-d lattice N
```
Combinatorial and Discrete Geometry, Release 10.3

sage: cones.rearrangement(2,2).rays()
N( 1, 0),
N( 1, -1),
N(-1, 1)
in 2-d lattice N

sage: cones.schur(3).rays()
N(1, -1, 0),
N(0, 1, -1)
in 3-d lattice N

sage: cones.trivial(3).rays()
Empty collection
in 3-d lattice N

To specify some other lattice, pass it as an argument to the function:

sage: K = cones.nonnegative_orthant(3)
sage: cones.schur(lattice=K.dual().lattice())
2-d cone in 3-d lattice M

For more information about these cones, see the documentation for the individual functions and the references therein.

sage.geometry.cone_catalog.nonnegative_orthant(ambient_dim=None, lattice=None)

The nonnegative orthant in ambient_dim dimensions, or living in lattice.

The nonnegative orthant consists of all componentwise-nonnegative vectors. It is the convex-conic hull of the standard basis.

INPUT:
• ambient_dim – a nonnegative integer (default: None); the dimension of the ambient space
• lattice – a toric lattice (default: None); the lattice in which the cone will live

If ambient_dim is omitted, then it will be inferred from the rank of lattice. If the lattice is omitted, then the default lattice of rank ambient_dim will be used.

A ValueError is raised if neither ambient_dim nor lattice are specified. It is also a ValueError to specify both ambient_dim and lattice unless the rank of lattice is equal to ambient_dim.

OUTPUT:
A ConvexRationalPolyhedralCone living in lattice and having ambient_dim standard basis vectors as its generators. Each generating ray has the integer ring as its base ring.

A ValueError can be raised if the inputs are incompatible or insufficient. See the INPUT documentation for details.

REFERENCES:
• Chapter 2 in [BV2009] (Examples 2.4, 2.14, and 2.23 in particular)

EXAMPLES:

sage: cones.nonnegative_orthant(3).rays()
N(1, 0, 0),
N(0, 1, 0),
N(0, 0, 1)
in 3-d lattice N
The rearrangement cone of order $p$ in $\text{ambient\_dim}$ dimensions, or living in $\text{lattice}$.

The rearrangement cone of order $p$ in $\text{ambient\_dim}$ dimensions consists of all vectors of length $\text{ambient\_dim}$ whose smallest $p$ components sum to a nonnegative number.

For example, the rearrangement cone of order one has its single smallest component nonnegative. This implies that all components are nonnegative, and that therefore the rearrangement cone of order one is the nonnegative orthant in its ambient space.

When $p$ and $\text{ambient\_dim}$ are equal, all components of the cone's elements must sum to a nonnegative number. In other words, the rearrangement cone of order $\text{ambient\_dim}$ is a half-space.

**INPUT:**

- $p$ – a nonnegative integer; the number of components to “rearrange”, between 1 and $\text{ambient\_dim}$ inclusive
- $\text{ambient\_dim}$ – a nonnegative integer (default: None); the dimension of the ambient space
- $\text{lattice}$ – a toric lattice (default: None); the lattice in which the cone will live

If $\text{ambient\_dim}$ is omitted, then it will be inferred from the rank of $\text{lattice}$. If the $\text{lattice}$ is omitted, then the default lattice of rank $\text{ambient\_dim}$ will be used.

A ValueError is raised if neither $\text{ambient\_dim}$ nor $\text{lattice}$ are specified. It is also a ValueError to specify both $\text{ambient\_dim}$ and $\text{lattice}$ unless the rank of $\text{lattice}$ is equal to $\text{ambient\_dim}$.

It is also a ValueError to specify a non-integer $p$.

**OUTPUT:**

A ConvexRationalPolyhedralCone representing the rearrangement cone of order $p$ living in $\text{lattice}$, with ambient dimension $\text{ambient\_dim}$. Each generating ray has the integer ring as its base ring.

A ValueError can be raised if the inputs are incompatible or insufficient. See the INPUT documentation for details.

**ALGORITHM:**

Suppose that the ambient space is of dimension $n$. The extreme directions of the rearrangement cone for $1 \leq p \leq n - 1$ are given by [Jeong2017] Theorem 5.2.3. When $2 \leq p \leq n - 2$ (that is, if we ignore $p = 1$ and $p = n - 1$), they consist of

- the standard basis $\{e_1, e_2, \ldots, e_n\}$ for the ambient space, and
- the $n$ vectors $(1, 1, \ldots, 1)^T - pe_i$ for $i = 1, 2, \ldots, n$.

Special cases are then given for $p = 1$ and $p = n - 1$ in the theorem. However in SageMath we don’t need conically-independent extreme directions. We only need a generating set, because the Cone() function will eliminate any redundant generators. And one can easily verify that the special-case extreme directions for $p = 1$ and $p = n - 1$ are contained in the conic hull of the $2n$ generators just described. The half space resulting from $p = n$ is also covered by this set of generators, so for all valid $p$ we simply take the conic hull of those $2n$ vectors.

**REFERENCES:**

- [GJ2016], Section 4
- [HS2010], Example 2.21
- [Jeong2017], Section 5.2

**EXAMPLES:**

The rearrangement cones of order one are nonnegative orthants:
When $p$ and $\text{ambient\_dim}$ are equal, the rearrangement cone is a half-space, so we expect its lineality to be one less than $\text{ambient\_dim}$ because it will contain a hyperplane but is not the entire space:

```sage
sage: cones.rearrangement(5, 5).lineality()
4
```

Jeong’s Proposition 5.2.1 [Jeong2017] states that all rearrangement cones are proper when $p$ is less than $\text{ambient\_dim}$:

```sage
sage: all( cones.rearrangement(p, ambient_dim).is_proper() 
....:     for ambient_dim in range(10) 
....:     for p in range(1, ambient_dim) )
True
```

Jeong’s Corollary 5.2.4 [Jeong2017] states that if $p = n - 1$ in an $n$-dimensional ambient space, then the Lyapunov rank of the rearrangement cone is $n$, and that for all other $p > 1$ its Lyapunov rank is one:

```sage
sage: all( cones.rearrangement(p, ambient_dim).lyapunov_rank() == ambient_dim 
....:     for ambient_dim in range(2, 10) 
....:     for p in range(2, ambient_dim-1) )
True
```

```
sage.geometry.cone_catalog.schur(ambient_dim=None, lattice=None)
The Schur cone in $\text{ambient\_dim}$ dimensions, or living in $\text{lattice}$.

The Schur cone in $n$ dimensions induces the majorization ordering on the ambient space. If $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for the space, then its generators are $\{e_i - e_{i+1} | 1 \leq i \leq n - 1\}$. Its dual is the downward monotonic cone.

INPUT:

- $\text{ambient\_dim}$ – a nonnegative integer (default: None): the dimension of the ambient space
- $\text{lattice}$ – a toric lattice (default: None): the lattice in which the cone will live

If $\text{ambient\_dim}$ is omitted, then it will be inferred from the rank of $\text{lattice}$. If the $\text{lattice}$ is omitted, then the default lattice of rank $\text{ambient\_dim}$ will be used.

A ValueError is raised if neither $\text{ambient\_dim}$ nor $\text{lattice}$ are specified. It is also a ValueError to specify both $\text{ambient\_dim}$ and $\text{lattice}$ unless the rank of $\text{lattice}$ is equal to $\text{ambient\_dim}$.

OUTPUT:

A ConvexRationalPolyhedralCone representing the Schur cone living in $\text{lattice}$, with ambient dimension $\text{ambient\_dim}$. Each generating ray has the integer ring as its base ring.

A ValueError can be raised if the inputs are incompatible or insufficient. See the INPUT documentation for details.

REFERENCES:

420 Chapter 2. Polyhedral computations
EXAMPLES:

Verify the claim [SS2016] that the maximal angle between any two generators of the Schur cone and the nonnegative orthant in dimension five is \((3/4) \pi\):

```sage
sage: # needs sage.rings.number_fields
sage: P = cones.schur(5)
sage: Q = cones.nonnegative_orthant(5)
sage: G = (g.change_ring(QQbar).normalized() for g in P)
sage: H = (h.change_ring(QQbar).normalized() for h in Q)
sage: actual = max(arccos(u.inner_product(v)) for u in G for v in H)
sage: expected = 3*pi/4
sage: abs(actual - expected).n() < 1e-12
True
```

The dual of the Schur cone is the “downward monotonic cone” [GS2010], whose elements’ entries are in non-increasing order:

```sage
sage: ambient_dim = ZZ.random_element(10)
sage: K = cones.schur(ambient_dim).dual()
sage: x = K.random_element()
sage: all( x[i] >= x[i+1] for i in range(ambient_dim-1) )
True
```

```

sage.geometry.cone_catalog.trivial(ambient_dim=None, lattice=None)

The trivial cone with no nonzero generators in ambient_dim dimensions, or living in lattice.

INPUT:

- ambient_dim – a nonnegative integer (default: None); the dimension of the ambient space
- lattice – a toric lattice (default: None); the lattice in which the cone will live

If ambient_dim is omitted, then it will be inferred from the rank of lattice. If the lattice is omitted, then the default lattice of rank ambient_dim will be used.

A ValueError is raised if neither ambient_dim nor lattice are specified. It is also a ValueError to specify both ambient_dim and lattice unless the rank of lattice is equal to ambient_dim.

OUTPUT:

A ConvexRationalPolyhedralCone representing the trivial cone with no nonzero generators living in lattice, with ambient dimension ambient_dim.

A ValueError can be raised if the inputs are incompatible or insufficient. See the INPUT documentation for details.

EXAMPLES:

Construct the trivial cone, containing only the origin, in three dimensions:

```sage
sage: cones.trivial(3)
0-d cone in 3-d lattice N
```

If a lattice is given, the trivial cone will live in that lattice:

```

2.5. Toric geometry
2.5.4 Rational polyhedral fans

This module was designed as a part of the framework for toric varieties \((\text{variety}, \text{fano}\_\text{variety})\). While the emphasis is on complete full-dimensional fans, arbitrary fans are supported. Work with distinct lattices. The default lattice is \(\text{ToricLattice} N\) of the appropriate dimension. The only case when you must specify lattice explicitly is creation of a 0-dimensional fan, where dimension of the ambient space cannot be guessed.

A **rational polyhedral fan** is a *finite* collection of *strictly* convex rational polyhedral cones, such that the intersection of any two cones of the fan is a face of each of them and each face of each cone is also a cone of the fan.

**AUTHORS:**

- Andrey Novoseltsev (2010-06-17): substantial improvement during review by Volker Braun.

**EXAMPLES:**

Use \(\text{Fan}()\) to construct fans “explicitly”:

```sage
sage: fan = Fan(cones=[(0,1), (1,2)],
....: rays=[(1,0), (0,1), (-1,0)])
sage: fan
Rational polyhedral fan in 2-d lattice N
```

In addition to giving such lists of cones and rays you can also create cones first using \(\text{Cone}()\) and then combine them into a fan. See the documentation of \(\text{Fan}()\) for details.

In 2 dimensions there is a unique maximal fan determined by rays, and you can use \(\text{Fan2d}()\) to construct it:

```sage
sage: fan2d = Fan2d(rays=[(1,0), (0,1), (-1,0)])
sage: fan2d.is_equivalent(fan)
True
```

But keep in mind that in higher dimensions the cone data is essential and cannot be omitted. Instead of building a fan from scratch, for this tutorial we will use an easy way to get two fans associated to lattice polytopes \(\text{FaceFan()}\) and \(\text{NormalFan()}\):

```sage
sage: fan1 = FaceFan(lattice_polytope.cross_polytope(3))
sage: fan2 = NormalFan(lattice_polytope.cross_polytope(3))
```

Given such “automatic” fans, you may wonder what are their rays and cones:

```sage
sage: fan1.rays()
M( 1, 0, 0),
M( 0, 1, 0),
M( 0, 0, 1),
M(-1, 0, 0),
M( 0, -1, 0),
M( 0, 0, -1)
in 3-d lattice M
sage: fan1.generating_cones()
(3-d cone of Rational polyhedral fan in 3-d lattice M,
```
The last output is not very illuminating. Let's try to improve it:

```sage
define the routine for cone in fan: print(cone.rays())
M(0, 1, 0),
M(0, 0, 1),
M(-1, 0, 0) in 3-d lattice M
M(0, 0, 1),
M(-1, 0, 0),
M(0, -1, 0) in 3-d lattice M
M(-1, 0, 0),
M(0, -1, 0),
M(0, 0, -1) in 3-d lattice M
M(0, 1, 0),
M(0, 0, 1),
M(-1, 0, 0) in 3-d lattice M
M(-1, 0, 0),
M(0, 0, -1) in 3-d lattice M
M(0, 1, 0),
M(0, 0, 1),
M(0, 0, -1) in 3-d lattice M
M(0, 1, 0),
M(0, 0, 1),
M(1, 0, 0) in 3-d lattice M
M(1, 0, 0),
M(0, 0, 1),
M(0, 0, -1) in 3-d lattice M
M(1, 0, 0),
M(1, 0, 0),
M(0, -1, 0) in 3-d lattice M
M(1, 0, 0),
M(1, 0, 0),
M(0, 0, -1) in 3-d lattice M
M(1, 0, 0),
M(0, 0, 1),
M(0, 0, -1) in 3-d lattice M
M(1, 0, 0),
M(0, 0, 1),
M(-1, 0, 0) in 3-d lattice M
M(1, 0, 0),
M(0, 0, 1),
M(0, -1, 0) in 3-d lattice M
M(1, 0, 0),
M(0, 0, 1),
M(0, 0, -1) in 3-d lattice M
```

You can also do

```sage
define the routine for cone in fan: print(cone.ambient_ray_indices())
(1, 2, 3)
(2, 3, 4)
(3, 4, 5)
(1, 3, 5)
(0, 1, 5)
(0, 1, 2)
(0, 2, 4)
(0, 4, 5)
```

to see indices of rays of the fan corresponding to each cone.

2.5. Toric geometry
While the above cycles were over “cones in fan”, it is obvious that we did not get ALL the cones: every face of every cone in a fan must also be in the fan, but all of the above cones were of dimension three. The reason for this behaviour is that in many cases it is enough to work with generating cones of the fan, i.e. cones which are not faces of bigger cones. When you do need to work with lower dimensional cones, you can easily get access to them using \texttt{cones()}:

```
sage: [cone.ambient_ray_indices() for cone in fan1.cones(2)]
[(0, 1), (0, 2), (1, 2), (1, 3), (2, 3), (0, 4),
 (2, 4), (3, 4), (1, 5), (3, 5), (4, 5), (0, 5)]
```

In fact, you do not have to type \texttt{.cones}:

```
sage: [cone.ambient_ray_indices() for cone in fan1(2)]
[(0, 1), (0, 2), (1, 2), (1, 3), (2, 3), (0, 4),
 (2, 4), (3, 4), (1, 5), (3, 5), (4, 5), (0, 5)]
```

You may also need to know the inclusion relations between all of the cones of the fan. In this case check out \texttt{cone_lattice()}:

```
sage: L = fan1.cone_lattice()
sage: L
Finite lattice containing 28 elements with distinguished linear extension
sage: L.bottom()
0-d cone of Rational polyhedral fan in 3-d lattice M
sage: L.top()
Rational polyhedral fan in 3-d lattice M
sage: cone = L.level_sets()[2][0]
sage: cone
2-d cone of Rational polyhedral fan in 3-d lattice M
sage: sorted(L.hasse_diagram().neighbors(cone))
[1-d cone of Rational polyhedral fan in 3-d lattice M,
 1-d cone of Rational polyhedral fan in 3-d lattice M,
 3-d cone of Rational polyhedral fan in 3-d lattice M,
 3-d cone of Rational polyhedral fan in 3-d lattice M]
```

You can check how “good” a fan is:

```
sage: fan1.is_complete()
True
sage: fan1.is_simplicial()
True
sage: fan1.is_smooth()
True
```

The face fan of the octahedron is really good! Time to remember that we have also constructed its normal fan:

```
sage: fan2.is_complete()
True
sage: fan2.is_simplicial()
False
sage: fan2.is_smooth()
False
```

This one does have some “problems,” but we can fix them:

```
sage: fan3 = fan2.make_simplicial()
sage: fan3.is_simplicial()
True
```

(continues on next page)
Note that we had to save the result of \texttt{make_simplicial()} in a new fan. Fans in Sage are immutable, so any operation that does change them constructs a new fan.

We can also make \texttt{fan3} smooth, but it will take a bit more work:

\begin{verbatim}
sage: cube = lattice_polytope.cross_polytope(3).polar()
sage: sk = cube.skeleton_points(2)
sage: rays = [cube.point(p) for p in sk]
sage: fan4 = fan3.subdivide(new_rays=rays)
sage: fan4.is_smooth()
True
\end{verbatim}

Let's see how “different” are \texttt{fan2} and \texttt{fan4}:

\begin{verbatim}
sage: fan2.ngenerating_cones() 6
sage: fan2.nrays() 8
sage: fan4.ngenerating_cones() # needs palp
48
sage: fan4.nrays() # needs palp
26
\end{verbatim}

Smoothness does not come for free!

Please take a look at the rest of the available functions below and their complete descriptions. If you need any features that are missing, feel free to suggest them. (Or implement them on your own and submit a patch to Sage for inclusion!)

\begin{verbatim}
class sage.geometry.fan.Cone_of_fan(ambient, ambient_ray_indices)

Bases: ConvexRationalPolyhedralCone

Construct a cone belonging to a fan.

Warning: This class does not check that the input defines a valid cone of a fan. You must not construct objects of this class directly.
\end{verbatim}

In addition to all of the properties of “regular” \texttt{cones}, such cones know their relation to the fan.

\textbf{INPUT}:

- \texttt{ambient} – fan whose cone is constructed;
- \texttt{ambient\_ray\_indices} – increasing list or tuple of integers, indices of rays of \texttt{ambient} generating this cone.

\textbf{OUTPUT}:

\texttt{cone} of \texttt{ambient}

\textbf{EXAMPLES}:

The intended way to get objects of this class is the following:
Combinatorial and Discrete Geometry, Release 10.3

```python
sage: # needs palp
sage: fan = toric_varieties.P1xP1().fan()
sage: cone = fan.generating_cone(0); cone
2-d cone of Rational polyhedral fan in 2-d lattice N
sage: cone.ambient_ray_indices()
(0, 2)
sage: cone.star_generator_indices()
(0,)
```

**star_generator_indices()**

Return indices of generating cones of the “ambient fan” containing self.

**OUTPUT:**

increasing tuple of integers

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
# needs palp
sage: cone = P1xP1.fan().generating_cone(0)
# needs palp
sage: cone.star_generator_indices()
(0,)
```

**star_generators()**

Return indices of generating cones of the “ambient fan” containing self.

**OUTPUT:**

increasing tuple of integers

**EXAMPLES:**

```python
sage: P1xP1 = toric_varieties.P1xP1()
# needs palp
sage: cone = P1xP1.fan().generating_cone(0)
# needs palp
sage: cone.star_generators()
(2-d cone of Rational polyhedral fan in 2-d lattice N,)
```

**sage.geometry.fan.FaceFan**(polytope, lattice=None)

Construct the face fan of the given rational polytope.

**INPUT:**

- **polytope** – a *polytope* over $\mathbb{Q}$ or a *lattice* *polytope*. A (not necessarily full-dimensional) polytope containing the origin in its relative interior.

- **lattice** – *ToricLattice*, $\mathbb{Z}^n$, or any other object that behaves like these. If not specified, an attempt will be made to determine an appropriate toric lattice automatically.

**OUTPUT:**

rational polyhedral fan

See also *NormalFan()*.

**EXAMPLES:**
Let’s construct the fan corresponding to the product of two projective lines:

```
sage: diamond = lattice_polytope.cross_polytope(2)
sage: P1xP1 = FaceFan(diamond)
sage: P1xP1.rays()
M( 1, 0),
M( 0, 1),
M(-1, 0),
M( 0, -1)
in 2-d lattice M
sage: for cone in P1xP1: print(cone.rays())
M(-1, 0),
M( 0, -1)
in 2-d lattice M
M( 0, 1),
M(-1, 0)
in 2-d lattice M
M(1, 0),
M(0, 1)
in 2-d lattice M
M(1, 0),
M(0, -1)
in 2-d lattice M
```

```
sage.geometry.fan.Fan(cones, rays=None, lattice=None, check=True, normalize=True, is_complete=None, virtual_rays=None, discard_faces=False, allow_arrangement=False)
```

Construct a rational polyhedral fan.

**Note:** Approximate time to construct a fan consisting of \( n \) cones is \( n^2/5 \) seconds. That is half an hour for 100 cones. This time can be significantly reduced in the future, but it is still likely to be \( \sim n^2 \) (with, say, \( /500 \) instead of \( /5 \)). If you know that your input does form a valid fan, use check=False option to skip consistency checks.

**INPUT:**

- **cones** – list of either `Cone` objects or lists of integers interpreted as indices of generating rays in `rays`. These must be only maximal cones of the fan, unless `discard_faces=True` or `allow_arrangement=True` option is specified;

- **rays** – list of rays given as list or vectors convertible to the rational extension of `lattice`. If cones are given by `Cone` objects `rays` may be determined automatically. You still may give them explicitly to ensure a particular order of rays in the fan. In this case you must list all rays that appear in `cones`. You can give “extra” ones if it is convenient (e.g. if you have a big list of rays for several fans), but all “extra” rays will be discarded;

- **lattice** – `ToricLattice`, \( \mathbb{Z}^n \), or any other object that behaves like these. If not specified, an attempt will be made to determine an appropriate toric lattice automatically;

- **check** – by default the input data will be checked for correctness (e.g. that intersection of any two given cones is a face of each), unless `allow_arrangement=True` option is specified. If you know for sure that the input is correct, you may significantly decrease construction time using check=False option;

- **normalize** – you can further speed up construction using normalize=False option. In this case `cones` must be a list of sorted tuples and `rays` must be immutable primitive vectors in `lattice`. In general, you should not use this option, it is designed for code optimization and does not give as drastic improvement in speed as the previous one;

- **is_complete** – every fan can determine on its own if it is complete or not, however it can take quite a bit of time for “big” fans with many generating cones. On the other hand, in some situations it is known in
advance that a certain fan is complete. In this case you can pass `is_complete=True` option to speed up some computations. You may also pass `is_complete=False` option, although it is less likely to be beneficial. Of course, passing a wrong value can compromise the integrity of data structures of the fan and lead to wrong results, so you should be very careful if you decide to use this option;

- **virtual_rays** – (optional, computed automatically if needed) a list of ray generators to be used for `virtual_rays()`;

- **discard_faces** – by default, the fan constructor expects the list of maximal cones, unless `allow_arrangement=True` option is specified. If you provide “extra” ones and leave `allow_arrangement=False` (default) and `check=True` (default), an exception will be raised. If you provide “extra” cones and set `allow_arrangement=False` (default) and `check=False`, you may get wrong results as assumptions on internal data structures will be invalid. If you want the fan constructor to select the maximal cones from the given input, you may provide `discard_faces=True` option (it works both for `check=True` and `check=False`).

- **allow_arrangement** – by default (`allow_arrangement=False`), the fan constructor expects that the intersection of any two given cones is a face of each. If `allow_arrangement=True` option is specified, then construct a rational polyhedral fan from the cone arrangement, so that the union of the cones in the polyhedral fan equals to the union of the given cones, and each given cone is the union of some cones in the polyhedral fan.

**OUTPUT:**

a fan

See also:

In 2 dimensions you can cyclically order the rays. Hence the rays determine a unique maximal fan without having to specify the cones, and you can use `Fan2d()` to construct this fan from just the rays.

**EXAMPLES:**

Let’s construct a fan corresponding to the projective plane in several ways:

```
sage: cone1 = Cone([(1,0), (0,1)])
sage: cone2 = Cone([(0,1), (-1,-1)])
sage: cone3 = Cone([(-1,-1), (1,0)])
sage: P2 = Fan([cone1, cone2, cone2])
Traceback (most recent call last):
...
ValueError: you have provided 3 cones, but only 2 of them are maximal!
Use discard_faces=True if you indeed need to construct a fan from these cones.
```

Oops! There was a typo and `cone2` was listed twice as a generating cone of the fan. If it was intentional (e.g. the list of cones was generated automatically and it is possible that it contains repetitions or faces of other cones), use `discard_faces=True` option:

```
sage: P2 = Fan([cone1, cone2, cone2], discard_faces=True)
sage: P2.ngenerating_cones()
2
```

However, in this case it was definitely a typo, since the fan of $\mathbb{P}^2$ has 3 maximal cones:

```
sage: P2 = Fan([cone1, cone2, cone3])
sage: P2.ngenerating_cones()
3
```

Looks better. An alternative way is
That may seem wrong, but it is not:

```sage
P2.is_equivalent(P2a)
```

True

See `is_equivalent()` for details.

Yet another way to construct this fan is

```sage
P2b = Fan(cones, rays, check=False)
P2b.ngenerating_cones()
```

3

```sage
P2a == P2b
```

True

If you try the above examples, you are likely to notice the difference in speed, so when you are sure that everything is correct, it is a good idea to use `check=False` option. On the other hand, it is usually **NOT** a good idea to use `normalize=False` option:

```sage
P2c = Fan(cones, rays, check=False, normalize=False)
```

Traceback (most recent call last):
...
AttributeError: 'tuple' object has no attribute 'parent'...

Yet another way is to use functions `FaceFan()` and `NormalFan()` to construct fans from lattice polytopes.

We have not yet used `lattice` argument, since if was determined automatically:

```sage
P2.lattice()
```

2-d lattice N

```sage
P2b.lattice()
```

2-d lattice N

However, it is necessary to specify it explicitly if you want to construct a fan without rays or cones:

```sage
Fan([], [])
```

Traceback (most recent call last):
...
`ValueError: you must specify the lattice when you construct a fan without rays and cones!`

```sage
F = Fan([], [], lattice=ToricLattice(2, "L"))
```

```sage
F.lattice_dim()
```

2

```sage
F.dim()
```

0

In the following examples, we test the `allow_arrangement=True` option. See [github issue #25122](https://github.com/sagemath/sage/issues/25122).
The intersection of the two cones is not a face of each. Therefore, they do not belong to the same rational polyhedral fan:

```
sage: c1 = Cone([(-2,-1,1), (-2,1,1), (2,1,1), (2,-1,1)])
sage: c2 = Cone([(-1,-2,1), (-1,2,1), (1,2,1), (1,-2,1)])
sage: c1.intersection(c2).is_face_of(c1)
False
sage: c1.intersection(c2).is_face_of(c2)
False
sage: Fan([c1, c2])
Traceback (most recent call last):
  ... ValueError: these cones cannot belong to the same fan!
```

Let’s construct the fan using `allow_arrangement=True` option:

```
sage: fan = Fan([c1, c2], allow_arrangement=True)
sage: fan.ngenerating_cones()
5
```

Another example where cone `c2` is inside cone `c1`:

```
sage: c1 = Cone([((4, 0, 0), (0, 4, 0), (0, 0, 4))])
sage: c2 = Cone([((2, 1, 1), (1, 2, 1), (1, 1, 2))])
sage: fan = Fan([c1, c2], allow_arrangement=True)
sage: fan.ngenerating_cones()
7
sage: fan.plot()  # needs sage.plot
```

Cones of different dimension:

```
sage: c1 = Cone([(1,0), (0,1)])
sage: c2 = Cone([(2,1)])
sage: c3 = Cone([-(-1,-2)])
sage: fan = Fan([c1, c2, c3], allow_arrangement=True)
sage: for cone in sorted(fan.generating_cones()): print(sorted(cone.rays()))
  [N(-1, -2)]
  [N(0, 1), N(1, 2)]
  [N(1, 0), N(2, 1)]
  [N(1, 2), N(2, 1)]
```

A 3-d cone and a 1-d cone:

```
sage: c3 = Cone([([0, 1, 1], [1, 0, 1], [0, -1, 1], [-1, 0, 1])])
sage: c1 = Cone([([0, 0, 1])])
sage: fan1 = Fan([c1, c3], allow_arrangement=True)
sage: fan1.plot()  # needs sage.plot
```

A 3-d cone and two 2-d cones:

```
sage: c2v = Cone([([0, 1, 1], [0, -1, 1])])
sage: c2h = Cone([([1, 0, 1], [-1, 0, 1])])
sage: fan2 = Fan([c2v, c2h, c3], allow_arrangement=True)
```
sage: fan2.is_simplicial()
True
sage: fan2.is_equivalent(fan1)
True

sage.geometry.fan.Fan2d(rays, lattice=None)
Construct the maximal 2-d fan with given rays.

In two dimensions we can uniquely construct a fan from just rays, just by cyclically ordering the rays and constructing as many cones as possible. This is why we implement a special constructor for this case.

INPUT:

• rays – list of rays given as list or vectors convertible to the rational extension of lattice. Duplicate rays are removed without changing the ordering of the remaining rays.

• lattice – ToricLattice, \( \mathbb{Z}^n \), or any other object that behaves like these. If not specified, an attempt will be made to determine an appropriate toric lattice automatically.

EXAMPLES:

sage: Fan2d([(0,1), (1,0)])
Rational polyhedral fan in 2-d lattice N
sage: Fan2d([], lattice=ToricLattice(2, 'myN'))
Rational polyhedral fan in 2-d lattice myN

The ray order is as specified, even if it is not the cyclic order:

sage: fan1 = Fan2d([(0,1), (1,0)])
sage: fan1.rays()
N(0, 1),
N(1, 0)
in 2-d lattice N
sage: fan2 = Fan2d([(1,0), (0,1)])
sage: fan2.rays()
N(1, 0),
N(0, 1)
in 2-d lattice N
sage: fan1 == fan2, fan1.is_equivalent(fan2)
(False, True)
sage: fan = Fan2d([(1,1), (-1,-1), (1,-1), (-1,1)])
sage: [cone.ambient_ray_indices() for cone in fan]
[(2, 1), (1, 3), (3, 0), (0, 2)]
sage: fan.is_complete()
True

sage.geometry.fan.NormalFan(polytope, lattice=None)
Construct the normal fan of the given rational polytope.

This returns the inner normal fan. For the outer normal fan, use NormalFan(-P).

INPUT:

• polytope – a full-dimensional polytope over \( \mathbb{Q} \) or:class:latticepolytope < sage.geometry.lattice.polytope.LatticePolytopeClass >.

2.5. Toric geometry 431
• lattice – ToricLattice, \( \mathbb{Z}^n \), or any other object that behaves like these. If not specified, an attempt will be made to determine an appropriate toric lattice automatically.

OUTPUT:

\textit{rational polyhedral fan}

See also \texttt{FaceFan()}.

EXAMPLES:

Let’s construct the fan corresponding to the product of two projective lines:

```
sage: square = LatticePolytope([[1,1], [-1,1], [-1,-1], (1,-1)])
sage: P1xP1 = NormalFan(square)
sage: P1xP1.rays()
N(1, 0),
N(0, 1),
N(-1, 0),
N(0, -1)
in 2-d lattice N

sage: for cone in P1xP1: print(cone.rays())
N(-1, 0),
N(0, -1)
in 2-d lattice N
N(1, 0),
N(0, -1)
in 2-d lattice N
N(1, 0),
N(0, -1)
in 2-d lattice N
N(0, 1),
N(-1, 0)
in 2-d lattice N

sage: cuboctahed = polytopes.cuboctahedron()
sage: NormalFan(cuboctahed)
Rational polyhedral fan in 3-d lattice N
```

\texttt{sage.geometry.fan.RationalPolyhedralFan}\( (\texttt{cones}, \texttt{rays}, \texttt{lattice}, \texttt{is_complete}=\texttt{None}, \texttt{virtual_rays}=\texttt{None})\)

\texttt{Bases: IntegralRayCollection, Callable, Container}

Create a rational polyhedral fan.

\textbf{Warning:} This class does not perform any checks of correctness of input nor does it convert input into the standard representation. Use \texttt{Fan()} to construct fans from “raw data” or \texttt{FaceFan()} and \texttt{NormalFan()} to get fans associated to polytopes.

Fans are immutable, but they cache most of the returned values.

\textbf{INPUT:}

• \texttt{cones} – list of generating cones of the fan, each cone given as a list of indices of its generating rays in \texttt{rays};

• \texttt{rays} – list of immutable primitive vectors in \texttt{lattice} consisting of exactly the rays of the fan (i.e. no “extra” ones);
• lattice — ToricLattice, $\mathbb{Z}^n$, or any other object that behaves like these. If None, it will be determined as parent() of the first ray. Of course, this cannot be done if there are no rays, so in this case you must give an appropriate lattice directly;

• is_complete – if given, must be True or False depending on whether this fan is complete or not. By default, it will be determined automatically if necessary;

• virtual_rays – if given, must be a list of immutable primitive vectors in lattice, see virtual_rays() for details. By default, it will be determined automatically if necessary.

OUTPUT:

rational polyhedral fan

Gale_transform()

Return the Gale transform of self.

OUTPUT:

A matrix over $\mathbb{Z}$

EXAMPLES:

```
sage: fan = toric_varieties.P1xP1().fan()  # needs palp
sage: fan.Gale_transform()  # needs palp
[ 1  1  0  0 -2]
[ 0  0  1  1 -2]
sage: _.base_ring()  # needs palp
Integer Ring
```

Stanley_Reisner_ideal( ring )

Return the Stanley-Reisner ideal.

INPUT:

• A polynomial ring in self.nrays() variables.

OUTPUT:

The Stanley-Reisner ideal in the given polynomial ring

EXAMPLES:

```
sage: fan = Fan([ [0,1,3], [3,4], [2,0], [1,2,4] ],
[-3, -2, 1], (0, 0, 1), (3, -2, 1), (-1, -1, 1), (1, -1, 1)])
sage: fan.Stanley_Reisner_ideal(PolynomialRing(QQ, 5, 'A, B, C, D, E'))
Ideal (A*B, A*C, B*D) of Multivariate Polynomial Ring in A, B, C, D, E over Rational Field
```

cartesian_product( other, lattice=None )

Return the Cartesian product of self with other.

INPUT:

• other — a rational polyhedral fan;

• lattice — (optional) the ambient lattice for the Cartesian product fan. By default, the direct sum of the ambient lattices of self and other is constructed.
OUTPUT:

a *fan* whose cones are all pairwise Cartesian products of the cones of *self* and *other*

EXAMPLES:

```
sage: K = ToricLattice(1, 'K')
sage: fan1 = Fan([[0],[1]], [[1,],[(-1,)]], lattice=K)
sage: L = ToricLattice(2, 'L')
sage: fan2 = Fan(rays=[(1,0), (0,1), (-1,-1)],
            cones=[[0,1], [1,2], [2,0]], lattice=L)
sage: fan1.cartesian_product(fan2)
Rational polyhedral fan in 3-d lattice K+L
sage: _.ngenerating_cones()
6
```

**common_refinement** *(other)*

Return the common refinement of this fan and *other*.

INPUT:

- *other* – a *fan* in the same *lattice* and with the same support as this fan

OUTPUT:

a *fan*

EXAMPLES:

Refining a fan with itself gives itself:

```
sage: F0 = Fan2d([(1,0), (0,1), (-1,0), (0,-1)])
sage: F0.common_refinement(F0) == F0
True
```

A more complex example with complete fans:

```
sage: F1 = Fan([[0],[1]], [[1,],[(-1,)]])
sage: F2 = Fan2d([(1,0), (1,1), (0,1), (-1,0), (0,-1)])
sage: F3 = F2.cartesian_product(F1)
sage: F4 = F1.cartesian_product(F2)
sage: FF = F3.common_refinement(F4)
sage: FF.ngenerating_cones()
13
sage: F3.ngenerating_cones()
10
sage: F4.ngenerating_cones()
10
```

An example with two non-complete fans with the same support:

```
sage: F5 = Fan2d([(1,0), (1,2), (0,1)])
sage: F6 = Fan2d([(1,0), (2,1), (0,1)])
sage: F5.common_refinement(F6).ngenerating_cones()
3
```

Both fans must live in the same lattice:

```
sage: F0.common_refinement(F1)
Traceback (most recent call last):
```

(continues on next page)
ValueError: the fans are not in the same lattice

complex(base_ring=Integer Ring, extended=False)

Return the chain complex of the fan.

To a $d$-dimensional fan $\Sigma$, one can canonically associate a chain complex $K^*$

$0 \longrightarrow \mathbb{Z}^{\Sigma(d)} \longrightarrow \mathbb{Z}^{\Sigma(d-1)} \longrightarrow \ldots \longrightarrow \mathbb{Z}^{\Sigma(0)} \longrightarrow 0$

where the leftmost non-zero entry is in degree 0 and the rightmost entry in degree $d$. See [Kly1990], eq. (3.2). This complex computes the homology of $|\Sigma| \subset N_\mathbb{R}$ with arbitrary support,

$H_i(K) = H_{d-i}(|\Sigma|, \mathbb{Z})_{\text{non-cpt}}$

For a complete fan, this is just the non-compactly supported homology of $\mathbb{R}^d$. In this case, $H_0(K) = \mathbb{Z}$ and 0 in all non-zero degrees.

For a complete fan, there is an extended chain complex

$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{\Sigma(d)} \longrightarrow \mathbb{Z}^{\Sigma(d-1)} \longrightarrow \ldots \longrightarrow \mathbb{Z}^{\Sigma(0)} \longrightarrow 0$

where we take the first $\mathbb{Z}$ term to be in degree -1. This complex is an exact sequence, that is, all homology groups vanish.

The orientation of each cone is chosen as in oriented_boundary().

INPUT:

- extended – Boolean (default:False). Whether to construct the extended complex, that is, including the $\mathbb{Z}$-term at degree -1 or not.
- base_ring – A ring (default: $\mathbb{Z}$). The ring to use instead of $\mathbb{Z}$.

OUTPUT:

The complex associated to the fan as a ChainComplex. This raises a ValueError if the extended complex is requested for a non-complete fan.

EXAMPLES:

```sage
# needs palp
sage: fan = toric_varieties.P(3).fan()
sage: K_normal = fan.complex(); K_normal
Chain complex with at most 4 nonzero terms over Integer Ring
sage: K_normal.homology()
{0: Z, 1: 0, 2: 0, 3: 0}
sage: K_extended = fan.complex(extended=True); K_extended
Chain complex with at most 5 nonzero terms over Integer Ring
sage: K_extended.homology()
{-1: 0, 0: 0, 1: 0, 2: 0, 3: 0}
```

Homology computations are much faster over $\mathbb{Q}$ if you do not care about the torsion coefficients:

```sage
sage: toric_varieties.P2_123().fan().complex(extended=True, base_ring=QQ)  # ...
Chain complex with at most 4 nonzero terms over Rational Field
sage: _.homology()
```

(continues on next page)
The extended complex is only defined for complete fans:

```python
sage: fan = Fan([Cone([(1,0)])])
sage: fan.is_complete()
False
sage: fan.complex(extended=True)
ValueError: The extended complex is only defined for complete fans!
```

The definition of the complex does not refer to the ambient space of the fan, so it does not distinguish a fan from the same fan embedded in a subspace:

```python
sage: K1 = Fan([Cone([-1,0]), Cone([1,0,0])]).complex()
sage: K2 = Fan([Cone([-1,0,0]), Cone([1,0])]).complex()
sage: K1 == K2
True
```

Things get more complicated for non-complete fans:

```python
sage: fan = Fan([Cone([(1,1,1)]),
....: Cone([-1,0,0, (0,1,0), (0,0,-1)])])
sage: fan.complex().homology()
{0: 0, 1: 0, 2: Z x Z, 3: 0}
sage: fan = Fan([Cone([(1,0,0), (0,1,0)]),
....: Cone([-1,0,0), (0,-1,0), (0,0,-1)])])
sage: fan.complex().homology()
{0: 0, 1: 0, 2: Z, 3: 0}
sage: fan = Fan([Cone([-1,0,0), (0,-1,0)])])
sage: fan.complex().homology()
{0: 0, 1: 0, 2: 0, 3: 0}
```

```python
cone_containing(*points)
```
Return the smallest cone of `self` containing all given points.

**INPUT:**

- either one or more indices of rays of `self`, or one or more objects representing points of the ambient space of `self`, or a list of such objects (you CANNOT give a list of indices).

**OUTPUT:**

A `cone of fan` whose ambient fan is `self`

**Note:** We think of the origin as of the smallest cone containing no rays at all. If there is no ray in `self` that contains all rays, a `ValueError` exception will be raised.

**EXAMPLES:**
Combinatorial and Discrete Geometry, Release 10.3

```
sage: cone1 = Cone([0,-1), (1,0)])
sage: cone2 = Cone([1,0), (0,1)])
sage: f = Fan([cone1, cone2])
sage: f.rays()
N(0, -1),
N(0, 1),
N(1, 0)
in 2-d lattice N
sage: f.cone_containing(0)  # ray index
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing(0, 1)  # ray indices
Traceback (most recent call last):
...
ValueError: there is no cone in
Rational polyhedral fan in 2-d lattice N
containing all of the given rays! Ray indices: [0, 1]
sage: f.cone_containing(0, 2)  # ray indices
2-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing((0,1))  # point
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing(((0,1)))  # point
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing((1,1))
2-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing((1,1), (1,0))
2-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing()
0-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing((0,0))
0-d cone of Rational polyhedral fan in 2-d lattice N
sage: f.cone_containing((-1,1))
Traceback (most recent call last):
...
ValueError: there is no cone in
Rational polyhedral fan in 2-d lattice N
containing all of the given points! Points: [N(-1, 1)]
```

**cone_lattice()**

Return the cone lattice of self.

This lattice will have the origin as the bottom (we do not include the empty set as a cone) and the fan itself as the top.

**OUTPUT:**

finite poset <sage.combinat.posets.posets.FinitePoset of cones of fan>, behaving like “regular” cones, but also containing the information about their relation to this fan, namely, the contained rays and containing generating cones. The top of the lattice will be this fan itself (which is not a cone of fan).

See also cones().

**EXAMPLES:**

Cone lattices can be computed for arbitrary fans:

```
sage: cone1 = Cone([1,0), (0,1)])
sage: cone2 = Cone([(-1,0)])
sage: fan = Fan([cone1, cone2])
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```python
sage: fan.rays()
N(-1, 0),
N( 0, 1),
N( 1, 0)
in 2-d lattice N
sage: for cone in fan: print(cone.ambient_ray_indices())
(1, 2)
(0,)
sage: L = fan.cone_lattice()
sage: L
Finite poset containing 6 elements with distinguished linear extension
```

These 6 elements are the origin, three rays, one two-dimensional cone, and the fan itself. Since we do add
the fan itself as the largest face, you should be a little bit careful with this last element:

```python
sage: for face in L: print(face.ambient_ray_indices())
Traceback (most recent call last):
  ... AttributeError: 'RationalPolyhedralFan'
oobject has no attribute 'ambient_ray_indices'
sage: L.top()
Rational polyhedral fan in 2-d lattice N
```

For example, you can do

```python
sage: for l in L.level_sets()[:-1]:
....:   print([[f.ambient_ray_indices() for f in l])
[[]]
[[0,), (1,), (2,)]]
[[1, 2]]
```

If the fan is complete, its cone lattice is atomic and coatomic and can (and will!) be computed in a much
more efficient way, but the interface is exactly the same:

```python
sage: fan = toric_varieties.P1xP1().fan() # needs palp
sage: L = fan.cone_lattice() # needs palp
sage: for l in L.level_sets()[:-1]: # needs palp
....:   print([[f.ambient_ray_indices() for f in l])
[[]]
[[0,), (1,), (2,), (3,)]
[[0, 2), (1, 2), (0, 3), (1, 3)]
```

Let’s also consider the cone lattice of a fan generated by a single cone:

```python
sage: fan = Fan([cone1])
sage: L = fan.cone_lattice()
sage: L
Finite poset containing 5 elements with distinguished linear extension
```

Here these 5 elements correspond to the origin, two rays, one generating cone of dimension two, and the
whole fan. While this single cone “is” the whole fan, it is consistent and convenient to distinguish them in the
cone lattice.

```python
cones (dim=None, codim=None)
```
Return the specified cones of self.

**INPUT:**

- `dim` – dimension of the requested cones;
- `codim` – codimension of the requested cones.

**Note:** You can specify at most one input parameter.

**OUTPUT:**

tuple of cones of self of the specified (co)dimension, if either `dim` or `codim` is given. Otherwise tuple of such tuples for all existing dimensions.

**EXAMPLES:**

```sage
cone1 = Cone([(1,0), (0,1)])
cone2 = Cone([(-1,0)])
fan = Fan([cone1, cone2])
fan(dim=0)
(0-d cone of Rational polyhedral fan in 2-d lattice N,)
fan(codim=2)
(0-d cone of Rational polyhedral fan in 2-d lattice N,)
sage: for cone in fan.cones(1): cone.ray(0)
N(-1, 0)
N(0, 1)
N(1, 0)
sage: fan.cones(2)
(2-d cone of Rational polyhedral fan in 2-d lattice N,)
```

You cannot specify both dimension and codimension, even if they “agree”:

```sage
fan(dim=1, codim=1)
Traceback (most recent call last):
  ...
ValueError: dimension and codimension cannot be specified together!
```

But it is OK to ask for cones of too high or low (co)dimension:

```sage
fan(-1)
()
fan(3)
()
fan(codim=4)
()
```

**contains (cone)**

Check if a given `cone` is equivalent to a cone of the fan.

**INPUT:**

- `cone` – anything.

**OUTPUT:**

False if `cone` is not a cone or if `cone` is not equivalent to a cone of the fan, True otherwise
**Note:** Recall that a fan is a (finite) collection of cones. A cone is contained in a fan if it is equivalent to one of the cones of the fan. In particular, it is possible that all rays of the cone are in the fan, but the cone itself is not.

If you want to know whether a point is in the support of the fan, you should use `support_contains()`.

**EXAMPLES:**

We first construct a simple fan:

```sage
cone1 = Cone([0,-1], [1,0])
cone2 = Cone([1,0], [0,1])
f = Fan([cone1, cone2])
```

Now we check if some cones are in this fan. First, we make sure that the order of rays of the input cone does not matter (check=False option ensures that rays of these cones will be listed exactly as they are given):

```sage
f.contains(Cone([1,0], [0,1], check=False))
True
f.contains(Cone([0,1], [1,0], check=False))
True
```

Now we check that a non-generating cone is in our fan:

```sage
f.contains(Cone([1,0]))
True
Cone([1,0]) in f      # equivalent to the previous command
True
```

Finally, we test some cones which are not in this fan:

```sage
f.contains(Cone([1,1]))
False
f.contains(Cone([1,0], [-1,0]))
True
```

A point is not a cone:

```sage
n = f.lattice()(1,1); n
N(1, 1)
f.contains(n)
False
```

**embed** (*cone*)

Return the cone equivalent to the given one, but sitting in self.

You may need to use this method before calling methods of `cone` that depend on the ambient structure, such as `ambient_ray_indices()` or `facet_of()`. The cone returned by this method will have `self` as ambient. If `cone` does not represent a valid cone of `self`, `ValueError` exception is raised.

**Note:** This method is very quick if `self` is already the ambient structure of `cone`, so you can use without extra checks and performance hit even if `cone` is likely to sit in `self` but in principle may not.

**INPUT:**

- `cone` - a cone.
OUTPUT:

a cone of fan, equivalent to cone but sitting inside self

EXAMPLES:

Let’s take a 3-d fan generated by a cone on 4 rays:

```
sage: f = Fan([Cone([(1,0,1), (0,1,1), (-1,0,1), (0,-1,1)])])
```

Then any ray generates a 1-d cone of this fan, but if you construct such a cone directly, it will not “sit” inside the fan:

```
sage: ray = Cone([(0,-1,1)])
sage: ray
1-d cone in 3-d lattice N
sage: ray.ambient_ray_indices()
(0,)
sage: ray.adjacent()
()
sage: ray.ambient()
1-d cone in 3-d lattice N
```

If we want to operate with this ray as a part of the fan, we need to embed it first:

```
sage: e_ray = f.embed(ray)
sage: e_ray
1-d cone of Rational polyhedral fan in 3-d lattice N
sage: e_ray.rays()
N(0, -1, 1)
in 3-d lattice N
sage: e_ray.is ray
False
sage: e_ray.is_equivalent(ray)
True
sage: e_ray.ambient_ray_indices()
(3,)
sage: e_ray.adjacent()
(1-d cone of Rational polyhedral fan in 3-d lattice N, 1-d cone of Rational polyhedral fan in 3-d lattice N)
sage: e_ray.ambient()
Rational polyhedral fan in 3-d lattice N
```

Not every cone can be embedded into a fixed fan:

```
sage: f.embed(Cone([(0,0,1)]))
Traceback (most recent call last):
...
ValueError: 1-d cone in 3-d lattice N does not belong
to Rational polyhedral fan in 3-d lattice N!
sage: f.embed(Cone([(1,0,1), (-1,0,1)]))
Traceback (most recent call last):
...
ValueError: 2-d cone in 3-d lattice N does not belong
to Rational polyhedral fan in 3-d lattice N!
```

```
f_vector()
Return the f-vector of the fan.
This is the tuple \((f_0, f_1, \ldots, f_d)\) where \(f_i\) is the number of cones of dimension \(i\).
```

2.5. Toric geometry 441
EXAMPLES:

```python
sage: F = ClusterAlgebra(['A',2]).cluster_fan()
sage: F.f_vector()
(1, 5, 5)
```

`generating_cone(n)`

Return the \( n \)-th generating cone of \( \text{self} \).

**INPUT:**

- \( n \) – integer, the index of a generating cone.

**OUTPUT:**

`cone of fan`

**EXAMPLES:**

```python
sage: fan = toric_varieties.P1xP1().fan()
sage: fan.generating_cone(0)
(2-d cone of Rational polyhedral fan in 2-d lattice N,
 2-d cone of Rational polyhedral fan in 2-d lattice N,
 2-d cone of Rational polyhedral fan in 2-d lattice N)
```

`generating_cones()`

Return generating cones of \( \text{self} \).

**OUTPUT:**

`tuple of cones of fan`

**EXAMPLES:**

```python
sage: fan = toric_varieties.P1xP1().fan()
sage: fan.generating_cones()
(2-d cone of Rational polyhedral fan in 2-d lattice N,
 1-d cone of Rational polyhedral fan in 2-d lattice N)
```

`is_complete()`

Check if \( \text{self} \) is complete.

A rational polyhedral fan is *complete* if its cones fill the whole space.

**OUTPUT:**

True if \( \text{self} \) is complete and False otherwise

**EXAMPLES:**
sage: fan = toric_varieties.P1xP1().fan()  # needs palp
sage: fan.is_complete()  # needs palp
True
sage: cone1 = Cone([(1,0), (0,1)])
sage: cone2 = Cone([(-1,0)])
sage: fan = Fan([cone1, cone2])
sage: fan.is_complete()
False

is_equivalent(other)
Check if self is “mathematically” the same as other.

INPUT:
• other - fan.

OUTPUT:
True if self and other define the same fans as collections of equivalent cones in the same lattice, False otherwise.

There are three different equivalences between fans $F_1$ and $F_2$ in the same lattice:

1. They have the same rays in the same order and the same generating cones in the same order. This is tested by $F1 == F2$.
2. They have the same rays and the same generating cones without taking into account any order. This is tested by $F1.is_equivalent(F2)$.
3. They are in the same orbit of $GL(n, \mathbb{Z})$ (and, therefore, correspond to isomorphic toric varieties). This is tested by $F1.is_isomorphic(F2)$.

Note that virtual_rays() are included into consideration for all of the above equivalences.

EXAMPLES:

sage: fan1 = Fan(cones=[(0,1), (1,2)],
....:     rays=[(1,0), (0,1), (-1,-1)],
....:     check=False)
sage: fan2 = Fan(cones=[(2,1), (0,2)],
....:     rays=[(1,0), (-1,-1), (0,1)],
....:     check=False)
sage: fan3 = Fan(cones=[(0,1), (1,2)],
....:     rays=[(1,0), (0,1), (-1,1)],
....:     check=False)
sage: fan1 == fan2
False
sage: fan1.is_equivalent(fan2)
True
sage: fan1 == fan3
False
sage: fan1.is_equivalent(fan3)
False

is_isomorphic(other)
Check if self is in the same $GL(n, \mathbb{Z})$-orbit as other.

There are three different equivalences between fans $F_1$ and $F_2$ in the same lattice:
1. They have the same rays in the same order and the same generating cones in the same order. This is tested by \( F_1 == F_2 \).

2. They have the same rays and the same generating cones without taking into account any order. This is tested by \( F_1\text{.is_equivalent}(F_2) \).

3. They are in the same orbit of \( GL(n, \mathbb{Z}) \) (and, therefore, correspond to isomorphic toric varieties). This is tested by \( F_1\text{.is_isomorphic}(F_2) \).

Note that \texttt{virtual\_rays()} are included into consideration for all of the above equivalences.

**INPUT:**

- other – a fan.

**OUTPUT:**

True if self and other are in the same \( GL(n, \mathbb{Z}) \)-orbit, False otherwise

**See also:**

If you want to obtain the actual fan isomorphism, use \texttt{isomorphism()}.

**EXAMPLES:**

Here we pick an \( SL(2, \mathbb{Z}) \) matrix \( m \) and then verify that the image fan is isomorphic:

```sage
rays = ((1, 1), (0, 1), (-1, -1), (1, 0))
cones = [(0,1), (1,2), (2,3), (3,0)]
fan1 = Fan(cones, rays)
m = matrix([[3, -2], [-1, 1]])
fan2 = Fan(cones, [vector(r)*m for r in rays])
fan1.is_isomorphic(fan2)
True
fan1.is_equivalent(fan2)
False
fan1 == fan2
False
```

These fans are “mirrors” of each other:

```sage
rays = ((1, 1), (0, 1), (-1, -1), (1, 0))
cones = [(0,1), (1,2), (2,3), (3,0)]
fan1 = Fan(cones, rays)
m = matrix([[3, -2], [-1, 1]])
fan2 = Fan(cones, [vector(r)*m for r in rays])
fan1.is_isomorphic(fan2)
True
fan1.is_equivalent(fan2)
False
fan1 == fan2
False
```

**is_simplicial()**

Check if self is simplicial.

A rational polyhedral fan is \texttt{simplicial} if all of its cones are, i.e. primitive vectors along generating rays of every cone form a part of a \textit{rational} basis of the ambient space.

**OUTPUT:**
True if self is simplicial and False otherwise

EXAMPLES:

```python
sage: fan = toric_varieties.P1xP1().fan()  # needs palp
sage: fan.is_simplicial()                  # needs palp
    True
sage: cone1 = Cone([(1,0), (0,1)])
    sage: cone2 = Cone([(-1,0)])
    sage: fan = Fan([cone1, cone2])
    sage: fan.is_simplicial()
    True
```

In fact, any fan in a two-dimensional ambient space is simplicial. This is no longer the case in dimension three:

```python
sage: fan = NormalFan(lattice_polytope.cross_polytope(3))
sage: fan.is_simplicial()
    False
sage: fan.generating_cone(0).nrays()
    4
```

`is_smooth (codim=None)`

Check if self is smooth.

A rational polyhedral fan is smooth if all of its cones are, i.e. primitive vectors along generating rays of every cone form a part of an integral basis of the ambient space. In this case the corresponding toric variety is smooth.

A fan in an $n$-dimensional lattice is smooth up to codimension $c$ if all cones of codimension greater than or equal to $c$ are smooth, i.e. if all cones of dimension less than or equal to $n - c$ are smooth. In this case the singular set of the corresponding toric variety is of dimension less than $c$.

INPUT:

- `codim` - codimension in which smoothness has to be checked, by default complete smoothness will be checked.

OUTPUT:

True if self is smooth (in codimension codim, if it was given) and False otherwise.

EXAMPLES:

```python
sage: fan = toric_varieties.P1xP1().fan()  # needs palp
sage: fan.is_smooth()                      # needs palp
    True
sage: cone1 = Cone([(1,0), (0,1)])
    sage: cone2 = Cone([(-1,0)])
    sage: fan = Fan([cone1, cone2])
    sage: fan.is_smooth()
    True
sage: fan = NormalFan(lattice_polytope.cross_polytope(2))
sage: fan.is_smooth()
    False
sage: fan.is_smooth(codim=1)
```

(continues on next page)
isomorphism(\textit{other})

Return a fan isomorphism from \textit{self} to \textit{other}.

\textbf{INPUT:}

- \textit{other} – fan.

\textbf{OUTPUT:}

A fan isomorphism. If no such isomorphism exists, a \texttt{FanNotIsomorphicError} is raised.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: rays = ((1, 1), (0, 1), (-1, -1), (3, 1))
sage: cones = [(0,1), (1,2), (2,3), (3,0)]
sage: fan1 = Fan(cones, rays)
sage: m = matrix([-2,3], [1,-1])
sage: fan2 = Fan(cones, [vector(r)*m for r in rays])

sage: fan1.isomorphism(fan2)
Fan morphism defined by the matrix
[-2  3]
[ 1 -1]
Domain fan: Rational polyhedral fan in 2-d lattice N
Codomain fan: Rational polyhedral fan in 2-d lattice N

sage: fan2.isomorphism(fan1)
Fan morphism defined by the matrix
[1 3]
[1 2]
Domain fan: Rational polyhedral fan in 2-d lattice N
Codomain fan: Rational polyhedral fan in 2-d lattice N

sage: fan1.isomorphism(toric_varieties.P2().fan())
# needs palp
Traceback (most recent call last):
...
FanNotIsomorphicError
\end{verbatim}

\textbf{linear_equivalence_ideal(\textit{ring})}

Return the ideal generated by linear relations.

\textbf{INPUT:}

- A polynomial ring in \textit{self}.\texttt{nrays()} variables.

\textbf{OUTPUT:}

Return the ideal, in the given \textit{ring}, generated by the linear relations of the rays. In toric geometry, this corresponds to rational equivalence of divisors.

\textbf{EXAMPLES:}
sage: fan = Fan([[0,1,3],[3,4],[2,0],[1,2,4]],
....:     [(-3, -2, 1), (0, 0, 1), (3, -2, 1), (-1, -1, 1), (1, -1, 1)])

sage: fan.linear_equivalence_ideal(PolynomialRing(QQ, 5, 'A, B, C, D, E'))
Ideal (-3*A + 3*C - D + E, -2*A - 2*C - D - E, A + B + C + D + E) of
Multivariate Polynomial Ring in A, B, C, D, E over Rational Field

make_simplicial(**kwds)

Construct a simplicial fan subdividing self.

It is a synonym for subdivide() with make_simplicial=True option.

INPUT:
• this functions accepts only keyword arguments. See subdivide() for documentation.

OUTPUT:

rational polyhedral fan

EXAMPLES:

sage: fan = NormalFan(lattice_polytope.cross_polytope(3))
sage: fan.is_simplicial()
False
sage: fan.ngenerating_cones()
6
sage: new_fan = fan.make_simplicial()
sage: new_fan.is_simplicial()
True
sage: new_fan.ngenerating_cones()
12

ngenerating_cones()

Return the number of generating cones of self.

OUTPUT:

integer

EXAMPLES:

sage: fan = toric_varieties.P1xP1().fan()
¬¬needs palp
sage: fan.ngenerating_cones()
¬¬needs palp
4
sage: cone1 = Cone([(1,0), (0,1)])
sage: cone2 = Cone([-1,0]])
sage: fan = Fan([cone1, cone2])
sage: fan.ngenerating_cones()
2

oriented_boundary(cone)

Return the facets bounding cone with their induced orientation.

INPUT:
• cone – a cone of the fan or the whole fan.

OUTPUT:
The boundary cones of \( \text{cone} \) as a formal linear combination of cones with coefficients \( \pm 1 \). Each summand is a facet of \( \text{cone} \) and the coefficient indicates whether their (chosen) orientation agrees or disagrees with the “outward normal first” boundary orientation. Note that the orientation of any individual cone is arbitrary. This method once and for all picks orientations for all cones and then computes the boundaries relative to that chosen orientation.

If \( \text{cone} \) is the fan itself, the generating cones with their orientation relative to the ambient space are returned. See \text{complex()} for the associated chain complex. If you do not require the orientation, use \text{cone. facets()} instead.

**EXAMPLES:**

```
sage: # needs palp
sage: fan = toric_varieties.P(3).fan()
sage: cone = fan[2][0]
sage: bdry = fan.oriented_boundary(cone); bdry
-1-d cone of Rational polyhedral fan in 3-d lattice N
+ 1-d cone of Rational polyhedral fan in 3-d lattice N
sage: bdry[0]
(-1, 1-d cone of Rational polyhedral fan in 3-d lattice N)
sage: bdry[1]
(1, 1-d cone of Rational polyhedral fan in 3-d lattice N)
sage: fan.oriented_boundary(bdry[0][1])
-0-d cone of Rational polyhedral fan in 3-d lattice N
sage: fan.oriented_boundary(bdry[1][1])
-0-d cone of Rational polyhedral fan in 3-d lattice N
```

If you pass the fan itself, this method returns the orientation of the generating cones which is determined by the order of the rays in \text{cone.ray_basis()}

```
sage: fan.oriented_boundary(fan)  # needs palp
-3-d cone of Rational polyhedral fan in 3-d lattice N
+ 3-d cone of Rational polyhedral fan in 3-d lattice N
- 3-d cone of Rational polyhedral fan in 3-d lattice N
+ 3-d cone of Rational polyhedral fan in 3-d lattice N
sage: [cone.rays().basis().matrix().det()  # needs palp
....: for cone in fan.generating_cones()]
[-1, 1, -1, 1]
```

A non-full dimensional fan:

```
sage: cone = Cone([(4,5)])
sage: fan = Fan([cone])
sage: fan.oriented_boundary(cone)
0-d cone of Rational polyhedral fan in 2-d lattice N
sage: fan.oriented_boundary(fan)
1-d cone of Rational polyhedral fan in 2-d lattice N
```

**plot(****options**)

Plot self.

**INPUT:**

- any options for toric plots (see \text{toric_plotter.options}), none are mandatory.

**OUTPUT:**
Combinatorial and Discrete Geometry, Release 10.3

a plot

EXAMPLES:
sage: fan = toric_varieties.dP6().fan()  # needs palp
sage: fan.plot()  # needs palp sage.plot
Graphics object consisting of 31 graphics primitives

primitive_collections()
Return the primitive collections.

OUTPUT:
Return the subsets \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} such that
- The points \{p_{i_1}, \ldots, p_{i_k}\} do not span a cone of the fan.
- If you remove any one \(p_{i_j}\) from the set, then they do span a cone of the fan.

Note: By replacing the multiindices \{i_1, \ldots, i_k\} of each primitive collection with the monomials \(x_{i_1} \cdots x_{i_k}\) one generates the Stanley-Reisner ideal in \(\mathbb{Z}[x_1, \ldots]\).

REFERENCES:
- [Bat1991]

EXAMPLES:
sage: fan = Fan([[0,1,3], [3,4], [2,0], [1,2,4]],
.........
[(-3, -2, 1), (0, 0, 1), (3, -2, 1), (-1, -1, 1), (1, -1, 1)])
sage: fan.primitive_collections()
[frozenset({0, 4}),
frozenset({2, 3}),
frozenset({0, 1, 2}),
frozenset({1, 3, 4})]

subdivide(new_rays=None, make_simplicial=False, algorithm='default', verbose=False)
Construct a new fan subdividing self.

INPUT:
- new_rays - list of new rays to be added during subdivision, each ray must be a list or a vector. May be empty or None (default);
- make_simplicial - if True, the returned fan is guaranteed to be simplicial, default is False;
- algorithm - string with the name of the algorithm used for subdivision. Currently there is only one available algorithm called "default";
- verbose - if True, some timing information may be printed during the process of subdivision.

OUTPUT:
rational polyhedral fan
Currently the “default” algorithm corresponds to iterative stellar subdivision for each ray in new_rays.

EXAMPLES:
sage: fan = NormalFan(lattice_polytope.cross_polytope(3))
sage: fan.is_simplicial()
False
sage: fan.ngenerating_cones()
6
sage: fan.nrays()
8
sage: new_fan = fan.subdivide(new_rays=[(1,0,0)])
sage: new_fan.is_simplicial()
False
sage: new_fan.ngenerating_cones()
9
sage: new_fan.nrays()
9

support_contains(*args)

Check if a point is contained in the support of the fan.

The support of a fan is the union of all cones of the fan. If you want to know whether the fan contains a given cone, you should use contains() instead.

INPUT:

- *args – an element of self.lattice() or something that can be converted to it (for example, a list of coordinates).

OUTPUT:

True if point is contained in the support of the fan, False otherwise

toric_variety(*args, **kwds)

Return the associated toric variety.

INPUT:

same arguments as ToricVariety()

OUTPUT:

a toric variety

This is equivalent to the command ToricVariety(self) and is provided only as a convenient alternative method to go from the fan to the associated toric variety.

EXAMPLES:

sage: Fan([Cone([(1,0)]), Cone([(0,1)])]).toric_variety()
2-d toric variety covered by 2 affine patches

vertex_graph()

Return the graph of 1- and 2-cones.

OUTPUT:

An edge-colored graph. The vertices correspond to the 1-cones (i.e. rays) of the fan. Two vertices are joined by an edge iff the rays span a 2-cone of the fan. The edges are colored by pairs of integers that classify the 2-cones up to $GL(2,\mathbb{Z})$ transformation, see classify_cone_2d().

EXAMPLES:
virtual_rays(*args)

Return (some of the) virtual rays of self.

Let $N$ be the $D$-dimensional lattice() of a $d$-dimensional fan $\Sigma$ in $\mathbb{N}_R$. Then the corresponding toric variety is of the form $X \times (\mathbb{C}^*)^{D-d}$. The actual rays() of $\Sigma$ give a canonical choice of homogeneous coordinates on $X$. This function returns an arbitrary but fixed choice of virtual rays corresponding to a (non-canonical) choice of homogeneous coordinates on the torus factor. Combinatorially primitive integral generators of virtual rays span the $D - d$ dimensions of $\mathbb{N}_Q$ “missed” by the actual rays. (In general addition of virtual rays is not sufficient to span $N$ over $\mathbb{Z}$.)

Note: You may use a particular choice of virtual rays by passing optional argument virtual_rays to the Fan() constructor.

INPUT:

- ray_list – a list of integers, the indices of the requested virtual rays. If not specified, all virtual rays of self will be returned.

OUTPUT:

a PointCollection of primitive integral ray generators. Usually (if the fan is full-dimensional) this will be empty.

EXAMPLES:

```python
sage: f = Fan([Cone([(1,0,1,0), (0,1,1,0)])])
sage: f.virtual_rays()
N(1, 0, 0, 0),
N(0, 0, 0, 1)
in 4-d lattice N

sage: f.rays()
N(1, 0, 1, 0),
N(0, 1, 1, 0)
in 4-d lattice N

sage: f.virtual_rays([0])
N(1, 0, 0, 0)
in 4-d lattice N
```
You can also give virtual ray indices directly, without packing them into a list:

```
sage: f.virtual_rays(0)
N(1, 0, 0, 0)
in 4-d lattice N
```

Make sure that github issue #16344 is fixed and one can compute the virtual rays of fans in non-saturated lattices:

```
sage: N = ToricLattice(1)
sage: B = N.submodule([2,]).basis()
sage: f = Fan([Cone([B[0]])])
sage: len(f.virtual_rays())
0
```

`sage.geometry.fan.discard_faces(cones)`

Return the cones of the given list which are not faces of each other.

**INPUT:**

- cones – a list of cones.

**OUTPUT:**

A list of cones, sorted by dimension in decreasing order

**EXAMPLES:**

Consider all cones of a fan:

```
sage: Sigma = toric_varieties.P2().fan()  #...
˓→needs palp
sage: cones = flatten(Sigma.cones())  #...
˓→needs palp
sage: len(cones)  #...
˓→needs palp
7
```

Most of them are not necessary to generate this fan:

```
sage: from sage.geometry.fan import discard_faces
sage: len(discard_faces(cones))  #...
˓→needs palp
3
sage: Sigma.ngenerating_cones()  #...
˓→needs palp
3
```

`sage.geometry.fan.is_Fan(x)`

Check if `x` is a Fan.

**INPUT:**

- `x` – anything.

**OUTPUT:**

True if `x` is a fan and False otherwise

**EXAMPLES:**
sage: from sage.geometry.fan import is_Fan
sage: is_Fan(1)
False
sage: fan = toric_varieties.P2().fan(); fan
# needs palp
Rational polyhedral fan in 2-d lattice N
sage: is_Fan(fan)
# needs palp
True

2.5.5 Morphisms between toric lattices compatible with fans

This module is a part of the framework for toric varieties (*variety*, *fano_variety*). Its main purpose is to provide support for working with lattice morphisms compatible with fans via *FanMorphism* class.

AUTHORS:

• Andrey Novoseltsev (2010-10-17): initial version.
• Andrey Novoseltsev (2011-04-11): added tests for injectivity/surjectivity, fibration, bundle, as well as some related methods.

EXAMPLES:

Let's consider the face and normal fans of the “diamond” and the projection to the $x$-axis:

```
sage: diamond = lattice_polytope.cross_polytope(2)
sage: face = FaceFan(diamond, lattice=ToricLattice(2))
sage: normal = NormalFan(diamond)
sage: N = face.lattice()
sage: H = End(N)
sage: phi = H([N.0, 0])
sage: phi
Free module morphism defined by the matrix
[1 0]
[0 0]
Domain: 2-d lattice N
Codomain: 2-d lattice N
```

```
sage: FanMorphism(phi, normal, face)
Traceback (most recent call last):
... 
ValueError: the image of generating cone #1 of the domain fan
is not contained in a single cone of the codomain fan!
```

Some of the cones of the normal fan fail to be mapped to a single cone of the face fan. We can rectify the situation in the following way:

```
sage: fm = FanMorphism(phi, normal, face, subdivide=True)
sage: fm
Fan morphism defined by the matrix
[1 0]
[0 0]
Domain fan: Rational polyhedral fan in 2-d lattice N
Codomain fan: Rational polyhedral fan in 2-d lattice N
```

```
sage: fm.domain_fan().rays()
N( 1, 1),
N( 1, -1),
```

(continues on next page)
As you see, it was necessary to insert two new rays (to prevent “upper” and “lower” cones of the normal fan from being mapped to the whole $x$-axis).

```python
class sage.geometry.fan_morphism.FanMorphism(morphism, domain_fan, codomain=None, subdivide=False, check=True, verbose=False)

Bases: FreeModuleMorphism

Create a fan morphism.

Let $\Sigma_1$ and $\Sigma_2$ be two fans in lattices $N_1$ and $N_2$ respectively. Let $\phi$ be a morphism (i.e. a linear map) from $N_1$ to $N_2$. We say that $\phi$ is compatible with $\Sigma_1$ and $\Sigma_2$ if every cone $\sigma_1 \in \Sigma_1$ is mapped by $\phi$ into a single cone $\sigma_2 \in \Sigma_2$, i.e. $\phi(\sigma_1) \subset \sigma_2$ ($\sigma_2$ may be different for different $\sigma_1$).

By a fan morphism we understand a morphism between two lattices compatible with specified fans in these lattices. Such morphisms behave in exactly the same way as “regular” morphisms between lattices, but:

• fan morphisms have a special constructor allowing some automatic adjustments to the initial fans (see below);
• fan morphisms are aware of the associated fans and they can be accessed via `codomain_fan()` and `domain_fan()`;
• fan morphisms can efficiently compute `image_cone()` of a given cone of the domain fan and `preimage_cones()` of a given cone of the codomain fan.

**INPUT:**

• `morphism` – either a morphism between domain and codomain, or an integral matrix defining such a morphism;
• `domain_fan` – a fan in the domain;
• `codomain` – (default: `None`) either a codomain lattice or a fan in the codomain. If the codomain fan is not given, the image fan (fan generated by images of generating cones) of `domain_fan` will be used, if possible;
• `subdivide` – (default: `False`) if True and `domain_fan` is not compatible with the codomain fan because it is too coarse, it will be automatically refined to become compatible (the minimal refinement is canonical, so there are no choices involved);
• `check` – (default: `True`) if False, given fans and morphism will be assumed to be compatible. Be careful when using this option, since wrong assumptions can lead to wrong and hard-to-detect errors. On the other hand, this option may save you some time;
• `verbose` – (default: `False`) if True, some information may be printed during construction of the fan morphism.

**OUTPUT:**

• a fan morphism.
EXAMPLES:

Here we consider the face and normal fans of the “diamond” and the projection to the $x$-axis:

```
sage: diamond = lattice_polytope.cross_polytope(2)
sage: face = FaceFan(diamond, lattice=ToricLattice(2))
sage: normal = NormalFan(diamond)
sage: N = face.lattice()
sage: H = End(N)
sage: phi = H([N.0, 0])
sage: phi
Free module morphism defined by the matrix
[1 0]
[0 0]
Domain: 2-d lattice N
Codomain: 2-d lattice N
```

Note, that since $\phi$ is compatible with these fans, the returned fan is exactly the same object as the initial domain_fan.

```
sage: fm = FanMorphism(phi, face, normal)
sage: fm.domain_fan()
is face
```

We had to subdivide two of the four cones of the normal fan, since they were mapped by $\phi$ into non-strictly convex cones.

It is possible to omit the codomain fan, in which case the image fan will be used instead of it:

```
sage: fm = FanMorphism(phi, face)
sage: fm.codomain_fan()  
Rational polyhedral fan in 2-d lattice N
sage: fm.codomain_fan().rays()
N( 1, 0),
N(-1, 0)
in 2-d lattice N
```

Now we demonstrate a more subtle example. We take the first quadrant as our domain fan. Then we divide the first quadrant into three cones, throw away the middle one and take the other two as our codomain fan. These fans are incompatible with the identity lattice morphism since the image of the domain fan is out of the support of the codomain fan:

```
sage: N = ToricLattice(2)
sage: phi = End(N).identity()
sage: F1 = Fan(cones=[(0,1)], rays=[(1,0), (0,1)])
sage: F2 = Fan(cones=[(0,1), (2,3)],
.........:     rays=[(1,0), (2,1), (1,2), (0,1)])
sage: FanMorphism(phi, F1, F2)
(continues on next page)
```
Traceback (most recent call last):
...
ValueError: the image of generating cone #0 of the domain fan is not contained in a single cone of the codomain fan!
sage: FanMorphism(phi, F1, F2, subdivide=True)
Traceback (most recent call last):
...
ValueError: morphism defined by
[1 0]
[0 1]
do not map
Rational polyhedral fan in 2-d lattice N into the support of
Rational polyhedral fan in 2-d lattice N!

The problem was detected and handled correctly (i.e. an exception was raised). However, the used algorithm requires extra checks for this situation after constructing a potential subdivision and this can take significant time. You can save about half the time using check=False option, if you know in advance that it is possible to make fans compatible with the morphism by subdividing the domain fan. Of course, if your assumption was incorrect, the result will be wrong and you will get a fan which does map into the support of the codomain fan, but is not a subdivision of the domain fan. You can test it on the example above:

sage: fm = FanMorphism(phi, F1, F2, subdivide=True,
...: check=False, verbose=True)
Placing ray images (... ms)
Computing chambers (... ms)
Number of domain cones: 1.
Number of chambers: 2.
Cone 0 sits in chambers 0 1 (... ms)
sage: fm.domain_fan().is_equivalent(F2)
True

codomain_fan (dim=None, codim=None)
Return the codomain fan of self.

INPUT:
• dim – dimension of the requested cones;
• codim – codimension of the requested cones.

OUTPUT:
• rational polyhedral fan if no parameters were given, tuple of cones otherwise.

EXAMPLES:

sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant = Fan([quadrant])
sage: quadrant_bl = quadrant.subdivide([(1,1)])
sage: fm = FanMorphism(identity_matrix(2), quadrant_bl, quadrant)
sage: fm.codomain_fan()
Rational polyhedral fan in 2-d lattice N
sage: fm.codomain_fan() is quadrant
True

domain_fan (dim=None, codim=None)
Return the codomain fan of self.

456 Chapter 2. Polyhedral computations
INPUT:
• \( \text{dim} \) – dimension of the requested cones;
• \( \text{codim} \) – codimension of the requested cones.

OUTPUT:
• rational polyhedral fan if no parameters were given, tuple of cones otherwise.

EXAMPLES:
```
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant = Fan([quadrant])
sage: quadrant_bl = quadrant.subdivide([(1,1)])
sage: fm = FanMorphism(identity_matrix(2), quadrant_bl, quadrant)
sage: fm.domain_fan()
Rational polyhedral fan in 2-d lattice N
sage: fm.domain_fan() is quadrant_bl
True
```

\textbf{factor()}
Factor \texttt{self} into injective \* birational \* surjective morphisms.

OUTPUT:
• a triple of \texttt{FanMorphism} \((\phi_s, \phi_b, \phi_i)\), such that \(\phi_s\) is surjective, \(\phi_b\) is birational, \(\phi_i\) is injective, and \(\texttt{self} \) is equal to \(\phi_i \circ \phi_b \circ \phi_s\).

Intermediate fans live in the saturation of the image of \texttt{self} as a map between lattices and are the image of the \texttt{domain_fan()} and the restriction of the \texttt{codomain_fan()}, i.e. if \texttt{self} maps \(\Sigma \to \Sigma'\), then we have factorization into

\[
\Sigma \to \Sigma_s \to \Sigma_i \hookrightarrow \Sigma.
\]

\textbf{Note:}
• \(\Sigma_s\) is the finest fan with the smallest support that is compatible with \texttt{self}: any fan morphism from \(\Sigma\) given by the same map of lattices as \texttt{self} factors through \(\Sigma_s\).
• \(\Sigma_i\) is the coarsest fan of the largest support that is compatible with \texttt{self}: any fan morphism into \(\Sigma'\) given by the same map of lattices as \texttt{self} factors though \(\Sigma_i\).

EXAMPLES:
We map an affine plane into a projective 3-space in such a way, that it becomes “a double cover of a chart of the blow up of one of the coordinate planes”:
```
sage: A2 = toric_varieties.A2()
sage: P3 = toric_varieties.P(3)  #...
→ needs palp
sage: m = matrix([[2,0,0], [1,1,0]])
sage: phi = A2.hom(m, P3)  #...
→ needs palp
sage: phi.as_polynomial_map()  #...
→ needs palp
Scheme morphism:
  From: 2-d affine toric variety
  To: 3-d CPR-Fano toric variety covered by 4 affine patches
```
(continues on next page)
Defn: Defined on coordinates by sending \([x : y]\) to
\([x^2 y : y : 1 : 1]\)

Now we will work with the underlying fan morphism:

```sage
# needs palp
sage: phi = phi.fan_morphism(); phi
Fan morphism defined by the matrix
[2 0 0]
[1 1 0]
Domain fan: Rational polyhedral fan in 2-d lattice N
Codomain fan: Rational polyhedral fan in 3-d lattice N
sage: phi.is_surjective(), phi.is_birational(), phi.isInjective()
(False, False, False)
sage: phi_i, phi_b, phi_s = phi.factor()
sage: phi_s.is_surjective(), phi_b.is_birational(), phi_i.is_injective()
(True, True, True)
sage: prod(phi.factor()) == phi
True
```

Double cover (surjective):

```sage
sage: A2.fan().rays()
N(1, 0),
N(0, 1)
in 2-d lattice N
sage: phi_s
# needs palp
Fan morphism defined by the matrix
[2 0]
[1 1]
Domain fan: Rational polyhedral fan in 2-d lattice N
Codomain fan: Rational polyhedral fan in Sublattice <N(1, 0, 0), N(0, 1, 0)>
sage: phi_s.codomain_fan().rays()
# needs palp
N(1, 0, 0),
N(1, 1, 0)
in Sublattice <N(1, 0, 0), N(0, 1, 0)>
```

Blowup chart (birational):

```sage
sage: phi_b
# needs palp
Fan morphism defined by the matrix
[1 0]
[0 1]
Domain fan: Rational polyhedral fan in Sublattice <N(1, 0, 0), N(0, 1, 0)>
Codomain fan: Rational polyhedral fan in Sublattice <N(1, 0, 0), N(0, 1, 0)>
sage: phi_b.codomain_fan().rays()
# needs palp
N(-1, -1, 0),
N( 0, 1, 0),
N( 1, 0, 0)
in Sublattice <N(1, 0, 0), N(0, 1, 0)>
```

Coordinate plane inclusion (injective):
Fan morphism defined by the matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
Domain fan: Rational polyhedral fan in Sublattice <N(1, 0, 0), N(0, 1, 0)>
Codomain fan: Rational polyhedral fan in 3-d lattice N

\[
\begin{align*}
\mathbf{N}(1, 0, 0), \\
\mathbf{N}(0, 1, 0), \\
\mathbf{N}(0, 0, 1), \\
\mathbf{N}(-1, -1, -1)
\end{align*}
\] in 3-d lattice N

**image_cone** (*cone*)

Return the cone of the codomain fan containing the image of *cone*.

**INPUT:**

- *cone* – a *cone* equivalent to a cone of the *domain_fan()* of self.

**OUTPUT:**

- a *cone* of the *codomain_fan()* of self.

**EXAMPLES:**

```python
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant = Fan([quadrant])
sage: quadrant_bl = quadrant.subdivide([(1,1)])
sage: fm = FanMorphism(identity_matrix(2), quadrant_bl, quadrant)
sage: fm.image_cone(Cone([(1,0)]))
1-d cone of Rational polyhedral fan in 2-d lattice N
sage: fm.image_cone(Cone([(1,1)]))
2-d cone of Rational polyhedral fan in 2-d lattice N
```

**index** (*cone*=*None*)

Return the index of self as a map between lattices.

**INPUT:**

- *cone* – (default: None) a *cone* of the *codomain_fan()* of self.

**OUTPUT:**

- an integer, infinity, or None.

If no cone was specified, this function computes the index of the image of *self* in the codomain. If a cone \(\sigma\) was given, the index of *self* over *sigma* is computed in the sense of Definition 2.1.7 of [HLY2002]: if \(\sigma'\) is any cone of the *domain_fan()* of *self* whose relative interior is mapped to the relative interior of \(\sigma\), it is the index of the image of \(\mathcal{N}'(\sigma')\) in \(\mathcal{N}(\sigma)\), where \(\mathcal{N}'\) and \(\mathcal{N}\) are domain and codomain lattices respectively. While that definition was formulated for the case of the finite index only, we extend it to the infinite one as well and return None if there is no \(\sigma'\) at all. See examples below for situations when such things happen. Note also that the index of *self* is the same as index over the trivial cone.

**EXAMPLES:**

```python
sage: # needs palp
sage: Sigma = toric_varieties.dP8().fan()
(sage, continues on next page)```
Infinite index in the last example indicates that the image has positive codimension in the codomain. Let’s look at the rays of our fans:

```
sage: Sigma_p.rays() # needs palp
N( 1),
N(-1)
in 1-d lattice N
sage: Sigma.rays() # needs palp
N( 1, 1),
N( 0, 1),
N(-1, -1),
N( 1, 0)
in 2-d lattice N
sage: xi.factor()[0].domain_fan().rays() # needs palp
N(-1, 0),
N( 1, 0)
in Sublattice <N(1, 0)>
```

We see that one of the rays of the fan of $\mathbb{P}1$ is mapped to a ray, while the other one to the interior of some 2-d cone. Both rays correspond to single points on $\mathbb{P}1$, yet one is mapped to the distinguished point of a torus invariant curve of $d\mathbb{P}8$ (with the rest of this curve being uncovered) and the other to a fixed point of $d\mathbb{P}8$ (thus completely covering this torus orbit in $d\mathbb{P}8$).

We should therefore expect the following behaviour: all indices over 1-d cones are None, except for one which is infinite, and all indices over 2-d cones are None, except for one which is $1$:

```
sage: [xi.index(cone) for cone in Sigma(1)] # needs palp
[None, None, None, +Infinity]
sage: [xi.index(cone) for cone in Sigma(2)] # needs palp
[None, 1, None, None]
```

is_birational()  
Check if self is birational.

OUTPUT:

- True if self is birational, False otherwise.

For fan morphisms this check is equivalent to self.index() == 1 and means that the corresponding map between toric varieties is birational.

EXAMPLES:
is_bundle()

Check if self is a bundle.

OUTPUT:

• True if self is a bundle, False otherwise.

Let $\phi : \Sigma \rightarrow \Sigma'$ be a fan morphism such that the underlying lattice morphism $\phi : N \rightarrow N'$ is surjective. Let $\Sigma_0$ be the kernel fan of $\phi$. Then $\phi$ is a bundle (or splitting) if there is a subfan $\hat{\Sigma}$ of $\Sigma$ such that the following two conditions are satisfied:

1. Cones of $\Sigma$ are precisely the cones of the form $\sigma_0 + \hat{\sigma}$, where $\sigma_0 \in \Sigma_0$ and $\hat{\sigma} \in \hat{\Sigma}$.

2. Cones of $\hat{\Sigma}$ are in bijection with cones of $\Sigma'$ induced by $\phi$ and $\phi$ maps lattice points in every cone $\hat{\sigma} \in \hat{\Sigma}$ bijectively onto lattice points in $\phi(\hat{\sigma})$.

If a fan morphism $\phi : \Sigma \rightarrow \Sigma'$ is a bundle, then $X_\Sigma$ is a fiber bundle over $X_{\Sigma'}$ with fibers $X_{\Sigma_0, N_0}$, where $N_0$ is the kernel lattice of $\phi$. See [CLS2011] for more details.

See also:

is_fibration(), kernel_fan().

EXAMPLES:

We consider several maps between fans of a del Pezzo surface and the projective line:
The first of these maps induces not only a fibration, but a fiber bundle structure. The second map is very similar, yet it fails to be a bundle, as its index is 2. The last map is not even a fibration.

**is_dominant()**

Return whether the fan morphism is dominant.

A fan morphism $\phi$ is dominant if it is surjective as a map of vector spaces. That is, $\phi_R : N_R \to N'_R$ is surjective.

If the domain fan is **complete**, then this implies that the fan morphism is **surjective**.

If the fan morphism is dominant, then the associated morphism of toric varieties is dominant in the algebraic-geometric sense (that is, surjective onto a dense subset).

**OUTPUT:**

Boolean.

**EXAMPLES:**

```sage
sage: P1 = toric_varieties.P1()
sage: A1 = toric_varieties.A1()
sage: phi = FanMorphism(matrix([[1]]), A1.fan(), P1.fan())
sage: phi.is_dominant()
True
sage: phi.is_surjective()
False
```

**is_fibration()**

Check if `self` is a fibration.

**OUTPUT:**

• **True** if `self` is a fibration, **False** otherwise.

A fan morphism $\phi : \Sigma \to \Sigma'$ is a **fibration** if for any cone $\sigma' \in \Sigma'$ and any primitive preimage cone $\sigma \in \Sigma$ corresponding to $\sigma'$ the linear map of vector spaces $\phi_R$ induces a bijection between $\sigma$ and $\sigma'$, and, in addition, $\phi$ is **dominant** (that is, $\phi_R : N_R \to N'_R$ is surjective).

If a fan morphism $\phi : \Sigma \to \Sigma'$ is a fibration, then the associated morphism between toric varieties $\tilde{\phi} : X_{\Sigma} \to X_{\Sigma'}$ is a fibration in the sense that it is surjective and all of its fibers have the same dimension, namely $\text{dim } X_{\Sigma} - \text{dim } X_{\Sigma'}$. These fibers do not have to be isomorphic, i.e. a fibration is not necessarily a fiber bundle. See [HLY2002] for more details.

**See also:**

`is_bundle()`, `primitive_preimage_cones()`.

**EXAMPLES:**

We consider several maps between fans of a del Pezzo surface and the projective line:

```sage
sage: # needs palp
sage: Sigma = toric_varieties.dP8().fan()
sage: Sigma_p = toric_varieties.P1().fan()
sage: phi = FanMorphism(matrix([[1], [-1]]), Sigma, Sigma_p)
sage: psi = FanMorphism(matrix([[2], [-2]]), Sigma, Sigma_p)
sage: xi = FanMorphism(matrix([[1, 0]]), Sigma_p, Sigma)
sage: phi.is_bundle()
True
sage: phi.is_fibration()
True
```

(continues on next page)
The first of these maps induces not only a fibration, but a fiber bundle structure. The second map is very similar, yet it fails to be a bundle, as its index is 2. The last map is not even a fibration.

**is_injective()**

Check if *self* is injective.

**OUTPUT:**
- True if *self* is injective, False otherwise.

Let \( \phi : \Sigma \rightarrow \Sigma' \) be a fan morphism such that the underlying lattice morphism \( \phi : N \rightarrow N' \) bijectively maps \( N \) to a saturated sublattice of \( N' \). Let \( \psi : \Sigma \rightarrow \Sigma'_0 \) be the restriction of \( \phi \) to the image. Then \( \phi \) is **injective** if the map between cones corresponding to \( \psi \) (injectively) maps each cone of \( \Sigma \) to a cone of the same dimension.

If a fan morphism \( \phi : \Sigma \rightarrow \Sigma' \) is injective, then the associated morphism between toric varieties \( \tilde{\phi} : X_\Sigma \rightarrow X_{\Sigma'} \) is injective.

**See also:**

`factor()`.

**EXAMPLES:**

Consider the fan of the affine plane:

```python
sage: A2 = toric_varieties.A(2).fan()
```

We will map several fans consisting of a single ray into the interior of the 2-cone:

```python
sage: Sigma = Fan([Cone([(1,1)])])
sage: m = identity_matrix(2)
sage: FanMorphism(m, Sigma, A2).is_injective()  # False
```

This morphism was not injective since (in the toric varieties interpretation) the 1-dimensional orbit corresponding to the ray was mapped to the 0-dimensional orbit corresponding to the 2-cone.

```python
sage: Sigma = Fan([Cone([(1,1)])])
sage: m = matrix([1,2,[1,1]])
sage: FanMorphism(m, Sigma, A2).is_injective()  # True
```

While the fans in this example are close to the previous one, here the ray corresponds to a 0-dimensional orbit.
Here the problem is that \( m \) maps the domain lattice to a non-saturated sublattice of the codomain. The corresponding map of the toric varieties is a two-sheeted cover of its image.

We also embed the affine plane into the projective one:

```python
sage: P2 = toric_varieties.P(2).fan()  # needs palp
sage: m = identity_matrix(2)

sage: FanMorphism(m, A2, P2).is_injective()  # needs palp
True
```

**is_surjective()**

Check if self is surjective.

OUTPUT:

- True if self is surjective, False otherwise.

A fan morphism \( \phi : \Sigma \to \Sigma' \) is **surjective** if the corresponding map between cones is surjective, i.e. for each cone \( \sigma' \in \Sigma' \) there is at least one preimage cone \( \sigma \in \Sigma \) such that the relative interior of \( \sigma \) is mapped to the relative interior of \( \sigma' \) and, in addition, \( \phi_R : \mathbb{N}_R \to \mathbb{N}'_R \) is surjective.

If a fan morphism \( \phi : \Sigma \to \Sigma' \) is surjective, then the associated morphism between toric varieties \( \tilde{\phi} : X_\Sigma \to X_{\Sigma'} \) is surjective.

See also:

- `is_bundle()`, `is_fibration()`, `preimage_cones()`, `is_complete()`.

**EXAMPLES:**

We check that the blow up of the affine plane at the origin is surjective:

```python
sage: A2 = toric_varieties.A(2).fan()

sage: B1 = A2.subdivide([[(1,1)]]

sage: m = identity_matrix(2)

sage: FanMorphism(m, B1, A2).is_surjective()
True
```

It remains surjective if we throw away “south and north poles” of the exceptional divisor:

```python
sage: FanMorphism(m, Fan(B1.cones(1)), A2).is_surjective()
True
```

But a single patch of the blow up does not cover the plane:

```python
sage: F = Fan([B1.generating_cone(0)])

sage: FanMorphism(m, F, A2).is_surjective()
False
```

**kernel_fan()**

Return the subfan of the domain fan mapped into the origin.

OUTPUT:
• a fan.

Note: The lattice of the kernel fan is the \texttt{kernel()} sublattice of \texttt{self}.

See also: \texttt{preimage_fan()}.

EXAMPLES:

\begin{verbatim}
sage: fan = Fan(rays=[(1,0), (1,1), (0,1)], cones=[(0,1), (1,2)])
sage: fm = FanMorphism(matrix(2, 1, [1,-1]), fan, ToricLattice(1))
sage: fm.kernel_fan()
Rational polyhedral fan in Sublattice <N(1, 1)>

sage: _.rays()
N(1, 1)
in Sublattice <N(1, 1)>

sage: fm.kernel_fan().cones()
((0-d cone of Rational polyhedral fan in Sublattice <N(1, 1)>,),
 (1-d cone of Rational polyhedral fan in Sublattice <N(1, 1)>,))
\end{verbatim}

\texttt{preimage_cones()}

Return cones of the domain fan whose \texttt{image_cone()} is \texttt{cone}.

INPUT:

• \texttt{cone} – a \texttt{cone} equivalent to a cone of the \texttt{codomain_fan()} of \texttt{self}.

OUTPUT:

• a tuple of \texttt{cones} of the \texttt{domain_fan()} of \texttt{self}, sorted by dimension.

See also: \texttt{preimage_fan()}.

EXAMPLES:

\begin{verbatim}
sage: quadrant = Cone([(1,0), (0,1)])
sage: quadrant = Fan([quadrant])
sage: quadrant_bl = quadrant.subdivide([(1,1)])
sage: fm = FanMorphism(identity_matrix(2), quadrant_bl, quadrant)
sage: fm.preimage_cones(Cone([(1,0)]))
(1-d cone of Rational polyhedral fan in 2-d lattice N,)

sage: fm.preimage_cones(Cone([(1,0), (0,1)]))
(1-d cone of Rational polyhedral fan in 2-d lattice N,
 2-d cone of Rational polyhedral fan in 2-d lattice N,
 2-d cone of Rational polyhedral fan in 2-d lattice N)
\end{verbatim}

\texttt{preimage_fan()}

Return the subfan of the domain fan mapped into \texttt{cone}.

INPUT:

• \texttt{cone} – a \texttt{cone} equivalent to a cone of the \texttt{codomain_fan()} of \texttt{self}.

OUTPUT:

• a fan.
Note: The preimage fan of cone consists of all cones of the `domain_fan()` which are mapped into cone, including those that are mapped into its boundary. So this fan is not necessarily generated by `preimage_cones()` of cone.

See also:

`kernel_fan()`, `preimage_cones()`.

**EXAMPLES:**

```python
sage: quadrant_cone = Cone([(1,0), (0,1)])
sage: quadrant_fan = Fan([quadrant_cone])
sage: quadrant_bl = quadrant_fan.subdivide([(1,1)])
sage: fm = FanMorphism(identity_matrix(2),
rs:
    quadrant_bl, quadrant_fan)
sage: fm.preimage_fan(Cone([(1,0)]).cones())
((0-d cone of Rational polyhedral fan in 2-d lattice N,),
(1-d cone of Rational polyhedral fan in 2-d lattice N,))
sage: len(fm.preimage_cones(quadrant_cone).ngenerating_cones())
2
sage: len(fm.preimage_cones(quadrant_cone))
3
```

**primitive_preimage_cones** (cone)

Return the primitive cones of the domain fan corresponding to cone.

**INPUT:**

• cone – a cone equivalent to a cone of the `codomain_fan()` of self.

**OUTPUT:**

• a cone.

Let $\phi : \Sigma \rightarrow \Sigma'$ be a fan morphism, let $\sigma \in \Sigma$, and let $\sigma' = \phi(\sigma)$. Then $\sigma$ is a primitive cone corresponding to $\sigma'$ if there is no proper face $\tau$ of $\sigma$ such that $\phi(\tau) = \sigma'$.

Primitive cones play an important role for fibration morphisms.

See also:

`is_fibration()`, `preimage_cones()`, `preimage_fan()`.

**EXAMPLES:**

Consider a projection of a del Pezzo surface onto the projective line:

```python
sage: Sigma = toric_varieties.dP6().fan()  # needs palp
sage: Sigma.rays()  # needs palp
N( 0, 1),
N(-1, 0),
N(-1, -1),
N( 0, -1),
N( 1, 0),
N( 1, 1)
in 2-d lattice N
sage: Sigma_p = toric_varieties.P1().fan()
```
Under this map, one pair of rays is mapped to the origin, one in the positive direction, and one in the negative one. Also three 2-dimensional cones are mapped in the positive direction and three in the negative one, so there are 5 preimage cones corresponding to either of the rays of the codomain fan $\Sigma_p$:

```
sage: len(phi.preimage_cones(Cone([(1,)])))  # needs palp
5
```

Yet only rays are primitive:

```
sage: phi.primitive_preimage_cones(Cone([(1,)]))  # needs palp
((1-d cone of Rational polyhedral fan in 2-d lattice N,)
 (1-d cone of Rational polyhedral fan in 2-d lattice N))
```

Since all primitive cones are mapped onto their images bijectively, we get a fibration:

```
sage: phi.is_fibration()  # needs palp
True
```

But since there are several primitive cones corresponding to the same cone of the codomain fan, this map is not a bundle, even though its index is 1:

```
sage: phi.is_bundle()  # needs palp
False
sage: phi.index()  # needs palp
1
```

**relative_star_generators** *(domain_cone)*

Return the relative star generators of *domain_cone*.

**INPUT:**

* domain_cone – a cone of the *domain_fan()* of self.

**OUTPUT:**

* star_generators() of *domain_cone* viewed as a cone of *preimage_fan()* of *image_cone()* of *domain_cone*.

**EXAMPLES:**

```
sage: A2 = toric_varieties.A(2).fan()
sage: B1 = A2.subdivide([(1,1)])
sage: f = FanMorphism(identity_matrix(2), B1, A2)
sage: for c1 in B1(1):
   ....:     print(f.relative_star_generators(c1))
(1-d cone of Rational polyhedral fan in 2-d lattice N,
 (1-d cone of Rational polyhedral fan in 2-d lattice N,
 (2-d cone of Rational polyhedral fan in 2-d lattice N,
 2-d cone of Rational polyhedral fan in 2-d lattice N)
```
2.5.6 Point collections

This module was designed as a part of framework for toric varieties (\texttt{variety, fano\_variety}).

AUTHORS:

- Andrey Novoseltsev (2012-03-06): additions and doctest changes while switching cones to use point collections.

EXAMPLES:

The idea behind \emph{point collections} is to have a container for points of the same space that

- behaves like a tuple \emph{without significant performance penalty}:

  \begin{verbatim}
  sage: c = Cone([[0,0,1], [1,0,1], [0,1,1], [1,1,1]]).rays()
  sage: c[1]
  N(1, 0, 1)
  sage: for point in c: point
  N(0, 0, 1)
  N(1, 0, 1)
  N(0, 1, 1)
  N(1, 1, 1)
  \end{verbatim}

- prints in a convenient way and with clear indication of the ambient space:

  \begin{verbatim}
  sage: c
  N(0, 0, 1),
  N(1, 0, 1),
  N(0, 1, 1),
  N(1, 1, 1)
  in 3-d lattice N
  \end{verbatim}

- allows (cached) access to alternative representations:

  \begin{verbatim}
  sage: c.set()
  frozenset({N(0, 0, 1), N(0, 1, 1), N(1, 0, 1), N(1, 1, 1)})
  \end{verbatim}

- allows introduction of additional methods:

  \begin{verbatim}
  sage: c.basis()
  N(0, 0, 1),
  N(1, 0, 1),
  N(0, 1, 1)
  in 3-d lattice N
  \end{verbatim}

Examples of natural point collections include ray and line generators of cones, vertices and points of polytopes, normals to facets, their subcollections, etc.

Using this class for all of the above cases allows for unified interface \emph{and} cache sharing. Suppose that $\Delta$ is a reflexive polytope. Then the same point collection can be linked as

1. vertices of $\Delta$;
2. facet normals of its polar $\Delta^\circ$;
3. ray generators of the face fan of $\Delta$;
4. ray generators of the normal fan of $\Delta$.

If all these objects are in use and, say, a matrix representation was computed for one of them, it becomes available to all others as well, eliminating the need to spend time and memory four times.
class sage.geometry.point_collection.PointCollection
    Bases: SageObject

    Create a point collection.

    **Warning:** No correctness check or normalization is performed on the input data. This class is designed for internal operations and you probably should not use it directly.

    Point collections are immutable, but cache most of the returned values.

    **INPUT:**
    - **points** – an iterable structure of immutable elements of module, if points are already accessible to you as a tuple, it is preferable to use it for speed and memory consumption reasons;
    - **module** – an ambient module for points. If None (the default), it will be determined as `parent()` of the first point. Of course, this cannot be done if there are no points, so in this case you must give an appropriate module directly.

    **OUTPUT:**
    - a point collection.

    **basis()**
    Return a linearly independent subset of points of self.

    **OUTPUT:**
    - a point collection giving a random (but fixed) choice of an $\mathbf{R}$-basis for the vector space spanned by the points of self.

    **EXAMPLES:**

    ```sage
    sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
    sage: c.basis()
    N(0, 0, 1),
    N(1, 0, 1),
    N(0, 1, 1)
    in 3-d lattice N
    ```

    Calling this method twice will always return exactly the same point collection:

    ```sage
    sage: c.basis().basis() is c.basis()
    True
    ```

    **cardinality()**
    Return the number of points in self.

    **OUTPUT:**
    - an integer.

    **EXAMPLES:**

    ```sage
    sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
    sage: c.cardinality()
    4
    ```
cartesian_product \(\text{other, module=None}\)

Return the Cartesian product of self with other.

**INPUT:**
- \text{other} – a \text{point collection};
- module – (optional) the ambient module for the result. By default, the direct sum of the ambient modules of self and other is constructed.

**OUTPUT:**
- a \text{point collection}.

**EXAMPLES:**

```
sage: c = Cone([\(0,0,1\), \(1,1,1\)]).rays()
sage: c.cartesian_product(c)
```

\[
\begin{array}{cccc}
N+N(0, 0, 1, 0, 0, 1), \\
N+N(1, 1, 1, 0, 0, 1), \\
N+N(0, 0, 1, 1, 1, 1), \\
N+N(1, 1, 1, 1, 1, 1)
\end{array}
\]
in 6-d lattice N+N

column_matrix()

Return a matrix whose columns are points of self.

**OUTPUT:**
- a \text{matrix}.

**EXAMPLES:**

```
sage: c = Cone([\(0,0,1\), \(1,0,1\), \(0,1,1\), \(1,1,1\)]).rays()
sage: c.column_matrix()
```

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

dim()

Return the dimension of the space spanned by points of self.

**Note:** You can use either \text{dim()} or \text{dimension()}.

**OUTPUT:**
- an integer.

**EXAMPLES:**

```
sage: c = Cone([\(0,0,1\), \(1,1,1\)]).rays()
sage: c.dimension()
2
sage: c.dim()
2
```

dimension()

Return the dimension of the space spanned by points of self.
**Note:** You can use either `dim()` or `dimension()`.

**OUTPUT:**
- an integer.

**EXAMPLES:**
```
sage: c = Cone([(0,0,1), (1,1,1)]).rays()
sage: c.dimension()
2
sage: c.dim()
2
```

**dual_module()**
Return the dual of the ambient module of `self`.

**OUTPUT:**
- a module. If possible (that is, if the ambient `module()` `M` of `self` has a `dual()` method), the dual module is returned. Otherwise, $R^n$ is returned, where $n$ is the dimension of $M$ and $R$ is its base ring.

**EXAMPLES:**
```
sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
sage: c.dual_module()
3-d lattice M
```

**index(**args**)**
Return the index of the first occurrence of `point` in `self`.

**INPUT:**
- `point` – a point of `self`;
- `start` – (optional) an integer, if given, the search will start at this position;
- `stop` – (optional) an integer, if given, the search will stop at this position.

**OUTPUT:**
- an integer if `point` is in `self[start:stop]`, otherwise a `ValueError` exception is raised.

**EXAMPLES:**
```
sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
sage: c.index((0,1,1))
Traceback (most recent call last):
...  
ValueError: tuple.index(x): x not in tuple
```

Note that this was not a mistake: the `tuple` `(0, 1, 1)` is not a point of `c`! We need to pass actual element of the ambient module of `c` to get their indices:
```
sage: N = c.module()
sage: c.index(N(0, 1, 1))
2
sage: c[2]
N(0, 1, 1)
```
**matrix()**

Return a matrix whose rows are points of self.

**OUTPUT:**

• a matrix.

**EXAMPLES:**

```python
sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
sage: c.matrix()
[0 0 1]
[1 0 1]
[0 1 1]
[1 1 1]
```

**module()**

Return the ambient module of self.

**OUTPUT:**

• a module.

**EXAMPLES:**

```python
sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
sage: c.module()
3-d lattice N
```

**static output_format (format=None)**

Return or set the output format for ALL point collections.

**INPUT:**

• **format** – (optional) if given, must be one of the strings
  
  – “default” – output one point per line with vertical alignment of coordinates in text mode, same as “tuple” for LaTeX;
  
  – “tuple” – output `tuple(self)` with lattice information;
  
  – “matrix” – output `matrix()` with lattice information;
  
  – “column matrix” – output `column_matrix()` with lattice information;
  
  – “separated column matrix” – same as “column matrix” for text mode, for LaTeX separate columns by lines (not shown by jsMath).

**OUTPUT:**

• a string with the current format (only if format was omitted).

This function affects both regular and LaTeX output.

**EXAMPLES:**

```python
sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
sage: c
N(0, 0, 1),
N(1, 0, 1),
N(0, 1, 1),
N(1, 1, 1)
in 3-d lattice N
```

(continues on next page)
sage: c.output_format()
'default'
sage: c.output_format("tuple")
sage: c
(N(0, 0, 1), N(1, 0, 1), N(0, 1, 1), N(1, 1, 1))
in 3-d lattice N
sage: c.output_format("matrix")
sage: c
[0 0 1]
[1 0 1]
[0 1 1]
[1 1 1]
in 3-d lattice N
sage: c.output_format("column matrix")
sage: c
[0 1 0 1]
[0 0 1 1]
[1 1 1 1]
in 3-d lattice N
sage: c.output_format("separated column matrix")
sage: c
[0 1 0 1]
[0 0 1 1]
[1 1 1 1]
in 3-d lattice N

Note that the last two outputs are identical, separators are only inserted in the LaTeX mode:

sage: latex(c)
\left(\begin{array}{r|r|r|r}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right)_{N}

Since this is a static method, you can call it for the class directly:

sage: from sage.geometry.point_collection import PointCollection
sage: PointCollection.output_format("default")
sage: c
N(0, 0, 1),
N(1, 0, 1),
N(0, 1, 1),
N(1, 1, 1)
in 3-d lattice N

set()

Return points of self as a frozenset.

OUTPUT:

• a frozenset.

EXAMPLES:

sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)]).rays()
sage: c.set()
frozenset({N(0, 0, 1), N(0, 1, 1), N(1, 0, 1), N(1, 1, 1)})
write_for_palp(f)
Write self into an open file f in PALP format.

INPUT:

- f – a file opened for writing.

EXAMPLES:

```python
sage: o = lattice_polytope.cross_polytope(3)
sage: from io import StringIO
sage: f = StringIO()
sage: o.vertices().write_for_palp(f)
sage: print(f.getvalue())
6 3
1 0 0
0 1 0
0 0 1
-1 0 0
0 -1 0
0 0 -1
```

sage.geometry.point_collection.is_PointCollection(x)
Check if x is a point collection.

INPUT:

- x – anything.

OUTPUT:

- True if x is a point collection and False otherwise.

EXAMPLES:

```python
sage: from sage.geometry.point_collection import is_PointCollection
sage: is_PointCollection(1)
False
sage: c = Cone([(0,0,1), (1,0,1), (0,1,1), (1,1,1)])
sage: is_PointCollection(c.rays())
True
```

sage.geometry.point_collection.read_palp_point_collection(f, lattice=None, permutation=False)
Read and return a point collection from an opened file.

Data must be in PALP format:

- the first input line starts with two integers m and n, the number of points and the number of components of each;
- the rest of the first line may contain a permutation;
- the next m lines contain n numbers each.

**Note:** If m < n, it is assumed (for compatibility with PALP) that the matrix is transposed, i.e. that each column is a point.

INPUT:

- f – an opened file with PALP output.
• **lattice** – the lattice for points. If not given, the toric lattice $M$ of dimension $n$ will be used.

• **permutation** – (default: False) if True, try to retrieve the permutation. This parameter makes sense only when PALP computed the normal form of a lattice polytope.

**OUTPUT:**

• a point collection, optionally followed by a permutation. None if EOF is reached.

**EXAMPLES:**

```sage
data = "3 2 regular
         1 2
         3 4
         5 6
         2 3 transposed
         1 2 3
         4 5 6"
sage: print(data)
3 2 regular
1 2
3 4
5 6
2 3 transposed
1 2 3
4 5 6
sage: from io import StringIO
sage: f = StringIO(data)
sage: from sage.geometry.point_collection import read_palp_point_collection
sage: read_palp_point_collection(f)
M(1, 2),
M(3, 4),
M(5, 6)
in 2-d lattice M
sage: read_palp_point_collection(f)
M(1, 4),
M(2, 5),
M(3, 6)
in 2-d lattice M
sage: read_palp_point_collection(f) is None
True
```

### 2.5.7 Toric plotter

This module provides a helper class `ToricPlotter` for producing plots of objects related to toric geometry. Default plotting objects can be adjusted using `options()` and reset using `reset_options()`.

**AUTHORS:**

• Andrey Novoseltsev (2010-10-03): initial version, using some code bits by Volker Braun.

**EXAMPLES:**

In most cases, this module is used indirectly, e.g.

```sage
fan = toric_varieties.dP6().fan()  # ...
 ← needs palp sage.graphs
fan.plot()  # ...
← needs palp sage.graphs sage.plot
```

You may change default plotting options as follows:
```python
sage: toric_plotter.options("show_rays")
True
sage: toric_plotter.options(show_rays=False)
sage: toric_plotter.options("show_rays")
False
sage: fan.plot()
#<needs palp sage.graphs sage.plot
Graphics object consisting of 19 graphics primitives
sage: toric_plotter.reset_options()
sage: toric_plotter.options("show_rays")
True
sage: fan.plot()
#<needs palp sage.graphs sage.plot
Graphics object consisting of 31 graphics primitives
```

```python
class sage.geometry.toric_plotter.ToricPlotter(all_options, dimension, generators=None)
    Bases: SageObject

    Create a toric plotter.

    INPUT:
    • all_options — a dictionary, containing any of the options related to toric objects (see options()) and any other options that will be passed to lower level plotting functions;
    • dimension — an integer (1, 2, or 3), dimension of toric objects to be plotted;
    • generators — (optional) a list of ray generators, see examples for a detailed explanation of this argument.

    OUTPUT:
    • a toric plotter.

    EXAMPLES:
    In most cases there is no need to create and use ToricPlotter directly. Instead, use plotting method of the object which you want to plot, e.g.

    ```python
    sage: fan = toric_varieties.dP6().fan()  #<needs palp sage.graphs
    sage: fan.plot()  #<needs palp sage.graphs sage.plot
    Graphics object consisting of 31 graphics primitives
    sage: print(fan.plot())  #<needs palp sage.graphs sage.plot
    Graphics object consisting of 31 graphics primitives
    ```

    If you do want to create your own plotting function for some toric structure, the anticipated usage of toric plotters is the following:

    • collect all necessary options in a dictionary;
    • pass these options and dimension to ToricPlotter;
    • call include_points() on ray generators and any other points that you want to be present on the plot (it will try to set appropriate cut-off bounds);
    • call adjust_options() to choose “nice” default values for all options that were not set yet and ensure consistency of rectangular and spherical cut-off bounds;
    • call set_rays() on ray generators to scale them to the cut-off bounds of the plot;
    • call appropriate plot_* functions to actually construct the plot.
```
For example, the plot from the previous example can be obtained as follows:

```
sage: # needs palp sage.graphs sage.plot
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: options = dict()    # use default for everything
sage: tp = ToricPlotter(options, fan.lattice().degree())
```

```
sage: result = tp.plot_lattice()
sage: result += tp.plot_rays()
sage: result += tp.plot_generators()
sage: result += tp.plot_walls(fan(2))
sage: result
```

```
Graphics object consisting of 31 graphics primitives
```

In most situations it is only necessary to include generators of rays, in this case they can be passed to the constructor as an optional argument. In the example above, the toric plotter can be completely set up using

```
sage: tp = ToricPlotter(options, fan.lattice().degree(), fan.rays())
```

All options are exposed as attributes of toric plotters and can be modified after constructions, however you will have to manually call `adjust_options()` and `set_rays()` again if you decide to change the plotting mode and/or cut-off bounds. Otherwise plots maybe invalid.

**adjust_options()**

Adjust plotting options.

This function determines appropriate default values for those options, that were not specified by the user, based on the other options. See `ToricPlotter` for a detailed example.

**OUTPUT:**

• none.

**include_points**(points, force=False)

Try to include points into the bounding box of self.

**INPUT:**

• points – a list of points;

• force – boolean (default: False). by default, only bounds that were not set before will be chosen to include points. Use force=True if you don’t mind increasing existing bounding box.

**OUTPUT:**

• none.

**EXAMPLES:**

```
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2)
sage: print(tp.radius)
None
```

```
sage: tp.include_points([(3, 4)])
sage: print(tp.radius)
5.5...
```

(continues on next page)
plot_generators()  
Plot ray generators.
Ray generators must be specified during construction or using set_rays() before calling this method.

OUTPUT:

• a plot.

EXAMPLES:

```python
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2, [(3,4)])
sage: tp.plot_generators()  # needs sage.plot
```

plot_labels(labels, positions)
Plot labels at specified positions.

INPUT:

• labels – a string or a list of strings;
• positions – a list of points.

OUTPUT:

• a plot.

EXAMPLES:

```python
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2)
sage: tp.plot_labels("u", [(1.5,0)])  # needs sage.plot
```

plot_lattice()
Plot the lattice (i.e. its points in the cut-off bounds of self).

OUTPUT:

• a plot.

EXAMPLES:

```python
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2)
sage: tp.plot_lattice()  # needs sage.plot
```
plot_points(points)
Plot given points.

INPUT:
• points – a list of points.

OUTPUT:
• a plot.

EXAMPLES:
```python
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2)
sage: tp.adjust_options()
sage: tp.plot_points([(1,0), (0,1)])
Graphics object consisting of 1 graphics primitive
```

plot_ray_labels()
Plot ray labels.
Usually ray labels are plotted together with rays, but in some cases it is desirable to output them separately.
Ray generators must be specified during construction or using set_rays() before calling this method.

OUTPUT:
• a plot.

EXAMPLES:
```python
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2, [(3,4)])
sage: tp.plot_ray_labels()
Graphics object consisting of 1 graphics primitive
```

plot_rays()
Plot rays and their labels.
Ray generators must be specified during construction or using set_rays() before calling this method.

OUTPUT:
• a plot.

EXAMPLES:
```python
sage: from sage.geometry.toric_plotter import ToricPlotter
sage: tp = ToricPlotter(dict(), 2, [(3,4)])
sage: tp.plot_rays()
Graphics object consisting of 2 graphics primitives
```

plot_walls(walls)
Plot walls, i.e. 2-d cones, and their labels.
Ray generators must be specified during construction or using set_rays() before calling this method and these specified ray generators will be used in conjunction with ambient_ray_indices() of walls.

INPUT:
• walls – a list of 2-d cones.

OUTPUT:

• a plot.

EXAMPLES:

```
sage: quadrant = Cone([(1,0), (0,1)])
sage: from sage.geometry.toric_plotter import ToricPlotter
toric_plotter
sage: tp = ToricPlotter(doctests, 2, quadrant.rays())
#…

# needs sage.plot
Graphics object consisting of 2 graphics primitives
```

Let’s also check that the truncating polyhedron is functioning correctly:

```
sage: tp = ToricPlotter({"mode": "box"}, 2, quadrant.rays())
#…

# needs sage.plot
Graphics object consisting of 2 graphics primitives
```

**set_rays** *(generators)*

Set up rays and their generators to be used by plotting functions.

As an alternative to using this method, you can pass **generators** to `ToricPlotter` constructor.

INPUT:

• **generators** - a list of primitive non-zero ray generators.

OUTPUT:

• none.

EXAMPLES:

```
sage: from sage.geometry.toric_plotter import ToricPlotter
toric_plotter
sage: tp = ToricPlotter(doctests, 2)
#…

# needs sage.plot
Traceback (most recent call last):
...
AttributeError: 'ToricPlotter' object has no attribute 'rays'...

# needs sage.plot
Graphics object consisting of 2 graphics primitives
```

`sage.geometry.toric_plotter.color_list(color, n)`

Normalize a list of $n$ colors.

INPUT:

• **color** – anything specifying a `Color`, a list of such specifications, or the string “rainbow”;
• **n** - an integer.

OUTPUT:

• a list of $n$ colors.
If color specified a single color, it is repeated \( n \) times. If it was a list of \( n \) colors, it is returned without changes. If it was “rainbow”, the rainbow of \( n \) colors is returned.

EXAMPLES:

```
sage: from sage.geometry.toric_plotter import color_list
color_list("grey", 1)
[RGB color (0.5019607843137255, 0.5019607843137255, 0.5019607843137255)]
sage: len(color_list("grey", 3))
3
sage: L = color_list("rainbow", 3)
L
[RGB color (1.0, 0.0, 0.0),
 RGB color (0.0, 1.0, 0.0),
 RGB color (0.0, 0.0, 1.0)]
```

```
sage: color_list(L, 3)
[RGB color (1.0, 0.0, 0.0),
 RGB color (0.0, 1.0, 0.0),
 RGB color (0.0, 0.0, 1.0)]
sage: color_list(L, 4)
Traceback (most recent call last):
... ValueError: expected 4 colors, got 3!
```

```
sage.geometry.toric_plotter.label_list (label, n, math_mode, index_set=None)
```

Normalize a list of \( n \) labels.

INPUT:

- label – None, a string, or a list of string;
- \( n \) – an integer;
- math_mode – boolean, if True, will produce LaTeX expressions for labels;
- index_set – a list of integers (default: range(\( n \))) that will be used as subscripts for labels.

OUTPUT:

- a list of \( n \) labels.

If label was a list of \( n \) entries, it is returned without changes. If label is None, a list of \( n \) None's is returned. If label is a string, a list of strings of the form \$\text{label}_{i}\$ is returned, where \( i \) ranges over index_set. (If math_mode=\text{False}, the form “label_i” is used instead.) If \( n=1 \), there is no subscript added, unless index_set was specified explicitly.

EXAMPLES:

```
sage: from sage.geometry.toric_plotter import label_list
sage: label_list("u", 3, False)
['u_0', 'u_1', 'u_2']
sage: label_list("u", 3, True)
['u_{0}', 'u_{1}', 'u_{2}']
sage: label_list("u", 1, True)
['u_{0}']
```

```
sage.geometry.toric_plotter.options (option=None, **kwds)
```

Get or set options for plots of toric geometry objects.
Note: This function provides access to global default options. Any of these options can be overridden by passing them directly to plotting functions. See also reset_options().

INPUT:
- None;
- option – a string, name of the option whose value you wish to get;
- keyword arguments specifying new values for one or more options.

OUTPUT:
- if there was no input, the dictionary of current options for toric plots;
- if option argument was given, the current value of option;
- if other keyword arguments were given, none.

Name Conventions
To clearly distinguish parts of toric plots, in options and methods we use the following name conventions:

Generator
A primitive integral vector generating a 1-dimensional cone, plotted as an arrow from the origin (or a line, if the head of the arrow is beyond cut-off bounds for the plot).

Ray
A 1-dimensional cone, plotted as a line from the origin to the cut-off bounds for the plot.

Wall
A 2-dimensional cone, plotted as a region between rays (in the above sense). Its exact shape depends on the plotting mode (see below).

Chamber
A 3-dimensional cone, plotting is not implemented yet.

Plotting Modes
A plotting mode mostly determines the shape of the cut-off region (which is always relevant for toric plots except for trivial objects consisting of the origin only). The following options are available:

Box
The cut-off region is a box with edges parallel to coordinate axes.

Generators
The cut-off region is determined by primitive integral generators of rays. Note that this notion is well-defined only for rays and walls, in particular you should plot the lattice on your own (plot_lattice()) will use box mode which is likely to be unsuitable). While this method may not be suitable for general fans, it is quite natural for fans of CPR-Fano toric varieties. <sage.schemes.toric.fano_variety.CPRFanoToricVariety_field

Round
The cut-off regions is a sphere centered at the origin.

Available Options
Default values for the following options can be set using this function:
- mode – “box”, “generators”, or “round”, see above for descriptions;
• show_lattice – boolean, whether to show lattice points in the cut-off region or not;
• show_rays – boolean, whether to show rays or not;
• show_generators – boolean, whether to show rays or not;
• show_walls – boolean, whether to show rays or not;
• generator_color – a color for generators;
• label_color – a color for labels;
• point_color – a color for lattice points;
• ray_color – a color for rays, a list of colors (one for each ray), or the string “rainbow”;
• wall_color – a color for walls, a list of colors (one for each wall), or the string “rainbow”;
• wall_alpha – a number between 0 and 1, the alpha-value for walls (determining their transparency);
• point_size – an integer, the size of lattice points;
• ray_thickness – an integer, the thickness of rays;
• generator_thickness – an integer, the thickness of generators;
• font_size – an integer, the size of font used for labels;
• ray_label – a string or a list of strings used for ray labels; use None to hide labels;
• wall_label – a string or a list of strings used for wall labels; use None to hide labels;
• radius – a positive number, the radius of the cut-off region for “round” mode;
• xmin, xmax, ymin, ymax, zmin, zmax – numbers determining the cut-off region for “box” mode. Note that you cannot exclude the origin - if you try to do so, bounds will be automatically expanded to include it;
• lattice_filter – a callable, taking as an argument a lattice point and returning True if this point should be included on the plot (useful, e.g. for plotting sublattices);
• wall_zorder, ray_zorder, generator_zorder, point_zorder, label_zorder – integers, z-orders for different classes of objects. By default all values are negative, so that you can add other graphic objects on top of a toric plot. You may need to adjust these parameters if you want to put a toric plot on top of something else or if you want to overlap several toric plots.

You can see the current default value of any options by typing, e.g.

```sage
toric_plotter.options("show_rays")
```

True

If the default value is None, it means that the actual default is determined later based on the known options. Note, that not all options can be determined in such a way, so you should not set options to None unless it was its original state. (You can always revert to this “original state” using `reset_options()`.)

EXAMPLES:
The following line will make all subsequent toric plotting commands to draw “rainbows” from walls:

```sage
toric_plotter.options(wall_color="rainbow")
```

If you prefer a less colorful output (e.g. if you need black-and-white illustrations for a paper), you can use something like this:

```sage
toric_plotter.options(wall_color="grey")
```
Reset options for plots of toric geometry objects.

OUTPUT:

• none.

EXAMPLES:

```
sage: toric_plotter.options("show_rays")
True
sage: toric_plotter.options(show_rays=False)
False
sage: toric_plotter.options("show_rays")
False
```

Now all toric plots will not show rays, unless explicitly requested. If you want to go back to “default defaults”, use this method:

```
sage: toric_plotter.reset_options()
sage: toric_plotter.options("show_rays")
True
```

```
plot = toric_plotter.sector(ray1, ray2, **extra_options)
Plot a sector between ray1 and ray2 centered at the origin.

Note: This function was intended for plotting strictly convex cones, so it plots the smaller sector between ray1 and ray2 and, therefore, they cannot be opposite. If you do want to use this function for bigger regions, split them into several parts.

Note: As of version 4.6 Sage does not have a graphic primitive for sectors in 3-dimensional space, so this function will actually approximate them using polygons (the number of vertices used depends on the angle between rays).

INPUT:

• ray1, ray2 – rays in 2- or 3-dimensional space of the same length;
• extra_options – a dictionary of options that should be passed to lower level plotting functions.

OUTPUT:

• a plot.

EXAMPLES:

```
sage: from sage.geometry.toric_plotter import sector
sage: sector((1,0), (0,1))  # needs sage.symbolic
Graphics object consisting of 1 graphics primitive
sage: sector((3,2,1), (1,2,3))  # needs sage.plot
Graphics3d Object
```
2.5.8 Groebner Fans

Sage provides much of the functionality of \textit{gfan}, which is a software package whose main function is to enumerate all reduced Groebner bases of a polynomial ideal. The reduced Groebner bases yield the maximal cones in the Groebner fan of the ideal. Several subcomputations can be issued and additional tools are included. Among these the highlights are:

- Commands for computing tropical varieties.
- Interactive walks in the Groebner fan of an ideal.
- Commands for graphical renderings of Groebner fans and monomial ideals.

AUTHORS:

- Anders Nedergaard Jensen: Wrote the \textit{gfan} C++ program, which implements algorithms many of which were invented by Jensen, Komei Fukuda, and Rekha Thomas. All the underlying hard work of the Groebner fans functionality of Sage depends on this C++ program.
- Tristram Bogart: the design of the Sage interface to \textit{gfan} is joint work with Tristram Bogart, who also supplied numerous examples.
- Marshall Hampton (2008-03-25): Rewrote various functions to use \textit{gfan-0.3}. This is still a work in progress, comments are appreciated on sage-devel@googlegroups.com (or personally at hamptonio@gmail.com).

EXAMPLES:

```sage
data
x, y = QQ['x, y'].gens()
sage: i = ideal(x^2 - y^2 + 1)
sage: g = i.groebner_fan()
sage: g.reduced_groebner_bases()
[[x^2 - y^2 + 1], [-x^2 + y^2 - 1]]
```

REFERENCES:


\texttt{class sage.rings.polynomial.groebner_fan.GroebnerFan(I, is_groebner_basis=False, symmetry=None, verbose=False)}

Bases: \texttt{SageObject}

This class is used to access capabilities of the program \textit{Gfan}.

In addition to computing Groebner fans, \textit{Gfan} can compute other things in tropical geometry such as tropical prevarieties.

INPUT:

- \texttt{I} – ideal in a multivariate polynomial ring
- \texttt{is_groebner_basis} – bool (default False). If True, then I.gens() must be a Groebner basis with respect to the standard degree lexicographic term order.
- \texttt{symmetry} – default: None; if not None, describes symmetries of the ideal
- \texttt{verbose} – default: False; if True, printout useful info during computations

EXAMPLES:
Here is an example of the use of the tropical_intersection command, and then using the RationalPolyhedralFan class to compute the Stanley-Reisner ideal of the tropical prevariety:

```sage
sage: R.<x,y,z> = QQ[]
sage: I = R.ideal([(x+y+z)^3-1,(x+y+z)^3-x,(x+y+z)-3])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: PF.rays()
[[-1, 0, 0], [0, -1, 0], [0, 0, -1], [1, 1, 1]]
sage: RPF = PF.to_RationalPolyhedralFan()
sage: RPF.Stanley_Reisner_ideal(PolynomialRing(QQ,4,A, B, C, D))
```

### buchberger()

Return a lexicographic reduced Groebner basis for the ideal.

**EXAMPLES:**

```sage
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: G = R.ideal([-z^3 + y^2, -z^3 + x]).groebner_fan()
sage: G.buchberger()
[-z^3 + y^2, -z^3 + x]
```

### characteristic()

Return the characteristic of the base ring.

**EXAMPLES:**

```sage
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: i1 = ideal(x*z + 6*y*z - z^2, x*y + 6*x*z + y*z - z^2, y^2 + x*z + y*z)
sage: gf = i1.groebner_fan()
sage: gf.characteristic()
0
```

### dimension_of_homogeneity_space()

Return the dimension of the homogeneity space.

**EXAMPLES:**

```sage
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()
sage: G.dimension_of_homogeneity_space()
0
```

### gfan(cmd='bases', I=None, format=None)

Return the gfan output as a string given an input cmd.

The default is to produce the list of reduced Groebner bases in gfan format.

**INPUT:**

```sage
```

---

**Chapter 2. Polyhedral computations**
Combinatorial and Discrete Geometry, Release 10.3

- cmd – string (default: 'bases'), GFan command
- I – ideal (default: None)
- format – bool (default: None), deprecated

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: gf = R.ideal([x^3-y,y^3-x-1]).groebner_fan()
sage: gf.gfan()
```

```
Q[x,y]
{{
y^9-1-y+3*y^3-3*y^6,
x+1-y^3}
,
{
x^3-y,
y^3-1-x}
,
{
x^9-1-x,\ny-n^3}\n
```

**homogeneity_space()**

Return the homogeneity space of a list of polynomials that define this Groebner fan.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()
sage: H = G.homogeneity_space()
```

**ideal()**

Return the ideal that was used to define this Groebner fan.

EXAMPLES:

```
sage: R.<x1,x2> = PolynomialRing(QQ,2)
sage: gf = R.ideal([x1^3-x2,x2^3-2*x1-2]).groebner_fan()
sage: gf.ideal()
Ideal (x1^3 - x2, x2^3 - 2*x1 - 2) of Multivariate Polynomial Ring in x1, x2 over Rational Field
```

**interactive(*args, **kwds)**

See the documentation for self[0].interactive(). This does not work with the notebook.

EXAMPLES:

```
sage: print("This is not easily doc-testable; please write a good one!")
This is not easily doc-testable; please write a good one!
```

**maximal_total_degree_of_a_groebner_basis()**

Return the maximal total degree of any Groebner basis.

EXAMPLES:

```
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()
sage: G.maximal_total_degree_of_a_groebner_basis()
4
```

**minimal_total_degree_of_a_groebner_basis()**

Return the minimal total degree of any Groebner basis.

EXAMPLES:
mixed_volume()

Return the mixed volume of the generators of this ideal.

This is not really an ideal property, it can depend on the generators used.

The generators must give a square system (as many polynomials as variables).

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]

sage: example_ideal = R.ideal([x^2-y-1, y^2-z-1, z^2-x-1])

sage: gf = example_ideal.groebner_fan()

sage: mv = gf.mixed_volume()

sage: mv
8
```

number_of_reduced_groebner_bases()

Return the number of reduced Groebner bases.

EXAMPLES:

```python
sage: R.<x,y> = PolynomialRing(QQ,2)

sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()

sage: G.number_of_reduced_groebner_bases()
3
```

number_of_variables()

Return the number of variables.

EXAMPLES:

```python
sage: R.<x,y> = PolynomialRing(QQ,2)

sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()

sage: G.number_of_variables()
2
```

polyhedralfan()

Return a polyhedral fan object corresponding to the reduced Groebner bases.

EXAMPLES:
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-1]).groebner_fan()
sage: pf = gf.polyhedralfan()
sage: pf.rays()
[[0, 0, 1], [0, 1, 0], [1, 0, 0]]

reduced_groebner_bases()

EXAMPLES:

sage: R.<x,y,z> = PolynomialRing(QQ, 3, order='lex')
sage: G = R.ideal([x^2*y - z, y^2*z - x, z^2*x - y]).groebner_fan()
sage: X = G.reduced_groebner_bases()
sage: len(X)
33
sage: X[0]
[z^15 - z, x - z^9, y - z^11]

sage: X[0].ideal()
Ideal (z^15 - z, x - z^9, y - z^11) of Multivariate Polynomial Ring in x, y, z over Rational Field

sage: X[:5]
[[z^15 - z, x - z^9, y - z^11],
 [y^2 - z^8, x - z^9, y*z^4 - z, -y + z^11],
 [y^3 - z^5, x - y^2*z, y^2*z^3 - y, y*z^4 - z, -y^2 + z^8],
 [y^4 - z^2, x - y^2*z, y^2*z^3 - y, y*z^4 - z, -y^3 + z^5],
 [y^9 - z, y^6*z - y, x - y^2*z, -y^4 + z^2]]

sage: R3.<x,y,z> = PolynomialRing(GF(2477),3)
sage: gf = R3.ideal([300*x^3-y,y^2-z,z^2-12]).groebner_fan()
sage: gf.reduced_groebner_bases()
[[z^2 - 12, y^2 - z, x^3 + 933*y],
 [y^4 - 12, x^3 + 933*y, -y^2 + z],
 [x^6 - 1062*z, z^2 - 12, -300*x^3 + y],
 [x^12 + 200, -300*x^3 + y, -828*x^6 + z]]

render (file=None, larger=False, shift=0, rgbcolor=(0, 0, 0), polyfill=True, scale_colors=True)

Render a Groebner fan as sage graphics or save as an xfig file.

More precisely, the output is a drawing of the Groebner fan intersected with a triangle. The corners of the triangle are (1,0,0) to the right, (0,1,0) to the left and (0,0,1) at the top. If there are more than three variables in the ring we extend these coordinates with zeros.

INPUT:

- **file** – a filename if you prefer the output saved to a file. This will be in xfig format.
- **shift** – shift the positions of the variables in the drawing. For example, with shift=1, the corners will be b (right), c (left), and d (top). The shifting is done modulo the number of variables in the polynomial ring. The default is 0.
- **larger** – bool (default: False); if True, make the triangle larger so that the shape of the Groebner region appears. Affects the xfig file but probably not the sage graphics.
- **rgbcolor** – This will not affect the saved xfig file, only the sage graphics produced.
- **polyfill** – Whether or not to fill the cones with a color determined by the highest degree in each reduced Groebner basis for that cone.
- **scale_colors** – If True, this will normalize color values to try to maximize the range

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```python
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: G = R.ideal([y^3 - x^2, y^2 - 13*x,z]).groebner_fan()
sage: test_render = G.render()  # needs sage.plot

sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: G = R.ideal([x^2*y - z, y^2*z - x, z^2*x - y]).groebner_fan()
sage: test_render = G.render(larger=True)  # needs sage.plot
```

**render3d** *(verbose=False)*

For a Groebner fan of an ideal in a ring with four variables, this function intersects the fan with the standard simplex perpendicular to (1,1,1,1), creating a 3d polytope, which is then projected into 3 dimensions. The edges of this projected polytope are returned as lines.

**EXAMPLES:**

```python
sage: R4.<w,x,y,z> = PolynomialRing(QQ,4)
sage: gf = R4.ideal([w^2-x,x^2-y,y^2-z,z^2-x]).groebner_fan()
sage: three_d = gf.render3d()  # needs sage.plot
```

**ring()**

Return the multivariate polynomial ring.

**EXAMPLES:**

```python
sage: R.<x1,x2> = PolynomialRing(QQ,2)
sage: gf = R.ideal([x1^3-x2,x2^3-x1-2]).groebner_fan()
sage: gf.ring()
Multivariate Polynomial Ring in x1, x2 over Rational Field
```

**tropical_basis**(check=True, verbose=False)

Return a tropical basis for the tropical curve associated to this ideal.

**INPUT:**

- **check** - bool (default: True); if True raises a ValueError exception if this ideal does not define a tropical curve (i.e., the condition that R/I has dimension equal to 1 + the dimension of the homogeneity space is not satisfied).

**EXAMPLES:**

```python
sage: R.<x,y,z> = PolynomialRing(QQ,3, order='lex')
sage: G = R.ideal([y^3-3*x^2, z^3-x-y-2*y^3+2*x^2]).groebner_fan()
sage: G
groebner fan of the ideal:
Ideal (-3*x^2 + y^3, 2*x^2 - x - 2*y^3 - y + z^3) of Multivariate Polynomial Ring in x, y, z over Rational Field
```

**tropical_intersection**(parameters=[], symmetry_generators=[], *args, **kwds)

Return information about the tropical intersection of the polynomials defining the ideal.

This is the common refinement of the outward-pointing normal fans of the Newton polytopes of the generators of the ideal. Note that some people use the inward-pointing normal fans.

**INPUT:**
parameters (optional) - a list of variables to be considered as parameters

• symmetry_generators (optional) - generators of the symmetry group

OUTPUT: a TropicalPrevariety object

EXAMPLES:

```sage
R.<x,y,z> = PolynomialRing(QQ,3)
sage: I = R.ideal(x*z + 6*y*z - z^2, x*y + 6*x*z + y*z - z^2, y^2 + x*z + y*z)
sage: gf = I.groebner_fan()
sage: pf = gf.tropical_intersection()
sage: pf.rays()
[[-2, 1, 1]]
```

```sage
R.<x,y,z> = PolynomialRing(QQ,3)
sage: f1 = x*y*z - 1
sage: f2 = f1*(x^2 + y^2 + z^2)
```

```sage
R.<x,y,z> = QQ[]
sage: I = R.ideal([(x+y+z)^2-1,(x+y+z)-x,(x+y+z)-3])
sage: GF = I.groebner_fan()
sage: TI = GF.tropical_intersection()
sage: TI.rays()
[[-1, 0, 0], [0, -1, -1], [1, 1, 1]]
```

weight_vectors()

Return the weight vectors corresponding to the reduced Groebner bases.

EXAMPLES:

```sage
r3.<x,y,z> = PolynomialRing(QQ,3)
sage: g = r3.ideal([x^3+y,y^3-z,z^2-x]).groebner_fan()
sage: g.weight_vectors()

[(3, 7, 1), (5, 1, 2), (7, 1, 4), (5, 1, 4), (1, 1, 1), (1, 4, 8), (1, 4, 10)]
```

```sage
r4.<x,y,z,w> = PolynomialRing(QQ,4)
sage: g4 = r4.ideal([x^3+y^3-z, z^2-x, z^2 - w]).groebner_fan()
sage: len(g4.weight_vectors())
23
```

class sage.rings.polynomial.groebner_fan.InitialForm(cone, rays, initial_forms)

Bases: SageObject

A system of initial forms from a polynomial system.

To each form is associated a cone and a list of polynomials (the initial form system itself).

This class is intended for internal use inside of the TropicalPrevariety class.

EXAMPLES:
sage: from sage.rings.polynomial.groebner_fan import InitialForm
sage: R.<x,y> = QQ[]
sage: inform = InitialForm([0], [[-1, 0]], [y^2 - 1, y^2 - 2, y^2 - 3])
sage: inform._cone
[0]

cone()

The cone associated with the initial form system.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: I = R.ideal([(x+y)^2-1,(x+y)^2-2,(x+y)^2-3])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: pfi0 = PF.initial_form_systems()[0]
sage: pfi0.cone()

initial_forms()

The initial forms (polynomials).

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: I = R.ideal([(x+y)^2-1,(x+y)^2-2,(x+y)^2-3])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: pfi0 = PF.initial_form_systems()[0]
sage: pfi0.initial_forms()

internal_ray()

A ray internal to the cone associated with the initial form system.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: I = R.ideal([(x+y)^2-1,(x+y)^2-2,(x+y)^2-3])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: pfi0 = PF.initial_form_systems()[0]
sage: pfi0.internal_ray()

rays()

The rays of the cone associated with the initial form system.

EXAMPLES:

sage: R.<x,y> = QQ[]
sage: I = R.ideal([(x+y)^2-1,(x+y)^2-2,(x+y)^2-3])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: pfi0 = PF.initial_form_systems()[0]
sage: pfi0.rays()

Chapter 2. Polyhedral computations
class sage.rings.polynomial.groebner_fan.PolyhedralCone(gfan_polyhedral_cone, ring=Rational Field)

Bases: SageObject

Convert polymake/gfan data on a polyhedral cone into a sage class.

Currently (18-03-2008) needs a lot of work.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf[0].groebner_cone()
sage: a.facets()
[[0, 0, 1], [0, 1, 0], [1, 0, 0]]
```

ambient_dim()

Return the ambient dimension of the Groebner cone.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf[0].groebner_cone()
sage: a.ambient_dim()
3
```

dim()

Return the dimension of the Groebner cone.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf[0].groebner_cone()
sage: a.dim()
3
```

facets()

Return the inward facet normals of the Groebner cone.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf[0].groebner_cone()
sage: a.facets()
[[0, 0, 1], [0, 1, 0], [1, 0, 0]]
```

lineality_dim()

Return the lineality dimension of the Groebner cone. This is just the difference between the ambient dimension and the dimension of the cone.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf[0].groebner_cone()
```

(continues on next page)
relative_interior_point()

Return a point in the relative interior of the Groebner cone.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf[0].groebner_cone()
sage: a.relative_interior_point()
[1, 1, 1]
```

class sage.rings.polynomial.groebner_fan.PolyhedralFan(gfan_polyhedral_fan, parameter_indices=None)

Bases: SageObject

Convert polymake/gfan data on a polyhedral fan into a sage class.

INPUT:

- `gfan_polyhedral_fan` - output from gfan of a polyhedral fan.

EXAMPLES:

```python
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: i2 = ideal(x*z + 6*y*z - z^2, x*y + 6*x*z + y*z - z^2, y^2 + x*z + y*z)
sage: gf2 = i2.groebner_fan(verbose=False)
sage: pf = gf2.polyhedralfan()
sage: pf.rays()
[[-1, 0, 1], [-1, 1, 0], [1, -2, 1], [1, 1, -2], [2, -1, -1]]
```

ambient_dim()

Return the ambient dimension of the Groebner fan.

EXAMPLES:

```python
sage: R3.<x,y,z> = PolynomialRing(QQ,3)
sage: gf = R3.ideal([x^8-y^4,y^4-z^2,z^2-2]).groebner_fan()
sage: a = gf.polyhedralfan()
sage: a.ambient_dim()
3
```

cones()

A dictionary of cones in which the keys are the cone dimensions. For each dimension, the value is a list of the cones, where each element consists of a list of ray indices.

EXAMPLES:

```python
sage: R.<x,y,z> = QQ[]
sage: f = 1+x+y+x*y
sage: I = R.ideal([f+z*f, 2*f+z*f, 3*f+z^2*f])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: PF.cones()
{1: [[0], [1], [2], [3], [4], [5]], 2: [[0, 1], [0, 2], [0, 3], [0, 4], [1, -2], [1, 3], [2, 4], [3, 4], [1, 5], [2, 5], [3, 5], [4, 5]]}
```
dim()

Return the dimension of the Groebner fan.

EXAMPLES:

```sage
R3.<x,y,z> = PolynomialRing(QQ,3)
gf = R3.ideal([x^8-y^4,z^2-2]).groebner_fan()
a = gf.polyhedralfan()
a.dim()
3
```

f_vector()

The f-vector of the fan.

EXAMPLES:

```sage
R.<x,y,z> = QQ[]
f = 1+x+y+x*y
I = R.ideal([f+z*f, 2*f+z*f, 3*f+z^2*f])
GF = I.groebner_fan()
PF = GF.tropical_intersection()
PF.f_vector()
[1, 6, 12]
```

is_simplicial()

Whether the fan is simplicial or not.

EXAMPLES:

```sage
R.<x,y,z> = QQ[]
f = 1+x+y+x*y
I = R.ideal([f+z*f, 2*f+z*f, 3*f+z^2*f])
GF = I.groebner_fan()
PF = GF.tropical_intersection()
PF.is_simplicial()
True
```

lineality_dim()

Return the lineality dimension of the fan. This is the dimension of the largest subspace contained in the fan.

EXAMPLES:

```sage
R3.<x,y,z> = PolynomialRing(QQ,3)
gf = R3.ideal([x^8-y^4,z^2-2]).groebner_fan()
a = gf.polyhedralfan()
a.lineality_dim()
0
```

maximal_cones()

A dictionary of the maximal cones in which the keys are the cone dimensions. For each dimension, the value is a list of the maximal cones, where each element consists of a list of ray indices.

EXAMPLES:

```sage
R.<x,y,z> = QQ[]
f = 1+x+y+x*y
I = R.ideal([f+z*f, 2*f+z*f, 3*f+z^2*f])
GF = I.groebner_fan()
```
sage: PF = GF.tropical_intersection()
sage: PF.maximal_cones()
{(2: [[0, 1], [0, 2], [0, 3], [0, 4], [1, 2], [1, 3], [2, 4], [3, 4], [1, 5],
     \rightarrow [2, 5], [3, 5], [4, 5]])}

rays()
A list of rays of the polyhedral fan.

EXAMPLES:

sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: i2 = ideal(x*z + 6*y*z - z^2, x*y + 6*x*z + y*z - z^2, y^2 + x*z + y*z)
sage: gf2 = i2.groebner_fan(verbos=False)
sage: pf = gf2.polyhedralfan()
sage: pf.rays()
[[-1, 0, 1], [-1, 1, 0], [1, -2, 1], [1, 1, -2], [2, -1, -1]]

to_RationalPolyhedralFan()
Converts to the RationalPolyhedralFan class, which is more actively maintained. While the information in each class is essentially the same, the methods and implementation are different.

EXAMPLES:

sage: R.<x,y,z> = QQ[]
sage: f = 1+x+y+x*y
sage: I = R.ideal([f+z*f, 2*f+z*f, 3*f+z^2*f])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: fan = PF.to_RationalPolyhedralFan()
sage: [tuple(q.facet_normals()) for q in fan]
[(M(0, -1, 0), M(-1, 0, 0)), (M(0, 0, -1), M(-1, 0, 0)), (M(0, 0, 1), M(-1, 0, 0)),
  (M(0, -1, 0), M(0, 1, 0), M(0, 0, -1)), (M(1, 0, 0), M(0, 0, 1)), (M(1, 0,
  \rightarrow 0), M(0, -1, 0)), (M(1, 0, 0), M(0, 0, -1)), (M(1, 0, 1), M(0, 0,
  \rightarrow 0), M(1, 0, 0), M(0, 0, -1))]

Here we use the RationalPolyhedralFan's Gale_transform method on a tropical prevariety.

sage: fan.Gale_transform()
[ 1 0 0 0 0 1 -2]
[ 0 1 0 0 1 0 -2]
[ 0 0 1 1 0 0 -2]

class sage.rings.polynomial.groebner_fan.ReducedGroebnerBasis(groebner_fan, gens, gfan_gens)

Bases: SageObject, list
A class for representing reduced Groebner bases as produced by gfan.

INPUT:

• groebner_fan - a GroebnerFan object from an ideal
• gens - the generators of the ideal
• gfan_gens - the generators as a gfan string

EXAMPLES:
sage: R.<a,b> = PolynomialRing(QQ,2)
sage: gf = R.ideal([a^2-b^2,b-a-1]).groebner_fan()
sage: from sage.rings.polynomial.groebner_fan import ReducedGroebnerBasis
sage: ReducedGroebnerBasis(gf[0],gf[0]._gfan_gens())
[b - 1/2, a + 1/2]

**groebner_cone** (*restrict=False*)

Return defining inequalities for the full-dimensional Groebner cone associated to this marked minimal reduced Groebner basis.

**INPUT:**

- *restrict* - bool (default: False); if True, add an inequality for each coordinate, so that the cone is restricted to the positive orthant.

**OUTPUT:** tuple of integer vectors

**EXAMPLES:**

```sage
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()
sage: poly_cone = G[1].groebner_cone()
sage: poly_cone.facets()
[[[[-1, 2], [1, -1]]]
sage: [g.groebner_cone().facets() for g in G]
[[[[0, 1], [1, -2]], [[-1, 2], [1, -1]], [[-1, 1], [1, 0]]]]
sage: G[1].groebner_cone(restrict=True).facets()
[[[-1, 2], [1, -1]]]
```

**ideal()**

Return the ideal generated by this basis.

**EXAMPLES:**

```sage
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: G = R.ideal([x - z^3, y^2 - 13*x]).groebner_fan()
sage: G[0].ideal()
Ideal (-13*z^3 + y^2, -z^3 + x) of Multivariate Polynomial Ring in x, y, z
˓
→ over Rational Field
```

**interactive** (*latex=False, flippable=False, wall=False, inequalities=False, weight=False*)

Do an interactive walk of the Groebner fan starting at this reduced Groebner basis.

**EXAMPLES:**

```sage
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: G = R.ideal([y^3 - x^2, y^2 - 13*x]).groebner_fan()
sage: G[0].interactive() # not tested
Initializing gfan interactive mode
**************************************************************************
* Press control-C to return to Sage *
**************************************************************************
....
```

**class** `sage.rings.polynomial.groebner_fan.TropicalPrevariety` *(gfan_polyhedral_fan, polynomial_system, poly_ring, parameters=None)*

2.5. Toric geometry 497
Bases: PolyhedralFan

This class is a subclass of the PolyhedralFan class, with some additional methods for tropical prevarieties.

INPUT:

- \texttt{gfan\_polyhedral\_fan} – output from \texttt{gfan} of a polyhedral fan.
- \texttt{polynomial\_system} – a list of polynomials
- \texttt{poly\_ring} – the polynomial ring of the list of polynomials
- \texttt{parameters} (optional) – a list of variables to be considered as parameters

EXAMPLES:

```sage
sage: R.<x,y,z> = QQ[]
sage: I = R.ideal([(x+y+z)^2-1,(x+y+z)-x,(x+y+z)-3])
sage: GF = I.groebner_fan()
sage: TI = GF.tropical_intersection()
sage: TI._polynomial_system
[x^2 + 2*x*y + y^2 + 2*x*z + 2*y*z + z^2 - 1, y + z, x + y + z - 3]
```

\texttt{initial\_form\_systems}()

Return a list of systems of initial forms for each cone in the tropical prevariety.

EXAMPLES:

```sage
sage: R.<x,y> = QQ[]
sage: I = R.ideal([(x+y)^2-1,(x+y)^2-2,(x+y)^2-3])
sage: GF = I.groebner_fan()
sage: PF = GF.tropical_intersection()
sage: pfi = PF.initial_form_systems()
sage: for q in pfi:
    ...:
    print(q.initial_forms())
[y^2 - 1, y^2 - 2, y^2 - 3]
[x^2 - 1, x^2 - 2, x^2 - 3]
[x^2 + 2*x*y + y^2, x^2 + 2*x*y + y^2, x^2 + 2*x*y + y^2]
```

\texttt{sage.rings.polynomial.groebner\_fan.ideal\_to\_gfan\_format(input\_ring, polys)}

Return the ideal in \texttt{gfan}'s notation.

EXAMPLES:

```sage
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: polys = [x^2*y - z, y^2*z - x, z^2*x - y]
sage: from sage.rings.polynomial.groebner\_fan import ideal\_to\_gfan\_format
sage: ideal\_to\_gfan\_format(R, polys)
'Q[x, y, z]{x^2*y-z,y^2*z-x,z^2*x-y}'
```

\texttt{sage.rings.polynomial.groebner\_fan.max\_degree(list\_of\_polys)}

Compute the maximum degree of a list of polynomials.

EXAMPLES:

```sage
sage: from sage.rings.polynomial.groebner\_fan import max\_degree
sage: R.<x,y> = PolynomialRing(QQ,2)
sage: p_list = [x^2-y,x*y^10-x]
```

(continues on next page)
sage: max_degree(p_list)
11.0

sage.rings.polynomial.groebner_fan.prefix_check(str_list)
Check if any strings in a list are prefixes of another string in the list.

EXAMPLES:

```
sage: from sage.rings.polynomial.groebner_fan import prefix_check
sage: prefix_check(['z1', 'z1z1'])
False
sage: prefix_check(['z1', 'zz1'])
True
```

sage.rings.polynomial.groebner_fan.ring_to_gfan_format(input_ring)
Converts a ring to gfan's format.

EXAMPLES:

```
sage: R.<w,x,y,z> = QQ[]
sage: from sage.rings.polynomial.groebner_fan import ring_to_gfan_format
sage: ring_to_gfan_format(R)
'Q[w, x, y, z]'
sage: R2.<x,y> = GF(2)[]
sage: ring_to_gfan_format(R2)
'Z/2Z[x, y]'```

sage.rings.polynomial.groebner_fan.verts_for_normal(normal, poly)
Return the exponents of the vertices of a Newton polytope that make up the supporting hyperplane for the given outward normal.

EXAMPLES:

```
sage: from sage.rings.polynomial.groebner_fan import verts_for_normal
sage: R.<x,y,z> = PolynomialRing(QQ,3)
sage: f1 = x*y*z - 1
sage: f2 = f1*(x^2 + y^2 + 1)
sage: verts_for_normal([1,1,1],f2)
[(3, 1, 1), (1, 3, 1)]
```

## 2.6 Base classes for polyhedra

### 2.6.1 Base class for polyhedra: Initialization and access to Vrepresentation and Hrepresentation

```
class sage.geometry.polyhedron.base0.Polyhedron_base0 (parent, Vrep, Hrep, Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False, **kwds)
```

Bases: `Element, Polyhedron`

Initialization and basic access for polyhedra.
See `sage.geometry.polyhedron.base.Polyhedron_base`.

**Hrep_generator()**

Return an iterator over the objects of the H-representation (inequalities or equations).

**EXAMPLES:**
```
sage: p = polytopes.hypercube(3)
sage: next(p.Hrep_generator())
An inequality (-1, 0, 0) x + 1 >= 0
```

**Hrepresentation(index=None)**

Return the objects of the H-representation. Each entry is either an inequality or a equation.

**INPUT:**
- `index` – either an integer or None

**OUTPUT:**
The optional argument is an index running from 0 to `self.n_Hrepresentation() - 1`. If present, the H-representation object at the given index will be returned. Without an argument, returns the list of all H-representation objects.

**EXAMPLES:**
```
sage: p = polytopes.hypercube(3, backend='field')
sage: p.Hrepresentation(0)
An inequality (-1, 0, 0) x + 1 >= 0
sage: p.Hrepresentation(0) == p.Hrepresentation()[0]
True
```

**Hrepresentation_str(separator='
', latex=False, style='>=', align=None, **kwds)**

Return a human-readable string representation of the Hrepresentation of this polyhedron.

**INPUT:**
- `separator` – a string. Default is "\n".
- `latex` – a boolean. Default is False.
- `style` – either "positive" (making all coefficients positive) or "\<=" or ">=". Default is ">=".
- `align` – a boolean or None. Default is None in which case align is True if separator is the newline character. If set, then the lines of the output string are aligned by the comparison symbol by padding blanks.

Keyword parameters of `repr_pretty()` are passed on:
- `prefix` – a string
- `indices` – a tuple or other iterable

**OUTPUT:**
A string.

**EXAMPLES:**
```
sage: P = polytopes.permutahedron(3)
sage: print(P.Hrepresentation_str())
x0 + x1 + x2 == 6
```
(continues on next page)
\begin{array}{rcl}
  x_{0} + x_{1} + x_{2} & = & 6 \\
  x_{0} + x_{1} & \geq & 3 \\
  5 & \geq & x_{0} + x_{1} \\
  x_{1} & \geq & 1 \\
  3 & \geq & x_{0} \\
  3 & \geq & x_{1} \\
\end{array}

\textbf{Vrep\_generator()} \\
Return an iterator over the objects of the V-representation (vertices, rays, and lines).

\textbf{EXAMPLES:}

\begin{verbatim}
sage: p = polytopes.cyclic_polytope(3,4) sage: vg = p.Vrep_generator() sage: next(vg) (continues on next page)
\end{verbatim}
A vertex at (0, 0, 0)

\[ \text{sage: next(vg)} \]

A vertex at (1, 1, 1)

**Vrepresentation** (*index=None*)

Return the objects of the V-representation. Each entry is either a vertex, a ray, or a line.

See `sage.geometry.polyhedron.constructor` for a definition of vertex/ray/line.

**INPUT:**

- *index* – either an integer or None

**OUTPUT:**

The optional argument is an index running from 0 to `self.n_Vrepresentation()-1`. If present, the V-representation object at the given index will be returned. Without an argument, returns the list of all V-representation objects.

**EXAMPLES:**

\[ \text{sage: p = polytopes.simplex(4, project=\text{True})} \]
\[ \text{sage: p.Vrepresentation(0)} \]
A vertex at (0.7071067812, 0.4082482905, 0.2886751346, 0.2236067977)
\[ \text{sage: p.Vrepresentation(0) == p.Vrepresentation()[0]} \] True

**backend**()

Return the backend used.

**OUTPUT:**

The name of the backend used for computations. It will be one of the following backends:

- *ppl* the Parma Polyhedra Library
- *cdd* CDD
- *normaliz* normaliz
- *polymake* polymake
- *field* a generic Sage implementation

**EXAMPLES:**

\[ \text{sage: triangle = Polyhedron(vertices = \{[1, 0], [0, 1], [1, 1]\})} \]
\[ \text{sage: triangle.backend()} \] 'ppl'
\[ \text{sage: D = polytopes.dodecahedron()} \]
\[ \text{# needs sage.rings.number_field}
\]
\[ \text{sage: D.backend()} \]
\[ \text{# needs sage.rings.number_field}
\]
\[ \text{'field'} \]
\[ \text{sage: P = Polyhedron([[1.23]])} \]
\[ \text{sage: P.backend()} \] 'cdd'

**base_extend**(base_ring, backend=None)

Return a new polyhedron over a larger base ring.

This method can also be used to change the backend.
INPUT:

- **base_ring** – the new base ring
- **backend** – the new backend, see `Polyhedron()`. If None (the default), attempt to keep the same backend. Otherwise, use the same defaulting behavior as described there.

OUTPUT:

The same polyhedron, but over a larger base ring and possibly with a changed backend.

**EXAMPLES:**

```sage
P = Polyhedron(vertices=[(1,0), (0,1)], rays=[(1,1)], base_ring=ZZ); P
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices... and 1 ray
```

```sage
P.base_extend(QQ)
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 2 vertices... and 1 ray
```

```sage
P.base_extend(QQ) == P
True
```

**base_ring()**

Return the base ring.

OUTPUT:

The ring over which the polyhedron is defined. Must be a sub-ring of the reals to define a polyhedron, in particular comparison must be defined. Popular choices are

- **ZZ** (the ring of integers, lattice polytope),
- **QQ** (exact arithmetic using gmp),
- **RDF** (double precision floating-point arithmetic), or
- **AA** (real algebraic field).

**EXAMPLES:**

```sage
triangle = Polyhedron(vertices = [[1,0],[0,1],[1,1]])
triangle.base_ring() == ZZ
True
```

**cdd_Hrepresentation()**

Write the inequalities/equations data of the polyhedron in cdd's H-representation format.

See also:

*write_cdd_Hrepresentation() – export the polyhedron as a H-representation to a file.*

**OUTPUT:** a string

**EXAMPLES:**

```sage
p = polytopes.hypercube(2)
sage: print(p.cdd_Hrepresentation())
H-representation
begin
  4 3 rational
  1 -1 0
  1 0 -1
  1 1 0
(continues on next page)
```
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```python
1 0 1
end

sage: triangle = Polyhedron(vertices=[[1,0], [0,1], [1,1]], base_ring=AA)  # needs sage.rings.number_field
sage: triangle.base_ring()  # needs sage.rings.number_field
Algebraic Real Field
sage: triangle.cdd_Hrepresentation()  # needs sage.rings.number_field
Traceback (most recent call last):
...
TypeError: the base ring must be ZZ, QQ, or RDF
```

cdd_Vrepresentation()
Write the vertices/rays/lines data of the polyhedron in cdd's V-representation format.

See also:
write_cdd_Vrepresentation() – export the polyhedron as a V-representation to a file.

OUTPUT: a string

EXAMPLES:

```python
sage: q = Polyhedron(vertices = [[1,1], [0,0], [1,0], [0,1]])
sage: print(q.cdd_Vrepresentation())
V-representation
begin
4 3 rational
1 0 0
1 0 1
1 1 0
1 1 1
end
```

change_ring(base_ring, backend=None)
Return the polyhedron obtained by coercing the entries of the vertices/lines/rays of this polyhedron into the given ring.

This method can also be used to change the backend.

INPUT:

- base_ring – the new base ring
- backend – the new backend or None (default), see Polyhedron(). If None (the default), attempt to keep the same backend. Otherwise, use the same defaulting behavior as described there.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[(1,0), (0,1)], rays=[(1,1)], base_ring=QQ); P
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 2 vertices...
and 1 ray
sage: P.change_ring(ZZ)
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices...
and 1 ray
sage: P.change_ring(ZZ) == P
```

(continues on next page)
\begin{verbatim}
True

sage: P = Polyhedron(vertices=[(-1.3,0), (0,2.3)], base_ring=RDF); P.
  \→ vertices()
(A vertex at (-1.3, 0.0), A vertex at (0.0, 2.3))

sage: P.change_ring(QQ).vertices()
(A vertex at (-13/10, 0), A vertex at (0, 23/10))

sage: P == P.change_ring(QQ)
True

sage: P.change_ring(ZZ)
Traceback (most recent call last):
  ...
TypeError: cannot change the base ring to the Integer Ring

sage: P = polytopes.regular_polygon(3); P
# needs sage.rings.number_field
A 2-dimensional polyhedron in AA^2 defined as the convex hull of 3 vertices

sage: P.vertices()
# needs sage.rings.number_field
(A vertex at (0.?e-16, 1.000000000000000?),
A vertex at (0.866025403784439?, -0.500000000000000?),
A vertex at (-0.866025403784439?, -0.500000000000000?))

sage: P.change_ring(QQ)
# needs sage.rings.number_field
Traceback (most recent call last):
  ...
TypeError: cannot change the base ring to the Rational Field

Warning: The base ring RDF should be used with care. As it is not an exact ring, certain computations may break or silently produce wrong results, for example changing the base ring from an exact ring into RDF may cause a loss of data:

sage: P = Polyhedron([[2/3,0], [666666666666666666667/10^16,0]], base_ring=AA);
  \→ P
# needs sage.rings.number_field
A 1-dimensional polyhedron in AA^2 defined as the convex hull of 2 vertices

sage: Q = P.change_ring(RDF); Q
# needs sage.rings.number_field
A 0-dimensional polyhedron in RDF^2 defined as the convex hull of 1 vertex

sage: P.n_vertices() == Q.n_vertices()
False
\end{verbatim}

**equation_generator()**

Return a generator for the linear equations satisfied by the polyhedron.

**EXAMPLES:**

sage: p = polytopes.regular_polygon(8,base_ring=RDF)
sage: p3 = Polyhedron(vertices = [x+[0] for x in p.vertices()], base_ring=RDF)
sage: next(p3.equation_generator())
An equation (0.0, 0.0, 1.0) x + 0.0 == 0

**equations()**

Return all linear constraints of the polyhedron.
OUTPUT:
A tuple of equations.

EXAMPLES:
```
sage: test_p = Polyhedron(vertices = [[1,2,3,4],[2,1,3,4],[4,3,2,1],[3,4,1,2]])
sage: test_p.equations()
(An equation (1, 1, 1, 1) x - 10 == 0,)
```

`equations_list()`
Return the linear constraints of the polyhedron. As with inequalities, each constraint is given as $[b-a1 -a2 \ldots an]$ where for variables $x1, x2, \ldots, xn$, the polyhedron satisfies the equation $b = a1*x1 + a2*x2 + \ldots + an*xn$.

Note: It is recommended to use `equations()` or `equation_generator()` instead to iterate over the list of `Equation` objects.

EXAMPLES:
```
sage: test_p = Polyhedron(vertices = [[1,2,3,4],[2,1,3,4],[4,3,2,1],[3,4,1,2]])
sage: test_p.equations_list()
[[-10, 1, 1, 1]]
```

`inequalities()`
Return all inequalities.

OUTPUT:
A tuple of inequalities.

EXAMPLES:
```
sage: p = Polyhedron(vertices = [[0,0,0],[0,0,1],[0,1,0],[1,0,0],[2,2,2]])
sage: p.inequalities()[0:3]
(An inequality (1, 0, 0) x + 0 >= 0,
 An inequality (0, 1, 0) x + 0 >= 0,
 An inequality (0, 0, 1) x + 0 >= 0)
sage: # needs sage.combinat
sage: p3 = Polyhedron(vertices=Permutations([1, 2, 3, 4]))
sage: ieqs = p3.inequalities()
sage: ieqs[0]
An inequality (0, 1, 1, 1) x - 6 >= 0
sage: list(_)
[-6, 0, 1, 1, 1]
```

`inequalities_list()`
Return a list of inequalities as coefficient lists.

Note: It is recommended to use `inequalities()` or `inequality_generator()` instead to iterate over the list of `Inequality` objects.

EXAMPLES:
sage: p = Polyhedron(vertices = [[0,0,0],[0,0,1],[0,1,0],[1,0,0],[2,2,2]])
sage: p.inequalities_list()[0:3]
[[0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]]

sage: # needs sage.combinat
sage: p3 = Polyhedron(vertices=Permutations([1, 2, 3, 4]))
sage: ieqs = p3.inequalities_list()
sage: ieqs[0]
[-6, 0, 1, 1, 1]
sage: ieqs[-1]
[-3, 0, 1, 0, 1]
sage: ieqs == [list(x) for x in p3.inequality_generator()]
True

inequality_generator()

Return a generator for the defining inequalities of the polyhedron.

OUTPUT:

A generator of the inequality Hrepresentation objects.

EXAMPLES:

sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[1,1]])
sage: for v in triangle.inequality_generator(): print(v)
An inequality (1, 1) x - 1 >= 0
An inequality (0, -1) x + 1 >= 0
An inequality (-1, 0) x + 1 >= 0

sage: [ v for v in triangle.inequality_generator() ]
[[(1, 1), -1], [(0, -1), 1], [(-1, 0), 1]]

is_compact()

Test for boundedness of the polytope.

EXAMPLES:

sage: p = polytopes.icosahedron()  # needs sage.rings.number_field
sage: p.is_compact()  # needs sage.rings.number_field
True
sage: p = Polyhedron(ieqs = [[0,1,0,0],[0,0,1,0],[0,0,0,1],[1,-1,0,0]])

sage: p.is_compact()
False

is Immutable()

Return True if the polyhedron is immutable, i.e. it cannot be modified in place.

EXAMPLES:

sage: p = polytopes.cube(backend='field')
sage: p.is_immutable()
True

2.6. Base classes for polyhedra
is_mutable()
Return True if the polyhedron is mutable, i.e. it can be modified in place.

EXAMPLES:

```python
sage: p = polytopes.cube(backend='field')
sage: p.is_mutable()
False
```

line_generator()
Return a generator for the lines of the polyhedron.

EXAMPLES:

```python
sage: pr = Polyhedron(rays=[[1,0],[-1,0],[0,1]], vertices=[[-1,-1]])
sage: next(pr.line_generator()).vector()
(1, 0)
```

lines()
Return all lines of the polyhedron.

OUTPUT:
A tuple of lines.

EXAMPLES:

```python
sage: p = Polyhedron(rays=[[1,0],[-1,0],[0,1],[1,1]], vertices=[[-2,-2],
    →[2,3]])
sage: p.lines()
(A line in the direction (1, 0),)
```

lines_list()
Return a list of lines of the polyhedron. The line data is given as a list of coordinates rather than as a Hrepresentation object.

**Note:** It is recommended to use line_generator() instead to iterate over the list of Line objects.

EXAMPLES:

```python
sage: p = Polyhedron(rays=[[1,0],[-1,0],[0,1],[1,1]], vertices=[[-2,-2],
    →[2,3]])
sage: p.lines_list()
[[[1, 0]]
sage: p.lines_list() == [list(x) for x in p.line_generator()]
True
```

n_Hrepresentation()
Return the number of objects that make up the H-representation of the polyhedron.

OUTPUT:
Integer.

EXAMPLES:
sage: p = polytopes.cross_polytope(4)
sage: p.n_Hrepresentation()
16
sage: p.n_Hrepresentation() == p.n_inequalities() + p.n_equations()
True

**n_Vrepresentation()**

Return the number of objects that make up the V-representation of the polyhedron.

**OUTPUT:**

Integer.

**EXAMPLES:**

sage: p = polytopes.simplex(4)
sage: p.n_Vrepresentation()
5
sage: p.n_Vrepresentation() == p.n_vertices() + p.n_rays() + p.n_lines()
True

**n_equations()**

Return the number of equations. The representation will always be minimal, so the number of equations is the codimension of the polyhedron in the ambient space.

**EXAMPLES:**

sage: p = Polyhedron(vertices = [[1,0,0],[0,1,0],[0,0,1]])
sage: p.n_equations()
1

**n_facets()**

Return the number of inequalities. The representation will always be minimal, so the number of inequalities is the number of facets of the polyhedron in the ambient space.

**EXAMPLES:**

sage: p = Polyhedron(vertices = [[1,0,0],[0,1,0],[0,0,1]])
sage: p.n_inequalities()
3
sage: p = Polyhedron(vertices = [[t,t^2,t^3] for t in range(6)])
sage: p.n_facets()
8

**n_inequalities()**

Return the number of inequalities. The representation will always be minimal, so the number of inequalities is the number of facets of the polyhedron in the ambient space.

**EXAMPLES:**

sage: p = Polyhedron(vertices = [[1,0,0],[0,1,0],[0,0,1]])
sage: p.n_inequalities()
3
sage: p = Polyhedron(vertices = [[t,t^2,t^3] for t in range(6)])
sage: p.n_facets()
8

---

2.6. Base classes for polyhedra

---

509
n_lines()

Return the number of lines. The representation will always be minimal.

EXAMPLES:

```
sage: p = Polyhedron(vertices = [[0,0]], rays=[[0,1],[0,-1]])
sage: p.n_lines()
1
```

n_rays()

Return the number of rays. The representation will always be minimal.

EXAMPLES:

```
sage: p = Polyhedron(vertices = [[1,0],[0,1]], rays=[[1,1]])
sage: p.n_rays()
1
```

n_vertices()

Return the number of vertices. The representation will always be minimal.

Warning: If the polyhedron has lines, return the number of vertices in the \texttt{V}representation. As the represented polyhedron has no 0-dimensional faces (i.e. vertices), \texttt{n\_vertices} corresponds to the number of \(k\)-faces, where \(k\) is the number of lines:

```
sage: P = Polyhedron(rays=[[1,0,0]],lines=[[0,1,0]])
sage: P.n_vertices()
1
sage: P.faces(0)
()
sage: P.f_vector()
(1, 0, 1)
sage: P = Polyhedron(rays=[[1,0,0]],lines=[[0,1,0],[0,1,1]])
sage: P.n_vertices()
1
sage: P.f_vector()
(1, 0, 0, 1, 1)
```

EXAMPLES:

```
sage: p = Polyhedron(vertices = [[1,0],[0,1],[1,1]], rays=[[1,1]])
sage: p.n_vertices()
2
```

ray_generator()

Return a generator for the rays of the polyhedron.

EXAMPLES:

```
sage: pi = Polyhedron(ieqs = [[1,1,0],[1,0,1]])
sage: pir = pi.ray_generator()
sage: [x.vector() for x in pir]
[(1, 0), (0, 1)]
```
**rays()**

Return a list of rays of the polyhedron.

**OUTPUT:**

A tuple of rays.

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs = [[0,0,0,1],[0,0,1,0],[1,1,0,0]])
sage: p.rays()
(A ray in the direction (1, 0, 0),
 A ray in the direction (0, 1, 0),
 A ray in the direction (0, 0, 1))
```

**rays_list()**

Return a list of rays as coefficient lists.

**Note:** It is recommended to use `rays()` or `ray_generator()` instead to iterate over the list of `Ray` objects.

**OUTPUT:**

A list of rays as lists of coordinates.

**EXAMPLES:**

```
sage: p = Polyhedron(ieqs = [[0,0,0,1],[0,0,1,0],[1,1,0,0]])
sage: p.rays_list()
[[1, 0, 0], [0, 1, 0], [0, 0, 1]]
sage: p.rays_list() == [list(r) for r in p.ray_generator()]
True
```

**vertex_generator()**

Return a generator for the vertices of the polyhedron.

**Warning:** If the polyhedron has lines, return a generator for the vertices of the Vrepresentation. However, the represented polyhedron has no 0-dimensional faces (i.e. vertices):

```
sage: P = Polyhedron(rays=[[1,0,0]],lines=[[0,1,0]])
sage: list(P.vertex_generator())
[0, 0, 0]
sage: P.faces(0)
() 
```

**EXAMPLES:**

```
sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[1,1]])
sage: for v in triangle.vertex_generator(): print(v) 
A vertex at (0, 1)
A vertex at (1, 0)
A vertex at (1, 1)
sage: v_gen = triangle.vertex_generator()
sage: next(v_gen) # the first vertex
A vertex at (0, 1)
```

(continues on next page)
sage: next(v_gen)  # the second vertex
A vertex at (1, 0)
sage: next(v_gen)  # the third vertex
A vertex at (1, 1)
sage: try:
    next(v_gen)  # there are only three vertices
except StopIteration:
    print("STOP")
STOP
sage: type(v_gen)
<... generator>

sage: [v for v in triangle.vertex_generator()]
[A vertex at (0, 1), A vertex at (1, 0), A vertex at (1, 1)]

vertices()

Return all vertices of the polyhedron.

OUTPUT:

A tuple of vertices.

Warning: If the polyhedron has lines, return the vertices of the Vrepresentation. However, the represented polyhedron has no 0-dimensional faces (i.e. vertices):

sage: P = Polyhedron(rays=[[1,0,0]], lines=[[0,1,0]])
sage: P.vertices()
(A vertex at (0, 0, 0),)
sage: P.faces(0)
()  

EXAMPLES:

sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[1,1]])
sage: triangle.vertices()
(A vertex at (0, 1), A vertex at (1, 0), A vertex at (1, 1))
sage: a_simplex = Polyhedron(ieqs = [
    [0,1,0,0,0], [0,0,1,0,0], [0,0,0,1,0], [0,0,0,0,1]
    ], eqns = [[1,-1,-1,-1,-1]])
sage: a_simplex.vertices()
(A vertex at (1, 0, 0, 0), A vertex at (0, 1, 0, 0),
A vertex at (0, 0, 1, 0), A vertex at (0, 0, 0, 1))

vertices_list()

Return a list of vertices of the polyhedron.

Note: It is recommended to use vertex_generator() instead to iterate over the list of Vertex objects.

Warning: If the polyhedron has lines, return the vertices of the Vrepresentation. However, the represented polyhedron has no 0-dimensional faces (i.e. vertices):

sage: P = Polyhedron(rays=[[1,0,0]], lines=[[0,1,0]])
sage: P.vertices_list()
[[0, 0, 0]]
vertices_matrix (base_ring=\text{None})

Return the coordinates of the vertices as the columns of a matrix.

INPUT:

- \text{base_ring} – A ring or None (default). The base ring of the returned matrix. If not specified, the base ring of the polyhedron is used.

OUTPUT:

A matrix over \text{base_ring} whose columns are the coordinates of the vertices. A \text{TypeError} is raised if the coordinates cannot be converted to \text{base_ring}.

\textbf{Warning:} If the polyhedron has lines, return the coordinates of the vertices of the Vrepresentation. However, the represented polyhedron has no 0-dimensional faces (i.e. vertices):

```
sage: P = Polyhedron(rays=[[1,0,0]],lines=[[0,1,0]])
sage: P.vertices_matrix()
[0]
[0]
[0]
sage: P.faces(0)
()
```

EXAMPLES:

```
sage: triangle = Polyhedron(vertices=[[1,0],[0,1],[1,1]])
sage: triangle.vertices_matrix()
[0 1 1]
[1 0 1]
sage: (triangle/2).vertices_matrix()
[ 0 1/2 1/2]
[1/2 0 1/2]
sage: (triangle/2).vertices_matrix(ZZ)
Traceback (most recent call last):
...TypeError: no conversion of this rational to integer
```
write_cdd_Hrepresentation(filename)
Export the polyhedron as a H-representation to a file.

INPUT:

- filename – the output file.

See also:
cdd_Hrepresentation() – return the H-representation of the polyhedron as a string.

EXAMPLES:

```python
sage: from sage.misc.temporary_file import tmp_filename
sage: filename = tmp_filename(ext='.ext')
sage: polytopes.cube().write_cdd_Hrepresentation(filename)
```

write_cdd_Vrepresentation(filename)
Export the polyhedron as a V-representation to a file.

INPUT:

- filename – the output file.

See also:
cdd_Vrepresentation() – return the V-representation of the polyhedron as a string.

EXAMPLES:

```python
sage: from sagemisc.temporary_file import tmp_filename
sage: filename = tmp_filename(ext='.ext')
```

2.6.2 Base class for polyhedra: Implementation of the ConvexSet_base API

Define methods that exist for convex sets, but not constructions such as dilation or product.

class sage.geometry.polyhedron.base1.Polyhedron_base1(parent, Vrep, Hrep,
Vrep_directory=None, Hrep_directory=None,
pref_rep=None, base_ring=None, **kwds)

Bases: Polyhedron_base0, ConvexSet_closed

Convex set methods for polyhedra, but not constructions such as dilation or product.

See sage.geometry.polyhedron.base.Polyhedron_base.

Hrepresentation_space()

Return the linear space containing the H-representation vectors.

OUTPUT:

A free module over the base ring of dimension ambient_dim() + 1.

EXAMPLES:

```python
sage: poly_test = Polyhedron(vertices = [[1,0,0,0],[0,1,0,0]])
sage: poly_test.Hrepresentation_space()
Ambient free module of rank 5 over the principal ideal domain Integer Ring
```
**Vrepresentation_space()**

Return the ambient free module.

**OUTPUT:**

A free module over the base ring of dimension `ambient_dim()`.

**EXAMPLES:**

```python
sage: poly_test = Polyhedron(vertices = [[1,0,0,0],[0,1,0,0]])
sage: poly_test.Vrepresentation_space()
Ambient free module of rank 4 over the principal ideal domain Integer Ring
sage: poly_test.ambient_space() is poly_test.Vrepresentation_space()
True
```

**a_maximal_chain()**

Return a maximal chain of the face lattice in increasing order.

Subclasses must provide an implementation of this method.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.base1 import Polyhedron_base1
sage: P = polytopes.cube()
sage: Polyhedron_base1.a_maximal_chain
<abstract method a_maximal_chain at ...>
```

**ambient(base_field=None)**

Return the ambient vector space.

It is the ambient free module (`Vrepresentation_space()`) tensored with a field.

**INPUT:**

- `base_field` – (default: the fraction field of the base ring) a field.

**EXAMPLES:**

```python
sage: poly_test = Polyhedron(vertices = [[1,0,0,0],[0,1,0,0]])
sage: poly_test.ambient_vector_space()
Vector space of dimension 4 over Rational Field
sage: poly_test.ambient_vector_space() is poly_test.ambient()
True
sage: poly_test.ambient_vector_space(AA)  # needs sage.rings.number_field
Vector space of dimension 4 over Algebraic Real Field
sage: poly_test.ambient_vector_space(RDF)  # needs sage.symbolic
Vector space of dimension 4 over Real Double Field
sage: poly_test.ambient_vector_space(SR)  # needs sage.symbolic
Vector space of dimension 4 over Symbolic Ring
```

**ambient_dim()**

Return the dimension of the ambient space.

**EXAMPLES:**

```python
sage: poly_test = Polyhedron(vertices = [[1,0,0,0],[0,1,0,0]])
sage: poly_test.ambient_dim()
4
```
ambient_space()

Return the ambient free module.

OUTPUT:

A free module over the base ring of dimension ambient_dim().

EXAMPLES:

```
sage: poly_test = Polyhedron(vertices = [[1,0,0,0],[0,1,0,0]])
sage: poly_test.Vrepresentation_space()
Ambient free module of rank 4 over the principal ideal domain Integer Ring
sage: poly_test.ambient_space() is poly_test.Vrepresentation_space()
True
```

ambient_vector_space(base_field=None)

Return the ambient vector space.

It is the ambient free module (Vrepresentation_space()) tensored with a field.

INPUT:

- base_field – (default: the fraction field of the base ring) a field.

EXAMPLES:

```
sage: poly_test = Polyhedron(vertices = [[1,0,0,0],[0,1,0,0]])
sage: poly_test.ambient_vector_space()
Vector space of dimension 4 over Rational Field
```

```
sage: poly_test.ambient_vector_space() is poly_test.ambient()
True
```

```
sage: poly_test.ambient_vector_space(AA) # needs sage.rings.number_field
Vector space of dimension 4 over Algebraic Real Field
```

```
sage: poly_test.ambient_vector_space(RDF)
Vector space of dimension 4 over Real Double Field
```

```
sage: poly_test.ambient_vector_space(SR) # needs sage.symbolic
Vector space of dimension 4 over Symbolic Ring
```

an_affine_basis()

Return points in self that form a basis for the affine span of self.

This implementation of the method an_affine_basis() for polytopes guarantees the following:

- All points are vertices.
- The basis is obtained by considering a maximal chain of faces in the face lattice and picking for each cover relation one vertex that is in the difference. Thus this method is independent of the concrete realization of the polytope.

For unbounded polyhedra, the result may contain arbitrary points of self, not just vertices.

EXAMPLES:

```
sage: P = polytopes.cube()
sage: P.an_affine_basis()
[ A vertex at (-1, -1, -1),
  A vertex at (1, -1, -1),
  A vertex at (1, -1, 1),
  (continues on next page) ]
```
Unbounded polyhedra:

```
sage: p = Polyhedron(vertices=[(0,0)], rays=[(1,0), (0,1)])
sage: p.an_affine_basis()
[1, 0, 0]
sage: p = Polyhedron(vertices=[(2,1)], rays=[(1,0), (0,1)])
sage: p.an_affine_basis()
[2, 1, 0]
sage: p = Polyhedron(vertices=[(2,1)], rays=[(1,0)], lines=[(0,1)])
[2, 1, 1, 0]
```

contains(point)

Test whether the polyhedron contains the given point.

See also:

interior_contains(), relative_interior_contains().

INPUT:

• point – coordinates of a point (an iterable)

OUTPUT:

Boolean.

EXAMPLES:

```
sage: P = Polyhedron(vertices=[1,1],[1,-1],[0,0])
sage: P.contains([1,0])
True
sage: P.contains(P.center()) # true for any convex set
True
```

As a shorthand, one may use the usual in operator:

```
sage: P.center() in P
True
sage: [-1,-1] in P
False
```

The point need not have coordinates in the same field as the polyhedron:

```
sage: # needs sage.symbolic
sage: ray = Polyhedron(vertices=[(0,0)], rays=[(1,0)], base_ring=QQ)
sage: ray.contains([sqrt(2)/3,0]) # irrational coordinates are ok
True
sage: a = var('a')
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

sage: ray.contains([a,0]) # a might be negative!
False
sage: assume(a>0)
sage: ray.contains([a,0])
True
sage: ray.contains(['hello', 'kitty']) # no common ring for coordinates
False

The empty polyhedron needs extra care, see github issue #10238:

sage: empty = Polyhedron(); empty
The empty polyhedron in ZZ^0
sage: empty.contains([])
False
sage: empty.contains([0]) # not a point in QQ^0
False
sage: full = Polyhedron(vertices=[()]); full
A 0-dimensional polyhedron in ZZ^0 defined as the convex hull of 1 vertex
sage: full.contains([])
True
sage: full.contains([0])
False

dim()

Return the dimension of the polyhedron.

OUTPUT:
-1 if the polyhedron is empty, otherwise a non-negative integer.

EXAMPLES:

sage: simplex = Polyhedron(vertices = [[1,0,0,0],[0,0,0,1],[0,1,0,0],[0,0,1,0]])
sage: simplex.dim()
3
sage: simplex.ambient_dim()
4

The empty set is a special case (github issue #12193):

sage: P1=Polyhedron(vertices=[[1,0,0],[0,1,0],[0,0,1]])
sage: P2=Polyhedron(vertices=[[2,0,0],[0,2,0],[0,0,2]])
sage: P12 = P1.intersection(P2)
sage: P12
The empty polyhedron in ZZ^3
sage: P12.dim()
-1

dimension()

Return the dimension of the polyhedron.

OUTPUT:
-1 if the polyhedron is empty, otherwise a non-negative integer.

EXAMPLES:
The empty set is a special case (github issue #12193):

```python
sage: P1=Polyhedron(vertices=[[1,0,0],[0,1,0],[0,0,1]])
sage: P2=Polyhedron(vertices=[[2,0,0],[0,2,0],[0,0,2]])
sage: P12 = P1.intersection(P2)
sage: P12
The empty polyhedron in ZZ^3
sage: P12.dim()
-1
```

The interior of `self`.

**OUTPUT:**

- either an empty polyhedron or an instance of `RelativeInterior`

**EXAMPLES:**

If the polyhedron is full-dimensional, the result is the same as that of `relative_interior()`:

```python
sage: P_full = Polyhedron(vertices=[[0,0],[1,1],[1,-1]])
sage: P_full.interior()
Relative interior of a 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
```

If the polyhedron is of strictly smaller dimension than the ambient space, its interior is empty:

```python
sage: P_lower = Polyhedron(vertices=[[0,1], [0,-1]])
sage: P_lower.interior()
The empty polyhedron in ZZ^2
```

**interior_contains**(point)

Test whether the interior of the polyhedron contains the given point.

**See also:**

`contains()`, `relative_interior_contains()`.

**INPUT:**

- point – coordinates of a point

**OUTPUT:**

True or False.

**EXAMPLES:**

```python
sage: P = Polyhedron(vertices=[[0,0],[1,1],[1,-1]])
sage: P.contains( [1,0] )
True
sage: P.interior_contains( [1,0] )
False
```
If the polyhedron is of strictly smaller dimension than the ambient space, its interior is empty:

```python
sage: P = Polyhedron(vertices=[[0,1],[0,-1]])
sage: P.contains([0,0])
True
sage: P.interior_contains([0,0])
False
```

The empty polyhedron needs extra care, see github issue #10238:

```python
sage: empty = Polyhedron(); empty
The empty polyhedron in ZZ^0
sage: empty.interior_contains([])
False
```

**is_empty()**

Test whether the polyhedron is the empty polyhedron

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[[1,0,0],[0,1,0],[0,0,1]]); P
A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: P.is_empty(), P.is_universe()
(False, False)

sage: Q = Polyhedron(vertices=()); Q
The empty polyhedron in ZZ^0
sage: Q.is_empty(), Q.is_universe()
(True, False)

sage: R = Polyhedron(lines=[(1,0),(0,1)]); R
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and
2 lines
sage: R.is_empty(), R.is_universe()
(False, True)
```

**is_relatively_open()**

Return whether self is relatively open.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[(1,0), (-1,0)]); P
A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: P.is_relatively_open()
False

sage: P0 = Polyhedron(vertices=[[1, 2]]); P0
A 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex
sage: P0.is_relatively_open()
True
```

(continues on next page)
sage: Empty = Polyhedron(ambient_dim=2); Empty
The empty polyhedron in ZZ^2
sage: Empty.is_relatively_open()
True

sage: Line = Polyhedron(vertices=[(1, 1)], lines=[(1, 0)]); Line
A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 1 line
sage: Line.is_relatively_open()
True

is_universe()

Test whether the polyhedron is the whole ambient space

OUTPUT:

Boolean.

EXAMPLES:

sage: P = Polyhedron(vertices=[[1,0,0],[0,1,0],[0,0,1]]); P
A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: P.is_empty(), P.is_universe()
(False, False)

sage: Q = Polyhedron(vertices=()); Q
The empty polyhedron in ZZ^0
sage: Q.is_empty(), Q.is_universe()
(True, False)

sage: R = Polyhedron(lines=[(1,0),(0,1)]); R
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and 2 lines
sage: R.is_empty(), R.is_universe()
(False, True)

relative_interior()

Return the relative interior of self.

EXAMPLES:

sage: P = Polyhedron(vertices=[(1,0), (-1,0)])
sage: ri_P = P.relative_interior(); ri_P
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: (0, 0) in ri_P
True
sage: (1, 0) in ri_P
False

sage: P0 = Polyhedron(vertices=[[1, 2]])
sage: P0.relative_interior() is P0
True

sage: Empty = Polyhedron(ambient_dim=2)
sage: Empty.relative_interior() is Empty
True
sage: Line = Polyhedron(vertices=[(1, 1)], lines=[(1, 0)])
sage: Line.relative_interior() is Line
True

relative_interior_contains(point)

Test whether the relative interior of the polyhedron contains the given point.

See also:
contains(), interior_contains().

INPUT:
• point – coordinates of a point

OUTPUT:
True or False

EXAMPLES:

sage: P = Polyhedron(vertices=[(1,0), (-1,0)])
sage: P.contains( (0,0) )
True
sage: P.interior_contains( (0,0) )
False
sage: P.relative_interior_contains( (0,0) )
True
sage: P.relative_interior_contains( (1,0) )
False

The empty polyhedron needs extra care, see github issue #10238:

sage: empty = Polyhedron(); empty
The empty polyhedron in ZZ^0
sage: empty.relative_interior_contains([])
False

representative_point()

Return a “generic” point.

See also:
sage.geometry.polyhedron.base.Polyhedron_base.center().

OUTPUT:
A point as a coordinate vector. The point is chosen to be interior if possible. If the polyhedron is not full-dimensional, the point is in the relative interior. If the polyhedron is zero-dimensional, its single point is returned.

EXAMPLES:

sage: p = Polyhedron(vertices=[(3,2)], rays=[(1,-1)])
sage: p.representative_point()
(4, 1)
sage: p.center()
(3, 2)
2.6.3 Base class for polyhedra: Methods related to lattice points

class sage.geometry.polyhedron.base2.Polyhedron_base2 (parent, Vrep, Hrep, Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False, **kwds)

Bases: Polyhedron_base1

Methods related to lattice points.

See sage.geometry.polyhedron.base.Polyhedron_base.

generating_function_of_integral_points (**kwds)

Return the multivariate generating function of the integral points of this polyhedron.

To be precise, this returns

\[ \sum_{\mathbf{r} \in \text{polyhedron} \cap \mathbb{Z}^d} y_0^{r_0} \cdots y_{d-1}^{r_{d-1}}. \]

This calls generating_function_of_integral_points(), so have a look at the documentation and examples there.

INPUT:

The following keyword arguments are passed to generating_function_of_integral_points():

- **split** – (default: False) a boolean or list
  - split=False computes the generating function directly, without any splitting.
  - When split is a list of disjoint polyhedra, then for each of these polyhedra, this polyhedron is intersected with it, its generating function computed and all these generating functions are summed up.
  - split=True splits into \( d! \) disjoint polyhedra.
- **result_as_tuple** – (default: None) a boolean or None
  This specifies whether the output is a (partial) factorization (result_as_tuple=False) or a sum of such (partial) factorizations (result_as_tuple=True). By default (result_as_tuple=None), this is automatically determined. If the output is a sum, it is represented as a tuple whose entries are the summands.
- **indices** – (default: None) a list or tuple
  If this is None, this is automatically determined.
- **name** – (default: 'y') a string
  The variable names of the Laurent polynomial ring of the output are this string followed by an integer.
- **names** – a list or tuple of names (strings), or a comma separated string
  name is extracted from names, therefore names has to contain exactly one variable name, and name and "names" cannot be specified both at the same time.
• **Factorization_sort** (default: False) and **Factorization_simplify** (default: True)—booleans

These are passed on to `sage.structure.factorization.Factorization` when creating the result.

• **sort_factors**—(default: False) a boolean

If set, then the factors of the output are sorted such that the numerator is first and only then all factors of the denominator. It is ensured that the sorting is always the same; use this for doctesting.

**OUTPUT:**

The generating function as a (partial) `Factorization` of the result whose factors are Laurent polynomials

The result might be a tuple of such factorizations (depending on the parameter `result_as_tuple`) as well.

**Note:** At the moment, only polyhedra with nonnegative coordinates (i.e. a polyhedron in the nonnegative orthant) are handled.

**EXAMPLES:**

```python
sage: # needs sage.combinat
sage: P2 = (Polyhedron(ieqs=[(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, -1)]),
...: Polyhedron(ieqs=[(0, -1, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0)]))

sage: P2[0].generating_function_of_integral_points(sort_factors=True)
1 * (-y0 + 1)^-1 * (-y1 + 1)^-1 * (-y0*y2 + 1)^-1

sage: P2[1].generating_function_of_integral_points(sort_factors=True)
1 * (-y1 + 1)^-1 * (-y2 + 1)^-1 * (-y0*y2 + 1)^-1

sage: (P2[0] & P2[1]).Hrepresentation()
(An equation (1, 0, -1) x + 0 == 0,
An inequality (1, 0, 0) x + 0 >= 0,
An inequality (0, 1, 0) x + 0 >= 0)

sage: (P2[0] & P2[1]).generating_function_of_integral_points(sort_factors=True)
1 * (-y1 + 1)^-1 * (-y0*y2 + 1)^-1

The number of integer partitions $1 \leq r_0 \leq r_1 \leq r_2 \leq r_3 \leq r_4$:

```python
sage: # needs sage.combinat
sage: P = Polyhedron(ieqs=[(-1, 1, 0, 0, 0, 0), (0, -1, 1, 0, 0, 0),
...: (0, 0, -1, 1, 0, 0), (0, 0, 0, -1, 1, 0),
...: (0, 0, 0, 0, -1, 1)])

sage: f = P.generating_function_of_integral_points(sort_factors=True); f
y0*y1*y2*y3*y4 * (-y4 + 1)^-1 * (-y3*y4 + 1)^-1 * (-y2*y3*y4 + 1)^-1 * (-y1*y2*y3*y4 + 1)^-1 * (-y0*y1*y2*y3*y4 + 1)^-1

sage: f = f.value()
sage: P.<z> = PowerSeriesRing(ZZ)
sage: c = f.subs({y: z for y in f.parent().gens()}); c
z^5 + z^6 + 2*z^7 + 3*z^8 + 5*z^9 + 7*z^10 + 10*z^11 + 13*z^12 + 18*z^13 + 23*z^14 + 30*z^15 + 37*z^16 + 47*z^17 + 57*z^18 + 70*z^19 + 84*z^20 + 101*z^21 + 119*z^22 + 141*z^23 + 164*z^24 + O(z^25)

sage: ([Partitions(k, length=5).cardinality() for k in range(5,20)] ==
...: c.truncate().coefficients(sparse=False)[5:20])
True
```
See also:

More examples can be found at `generating_function_of_integral_points()`.

**get_integral_point**(index, **kwds)

Return the index-th integral point in this polyhedron.

This is equivalent to `sorted(self.integral_points())[index]`. However, so long as `integral_points_count()` does not need to enumerate all integral points, neither does this method. Hence it can be significantly faster. If the polyhedron is not compact, a `ValueError` is raised.

**INPUT:**

- index – integer. The index of the integral point to be found. If this is not in `[0, self.integral_point_count())`, an `IndexError` is raised.
- **kwds** – optional keyword parameters that are passed to `integral_points_count()`.

**ALGORITHM:**

The function computes each of the components of the requested point in turn. To compute \( x_i \), the \( i \)th component, it bisects the upper and lower bounds on \( x_i \) given by the bounding box. At each bisection, it uses `integral_points_count()` to determine on which side of the bisecting hyperplane the requested point lies.

See also:

`integral_points_count()`.

**EXAMPLES:**

```python
sage: P = Polyhedron(vertices=[(-1,-1),(1,0),(1,1),(0,1)])
sage: P.get_integral_point(1)
(0, 0)
sage: P.get_integral_point(4)
(1, 1)
sage: sorted(P.integral_points())
[(-1, -1), (0, 0), (0, 1), (1, 0), (1, 1)]
sage: P.get_integral_point(5)
Traceback (most recent call last):
  ... IndexError: ...
```

```python
sage: Q = Polyhedron([(1,3), (2, 7), (9, 77)])
sage: [Q.get_integral_point(i) for i in range(Q.integral_points_count())] ==
    sorted(Q.integral_points())
True
```

```python
sage: R = Polyhedron(vertices=[[1/2, 1/3]], rays=[[1, 1]])
sage: R.get_integral_point(0)
Traceback (most recent call last):
  ... ValueError: ...
```
Combinatorial and Discrete Geometry, Release 10.3

**h_star_vector()**

Return the $h^*$-vector of the lattice polytope.

The $h^*$-vector records the coefficients of the polynomial in the numerator of the Ehrhart series of a lattice polytope.

**INPUT:**

- `self` - A lattice polytope.

**OUTPUT:**

A list whose entries give the $h^*$-vector.

**EXAMPLES:**

The $h^*$-vector of a unimodular simplex $S$ (a simplex with volume $= \frac{1}{\text{dim}(S)!}$) is always 1. Here we test this on simplices up to dimension 3:

```sage
sage: # optional - pynormaliz
sage: s1 = polytopes.simplex(1,backend='normaliz')
sage: s2 = polytopes.simplex(2,backend='normaliz')
sage: s3 = polytopes.simplex(3,backend='normaliz')
sage: [s1.h_star_vector(), s2.h_star_vector(), s3.h_star_vector()]
[[1], [1], [1]]
```

For a less trivial example, we compute the $h^*$-vector of the $0/1$-cube, which has the Eulerian numbers $(3, i)$ for $i \in [0, 2]$ as an $h^*$-vector:

```sage
sage: cube = polytopes.cube(intervals=zero_one, backend='normaliz') # optional - pynormaliz
sage: cube.h_star_vector() # optional - pynormaliz
[1, 4, 1]
sage: from sage.combinat.combinat import eulerian_number
sage: [eulerian_number(3,i) for i in range(3)]
[1, 4, 1]
```

**integral_points**(threshold=100000)

Return the integral points in the polyhedron.

Uses either the naive algorithm (iterate over a rectangular bounding box) or triangulation + Smith form.

**INPUT:**

- `threshold` - integer (default: 100000). Use the naive algorithm as long as the bounding box is smaller than this.

**OUTPUT:**

The list of integral points in the polyhedron. If the polyhedron is not compact, a `ValueError` is raised.

**EXAMPLES:**

```sage
sage: Polyhedron(vertices=[(-1,-1),(1,0),(1,1),(0,1)]).integral_points()
((-1, -1), (0, 0), (0, 1), (1, 0), (1, 1))
sage: simplex = Polyhedron([[1,2,3], (2,3,7), (-2,-3,-11)])
sage: simplex.integral_points()
((-2, -3, -11), (0, 0, -2), (1, 2, 3), (2, 3, 7))
```

The polyhedron need not be full-dimensional:
Here is a simplex where the naive algorithm of running over all points in a rectangular bounding box no longer works fast enough:

```python
sage: v = [(1,0,7,-1), (-2,-2,4,-3), (0,0,-1,4), (2,9,0,-5), (-2,-1,5,1)]
sage: simplex = Polyhedron(v); simplex
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 5 vertices
sage: len(simplex.integral_points())
49
```

A case where rounding in the right direction goes a long way:

```python
sage: P = 1/10*polytopes.hypercube(14, backend='field')
sage: P.integral_points()
((0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),)
```

Finally, the 3-d reflexive polytope number 4078:

```python
sage: v = [(1,0,0), (0,1,0), (0,0,1), (0,0,-1), (0,-2,1),
       (-1,2,-1), (-1,2,-2), (-1,1,-2), (-1,-1,2), (-1,-3,2)]
sage: P = Polyhedron(v)
sage: pts1 = P.integral_points() # Sage's own code
sage: all(P.contains(p) for p in pts1)
True
sage: pts2 = LatticePolytope(v).points() # needs palp
sage: for p in pts1: p.set_immutable()
```

```
integral_points_count(**kwds)

Return the number of integral points in the polyhedron.

This generic version of this method simply calls integral_points().

EXAMPLES:
```
We shrink the polyhedron a little bit:

```sage
Q = P*(8/9)
sage: Q.integral_points_count()
1
```

Same for a polyhedron whose coordinates are not rationals. Note that the answer is an integer even though there are no guarantees for exactness:

```sage
Q = P*RDF(8/9)
sage: Q.integral_points_count()
1
```

Unbounded polyhedra (with or without lattice points) are not supported:

```sage
P = Polyhedron(vertices=[[1/2, 1/3]], rays=[[1, 1]])
sage: P.integral_points_count()
Traceback (most recent call last):
...,
NotImplementedError: ...
```

```sage
P = Polyhedron(vertices=[[1, 1]], rays=[[1, 1]])
sage: P.integral_points_count()
Traceback (most recent call last):
...,
NotImplementedError: ...
```

**is_lattice_polytope()**

Return whether the polyhedron is a lattice polytope.

**OUTPUT:**

True if the polyhedron is compact and has only integral vertices, False otherwise.

**EXAMPLES:**

```sage
sage: polytopes.cross_polytope(3).is_lattice_polytope()
True
sage: polytopes.regular_polygon(5).is_lattice_polytope()  # needs sage.rings.number_field
False
```

**lattice_polytope**(envelope=False)

Return an encompassing lattice polytope.

**INPUT:**

- envelope - boolean (default: False). If the polyhedron has non-integral vertices, this option decides whether to return a strictly larger lattice polytope or raise a `ValueError`. This option has no effect if the polyhedron has already integral vertices.

**OUTPUT:**

A `LatticePolytope`. If the polyhedron is compact and has integral vertices, the lattice polytope equals the polyhedron. If the polyhedron is compact but has at least one non-integral vertex, a strictly larger lattice polytope is returned.

If the polyhedron is not compact, a `NotImplementedError` is raised.

If the polyhedron is not integral and `envelope=False`, a `ValueError` is raised.

**ALGORITHM:**
For each non-integral vertex, a bounding box of integral points is added and the convex hull of these integral points is returned.

EXAMPLES:

First, a polyhedron with integral vertices:

```sage
P = Polyhedron(vertices=[(1, 0), (0, 1), (-1, 0), (0, -1)])
lp = P.lattice_polytope(); lp
```

```
2-d reflexive polytope... in 2-d lattice M
```

```sage
lp.vertices()
```

```
M(-1, 0),
M( 0, -1),
M( 0, 1),
M( 1, 0)
in 2-d lattice M
```

Here is a polyhedron with non-integral vertices:

```sage
P = Polyhedron( vertices = [(1/2, 1/2), (0, 1), (-1, 0), (0, -1)])
lp = P.lattice_polytope()
```

```
Traceback (most recent call last):
...
ValueError: Some vertices are not integral. You probably want to add the argument "envelope=True" to compute an enveloping lattice polytope.
```

```sage
lp = P.lattice_polytope(True)
```

```sage
lp.vertices()
```

```
M(-1, 0),
M( 0, -1),
M( 1, 1),
M( 0, 1),
M( 1, 0)
in 2-d lattice M
```

random_integral_point(**kwds)

Return an integral point in this polyhedron chosen uniformly at random.

INPUT:

- **kwds** – optional keyword parameters that are passed to get_integral_point().

OUTPUT:

The integral point in the polyhedron chosen uniformly at random. If the polyhedron is not compact, a ValueError is raised. If the polyhedron does not contain any integral points, an EmptySetError is raised.

See also:

get_integral_point().

EXAMPLES:

```sage
P = Polyhedron(vertices=[(-1,-1),(1,0),(0,1)])
```

(continues on next page)
2.6.4 Base class for polyhedra: Methods regarding the combinatorics of a polyhedron

Excluding methods relying on sage.graphs.

```python
class sage.geometry.polyhedron.base3.Polyhedron_base3(parent, Vrep, Hrep,
Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False,
**kwds)
```

Methods related to the combinatorics of a polyhedron.

See `sage.geometry.polyhedron.base.Polyhedron_base`.

```python
def a_maximal_chain()
    Return a maximal chain of the face lattice in increasing order.
```

EXAMPLES:

```python
sage: P = polytopes.cube()
sage: P.a_maximal_chain()
[A -1-dimensional face of a Polyhedron in ZZ^3, A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1_ vertex, A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2_ vertices, A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4_]
```
Combinatorial and Discrete Geometry, Release 10.3

adjacency_matrix(algorithm=None)

Return the binary matrix of vertex adjacencies.

INPUT:

- algorithm – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

EXAMPLES:

```python
sage: polytopes.simplex(4).vertex_adjacency_matrix()
[0 1 1 1 1]
[1 0 1 1 1]
[1 1 0 1 1]
[1 1 1 0 1]
[1 1 1 1 0]
```

The rows and columns of the vertex adjacency matrix correspond to the Vrepresentation() objects: vertices, rays, and lines. The \((i, j)\) matrix entry equals 1 if the \(i\)-th and \(j\)-th V-representation object are adjacent.

Two vertices are adjacent if they are the endpoints of an edge, that is, a one-dimensional face. For unbounded polyhedra this clearly needs to be generalized and we define two V-representation objects (see sage.geometry.polyhedron.constructor) to be adjacent if they together generate a one-face. There are three possible combinations:

- Two vertices can bound a finite-length edge.
- A vertex and a ray can generate a half-infinite edge starting at the vertex and with the direction given by the ray.
- A vertex and a line can generate an infinite edge. The position of the vertex on the line is arbitrary in this case, only its transverse position matters. The direction of the edge is given by the line generator.

For example, take the half-plane:
Its (non-unique) V-representation consists of a vertex, a ray, and a line. The only edge is spanned by the vertex and the line generator, so they are adjacent:

```
sage: half_plane.Vrepresentation()
(A line in the direction (0, 1), A ray in the direction (1, 0), A vertex at...
  -> (0, 0))
sage: half_plane.vertex_adjacency_matrix()
[0 0 1]
[0 0 0]
[1 0 0]
```

In one dimension higher, that is for a half-space in 3 dimensions, there is no one-dimensional face. Hence nothing is adjacent:

```
sage: Polyhedron(ieqs=[[0,1,0,0]]).vertex_adjacency_matrix()
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

EXAMPLES:

In a bounded polygon, every vertex has precisely two adjacent ones:

```
sage: P = Polyhedron(ieqs=[[0,1,0,0]], vertices=[[1,0], [0,1], [0,1], [0,1]])
sage: for v in P.Vrep_generator():
    print("{} {} ".format(P.adjacency_matrix().row(v.index()), v))
(0, 1, 0, 1) A vertex at (0, 1)
(1, 0, 1, 0) A vertex at (1, 0)
(0, 1, 0, 1) A vertex at (0, 1)
(1, 0, 1, 0) A vertex at (1, 0)
```

If the V-representation of the polygon contains vertices and one ray, then each V-representation object is adjacent to two V-representation objects:

```
sage: P = Polyhedron(ieqs=[[0,1,0,0]], vertices=[[1,0], [0,1], [0,1], [0,1]],
                  rays=[[0,1]])
sage: for v in P.Vrep_generator():
    print("{} {} ".format(P.adjacency_matrix().row(v.index()), v))
(0, 1, 0, 0, 1) A ray in the direction (0, 1)
(1, 0, 1, 0, 0) A vertex at (0, 1)
(0, 1, 0, 1, 0) A vertex at (1, 0)
(0, 0, 1, 0, 1) A vertex at (3, 0)
(1, 0, 1, 0) A vertex at (4, 1)
```

If the V-representation of the polygon contains vertices and two distinct rays, then each vertex is adjacent to two V-representation objects (which can now be vertices or rays). The two rays are not adjacent to each other:

```
sage: P = Polyhedron(ieqs=[[0,1,0,0]], vertices=[[1,0], [0,1], [0,1], [0,1]],
                  rays=[[0,1], [1,1]])
sage: for v in P.Vrep_generator():
    print("{} {} ".format(P.adjacency_matrix().row(v.index()), v))
(0, 1, 0, 0, 0) A ray in the direction (0, 1)
(0, 1, 0, 1, 0) A ray in the direction (1, 0)
```

(continues on next page)
The vertex adjacency matrix has base ring integers. This way one can express various counting questions:

```sage
sage: P = polytopes.cube()
sage: Q = P.stack(P.faces(2)[0])
sage: M = Q.vertex_adjacency_matrix()
sage: sum(M)
(4, 4, 3, 3, 4, 4, 4, 3, 3)
sage: G = Q.vertex_graph()
# needs sage.graphs
sage: G.degree()
# needs sage.graphs
[4, 4, 3, 3, 4, 4, 4, 3, 3]
```

**bounded_edges()**

Return the bounded edges (excluding rays and lines).

OUTPUT:

A generator for pairs of vertices, one pair per edge.

EXAMPLES:

```sage
sage: p = Polyhedron(vertices=[[1,0],[0,1]], rays=[[1,0],[0,1]])
sage: [ e for e in p.bounded_edges() ]
[(A vertex at (0, 1), A vertex at (1, 0))]
sage: for e in p.bounded_edges(): print(e)
(A vertex at (0, 1), A vertex at (1, 0))
```

**combinatorial_polyhedron()**

Return the combinatorial type of self.

See `sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron`.

EXAMPLES:

```sage
sage: polytopes.cube().combinatorial_polyhedron()
A 3-dimensional combinatorial polyhedron with 6 facets
sage: polytopes.cyclic_polytope(4,10).combinatorial_polyhedron()
A 4-dimensional combinatorial polyhedron with 35 facets
sage: Polyhedron(rays=[[0,1],[1,0]]).combinatorial_polyhedron()
A 2-dimensional combinatorial polyhedron with 2 facets
```

**f_vector**

Return the f-vector.

INPUT:

- `num_threads` – integer (optional); specify the number of threads; otherwise determined by `ncpus()`
- `parallelization_depth` – integer (optional); specify how deep in the lattice the parallelization is done
algorithm -- string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' -- start with the facets
  - 'dual' -- start with the vertices
  - None -- choose automatically

OUTPUT:
Return a vector whose $i$-th entry is the number of $i-2$-dimensional faces of the polytope.

Note: The vertices as given by vertices() do not need to correspond to 0-dimensional faces. If a polyhedron contains $k$ lines they correspond to $k$-dimensional faces. See example below.

EXAMPLES:

```
sage: p = Polyhedron(vertices=[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1], [0, 0, 0]])
sage: p.f_vector()
(1, 7, 12, 7, 1)
sage: polytopes.cyclic_polytope(4,10).f_vector()
(1, 10, 45, 70, 35, 1)
sage: polytopes.hypercube(5).f_vector()
(1, 32, 80, 80, 40, 10, 1)
```

Polyhedra with lines do not have 0-faces:

```
sage: Polyhedron(ieqs=[[1,-1,0,0],[1,1,0,0]]).f_vector()
(1, 0, 2, 1)
```

However, the method Polyhedron_base.vertices() returns two points that belong to the Vrepresentation:

```
sage: P = Polyhedron(ieqs=[[1,-1,0],[1,1,0]])
sage: P.vertices()
(A vertex at (1, 0), A vertex at (-1, 0))
sage: P.f_vector()
(1, 0, 2, 1)
```

face_generator (face_dimension=None, algorithm=None, **kwds)
Return an iterator over the faces of given dimension.

If dimension is not specified return an iterator over all faces.

INPUT:

  - face_dimension -- integer (default None), yield only faces of this dimension if specified
  - algorithm -- string (optional); specify whether to start with facets or vertices:
    - 'primal' -- start with the facets
    - 'dual' -- start with the vertices
    - None -- choose automatically

OUTPUT:
A `FaceIterator_geom`. This class iterates over faces as `PolyhedronFace`. See `face` for details. The order is random but fixed.

EXAMPLES:

```python
sage: P = polytopes.cube()
sage: it = P.face_generator()
sage: it
iterator over the faces of a 3-dimensional polyhedron in ZZ^3
sage: list(it)
[A 3-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 8 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 vertices,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1 vertex,
] (continues on next page)
```
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2... vertices,
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1... vertex,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2... vertices]

```
sage: P = polytopes.hypercube(4)
sage: list(P.face_generator(2))[:4]
```

```
[4 vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^4$ defined as the convex hull of...
4 vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^4$ defined as the convex hull of...
4 vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^4$ defined as the convex hull of...
4 vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^4$ defined as the convex hull of...
4 vertices]
```

If a polytope has more facets than vertices, the dual mode is chosen:

```
sage: P = polytopes.cross_polytope(3)
sage: list(P.face_generator())
```

```
[6 vertices,
A -1-dimensional face of a Polyhedron in $\mathbb{Z}^3$,
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1...
vertex,
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1...
vertex,
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1...
vertex,
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1...
vertex,
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1...
vertex,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 3...
vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 3...
vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 3...
vertices,
A 2-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 3...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 2...
vertices,
]```
The face iterator can also be slightly modified. In non-dual mode we can skip subfaces of the current (proper) face:

```python
sage: P = polytopes.cube()
sage: it = P.face_generator(algorithm='primal')
sage: _ = next(it), next(it)
sage: face = next(it)
sage: face.ambient_H_indices()
(5,)
sage: it.ignore_subfaces()
sage: face = next(it)
sage: face.ambient_H_indices()
(4,)
sage: [face.ambient_H_indices() for face in it]
[(3,), (2,), (1,), (0,), (2, 3), (1, 3), (1, 2, 3), (1, 2), (0, 2), (0, 1, 2), (0, 1)]
```

In dual mode we can skip supfaces of the current (proper) face:

```python
sage: P = polytopes.cube()
sage: it = P.face_generator(algorithm='dual')
sage: _ = next(it), next(it)
sage: face = next(it)
sage: face.ambient_V_indices()
(7,)
```

(continues on next page)
In non-dual mode, we cannot skip supfaces:

```
sage: it = P.face_generator(algorithm='primal')
sage: _ = next(it), next(it)
sage: next(it)
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4
→vertices
sage: it.ignore_supfaces()
Traceback (most recent call last):
  ...
ValueError: only possible when in dual mode
```

In dual mode, we cannot skip subfaces:

```
sage: it = P.face_generator(algorithm='dual')
sage: _ = next(it), next(it)
sage: next(it)
A 0-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 1
→vertex
sage: it.ignore_subfaces()
Traceback (most recent call last):
  ...
ValueError: only possible when not in dual mode
```

We can only skip sub-/supfaces of proper faces:

```
sage: it = P.face_generator(algorithm='primal')
sage: next(it)
A 3-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 8
→vertices
sage: it.ignore_subfaces()
Traceback (most recent call last):
  ...
ValueError: iterator not set to a face yet
```

```
See also:

*FaceIterator_geom*.

ALGORITHM:

See *FaceIterator*.

**faces** (*face_dimension*)

Return the faces of given dimension

INPUT:

- *face_dimension* – integer. The dimension of the faces whose representation will be returned.

OUTPUT:

A tuple of *PolyhedronFace*. See module *sage.geometry.polyhedron.face* for details. The order is random but fixed.

See also:

*face_generator()*,*facet()*.

EXAMPLES:

Here we find the vertex and face indices of the eight three-dimensional facets of the four-dimensional hypercube:

```python
sage: p = polytopes.hypercube(4)
sage: list(f.ambient_V_indices() for f in p.faces(3))

[(0, 5, 6, 7, 8, 9, 14, 15),
 (1, 4, 5, 6, 10, 13, 14, 15),
 (1, 2, 6, 7, 8, 10, 11, 15),
 (8, 9, 10, 11, 12, 13, 14, 15),
 (0, 3, 4, 5, 9, 12, 13, 14),
 (0, 2, 3, 7, 8, 9, 11, 12),
 (1, 2, 3, 4, 10, 11, 12, 13),
 (0, 1, 2, 3, 4, 5, 6, 7)]
```

```python
sage: face = p.faces(3)[3]
sage: face.ambient_Hrepresentation()
(An inequality (1, 0, 0, 0) x + 1 >= 0,)
sage: face.vertices()
(A vertex at (-1, -1, 1, -1),
 A vertex at (-1, -1, 1, 1),
 A vertex at (-1, 1, -1, -1),
 A vertex at (-1, 1, 1, -1),
 A vertex at (-1, 1, 1, 1),
 A vertex at (-1, 1, -1, 1),
 A vertex at (-1, -1, 1, 1),
 A vertex at (-1, -1, -1, 1),
 A vertex at (-1, -1, -1, -1))
```

You can use the *index()* method to enumerate vertices and inequalities:

```python
sage: def get_idx(rep): return rep.index()
sage: [get_idx(_) for _ in face.ambient_Hrepresentation()]
[4]
sage: [get_idx(_) for _ in face.ambient_Vrepresentation()]
[8, 9, 10, 11, 12, 13, 14, 15]
sage: [ [get_idx(_), get_idx(_)] for _ in face.ambient_Vrepresentation()]
```

(continues on next page)
facet_adjacency_matrix(algorithm=None)

Return the adjacency matrix for the facets.

INPUT:

- algorithm – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

EXAMPLES:

```
sage: s4 = polytopes.simplex(4, project=True)
sage: s4.facet_adjacency_matrix()
[0 1 1 1 1]
[1 0 1 1 1]
[1 1 0 1 1]
[1 1 1 0 1]
[1 1 1 1 0]
sage: p = Polyhedron(vertices=[(0,0),(1,0),(0,1)])
sage: p.facet_adjacency_matrix()
[0 1 1]
[1 0 1]
[1 1 0]
```

The facet adjacency matrix has base ring integers. This way one can express various counting questions:

```
sage: P = polytopes.cube()
sage: Q = P.stack(P.faces(2)[0])
sage: M = Q.facet_adjacency_matrix()
sage: sum(M)
(4, 4, 4, 3, 3, 3, 3, 4)
```

facets()

Return the facets of the polyhedron.

Facets are the maximal nontrivial faces of polyhedra. The empty face and the polyhedron itself are trivial.

A facet of a \(d\)-dimensional polyhedron is a face of dimension \(d - 1\). For \(d \neq 0\) the converse is true as well.

OUTPUT:

A tuple of \texttt{PolyhedronFace}. See \texttt{face} for details. The order is random but fixed.
See also:

facets()  

EXAMPLES:  

Here we find the eight three-dimensional facets of the four-dimensional hypercube:

```
sage: p = polytopes.hypercube(4)
sage: p.facets()
(A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...)
```

This is the same result as explicitly finding the three-dimensional faces:

```
sage: dim = p.dimension()
sage: p.faces(dim-1)
(A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...
A 3-dimensional face of a Polyhedron in ZZ^4 defined as the convex hull of 8...)
```

The 0-dimensional polyhedron does not have facets:

```
sage: P = Polyhedron([[0]])
sage: P.facets()
()
```

`greatest_common_subface_of_Hrep(*Hrepresentatives)`  

Return the largest face that is contained in Hrepresentatives.

INPUT:
• **Hrepresentatives** – facets or indices of Hrepresentatives; the indices are assumed to be the indices of the *Hrepresentation()*

**OUTPUT:** a *PolyhedronFace*

**EXAMPLES:**

```python
sage: P = polytopes.permutahedron(5)
sage: P.meet_of_Hrep()
A 4-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 120 vertices

sage: P.meet_of_Hrep(1)
A 3-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 24 vertices

sage: P.meet_of_Hrep(4)
A 3-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 12 vertices

sage: P.meet_of_Hrep(1, 3, 7)
A 1-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 2 vertices

sage: P.meet_of_Hrep(0).
A 4-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 120 vertices

The input is flexible:

```python
sage: P.meet_of_Hrep(P.facets()[0], P.inequalities()[2], 7)
A 1-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 2 vertices

The **Hrepresentatives** must belong to *self*:

```python
sage: P = polytopes.cube(backend='ppl')
sage: Q = polytopes.cube(backend='field')
sage: f = P.facets()[0]
sage: P.meet_of_Hrep(f)
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices

sage: Q.meet_of_Hrep(f)
Traceback (most recent call last):
  ...
ValueError: not a facet of `self`

sage: f = P.inequalities()[0]
sage: P.meet_of_Hrep(f)
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices

sage: Q.meet_of_Hrep(f)
Traceback (most recent call last):
  ...
ValueError: not a facet of `self`
```

**incidence_matrix()**

Return the incidence matrix.
Note: The columns correspond to inequalities/equations in the order \texttt{Hrepresentation()}, the rows correspond to vertices/rays/lines in the order \texttt{Vrepresentation()}.

See also:

\texttt{slack_matrix()}.  

EXAMPLES:

```
sage: p = polytopes.cuboctahedron()
sage: p.incidence_matrix()
[0 0 1 1 0 1 0 0 0 0 1 0 0 0]
[0 0 0 1 0 0 1 0 1 0 1 0 0 0]
[0 0 1 1 0 0 1 0 0 0 0 0 0 0]
[1 0 0 1 1 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 1 1 0 0 0 0]
[0 0 1 0 0 1 0 1 0 0 1 0 0 0]
[1 0 0 0 0 1 0 0 0 1 0 0 1 0]
[1 0 0 1 0 0 1 0 0 0 0 0 0 1]
[0 1 0 0 0 1 0 0 1 0 1 0 0 0]
[0 1 0 0 1 0 0 0 0 1 0 0 1 0]
[1 0 0 0 0 0 1 0 0 1 0 1 0 0]
[1 1 0 0 0 0 0 0 0 0 0 0 1 1]
sage: v = p.Vrepresentation(0)
sage: v
A vertex at (-1, -1, 0)
sage: h = p.Hrepresentation(2)
sage: h
An inequality (1, 1, -1) x + 2 >= 0
sage: h.eval(v)  # evaluation (1, 1, -1) * (-1/2, -1/2, 0) + 1
0
sage: h*v  # same as h.eval(v)
0
sage: p.incidence_matrix() [0,2]  # this entry is (v,h)
1
sage: h.contains(v)
True
sage: p.incidence_matrix() [2,0]  # note: not symmetric
0
```

The incidence matrix depends on the ambient dimension:

```
sage: simplex = polytopes.simplex(); simplex
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 4 vertices
sage: simplex.incidence_matrix()
[1 1 1 0]
[1 1 0 1]
[1 0 1 1]
[sage: simplex = simplex.affine_hull_projection(); simplex
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 4 vertices
sage: simplex.incidence_matrix()
[1 1 1 0]
[1 1 0 1]
[1 0 1 1]
[0 1 1 1]
```

An incidence matrix does not determine a unique polyhedron:
Combination and Discrete Geometry, Release 10.3

```python
sage: P = Polyhedron(vertices=[[0,1],[1,1],[1,0]])
sage: P.incidence_matrix()
[1 1 0]
[1 0 1]
[0 1 1]
sage: Q = Polyhedron(vertices=[[0,1], [1,0]], rays=[[1,1]])
sage: Q.incidence_matrix()
[1 1 0]
[1 0 1]
[0 1 1]

An example of two polyhedra with isomorphic face lattices but different incidence matrices:

```python
sage: Q.incidence_matrix()
[1 1 0]
[1 0 1]
[0 1 1]
sage: R = Polyhedron(vertices=[[0,1], [1,0]], rays=[[1,3/2], [3/2,1]])
sage: R.incidence_matrix()
[1 1 0]
[1 0 1]
[0 1 0]
[0 0 1]
```

The incidence matrix has base ring integers. This way one can express various counting questions:

```python
sage: P = polytopes.twenty_four_cell()
sage: M = P.incidence_matrix()
sage: sum(sum(x) for x in M) == P.flag_f_vector(0, 3)  # needs sage.combinat
True
```

`is_bipyramid`(certificate=False)

Test whether the polytope is combinatorially equivalent to a bipyramid over some polytope.

**INPUT:**

- certificate – boolean (default: False); specifies whether to return two vertices of the polytope which are the apices of a bipyramid, if found

**OUTPUT:**

If certificate is True, returns a tuple containing:

1. Boolean.

2. None or a tuple containing:
   - The first apex.
   - The second apex.

If certificate is False returns a boolean.

**EXAMPLES:**

```python
sage: P = polytopes.octahedron()
sage: P.is_bipyramid()
```

(continues on next page)
True
sage: P.is_bipyramid(certificate=True)
(True, [A vertex at (1, 0, 0), A vertex at (-1, 0, 0)])
sage: Q = polytopes.cyclic_polytope(3,7)
sage: Q.is_bipyramid()
False
sage: R = Q.bipyramid()
sage: R.is_bipyramid(certificate=True)
(True, [A vertex at (1, 3, 13, 63), A vertex at (-1, 3, 13, 63)])

is_lawrence_polytope()
Return True if self is a Lawrence polytope.

A polytope is called a Lawrence polytope if it has a centrally symmetric (normalized) Gale diagram.

EXAMPLES:
sage: P = polytopes.hypersimplex(5,2)
sage: L = P.lawrence_polytope()
sage: L.is_lattice_polytope()
True
sage: egyptian_pyramid = polytopes.regular_polygon(4).pyramid()
# needs sage.number_field
sage: egyptian_pyramid.is_lawrence_polytope()
# needs sage.number_field
True
sage: polytopes.octahedron().is_lawrence_polytope()
False

REFERENCES:
For more information, see [BaSt1990].

is_neighborhood (k=None)
Return whether the polyhedron is neighborly.

If the input k is provided, then return whether the polyhedron is k-neighborly

A polyhedron is neighborly if every set of n vertices forms a face for n up to floor of half the dimension of the polyhedron. It is k-neighborly if this is true for n up to k.

INPUT:

• k – the dimension up to which to check if every set of k vertices forms a face. If no k is provided, check up to floor of half the dimension of the polyhedron.

OUTPUT:

• True if every set of up to k vertices forms a face,
• False otherwise

See also:
neighborliness()

EXAMPLES:
Cyclic polytopes are neighborly:

```
sage: all(polytopes.cyclic_polytope(i, i + 1 + j).is_neighborly() for i in range(5) for j in range(3))
True
```

The neighborliness of a polyhedron equals floor of dimension half (or larger in case of a simplex) if and only if the polyhedron is neighborly:

```
sage: testpolys = [polytopes.cube(), polytopes.cyclic_polytope(6, 9), polytopes.simplex(6)]
sage: [(P.neighborliness() >= P.dim() // 2) == P.is_neighborly() for P in testpolys]
[True, True, True]
```

**is_prism** *(certificate=False)*

Test whether the polytope is combinatorially equivalent to a prism of some polytope.

**INPUT:**

- certificate – boolean (default: False); specifies whether to return two facets of the polytope which are the bases of a prism, if found

**OUTPUT:**

If certificate is True, returns a tuple containing:

1. Boolean.

2. None or a tuple containing:
   a. List of the vertices of the first base facet.
   b. List of the vertices of the second base facet.

If certificate is False returns a boolean.

**EXAMPLES:**

```
sage: P = polytopes.cube()
sage: P.is_prism()
True
sage: P.is_prism(certificate=True)
(True,
 [A vertex at (1, -1, -1),
  A vertex at (1, -1, 1),
  A vertex at (-1, -1, 1),
  A vertex at (-1, -1, -1)),
 [A vertex at (1, 1, -1),
  A vertex at (1, 1, 1),
  A vertex at (-1, 1, -1),
  A vertex at (-1, 1, 1))]
sage: Q = polytopes.cyclic_polytope(3, 8)
```
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```
sage: Q.is_prism()
False
sage: R = Q.prism()
sage: R.is_prism(certificate=True)
(True,
 [(A vertex at (0, 3, 9, 27),
  A vertex at (0, 6, 36, 216),
  A vertex at (0, 0, 0, 0),
  A vertex at (0, 7, 49, 343),
  A vertex at (0, 5, 25, 125),
  A vertex at (0, 1, 1, 1),
  A vertex at (0, 2, 4, 8),
  A vertex at (0, 4, 16, 64)),
 (A vertex at (1, 6, 36, 216),
  A vertex at (1, 0, 0, 0),
  A vertex at (1, 7, 49, 343),
  A vertex at (1, 5, 25, 125),
  A vertex at (1, 1, 1, 1),
  A vertex at (1, 2, 4, 8),
  A vertex at (1, 4, 16, 64),
  A vertex at (1, 3, 9, 27))]
```

**is_pyramid** *(certificate=False)*

Test whether the polytope is a pyramid over one of its facets.

**INPUT:**

- certificate – boolean (default: False); specifies whether to return a vertex of the polytope which is the apex of a pyramid, if found

**OUTPUT:**

If certificate is True, returns a tuple containing:

1. Boolean.
2. The apex of the pyramid or None.

If certificate is False returns a boolean.

**EXAMPLES:**

```
sage: P = polytopes.simplex(3)
sage: P.is_pyramid()
True
sage: P.is_pyramid(certificate=True)
(True, A vertex at (1, 0, 0, 0))
sage: egyptian_pyramid = polytopes.regular_polygon(4).pyramid()  # needs sage.rings.number_field
sage: egyptian_pyramid.is_pyramid()  # needs sage.rings.number_field
True
sage: Q = polytopes.octahedron()
sage: Q.is_pyramid()
False
```

For the 0-dimensional polyhedron, the output is True, but it cannot be constructed as a pyramid over the empty polyhedron:

2.6. Base classes for polyhedra 547
is_simple()  
Test for simplicity of a polytope.

See Wikipedia article Simple_polytope

EXAMPLES:

```python
sage: p = Polyhedron([[0,0,0],[1,0,0],[0,1,0],[0,0,1]])
sage: p.is_simple()
True
sage: p = Polyhedron([[0,0,0],[4,4,0],[4,0,0],[0,4,0],[2,2,2]])
sage: p.is_simple()
False
```

is_simplex()  
Return whether the polyhedron is a simplex.

A simplex is a bounded polyhedron with \(d + 1\) vertices, where \(d\) is the dimension.

EXAMPLES:

```python
sage: Polyhedron([[0,0,0], (1,0,0), (0,1,0)]).is_simplex()
True
sage: polytopes.simplex(3).is_simplex()
True
sage: polytopes.hypercube(3).is_simplex()
False
```

is_simplicial()  
Tests if the polytope is simplicial

A polytope is simplicial if every facet is a simplex.

See Wikipedia article Simplicial_polytope

EXAMPLES:

```python
sage: p = polytopes.hypercube(3)
sage: p.is_simplicial()
False
sage: q = polytopes.simplex(5, project=True)
sage: q.is_simplicial()
True
sage: p = Polyhedron([[0,0,0],[1,0,0],[0,1,0],[0,0,1]])
sage: p.is_simplicial()
True
sage: q = Polyhedron([[1,1,1],[-1,1,1],[1,-1,1],[-1,-1,1],[1,1,-1]])
sage: q.is_simplicial()
False
sage: P = polytopes.simplex(); P
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 4 vertices
```
sage: P.is_simplicial()
True

The method is not implemented for unbounded polyhedra:

```python
sage: p = Polyhedron(vertices=[(0,0)],rays=[(1,0),(0,1)])
sage: p.is_simplicial()
Traceback (most recent call last):
...  
NotImplementedError: this function is implemented for polytopes only
```

## join_of_Vrep(*Vrepresentatives)

Return the smallest face that contains Vrepresentatives.

**INPUT:**

- Vrepresentatives – vertices/rays/lines of self or indices of such

**OUTPUT:** a *PolyhedronFace*

**Note:** In the case of unbounded polyhedra, the join of rays etc. may not be well-defined.

**EXAMPLES:**

```python
sage: P = polytopes.permutahedron(5)
sage: P.join_of_Vrep(1)
A 0-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 1 → vertex
sage: P.join_of_Vrep()
A -1-dimensional face of a Polyhedron in ZZ^5
sage: P.join_of_Vrep(0,12,13).ambient_V_indices()
(0, 12, 13, 68)
```

The input is flexible:

```python
sage: P.join_of_Vrep(2, P.vertices()[3], P.Vrepresentation(4))
A 2-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 6 → vertices
```

```python
sage: P = polytopes.cube()
sage: a, b = P.faces(0)[:2]
sage: P.join_of_Vrep(a, b)
A 1-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 2 → vertices
```

In the case of an unbounded polyhedron, the join may not be well-defined:

```python
sage: P = Polyhedron(vertices=[[1,0], [0,1]], rays=[[1,1]])
sage: P.join_of_Vrep()
A 0-dimensional face of a Polyhedron in QQ^2 defined as the convex hull of 1 → vertex
sage: P.join_of_Vrep(0,1)
A 1-dimensional face of a Polyhedron in QQ^2 defined as the convex hull of 2 → vertices
sage: P.join_of_Vrep(0,2)
```

(continues on next page)
A 1-dimensional face of a Polyhedron in $\mathbb{Q}^2$ defined as the convex hull of 1 vertex and 1 ray
```
sage: P.join_of_Vrep(1,2)
```
A 1-dimensional face of a Polyhedron in $\mathbb{Q}^2$ defined as the convex hull of 1 vertex and 1 ray
```
sage: P.join_of_Vrep(2)
```
Traceback (most recent call last):
  ... ValueError: the join is not well-defined

The **Vrepresentatives** must be of self:
```
sage: P = polytopes.cube(backend='ppl')
sage: Q = polytopes.cube(backend='field')
sage: v = P.vertices()[0]
sage: P.join_of_Vrep(v)
```
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1 vertex
```
sage: Q.join_of_Vrep(v)
```
Traceback (most recent call last):
  ... ValueError: not a Vrepresentative of ``self``
```
sage: f = P.faces(0)[0]
sage: P.join_of_Vrep(v)
```
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^3$ defined as the convex hull of 1 vertex
```
sage: Q.join_of_Vrep(v)
```
Traceback (most recent call last):
  ... ValueError: not a Vrepresentative of ``self``

**least_common_superface_of_Vrep** (*Vrepresentatives*)
Return the smallest face that contains Vrepresentatives.

**INPUT:**
- Vrepresentatives – vertices/rays/lines of self or indices of such

**OUTPUT:** a PolyhedronFace

**Note:** In the case of unbounded polyhedra, the join of rays etc. may not be well-defined.

**EXAMPLES:**
```
sage: P = polytopes.permutahedron(5)
sage: P.join_of_Vrep(1)
```
A 0-dimensional face of a Polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex
```
sage: P.join_of_Vrep()
```
A 1-dimensional face of a Polyhedron in $\mathbb{Z}^5$
```
sage: P.join_of_Vrep(0,12,13).ambient_V_indices()
```
(0, 12, 13, 68)

The input is flexible:
The \texttt{Vrepresentatives} must be of \texttt{self}:

\begin{verbatim}
sage: P = polytopes.cube()
sage: Q = polytopes.cube(backend='field')
sage: v = P.vertices()[0]
sage: P.join_of_Vrep(v)
sage: Q.join_of_Vrep(v)
sage: f = P.faces(0)[0]
sage: P.join_of_Vrep(v)
sage: Q.join_of_Vrep(v)

Traceback (most recent call last):
... ValueError: not a Vrepresentative of `\texttt{self}`
\end{verbatim}

\texttt{meet_of_Hrep(*Hrepresentatives)}

Return the largest face that is contained in \texttt{Hrepresentatives}.

\textbf{INPUT:}

- \texttt{Hrepresentatives} – facets or indices of \texttt{Hrepresentatives}; the indices are assumed to be the indices of the \texttt{Hrepresentation()}

\section*{2.6. Base classes for polyhedra}
COMBINATORIAL AND DISCRETE GEOMETRY

OUTPUT: a \texttt{PolyhedronFace}

EXAMPLES:

\begin{verbatim}
sage: P = polytopes.permutahedron(5)
sage: P.meet_of_Hrep()
A 4-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 120 vertices
sage: P.meet_of_Hrep(1)
A 3-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 24 vertices
sage: P.meet_of_Hrep(4)
A 3-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 12 vertices
sage: P.meet_of_Hrep(1,3,7)
A 1-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 2 vertices
sage: P.meet_of_Hrep(1,3,7).ambient_H_indices()
(0, 1, 3, 7)
\end{verbatim}

The indices are the indices of the \texttt{Hrepresentation()}. 0 corresponds to an equation and is ignored:

\begin{verbatim}
sage: P.meet_of_Hrep(0)
A 4-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 120 vertices
\end{verbatim}

The input is flexible:

\begin{verbatim}
sage: P.meet_of_Hrep(P.facets()[0], P.inequalities()[2], 7)
A 1-dimensional face of a Polyhedron in ZZ^5 defined as the convex hull of 2 vertices
\end{verbatim}

The \texttt{Hrepresentatives} must belong to \texttt{self}:

\begin{verbatim}
sage: P = polytopes.cube(backend='ppl')
sage: Q = polytopes.cube(backend='field')
sage: f = P.facets()[0]
sage: P.meet_of_Hrep(f)
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices
sage: Q.meet_of_Hrep(f)
Traceback (most recent call last):
  ...
ValueError: not a facet of ''self''
sage: f = P.inequalities()[0]
sage: P.meet_of_Hrep(f)
A 2-dimensional face of a Polyhedron in ZZ^3 defined as the convex hull of 4 vertices
sage: Q.meet_of_Hrep(f)
Traceback (most recent call last):
  ...
ValueError: not a facet of ''self''
\end{verbatim}

\texttt{neighborliness()}

Return the largest \( k \), such that the polyhedron is \( k \)-neighborly.

A polyhedron is \( k \)-neighborly if every set of \( n \) vertices forms a face for \( n \) up to \( k \).

In case of the \( d \)-dimensional simplex, it returns \( d + 1 \).
See also:

`is_neighborly()`

**EXAMPLES:**

```
sage: cube = polytopes.cube()
sage: cube.neighborliness()
1
sage: P = Polyhedron(); P
The empty polyhedron in ZZ^0
sage: P.neighborliness()
0
sage: P = Polyhedron([[0]]); P
A 0-dimensional polyhedron in ZZ^1 defined as the convex hull of 1 vertex
sage: P.neighborliness()
1
sage: S = polytopes.simplex(5); S
A 5-dimensional polyhedron in ZZ^6 defined as the convex hull of 6 vertices
sage: S.neighborliness()
6
sage: C = polytopes.cyclic_polytope(7,10); C
A 7-dimensional polyhedron in QQ^7 defined as the convex hull of 10 vertices
sage: C.neighborliness()
3
sage: C = polytopes.cyclic_polytope(6,11); C
A 6-dimensional polyhedron in QQ^6 defined as the convex hull of 11 vertices
sage: C.neighborliness()
3
sage: [polytopes.cyclic_polytope(5,n).neighborliness() for n in range(6,10)]
[6, 2, 2, 2]
```

`simpliciality()`

Return the largest integer \( k \) such that the polytope is \( k \)-simplicial.

A polytope is \( k \)-simplicial, if every \( k \)-face is a simplex. If `self` is a simplex, returns its dimension.

**EXAMPLES:**

```
sage: polytopes.cyclic_polytope(10,4).simpliciality()
3
sage: polytopes.hypersimplex(5,2).simpliciality()
2
sage: polytopes.cross_polytope(4).simpliciality()
3
sage: polytopes.simplex(3).simpliciality()
3
sage: polytopes.simplex(1).simpliciality()
1
```

The method is not implemented for unbounded polyhedra:

```
sage: p = Polyhedron(vertices=[[0,0]],rays=[[(1,0),(0,1)]]
sage: p.simpliciality()
Traceback (most recent call last):
...
NotImplementedError: this function is implemented for polytopes only
```

`simplicity()`

Return the largest integer \( k \) such that the polytope is \( k \)-simple.
A polytope $P$ is $k$-simple, if every $(d-1-k)$-face is contained in exactly $k+1$ facets of $P$ for $1 \leq k \leq d-1$. Equivalently it is $k$-simple if the polar/dual polytope is $k$-simplicial. If `self` is a simplex, it returns its dimension.

**EXAMPLES:**

```
sage: polytopes.hypersimplex(4,2).simplicity()
1
sage: polytopes.hypersimplex(5,2).simplicity()
2
sage: polytopes.hypersimplex(6,2).simplicity()
3
sage: polytopes.simplex(3).simplicity()
3
sage: polytopes.simplex(1).simplicity()
1
```

The method is not implemented for unbounded polyhedra:

```
sage: p = Polyhedron(vertices=[(0,0)],rays=[(1,0),(0,1)])
sage: p.simplicity()
Traceback (most recent call last):
...  
NotImplementedError: this function is implemented for polytopes only
```

### slack_matrix()

Return the slack matrix.

The entries correspond to the evaluation of the Hrepresentation elements on the Vrepresentation elements.

**Note:** The columns correspond to inequalities/equations in the order `Hrepresentation()`, the rows correspond to vertices/rays/lines in the order `Vrepresentation()`.

**See also:**

`incidence_matrix()`.

**EXAMPLES:**

```
sage: P = polytopes.cube()
sage: P.slack_matrix()
[0 2 2 2 0 0]
[0 0 2 2 0 2]
[0 0 0 2 2 2]
[0 2 0 2 2 0]
[2 2 0 2 0 2]
[2 2 2 0 0 0]
[2 0 2 0 2 2]
[2 0 0 2 2 2]
```

```
sage: P = polytopes.cube(intervals='zero_one')
sage: P.slack_matrix()
[0 1 1 1 0 0]
[0 0 1 1 0 1]
[0 0 0 1 1 1]
[0 1 0 1 1 0]
[1 1 0 0 1 0]
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

```
sage: # needs sage.rings.number_field
sage: P = polytopes.dodecahedron().faces(2)[0].as_polyhedron()
sage: P.slack_matrix()
\[
\begin{bmatrix}
\frac{1}{2} \sqrt{5} - \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} \sqrt{5} - \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} \sqrt{5} - \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} \sqrt{5} - \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} \sqrt{5} - \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} \sqrt{5} - \frac{1}{2} \\
\end{bmatrix}
\]
```

```
sage: P = polyhedron(rays=[[1, 0], [0, 1]])
sage: P.slack_matrix()
\[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]
```

`vertex_adjacency_matrix(algorithm=None)`

Return the binary matrix of vertex adjacencies.

INPUT:

- `algorithm` -- string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

EXAMPLES:

```
sage: polytopes.simplex(4).vertex_adjacency_matrix()
\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
```

The rows and columns of the vertex adjacency matrix correspond to the Vrepresentation() objects: vertices, rays, and lines. The (i, j) matrix entry equals 1 if the i-th and j-th V-representation object are adjacent.

Two vertices are adjacent if they are the endpoints of an edge, that is, a one-dimensional face. For unbounded polyhedra this clearly needs to be generalized and we define two V-representation objects (see sage.geometry.polyhedron.constructor) to be adjacent if they together generate a one-face. There are three possible combinations:

- Two vertices can bound a finite-length edge.
- A vertex and a ray can generate a half-infinite edge starting at the vertex and with the direction given by the ray.
• A vertex and a line can generate an infinite edge. The position of the vertex on the line is arbitrary in this case, only its transverse position matters. The direction of the edge is given by the line generator.

For example, take the half-plane:

```sage
half_plane = Polyhedron(ieqs=[(0,1,0)])
half_plane.Hrepresentation()
(An inequality (1, 0) x + 0 >= 0,)
```

Its (non-unique) V-representation consists of a vertex, a ray, and a line. The only edge is spanned by the vertex and the line generator, so they are adjacent:

```sage
half_plane.Vrepresentation()
(A line in the direction (0, 1), A ray in the direction (1, 0), A vertex at (0, 0))
```

```sage
half_plane.vertex_adjacency_matrix()
[0 0 1]
[0 0 0]
[1 0 0]
```

In one dimension higher, that is for a half-space in 3 dimensions, there is no one-dimensional face. Hence nothing is adjacent:

```sage
Polyhedron(ieqs=[(0,1,0,0)]).vertex_adjacency_matrix()
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
[0 0 0 0]
```

**EXAMPLES:**

In a bounded polygon, every vertex has precisely two adjacent ones:

```sage
P = Polyhedron(vertices=[(0, 1), (1, 0), (3, 0), (4, 1)])
for v in P.Vrep_generator():
    print("{} {}", format(P.adjacency_matrix().row(v.index()), v))
(0, 1, 0, 1) A vertex at (0, 1)
(1, 0, 1, 0) A vertex at (1, 0)
(0, 1, 0, 1) A vertex at (3, 0)
(1, 0, 1, 0) A vertex at (4, 1)
```

If the V-representation of the polygon contains vertices and one ray, then each V-representation object is adjacent to two V-representation objects:

```sage
P = Polyhedron(vertices=[(0, 1), (1, 0), (3, 0), (4, 1)],
    rays=[(0,1)])
for v in P.Vrep_generator():
    print("{} {}", format(P.adjacency_matrix().row(v.index()), v))
(0, 1, 0, 1) A ray in the direction (0, 1)
(1, 0, 1, 0) A vertex at (0, 1)
(0, 1, 0, 1) A vertex at (1, 0)
(0, 0, 1, 0) A vertex at (3, 0)
(1, 0, 1, 0) A vertex at (4, 1)
```

If the V-representation of the polygon contains vertices and two distinct rays, then each vertex is adjacent to two V-representation objects (which can now be vertices or rays). The two rays are not adjacent to each other:
```python
sage: P = Polyhedron(vertices=[(0, 1), (1, 0), (3, 0), (4, 1)],
                 rays=[(0,1), (1,1)])

sage: for v in P.Vrep_generator():
    print("{} {}\n{} {}\n{} {}\n{} {}\n{} {}").format(P.adjacency_matrix().row(v.index()), v)
(0, 1, 0, 0, 0) A ray in the direction (0, 1)
(1, 0, 1, 0, 0) A vertex at (0, 1)
(0, 1, 0, 0, 1) A vertex at (1, 0)
(0, 0, 0, 0, 1) A ray in the direction (1, 1)
(0, 0, 1, 1, 0) A vertex at (3, 0)
```

The vertex adjacency matrix has base ring integers. This way one can express various counting questions:

```python
sage: P = polytopes.cube()
sage: Q = P.stack(P.faces(2)[0])
sage: M = Q.vertex_adjacency_matrix()
sage: sum(M)
(4, 4, 3, 3, 4, 4, 4, 3, 3)
sage: G = Q.vertex_graph()
# needs sage.graphs
sage: G.degree()
# needs sage.graphs
[4, 4, 3, 3, 4, 4, 4, 3, 3]
```

### 2.6.5 Base class for polyhedra: Graph-theoretic methods

Define methods relying on `sage.graphs`.

```python
class sage.geometry.polyhedron.base4.Polyhedron_base4(parent, Vrep, Hrep,
                                      Vrep_minimal=None,
                                      Hrep_minimal=None,
                                      pref_rep=None, mutable=False,
                                      **kwds)

Bases: Polyhedron_base3

Methods relying on sage.graphs.

See sage.geometry.polyhedron.base.Polyhedron_base.

combinatorial_automorphism_group(vertex_graph_only=False)

Computes the combinatorial automorphism group.

If `vertex_graph_only` is True, the automorphism group of the vertex-edge graph of the polyhedron
is returned. Otherwise the automorphism group of the vertex-facet graph, which is isomorphic to the auto-
morphism group of the face lattice is returned.

INPUT:

- `vertex_graph_only` – boolean (default: False); whether to return the automorphism group of
  the vertex edges graph or of the lattice

OUTPUT:

A PermutationGroup that is isomorphic to the combinatorial automorphism group is returned.

- if `vertex_graph_only` is True: The automorphism group of the vertex-edge graph of the poly-
  hedron
```

2.6. Base classes for polyhedra 557
• if `vertex_graph_only` is `False` (default): The automorphism group of the vertex-facet graph of the polyhedron, see `vertex_facet_graph()`. This group is isomorphic to the automorphism group of the face lattice of the polyhedron.

**NOTE:**

Depending on `vertex_graph_only`, this method returns groups that are not necessarily isomorphic, see the examples below.

**See also:**

`is_combinatorially_isomorphic()`, `graph()`, `vertex_facet_graph()`.

**EXAMPLES:**

```python
sage: quadrangle = Polyhedron(vertices=[(0,0),(1,0),(0,1),(2,3)])
sage: quadrangle.combinatorial_automorphism_group().is_isomorphic(groups.permutation.Dihedral(4))
True
sage: quadrangle.restricted_automorphism_group()
Permutation Group with generators [()]

Permutations of the vertex graph only exchange vertices with vertices:

```python
sage: P = Polyhedron(vertices=[(1,0), (1,1)], rays=[(1,0)])
sage: P.combinatorial_automorphism_group(vertex_graph_only=True)
Permutation Group with generators [(A vertex at (1,0),A vertex at (1,1))]

This shows an example of two polytopes whose vertex-edge graphs are isomorphic, but their face lattices are not isomorphic:

```python
sage: Q = polytopes.cyclic_polytope(4,8)
sage: C = Q.combinatorial_automorphism_group(vertex_graph_only=True).is_isomorphic(Q.combinatorial_automorphism_group(vertex_graph_only=True))
True
sage: C.combinatorial_automorphism_group(vertex_graph_only=False).is_isomorphic(Q.combinatorial_automorphism_group(vertex_graph_only=False))
False
```
The automorphism group of the face lattice is isomorphic to the combinatorial automorphism group:

```
sage: CG = C.hasse_diagram().automorphism_group()
sage: C.combinatorial_automorphism_group().is_isomorphic(CG)
True
sage: QG = Q.hasse_diagram().automorphism_group()
sage: Q.combinatorial_automorphism_group().is_isomorphic(QG)
True
```

### face_lattice()

Return the face-lattice poset.

**OUTPUT:**

A `FinitePoset`. Elements are given as `PolyhedronFace`.

In the case of a full-dimensional polytope, the faces are pairs (vertices, inequalities) of the spanning vertices and corresponding saturated inequalities. In general, a face is defined by a pair (V-rep. objects, H-rep. objects). The V-representation objects span the face, and the corresponding H-representation objects are those inequalities and equations that are saturated on the face.

The bottom-most element of the face lattice is the “empty face”. It contains no V-representation object. All H-representation objects are incident.

The top-most element is the “full face”. It is spanned by all V-representation objects. The incident H-representation objects are all equations and no inequalities.

In the case of a full-dimensional polytope, the “empty face” and the “full face” are the empty set (no vertices, all inequalities) and the full polytope (all vertices, no inequalities), respectively.

**ALGORITHM:**

See `sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator`.

**Note:** The face lattice is not cached, as long as this creates a memory leak, see [github issue #28982](https://github.com/sagemath/sage/issues/28982).

**EXAMPLES:**

```
sage: square = polytopes.hypercube(2)
sage: fl = square.face_lattice(); fl
Finite lattice containing 10 elements
sage: list(f.ambient_V_indices() for f in fl)
[(0,), (1,), (0, 1), (2,), (1, 2), (3,), (0, 3), (2, 3), (0, 1, 2, 3)]
sage: poset_element = fl[5]
sage: a_face = poset_element
sage: a_face
A 1-dimensional face of a Polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: a_face.ambient_V_indices()
(1, 2)
sage: set(a_face.ambient_Vrepresentation()) == set([square.Vrepresentation(1), square.Vrepresentation(2)])
True
sage: a_face.ambient_Hrepresentation()
(A vertex at (1, 1), A vertex at (-1, 1))
sage: a_face.ambient_Hrepresentation()
(An inequality (0, -1) x + 1 >= 0,)
```

A more complicated example:
sage: c5_10 = Polyhedron(vertices = [[i,i^2,i^3,i^4,i^5] for i in range(1,˓→11)])
sage: c5_10_fl = c5_10.face_lattice()
sage: [len(x) for x in c5_10_fl.level_sets()]
[1, 10, 45, 100, 105, 42, 1]

Note that if the polyhedron contains lines then there is a dimension gap between the empty face and the first non-empty face in the face lattice:

sage: line = Polyhedron(vertices=[[0,]], lines=[[1,]])
sage: [ fl.dim() for fl in line.face_lattice() ]
[-1, 1]

flag_f_vector(*args)

Return the flag f-vector.

For each $-1 < i_0 < \cdots < i_n < d$ the flag f-vector counts the number of flags $F_0 \subset \cdots \subset F_n$ with $F_j$ of dimension $i_j$ for each $0 \leq j \leq n$, where $d$ is the dimension of the polyhedron.

INPUT:

* args – integers (optional); specify an entry of the flag-f-vector; must be an increasing sequence of integers

OUTPUT:

* a dictionary, if no arguments were given
* an Integer, if arguments were given

EXAMPLES:

Obtain the entire flag-f-vector:

sage: P = polytopes.twenty_four_cell()
sage: P.flag_f_vector()
{(-1,): 1, (0,): 24, (0, 1): 192, (0, 1, 2): 576, (0, 1, 2, 3): 1152, (0, 1, 3): 576, (0, 2): 288, (0, 2, 3): 576, (0, 3): 144, (1,): 96, (1, 2): 288, (1, 2, 3): 576, (1, 3): 288, (2,): 96, (2, 3): 192, (3,): 24, (4,): 1}

Specify an entry:

sage: P.flag_f_vector(0,3)
144
sage: P.flag_f_vector(2)
96
Leading $-1$ and trailing entry of dimension are allowed:

```python
sage: P.flag_f_vector(-1,0,3)
144
sage: P.flag_f_vector(-1,0,3,4)
144
```

One can get the number of trivial faces:

```python
sage: P.flag_f_vector(-1)
1
sage: P.flag_f_vector(4)
1
```

Polyhedra with lines, have 0 entries accordingly:

```python
sage: P = (Polyhedron(lines=[[1]]) * polytopes.cross_polytope(3))
sage: P.flag_f_vector()
{(-1,): 1,
 (0, 1): 0,
 (0, 1, 2): 0,
 (0, 1, 3): 0,
 (0, 2): 0,
 (0, 2, 3): 0,
 (0, 3): 0,
 (0,): 0,
 (1, 2): 24,
 (1, 2, 3): 48,
 (1, 3): 24,
 (1,): 6,
 (2, 3): 24,
 (2,): 12,
 (3,): 8,
 4: 1}
```

If the arguments are not strictly increasing or out of range, a key error is raised:

```python
sage: P.flag_f_vector(-1,0,3,6)
Traceback (most recent call last):
  ...
KeyError: (0, 3, 6)
sage: P.flag_f_vector(-1,3,0)
Traceback (most recent call last):
  ...
KeyError: (3, 0)
```

**graph (**kwds**)

Return a graph in which the vertices correspond to vertices of the polyhedron, and edges to edges.

**INPUT:**

- names – boolean (default: True); if False, then the nodes of the graph are labeled by the indices of the Vrepresentation
- algorithm – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
Note: The graph of a polyhedron with lines has no vertices, as the polyhedron has no vertices (0-faces). The method `vertices()` returns the defining points in this case.

EXAMPLES:

```sage
g3 = polytopes.hypercube(3).vertex_graph(); g3
Graph on 8 vertices
sage: g3.automorphism_group().cardinality()
48
sage: s4 = polytopes.simplex(4).vertex_graph(); s4
Graph on 5 vertices
sage: s4.is_eulerian()
True
```

The graph of an unbounded polyhedron is the graph of the bounded complex:

```sage
open_triangle = Polyhedron(vertices=[[1,0], [0,1]], ....:
...: rays =[[1,1]])
sage: open_triangle.vertex_graph()
Graph on 2 vertices
```

The graph of a polyhedron with lines has no vertices:

```sage
line = Polyhedron(lines=[[0,1]])
sage: line.vertex_graph()
Graph on 0 vertices
```

`hasse_diagram()`

Return the Hasse diagram of the face lattice of `self`.

This is the Hasse diagram of the poset of the faces of `self`.

OUTPUT: a directed graph

EXAMPLES:

```sage
# needs sage.rings.number_field
P = polytopes.regular_polygon(4).pyramid()
D = P.hasse_diagram(); D
Digraph on 20 vertices
sage: D.degree_polynomial()
x^5 + x^4*y + x*y^4 + y^5 + 4*x^3*y + 8*x^2*y^2 + 4*x*y^3
```

Faces of a mutable polyhedron are not hashable. Hence those are not suitable as vertices of the hasse diagram. Use the combinatorial polyhedron instead:

```sage
# needs sage.rings.number_field
sage: P = polytopes.regular_polygon(4).pyramid()
sage: parent = P.parent()
sage: parent = parent.change_ring(QQ, backend='ppl')
sage: Q = parent._element_constructor_(P, mutable=True)
sage: Q.hasse_diagram()
Traceback (most recent call last):
...
```

(continues on next page)
TypeError: mutable polyhedra are unhashable
sage: C = Q.combinatorial_polyhedron()
sage: D = C.hasse_diagram()
sage: set(D.vertices(sort=False)) == set(range(20))
True
sage: def index_to_combinatorial_face(n):
    return C.face_by_face_lattice_index(n)
sage: D.relabel(index_to_combinatorial_face, inplace=True)
sage: D.vertices(sort=True)
[A -1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 0-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 1-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 2-dimensional face of a 3-dimensional combinatorial polyhedron,
 A 3-dimensional face of a 3-dimensional combinatorial polyhedron]
sage: D.degree_polynomial()
x^5 + x^4*y + x*y^4 + y^5 + 4*x^3*y + 8*x^2*y^2 + 4*x*y^3

is_combinatorially_isomorphic(other, algorithm='bipartite_graph')

Return whether the polyhedron is combinatorially isomorphic to another polyhedron.

We only consider bounded polyhedra. By definition, they are combinatorially isomorphic if their face lattices are isomorphic.

INPUT:

- other – a polyhedron object
- algorithm (default = 'bipartite_graph') – the algorithm to use. The other possible value is 'face_lattice'.

OUTPUT:

- True if the two polyhedra are combinatorially isomorphic
- False otherwise

See also:

combinatorial_automorphism_group(), vertex_facet_graph().

REFERENCES:

For the equivalence of the two algorithms see [KK1995], p. 877-878

EXAMPLES:

The square is combinatorially isomorphic to the 2-dimensional cube:
All the faces of the 3-dimensional permutahedron are either combinatorially isomorphic to a square or a hexagon:

```python
sage: H = polytopes.regular_polygon(6)  # needs sage.rings.number_field
sage: S = polytopes.hypercube(2)
sage: P = polytopes.permutahedron(4)
sage: all(F.as_polyhedron().is_combinatorially_isomorphic(S)  # needs sage.rings.number_field
....:     or F.as_polyhedron().is_combinatorially_isomorphic(H)
....:     for F in P.faces(2))
```

Checking that a regular simplex intersected with its reflection through the origin is combinatorially isomorphic to the intersection of a cube with a hyperplane perpendicular to its long diagonal:

```python
sage: def simplex_intersection(k):
....:     S1 = Polyhedron([vector(v)-vector(polytopes.simplex(k).center()) for v in polytopes.simplex(k).vertices_list()])
....:     S2 = Polyhedron([-vector(v) for v in S1.vertices_list()])
....:     return S1.intersection(S2)
sage: def cube_intersection(k):
....:     C = polytopes.hypercube(k+1)
....:     H = Polyhedron(eqns=[[0]+[1 for i in range(k+1)]])
....:     return C.intersection(H)
sage: [simplex_intersection(k).is_combinatorially_isomorphic(cube_intersection(k)) for k in range(2,5)]
[True, True, True]
sage: simplex_intersection(2).is_combinatorially_isomorphic(polytopes.regular_polygon(6))  # needs sage.rings.number_field
True
sage: simplex_intersection(3).is_combinatorially_isomorphic(polytopes.octahedron())
```

Two polytopes with the same \( f \)-vector, but different combinatorial types:

```python
sage: P = Polyhedron([[-605520/1525633, -605520/1525633, -1261500/1525633, -52200/1525633, 11833/1525633],
    [-720/1769, -600/1769, 1500/1769, 0, -31/1769],
    [-216/749, 240/749, -240/749, -432/749, 461/749],
    [-50/181, 50/181, 60/181, -100/181, -119/181],
    [-32/51, -16/51, -4/51, 12/17, 1/17],
    [1, 0, 0, 0, 0], [16/129, 128/129, 0, 0, 1/129],
    [64/267, -128/267, 24/89, -128/267, 57/89],
    [1200/3953, -1200/3953, -1440/3953, -360/3953, -3247/3953],
    [1512/5597, 1512/5597, 588/5597, 4704/5597, 2069/5597]])
sage: C = polytopes.cyclic_polytope(5,10)
sage: C.f_vector() == P.f_vector(); C.f_vector()
(1, 10, 45, 100, 105, 42, 1)
```

(continues on next page)
sage: S = polytopes.simplex(3)
sage: S = S.face_truncation(S.faces(0)[3])
sage: S = S.face_truncation(S.faces(0)[4])
sage: S = S.face_truncation(S.faces(0)[5])
sage: T = polytopes.simplex(3)
sage: T = T.face_truncation(T.faces(0)[3])
sage: T = T.face_truncation(T.faces(0)[4])
sage: T = T.face_truncation(T.faces(0)[4])
sage: T.is_combinatorially_isomorphic(S)
False
sage: T.f_vector(), S.f_vector()
((1, 10, 15, 7, 1), (1, 10, 15, 7, 1))
sage: C = polytopes.hypercube(5)
sage: C.is_combinatorially_isomorphic(C)
True
sage: C.is_combinatorially_isomorphic(C, algorithm='magic')
Traceback (most recent call last):
  ... RuntimeError: 'algorithm' must be 'bipartite graph' or 'face_lattice'
sage: G = Graph()
sage: C.is_combinatorially_isomorphic(G)
Traceback (most recent call last):
  ... AssertionError: input 'other' must be a polyhedron
sage: H = Polyhedron(eqns=[[0,1,1,1,1]]); H
A 3-dimensional polyhedron in QQ^4 defined as the convex hull of 1 vertex and 3 lines
sage: C.is_combinatorially_isomorphic(H)
Traceback (most recent call last):
  ... AssertionError: polyhedron 'other' must be bounded

is_self_dual()
Return whether the polytope is self-dual.
A polytope is self-dual if its face lattice is isomorphic to the face lattice of its dual polytope.

EXAMPLES:

sage: polytopes.simplex().is_self_dual()
True
sage: polytopes.twenty_four_cell().is_self_dual()
True
sage: polytopes.cube().is_self_dual()
False
sage: polytopes.hypersimplex(5,2).is_self_dual() # needs sage.combinat
False
sage: P = Polyhedron(vertices=[[1/2, 1/3]], rays=[[1, 1]]).is_self_dual()
Traceback (most recent call last):
  ... ValueError: polyhedron has to be compact

restricted_automorphism_group(output='abstract')
Return the restricted automorphism group.

First, let the linear automorphism group be the subgroup of the affine group $AGL(d, \mathbb{R}) = GL(d, \mathbb{R}) \ltimes \mathbb{R}^d$ preserving the $d$-dimensional polyhedron. The affine group acts in the usual way $\vec{x} \mapsto A\vec{x} + b$ on the ambient space.

The restricted automorphism group is the subgroup of the linear automorphism group generated by permutations of the generators of the same type. That is, vertices can only be permuted with vertices, ray generators with ray generators, and line generators with line generators.

For example, take the first quadrant

$$ Q = \left\{ (x, y) \middle| x \geq 0, \ y \geq 0 \right\} \subset \mathbb{Q}^2 $$

Then the linear automorphism group is

$$ \text{Aut}(Q) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{Q}_{>0} \right\} \subset GL(2, \mathbb{Q}) \subset E(d) $$

Note that there are no translations that map the quadrant $Q$ to itself, so the linear automorphism group is contained in the general linear group (the subgroup of transformations preserving the origin). The restricted automorphism group is

$$ \text{Aut}(Q) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}_2 $$

**INPUT:**

- output – how the group should be represented:
  - "abstract" (default) – return an abstract permutation group without further meaning.
  - "permutation" – return a permutation group on the indices of the polyhedron generators. For example, the permutation $(0,1)$ would correspond to swapping self.Vrepresentation(0) and self.Vrepresentation(1).
  - "matrix" – return a matrix group representing affine transformations. When acting on affine vectors, you should append a 1 to every vector. If the polyhedron is not full dimensional, the returned matrices act as the identity on the orthogonal complement of the affine space spanned by the polyhedron.
  - "matrixlist" – like matrix, but return the list of elements of the matrix group. Useful for fields without a good implementation of matrix groups or to avoid the overhead of creating the group.

**OUTPUT:**

- For output="abstract" and output="permutation": a PermutationGroup.
- For output="matrix": a MatrixGroup().
- For output="matrixlist": a list of matrices.

**REFERENCES:**

- [BSS2009]

**EXAMPLES:**

A cross-polytope example:
sage: P = polytopes.cross_polytope(3)
sage: P.restricted_automorphism_group() == PermutationGroup([[3,4], [2,3], (4,5), [1,2], (5,6), [1,6]])
True
sage: P.restricted_automorphism_group(output="permutation") ==
PermutationGroup([[2,3], [1,2], [3,4], [1,4], [0,1], [4,5], [0,5]])
True
sage: mgens = [[[1,0,0,0], [0,1,0,0], [0,0,-1,0], [0,0,0,1]], [[1,0,0,0], [0,0,1,0], [0,1,0,0], [0,0,0,1]], [[1,0,0,0], [0,0,1,0], [0,1,0,0], [0,0,0,1]], [[0,1,0,0], [1,0,0,0], [0,0,1,0], [0,0,0,1]]]
We test groups for equality in a fool-proof way; they can have different generators, etc:
sage: poly_g = P.restricted_automorphism_group(output="matrix")
sage: matrix_g = MatrixGroup([matrix(QQ,t) for t in mgens])
sage: all(t.matrix() in poly_g for t in matrix_g.gens())
True
sage: all(t.matrix() in matrix_g for t in poly_g.gens())
True

24-cell example:
sage: P24 = polytopes.twenty_four_cell()
sage: AutP24 = P24.restricted_automorphism_group()
sage: PermutationGroup(
....:   (1,20,2,24,5,23) (3,18,10,19,4,14) (6,21,11,22,7,15) (8,12,16,17,13,9)
....:   (1,21,8,24,4,17) (2,11,6,15,9,13) (3,20) (5,22) (10,16,12,23,14,19))
....: ).is_isomorphic(AutP24)
True
sage: AutP24.order()
1152

Here is the quadrant example mentioned in the beginning:
sage: P = Polyhedron(rays=[(1,0),(0,1)])
sage: P.Vrepresentation()
(A vertex at (0, 0), A ray in the direction (0, 1), A ray in the direction (1, 0))
sage: P.restricted_automorphism_group(output="permutation")
Permutation Group with generators [(1,2)]

Also, the polyhedron need not be full-dimensional:
sage: P = Polyhedron(vertices=[(1,2,3,4,5),(7,8,9,10,11)])
sage: P.restricted_automorphism_group()
Permutation Group with generators [(1,2)]
sage: G = P.restricted_automorphism_group(output="matrixlist")
sage: G
([1  0  0  0  0] [-87/55  -82/55  -2/5  38/55  98/55  12/11]
[0  1  0  0  0] [-142/55  -27/55  -2/5  38/55  98/55  12/11]
[0  0  1  0  0] [-142/55  -82/55  3/5  38/55  98/55  12/11]
[0  0  0  1  0] [-142/55  -82/55  -2/5  93/55  98/55  12/11]
[0  0  0  0  1] [-142/55  -82/55  -2/5  38/55  153/55  12/11]
[0  0  0  0  0], [ 0  0  0  0  0  0  1])
sage: g = AffineGroup(5, QQ)(G[1])
(continues on next page)
Affine transformations do not change the restricted automorphism group. For example, any non-degenerate triangle has the dihedral group with 6 elements, $D_6$, as its automorphism group:

```
sage: initial_points = [vector([1,0]), vector([0,1]), vector([-2,-1])]
sage: points = initial_points
sage: Polyhedron(vertices=points).restricted_automorphism_group()
Permutation Group with generators [(2,3), (1,2)]
sage: points = [pt - initial_points[0] for pt in initial_points]
sage: Polyhedron(vertices=points).restricted_automorphism_group()
Permutation Group with generators [(2,3), (1,2)]
sage: points = [pt - initial_points[1] for pt in initial_points]
sage: Polyhedron(vertices=points).restricted_automorphism_group()
Permutation Group with generators [(2,3), (1,2)]
sage: points = [pt - 2*initial_points[1] for pt in initial_points]
sage: Polyhedron(vertices=points).restricted_automorphism_group()
Permutation Group with generators [(2,3), (1,2)]
```

The `output="matrixlist"` can be used over fields without a complete implementation of matrix groups:

```
sage: P = polytopes.dodecahedron(); P
A 3-dimensional polyhedron in (Number Field in sqrt5 with defining polynomial $x^2 - 5$ with sqrt5 = 2.236067977499790?)^3 defined as the convex hull of 20 vertices
sage: G = P.restricted_automorphism_group(output="matrixlist")
sage: len(G)
120
```

Floating-point computations are supported with a simple fuzzy zero implementation:

```
sage: P = Polyhedron(vertices=[(1/3,0,0,1),(0,1/4,0,1),(0,0,1/5,1)], base_ring=RDF)
sage: P.restricted_automorphism_group()
Permutation Group with generators [(2,3), (1,2)]
sage: len(P.restricted_automorphism_group(output="matrixlist"))
6
```

**vertex_digraph**(f, increasing=True)

Return the directed graph of the polyhedron according to a linear form.
The underlying undirected graph is the graph of vertices and edges.

INPUT:

• \( f \) – a linear form. The linear form can be provided as:
  – a vector space morphism with one-dimensional codomain, (see `sage.modules.vector_space_morphism.linear_transformation()` and `sage.modules.vector_space_morphism.VectorSpaceMorphism`)
  – a vector; in this case the linear form is obtained by duality using the dot product: \( f(v) = v \cdot \text{dot_product}(f) \).

• increasing – boolean (default True) whether to orient edges in the increasing or decreasing direction.

By default, an edge is oriented from \( v \) to \( w \) if \( f(v) \leq f(w) \).

If \( f(v) = f(w) \), then two opposite edges are created.

EXAMPLES:

```python
sage: penta = Polyhedron([[0,0],[1,0],[0,1],[1,2],[3,2]])
sage: G = penta.vertex_digraph(vector([1,1])); G
Digraph on 5 vertices
sage: G.sinks()
[A vertex at (3, 2)]
sage: A = matrix(ZZ, [[1], [-1]])
sage: f = linear_transformation(A)
sage: G = penta.vertex_digraph(f); G
Digraph on 5 vertices
sage: G.is_directed_acyclic()
False
```

See also:

`vertex_graph()`

`vertex_facet_graph(labels=True)`

Return the vertex-facet graph.

This function constructs a directed bipartite graph. The nodes of the graph correspond to the vertices of the polyhedron and the facets of the polyhedron. There is a directed edge from a vertex to a face if and only if the vertex is incident to the face.

INPUT:

• labels – boolean (default: True); decide how the nodes of the graph are labelled. Either with the original vertices/facets of the Polyhedron or with integers.

OUTPUT:

• a bipartite DiGraph. If labels is True, then the nodes of the graph will actually be the vertices and facets of self, otherwise they will be integers.

See also:

`combinatorial_automorphism_group()`, `is_combinatorially_isomorphic()`.

EXAMPLES:
sage: P = polytopes.cube()
sage: G = P.vertex_facet_graph(); G
Digraph on 14 vertices
sage: G.vertices(sort=True, key=lambda v: str(v))
[A vertex at (-1, -1, -1),
 A vertex at (-1, -1, 1),
 A vertex at (-1, 1, -1),
 A vertex at (-1, 1, 1),
 A vertex at (1, -1, -1),
 A vertex at (1, -1, 1),
 A vertex at (1, 1, -1),
 A vertex at (1, 1, 1),
 An inequality (-1, 0, 0) x + 1 >= 0,
 An inequality (0, -1, 0) x + 1 >= 0,
 An inequality (0, 0, -1) x + 1 >= 0,
 An inequality (0, 0, 1) x + 1 >= 0,
 An inequality (1, 0, 0) x + 1 >= 0]
sage: G.automorphism_group().is_isomorphic(P.hasse_diagram().automorphism_group())
True
sage: O = polytopes.octahedron(); O
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
sage: O.vertex_facet_graph()
Digraph on 14 vertices
sage: H = O.vertex_facet_graph()
sage: G.is_isomorphic(H)
False
sage: G2 = copy(G)
sage: G2.reverse_edges(G2.edges(sort=True))
sage: G2.is_isomorphic(H)
True

vertex_graph(**kwds)

Return a graph in which the vertices correspond to vertices of the polyhedron, and edges to edges.

INPUT:

- **names** – boolean (default: True); if False, then the nodes of the graph are labeled by the indices of the Vrepresentation
- **algorithm** – string (optional); specify whether the face generator starts with facets or vertices:
  - 'primal' – start with the facets
  - 'dual' – start with the vertices
  - None – choose automatically

Note: The graph of a polyhedron with lines has no vertices, as the polyhedron has no vertices (0-faces). The method `vertices()` returns the defining points in this case.

EXAMPLES:

sage: g3 = polytopes.hypercube(3).vertex_graph(); g3
Graph on 8 vertices
sage: g3.automorphism_group().cardinality()
(continues on next page)
The graph of an unbounded polyhedron is the graph of the bounded complex:

\[
sage:\text{open\_triangle} = \text{Polyhedron(}
\text{vertices=}[[1,0], [0,1]], \text{rays=}=[[1,1]])\text{; open\_triangle.vertex\_graph()}
\]
Graph on 2 vertices

The graph of a polyhedron with lines has no vertices:

\[
sage:\text{line} = \text{Polyhedron(}
\text{lines=}=[[0,1]])
\]
\[
sage:\text{line.vertex\_graph()}
\]
Graph on 0 vertices

### 2.6.6 Base class for polyhedra: Methods for constructing new polyhedra

Except for affine hull and affine hull projection.

**class** `sage.geometry.polyhedron.base5.Polyhedron_base5` *(parent, Vrep, Hrep, Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False, **kwds)*

Bases: `Polyhedron_base4`

Methods constructing new polyhedra except for affine hull and affine hull projection.

See `sage.geometry.polyhedron.base.Polyhedron_base`.

**bipyramid()**

Return a polyhedron that is a bipyramid over the original.

**EXAMPLES:**

\[
sage:\text{octahedron} = \text{polytopes.cross\_polytope(3)}
\]
\[
sage:\text{cross\_poly\_4d} = \text{octahedron.bipyramid()}
\]
\[
sage:\text{cross\_poly\_4d.n\_vertices()}
\]

\[
q = \{\text{list(v) for v in cross\_poly\_4d.vertex\_generator()}; q
\]
\[
[[1, 0, 0, 0],
[0, 1, 0, 0],
[0, 0, 1, 0],
[0, 0, 0, 1],
[0, 0, 0, 1],
[0, 0, 1, 0],
[0, 0, 0, 1],
[0, 0, 0, 0]]
\]

Now check that bipyramids of cross-polytopes are cross-polytopes:
sage: q2 = [list(v) for v in polytopes.cross_polytope(4).vertex_generator()]
sage: [v in q2 for v in q]
[True, True, True, True, True, True, True, True]

**cartesian_product** *(other)*

Return the Cartesian product.

**INPUT:**

- *other* – a *Polyhedron_base*

**OUTPUT:**

The Cartesian product of *self* and *other* with a suitable base ring to encompass the two.

**EXAMPLES:**

```
sage: P1 = Polyhedron([[0], [1]], base_ring=ZZ)
sage: P2 = Polyhedron([[0], [1]], base_ring=QQ)
sage: P1.product(P2)
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices
```

The Cartesian product is the product in the semiring of polyhedra:

```
sage: P1 * P1
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: P1 * P2
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices
sage: P2 * P2
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices
sage: 2 * P1
A 1-dimensional polyhedron in ZZ^1 defined as the convex hull of 2 vertices
sage: P1 * 2.0
A 1-dimensional polyhedron in RDF^1 defined as the convex hull of 2 vertices
```

An alias is **cartesian_product()**:

```
sage: P1.cartesian_product(P2) == P1.product(P2)
True
```

**convex_hull** *(other)*

Return the convex hull of the set-theoretic union of the two polyhedra.

**INPUT:**

- *other* – a *Polyhedron*

**OUTPUT:**

The convex hull.

**EXAMPLES:**

```
sage: a_simplex = polytopes.simplex(3, project=True)
sage: verts = a_simplex.vertices()
sage: verts = [x[0]*3/5 + x[1]*4/5, -x[0]*4/5 + x[1]*3/5, x[2]] for x in verts]
sage: another_simplex = Polyhedron(verts=verts)
sage: simplex_union = a_simplex.convex_hull(another_simplex)
sage: simplex_union.n_vertices()
7
```
dilation(\textit{scalar})

Return the dilated (uniformly stretched) polyhedron.

INPUT:

\begin{itemize}
  \item \textit{scalar} – A scalar, not necessarily in base\_ring()
\end{itemize}

OUTPUT:

The polyhedron dilated by that scalar, possibly coerced to a bigger base ring.

EXAMPLES:

\begin{verbatim}sage: p = Polyhedron(\text{vertices}=[[t, t^2, t^3] \text{ for } t \text{ in } \text{srange}(2, 6)]) sage: next(p.vertex\_generator()) A vertex at (2, 4, 8) sage: p2 = p.dilation(2) sage: next(p2.vertex\_generator()) A vertex at (4, 8, 16) sage: p.dilation(2) == p * 2 True\end{verbatim}

direct\_sum(\textit{other})

Return the direct sum of \textit{self} and \textit{other}.

The direct sum of two polyhedrons is the subdirect sum of the two, when they have the origin in their interior. To avoid checking if the origin is contained in both, we place the affine subspace containing \textit{other} at the center of \textit{self}.

INPUT:

\begin{itemize}
  \item \textit{other} – a \texttt{Polyhedron\_base}
\end{itemize}

EXAMPLES:

\begin{verbatim}sage: P1 = Polyhedron(\text{faces}=[[1], [2]], \text{base\_ring}=\text{ZZ}) sage: P2 = Polyhedron(\text{faces}=[[3], [4]], \text{base\_ring}=\text{QQ}) sage: ds = P1.direct\_sum(P2);ds A 2-dimensional polyhedron in \text{QQ}\^2 defined as the convex hull of 4 vertices sage: ds.vertices() (A vertex at (1, 0),
  A vertex at (2, 0),
  A vertex at (3/2, -1/2),
  A vertex at (3/2, 1/2))\end{verbatim}

See also:

\begin{verbatim}join() subdirect\_sum()\end{verbatim}

face\_split(\textit{face})

Return the face splitting of the face \textit{face}.

Splitting a face correspond to the bipyramid (see bipyramid()) of \textit{self} where the two new vertices are placed above and below the center of \textit{face} instead of the center of the whole polyhedron. The two new vertices are placed in the new dimension at height $-1$ and $1$.

INPUT:

\begin{itemize}
  \item \textit{face} – a \texttt{Polyhedron\_Face} or a \texttt{Vertex}
\end{itemize}

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```
sage: # needs sage.rings.number_field
sage: pentagon = polytopes.regular_polygon(5)
sage: f = pentagon.faces(1)[0]
sage: fsplit_pentagon = pentagon.face_split(f)
sage: fsplit_pentagon.f_vector()
(1, 7, 14, 9, 1)
```

See also:

`one_point_suspension()`

```
def face_truncation(face, linear_coefficients=None, cut_frac=None):
    """Return a new polyhedron formed by truncating a face by an hyperplane."

    By default, the normal vector of the hyperplane used to truncate the polyhedron is obtained by taking the barycenter vector of the cone corresponding to the truncated face in the normal fan of the polyhedron. It is possible to change the direction using the option linear_coefficients.

    To determine how deep the truncation is done, the method uses the parameter cut_frac. By default it is equal to \( \frac{1}{3} \). Once the normal vector of the cutting hyperplane is chosen, the vertices of polyhedron are evaluated according to the corresponding linear function. The parameter \( \frac{1}{3} \) means that the cutting hyperplane is placed \( \frac{1}{3} \) of the way from the vertices of the truncated face to the next evaluated vertex.

    INPUT:

    * face -- a PolyhedronFace
    * linear_coefficients -- tuple of integer. Specifies the coefficient of the normal vector of the cutting hyperplane used to truncate the face. The default direction is determined using the normal fan of the polyhedron.
    * cut_frac -- number between 0 and 1. Determines where the hyperplane cuts the polyhedron. A value close to 0 cuts very close to the face, whereas a value close to 1 cuts very close to the next vertex (according to the normal vector of the cutting hyperplane). Default is \( \frac{1}{3} \).

    OUTPUT:

    A Polyhedron object, truncated as described above.

    EXAMPLES:

    `````````
A vertex at (-1, 1, -1),
A vertex at (-1, -1/3, -1),
A vertex at (-1/3, -1, -1),
A vertex at (-1, -1, -1/3))
sage: vertex_trunc2 = Cube.face_truncation(Cube.faces(0)[0], cut_frac=1/2)
sage: vertex_trunc2.f_vector()
(1, 10, 15, 7, 1)
sage: tuple(f.ambient_V_indices() for f in vertex_trunc2.faces(2))
((4, 5, 6, 7, 9),
 (0, 3, 4, 8, 9),
 (0, 1, 6, 7, 8),
 (7, 8, 9),
 (2, 3, 4, 5),
 (1, 2, 5, 6),
 (0, 1, 2, 3))
sage: vertex_trunc2.vertices()
(A vertex at (1, -1, -1),
 A vertex at (1, 1, -1),
 A vertex at (1, 1, 1),
 A vertex at (1, -1, 1),
 A vertex at (-1, -1, 1),
 A vertex at (-1, 1, 1),
 A vertex at (-1, 1, -1),
 A vertex at (-1, 0, -1),
 A vertex at (0, -1, -1),
 A vertex at (-1, -1, 0))
sage: vertex_trunc3 = Cube.face_truncation(Cube.faces(0)[0], cut_frac=0.3)
sage: vertex_trunc3.vertices()
(A vertex at (-1.0, -1.0, 1.0),
 A vertex at (-1.0, 1.0, -1.0),
 A vertex at (-1.0, 1.0, 1.0),
 A vertex at (1.0, 1.0, -1.0),
 A vertex at (1.0, 1.0, 1.0),
 A vertex at (-1.0, -1.0, 1.0),
 A vertex at (1.0, -1.0, 1.0),
 A vertex at (-0.4, -1.0, -1.0),
 A vertex at (-1.0, -0.4, -1.0),
 A vertex at (-1.0, -1.0, -0.4))
sage: edge_trunc = Cube.face_truncation(Cube.faces(1)[11])
sage: edge_trunc.f_vector()
(1, 10, 15, 7, 1)
sage: tuple(f.ambient_V_indices() for f in edge_trunc.faces(2))
((0, 5, 6, 7),
 (1, 4, 5, 6, 8),
 (6, 7, 8, 9),
 (0, 2, 3, 7, 9),
 (1, 2, 8, 9),
 (0, 3, 4, 5),
 (1, 2, 3, 4))
sage: face_trunc = Cube.face_truncation(Cube.faces(2)[2])
sage: face_trunc.vertices()
(A vertex at (1, -1, -1),
 A vertex at (1, 1, -1),
 A vertex at (1, 1, 1),
 A vertex at (1, -1, 1),
 A vertex at (-1/3, -1, 1),
 A vertex at (-1/3, 1, 1),
A vertex at (-1/3, 1, -1),
A vertex at (-1/3, -1, -1))
sage: face_trunc.face_lattice().is_isomorphic(Cube.face_lattice()) # needs sage.combinat sage.graphs
True

**intersection** *(other)*

Return the intersection of one polyhedron with another.

**INPUT:**

- **other** – a Polyhedron

**OUTPUT:**

The intersection.

Note that the intersection of two \(\mathbb{Z}\)-polyhedra might not be a \(\mathbb{Z}\)-polyhedron. In this case, a \(\mathbb{Q}\)-polyhedron is returned.

**EXAMPLES:**

```
sage: cube = polytopes.hypercube(3)
sage: oct = polytopes.cross_polytope(3)
sage: cube.intersection(oct*2)
A 3-dimensional polyhedron in \(\mathbb{Z}^3\) defined as the convex hull of 12 vertices
```

As a shorthand, one may use:

```
sage: cube & oct*2
A 3-dimensional polyhedron in \(\mathbb{Z}^3\) defined as the convex hull of 12 vertices
```

The intersection of two \(\mathbb{Z}\)-polyhedra is not necessarily a \(\mathbb{Z}\)-polyhedron:

```
sage: P = Polyhedron([[0,0],[1,1]], base_ring=ZZ)
sage: Q = Polyhedron([[0,0],[1,0]], base_ring=ZZ)
sage: P.intersection(Q)
A 0-dimensional polyhedron in \(\mathbb{Q}^2\) defined as the convex hull of 1 vertex
```

**join** *(other)*

Return the join of self and other.

The join of two polyhedra is obtained by first placing the two objects in two non-intersecting affine subspaces \(V\) and \(W\) whose affine hull is the whole ambient space, and finally by taking the convex hull of their union. The dimension of the join is the sum of the dimensions of the two polyhedron plus 1.

**INPUT:**

- **other** – a polyhedron

**EXAMPLES:**

```
sage: P1 = Polyhedron([[0],[1]], base_ring=ZZ)
sage: P2 = Polyhedron([[0],[1]], base_ring=QQ)
sage: P1.join(P2)
```

(continues on next page)
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 4 vertices
sage: P1.join(P1)
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 4 vertices
sage: P2.join(P2)
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 4 vertices

An unbounded example:
sage: R1 = Polyhedron(rays=[[1]])
sage: R1.join(R1)

**lawrence_extension**(v)
Return the Lawrence extension of self on the point v.

Let P be a polytope and v be a vertex of P or a point outside P. The Lawrence extension of P on v is the convex hull of (v, 1), (v, 2) and (u, 0) for all vertices u in P other than v if v is a vertex.

**INPUT:**
- v – a vertex of self or a point outside it

**EXAMPLES:**

sage: P = polytopes.cube()
sage: P.lawrence_extension(P.vertices()[0])
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 9 vertices
sage: P.lawrence_extension([-1,-1,-1])
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 9 vertices

**REFERENCES:**
For more information, see Section 6.6 of [Zie2007].

**lawrence_polytope()**
Return the Lawrence polytope of self.

Let P be a d-polytope in R^r with n vertices. The Lawrence polytope of P is the polytope whose vertices are the columns of the following (r+n)-by-2n matrix.

\[
\begin{pmatrix}
V & V \\
I_n & 2I_n
\end{pmatrix},
\]

where V is the r-by-n vertices matrix of P.

**EXAMPLES:**

sage: P = polytopes.octahedron()
sage: L = P.lawrence_polytope(); L
A 9-dimensional polyhedron in ZZ^9 defined as the convex hull of 12 vertices
sage: V = P.vertices_list()
sage: for i, v in enumerate(V):
....: v = v + i*[0]
....: P = P.lawrence_extension(v)
sage: P == L
True

**REFERENCES:**
For more information, see Section 6.6 of [Zie2007].

**linear_transformation**(*linear_transf*, *new_base_ring=None*)

Return the linear transformation of self.

**INPUT:**

- *linear_transf* – a matrix, not necessarily in *base_ring()*
- *new_base_ring* – ring (optional); specify the new base ring; may avoid coercion failure

**OUTPUT:**

The polyhedron transformed by that matrix, possibly coerced to a bigger base ring.

**EXAMPLES:**

```python
sage: b3 = polytopes.Birkhoff_polytope(3)
sage: proj_mat = matrix([[0,1,0,0,0,0,0,0,0],
                      [0,0,0,1,0,0,0,0,0],
                      ..:
                      [0,0,0,0,1,0,0,0,0],[0,0,0,0,0,0,1,0,0]])
sage: b3_proj = proj_mat * b3; b3_proj
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 5 vertices
```

```python
sage: b3 = polytopes.Birkhoff_polytope(3)
sage: proj_mat = matrix([[0,1,0,0,0,0,0,0,0],
                      [0,0,0,1,0,0,0,0,0],
                      ..:
                      [0,0,0,0,1,0,0,0,0],[0,0,0,0,0,0,1,0,0]])
sage: b3_proj = proj_mat * b3; b3_proj
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 5 vertices
```

Specifying the new base ring may avoid coercion failure:

```python
sage: # needs sage.rings.number_field
sage: K.<sqrt2> = QuadraticField(2)
sage: L.<sqrt3> = QuadraticField(3)
sage: P = polytopes.cube()*sqrt2
sage: M = matrix([[sqrt3, 0, 0],
              [0, sqrt3, 0],
              [0, 0, 1]])
sage: P.linear_transformation(M, new_base_ring=K.composite_fields(L)[0])
A 3-dimensional polyhedron in
(Number Field in sqrt2sqrt3 with defining polynomial x^4 - 10*x^2 + 1
with sqrt2sqrt3 = 0.3178372451957823?)^3
defined as the convex hull of 8 vertices
```

Linear transformation without specified new base ring fails in this case:

```python
sage: M*P
˓
needs sage.rings.number_field
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *:
'Full MatrixSpace of 3 by 3 dense matrices over Number Field in sqrt3
with defining polynomial x^2 - 3 with sqrt3 = 1.732050807568878?' and
'Full MatrixSpace of 3 by 8 dense matrices over Number Field in sqrt2
with defining polynomial x^2 - 2 with sqrt2 = 1.414213562373095?'
```

**minkowski_difference**(*other*)

Return the Minkowski difference.
Minkowski subtraction can equivalently be defined via Minkowski addition (see `minkowski_sum()`) or as set-theoretic intersection via

\[ X \ominus Y = (X^c + Y)^c = \bigcap_{y \in Y} (X - y) \]

where superscript “c” means the complement in the ambient vector space. The Minkowski difference of convex sets is convex, and the difference of polyhedra is again a polyhedron. We only consider the case of polyhedra in the following. Note that it is not quite the inverse of addition. In fact:

- \((X + Y) - Y = X\) for any polyhedra \(X, Y\).
- \((X - Y) + Y \subseteq X\)
- \((X - Y) + Y = X\) if and only if \(Y\) is a Minkowski summand of \(X\).

**INPUT:**

- `other` – a `Polyhedron_base`

**OUTPUT:**

The Minkowski difference of `self` and `other`. Also known as Minkowski subtraction of `other` from `self`.

**EXAMPLES:**

```python
sage: X = polytopes.hypercube(3)
sage: Y = Polyhedron(vertices=[(0,0,0), (0,0,1), (0,1,0), (1,0,0)]) / 2
sage: (X+Y)-Y == X
True
sage: (X-Y)+Y < X
True
```

The polyhedra need not be full-dimensional:

```python
sage: X2 = Polyhedron(vertices=[(-1,-1,0), (1,-1,0), (-1,1,0), (1,1,0)])
sage: Y2 = Polyhedron(vertices=[(0,0,0), (0,1,0), (1,0,0)]) / 2
sage: (X2+Y2)-Y2 == X2
True
sage: (X2-Y2)+Y2 < X2
True
```

Minus sign is really an alias for `minkowski_difference()`

```python
sage: four_cube = polytopes.hypercube(4)
sage: four_simplex = Polyhedron(vertices=[[0, 0, 0, 1], [0, 0, 1, 0],
                                           [0, 1, 0, 0], [1, 0, 0, 0]])
sage: four_cube - four_simplex
A 4-dimensional polyhedron in QQ^4 defined as the convex hull of 16 vertices
sage: four_cube.minkowski_difference(four_simplex) == four_cube - four_simplex
True
```

Coercion of the base ring works:

```python
sage: poly_spam = Polyhedron([[3,4,5,2], [1,0,0,1], [0,0,0,0],
                            [0,4,3,2], [-3,-3,-3,-3]], base_ring=ZZ)
sage: poly_eggs = Polyhedron([[5,4,5,4], [-4,5,-4,5],
                            [4,-5,4,-5], [0,0,0,0]], base_ring=QQ) / 100
sage: poly_spam - poly_eggs
A 4-dimensional polyhedron in QQ^4 defined as the convex hull of 5 vertices
```
minkowski_sum(other)
Return the Minkowski sum.

Minkowski addition of two subsets of a vector space is defined as

\[ X \oplus Y = \bigcup_{y \in Y} (X + y) = \bigcup_{x \in X, y \in Y} (x + y) \]

See minkowski_difference() for a partial inverse operation.

INPUT:

- other – a Polyhedron_base

OUTPUT:

The Minkowski sum of self and other

EXAMPLES:

```
sage: X = polytopes.hypercube(3)
sage: Y = Polyhedron(plexes=[(0,0,0), (0,0,1/2), (0,1/2,0), (1/2,0,0)])
sage: X+Y
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 13 vertices
sage: four_cube = polytopes.hypercube(4)
sage: four_simplex = Polyhedron(plexes=[(0,0,0,1,0,0), (1,0,0,0,0,0), ...
.....:   (0,0,0,0,0,0), (1,0,0,0,0,0)])
sage: four_cube + four_simplex
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 36 vertices
sage: four_cube.minkowski_sum(four_simplex) == four_cube + four_simplex
True
sage: poly_spam = Polyhedron([[3,4,5,2], [1,0,0,1], [0,0,0,0], ...
.....:   [0,4,3,2], [-3,-3,-3,-3]], base_ring=ZZ)
sage: poly_eggs = Polyhedron([[5,4,5,4], [-4,5,-4,5], ...
.....:   [4,-5,4,-5], [0,0,0,0]], base_ring=QQ)
sage: poly_spam + poly_spam + poly_eggs
A 4-dimensional polyhedron in QQ^4 defined as the convex hull of 12 vertices
```

one_point_suspension(vertex)
Return the one-point suspension of self by splitting the vertex vertex.

The resulting polyhedron has one more vertex and its dimension increases by one.

INPUT:

- vertex – a Vertex of self

EXAMPLES:

```
sage: cube = polytopes.cube()
sage: v = cube.vertices()[0]
sage: ops_cube = cube.one_point_suspension(v)
sage: ops_cube.f_vector()
(1, 9, 24, 24, 9, 1)
sage: # needs sage.rings.number_field
sage: pentagon = polytopes.regular_polygon(5)
sage: v = pentagon.vertices()[0]
sage: ops_pentagon = pentagon.one_point_suspension(v)
```

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

sage: ops_pentagon.f_vector()
(1, 6, 12, 8, 1)

It works with a polyhedral face as well:

sage: vv = cube.faces(0)[1]
sage: ops_cube2 = cube.one_point_suspension(vv)
sage: ops_cube == ops_cube2
True

See also:

face_split()

polar (in_affine_span=False)
Return the polar (dual) polytope.

The original vertices are translated so that their barycenter is at the origin, and then the vertices are used as
the coefficients in the polar inequalities.

The polytope must be full-dimensional, unless in_affine_span is True. If in_affine_span is True, then the operation will be performed in the linear/affine span of the polyhedron (after translation).

EXAMPLES:

sage: p = Polyhedron(vertices=[[0,0,1], [0,1,0], [1,0,0], [0,0,0], [1,1,1]],
....:     base_ring=QQ); p
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 5 vertices
sage: p.polar()
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 6 vertices
sage: cube = polytopes.hypercube(3)
sage: octahedron = polytopes.cross_polytope(3)
sage: cube_dual = cube.polar()
sage: octahedron == cube_dual
True

in_affine_span somewhat ignores equations, performing the polar in the spanned subspace (after trans-
lating barycenter to origin):

sage: P = polytopes.simplex(3, base_ring=QQ)
sage: P.polar(in_affine_span=True)
A 3-dimensional polyhedron in QQ^4 defined as the convex hull of 4 vertices

Embedding the polytope in a higher dimension, commutes with polar in this case:

sage: point = Polyhedron([[0]])
sage: P = polytopes.cube().change_ring(QQ)
sage: (P*point).polar(in_affine_span=True) == P.polar()*point
True

prism()
Return a prism of the original polyhedron.

EXAMPLES:
sage: square = polytopes.hypercube(2)
sage: cube = square.prism()
sage: cube
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: hypercube = cube.prism()
sage: hypercube.n_vertices()
16

**product**(other)

Return the Cartesian product.

**INPUT:**

- other — a `Polyhedron_base`

**OUTPUT:**

The Cartesian product of self and other with a suitable base ring to encompass the two.

**EXAMPLES:**

sage: P1 = Polyhedron([[0], [1]], base_ring=ZZ)
sage: P2 = Polyhedron([[0], [1]], base_ring=QQ)
sage: P1.product(P2)
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices

The Cartesian product is the product in the semiring of polyhedra:

sage: P1 * P1
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: P1 * P2
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices
sage: P2 * P2
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 4 vertices
sage: 2 * P1
A 1-dimensional polyhedron in ZZ^1 defined as the convex hull of 2 vertices
sage: P1 * 2.0
A 1-dimensional polyhedron in RDF^1 defined as the convex hull of 2 vertices

An alias is `cartesian_product()`:

sage: P1.cartesian_product(P2) == P1.product(P2)
True

**pyramid()**

Return a polyhedron that is a pyramid over the original.

**EXAMPLES:**

sage: square = polytopes.hypercube(2); square
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: egyptian_pyramid = square.pyramid(); egyptian_pyramid
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 5 vertices
sage: egyptian_pyramid.n_vertices()
5
sage: for v in egyptian_pyramid.vertex_generator(): print(v)
A vertex at (0, -1, -1)
A vertex at (0, -1, 1)
A vertex at (0, 1, -1)
(continues on next page)
stack \( (\text{face}, \text{position}=\text{None}) \)

Return a new polyhedron formed by stacking onto a face. Stacking a face adds a new vertex located slightly outside of the designated face.

**INPUT:**

- \( \text{face} \) – a PolyhedronFace
- \( \text{position} \) – a positive number. Determines a relative distance from the barycenter of \( \text{face} \). A value close to 0 will place the new vertex close to the face and a large value further away. Default is 1. If the given value is too large, an error is returned.

**OUTPUT:**

A Polyhedron object

**EXAMPLES:**

```
sage: cube = polytopes.cube()
sage: square_face = cube.facets()[2]
sage: stacked_square = cube.stack(square_face)
sage: stacked_square.f_vector()
(1, 9, 16, 9, 1)

sage: edge_face = cube.faces(1)[3]
sage: stacked_edge = cube.stack(edge_face)
sage: stacked_edge.f_vector()
(1, 9, 17, 10, 1)

sage: cube.stack(cube.faces(0)[0])
Traceback (most recent call last):
  ...:
ValueError: cannot stack onto a vertex

sage: stacked_square_half = cube.stack(square_face, position=1/2)
sage: stacked_square_half.f_vector()
(1, 9, 16, 9, 1)

sage: stacked_square_large = cube.stack(square_face, position=10)
Traceback (most recent call last):
  ...:
ValueError: the chosen position is too large
```

(continues on next page)
It is possible to stack on unbounded faces:

```python
sage: Q = Polyhedron(vertices=[[0,1], [1,0]], rays=[[1,1]])
sage: E = Q.faces(1)
sage: Q.stack(E[0],1/2).Vrepresentation()
(A vertex at (0, 1),
 A vertex at (1, 0),
 A ray in the direction (1, 1),
 A vertex at (2, 0))
sage: Q.stack(E[1],1/2).Vrepresentation()
(A vertex at (0, 1),
 A vertex at (0, 2),
 A vertex at (1, 0),
 A ray in the direction (1, 1))
sage: Q.stack(E[2],1/2).Vrepresentation()
(A vertex at (0, 0),
 A vertex at (0, 1),
 A vertex at (1, 0),
 A ray in the direction (1, 1))
```

Stacking requires a proper face:

```python
sage: Q.stack(Q.faces(2)[0])
Traceback (most recent call last):
 ...
ValueError: can only stack on proper face
```

**subdirect_sum(other)**

Return the subdirect sum of `self` and `other`.

The subdirect sum of two polyhedron is a projection of the join of the two polytopes. It is obtained by placing the two objects in orthogonal subspaces intersecting at the origin.

INPUT:

- **other** — a `Polyhedron_base`

**EXAMPLES:**

```python
sage: P1 = Polyhedron([[1], [2]], base_ring=ZZ)
sage: P2 = Polyhedron([[3], [4]], base_ring=QQ)
sage: sds = P1.subdirect_sum(P2); sds
A 2-dimensional polyhedron in QQ^2
defined as the convex hull of 4 vertices
```
sage: sds.vertices()
(A vertex at (0, 3),
A vertex at (0, 4),
A vertex at (1, 0),
A vertex at (2, 0))

See also:

join() direct_sum()

translation(displacement)

Return the translated polyhedron.

INPUT:

• displacement – a displacement vector or a list/tuple of coordinates that determines a displacement vector

OUTPUT:

The translated polyhedron.

EXAMPLES:

sage: P = Polyhedron([[0, 0], [1, 0], [0, 1]], base_ring=ZZ)

sage: P.translation([2, 1])
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices

sage: P.translation(vector(QQ, [2, 1]))
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices

truncation(cut_frac=None)

Return a new polyhedron formed from two points on each edge between two vertices.

INPUT:

• cut_frac – integer, how deeply to cut into the edge. Default is $\frac{1}{3}$.

OUTPUT:

A Polyhedron object, truncated as described above.

EXAMPLES:

sage: cube = polytopes.hypercube(3)

sage: trunc_cube = cube.truncation()

sage: trunc_cube.n_vertices()
24

sage: trunc_cube.n_inequalities()
14

wedge(face, width=1)

Return the wedge over a face of the polytope self.

The wedge over a face $F$ of a polytope $P$ with width $w \neq 0$ is defined as:

$$(P \times \mathbb{R}) \cap \{a^\top x + |wx_{d+1}| \leq b\}$$

where $\{x|a^\top x = b\}$ is a supporting hyperplane defining $F$.

INPUT:

• face – a PolyhedronFace of self, the face which we take the wedge over
- width – a nonzero number (default: 1); specifies how wide the wedge will be

**OUTPUT:**

A (bounded) polyhedron

**EXAMPLES:**

```
sage: # needs sage.rings.number_field
sage: P_4 = polytopes.regular_polygon(4)
sage: W1 = P_4.wedge(P_4.faces(1)[0]); W1
A 3-dimensional polyhedron in AA^3 defined as the convex hull of 6 vertices
sage: triangular_prism = polytopes.regular_polygon(3).prism()
sage: W1.is_combinatorially_isomorphic(triangular_prism) #_
→needs sage.graphs
True
	sage: Q = polytopes.hypersimplex(4,2)
sage: W2 = Q.wedge(Q.faces(2)[7]); W2
A 4-dimensional polyhedron in QQ^5 defined as the convex hull of 9 vertices
sage: W2.vertices()
(A vertex at (1, 1, 0, 0, 1),
 A vertex at (1, 1, 0, 0, -1),
 A vertex at (1, 0, 1, 0, 1),
 A vertex at (1, 0, 1, 0, -1),
 A vertex at (1, 0, 0, 1, 1),
 A vertex at (1, 0, 0, 1, -1),
 A vertex at (0, 1, 1, 0, 0),
 A vertex at (0, 1, 1, 0, 1),
 A vertex at (0, 1, 0, 1, 0))

sage: W3 = Q.wedge(Q.faces(1)[11]); W3
A 4-dimensional polyhedron in QQ^5 defined as the convex hull of 10 vertices
sage: W3.vertices()
(A vertex at (1, 1, 0, 0, -2),
 A vertex at (1, 1, 0, 0, 2),
 A vertex at (1, 0, 1, 0, -2),
 A vertex at (1, 0, 1, 0, 2),
 A vertex at (1, 0, 0, 1, 1),
 A vertex at (1, 0, 0, 1, -1),
 A vertex at (0, 1, 0, 1, 0),
 A vertex at (0, 1, 0, 1, 1),
 A vertex at (0, 1, 1, 0, 0),
 A vertex at (0, 1, 1, 0, -1))

sage: C_3_7 = polytopes.cyclic_polytope(3,7)
sage: P_6 = polytopes.regular_polygon(6) #_
→needs sage.rings.number_field

```

**REFERENCES:**

For more information, see Chapter 15 of [HoDaCG17].
2.6.7 Base class for polyhedra: Methods for plotting and affine hull projection

```python
class sage.geometry.polyhedron.base6.Polyhedron_base6(parent, Vrep, Hrep,
    Vrep_minimal=None,
    Hrep_minimal=None,
    pref_rep=None, mutable=False, **kwds)
```

Bases: `Polyhedron_base5`

Methods related to plotting including affine hull projection.

**affine_hull** (*args, **kwds**)

Return the affine hull of `self` as a polyhedron.

**affine_hull_manifold** (*name=None, latex_name=None, start_index=0, ambient_space=None, ambient_chart=None, names=None, **kwds**)

Return the affine hull of `self` as a manifold.

If `self` is full-dimensional, it is just the ambient Euclidean space. Otherwise, it is a Riemannian submanifold of the ambient Euclidean space.

**INPUT:**

- `ambient_space` – a `EuclideanSpace` of the ambient dimension (default: the manifold of `ambient_chart`, if provided; otherwise, a new instance of `EuclideanSpace`).
- `ambient_chart` – a chart on `ambient_space`.
- `names` – names for the coordinates on the affine hull.
- **optional arguments accepted by** `affine_hull_projection()`.

The default chart is determined by the optional arguments of `affine_hull_projection()`.

**EXAMPLES:**

```python
sage: half_plane_in_space = Polyhedron(ieqs=[(0,1,0,0)], eqns=[(0,0,0,1)])
sage: half_plane_in_space.affine_hull().Hrepresentation()
(An equation (0, 0, 1) x + 0 == 0,)
sage: polytopes.cube().affine_hull().is_universe()
True
```

```python
sage: # needs sage.symbolic
sage: triangle = Polyhedron([(1, 0, 0), (0, 1, 0), (0, 0, 1)]); triangle
A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: A = triangle.affine_hull_manifold(name='A'); A
2-dimensional Riemannian submanifold A embedded in the Euclidean space E^3
sage: A.embedding().display()
A → E^3
  (x0, x1) ↦ (x, y, z) = (t0 + x0, t0 + x1, t0 - x0 - x1 + 1)
sage: A.embedding().inverse().display()
E^3 → A
  (x, y, z) ↦ (x0, x1) = (x, y)
sage: A.adapted_chart()
[Chart (E^3, (x0_E3, x1_E3, t0_E3))]
sage: A.normal().display()
n = 1/3*sqrt(3) e_x + 1/3*sqrt(3) e_y + 1/3*sqrt(3) e_z
```

(continues on next page)
sage: A.induced_metric()  # Need to call this before volume_form
Riemannian metric gamma on the
2-dimensional Riemannian submanifold A embedded in the Euclidean space E^3

sage: A.volume_form()
2-form eps_gamma on the
2-dimensional Riemannian submanifold A embedded in the Euclidean space E^3

Orthogonal version:

sage: A = triangle.affine_hull_manifold(name='A', orthogonal=True); A  # needs sage.symbolic
2-dimensional Riemannian submanifold A embedded in the Euclidean space E^3

sage: A.embedding().display()  # needs sage.symbolic
A → E^3
(x0, x1) ↦ (x, y, z) = (t0 - 1/2*x0 - 1/3*x1 + 1, t0 + 1/2*x0 - 1/3*x1, t0 + 2/3*x1)

sage: A.embedding().inverse().display()  # needs sage.symbolic
E^3 → A
(x, y, z) ↦ (x0, x1) = (-x + y + 1, -1/2*x - 1/2*y + z + 1/2)

Arrangement of affine hull of facets:

sage: cube = polytopes.cube(); cube
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices

sage: cube.affine_hull_manifold()  # needs sage.symbolic
Euclidean space E^3

**affine_hull_projection** (as_polyhedron, as_affine_map=None, orthogonal=False, orthonormal=False, extend=False, minimal=False, return_all_data=False, as_convex_set=False)

Return the polyhedron projected into its affine hull.

Each polyhedron is contained in some smallest affine subspace (possibly the entire ambient space) – its affine hull. We provide an affine linear map that projects the ambient space of the polyhedron to the standard Euclidean space of dimension of the polyhedron, which restricts to a bijection from the affine hull.

The projection map is not unique; some parameters control the choice of the map. Other parameters control the output of the function.
INPUT:

- `as_polyhedron` (or `as_convex_set`) – (boolean or the default None) and
- `as_affine_map` – (boolean, default False) control the output

The default `as_polyhedron=None` translates to `as_polyhedron=not as_affine_map`, therefore to `as_polyhedron=True` if nothing is specified.

If exactly one of either `as_polyhedron` or `as_affine_map` is set, then either a polyhedron or the affine transformation is returned. The affine transformation sends the embedded polytope to a full-dimensional one. It is given as a pair `(A, b)`, where `A` is a linear transformation and `b` is a vector, and the affine transformation sends `v` to `A(v) + b`.

If both `as_polyhedron` and `as_affine_map` are set, then both are returned, encapsulated in an instance of `AffineHullProjectionData`.

- `return_all_data` – (boolean, default False)
  If set, then `as_polyhedron` and `as_affine_map` will set (possibly overridden) and additional (internal) data concerning the transformation is returned. Everything is encapsulated in an instance of `AffineHullProjectionData` in this case.

- `orthogonal` – boolean (default: False); if True, provide an orthogonal transformation.
- `orthonormal` – boolean (default: False); if True, provide an orthonormal transformation. If the base ring does not provide the necessary square roots, the `extend` parameter needs to be set to True.
- `extend` – boolean (default: False); if True, allow base ring to be extended if necessary. This becomes relevant when requiring an orthonormal transformation.
- `minimal` – boolean (default: False); if True, when doing an extension, it computes the minimal base ring of the extension, otherwise the base ring is `AA`.

OUTPUT:

A full-dimensional polyhedron or an affine transformation, depending on the parameters `as_polyhedron` and `as_affine_map`, or an instance of `AffineHullProjectionData` containing all data (parameter `return_all_data`).

If the output is an instance of `AffineHullProjectionData`, the following fields may be set:

- `image` – the projection of the original polyhedron
- `projection_map` – the affine map as a pair whose first component is a linear transformation and its second component a shift; see above.
- `section_map` – an affine map as a pair whose first component is a linear transformation and its second component a shift. It maps the codomain of `affine_map` to the affine hull of `self`. It is a right inverse of `projection_map`.

Note that all of these data are compatible.

Todo:

- make the parameters `orthogonal` and `orthonormal` work with unbounded polyhedra.

EXAMPLES:

```
sage: triangle = Polyhedron([[1,0,0), (0,1,0), (0,0,1)]); triangle
A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 3 vertices
sage: triangle.affine_hull_projection()
```
A 2-dimensional polyhedron in \(\mathbb{Z}^2\) defined as the convex hull of 3 vertices

```python
sage: half3d = Polyhedron(vertices=[(3,2,1)], rays=[(1,0,0)])
```

```python
sage: half3d.affine_hull_projection().Vrepresentation()
```

(A ray in the direction (1), A vertex at (3))

The resulting affine hulls depend on the parameter `orthogonal` and `orthonormal`:

```python
sage: L = Polyhedron([[1,0], [0,1]]); L
```

A 1-dimensional polyhedron in \(\mathbb{Z}^2\) defined as the convex hull of 2 vertices

```python
sage: A = L.affine_hull_projection(); A
```

An affine hull of 2 vertices

```python
sage: A.affine_hull_projection(orthogonal=True); A
```

A 1-dimensional polyhedron in \(\mathbb{Q}^1\) defined as the convex hull of 2 vertices

```python
sage: A.affine_hull_projection(orthonormal=True)
```

Traceback (most recent call last):
...

ValueError: the base ring needs to be extended; try with "extend=True"

```python
sage: A.affine_hull_projection(orthogonal=True, extend=True); A
```

More generally:

```python
sage: S = polytopes.simplex(); S
```

A 3-dimensional polyhedron in \(\mathbb{Z}^4\) defined as the convex hull of 4 vertices

```python
sage: S.vertices()
```

(A vertex at (0, 0, 0, 1),
A vertex at (0, 0, 1, 0),
A vertex at (0, 1, 0, 0),
A vertex at (1, 0, 0, 0))

```python
sage: A = S.affine_hull_projection(); A
```

A 3-dimensional polyhedron in \(\mathbb{Z}^3\) defined as the convex hull of 4 vertices

```python
sage: A.affine_hull_projection(orthogonal=True)
```

A 3-dimensional polyhedron in \(\mathbb{Q}^3\) defined as the convex hull of 4 vertices

```python
sage: A.affine_hull_projection(orthogonal=True, extend=True)
```

A 3-dimensional polyhedron in \(\mathbb{A}^3\) defined as the convex hull of 4 vertices

(continues on next page)
With the parameter `minimal` one can get a minimal base ring:

```
sage: # needs sage.rings.number_field
sage: s = polytopes.simplex(3)
sage: s_AA = s.affine_hull_projection(orthonormal=True, extend=True)
sage: s_AA.base_ring()
Algebraic Real Field
sage: s_full = s.affine_hull_projection(orthonormal=True, extend=True, minimal=True)
sage: s_full.base_ring()
Number Field in a with defining polynomial y^4 - 4*y^2 + 1
with a = 0.5176380902050415?
```

More examples with the `orthonormal` parameter:

```
sage: P = polytopes.permutahedron(3); P
A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices
sage: set([F.as_polyhedron().affine_hull_projection(orthonormal=True, extend=True).volume() for F in P.affine_hull_projection().faces(1)]) == {1, sqrt(AA(2))}
True
sage: set([F.as_polyhedron().affine_hull_projection(orthonormal=True, extend=True).volume() for F in P.affine_hull_projection(orthonormal=True, extend=True).faces(1)]) == {sqrt(AA(2))}
True

sage: # needs sage.rings.number_field
sage: D = polytopes.dodecahedron()
sage: F = D.faces(2)[0].as_polyhedron()
sage: F.affine_hull_projection(orthogonal=True)
A 2-dimensional polyhedron in
(Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?)^2
defined as the convex hull of 5 vertices
sage: F.affine_hull_projection(orthonormal=True, extend=True)
A 2-dimensional polyhedron in AA^2 defined as the convex hull of 5 vertices
```

```
sage: # needs sage.rings.number_field
sage: K.<sqrt2> = QuadraticField(2)
sage: P = Polyhedron( [2*[K.zero()],2*[sqrt2]] ); P
A 1-dimensional polyhedron in
(Number Field in sqrt2 with defining polynomial x^2 - 2
with sqrt2 = 1.414213562373095?)^2
defined as the convex hull of 2 vertices
sage: P.vertices()
(A vertex at (0, 0), A vertex at (sqrt2, sqrt2))
sage: A = P.affine_hull_projection(orthonormal=True); A
A 1-dimensional polyhedron in
```

(continues on next page)
(Number Field in sqrt2 with defining polynomial x^2 - 2
with sqrt2 = 1.414213562373095?)^1
defined as the convex hull of 2 vertices
sage: A.vertices()
(A vertex at (0), A vertex at (2))

sage: # needs sage.rings.number_field
sage: K.<sqrt3> = QuadraticField(3)
sage: P = Polyhedron([2*[K.zero()], 2*[sqrt3]]); P
A 1-dimensional polyhedron in
(Number Field in sqrt3 with defining polynomial x^2 - 3
with sqrt3 = 1.732050807568878?)^2
defined as the convex hull of 2 vertices
sage: P.vertices()
(A vertex at (0, 0), A vertex at (sqrt3, sqrt3))
sage: A = P.affine_hull_projection(orthonormal=True)
Traceback (most recent call last):
...
ValueError: the base ring needs to be extended; try with "extend=True"
sage: A = P.affine_hull_projection(orthonormal=True, extend=True); A
A 1-dimensional polyhedron in AA^1 defined as the convex hull of 2 vertices
sage: A.vertices()
(A vertex at (0), A vertex at (2.449489742783178?))
sage: sqrt(6).n()
2.44948974278318
The affine hull is combinatorially equivalent to the input:

sage: P.is_combinatorially_isomorphic(P.affine_hull_projection())  # needs sage.rings.number_field
True
sage: P.is_combinatorially_isomorphic(P.affine_hull_projection(...)  # needs sage.rings.number_field
....: orthogonal=True))
True
sage: P.is_combinatorially_isomorphic(P.affine_hull_projection(...)  # needs sage.rings.number_field
....: orthonormal=True, extend=True))
True

The orthonormal=True parameter preserves volumes; it provides an isometric copy of the polyhedron:

sage: # needs sage.rings.number_field
sage: Pentagon = polytopes.dodecahedron().faces(2)[0].as_polyhedron()
sage: P = Pentagon.affine_hull_projection(orthonormal=True, extend=True)
sage: _, c = P.is_inscribed(certificate=True)
sage: c
(0.4721359549995794?, 0.6498393924658126?)
sage: circumradius = (c - vector(P.vertices()[0])).norm()
sage: p = polytopes.regular_polygon(5)
sage: p.volume()
2.377641290737884?
sage: P.volume()
1.53406271079097?
sage: p.volume()*circumradius^2
1.534062710790965?
sage: P.volume() == p.volume()*circumradius^2
...
One can also use orthogonal parameter to calculate volumes; in this case we don’t need to switch base rings. One has to divide by the square root of the determinant of the linear part of the affine transformation times its transpose:

```python
sage: # needs sage.rings.number_field
sage: Pentagon = polytopes.dodecahedron().faces(2)[0].as_polyhedron()
    # orthogonal=True, extend=True
sage: Pnormal = Pentagon.affine_hull_projection(orthonormal=True)
sage: Pgonal = Pentagon.affine_hull_projection(orthogonal=True)

sage: Adet = (A.matrix().transpose()*A.matrix()).det()

sage: Pnormal.volume()   # 1.53406271079097

sage: Pgonal.volume()/Adet.sqrt()    # -80*(55*sqrt(5) - 123)/sqrt(-6368*sqrt(5) + 14240)

sage: AA(Pgonal.volume()^2) == (Pnormal.volume()^2)*AA(Adet)    # True
```

Another example with as_affine_map=True:

```python
sage: # needs sage.combinat sage.rings.number_field
sage: P = polytopes.permutahedron(4)
    # orthogonal=True, extend=True
sage: A, b = P.affine_hull_projection(orthogonal=True, extend=True)

sage: A(P.center()) + b == Q.center()    # True
```

For unbounded, non-full-dimensional polyhedra, the orthogonal=True and orthonormal=True is not implemented:

```python
sage: P = Polyhedron(ieqs=[[0, 1, 0], [0, 0, 1], [0, 0, -1]]); P
A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and... ->1 ray

sage: P.is_compact()    # False
sage: P.is_full_dimensional()    # False
sage: P.affine_hull_projection(orthogonal=True)
Traceback (most recent call last):
... NotImplementedError: "orthogonal=True" and "orthonormal=True" work only for compact polyhedra
```

Setting as_affine_map to True without orthogonal or orthonormal set to True:
Combinatorial and Discrete Geometry, Release 10.3

```
sage: S = polytopes.simplex()
sage: S.affine_hull_projection(as_affine_map=True)
(Vector space morphism represented by the matrix:
[1 0 0]
[0 1 0]
[0 0 1]
[0 0 0]
Domain: Vector space of dimension 4 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field,
(0, 0, 0))
```

If the polyhedron is full-dimensional, it is returned:

```
sage: polytopes.cube().affine_hull_projection()
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: polytopes.cube().affine_hull_projection(as_affine_map=True)
(Vector space morphism represented by the matrix:
[1 0 0]
[0 1 0]
[0 0 1]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field,
(0, 0, 0))
```

Return polyhedron and affine map:

```
sage: S = polytopes.simplex(2)
sage: data = S.affine_hull_projection(orthogonal=True,
....: as_polyhedron=True,
....: as_affine_map=True); data
AffineHullProjectionData(image=A 2-dimensional polyhedron in QQ^2
defined as the convex hull of 3 vertices,
projection_linear_map=Vector space morphism represented by the matrix:
[ -1 -1/2
 [ 1 -1/2
 [ 0 1
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field,
projection_translation=(1, 1/2),
section_linear_map=Vector space morphism represented by the matrix:
[-1/2 1/2 0
(continues on next page)
```
The section map is a right inverse of the projection map:

```python
sage: mat = data.section_linear_map.matrix().transpose()
sage: data.image.linear_transformation(mat) + data.section_translation == S
True
```

Same without `orthogonal=True`:

```python
sage: data = S.affine_hull_projection(return_all_data=True); data
AffineHullProjectionData(image=A 2-dimensional polyhedron in ZZ^2
defined as the convex hull of 3 vertices,
projection_linear_map=Vector space morphism represented by the matrix:
[ 1 0 ]
[ 0 1 ]
Domain: Vector space of dimension 3 over Rational Field
Codomain: Vector space of dimension 2 over Rational Field,
projection_translation=(0, 0),
section_linear_map=Vector space morphism represented by the matrix:
[ 1 0 -1 ]
[ 0 1 -1 ]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 3 over Rational Field,
section_translation=(0, 0, 1))
sage: mat = data.section_linear_map.matrix().transpose()
sage: data.image.linear_transformation(mat) + data.section_translation == S
True
```

```python
sage: P0 = Polyhedron(....: ieqs=[(0, -1, 0, 1, 1, 1), (0, 1, 1, 0, -1, -1), (0, -1, 1, 1, 0, -1), (0, 1, 0, 0, 0, 0), (0, 0, 1, 1, -1, -1), (0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 0, 0)])
sage: P = P0.intersection(Polyhedron(eqns=[(-1, 1, 1, 1, 1, 1)]))
sage: P.dim()
4
sage: P.affine_hull_projection(orthogonal=True, as_affine_map=True)[0]
Vector space morphism represented by the matrix:
[ 0 0 0 1/3]
[ -2/3 -1/6 0 -1/12]
[ 1/3 -1/6 1/2 -1/12]
[ 0 1/2 0 -1/12]
[ 1/3 -1/6 -1/2 -1/12]
Domain: Vector space of dimension 5 over Rational Field
Codomain: Vector space of dimension 4 over Rational Field
```

```python
gale_transform()  
Return the Gale transform of a polytope as described in the reference below.
```

```
OUTPUT:
```

A list of vectors, the Gale transform. The dimension is the dimension of the affine dependencies of the vertices
EXAMPLES:

This is from the reference, for a triangular prism:

```
sage: p = Polyhedron(vertices = [[0,0],[0,1],[1,0]])
sage: p2 = p.prism()
sage: p2.gale_transform()
((-1, 0), (0, -1), (1, 1), (-1, -1), (1, 0), (0, 1))
```

REFERENCES:


See also:

`gale_transform_to_polyhedron()`.

`plot(point=None, line=None, polygon=None, wireframe='blue', fill='green', position=None, orthonormal=True, **kwds)`

Return a graphical representation.

INPUT:

- `point`, `line`, `polygon` – Parameters to pass to point (0d), line (1d), and polygon (2d) plot commands. Allowed values are:
  - A Python dictionary to be passed as keywords to the plot commands.
  - A string or triple of numbers: The color. This is equivalent to passing the dictionary `{'color': ..}`.
  - `False`: Switches off the drawing of the corresponding graphics object
- `wireframe`, `fill` – Similar to `point`, `line`, and `polygon`, but `fill` is used for the graphics objects in the dimension of the polytope (or of dimension 2 for higher dimensional polytopes) and `wireframe` is used for all lower-dimensional graphics objects (default: ‘green’ for `fill` and ‘blue’ for `wireframe`)
- `position` – positive number; the position to take the projection point in Schlegel diagrams.
- `orthonormal` – Boolean (default: True); whether to use orthonormal projections.
- `**kwds` – optional keyword parameters that are passed to all graphics objects.

OUTPUT:

A (multipart) graphics object.

EXAMPLES:

```
sage: square = polytopes.hypercube(2)
sage: point = Polyhedron([[1,1]])
sage: line = Polyhedron([[1,1],[2,1]])
sage: cube = polytopes.hypercube(3)
sage: hypercube = polytopes.hypercube(4)
```

By default, the wireframe is rendered in blue and the fill in green:

```
sage: # needs sage.plot
sage: square.plot()
Graphics object consisting of 6 graphics primitives
```

(continues on next page)
sage: point.plot()
Graphics object consisting of 1 graphics primitive
sage: line.plot()
Graphics object consisting of 2 graphics primitives
sage: cube.plot()
Graphics3d Object
sage: hypercube.plot()
Graphics3d Object

Draw the lines in red and nothing else:

sage: # needs sage.plot
sage: square.plot(point=False, line='red', polygon=False)
Graphics object consisting of 4 graphics primitives
sage: point.plot(point=False, line='red', polygon=False)
Graphics object consisting of 0 graphics primitives
sage: line.plot(point=False, line='red', polygon=False)
Graphics object consisting of 1 graphics primitive
sage: cube.plot(point=False, line='red', polygon=False)
Graphics3d Object
sage: hypercube.plot(point=False, line='red', polygon=False)
Graphics3d Object

Draw points in red, no lines, and a blue polygon:

sage: # needs sage.plot
sage: square.plot(point={'color':'red'}, line=False, polygon=(0,0,1))
Graphics object consisting of 2 graphics primitives
sage: point.plot(point={'color':'red'}, line=False, polygon=(0,0,1))
Graphics object consisting of 1 graphics primitive
sage: line.plot(point={'color':'red'}, line=False, polygon=(0,0,1))
Graphics object consisting of 1 graphics primitive
sage: cube.plot(point={'color':'red'}, line=False, polygon=(0,0,1))
Graphics3d Object
sage: hypercube.plot(point={'color':'red'}, line=False, polygon=(0,0,1))
Graphics3d Object

If we instead use the fill and wireframe options, the coloring depends on the dimension of the object:

sage: # needs sage.plot
sage: square.plot(fill='green', wireframe='red')
Graphics object consisting of 6 graphics primitives
sage: point.plot(fill='green', wireframe='red')
Graphics object consisting of 1 graphics primitive
sage: line.plot(fill='green', wireframe='red')
Graphics object consisting of 1 graphics primitive
sage: cube.plot(fill='green', wireframe='red')
Graphics3d Object
sage: hypercube.plot(fill='green', wireframe='red')
Graphics3d Object

It is possible to draw polyhedra up to dimension 4, no matter what the ambient dimension is:

sage: hcube = polytopes.hypercube(5)
sage: facet = hcube.facets()[0].as_polyhedron(); facet
A 4-dimensional polyhedron in ZZ^5 defined as the convex hull of 16 vertices
sage: facet.plot()
For a 3d plot, we may draw the polygons with rainbow colors, using any of the following ways:

```python
sage: cube.plot(polygon='rainbow')  #...
Graphics3d Object
sage: cube.plot(polygon={'color':'rainbow'})  #...
Graphics3d Object
sage: cube.plot(fill='rainbow')  #...
Graphics3d Object
```

For a 3d plot, the size of a point, the thickness of a line and the width of an arrow are controlled by the respective parameters:

```python
sage: prism = Polyhedron(vertices=[[0,0,0],[1,0,0],[0,1,0]], rays=[[0,0,1]])
sage: prism.plot(size=20, thickness=30, width=1)  #...
Graphics3d Object
sage: prism.plot(point={'size':20, 'color':'black'},
               line={'thickness':30, 'width':1, 'color':'black'},
               polygon='rainbow')
```

**projection** (*projection=None*)

Return a projection object.

INPUT:

- `proj` - a projection function

OUTPUT:

The identity projection. This is useful for plotting polyhedra.

See also:

`schlegel_projection()` for a more interesting projection.

EXAMPLES:

```python
sage: p = polytopes.hypercube(3)
sage: proj = p.projection()
sage: proj
The projection of a polyhedron into 3 dimensions
```

**render_solid** (**kwds**)

Return a solid rendering of a 2- or 3-d polytope.

EXAMPLES:

```python
sage: p = polytopes.hypercube(3)
sage: p_solid = p.render_solid(opacity=.7)  #...
Graphics3d Object
sage: type(p_solid)  #...
```

(continues on next page)
needs sage.plot
<class 'sage.plot.plot3d.index_face_set.IndexFaceSet'>

render_wireframe(**kwds)
For polytopes in 2 or 3 dimensions, return the edges as a list of lines.

EXAMPLES:

```
sage: p = Polyhedron([[1,2],[1,1],[0,0]])
sage: p_wireframe = p.render_wireframe()  # needs sage.plot
sage: p_wireframe._objects  # needs sage.plot
[Line defined by 2 points, Line defined by 2 points, Line defined by 2 points]
```

schlegel_projection(facet=None, position=None)
Return the Schlegel projection.

- The facet is orthonormally transformed into its affine hull.
- The position specifies a point coming out of the barycenter of the facet from which the other vertices will be projected into the facet.

INPUT:

- facet – a PolyhedronFace. The facet into which the Schlegel diagram is created. The default is the first facet.
- position – a positive number. Determines a relative distance from the barycenter of facet. A value close to 0 will place the projection point close to the facet and a large value further away. Default is 1. If the given value is too large, an error is returned.

OUTPUT:
A Projection object.

EXAMPLES:

```
sage: p = polytopes.hypercube(3)
sage: sch_proj = p.schlegel_projection()
sage: schlegel_edge_indices = sch_proj.lines
sage: len([x for x in schlegel_edges if x[0][0] > 0])
8
```

The Schlegel projection preserves the convexity of facets, see github issue #30015:

```
sage: fcube = polytopes.hypercube(4)
sage: tfcube = fcube.face_truncation(fcube.faces(0)[0])
sage: tfcube.facets()[-1]  # needs sage.plot
A 3-dimensional face of a Polyhedron in QQ^4 defined as the convex hull of 8...→vertices
sage: sp = tfcube.schlegel_projection(tfcube.facets()[-1])  # needs sage.plot
Graphics3d Object
```

The same truncated cube but see inside the tetrahedral facet:
A 3-dimensional face of a Polyhedron in QQ^4 defined as the convex hull of 4 vertices

A different values of position changes the projection:

A value which is too large give a projection point that sees more than one facet resulting in a error:

show(**kwds)

This method attempts to display the graphics immediately, without waiting for the currently running code (if any) to return to the command line. Be careful, calling it from within a loop will potentially launch a large number of external viewer programs.

INPUT:

• kwds – optional keyword arguments. See plot() for the description of available options.

OUTPUT:

This method does not return anything. Use plot() if you want to generate a graphics object that can be saved or further transformed.

EXAMPLES:

Return a tikz picture of self as a string or as a TikzPicture according to a projection view and an angle angle obtained via the threejs viewer. self must be bounded.

INPUT:

• view - list (default: [0,0,1]) representing the rotation axis (see note below).

• angle - integer (default: 0) angle of rotation in degree from 0 to 360 (see note below).
• scale - integer (default: 1) specifying the scaling of the tikz picture.
• edge_color - string (default: ‘blue!95!black’) representing colors which tikz recognize.
• facet_color - string (default: ‘blue!95!black’) representing colors which tikz recognize.
• vertex_color - string (default: ‘green’) representing colors which tikz recognize.
• opacity - real number (default: 0.8) between 0 and 1 giving the opacity of the front facets.
• axis - Boolean (default: False) draw the axes at the origin or not.
• output_type - string (default: None), valid values are None (deprecated), 'LatexExpr' and 'TikzPicture', whether to return a LatexExpr object (which inherits from Python str) or a TikzPicture object from module sage.misc.latex_standalone

OUTPUT:
• LatexExpr object or TikzPicture object

Note: This is a wrapper of a method of the projection object self.projection(). See tikz() for more detail.

The inputs view and angle can be obtained by visualizing it using .show(aspect_ratio=1). This will open an interactive view in your default browser, where you can rotate the polytope. Once the desired view angle is found, click on the information icon in the lower right-hand corner and select Get Viewpoint. This will copy a string of the form '[x,y,z],angle' to your local clipboard. Go back to Sage and type Img = P.tikz([x,y,z],angle).

The inputs view and angle can also be obtained from the viewer Jmol:

1) Right click on the image
2) Select `''Console``
3) Select the tab `''State''`
4) Scroll to the line `''moveto''`

It reads something like:

moveto 0.0 {x y z angle} Scale

The view is then [x,y,z] and angle is angle. The following number is the scale.

Jmol performs a rotation of angle degrees along the vector [x,y,z] and show the result from the z-axis.

EXAMPLES:

sage: # needs sage.plot
sage: co = polytopes.cuboctahedron()
sage: Img = co.tikz([0, 0, 1], 0, output_type='TikzPicture')
sage: Img
\documentclass[tikz]{standalone}
\begin{document}
\begin{tikzpicture}
...
When output type is a `sage.misc.latex_standalone.TikzPicture`:

```sage
# needs sage.plot
co = polytopes.cuboctahedron()
t = co.tikz([674, 108, -731], 112, output_type='TikzPicture'); t
\documentclass{tikz}(standalone)
\begin{document}
\begin{tikzpicture}
  \x={(0.249656cm, -0.577639cm)},
  \y={(0.777700cm, -0.358578cm)},
  \z={(-0.576936cm, -0.733318cm)},
  scale=1.000000,
  \node[vertex] at (1.00000, 0.00000, 1.00000) {};
  \node[vertex] at (1.00000, 1.00000, 0.00000) {};
  %
  %
  \end{tikzpicture}
\end{document}
sage: path_to_file = t.pdf() # not tested
```

2.6.8 Base class for polyhedra: Methods for triangulation and volume computation

```python
class sage.geometry.polyhedron.base7.Polyhedron_base7(parent, Vrep, Hrep,
Vrep_minimal=None,
Hrep_minimal=None,
pref_rep=None, mutable=False,
**kwds)
```

Bases: `Polyhedron_base6`

Methods related to triangulation and volume.
**centroid** \((\text{engine}='auto', **\text{kwds})\)

Return the center of the mass of the polytope.

The mass is taken with respect to the induced Lebesgue measure, see \uppercase{volume()}. If the polyhedron is not compact, a \uppercase{NotImplementedError} is raised.

**INPUT:**

- \text{engine} -- either 'auto' (default), 'internal', 'TOPCOM', or 'normaliz'. The 'internal' and 'TOPCOM' instruct this package to always use its own triangulation algorithms or TOPCOM's algorithms, respectively. By default ('auto'), TOPCOM is used if it is available and internal routines otherwise.

- **\text{kwds}** -- keyword arguments that are passed to the triangulation engine (see \uppercase{triangulate()}).

**OUTPUT:** The centroid as vector.

**ALGORITHM:**

We triangulate the polytope and find the barycenter of the simplices. We add the individual barycenters weighted by the fraction of the total mass.

**EXAMPLES:**

```sage
sage: P = polytopes.hypercube(2).pyramid()
sage: P.centroid()
(1/4, 0, 0)
sage: P = polytopes.associahedron([A, 2])  # needs sage.combinat
sage: P.centroid()  # needs sage.combinat
(2/21, 2/21)
sage: P = polytopes.permutahedron(4, backend='normaliz')  # optional
sage: P.centroid()  # optional
(5/2, 5/2, 5/2, 5/2)
```

The method is not implemented for unbounded polyhedra:

```sage
sage: P = Polyhedron(vertices=[(0, 0)], rays=[(1, 0), (0, 1)])
sage: P.centroid()
Traceback (most recent call last):
  ...
NotImplementedError: the polyhedron is not compact
```

The centroid of an empty polyhedron is not defined:

```sage
sage: P = Polyhedron()
P.centroid()
Traceback (most recent call last):
  ...
ZeroDivisionError: rational division by zero
```

**integrate** \((\text{function}, \text{measure}='\text{ambient}', **\text{kwds})\)

Return the integral of \text{function} over this polytope.

**INPUT:**

- \text{self} -- Polyhedron
• `function` — a multivariate polynomial or a valid LattE description string for polynomials

• `measure` — string, the measure to use

  Allowed values are:

  – `ambient` (default): Lebesgue measure of ambient space,

  – `induced`: Lebesgue measure of the affine hull,

  – `induced_nonnormalized`: Lebesgue measure of the affine hull without the normalization by

  \( \sqrt{\det(A^\top A)} \) (with \( A \) being the affine transformation matrix; see `affine_hull()`).

• `**kwds` — additional keyword arguments that are passed to the engine

**OUTPUT:**

The integral of the polynomial over the polytope

**Note:** The polytope triangulation algorithm is used. This function depends on LattE (i.e., the `latte_int` optional package).

**EXAMPLES:**

```sage
sage: P = polytopes.cube()
sage: x, y, z = polygens(QQ, 'x, y, z')
sage: P.integrate(x^2*y^2*z^2)  # optional ~
        → latte_int
        8/27
```

If the polyhedron has floating point coordinates, an inexact result can be obtained if we transform to rational coordinates:

```sage
sage: P = 1.4142*polytopes.cube()
sage: P_QQ = Polyhedron(vertices=[QQ(vi) for vi in v for v in P.vertex_generator()])
sage: RDF(P_QQ.integrate(x^2*y^2*z^2))  # optional ~
        → latte_int
        6.703841212195228
```

Integral over a non full-dimensional polytope:

```sage
sage: x, y = polygens(QQ, 'x, y')
sage: P = Polyhedron(vertices=[[0,0], [1,1]])
sage: P.integrate(x*y)
0
sage: ixy = P.integrate(x*y, measure='induced'); ixy  # optional ~
        → latte_int
        0.4714045207910317?
sage: ixy.parent()  # optional ~
        → latte_int
        Algebraic Real Field
```

Convert to a symbolic expression:

```sage
sage: ixy.radical_expression()  # optional ~
        → latte_int
        1/3*sqrt(2)
```

Another non full-dimensional polytope integration:
sage: R.<x, y, z> = QQ[]
sage: P = polytopes.simplex(2)
sage: V = AA(P.volume(measure='induced'))  # needs sage.rings.number_field
sage: V.radical_expression()  # needs sage.rings.number_field sage.symbolic
1/2*sqrt(3)
sage: P.integrate(R(1), measure='induced') == V  # optional ~
    # latte_int, needs sage.rings.number_field sage.symbolic
True

Computing the mass center:

sage: (P.integrate(x, measure='induced'))
    # optional ~
    # latte_int, needs sage.rings.number_field sage.symbolic
    ....: / V).radical_expression()
1/3
sage: (P.integrate(y, measure='induced'))  # optional ~
    # latte_int, needs sage.rings.number_field sage.symbolic
    ....: / V).radical_expression()
1/3
sage: (P.integrate(z, measure='induced'))  # optional ~
    # latte_int, needs sage.rings.number_field sage.symbolic
    ....: / V).radical_expression()
1/3

triangulate (engine='auto', connected=True, fine=False, regular=None, star=None)

Return a triangulation of the polytope.

INPUT:

- engine – either ‘auto’ (default), ‘internal’, ‘TOPCOM’, or ‘normaliz’. The ‘internal’ and ‘TOPCOM’ instruct this package to always use its own triangulation algorithms or TOPCOM’s algorithms, respectively. By default (‘auto’), TOPCOM is used if it is available and internal routines otherwise.

The remaining keyword parameters are passed through to the PointConfiguration constructor:

- connected – boolean (default: True). Whether the triangulations should be connected to the regular triangulations via bistellar flips. These are much easier to compute than all triangulations.

- fine – boolean (default: False). Whether the triangulations must be fine, that is, make use of all points of the configuration.

- regular – boolean or None (default: None). Whether the triangulations must be regular. A regular triangulation is one that is induced by a piecewise-linear convex support function. In other words, the shadows of the faces of a polyhedron in one higher dimension.
  - True: Only regular triangulations.
  - False: Only non-regular triangulations.
  - None (default): Both kinds of triangulation.

- star – either None (default) or a point. Whether the triangulations must be star. A triangulation is star if all maximal simplices contain a common point. The central point can be specified by its index (an integer) in the given points or by its coordinates (anything iterable.)

OUTPUT:

A triangulation of the convex hull of the vertices as a Triangulation. The indices in the triangulation correspond to the Vrepresentation() objects.
EXAMPLES:

```python
sage: cube = polytopes.hypercube(3)
sage: triangulation = cube.triangulate(
    ....:   engine='internal')  # to make doctest independent of TOPCOM
sage: triangulation
(<0,1,2,7>, <0,1,5,7>, <0,2,3,7>, <0,3,4,7>, <0,4,5,7>, <1,5,6,7>)
sage: simplex_indices = triangulation[0]; simplex_indices
(0, 1, 2, 7)
sage: simplex_vertices = [cube.Vrepresentation(i) for i in simplex_indices]
sage: simplex_vertices
[0, 1, 2, 7]
sage: simplex_vertices[0].Vrepresentation()  # A vertex at (1, -1, -1),
A vertex at (1, 1, -1),
A vertex at (1, 1, 1),
A vertex at (-1, 1, 1)
sage: Polyhedron(simplex_vertices)
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 4 vertices
```

It is possible to use 'normaliz' as an engine. For this, the polyhedron should have the backend set to normaliz:

```python
sage: P = Polyhedron(vertices=[[0,0,1], [1,0,1], [0,1,1], [1,1,1]],
    ....:   backend='normaliz')
sage: P.triangulate(engine='normaliz')  # optional -- pynormaliz
(<0,1,2>, <1,2,3>)
```

The normaliz engine can triangulate pointed cones:

```python
sage: C1 = Polyhedron(rays=[[0,0,1], [1,0,1], [0,1,1], [1,1,1]],
    ....:   backend='normaliz')
sage: C1.triangulate(engine='normaliz:')
(<0,1,2>, <1,2,3>)
sage: C2 = Polyhedron(rays=[[1,0,1], [0,0,1], [0,1,1], [1,1,10/9]],
    ....:   backend='normaliz')
sage: C2.triangulate(engine='normaliz')
(<0,1,2>, <1,2,3>)
```

They can also be affine cones:

```python
sage: K = Polyhedron(vertices=[[1,1,1]],
    ....:   rays=[[1,0,0], [0,1,0], [1,1,-1], [1,1,1]],
    ....:   backend='normaliz')
sage: K.triangulate(engine='normaliz')  # optional -- pynormaliz
```

(continues on next page)
volume (measure='ambient', engine='auto', **kwds)

Return the volume of the polytope.

INPUT:

• measure – string. The measure to use. Allowed values are:
  - ambient (default): Lebesgue measure of ambient space (volume)
  - induced: Lebesgue measure of the affine hull (relative volume)
  - induced_rational: Scaling of the Lebesgue measure for rational polytopes, such that the unit hypercube has volume 1
  - induced_lattice: Scaling of the Lebesgue measure, such that the volume of the hypercube is factorial(n)

• engine – string. The backend to use. Allowed values are:
  - 'auto' (default): choose engine according to measure
  - 'internal': see triangulate()
  - 'TOPCOM': see triangulate()
  - 'lrs': use David Avis’s lrs program (optional)
  - 'latte': use LattE integrale program (optional)
  - 'normaliz': use Normaliz program (optional)

• **kwds – keyword arguments that are passed to the triangulation engine

OUTPUT:

The volume of the polytope

EXAMPLES:

```
sage: polytopes.hypercube(3).volume()
8
sage: (polytopes.hypercube(3)*2).volume()
64
sage: polytopes.twenty_four_cell().volume()
2
```

Volume of the same polytopes, using the optional package Lrslib (which requires a rational polytope):

```
sage: I3 = polytopes.hypercube(3)
sage: I3.volume(engine='lrs') # optional ~
8
sage: C24 = polytopes.twenty_four_cell()
sage: C24.volume(engine='lrs') # optional ~
2
```

If the base ring is exact, the answer is exact:
When considering lower-dimensional polytopes, we can ask for the ambient (full-dimensional), the induced measure (of the affine hull) or, in the case of lattice polytopes, for the induced rational measure. This is controlled by the parameter measure. Different engines may have different ideas on the definition of volume of a lower-dimensional object:

```python
code
sage: P = Polyhedron([0, 0], [1, 1])
sage: P.volume() 0
sage: P.volume(measure='induced') 1
```

```python
code
sage: S = polytopes.regular_polygon(6); S
A 2-dimensional polyhedron in AA^2 defined as the convex hull of 6 vertices
sage: edge = S.faces(1)[0].as_polyhedron()
sage: edge.vertices()
(A vertex at (0.866025403784439?, 1/2), A vertex at (0, 1))
sage: edge.volume() 0
sage: edge.volume(measure='induced') 1
```

```python
code
sage: P = Polyhedron([[0,0], [1,0], [1,1]])
sage: P.volume() 0
sage: P.volume(measure='induced') 2.598076211353316?
```

```python
code
sage: P = Polyhedron(backend=normaliz, vertices=[(1,0,0), (0,0,1),
                                              (-1,1,1), (-1,2,0)])
```

(continues on next page)
The same polytope without normaliz backend:

```
sage: P = Polyhedron(vertices=[[1,0,0], [0,0,1], [-1,1,1], [-1,2,0]])
sage: P.volume(measure='induced_lattice', engine='latte')  # optional --
˓→latte_int
3

sage: # needs sage.groups sage.rings.number_field
sage: Dexact = polytopes.dodecahedron()
sage: F0 = Dexact.faces(2)[0].as_polyhedron()
sage: v = F0.volume(measure='induced', engine='internal'); v
1.53406271079097?
sage: F4 = Dexact.faces(2)[4].as_polyhedron()
sage: v = F4.volume(measure='induced', engine='internal'); v
1.53406271079097?
sage: RDF(v)  # abs tol 1e-9
1.53406271079044
```

```
sage: # needs sage.groups
sage: Dinexact = polytopes.dodecahedron(exact=False)
sage: F2 = Dinexact.faces(2)[2].as_polyhedron()
sage: w = F2.volume(measure='induced', engine='internal'); w
1.5340627082974878
```

```
sage: all(polytopes.simplex(d).volume(measure='induced')  #_
˓→needs sage.rings.number_field sage.symbolic
.....: == sqrt(d+1)/factorial(d)
.....: for d in range(1,5))
True
```

```
sage: I = Polyhedron([[3, 0], [0, 9]])
sage: I.volume(measure='induced')  #_
˓→needs sage.rings.number_field
9.48683298050514?
```

```
sage: T = Polyhedron([[3, 0, 0], [0, 4, 0], [0, 0, 5]])
sage: T.volume(measure='induced')  #_
˓→needs sage.rings.number_field
13.86542462386205?
```

```
sage: Q = Polyhedron(vertices=[[0, 0, 1, 1], [0, 1, 1, 0], [1, 1, 0, 0]])
sage: Q.volume(measure='induced')  # optional --
˓→latte_int
1/2
```

The volume of a full-dimensional unbounded polyhedron is infinity:
The volume of a non full-dimensional unbounded polyhedron depends on the measure used:

```
sage: P = Polyhedron(ieqs = [[1,1,1], [-1,-1,-1], [3,1,0]]); P
A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and
  → 1 ray
sage: P.volume()
0
sage: P.volume(measure='induced')
+Infinity
sage: P.volume(measure='ambient')
0
sage: P.volume(measure='induced_rational')
# optional ~
  → pynormaliz
+Infinity
sage: P.volume(measure='induced_rational',engine='latte')
+Infinity
```

The volume in 0-dimensional space is taken by counting measure:

```
sage: P = Polyhedron(vertices=([[]])); P
A 0-dimensional polyhedron in ZZ^0 defined as the convex hull of 1 vertex
sage: P.volume()
1
sage: P = Polyhedron(vertices=[]); P
The empty polyhedron in ZZ^0
sage: P.volume()
0
```

### 2.6.9 Base class for polyhedra: Miscellaneous methods

class sage.geometry.polyhedron.base.Polyhedron_base

```
 Bases: Polyhedron_base
```

Base class for Polyhedron objects

**INPUT:**

- `parent` – the parent, an instance of Polyhedra.
- `Vrep` – a list [vertices, rays, lines] or None. The V-representation of the polyhedron. If None, the polyhedron is determined by the H-representation.
- `Hrep` – a list [ieqs, eqns] or None. The H-representation of the polyhedron. If None, the polyhedron is determined by the V-representation.
- `Vrep_minimal` (optional) – see below
- `Hrep_minimal` (optional) – see below
- `pref_rep` – string (default: None); one of Vrep or Hrep to pick this in case the backend cannot initialize from complete double description

---

The volume of a non full-dimensional unbounded polyhedron depends on the measure used:

```
sage: P = Polyhedron(vertices=[[1, 0], [0, 1]], rays=[[1, 1]])
sage: P.volume()
+Infinity
```

The volume of a non full-dimensional unbounded polyhedron depends on the measure used:

```
sage: P = Polyhedron(ieqs = [[1,1,1], [-1,-1,-1], [3,1,0]]); P
A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and
  → 1 ray
sage: P.volume()
0
sage: P.volume(measure='induced')
+Infinity
sage: P.volume(measure='ambient')
0
sage: P.volume(measure='induced_rational')
# optional ~
  → pynormaliz
+Infinity
sage: P.volume(measure='induced_rational',engine='latte')
+Infinity
```

The volume in 0-dimensional space is taken by counting measure:

```
sage: P = Polyhedron(vertices=([[]])); P
A 0-dimensional polyhedron in ZZ^0 defined as the convex hull of 1 vertex
sage: P.volume()
1
sage: P = Polyhedron(vertices=[]); P
The empty polyhedron in ZZ^0
sage: P.volume()
0
```

### 2.6.9 Base class for polyhedra: Miscellaneous methods

class sage.geometry.polyhedron.base.Polyhedron_base

```
 Bases: Polyhedron_base
```

Base class for Polyhedron objects

**INPUT:**

- `parent` – the parent, an instance of Polyhedra.
- `Vrep` – a list [vertices, rays, lines] or None. The V-representation of the polyhedron. If None, the polyhedron is determined by the H-representation.
- `Hrep` – a list [ieqs, eqns] or None. The H-representation of the polyhedron. If None, the polyhedron is determined by the V-representation.
- `Vrep_minimal` (optional) – see below
- `Hrep_minimal` (optional) – see below
- `pref_rep` – string (default: None); one of Vrep or Hrep to pick this in case the backend cannot initialize from complete double description

---

The volume of a non full-dimensional unbounded polyhedron depends on the measure used:

```
sage: P = Polyhedron(ieqs = [[1,1,1], [-1,-1,-1], [3,1,0]]); P
A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and
  → 1 ray
sage: P.volume()
0
sage: P.volume(measure='induced')
+Infinity
sage: P.volume(measure='ambient')
0
sage: P.volume(measure='induced_rational')
# optional ~
  → pynormaliz
+Infinity
sage: P.volume(measure='induced_rational',engine='latte')
+Infinity
```

The volume in 0-dimensional space is taken by counting measure:

```
sage: P = Polyhedron(vertices=([[]])); P
A 0-dimensional polyhedron in ZZ^0 defined as the convex hull of 1 vertex
sage: P.volume()
1
sage: P = Polyhedron(vertices=[]); P
The empty polyhedron in ZZ^0
sage: P.volume()
0
```

### 2.6.9 Base class for polyhedra: Miscellaneous methods

class sage.geometry.polyhedron.base.Polyhedron_base

```
 Bases: Polyhedron_base
```

Base class for Polyhedron objects

**INPUT:**

- `parent` – the parent, an instance of Polyhedra.
- `Vrep` – a list [vertices, rays, lines] or None. The V-representation of the polyhedron. If None, the polyhedron is determined by the H-representation.
- `Hrep` – a list [ieqs, eqns] or None. The H-representation of the polyhedron. If None, the polyhedron is determined by the V-representation.
- `Vrep_minimal` (optional) – see below
- `Hrep_minimal` (optional) – see below
- `pref_rep` – string (default: None); one of Vrep or Hrep to pick this in case the backend cannot initialize from complete double description

---

The volume of a non full-dimensional unbounded polyhedron depends on the measure used:

```
sage: P = Polyhedron(ieqs = [[1,1,1], [-1,-1,-1], [3,1,0]]); P
A 1-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and
  → 1 ray
sage: P.volume()
0
sage: P.volume(measure='induced')
+Infinity
sage: P.volume(measure='ambient')
0
sage: P.volume(measure='induced_rational')
# optional ~
  → pynormaliz
+Infinity
sage: P.volume(measure='induced_rational',engine='latte')
+Infinity
```

The volume in 0-dimensional space is taken by counting measure:

```
sage: P = Polyhedron(vertices=([[]])); P
A 0-dimensional polyhedron in ZZ^0 defined as the convex hull of 1 vertex
sage: P.volume()
1
sage: P = Polyhedron(vertices=[]); P
The empty polyhedron in ZZ^0
sage: P.volume()
0
```
• mutable – ignored

If both \texttt{Vrep} and \texttt{Hrep} are provided, then \texttt{Vrep\_minimal} and \texttt{Hrep\_minimal} must be set to \texttt{True}.

\textbf{barycentric\_subdivision} \texttt{(subdivision\_frac=None)}

Return the barycentric subdivision of a compact polyhedron.

\textbf{DEFINITION:}

The barycentric subdivision of a compact polyhedron is a standard way to triangulate its faces in such a way that maximal faces correspond to flags of faces of the starting polyhedron (i.e. a maximal chain in the face lattice of the polyhedron). As a simplicial complex, this is known as the order complex of the face lattice of the polyhedron.

\textbf{REFERENCE:}

See Wikipedia article Barycentric\_subdivision


\textbf{INPUT:}

• \texttt{subdivision\_frac – number}. Gives the proportion how far the new vertices are pulled out of the polytope. Default is $\frac{1}{3}$ and the value should be smaller than $\frac{1}{2}$. The subdivision is computed on the polar polyhedron.

\textbf{OUTPUT:}

A Polyhedron object, subdivided as described above.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P = polytopes.hypercube(3)
sage: P.barycentric_subdivision()
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 26 vertices

sage: P = Polyhedron(vertices=[(0,0,0),(0,1,0),(1,0,0),(0,0,1)])
sage: P.barycentric_subdivision()
A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 14 vertices

sage: P = Polyhedron(vertices=[(0,1,0),(0,0,1),(1,0,0)])
sage: P.barycentric_subdivision()
A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 6 vertices

sage: P = polytopes.regular_polygon(4, base_ring=QQ)
\texttt{# needs sage.rings.number_field}
sage: P.barycentric_subdivision()
\texttt{# needs sage.rings.number_field}
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 8 vertices
\end{verbatim}

\textbf{boundary\_complex}()

Return the simplicial complex given by the boundary faces of \texttt{self}, if it is simplicial.

\textbf{OUTPUT:}

A (spherical) simplicial complex

\textbf{EXAMPLES:}

The boundary complex of the octahedron:
The polyhedron should be simplicial:

```
sage: c = polytopes.cube()
sage: c.boundary_complex()
Traceback (most recent call last):
  ... Not ImplementedError: this function is only implemented for simplicial...
```

**bounding_box** *(integral=False, integral_hull=False)*

Return the coordinates of a rectangular box containing the non-empty polytope.

**INPUT:**

- `integral` – Boolean (default: False). Whether to only allow integral coordinates in the bounding box.

- `integral_hull` – Boolean (default: False). If True, return a box containing the integral points of the polytope, or None, None if it is known that the polytope has no integral points.

**OUTPUT:**

A pair of tuples `(box_min, box_max)` where `box_min` are the coordinates of a point bounding the coordinates of the polytope from below and `box_max` bounds the coordinates from above.

**EXAMPLES:**

```
sage: Polyhedron([(1/3, 2/3), (2/3, 1/3)]).bounding_box()
((1/3, 1/3), (2/3, 2/3))
sage: Polyhedron([(1/3, 2/3), (2/3, 1/3)]).bounding_box(integral=True)
((0, 0), (1, 1))
sage: Polyhedron([(1/3, 2/3), (2/3, 1/3)]).bounding_box(integral_hull=True)
(None, None)
sage: Polyhedron([(1/3, 2/3), (3/3, 4/3)]).bounding_box(integral_hull=True)
((1, 1), (1, 1))
sage: polytopes.buckyball(exact=False).bounding_box()  #...
"""
**center()**

Return the average of the vertices.

See also:

`sage.geometry.polyhedron.base1.Polyhedron_base1.representative_point()`.

**OUTPUT:**

The center of the polyhedron. All rays and lines are ignored. Raises a `ZeroDivisionError` for the empty polytope.

**EXAMPLES:**

```sage
p = polytopes.hypercube(3)
p = p + vector([1,0,0])
p.center()  # (1, 0, 0)
```

**face_fan()**

Return the face fan of a compact rational polyhedron.

**OUTPUT:**

A fan of the ambient space as a `RationalPolyhedralFan`.

See also:

`normal_fan()`.

**EXAMPLES:**

```sage
T = polytopes.cuboctahedron()
T.face_fan()  # Rational polyhedral fan in 3-d lattice M
```

The polytope should contain the origin in the interior:

```sage
P = Polyhedron(vertices=[[1/2, 1], [1, 1/2]])
P.face_fan()  # Traceback (most recent call last):
...  # ValueError: face fans are defined only for polytopes containing the origin as an interior point!
```

```sage
Q = Polyhedron(vertices=[[-1, 1/2], [1, -1/2]])
Q.contains([0,0])  # True
Q.face_fan(); FF  # Rational polyhedral fan in 2-d lattice M
```

The polytope has to have rational coordinates:

```sage
S = polytopes.dodecahedron()  # needs sage.rings.number_field
S.face_fan()  # needs sage.rings.number_field
Traceback (most recent call last):
...  # NotImplementedError: face fan handles only polytopes over the rationals
```
REFERENCES:
For more information, see Chapter 7 of [Zie2007].

**hyperplane_arrangement()**

Return the hyperplane arrangement defined by the equations and inequalities.

**OUTPUT:**

A hyperplane arrangement consisting of the hyperplanes defined by the \texttt{Hrepresentation()}. If the polytope is full-dimensional, this is the hyperplane arrangement spanned by the facets of the polyhedron.

**EXAMPLES:**

```python
sage: p = polytopes.hypercube(2)
sage: p.hyperplane_arrangement()
Arrangement <-t0 + 1 | -t1 + 1 | t1 + 1 | t0 + 1>
```

**is_inscribed**

This function tests whether the vertices of the polyhedron are inscribed on a sphere.

The polyhedron is expected to be compact and full-dimensional. A full-dimensional compact polytope is inscribed if there exists a point in space which is equidistant to all its vertices.

**ALGORITHM:**

The function first computes the circumsphere of a full-dimensional simplex with vertices of \texttt{self}. It is found by lifting the points on a paraboloid to find the hyperplane on which the circumsphere is lifted. Then, it checks if all other vertices are equidistant to the circumcenter of that simplex.

**INPUT:**

- \texttt{certificate} – (default: \texttt{False}) boolean; specifies whether to return the circumcenter, if found.

**OUTPUT:**

If \texttt{certificate} is true, returns a tuple containing:

1. Boolean.
2. The circumcenter of the polytope or None.

If \texttt{certificate} is false:

- a Boolean.

**EXAMPLES:**

```python
sage: q = Polyhedron(vertices=[[1,1,1,1],[-1,-1,1,1],[1,-1,-1,1],
....: [-1,1,-1,1],[1,1,1,-1],[-1,-1,1,-1],
....: [1,-1,-1,-1],[-1,1,-1,-1],[0,0,10/13,-24/13],
....: [0,0,-10/13,-24/13]])
sage: q.is_inscribed(certificate=True)
(True, (0, 0, 0, 0))
sage: cube = polytopes.cube()
sage: cube.is_inscribed()
True
sage: translated_cube = Polyhedron(verteces=[v.vector() + vector([1,2,3])
....: for v in cube.vertices()])
sage: translated_cube.is_inscribed(certificate=True)
(True, (1, 2, 3))
```

(continues on next page)
The method is not implemented for non-full-dimensional polytope or unbounded polyhedra:

```python
sage: square = Polyhedron(verts=[[1,0,0],[0,1,0],[1,1,0],[0,0,0]])
sage: square.is_inscribed()
Traceback (most recent call last):
...  
NotImplementedError: this function is implemented for full-dimensional polyhedra only
```

is_minkowski_summand(Y)
Test whether Y is a Minkowski summand.
See minkowski_sum().

OUTPUT: 
Boolean. Whether there exists another polyhedron Z such that self can be written as Y ⊕ Z.

EXAMPLES:

```python
sage: A = polytopes.hypercube(2)
sage: B = Polyhedron(verts=[(0,1), (1/2,1)])
sage: C = Polyhedron(verts=[(1,1)])
```

```python
sage: A.is_minkowski_summand(B)  
True
sage: A.is_minkowski_summand(C)  
True
sage: B.is_minkowski_summand(C)  
True
sage: B.is_minkowski_summand(A)  
False
sage: C.is_minkowski_summand(A)  
False
sage: C.is_minkowski_summand(B)  
False
```

normal_fan(direction='inner')
Return the normal fan of a compact full-dimensional rational polyhedron.

This returns the inner normal fan of self. For the outer normal fan, use direction='outer'.

INPUT:

- direction – either 'inner' (default) or 'outer'; if set to 'inner', use the inner normal vectors to span the cones of the fan, if set to 'outer', use the outer normal vectors.

OUTPUT:

A complete fan of the ambient space as a RationalPolyhedralFan.
See also:

\textit{face\_fan()}.  

**EXAMPLES:**

\begin{verbatim}
sage: S = Polyhedron(\text{vertices}=[[0, 0], [1, 0], [0, 1]])
sage: S.normal_fan()
Rational polyhedral fan in 2-d lattice \(N\)

sage: C = polytopes.hypercube(4)
sage: NF = C.normal_fan(); NF
Rational polyhedral fan in 4-d lattice \(N\)
\end{verbatim}

Currently, it is only possible to get the normal fan of a bounded rational polytope:

\begin{verbatim}
sage: P = Polyhedron(\text{rays}=[[1, 0], [0, 1]])
sage: P.normal_fan()
Traceback (most recent call last):
...
NotImplementedError: the normal fan is only supported for polytopes (compact... →polyhedra).

sage: Q = Polyhedron(\text{vertices}=[[1, 0, 0], [0, 1, 0], [0, 0, 1]])
sage: Q.normal_fan()
Traceback (most recent call last):
...
ValueError: the normal fan is only defined for full-dimensional polytopes

sage: R = Polyhedron(\text{vertices}=[[0, 0], [2, 0], [0, 2], [2, 1], [1, 2]])
sage: P.normal_fan(direction=None)
Traceback (most recent call last):
...
TypeError: the direction should be 'inner' or 'outer'

sage: inner_nf = P.normal_fan()
sage: inner_nf.rays()
\(N( 1, 0),\)
\(N( 0, -1),\)
\(N( 0, 1),\)
\(N(-1, 0),\)
\(N(-1, -1)\)
in 2-d lattice \(N\)

sage: outer_nf = P.normal_fan(direction='outer')
sage: outer_nf.rays()
\(N( 1, 0),\)
\(N( 1, 1),\)
\(N( 0, 1),\)
\end{verbatim}

(continues on next page)
REFERENCES:
For more information, see Chapter 7 of [Zie2007].

permutations_to_matrices (conj_class_reps, acting_group=None, additional_elts=None)

Return a dictionary between different representations of elements in the acting_group, with group elements represented as permutations of the vertices of this polytope (keys) or matrices (values).

The dictionary has entries for the generators of the acting_group and the representatives of conjugacy classes in conj_class_reps. By default, the acting_group is the restricted_automorphism_group() of the polytope. Each element in additional_elts also becomes a key.

INPUT:
- conj_class_reps – list. A list of representatives of the conjugacy classes of the acting_group.
- acting_group – a subgroup of polytope's restricted_automorphism_group().
- additional_elts – list (default=None). A subset of the restricted_automorphism_group() of the polytope expressed as permutations.

OUTPUT:
A dictionary between elements of the acting_group expressed as permutations (keys) and matrices (values).

EXAMPLES:
This example shows the dictionary between permutations and matrices for the generators of the restricted_automorphism_group of the ±1 2-dimensional square. The permutations are written in terms of the vertices of the square:

sage: # optional - pynormaliz, needs sage.groups
sage: square = Polyhedron(backend=normaliz)
sage: square.vertices()
(A vertex at (-1, -1),
 A vertex at (-1, 1),
 A vertex at (1, -1),
 A vertex at (1, 1))
sage: aut_square = square.restricted_automorphism_group(output='permutation')
sage: conj_reps = aut_square.conjugacy_classes_representatives()
sage: gens_dict = square.permutations_to_matrices(conj_reps)
sage: rotation_180 = aut_square([(0,3),(1,2)])

This example tests the functionality for additional elements:
sage: # needs sage.groups sage.rings.real_mpfr
sage: C = polytopes.cross_polytope(2)
sage: G = C.restricted_automorphism_group(output='permutation')
sage: conj_reps = G.conjugacy_classes_representatives()
sage: add_elt = G([(0, 2, 3, 1)])
sage: dict = C.permutations_to_matrices(conj_reps,
....: additional_elts=[add_elt])
sage: dict[add_elt]
[ 0 1 0]
[-1 0 0]
[ 0 0 1]

radius()

Return the maximal distance from the center to a vertex. All rays and lines are ignored.

OUTPUT:

The radius for a rational polyhedron is, in general, not rational. Use radius_square() if you need a rational distance measure.

EXAMPLES:

sage: p = polytopes.hypercube(4)
sage: p.radius()
2

radius_square()

Return the square of the maximal distance from the center() to a vertex. All rays and lines are ignored.

OUTPUT:

The square of the radius, which is in base_ring().

EXAMPLES:

sage: p = polytopes.permutahedron(4, project=False)
sage: p.radius_square()
5

to_linear_program(solver=None, return_variable=False, base_ring=None)

Return a linear optimization problem over the polyhedron in the form of a MixedIntegerLinearProgram.

INPUT:

- solver – select a solver (MIP backend). See the documentation of for MixedIntegerLinearProgram. Set to None by default.
- return_variable – (default: False) If True, return a tuple (p, x), where p is the MixedIntegerLinearProgram object and x is the vector-valued MIP variable in this problem, indexed from 0. If False, only return p.
- base_ring – select a field over which the linear program should be set up. Use RDF to request a fast inexact (floating point) solver even if self is exact.

Note that the MixedIntegerLinearProgram object will have the null function as an objective to be maximized.

See also:

polyhedron() – return the polyhedron associated with a MixedIntegerLinearProgram object.
EXAMPLES:

Exact rational linear program:

```
sage: p = polytopes.cube()
sage: p.to_linear_program()
Linear Program (no objective, 3 variables, 6 constraints)
sage: lp, x = p.to_linear_program(return_variable=True)
sage: lp.set_objective(2*x[0] + 1*x[1] + 39*x[2])
sage: lp.solve()
42
sage: lp.get_values(x[0], x[1], x[2])
[1, 1, 1]
```

Floating-point linear program:

```
sage: lp, x = p.to_linear_program(return_variable=True, base_ring=RDF)
sage: lp.set_objective(2*x[0] + 1*x[1] + 39*x[2])
sage: lp.solve()
42.0
```

Irrational algebraic linear program over an embedded number field:

```
sage: # needs sage.rings.number_field
sage: p = polytopes.icosahedron()
sage: lp, x = p.to_linear_program(return_variable=True)
sage: lp.set_objective(x[0] + x[1] + x[2])
sage: lp.solve()
1/4*sqrt5 + 3/4
```

Same example with floating point:

```
sage: lp, x = p.to_linear_program(return_variable=True, base_ring=RDF)
sage: lp.set_objective(x[0] + x[1] + x[2])
sage: lp.solve() # tol 1e-5
1.3090169943749475
```

Same example with a specific floating point solver:

```
sage: lp, x = p.to_linear_program(return_variable=True, solver='GLPK')
sage: lp.set_objective(x[0] + x[1] + x[2])
sage: lp.solve() # tol 1e-8
1.3090169943749475
```

Irrational algebraic linear program over AA:

```
sage: # needs sage.rings.number_field
sage: p = polytopes.icosahedron(base_ring=AA)
sage: lp, x = p.to_linear_program(return_variable=True)
sage: lp.set_objective(x[0] + x[1] + x[2])
sage: lp.solve() # long time
1.309016994374948?
```

```
sage.geometry.polyhedron.base.is_Polyhedron(X)
Test whether X is a Polyhedron.
```

INPUT:
• X – anything.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: p = polytopes.hypercube(2)
sage: from sage.geometry.polyhedron.base import is_Polyhedron
sage: is_Polyhedron(p)
True
sage: is_Polyhedron(123456)
False
```

### 2.6.10 Base class for polyhedra over Q

```python
class sage.geometry.polyhedron.base_QQ.Polyhedron_QQ (parent, Vrep, Hrep, Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False, **kwds)
```

**Bases:** `Polyhedron_base`

Base class for Polyhedra over Q

**Hstar_function** *(acting_group=None, output=None)*

Return $H^*$ as a rational function in $t$ with coefficients in the ring of class functions of the `acting_group` of this polytope.

Here, $H^*(t) = \sum_{m} \chi_{m} t^{m} \det(\text{Id} - \rho(t))$. The irreducible characters of `acting_group` form an orthonormal basis for the ring of class functions with values in $\mathbb{C}$. The coefficients of $H^*(t)$ are expressed in this basis.

**INPUT:**

- **acting_group** – (default=None) a permgroup object. A subgroup of the polytope's restricted_automorphism_group. If None, it is set to the full restricted_automorphism_group of the polytope. The acting group should always use `output='permutation'`.
- **output** – string. an output option. The allowed values are:
  - None (default): returns the rational function $H^*(t)$. $H^*$ is a rational function in $t$ with coefficients in the ring of class functions.
  - 'e_series_list': Returns a list of the ehrhart_series for the fixed_subpolytopes of each conjugacy class representative.
  - 'determinant_vec': Returns a list of the determinants of $\text{Id} - \rho * t$ for each conjugacy class representative.
  - 'Hstar_as_lin_comb': Returns a vector of the coefficients of the irreducible representations in the expression of $H^*$.
  - 'prod_det_es': Returns a vector of the product of determinants and the Ehrhart series.
  - 'complete': Returns a list with Hstar, Hstar_as_lin_comb, character table of the acting group, and whether Hstar is effective.

**OUTPUT:**
The default output is the rational function $H^*$. $H^*$ is a rational function in $t$ with coefficients in the ring of class functions. There are several output options to see the intermediary outputs of the function.

EXAMPLES:

The $H^*$-polynomial of the standard $(d - 1)$-dimensional simplex $S = \text{conv}(e_1, \ldots, e_d)$ under its restricted_automorphism_group() is equal to $1 = \chi_{\text{trivial}}$ (Prop 6.1 [Stap2011]). Here is the computation for the 3-dimensional standard simplex:

```
sage: # optional - pynormaliz
sage: S = polytopes.simplex(3, backend='normaliz'); S
A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 4 vertices
sage: G = S.restricted_automorphism_group(output='permutation')
True
sage: Hstar = S._Hstar_function_normaliz(G); Hstar
chi_4
sage: G.character_table()
\[ 1 \ -1 \ 1 \ 1 \ -1 \\
3 \ -1 \ 0 \ -1 \ 1 \\
2 \ 0 \ -1 \ 2 \ 0 \\
3 \ 1 \ 0 \ -1 \ -1 \\
1 \ 1 \ 1 \ 1 \ 1 \]
```

The next example is Example 7.6 in [Stap2011], and shows that $H^*$ is not always a polynomial. Let $P$ be the polytope with vertices $\pm(0,0,1), \pm(1,0,1), \pm(0,1,1), \pm(1,1,1)$ and let $G = \mathbb{Z}/2\mathbb{Z}$ act on $P$ as follows:

```
sage: # optional - pynormaliz
sage: P = Polyhedron(vertices=[[0,0,1], [0,0,-1], [1,0,1], ....: [-1,0,-1], [0,1,1], ....: [0,-1,-1], [1,1,1], [-1,-1,-1]], ....: backend='normaliz')
sage: K = P.restricted_automorphism_group(output='permutation')
sage: G = K.subgroup(gens=[K([(0,2),(1,3),(4,6),(5,7)])])
sage: conj_reps = G.conjugacy_classes_representatives()
sage: Dict = P.permutations_to_matrices(conj_reps, ....: acting_group=G)
sage: list(Dict.keys())[0]
(0,2) (1,3) (4,6) (5,7)
sage: list(Dict.values())[0]
[-1 0 1 0] \\
[ 0 1 0 0] \\
[ 0 0 1 0] \\
[ 0 0 0 1]
sage: len(G)
2
sage: G.character_table()
\[ 1 \ 1 \]
\[ 1 \ -1 \]
```

Then we calculate the rational function $H^*(t)$:

```
sage: Hst = P._Hstar_function_normaliz(G); Hst  # optional --
˓→pynormaliz
(\chi_0*t^4 + (3*\chi_0 + 3*\chi_1)*t^3
+ (8*\chi_0 + 2*\chi_1)*t^2 + (3*\chi_0 + 3*\chi_1)*t + \chi_0)/(t + 1)
```
To see the exact as written in [Stap2011], we can format it as 'Hstar_as_lin_comb'. The first coordinate is the coefficient of the trivial character; the second is the coefficient of the sign character:

```
sage: lin = P._Hstar_function_normaliz(˓→pynormaliz
.....: G, output='Hstar_as_lin_comb'); lin
((t^4 + 3*t^3 + 8*t^2 + 3*t + 1)/(t + 1),
  (3*t^3 + 2*t^2 + 3*t)/(t + 1))
```

ehrhart_polynomial

Return the Ehrhart polynomial of this polyhedron.

The polyhedron must be a lattice polytope. Let $P$ be a lattice polytope in $\mathbb{R}^d$ and define $L(P, t) = \#(tP \cap \mathbb{Z}^d)$. Then E. Ehrhart proved in 1962 that $L$ coincides with a rational polynomial of degree $d$ for integer $t$. $L$ is called the Ehrhart polynomial of $P$. For more information see the Wikipedia article Ehrhart_polynomial.

The Ehrhart polynomial may be computed using either LattE Integrale or Normaliz by setting `engine` to 'latte' or 'normaliz' respectively.

**INPUT:**

- `engine` – string; The backend to use. Allowed values are:
  - None (default); When no input is given the Ehrhart polynomial is computed using LattE Integrale (optional)
  - 'latte'; use LattE integrale program (optional)
  - 'normaliz'; use Normaliz program (optional package pynormaliz). The backend of `self` must be set to 'normaliz'.
- `variable` – string (default: 't'); The variable in which the Ehrhart polynomial should be expressed.
- When the `engine` is 'latte', the additional input values are:
  - `verbose` - boolean (default: False); If True, print the whole output of the LattE command.

The following options are passed to the LattE command, for details consult the LattE documentation:

- `dual` - boolean; triangulate and signed-decompose in the dual space
- `irrational_primal` - boolean; triangulate in the dual space, signed-decompose in the primal space using irrationalization.
- `irrational_all_primal` - boolean; triangulate and signed-decompose in the primal space using irrationalization.
- `maxdet` - integer; decompose down to an index (determinant) of `maxdet` instead of index 1 (unimodular cones).
- `no_decomposition` - boolean; do not signed-decompose simplicial cones.
- `compute_vertex_cones` - string; either 'cdd' or 'lrs' or '4ti2'
- `smith_form` - string; either 'ilio' or 'lidia'
- `dualization` - string; either 'cdd' or '4ti2'
- `triangulation` - string; 'cddlib', '4ti2' or 'topcom'
- `triangulation_max_height` - integer; use a uniform distribution of height from 1 to this number
OUTPUT:
A univariate polynomial in variable over a rational field.

See also:
latte the interface to LattE Integrale PyNormaliz

EXAMPLES:
To start, we find the Ehrhart polynomial of a three-dimensional simplex, first using engine='latte'. Leaving the engine unspecified sets the engine to 'latte' by default:

```python
sage: simplex = Polyhedron(vertices=[(0,0,0),(3,3,3),(-3,2,1),(1,-1,-2)])
sage: simplex = simplex.change_ring(QQ)
sage: poly = simplex.ehrhart_polynomial(engine='latte')  # optional -
              latte_int
sage: poly
7/2*t^3 + 2*t^2 - 1/2*t + 1  # optional -
              latte_int
sage: poly(1)  # optional -
6
sage: len(simplex.integral_points())  # optional -
6
sage: poly(2)  # optional -
36
sage: len((2*simplex).integral_points())  # optional -
36
```

Now we find the same Ehrhart polynomial, this time using engine='normaliz'. To use the Normaliz engine, the simplex must be defined with backend='normaliz':

```python
sage: # optional - pynormaliz
sage: simplex = Polyhedron(vertices=[(0,0,0), (3,3,3),
                                  (-3,2,1), (1,-1,-2)],
                                  backend='normaliz')
sage: simplex = simplex.change_ring(QQ)
sage: poly = simplex.ehrhart_polynomial(engine='normaliz')
sage: poly
7/2*t^3 + 2*t^2 - 1/2*t + 1
```

If the engine='normaliz', the backend should be 'normaliz', otherwise it returns an error:

```python
sage: simplex = Polyhedron(vertices=[(0,0,0), (3,3,3),
                                  (-3,2,1), (1,-1,-2)])
```

The polyhedron should be compact:

```python
sage: C = Polyhedron(rays=[[1,2], [2,1]],  # optional -
                   backend='normaliz')
```

(continues on next page)
The polyhedron should have integral vertices:

```python
sage: L = Polyhedron(vertices=[[0], [1/2]])
sage: L.ehrhart_polynomial()
Traceback (most recent call last):
...  
TypeError: the polytope has nonintegral vertices, use ehrhart_quasipolynomial...
```

\texttt{ehrhart\_quasipolynomial}(variable='t', engine=None, verbose=False, dual=None, irrational_primal=None, irrational_all_primal=None, maxdet=None, no_decomposition=None, compute_vertex_cones=None, smith_form=None, dualization=None, triangulation=None, triangulation_max_height=None, **kwargs)

Compute the Ehrhart quasipolynomial of this polyhedron with rational vertices.

If the polyhedron is a lattice polytope, returns the Ehrhart polynomial, a univariate polynomial in variable over a rational field. If the polyhedron has rational, nonintegral vertices, returns a tuple of polynomials in variable over a rational field. The Ehrhart counting function of a polytope $P$ with rational vertices is given by a quasipolynomial. That is, there exists a positive integer $l$ and $l$ polynomials $ehr_{P,i}$ for $i \in \{1, \ldots, l\}$ such that if $t$ is equivalent to $i$ mod $l$ then $tP \cap \mathbb{Z}^d = ehr_{P,i}(t)$.

\textbf{INPUT:}

- \texttt{variable} – string (default: 't'); The variable in which the Ehrhart polynomial should be expressed.
- \texttt{engine} – string; The backend to use. Allowed values are:
  - None (default); When no input is given the Ehrhart polynomial is computed using Normaliz (optional)
  - 'latte'; use LattE Integrale program (requires optional package ‘latte_int’)
  - 'normaliz'; use the Normaliz program (requires optional package ‘pynormaliz’). The backend of self must be set to 'normaliz'.
- When the engine is 'latte', the additional input values are:
  - \texttt{verbose} - boolean (default: False); If True, print the whole output of the LattE command.

The following options are passed to the LattE command, for details consult the LattE documentation:

- \texttt{dual} – boolean; triangulate and signed-decompose in the dual space
- \texttt{irrational\_primal} – boolean; triangulate in the dual space, signed-decompose in the primal space using irrationalization.
- \texttt{irrational\_all\_primal} – boolean; triangulate and signed-decompose in the primal space using irrationalization.
- \texttt{maxdet} – integer; decompose down to an index (determinant) of maxdet instead of index 1 (unimodular cones).
- \texttt{no\_decomposition} – boolean; do not signed-decompose simplicial cones.
- compute_vertex_cones - string; either 'cdd' or 'lrs' or '4ti2'
- smith_form - string; either 'ilio' or 'lidia'
- dualization - string; either 'cdd' or '4ti2'
- triangulation - string; 'cddlib', '4ti2' or 'topcom'
- triangulation_max_height - integer; use a uniform distribution of height from 1 to this number

OUTPUT:

A univariate polynomial over a rational field or a tuple of such polynomials.

See also:

latte the interface to LattE Integrale PyNormaliz

Warning: If the polytope has rational, non integral vertices, it must have backend='normaliz'.

EXAMPLES:

As a first example, consider the line segment [0,1/2]. If we dilate this line segment by an even integral factor \( k \), then the dilated line segment will contain \( k/2 + 1 \) lattice points. If \( k \) is odd then there will be \( k/2 + 1/2 \) lattice points in the dilated line segment. Note that it is necessary to set the backend of the polytope to 'normaliz':

```python
sage: line_seg = Polyhedron(vertices=[[0], [1/2]], backend='normaliz');
A 1-dimensional polyhedron in QQ^1 defined as the convex hull of 2 vertices
sage: line_seg.ehrhart_quasipolynomial() # optional ─
(1/2*t + 1, 1/2*t + 1/2)
```

For a more exciting example, let us look at the subpolytope of the 3 dimensional permutahedron fixed by the reflection across the hyperplane \( x_1 = x_4 \):

```python
sage: verts = [[3/2, 3, 4, 3/2],
.......: [3/2, 4, 3, 3/2],
.......: [5/2, 1, 4, 5/2],
.......: [5/2, 4, 1, 5/2],
.......: [7/2, 1, 2, 7/2],
.......: [7/2, 2, 1, 7/2]]
sage: subpoly = Polyhedron(verts=verts, backend='normaliz')
sage: eq = subpoly.ehrhart_quasipolynomial();
(4*t^2 + 3*t + 1, 4*t^2 + 2*t)
sage: even_ep = eq[0] # optional ─
sage: odd_ep = eq[1] # optional ─
sage: even_ep(2) # optional ─
(continues on next page)
```
A polytope with rational nonintegral vertices must have `backend='normaliz'`:

```sage
ts = 2*subpoly
sage: ts.integral_points_count()                          # optional --
23
```

The polyhedron should be compact:

```sage
odd_ep(1)
```

If the polytope happens to be a lattice polytope, the Ehrhart polynomial is returned:

```sage
simplex = Polyhedron(vertices=[(0,0,0), (2,2,2), (-2,2,-2), (2,-2,2), (-2,-2,-2)],
                      backend='normaliz')
sage: simplex.ehrhart_quasipolynomial()                     # optional --
7/2*t^3 + 2*t^2 - 1/2*t + 1
```

**fixed_subpolytope** *(vertex_permutation)*

Return the fixed subpolytope of this polytope by the cyclic action of `vertex_permutation`.

The fixed subpolytope of this polytope under the `vertex_permutation` is the subset of this polytope that is fixed pointwise.

**INPUT:**

- `vertex_permutation` – permutation; a permutation of the vertices of `self`.

**OUTPUT:**
A subpolytope of `self`.

**Note:** The vertex permutation is obtained as a permutation of the vertices represented as a permutation. For example, `vertex_permutation = self.restricted_automorphism_group(output='permutation')`.

Requiring a lattice polytope as opposed to a rational polytope as input is purely conventional.

**EXAMPLES:**

The fixed subpolytopes of the cube can be obtained as follows:

```python
sage: Cube = polytopes.cube(backend='normaliz')  # optional ~
    # pynormaliz
sage: AG = Cube.restricted_automorphism_group(  # optional ~
    # pynormaliz
    ....: output='permutation')

sage: reprs = AG.conjugacy_classes_representatives()  # optional ~
    # pynormaliz
sage: Cube.fixed_subpolytope(vertex_permutation=reprs[0])  # optional ~
    # pynormaliz
```

The fixed subpolytope of the identity element of the group is the entire cube:

```python
sage: reprs[0]  # optional ~
    # pynormaliz
()

sage: Cube.fixed_subpolytope(vertex_permutation=reprs[0])  # optional ~
    # pynormaliz
```

A 3-dimensional polyhedron in QQ^3 defined as the convex hull of 8 vertices

```python
sage: _.vertices()  # optional ~
    # pynormaliz
(A vertex at (-1, -1, -1),
A vertex at (-1, -1, 1),
A vertex at (-1, 1, -1),
A vertex at (-1, 1, 1),
A vertex at (1, -1, -1),
A vertex at (1, -1, 1),
A vertex at (1, 1, -1),
A vertex at (1, 1, 1))
```

You can obtain non-trivial examples:

```python
sage: G = AG([(0,1),(2,3),(4,5),(6,7)])  # optional ~
    # pynormaliz

sage: fsp = Cube.fixed_subpolytope(G); fsp  # optional ~
    # pynormaliz
```

A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 4 vertices

```python
sage: fsp.vertices()  # optional ~
    # pynormaliz
```

(A vertex at (-1, -1, 0),
A vertex at (-1, 1, 0),
A vertex at (1, -1, 0),
A vertex at (1, 1, 0))

The next example shows that `fixed_subpolytope()` works for rational polytopes:

```python
sage: # optional ~ pynormaliz
sage: P = Polyhedron(vertices=[[0], [1/2]],

(continues on next page)
fixed_subpolytopes (conj_class_reps)

Return the fixed subpolytopes of this polytope under the actions of the given conjugacy class representatives. The conj_class_reps are representatives of the conjugacy classes of a subgroup of the automorphism group of this polytope. For an element of the automorphism group, the fixed subpolytope is the subset of this polytope that is fixed pointwise.

INPUT:

- conj_class_reps – a list of representatives of the conjugacy classes of the subgroup of the restricted_automorphism_group() of the polytope. Each element is written as a permutation of the vertices of the polytope.

OUTPUT:

A dictionary where the elements of conj_class_reps are keys and the fixed subpolytopes are values.

Note: Two elements in the same conjugacy class fix lattice-isomorphic subpolytopes.

EXAMPLES:

Here is an example for the square:

```
sage: # optional - pynormaliz, needs sage.groups
sage: p = polytopes.hypercube(2, backend='normaliz'); p
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices
sage: aut_p = p.restricted_automorphism_group(
    ....:     output='permutation')
sage: aut_p.order()
8
sage: conj_list = aut_p.conjugacy_classes_representatives()
sage: fixedpolytopes_dict = p.fixed_subpolytopes(conj_list)
sage: fixedpolytopes_dict[aut_p([(0,3),(1,2)])]
A 0-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex
```

integral_points_count (verbose=False, use_Hrepresentation=False,
    explicit Enumeration_threshold=1000, preprocess=True, **kwds)

Return the number of integral points in the polyhedron.

This method uses the optional package latte_int if an estimate for lattice points based on bounding boxes exceeds explicit Enumeration_threshold.

INPUT:
• **verbose** – (boolean; **False** by default) whether to display verbose output.

• **use_Hrepresentation** – (boolean; **False** by default) – whether to send the H or V representation to **LattE**

• **preprocess** – (boolean; **True** by default) whether, if the integral hull is known to lie in a coordinate hyperplane, to tighten bounds to reduce dimension

**See also:**

**latte** the interface to **LattE** interfaces

**EXAMPLES:**

```python
sage: P = polytopes.cube()
sage: P.integral_points_count()
27
sage: P.integral_points_count(explicit Enumeration_threshold=0)  # optional -...
˓→latte_int
27
```

We enlarge the polyhedron to force the use of the generating function methods implemented in **LattE integrale**, rather than explicit enumeration:

```python
sage: (1000000000*P).integral_points_count(Verbose=True)  # optional -...
˓→latte_int
This is LattE integrale...
...
Total time:...
800000012000000000600000001
```

We shrink the polyhedron a little bit:

```python
sage: Q = P*(8/9)
sage: Q.integral_points_count()
1
sage: Q.integral_points_count(explicit Enumeration_threshold=0)
1
```

Unbounded polyhedra (with or without lattice points) are not supported:

```python
sage: P = Polyhedron(vertices=[[1/2, 1/3]], rays=[[1, 1]])
sage: P.integral_points_count()
Traceback (most recent call last):
...
NotImplementedError: ...
sage: P = Polyhedron(vertices=[[1, 1]], rays=[[1, 1]])
sage: P.integral_points_count()
Traceback (most recent call last):
...
NotImplementedError: ...
```

“Fibonacci” knapsacks (preprocessing helps a lot):

```python
sage: def fibonacci_knapsack(d, b, backend=None):
    .....:  lp = MixedIntegerLinearProgram(base_ring=QQ)
    .....:  x = lp.new_variable(nonnegative=True)
    .....:  lp.add_constraint(lp.sum(fibonacci(i+3)*x[i] for i in range(d)) <=...
˓→b)
```

(continues on next page)
is_effective \( (H_{\text{star}}, H_{\text{star \_as \_lin \_comb}}) \)

Test for the effectiveness of the \( H_{\text{star}} \) series of this polytope.

The \( H_{\text{star}} \) series of the polytope is determined by the action of a subgroup of the polytope’s restricted_automorphism_group(). The \( H_{\text{star}} \) series is effective if it is a polynomial in \( t \) and the coefficient of each \( t^i \) is an effective character in the ring of class functions of the acting group. A character \( \rho \) is effective if the coefficients of the irreducible representations in the expression of \( \rho \) are non-negative integers.

INPUT:

- \( H_{\text{star}} \) — a rational function in \( t \) with coefficients in the ring of class functions.
- \( H_{\text{star \_as \_lin \_comb}} \) — vector. The coefficients of the irreducible representations of the acting group in the expression of \( H_{\text{star}} \) as a linear combination of irreducible representations with coefficients in the field of rational functions in \( t \).

OUTPUT:

Boolean. Whether the \( H_{\text{star}} \) series is effective.

See also:

\( H_{\text{star \_function}}() \)

EXAMPLES:

The \( H^* \) series of the two-dimensional permutahedron under the action of the symmetric group is effective:

\[
\text{sage: } \text{# optional - pynormaliz} \\
\text{sage: } p3 = \text{polytopes.permutahedron}(3, \text{backend='normaliz'}) \\
\text{sage: } G = p3.\text{restricted_automorphism\_group(} \\
\text{\textbackslash ...: output='permutation'}) \\
\text{sage: } \text{reflection12 = G(\{(0,2),(1,4),(3,5)\})} \\
\text{sage: } \text{reflection23 = G(\{(0,1),(2,3),(4,5)\})} \\
\text{sage: } S3 = G.\text{subgroup(gens=[reflection12, reflection23])} \\
\text{sage: } S3.\text{is_isomorphic}()\text{\( (\text{SymmetricGroup} (3)\))} \\
\text{True} \\
\text{sage: } Hstar = p3.\text{Hstar\_function}(S3) \\
\text{sage: } Hlin = p3.\text{Hstar\_function}(S3, \text{...: output='Hstar\_as\_lin\_comb'}) \\
\text{sage: } p3.\text{is\_effective}(Hstar, Hlin) \\
\text{True}
\]

If the \( H^* \)-series is not polynomial, then it is not effective:

\[
\text{sage: } \text{# optional - pynormaliz} \\
\text{sage: } P = \text{Polyhedron(\text{vertices}=[\{0,0,1\}, \{0,0,-1\}, \{1,0,1\}, \text{...: [-1,0,-1], \{0,1,1\}, \text{...: [0,-1,-1], \{1,1,1\}, [-1,-1,-1]], \text{...: backend='normaliz'})} \\
\text{sage: } G = P.\text{restricted_automorphism\_group(} \\
\text{\textbackslash ...: output='permutation'}) \\
\text{sage: } H = G.\text{subgroup(gens=[G(\{(0,2),(1,3),(4,6),(5,7)\})])} \\
\]

(continues on next page)
sage: Hstar = P.Hstar_function(H); Hstar
(chi_0*t^4 + (3*chi_0 + 3*chi_1)*t^3 + (8*chi_0 + 2*chi_1)*t^2 + (3*chi_0 + 3*chi_1)*t + chi_0)/(t + 1)
sage: Hstar_lin = P.Hstar_function(H,
....:     output='Hstar_as_lin_comb')
sage: P.is_effective(Hstar, Hstar_lin)
False

2.6.11 Base class for polyhedra over $Z$

class sage.geometry.polyhedron.base_ZZ.Polyhedron_ZZ (parent, Vrep, Hrep,
Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False, **kwds)

Bases: Polyhedron_QQ
Base class for Polyhedra over $Z$

ehrhart_polynomial (engine=None, variable='t', verbose=False, dual=None, irrational_primal=None,
irrational_all_primal=None, maxdet=None, no_decomposition=None,
compute_vertex_cones=None, smith_form=None, dualization=None,
triangulation=None, triangulation_max_height=None, **kwds)

Return the Ehrhart polynomial of this polyhedron.

Let $P$ be a lattice polytope in $\mathbb{R}^d$ and define
$L(P, t) = \#(tP \cap \mathbb{Z}^d)$. Then E. Ehrhart proved in 1962 that
$L$ coincides with a rational polynomial of degree $d$ for integer $t$. $L$ is called the Ehrhart polynomial of $P$. For
more information see the Wikipedia article Ehrhart_polynomial.

The Ehrhart polynomial may be computed using either LattE Integrale or Normaliz by setting engine to
'latte' or 'normaliz' respectively.

INPUT:

- engine – string: The backend to use. Allowed values are:
  - None (default); When no input is given the Ehrhart polynomial is computed using LattE Integrale
    (optional)
  - 'latte'; use LattE integrale program (optional)
  - 'normaliz'; use Normaliz program (optional). The backend of self must be set to 'normaliz'.
- variable – string (default: 't'); The variable in which the Ehrhart polynomial should be expressed.
- When the engine is 'latte' or None, the additional input values are:
  - verbose - boolean (default: False); if True, print the whole output of the LattE command.

The following options are passed to the LattE command, for details consult the LattE documentation:

- dual - boolean; triangulate and signed-decompose in the dual space
- irrational_primal - boolean; triangulate in the dual space, signed-decompose in the primal space
  using irrationalization.
- irrational_all_primal - boolean; Triangulate and signed-decompose in the primal space
  using irrationalization.
- maxdet - integer; decompose down to an index (determinant) of maxdet instead of index 1
  (unimodular cones).
OUTPUT:
The Ehrhart polynomial as a univariate polynomial in variable over a rational field.

See also:
latte the interface to LattE Integrale PyNormaliz

EXAMPLES:
To start, we find the Ehrhart polynomial of a three-dimensional simplex, first using engine='latte'. Leaving the engine unspecified sets the engine to 'latte' by default:

```
sage: simplex = Polyhedron(vertices=[(0,0,0),(3,3,3),(-3,2,1),(1,-1,-2)])
sage: poly = simplex.ehrhart_polynomial(engine = 'latte')  # optional - latte
sage: poly
7/2*t^3 + 2*t^2 - 1/2*t + 1
sage: poly(1)  # optional - latte
6
sage: len(simplex.integral_points())
6
sage: poly(2)  # optional - latte
36
sage: len((2*simplex).integral_points())
36
```

Now we find the same Ehrhart polynomial, this time using engine='normaliz'. To use the Normaliz engine, the simplex must be defined with backend='normaliz':

```
sage: simplex = Polyhedron(vertices=[(0,0,0),(3,3,3),(-3,2,1),(1,-1,-2)], backend='normaliz')  # optional - pynormaliz
sage: poly = simplex.ehrhart_polynomial(engine='normaliz')  # optional - pynormaliz
sage: poly
7/2*t^3 + 2*t^2 - 1/2*t + 1
sage: len(simplex.integral_points())
6
sage: len((2*simplex).integral_points())
36
```

If the engine='normaliz', the backend should be 'normaliz', otherwise it returns an error:

```
sage: simplex = Polyhedron(vertices=[(0,0,0),(3,3,3),(-3,2,1),(1,-1,-2)])
sage: simplex.ehrhart_polynomial(engine='normaliz')
Traceback (most recent call last):
...
TypeError: The polyhedron's backend should be 'normaliz'
```
Now we find the Ehrhart polynomials of the unit hypercubes of dimensions three through six. They are computed first with engine='latte' and then with engine='normaliz'. The degree of the Ehrhart polynomial matches the dimension of the hypercube, and the coefficient of the leading monomial equals the volume of the unit hypercube:

```python
sage: # optional - latte_int
sage: from itertools import product
sage: def hypercube(d):
....:     return Polyhedron(vertices=list(product([0,1],repeat=d)))

sage: hypercube(3).ehrhart_polynomial()
t^3 + 3*t^2 + 3*t + 1

sage: hypercube(4).ehrhart_polynomial()
t^4 + 4*t^3 + 6*t^2 + 4*t + 1

sage: hypercube(5).ehrhart_polynomial()
t^5 + 5*t^4 + 10*t^3 + 10*t^2 + 5*t + 1

sage: hypercube(6).ehrhart_polynomial()
t^6 + 6*t^5 + 15*t^4 + 20*t^3 + 15*t^2 + 6*t + 1

sage: # optional - pynormaliz
sage: from itertools import product
sage: def hypercube(d):
....:     return Polyhedron(vertices=list(product([0,1],repeat=d)),backend=˓→'normaliz')

sage: hypercube(3).ehrhart_polynomial(engine='normaliz')
t^3 + 3*t^2 + 3*t + 1

sage: hypercube(4).ehrhart_polynomial(engine='normaliz')
t^4 + 4*t^3 + 6*t^2 + 4*t + 1

sage: hypercube(5).ehrhart_polynomial(engine='normaliz')
t^5 + 5*t^4 + 10*t^3 + 10*t^2 + 5*t + 1

sage: hypercube(6).ehrhart_polynomial(engine='normaliz')
t^6 + 6*t^5 + 15*t^4 + 20*t^3 + 15*t^2 + 6*t + 1

An empty polyhedron:

```python
sage: p = Polyhedron(ambient_dim=3, vertices=[])
sage: p.ehrhart_polynomial()
0
sage: parent(_)
Univariate Polynomial Ring in t over Rational Field
```

The polyhedron should be compact:

```python
sage: C = Polyhedron(rays=[[1,2],[2,1]])
sage: C.ehrhart_polynomial()
Traceback (most recent call last):
...
ValueError: Ehrhart polynomial only defined for compact polyhedra
```

**fibration_generator**(dim)

Generate the lattice polytope fibrations.

For the purposes of this function, a lattice polytope fiber is a sub-lattice polytope. Projecting the plane spanned by the subpolytope to a point yields another lattice polytope, the base of the fiberation.

**INPUT:**

- dim – integer. The dimension of the lattice polytope fiber.

**OUTPUT:**
A generator yielding the distinct lattice polytope fibers of given dimension.

EXAMPLES:

```
sage: P = Polyhedron(toric_varieties.P4_11169().fan().rays(), base_ring=ZZ)  # needs paip sage.graphs
sage: list(P.fibration_generator(2))  # needs paip sage.graphs
[A 2-dimensional polyhedron in ZZ^4 defined as the convex hull of 3 vertices]
```

**find_translation** *(translated_polyhedron)*

Return the translation vector to *translated_polyhedron*.

**INPUT:**

- *translated_polyhedron* — a polyhedron.

**OUTPUT:**

A \(\mathbb{Z}\)-vector that translates *self* to *translated_polyhedron*. A `ValueError` is raised if *translated_polyhedron* is not a translation of *self*, this can be used to check that two polyhedra are not translates of each other.

**EXAMPLES:**

```
sage: X = polytopes.cube()
sage: X.find_translation(X + vector([2,3,5]))
(2, 3, 5)
sage: X.find_translation(2*X)
Traceback (most recent call last):
  ... ValueError: polyhedron is not a translation of self
```

**has_IP_property**

Test whether the polyhedron has the IP property.

The IP (interior point) property means that

- *self* is compact (a polytope).
- *self* contains the origin as an interior point.

This implies that

- *self* is full-dimensional.
- The dual polyhedron is again a polytope (that is, a compact polyhedron), though not necessarily a lattice polytope.

**EXAMPLES:**

```
sage: Polyhedron([(-1,-1),(1,0),(0,1)], base_ring=ZZ).has_IP_property()
True
sage: Polyhedron([(0,0),(1,0),(0,1)], base_ring=ZZ).has_IP_property()
False
sage: Polyhedron([(1,1),(1,0),(0,1)], base_ring=ZZ).has_IP_property()
False
```

**REFERENCES:**

- [PALP]
is_lattice_polytope()  
Return whether the polyhedron is a lattice polytope.  

OUTPUT:  
True if the polyhedron is compact and has only integral vertices, False otherwise. 

EXAMPLES:  

```sage  
polytopes.cross_polytope(3).is_lattice_polytope()  
True  
polytopes.regular_polygon(5).is_lattice_polytope()  
False  
```

is_reflexive()  
A lattice polytope is reflexive if it contains the origin in its interior and its polar with respect to the origin is a lattice polytope. 
Equivalently, it is reflexive if it is of the form \( \{ x \in \mathbb{R}^d : Ax \leq 1 \} \) for some integer matrix \( A \) and \( d \) the ambient dimension.  

EXAMPLES:  

```sage  
p = Polyhedron(vertices=[(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)], base_ring=ZZ)  
p.is_reflexive()  
True  
polytopes.hypercube(4).is_reflexive()  
True  
p = Polyhedron(vertices=[(1,0), (0,2), (-1,0), (0,-1)], base_ring=ZZ)  
p.is_reflexive()  
False  
p = Polyhedron(vertices=[(1,0), (0,2), (-1,0)], base_ring=ZZ)  
p.is_reflexive()  
False  
```

An error is raised, if the polyhedron is not compact:  

```sage  
p = Polyhedron(rays=[(1,)], base_ring=ZZ)  
p.is_reflexive()  
Traceback (most recent call last):  
...  
ValueError: the polyhedron is not compact  
```

minkowski_decompositions()  
Return all Minkowski sums that add up to the polyhedron. 

OUTPUT:  
A tuple consisting of pairs \((X, Y)\) of \( \mathbb{Z}\)-polyhedra that add up to self. All pairs up to exchange of the summands are returned, that is, \((Y, X)\) is not included if \((X, Y)\) already is.  

EXAMPLES:  

```sage  
square = Polyhedron(  
vertices=[[0,0],[1,0],[0,1],[1,1]])  
square.minkowski_decompositions()  
((A 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex,  
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4..  
(continues on next page)  

2.6. Base classes for polyhedra 635
(continued from previous page)

Example from [http://cgi.di.uoa.gr/~amantzaf/geo/](http://cgi.di.uoa.gr/~amantzaf/geo/)

```python
sage: Q = Polyhedron(vertices=[(4,0), (6,0), (0,3), (4,3)])
sage: R = Polyhedron(vertices=[(0,0), (5,0), (8,4), (3,2)])
sage: (Q+R).minkowski_decompositions()
((A 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 7
→vertices),
(A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4
→vertices,
(A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 7
→vertices),
(A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4
→vertices,
(A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 7
→vertices),
(A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 5
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4
→vertices),
(A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 7
→vertices),
(A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 5
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3
→vertices),
(A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 7
→vertices),
(A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 2
→vertices,
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 6
→vertices))
```

```python
sage: [ len(square.dilation(i).minkowski_decompositions())  
      for i in range(6) ]
[1, 2, 5, 8, 13, 18]
sage: [ integer_ceil((i^2 + 2*i - 1) / 2) + 1  
      for i in range(10) ]
[1, 2, 5, 8, 13, 18, 25, 32, 41, 50]
```

*normal_form*(algorithm='palp_native', permutation=False)

Return the normal form of vertices of the lattice polytope *self*.

**INPUT:**

- **algorithm** – must be "palp_native", the default.
- **permutation** – boolean (default: False); if True, the permutation applied to vertices to obtain the normal form is returned as well.
For more more detail, see \texttt{normal\_form()}. 

**EXAMPLES:**

We compute the normal form of the “diamond”:

```python
sage: d = Polyhedron([(1,0), (0,1), (-1,0), (0,-1)])
sage: d.vertices()
(A vertex at (-1, 0),
 A vertex at (0, -1),
 A vertex at (0, 1),
 A vertex at (1, 0))
sage: d.normal_form() #→ needs sage.groups
[(1, 0), (0, 1), (0, -1), (-1, 0)]
sage: d.lattice_polytope().normal_form("palp\_native") #→ needs sage.groups
M( 1, 0),
M( 0, 1),
M( 0, -1),
M(-1, 0)
in 2-d lattice M
```

Using \texttt{permutation=True}:

```python
sage: d.normal_form(permutation=True) #→ needs sage.groups
([(1, 0), (0, 1), (0, -1), (-1, 0)], ())
```

It is not possible to compute normal forms for polytopes which do not span the space:

```python
sage: p = Polyhedron([(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)])
sage: p.normal_form()
Traceback (most recent call last):
... ValueError: normal form is not defined for lower-dimensional polyhedra, got A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 4 vertices
```

The normal form is also not defined for unbounded polyhedra:

```python
sage: p = Polyhedron(vertices=[[1, 1]], rays=[[1, 0], [0, 1]], base_ring=ZZ)
sage: p.normal_form()
Traceback (most recent call last):
... ValueError: normal form is not defined for unbounded polyhedra, got A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and...
```

See \texttt{github issue #15280} for proposed extensions to these cases.

\texttt{polar()}

Return the polar (dual) polytope.

The polytope must have the IP-property (see \texttt{has\_IP\_property()}), that is, the origin must be an interior point. In particular, it must be full-dimensional.

**OUTPUT:**

The polytope whose vertices are the coefficient vectors of the inequalities of \texttt{self} with inhomogeneous term normalized to unity.
EXAMPLES:

```sage
p = Polyhedron(vertices=[(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)], base_ring=ZZ)
p.polar()
```

A 3-dimensional polyhedron in \( \mathbb{Z}^3 \) defined as the convex hull of 4 vertices
```
type(_)
```

\(<\text{class} 'sage.geometry.polyhedron.parent.Polyhedra_ZZ_ppl_with_category.element_class'>\)
```
p.polar().base_ring()
```

Integer Ring

2.6.12 Base class for polyhedra over RDF

```sage
class sage.geometry.polyhedron.base_RDF.Polyhedron_RDF (parent, Vrep, Hrep, 
Vrep_minimal=None, 
Hrep_minimal=None, 
pref_rep=None, mutable=False, 
**kwds)
```

Bases: `Polyhedron_base`

Base class for polyhedra over RDF.

2.7 Backends for Polyhedra

2.7.1 The cdd backend for polyhedral computations

```sage
class sage.geometry.polyhedron.backend_cdd.Polyhedron_QQ_cdd (parent, Vrep, Hrep, 
**kwds)
```

Polyhedra over QQ with cdd

INPUT:
- `parent` – the parent, an instance of `Polyhedra`.
- `Vrep` – a list `[vertices, rays, lines]` or None.
- `Hrep` – a list `[ieqs, eqns]` or None.

EXAMPLES:

```sage
from sage.geometry.polyhedron.parent import Polyhedra
parent = Polyhedra(QQ, 2, backend='cdd')
from sage.geometry.polyhedron.backend_cdd import Polyhedron_QQ_cdd
Polyhedron_QQ_cdd(parent, [ [(1,0),(0,1),(0,0)], [], []], None,
verbose=False)
```

A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 3 vertices

```sage
class sage.geometry.polyhedron.backend_cdd.Polyhedron_cdd (parent, Vrep, Hrep, 
Vrep_minimal=None, 
Hrep_minimal=None, 
pref_rep=None, 
mutable=False, **kwds)
```

638 Chapter 2. Polyhedral computations
2.7.2 The cdd backend for polyhedral computations, floating point version

```python
class sage.geometry.polyhedron.backend_cdd_rdf.Polyhedron_RDF_cdd(parent, Vrep, Hrep, **kwds):
    Bases: Polyhedron_cdd, Polyhedron_RDF

Polyhedra over RDF with cdd

INPUT:

- ambient_dim – integer. The dimension of the ambient space.
- Vrep – a list [vertices, rays, lines] or None.
- Hrep – a list [ieqs, eqns] or None.

EXAMPLES:
```

2.7.3 The Python backend

While slower than specialized C/C++ implementations, the implementation is general and works with any exact field in Sage that allows you to define polyhedra.

```
class sage.geometry.polyhedron.backend_field.Polyhedron_field(parent, Vrep, Hrep, **kwds):
    Bases: Polyhedron_RDF
```

2.7. Backends for Polyhedra 639
Combinatorial and Discrete Geometry, Release 10.3

Bases: Polyhedron_base

Polyhedra over all fields supported by Sage

INPUT:

- Vrep – a list [vertices, rays, lines] or None.
- Hrep – a list [ieqs, eqns] or None.

EXAMPLES:

```python
sage: p = Polyhedron(vertices=[[0,0],[AA(2).sqrt(),0],[0,AA(3).sqrt()]],
           rays=[[1,1]], lines=[], backend='field', base_ring=AA)
```

2.7.4 The Python backend, using number fields internally

```python
class sage.geometry.polyhedron.backend_number_field.Polyhedron_number_field(parent, Vrep, Hrep, Vrep_minimal=None, Hrep_minimal=None, pref_rep=None, mutable=False, **kwds)
```

Bases: Polyhedron_field, Polyhedron_base_number_field

Polyhedra whose data can be converted to number field elements

All computations are done internally using a fixed real embedded number field, which is determined automatically.

INPUT:

- Vrep – a list [vertices, rays, lines] or None.
- Hrep – a list [ieqs, eqns] or None.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[[1], [sqrt(2)]], backend='number_field'); P
```

(continues on next page)
2.7.5 The Normaliz backend for polyhedral computations

Note: This backend requires PyNormaliz. To install PyNormaliz, type `sage -i pynormaliz` in the terminal.

AUTHORS:

- Matthias Köppe (2016-12): initial version
- Jean-Philippe Labbé (2019-04): Expose normaliz features and added functionalities

```python
class sage.geometry.polyhedron.backend_normaliz.Polyhedron_QQ_normaliz(parent, Vrep, Hrep, normaliz_cone=None, normaliz_data=None, internal_base_ring=None, **kwds):
    Bases: Polyhedron_normaliz, Polyhedron_QQ

Polyhedra over $\mathbb{Q}$ with normaliz.

INPUT:

- `Vrep` - a list `[vertices, rays, lines]` or `None`
- `Hrep` - a list `[ieqs, eqns]` or `None`

EXAMPLES:

```python
sage: p = Polyhedron(vertices=[[0,0], [1,0], [0,1]],
                      rays=[[1,1]], lines=[],
```
```
ehrhart_series (\texttt{variable='t'} )

Return the Ehrhart series of a compact rational polyhedron.

The Ehrhart series is the generating function where the coefficient of $t^k$ is number of integer lattice points inside the $k$-th dilation of the polytope.

**INPUT:**

- \begin{itemize}
  - \texttt{variable} - string (default: 't')
\end{itemize}

**OUTPUT:**

A rational function.

**EXAMPLES:**

\begin{verbatim}
sage: S = Polyhedron(\texttt{vertices=[[0,1],[1,0]]})
sage: ES = S.ehrhart_series()
sage: ES.numerator() 1
sage: ES.denominator().factor() \((t - 1)^2\)
sage: C = Polyhedron(\texttt{vertices=[[0,0],[0,1],[1,0],[1,1]]})
sage: ES = C.ehrhart_series()
sage: ES.numerator() \(t^2 + 4*t + 1\)
sage: ES.denominator().factor() \((t - 1)^4\)
\end{verbatim}

The following example is from the Normaliz manual contained in the file \texttt{rational.in}:

\begin{verbatim}
sage: rat_poly = Polyhedron(\texttt{vertices=[[1/2,1/2],[-1/3,-1/3],[1/4,-1/2]]})
sage: ES = rat_poly.ehrhart_series()
sage: ES.numerator() 2*t^6 + 3*t^5 + 4*t^4 + 3*t^3 + t^2 + t + 1
sage: ES.denominator().factor() \((-1) * (t + 1)^2 * (t - 1)^3 * (t^2 + 1) * (t^2 + t + 1)\)
\end{verbatim}

The polyhedron should be compact:

\begin{verbatim}
sage: C = Polyhedron(\texttt{rays=[[1,2],[2,1]]})
Traceback (most recent call last):
... NotImplementedError: Ehrhart series can only be computed for compact...
\end{verbatim}

See also:

hilbert_series()
hilbert_series (grading, variable='t')

Return the Hilbert series of the polyhedron with respect to grading.

INPUT:

- grading – vector. The grading to use to form the Hilbert series
- variable – string (default: 't')

OUTPUT:

A rational function.

EXAMPLES:

```
sage: C = Polyhedron(backend='normaliz',
       ....: rays=[[0,0,1], [0,1,1], [1,0,1], [1,1,1]])
sage: HS = C.hilbert_series([1,1,1])
sage: HS.numerator()
t^2 + 1
sage: HS.denominator().factor()
(-1) * (t + 1) * (t - 1)^3 * (t^2 + t + 1)
```

By changing the grading, you can get the Ehrhart series of the square lifted at height 1:

```
sage: C.hilbert_series([0,0,1])
(t + 1)/(-t^3 + 3*t^2 - 3*t + 1)
```

Here is an example 2cone.in from the Normaliz manual:

```
sage: C = Polyhedron(backend='normaliz', rays=[[1,3], [2,1]])
sage: HS = C.hilbert_series([1,1])
sage: HS.numerator()
t^5 + t^4 + t^3 + t^2 + 1
sage: HS.denominator().factor()
(t + 1) * (t - 1)^2 * (t^2 + 1) * (t^2 + t + 1)
```

Here is the magic square example form the Normaliz manual:

```
sage: eq = [[0,1,1,1,-1,-1,-1, 0, 0, 0],
       ....: [0,0,1,1,-1, 0, 0,-1,-1,-1],
       ....: [0,1,0,1, 0,-1, 0, 0,-1, 0],
       ....: [0,1,1,0, 0,-1,-1, 0, 0,-1],
       ....: [0,0,1,1, 0,-1, 0, 0,-1, 0],
       ....: [0,1,1,0, 0,-1, 0, 0,-1, 0]]
sage: magic_square = (Polyhedron(eqns=eq, backend='normaliz')
       ....:     & Polyhedron(rays=identity_matrix(9).rows()))
sage: grading = [1,1,1,0,0,0,0,0,0]
sage: magic_square.hilbert_series(grading)
(t^6 + 2*t^3 + 1)/(-t^9 + 3*t^6 - 3*t^3 + 1)
```

See also:

ehrhart_series()
**integral_points** *(threshold=10000)*

Return the integral points in the polyhedron.

Uses either the naive algorithm (iterate over a rectangular bounding box) or triangulation + Smith form.

**INPUT:**

- **threshold** — integer (default: 10000); use the naive algorithm as long as the bounding box is smaller than this

**OUTPUT:**

The list of integral points in the polyhedron. If the polyhedron is not compact, a `ValueError` is raised.

**EXAMPLES:**

```sage
definitions:
    Polyhedron(vertices=[(-1,-1), (1,0), (1,1), (0,1)],
                backend='normaliz').integral_points()
((-1, -1), (0, 0), (0, 1), (1, 0), (1, 1))

sage: simplex = Polyhedron([(1,2,3), (2,3,7), (-2,-3,-11)],
                        backend='normaliz')

sage: simplex.integral_points()
((-2, -3, -11), (0, 0, -2), (1, 2, 3), (2, 3, 7))

The polyhedron need not be full-dimensional:

```sage
definitions:
    Polyhedron([[1,2,3,5], (2,3,7,5), (-2,-3,-11,5)],
                backend='normaliz').integral_points()
((-2, -3, -11, 5), (0, 0, -2, 5), (1, 2, 3, 5), (2, 3, 7, 5))

sage: point = Polyhedron([[2,3,7]],
                        backend='normaliz')

sage: point.integral_points()
((2, 3, 7),)

sage: empty = Polyhedron(backend='normaliz')

sage: empty.integral_points()
()```

Here is a simplex where the naive algorithm of running over all points in a rectangular bounding box no longer works fast enough:

```sage
definitions:
    v = [(1,0,7,-1), (-2,-2,4,-3), (-1,-1,-1,4), (2,9,0,-5), (-2,-1,5,1)]
    simplex = Polyhedron(v, backend='normaliz'); simplex
A 4-dimensional polyhedron in ZZ^4 defined as the convex hull of 5 vertices

sage: len(simplex.integral_points())
49```

A rather thin polytope for which the bounding box method would be a very bad idea (note this is a rational (non-lattice) polytope, so the other backends use the bounding box method):

```sage
definitions:
    P = Polyhedron(vertices=((0, 0), (178933,37121))) + 1/1000*polytopes.
    →hypercube(2)
    P = Polyhedron(backend='normaliz')
    P = Polyhedron(backend='normaliz')

sage: len(P.integral_points())
434 ```
Finally, the 3-d reflexive polytope number 4078:

```
sage: v = [(1,0,0), (0,1,0), (0,0,1), (0,0,-1), (0,-2,1),
       ....:  (-1,2,-1), (-1,2,-2), (-1,1,-2), (-1,-1,2), (-1,-3,2)]
sage: P = Polyhedron(v, backend='normaliz')
sage: pts1 = P.integral_points()
sage: all(P.contains(p) for p in pts1)
True
sage: pts2 = LatticePolytope(v).points()  # needs palp
sage: for p in pts1: p.set_immutable()
```

```
sage: set(pts1) == set(pts2)  # needs palp
True
sage: timeit(Polyhedron(v, backend='normaliz').integral_points())  # not tested - random
625 loops, best of 3: 1.41 ms per loop
sage: timeit(LatticePolytope(v).points())  # not tested - random
25 loops, best of 3: 17.2 ms per loop
```

**integral_points_generators()**

Return the integral points generators of the polyhedron.

Every integral point in the polyhedron can be written as a (unique) non-negative linear combination of integral points contained in the three defining parts of the polyhedron: the integral points (the compact part), the recession cone, and the lineality space.

**OUTPUT:**

A tuple consisting of the integral points, the Hilbert basis of the recession cone, and an integral basis for the lineality space.

**EXAMPLES:**

Normaliz gives a nonnegative integer basis of the lineality space:

```
sage: P = Polyhedron(backend='normaliz', lines=[[2,2]])
sage: P.integral_points_generators()
(((0, 0),), (), ((1, 1),))
```

A recession cone generated by two rays:

```
sage: C = Polyhedron(backend='normaliz', rays=[[1,2], [2,1]])
sage: C.integral_points_generators()
(((0, 0),), ((1, 1), (1, 2), (2, 1)), ())
```

Empty polyhedron:

```
sage: P = Polyhedron(backend='normaliz')
sage: P.integral_points_generators()
(((), (), ()))
```
class sage.geometry.polyhedron.backend_normaliz.Polyhedron_ZZ_normaliz(parent, Vrep, Hrep, normaliz_cone=None, normaliz_data=None, internal_base_ring=None, **kwds)

Bases: Polyhedron_QQ_normaliz, Polyhedron_ZZ

Polyhedra over Z with normaliz.

INPUT:

• Vrep – a list [vertices, rays, lines] or None
• Hrep – a list [ieqs, eqns] or None

EXAMPLES:

```
sage: p = Polyhedron(vertices=[(0,0), (1,0), (0,1)],
....: rays=[(1,1)], lines=[],
....: backend='normaliz', base_ring=ZZ)
sage: TestSuite(p).run()
```

class sage.geometry.polyhedron.backend_normaliz.Polyhedron_normaliz(parent, Vrep, Hrep, normaliz_cone=None, normaliz_data=None, internal_base_ring=None, **kwds)

Bases: Polyhedron_base_number_field

Polyhedra with normaliz

INPUT:

• parent – Polyhedra the parent
• Vrep – a list [vertices, rays, lines] or None; the V-representation of the polyhedron; if None, the polyhedron is determined by the H-representation
• Hrep – a list [ieqs, eqns] or None; the H-representation of the polyhedron; if None, the polyhedron is determined by the V-representation
• normaliz_cone – a PyNormaliz wrapper of a normaliz cone

Only one of Vrep, Hrep, or normaliz_cone can be different from None.

EXAMPLES:

```
sage: p = Polyhedron(vertices=[(0,0), (1,0), (0,1)],
....: rays=[(1,1)], lines=[],
....: backend='normaliz')
sage: TestSuite(p).run()
```

Two ways to get the full space:
A lower-dimensional affine cone; we test that there are no mysterious inequalities coming in from the homogenization:

```python
sage: P = Polyhedron(vertices=[(1, 1)], rays=[(0, 1)],
            ....: backend='normaliz')
sage: P.n_inequalities()
1
sage: P.equations()
(An equation (1, 0) x - 1 == 0,)
```

The empty polyhedron:

```python
sage: P = Polyhedron(ieqs=[[−2, 1, 1], [−3, −1, −1], [−4, 1, −2]],
            ....: backend='normaliz')
sage: P
The empty polyhedron in QQ^2
sage: P.Vrepresentation()
()  # needs sage.plot
sage: P.Hrepresentation()
(An equation -1 == 0,)
```

`integral_hull()`

Return the integral hull in the polyhedron.

This is a new polyhedron that is the convex hull of all integral points.

EXAMPLES:

Unbounded example from Normaliz manual, “a dull polyhedron”:

```python
sage: P = Polyhedron(ieqs=[[1, 0, 2], [3, 0, -2], [3, 2, -2]],
            ....: backend='normaliz')
sage: PI = P.integral_hull()
sage: P.plot(color=yellow) + PI.plot(color=green) # needs sage.plot
Graphics object consisting of 10 graphics primitives
sage: PI.Vrepresentation()
(A vertex at (-1, 0),
 A vertex at (0, 1),
 A ray in the direction (1, 0))
```

Nonpointed case:

```python
sage: P = Polyhedron(vertices=[[1/2, 1/3]], rays=[(1, 1)],
            ....: lines=[[-1, 1]], backend='normaliz')
sage: PI = P.integral_hull()
sage: PI.Vrepresentation()
(A vertex at (1, 0),
 A ray in the direction (1, 0),
 A line in the direction (1, -1))
```

Empty polyhedron:
2.7.6 The polymake backend for polyhedral computations

Note: This backend requires polymake. To install it, type `sage -i polymake` in the terminal.

AUTHORS:
- Matthias Köppe (2017-03): initial version

```python
class sage.geometry.polyhedron.backend_polymake.Polyhedron_QQ_polymake(
    parent, Vrep, Hrep, polymake_polytope=None, **kwds)

Bases: Polyhedron_polymake, Polyhedron_QQ
Polyhedra over \( \mathbb{Q} \) with polymake.

INPUT:
- `Vrep` — a list `[vertices, rays, lines]` or `None`
- `Hrep` — a list `[ieqs, eqns]` or `None`

EXAMPLES:
```
sage: p = Polyhedron(vertices=[(0,0),(1,0),(0,1)],
    rays=[(1,1)], lines=[],
    backend='polymake', base_ring=QQ) # optional -
    ....: rays=[(1,1)], lines=[],
    ....: backend='polymake', base_ring=QQ)
sage: TestSuite(p).run() # optional -
    ....: jupyter
```
```

class sage.geometry.polyhedron.backend_polymake.Polyhedron_ZZ_polymake(
    parent, Vrep, Hrep, polymake_polytope=None, **kwds)

Bases: Polyhedron_polymake, Polyhedron_ZZ
Polyhedra over \( \mathbb{Z} \) with polymake.

INPUT:
- `Vrep` — a list `[vertices, rays, lines]` or `None`
- `Hrep` — a list `[ieqs, eqns]` or `None`

EXAMPLES:
```
class sage.geometry.polyhedron.backend_polymake.Polyhedron_polymake(parent, Vrep, Hrep, polymake_polytope=None, **kwds)

Bases: Polyhedron_base

Polyhedra with polymake

INPUT:

• parent – Polyhedra the parent
• Vrep – a list [vertices, rays, lines] or None; the V-representation of the polyhedron; if None, the polyhedron is determined by the H-representation
• Hrep – a list [ieqs, eqns] or None; the H-representation of the polyhedron; if None, the polyhedron is determined by the V-representation
• polymake_polytope – a polymake polytope object

Only one of Vrep, Hrep, or polymake_polytope can be different from None.

EXAMPLES:

A lower-dimensional affine cone; we test that there are no mysterious inequalities coming in from the homogenization:

The empty polyhedron:

It can also be obtained differently:
Combinatorial and Discrete Geometry, Release 10.3

sage: # optional - jupymake
sage: P=Polyhedron(ieqs=[[-2, 1, 1], [-3, -1, 1], [-4, 1, -2]],
.....: backend='polymake')
sage: P
The empty polyhedron in QQ^2
sage: P.Vrepresentation()
()  
sage: P.Hrepresentation()
(An equation -1 == 0,)

The full polyhedron:

sage: Polyhedron(eqs=[[0, 0, 0]], backend='polymake')  # optional - jupymake
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 lines
sage: Polyhedron(ieqs=[[0, 0, 0]], backend='polymake')  # optional - jupymake
A 2-dimensional polyhedron in QQ^2 defined as the convex hull of 1 vertex and 2 lines

Quadratic fields work:

sage: V = polytopes.dodecahedron().vertices_list()  # needs sage.rings.number_field
sage: Polyhedron(vertices=V, backend='polymake')  # optional - jupymake,
A 3-dimensional polyhedron in (Number Field in sqrt5 with defining polynomial x^2 - 5
with sqrt5 = 2.236067977499790?)^3
defined as the convex hull of 20 vertices

2.7.7 The PPL (Parma Polyhedra Library) backend for polyhedral computations

class sage.geometry.polyhedron.backend_ppl.Polyhedron_QQ_ppl (parent, Vrep, Hrep,
ppl_polyhedron=None, mutable=False, **kwds)

Bases: Polyhedron_ppl, Polyhedron_QQ

Polyhedra over Q with ppl

INPUT:

• Vrep - a list [vertices, rays, lines] or None.
• Hrep - a list [ieqs, eqns] or None.

EXAMPLES:

sage: p = Polyhedron(vertices=[(0,0),(1,0),(0,1)], rays=[(1,1)], lines=[],
.....: backend='ppl', base_ring=QQ)
sage: TestSuite(p).run()

class sage.geometry.polyhedron.backend_ppl.Polyhedron ZZ_ppl (parent, Vrep, Hrep,
ppl_polyhedron=None, mutable=False, **kwds)

Bases: Polyhedron_ppl, Polyhedron ZZ
Polyhedra over $\mathbb{Z}$ with ppl

INPUT:

- $Vrep$ – a list [vertices, rays, lines] or None.
- $Hrep$ – a list [ieqs, eqns] or None.

EXAMPLES:

```python
sage: p = Polyhedron(vertices=[(0,0),(1,0),(0,1)], rays=[(1,1)], lines=[], backend='ppl', base_ring=ZZ)
sage: TestSuite(p).run()
```

```python
class sage.geometry.polyhedron.backend_ppl.Polyhedron_ppl(parent, Vrep, Hrep, ppl_polyhedron=None, mutable=False, **kwds)
```

Bases: Polyhedron_mutable

Polyhedra with ppl

INPUT:

- $Vrep$ – a list [vertices, rays, lines] or None.
- $Hrep$ – a list [ieqs, eqns] or None.

EXAMPLES:

```python
sage: p = Polyhedron(vertices=[(0,0),(1,0),(0,1)], rays=[(1,1)], lines=[], backend='ppl')
sage: TestSuite(p).run()
```

```python
Hrepresentation (index=None)

Return the objects of the H-representation. Each entry is either an inequality or a equation.

INPUT:

- index – either an integer or None

OUTPUT:

The optional argument is an index running from 0 to self.n_Hrepresentation()-1. If present, the H-representation object at the given index will be returned. Without an argument, returns the list of all H-representation objects.

EXAMPLES:

```python
sage: p = polytopes.hypercube(3)
sage: p.Hrepresentation(0)
An inequality (-1, 0, 0) x + 1 >= 0
sage: p.Hrepresentation(0) == p.Hrepresentation()[0]
True
```

```python
sage: P = p.parent()
sage: p = P._element_constructor_(p, mutable=True)
sage: p.Hrepresentation(0)
An inequality (0, 0, -1) x + 1 >= 0
sage: p._clear_cache()
sage: p.Hrepresentation(0)
An inequality (0, 0, -1) x + 1 >= 0
sage: TestSuite(p).run()
```
**Vrepresentation** *(index=None)*

Return the objects of the V-representation. Each entry is either a vertex, a ray, or a line.

See `sage.geometry.polyhedron.constructor` for a definition of vertex/ray/line.

**INPUT:**

- `index` – either an integer or `None`

**OUTPUT:**

The optional argument is an index running from 0 to `self.n_Vrepresentation()` - 1. If present, the V-representation object at the given index will be returned. Without an argument, returns the list of all V-representation objects.

**EXAMPLES:**

```python
sage: p = polytopes.cube()
sage: p.Vrepresentation(0)
A vertex at (1, -1, -1)
sage: P = p.parent()
sage: p = P._element_constructor_(p, mutable=True)
sage: p.Vrepresentation(0)
A vertex at (-1, -1, -1)
sage: p._clear_cache()
sage: p.Vrepresentation(0)
A vertex at (-1, -1, -1)
sage: TestSuite(p).run()
```

**set_immutable()**

Make this polyhedron immutable. This operation cannot be undone.

**EXAMPLES:**

```python
sage: p = Polyhedron([[1, 1]], mutable=True)
sage: p.is_mutable()
True
sage: hasattr(p, "_Vrepresentation")
False
sage: p.set_immutable()
sage: hasattr(p, "_Vrepresentation")
True
```

### 2.7.8 Double Description Algorithm for Cones

This module implements the double description algorithm for extremal vertex enumeration in a pointed cone following [FP1996]. With a little bit of preprocessing (see `double_description_inhomogeneous`) this defines a backend for polyhedral computations. But as far as this module is concerned, `inequality` always means without a constant term and the origin is always a point of the cone.

**EXAMPLES:**

```python
sage: from sage.geometry.polyhedron.double_description import StandardAlgorithm
sage: A = matrix(QQ, 
[(1,0,1), (0,1,1), (-1,-1,1)])
sage: alg = StandardAlgorithm(A); alg
Pointed cone with inequalities
(continues on next page)```
(1, 0, 1)
(0, 1, 1)
(-1, -1, 1)
sage: DD, _ = alg.initial_pair(); DD
Double description pair \((A, R)\) defined by
\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
-1 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
2/3 & -1/3 & -1/3 \\
-1/3 & 2/3 & -1/3 \\
1/3 & 1/3 & 1/3 \\
\end{pmatrix}
\]
The implementation works over any exact field that is embedded in \(\mathbb{R}\), for example:

```
sage: from sage.geometry.polyhedron.double_description import StandardAlgorithm
sage: A = matrix(AA, [(1,0,1), (0,1,1), (-AA(2).sqrt(),-AA(3).sqrt(),1),
# needs sage.rings.number_field....:
(-AA(3).sqrt(),-AA(2).sqrt(),1)])
sage: alg = StandardAlgorithm(A)
sage: alg.run().R
```

```
[(-0.4177376677004119?, 0.582262332295881?, 0.4177376677004119?),
 (-0.2411809548974793?, -0.2411809548974793?, 0.2411809548974793?),
 (0.07665629029830300?, 0.2411809548974793?, 0.2411809548974793?),
 (0.582262332295881?, -0.4177376677004119?, 0.4177376677004119?)]
```

**class** `sage.geometry.polyhedron.double_description.DoubleDescriptionPair(problem, A_rows, R_cols)`

Bases: `object`

Base class for a double description pair \((A, R)\)

**Warning:** You should use the `Problem.initial_pair()` or `Problem.run()` to generate double description pairs for a set of inequalities, and not generate `DoubleDescriptionPair` instances directly.

**INPUT:**

- `problem` – instance of `Problem`.
- `A_rows` – list of row vectors of the matrix \(A\). These encode the inequalities.
- `R_cols` – list of column vectors of the matrix \(R\). These encode the rays.

**R_by_sign\((a)\)**

Classify the rays into those that are positive, zero, and negative on \(a\).

**INPUT:**


**OUTPUT:**

A triple consisting of the rays (columns of \(R\)) that are positive, zero, and negative on \(a\). In that order.

**EXAMPLES:**

```sage
define_problem():
    from sage.geometry.polyhedron.double_description import *
    return Problem()

sage: from sage.geometry.polyhedron.double_description import *
    StandardAlgorithm
sage: A = matrix(QQ, [(1,0,1), (0,1,1), (-1,-1,1)])
```
sage: DD, _ = StandardAlgorithm(A).initial_pair()
sage: DD.R_by_sign(vector([1,-1,0]))
([(2/3, -1/3, 1/3)], [(-1/3, -1/3, 1/3)], [(-1/3, 2/3, 1/3)])
sage: DD.R_by_sign(vector([1,1,1]))
([(2/3, -1/3, 1/3), (-1/3, 2/3, 1/3)], [], [(-1/3, -1/3, 1/3)])

are_adjacent (r1, r2)
Return whether the two rays are adjacent.

INPUT:
• r1, r2 – two rays.

OUTPUT:
Boolean. Whether the two rays are adjacent.

EXAMPLES:

sage: from sage.geometry.polyhedron.double_description import *

sage: A = matrix(QQ, [(0,1,0), (1,0,0), (0,-1,1), (-1,0,1)])
sage: DD = StandardAlgorithm(A).run()
sage: DD.are_adjacent(DD.R[0], DD.R[1])
True
sage: DD.are_adjacent(DD.R[0], DD.R[2])
True
sage: DD.are_adjacent(DD.R[0], DD.R[3])
False

cone ()
Return the cone defined by A.

This method is for debugging only. Assumes that the base ring is Q.

OUTPUT:
The cone defined by the inequalities as a Polyhedron(), using the PPL backend.

EXAMPLES:

sage: from sage.geometry.polyhedron.double_description import *

sage: A = matrix(QQ, [(1,0,1), (0,1,1), (-1,-1,1)])
sage: DD, _ = StandardAlgorithm(A).initial_pair()
sage: DD.cone().Hrepresentation()
(An inequality (-1, -1, 1) x + 0 >= 0,
An inequality (0, 1, 1) x + 0 >= 0,
An inequality (1, 0, 1) x + 0 >= 0)

dual ()
Return the dual.

OUTPUT:
For the double description pair \((A, R)\) this method returns the dual double description pair \((R^T, A^T)\)

EXAMPLES:
```python
sage: from sage.geometry.polyhedron.double_description import Problem
sage: A = matrix(QQ, [(0,1,0), (1,0,0), (0,-1,1), (-1,0,1)])
sage: DD, _ = Problem(A).initial_pair()
sage: DD
Double description pair (A, R) defined by
[ 0 1 0]  [0 1 0]
A = [ 1 0 0], R = [1 0 0]
[ 0 -1 1]  [1 0 1]
```

```python
sage: DD.dual()
Double description pair (A, R) defined by
[0 1 1]  [ 0 1 0]
A = [1 0 0], R = [ 1 0 -1]
[0 0 1]  [ 0 0 1]
```

### first_coordinate_plane()

Restrict to the first coordinate plane.

**OUTPUT:**

A new double description pair with the constraint $x_0 = 0$ added.

**EXAMPLES:**

```python
sage: A = matrix([[(1, 1), (-1, 1)]])
sage: from sage.geometry.polyhedron.double_description import StandardAlgorithm
sage: DD, _ = StandardAlgorithm(A).initial_pair()
sage: DD
Double description pair (A, R) defined by
A = [ 1 1], R = [ 1/2 -1/2]
[-1 1]  [ 1/2 1/2]

sage: DD.first_coordinate_plane()
Double description pair (A, R) defined by
A = [ 1 1], R = [ 0]
[-1 0]  [1/2]
[ 1 0]
```

### inner_product_matrix()

Return the inner product matrix between the rows of $A$ and the columns of $R$.

**OUTPUT:**

A matrix over the base ring. There is one row for each row of $A$ and one column for each column of $R$.

**EXAMPLES:**

```python
sage: A = matrix(QQ, [(1,0,1), (0,1,1), (-1,-1,1)])
sage: alg = StandardAlgorithm(A)
sage: DD, _ = alg.initial_pair()
sage: DD.inner_product_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

### is_extremal(ray)

Test whether the ray is extremal.

2.7. Backends for Polyhedra
EXAMPLES:

```python
gate: from sage.geometry.polyhedron.double_description import StandardAlgorithm
gate: A = matrix(QQ, [(0,1,0), (1,0,0), (0,-1,1), (-1,0,1)])
gate: DD = StandardAlgorithm(A).run()
gate: DD.is_extremal(DD.R[0])
True
```

**matrix_space** *(nrows, ncols)*

Return a matrix space of size `nrows` and `ncols` over the base ring of `self`.

These matrix spaces are cached to avoid their creation in the very demanding `add_inequality()` and more precisely `are_adjacent()`.

EXAMPLES:

```python
gate: from sage.geometry.polyhedron.double_description import Problem
gate: A = matrix(QQ, [(1,0,1), (0,1,1), (-1,-1,1)])
gate: DD, _ = Problem(A).initial_pair()
gate: DD.matrix_space(2,2)
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
gate: DD.matrix_space(3,2)
Full MatrixSpace of 3 by 2 dense matrices over Rational Field
```

**verify()**

Validate the double description pair.

This method used the PPL backend to check that the double description pair is valid. An assertion is triggered if it is not. Does nothing if the base ring is not `QQ`.

EXAMPLES:

```python
gate: from sage.geometry.polyhedron.double_description import DoubleDescriptionPair, Problem
gate: A = matrix(QQ, [(1,0,1), (0,1,1), (-1,-1,1)])
gate: alg = Problem(A)
gate: DD = DoubleDescriptionPair(alg,
...:   [(1, 0, 3), (0, 1, 1), (-1, -1, 1)],
...:   [(2/3, -1/3, 1/3), (-1/3, 2/3, 1/3), (-1/3, -1/3, 1/3)])
gate: DD.verify()
Traceback (most recent call last):
...
  assert A_cone == R_cone
AssertionError
```

**zero_set** *(ray)*

Return the zero set (active set) \(Z(r)\).

INPUT:
• ray – a ray vector.

OUTPUT:
A set containing the inequality vectors that are zero on ray.

EXAMPLES:

```python
sage: from sage.geometry.polyhedron.double_description import Problem
sage: A = matrix(QQ, [(1,0,1), (0,1,1), (-1,-1,1)])

sage: DD, _ = Problem(A).initial_pair()

sage: r = DD.R[0]; r
(2/3, -1/3, 1/3)

sage: DD.zero_set(r)
{(-1, -1, 1), (0, 1, 1)}
```

class sage.geometry.polyhedron.double_description.Problem(A)
Base class for implementations of the double description algorithm
It does not make sense to instantiate the base class directly, it just provides helpers for implementations.

INPUT:
• A – a matrix. The rows of the matrix are interpreted as homogeneous inequalities \(Ax \geq 0\). Must have maximal rank.

A()
Return the rows of the defining matrix \(A\).

OUTPUT:
The matrix \(A\) whose rows are the inequalities.

EXAMPLES:

```python
sage: A = matrix([[1, 1], [-1, 1]])

sage: from sage.geometry.polyhedron.double_description import Problem

sage: Problem(A).A()
[(1, 1), (-1, 1)]
```

A_matrix()
Return the defining matrix \(A\).

OUTPUT:
Matrix whose rows are the inequalities.

EXAMPLES:

```python
sage: A = matrix([[1, 1], [-1, 1]])

sage: from sage.geometry.polyhedron.double_description import Problem

sage: Problem(A).A_matrix()
[[ 1  1]
 [ 1  1]]
```

base_ring()
Return the base field.

OUTPUT:
A field.
EXAMPLES:

```sage
A = matrix(AA, [(1, 1), (-1, 1)])
Problem(A).base_ring() # needs sage.rings.number_field
```

`dim()`

Return the ambient space dimension.

**OUTPUT:**

Integer. The ambient space dimension of the cone.

**EXAMPLES:**

```sage
A = matrix(QQ, [(1, 1), (-1, 1)])
Problem(A).dim()
```

`initial_pair()`

Return an initial double description pair.

Picks an initial set of rays by selecting a basis. This is probably the most efficient way to select the initial set.

**INPUT:**

- `pair_class` – subclass of `DoubleDescriptionPair`.

**OUTPUT:**

A pair consisting of a `DoubleDescriptionPair` instance and the tuple of remaining unused inequalities.

**EXAMPLES:**

```sage
A = matrix([(-1, 1), (-1, 2), (1/2, -1/2), (1/2, 2)])
DD, remaining = Problem(A).initial_pair()
DD.verify()
remaining
```

`pair_class`

alias of `DoubleDescriptionPair`

**class** `sage.geometry.polyhedron.double_description.StandardAlgorithm(A)`

Bases: `Problem`

Standard implementation of the double description algorithm

See [FP1996] for the definition of the “Standard Algorithm”.

**EXAMPLES:**

```sage
A = matrix(QQ, [(1, 1), (-1, 1)])
StandardAlgorithm(A).run()
```

Chapter 2. Polyhedral computations
pair_class
    alias of StandardDoubleDescriptionPair

run()
Run the Standard Algorithm.

OUTPUT:
A double description pair \((A, R)\) of all inequalities as a DoubleDescriptionPair. By virtue of the double description algorithm, the columns of \(R\) are the extremal rays.

EXAMPLES:

```sage
from sage.geometry.polyhedron.double_description import StandardAlgorithm
A = matrix(QQ, [(0,1,0), (1,0,0), (0,-1,1), (-1,0,1)])
StandardAlgorithm(A).run()
```

Double description pair \((A, R)\) defined by
\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & -1 & 1 \\
-1 & 0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

class sage.geometry.polyhedron.double_description.StandardDoubleDescriptionPair

Bases: DoubleDescriptionPair

Double description pair for the “Standard Algorithm”.

See StandardAlgorithm.

add_inequality(a)
Add the inequality \(a\) to the matrix \(A\) of the double description.

INPUT:

- \(a\) – vector. An inequality.

EXAMPLES:

```sage
A = matrix([(-1, 1, 0), (-1, 2, 1), (1/2, -1/2, -1)])
from sage.geometry.polyhedron.double_description import StandardAlgorithm
DD, _ = StandardAlgorithm(A).initial_pair()
DD.add_inequality(vector([1,0,0]))
```

Double description pair \((A, R)\) defined by
\[
\begin{bmatrix}
-1 & 1 & 0 \\
-1 & 2 & 1 \\
1/2 & -1/2 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & -1 & -1/2 & -2 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

sage.geometry.polyhedron.double_description.random_inequalities(d, n)
Random collections of inequalities for testing purposes.

INPUT:

- \(d\) – integer. The dimension.
• $n$ – integer. The number of random inequalities to generate.

OUTPUT:

A random set of inequalities as a StandardAlgorithm instance.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.double_description import random_inequalities
sage: P = random_inequalities(5, 10)
sage: P.run().verify()
```

### 2.7.9 Double Description for Arbitrary Polyhedra

This module is part of the python backend for polyhedra. It uses the double description method for cones double_description to find minimal H/V-representations of polyhedra. The latter works with cones only. This is sufficient to treat general polyhedra by the following construction: Any polyhedron can be embedded in one dimension higher in the hyperplane $(1, *, *, ..., *)$. The cone over the embedded polyhedron will be called the homogenized cone in the following. Conversely, intersecting the homogenized cone with the hyperplane $x_0 = 1$ gives you back the original polyhedron.

While slower than specialized C/C++ implementations, the implementation is general and works with any field in Sage that allows you to define polyhedra.

**Note:** If you just want polyhedra over arbitrary fields then you should just use the Polyhedron() constructor.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.double_description_inhomogeneous import Hrep2Vrep
    ....: Vrep2Hrep
sage: Hrep2Vrep(QQ, 2, [{(1, 2, 3), (2, 4, 3)}], [])
[-1/2|-1/2 1/2]
[ 0| 2/3 -1/3]
```

Note that the columns of the printed matrix are the vertices, rays, and lines of the minimal V-representation. Dually, the rows of the following are the inequalities and equations:

```
sage: Vrep2Hrep(QQ, 2, [(-1/2, 0)], [(-1/2, 2/3), (1/2, -1/3)], [])
[1 2 3]
[2 4 3]
[------]
```

**class** sage.geometry.polyhedron.double_description_inhomogeneous.Hrep2Vrep**(base_ring, dim, inequalities, equations)**

**Bases:** PivotedInequalities

Convert H-representation to a minimal V-representation.

**INPUT:**

- **base_ring** – a field.
- **dim** – integer. The ambient space dimension.
• inequalities – list of inequalities. Each inequality is given as constant term, dim coefficients.

• equations – list of equations. Same notation as for inequalities.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.double_description_inhomogeneous import Hrep2Vrep

sage: Hrep2Vrep(QQ, 2, [(1,2,3), (2,4,3)], [])
[-1/2 -1/2 1/2]
[ 0| 2/3 -1/3]

sage: Hrep2Vrep(QQ, 2, [(1,2,3), (2,-2,-3)], [])
[ 1 -1/2 1]
[ 0 0 -2/3]

sage: Hrep2Vrep(QQ, 2, [(1,2,3), (2,2,3)], [])
[-1/2 1/2 1]
[ 0| 0 -2/3]

sage: Hrep2Vrep(QQ, 2, [(1,2,3), (2,4,3), (5,-1,-2)], [])
[-19/5 -1/2 2/33 1/11]
[ 22/5 0 -1/3 -2/33]

sage: Hrep2Vrep(QQ, 2, [(0,2,3), (0,4,3), (0,-1,-2)], [])
[ 0 1/2 1/3]
[ 0 -1/3 -1/6]

sage: Hrep2Vrep(QQ, 2, [(1,2,3), (7,8,9)])

sage: H = Hrep2Vrep(QQ, 2, [[1,2]], [])

sage: H.verify([[1,2]], [])
# universe

sage: H = Hrep2Vrep(QQ, 2, [[-1,0,0]], [])

sage: H.verify([], [])
# universe
```

`verify (inequalities, equations)`

Compare result to PPL if the base ring is QQ.

This method is for debugging purposes and compares the computation with another backend if available.

INPUT:

• inequalities, equations – see `Hrep2Vrep`.

EXAMPLES:

```
sage: from sage.geometry.polyhedron.double_description_inhomogeneous import Hrep2Vrep

sage: H = Hrep2Vrep(QQ, 1, [[1,2]], [])

sage: H.verify([[1,2]], [])
```

class `sage.geometry.polyhedron.double_description_inhomogeneous.PivotedInequalities(base_ring, dim)`

Bases: `sage.geometry.polyhedron.double_description_inhomogeneous.Inequalities`

Base class for inequalities that may contain linear subspaces

INPUT:
• `base_ring` – a field.
• `dim` – integer. The ambient space dimension.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.double_description_inhomogeneous import PivotedInequalities
sage: piv = PivotedInequalities(QQ, 2)
sage: piv._pivot_inequalities(matrix([[(1,1,3), (5,5,7)]]))
[1 3]
[5 7]
sage: piv._pivots
(0, 2)
sage: piv._linear_subspace
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[ 1 -1  0]
```

**class** `sage.geometry.polyhedron.double_description_inhomogeneous.Vrep2Hrep` (`base_ring`, `dim`, `vertices`, `rays`, `lines`)

**Bases:** `PivotedInequalities`

Convert V-representation to a minimal H-representation.

**INPUT:**
• `base_ring` – a field.
• `dim` – integer. The ambient space dimension.
• `vertices` – list of vertices. Each vertex is given as list of `dim` coordinates.
• `rays` – list of rays. Each ray is given as list of `dim` coordinates, not all zero.
• `lines` – list of line generators. Each line is given as list of `dim` coordinates, not all zero.

**EXAMPLES:**

```
sage: from sage.geometry.polyhedron.double_description_inhomogeneous import Vrep2Hrep
sage: Vrep2Hrep(QQ, 2, [(-1/2,0)], [(-1/2,2/3), (1/2,-1/3)], [])
[1 2 3]
[-----]
sage: Vrep2Hrep(QQ, 2, [(1,0), (-1/2,0)], [], [(1,-2/3)])
[ 1/3 2/3 1]
[ 2/3 -2/3 -1]
[------------]
sage: Vrep2Hrep(QQ, 2, [(-1/2,0)], [(1/2,0)], [(1,-2/3)])
[1 2 3]
[-----]
sage: Vrep2Hrep(QQ, 2, [(1,1), (0,4), (-2,-3)], [], [])
[8/13 7/13 -2/13]

(continues on next page)```
Combinatorial and Discrete Geometry, Release 10.3

verify (vertices, rays, lines)

Compare result to PPL if the base ring is QQ.

This method is for debugging purposes and compares the computation with another backend if available.

INPUT:

• vertices, rays, lines – see \texttt{Vrep2Hrep}.

EXAMPLES:

\begin{verbatim}
sage: from sage.geometry.polyhedron.double_description_inhomogeneous import ...
    \rightarrow \text{Vrep2Hrep}
sage: vertices = \([-1/2, 0]\]
sage: rays = \([-1/2, 2/3], (1/2, -1/3)\]
sage: lines = []
sage: V2H = Vrep2Hrep(QQ, 2, vertices, rays, lines)
sage: V2H.verify(vertices, rays, lines)
\end{verbatim}
3.1 Triangulations of a point configuration

A point configuration is a finite set of points in Euclidean space or, more generally, in projective space. A triangulation is a simplicial decomposition of the convex hull of a given point configuration such that all vertices of the simplices end up lying on points of the configuration. That is, there are no new vertices apart from the initial points.

Note that points that are not vertices of the convex hull need not be used in the triangulation. A triangulation that does make use of all points of the configuration is called fine, and you can restrict yourself to such triangulations if you want. See \texttt{PointConfiguration} and \texttt{restrict_to_fine_triangulations()} for more details.

Finding a single triangulation and listing all connected triangulations is implemented natively in this package. However, for more advanced options [TOPCOM] needs to be installed. It is available as an optional package for Sage, and you can install it with the shell command

\begin{verbatim}
sage -i topcom
\end{verbatim}

\textbf{Note:} TOPCOM and the internal algorithms tend to enumerate triangulations in a different order. This is why we always explicitly specify the engine as \texttt{engine='topcom'} or \texttt{engine='internal'} in the doctests. In your own applications, you do not need to specify the engine. By default, TOPCOM is used if it is available and the internal algorithms are used otherwise.

\textbf{EXAMPLES:}

First, we select the internal implementation for enumerating triangulations:

\begin{verbatim}
sage: PointConfiguration.set_engine('internal')  # to make doctests independent of...
\end{verbatim}

A 2-dimensional point configuration:

\begin{verbatim}
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]]); p
A point configuration in affine 2-space over Integer Ring consisting of 5 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
\end{verbatim}

A triangulation of it:

\begin{verbatim}
sage: t = p.triangulate(); t  # a single triangulation
(\<1,3,4\>, \<2,3,4\>)
sage: len(t)
2
\end{verbatim}

(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

List triangulations of it:

```
sage: list(p.triangulations())
[(<1,3,4>, <2,3,4>),
 (<0,1,3>, <0,1,4>, <0,2,3>, <0,2,4>),
 (<1,2,3>, <1,2,4>),
 (<0,1,2>, <0,1,4>, <0,2,4>, <1,2,3>)]
sage: p_fine = p.restrict_to_fine_triangulations(); p_fine
A point configuration in affine 2-space over Integer Ring consisting
of 5 points. The triangulations of this point configuration are
assumed to be connected, fine, not necessarily regular.
sage: list(p_fine.triangulations())
[(<0,1,3>, <0,1,4>, <0,2,3>, <0,2,4>),
 (\ldots)]
```

Graphics object consisting of 12 graphics primitives
A 3-dimensional point configuration:

```python
sage: p = [[0,-1,-1], [0,0,1], [0,1,0],
       [1,-1,-1], [1,0,1], [1,1,0]]
sage: points = PointConfiguration(p)
sage: triang = points.triangulate()
sage: triang.plot(axes=False)  # needs sage.plot
```

![Triangulation plot](Image)

The standard example of a non-regular triangulation (requires TOPCOM):

```python
sage: # optional - topcom
sage: PointConfiguration.set_engine('topcom')
```

```python
sage: p = PointConfiguration([[-1,-5/9], [0,10/9], [1,-5/9],
                          [-2,-10/9], [0,20/9], [2,-10/9]])
sage: p_regular = p.restrict_to_regular_triangulations(True)
sage: regular = p_regular.triangulations_list()
sage: p_nonregular = p.restrict_to_regular_triangulations(False)
sage: nonregular = p_nonregular.triangulations_list()
sage: len(regular)
16
sage: len(nonregular)
2
```

```python
sage: nonregular[0].plot(aspect_ratio=1, axes=False)  # needs sage.plot
```

```
```

Note that the points need not be in general position. That is, the points may lie in a hyperplane and the linear dependencies will be removed before passing the data to TOPCOM which cannot handle it:

```python
sage: points = [[0,0,0,1], [0,3,0,1], [3,0,0,1], [0,0,1,1],
            [0,3,1,1], [3,0,1,1], [1,1,2,1]]
sage: points = [p + [1,2,3] for p in points]
sage: pc = PointConfiguration(points)
sage: pc.ambient_dim()
7
```

(continues on next page)
AUTHORS:

- Volker Braun: initial version, 2010
- Josh Whitney: added functionality for computing volumes and secondary polytopes of PointConfigurations
- Marshall Hampton: improved documentation and doctest coverage
- Volker Braun: Cythonized parts of it, added a C++ implementation of the bistellar flip algorithm to enumerate all connected triangulations.
- Volker Braun 2011: switched the triangulate() method to the placing triangulation (faster).

```python
sage: pc.dim()
sage: pc.triangulate()
(0, 1, 2, 6), (0, 1, 3, 6), (0, 2, 3, 6), (1, 2, 4, 6), (1, 3, 4, 6), (2, 3, 5, 6), (2, 4, 5, 6)
sage: _ in pc.triangulations()
True
sage: len(pc.triangulations_list())
26
```

class `sage.geometry.triangulation.point_configuration.PointConfiguration` (`points`,
connected, fine, regular, star, defined_affine)

Bases: `UniqueRepresentation, PointConfiguration_base`

A collection of points in Euclidean (or projective) space.

This is the parent class for the triangulations of the point configuration. There are a few options to specifically select what kind of triangulations are admissible.

INPUT:

The constructor accepts the following arguments:

- `points` – the points. Technically, any iterable of iterables will do. In particular, a `PointConfiguration` can be passed.
- `projective` – boolean (default: False). Whether the point coordinates should be interpreted as projective (True) or affine (False) coordinates. If necessary, points are projectivized by setting the last homogeneous coordinate to one and/or affine patches are chosen internally.
- `connected` – boolean (default: True). Whether the triangulations should be connected to the regular triangulations via bistellar flips. These are much easier to compute than all triangulations.
- `fine` – boolean (default: False). Whether the triangulations must be fine, that is, make use of all points of the configuration.
- `regular` – boolean or None (default: None). Whether the triangulations must be regular. A regular triangulation is one that is induced by a piecewise-linear convex support function. In other words, the shadows of the faces of a polyhedron in one higher dimension.
  - True: Only regular triangulations.
  - False: Only non-regular triangulations.
– None (default): Both kinds of triangulation.

• star – either None or a point. Whether the triangulations must be star. A triangulation is star if all maximal simplices contain a common point. The central point can be specified by its index (an integer) in the given points or by its coordinates (anything iterable.)

EXAMPLES:

```sage
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]]);
A point configuration in affine 2-space over Integer Ring
consisting of 5 points. The triangulations of this point
configuration are assumed to be connected, not necessarily fine,
not necessarily regular.
sage: p.triangulate() # a single triangulation
(<1,3,4>, <2,3,4>)
```

Element

alias of Triangulation

Gale_transform(points=None)

Return the Gale transform of self.

INPUT:

• points – a tuple of points or point indices or None (default). A subset of points for which to compute the Gale transform. By default, all points are used.

OUTPUT:

A matrix over base_ring().

EXAMPLES:

```sage
sage: pc = PointConfiguration([(0,0), (1,0), (0,1), (1,1), (0,1)])
sage: pc.Gale_transform()  
[ 1 -1 0 1 -1]
[ 0 0 1 -2 1]
sage: pc.Gale_transform((0,1,3,4))
[ 1 -1 1 -1]
sage: points = (pc.point(0), pc.point(1), pc.point(3), pc.point(4))
sage: pc.Gale_transform(points)
[ 1 -1 1 -1]
```

an_element()

Synonymous for triangulate().

bistellar_flips()

Return the bistellar flips.

OUTPUT:

The bistellar flips as a tuple. Each flip is a pair \((T_+, T_-)\) where \(T_+\) and \(T_-\) are partial triangulations of the point configuration.

EXAMPLES:

```sage
sage: pc = PointConfiguration([(0,0), (1,0), (0,1), (1,1)])
sage: pc.bistellar_flips()
(((<0,1,3>, <0,2,3>), (<0,1,2>, <1,2,3>)),)
```

(continues on next page)
sage: Tpos, Tneg = pc.bistellar_flips()[0]
sage: Tpos.plot(axes=False)  # needs sage.plot
Graphics object consisting of 11 graphics primitives
sage: Tneg.plot(axes=False)  # needs sage.plot
Graphics object consisting of 11 graphics primitives

The 3d analog:

sage: pc = PointConfiguration([(0,0,0),(0,2,0),(0,0,2),(-1,0,0),(1,1,1)])
sage: pc.bistellar_flips()
(((<0,1,2,3>, <0,1,2,4>), (<0,1,3,4>, <0,2,3,4>, <1,2,3,4>)),)

A 2d flip on the base of the pyramid over a square:

sage: pc = PointConfiguration([(0,0,0),(0,2,0),(0,0,2),(0,2,2),(1,1,1)])
sage: pc.bistellar_flips()  
(((<0,1,3>, <0,2,3>), (<0,1,2>, <1,2,3>)),)

sage: Tpos, Tneg = pc.bistellar_flips()[0]  
sage: Tpos.plot(axes=False)  # needs sage.plot
Graphics3d Object

\textbf{circuits()}  
Return the circuits of the point configuration.
Roughly, a circuit is a minimal linearly dependent subset of the points. That is, a circuit is a partition
\[
\{0, 1, \ldots, n - 1\} = C_+ \cup C_0 \cup C_-
\]
such that there is an (unique up to an overall normalization) affine relation
\[
\sum_{i \in C_+} \alpha_i \vec{p}_i = \sum_{j \in C_-} \alpha_j \vec{p}_j
\]
with all positive (or all negative) coefficients, where \(\vec{p}_i = (p_1, \ldots, p_k, 1)\) are the projective coordinates of the \(i\)-th point.

\textbf{OUTPUT:}
The list of (unsigned) circuits as triples \((C_+, C_0, C_-)\). The swapped circuit \((C_-, C_0, C_+)\) is not returned separately.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: p = PointConfiguration([(0,0), (+1,0), (-1,0), (0,+1), (0,-1)])
sage: sorted(p.circuits())
[((0,), (1, 2), (3, 4)), ((0,), (3, 4), (1, 2)), ((1, 2), (0,), (3, 4))]
\end{verbatim}

\textbf{circuits_support()}  
A generator for the supports of the circuits of the point configuration.
See \textbf{circuits()} for details.

\textbf{OUTPUT:}
A generator for the supports \(C_- \cup C_+\) (returned as a Python tuple) for all circuits of the point configuration.

\textbf{EXAMPLES:}

\begin{verbatim}
```
sage: p = PointConfiguration([(0,0), (+1,0), (-1,0), (0,+1), (0,-1)])
sage: sorted(p.circuits_support())
[(0, 1, 2), (0, 3, 4), (1, 2, 3, 4)]

contained_simplex (large=True, initial_point=None, point_order=None)

Return a simplex contained in the point configuration.

INPUT:

- large – boolean. Whether to attempt to return a large simplex.
- initial_point – a Point or None (default). A specific point to start with when picking the simplex vertices.
- point_order – a list or tuple of (some or all) Point or None (default).

OUTPUT:

A tuple of points that span a simplex of dimension dim(). If large==True, the simplex is constructed by successively picking the farthest point. This will ensure that the simplex is not unnecessarily small, but will in general not return a maximal simplex. If a point_order is specified, the simplex is greedily constructed by considering the points in this order. The large option and initial_point is ignored in this case. The point_order may contain only a subset of the points; in this case, the dimension of the simplex will be the dimension of this subset.

EXAMPLES:

sage: pc = PointConfiguration([(0,0), (1,0), (2,1), (1,1), (0,1)])
sage: pc.contained_simplex()  # Output: (P(0, 1), P(2, 1), P(1, 0))
sage: pc.contained_simplex(large=False)  # Output: (P(0, 1), P(1, 1), P(1, 0))
sage: pc.contained_simplex(initial_point=pc.point(2))  # Output: (P(2, 1), P(0, 0), P(1, 0))

sage: pc = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]])
sage: pc.contained_simplex()  # Output: (P(-1, -1), P(1, 1), P(0, 1))
sage: pc.contained_simplex(point_order=[pc[1], pc[3], pc[4], pc[2], pc[0]])  # Output: (P(0, 1), P(1, 1), P(-1, -1))

Lower-dimensional example:

sage: pc.contained_simplex(point_order=[pc[0], pc[3], pc[4]])  # Output: (P(0, 0), P(1, 1))

convex_hull ()

Return the convex hull of the point configuration.

EXAMPLES:

sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]])
sage: p.convex_hull()  # Output: A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 4 vertices

distance (x, y)

Returns the distance between two points.

INPUT:
• \(x, y\) – two points of the point configuration.

OUTPUT:
The distance between \(x\) and \(y\), measured either with `distance_affine()` or `distance_FS()` depending on whether the point configuration is defined by affine or projective points. These are related, but not equal to the usual flat and Fubini-Study distance.

EXAMPLES:

```python
sage: pc = PointConfiguration([(0,0), (1,0), (2,1), (1,2), (0,1)])
sage: [pc.distance(pc.point(0), p) for p in pc.points()]
[0, 1, 5, 5, 1]

sage: pc = PointConfiguration([(0,0,1), (1,0,1), (2,1,1), (1,2,1), (0,1,1)],
                          projective=True)
sage: [pc.distance(pc.point(0), p) for p in pc.points()]
[0, 1/2, 5/6, 5/6, 1/2]
```

distance_FS \((x, y)\)

Returns the distance between two points.

The distance function used in this method is \(1 - \cos d_{FS}(x, y)^2\), where \(d_{FS}\) is the Fubini-Study distance of projective points. Recall the Fubini-Study distance function

\[
d_{FS}(x, y) = \arccos \sqrt{\frac{(x \cdot y)^2}{|x|^2 |y|^2}}
\]

INPUT:

• \(x, y\) – two points of the point configuration.

OUTPUT:
The distance \(1 - \cos d_{FS}(x, y)^2\). Note that this distance lies in the same field as the entries of \(x, y\). That is, the distance of rational points will be rational and so on.

EXAMPLES:

```python
sage: pc = PointConfiguration([(0,0,1), (1,0,1), (2,1,1), (1,2,1), (0,1,1)],
                          projective=True)
sage: [pc.distance_FS(pc.point(0), p) for p in pc.points()]
[0, 1/2, 5/6, 5/6, 1/2]
```

distance_affine \((x, y)\)

Returns the distance between two points.

The distance function used in this method is \(d_{aff}(x, y)^2\), the square of the usual affine distance function

\[
d_{aff}(x, y) = |x - y|
\]

INPUT:

• \(x, y\) – two points of the point configuration.

OUTPUT:
The metric distance-square \(d_{aff}(x, y)^2\). Note that this distance lies in the same field as the entries of \(x, y\). That is, the distance of rational points will be rational and so on.

EXAMPLES:
sage: pc = PointConfiguration([(0,0),(1,0),(2,1),(1,2),(0,1)])
sage: [pc.distance_affine(pc.point(0), p) for p in pc.points()]
[0, 1, 5, 5, 1]

exclude_points(point_idx_list)

Return a new point configuration with the given points removed.

INPUT:

- point_idx_list – a list of integers. The indices of points to exclude.

OUTPUT:

A new PointConfiguration with the given points removed.

EXAMPLES:

sage: p = PointConfiguration([[-1,0], [0,0], [1,-1], [1,0], [1,1]])
sage: list(p)
[P(-1, 0), P(0, 0), P(1, -1), P(1, 0), P(1, 1)]
sage: q = p.exclude_points([3])
sage: list(q)
[P(-1, 0), P(0, 0), P(1, -1), P(1, 1)]
sage: p.exclude_points(p.face_interior(codim=1)).points()
(P(-1, 0), P(0, 0), P(1, -1), P(1, 1))

face_codimension(point)

Return the smallest \(d \in \mathbb{Z}\) such that point is contained in the interior of a codimension-\(d\) face.

EXAMPLES:

sage: triangle = PointConfiguration([[-1,0], [0,0], [1,-1], [1,0], [1,1]])
sage: triangle.point(2)
P(1, 0)
sage: triangle.face_codimension(2)
1
sage: triangle.face_codimension([1,0])
1

This also works for degenerate cases like the tip of the pyramid over a square (which saturates four inequalities):

sage: pyramid = PointConfiguration([[1,0,0], [0,1,1], [0,1,-1],
                                  ....: [0,-1,-1], [0,-1,1]])
sage: pyramid.face_codimension(0)
3

face_interior(dim=None, codim=None)

Return points by the codimension of the containing face in the convex hull.

EXAMPLES:

sage: triangle = PointConfiguration([[-1,0], [0,0], [1,-1], [1,0], [1,1]])
sage: triangle.face_interior()  
((1,), (3,), (0, 2, 4))
sage: triangle.face_interior(dim=0)  # the vertices of the convex hull
(0, 2, 4)
sage: triangle.face_interior(codim=1)  # interior of facets
(3,)
**farthest_point**(points, among=None)

Return the point with the most distance from points.

INPUT:

- points – a list of points.
- among – a list of points or None (default). The set of points from which to pick the farthest one. By default, all points of the configuration are considered.

OUTPUT:

A *Point* with largest minimal distance from all given points.

EXAMPLES:

```python
sage: pc = PointConfiguration([(0,0), (1,0), (1,1), (0,1)])
sage: pc.farthest_point([pc.point(0)])
P(1, 1)
```

**lexicographic_triangulation()**

Return the lexicographic triangulation.

The algorithm was taken from [PUNTOS].

EXAMPLES:

```python
sage: p = PointConfiguration([(0,0), (+1,0), (-1,0), (0,+1), (0,-1)])
sage: p.lexicographic_triangulation()
(<1,3,4>, <2,3,4>)
```

**placing_triangulation**(point_order=None)

Construct the placing (pushing) triangulation.

INPUT:

- point_order – list of points or integers. The order in which the points are to be placed. If not given, the points will be placed in some arbitrary order that attempts to produce a small number of simplices.

OUTPUT:

A *Triangulation*.

EXAMPLES:

```python
sage: pc = PointConfiguration([(0,0), (1,0), (2,1), (1,2), (0,1)])
sage: pc.placing_triangulation()
(<0,1,2>, <0,2,4>, <2,3,4>)
sage: pc.placing_triangulation(point_order=(3,2,1,4,0))
(<0,1,4>, <1,2,3>, <1,3,4>)
sage: pc.placing_triangulation(point_order=[pc[1], pc[3], pc[4], pc[0]])
(<0,1,4>, <1,3,4>)
sage: U = matrix(
....:   [ 0, 0, 0, 0, 0, 2, 4,-1, 1, 1, 0, 0, 1, 0],
....:   [ 0, 0, 0, 1, 0, 0,-1, 0, 0, 0, 0, 0, 0, 0],
....:   [ 0, 2, 0, 0, 0, 0,-1, 0, 1, 0, 1, 0, 0, 1],
....:   [ 0, 1, 1, 0, 0, 1, 0,-2, 1, 0, 0,-1, 1, 1],
....:   [ 0, 0, 0, 0, 0, 1, 0,-1, 0, 0, 0, 0, 0, 0]
....: )
sage: p = PointConfiguration(U.columns())
sage: triangulation = p.placing_triangulation(); triangulation
```
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

\[\begin{align*}
&<0,2,3,4,6,7>, <0,2,3,4,6,12>, <0,2,3,4,7,13>, <0,2,3,4,12,13>, \\
&<0,2,3,6,7,13>, <0,2,3,6,12,13>, <0,2,4,6,7,13>, <0,2,4,6,12,13>, \\
&<0,3,4,6,7,12>, <0,3,4,7,12,13>, <0,3,6,7,12,13>, <0,4,6,7,12,13>, \\
&<1,3,4,5,6,12>, <1,3,4,6,11,12>, <1,3,4,7,11,13>, <1,3,4,11,12,13>, \\
&<1,3,6,7,11,13>, <1,3,6,11,12,13>, <1,4,6,7,11,13>, <1,4,6,11,12,13>, \\
&<3,4,6,7,11,12>, <3,4,7,11,12,13>, <3,6,7,11,12,13>, <4,6,7,11,12,13>
\end{align*}\]

sage: sum(p.volume(t) for t in triangulation)
42

sage: p0 = PointConfiguration([[0,0], [+1,0], (-1,0), (0,+1), (0,-1)])

sage: p0.pushing_triangulation(point_order=[1,2,0,3,4])
(<1,2,3>, <1,2,4>)

sage: p0.pushing_triangulation(point_order=[0,1,2,3,4])
(<0,1,3>, <0,1,4>, <0,2,3>, <0,2,4>)

The same triangulation with renumbered points 0->4, 1->0, etc:

sage: p1 = PointConfiguration([[+1,0], (-1,0), (0,+1), (0,-1), (0,0)])

sage: p1.pushing_triangulation(point_order=[4,0,1,2,3])
(<0,2,4>, <0,3,4>, <1,2,4>, <1,3,4>)

\[\text{plot (**kwds)}\]

Produce a graphical representation of the point configuration.

\[\text{EXAMPLES:}\]

sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]])

sage: p.plot(axes=False) # needs sage.plot

\[\text{Graphics object consisting of 5 graphics primitives}\]

\[\text{positive_circuits (**negative)}\]

Returns the positive part of circuits with fixed negative part.

A circuit is a pair \((C_+, C_-)\), each consisting of a subset (actually, an ordered tuple) of point indices.

\[\text{INPUT:}\]

\[\bullet \ \text{**negative} = \text{integer. The indices of points.}\]

\[\text{OUTPUT:}\]

A tuple of all circuits with \(C_- = \text{negative}\).
**pushing_triangulation** *(point_order=None)*

Construct the placing (pushing) triangulation.

**INPUT:**

- `point_order` – list of points or integers. The order in which the points are to be placed. If not given, the points will be placed in some arbitrary order that attempts to produce a small number of simplices.

**OUTPUT:**

A Triangulation.

**EXAMPLES:**

```sage
c = PointConfiguration([(0,0), (1,0), (2,1), (1,2), (0,1)])
c.placing_triangulation()  # (1,2,3), (1,2,4)
c.placing_triangulation(point_order=[1,2,0,3,4])  # (0,1,3), (0,1,4), (0,2,3), (0,2,4)
```

The same triangulation with renumbered points 0->4, 1->0, etc:

```sage
d = PointConfiguration([(1,0), (-1,0), (0,1), (0,-1), (0,0)])
d.placing_triangulation(point_order=[4,0,1,2,3])  # (0,2,4), (0,3,4), (1,2,4), (1,3,4)
```
**restrict_to_connected_triangulations** *(connected=True)*

Restrict to connected triangulations.

**NOTE:**
Finding non-connected triangulations requires the optional TOPCOM package.

**INPUT:**

- **connected** – boolean. Whether to restrict to triangulations that are connected by bistellar flips to the regular triangulations.

**OUTPUT:**

A new *PointConfiguration* with the same points, but whose triangulations will all be in the connected component. See *PointConfiguration* for details.

**EXAMPLES:**

```sage
c = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]]);
c
A point configuration in affine 2-space over Integer Ring consisting of 5 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
c
len(c.triangulations_list())
4

sage: c.set_engine('topcom')
sage: c = c.restrict_to_connected_triangulations(connected=False) # optional - topcom
sage: len(c.triangulations_list())
4
sage: c = c.restrict_to_connected_triangulations(connected=True) # optional - topcom
True
sage: c.set_engine('internal')
```

**restrict_to_fine_triangulations** *(fine=True)*

Restrict to fine triangulations.

**INPUT:**

- **fine** – boolean. Whether to restrict to fine triangulations.

**OUTPUT:**

A new *PointConfiguration* with the same points, but whose triangulations will all be fine. See *PointConfiguration* for details.

**EXAMPLES:**

```sage
c = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]])
c
A point configuration in affine 2-space over Integer Ring consisting of 5 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
c
len(c.triangulations_list())
4
sage: c = c.restrict_to_fine_triangulations()
```

(continues on next page)
restrict_to_regular_triangulations (regular=True)

Restrict to regular triangulations.

NOTE:

Regularity testing requires the optional TOPCOM package.

INPUT:

• regular – True, False, or None. Whether to restrict to regular triangulations, irregular triangulations, or lift any restrictions on regularity.

OUTPUT:

A new PointConfiguration with the same points, but whose triangulations will all be regular as specified. See PointConfiguration for details.

EXAMPLES:

```
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]]); p
A point configuration in affine 2-space over Integer Ring consisting of 5 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
sage: len(p.triangulations_list())
4
sage: p == p_fine.restrict_to_fine_triangulations(fine=False)
True
sage: p_regular = p.restrict_to_regular_triangulations() # optional - topcom
sage: len(p_regular.triangulations_list()) # optional - topcom
4
sage: p == p_regular.restrict_to_regular_triangulations(regular=None) # optional - topcom
True
```

restrict_to_star_triangulations (star)

Restrict to star triangulations with the given point as the center.

INPUT:

• origin – None or an integer or the coordinates of a point. An integer denotes the index of the central point. If None is passed, any restriction on the starshape will be removed.

OUTPUT:

A new PointConfiguration with the same points, but whose triangulations will all be star. See PointConfiguration for details.

EXAMPLES:

```
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]]);
p
sage: len(list(p.triangulations()))
4
sage: p_star = p.restrict_to_star_triangulations(0)
sage: p_star is p.restrict_to_star_triangulations((0,0))
```
sage: p_star.triangulations_list()
[(<0,1,3>, <0,1,4>, <0,2,3>, <0,2,4>)]

sage: p_newstar = p_star.restrict_to_star_triangulations(1)  # pick different origin
sage: p_newstar.triangulations_list()
[(<1,2,3>, <1,2,4>)]

sage: p == p_star.restrict_to_star_triangulations(star=None)
True

**restricted_automorphism_group()**

Return the restricted automorphism group.

First, let the linear automorphism group be the subgroup of the affine group \(AGL(d, \mathbb{R}) = GL(d, \mathbb{R}) \ltimes \mathbb{R}^d\) preserving the \(d\)-dimensional point configuration. The affine group acts in the usual way \(\vec{x} \mapsto A\vec{x} + b\) on the ambient space.

The restricted automorphism group is the subgroup of the linear automorphism group generated by permutations of points. See [BSS2009] for more details and a description of the algorithm.

**OUTPUT:**

A `PermutationGroup` that is isomorphic to the restricted automorphism group is returned.

Note that in Sage, permutation groups always act on positive integers while lists etc. are indexed by nonnegative integers. The indexing of the permutation group is chosen to be shifted by \(+1\). That is, the transposition \((i,j)\) in the permutation group corresponds to exchange of `self[i-1]` and `self[j-1]`.

**EXAMPLES:**

```python
sage: pyramid = PointConfiguration([[1,0,0], [0,1,1], [0,1,-1], .... [0,-1,-1], [0,-1,1]])
```

```python
sage: G = pyramid.restricted_automorphism_group()  # needs sage.graphs sage.groups
sage: G == PermutationGroup([(3,5)], [(2,3),(4,5)], [(2,4)])  # needs sage.graphs sage.groups
True
```

The square with an off-center point in the middle. Note that the middle point breaks the restricted automorphism group \(D_4\) of the convex hull:

```python
sage: square = PointConfiguration([(3/4,3/4), (1,1), (1,-1), (-1,-1), (-1,1)])
```

```python
sage: square.restricted_automorphism_group()  # needs sage.graphs sage.groups
Permutation Group with generators [(3,5)]
```

**secondary_polytope()**

Calculate the secondary polytope of the point configuration.

For a definition of the secondary polytope, see [GKZ1994] page 220 Definition 1.6.

Note that if you restricted the admissible triangulations of the point configuration then the output will be the corresponding face of the whole secondary polytope.
OUTPUT:

The secondary polytope of the point configuration as an instance of `Polyhedron_base`.

EXAMPLES:

```python
sage: p = PointConfiguration([[0,0], [1,0], [2,1], [1,2], [0,1]])
sage: poly = p.secondary_polytope()
sage: poly.vertices_matrix()
[1 1 3 3 5]
[3 5 1 4 1]
[4 2 5 2 4]
[2 4 2 5 4]
[5 3 4 1 1]
sage: poly.Vrepresentation()
(A vertex at (1, 3, 4, 2, 5),
 A vertex at (1, 5, 2, 4, 3),
 A vertex at (3, 1, 5, 2, 4),
 A vertex at (3, 4, 2, 5, 1),
 A vertex at (5, 1, 4, 4, 1))
sage: poly.Hrepresentation()
(An equation (0, 0, 1, 2, 1) x - 13 == 0,
 An equation (1, 0, 0, 2, 2) x - 15 == 0,
 An equation (0, 1, 0, -3, -2) x + 13 == 0,
 An inequality (0, 0, 0, -1, -1) x + 7 >= 0,
 An inequality (0, 0, 0, 1, 0) x - 2 >= 0,
 An inequality (0, 0, 0, -2, -1) x + 11 >= 0,
 An inequality (0, 0, 0, 3, 2) x - 14 >= 0)
```

```python
class method set_engine (engine='auto')
```

Set the engine used to compute triangulations.

INPUT:

- `engine`: either 'auto' (default), 'internal', or 'topcom'. The latter two instruct this package to always use its own triangulation algorithms or TOPCOM's algorithms, respectively. By default ('auto'), internal routines are used.

EXAMPLES:

```python
sage: # optional - topcom
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]])
sage: p.set_engine('internal')  # to make doctests independent of TOPCOM
sage: p.triangulate()
(<1,3,4>, <2,3,4>)
sage: p.set_engine('topcom')
sage: p.triangulate()
(<0,1,2>, <0,1,4>, <0,2,4>, <1,2,3>)
sage: p.set_engine('internal')
```

```python
star_center ()
```

Return the center used for star triangulations.

See also:

`restrict_to_star_triangulations()`.

OUTPUT:

A `Point` if a distinguished star central point has been fixed. `ValueError` exception is raised otherwise.
Combinatorial and Discrete Geometry, Release 10.3

EXAMPLES:

\begin{verbatim}
sage: pc = PointConfiguration([(1,0), (-1,0), (0,1), (0,2)], star=(0,1)); pc
A point configuration in affine 2-space over Integer Ring consisting of 4 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular, and star with center P(0, 1).
sage: pc.star_center()
P(0, 1)
sage: pc_nostar = pc.restrict_to_star_triangulations(None); pc_nostar
A point configuration in affine 2-space over Integer Ring consisting of 4 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
sage: pc_nostar.star_center()
Traceback (most recent call last):
... ValueError: The point configuration has no star center defined.
\end{verbatim}

\texttt{triangulate} (\texttt{verbose=False})

Return one (in no particular order) triangulation.

INPUT:

\begin{itemize}
\item \texttt{verbose} – boolean. Whether to print out the TOPCOM interaction, if any.
\end{itemize}

OUTPUT:

A \texttt{Triangulation} satisfying all restrictions imposed. This raises a \texttt{ValueError} if no such triangulation exists.

EXAMPLES:

\begin{verbatim}
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1], [-1,-1]])
sage: p.triangulate()
(<1,3,4>, <2,3,4>)
sage: list( p.triangulate() )
[(1, 3, 4), (2, 3, 4)]

Using TOPCOM yields a different, but equally good, triangulation:

\begin{verbatim}
sage: # optional - topcom sage: p.set_engine('topcom') sage: p.triangulate()
(<0,1,2>, <0,1,4>, <0,2,4>, <1,2,3>) sage: list(p.triangulate())
[(0, 1, 2), (0, 1, 4), (0, 2, 4), (1, 2, 3)]
\end{verbatim}

\texttt{triangulations} (\texttt{verbose=False})

Returns all triangulations.

\begin{itemize}
\item \texttt{verbose} – boolean (default: False). Whether to print out the TOPCOM interaction, if any.
\end{itemize}

OUTPUT:

A generator for the triangulations satisfying all the restrictions imposed. Each triangulation is returned as a \texttt{Triangulation} object.

EXAMPLES:
triangulations_list (verbose=False)

Return all triangulations.

INPUT:

• verbose – boolean. Whether to print out the TOPCOM interaction, if any.

OUTPUT:

A list of triangulations (see Triangulation) satisfying all restrictions imposed previously.

EXAMPLES:
```python
sage: p = PointConfiguration([[0,0], [0,1], [1,0], [1,1]])
sage: p.triangulations_list()
[(<0,1,2>, <1,2,3>), (<0,1,3>, <0,2,3>)]
sage: list(map(list, p.triangulations_list()))
[[[0, 1, 2], [1, 2, 3]], [[0, 1, 3], [0, 2, 3]]]
sage: p.set_engine('topcom')
sage: p.triangulations_list()
# optional - topcom
[(<0,1,2>, <1,2,3>), (<0,1,3>, <0,2,3>)]
sage: p.set_engine('internal')
```

```python
volume (simplex=None)
Find \( n! \) times the \( n \)-volume of a simplex of dimension \( n \).

INPUT:
- `simplex` (optional argument) – a simplex from a triangulation \( T \) specified as a list of point indices.

OUTPUT:
- If a simplex was passed as an argument: \( n! \) * (volume of simplex).
- Without argument: \( n! \) * (the total volume of the convex hull).

EXAMPLES:
The volume of the standard simplex should always be 1:
```
sage: p = PointConfiguration([[0,0], [1,0], [0,1], [1,1]])
sage: p.volume([0,1,2]) 1
sage: simplex = p.triangulate()[0]  # first simplex of triangulation
sage: p.volume(simplex) 1
```
The square can be triangulated into two minimal simplices, so in the “integral” normalization its volume equals two:
```
sage: p.volume() 2
```

Note: We return \( n! \) * (metric volume of the simplex) to ensure that the volume is an integer. Essentially, this normalizes things so that the volume of the standard \( n \)-simplex is 1. See [GKZ1994] page 182.

### 3.2 Base classes for triangulations

We provide (fast) cython implementations here.

AUTHORS:
- Volker Braun (2010-09-14): initial version.

```python
class sage.geometry.triangulation.base.ConnectedTriangulationsIterator
    Bases: sageObject

    A Python shim for the C++-class ‘triangulations’
```

INPUT:
• **point_configuration** – a *PointConfiguration*.

• **seed** – a regular triangulation or *None* (default). In the latter case, a suitable triangulation is generated automatically. Otherwise, you can explicitly specify the seed triangulation as
  
  – A *Triangulation* object, or
  
  – an iterable of iterables specifying the vertices of the simplices, or
  
  – an iterable of integers, which are then considered the enumerated simplices (see *simplex_to_int()*).

• **star** – either *None* (default) or an integer. If an integer is passed, all returned triangulations will be star with respect to the

• **fine** – boolean (default: *False*). Whether to return only fine triangulations, that is, simplicial decompositions that make use of all the points of the configuration.

**OUTPUT:**

An iterator. The generated values are tuples of integers, which encode simplices of the triangulation. The output is a suitable input to *Triangulation*.

**EXAMPLES:**

```python
sage: p = PointConfiguration([[0,0],[0,1],[1,0],[1,1],[-1,-1]])
sage: from sage.geometry.triangulation.base import ConnectedTriangulationsIterator
sage: ci = ConnectedTriangulationsIterator(p)
sage: next(ci)
(9, 10)
sage: next(ci)
(2, 3, 4, 5)
sage: next(ci)
(7, 8)
sage: next(ci)
(1, 3, 5, 7)
sage: next(ci)
Traceback (most recent call last):
  ... StopIteration
```

You can reconstruct the triangulation from the compressed output via:

```python
sage: from sage.geometry.triangulation.element import Triangulation
sage: Triangulation((2, 3, 4, 5), p)
(<0,1,3>, <0,1,4>, <0,2,3>, <0,2,4>)
```

How to use the restrictions:

```python
sage: ci = ConnectedTriangulationsIterator(p, fine=True)
sage: list(ci)
[(2, 3, 4, 5), (1, 3, 5, 7)]
sage: ci = ConnectedTriangulationsIterator(p, star=1)
sage: list(ci)
[(7, 8)]
sage: ci = ConnectedTriangulationsIterator(p, star=1, fine=True)
sage: list(ci)
[]
```

**class** *sage.geometry.triangulation.base.Point*

Bases: *SageObject*

A point of a point configuration.
Note that the coordinates of the points of a point configuration are somewhat arbitrary. What counts are the abstract linear relations between the points, for example encoded by the `circuits()`.

**Warning:** You should not create `Point` objects manually. The constructor of `PointConfiguration_base` takes care of this for you.

**INPUT:**

- `point_configuration` – `PointConfiguration_base`. The point configuration to which the point belongs.
- `i` – integer. The index of the point in the point configuration.
- `projective` – the projective coordinates of the point.
- `affine` – the affine coordinates of the point.
- `reduced` – the reduced (with linearities removed) coordinates of the point.

**EXAMPLES:**

```
sage: pc = PointConfiguration([[(0,0)]])
sage: from sage.geometry.triangulation.base import Point
sage: Point(pc, 123, (0,0,1), (0,0), ())  # P(0, 0)
```

**affine()**

Return the affine coordinates of the point in the ambient space.

**OUTPUT:**

A tuple containing the coordinates.

**EXAMPLES:**

```
sage: pc = PointConfiguration([[10, 0, 1], [10, 0, 0], [10, 2, 3]])
sage: p = pc.point(2); p  # P(10, 2, 3)
sage: p.affine()  # (10, 2, 3)
sage: p.projective()  # (10, 2, 3, 1)
sage: p.reduced_affine()  # (2, 2)
sage: p.reduced_projective()  # (2, 2)
sage: p.reduced_affine_vector()  # (2, 2)
```

**index()**

Return the index of the point in the point configuration.

**EXAMPLES:**

```
sage: pc = PointConfiguration([[(0, 1), [0, 0], [1, 0]]])
sage: p = pc.point(2); p  # P(1, 0)
sage: p.index()  # 2
```
point_configuration()

Return the point configuration to which the point belongs.

OUTPUT:

A :class:`PointConfiguration`.

EXAMPLES:

```
sage: pc = PointConfiguration([ (0,0), (1,0), (0,1) ])
sage: p = pc.point(0)
sage: p == pc.point(0)
True
sage: p.point_configuration() == pc
True
```

projective()

Return the projective coordinates of the point in the ambient space.

OUTPUT:

A tuple containing the coordinates.

EXAMPLES:

```
sage: pc = PointConfiguration([10, 0, 1], [10, 0, 0], [10, 2, 3])
sage: p = pc.point(2); p
P(10, 2, 3)
sage: p.affine()
(10, 2, 3)
sage: p.projective()
(10, 2, 3, 1)
sage: p.reduced_affine()
(2, 2)
sage: p.reduced_projective()
(2, 2, 1)
sage: p.reduced_affine_vector()
(2, 2)
```

reduced_affine()

Return the affine coordinates of the point on the hyperplane spanned by the point configuration.

OUTPUT:

A tuple containing the coordinates.

EXAMPLES:

```
sage: pc = PointConfiguration([10, 0, 1], [10, 0, 0], [10, 2, 3])
sage: p = pc.point(2); p
P(10, 2, 3)
sage: p.affine()
(10, 2, 3)
sage: p.projective()
(10, 2, 3, 1)
sage: p.reduced_affine()
(2, 2)
sage: p.reduced_projective()
(2, 2, 1)
sage: p.reduced_affine_vector()
(2, 2)
```
reduced_affine_vector()

Return the affine coordinates of the point on the hyperplane spanned by the point configuration.

OUTPUT:

A tuple containing the coordinates.

EXAMPLES:

```
sage: pc = PointConfiguration([[10, 0, 1], [10, 0, 0], [10, 2, 3]])
sage: p = pc.point(2); p
P(10, 2, 3)
sage: p.affine()
(10, 2, 3)
sage: p.projective()
(10, 2, 3, 1)
sage: p.reduced_affine()
(2, 2)
sage: p.reduced_projective()
(2, 2, 1)
sage: p.reduced_affine_vector()
(2, 2)
```

reduced_projective()

Return the projective coordinates of the point on the hyperplane spanned by the point configuration.

OUTPUT:

A tuple containing the coordinates.

EXAMPLES:

```
sage: pc = PointConfiguration([[10, 0, 1], [10, 0, 0], [10, 2, 3]])
sage: p = pc.point(2); p
P(10, 2, 3)
sage: p.affine()
(10, 2, 3)
sage: p.projective()
(10, 2, 3, 1)
sage: p.reduced_affine()
(2, 2)
sage: p.reduced_projective()
(2, 2, 1)
sage: p.reduced_affine_vector()
(2, 2)
```

reduced_projective_vector()

Return the affine coordinates of the point on the hyperplane spanned by the point configuration.

OUTPUT:

A tuple containing the coordinates.

EXAMPLES:

```
sage: pc = PointConfiguration([[10, 0, 1], [10, 0, 0], [10, 2, 3]])
sage: p = pc.point(2); p
P(10, 2, 3)
sage: p.affine()
(10, 2, 3)
```

(continues on next page)
sage: p.projective()
(10, 2, 3, 1)
sage: p.reduced_affine()
(2, 2)
sage: p.reduced_projective()
(2, 2, 1)
sage: p.reduced_affine_vector()
(2, 2)
sage: type(p.reduced_affine_vector())
<class 'sage.modules.vector_rational_dense.Vector_rational_dense'>

```
class sage.geometry.triangulation.base.PointConfiguration_base

Bases: Parent

The cython abstract base class for PointConfiguration.

**Warning:** You should not instantiate this base class, but only its derived class `PointConfiguration`.

ambient_dim()

Return the dimension of the ambient space of the point configuration.

See also dimension()

**EXAMPLES:**

```
sage: p = PointConfiguration([[0,0,0]])
sage: p.ambient_dim()
3
sage: p.dim()
0
```

base_ring()

Return the base ring, that is, the ring containing the coordinates of the points.

**OUTPUT:**

A ring.

**EXAMPLES:**

```
sage: p = PointConfiguration([[0,0]])
sage: p.base_ring()
Integer Ring
sage: p = PointConfiguration([[1/2,3]])
sage: p.base_ring()
Rational Field
sage: p = PointConfiguration([[0.2, 5]])
sage: p.base_ring()
Real Field with 53 bits of precision
```

dim()

Return the actual dimension of the point configuration.

See also `ambient_dim()`
EXAMPLES:

```sage
sage: p = PointConfiguration([[0,0,0]])
sage: p.ambient_dim()
3
sage: p.dim()
0
```

**int_to_simplex**(s)

Reverse the enumeration of possible simplices in `simplex_to_int()`.

The enumeration is compatible with [PUNTOS].

**INPUT:**

- • s – int. An integer that uniquely specifies a simplex.

**OUTPUT:**

An ordered tuple consisting of the indices of the vertices of the simplex.

**EXAMPLES:**

```sage
sage: U=matrix([....:      [ 0, 0, 0, 0, 0, 2, 4,-1, 1, 1, 0, 0, 1, 0],
....:      [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
....:      [ 0, 2, 0, 0, 0, 0, 0,-1, 0, 1, 0, 1, 0, 1],
....:      [ 0, 1, 1, 0, 0, 0, 0, 0,-2, 1, 0, 0, 0, 1],
....:      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].
....:      ])
sage: pc = PointConfiguration(U.columns())
sage: pc.simplex_to_int([1,3,4,7,10,13])
1678
sage: pc.int_to_simplex(1678)
(1, 3, 4, 7, 10, 13)
```

**is_affine**()

Return whether the configuration is defined by affine points.

**OUTPUT:**

Boolean. If true, the homogeneous coordinates all have 1 as their last entry.

**EXAMPLES:**

```sage
sage: p = PointConfiguration([(0.2, 5), (3, 0.1)])
sage: p.is_affine()
True
sage: p = PointConfiguration([(0.2, 5, 1), (3, 0.1, 1)], projective=True)
sage: p.is_affine()
False
```

**n_points**()

Return the number of points.

Same as `len(self)`.

**EXAMPLES:**

```sage
sage: p = PointConfiguration([(0,1), (0,1), (0,1), (0,1), (0,1), (0,1)])
sage: p.n_points()
6
```
sage: p = PointConfiguration([[0,0],[0,1],[1,0],[1,1],[-1,-1]])
sage: p
A point configuration in affine 2-space over Integer Ring
consisting of 5 points. The triangulations of this point
configuration are assumed to be connected, not necessarily
fine, not necessarily regular.
sage: len(p)
5
sage: p.n_points()
5

point (i)
Return the i-th point of the configuration.

Same as __getitem__() 

INPUT:

• i – integer.

OUTPUT:
A point of the point configuration.

EXAMPLES:

sage: pconfig = PointConfiguration([[0,0],[0,1],[1,0],[1,1],[-1,-1]])
sage: list(pconfig)
[Point(0, 0), Point(0, 1), Point(1, 0), Point(1, 1), Point(-1, -1)]
sage: [ p for p in pconfig.points() ]
[Point(0, 0), Point(0, 1), Point(1, 0), Point(1, 1), Point(-1, -1)]
sage: pconfig.point(0)
Point(0, 0)
sage: pconfig[0]
Point(0, 0)
sage: pconfig.point(1)
Point(0, 1)
sage: pconfig.point( pconfig.n_points()-1 )
Point(-1, -1)

points ()
Return a list of the points.

OUTPUT:
A list of the points. See also the __iter__() method, which returns the corresponding generator.

EXAMPLES:

sage: pconfig = PointConfiguration([[0,0],[0,1],[1,0],[1,1],[-1,-1]])
sage: list(pconfig)
[Point(0, 0), Point(0, 1), Point(1, 0), Point(1, 1), Point(-1, -1)]
sage: [ p for p in pconfig.points() ]
[Point(0, 0), Point(0, 1), Point(1, 0), Point(1, 1), Point(-1, -1)]
sage: pconfig.point(0)
Point(0, 0)
sage: pconfig.point(1)
Point(0, 1)
sage: pconfig.point( pconfig.n_points()-1 )
Point(-1, -1)
**reduced_affine_vector_space()**

Return the vector space that contains the affine points.

**OUTPUT:**

A vector space over the fraction field of `base_ring()`.

**EXAMPLES:**

```python
sage: p = PointConfiguration([[0,0,0], [1,2,3]])
sage: p.base_ring()
Integer Ring
sage: p.reduced_affine_vector_space()
Vector space of dimension 1 over Rational Field
sage: p.reduced_projective_vector_space()
Vector space of dimension 2 over Rational Field
```

**reduced_projective_vector_space()**

Return the vector space that is spanned by the homogeneous coordinates.

**OUTPUT:**

A vector space over the fraction field of `base_ring()`.

**EXAMPLES:**

```python
sage: p = PointConfiguration([[0,0,0], [1,2,3]])
sage: p.base_ring()
Integer Ring
sage: p.reduced_affine_vector_space()
Vector space of dimension 1 over Rational Field
sage: p.reduced_projective_vector_space()
Vector space of dimension 2 over Rational Field
```

**simplex_to_int**(simplex)

Return an integer that uniquely identifies the given simplex.

See also the inverse method `int_to_simplex()`.

The enumeration is compatible with [PUNTOS].

**INPUT:**

- `simplex` – iterable, for example a list. The elements are the vertex indices of the simplex.

**OUTPUT:**

An integer that uniquely specifies the simplex.

**EXAMPLES:**

```python
sage: U=matrix([...:
               [ 0, 0, 0, 0, 0, 2, 4,-1, 1, 1, 0, 0, 1, 0],
               [ 0, 0, 1, 0, 0,-1, 0, 0, 0, 0, 0, 0, 0, 0],
               [ 0, 2, 0, 0, 0,-1, 0, 1, 0, 0, 0, 0, 0, 0],
               [ 0, 1, 1, 0, 0,-2, 1, 0, 0, 1, 0, 0, 0, 0],
               [ 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0, 0, 0, 0]]

sage: pc = PointConfiguration(U.columns())
sage: pc.simplex_to_int([1,3,4,7,10,13])
1678
```

(continues on next page)
3.3 A triangulation

In Sage, the \texttt{PointConfiguration} and \texttt{Triangulation} satisfy a parent/element relationship. In particular, each triangulation refers back to its point configuration. If you want to triangulate a point configuration, you should construct a point configuration first and then use one of its methods to triangulate it according to your requirements. You should never have to construct a \texttt{Triangulation} object directly.

\textbf{EXAMPLES:}

First, we select the internal implementation for enumerating triangulations:

\begin{verbatim}
  sage: PointConfiguration.set_engine('internal')  # to make doctests independent of...
  ⟷ TOPCOM
\end{verbatim}

Here is a simple example of how to triangulate a point configuration:

\begin{verbatim}
  sage: p = [[0,-1,-1],[0,0,1],[0,1,0], [1,-1,-1],[1,0,1],[1,1,0]]
  sage: points = PointConfiguration(p)
  sage: triang = points.triangulate(); triang
  (<0,1,2,5>, <0,1,3,5>, <1,3,4,5>)
  sage: triang.plot(axes=False)  #...
  ⟷ needs sage.plot
  Graphics3d Object
\end{verbatim}

See \texttt{sage.geometry.triangulation.point_configuration} for more details.

\begin{verbatim}
  class sage.geometry.triangulation.element.Triangulation(triangulation, parent, check=True)

  Bases: Element

  A triangulation of a \texttt{PointConfiguration}.

  Warning: You should never create \texttt{Triangulation} objects manually. See \texttt{triangulate()} and \texttt{triangulations()} to triangulate point configurations.

  adjacency_graph()

  Return a graph showing which simplices are adjacent in the triangulation.

  OUTPUT:

  A graph consisting of vertices referring to the simplices in the triangulation, and edges showing which simplices are adjacent to each other.

  See also:

  • To obtain the triangulation’s 1-skeleton, use \texttt{SimplicialComplex.graph()} through \texttt{MyTriangulation.simplicial_complex().graph()}.

  AUTHORS:

  • Stephen Farley (2013-08-10): initial version
\end{verbatim}
EXAMPLES:

```
sage: p = PointConfiguration([[1,0,0], [0,1,0], [0,0,1], [-1,0,1],
....:                          [1,0,-1], [-1,0,0], [0,-1,0], [0,0,-1]])
sage: t = p.triangulate()
sage: t.adjacency_graph()  # needs sage.graphs
Graph on 8 vertices
```

**boundary**

Return the boundary of the triangulation.

**OUTPUT:**

The outward-facing boundary simplices (of dimension \(d - 1\)) of the \(d\)-dimensional triangulation as a set. Each boundary is returned by a tuple of point indices.

**EXAMPLES:**

```
sage: triangulation = polytopes.cube().triangulate(engine='internal')
sage: triangulation
(<0,1,2,7>, <0,1,5,7>, <0,2,3,7>, <0,3,4,7>, <0,4,5,7>, <1,5,6,7>)
sage: triangulation.boundary()
frozenset({(0, 1, 2),
            (0, 1, 5),
            (0, 2, 3),
            (0, 3, 4),
            (0, 4, 5),
            (1, 2, 7),
            (1, 5, 6),
            (1, 6, 7),
            (2, 3, 7),
            (3, 4, 7),
            (4, 5, 7),
            (5, 6, 7)})
sage: triangulation.interior_facets()
frozenset({(0, 1, 7), (0, 2, 7), (0, 3, 7), (0, 4, 7), (0, 5, 7), (1, 5, 7)})
```

**boundary_polyhedral_complex(**kwds**)**

Return the boundary of \(self\) as a PolyhedralComplex.

**OUTPUT:**

A PolyhedralComplex whose maximal cells are the simplices of the boundary of \(self\).

**EXAMPLES:**

```
sage: P = polytopes.cube()
sage: pc = PointConfiguration(P.vertices())
sage: T = pc.placing_triangulation(); T
(<0,1,2,7>, <0,1,5,7>, <0,2,3,7>, <0,3,4,7>, <0,4,5,7>, <1,5,6,7>)
sage: bd_C = T.boundary_polyhedral_complex(); bd_C
Polyhedral complex with 12 maximal cells
sage: [P.vertices_list() for P in bd_C.maximal_cells_sorted()]  # needs sage.graphs
[[]
, []
, []
, []
, []
, []
, []
, []
, []
, []
, []
, []
]
```
It is a subcomplex of self as a polyhedral_complex():

```
sage: C = T.polyhedral_complex()  # needs sage.graphs
sage: bd_C.is_subcomplex(C)  # needs sage.graphs
True
```

boundary_simplicial_complex()
Return the boundary of self as an (abstract) simplicial complex.

OUTPUT:
A SimplicialComplex.

EXAMPLES:
```
sage: p = polytopes.cuboctahedron()
sage: triangulation = p.triangulate(engine='internal')
sage: bd_sc = triangulation.boundary_simplicial_complex(); bd_sc  # needs sage.graphs
Simplicial complex with 12 vertices and 20 facets
```

The boundary of every convex set is a topological sphere, so it has spherical homology:
```
sage: bd_sc.homology()  # needs sage.graphs
{0: 0, 1: 0, 2: ℤ}
```

It is a subcomplex of self as a simplicial_complex():
```
sage: sc = triangulation.simplicial_complex()  # needs sage.graphs
sage: all(f in sc for f in bd_sc.maximal_faces())  # needs sage.graphs
True
```

enumerate_simplices()
Return the enumerated simplices.

OUTPUT:
A tuple of integers that uniquely specifies the triangulation.

EXAMPLES:
```
sage: pc = PointConfiguration(matrix([ 0, 0, 0, 0, 0, 2, 4,-1, 1, 1, 0, 0, 1, 0], 694 Chapter 3. Triangulations

(continues on previous page)
sage: triangulation = pc.lexicographic_triangulation()
sage: triangulation.enumerate_simplices()

(1678, 1688, 1769, 1779, 1895, 1905, 2112, 2143, 2234, 2360, 2555, 2580, 2610, 2626, 2650, 2652, 2654, 2661, 2663, 2667, 2685, 2755, 2757, 2759, 2766, 2768, 2772, 2811, 2881, 2883, 2885, 2892, 2894, 2898)

You can recreate the triangulation from this list by passing it to the constructor:

sage: from sage.geometry.triangulation.point_configuration import...–>Triangulation
sage: Triangulation([1678, 1688, 1769, 1779, 1895, 1905, 2112, 2143,...,2685, 2755, 2757, 2759, 2766, 2768, 2772, 2811, 2881, 2883, 2885, 2892, 2894, 2898], pc)

fan (origin=None)

Construct the fan of cones over the simplices of the triangulation.

INPUT:

• origin=None (default) or coordinates of a point. The common apex of all cones of the fan. If None, the triangulation must be a star triangulation and the distinguished central point is used as the origin.

OUTPUT:

A RationalPolyhedralFan. The coordinates of the points are shifted so that the apex of the fan is the origin of the coordinate system.

Note: If the set of cones over the simplices is not a fan, a suitable exception is raised.

EXAMPLES:

sage: pc = PointConfiguration([(0,0), (1,0), (0,1), (-1,-1)], star=0,˓→fine=True)
sage: triangulation = pc.triangulate()
sage: fan = triangulation.fan(); fan
Rational polyhedral fan in 2-d lattice N

Toric diagrams (the \( \mathbb{Z}_3 \) hyperconifold):
Combinatorial and Discrete Geometry, Release 10.3

\begin{verbatim}
sage: vertices=[(0, 1, 0), (0, 3, 1), (0, 2, 3), (0, 0, 2)]
sage: interior=[(0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2)]
sage: points = vertices + interior
sage: pc = PointConfiguration(points, fine=True)
sage: triangulation = pc.triangulate()
sage: fan = triangulation.fan((-1,0,0)); fan
Rational polyhedral fan in 3-d lattice N
sage: fan.rays()
N(1, 1, 0),
N(1, 3, 1),
N(1, 2, 3),
N(1, 0, 2),
N(1, 1, 1),
N(1, 1, 2),
N(1, 2, 1),
N(1, 2, 2)
in 3-d lattice N

gkz_phi()
Calculate the GKZ phi vector of the triangulation.
The phi vector is a vector of length equals to the number of points in the point configuration. For a fixed
triangulation \( T \), the entry corresponding to the \( i \)-th point \( p_i \) is
\[
\phi_T(p_i) = \sum_{t \in T, \exists p_i} Vol(t)
\]
that is, the total volume of all simplices containing \( p_i \). See also [GKZ1994] page 220 equation 1.4.

OUTPUT:
The phi vector of self.

EXAMPLES:

\begin{verbatim}
sage: p = PointConfiguration([[0,0],[1,0],[2,1],[1,2],[0,1]])
sage: p.triangulate().gkz_phi()
(3, 1, 5, 2, 4)
sage: p.lexicographic_triangulation().gkz_phi()
(1, 3, 4, 2, 5)
\end{verbatim}

interior_facets()
Return the interior facets of the triangulation.

OUTPUT:
The inward-facing boundary simplices (of dimension \( d - 1 \)) of the \( d \)-dimensional triangulation as a set. Each
boundary is returned by a tuple of point indices.

EXAMPLES:

\begin{verbatim}
sage: triangulation = polytopes.cube().triangulate(engine='internal')
sage: triangulation
(<0,1,2,7>, <0,1,5,7>, <0,2,3,7>, <0,3,4,7>, <0,4,5,7>, <1,5,6,7>)
sage: triangulation.boundary()
frozenset({(0, 1, 2),
    (0, 1, 5),
    (0, 2, 3),
    (0, 3, 4),
    (0, 4, 5),
    (0, 4, 6),
    (0, 5, 6),
    (1, 2, 3),
    (1, 2, 4),
    (1, 3, 4),
    (1, 3, 5),
    (1, 4, 5),
    (1, 4, 6),
    (1, 5, 6),
    (2, 3, 4),
    (2, 3, 5),
    (2, 4, 5),
    (2, 4, 6),
    (2, 5, 6),
    (3, 4, 5),
    (3, 4, 6),
    (3, 5, 6),
    (4, 5, 6)})
\end{verbatim}
\end{verbatim}

(continues on next page)
normal_cone()

Return the (closure of the) normal cone of the triangulation.

Recall that a regular triangulation is one that equals the "crease lines" of a convex piecewise-linear function. This support function is not unique, for example, you can scale it by a positive constant. The set of all piecewise-linear functions with fixed creases forms an open cone. This cone can be interpreted as the cone of normal vectors at a point of the secondary polytope, which is why we call it normal cone. See [GKZ1994] Section 7.1 for details.

OUTPUT:

The closure of the normal cone. The $i$-th entry equals the value of the piecewise-linear function at the $i$-th point of the configuration.

For an irregular triangulation, the normal cone is empty. In this case, a single point (the origin) is returned.

EXAMPLES:

```
sage: triangulation = polytopes.hypercube(2).triangulate(engine='internal')
sage: triangulation
(<0,1,3>, <1,2,3>)
sage: N = triangulation.normal_cone(); N
4-d cone in 4-d lattice
sage: N.rays()
( 0, 0, 0, -1),
( 0, 0, 1, 1),
( 0, 0, -1, -1),
( 1, 0, 0, 1),
(-1, 0, 0, -1),
( 0, 1, 0, -1),
( 0, -1, 0, 1)
in Ambient free module of rank 4
over the principal ideal domain Integer Ring
sage: N.dual().rays()
(1, -1, 1, -1)
in Ambient free module of rank 4
over the principal ideal domain Integer Ring
```

plot(**kwds)

Produce a graphical representation of the triangulation.

EXAMPLES:

```
sage: p = PointConfiguration([[0,0],[0,1],[1,0],[1,1],[-1,-1]])
sage: triangulation = p.triangulate()
sage: triangulation
(<1,3,4>, <2,3,4>)
```

(continues on next page)
point_configuration()

Returns the point configuration underlying the triangulation.

EXAMPLES:

```python
sage: pconfig = PointConfiguration([[0,0],[0,1],[1,0]])
sage: pconfig
A point configuration in affine 2-space over Integer Ring consisting of 3 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
sage: triangulation = pconfig.triangulate()
sage: triangulation
(<0,1,2>)
sage: triangulation.point_configuration()
A point configuration in affine 2-space over Integer Ring consisting of 3 points. The triangulations of this point configuration are assumed to be connected, not necessarily fine, not necessarily regular.
sage: pconfig == triangulation.point_configuration()
True
```

polyhedral_complex(**kwds)

Return self as a PolyhedralComplex.

OUTPUT:

A PolyhedralComplex whose maximal cells are the simplices of the triangulation.

EXAMPLES:

```python
sage: P = polytopes.cube()
sage: pc = PointConfiguration(P.vertices())
sage: T = pc.placing_triangulation(); T
(<0,1,2,7>, <0,1,5,7>, <0,2,3,7>, <0,3,4,7>, <0,4,5,7>, <1,5,6,7>)
sage: C = T.polyhedral_complex(); C
A polyhedral complex with 6 maximal cells
sage: [P.vertices_list() for P in C.maximal_cells_sorted()]
```

simplicial_complex()

Return self as an (abstract) simplicial complex.

OUTPUT:

A SimplicialComplex.

EXAMPLES:
Any convex set is contractable, so its reduced homology groups vanish:

```
sage: sc.homology()

{0: 0, 1: 0, 2: 0, 3: 0}
```

`sage.geometry.triangulation.element.triangulation_render_2d(triangulation, **kwds)`

Return a graphical representation of a 2-d triangulation.

**INPUT:**

- `triangulation` – a `Triangulation`
- `**kwds` – keywords that are passed on to the graphics primitives

**OUTPUT:**

A 2-d graphics object.

**EXAMPLES:**

```
sage: points = PointConfiguration([[0,0],[0,1],[1,0],[1,1],[-1,-1]])
sage: triang = points.triangulate()
sage: triang.plot(axes=False, aspect_ratio=1)  # indirect doctest

Graphics object consisting of 12 graphics primitives
```

`sage.geometry.triangulation.element.triangulation_render_3d(triangulation, **kwds)`

Return a graphical representation of a 3-d triangulation.

**INPUT:**

- `triangulation` – a `Triangulation`
- `**kwds` – keywords that are passed on to the graphics primitives

**OUTPUT:**

A 3-d graphics object.

**EXAMPLES:**

```
sage: p = [[0,-1,-1],[0,0,1],[0,1,0], [1,-1,-1],[1,0,1],[1,1,0]]
sage: points = PointConfiguration(p)
sage: triang = points.triangulate()
sage: triang.plot(axes=False)  # indirect doctest

Graphics3d Object
```
4.1 Abstract base classes for classes in geometry

class sage.geometry.abc.ConvexRationalPolyhedralCone
   Bases: object
   Abstract base class for ConvexRationalPolyhedralCone

   This class is defined for the purpose of isinstance tests. It should not be instantiated.

   EXAMPLES:

   sage: import sage.geometry.abc
   sage: C = cones.nonnegative_orthant(2)
   sage: isinstance(C, sage.geometry.abc.ConvexRationalPolyhedralCone)
   True

   By design, there is a unique direct subclass:

   sage: sage.geometry.abc.ConvexRationalPolyhedralCone.__subclasses__()  
   [[<class 'sage.geometry.cone.ConvexRationalPolyhedralCone'>]]
   sage: len(sage.geometry.abc.Polyhedron.__subclasses__()) <= 1
   True

class sage.geometry.abc.LatticePolytope
   Bases: object
   Abstract base class for LatticePolytopeClass

   This class is defined for the purpose of isinstance tests. It should not be instantiated.

   EXAMPLES:

   sage: import sage.geometry.abc
   sage: P = LatticePolytope([(1,2,3), (4,5,6)])
   sage: isinstance(P, sage.geometry.abc.LatticePolytope)
   True

   By design, there is a unique direct subclass:
class sage.geometry.abc.Polyhedron

Bases: object

Abstract base class for Polyhedron_base

This class is defined for the purpose of isinstance tests. It should not be instantiated.

EXAMPLES:

```python
sage: import sage.geometry.abc
sage: P = polytopes.cube()
# needs sage.geometry.polyhedron
sage: isinstance(P, sage.geometry.abc.Polyhedron)  # needs sage.geometry.polyhedron
True
```

By design, there is a unique direct subclass:

```python
sage: sage.geometry.abc.Polyhedron.__subclasses__()  # needs sage.geometry.polyhedron
[<class 'sage.geometry.polyhedron.base0.Polyhedron_base0'>]
sage: len(sage.geometry.abc.Polyhedron.__subclasses__()) <= 1
True
```

4.2 Convex Sets

class sage.geometry.convex_set.AffineHullProjectionData(image: Any | None = None, projection_linear_map: Any | None = None, projection_translation: Any | None = None, section_linear_map: Any | None = None, section_translation: Any | None = None)

Bases: object

image: Any = None

projection_linear_map: Any = None

projection_translation: Any = None

section_linear_map: Any = None

section_translation: Any = None
class sage.geometry.convex_set.ConvexSet_base
Bases: SageObject, Set_base

Abstract base class for convex sets.

affine_hull (*args, **kwds)

Return the affine hull of self as a polyhedron.

EXAMPLES:

```
sage: from sage.geometry.convex_set import ConvexSet_compact
sage: class EmbeddedDisk(ConvexSet_compact):
    ....: def an_affine_basis(self):
    ....:     return [vector([1, 0, 0]), vector([1, 1, 0]), vector([1, 0, 1])]
sage: O = EmbeddedDisk()
sage: O.dim()
2
sage: O.affine_hull()
A 2-dimensional polyhedron in QQ^3 defined as the convex hull of 1 vertex and
-> 2 lines
```

affine_hull_projection (as_convex_set=None, as_affine_map=False, orthogonal=False, orthonormal=False, extend=False, minimal=False, return_all_data=False, **kwds)

Return self projected into its affine hull.

Each convex set is contained in some smallest affine subspace (possibly the entire ambient space) – its affine hull. We provide an affine linear map that projects the ambient space of the convex set to the standard Euclidean space of dimension of the convex set, which restricts to a bijection from the affine hull.

The projection map is not unique; some parameters control the choice of the map. Other parameters control the output of the function.

EXAMPLES:

```
sage: P = Polyhedron(vertices=[[1, 0], [0, 1]])
sage: ri_P = P.relative_interior(); ri_P
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex...
-> hull of 2 vertices
sage: ri_P.affine_hull_projection(as_affine_map=True)
(Vector space morphism represented by the matrix:
[1]
[0]
Domain: Vector space of dimension 2 over Rational Field
Codomain: Vector space of dimension 1 over Rational Field,
(0))
sage: P_aff = P.affine_hull_projection(); P_aff
A 1-dimensional polyhedron in ZZ^1 defined as the convex hull of 2 vertices
sage: ri_P_aff = ri_P.affine_hull_projection(); ri_P_aff
Relative interior of a 1-dimensional polyhedron in QQ^1 defined as the convex...
-> hull of 2 vertices
sage: ri_P_aff.closure() == P_aff
True
```

ambient()

Return the ambient convex set or space.

The default implementation delegates to ambient_vector_space().

EXAMPLES:
sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:    def ambient_vector_space(self, base_field=None):
    ....:        return (base_field or QQ)^2001
sage: ExampleSet().ambient()
Vector space of dimension 2001 over Rational Field

ambient_dim()  
Return the dimension of the ambient convex set or space.  
The default implementation obtains it from ambient().  

EXAMPLES:

sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:    def ambient(self):
    ....:        return QQ^7
sage: ExampleSet().ambient_dim()
7

ambient_dimension()  
Return the dimension of the ambient convex set or space.  
This is the same as ambient_dim().  

EXAMPLES:

sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:    def ambient_dim(self):
    ....:        return 91
sage: ExampleSet().ambient_dimension()
91

ambient_vector_space(base_field=None)  
Return the ambient vector space.  
Subclasses must provide an implementation of this method.  
The default implementations of ambient(), ambient_dim(), ambient_dimension() use this method.  

EXAMPLES:

sage: from sage.geometry.convex_set import ConvexSet_base
sage: C = ConvexSet_base()
sage: C.ambient_vector_space()
Traceback (most recent call last):
...
NotImplementedError: <abstract method ambient_vector_space at ...

an_affine_basis()  
Return points that form an affine basis for the affine hull.  
The points are guaranteed to lie in the topological closure of self.  

EXAMPLES:
```python
sage: from sage.geometry.convex_set import ConvexSet_base
sage: C = ConvexSet_base()
sage: C.an_affine_basis()
Traceback (most recent call last):
...  
TypeError: 'NotImplementedType' object is not callable
```

**an_element()**

Return a point of self.

If self is empty, an EmptySetError will be raised.

The default implementation delegates to _some_elements_().

**EXAMPLES:**

```python
sage: from sage.geometry.convex_set import ConvexSet_compact
sage: class BlueBox(ConvexSet_compact):
    ....:     def _some_elements_(self):
    ....:         yield 'blue'
    ....:         yield 'cyan'
sage: BlueBox().an_element()
'blue'
```

**cardinality()**

Return the cardinality of this set.

**OUTPUT:**

Either an integer or Infinity.

**EXAMPLES:**

```python
sage: p = LatticePolytope([], lattice=ToricLattice(3).dual()); p
-1-d lattice polytope in 3-d lattice M
sage: p.cardinality()
0
sage: q = Polyhedron(ambient_dim=2); q
The empty polyhedron in ZZ^2
sage: q.cardinality()
0
sage: r = Polyhedron(rays=[(1, 0)]); r
A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and...
˓→1 ray
sage: r.cardinality()
+Infinity
```

**cartesian_product**(other)

Return the Cartesian product.

**INPUT:**

- *other* – another convex set

**OUTPUT:**

The Cartesian product of self and other.

**closure()**

Return the topological closure of self.
EXAMPLES:

```python
sage: from sage.geometry.convex_set import ConvexSet_closed
sage: C = ConvexSet_closed()
```

```
sage: C.is(C)
True
```

codim()

Return the codimension of `self` in `self.ambient()`.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[(1,2,3)], rays=[(1,0,0)])
sage: P.codimension()
2
```

An alias is `codim()`:

```python
sage: P.codim()
2
```

codimension()  

Return the codimension of `self` in `self.ambient()`.

EXAMPLES:

```python
sage: P = Polyhedron(vertices=[(1,2,3)], rays=[(1,0,0)])
sage: P.codimension()
2
```

An alias is `codim()`:

```python
sage: P.codim()
2
```

contains (point)

Test whether `self` contains the given point.

INPUT:

- `point` – a point or its coordinates
dilation (scalar)

Return the dilated (uniformly stretched) set.

INPUT:

- `scalar` – A scalar, not necessarily in `base_ring()`

EXAMPLES:

```python
sage: from sage.geometry.convex_set import ConvexSet_compact
sage: class GlorifiedPoint(ConvexSet_compact):
    ....:    def __init__(self, p):
    ....:        self._p = p
    ....:    def ambient_vector_space(self):
    ....:        return self._p.parent().vector_space()
    ....:    def linear_transformation(self, linear_transf):
    ....:        return GlorifiedPoint(linear_transf * self._p)
```

(continues on next page)
dim()  
Return the dimension of self.

Subclasses must provide an implementation of this method or of the method \texttt{an_affine_basis()}.

dimension()  
Return the dimension of self.

This is the same as \texttt{dim()}.

EXAMPLES:

```python
sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:     def dim(self):
    ....:         return 42
sage: ExampleSet().dimension()
42
```

interior()  
Return the topological interior of self.

EXAMPLES:

```python
sage: from sage.geometry.convex_set import ConvexSet_open
sage: C = ConvexSet_open()
sage: C.interior() is C
True
```

intersection\ (other)  
Return the intersection of self and other.

INPUT:

• other – another convex set

OUTPUT:

The intersection.

is_closed()  
Return whether self is closed.

The default implementation of this method only knows that the empty set, a singleton set, and the ambient space are closed.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:     def dim(self):
    ....:         return 0
```
is_compact()  
Return whether self is compact.

The default implementation of this method only knows that a non-closed set cannot be compact, and that the empty set and a singleton set are compact.

OUTPUT:
Boolean.

Example:
```
sage: from sage.geometry.convex_set import ConvexSet_base sage: class ExampleSet(ConvexSet_base): ...:     def dim(self): ...:         return 0 sage: ExampleSet().is_compact() True
```

is_empty()  
Test whether self is the empty set.

OUTPUT:
Boolean.

EXAMPLES:
```
sage: p = LatticePolytope([], lattice=ToricLattice(3).dual()); p  
-1-d lattice polytope in 3-d lattice M
sage: p.is_empty() True
```

is_finite()  
Test whether self is a finite set.

OUTPUT:
Boolean.

EXAMPLES:
```
sage: p = LatticePolytope([], lattice=ToricLattice(3).dual()); p  
-1-d lattice polytope in 3-d lattice M
sage: p.is_finite() True
sage: q = Polyhedron(ambient_dim=2); q  
The empty polyhedron in ZZ^2
sage: q.is_finite() True
sage: r = Polyhedron(rays=[(1, 0)]); r  
A 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex and  
→-1 ray
sage: r.is_finite() False
```

is_full_dimensional()  
Return whether self is full dimensional.

OUTPUT:
Boolean. Whether the polyhedron is not contained in any strict affine subspace.

EXAMPLES:
Combinatorial and Discrete Geometry, Release 10.3

```sage
sage: c = Cone([(1,0)])
sage: c.is_full_dimensional()
False

sage: polytopes.hypercube(3).is_full_dimensional()
True
sage: Polyhedron(vertices=[(1,2,3)], rays=[(1,0,0)]).is_full_dimensional()
False
```

**is_open()**

Return whether self is open.

The default implementation of this method only knows that the empty set and the ambient space are open.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```sage
sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:     def is_empty(self):
    ....:         return False
    ....:     def is_universe(self):
    ....:         return True
sage: ExampleSet().is_open()
True
```

**is_relatively_open()**

Return whether self is relatively open.

The default implementation of this method only knows that open sets are also relatively open, and in addition singletons are relatively open.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```sage
sage: from sage.geometry.convex_set import ConvexSet_base
sage: class ExampleSet(ConvexSet_base):
    ....:     def is_open(self):
    ....:         return True
sage: ExampleSet().is_relatively_open()
True
```

**is_universe()**

Test whether self is the whole ambient space.

**OUTPUT:**

Boolean.

**linear_transformation(linear_transf)**

Return the linear transformation of self.

**INPUT:**

- `linear_transf` – a matrix

4.2. Convex Sets 709
relative_interior()
Return the relative interior of self.
EXAMPLES:

```
sage: from sage.geometry.convex_set import ConvexSet_relatively_open
sage: C = ConvexSet_relatively_open()
sage: C.relative_interior() is C
True
```

representative_point()
Return a “generic” point of self.
OUTPUT:
A point in the relative interior of self as a coordinate vector.
EXAMPLES:

```
sage: C = Cone([[1, 2, 0], [2, 1, 0]])
sage: C.representative_point()
(1, 1, 0)
```

some_elements()
Return a list of some points of self.
If self is empty, an empty list is returned; no exception will be raised.
The default implementation delegates to _some_elements_.
EXAMPLES:

```
sage: from sage.geometry.convex_set import ConvexSet_compact
sage: class BlueBox(ConvexSet_compact):
    ....:     def _some_elements_(self):
    ....:         yield 'blue'
    ....:         yield 'cyan'
sage: BlueBox().some_elements()
['blue', 'cyan']
```

translation(displacement)
Return the translation of self by a displacement vector.
INPUT:
• displacement – a displacement vector or a list/tuple of coordinates that determines a displacement vector

class sage.geometry.convex_set.ConvexSet_closed
Bases: ConvexSet_base
Abstract base class for closed convex sets.

is_closed()
Return whether self is closed.
OUTPUT:
Boolean.
EXAMPLES:
```
sage: hcube = polytopes.hypercube(5)
sage: hcube.is_closed()
True
```

**is_open()**

Return whether `self` is open.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: hcube = polytopes.hypercube(5)
sage: hcube.is_open()
False
sage: zerocube = polytopes.hypercube(0)
sage: zerocube.is_open()
True
```

class `sage.geometry.convex_set.ConvexSet_compact`

Bases: `ConvexSet_closed`

Abstract base class for compact convex sets.

**is_compact()**

Return whether `self` is compact.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: cross3 = lattice_polytope.cross_polytope(3)
sage: cross3.is_compact()
True
```

**is_relatively_open()**

Return whether `self` is open.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: hcube = polytopes.hypercube(5)
sage: hcube.is_open()
False
sage: zerocube = polytopes.hypercube(0)
sage: zerocube.is_open()
True
```

**is_universe()**

Return whether `self` is the whole ambient space

**OUTPUT:**
Boolean.

EXAMPLES:

```python
sage: cross3 = lattice_polytope.cross_polytope(3)
sage: cross3.is_universe()
False
sage: point0 = LatticePolytope([[0]]); point0
0-d reflexive polytope in 0-d lattice M
sage: point0.is_universe()
True
```

class sage.geometry.convex_set.ConvexSet_open

Bases: ConvexSet_relatively_open

Abstract base class for open convex sets.

```python
... class OpenBall(ConvexSet_open):
    .... def dim(self):
    ....     return 3
    .... def is_universe(self):
    ....     return False
sage: OpenBall().is_closed()
False
```

```python
is_open()
```

Return whether `self` is open.

OUTPUT:

Boolean.

EXAMPLES:

```python
sage: from sage.geometry.convex_set import ConvexSet_open
sage: b = ConvexSet_open()
sage: b.is_open()
True
```

class sage.geometry.convex_set.ConvexSet_relatively_open

Bases: ConvexSet_base

Abstract base class for relatively open convex sets.

```python
... is_open()
```

Return whether `self` is open.

OUTPUT:

Boolean.

EXAMPLES:
```python
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior()
sage: ri_segment.is_open()
False
```

```python
is_relatively_open()
Return whether self is relatively open.

OUTPUT:
Boolean.

EXAMPLES:
```
```python
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior()
sage: ri_segment.is_relatively_open()
True
```

## 4.3 Linear Expressions

A linear expression is just a linear polynomial in some (fixed) variables (allowing a nonzero constant term). This class only implements linear expressions for others to use.

EXAMPLES:
```
```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ); L
Module of linear expressions in variables x, y, z over Rational Field
sage: x + 2*y + 3*z + 4
x + 2*y + 3*z + 4
sage: L(4)
0*x + 0*y + 0*z + 4
```

You can also pass coefficients and a constant term to construct linear expressions:
```
```python
sage: L([[1, 2, 3], 4])
x + 2*y + 3*z + 4
sage: L([(1, 2, 3), 4])
x + 2*y + 3*z + 4
sage: L([4, 1, 2, 3])  # note: constant is first in single-tuple notation
x + 2*y + 3*z + 4
```

The linear expressions are a module over the base ring, so you can add them and multiply them with scalars:
```
```python
sage: m = x + 2*y + 3*z + 4
sage: 2*m
2*x + 4*y + 6*z + 8
sage: m+m
2*x + 4*y + 6*z + 8
sage: m-m
0*x + 0*y + 0*z + 0
```

```python
class sage.geometry.linear_expression.LinearExpression (parent, coefficients, constant, check=True)
Bases: ModuleElement
```
A linear expression.

A linear expression is just a linear polynomial in some (fixed) variables.

EXAMPLES:

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: m = L([1, 2, 3], 4); m
x + 2*y + 3*z + 4
sage: m2 = L((1, 2, 3), 4); m2
x + 2*y + 3*z + 4
sage: m3 = L([4, 1, 2, 3]); m3  # note: constant is first in single-tuple notation
x + 2*y + 3*z + 4
sage: m == m2
True
sage: m2 == m3
True
sage: L.zero()
0*x + 0*y + 0*z + 0
sage: a = L([12, 2/3, -1], -2)
sage: a - m
11*x - 4/3*y - 4*z - 6
sage: LZ.<x,y,z> = LinearExpressionModule(ZZ)
sage: a - LZ([2, -1, 3], 1)
10*x + 5/3*y - 4*z - 3
```

**A()**

Return the coefficient vector.

**OUTPUT:**

The coefficient vector of the linear expression.

**EXAMPLES:**

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: linear = L([1, 2, 3], 4); linear
x + 2*y + 3*z + 4
sage: linear.A()
(1, 2, 3)
sage: linear.b()
4
```

**b()**

Return the constant term.

**OUTPUT:**

The constant term of the linear expression.

**EXAMPLES:**

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: linear = L([1, 2, 3], 4); linear
x + 2*y + 3*z + 4
sage: linear.A()
(1, 2, 3)
sage: linear.b()
4
```

(continues on next page)
change_ring (base_ring)
Change the base ring of this linear expression.

INPUT:

• base_ring – a ring; the new base ring

OUTPUT:
A new linear expression over the new base ring.

EXAMPLES:

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: a = x + 2*y + 3*z + 4; a
x + 2*y + 3*z + 4
sage: a.change_ring(RDF)
1.0*x + 2.0*y + 3.0*z + 4.0
```

coefficients ()
Return all coefficients.

OUTPUT:
The constant (as first entry) and coefficients of the linear terms (as subsequent entries) in a list.

EXAMPLES:

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: linear = L([1, 2, 3], 4); linear
x + 2*y + 3*z + 4
sage: linear.coefficients()
[4, 1, 2, 3]
```

constant_term ()
Return the constant term.

OUTPUT:
The constant term of the linear expression.

EXAMPLES:

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: linear = L([1, 2, 3], 4); linear
x + 2*y + 3*z + 4
sage: linear.A()
(1, 2, 3)
sage: linear.b()
4
```
dense_coefficient_list()
Return all coefficients.

OUTPUT:
The constant (as first entry) and coefficients of the linear terms (as subsequent entries) in a list.

EXAMPLES:
```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: linear = L([1, 2, 3], 4); linear
x + 2*y + 3*z + 4
sage: linear.coefficients()
[4, 1, 2, 3]
```

evaluate(point)
Evaluate the linear expression.

INPUT:
- point – list/tuple/iterable of coordinates; the coordinates of a point

OUTPUT:
The linear expression $Ax + b$ evaluated at the point $x$.

EXAMPLES:
```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y> = LinearExpressionModule(QQ)
sage: ex = 2*x + 3* y + 4
sage: ex.evaluate([1,1])
9
sage: ex([1,1])    # syntactic sugar
9
sage: ex([pi, e])  # needs sage.symbolic
2*pi + 3*e + 4
```

monomial_coefficients(copy=True)
Return a dictionary whose keys are indices of basis elements in the support of self and whose values are the corresponding coefficients.

INPUT:
- copy – ignored

EXAMPLES:
```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: linear = L([1, 2, 3], 4)
sage: sorted(linear.monomial_coefficients().items(), key=lambda x: str(x[0]))
[(0, 1), (1, 2), (2, 3), (b, 4)]
```

class sage.geometry.linear_expression.LinearExpressionModule(base_ring, names=())
Bases: Parent, UniqueRepresentation
The module of linear expressions.
This is the module of linear polynomials which is the parent for linear expressions.
EXAMPLES:

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
sage: L
Module of linear expressions in variables x, y, z over Rational Field
sage: L.an_element()
x + 0*y + 0*z + 0
```

**Element**

alias of `LinearExpression`

**ambient_module()**

Return the ambient module.

See also:

`ambient_vector_space()`

**OUTPUT:**

The domain of the linear expressions as a free module over the base ring.

EXAMPLES:

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
sage: L.ambient_module()
Vector space of dimension 3 over Rational Field
sage: M = LinearExpressionModule(ZZ, ('r', 's'))
sage: M.ambient_module()
Ambient free module of rank 2 over the principal ideal domain Integer Ring
sage: M.ambient_vector_space()
Vector space of dimension 2 over Rational Field
```

**ambient_vector_space()**

Return the ambient vector space.

See also:

`ambient_module()`

**OUTPUT:**

The vector space (over the fraction field of the base ring) where the linear expressions live.

EXAMPLES:

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
sage: L.ambient_vector_space()
Vector space of dimension 3 over Rational Field
sage: M = LinearExpressionModule(ZZ, ('r', 's'))
sage: M.ambient_module()
Ambient free module of rank 2 over the principal ideal domain Integer Ring
sage: M.ambient_vector_space()
Vector space of dimension 2 over Rational Field
```

**basis()**

Return a basis of `self`.

EXAMPLES:
```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
```
```
sage: list(L.basis())
[x + 0*y + 0*z + 0,
 0*x + y + 0*z + 0,
 0*x + 0*y + z + 0,
 0*x + 0*y + 0*z + 1]
```

The `change_ring` function:

- **Description**: Returns a new module with a changed base ring.
- **Input**:
  - `base_ring`: A ring; the new base ring
- **Output**: A new linear expression over the new base ring.

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: M.<y> = LinearExpressionModule(ZZ)
sage: L = M.change_ring(QQ); L
Module of linear expressions in variable y over Rational Field
```

The `gen` function:

- **Description**: Returns the \(i\)-th generator.
- **Input**: \(i\) – integer
- **Output**: A linear expression.

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, (x, y, z))
sage: L.gen(0)
x + 0*y + 0*z + 0
```

The `gens` function:

- **Description**: Returns the generators of `self`.
- **Output**: A tuple of linear expressions, one for each linear variable.

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
sage: L.gens()
(x + 0*y + 0*z + 0, 0*x + y + 0*z + 0, 0*x + 0*y + z + 0)
```

The `ngens` function:

- **Description**: Returns the number of linear variables.
- **Output**:

```python
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
sage: L.ngens()
(3)
```
An integer.

EXAMPLES:

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L = LinearExpressionModule(QQ, ('x', 'y', 'z'))
sage: L.ngens()
3
```

`random_element()`  
Return a random element.

EXAMPLES:

```
sage: from sage.geometry.linear_expression import LinearExpressionModule
sage: L.<x,y,z> = LinearExpressionModule(QQ)
sage: L.random_element() in L
True
```

### 4.4 Newton Polygons

This module implements finite Newton polygons and infinite Newton polygons having a finite number of slopes (and hence a last infinite slope).

```python
class sage.geometry.newton_polygon.NewtonPolygon_element(polyhedron, parent)
Bases: Element

Class for infinite Newton polygons with last slope.

`last_slope()`  
Returns the last (infinite) slope of this Newton polygon if it is infinite and `+Infinity` otherwise.

EXAMPLES:
```
sage: from sage.geometry.newton_polygon import NewtonPolygon
sage: NP1 = NewtonPolygon([[0,0], [1,1], [2,8], [3,5]], last_slope=3)
sage: NP1.last_slope()
3
sage: NP2 = NewtonPolygon([[0,0], [1,1], [2,5]])
sage: NP2.last_slope()
+Infinity
```

We check that the last slope of a sum (resp. a product) is the minimum of the last slopes of the summands (resp. the factors):
```
sage: (NP1 + NP2).last_slope()
3
sage: (NP1 * NP2).last_slope()
3
```

`plot(**kwargs)`  
Plot this Newton polygon.

**Note:** All usual rendering options (color, thickness, etc.) are available.
Examples:

```python
sage: from sage.geometry.newton_polygon import NewtonPolygon
sage: NP = NewtonPolygon([(0,0), (1,1), (2,6)])
```

```
# needs sage.plot
polygon = NP.plot()
```

reverse (degree=None)

Returns the symmetric of self

**Input:**

- degree – an integer (default: the top right abscissa of this Newton polygon)

**Output:**

The image this Newton polygon under the symmetry ‘(x,y) mapsto (degree-x, y)’

Examples:

```python
sage: from sage.geometry.newton_polygon import NewtonPolygon
sage: NP = NewtonPolygon([(0,0), (1,1), (2,5)])
```

```
sage: NP2 = NP.reverse(); NP2
Finite Newton polygon with 3 vertices: (0, 5), (1, 1), (2, 0)
```

We check that the slopes of the symmetric Newton polygon are the opposites of the slopes of the original Newton polygon:

```
sage: NP.slopes()
[1, 4]
sage: NP2.slopes()
[-4, -1]
```

slopes (repetition=True)

Returns the slopes of this Newton polygon

**Input:**

- repetition – a boolean (default: True)

**Output:**

The consecutive slopes (not including the last slope if the polygon is infinity) of this Newton polygon.

If repetition is True, each slope is repeated a number of times equal to its length. Otherwise, it appears only one time.

Examples:

```python
sage: from sage.geometry.newton_polygon import NewtonPolygon
sage: NP = NewtonPolygon([(0,0), (1,1), (3,6)]); NP
```

```
Finite Newton polygon with 3 vertices: (0, 0), (1, 1), (3, 6)
```

```
sage: NP.slopes()
[1, 5/2, 5/2]
sage: NP.slopes(repetition=False)
[1, 5/2]
```
vertices (copy=True)

Returns the list of vertices of this Newton polygon

INPUT:
• copy – a boolean (default: True)

OUTPUT:
The list of vertices of this Newton polygon (or a copy of it if copy is set to True)

EXAMPLES:

```
sage: from sage.geometry.newton_polygon import NewtonPolygon
sage: NP = NewtonPolygon([(0,0), (1,1), (2,5)]); NP
Finite Newton polygon with 3 vertices: (0, 0), (1, 1), (2, 5)

sage: v = NP.vertices(); v
[(0, 0), (1, 1), (2, 5)]
```

class sage.geometry.newton_polygon.ParentNewtonPolygon

Construct a Newton polygon.

INPUT:
• arg – a list/tuple/iterable of vertices or of slopes. Currently, slopes must be rational numbers.
• sort_slopes – boolean (default: True). Specifying whether slopes must be first sorted
• last_slope – rational or infinity (default: Infinity). The last slope of the Newton polygon

OUTPUT:
The corresponding Newton polygon.

Note: By convention, a Newton polygon always contains the point at infinity (0, ∞). These polygons are attached to polynomials or series over discrete valuation rings (e.g. padics).

EXAMPLES:
We specify here a Newton polygon by its vertices:

```
sage: from sage.geometry.newton_polygon import NewtonPolygon
sage: NewtonPolygon([(0,0), (1,1), (3,5)])
Finite Newton polygon with 3 vertices: (0, 0), (1, 1), (3, 5)
```

We note that the convex hull of the vertices is automatically computed:

```
sage: NewtonPolygon([(0,0), (1,1), (2,8), (3,5)])
Finite Newton polygon with 3 vertices: (0, 0), (1, 1), (3, 5)
```

Note that the value +Infinity is allowed as the second coordinate of a vertex:

```
sage: NewtonPolygon([(0,0), (1,Infinity), (2,8), (3,5)])
Finite Newton polygon with 2 vertices: (0, 0), (3, 5)
```

If last_slope is set, the returned Newton polygon is infinite and ends with an infinite line having the specified slope:
sage: NewtonPolygon([ (0,0), (1,1), (2,8), (3,5) ], last_slope=3)
Infinite Newton polygon with 3 vertices: (0, 0), (1, 1), (3, 5) ending by an infinite line of slope 3

Specifying a last slope may discard some vertices:

sage: NewtonPolygon([ (0,0), (1,1), (2,8), (3,5) ], last_slope=3/2)
Infinite Newton polygon with 2 vertices: (0, 0), (1, 1) ending by an infinite line of slope 3/2

Next, we define a Newton polygon by its slopes:

sage: NP = NewtonPolygon([0, 1/2, 1/2, 2/3, 2/3, 2/3, 1, 1])
sage: NP
Finite Newton polygon with 5 vertices: (0, 0), (1, 0), (3, 1), (6, 3), (8, 5)
sage: NP.slopes()
[0, 1/2, 1/2, 2/3, 2/3, 2/3, 1, 1]

By default, slopes are automatically sorted:

sage: NP2 = NewtonPolygon([0, 1, 1/2, 2/3, 1/2, 2/3, 1, 2/3])
sage: NP2
Finite Newton polygon with 5 vertices: (0, 0), (1, 0), (3, 1), (6, 3), (8, 5)
sage: NP == NP2
True

except if the contrary is explicitly mentioned:

sage: NewtonPolygon([0, 1, 1/2, 2/3, 1/2, 2/3, 1, 2/3], sort_slopes=False)
Finite Newton polygon with 4 vertices: (0, 0), (1, 0), (6, 10/3), (8, 5)

Slopes greater that or equal last_slope (if specified) are discarded:

sage: NP = NewtonPolygon([0, 1/2, 1/2, 2/3, 2/3, 2/3, 1, 2/3], last_slope=2/3)
sage: NP
Infinite Newton polygon with 3 vertices: (0, 0), (1, 0), (3, 1) ending by an infinite line of slope 2/3
sage: NP.slopes()
[0, 1/2, 1/2]

Be careful, do not confuse Newton polygons provided by this class with Newton polytopes. Compare:

sage: NP = NewtonPolygon([ (0,0), (1,45), (3,6) ]); NP
Finite Newton polygon with 2 vertices: (0, 0), (3, 6)
sage: x, y = polygen(QQ,'x, y')
sage: p = 1 + x*y**45 + x**3*y**6
sage: p.newton_polytope()
A 2-dimensional polyhedron in ZZ^2 defined as the convex hull of 3 vertices
sage: p.newton_polytope().vertices()
(A vertex at (0, 0), A vertex at (1, 45), A vertex at (3, 6))

Element

alias of NewtonPolygon_element
4.5 Relative Interiors of Polyhedra and Cones

```python
class sage.geometry.relative_interior.RelativeInterior(polyhedron):
    Bases: ConvexSet_relatively_open

    The relative interior of a polyhedron or cone.
    This class should not be used directly. Use methods relative_interior(), interior(), relative_interior(), interior() instead.

    EXAMPLES:
    sage: segment = Polyhedron([[1, 2], [3, 4]])
    sage: segment.relative_interior()
    Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
    sage: octant = Cone(((1,0), (0,1), (0,0,1)))
    sage: octant.relative_interior()
    Relative interior of 3-d cone in 3-d lattice N

    ambient()
    Return the ambient convex set or space.

    EXAMPLES:
    sage: segment = Polyhedron([[1, 2], [3, 4]])
    sage: ri_segment = segment.relative_interior(); ri_segment
    Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
    sage: ri_segment.ambient()
    Vector space of dimension 2 over Rational Field

    ambient_dim()
    Return the dimension of the ambient space.

    EXAMPLES:
    sage: segment = Polyhedron([[1, 2], [3, 4]])
    sage: segment.ambient_dim()
    2
    sage: ri_segment = segment.relative_interior(); ri_segment
    Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
    sage: ri_segment.ambient_dim()
    2

    ambient_vector_space(base_field=None)
    Return the ambient vector space.

    EXAMPLES:
    sage: segment = Polyhedron([[1, 2], [3, 4]])
    sage: ri_segment = segment.relative_interior(); ri_segment
    Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
    sage: ri_segment.ambient_vector_space()
    Vector space of dimension 2 over Rational Field
```
an_affine_basis()
Return points that form an affine basis for the affine hull.
The points are guaranteed to lie in the topological closure of self.

EXAMPLES:

```
sage: segment = Polyhedron([[1, 0], [0, 1]])
sage: segment.relative_interior().an_affine_basis()
[A vertex at (1, 0), A vertex at (0, 1)]
```

closure()
Return the topological closure of self.

EXAMPLES:

```
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: ri_segment.closure() is segment
True
```

dilation(scalar)
Return the dilated (uniformly stretched) set.

INPUT:

• scalar – A scalar

EXAMPLES:

```
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: A = ri_segment.dilation(2); A
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: A.closure().vertices()
(A vertex at (2, 4), A vertex at (6, 8))
sage: B = ri_segment.dilation(-1/3); B
Relative interior of a 1-dimensional polyhedron in QQ^2 defined as the convex hull of 2 vertices
sage: B.closure().vertices()
(A vertex at (-1, -4/3), A vertex at (-1/3, -2/3))
sage: C = ri_segment.dilation(0); C
A 0-dimensional polyhedron in ZZ^2 defined as the convex hull of 1 vertex
sage: C.vertices()
(A vertex at (0, 0),)
```

dim()
Return the dimension of self.

EXAMPLES:

```
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: segment.dim()
1
```

(continues on next page)
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of
a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: ri_segment.dim()
1

interior()

Return the interior of self.

EXAMPLES:

```
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: ri_segment.interior()
The empty polyhedron in ZZ^2
sage: octant = Cone([(1,0,0), (0,1,0), (0,0,1)])
sage: ri_octant = octant.relative_interior(); ri_octant
Relative interior of 3-d cone in 3-d lattice N
sage: ri_octant.interior() is ri_octant
True
```

is_closed()

Return whether self is closed.

OUTPUT:

Boolean.

EXAMPLES:

```
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: ri_segment.is_closed()
False
```

is_universe()

Return whether self is the whole ambient space

OUTPUT:

Boolean.

EXAMPLES:

```
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: ri_segment.is_universe()
False
```

linear_transformation(linear_transf, **kwds)

Return the linear transformation of self.
By [Roc1970], Theorem 6.6, the linear transformation of a relative interior is the relative interior of the linear transformation.

INPUT:

- `linear_transf` – a matrix
- `**kwds` – passed to the `linear_transformation()` method of the closure of `self`.

EXAMPLES:

```python
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: T = matrix([[1, 1]])
sage: A = ri_segment.linear_transformation(T); A
Relative interior of a 1-dimensional polyhedron in ZZ^1 defined as the convex hull of 2 vertices
sage: A.closure().vertices()
(A vertex at (3), A vertex at (7))
```

`relative_interior()`

Return the relative interior of `self`.

As `self` is already relatively open, this method just returns `self`.

EXAMPLES:

```python
sage: segment = Polyhedron([[1, 2], [3, 4]])
sage: ri_segment = segment.relative_interior(); ri_segment
Relative interior of a 1-dimensional polyhedron in ZZ^2 defined as the convex hull of 2 vertices
sage: ri_segment.relative_interior() is ri_segment
True
```

`representative_point()`

Return a “generic” point of `self`.

OUTPUT:

A point in `self` (thus, in the relative interior of `self`) as a coordinate vector.

EXAMPLES:

```python
sage: C = Cone([[1, 2, 0], [2, 1, 0]])
sage: C.relative_interior().representative_point()
(1, 1, 0)
```

`translation()`

Return the translation of `self` by a displacement vector.

INPUT:

- `displacement` – a displacement vector or a list/tuple of coordinates that determines a displacement vector

EXAMPLES:
4.6 Ribbon Graphs

This file implements objects called ribbon graphs. These are graphs together with a cyclic ordering of the darts adjacent to each vertex. This data allows us to unambiguously “thicken” the ribbon graph to an orientable surface with boundary. Also, every orientable surface with non-empty boundary is the thickening of a ribbon graph.

AUTHORS:

• Pablo Portilla (2016)

class sage.geometry.ribbon_graph.RibbonGraph(sigma, rho)

Bases: SageObject, UniqueRepresentation

A ribbon graph codified as two elements of a certain permutation group.

A comprehensive introduction on the topic can be found in the beginning of [GGD2011] Chapter 4. More concretely, we will use a variation of what is called in the reference “The permutation representation pair of a dessin”. Note that in that book, ribbon graphs are called “dessins d’enfant”. For the sake on completeness we reproduce an adapted version of that introduction here.

Brief introduction

Let $\Sigma$ be an orientable surface with non-empty boundary and let $\Gamma$ be the topological realization of a graph that is embedded in $\Sigma$ in such a way that the graph is a strong deformation retract of the surface.

Let $v(\Gamma)$ be the set of vertices of $\Gamma$, suppose that these are white vertices. Now we mark black vertices in an interior point of each edge. In this way we get a bipartite graph where all the black vertices have valency 2 and there is no restriction on the valency of the white vertices. We call the edges of this new graph darts (sometimes they are also called half edges of the original graph). Observe that each edge of the original graph is formed by two darts.

Given a white vertex $v \in v(\Gamma)$, let $d(v)$ be the set of darts adjacent to $v$. Let $D(\Gamma)$ be the set of all the darts of $\Gamma$ and suppose that we enumerate the set $D(\Gamma)$ and that it has $n$ elements.

With the orientation of the surface and the embedding of the graph in the surface we can produce two permutations:

• A permutation that we denote by $\sigma$. This permutation is a product of as many cycles as white vertices (that is vertices in $\Gamma$). For each vertex consider a small topological circle around it in $\Sigma$. This circle intersects each adjacent dart once. The circle has an orientation induced by the orientation on $\Sigma$ and so defines a cycle that sends the number associated to one dart to the number associated to the next dart in the positive orientation of the circle.

• A permutation that we denote by $\rho$. This permutation is a product of as many 2-cycles as edges has $\Gamma$. It just tells which two darts belong to the same edge.
Abstract definition

Consider a graph $\Gamma$ (not a priori embedded in any surface). Now we can again consider one vertex in the interior of each edge splitting each edge in two darts. We label the darts with numbers.

We say that a ribbon structure on $\Gamma$ is a set of two permutations $(\sigma, \rho)$. Where $\sigma$ is formed by as many disjoint cycles as vertices had $\Gamma$. And each cycle is a cyclic ordering of the darts adjacent to a vertex. The permutation $\rho$ just tell us which two darts belong to the same edge.

For any two such permutations there is a way of “thickening” the graph to a surface with boundary in such a way that the surface retracts (by a strong deformation retract) to the graph and hence the graph is embedded in the surface in a such a way that we could recover $\sigma$ and $\rho$.

INPUT:

- $\sigma$ – a permutation a product of disjoint cycles of any length; singletons (vertices of valency 1) need not be specified
- $\rho$ – a permutation which is a product of disjoint 2-cycles

Alternatively, one can pass in 2 integers and this will construct a ribbon graph with genus $\sigma$ and $\rho$ boundary components. See make_ribbon().

One can also construct the bipartite graph modeling the corresponding Brieskorn-Pham singularity by passing 2 integers and the keyword bipartite=True. See bipartite_ribbon_graph().

EXAMPLES:

Consider the ribbon graph consisting of just 1 edge and 2 vertices of valency 1:

```sage
sage: s0 = PermutationGroupElement('(1)(2)')
sage: r0 = PermutationGroupElement('(1,2)')
sage: R0 = RibbonGraph(s0, r0); R0
Ribbon graph of genus 0 and 1 boundary components
```

Consider a graph that has 2 vertices of valency 3 (and hence 3 edges). That is represented by the following two permutations:

```sage
sage: s1 = PermutationGroupElement('(1,3,5)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)')
sage: R1 = RibbonGraph(s1, r1); R1
Ribbon graph of genus 1 and 1 boundary components
```

By drawing the picture in a piece of paper, one can see that its thickening has only 1 boundary component. Since the thickening is homotopically equivalent to the graph and the graph has Euler characteristic $-1$, we find that the thickening has genus 1:

```sage
sage: R1.number_boundaries()
1
sage: R1.genus()
1
```

The following example corresponds to the complete bipartite graph of type $(2, 3)$, where we have added one more edge $(8, 15)$ that ends at a vertex of valency 1. Observe that it is not necessary to specify the vertex (15) of valency 1 when we define sigma:

```sage
sage: s2 = PermutationGroupElement('(1,3,5,8)(2,4,6)')
sage: r2 = PermutationGroupElement('(1,2)(3,4)(5,6)(8,15)')
sage: R2 = RibbonGraph(s2, r2); R2
Ribbon graph of genus 1 and 1 boundary components
```
This example is constructed by taking the bipartite graph of type $(3,3)$:

```
sage: s3 = PermutationGroupElement('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)')
sage: r3 = PermutationGroupElement('(1,16)(2,13)(3,10)(4,17)(5,14)(6,11)(7,18)(8,15)(9,12)')
sage: R3 = RibbonGraph(s3, r3); R3
Ribbon graph of genus 1 and 3 boundary components
```

The labeling of the darts can omit some numbers:

```
sage: s4 = PermutationGroupElement('(3,5,10,12)')
sage: r4 = PermutationGroupElement('(3,10)(5,12)')
sage: R4 = RibbonGraph(s4, r4); R4
Ribbon graph of genus 1 and 1 boundary components
```

The next example is the complete bipartite graph of type $(3,3)$, where we have added an edge that ends at a vertex of valency 1:

```
sage: s5 = PermutationGroupElement('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18,19)')
sage: R5 = RibbonGraph(s5, r5); R5
Ribbon graph of genus 1 and 3 boundary components
```

```
sage: C = R5.contract_edge(9); C
Ribbon graph of genus 1 and 3 boundary components
```

```
sage: C.sigma()
(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)
sage: C.rho()
(1,16)(2,13)(3,10)(4,17)(5,14)(6,11)(7,18)(8,15)(9,12)
sage: S = R5.reduced(); S
Ribbon graph of genus 1 and 3 boundary components
```

```
sage: S.sigma()
(5,6,8,9,14,15,11,12)
sage: S.rho()
(5,14)(5,6,8,9,14)(6,11)(7,8,15)(6,11)
sage: R5.boundary()
[[1, 16, 17, 4, 5, 14, 15, 8, 9, 12, 10, 3],
  [2, 13, 14, 5, 6, 11, 12, 9, 7, 18, 19, 20, 20, 19, 16, 1],
  [3, 10, 11, 6, 4, 17, 18, 7, 8, 15, 13, 2]]
sage: S.boundary()
[[5, 14, 15, 8, 9, 12], [6, 11, 12, 9, 14, 5], [8, 15, 11, 6]]
sage: R5.homology_basis()

```

```
We construct a ribbon graph corresponding to a genus 0 surface with 5 boundary components:
```

```
```
```
```
We construct the Brieskorn-Pham singularity of type \((2, 3)\):

\[
\text{sage}: \text{B23 = RibbonGraph(2, 3, bipartite=\text{True})}; \text{B23}
\]

Ribbon graph of genus 1 and 1 boundary components

\[
\text{sage}: \text{B23.sigma()}
\]

\((1,2,3) (4,5,6) (7,8) (9,10) (11,12)\)

\[
\text{sage}: \text{B23.rho()}
\]

\((1,8) (2,10) (3,12) (4,7) (5,9) (6,11)\)

\section*{boundary()}

Return the labeled boundaries of \textit{self}.

If you cut the thickening of the graph along the graph, you get a collection of cylinders (recall that the graph was a strong deformation retract of the thickening). In each cylinder one of the boundary components has a labelling of its edges induced by the labelling of the darts.

\section*{OUTPUT:}

A list of lists. The number of inner lists is the number of boundary components of the surface. Each list in the list consists of an ordered tuple of numbers, each number comes from the number assigned to the corresponding dart before cutting.

\section*{EXAMPLES:}

We start with a ribbon graph whose thickening has one boundary component. We compute its labeled boundary, then reduce it and compute the labeled boundary of the reduced ribbon graph:

\[
\text{sage}: \text{s1 = PermutationGroupElement\('(1,3,5) (2,4,6)'\)}
\]

\[
\text{sage}: \text{r1 = PermutationGroupElement\('(1,2) (3,4) (5,6)'\)}
\]

\[
\text{sage}: \text{R1 = RibbonGraph(s1,r1)}; \text{R1}
\]

Ribbon graph of genus 1 and 1 boundary components

\[
\text{sage}: \text{R1.boundary()}
\]

\[
[[1, 2, 4, 3, 5, 6, 2, 1, 3, 4, 6, 5]]
\]

\[
\text{sage}: \text{H1 = R1.reduced()}; \text{H1}
\]

Ribbon graph of genus 1 and 1 boundary components

\[
\text{sage}: \text{H1.sigma()}
\]

\((3,5,4,6)\)

\[
\text{sage}: \text{H1.rho()}
\]

\((3,4) (5,6)\)

\[
\text{sage}: \text{H1.boundary()}
\]

\[
[[3, 4, 6, 5, 4, 3, 5, 6]]
\]

We now consider a ribbon graph whose thickening has 3 boundary components. Also observe that in one of the labeled boundary components, a numbers appears twice in a row. That is because the ribbon graph has a vertex of valency 1:

\[
\text{sage}: \text{s2=PermutationGroupElement\('(1,2,3) (4,5,6) (7,8,9) (10,11,12) (13,14, 15) (16,17,18,19)'\)}
\]

\[
\text{sage}: \text{r2=PermutationGroupElement\('(1,16) (2,13) (3,10) (4,17) (5,14) (6,11) (7, 18) (8,15) (9,12) (19,20)'\)}
\]

\[
\text{sage}: \text{R2 = RibbonGraph(s2,r2)}
\]

(continues on next page)
contract_edge \( (k) \)

Return the ribbon graph resulting from the contraction of the \( k \)-th edge in \( \text{self} \).

For a ribbon graph \( (\sigma, \rho) \), we contract the edge corresponding to the \( k \)-th transposition of \( \rho \).

**INPUT:**

- \( k \) – non-negative integer; the position in \( \rho \) of the transposition that is going to be contracted

**OUTPUT:**

- a ribbon graph resulting from the contraction of that edge

**EXAMPLES:**

We start again with the one-holed torus ribbon graph:

```sage
sage: s1 = PermutationGroupElement('(1,3,5)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)')
sage: R1 = RibbonGraph(s1,r1); R1
Ribbon graph of genus 1 and 1 boundary components
sage: S1 = R1.contract_edge(1); S1
Ribbon graph of genus 1 and 1 boundary components
sage: S1.sigma()
(1,6,2,5)
sage: S1.rho()
(1,2)(5,6)
```

However, this ribbon graphs is formed only by loops and hence it cannot be longer reduced, we get an error if we try to contract a loop:

```sage
sage: S1.contract_edge(1)
Traceback (most recent call last):
  ... ValueError: the edge is a loop and cannot be contracted
```

In this example, we consider a graph that has one edge \((19,20)\) such that one of its ends is a vertex of valency 1. This is the vertex \((20)\) that is not specified when defining \( \sigma \). We contract precisely this edge and get a ribbon graph with no vertices of valency 1:

```sage
sage: s2 = PermutationGroupElement('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,
˓→15)(16,17,18,19)')
sage: r2 = PermutationGroupElement('(1,16)(2,13)(3,10)(4,17)(5,14)(6,11)(7,
˓→18)(8,15)(9,12)(19,20)')
sage: R2 = RibbonGraph(s2,r2); R2
Ribbon graph of genus 1 and 3 boundary components
sage: R2.sigma()
(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18,19)
sage: R2c = R2.contract_edge(9); R2; R2c.sigma(); R2c.rho()
Ribbon graph of genus 1 and 3 boundary components
(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18,19)
(1,16)(2,13)(3,10)(4,17)(5,14)(6,11)(7,18)(8,15)(9,12)
```
**extrude_edge** (*vertex, dart1, dart2*)

Return a ribbon graph resulting from extruding an edge from a vertex, pulling from it, all darts from dart1 to dart2 including both.

**INPUT:**

- *vertex* — the position of the vertex in the permutation \( \sigma \), which must have valency at least 2
- *dart1* — the position of the first in the cycle corresponding to *vertex*
- *dart2* — the position of the second dart in the cycle corresponding to *vertex*

**OUTPUT:**

A ribbon graph resulting from extruding a new edge that pulls from *vertex* a new vertex that is, now, adjacent to all the darts from dart1 to dart2 (not including dart2) in the cyclic ordering given by the cycle corresponding to *vertex*. Note that dart1 may be equal to dart2 allowing thus to extrude a contractible edge from a vertex.

**EXAMPLES:**

We try several possibilities in the same graph:

```python
sage: s1 = PermutationGroupElement('[(1,3,5)(2,4,6)]
sage: r1 = PermutationGroupElement('[(1,2)(3,4)(5,6)]
sage: R1 = RibbonGraph(s1, r1); R1
Ribbon graph of genus 1 and 1 boundary components
sage: E1 = R1.extrude_edge(1, 1, 2); E1
Ribbon graph of genus 1 and 1 boundary components
sage: E1.sigma()
[(1,3,5)(2,8,6)(4,7)]
sage: E1.rho()
[(1,2)(3,4)(5,6)(7,8)]
sage: E2 = R1.extrude_edge(1, 1, 3); E2
Ribbon graph of genus 1 and 1 boundary components
sage: E2.sigma()
[(1,3,5)(2,8)(4,6,7)]
sage: E2.rho()
[(1,2)(3,4)(5,6)(7,8)]
```

We can also extrude a contractible edge from a vertex. This new edge will end at a vertex of valency 1:

```python
sage: E1p = R1.extrude_edge(0, 0, 0); E1p
Ribbon graph of genus 1 and 1 boundary components
sage: E1p.sigma()
[(1,3,5,8)(2,4,6)]
sage: E1p.rho()
[(1,2)(3,4)(5,6)(7,8)]
```

In the following example we first extrude one edge from a vertex of valency 3 generating a new vertex of valency 2. Then we extrude a new edge from this vertex of valency 2:

```python
sage: s1 = PermutationGroupElement('[(1,3,5)(2,4,6)]
sage: r1 = PermutationGroupElement('[(1,2)(3,4)(5,6)]
sage: R1 = RibbonGraph(s1, r1); R1
Ribbon graph of genus 1 and 1 boundary components
sage: E1 = R1.extrude_edge(0, 0, 1); E1
Ribbon graph of genus 1 and 1 boundary components
sage: E1.sigma()
[(1,7)(2,4,6)(3,5,8)]
```

(continues on next page)
sage: E1.rho()
(1,2)(3,4)(5,6)(7,8)
sage: F1 = E1.extrude_edge(0,0,1); F1
Ribbon graph of genus 1 and 1 boundary components
sage: F1.sigma()
(1,9)(2,4,6)(3,5,8)(7,10)
sage: F1.rho()
(1,2)(3,4)(5,6)(7,8)(9,10)

\textbf{genus()}

Return the genus of the thickening of \texttt{self}.

\textbf{OUTPUT:}

- \( g \) – non-negative integer representing the genus of the thickening of the ribbon graph

\textbf{EXAMPLES:}

```python
sage: s1 = PermutationGroupElement('(1,3,5)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)')
sage: R1 = RibbonGraph(s1,r1)
sage: R1.genus()
1
sage: s3 = PermutationGroupElement('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15,\rightarrow 16)(17,18,19,20)(21,22,23,24)')
sage: r3 = PermutationGroupElement('(1,21)(2,17)(3,13)(4,22)(7,23)(5,18)(6,
\rightarrow 14)(8,19)(9,15)(10,24)(11,20)(12,16)')
sage: R3 = RibbonGraph(s3,r3); R3.genus()
3
```

\textbf{homology_basis()}

Return an oriented basis of the first homology group of the graph.

\textbf{OUTPUT:}

- A 2-dimensional array of ordered edges in the graph (given by pairs). The length of the first dimension is \( \mu \). Each row corresponds to an element of the basis and is a circle contained in the graph.

\textbf{EXAMPLES:}

```python
sage: R = RibbonGraph(0,6); R
Ribbon graph of genus 0 and 6 boundary components
sage: R.mu()
5
sage: R.homology_basis()
[[[3, 4], [2, 1]],
[[5, 6], [2, 1]],
[[7, 8], [2, 1]],
[[9, 10], [2, 1]],
[[11, 12], [2, 1]]]

sage: R = RibbonGraph(1,1); R
Ribbon graph of genus 1 and 1 boundary components
sage: R.mu()
2
sage: R.homology_basis()
[[[2, 5], [4, 1]], [[3, 6], [4, 1]]]
```

\textbf{4.6. Ribbon Graphs}
sage: H = R.reduced(); H
Ribbon graph of genus 1 and 1 boundary components
sage: H.sigma()
(2,3,5,6)
sage: H.rho()
(2,5)(3,6)
sage: H.homology_basis()
[[[2, 5]], [[3, 6]]]

sage: s3 = PermutationGroupElement('(1,2,3,4,5,6,7,8,9,10,11,27,25,23)(12,24,26,28,13,14,15,16,17,18,19,20,21,22)')
sage: R3 = RibbonGraph(s3,r3); R3
Ribbon graph of genus 5 and 4 boundary components
sage: R3.mu()
13
sage: R3.homology_basis()
[[[2, 13]], [[3, 14]], [[4, 15]], [[5, 16]], [[6, 17]], [[7, 18]], [[8, 19]], [[9, 20]], [[10, 21]], [[11, 22]], [[23, 24]], [[25, 26]], [[27, 28]]]

make_generic()

Return a ribbon graph equivalent to self but where every vertex has valency 3.

OUTPUT:
• a ribbon graph that is equivalent to self but is generic in the sense that all vertices have valency 3

EXAMPLES:

```
sage: R = RibbonGraph(1,3); R
Ribbon graph of genus 1 and 3 boundary components
sage: R.sigma()
(1,2,3,9,7)(4,8,10,5,6)
sage: R.rho()
(1,4)(2,5)(3,6)(7,8)(9,10)
sage: G = R.make_generic(); G
Ribbon graph of genus 1 and 3 boundary components
sage: G.sigma()
(2,3,11)(5,6,13)(7,8,15)(9,16,17)(10,14,19)(12,18,21)(20,22)
sage: G.rho()
(2,5)(3,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)
sage: R.genus() == G.genus() and R.number_boundaries() == G.number_boundaries()
True
```

```
sage: R = RibbonGraph(5,4); R
Ribbon graph of genus 5 and 4 boundary components
sage: R.sigma()
(1,2,3,4,5,6,7,8,9,10,11,27,25,23)(12,24,26,28,13,14,15,16,17,18,19,20,21,22)
sage: R.rho()
sage: G = R.reduced(); G
Ribbon graph of genus 5 and 4 boundary components
sage: G.sigma()
(2,3,4,5,6,7,8,9,10,11,27,25,23,24,26,28,13,14,15,16,17,18,19,20,21,22)
sage: G.rho()
sage: G.genus() == R.genus() and G.number_boundaries() == R.number_boundaries()
True
```

```
sage: R = RibbonGraph(0,6); R
Ribbon graph of genus 0 and 6 boundary components
sage: R.sigma()
(1,11,9,7,5,3)(2,4,6,8,10,12)
sage: R.rho()
(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)
sage: G = R.reduced(); G
Ribbon graph of genus 0 and 6 boundary components
sage: G.sigma()
(3,4,6,8,10,12,11,9,7,5)
sage: G.rho()
(3,4)(5,6)(7,8)(9,10)(11,12)
sage: G.genus() == R.genus() and G.number_boundaries() == R.number_boundaries()
True
```

\[ \mu() \]

Return the rank of the first homology group of the thickening of the ribbon graph.

EXAMPLES:
sage: s1 = PermutationGroupElement('(1,3,5)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)')
sage: R1 = RibbonGraph(s1,r1); R1
Ribbon graph of genus 1 and 1 boundary components
sage: R1.mu()
2

normalize()

Return an equivalent graph such that the enumeration of its darts exhausts all numbers from 1 to the number of darts.

OUTPUT:

- a ribbon graph equivalent to self such that the enumeration of its darts exhausts all numbers from 1 to the number of darts.

EXAMPLES:

sage: s0 = PermutationGroupElement('(1,22,3,4,5,6,7,15)(8,16,9,10,11,12,13,14)
˓→')
sage: r0 = PermutationGroupElement('(1,8)(22,9)(3,10)(4,11)(5,12)(6,13)(7,14)(15,16)')
sage: R0 = RibbonGraph(s0,r0); R0
Ribbon graph of genus 3 and 2 boundary components
sage: RN0 = R0.normalize(); RN0; RN0.sigma(); RN0.rho()
Ribbon graph of genus 3 and 2 boundary components
(1,16,2,3,4,5,6,14)(7,15,8,9,10,11,12,13)
(1,7)(2,9)(3,10)(4,11)(5,12)(6,13)(8,16)(14,15)

sage: s1 = PermutationGroupElement('(5,10,12)(30,34,78)')
sage: r1 = PermutationGroupElement('(5,30)(10,34)(12,78)')
sage: R1 = RibbonGraph(s1,r1); R1
Ribbon graph of genus 1 and 1 boundary components
sage: RN1 = R1.normalize(); RN1; RN1.sigma(); RN1.rho()
Ribbon graph of genus 1 and 1 boundary components
(1,2,3)(4,5,6)
(1,4)(2,5)(3,6)

number_boundaries()

Return number of boundary components of the thickening of the ribbon graph.

EXAMPLES:

The first example is the ribbon graph corresponding to the torus with one hole:

sage: s1 = PermutationGroupElement('(1,3,5)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)')
sage: R1 = RibbonGraph(s1,r1)
sage: R1.number_boundaries()
1

This example is constructed by taking the bipartite graph of type (3, 3):

sage: s2 = PermutationGroupElement('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,
˓→15)(16,17,18)')
sage: r2 = PermutationGroupElement('(1,16)(2,13)(3,10)(4,17)(5,14)(6,11)(7,
˓→18)(8,15)(9,12)')
sage: R2 = RibbonGraph(s2,r2)
(continues on next page)
Combinatorial and Discrete Geometry, Release 10.3

(continued from previous page)

```python
sage: R2.number_boundaries()
3
```

**reduced()**

Return a ribbon graph with 1 vertex and $\mu$ edges (where $\mu$ is the first betti number of the graph).

**OUTPUT:**

- a ribbon graph whose $\sigma$ permutation has only 1 non-singleton cycle and whose $\rho$ permutation is a product of $\mu$ disjoint 2-cycles

**EXAMPLES:**

```python
sage: s1 = PermutationGroupElement('1,3,5)(2,4,6)')
sage: r1 = PermutationGroupElement('1,2)(3,4)(5,6)')
sage: R1 = RibbonGraph(s1,r1); R1
Ribbon graph of genus 1 and 1 boundary components
sage: G1 = R1.reduced(); G1
Ribbon graph of genus 1 and 1 boundary components
sage: G1.sigma()
(3,5,4,6)
sage: G1.rho()
(3,4)(5,6)

sage: s2 = PermutationGroupElement('1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18,19)')
sage: r2 = PermutationGroupElement('1,16)(2,13)(3,10)(4,22)(5,18)(6,14)(7,12)(8,15)(9,20)')
sage: R2 = RibbonGraph(s2,r2); R2
Ribbon graph of genus 1 and 3 boundary components
sage: G2 = R2.reduced(); G2
Ribbon graph of genus 1 and 3 boundary components
sage: G2.sigma()
(5,6,8,9,14,15,11,12)
sage: G2.rho()
(5,14)(6,11)(8,15)(9,12)

sage: s3 = PermutationGroupElement('1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24)')
sage: R3 = RibbonGraph(s3,r3); R3
Ribbon graph of genus 3 and 1 boundary components
sage: G3 = R3.reduced(); G3
Ribbon graph of genus 3 and 1 boundary components
sage: G3.sigma()
(5,6,8,9,11,12,18,19,20,14,15,16)
sage: G3.rho()
(5,18)(6,14)(8,19)(9,15)(11,20)(12,16)
```

**rho()**

Return the permutation $\rho$ of self.

**EXAMPLES:**

```python
sage: s1 = PermutationGroupElement('1,3,5,8)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)(8,15)')
(continues on next page)```
Combinatorial and Discrete Geometry, Release 10.3

sage: R = RibbonGraph(s1, r1)
sage: R.rho()
(1,2)(3,4)(5,6)(8,15)

sigma()

Return the permutation $\sigma$ of self.

EXAMPLES:

sage: s1 = PermutationGroupElement('(1,3,5,8)(2,4,6)')
sage: r1 = PermutationGroupElement('(1,2)(3,4)(5,6)(8,15)')
sage: R = RibbonGraph(s1, r1)
sage: R.sigma()
(1,3,5,8)(2,4,6)

sage.geometry.ribbon_graph.bipartite_ribbon_graph(p, q)

Return the bipartite graph modeling the corresponding Brieskorn-Pham singularity.

Take two parallel lines in the plane, and consider $p$ points in one of them and $q$ points in the other. Join with a line each point from the first set with every point with the second set. The resulting is a planar projection of the complete bipartite graph of type $(p, q)$. If you consider the cyclic ordering at each vertex induced by the positive orientation of the plane, the result is a ribbon graph whose associated orientable surface with boundary is homeomorphic to the Milnor fiber of the Brieskorn-Pham singularity $x^p + y^q$. It satisfies that it has $\gcd(p, q)$ number of boundary components and genus $(pq - p - q - \gcd(p, q) - 2)/2$.

INPUT:

- $p$ – a positive integer
- $q$ – a positive integer

EXAMPLES:

sage: B23 = RibbonGraph(2,3,bipartite=True); B23; B23.sigma(); B23.rho()
Ribbon graph of genus 1 and 1 boundary components
(1,2,3)(4,5,6)(7,8)(9,10)(11,12)
(1,8)(2,10)(3,12)(4,7)(5,9)(6,11)

sage: B32 = RibbonGraph(3,2,bipartite=True); B32; B32.sigma(); B32.rho()
Ribbon graph of genus 1 and 1 boundary components
(1,2)(3,4)(5,6)(7,8,9)(10,11,12)
(1,9)(2,12)(3,8)(4,11)(5,7)(6,10)

sage: B33 = RibbonGraph(3,3,bipartite=True); B33; B33.sigma(); B33.rho()
Ribbon graph of genus 1 and 3 boundary components
(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)
(1,12)(2,15)(3,18)(4,11)(5,14)(6,17)(7,10)(8,13)(9,16)

sage: B24 = RibbonGraph(2,4,bipartite=True); B24; B24.sigma(); B24.rho()
Ribbon graph of genus 1 and 2 boundary components
(1,2,3,4)(5,6,7,8)(9,10)(11,12)(13,14)(15,16)
(1,10)(2,12)(3,14)(4,16)(5,9)(6,11)(7,13)(8,15)

sage: B47 = RibbonGraph(4,7, bipartite=True); B47; B47.sigma(); B47.rho()
Ribbon graph of genus 9 and 1 boundary components
(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21)(22,23,24,25,26,27,28)
(29,30,31,32)(33,34,35,36)(37,38,39,40)(41,42,43,44)(45,46,47,48)(49,50,51,52)
(53,54,55,56)
sage.geometry.ribbon_graph.make_ribbon(g, r)

Return a ribbon graph whose thickening has genus $g$ and $r$ boundary components.

**INPUT:**

- $g$ – non-negative integer representing the genus of the thickening
- $r$ – positive integer representing the number of boundary components of the thickening

**OUTPUT:**

- a ribbon graph that has 2 vertices (two non-trivial cycles in its sigma permutation) of valency $2g + r$ and it has $2g + r$ edges (and hence $4g + 2r$ darts)

**EXAMPLES:**

```
sage: from sage.geometry.ribbon_graph import make_ribbon
sage: R = make_ribbon(0, 1); R
Ribbon graph of genus 0 and 1 boundary components
sage: R.sigma()
()       # (1,2)
sage: R.rho()
(1,2)
sage: R = make_ribbon(0, 5); R
Ribbon graph of genus 0 and 5 boundary components
sage: R.sigma()
(1,9,7,5,3) (2,4,6,8,10)
sage: R.rho()
(1,2) (3,4) (5,6) (7,8) (9,10)
sage: R = make_ribbon(1, 1); R
Ribbon graph of genus 1 and 1 boundary components
sage: R.sigma()
(1,2,3) (4,5,6)
sage: R.rho()
(1,4) (2,5) (3,6)
sage: R = make_ribbon(7, 3); R
Ribbon graph of genus 7 and 3 boundary components
sage: R.sigma()
(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,33,31)(16,32,34,17,18,19,20,21,22,23,24,25,26,27,28,29,30)
sage: R.rho()
(1,16) (2,17) (3,18) (4,19) (5,20) (6,21) (7,22) (8,23) (9,24) (10,25) (11,26) (12,27) (13,28) (14,29) (15,30) (31,32) (33,34)
```
4.7 Pseudolines

This module gathers everything that has to do with pseudolines, and for a start a \texttt{PseudolineArrangement} class that can be used to describe an arrangement of pseudolines in several different ways, and to translate one description into another, as well as to display \textit{Wiring diagrams} via the \texttt{show} method.

In the following, we try to stick to the terminology given in [Fe1997], which can be checked in case of doubt. And please fix this module's documentation afterwards :-)

\textbf{Definition}

A \textit{pseudoline} cannot be defined by itself, though it can be thought of as a \(x\)-monotone curve in the plane. A \textit{set} of pseudolines, however, represents a set of such curves that pairwise intersect exactly once (and hence mimic the behaviour of straight lines in general position). We also assume that those pseudolines are in general position, that is that no three of them cross at the same point.

The present class is made to deal with a combinatorial encoding of a pseudolines arrangement, that is the ordering in which a pseudoline \(l_i\) of an arrangement \(l_0, \ldots, l_{n-1}\) crosses the \(n-1\) other lines.

\begin{tabular}{|c|}
\hline
\textbf{Warning:} It is assumed through all the methods that the given lines are numbered according to their \(y\)-coordinate on the vertical line \(x = -\infty\). For instance, it is not possible that the first transposition be \((0, 2)\) (or equivalently that the first line \(l_0\) crosses is \(l_2\) and conversely), because one of them would have to cross \(l_1\) first. \\
\hline
\end{tabular}

4.7.1 Encodings

\textbf{Permutations}

An arrangement of pseudolines can be described by a sequence of \(n\) lists of length \(n-1\), where the \(i\) list is a permutation of \(\{0, \ldots, n-1\}\setminus i\) representing the ordering in which the \(i\) th pseudoline meets the other ones.

\begin{verbatim}
sage: from sage.geometry.pseudolines import PseudolineArrangement
gsage: permutations = [[3, 2, 1], [3, 2, 0], [3, 1, 0], [2, 1, 0]]
sgage: p = PseudolineArrangement(permutations)
sgage: p
Arrangement of pseudolines of size 4
sage: p.show()  # needs sage.plot
\end{verbatim}

\textbf{Sequence of transpositions}

An arrangement of pseudolines can also be described as a sequence of \(\binom{n}{2}\) transpositions (permutations of two elements). In this sequence, the transposition \((2, 3)\) appears before \((8, 2)\) if \(l_2\) crosses \(l_3\) before it crosses \(l_8\). This encoding is easy to obtain by reading the wiring diagram from left to right (see the \texttt{show} method).

\begin{verbatim}
sage: from sage.geometry.pseudolines import PseudolineArrangement
gsage: transpositions = [(3, 2), (3, 1), (0, 3), (2, 1), (0, 2), (0, 1)]
sgage: p = PseudolineArrangement(transpositions)
sgage: p
Arrangement of pseudolines of size 4
sage: p.show()  # needs sage.plot
\end{verbatim}

Note that this ordering is not necessarily unique.

\textbf{Felsner's Matrix}
Felser gave an encoding of an arrangement of pseudolines that takes $n^2$ bits instead of the $n^2 \log(n)$ bits required by the two previous encodings.

Instead of storing the permutation $[3, 2, 1]$ to remember that line $l_0$ crosses $l_3$ then $l_2$ then $l_1$, it is sufficient to remember the positions for which each line $l_i$ meets a line $l_j$ with $j < i$. As $l_0$ – the first of the lines – can only meet pseudolines with higher index, we can store $[0, 0, 0]$ instead of $[3, 2, 1]$ stored previously. For $l_1$’s permutation $[3, 2, 0]$ we only need to remember that $l_1$ first crosses 2 pseudolines of higher index, and then a pseudoline with smaller index, which yields the bit vector $[0, 0, 1]$. Hence we can transform the list of permutations above into a list of $n$ bit vectors of length $n - 1$, that is

$\begin{array}{cccc}
3 & 2 & 1 & 0 & 0 & 0 \\
3 & 2 & 0 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 & 1 & 1
\end{array}$

In order to go back from Felser’s matrix to an encoding by a sequence of transpositions, it is sufficient to look for occurrences of $0 \ 1$ in the first column of the matrix, as it corresponds in the wiring diagram to a line going up while the line immediately above it goes down – those two lines cross. Each time such a pattern is found it yields a new transposition, and the matrix can be updated so that this pattern disappears. A more detailed description of this algorithm is given in [Fe1997].

```
sage: from sage.geometry.pseudolines import PseudolineArrangement
sage: felsner_matrix = [[0, 0, 0], [0, 0, 1], [0, 1, 1], [1, 1, 1]]
sage: p = PseudolineArrangement(felsner_matrix)
sage: p
Arrangement of pseudolines of size 4
```

### 4.7.2 Example

Let us define in the plane several lines $l_i$ of equation $y = ax + b$ by picking a coefficient $a$ and $b$ for each of them. We make sure that no two of them are parallel by making sure all of the $a$ chosen are different, and we avoid a common crossing of three lines by adding a random noise to $b$:

```
sage: n = 20
sage: l = sorted(zip(Subsets(20*n, n).random_element(),
.....: [randint(0, 20*n) + random() for i in range(n)]))
```

We can now compute for each $i$ the order in which line $i$ meets the other lines:

```
sage: permutations = [[0..i-1] + [i+1..n-1] for i in range(n)]
sage: def a(x): return l[x][0]
sage: def b(x): return l[x][1]
sage: for i, perm in enumerate(permutations):
.....: perm.sort(key=lambda j: (b(j)-b(i))/(a(i)-a(j)))
```

And finally build the line arrangement:

```
sage: from sage.geometry.pseudolines import PseudolineArrangement
sage: p = PseudolineArrangement(permutations)
sage: print(p)
```

(continues on next page)
Arrangement of pseudolines of size 20

```python
sage: p.show(figsize=[20,8])
# needs sage.combinat sage.plot
```

**Author**

Nathann Cohen

### 4.7.3 Methods

```python
class sage.geometry.pseudolines.PseudolineArrangement (seq, encoding='auto')
```

Creates an arrangement of pseudolines.

**INPUT:**

- `seq` (a sequence describing the line arrangement). It can be:
  - A list of \( n \) permutations of size \( n - 1 \).
  - A list of \( \binom{n}{2} \) transpositions
  - A Felsner matrix, given as a sequence of \( n \) binary vectors of length \( n - 1 \).
- `encoding` (information on how the data should be interpreted), and can assume any value among 'transpositions', 'permutations', 'Felsner' or 'auto'. In the latter case, the type will be guessed (default behaviour).

**Note:**

- The pseudolines are assumed to be integers \( 0, \ldots, n - 1 \).
- For more information on the different encodings, see the `pseudolines` module’s documentation.

**felsner_matrix()**

Return a Felsner matrix describing the arrangement.

See the `pseudolines` module’s documentation for more information on this encoding.

**EXAMPLES:**

```python
sage: from sage.geometry.pseudolines import PseudolineArrangement
sage: permutations = [[3, 2, 1], [3, 2, 0], [3, 1, 0], [2, 1, 0]]
```

```python
sage: p = PseudolineArrangement(permutations)
```

```python
sage: p.felsner_matrix()
[[0, 0, 0], [0, 0, 1], [0, 1, 1], [1, 1, 1]]
```

**permutations()**

Return the arrangements as \( n \) permutations of size \( n - 1 \).

See the `pseudolines` module’s documentation for more information on this encoding.

**EXAMPLES:**

```python
sage: PseudolineArrangement(permutations)
```
sage: from sage.geometry.pseudolines import PseudolineArrangement
sage: permutations = [[3, 2, 1], [3, 2, 0], [3, 1, 0], [2, 1, 0]]
sage: p = PseudolineArrangement(permutations)
sage: p.permutations()
[[3, 2, 1], [3, 2, 0], [3, 1, 0], [2, 1, 0]]

**show(****args**

Displays the pseudoline arrangement as a wiring diagram.

**INPUT:**

- **args** – any arguments to be forwarded to the show method. In particular, to tune the dimensions, use the figsize argument (example below).

**EXAMPLES:**

```
sage: from sage.geometry.pseudolines import PseudolineArrangement
data: permutations = [[3, 2, 1], [3, 2, 0], [3, 1, 0], [2, 1, 0]]
sage: p = PseudolineArrangement(permutations)
sage: p.show(figsize=[7,5])  #...
```

```
needs sage.plot
```

**transpositions()**

Return the arrangement as \( (\binom{n}{2}) \) transpositions.

See the pseudolines module’s documentation for more information on this encoding.

**EXAMPLES:**

```
sage: from sage.geometry.pseudolines import PseudolineArrangement
data: permutations = [[3, 2, 1], [3, 2, 0], [3, 1, 0], [2, 1, 0]]
sage: p1 = PseudolineArrangement(permutations)
sage: transpositions = [(3, 2), (3, 1), (0, 3), (2, 1), (0, 2), (0, 1)]
sage: p2 = PseudolineArrangement(transpositions)
sage: p1 == p2
True
sage: p1.transpositions()
[(3, 2), (3, 1), (0, 3), (2, 1), (0, 2), (0, 1)]
sage: p2.transpositions()
[(3, 2), (3, 1), (0, 3), (2, 1), (0, 2), (0, 1)]
```

### 4.8 Voronoi diagram

This module provides the class `VoronoiDiagram` for computing the Voronoi diagram of a finite list of points in \( \mathbb{R}^d \).

```
class sage.geometry.voronoi_diagram.VoronoiDiagram(points)
```

**Bases:** `SageObject`

Base class for the Voronoi diagram.

Compute the Voronoi diagram of a list of points.

**INPUT:**

- `points` – a list of points. Any valid input for the `PointConfiguration` will do.

**OUTPUT:**
An instance of the VoronoiDiagram class.

EXAMPLES:

Get the Voronoi diagram for some points in $\mathbb{R}^3$:

```
sage: V = VoronoiDiagram([[1, 3, .3], [2, -2, 1], [-1, 2, -.1]]); V
The Voronoi diagram of 3 points of dimension 3 in the Real Double Field
```

```
sage: VoronoiDiagram([])
The empty Voronoi diagram.
```

Get the Voronoi diagram of a regular pentagon in $\mathbb{A}^2$. All cells meet at the origin:

```
sage: DV = VoronoiDiagram([AA(c) for c in v] #...
    for v in polytopes.regular_polygon(5).vertices_.list()); DV
The Voronoi diagram of 5 points of dimension 2 in the Algebraic Real Field
```

```
sage: all(P.contains([0, 0]) for P in DV.regions().values()) #...
True
```

```
sage: any(P.interior_contains([0, 0]) for P in DV.regions().values()) #...
False
```

If the vertices are not converted to $\mathbb{A}^2$ before, the method throws an error:

```
sage: polytopes.dodecahedron().vertices_list()[0][0].parent() #...
Number Field in sqrt5 with defining polynomial x^2 - 5 with sqrt5 = 2.
\text{236079774997990?}
```

```
sage: VoronoiDiagram(polytopes.dodecahedron().vertices_list()) #...
Traceback (most recent call last):
...
NotImplementedError: Base ring of the Voronoi diagram must be one of QQ, RDF, AA.
```

ALGORITHM:

We use hyperplanes tangent to the paraboloid one dimension higher to get a convex polyhedron and then project back to one dimension lower.

Todo:

- The dual construction: Delaunay triangulation
- improve 2d-plotting
- implement 3d-plotting
- more general constructions, like Voronoi diagrams with weights (power diagrams)

REFERENCES:

- [Mat2002] Ch.5.7, p.118.

AUTHORS:

- Moritz Firsching (2012-09-21)
ambient_dim()

Return the ambient dimension of the points.

EXAMPLES:

```sage
V = VoronoiDiagram([[.5, 3], [2, 5], [4, 5], [4, -1]])
sage: V.ambient_dim()
2
```

```sage
V = VoronoiDiagram([[1, 2, 3, 4, 5, 6]]); V.ambient_dim()
```

base_ring()

Return the base_ring of the regions of the Voronoi diagram.

EXAMPLES:

```sage
V = VoronoiDiagram([[1, 3, 1], [2, -2, 1], [-1, 2, 1/2]]); V.base_ring()
Rational Field
```

```sage
V = VoronoiDiagram([[1, 3.14], [2, -2/3], [-1, 22]]); V.base_ring()
Real Double Field
```

```sage
V = VoronoiDiagram([[1, 3], [2, 4]]); V.base_ring()
Rational Field
```

plot (cell_colors=None, **kwds)

Return a graphical representation for 2-dimensional Voronoi diagrams.

INPUT:

- cell_colors – (default: None) provide the colors for the cells, either as dictionary. Randomly colored cells are provided with None.
- **kwds – optional keyword parameters, passed on as arguments for plot().

OUTPUT:

A graphics object.

EXAMPLES:

```sage
# needs sage.plot
sage: P = [[0.671, 0.650], [0.258, 0.767], [0.562, 0.406],
....: [0.254, 0.709], [0.493, 0.879]]
sage: V = VoronoiDiagram(P); S=V.plot()
sage: show(S, xmin=0, xmax=1, ymin=0, ymax=1, aspect_ratio=1, axes=false)
```

```sage
sage: S = V.plot(cell_colors={0: red, 1: blue, 2: green,
....: 3: 'white', 4: 'yellow'})
sage: show(S, xmin=0, xmax=1, ymin=0, ymax=1, aspect_ratio=1, axes=false)
```

```sage
sage: S = V.plot(cell_colors=[red, blue, red, white, white])
sage: show(S, xmin=0, xmax=1, ymin=0, ymax=1, aspect_ratio=1, axes=false)
```

```sage
sage: S = V.plot(cell_colors=something else)
Traceback (most recent call last):
  ... AssertionError: 'cell_colors' must be a list or a dictionary
```

Trying to plot a Voronoi diagram of dimension other than 2 gives an error:

```sage
sage: VoronoiDiagram([[1, 2, 3], [6, 5, 4]]).plot() 
    # needs sage.plot
sage: V.plot()
Traceback (most recent call last):
```

(continues on next page)
points()

Return the input points (as a PointConfiguration).

EXAMPLES:

```python
sage: V = VoronoiDiagram([ [.5, 3], [2, 5], [4, 5], [4, -1]])
V.points()
A point configuration in affine 2-space over Real Field
with 53 bits of precision consisting of 4 points.
The triangulations of this point configuration are
assumed to be connected, not necessarily fine,
not necessarily regular.
```

regions()

Return the Voronoï regions of the Voronoi diagram as a dictionary of polyhedra.

EXAMPLES:

```python
sage: V = VoronoiDiagram([ [1, 3, .3], [2, -2, 1], [-1, 2, -.1]])
sage: P = V.points()
sage: V.regions() == {P[0]: Polyhedron(base_ring=RDF, lines=[[(-RDF(0.375),
-→RDF(0.13888888900000001), RDF(1.5277777799999999))]],
....: rays=[(RDF(9), -RDF(1),
-→RDF(20)), (RDF(4.5), RDF(1), -RDF(25))],
....: vertices=[(-RDF(1.1074999999999999), RDF(1.149444444), RDF(9.0138888900000004))]),
....: P[1]: Polyhedron(base_ring=RDF, lines=[(-RDF(0.375),
-→RDF(0.13888888900000001), RDF(1.5277777799999999))]],
....: rays=[(RDF(9), -RDF(1),
-→RDF(20)), (-RDF(2.25), -RDF(1), RDF(2.5))],
....: vertices=[(-RDF(1.1074999999999999), RDF(1.149444444), RDF(9.0138888900000004))]),
....: P[2]: Polyhedron(base_ring=RDF, lines=[(-RDF(0.375),
-→RDF(0.13888888900000001), RDF(1.5277777799999999))]],
....: rays=[(RDF(4.5), RDF(1),
-→RDF(25)), (-RDF(2.25), -RDF(1), RDF(2.5))],
....: vertices=[(-RDF(1.1074999999999999), RDF(1.149444444), RDF(9.0138888900000004))])
True
```
CHAPTER FIVE

HELPER FUNCTIONS

5.1 Find isomorphisms between fans

```
except FileNotFoundError
sage.geometry.fan_isomorphism.FanNotIsomorphicError
    Bases: Exception
    Exception to return if there is no fan isomorphism
sage.geometry.fan_isomorphism.fan_2d_cyclically_ordered_rays(fan)
    Return the rays of a 2-dimensional fan in cyclic order.
    INPUT:
    • fan – a 2-dimensional fan.
    OUTPUT:
    A PointCollection containing the rays in one particular cyclic order.
    EXAMPLES:

    sage: rays = ((1, 1), (-1, -1), (-1, 1), (1, -1))
    sage: cones = [(0, 2), (2, 1), (1, 3), (3, 0)]
    sage: fan = Fan(cones, rays)
    sage: fan.rays()
    N( 1, 1),
    N(-1, -1),
    N(-1, 1),
    N( 1, -1)
    in 2-d lattice N
    sage: from sage.geometry.fan_isomorphism import fan_2d_cyclically_ordered_rays
    sage: fan_2d_cyclically_ordered_rays(fan)
    N(-1, -1),
    N(-1, 1),
    N( 1, 1),
    N( 1, -1)
    in 2-d lattice N
```

```
sage.geometry.fan_isomorphism.fan_2d_echelon_form(fan)
    Return echelon form of a cyclically ordered ray matrix.
    INPUT:
    • fan – a fan.
    OUTPUT:
```
A matrix. The echelon form of the rays in one particular cyclic order.

EXAMPLES:

```python
sage: fan = toric_varieties.P2().fan()  #...
→ needs palp sage.graphs
sage: from sage.geometry.fan_isomorphism import fan_2d_echelon_form
sage: fan_2d_echelon_form(fan)  #...
→ needs palp sage.graphs
[ 1 0 -1]
[ 0 1 -1]
```

`sage.geometry.fan_isomorphism.fan_2d_echelon_forms(fan)`
Return echelon forms of all cyclically ordered ray matrices.

Note that the echelon form of the ordered ray matrices are unique up to different cyclic orderings.

INPUT:

• `fan` -- a fan.

OUTPUT:

A set of matrices. The set of all echelon forms for all different cyclic orderings.

EXAMPLES:

```python
sage: fan = toric_varieties.P2().fan()  #...
→ needs palp sage.graphs
sage: from sage.geometry.fan_isomorphism import fan_2d_echelon_forms
sage: fan_2d_echelon_forms(fan)  #...
→ needs palp sage.graphs
frozenset({[ 1 0 -1]
[ 0 1 -1]})
```

```python
sage: fan = toric_varieties.dP7().fan()  #...
→ needs palp sage.graphs
sage: sorted(fan_2d_echelon_forms(fan))  #...
→ needs palp sage.graphs
[
[ 1 0 -1 -1 0] [ 1 0 -1 -1 0] [ 1 0 -1 0 1] [ 1 0 -1 0 1]
[ 0 1 0 -1 -1], [ 0 1 1 0 -1], [ 0 1 1 0 -1], [ 0 1 0 -1 -1],
[ 1 0 -1 0 1]
[ 0 1 1 -1 -1]
]
```

`sage.geometry.fan_isomorphism.fan_isomorphic_necessary_conditions(fan1, fan2)`
Check necessary (but not sufficient) conditions for the fans to be isomorphic.

INPUT:

• `fan1, fan2` -- two fans.

OUTPUT:

Boolean. `False` if the two fans cannot be isomorphic. `True` if the two fans may be isomorphic.

EXAMPLES:
sage: fan1 = toric_varieties.P2().fan()  # needs palp sage.graphs
sage: fan2 = toric_varieties.dP8().fan()  # needs palp sage.graphs
sage: from sage.geometry.fan_isomorphism import fan_isomorphic_necessary_conditions
sage: fan_isomorphic_necessary_conditions(fan1, fan2)  # needs palp sage.graphs
False

sage.geometry.fan_isomorphism.fan_isomorphism_generator(fan1, fan2)
Iterate over the isomorphisms from fan1 to fan2.

ALGORITHM:
The sage.geometry.fan.Fan.vertex_graph() of the two fans is compared. For each graph isomorphism, we attempt to lift it to an actual isomorphism of fans.

INPUT:
• fan1, fan2 – two fans.

OUTPUT:
Yields the fan isomorphisms as matrices acting from the right on rays.

EXAMPLES:

sage: fan = toric_varieties.P2().fan()  # needs palp sage.graphs
sage: from sage.geometry.fan_isomorphism import fan_isomorphism_generator
sage: sorted(fan_isomorphism_generator(fan, fan))  # needs palp sage.graphs
[[[-1 -1] [-1 -1] [ 0 1] [ 1 0] [1 0],
  [ 0 1], [ 1 0], [-1 -1], [1 0], [-1 -1], [0 1]]

sage: m1 = matrix([[1, 0], [0, -5], [-3, 4]])

sage: m2 = matrix([[3, 0], [1, 0], [-2, 1]])

sage: m1.elementary_divisors() == m2.elementary_divisors() == [1, 1, 0]
True

sage: fan1 = Fan([Cone([m1*vector([23, 14]), m1*vector([-1, -14])]), Cone([m1*vector([-1, 14]), m1*vector([-100, -5])])])

sage: fan2 = Fan([Cone([m2*vector([23, 14]), m2*vector([-1, 14])]), Cone([m2*vector([-1, 14]), m2*vector([-100, -5])])])

sage: sorted(fan_isomorphism_generator(fan1, fan2))  # needs sage.graphs
[[[-12  1 -5]
  [-4  0 -1]
  [-5  0 -1]],

sage: m0 = identity_matrix(ZZ, 2)

sage: m1 = matrix([[1, 0], [0, -5], [-3, 4]])

sage: m2 = matrix([[3, 0], [1, 0], [-2, 1]])

sage: m1.elementary_divisors() == m2.elementary_divisors() == [1, 1, 0]
True

sage: fan0 = Fan([Cone([m0*vector([1, 0]), m0*vector([1, 1])]),

(continues on next page)
sage.geometry.fan_isomorphism.find_isomorphism(fan1, fan2, check=False)

Find an isomorphism of the two fans.

INPUT:

- fan1, fan2 – two fans.
- check – boolean (default: False). Passed to the fan morphism constructor, see FanMorphism().

OUTPUT:

A fan isomorphism. If the fans are not isomorphic, a FanNotIsomorphicError is raised.

EXAMPLES:

```python
sage: rays = ((1, 1), (0, 1), (-1, -1), (3, 1))
sage: cones = [(0,1), (1,2), (2,3), (3,0)]
sage: fan1 = Fan(cones, rays)
sage: m = matrix([[-2,3], [1,-1]])
sage: m.det() == -1
True
sage: fan2 = Fan(cones, [vector(r)*m for r in rays])
sage: from sage.geometry.fan_isomorphism import find_isomorphism
```

(continues on next page)
5.2 Construction of finite atomic and coatomic lattices from incidences

This module provides the function `lattice_from_incidences()` for computing finite atomic and coatomic lattices in the sense of partially ordered sets where any two elements have meet and joint. For example, the face lattice of a polyhedron.

```python
sage.geometry.hasse_diagram.lattice_from_incidences(atom_to_coatoms, coatom_to_atoms, face_constructor=None, required_atoms=None, key=None, **kwds)
```

Compute an atomic and coatomic lattice from the incidence between atoms and coatoms.

**INPUT:**

- `atom_to_coatoms` – list, `atom_to_coatom[i]` should list all coatoms over the i-th atom;
- `coatom_to_atoms` – list, `coatom_to_atom[i]` should list all atoms under the i-th coatom;
- `face_constructor` – function or class taking as the first two arguments sorted tuple of integers and any keyword arguments. It will be called to construct a face over atoms passed as the first argument and under coatoms passed as the second argument. Default implementation will just return these two tuples as a tuple;
- `required_atoms` – list of atoms (default: None). Each non-empty “face” requires at least one of the specified atoms present. Used to ensure that each face has a vertex.
- `key` – any hashable value (default: None). It is passed down to `FinitePoset`.
- all other keyword arguments will be passed to `face_constructor` on each call.

**OUTPUT:**

- `finite poset` with elements constructed by `face_constructor`. 

---

5.2. Construction of finite atomic and coatomic lattices from incidences 751
**Note:** In addition to the specified partial order, finite posets in Sage have internal total linear order of elements which extends the partial one. This function will try to make this internal order to start with the bottom and atoms in the order corresponding to `atom_to_coatoms` and to finish with coatoms in the order corresponding to `coatom_to_atoms` and the top. This may not be possible if atoms and coatoms are the same, in which case the preference is given to the first list.

**ALGORITHM:**

The detailed description of the used algorithm is given in [KP2002].

The code of this function follows the pseudo-code description in the section 2.5 of the paper, although it is mostly based on frozen sets instead of sorted lists - this makes the implementation easier and should not cost a big performance penalty. (If one wants to make this function faster, it should be probably written in Cython.)

While the title of the paper mentions only polytopes, the algorithm (and the implementation provided here) is applicable to any atomic and coatomliclattice if both incidences are given, see Section 3.4.

In particular, this function can be used for strictly convex cones and complete fans.

**REFERENCES:** [KP2002]

**AUTHORS:**


**EXAMPLES:**

Let us construct the lattice of subsets of \{0, 1, 2\}. Our atoms are \{0\}, \{1\}, and \{2\}, while our coatoms are \{0, 1\}, \{0, 2\}, and \{1, 2\}. Then incidences are

```python
sage: atom_to_coatoms = [(0,1), (0,2), (1,2)]
sage: coatom_to_atoms = [(0,1), (0,2), (1,2)]
```

and we can compute the lattice as

```python
sage: from sage.geometry.cone import lattice_from_incidences
sage: L = lattice_from_incidences(atom_to_coatoms, coatom_to_atoms); L
Finite lattice containing 8 elements with distinguished linear extension
sage: for level in L.level_sets(): print(level)
[(), (0, 1, 2))]
[((0,), (0, 1)), ((1,), (0, 2)), ((2,), (1, 2))]
[((0, 1), (0,)), ((0, 2), (1,)), ((1, 2), (2,))]
[((0, 1, 2), ())]```

For more involved examples see the source code of `sage.geometry.cone.ConvexRationalPolyhedralCone.face_lattice()` and `sage.geometry.fan.RationalPolyhedralFan._compute_cone_lattice()`. 
5.3 MISSING TITLE

5.4 Helper Functions For Freeness Of Hyperplane Arrangements

This contains the algorithms to check for freeness of a hyperplane arrangement. See `sage.geometry.hyperplane_arrangement.HyperplaneArrangementElement.is_free()` for details.

**Note:** This could be extended to a freeness check for more general modules over a polynomial ring.

```python
sage.geometry.hyperplane_arrangement.check_freeness.construct_free_chain(A)
```
Construct the free chain for the hyperplanes \( A \).

**ALGORITHM:**
We follow Algorithm 6.5 in [BC2012].

**INPUT:**
- \( A \) – a hyperplane arrangement

**EXAMPLES:**

```python
sage: from sage.geometry.hyperplane_arrangement.check_freeness import construct_free_chain
sage: H.<x,y,z> = HyperplaneArrangements(QQ)
sage: A = H(z, y+z, x+y+z)
sage: construct_free_chain(A)
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & z & -1 \\
0 & y & 1 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 \\
y + z & 0 & -1 \\
x & 0 & 1
\end{bmatrix}
\]
```

```python
sage.geometry.hyperplane_arrangement.check_freeness.less_generators(X)
```
Reduce the generator matrix of the module defined by \( X \).

This is Algorithm 6.4 in [BC2012] and relies on the row syzygies of the matrix \( X \).

**EXAMPLES:**

```python
sage: from sage.geometry.hyperplane_arrangement.check_freeness import less_generators
sage: R.<x,y,z> = QQ[]
sage: m = matrix([[1, 0, 0], [0, z, -1], [0, y, 1]])
sage: less_generators(m)
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & z & -1 \\
0 & y & 1
\end{bmatrix}
```

5.3. MISSING TITLE 753
CHAPTER SIX

INDICES AND TABLES

• Index
• Module Index
• Search Page
g
sage.geometry.abc, 701
sage.geometry.cone, 358
sage.geometry.cone_catalog, 417
sage.geometry.convex_set, 702
sage.geometry.fan, 422
sage.geometry.fan_isomorphism, 747
sage.geometry.fan_morphism, 453
sage.geometry.hasse_diagram, 751
sage.geometry.hyperplane_arrangement.affine_subspace, 56
sage.geometry.hyperplane_arrangement.arrangement, 3
sage.geometry.hyperplane_arrangement.check_freeness, 753
sage.geometry.hyperplane_arrangement.hyperplane, 49
sage.geometry.hyperplane_arrangement.library, 41
sage.geometry.hyperplane_arrangement.plot, 59
sage.geometry.integral_points, 753
sage.geometry.lattice_polytope, 165
sage.geometry.linear_expression, 713
sage.geometry.newton_polygon, 719
sage.geometry.point_collection, 468
sage.geometry.polyhedral_complex, 317
sage.geometry.polyhedron.backend_cdd, 638
sage.geometry.polyhedron.backend_cdd_rdf, 639
sage.geometry.polyhedron.backend_field, 639
sage.geometry.polyhedron.backend_normaliz, 641
sage.geometry.polyhedron.backend_number_field, 640
sage.geometry.polyhedron.backend_polytope, 648
sage.geometry.polyhedron.backend_ppl, 650
sage.geometry.polyhedron.base, 610
sage.geometry.polyhedron.base0, 499
sage.geometry.polyhedron.base1, 514
sage.geometry.polyhedron.base2, 523
sage.geometry.polyhedron.base3, 530
sage.geometry.polyhedron.base4, 557
sage.geometry.polyhedron.base5, 571
sage.geometry.polyhedron.base6, 587
sage.geometry.polyhedron.base7, 602
sage.geometry.polyhedron.base_QQ, 620
sage.geometry.polyhedron.base_RDF, 638
sage.geometry.polyhedron.base_ZZ, 631
sage.geometry.polyhedron.cdd_file_format, 162
sage.geometry.polyhedron.combinatorial, 246
sage.geometry.polyhedron.combinatorial_backend, 280
sage.geometry.polyhedron.combinatorial_polyhedron.conversions, 313
sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator, 293
sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces, 310
sage.geometry.polyhedron.combinatorial_polyhedron.polyhedron_face_lattice, 290
sage.geometry.polyhedron.constructor, 111
sage.geometry.polyhedron.double_description, 652
sage.geometry.polyhedron.double_description_inhomogeneous, 660
sage.geometry.polyhedron.face, 151
sage.geometry.polyhedron.generating_function, 241
sage.geometry.polyhedron.lattice_euclidean_group_element, 221
sage.geometry.polyhedron.library, 65

PYTHON MODULE INDEX
sage.geometry.polyhedron.modules.formal_polyhedra_module, 163
sage.geometry.polyhedron.palp_database, 223
sage.geometry.polyhedron.parent, 119
sage.geometry.polyhedron.plot, 139
sage.geometry.polyhedron.ppl_lattice_polygon, 224
sage.geometry.polyhedron.ppl_lattice_polytope, 228
sage.geometry.polyhedron.representation, 126
sage.geometry.pseudolines, 740
sage.geometry.relative_interior, 723
sage.geometry.ribbon_graph, 727
sage.geometry.toric_lattice, 342
sage.geometry.toric_plotter, 475
sage.geometry.triangulation.base, 683
sage.geometry.triangulation.element, 692
sage.geometry.triangulation.point_configuration, 665
sage.geometry.voronoi_diagram, 743

sage.rings.polynomial.groebner_fan, 485
INDEX

A

A() (sage.geometry.linear_expression.LinearExpression method), 714

A() (sage.geometry.polyhedron.double_description.Problem method), 657

A() (sage.geometry.polyhedron.representation.Hrepresentation method), 127

A_matrix() (sage.geometry.polyhedron.double_description.Problem method), 657

a_maximal_chain() (sage.geometry.polyhedron.base1.Polyhedron_base1 method), 515

a_maximal_chain() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 530

a_maximal_chain() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 251

add_cell() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 321

add_hyperplane() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 9

add_inequality() (sage.geometry.polyhedron.double_description.StandardDoubleDescriptionPair method), 659

adjacency_graph() (sage.geometry.triangulation.element.Triangulation method), 692

adjacency_matrix() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 531

adjacent() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 367

adjacent() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 168

adjacent() (sage.geometry.polyhedron.representation.Hrepresentation method), 127

adjacent() (sage.geometry.polyhedron.representation.Vrepresentation method), 137

adjust_options() (sage.geometry.toric_plotter.ToricPlotter method), 477

affine() (sage.geometry.triangulation.base.Point method), 685

affine_hull() (sage.geometry.polyhedron.base6.Polyhedron_base6 method), 587

affine_hull_manifold() (sage.geometry.polyhedron.base6.Polyhedron_base6 method), 587

affine_hull_projection() (sage.geometry.convex_set.ConvexSet_base method), 703

affine_hull_projection() (sage.geometry.polyhedron.base6.Polyhedron_base6 method), 588

affine_lattice_polytope() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 230

affine_space() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 230

affine_tangent_cone() (sage.geometry.polyhedron.face.PolyhedronFace method), 152

affine_transform() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 168

AffineHullProjectionData (class in sage.geometry.convex_set), 702

AffineSubspace (class in sage.geometry.hyperplane_arrangement.affine_subspace), 57

alexander_whitney() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 322

all_cached_data() (in module sage.geometry.lattice_polytope), 213

all_facet_equations() (in module sage.geometry.lattice_polytope), 213

all_nef_partitions() (in module sage.geometry.lattice_polytope), 214

all_points() (in module sage.geometry.lattice_polytope), 214

all_polars() (in module sage.geometry.lattice_polytope), 215

ambient() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 368

ambient() (sage.geometry.convex_set.ConvexSet_base method), 703

ambient() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 170

ambient() (sage.geometry.polyhedron.base1.Polyhe-
ambient() (sage.geometry.polyhedron.face.PolyhedronFace method), 153
ambient() (sage.geometry.relative_interior.RelativeInterior method), 723
ambient_dim() (sage.geometry.cone.IntegralRayCollection method), 407
ambient_dim() (sage.geometry.convex_set.ConvexSet_base method), 704
ambient_dim() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 170
ambient_dim() (sage.geometry.polyhedron_base1.Polyhedron_base1 method), 515
ambient_dim() (sage.geometry.polyhedron.face.PolyhedronFace method), 155
ambient_dim() (sage.geometry.polyhedron_parent.Polyhedra_base method), 122
ambient_dim() (sage.geometry.relative_interior.RelativeInterior method), 723
ambient_dim() (sage.geometry.triangulation_base.PointConfiguration_base method), 688
ambient_dim() (sage.geometry.voronoi_diagram.VoronoiDiagram method), 745
ambient_dim() (sage.rings.polynomial.groebner_fan.PolyhedralCone method), 493
ambient_dim() (sage.rings.polynomial.groebner_fan.PolyhedralFan method), 494
ambient_dimension() (sage.geometry.convex_set.ConvexSet_base method), 704
ambient_dimension() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 322
ambient_dimension() (sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face.CombinatorialFace method), 285
ambient_facet_indices() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 170
ambient_H_indices() (sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face.CombinatorialFace method), 282
ambient_H_indices() (sage.geometry.polyhedron.face.PolyhedronFace method), 153
ambient_Hrepresentation() (sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face.CombinatorialFace method), 283
ambient_Hrepresentation() (sage.geometry.polyhedron.face.PolyhedronFace method), 153
ambient_module() (sage.geometry.linear_expression.LinearExpressionModule method), 717
ambient_module() (sage.geometry.toric_lattice.ToricLattice_ambient method), 346
ambient_ordered_point_indices() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 171
ambient_point_indices() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 171
ambient_ray_indices() (sage.geometry.convex_set.ConvexRationalPolyhedralCone method), 369
ambient_space() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangements method), 39
ambient_space() (sage.geometry.polyhedron_base1.Polyhedron_base1 method), 515
ambient_space() (sage.geometry.polyhedron.parent.Polyhedra_base method), 122
ambient_space() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 231
ambient_V_indices() (sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face.CombinatorialFace method), 284
ambient_V_indices() (sage.geometry.polyhedron.face.PolyhedronFace method), 154
ambient_vector_space() (sage.geometry.cone.IntegralRayCollection method), 407
ambient_vector_space() (sage.geometry.convex_set.ConvexSet_base method), 704
ambient_vector_space() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 172
ambient_vector_space() (sage.geometry.linear_expression.LinearExpressionModule method), 717
ambient_vector_space() (sage.geometry.polyhedron_base1.Polyhedron_base1 method), 516
ambient_vector_space() (sage.geometry.polyhedron.face.PolyhedronFace method), 155
ambient_vector_space() (sage.geometry.relative_interior.RelativeInterior method), 723
ambient_vertex_indices() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 172
ambient_Vrepresentation() (sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face.CombinatorialFace method), 284
ambient_Vrepresentation() (sage.geometry.polyhedron.face.PolyhedronFace method), 154
AmbientVectorSpace (class in sage.geometry.hyperplane_arrangement.hyperplane), 50
an_affine_basis() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 369
bounding_box() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 232
braid() (sage.geometry.hyperplane_arrangement_library.HyperplaneArrangementLibrary method), 47
buchberger() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 486
buckyball() (sage.geometry.polyhedron.library.Polytopes method), 68

cantellated_one_hundred_twenty_cell() (sage.geometry.polyhedron.library.Polytopes method), 69
cantellated_six_hundred_cell() (sage.geometry.polyhedron.Library.Polytopes method), 69
cantitruncated_one_hundred_twenty_cell() (sage.geometry.polyhedron.library.Polytopes method), 70
cantitruncated_six_hundred_cell() (sage.geometry.polyhedron.library.Polytopes method), 70
cardinality() (sage.geometry.convex_set.ConvexSet_base method), 705
cartesian_product() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 369
cartesian_product() (sage.geometry.cone.IntegralRayCollection method), 407
cartesian_product() (sage.geometry.convex_set.ConvexSet_base method), 705
cartesian_product() (sage.geometry.fan.RationalPolyhedralFan method), 433
cartesian_product() (sage.geometry.point_collection.PointCollection method), 469
cartesian_product() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 572
Catalan() (sage.geometry.hyperplane_arrangement_library.HyperplaneArrangementLibrary method), 41
cdd_Hrepresentation() (in module sage.geometry.polyhedron.cdd_file_format), 162
cdd_Hrepresentation() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 503
cdd_Vrepresentation() (in module sage.geometry.polyhedron.cdd_file_format), 162
characteristic() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 11
closure() (sage.geometry.convex_set.ConvexSet_base method), 705
cells() (sage.geometry.polyhedron.combinatorial_polyhedron.PolyhedralComplex method), 323
cells_list_to_cells_dict() (in module sage.geometry.polyhedron.polar_complex), 341
cells_sorted() (sage.geometry.polyhedron_complex.PolyhedralComplex method), 324
center() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 10
center() (sage.geometry.polyhedron.base.Polyhedron_base method), 612
centroid() (sage.geometry.polyhedron.base7.Polyhedron_base7 method), 602
chain_complex() (sage.geometry.polyhedron_complex.PolyhedralComplex method), 324
change_ring() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 11
change_ring() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedra_base method), 612
change_ring() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedra_base0 method), 504
choose_algorithm_to_compute_edges_or_ridges() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron method), 253
circuits() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 670
circuits_support() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 670
classify_cone_2d() (in module sage.geometry.cone), 413
closed_faces() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 11
closure() (sage.geometry.polyhedron.combinatorial_polyhedron.PolyhedralComplex method), 323
closure() (sage.geometry.relative_interior.RelativeInterior method), 724
Index
dual_module() (sage.geometry.point_collection.PointCollection method), 471

E
edge_polytope() (sage.geometry.polyhedron.library.Polytopes static method), 73
dges() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 176
dges() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 255
ehrrart_polynomial() (sage.geometry.polyhedron_base.QQ.Polyhedron_QQ method), 622
ehrrart_polynomial() (sage.geometry.polyhedron_base.ZZ.Polyhedron_ZZ method), 631
ehrrart_quasipolynomial() (sage.geometry.polyhedron_base.QQ.Polyhedron_QQ method), 624
ehrrart_series() (sage.geometry.polyhedron_backend_normaliz.Polyhedron_QQ_normaliz method), 642
Element (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangements attribute), 39
Element (sage.geometry.hyperplane_arrangement.hyperplane.AmbientVectorSpace attribute), 51
Element (sage.geometry.linear_expression.LinearExpressionModule attribute), 717
Element (sage.geometry.newton_polygon.ParentNewtonPolygon attribute), 722
Element (sage.geometry.polyhedron.parent.Polyhedra_field attribute), 125
Element (sage.geometry.polyhedron.parent.Polyhedra_normaliz attribute), 125
Element (sage.geometry.polyhedron.parent.Polyhedra_number_field attribute), 125
Element (sage.geometry.polyhedron.parent.Polyhedra_polymake attribute), 125
Element (sage.geometry.polyhedron.parent.Polyhedra_QQ_cdd attribute), 120
Element (sage.geometry.polyhedron.parent.Polyhedra_QQ_normaliz attribute), 120
Element (sage.geometry.polyhedron.parent.Polyhedra_QQ_ppl attribute), 120
Element (sage.geometry.polyhedron.parent.Polyhedra_RDF_cdd attribute), 120
Element (sage.geometry.polyhedron.parent.Polyhedra_ZZ_normaliz attribute), 121
Element (sage.geometry.polyhedron.parent.Polyhedra_ZZ_ppl attribute), 121
Element (sage.geometry.toric_lattice.ToricLattice_ambient attribute), 346
Element (sage.geometry.toric_lattice.ToricLattice_generic attribute), 347
Element (sage.geometry.toric_lattice.ToricLattice_quotient attribute), 352
Element (sage.geometry.triangulation.point_configuration.PointConfiguration attribute), 669
embed() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 374
embed() (sage.geometry.fan.RationalPolyhedralFan method), 440
embed_in_reflexive_polytope() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 233
empty() (sage.geometry.polyhedron.parent.Polyhedra_base method), 123
enumerate_simplices() (sage.geometry.triangulation.element.Triangulation method), 694
Equation (class in sage.geometry.polyhedron.representation), 126
EQUATION (sage.geometry.polyhedron.representation.PolyhedronRepresentation attribute), 133
equation_generator() (sage.geometry.polyhedron_base0.Polyhedron_base0 method), 505
equations() (sage.geometry.polyhedron_base0.Polyhedron_base0 method), 505
equations_list() (sage.geometry.polyhedron_base0.Polyhedron_base0 method), 506
essentialization() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 19
eval() (sage.geometry.polyhedron.representation.Hrepresentation method), 127
evaluate() (sage.geometry.linear_expression.LinearExpression method), 716
evaluated_on() (sage.geometry.polyhedron.representation.Line method), 132
evaluated_on() (sage.geometry.polyhedron.representation.Ray method), 134
evaluated_on() (sage.geometry.polyhedron.representation.Vertex method), 135
exclude_points() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 673
exploded_plot() (in module sage.geometry.polyhedral_complex), 341
extrude_edge() (sage.geometry.ribbon_graph.RibbonGraph method), 731
F
f_vector() (sage.geometry.fan.RationalPolyhedralFan method), 441
f_vector() (sage.geometry.polyhedron_base3.Polyhedron_base3 method), 533
f_vector() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 256
f_vector() (sage.rings.polynomial.groebner_fan.PolyhedralFan method), 495
face_by_face_lattice_index() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 256
face_codimension() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 673
face_fan() (sage.geometry.polyhedron.base.Polyhedron_base method), 613
face_generator() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 534
face_interior() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 673
face_iter() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 259
face_lattice() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 375
face_lattice() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 176
face_lattice() (sage.geometry.polyhedron.base4.Polyhedron_base4 method), 559
face_lattice() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 260
face_poset() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 327
face_product() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 19
face_semigroup_algebra() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 20
face_split() (sage.geometry.polyhedron.base5.Polyhedron base5 method), 573
face_truncation() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 574
face_vector() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 21
FaceFan() (in module sage.geometry.fan), 426
FaceIterator (class in sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator), 295
FaceIterator_base (class in sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator), 299
FaceIterator_geom (class in sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator), 306
faces() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 377
faces() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 178
faces() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 539
facet_adjacency_matrix() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 540
facet_adjacency_matrix() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 261
facet_constant() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 179
facet_constants() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 180
facet_graph() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 261
facet_normal() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 180
facet_normals() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 378
facet_normals() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 181
facet_of() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 379
facet_of() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 182
facets() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 380
facets() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 182
facets() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 540
facets() (sage.geometry.polyhedron.combinatorial_polyhedron_base.CombinatorialPolyhedron method), 262
facets() (sage.rings.polynomial.groebner_fan.PolyhedralCone method), 493
facets_tuple_to_bit_rep_of_facets() (in module sage.geometry.polyhedron.combinatorial_polyhedron.conversions), 315
facets_tuple_to_bit_rep_of_Vrep() (in module sage.geometry.polyhedron.combinatorial_polyhedron.conversions), 314
factor() (sage.geometry.fan_morphism.FanMorphism method), 457
Fan() (in module sage.geometry.fan), 427
fan() (sage.geometry.triangulation.element.Triangulation method), 695
Fan2d() (in module sage.geometry.fan), 431
fan_2d_cyclically_ordered_rays() (in module sage.geometry.fan.isomorphism), 747
Index 769

graph() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 264
graphical() (sage.geometry.hyperplane_arrangement.library.HyperplaneArrangementLibrary method), 47
great_rhombicuboctahedron() (sage.geometry.polyhedron.library.Polytopes method), 82
greatest_common_subface_of_Hrep() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 82
groebner_cone() (sage.rings.polynomial.groebner_fan.ReducedGroebnerBasis method), 497
GroebnerFan (class in sage.rings.polynomial.groebner_fan), 485

H

h_star_vector() (sage.geometry.polyhedron.base2.Polyhedron_base2 method), 525
has_good_reduction() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 22
has_IP_property() (sage.geometry.polyhedron.base ZZ.Polyhedron ZZ method), 634
has_IP_property() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL method), 234
hasse_diagram() (sage.geometry.polyhedron.base4.Polyhedron_base4 method), 562
hasse_diagram() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 264
Hilbert_basis() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 364
Hilbert_coefficients() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 366
hilbert_series() (sage.geometry.polyhedron.backend_normaliz.Polyhedron QQ_normaliz method), 642
hodge_numbers() (sage.geometry.lattice_polytope.NefPartition method), 207
homogeneity_space() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 487
homogeneous_vector() (sage.geometry.polyhedron.representation.Line method), 132
homogeneous_vector() (sage.geometry.polyhedron.representation.Ray method), 134
homogeneous_vector() (sage.geometry.polyhedron.representation.Vertex method), 136
homology_basis() (sage.geometry.ribbon_graph.RibbonGraph method), 733
Hrep2Vrep (class in sage.geometry.polyhedron.double_description_inhomogeneous), 660
Hrep_generator() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 500
Hrepresentation (class in sage.geometry.polyhedron.representation), 127
Hrepresentation() (sage.geometry.polyhedron.backend_ppl.Polyhedron_ppl method), 651
Hrepresentation() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 500
Hrepresentation() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 250
Hrepresentation_space() (sage.geometry.polyhedron.base1.Polyhedron_base1 method), 514
Hrepresentation_space() (sage.geometry.polyhedron.parent.Polyhedra_base method), 121
Hrepresentation_str() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 500
Hstar_function() (sage.geometry.polyhedron.base QQ.Polyhedron QQ method), 620
hypercube() (sage.geometry.polyhedron.library.Polytopes method), 82
Hyperplane (class in sage.geometry.hyperplane_arrangement.hyperplane), 51
hyperplane_arrangement() (sage.geometry.polyhedron.base.Polyhedron_base method), 614
HyperplaneArrangementElement (class in sage.geometry.hyperplane_arrangement.arrangement), 8
HyperplaneArrangementLibrary (class in sage.geometry.hyperplane_arrangement.library), 41
HyperplaneArrangements (class in sage.geometry.hyperplane_arrangement.arrangement), 39
hyperplanes() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 22
hypersimplex() (sage.geometry.polyhedron.library.Polytopes method), 83

I

icosahedron() (sage.geometry.polyhedron.library.Polytopes method), 84
icosidodecahedron() (sage.geometry.polyhedron.library.Polytopes method), 85
icosidodecahedron_V2() (sage.geometry.polyhedron.library.Polytopes method), 85
ideal() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 487
ideal() (sage.rings.polynomial.groebner_fan.ReducedGroebnerBasis method), 497
Ish() (sage.geometry.hyperplane_arrangement.library.HyperplaneArrangementLibrary method), 43
IshB() (sage.geometry.hyperplane_arrangement.library.HyperplaneArrangementLibrary method), 44
isomorphism() (sage.geometry.fan.RationalPolyhedralFan method), 446

J
join() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 333
join() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 576
join_of_Vrep() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 271
join_of_Vrep() (sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.FaceIterator_base method), 301

K
kernel_fan() (sage.geometry.fan_morphism.FanMorphism method), 464
Kirkman_icosahedron() (sage.geometry.polyhedron.library.Polytopes method), 67

L
label_list() (in module sage.geometry.toric_plotter), 481
last_slope() (sage.geometry.newton_polygon.NewtonPolygon_element method), 719
lattice() (sage.geometry.cone.IntegralRayCollection method), 410
lattice() (sage.geometry.lattice_polytope.LatticePolytopeMethod), 186
lattice_automorphism_group() (sage.geometry.polyhedron.ppl_LatticePolytope.PPL_class method), 237
lattice_dim() (sage.geometry.cone.IntegralRayCollection method), 410
lattice_dim() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 186
lattice_from_incidence() (in module sage.geometry.hasse_diagram), 751
lattice_polytope() (sage.geometry.polyhedron.base2.Polyhedron_base2 method), 528
LatticeEuclideanGroupElement (class in sage.geometry.polyhedron.lattice_euclidean_group_element), 221

LatticePolygon_PPL_class (class in sage.geometry.polyhedron.ppl_LatticePolygon), 224
LatticePolytope (class in sage.geometry.abc), 701
LatticePolytope() (in module sage.geometry.lattice_polytope), 166
LatticePolytope_PPL() (in module sage.geometry.polyhedron.ppl_LatticePolytope), 229
LatticePolytope_PPL_class (class in sage.geometry.polyhedron.ppl_LatticePolytope), 230
LatticePolytopeClass (class in sage.geometry.lattice_polytope), 167
LatticePolytopeError, 222
LatticePolytopeNoEmbeddingError, 222
LatticePolytopesNotIsomorphicError, 222
lawrence_extension() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 577
lawrence_polytope() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 577
least_common_surface_of_Vrep() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 550
legend_3d() (in module sage.geometry.hyperplane_arrangement.plot), 60
less_generators() (in module sage.geometry.hyperplane_arrangement.check_freeness), 753
lexicographic_triangulation() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 674
Line (class in sage.geometry.polyhedron.representation), 132
LINE (sage.geometry.polyhedron.representation.PolyhedronRepresentation attribute), 133
line_generator() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 508
line_generator() (sage.geometry.polyhedron.face.PolyhedronFace method), 157
lineality() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 389
lineality_dim() (sage.rings.polynomial.groebner_fan.PolyhedralFan method), 493
lineality_dim() (sage.rings.polynomial.groebner_fan.PolyhedralFan method), 495
linear_equivalence_ideal() (sage.geometry.fan.RationalPolyhedralFan method), 446
linear_part() (sage.geometry.hyperplane_arrangement.affine_subspace.AffineSubspace method), 58
linear_part() (sage.geometry.hyperplane_arrangement.hyperplane.Hyperplane method), 53
linear_part() (sage.geometry.hyperplane_arrangement.hyperplane.Hyperplane method), 52
linear_part_projection() (sage.geometry.hyperplane_arrangement.hyperplane.Hyperplane method), 53
linear_subspace() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 390
### Combinatorial and Discrete Geometry

<table>
<thead>
<tr>
<th>Function</th>
<th>Class/Module</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear_transformation()</td>
<td>sage.geometry.convex_set.ConvexSet_base method</td>
</tr>
<tr>
<td>linear_transformation()</td>
<td>sage.geometry.polyhedron.base5.Polyhedron_base5 method</td>
</tr>
<tr>
<td>linear_transformation()</td>
<td>sage.geometry.relative_interior.RelativeInterior method</td>
</tr>
<tr>
<td>LinearExpression</td>
<td>class in sage.geometry.linear_expression</td>
</tr>
<tr>
<td>LinearExpressionModule</td>
<td>class in sage.geometry.linear_expression</td>
</tr>
<tr>
<td>linearly_independent_vertices()</td>
<td>sage.geometry.lattice_polytope.LatticePolytopeClass method</td>
</tr>
<tr>
<td>LinearExpression</td>
<td>class in sage.geometry.linear_expression</td>
</tr>
<tr>
<td>LinearExpressionModule</td>
<td>class in sage.geometry.linear_expression</td>
</tr>
<tr>
<td>lines()</td>
<td>sage.geometry.cone.ConvexRationalPolyhedralCone method</td>
</tr>
<tr>
<td>lines()</td>
<td>sage.geometry.polyhedron.base0.Polyhedron_base0 method</td>
</tr>
<tr>
<td>lines()</td>
<td>sage.geometry.polyhedron.face.PolyhedronFace method</td>
</tr>
<tr>
<td>lines_list()</td>
<td>sage.geometry.polyhedron.base0.Polyhedron_base0 method</td>
</tr>
<tr>
<td>linial()</td>
<td>sage.geometry.hyperplane_arrangement.Library.HyperplaneArrangementLibrary method</td>
</tr>
<tr>
<td>list()</td>
<td>sage.geometry.polyhedron.parent.Polyhedral_base method</td>
</tr>
<tr>
<td>ListOffaces</td>
<td>class in sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces</td>
</tr>
<tr>
<td>lyapunov_like_basis()</td>
<td>sage.geometry.cone.ConvexRationalPolyhedralCone method</td>
</tr>
<tr>
<td>lyapunov_rank()</td>
<td>sage.geometry.cone.ConvexRationalPolyhedralCone method</td>
</tr>
</tbody>
</table>

### M

<table>
<thead>
<tr>
<th>Function</th>
<th>Class/Module</th>
</tr>
</thead>
<tbody>
<tr>
<td>make_generic()</td>
<td>sage.geometry.ribbon_graph.RibbonGraph method</td>
</tr>
<tr>
<td>make_parent()</td>
<td>in module sage.geometry.hyperplane_arrangement.library</td>
</tr>
<tr>
<td>make_ribbon()</td>
<td>in module sage.geometry.ribbon_graph</td>
</tr>
<tr>
<td>make_simplicial()</td>
<td>sage.geometry.fan.RationalPolyhedralFan method</td>
</tr>
<tr>
<td>matrix()</td>
<td>sage.geometry.point_collection.PointCollection method</td>
</tr>
<tr>
<td>matrix()</td>
<td>sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces.ListOfFaces method</td>
</tr>
<tr>
<td>matrix_space()</td>
<td>sage.geometry.polyhedron.double_description.DoubleDescriptionPair method</td>
</tr>
<tr>
<td>matroid()</td>
<td>sage.geometry.hyperplane_arrangement.HyperplaneArrangementElement method</td>
</tr>
<tr>
<td>max_degree()</td>
<td>in module sage.rings.polynomial.groebner_fan</td>
</tr>
<tr>
<td>maximal_cell_iterator()</td>
<td>sage.geometry.polyhedral_complex.PolyhedralComplex method</td>
</tr>
<tr>
<td>maximal_cells()</td>
<td>sage.geometry.polyhedral_complex.PolyhedralComplex method</td>
</tr>
<tr>
<td>maximal_cells_sorted()</td>
<td>sage.geometry.polyhedral_complex.PolyhedralComplex method</td>
</tr>
<tr>
<td>maximal_cones()</td>
<td>sage.rings.polynomial.groebner_fan.PolyhedronFan method</td>
</tr>
<tr>
<td>maximal_total_degree_of_a_groebner_basis()</td>
<td>sage.rings.polynomial.groebner_fan.GroebnerFan method</td>
</tr>
<tr>
<td>meet_of_Hrep()</td>
<td>sage.geometry.polyhedron.base3.Polyhedron_base3 method</td>
</tr>
<tr>
<td>meet_of_Hrep()</td>
<td>sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method</td>
</tr>
<tr>
<td>meet_of_Hrep()</td>
<td>sage.geometry.polyhedron.combinatorial_polyhedron_face_iterator.FacetIterator_base method</td>
</tr>
<tr>
<td>minimal_generated_number()</td>
<td>sage.geometry.hyperplane_arrangement.HyperplaneArrangementElement method</td>
</tr>
<tr>
<td>minimal_total_degree_of_a_groebner_basis()</td>
<td>sage.rings.polynomial.groebner_fan.GroebnerFan method</td>
</tr>
<tr>
<td>minkowski_decompositions()</td>
<td>sage.geometry.polyhedron.base_ZZ.Polyhedron_ZZ method</td>
</tr>
<tr>
<td>minkowski_difference()</td>
<td>sage.geometry.polyhedron.base5.Polyhedron_base5 method</td>
</tr>
<tr>
<td>minkowski_sum()</td>
<td>in module sage.geometry.lattice_polytope</td>
</tr>
<tr>
<td>minkowski_sum()</td>
<td>sage.geometry.polyhedron.base5.Polyhedron_base5 method</td>
</tr>
<tr>
<td>mixed_volume()</td>
<td>(sage.rings.polynomial.groebner_fan.GroebnerFan method)</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.abc</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.cone</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.cone_catalog</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.convex_set</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.fan</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.fan_isomorphism</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.fan_morphism</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.hasse_diagram</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.hyperplane_arrangement.affine_subspace</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.hyperplane_arrangement.arrangement.affine_subspace</td>
</tr>
<tr>
<td>module</td>
<td>sage.geometry.hyperplane_arrangement.arrangement.CheckFreeness</td>
</tr>
</tbody>
</table>

Index 775
sage.geometry.hyperplane_arrangement.hyperplane, 49
sage.geometry.hyperplane_arrangement.library, 41
sage.geometry.hyperplane_arrangement.plot, 59
sage.geometry.integral_points, 753
sage.geometry.lattice_polytope, 165
sage.geometry.linear_expression, 713
sage.geometry.newton_polygon, 719
sage.geometry.point_collection, 468
sage.geometry.polyhedral_complex, 317
sage.geometry.polyhedronbackend_cdd, 638
sage.geometry.polyhedronbackend_cdd_rdf, 639
sage.geometry.polyhedronbackend_field, 639
sage.geometry.polyhedronbackend_normaliz, 641
sage.geometry.polyhedronbackend_number_field, 640
sage.geometry.polyhedronbackend_polytope, 640
sage.geometry.polyhedronbackend_polymake, 648
sage.geometry.polyhedronbackend_ppl, 650
sage.geometry.polyhedron.base, 610
sage.geometry.polyhedron.base0, 499
sage.geometry.polyhedron.base1, 514
sage.geometry.polyhedron.base2, 523
sage.geometry.polyhedron.base3, 530
sage.geometry.polyhedron.base4, 557
sage.geometry.polyhedron.base5, 571
sage.geometry.polyhedron.base6, 587
sage.geometry.polyhedron.base7, 602
sage.geometry.polyhedron.base_QQ, 620
sage.geometry.polyhedron.base_RDF, 638
sage.geometry.polyhedron.base_ZZ, 631
sage.geometry.polyhedron.cdd_file_format, 162
sage.geometry.polyhedron.combinatorial_polyhedron.base, 246
sage.geometry.polyhedron.combinatorial_polyhedron.combinatorial_face, 280
sage.geometry.polyhedron.combinatorial_polyhedron.conversions, 313
sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator, 293
sage.geometry.polyhedron.combinatorial_polyhedron.list_of_faces, 310
sage.geometry.polyhedron.combinatorial_polyhedron.polyhedron_face_lattice, 290
sage.geometry.polyhedron.constructor, 111
sage.geometry.polyhedron.double_description, 652
sage.geometry.polyhedron.double_description_inhomogeneous, 660
sage.geometry.polyhedron.face, 151
sage.geometry.polyhedron.generating_function, 241
sage.geometry.polyhedron.lattice_euclidean_group_element, 221
sage.geometry.polyhedron.library, 65
sage.geometry.polyhedron.modules.formal_polyhedra_module, 163
sage.geometry.polyhedron.palp_database, 223
sage.geometry.polyhedron.parent, 119
sage.geometry.polyhedron.plot, 139
sage.geometry.polyhedron.ppl_lattice_polytope, 224
sage.geometry.polyhedron.ppl_lattice_polytope, 228
sage.geometry.polyhedron.representation, 126
sage.geometry.pseudolines, 740
sage.geometry.relative_interior, 723
sage.geometry.ribbon_graph, 727
sage.geometry.toric_lattice, 342
sage.geometry.toric_plotter, 475
sage.geometry.triangulation.plotter, 475
sage.geometry.triangulation.base, 683
sage.geometry.triangulation.element, 692
sage.geometry.triangulation.point_configuration, 665
sage.geometry.voronoi_diagram, 743
sage.rings.polynomial.groebner_fan, 485
module() (sage.geometry.point_collection.PointCollection method), 472
monomial_coefficients() (sage.geometry.linear_expression.LinearExpression method), 716
mu() (sage.geometry.ribbon_graph.RibbonGraph method),
normal_fan() (sage.geometry.polyhedron.base.Polyhedron_base method), 615
normal_form() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 189
normal_form() (sage.geometry.polyhedron.base.ZZ.Polyhedron_ZZ method), 636
NormalFan() (in module sage.geometry.fan), 431
normalize() (sage.geometry.oriented_boundary().RibbonGraph method), 736
normalize_rays() (in module sage.geometry.oriented_boundary(), 414
nparts() (sage.geometry.lattice_polytope.NefPartition method), 209
npoints() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 194
nrays() (sage.geometry.oriented_boundary().IntegralRayCollection method), 411
number_boundaries() (sage.geometry.oriented_boundary().RibbonGraph method), 736
number_of_reduced_groebner_bases() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 488
number_of_variables() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 488
nvertices() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 194

O
octahedron() (sage.geometry.polyhedron.library.Polytopes method), 86
omnitruncated_one_hundred_twenty_cell() (sage.geometry.polyhedron.library.Polytopes method), 86
omnitruncated_six_hundred_cell() (sage.geometry.polyhedron.library.Polytopes method), 87
one_hundred_twenty_cell() (sage.geometry.polyhedron.library.Polytopes method), 87
one_point_suspension() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 580
only_subfaces() (sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.FaceIterator_base method), 304
only_subfaces() (sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.FaceIterator_base method), 305
options() (in module sage.geometry.toric_plotter), 481
ordered_vertices() (sage.geometry.polyhedron.ppl_lattice_polygon.LatticePolygon_PPL_class method), 225
oriented_boundary() (sage.geometry.fan.RationalPolyhedralFan method), 447
origin() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 194
orlik_solomon_algebra() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 29
orlik_terao_algebra() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 29
orthogonal_projection() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 54
orthogonal_sublattice() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 394
outer_normal() (sage.geometry.polyhedron.representation.Inequality method), 131
output_format() (sage.geometry.point_representation.Inequality method), 472

P
P (sage.geometry.polyhedron.combinatorial_polyhedron.face_iterator.FaceterIterator_geon attribute), 309
pair_class (sage.geometry.polyhedron.double_description.Problem attribute), 658
pair_class (sage.geometry.polyhedron.double_description.StandardAlgorithm attribute), 658
PALPreader (class in sage.geometry.polyhedron.palp_database), 223
parallelotope() (sage.geometry.polyhedron.library.Polytopes method), 88
parent() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 195
ParentNewtonPolygon (class in sage.geometry.newton_polygon), 721
part() (sage.geometry.lattice_polytope.NefPartition method), 209
part_of() (sage.geometry.lattice_polytope.NefPartition method), 210
part_of_point() (sage.geometry.lattice_polytope.NefPartition method), 210
parts() (sage.geometry.lattice_polytope.NefPartition method), 211
pentakis_dodecahedron() (sage.geometry.polyhedron.library.Polytopes method), 88
permutahedron() (sage.geometry.polyhedron.library.Polytopes method), 89
permutations() (sage.geometry.pseudolines.PseudolineArrangement method), 742
permutations_to_matrices() (sage.geometry.polyhedron.base.Polyhedron_base method), 617
PivotedInequalities (class in sage.geometry.polyhedron.double_description_inhomogeneous), 661
sage.geometry.polyhedron.palp_database
module, 223
sage.geometry.polyhedron.parent
module, 119
sage.geometry.polyhedron.plot
module, 139
sage.geometry.polyhedron.ppl_lattice_polygon
module, 224
sage.geometry.polyhedron.ppl_lattice_polytope
module, 228
sage.geometry.polyhedron.representation
module, 126
sage.geometry.pseudolines
module, 740
sage.geometry.relative_interior
module, 723
sage.geometry.ribbon_graph
module, 727
sage.geometry.toric_lattice
module, 342
sage.geometry.toric_plotter
module, 475
sage.geometry.triangulation.base
module, 683
sage.geometry.triangulation.element
module, 692
sage.geometry.triangulation.point_configuration
module, 665
sage.geometry.voronoi_diagram
module, 743
sage.rings.polynomial.groebner_fan
module, 485
saturation() (sage.geometry.toric_lattice.ToricLattice_generic method), 350
schlegel() (sage.geometry.polyhedron.plot.Projection method), 144
schlegel_projection() (sage.geometry.polyhedron.base6.Polyhedron_base6 method), 599
schur() (in module sage.geometry.cone_catalog), 420
secondary_polytope() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 679
section_linear_map (sage.geometry.convex_set.AffineHullProjectionData attribute), 702
section_translation (sage.geometry.convex_set.AffineHullProjectionData attribute), 702
sector() (in module sage.geometry.toric_plotter), 484
semigroup_generators() (sage.geometry.triangulation.base3.Polyhedron_base3 method), 553
simplex() (sage.geometry.polyhedron.library.Polytopes method), 94
simplex_to_int() (sage.geometry.triangulation.base.PointConfiguration_base method), 691
simplicial_complex() (sage.geometry.triangulation.element.Triangulation method), 698
simpliciality() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 553
simpliciality() (sage.geometry.polyhedron.combinatorial_polyhedron.base.CombinatorialPolyhedron method), 276
simplicity() (sage.geometry.polyhedron.library.Polytopes method), 95
skeleton() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 200
skeleton_points() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 200
skeleton_show() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 200
skip_palp_matrix() (in module sage.geometry.lattice_polytope), 219
slack_matrix() (sage.geometry.polyhedron.ppl_polytope.PolyhedronBase3 method), 554
slopes() (sage.geometry.newton_polygon.NewtonPolygon_element method), 720
small_rhombicuboctahedron() (sage.geometry.polyhedron.library.Polytopes method), 96
snub_cube() (sage.geometry.polyhedron.library.Polytopes method), 97
snub_dodecahedron() (sage.geometry.polyhedron.library.Polytopes method), 98
solid_restriction() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 402
some_elements() (sage.geometry.convex_set.ConvexSet_base method), 710
some_elements() (sage.geometry.polyhedron.parent.Polyhedra_base3 method), 124
span() (sage.geometry.cone.IntegralRayCollection method), 412
span() (sage.geometry.toric_lattice.ToricLattice_generic method), 350
span_of_basis() (sage.geometry.toric_lattice.ToricLattice_generic method), 350
stack() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 583
stacking_locus() (sage.geometry.polyhedron.face.PolyhedronFace method), 160
StandardAlgorithm (class in sage.geometry.polyhedron_double_description), 658
StandardDoubleDescriptionPair (class in sage.geometry.polyhedron_double_description), 659
Stanley_Reisner_ideal() (sage.geometry.fan.RationalPolyhedralFan method), 433
star_center() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 680
star_generator_indices() (sage.geometry.fan.Cone_of_fan method), 426
star_generators() (sage.geometry.fan.Cone_of_fan method), 426
stereographic() (sage.geometry.polyhedron.plot.Projection method), 145
stratify() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 339
strict_quotient() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 403
sub_polytope_generator() (sage.geometry.polyhedron.ppl_polytope.LatticePolytope_PPL_class method), 240
sub_polytopes() (sage.geometry.polyhedron.ppl_polytope.LatticePolygon_PPL_class method), 226
sub_reflexive_polygons() (in module sage.geometry.polyhedron.ppl_polytope), 227
subdirect_sum() (sage.geometry.polyhedron.polytope.Polytope_PPL_class method), 584
subdivide() (sage.geometry.fan.RationalPolyhedralFan method), 449
subdivide() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 339
sublattice() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 404
sublattice_complement() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 405
sublattice_quotient() (sage.geometry.cone.ConvexRationalPolyhedralCone method), 406
subpolygons_of_polar_P1xP1() (in module sage.geometry.polyhedron.ppl_lattice_polygon), 227
subpolygons_of_polar_P2() (in module sage.geometry.polyhedron.ppl_lattice_polygon), 228
subpolygons_of_polar_P2_112() (in module sage.geometry.polyhedron.ppl_lattice_polygon), 228
support_contains() (sage.geometry.fan.RationalPolyhedralFan method), 450
symmetric_edge_polytope() (sage.geometry.polyhedron.library.Polytopes static method), 98
symmetric_space() (sage.geometry.hyperplane_arrangement.hyperplane.AmbientVectorSpace method), 51
T

tetrahedron() (sage.geometry.polyhedron.library.Polytopes method), 101
tikz() (sage.geometry.polyhedron_base6.Polyhedron_base6 method), 600
tikz() (sage.geometry.polyhedron.plot.Projection method), 145
to_linear_program() (sage.geometry.polyhedron_base6.Polyhedron_base6 method), 618
to_RationalPolyhedralFan() (sage.rings.polynomial.groebner_fan.PolyhedralFan method), 496
to_symmetric_space() (sage.geometry.hyperplane_arrangement.hyperplane.Hyperplane method), 56
Index 787

trivial() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 490

trivial() (in module sage.geometry.cone_catalog), 421
tropical_basis() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 490
tropical_intersection() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 490

TropicalPrevariety (class in sage.rings.polynomial.groebner_fan), 497

truncated_cube() (sage.geometry.polyhedron.library.Polytopes method), 101

truncated_dodecahedron() (sage.geometry.polyhedron.library.Polytopes method), 102

truncated_icosidodecahedron() (sage.geometry.polyhedron.library.Polytopes method), 103

truncated_octahedron() (sage.geometry.polyhedron.library.Polytopes method), 103

truncated_one_hundred_twenty_cell() (sage.geometry.polyhedron.library.Polytopes method), 104

truncated_six_hundred_cell() (sage.geometry.polyhedron.library.Polytopes method), 104

truncated_tetrahedron() (sage.geometry.polyhedron.library.Polytopes method), 105

truncation() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 585
twenty_four_cell() (sage.geometry.polyhedron.library.Polytopes method), 105
type() (sage.geometry.polyhedron.representation.Equation method), 126
type() (sage.geometry.polyhedron.representation.Inequality method), 131
type() (sage.geometry.polyhedron.representation.Line method), 132
type() (sage.geometry.polyhedron.representation.Ray method), 135
type() (sage.geometry.polyhedron.representation.Vertex method), 136

unbounded_regions() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 36

union() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 36

union() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 340

union_as_polyhedron() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 341

universe() (sage.geometry.polyhedron.parent.Polyhedra_base method), 124

varchenko_matrix() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 37

vector() (sage.geometry.polyhedron.representation.PolyhedronRepresentation method), 134

verify() (sage.geometry.polyhedron.double_description_inhomogeneous.Hrep2Vrep method), 661
vertices() (sage.geometry.polyhedron.double_description_inhomogeneous.Vrep2Hrep method), 663
verify() (sage.geometry.polyhedron.double_description_DoubleDescriptionPair method), 656

Vertex (class in sage.geometry.polyhedron.representation), 135

VERTEX (sage.geometry.polyhedron.representation.PolyhedronRepresentation attribute), 133
vertex() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 202
vertex_adjacency_matrix() (sage.geometry.polyhedron.base3.Polyhedron_base3 method), 555
vertex_adjacency_matrix() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron method), 277
vertex_digraph() (sage.geometry.polyhedron.base4.Polyhedron_base4 method), 568
vertex_facet_graph() (sage.geometry.polyhedron.base4.Polyhedron_base4 method), 569
vertex_facet_graph() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron_base0.Polyhedron method), 278
vertex_facet_pairing_matrix() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron_base0.Polyhedron method), 278
vertex_generator() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 511
vertex_generator() (sage.geometry.polyhedron.face.PolyhedronFace method), 161
vertex_graph() (sage.geometry.fan.RationalPolyhedralFan method), 450
vertex_graph() (sage.geometry.polyhedron.base4.Polyhedron_base4 method), 570
vertex_graph() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron_base0.Polyhedron method), 279
vertices() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 37
vertices() (sage.geometry.lattice_polytope.LatticePolytopeClass method), 202
vertices() (sage.geometry.newton_polygon.NewtonPolygon_element method), 720
vertices() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 512
vertices() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron_base0.Polyhedron method), 279
vertices() (sage.geometry.polyhedron.face.PolyhedronFace method), 161
vertices() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 240
vertices_list() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 512
vertices_matrix() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 513
vertices_saturating() (sage.geometry.polyhedron.ppl_lattice_polytope.LatticePolytope_PPL_class method), 241

verts_for_normal() (in module sage.rings.polynomial.groebner_fan), 499

virtual_rays() (sage.geometry.fan.RationalPolyhedralFan method), 451

Volume() (sage.geometry.polyhedron.base7.Polyhedron_base7 method), 607
Volume() (sage.geometry.triangulation.point_configuration.PointConfiguration method), 683

VoronoiDiagram (class in sage.geometry.voronoi_diagram), 743

Vrep2Hrep (class in sage.geometry.polyhedron.double_description_inhomogeneous), 662

Vrep_generator() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 501

Vrepresentation (class in sage.geometry.polyhedron.representation), 137
Vrepresentation() (sage.geometry.polyhedron.backend_ppl.Polyhedron_ppl method), 651
Vrepresentation() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 502
Vrepresentation() (sage.geometry.polyhedron.combinatorial_polyhedron.Polyhedron_base0.Polyhedron method), 250
Vrepresentation_space() (sage.geometry.polyhedron.base1.Polyhedron_base1 method), 514
Vrepresentation_space() (sage.geometry.polyhedron.parent.Polyhedra_base method), 121

W

wedge() (sage.geometry.polyhedral_complex.PolyhedralComplex method), 341
wedge() (sage.geometry.polyhedron.base5.Polyhedron_base5 method), 585
weight_vectors() (sage.rings.polynomial.groebner_fan.GroebnerFan method), 491

whitney_data() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 38
whitney_number() (sage.geometry.hyperplane_arrangement.arrangement.HyperplaneArrangementElement method), 38
write_cdd_Hrepresentation() (sage.geometry.polyhedron.base0.Polyhedron_base0 method), 513
write_cdd_Vrepresentation()  (sage.geometry.polyhedron.base0.Polyhedron_base0
method), 514
write_for_palp()  (sage.geometry.point_collection.PointCollection method), 473
write_palp_matrix()  (in module sage.geometry.lattice_polytope), 220

Z
Z_operators_gens()  (sage.geometry.cone.ConvexRationalPolyhedralCone method), 367
zero()  (sage.geometry.polyhedron.parent.Polyhedra_base method), 125
zero_set()  (sage.geometry.polyhedron.double_description.DoubleDescriptionPair method), 656
zero_sum_projection()  (in module sage.geometry.polyhedron.library), 111
zonotope()  (sage.geometry.polyhedron.library.Polytopes method), 106