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1.1 Catalog of Cellular Automata

The `cellular_automata` object may be used to access examples of various cellular automata currently implemented in Sage. Using tab-completion on this object is an easy way to discover and quickly create the cellular automata that are available (as listed here).

Let <tab> indicate pressing the tab key. So begin by typing `cellular_automata.<tab>` to see the currently implemented named cellular automata.

- `cellular_automata.Elementary`
- `cellular_automata.GraftalLace`
- `cellular_automata.PeriodicSoliton`
- `cellular_automata.Soliton`

1.2 Elementary Cellular Automata

AUTHORS:

- Travis Scrimshaw (2018-07-07): Initial version

```python
class sage.dynamics.cellular_automata.elementary.ElementaryCellularAutomata (rule, width=None, initial_state=None, boundary=0, 0)
```

Bases: `sage.structure.sage_object.SageObject`

Elementary cellular automata.

An elementary cellular automaton is a 1-dimensional cellular deterministic automaton with two possible values: $X := \{0, 1\}$. A state is therefore a sequence $s \in X^n$, and the evolution of a state $s \rightarrow s'$ is given for $s'_i$ by looking at the values at positions $s_{i-1}, s_i, s_{i+1}$ and is determined by the rule $0 \leq r \leq 255$ as follows. Consider the binary representation $r = b_7b_6b_5b_4b_3b_2b_1b_0$. Then, we define $s'_i = b_j$, where $j = s_{i-1}s_is_{i+1}$ is the corresponding binary representation. In other words, the value $s'_i$ is given according to the following table:

<table>
<thead>
<tr>
<th>111</th>
<th>110</th>
<th>101</th>
<th>100</th>
<th>011</th>
<th>010</th>
<th>001</th>
<th>000</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>110</td>
<td>101</td>
<td>100</td>
<td>011</td>
<td>010</td>
<td>001</td>
<td>000</td>
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</table>

We consider the boundary values of $s_0 = s_{n+1} = 0$. 
INPUT:
- **rule** – an integer between 0 and 255
- **width** – (optional) the width of the ECA
- **initial_state** – (optional) the initial state given as a list of 0 and 1
- **boundary** – (default: (0, 0)) a tuple of the left and right boundary conditions respectively or None for periodic boundary conditions

Either `width` or `initial_state` must be given. If `width` is less than the length of `initial_state`, then `initial_state` has 0 prepended so the resulting list has length `width`. If only `width` is given, then the initial state is constructed randomly.

The boundary conditions can either be 0, 1, or a function that takes an integer \( n \) corresponding to the state and outputs either 0 or 1.

EXAMPLES:

We construct an example with rule \( r = 90 \) using \( n = 20 \). The initial state consists of a single 1 in the rightmost entry:

```
sage: ECA = cellular_automata.Elementary(90, width=20, initial_state=[1])
sage: ECA.evolve(20)
sage: ascii_art(ECA)
```

(continues on next page)
Now we consider the right boundary as being 1 on every third value:

```python
sage: def rbdry(n):
    return 1 if n % 3 == 0 else 0
sage: ECA = cellular_automata.Elementary(90, width=20, initial_state=[1],
    boundary=(0, rbdry))
sage: ECA.evolve(20)
sage: ascii_art(ECA)
```

Lastly we consider it with periodic boundary condition:

```python
sage: ECA = cellular_automata.Elementary(90, width=20, initial_state=[1],
    boundary=None)
sage: ECA.evolve(20)
sage: ascii_art(ECA)
```

(continues on next page)
We show the local evolution rules for rule 110:

```
sage: for t in cartesian_product([[0,1],[0,1],[0,1]]):
    ....:    ECA = cellular_automata.Elementary(110, list(t))
    ....:    ECA.print_states(2)
    ....:    print('#')
```

We construct an elementary cellular automaton with a random initial state with $n = 15$ and see the state after 50 evolutions:

```
sage: ECA = cellular_automata.Elementary(26, width=25)
sage: ECA.print_state(50)  # random
```

We construct and plot a larger example with rule 60:
With periodic boundary condition for rule 90:

```python
sage: ECA = cellular_automata.Elementary(90, initial_state=[1]+[0]*254+[1], boundary=None)
sage: ECA.evolve(256)
sage: ECA.plot()
Graphics object consisting of 1 graphics primitive
```

REFERENCES:

Wikipedia article Elementary_cellular_automaton

evolve

```python
 Evolve self.

INPUT:

• number – (optional) the number of times to perform the evolution

EXAMPLES:
```
sage: ECA = cellular_automata.Elementary(110, [1,0,0,1,1,0,1,0,1,1])
sage: ascii_art(ECA)
X  XX  X  X
sage: ECA.evolve()
sage: ascii_art(ECA)
X  XX  X  X
X  XXX  XXX
sage: ECA.evolve(10)
sage: ascii_art(ECA)
X  XX  X  X
X  XXX  XXX
XX  XXXX  X
X  XXX  X  XX
XX  XXXX  X
X  XXXX  X
X  XXX  X
X  X  X

plot (number=None)
Return a plot of self.

INPUT:

• number – the number of states to plot

EXAMPLES:

sage: ECA = cellular_automata.Elementary(110, width=256)
sage: ECA.evolve(256)
sage: ECA.plot()
Graphics object consisting of 1 graphics primitive

print_state (number=None)
Print the state number.

INPUT:

• number – (default: the current state) the state to print

EXAMPLES:

sage: ECA = cellular_automata.Elementary(110, width=10, initial_state=[1,0,0,1,1,0,1])
....:
ECA.print_state(15)
X  X  XXX
sage: ECA.print_state(10)
X  X  XX
sage: ECA.print_state(20)
X  XXX
sage: for i in range(11):
....:
  ECA.print_state(i)
  X  XX  X
  XX  XXXX
  XXXX  X
  XX  X  X
(continues on next page)
print_states \( \text{(number=None)} \)

Print the first \( \text{num} \) states of \( \text{self} \).

\textbf{Note:} If the number of states computed for \( \text{self} \) is less than \( \text{num} \), then this evolves the system using the default time evolution.

\textbf{INPUT:}

\begin{itemize}
  \item \textbf{number} – the number of states to print
\end{itemize}

\textbf{EXAMPLES:}

```python
sage: ECA = cellular_automata.Elementary(110, width=10,
....: initial_state=[1,0,1,1,0,1])
sage: ECA.print_states(10)
```

\begin{verbatim}
XX XX XXX
XX XXXXX X
XXXX XXX
X X XX X
XX XX XXX
XX XXXXX X
XXXX XXX
X X XX X
XX XXXXX
XXXXXXX X
XX XX XXX
```

### 1.3 Graftal Lace Cellular Automata

\textbf{AUTHORS:}

\begin{itemize}
  \item Travis Scrimshaw (2020-04-30): Initial version
\end{itemize}

**class** \texttt{sage.dynamics.cellular_automata.glca.GraftalLaceCellularAutomata} \( \text{(rule)} \)

Graftal Lace Cellular Automata (GLCA).

A GLCA is a deterministic cellular automaton whose rule is given by an 8-digit octal number \( r_7 \cdots r_0 \). For a node \( s_i \), let \( b_k \), for \( k = -1,0,1 \) denote if there is an edge from \( s_i \) to \( s'_i+k \), where \( s'_j \) is the previous row. We determine the value at \( t_{i+k} \) by considering the value of \( r_m \), where the binary representation of \( m \) is \( b_{-1} b_0 b_1 \). If \( r_m \) has a binary representation of \( b'_{-1} b'_0 b'_1 \{1\}' \), then we add \( b'_k \) to \( t_{i+k} \).

\textbf{INPUT:}

\begin{itemize}
  \item \textbf{rule} – a list of length 8 with integer entries \( 0 \leq x < 8 \)
\end{itemize}

\textbf{EXAMPLES:}
```python
sage: G = cellular_automata.GraftalLace([0,2,5,4,7,2,3,3])
sage: G.evolve(3)
sage: ascii_art(G)
```
```
    o |
    o |
    o o o
    /| /|
    o o o o o
    /| /|
    o o o o o o
```
```
sage: G = cellular_automata.GraftalLace([3,0,3,4,7,6,3,1])
sage: G.evolve(3)
sage: ascii_art(G)
```
```
    o |
    o |
    / |
    o o o
    / |
    o o o o o
    / |
    o o o o o o
```
```
sage: G = cellular_automata.GraftalLace([2,0,3,3,6,0,2,7])
sage: G.evolve(2)
sage: ascii_art(G)
```
```
    o |
    o |
    / |
    o o o
    / |
    o o o o o
    / |
    o o o o o o
```
```
REFERENCES:

• [Kas2018]

**evolve**(number=None)

Evolve self.

**INPUT:**

• number – (default: 1) the number of times to perform the evolution

**EXAMPLES:**

```python
sage: G = cellular_automata.GraftalLace([5,1,2,5,4,3,5,0])
sage: ascii_art(G)
```
```
    o |
    o |
```
```
sage: G.evolve(2)
sage: ascii_art(G)
```
```
    o |
    o |
    / |
    o o o
    / |
    o o o o
    / |
    o o o o o
```
```
(continues on next page)
```
sage: G = cellular_automata.GraftalLace([0,2,1,4,7,2,3,0])
sage: G.evolve(3)
sage: ascii_art(G)
  o
  |  
  o
  |
  o o o
  |
  o o o o o
  |
  o o o o o o o

plot (number=None)
Return a plot of self.

INPUT:

• number – the number of states to plot

EXAMPLES:

sage: G = cellular_automata.GraftalLace([5,1,2,5,4,5,5,0])
sage: G.evolve(20)
sage: G.plot()
Graphics object consisting of 865 graphics primitives

print_states (number=None, use_unicode=False)
Print the first num states of self.

Note: If the number of states computed for self is less than num, then this evolves the system using the default time evolution.

INPUT:

• number – the number of states to print

EXAMPLES:

sage: G = cellular_automata.GraftalLace([5,1,2,5,4,5,5,0])
sage: G.evolve(2)
sage: G.print_states()
  o
  |
  o
  /\  
 o o o
 / \ / \  
 o o o o o o o
sage: G.evolve(20)
sage: G.print_states(3)
  o
  |
  o
  /\  
 o o o
(continues on next page)

1.3. Graftal Lace Cellular Automata 11
1.4 Soliton Cellular Automata

AUTHORS:

• Travis Scrimshaw (2017-06-30): Initial version
• Travis Scrimshaw (2018-02-03): Periodic version

class sage.dynamics.cellular_automata.solitons.PeriodicSolitonCellularAutomata(initial_state, cartan_type=2, vacuum=1)

Bases: sage.dynamics.cellular_automata.solitons.SolitonCellularAutomata

A periodic soliton cellular automata.

Fix some $r \in I_0$. A periodic soliton cellular automata is a SolitonCellularAutomata with a state being a fixed number of tensor factors $p = p_f \otimes \cdots \otimes p_1 \otimes p_0$ and the time evolution $T_s$ is defined by

$$R(p \otimes u) = u \otimes T_s(p),$$

for some element $u \in B^{r,s}$.

INPUT:

• initial_state – the list of elements, can also be a string when vacuum is 1 and $n$ is $sl_n$
• cartan_type – (default: 2) the value $n$, for $sl_n$, or a Cartan type
• $r$ – (default: 1) the node index $r$; typically this corresponds to the height of the vacuum element

EXAMPLES:

The construction and usage is the same as for SolitonCellularAutomata:

```
sage: P = PeriodicSolitonCellularAutomata('1123334112411114231141123112', 4)
sage: P.evolve()
sage: P
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
..23334...24....423...4...23..2
evolutions: [(1, 31)]
current state:
34......24....243....4223.2333.4
sage: P.evolve(carrier_capacity=2)
sage: P.evolve(carrier_index=2)
sage: P.evolve(carrier_index=2, carrier_capacity=3)
sage: P.print_states(10)
t: 0
...23334...24....423...4...23..2
t: 1
34......24....243....4223.2333.4
t: 2
......24....24.3.....4223.2333.4
```

(continues on next page)
Using $r = 2$ in type $A_3^{(1)}$:

```sage
sage: initial = [[2,1],[2,1],[2,1],[4,1],[4,1],[2,1],[2,1],[2,1],[4,1],[4,1],[2,1],[2,1]]
sage: P = PeriodicSolitonCellularAutomata(initial, 4, 2)
sage: P.print_states(10)
```

We do some examples in other types:

```sage
sage: initial = [[1],[2],[2],[1],[1],[1],[3],[1],['E'],['1'],['1']]
sage: P = PeriodicSolitonCellularAutomata(initial, ['D',4,3])
sage: P.print_states(10)
```
sage: P = PeriodicSolitonCellularAutomata([[3], [2], [1], [1], [-2]], ['C', 2, 1])
sage: P.print_state_evolution(0)

3 2 1 1 -2
| | | | | |
11112 ---+-- 11112 ---+-- 11112 ---+-- 11112 ---+-- 11112
| | | | | |
2 1 -2 -2 1

REFERENCES:

• [KTT2006]
• [KS2006]
• [YT2002]
• [YYT2003]

evolve (carrier_capacity=None, carrier_index=None, number=None)

Evolve self.

Time evolution $T_s$ of a SCA state $p$ is determined by

$u \otimes T_s(p) = R(p \otimes u),$

where $u$ is some element in $B_{r,s}$.  

INPUT:

• carrier_capacity – (default: the number of balls in the system) the size $s$ of carrier
• carrier_index – (default: the vacuum index) the index $r$ of the carrier
• number – (optional) the number of times to perform the evolutions

To perform multiple evolutions of the SCA, carrier_capacity and carrier_index may be lists of the same length.

**Warning:** Time evolution is only guaranteed to result in a solution when the carrier_index is the defining $r$ of the SCA. If no solution is found, then this will raise an error.

EXAMPLES:

sage: P = PeriodicSolitonCellularAutomata('1241133214131221122', 4)
sage: P.evolve()
sage: P.print_state(0)

(continued from previous page)

sage: P = PeriodicSolitonCellularAutomata('12411132121', 4)
sage: P.evolve(carrier_index=2, carrier_capacity=3)
sage: P.state_evolution(0)
[[[1, 1, 1], [2, 2, 4]],
 [[1, 1, 2], [2, 2, 4]],
 [[1, 1, 3], [2, 2, 4]],
 [[1, 1, 1], [2, 2, 3]],
 [[1, 1, 1], [2, 2, 3]],
 [[1, 1, 1], [2, 2, 3]],
 [[1, 1, 2], [2, 2, 3]],
 [[1, 1, 1], [2, 2, 2]],
 [[1, 1, 1], [2, 2, 3]],
 [[1, 1, 1], [2, 2, 3]],
 [[1, 1, 1], [2, 2, 4]],
 [[1, 1, 1], [2, 2, 4]]]

class sage.dynamics.cellular_automata.solitons.SolitonCellularAutomata

Bases: sage.structure.sage_object.SageObject

Soliton cellular automata.

Fix an affine Lie algebra \( g \) with index \( I \) and classical index set \( I_0 \). Fix some \( r \in I_0 \). A soliton cellular automaton (SCA) is a discrete (non-linear) dynamical system given as follows. The states are given by elements of a semi-infinite tensor product of Kirillov-Reshetikhin crystals \( B^r \), where only a finite number of factors are not the

1.4. Soliton Cellular Automata 15
maximal element \( u \), which we will call the vacuum. The time evolution \( T_s \) is defined by

\[
R(p \otimes u_s) = u_s \otimes T_s(p),
\]

where \( p = \cdots \otimes p_3 \otimes p_2 \otimes p_1 \otimes p_0 \) is a state and \( u_s \) is the maximal element of \( B^{r:s} \). In more detail, we have \( R(p_i \otimes u^{(i)}) = u^{(i+1)} \otimes p_i \) with \( u^{(0)} = u_s \) and \( T_s(p) = \cdots \otimes p_1 \otimes p_0 \). This is well-defined since \( R(u \otimes u_s) = u_s \otimes u \) and \( u^{(k)} = u_s \) for all \( k \gg 1 \).

INPUT:

- initial_state – the list of elements, can also be a string when vacuum is 1 and \( n \) is \( sl_n \)
- cartan_type – (default: 2) the value \( n \), for \( sl_n \), or a Cartan type
- \( r \) – (default: 1) the node index \( r \); typically this corresponds to the height of the vacuum element

EXAMPLES:

We first create an example in \( sl_4 \) (type \( A_3 \)):

```
sage: B = SolitonCellularAutomata('341111122411112223', 4)
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
  initial state:
  34......224....2223
  evolutions: []
  current state:
  34......224....2223
```

We then apply an standard evolution:

```
sage: B.evolve()
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
  initial state:
  34......224....2223
  evolutions: [(1, 19)]
  current state:
  .................34.....224...2223....
```

Next, we apply a smaller carrier evolution. Note that the soliton of size 4 moves only 3 steps:

```
sage: B.evolve(3)
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
  initial state:
  34......224....2223
  evolutions: [(1, 19), (1, 3)]
  current state:
  .................34....224...2223....
```

We can also use carriers corresponding to non-vacuum indices. In these cases, the carrier might not return to its initial state, which results in a message being displayed about the resulting state of the carrier:

```
sage: B.evolve(carrier_capacity=7, carrier_index=3)
Last carrier:
  1 1 1 1 1 1 1
  2 2 2 2 2 3 3
  3 3 3 3 3 4 4
sage: B
```

(continues on next page)
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolutions: [(1, 19), (1, 3), (3, 7)]
current state:
.....................23....222....2223.......

\texttt{sage}: B.evolve(carrier\_capacity=3, carrier\_index=2)
Last carrier:
1  1  1
2  2  3
\texttt{sage}: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolutions: [(1, 19), (1, 3), (3, 7), (2, 3)]
current state:
.........................22.....223...2222........

To summarize our current evolutions, we can use \texttt{print\_states()}:

\texttt{sage}: B.print\_states(5)
t: 0
.........................34......224....2223
 t: 1
.........................34......224...2223....
 t: 2
.........................34....224...2223....
 t: 3
.........................23....222....2223....
 t: 4
.........................22.....223...2222........

To run the SCA further under the standard evolutions, one can use \texttt{print\_states()} or \texttt{latex\_states()}:

\texttt{sage}: B.print\_states(15)
t: 0
..............................................34......224....2223
 t: 1
..............................................34......224...2223....
 t: 2
..............................................34....224...2223....
 t: 3
..............................................23....222....2223....
 t: 4
..............................................22.....223...2222........
 t: 5
..............................................22....223..2222...........
 t: 6
..............................................2222..23...222................
 t: 7
..............................................2222....23..222................
 t: 8
..............................................2222......23.222........................
 t: 9
..............................................2222......23.222................
 t: 10
..............................................2222......23.222................

(continues on next page)
Next, we use $r = 2$ in type $A_3$. Here, we give the data as lists of values corresponding to the entries of the column of height $2$ from the largest entry to smallest. Our columns are drawn in French convention:

```python
sage: B = SolitonCellularAutomata([[4,1],[4,1],[2,1],[2,1],[2,1],[2,1],[3,1],[3,1],[3,2]], 4, 2)
```

We perform 3 evolutions and obtain the following:

```python
sage: B.evolve(number=3)
sage: B
```

Soliton cellular automata of type ['A', 3, 1] and vacuum = 2
initial state:
44 333
11....112
evolutions: [(2, 9), (2, 9), (2, 9)]
current state:
44 333
...11.112........

We construct Example 2.9 from [LS2017]:

```python
sage: B = SolitonCellularAutomata([[2],[-3],[1],[1],[1],[4],[0],[-2],....:[1],[1],[1],[1],[3],[-4],[-3],[-3],[1]], ['D',5,2])
sage: B.print_states(10)
t: 0 _ _ ___
 ..........................23...402....3433.
t: 1 _ _ ___
 ..........................23.402..3433....
t: 2 _ _ ___
 ........................23.402..3433........
t: 3 _ _ ___
 ........................243.02.3433........
t: 4 _ _ ___
 ........................2403..42333........
t: 5 _ _ ___
 ........................2403...442333.....
t: 6 _ _ ___
 ........................2403....442433.....
t: 7 _ _ ___
 ........................2403......442433....
t: 8 _ _ ___
 ........................2403.....442433.....
t: 9 _ _ ___
 ........................2403......442433.....
```

Example 3.4 from [LS2017]:

```python
```

---

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(continued from previous page)
Example 3.12 from [LS2017]:

```
sage: B = SolitonCellularAutomata([[-1,3,2], [3,2,1], [3,2,1], [-3,2,1],
....:                              [-2,-3,1]], ['B',3,1], 3)
sage: B.print_states(6)
```

```
-1 -3-2
 3 2-3
 2 . 1 1
 . . . . . . . . . . . . . . . . . .

-3-1
 . . . . . . 1-3 . . . . . . . . . .

-3-1 -3
 . . . . . . 1 3 . . . . . . . . . .

-3-1 -3
 . . . . . . 1 3 . . . . . . . . . .
```

Example 4.12 from [LS2017]:

```
sage: K = crystals.KirillovReshetikhin(['E',6,1], 1,1, 'KR')
sage: u = K.module_generators[0]
sage: x = u.f_string([1,3,4,5])
sage: y = u.f_string([1,3,4,2,5,6])
sage: a = u.f_string([1,3,4,2])
sage: B = SolitonCellularAutomata([a, u,u, u, x,y], ['E',6,1], 1)
sage: B
```

(continues on next page)
Soliton cellular automata of type ['E', 6, 1] and vacuum = 1

initial state:
(-2, 5) . . . (-5, 2, 6) (-2, -6, 4)
evolution: []
current state:
(-2, 5) . . . (-5, 2, 6) (-2, -6, 4)
sage: B.print_states(8)
t: 0 ...
t: 7 ... (-2, 5) (-2, -5, 4, 6) ... (-6, 2) ...

\texttt{evolve}(\texttt{carrier\_capacity}=None, \texttt{carrier\_index}=None, \texttt{number}=None)

Evolve self.

Time evolution $T_s$ of a SCA state $p$ is determined by

\[ u_{r,s} \otimes T_s(p) = R(p \otimes u_{r,s}) , \]

where $u_{r,s}$ is the maximal element of $B^{r,s}$.

INPUT:

- \texttt{carrier\_capacity} – (default: the number of balls in the system) the size $s$ of carrier
- \texttt{carrier\_index} – (default: the vacuum index) the index $r$ of the carrier
- \texttt{number} – (optional) the number of times to perform the evolutions

To perform multiple evolutions of the SCA, \texttt{carrier\_capacity} and \texttt{carrier\_index} may be lists of the same length.

EXAMPLES:

\begin{verbatim}
sage: B = SolitonCellularAutomata('3411111122411112223', 4)
sage: for k in range(10):
....:     B.evolve()
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolution: [(1, 19), (1, 19), (1, 19), (1, 19), (1, 19),
(1, 19), (1, 19), (1, 19), (1, 19)]
current state:
......2344.......222....23...............................
sage: B.reset()
sage: B.evolve(number=10); B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolution: [(1, 19), (1, 19), (1, 19), (1, 19), (1, 19),
(1, 19), (1, 19), (1, 19), (1, 19)]
current state:
......2344.......222....23...............................
sage: B.reset()
sage: B.evolve(carrier_capacity=[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]); B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
\end{verbatim}
evolutions: [(1, 1), (1, 2), (1, 3), (1, 4), (1, 5),
(1, 6), (1, 7), (1, 8), (1, 9), (1, 10)]
current state:
........2344....222..23..............................
sage: B.reset()
sage: B.evolve(carrier_index=[1, 2, 3])
Last carrier:
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolutuions: [(1, 19), (2, 19), (3, 19)]
current state:
.................................22......223...2222....
sage: B.reset()
sage: B.evolve(carrier_capacity=[1, 2, 3], carrier_index=[1, 2, 3])
Last carrier:
1 1
3 4
Last carrier:
1 1 1
2 2 3
3 3 4
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolutuions: [(1, 1), (2, 2), (3, 3)]
current state:
.....22.......223....2222..
sage: B.reset()
sage: B.evolve(1, 2, number=3)
Last carrier:
1
3
Last carrier:
1
4
Last carrier:
1
3
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224....2223
evolutuions: [(2, 1), (2, 1), (2, 1)]
current state:
.24......222.....2222.

latex_state_evolution (num, scale=1)
Return a latex version of the evolution process of the state num.
See also:

\texttt{state_evolution()}, \texttt{print_state_evolution()}

EXAMPLES:

```python
sage: B = SolitonCellularAutomata('113123', 3)
sage: B.evolve(3)
sage: B.latex_state_evolution(0)
\begin{tikzpicture}[scale=1]
  \node (i0) at (0.0,0.9) {$1$};
  \node (i1) at (2.48,0.9) {$1$};
  \node (i2) at (4.96,0.9) {$3$};
  \draw[->] (i0) -- (i1);
  \draw[->] (i1) -- (i2);
  \draw[->] (i2) -- (i0);
\end{tikzpicture}
sage: B.latex_state_evolution(1)
\begin{tikzpicture}[scale=1]
  \node (i0) at (0.0,0.9) {$1$};
  \node (i1) at (2.48,0.9) {$1$};
  \node (i2) at (4.96,0.9) {$3$};
  \draw[->] (i0) -- (i1);
  \draw[->] (i1) -- (i2);
  \draw[->] (i2) -- (i0);
\end{tikzpicture}
```

latex\_states (\texttt{num=\texttt{None}}, \texttt{as\_array=\texttt{True}}, \texttt{box\_width=\texttt{'}5pt\texttt{')}})

Return a latex version of the states.

INPUT:

- \texttt{num} – the number of states
- \texttt{as\_array} (default: True) if True, then the states are placed inside of an array; if False, then the states are given as a word
- \texttt{box\_width} – (default: '5pt') the width of the . used to represent the vacuum state when \texttt{as\_array} is True

If \texttt{as\_array} is \texttt{False}, then the vacuum element is printed in a gray color. If \texttt{as\_array} is \texttt{True}, then the vacuum is given as .

Use the \texttt{box\_width} to help create more even spacing when a column in the output contains only vacuum elements.

EXAMPLES:

```python
sage: B = SolitonCellularAutomata('411122', 4)
sage: B.latex_states(8)
{\arraycolsep=0.5pt \begin{array}{c|ccccccccccccccccccc}
  t = 0 & \cdots & ... & \makebox[5pt]{.} & 4 & \makebox[5pt]{.} & \makebox[5pt]{.} & 2 & 2 \\
  t = 1 & \cdots & ... & 4 & \makebox[5pt]{.} & \makebox[5pt]{.} & 2 & 2 & ... \\
  t = 2 & \cdots & ... & 4 & \makebox[5pt]{.} & 2 & 2 & ... \\
  t = 3 & \cdots & ... & 4 & 2 & 2 & ... \\
  t = 4 & \cdots & ... & 2 & 4 & 2 & 2 & ... \\
  t = 5 & \cdots & ... & 2 & 4 & \makebox[5pt]{.} & 2 & ... \\
  t = 6 & \cdots & ... & 2 & 4 & \makebox[5pt]{.} & \makebox[5pt]{.} & 2 & ... \\
  \end{array}}
sage: B = SolitonCellularAutomata('511122', 5)
```

(continues on next page)
print_state\hspace{1em}(\texttt{num=\texttt{\textbf{None}}}, \texttt{vacuum\_letter='.'}, \texttt{remove\_trailing\_vacuums=\texttt{False}})\\
Print the state \texttt{num}.
\textbf{INPUT:}\\
\hspace{1em}• \texttt{num} – (default: the current state) the state to print\\
\hspace{1em}• \texttt{vacuum\_letter} – (default: ‘.’) the letter to print for the vacuum\\
\hspace{1em}• \texttt{remove\_trailing\_vacuums} – (default: \texttt{False}) if \texttt{True} then this does not print the vacuum\\
\hspace{1em}letters at the right end of the state

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = SolitonCellularAutomata('3411111122411112223', 4)
sage: B.print_state()
34......224....2223
sage: B.evolve(number=2)
sage: B.print_state(vacuum_letter=',')
,,,,,,,,,,,,,,,,,,,,34,,,,224,,2223,,,,,,,,
sage: B.print_state(10, '_')
______2344_______222____23_______________________________
sage: B.print_state(10, '_', True)
______2344_______222____23
\end{verbatim}

print_state_evolution\hspace{1em}(\texttt{\textbf{num}})\\
Print the evolution process of the state \texttt{num}.

See also:

\texttt{\textbf{state\_evolution()}}, \texttt{\textbf{latex\_state\_evolution()}}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: B = SolitonCellularAutomata('1113123', 3)
sage: B.evolve(3)
sage: B.evolve(3)
sage: B.print_state_evolution(0)
1 1 1 3 1 2 3
   |   |   |   |   |   |
111 ---- 111 ---- 111 ---- 113 ---- 112 ---- 123 ---- 113 ---- 111
   |   |   |   |   |   |
1 1 1 3 2 3 1 1
sage: B.print_state_evolution(1)
1 1 1 3 2 3 1 1
\end{verbatim}
print_states (num=None, vacuum_letter=\'.\')

Print the first num states of self.

**Note:** If the number of states computed for self is less than num, then this evolves the system using the default time evolution.

**INPUT:**
- num – the number of states to print

**EXAMPLES:**

```sage
B = SolitonCellularAutomata([[2],[-1],[1],[1],[1],[1],[2],[2],[3],[2],[1],[1],[1],[1],[1],[1],[2],[3],[3],[-3],[-2]], ['C',3,1])
sage: B.print_states(7)
t: 0 _ _ _ __
   ...............21...2232..21......23332
   ...............21...2232...21....23332....
   ...............21..2232....21..23332..........
   ...............221..232...2231..332............
   ...............221...232.2231....332................
   ...............221...2321223......332................
   ...............221...321...321...321.....332............
   ...............221...321...321...321.....332............

sage: B.print_states(9, ' ')
t: 0 _ _ _
   2 32 223 2333
   2 32 223 2333
   2 32 223 2333
   23 2223 2333
   23 2133 2333
   2233 2223 333
   2233 2223 333
   2233 23223 3
   2233 232 23 3
```

(continues on next page)
Example 4.13 from [Yamada2007]:

```python
sage: B = SolitonCellularAutomata([[3], [3], [1], [1], [1], [1], [2], [2], [2]], ['D', 4, 3])
sage: B.print_states(15)
t: 0
....................................33....222
..................................33...222...
t: 2
................................33..222......
t: 3
..............................33.222..........
t: 4
.........................................33222.2
...............................................30222...........
t: 6
..............................332222...........
t: 7
...............................................33222.2
...............................................33222.
t: 9
...............................................33222.
t: 10

(continues on next page)
```
Example 4.14 from [Yamada2007]:

```
sage: B = SolitonCellularAutomata([[3],[1],[1],[1],[2],[3],[1],[1],[1],[2],
    ..., [3],[3]], ['D',4,3])
sage: B.print_states(15)
t: 0  
...2302..............................
t: 1  
...23322............................
t: 2  
...23322............................
t: 3  
...23322............................
t: 4  
...23322............................
t: 5  
...23322............................
t: 6  
...23322............................
t: 7  
...23322............................
t: 8  
...23322............................
t: 9  
...23322............................
t: 10 
...23322............................
t: 11 
...23322............................
t: 12 
...23322............................
t: 13 
...23322............................
t: 14 
...23322............................
```

```
reset()
Reset self back to the initial state.

EXAMPLES:
```
sage: B = SolitonCellularAutomata('34111111224', 4)
sage: B
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
  initial state:
  34......224
evolutuions: []
current state:
  34......224
sage: B.evolve()
sage: B.evolve()
```
Soliton cellular automata of type ['A', 3, 1] and vacuum = 1
initial state:
34......224
evolutions: [(1, 11), (1, 11), (1, 11), (1, 11)]
current state:
...34..224..........
CHAPTER TWO

PLOTTING OF MANDELBROT AND JULIA SETS

2.1 Mandelbrot and Julia sets

Plots the Mandelbrot and Julia sets for general polynomial maps in the complex plane.

The Mandelbrot set is the set of complex numbers \( c \) for which the map \( f_c(z) \) does not diverge when iterated from \( z = 0 \). This set of complex numbers can be visualized by plotting each value for \( c \) in the complex plane. The Mandelbrot set is often an example of a fractal when plotted in the complex plane. For general one parameter families of polynomials, the mandelbrot set is the parameter values for which the orbits of all critical points remains bounded.

The Julia set for a given parameter \( c \) is the set of complex numbers for which the function \( f_c(z) \) is bounded under iteration.

AUTHORS:

- Ben Barros

\[
\text{sage.dynamics.complex_dynamics.mandel_julia.external_ray}(\theta, **\text{kwds})
\]

Draws the external ray(s) of a given angle (or list of angles) by connecting a finite number of points that were approximated using Newton’s method. The algorithm used is described in a paper by Tomoki Kawahira.

REFERENCE:

[Kaw2009]

INPUT:

- \( \theta \) – double or list of doubles, angles between 0 and 1 inclusive.

kwds:

- \( \text{image} \) – 24-bit RGB image (optional - default: None) user specified image of Mandelbrot set.
- \( D \) – long (optional - default: 25) depth of the approximation. As \( D \) increases, the external ray gets closer to the boundary of the Mandelbrot set. If the ray doesn’t reach the boundary of the Mandelbrot set, increase \( D \).
- \( S \) – long (optional - default: 10) sharpness of the approximation. Adjusts the number of points used to approximate the external ray (number of points is equal to \( S \times D \)). If ray looks jagged, increase \( S \).
- \( R \) – long (optional - default: 100) radial parameter. If \( R \) is large, the external ray reaches sufficiently close to infinity. If \( R \) is too small, Newton’s method may not converge to the correct ray.
- \( \text{prec} \) – long (optional - default: 300) specifies the bits of precision used by the Complex Field when using Newton’s method to compute points on the external ray.
- \( \text{ray_color} \) – RGB color (optional - default: [255, 255, 255]) color of the external ray(s).
**OUTPUT:**
24-bit RGB image of external ray(s) on the Mandelbrot set.

**EXAMPLES:**

```
sage: external_ray(1/3)
500x500px 24-bit RGB image
```

```
sage: external_ray(0.6, ray_color=[255, 0, 0])
500x500px 24-bit RGB image
```

```
sage: external_ray([0, 0.2, 0.4, 0.7])
500x500px 24-bit RGB image
```

```
sage: external_ray([i/5 for i in range(1,5)])
500x500px 24-bit RGB image
```

**WARNING:**
If you are passing in an image, make sure you specify which parameters to use when drawing the external ray. For example, the following is incorrect:

```
sage: M = mandelbrot_plot(x_center=0)  # not tested
sage: external_ray(5/7, image=M)      # not tested
500x500px 24-bit RGB image
```

To get the correct external ray, we adjust our parameters:

```
sage: M = mandelbrot_plot(x_center=0)
sage: external_ray(5/7, x_center=0, image=M)
500x500px 24-bit RGB image
```

**Todo:** The `copy()` function for bitmap images needs to be implemented in Sage.

---

`sage.dynamics.complex_dynamics.mandel_julia.julia_plot(f=None, **kwds)`

Plots the Julia set of a given polynomial $f$. Users can specify whether they would like to display the Mandelbrot side by side with the Julia set with the `mandelbrot` argument. If $f$ is not specified, this method defaults to $f(z) = z^2 - 1$.

The Julia set of a polynomial $f$ is the set of complex numbers $z$ for which the function $f(z)$ is bounded under iteration. The Julia set can be visualized by plotting each point in the set in the complex plane. Julia sets are examples of fractals when plotted in the complex plane.

**ALGORITHM:**

Let $R_c = (1 + \sqrt{1 + 4|c|})/2$ if the polynomial is of the form $f(z) = z^2 + c$; otherwise, let $R_c = 2$. For every $p \in C$, if $|f^k(p)| > R_c$ for some $k \geq 0$, then $f^k(p) \to \infty$. Let $N$ be the maximum number of iterations. Compute the first $N$ points on the orbit of $p$ under $f$. If for any $k < N$, $|f^k(p)| > R_c$, we stop the iteration and assign a color to the point $p$ based on how quickly $p$ escaped to infinity under iteration of $f$. If $|f^i(p)| \leq R_c$ for all $i \leq N$, we assume $p$ is in the Julia set and assign the point $p$ the color black.

**INPUT:**

- $f$ – input polynomial (optional - default: $z^2 - 1$).
- `period` – list (optional - default: `None`), returns the Julia set for a random $c$ value with the given (formal) cycle structure.
• mandelbrot – boolean (optional - default: True), when set to True, an image of the Mandelbrot set is appended to the right of the Julia set.
• point_color – RGB color (optional - default: 'tomato'), color of the point $c$ in the Mandelbrot set (any valid input for Color).
• x_center – double (optional - default: -1.0), Real part of center point.
• y_center – double (optional - default: 0.0), Imaginary part of center point.
• image_width – double (optional - default: 4.0), width of image in the complex plane.
• max_iteration – long (optional - default: 500), maximum number of iterations the map $f(z)$.
• pixel_count – long (optional - default: 500), side length of image in number of pixels.
• base_color – hex color (optional - default: 'steelblue'), color used to determine the coloring of set (any valid input for Color).
• level_sep – long (optional - default: 1), number of iterations between each color level.
• number_of_colors – long (optional - default: 30), number of colors used to plot image.
• interact – boolean (optional - default: False), controls whether plot will have interactive functionality.

OUTPUT:
24-bit RGB image of the Julia set in the complex plane.

Todo:  Implement the side-by-side Mandelbrot-Julia plots for general one-parameter families of polynomials.

EXAMPLES:
The default $f$ is $z^2 - 1$:

```
sage: julia_plot()
1001x500px 24-bit RGB image
```

To display only the Julia set, set mandelbrot to False:

```
sage: julia_plot(mandelbrot=False)
500x500px 24-bit RGB image
```

```
sage: R.<z> = CC[]
sage: f = z^3 - z + 1
sage: julia_plot(f)
500x500px 24-bit RGB image
```

To display an interactive plot of the Julia set in the Notebook, set interact to True. (This is only implemented for polynomials of the form $f = z^2 + c$):

```
sage: julia_plot(interact=True)
interactive(children=(FloatSlider(value=-1.0, description=u'Real c'...
```

```
sage: R.<z> = CC[]
sage: f = z^2 + 1/2
sage: julia_plot(f,interact=True)
interactive(children=(FloatSlider(value=0.5, description=u'Real c'...
```
To return the Julia set of a random $c$ value with (formal) cycle structure $(2, 3)$, set `period = [2, 3]`:

```sage
sage: julia_plot(period=[2,3])
```

To return all of the Julia sets of $c$ values with (formal) cycle structure $(2, 3)$:

```sage
sage: period = [2,3]  # not tested
.....: R.<c> = QQ[]
.....: P.<x,y> = ProjectiveSpace(R,1)
.....: f = DynamicalSystem([x^2+c*y^2, y^2])
.....: L = f.dynatomic_polynomial(period).subs({x:0,y:1}).roots(ring=CC)
.....: c_values = [k[0] for k in L]
.....: for c in c_values:
.....:    julia_plot(c)
```

Polynomial maps can be defined over a polynomial ring or a fraction field, so long as $f$ is polynomial:

```sage
sage: R.<z> = CC[]
sage: f = z^2 - 1
sage: julia_plot(f)
```

```sage
sage: R.<z> = CC[]
sage: K = R.fraction_field(); z = K.gen()
sage: f = z^2-1
sage: julia_plot(f)
```

Interact functionality is not implemented if the polynomial is not of the form $f = z^2 + c$:

```sage
sage: R.<z> = CC[]
sage: f = z^3 + 1
sage: julia_plot(f, interact=True)
Traceback (most recent call last):
...
NotImplementedError: The interactive plot is only implemented for ...
```

`sage.dynamics.complex_dynamics.mandel_julia.kneading_sequence(theta)`

Determines the kneading sequence for an angle theta in $\mathbb{R}/\mathbb{Z}$ which is periodic under doubling. We use the definition for the kneading sequence given in Definition 3.2 of [LS1994].

**INPUT:**
- `theta` – a rational number with odd denominator

**OUTPUT:**
a string representing the kneading sequence of theta in $\mathbb{R}/\mathbb{Z}$

**REFERENCES:**
[LS1994]

**EXAMPLES:**

```sage
sage: kneading_sequence(0)
"*"
```

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Since 1/3 and 7/3 are the same in RR/ZZ, they have the same kneading sequence:

```
sage: kneading_sequence(7/3)
'1*'
```

We can also use (finite) decimal inputs, as long as the denominator in reduced form is odd:

```
sage: kneading_sequence(1.2)
'110*'
```

Since rationals with even denominator are not periodic under doubling, we have not implemented kneading sequences for such rationals:

```
sage: kneading_sequence(1/4)
Traceback (most recent call last):
  ... ValueError: input must be a rational number with odd denominator
```

The family \( f_c(z) \) must have parent \( \mathbb{R} \) of the form \( \mathbb{R} < z, c > = \mathbb{C} \). [Dev2005]

### mandelbrot_plot

Plot of the Mandelbrot set for a one parameter family of polynomial maps.

**REFERENCE:**

[Dev2005]

**INPUT:**

- \( f \) – map (optional - default: \( z^2 + c \)), polynomial family used to plot the Mandelbrot set.
- parameter – variable (optional - default: \( c \)), parameter variable used to plot the Mandelbrot set.
- x_center – double (optional - default: \(-1.0\)), Real part of center point.
- y_center – double (optional - default: \(0.0\)), Imaginary part of center point.
- image_width – double (optional - default: \(4.0\)), width of image in the complex plane.
- max_iteration – long (optional - default: \(500\)), maximum number of iterations the map \( f_c(z) \).
- pixel_count – long (optional - default: \(500\)), side length of image in number of pixels.
- base_color – RGB color (optional - default: \([40, 40, 40]\)) color used to determine the coloring of set.
- level_sep – long (optional - default: \(1\)) number of iterations between each color level.
- number_of_colors – long (optional - default: \(30\)) number of colors used to plot image.
- interact – boolean (optional - default: \(False\)), controls whether plot will have interactive functionality.

**OUTPUT:**

24-bit RGB image of the Mandelbrot set in the complex plane.

**EXAMPLES:**

```
sage: mandelbrot_plot()
500x500px 24-bit RGB image
```
To display an interactive plot of the Mandelbrot in the Notebook, set `interact` to True. (This is only implemented for $z^2 + c$):

```sage
sage: mandelbrot_plot(interact=True)
interactive(children=(FloatSlider(value=0.0, description=u'Real center', max=1.0, min=-1.0, step=1e-05),
FloatSlider(value=0.0, description=u'Imag center', max=1.0, min=-1.0, step=1e-05),
FloatSlider(value=4.0, description=u'Width', max=4.0, min=1e-05, step=1e-05),
IntSlider(value=500, description=u'Iterations', max=1000),
IntSlider(value=500, description=u'Pixels', max=1000, min=10),
IntSlider(value=1, description=u'Color sep', max=20, min=1),
IntSlider(value=30, description=u'# Colors', min=1),
ColorPicker(value='#ff6347', description=u'Base color'), Output()),
_dom_classes=(u'widget-interact',))
```

```sage
sage: mandelbrot_plot(interact=True, x_center=-0.75, y_center=0.25,
....: image_width=1/2, number_of_colors=75)
interactive(children=(FloatSlider(value=-0.75, description=u'Real center', max=1.0, min=-1.0, step=1e-05),
FloatSlider(value=0.25, description=u'Imag center', max=1.0, min=-1.0, step=1e-05),
FloatSlider(value=0.5, description=u'Width', max=4.0, min=1e-05, step=1e-05),
IntSlider(value=500, description=u'Iterations', max=1000),
IntSlider(value=500, description=u'Pixels', max=1000, min=10),
IntSlider(value=1, description=u'Color sep', max=20, min=1),
IntSlider(value=75, description=u'# Colors', min=1),
ColorPicker(value='#ff6347', description=u'Base color'), Output()),
_dom_classes=(u'widget-interact',))
```

Polynomial maps can be defined over a multivariate polynomial ring or a univariate polynomial ring tower:

```sage
sage: R.<z,c> = CC[]
sage: f = z^2 + c
sage: mandelbrot_plot(f)
```

```sage
sage: B.<c> = CC[]
sage: R.<z> = B[]
sage: f = z^5 + c
sage: mandelbrot_plot(f)
```

When the polynomial is defined over a multivariate polynomial ring it is necessary to specify the parameter variable (default parameter is $c$):

```sage
sage: R.<a,b> = CC[]
sage: f = a^2 + b^3
(continues on next page)```
Interact functionality is not implemented for general polynomial maps:

```
sage: R.<z,c> = CC[]
sage: f = z^3 + c
sage: mandelbrot_plot(f, interact=True)
Traceback (most recent call last):
...  
NotImplementedError: Interact only implemented for z^2 + c
```
A *discrete dynamical system* (henceforth DDS) is a pair \((X, \phi)\) of a set \(X\) and a map \(\phi : X \to X\). (This is one of several things known as a “discrete dynamical system” in mathematics.)

This file implements the following classes for discrete dynamical systems:

- **DiscreteDynamicalSystem**: general discrete dynamical system, as above. Inherit from this class if the ground set of your DDS is infinite or large enough that you want to avoid it getting stored as a list. See the doc of this class for further details.

- **FiniteDynamicalSystem**: finite discrete dynamical system. This can be instantiated by calling `DiscreteDynamicalSystem` with the parameter `is_finite` set to `True`.

- **InvertibleDiscreteDynamicalSystem**: invertible discrete dynamical system. This implements an `inverse_evolution` method for \(\phi^{-1}\) (the default implementation simply applies \(\phi\) over and over until the original value is revisited; the last value before that is then taken to be the result). This can be instantiated by calling `DiscreteDynamicalSystem` with the parameter `inverse` provided.

- **InvertibleFiniteDynamicalSystem**: invertible finite discrete dynamical system. This can be instantiated by calling `DiscreteDynamicalSystem` with the parameter `is_finite` set to `True` and the parameter `inverse` provided.

Todo:

- Implement some more functionality for homomesy and invariance testing: Checking invariance on a sublist; computing the first \(k\) entries of an orbit (useful when orbits can be too large); orbits_iterator (for when there are too many orbits to list); etc.

- Further examples for non-auto functionality: e.g., infection on a chessboard; Conway’s game of life.

- Subclasses for DDSes whose ground set is an enumerated set. Should we have those?

- Implement caching for orbits (can be useful: some DDSes have a complicated evolution that shouldn’t be recomputed every time). Does this require a whole new class?

- Further functionality for non-invertible DDSes: `is_recurrent`, `recurrent_entries`, `idempotent_power`, etc.

- Wrap (some of) the `cyclic_sieving_phenomenon.py` methods (`sage.combinat.cyclic_sieving_phenomenon`).

- Interact with sage.dynamics. This requires someone who knows the latter part of the Sage library well.
class sage.dynamics.finite_dynamical_system.DiscreteDynamicalSystem(X, phi, cache_orbits=False, create_tuple=False)

Bases: sage.structure.sage_object.SageObject

A discrete dynamical system.

A discrete dynamical system (henceforth DDS) is a pair \((X, \phi)\) of a set \(X\) and a map \(\phi : X \rightarrow X\). This set \(X\) is called the ground set of the DDS, while the map \(\phi\) is called the evolution of the DDS.

A discrete dynamical system (short: DDS) is a pair \((X, \phi)\) of a set \(X\) and a map \(\phi : X \rightarrow X\). (This is one of several things known as a “discrete dynamical system” in mathematics.) Thus, a DDS is the same as an endomorphism of a set. The DDS is said to be finite if \(X\) is finite. The DDS is said to be invertible if the map \(\phi\) is invertible. The set \(X\) is called the ground set of the DDS; the map \(\phi\) is called the evolution of the DDS; the inverse map \(\phi^{-1}\) (when it exists) is called the inverse evolution of the DDS.

Given a DDS \((X, \phi)\), we can study

• its orbits (i.e., the lists \((s, \phi(s), \phi^2(s), \phi^3(s), \ldots)\) for \(s \in X\)),
• its invariants (i.e., maps \(f : X \rightarrow Y\) satisfying \(f \circ \phi = f\)),
• its cycles (i.e., lists \((u_1, u_2, \ldots, u_k)\) of elements of \(X\) such that \(\phi(u_i) = u_{i+1}\) for each \(i \leq k\), where we set \(u_{k+1} = u_1\)),
• its homomesies (i.e., maps \(h : X \rightarrow A\) to a \(Q\)-vector space \(A\) such that the average of the values of \(h\) on each cycle is the same),

and various other features. (Some of these require \(X\) to be finite or at least to have finite orbits.)

INPUT:

• \(X\) – set, list, tuple, or another iterable, or None (default: None); the ground set for the DDS. This can be None (in which case Sage will not know the ground set, but can still apply evolution to any elements that are provided to it). Make sure to set the create_tuple argument to True if the \(X\) you provide is an iterator or a list, as otherwise your \(X\) would be exposed (and thus subject to mutation or exhaustion).

• \(\phi\) – function, or callable that acts like a function; the evolution of the DDS.

• cache_orbits – boolean (default: False); whether or not the orbits should be cached once they are computed. This currently does nothing, as we are not caching orbits yet.

• create_tuple – boolean (default: False); whether or not the input \(X\) should be translated into a tuple. Set this to True to prevent mutation if \(X\) is a list, and to prevent exhaustion if \(X\) is an iterator.

• inverse – function, or callable that acts like a function, or boolean or None (default: None); the inverse evolution of the DDS, if the DDS is invertible. Set this to None or False if the DDS is not invertible (or you don’t want Sage to treat it as such). Alternatively, by setting this argument to True, you can signal that the DDS is invertible without providing the inverse evolution. (In this case, Sage will compute the inverse, assuming the orbits to be finite.)

• isFinite – boolean or None (default: None); whether the DDS is finite. The default option None leaves this to Sage to decide. Only set this to True if you provide the ground set \(X\).

EXAMPLES:

The following discrete dynamical system is neither finite nor invertible:

```
sage: D = DiscreteDynamicalSystem(NN, lambda x: x + 2)
sage: D.ground_set()  
Non negative integer semiring
sage: D.evolution()(5)
```
The necessity of `create_tuple=True`:

```python
sage: X = [0, 1, 2, 3, 4]
sage: D_wrong = DiscreteDynamicalSystem(X, lambda x: (x**3) % 5)
sage: D_right = DiscreteDynamicalSystem(X, lambda x: (x**3) % 5, create_tuple=True)
sage: X[4] = 666  # evil
sage: D_wrong.ground_set()
[0, 1, 2, 3, 666]
sage: D_right.ground_set()
(0, 1, 2, 3, 4)
```

Here is an invertible (but infinite) discrete dynamical system whose orbits are finite:

```python
sage: D = DiscreteDynamicalSystem(NN, lambda x: (x + 2 if x % 6 < 4 else x - 4), inverse=True)
sage: D.ground_set()
Non negative integer semiring
sage: D.evolution()(5)
1
sage: D.evolution()(1)
3
sage: D.evolution()(3)
5
sage: D.evolution_power(2)(5)
3
sage: D.evolution_power(3)(5)
5
sage: D.evolution_power(-2)(5)
1
sage: D.inverse_evolution()(4)
2
sage: D.orbit(3)
[3, 5, 1]
```

Setting the `inverse` parameter to `None` or `False` would give the same system without the functionality that relies on invertibility:

```python
sage: D = DiscreteDynamicalSystem(NN, lambda x: (x + 2 if x % 6 < 4 else x - 4), inverse=False)
sage: D.ground_set()
Non negative integer semiring
sage: D.evolution()(5)
1
sage: D.inverse_evolution()(4)
Traceback (most recent call last):
  ... AttributeError: 'DiscreteDynamicalSystem' object has no attribute 'inverse_evolution'
sage: D.orbit(3)
(continues on next page)
```
Next, let us try out a finite non-invertible DDS:

```python
sage: D = DiscreteDynamicalSystem(tuple(range(13)), lambda x: (x**2) % 13)
sage: D.ground_set()
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)
sage: D.evolution()(4)
3
sage: D.orbit(4)
[4, 3, 9]
sage: D.orbit(1)
[1]
sage: D.orbit(3)
[3, 9]
```

Note that the finiteness is automatically being inferred here, since the (finite) tuple `tuple(range(13))` has been provided as the ground set.

Finally, here is a finite invertible DDS:

```python
sage: X = cartesian_product([[0, 1]]*8)
sage: Y = [s for s in X if sum(s) == 4]
sage: rot = lambda s : s[1:] + (s[0],)
sage: D = DiscreteDynamicalSystem(Y, rot, inverse=True)
sage: D.evolution()((0, 1, 1, 0, 1, 0, 0, 1))
(1, 1, 0, 1, 0, 0, 1, 0)
sage: D.inverse_evolution()((0, 1, 1, 0, 1, 0, 0, 1))
(1, 0, 1, 1, 0, 1, 0, 0)
sage: sorted(D.orbit_lengths())
[2, 4, 8, 8, 8, 8, 8, 8, 8, 8]
```

We could have just as well provided its inverse explicitly:

```python
sage: rot7 = lambda s: (s[-1],) + s[:-1]
sage: D = DiscreteDynamicalSystem(Y, rot, inverse=rot7)
sage: D.evolution()((0, 1, 1, 0, 1, 0, 0, 1))
(1, 1, 0, 1, 0, 0, 1, 0)
sage: D.inverse_evolution()((0, 1, 1, 0, 1, 0, 0, 1))
(1, 0, 1, 1, 0, 1, 0, 0)
```

### evolution()

Return the evolution of self.
EXAMPLES:

```python
sage: D = DiscreteDynamicalSystem([1, 3, 4], lambda x: (3 if x == 4 else 1), create_tuple=True)
sage: ev = D.evolution()
sage: ev(1)
1
sage: ev(4)
3
```

`evolution_power(n)`

Return the \(n\)-th power (with respect to composition) of the evolution of self.

This requires \(n\) to be a nonnegative integer.

EXAMPLES:

```python
sage: D = DiscreteDynamicalSystem(range(10), lambda x: (x + 3) % 10, create_tuple=True)
sage: ev3 = D.evolution_power(3)
sage: ev3(1)
0
sage: ev3(2)
1
sage: ev0 = D.evolution_power(0)
sage: ev0(1)
1
sage: ev0(2)
2
sage: D.evolution_power(-1)
Traceback (most recent call last):
...
ValueError: the \(n\)-th power of evolution is only defined for nonnegative integers \(n\)
```

`ground_set()`

Return the ground set of self.

This will return None if no ground set was provided in the construction of self.

**Warning:** Unless self has been constructed with the create_tuple parameter set to True, this method will return whatever ground set was provided to the constructor. In particular, if a list was provided, then this precise list will be returned; mutating this list will then corrupt self.

EXAMPLES:

```python
sage: D = DiscreteDynamicalSystem([1, 3, 4], lambda x: (3 if x == 4 else 1), create_tuple=True)
sage: D.ground_set()
(1, 3, 4)
```

`is_homomesic(h, average=None, find_average=False, elements=None)`

Check if \(h\) (a map from the ground set of self to a \(\mathbb{Q}\)-vector space) is homomesic with respect to self.

If the optional argument average is provided, then this also checks that the averages are equal to average.
If the optional argument `find_average` is set to `True`, then this method returns the average of \( h \) in case \( h \) is homomesic (instead of returning `True`).

If the optional argument `elements` (an iterable of elements of the ground set of `self`) is provided, then this method only checks homomesy for the cycles in the orbits of the elements given in the list `elements`. Note that `elements` must be provided if the ground set of `self` is infinite (or cannot be iterated through for any other reason), since there is no way to check all the cycles in this case.

This method will fail to terminate if any element of `elements` has an infinite orbit.

Let us recall the definition of homomesy: Let \((X, \phi)\) be a DDS. A cycle of \((X, \phi)\) is a finite list \( u = (u_1, u_2, \ldots, u_k) \) of elements of \( X \) such that \( \phi(u_i) = u_{i+1} \) for each \( i \leq k \), where we set \( u_{k+1} = u_1 \). Note that any element of \( X \) whose orbit is finite has a cycle in its orbit. Now, let \( h \) be a map from \( X \) to a \( \mathbb{Q} \)-vector space \( A \). If \( u = (u_1, u_2, \ldots, u_k) \) is any cycle of \((X, \phi)\), then the average of \( h \) on this cycle is defined to be the element \( (h(u_1) + h(u_2) + \cdots + h(u_k))/k \) of \( A \). We say that \( h \) is homomesic (with respect to the DDS \((X, \phi)\)) if and only if the averages of \( h \) on all cycles of \((X, \phi)\) are equal.

**EXAMPLES:**

```
sage: W = Words(2, 5)
sage: F = DiscreteDynamicalSystem(W, lambda x: x[1:] + Word([x[0]]), is_finite=True, inverse=True)
sage: F.is_homomesic(lambda w: sum(w))
False
sage: F.is_homomesic(lambda w: 1, average=1)
True
sage: F.is_homomesic(lambda w: 1, average=0)
False
sage: F.is_homomesic(lambda w: 1)
True
sage: F.is_homomesic(lambda w: 1, find_average=True)
1
sage: F.is_homomesic(lambda w: w[0] - w[1], average=0)
True
sage: F.is_homomesic(lambda w: w[0] - w[1], find_average=True)
0
```

Now, let us check homomesy restricted to specific cycles:

```
sage: F = finite_dynamical_systems.bitstring_rotation(7)
sage: descents = lambda x: sum(1 for i in range(6) if x[i] > x[i+1])
sage: F.is_homomesic(descents)
False
sage: F.is_homomesic(descents, elements=[(1, 0, 1, 0, 0, 0, 0), (1, 0, 0, 1, 0, 0, 0)])
True
sage: F.is_homomesic(descents, elements=[(1, 0, 1, 0, 0, 0, 0), (1, 1, 0, 0, 0, 0, 0)])
False
sage: F.is_homomesic(descents, elements=[(1, 0, 1, 0, 0, 0, 0)])
True
sage: F.is_homomesic(descents, elements=[])  # True
```

And here is a non-invertible finite dynamical system:

```
sage: F = finite_dynamical_systems.one_line([9, 1, 1, 6, 5, 4, 5, 5, 1])
sage: F.is_homomesic(lambda i: i)
True
```

(continues on next page)
sage: F.is_homomesic(lambda i: i % 2)
False
sage: F.is_homomesic(lambda i: i % 2, elements=[2, 9, 7])
True
sage: F.is_homomesic(lambda i: i % 2, elements=[2, 9, 4])
False
sage: F.is_homomesic(lambda i: i % 2, elements=[2, 9, 5, 7, 8, 2])
True

orbit (x, preperiod=False)
Return the orbit of the element x of the ground set of self under the evolution \( \phi \) of self.

This orbit is a list beginning with \( x \) and ending with the last element that is not a repetition of a previous element. If the orbit is infinite, then this method does not terminate!

If the optional argument preperiod is set to True, then this method returns a pair \((o, k)\), where \( o \) is the orbit of self, while \( k \) is the smallest nonnegative integer such that \( \phi^k(x) \in \{ \phi^i(x) \mid i > k \} \).

The orbit of the element \( x \) is also called the “rho” of \( x \), due to its shape when it is depicted as a directed graph.

EXAMPLES:

```python
sage: D = DiscreteDynamicalSystem(tuple(range(11)), lambda x: (x**2) % 11)
sage: D.orbit(6)
[6, 3, 9, 4, 5]
sage: D.orbit(6, preperiod=True)
([6, 3, 9, 4, 5], 1)
sage: D.orbit(3)
[3, 9, 4, 5]
sage: D.orbit(3, preperiod=True)
([3, 9, 4, 5], 0)
sage: D.orbit(9)
[9, 4, 5, 3]
sage: D.orbit(0)
[0]
```

class sage.dynamics.finite_dynamical_system.FiniteDynamicalSystem (X, \( \phi \),
cache_orbits=False,
create_tuple=False)

Bases: sage.dynamics.finite_dynamical_system.DiscreteDynamicalSystem

A finite discrete dynamical system.

A finite discrete dynamical system (henceforth FDDS) is a pair \((X, \phi)\) of a finite set \( X \) and a map \( \phi : X \to X \). This set \( X \) is called the ground set of the FDDS, while the map \( \phi \) is called the evolution of the FDDS.

The ground set \( X \) should always be provided as an iterable when defining a `FiniteDynamicalSystem`.

EXAMPLES:

```python
sage: from sage.dynamics.finite_dynamical_system import FiniteDynamicalSystem
sage: D = FiniteDynamicalSystem(tuple(range(11)), lambda x: (x**2) % 11)
sage: D.ground_set()
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
sage: D.evolution()(4)
5
sage: D.orbit(4)
```
sage: D.orbit(1)
[1]
sage: D.orbit(2)
[2, 4, 5, 3, 9]

sage: X = cartesian_product([[0, 1]]*8)
sage: Y = [s for s in X if sum(s) == 4]
sage: rot = lambda s: s[1:] + (0,)
sage: D = FiniteDynamicalSystem(Y, rot)
sage: D.evolution()((1, 1, 1, 0, 1, 0, 0, 1))
(1, 1, 0, 1, 0, 0, 1, 0)

cycles()

Return a list of all cycles of self, up to cyclic rotation.

We recall the definition of cycles: Let \((X, \phi)\) be a DDS. A cycle of \((X, \phi)\) is a finite list \(u = (u_1, u_2, \ldots, u_k)\) of elements of \(X\) such that \(\phi(u_i) = u_{i+1}\) for each \(i \leq k\), where we set \(u_{k+1} = u_1\). Note that any element of \(X\) whose orbit is finite has a cycle in its orbit.

EXAMPLES:

sage: BS = finite_dynamical_systems.bulgarian_solitaire
sage: BS(8).cycles()
[[[4, 3, 1], [3, 3, 2], [3, 2, 2, 1], [4, 2, 1, 1]],
 [[[4, 2, 2], [3, 3, 1, 1]]]
sage: BS(6).cycles()
[[[3, 2, 1]]]
sage: D = DiscreteDynamicalSystem(tuple(range(6)),
lambda x: (x + 2) % 6)
sage: D.cycles()
[[[5, 1, 3], [4, 0, 2]]]
sage: D = DiscreteDynamicalSystem(tuple(range(6)),
lambda x: (x ** 2) % 6)
sage: D.cycles()
[[[1], [4], [3], [0]]]
sage: D = DiscreteDynamicalSystem(tuple(range(11)),
lambda x: (x ** 2 - 1) % 11)
sage: D.cycles()
[[[10, 0], [8], [4]]]
sage: F = finite_dynamical_systems.one_line([4, 7, 2, 6, 2, 10, 9, 11, 5, 6, ˓→12, 12, 12, 6])
sage: F.cycles()
[[[6, 10], [12], [9, 5, 2, 7]]]

is_invariant (f)

Check if \(f\) is an invariant of self.

Let \((X, \phi)\) be a discrete dynamical system. Let \(Y\) be any set. Let \(f : X \to Y\) be any map. Then, we say that \(f\) is an invariant of \((X, \phi)\) if and only if \(f \circ \phi = f\).

EXAMPLES:

sage: W = Words(2, 5)
sage: F = DiscreteDynamicalSystem(W, lambda x: x[1:] + Word([x[0]]), is_ ˓→finite=True)
sage: F.is_invariant(lambda w: sum(w))
True
sage: F.is_invariant(lambda w: 1)
True
sage: F.is_invariant(lambda w: w[0] - w[1])
False
sage: F.is_invariant(lambda w: sum(i**2 for i in w))
True

Invariants and non-invariants of a permutation:

sage: F = finite_dynamical_system.permutation([3, 4, 5, 6, 1, 2])
sage: F.is_invariant(lambda i: i % 2)
True
sage: F.is_invariant(lambda i: i % 3)
False
sage: F.is_invariant(lambda i: i > 1)
False
sage: F.is_invariant(lambda i: i % 2 == 0)
True

class sage.dynamics.finite_dynamical_system.InvertibleDiscreteDynamicalSystem(X, phi, inverse=None, cache_orbits=False, create_tuple=False)

Bases: sage.dynamics.finite_dynamical_system.DiscreteDynamicalSystem

An invertible discrete dynamical system.

A discrete dynamical system (henceforth DDS) is a pair \((X, \phi)\) of a set \(X\) and a map \(\phi : X \to X\). This set \(X\) is called the ground set of the DDS, while the map \(\phi\) is called the evolution of the DDS. An invertible DDS is a DDS \((X, \phi)\) whose evolution \(\phi\) is invertible. In that case, \(\phi^{-1}\) is called the inverse evolution of the DDS.

See DiscreteDynamicalSystem for details.

INPUT:

- \(X\) – set, list, tuple, or another iterable, or None; the ground set for the DDS. This can be None in case of a DiscreteDynamicalSystem or a InvertibleDiscreteDynamicalSystem. Make sure to set the create_tuple argument to True if you provide an iterator or a list for \(X\), as otherwise the input would be exposed.
- \(\phi\) – function, or callable that acts like a function; the evolution of the DDS.
- \(\text{inverse}\) – function, or callable that acts like a function; the inverse evolution of the DDS. (A default implementation is implemented when this argument is not provided; but it assumes the orbits to be finite.)
- \(\text{cache_orbits}\) – boolean (default: False); whether or not the orbits should be cached once they are computed.
- \(\text{create_tuple}\) – boolean (default: False); whether or not the input \(X\) should be translated into a tuple (set this to True to prevent mutation if \(X\) is a list, and to prevent exhaustion if \(X\) is an iterator).

EXAMPLES:

sage: from sage.dynamics.finite_dynamical_system import InvertibleDiscreteDynamicalSystem
sage: D = InvertibleDiscreteDynamicalSystem(NN, lambda x: (x + 2 if x % 4 < 2 else x - 2))
sage: D.ground_set()  
Non negative integer semiring
sage: D.evolution()(5)  
7
sage: D.evolution()(6)  
4
sage: D.evolution()(4)  
6
sage: D.inverse_evolution()(4)  
6

The necessity of \texttt{create\_tuple=True}:

\begin{Verbatim}
\texttt{sage: X = [0, 1, 2, 3, 4]}
\texttt{sage: D\_wrong = InvertibleDiscreteDynamicalSystem(X, \texttt{lambda} x: (x**3) \% 5)}
\texttt{sage: D\_right = InvertibleDiscreteDynamicalSystem(X, \texttt{lambda} x: (x**3) \% 5, create\_\rightarrow\texttt{tuple=True})}
\texttt{sage: X[4] = 666 \# evil}
\texttt{sage: D\_wrong.ground_set()}  \[0, 1, 2, 3, 666]\n\texttt{sage: D\_right.ground_set()}  \(0, 1, 2, 3, 4)
\end{Verbatim}

\texttt{evolution\_power (n)}

Return the $n$-th power (with respect to composition) of the evolution of \texttt{self}.

This requires $n$ to be an integer.

\textbf{EXAMPLES:}

\begin{Verbatim}
\texttt{sage: D = DiscreteDynamicalSystem(range(10), \texttt{lambda} x: (x + 3) \% 10, create\_\rightarrow\texttt{tuple=True}, inverse=True)}
\texttt{sage: ev3 = D.evolution\_power(3)}
\texttt{sage: ev3(1)}  \(0)
\texttt{sage: ev3(2)}  \(1)
\texttt{sage: ev0 = D.evolution\_power(0)}
\texttt{sage: ev0(1)}  \(1)
\texttt{sage: ev0(2)}  \(2)
\texttt{sage: evm1 = D.evolution\_power(-1)}
\texttt{sage: evm1(1)}  \(8)
\texttt{sage: evm1(2)}  \(9)
\texttt{sage: evm2 = D.evolution\_power(-2)}
\texttt{sage: evm2(1)}  \(5)
\texttt{sage: evm2(2)}  \(6)
\end{Verbatim}

\texttt{inverse\_evolution()}  

Return the inverse evolution of \texttt{self} (as a map from the ground set of \texttt{self} to itself).

\textbf{EXAMPLES:}
```python
sage: D = DiscreteDynamicalSystem(tuple(range(8)), lambda x: (x + 2) % 8, inverse=True)
sage: D.inverse_evolution()(1)
7
sage: D.inverse_evolution()(3)
1

sage: D = DiscreteDynamicalSystem(ZZ, lambda x: (x + 2) % 8, inverse=True)
sage: D.inverse_evolution()(1)
7
sage: D.inverse_evolution()(3)
1
```

**inverse_evolution_default**(x)

Return the inverse evolution of `self`, applied to the element `x` of the ground set of `self`.

This is the default implementation, assuming that the orbit of `x` is finite.

**EXAMPLES:**

```python
sage: D = DiscreteDynamicalSystem(tuple(range(8)), lambda x: (x + 2) % 8, inverse=True)
sage: D.inverse_evolution_default(1)
7
sage: D.inverse_evolution_default(3)
1
```

**orbit**(x, preperiod=False)

Return the orbit of the element `x` of the ground set of `self`.

This orbit is a list beginning with `x` and ending with the last element until `x` reappears. If `x` never reappears, then this will not terminate!

If the optional argument `preperiod` is set to `True`, then this method returns a pair `(o, k)`, where `o` is the orbit of `self`, while `k` is the smallest nonnegative integer such that \(\phi^k(x) \in \{\phi^i(x) \mid i > k\}\). Note that `k` is necessarily 0, since the DDS `self` is invertible!

**EXAMPLES:**

```python
sage: D = DiscreteDynamicalSystem(tuple(range(8)), lambda x: (x + 2) % 8, inverse=True)
sage: D.ground_set()
(0, 1, 2, 3, 4, 5, 6, 7)
sage: D.orbit(2)
[2, 4, 6, 0]
sage: D.orbit(5)
[5, 7, 1, 3]
sage: D.orbit(5, preperiod=True)
([5, 7, 1, 3], 0)
```

sage: D = DiscreteDynamicalSystem(ZZ, lambda x: (x + 2) % 8, inverse=True)
sage: D.ground_set()
Integer Ring
sage: D.orbit(2)
[2, 4, 6, 0]
sage: D.orbit(5)
[5, 7, 1, 3]
sage: D.orbit(5, preperiod=True)
([5, 7, 1, 3], 0)
```
**verify_inverse_evolution**(x=None)
Verify that the composition of evolution and inverse evolution on self is the identity (both ways).

The optional argument x, if provided, restricts the testing to the element x only. Otherwise, all elements of the ground set are tested (if they can be enumerated).

This is mostly used to check the correctness of self-implemented inverse evolution methods.

**EXAMPLES:**

```python
sage: D = DiscreteDynamicalSystem(tuple(range(8)), lambda x: (x + 2) % 8, ..., inverse=True)
sage: D.verify_inverse_evolution()
True
sage: D.verify_inverse_evolution(3)
True
sage: fake_inverse = lambda x: x
sage: D = DiscreteDynamicalSystem(tuple(range(8)), lambda x: (x + 2) % 8, ..., inverse=fake_inverse)
sage: D.verify_inverse_evolution()
False
sage: D.verify_inverse_evolution(3)
False
```

**class** sage.dynamics.finite_dynamical_system.InvertibleFiniteDynamicalSystem(X, phi, inverse=None, cache_orbits=False, create_tuple=False)

Bases: sage.dynamics.finite_dynamical_system.InvertibleDiscreteDynamicalSystem,
        sage.dynamics.finite_dynamical_system.FiniteDynamicalSystem

An invertible finite discrete dynamical system.

A finite discrete dynamical system (henceforth FDDS) is a pair \((X, \phi)\) of a finite set \(X\) and a map \(\phi : X \rightarrow X\). This set \(X\) is called the ground set of the FDDS, while the map \(\phi\) is called the evolution of the FDDS. An FDDS \((X, \phi)\) is called invertible if the map \(\phi\) is invertible; in this case, \(\phi^{-1}\) is called the inverse evolution of the FDDS.

The ground set \(X\) should always be provided as an iterable when defining a FiniteDynamicalSystem.

**EXAMPLES:**

```python
sage: from sage.dynamics.finite_dynamical_system import InvertibleFiniteDynamicalSystem
sage: D = InvertibleFiniteDynamicalSystem(tuple(range(5)), lambda x: (x + 2) % 5)
sage: D.ground_set()
(0, 1, 2, 3, 4)
sage: D.evolution()(4)
1
sage: D.orbits()
[[4, 1, 3, 0, 2]]
sage: D.inverse_evolution()(2)
0
sage: D.inverse_evolution()(1)
4
sage: D.evolution_power(-1)(1)
4
sage: D.evolution_power(-2)(1)
(continues on next page)```
sage: X = cartesian_product([[0, 1]]*8)
sage: Y = [s for s in X if sum(s) == 4]
sage: rot = lambda s: s[1:] + (s[0],)
sage: D = InvertibleFiniteDynamicalSystem(Y, rot)
sage: D.evolution()((0, 1, 1, 0, 1, 0, 0, 1))
(1, 1, 0, 1, 0, 0, 1, 0)
sage: D.inverse_evolution()((0, 1, 1, 0, 1, 0, 0, 1))
(1, 0, 1, 1, 0, 1, 0, 0)
sage: sorted(D.orbit_lengths())
[2, 4, 8, 8, 8, 8, 8, 8, 8, 8]

cycles()
Return a list of all cycles of self, up to cyclic rotation.

We recall the definition of cycles: Let \((X, \phi)\) be a DDS. A cycle of \((X, \phi)\) is a finite list \(u = (u_1, u_2, \ldots, u_k)\) of elements of \(X\) such that \(\phi(u_i) = u_{i+1}\) for each \(i \leq k\), where we set \(u_{k+1} = u_1\). Note that any element of \(X\) whose orbit is finite has a cycle in its orbit.

Since self is invertible, the cycles of self are the same as its orbits.

EXAMPLES:

sage: D = DiscreteDynamicalSystem(tuple(range(6)), lambda x: (x + 2) % 6, inverse=True)
sage: D.cycles()
[[5, 1, 3], [4, 0, 2]]
sage: D = DiscreteDynamicalSystem(tuple(range(6)), lambda x: (x + 3) % 6, inverse=True)
sage: D.cycles()
[[5, 2], [4, 1], [3, 0]]

orbit_lengths()
Return a list of the lengths of all orbits of self.

EXAMPLES:

sage: D = DiscreteDynamicalSystem(tuple(range(6)), lambda x: (x + 2) % 6, inverse=True)
sage: D.orbit_lengths()
[3, 3]
sage: D = DiscreteDynamicalSystem(tuple(range(6)), lambda x: (x + 3) % 6, inverse=True)
sage: D.orbit_lengths()
[2, 2, 2]

orbits()
Return a list of all orbits of self, up to cyclic rotation.

EXAMPLES:

sage: D = DiscreteDynamicalSystem(tuple(range(6)), lambda x: (x + 2) % 6, inverse=True)
sage: D.orbits()
[[5, 1, 3], [4, 0, 2]]
sage: D = DiscreteDynamicalSystem(tuple(range(6)), lambda x: (x + 3) % 6, inverse=True)

(continues on next page)
\begin{verbatim}
sage: D.orbits()
[[5, 2], [4, 1], [3, 0]]
\end{verbatim}
SANDPILES

Functions and classes for mathematical sandpiles.

Version: 2.4

AUTHOR:

- David Perkinson (June 4, 2015) Upgraded from version 2.3 to 2.4.

MAJOR CHANGES

1. Eliminated dependence on 4ti2, substituting the use of Polyhedron methods. Thus, no optional packages are necessary.
2. Fixed bug in Sandpile.__init__ so that now multigraphs are handled correctly.
3. Created sandpiles to handle examples of Sandpiles in analogy with graphs.simplicial_complexes, and polytopes. In the process, we implemented a much faster way of producing the sandpile grid graph.
4. Added support for open and closed sandpile Markov chains.
5. Added support for Weierstrass points.
6. Implemented the Cori-Le Borgne algorithm for computing ranks of divisors on complete graphs.

NEW METHODS

- Sandpile: avalanche_polynomial, genus, group_gens, help, jacobian_representatives, markov_chain, p-card_representatives, smith_form, stable_configs, stationary_density, tutte_polynomial.
- SandpileConfig: burst_size, help.
- SandpileDivisor: help, is_linearly_equivalent, is_q_reduced, is_weierstrass_pt, polytope, polytope_integer_pts, q_reduced, rank, simulate_threshold, stabilize, weierstrass_div, weierstrass_gap_seq, weierstrass_pts, weierstrass_rank_seq.

MINOR CHANGES

- The sink argument to Sandpile.__init__ now defaults to the first vertex.
- A SandpileConfig or SandpileDivisor may now be multiplied by an integer.
- Sped up __add__ method for SandpileConfig and SandpileDivisor.
- Enhanced string representation of a Sandpile (via __repr__ and the name methods).
- Recurrents for complete graphs and cycle graphs are computed more quickly.
- The stabilization code for SandpileConfig has been made more efficient.
- Added optional probability distribution arguments to add_random methods.
EXAMPLES:

For general help, enter `Sandpile.help()`, `SandpileConfig.help()`, and `SandpileDivisor.help()`.

Miscellaneous examples appear below.

A weighted directed graph given as a Python dictionary:

```python
sage: from sage.sandpiles import *
sage: g = {0: {},
....: 1: {0: 1, 2: 1, 3: 1},
....: 2: {1: 1, 3: 1, 4: 1},
....: 3: {1: 1, 2: 1, 4: 1},
....: 4: {2: 1, 3: 1})
```

The associated sandpile with 0 chosen as the sink:

```python
sage: S = Sandpile(g,0)
```

Or just:

```python
sage: S = Sandpile(g)
```

A picture of the graph:

```python
sage: S.show() # long time
```

The relevant Laplacian matrices:

```python
sage: S.laplacian()
[ 0 0 0 0 0]
[-1 3 -1 -1 0]
[ 0 -1 3 -1 -1]
[ 0 -1 -1 3 -1]
[ 0 0 -1 -1 2]
sage: S.reduced_laplacian()
[ 3 -1 -1 0]
[-1 3 -1 -1]
[-1 -1 3 -1]
[ 0 -1 -1 2]
```

The number of elements of the sandpile group for S:

```python
sage: S.group_order()
8
```
The structure of the sandpile group:

```sage
S.invariant_factors()
[1, 1, 1, 8]
```

The elements of the sandpile group for S:

```sage
S.recurrents()
[(1: 2, 2: 2, 3: 2, 4: 1),
 (1: 2, 2: 2, 3: 2, 4: 0),
 (1: 2, 2: 1, 3: 2, 4: 0),
 (1: 2, 2: 2, 3: 0, 4: 1),
 (1: 2, 2: 0, 3: 2, 4: 1),
 (1: 2, 2: 2, 3: 1, 4: 0),
 (1: 2, 2: 1, 3: 2, 4: 1),
 (1: 2, 2: 2, 3: 1, 4: 1)]
```

The maximal stable element (2 grains of sand on vertices 1, 2, and 3, and 1 grain of sand on vertex 4:

```sage
S.max_stable()
{1: 2, 2: 2, 3: 2, 4: 1}
S.max_stable().values()
[2, 2, 2, 1]
```

The identity of the sandpile group for S:

```sage
S.identity()
{1: 2, 2: 2, 3: 2, 4: 0}
```

An arbitrary sandpile configuration:

```sage
c = SandpileConfig(S, [1, 0, 4, -3])
c.equivalent_recurrent()
{1: 2, 2: 2, 3: 2, 4: 0}
```

Some group operations:

```sage
m = S.max_stable()
i = S.identity()
m.values()
[2, 2, 2, 1]
i.values()
[2, 2, 2, 0]
m + i  # coordinate-wise sum
{1: 4, 2: 4, 3: 4, 4: 1}
m - i
{1: 0, 2: 0, 3: 0, 4: 1}
m  &  i  # add, then stabilize
{1: 2, 2: 2, 3: 2, 4: 1}
e = m + m
e
{1: 4, 2: 4, 3: 4, 4: 2}
~e  # stabilize
{1: 2, 2: 2, 3: 2, 4: 0}
a = ~m
a  &  m
{1: 0, 2: 0, 3: 0, 4: 0}
a * m  # add, then find the equivalent recurrent
```
Firing an unstable vertex returns resulting configuration:

```
sage: c = S.max_stable() + S.identity()
sage: c.fire_vertex(1)
{1: 1, 2: 5, 3: 5, 4: 1}
sage: c
{1: 4, 2: 4, 3: 4, 4: 1}
```

Fire all unstable vertices:

```
sage: c.unstable()
[1, 2, 3]
sage: c.fire_unstable()
{1: 3, 2: 3, 3: 3, 4: 3}
```

Stabilize c, returning the resulting configuration and the firing vector:

```
sage: c.stabilize(True)
[{1: 2, 2: 2, 3: 2, 4: 1}, {1: 6, 2: 8, 3: 8, 4: 8}]
sage: c
{1: 4, 2: 4, 3: 4, 4: 1}
sage: S.max_stable() & S.identity() == c.stabilize()
True
```

The number of superstable configurations of each degree:

```
sage: S.h_vector()
[1, 3, 4]
sage: S.postulation()
2
```

the saturated homogeneous toppling ideal:

```
sage: S.ideal()
Ideal (x1 - x0, x3*x2 - x0^2, x4^2 - x0^2, x2^3 - x4*x3*x0, x4*x2^2 - x3^2*x0, x3^3 - x4*x2*x0, x4*x3^2 - x2^2*x0) of Multivariate Polynomial Ring in x4, x3, x2, x1, x0 over Rational Field
```

its minimal free resolution:

```
sage: S.resolution()
'R^1 <-- R^7 <-- R^15 <-- R^13 <-- R^4'
```

and its Betti numbers:

```
sage: S.betti()
0 1 2 3 4
------------------------------------
```

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Some various ways of creating Sandpiles:

```python
sage: S = sandpiles.Complete(4) # for more options enter `''sandpile.TAB''`
sage: S = sandpiles.Wheel(6)
```

A multidigraph with loops (vertices 0, 1, 2; for example, there is a directed edge from vertex 2 to vertex 1 of weight 3, which can be thought of as three directed edges of the form (2,3). There is also a single loop at vertex 2 and an edge (2,0) of weight 2):

```python
sage: S = Sandpile({0:[1,2], 1:[0,0,2], 2:[0,0,1,1,1,2], 3:[2]})
```

Using the graph library (vertex 1 is specified as the sink; omitting this would make the sink vertex 0 by default):

```python
sage: S = Sandpile(graphs.PetersenGraph(),1)
```

Distribution of avalanche sizes:

```python
sage: S = sandpiles.Grid(10,10)
sage: m = S.max_stable()
sage: a = []
sage: for i in range(1000):
    m = m.add_random()
    m, f = m.stabilize(True)
    a.append(sum(f.values()))
sage: p = list_plot([[log(i+1),log(a.count(i))] for i in [0..max(a)] if a.count(i)])
sage: t = text("Distribution of avalanche sizes", (2,2), rgbcolor=(1,0,0))
sage: show(p+t,axes_labels=['log(N)','log(D(N))'])
```

Working with sandpile divisors:

```python
sage: S = sandpiles.Complete(4)
sage: D = SandpileDivisor(S, [0,0,0,5])
sage: E = D.stabilize(); E
{0: 1, 1: 1, 2: 1, 3: 2}
sage: D.is_linearly_equivalent(E)
True
sage: D.q_reduced()
{0: 4, 1: 0, 2: 0, 3: 1}
sage: S = sandpiles.Complete(4)
sage: D = SandpileDivisor(S, [0,0,0,5])
sage: E = D.stabilize(); E
{0: 1, 1: 1, 2: 1, 3: 2}
sage: D.is_linearly_equivalent(E)
True
sage: D.q_reduced()
{0: 4, 1: 0, 2: 0, 3: 1}
sage: D.rank()
2
sage: sorted(D.effective_div(), key=str)
```

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\[(0: 0, 1: 0, 2: 0, 3: 5),
(0: 0, 1: 0, 2: 4, 3: 1),
(0: 0, 1: 4, 2: 0, 3: 1),
(0: 1, 1: 1, 2: 1, 3: 2),
(0: 4, 1: 0, 2: 0, 3: 1)\]
sage: sorted(D.effective_div(False))
[[0, 0, 0, 5], [0, 0, 4, 1], [0, 4, 0, 1], [1, 1, 1, 2], [4, 0, 0, 1]]
sage: D.rank()
2
sage: D.rank(True)
(2, {0: 2, 1: 1, 2: 0, 3: 0})
E = D.rank(True)[1]  # E proves the rank is not 3
sage: E.values()
[2, 1, 0, 0]
sage: E.deg()
3
sage: rank(D - E)
-1
sage: (D - E).effective_div()
[]
sage: D.weierstrass_pts()
(0, 1, 2, 3)
sage: D.weierstrass_rank_seq(0)
(2, 1, 0, 0, 0, -1)
sage: D.weierstrass_pts()
(0, 1, 2, 3)
sage: D.weierstrass_rank_seq(0)
(2, 1, 0, 0, 0, -1)

\textbf{class} \texttt{sage.sandpiles.sandpile.Sandpile}(g, sink=None)
\textbf{Bases:} \texttt{sage.graphs.digraph.DiGraph}

Class for Dhar's abelian sandpile model.

\textbf{all\_k\_config}(k)

The constant configuration with all values set to \(k\).

\textbf{INPUT:}

k – integer

\textbf{OUTPUT:}

SandpileConfig

\textbf{EXAMPLES:}

\texttt{sage: s = sandpiles.Diamond()}
\texttt{sage: s.all_k_config(7)}
\{1: 7, 2: 7, 3: 7\}

\textbf{all\_k\_div}(k)

The divisor with all values set to \(k\).

\textbf{INPUT:}

k – integer

\textbf{OUTPUT:}

SandpileDivisor
EXAMPLES:

```python
sage: S = sandpiles.House()
sage: S.all_k_div(7)
{0: 7, 1: 7, 2: 7, 3: 7, 4: 7}
```

`avalanche_polynomial(multivariable=True)`

The avalanche polynomial. See NOTE for details.

**INPUT:**
- `multivariable` – (default: True) boolean

**OUTPUT:**
- polynomial

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(4)
sage: s.avalanche_polynomial()
9*x0*x1*x2 + 2*x0*x1 + 2*x0*x2 + 2*x1*x2 + 3*x0 + 3*x1 + 3*x2 + 24
sage: s.avalanche_polynomial(False)
9*x0^3 + 6*x0^2 + 9*x0 + 24
```

**Note:** For each nonsink vertex \( v \), let \( x_v \) be an indeterminate. If \( (r, v) \) is a pair consisting of a recurrent \( r \) and nonsink vertex \( v \), then for each nonsink vertex \( w \), let \( n_w \) be the number of times vertex \( w \) fires in the stabilization of \( r + v \). Let \( M(r, v) \) be the monomial \( \prod_{w} x_v^{n_w} \), i.e., the exponent records the vector of \( n_w \) as \( w \) ranges over the nonsink vertices. The avalanche polynomial is then the sum of \( M(r, v) \) as \( r \) ranges over the recurrents and \( v \) ranges over the nonsink vertices. If `multivariable` is False, then set all the indeterminates equal to each other (and, thus, only count the number of vertex firings in the stabilizations, forgetting which particular vertices fired).

`betti(verbosetrue)`

The Betti table for the homogeneous toppling ideal. If `verbose` is True, it prints the standard Betti table, otherwise, it returns a less formatted table.

**INPUT:**
- `verbose` – (default: True) boolean

**OUTPUT:**
- Betti numbers for the sandpile

**EXAMPLES:**

```python
sage: S = sandpiles.Diamond()
sage: S.betti()
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0:</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1:</td>
<td>-</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>2:</td>
<td>-</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>total:</td>
<td>1</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>
sage: S.betti(False)
[1, 6, 9, 4]
```
betti_complexes()
The support-complexes with non-trivial homology. (See NOTE.)

OUTPUT:
list (of pairs [divisors, corresponding simplicial complex])

EXAMPLES:

```python
sage: S = Sandpile({0:{}, 1:{0: 1, 2: 1, 3: 4}, 2:{3: 5}, 3:{1: 1, 2: 1}}, 0)
sage: p = S.betti_complexes()
sage: p[0]
[{-8: 1, 1: 5, 2: 4, 3: 1}, Simplicial complex with vertex set (1, 2, 3) and facets {{3}, {1, 2}}]
sage: S.resolution()
'\texttt{R}^1 \leftarrow \texttt{R}^5 \leftarrow \texttt{R}^5 \leftarrow \texttt{R}^1'
sage: S.betti()
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\hline
\vdots & & & \\
0 & 1 & - & - \\
1 & - & 5 & 5 & - \\
2 & - & - & - & 1 \\
\hline
\text{total} & 1 & 5 & 5 & 1
\end{array}
sage: len(p)
11
sage: p[0][1].homology()
{0: 2, 1: 0}
sage: p[-1][1].homology()
{0: 0, 1: 0, 2: 2}
```

Note: A support-complex is the simplicial complex formed from the supports of the divisors in a linear system.

burning_config()
The minimal burning configuration.

OUTPUT:
dict (configuration)

EXAMPLES:

```python
sage: g = {0:{}, 1:{0: 1, 3: 1, 4: 1}, 2:{0: 1, 3: 1, 5: 1}, ...
....: 3:{2: 1, 5: 1}, 4:{1: 1, 3: 1}, 5:{2: 1, 3: 1}}
sage: S = Sandpile(g, 0)
sage: S.burning_config()
{1: 2, 2: 0, 3: 1, 4: 1, 5: 0}
sage: S.burning_config().values()
[2, 0, 1, 1, 0]
sage: S.burning_script()
{1: 1, 2: 3, 3: 5, 4: 1, 5: 4}
sage: script = S.burning_script().values()
sage: script
[1, 3, 5, 1, 4]
sage: matrix(script)*S.reduced_laplacian()
[2 0 1 1 0]
```
Note: The burning configuration and script are computed using a modified version of Speer’s script algorithm. This is a generalization to directed multigraphs of Dhar’s burning algorithm.

A burning configuration is a nonnegative integer-linear combination of the rows of the reduced Laplacian matrix having nonnegative entries and such that every vertex has a path from some vertex in its support. The corresponding burning script gives the integer-linear combination needed to obtain the burning configuration. So if \( b \) is the burning configuration, \( \sigma \) is its script, and \( \tilde{L} \) is the reduced Laplacian, then \( \sigma \cdot \tilde{L} = b \). The minimal burning configuration is the one with the minimal script (its components are no larger than the components of any other script for a burning configuration).

The following are equivalent for a configuration \( c \) with burning configuration \( b \) having script \( \sigma \):

- \( c \) is recurrent;
- \( c + b \) stabilizes to \( c \);
- the firing vector for the stabilization of \( c + b \) is \( \sigma \).

**burning_script()**

A script for the minimal burning configuration.

OUTPUT:

dict

EXAMPLES:

```
sage: g = {0:{}, 1:{0:1,3:1,4:1}, 2:{0:1,3:1,5:1},
....:  3:{2:1,5:1}, 4:{1:1,3:1}, 5:{2:1,3:1}}
sage: S = Sandpile(g,0)
sage: S.burning_config()
{1: 2, 2: 0, 3: 1, 4: 1, 5: 0}
sage: S.burning_config().values()
[2, 0, 1, 1, 0]
sage: S.burning_script()
{1: 1, 2: 3, 3: 5, 4: 1, 5: 4}
sage: script = S.burning_script().values()
sage: script
[1, 3, 5, 1, 4]
sage: matrix(script)*S.reduced_laplacian()
[2 0 1 1 0]
```

Note: The burning configuration and script are computed using a modified version of Speer’s script algorithm. This is a generalization to directed multigraphs of Dhar’s burning algorithm.

A burning configuration is a nonnegative integer-linear combination of the rows of the reduced Laplacian matrix having nonnegative entries and such that every vertex has a path from some vertex in its support. The corresponding burning script gives the integer-linear combination needed to obtain the burning configuration. So if \( b \) is the burning configuration, \( \sigma \) is its script, and \( \tilde{L} \) is the reduced Laplacian, then \( \sigma \cdot \tilde{L} = b \). The minimal burning configuration is the one with the minimal script (its components are no larger than the components of any other script for a burning configuration).

The following are equivalent for a configuration \( c \) with burning configuration \( b \) having script \( \sigma \):

- \( c \) is recurrent;
- \( c + b \) stabilizes to \( c \);
- the firing vector for the stabilization of \( c + b \) is \( \sigma \).
canonical_divisor()

The canonical divisor. This is the divisor with \( \deg(v) - 2 \) grains of sand on each vertex (not counting loops). Only for undirected graphs.

**OUTPUT:**

SandpileDivisor

**EXAMPLES:**

```python
sage: S = sandpiles.Complete(4)
sage: S.canonical_divisor()
{0: 1, 1: 1, 2: 1, 3: 1}
sage: s = Sandpile({0:[1,1],1:[0,0,1,1,1],0}
sage: s.canonical_divisor()  # loops are disregarded
{0: 0, 1: 0}
```

**Warning:** The underlying graph must be undirected.

dict()

A dictionary of dictionaries representing a directed graph.

**OUTPUT:**

dict

**EXAMPLES:**

```python
sage: S = sandpiles.Diamond()
sage: S.dict()
{0: {1: 1, 2: 1},
  1: {0: 1, 2: 1, 3: 1},
  2: {0: 1, 1: 1, 3: 1},
  3: {1: 1, 2: 1}}
sage: S.sink()
0
```

genus()

The genus: \((\# \text{ non-loop edges}) - (\# \text{ vertices}) + 1\). Only defined for undirected graphs.

**OUTPUT:**

integer

**EXAMPLES:**

```python
sage: sandpiles.Complete(4).genus()
3
sage: sandpiles.Cycle(5).genus()
1
```

grobner()

A Groebner basis for the homogeneous toppling ideal. It is computed with respect to the standard sandpile ordering (see ring).

**OUTPUT:**

Groebner basis
EXAMPLES:

```
sage: S = sandpiles.Diamond()
sage: S.groebner()
[x^3*x^2*x^2 - x^2*x^1*x^0, x^2*x^1*x^1 - x^2*x^1*x^0, x^3*x^1*x^2 - x^2*x^2*x^0, x^1*x^3 - x^3*x^2*x^0, x^3^2 → x^0*x^2, x^2*x^1 - x^0*x^2]
```

**group_gens (verbose=True)**

A minimal list of generators for the sandpile group. If `verbose` is `False` then the generators are represented as lists of integers.

**INPUT:**

- `verbose` – (default: True) boolean

**OUTPUT:**

- list of SandpileConfig (or of lists of integers if `verbose` is `False`)

**EXAMPLES:**

```
sage: s = sandpiles.Cycle(5)
sage: s.group_gens()
[{1: 1, 2: 1, 3: 1, 4: 0}]
sage: s.group_gens()[0].order()
5
sage: s = sandpiles.Complete(5)
sage: s.group_gens(False)
[[2, 2, 3, 2], [2, 3, 2, 2], [3, 2, 2, 2]]
sage: [i.order() for i in s.group_gens()]
[5, 5, 5]
sage: s.invariant_factors()
[1, 5, 5, 5]
```

**group_order ()**

The size of the sandpile group.

**OUTPUT:**

- integer

**EXAMPLES:**

```
sage: S = sandpiles.House()
sage: S.group_order()
11
```

**h_vector ()**

The number of superstable configurations in each degree. Equivalently, this is the list of first differences of the Hilbert function of the (homogeneous) toppling ideal.

**OUTPUT:**

- list of nonnegative integers

**EXAMPLES:**

```
sage: s = sandpiles.Grid(2,2)
sage: s.hilbert_function()
[1, 5, 15, 35, 66, 106, 146, 178, 192]
sage: s.h_vector()
[1, 4, 10, 20, 31, 40, 40, 32, 14]
```
**static help** *(verbose=True)*

List of Sandpile-specific methods (not inherited from *Graph*). If verbose, include short descriptions.

**INPUT:**

verbose – (default: True) boolean

**OUTPUT:**

printed string

**EXAMPLES:**

```
sage: Sandpile.help() # long time
For detailed help with any method FOO listed below, enter "Sandpile.FOO?" or enter "S.FOO?" for any Sandpile S.

all_k_config -- The constant configuration with all values set to k.
all_k_div -- The divisor with all values set to k.
avalance_polynomial -- The avalanche polynomial.
betti -- The Betti table for the homogeneous toppling ideal.
betti_complexes -- The support-complexes with non-trivial homology.
burning_config -- The minimal burning configuration.
burning_script -- A script for the minimal burning configuration.
canonical_divisor -- The canonical divisor.
dict -- A dictionary of dictionaries representing a directed graph.
genus -- The genus: (# non-loop edges) - (# vertices) + 1.
groebner -- A Groebner basis for the homogeneous toppling ideal.
group_gens -- A minimal list of generators for the sandpile group.
group_order -- The size of the sandpile group.
h_vector -- The number of superstable configurations in each degree.
help -- List of Sandpile-specific methods (not inherited from "Graph").
hilbert_function -- The Hilbert function of the homogeneous toppling ideal.
ideal -- The saturated homogeneous toppling ideal.
identity -- The identity configuration.
in_degree -- The in-degree of a vertex or a list of all in-degrees.
invariant_factors -- The invariant factors of the sandpile group.
is_undirected -- Is the underlying graph undirected?
jacobian_representatives -- Representatives for the elements of the Jacobian group.
laplacian -- The Laplacian matrix of the graph.
markov_chain -- The sandpile Markov chain for configurations or divisors.
max_stable -- The maximal stable configuration.
max_stable_div -- The maximal stable divisor.
max_superstables -- The maximal superstable configurations.
min_recurrents -- The minimal recurrent elements.
nonsink_vertices -- The nonsink vertices.
nonspecial_divisors -- The nonspecial divisors.
out_degree -- The out-degree of a vertex or a list of all out-degrees.
```

(continues on next page)
picard_representatives -- Representatives of the divisor classes of degree \( d \) in the Picard group.
points -- Generators for the multiplicative group of zeros of the sandpile ideal.
postulation -- The postulation number of the toppling ideal.
recurrents -- The recurrent configurations.
reduced_laplacian -- The reduced Laplacian matrix of the graph.
reorder_vertices -- A copy of the sandpile with vertex names permuted.
resolution -- A minimal free resolution of the homogeneous toppling ideal.
ring -- The ring containing the homogeneous toppling ideal.
show -- Draw the underlying graph.
show3d -- Draw the underlying graph.
sink -- The sink vertex.
smith_form -- The Smith normal form for the Laplacian.
solve -- Approximations of the complex affine zeros of the sandpile ideal.
stable_configs -- Generator for all stable configurations.
stationary_density -- The stationary density of the sandpile.
superstables -- The superstable configurations.
symmetric_recurrents -- The symmetric recurrent configurations.
tutte_polynomial -- The Tutte polynomial of the underlying graph.
unsaturated_ideal -- The unsaturated, homogeneous toppling ideal.
version -- The version number of Sage Sandpiles.
zero_config -- The all-zero configuration.
zero_div -- The all-zero divisor.

hilbert_function()

The Hilbert function of the homogeneous toppling ideal.

INPUT:

tuple of nonnegative integers

EXAMPLES:

```sage```
s = sandpiles.Wheel(5)
s.hilbert_function()
```
[1, 5, 15, 31, 45]
```

```sage```
s.h_vector()
```
[1, 4, 10, 16, 14]
```

ideal(gens=False)

The saturated homogeneous toppling ideal. If gens is True, the generators for the ideal are returned instead.

INPUT:

gens - (default: False) boolean

OUTPUT:

ideal or, optionally, the generators of an ideal

EXAMPLES:

```sage```
S = sandpiles.Diamond()
S.ideal()
```

Ideal \((x_2 x_1 - x_0^2, x_3^2 - x_0^2, x_1^3 - x_3 x_2^2 + x_0, x_3 x_1^2 - x_2^2 x_0, x_2^3 - x_3 x_1 x_0, x_3 x_2^2 - x_1^2 x_0)\) of Multivariate Polynomial Ring in \(x_3, x_2, x_1, x_0\) over Rational Field

```python
sage: S.ideal(True)
[x2*x1 - x0^2, x3^2 - x0^2, x1^3 - x3*x2^2 + x0, x3*x1^2 - x2^2*x0, x2^3 - x3*x1*x0, x3*x2^2 - x1^2*x0]
```

```python
sage: S.ideal().gens()  # another way to get the generators
[x2*x1 - x0^2, x3^2 - x0^2, x1^3 - x3*x2^2 + x0, x3*x1^2 - x2^2*x0, x2^3 - x3*x1*x0, x3*x2^2 - x1^2*x0]
```

**identity** \((\text{verbose}=\text{True})\)

The identity configuration. If \(\text{verbose}\) is \(\text{False}\), the configuration are converted to a list of integers.

**INPUT:**

\(\text{verbose}-(\text{default: True})\) boolean

**OUTPUT:**

SandpileConfig or a list of integers If \(\text{verbose}\) is \(\text{False}\), the configuration are converted to a list of integers.

**EXAMPLES:**

```python
sage: s = sandpiles.Diamond()
sage: s.identity()
{1: 2, 2: 2, 3: 0}
sage: s.identity(False)
[2, 2, 0]
sage: s.identity() & s.max_stable() == s.max_stable()
True
```

**in_degree** \((v=\text{None})\)

The in-degree of a vertex or a list of all in-degrees.

**INPUT:**

\(v-(\text{optional})\) vertex name

**OUTPUT:**

integer or dict

**EXAMPLES:**

```python
sage: s = sandpiles.House()
sage: s.in_degree()
{0: 2, 1: 2, 2: 3, 3: 3, 4: 2}
sage: s.in_degree(2)
3
```

**invariant_factors**

The invariant factors of the sandpile group.

**OUTPUT:**

list of integers

**EXAMPLES:**
sage: s = sandpiles.Grid(2,2)
sage: s.invariant_factors()
[1, 1, 8, 24]

is_undirected()
Is the underlying graph undirected? True if \((u, v)\) is an edge if and only if \((v, u)\) is an edge, each edge with the same weight.

OUTPUT:
boolean

EXAMPLES:

sage: sandpiles.Complete(4).is_undirected()
True
sage: s = Sandpile({0:[1,2], 1:[0,2], 2:[0]}, 0)
sage: s.is_undirected()
False

jacobian_representatives (verbose=True)
Representatives for the elements of the Jacobian group. If verbose is False, then lists representing the divisors are returned.

INPUT:
verbose – (default: True) boolean

OUTPUT:
list of SandpileDivisor (or of lists representing divisors)

EXAMPLES:
For an undirected graph, divisors of the form \(s - \deg(s) \cdot \text{sink}\) as \(s\) varies over the superstables forms a distinct set of representatives for the Jacobian group:

sage: s = sandpiles.Complete(3)
sage: s.superstables(False)
[[0, 0], [0, 1], [1, 0]]
sage: s.jacobian_representatives(False)
[[0, 0, 0], [-1, 0, 1], [-1, 1, 0]]

If the graph is directed, the representatives described above may be equivalent modulo the rowspan of the Laplacian matrix:

sage: s = Sandpile({0: {1: 1, 2: 2}, 1: {0: 2, 2: 4}, 2: {0: 4, 1: 2}}, 0)
sage: s.group_order()
28
sage: s.jacobian_representatives()
[[0: -5, 1: 3, 2: 2], (0: -4, 1: 3, 2: 1)]

Let \(\tau\) be the nonnegative generator of the kernel of the transpose of the Laplacian, and let \(\tau_u\) be its sink component, then the sandpile group is isomorphic to the direct sum of the cyclic group of order \(\tau_u\) and the Jacobian group. In the example above, we have:

sage: s.laplacian().left_kernel()
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[14 5 8]
Note: The Jacobian group is the set of all divisors of degree zero modulo the integer rowspan of the Laplacian matrix.

**laplacian()**
The Laplacian matrix of the graph. Its rows encode the vertex firing rules.

**OUTPUT:**
matrix

**EXAMPLES:**

```python
sage: G = sandpiles.Diamond()
sage: G.laplacian()
[ 2 -1 -1 0]
[-1 3 -1 -1]
[-1 -1 3 -1]
[ 0 -1 -1 2]
```

**Warning:** The function `laplacian_matrix` should be avoided. It returns the indegree version of the Laplacian.

**markov_chain(state, distrib=None)**
The sandpile Markov chain for configurations or divisors. The chain starts at `state`. See NOTE for details.

**INPUT:**
- `state` – SandpileConfig, SandpileDivisor, or list representing one of these
- `distrib` – (optional) list of nonnegative numbers summing to 1 (representing a prob. dist.)

**OUTPUT:**
generator for Markov chain (see NOTE)

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(4)
sage: m = s.markov_chain([0,0,0])
sage: next(m)  # random
{1: 0, 2: 0, 3: 0}
sage: next(m).values()  # random
[0, 0, 0]
sage: next(m).values()  # random
[0, 0, 0]
sage: next(m).values()  # random
[0, 1, 0]
sage: next(m).values()  # random
[0, 2, 0]
sage: next(m).values()  # random
[0, 2, 1]
sage: next(m).values()  # random
[1, 2, 1]
sage: next(m).values()  # random
```
Note: The closed sandpile Markov chain has state space consisting of the configurations on a sandpile. It transitions from a state by choosing a vertex at random (according to the probability distribution `distrib`), dropping a grain of sand at that vertex, and stabilizing. If the chosen vertex is the sink, the chain stays at the current state.

The open sandpile Markov chain has state space consisting of the recurrent elements, i.e., the state space is the sandpile group. It transitions from the configuration $c$ by choosing a vertex $v$ at random according to `distrib`. The next state is the stabilization of $c + v$. If $v$ is the sink vertex, then the stabilization of $c + v$ is defined to be $c$.

Note that in either case, if `distrib` is specified, its length is equal to the total number of vertices (including the sink).

REFERENCES:

- [Lev2014]

**max_stable()**

The maximal stable configuration.

OUTPUT:

SandpileConfig (the maximal stable configuration)

EXAMPLES:

```python
sage: S = sandpiles.House()
sage: S.max_stable()
{1: 1, 2: 2, 3: 2, 4: 1}
```

**max_stable_div()**

The maximal stable divisor.

OUTPUT:

SandpileDivisor (the maximal stable divisor)
EXAMPLES:

```python
sage: s = sandpiles.Diamond()
sage: s.max_stable_div()
{0: 1, 1: 2, 2: 2, 3: 1}
sage: s.out_degree()
{0: 2, 1: 3, 2: 3, 3: 2}
```

**max_superstables** *(verbose=True)*

The maximal superstable configurations. If the underlying graph is undirected, these are the superstables of highest degree. If **verbose** is *False*, the configurations are converted to lists of integers.

**INPUT:**

**verbose** – *(default: True)* boolean

**OUTPUT:**

tuple of SandpileConfig

**EXAMPLES:**

```python
sage: s = sandpiles.Diamond()
sage: s.superstables(False)
[[0, 0, 0],
 [0, 0, 1],
 [1, 0, 1],
 [0, 2, 0],
 [2, 0, 0],
 [0, 1, 1],
 [1, 0, 0],
 [0, 0, 0]]
sage: s.max_superstables(False)
[[1, 0, 1], [0, 2, 0], [2, 0, 0], [0, 1, 1]]
sage: s.h_vector()
[1, 3, 4]
```

**min_recurrences** *(verbose=True)*

The minimal recurrent elements. If the underlying graph is undirected, these are the recurrent elements of least degree. If **verbose** is *False*, the configurations are converted to lists of integers.

**INPUT:**

**verbose** – *(default: True)* boolean

**OUTPUT:**

list of SandpileConfig

**EXAMPLES:**

```python
sage: s = sandpiles.Diamond()
sage: s.recurrents(False)
[[2, 2, 1],
 [2, 2, 0],
 [1, 2, 0],
 [2, 0, 1],
 [0, 2, 1],
 [2, 1, 0],
 [1, 2, 1],
 [2, 1, 1]]
(...continues on next page)
```
\begin{verbatim}
sage: s.min_recurrents(False)
[[1, 2, 0], [2, 0, 1], [0, 2, 1], [2, 1, 0]]
sage: [i.deg() for i in s.recurrents()]
[5, 4, 3, 3, 3, 3, 4, 4]
\end{verbatim}

\textbf{nonsink_vertices()} \\
The nonsink vertices. \\
\textbf{OUTPUT:} \\
list of vertices \\
\textbf{EXAMPLES:}

\begin{verbatim}
sage: s = sandpiles.Grid(2,3)
sage: s.nonsink_vertices()
[(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)]
\end{verbatim}

\textbf{nonspecial_divisors}(\texttt{verbose=True}) \\
The nonspecial divisors. Only for undirected graphs. (See \textbf{NOTE}.) \\
\textbf{INPUT:} \\
\texttt{verbose} – (default: \texttt{True}) boolean \\
\textbf{OUTPUT:} \\
list (of divisors) \\
\textbf{EXAMPLES:}

\begin{verbatim}
sage: S = sandpiles.Complete(4)
sage: ns = S.nonspecial_divisors()
sage: D = ns[0]
sage: D.values()
[-1, 0, 1, 2]
sage: D.deg()
2
sage: [i.effective_div() for i in ns]
[[], [], [], [], [], []]
\end{verbatim}

\textbf{Note:} The “nonspecial divisors” are those divisors of degree $g - 1$ with empty linear system. The term is only defined for undirected graphs. Here, $g = |E| - |V| + 1$ is the genus of the graph (not counting loops as part of $|E|$). If \texttt{verbose} is \texttt{False}, the divisors are converted to lists of integers.

\textbf{Warning:} The underlying graph must be undirected.

\textbf{out_degree}(\texttt{v=None}) \\
The out-degree of a vertex or a list of all out-degrees. \\
\textbf{INPUT:} \\
\texttt{v} - (optional) vertex name \\
\textbf{OUTPUT:} \\
ninteger or dict
EXEMPLARY:

```python
sage: s = sandpiles.House()
sage: s.out_degree()
{0: 2, 1: 2, 2: 3, 3: 3, 4: 2}
sage: s.out_degree(2)
3
```

**picard_representatives** *(d, verbose=True)*

Representatives of the divisor classes of degree \(d\) in the Picard group. (Also see the documentation for `jacobian_representatives`.)

**INPUT:**

- \(d\) – integer
- \(verbose\) – (default: True) boolean

**OUTPUT:**

list of SandpileDivisors (or lists representing divisors)

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(3)
sage: s.superstables(False)
[[0, 0], [0, 1], [1, 0]]
sage: s.jacobian_representatives(False)
[[0, 0, 0], [-1, 0, 1], [-1, 1, 0]]
sage: s.picard_representatives(3,False)
[[3, 0, 0], [2, 0, 1], [2, 1, 0]]
```

**points()**

Generators for the multiplicative group of zeros of the sandpile ideal.

**OUTPUT:**

list of complex numbers

**EXAMPLES:**

The sandpile group in this example is cyclic, and hence there is a single generator for the group of solutions.

```python
sage: S = sandpiles.Complete(4)
sage: S.points()
[[1, I, -I], [I, 1, -I]]
```

**postulation()**

The postulation number of the toppling ideal. This is the largest weight of a superstable configuration of the graph.

**OUTPUT:**

nonnegative integer

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(4)
sage: s.postulation()
3
```

**recurrents**(verbose=True)

The recurrent configurations. If `verbose` is `False`, the configurations are converted to lists of integers.
INPUT:

verbose – (default: True) boolean

OUTPUT:

list of recurrent configurations

EXAMPLES:

```sage
r = Sandpile(graphs.HouseXGraph(),0).recurrents()
r[:3]
```

```
{{1: 2, 2: 3, 3: 3, 4: 1}, {1: 1, 2: 3, 3: 3, 4: 0}, {1: 1, 2: 3, 3: 3, 4: 1}}
```

```sage
sandpiles.Complete(4).recurrents(False)
```

```
[[2, 2, 2],
 [2, 2, 1],
 [2, 1, 2],
 [1, 2, 2],
 [2, 2, 0],
 [2, 0, 2],
 [0, 2, 2],
 [2, 1, 1],
 [1, 2, 1],
 [1, 1, 2],
 [2, 1, 0],
 [2, 0, 1],
 [1, 2, 0],
 [1, 0, 2],
 [0, 2, 1],
 [0, 1, 2]]
```

```sage
sandpiles.Cycle(4).recurrents(False)
```

```
[[1, 1, 1],
 [0, 1, 1],
 [1, 0, 1],
 [1, 1, 0]]
```

reduced_laplacian()

The reduced Laplacian matrix of the graph.

OUTPUT:

matrix

EXAMPLES:

```sage
S = sandpiles.Diamond()
S.laplacian()
```

```
[ 2 -1 -1 0]
[-1 3 -1 -1]
[-1 -1 3 -1]
[ 0 -1 -1 2]
```

```sage
S.reduced_laplacian()
```

```
[ 3 -1 -1]
[-1 3 -1]
[-1 -1 2]
```

Note: This is the Laplacian matrix with the row and column indexed by the sink vertex removed.

reorder_vertices()

A copy of the sandpile with vertex names permuted.

After reordering, vertex $u$ comes before vertex $v$ in the list of vertices if $u$ is closer to the sink.
OUTPUT:
Sandpile

EXAMPLES:

```
sage: S = Sandpile({0: [1], 2: [0, 1], 1: [2]})
sage: S.dict()
{0: {1: 1}, 1: {2: 1}, 2: {0: 1, 1: 1}}
sage: T = S.reorder_vertices()
The vertices 1 and 2 have been swapped:
sage: T.dict()
{0: {1: 1}, 1: {0: 1, 2: 1}, 2: {0: 1}}
```

```
resolution (verbose=False)
A minimal free resolution of the homogeneous toppling ideal. If verbose is True, then all of
the mappings are returned. Otherwise, the resolution is summarized.

INPUT:
verbose – (default: False) boolean

OUTPUT:
free resolution of the toppling ideal

EXAMPLES:

```
sage: S = Sandpile({0: {}, 1: {0: 1, 2: 1, 3: 4}, 2: {3: 5}, 3: {1: 1, 2: 1}}, → 0)
sage: S.resolution()  # a Gorenstein sandpile graph
'R^1 <-- R^5 <-- R^5 <-- R^1'
sage: S.resolution(True)
[ [ 1^2 - x3*x0 x3*x1 - x2*x0 x3^2 - x2*x1 x2*x3 - x0^2 x2^2 - x1*x0],
  [ x3 x2 0 x0 0] [ x2^2 - x1*x0]
[ -x1 -x3 x2 0 -x0] [-x2*x3 + x0^2]
[ x0 x1 0 x2 0] [-x3^2 + x2*x1]
[ 0 0 -x1 -x3 x2] [x3*x1 - x2*x0]
[ 0 0 x0 x1 -x3], [ x1^2 - x3*x0]
]
sage: r = S.resolution(True)
sage: r[0]*r[1]
[0 0 0 0 0]
sage: r[1]*r[2]
[0]
[0]
[0]
[0]
```

```
ring ()
The ring containing the homogeneous toppling ideal.

OUTPUT:
ring

EXAMPLES:
```
sage: S = sandpiles.Diamond()
sage: S.ring()
Multivariate Polynomial Ring in x3, x2, x1, x0 over Rational Field
sage: S.ring().gens()
(x3, x2, x1, x0)

Note: The indeterminate \(x_i\) corresponds to the \(i\)-th vertex as listed by the method \(\text{vertices}\). The term-ordering is \text{degrevlex} with indeterminates ordered according to their distance from the sink (larger indeterminates are further from the sink).

\textbf{show(**kwds)}

Draw the underlying graph.

\textbf{INPUT:}

\texttt{kwds} – (optional) arguments passed to the show method for Graph or DiGraph

\textbf{EXAMPLES:}

\begin{verbatim}
sage: S = Sandpile({0:[], 1:[0,3,4], 2:[0,3,5], 3:[2,5], 4:[1,1], 5:[2,4]})
sage: S.show()
sage: S.show(graph_border=True, edge_labels=True)
\end{verbatim}

\textbf{show3d(**kwds)}

Draw the underlying graph.

\textbf{INPUT:}

\texttt{kwds} – (optional) arguments passed to the show method for Graph or DiGraph

\textbf{EXAMPLES:}

\begin{verbatim}
sage: S = sandpiles.House()
sage: S.show3d() # long time
\end{verbatim}

\textbf{sink()}

The sink vertex.

\textbf{OUTPUT:}

sink vertex

\textbf{EXAMPLES:}

\begin{verbatim}
sage: G = sandpiles.House()
sage: G.sink()
0
sage: H = sandpiles.Grid(2,2)
sage: H.sink()
(0, 0)
sage: type(H.sink())
<... 'tuple'>
\end{verbatim}

\textbf{smith_form()}

The Smith normal form for the Laplacian. In detail: a list of integer matrices \(D, U, V\) such that \(ULV = D\) where \(L\) is the transpose of the Laplacian, \(D\) is diagonal, and \(U\) and \(V\) are invertible over the integers.

\textbf{OUTPUT:}
list of integer matrices

EXAMPLES:

```
sage: s = sandpiles.Complete(4)
sage: D,U,V = s.smith_form()
sage: D
[1 0 0 0]
[0 4 0 0]
[0 0 4 0]
[0 0 0 0]
sage: U*s.laplacian()*V == D  # Laplacian symmetric => transpose not necessary
True
```

`solve()`

Approximations of the complex affine zeros of the sandpile ideal.

OUTPUT:

list of complex numbers

EXAMPLES:

```
sage: S = Sandpile({0: {}, 1: {2: 2}, 2: {0: 4, 1: 1}}, 0)
sage: S.solve()

[-0.707107000000000 + 0.707107000000000*I,
 0.707107000000000 + 0.707107000000000*I],
[-0.707107000000000 - 0.707107000000000*I,
 0.707107000000000 - 0.707107000000000*I],
[-I, -I],
[I, I],
[0.707107000000000 + 0.707107000000000*I,
 -0.707107000000000 - 0.707107000000000*I],
[0.707107000000000 - 0.707107000000000*I,
 -0.707107000000000 + 0.707107000000000*I],
[1, 1],
[-1, -1]
sage: len(_)
8
sage: S.group_order()
8
```

Note: The solutions form a multiplicative group isomorphic to the sandpile group. Generators for this group are given exactly by `points()`.

`stable_configs(smax=None)`

Generator for all stable configurations. If `smax` is provided, then the generator gives all stable configurations less than or equal to `smax`. If `smax` does not represent a stable configuration, then each component of `smax` is replaced by the corresponding component of the maximal stable configuration.

INPUT:

`smax` – (optional) SandpileConfig or list representing a SandpileConfig

OUTPUT:

generator for all stable configurations

EXAMPLES:
stationary_density()

The stationary density of the sandpile.

OUTPUT:

rational number

EXAMPLES:

```
sage: s = sandpiles.Complete(3)
sage: s.stationary_density()
10/9
sage: s = Sandpile(digraphs.DeBruijn(2,2),'00')
sage: s.stationary_density()
9/8
```

Note: The stationary density of a sandpile is the sum \( \sum_c (\deg(c) + \deg(s)) \) where \( \deg(s) \) is the degree of the sink and the sum is over all recurrent configurations.

REFERENCES:

• [Lev2014]

superstables (verbose=True)

The superstable configurations. If verbose is False, the configurations are converted to lists of integers. Superstables for undirected graphs are also known as G-parking functions.

INPUT:

verbose – (default: True) boolean

OUTPUT:

list of SandpileConfig

EXAMPLES:

```
sage: sp = Sandpile(graphs.HouseXGraph(),0).superstables()
sage: sp[:3]

{{1: 0, 2: 0, 3: 0, 4: 0}, {1: 1, 2: 0, 3: 0, 4: 1}, {1: 1, 2: 0, 3: 0, 4: 0}}
sage: sandpiles.Complete(4).superstables(False)

[[0, 0, 0],
 [0, 0, 1],
 [0, 1, 0],
 [1, 0, 0],
 [0, 0, 2],
 [0, 2, 0],
 [2, 0, 0],
 [0, 1, 1],
```
sandpiles.Cycle(4).superstables(False)
[[[0, 0, 0], [1, 0, 0], [0, 1, 0], [0, 0, 1]]

sage: sandpiles.Cycle(4).superstables(False)
[[[0, 0, 0], [1, 0, 0], [0, 1, 0], [0, 0, 1]]


cycle

symmetric_recurrents (orbits)
The symmetric recurrent configurations.

INPUT:

orbits - list of lists partitioning the vertices

OUTPUT:

list of recurrent configurations

EXAMPLES:

sage: S = Sandpile({0: {}, ....: 1: {0: 1, 2: 1, 3: 1}, ....: 2: {1: 1, 3: 1, 4: 1}, ....: 3: {1: 1, 2: 1, 4: 1}, ....: 4: {2: 1, 3: 1}})
sage: S.symmetric_recurrents([[1], [2, 3], [4]])
[[1: 2, 2: 2, 3: 2, 4: 1], [1: 2, 2: 2, 3: 2, 4: 0]]
sage: S.recurrents()
[[1: 2, 2: 2, 3: 2, 4: 1], [1: 2, 2: 2, 3: 2, 4: 0], [1: 2, 2: 1, 3: 2, 4: 0], [1: 2, 2: 2, 3: 1, 4: 0], [1: 2, 2: 1, 3: 2, 4: 1], [1: 2, 2: 2, 3: 1, 4: 1]]

Note: The user is responsible for ensuring that the list of orbits comes from a group of symmetries of the underlying graph.

tutte_polynomial ()
The Tutte polynomial of the underlying graph. Only defined for undirected sandpile graphs.

OUTPUT:

polynomial

EXAMPLES:

sage: s = sandpiles.Complete(4)
sage: s.tutte_polynomial()
x^3 + y^3 + 3*x^2 + 4*x*y + 3*y^2 + 2*x + 2*y
unsaturated_ideal()
The unsaturated, homogeneous toppling ideal.

OUTPUT:
ideal

EXAMPLES:

```sage
s = sandpiles.Diamond()
sage: S.ideal().gens()
[x1^3 - x3*x2*x0, x2^3 - x3*x1*x0, x3^2 - x2*x1]
sage: S.unsaturated_ideal().gens()
[x1^3 - x3*x2*x0, x2^3 - x3*x1*x0, x3^2 - x2*x1]
```

static version()
The version number of Sage Sandpiles.

OUTPUT:
string

EXAMPLES:

```sage
S = sandpiles.Complete(3)
sage: S.version()
Sage Sandpiles Version 2.4
sage: S.unsaturated_ideal().gens()
[x1^3 - x3*x2*x0, x2^3 - x3*x1*x0, x3^2 - x2*x1]
```

zero_config()
The all-zero configuration.

OUTPUT:
SandpileConfig

EXAMPLES:

```sage
s = sandpiles.Diamond()
sage: s.zero_config()
{1: 0, 2: 0, 3: 0}
```

zero_div()
The all-zero divisor.

OUTPUT:
SandpileDivisor

EXAMPLES:

```sage
S = sandpiles.House()
sage: S.zero_div()
{0: 0, 1: 0, 2: 0, 3: 0, 4: 0}
```
class sage.sandpiles.sandpile.SandpileConfig(S,c)
    Bases: dict
    Class for configurations on a sandpile.

    add_random(distrib=None)
    Add one grain of sand to a random vertex. Optionally, a probability distribution, distrib, may be placed
    on the vertices or the nonsink vertices. See NOTE for details.

    INPUT:
    
distrib – (optional) list of nonnegative numbers summing to 1 (representing a prob. dist.)

    OUTPUT:
    SandpileConfig

    EXAMPLES:

    sage: s = sandpiles.Complete(4)
sage: c = s.zero_config()
sage: c.add_random() # random
{1: 0, 2: 1, 3: 0}
sage: c
{1: 0, 2: 0, 3: 0}
sage: c.add_random([0.1,0.1,0.8]) # random
{1: 0, 2: 0, 3: 1}
sage: c.add_random([0.7,0.1,0.1,0.1]) # random
{1: 0, 2: 0, 3: 0}

We compute the “sizes” of the avalanches caused by adding random grains of sand to the maximal stable
configuration on a grid graph. The function stabilize() returns the firing vector of the stabilization,
a dictionary whose values say how many times each vertex fires in the stabilization.:  

    sage: S = sandpiles.Grid(10,10)
sage: m = S.max_stable()
sage: a = []
sage: for i in range(1000):
      m = m.add_random()
      m, f = m.stabilize(True)
      a.append(sum(f.values()))
sage: p = list_plot([[log(i+1),log(a.count(i))] for i in [0..max(a)] if a.
                        count(i)])
sage: p.axes_labels(['log(N)','log(D(N))'])
sage: t = text("Distribution of avalanche sizes", (2,2), rgbcolor=(1,0,0))
sage: show(p+t,axes_labels=['log(N)','log(D(N))']) # long time

Note: If distrib is None, then the probability is the uniform probability on the nonsink vertices. Otherwise, there are two possibilities:

(i) the length of distrib is equal to the number of vertices, and distrib represents a probability
distribution on all of the vertices. In that case, the sink may be chosen at random, in which case, the
configuration is unchanged.

(ii) Otherwise, the length of distrib must be equal to the number of nonsink vertices, and distrib
represents a probability distribution on the nonsink vertices.
**Warning:** If \( \text{distrib} \neq \text{None} \), the user is responsible for assuring the sum of its entries is 1 and that its length is equal to the number of sink vertices or the number of nonsink vertices.

\[
burst\_size(v)
\]

The burst size of the configuration with respect to the given vertex.

**INPUT:**

\( v \) – vertex

**OUTPUT:**

integer

**EXAMPLES:**

```sage
s = sandpiles.Diamond()
sage: [i.burst_size(0) for i in s.recurrents()]
[1, 1, 1, 1, 1, 1, 1, 1]
sage: [i.burst_size(1) for i in s.recurrents()]
[0, 0, 1, 2, 1, 2, 0, 2]
```

**Note:** To define \( c.\text{burst}(v) \), if \( v \) is not the sink, let \( c' \) be the unique recurrent for which the stabilization of \( c' + v \) is \( c \). The burst size is then the amount of sand that goes into the sink during this stabilization. If \( v \) is the sink, the burst size is defined to be 1.

**REFERENCES:**

• [Lev2014]

\[
deg()
\]

The degree of the configuration.

**OUTPUT:**

integer

**EXAMPLES:**

```sage
S = sandpiles.Complete(3)
sage: c = SandpileConfig(S, [1,2])
sage: c.deg()
3
```

\[
dualize()
\]

The difference with the maximal stable configuration.

**OUTPUT:**

SandpileConfig

**EXAMPLES:**

```sage
S = sandpiles.Cycle(3)
sage: c = SandpileConfig(S, [1,2])
sage: S.max_stable()
{1: 1, 2: 1}
sage: c.dualize()
{1: 0, 2: -1}
```
equivalent_recurrent (with_firing_vector=False)
The recurrent configuration equivalent to the given configuration. Optionally, return the corresponding firing vector.

INPUT:
with_firing_vector – (default: False) boolean

OUTPUT:
SandpileConfig or [SandpileConfig, firing_vector]

EXAMPLES:

```
sage: S = sandpiles.Diamond()
sage: c = SandpileConfig(S, [0,0,0])
sage: c.equivalent_recurrent() == S.identity()
True
sage: x = c.equivalent_recurrent(True)
sage: r = vector([x[0][v] for v in S.nonsink_vertices()])
sage: f = vector([x[1][v] for v in S.nonsink_vertices()])
sage: cv = vector(c.values())
sage: r == cv - f*S.reduced_laplacian()
True
```

Note: Let $L$ be the reduced Laplacian, $c$ the initial configuration, $r$ the returned configuration, and $f$ the firing vector. Then $r = c - f \cdot L$.

equivalent_superstable (with_firing_vector=False)
The equivalent superstable configuration. Optionally, return the corresponding firing vector.

INPUT:
with_firing_vector – (default: False) boolean

OUTPUT:
SandpileConfig or [SandpileConfig, firing_vector]

EXAMPLES:

```
sage: S = sandpiles.Diamond()
sage: m = S.max_stable()
sage: m.equivalent_superstable().is_superstable()
True
sage: x = m.equivalent_superstable(True)
sage: s = vector(x[0].values())
sage: f = vector(x[1].values())
sage: mv = vector(m.values())
sage: s == mv - f*S.reduced_laplacian()
True
```

Note: Let $L$ be the reduced Laplacian, $c$ the initial configuration, $s$ the returned configuration, and $f$ the
firing vector. Then \( s = c - f \cdot L \).

**fire_script** \((\text{sigma})\)
Fire the given script. In other words, fire each vertex the number of times indicated by \( \text{sigma} \).

**INPUT:**
\( \text{sigma} \) – SandpileConfig or (list or dict representing a SandpileConfig)

**OUTPUT:**
SandpileConfig

**EXAMPLES:**
```python
sage: S = sandpiles.Cycle(4)
sage: c = SandpileConfig(S, [1,2,3])
sage: c.unstable()
[2, 3]
sage: c.fire_script(SandpileConfig(S,[0,1,1]))
{1: 2, 2: 1, 3: 2}
sage: c.fire_script(SandpileConfig(S,[2,0,0])) == c.fire_vertex(1).fire_vertex(1)
True
```

**fire_unstable**
Fire all unstable vertices.

**OUTPUT:**
SandpileConfig

**EXAMPLES:**
```python
sage: S = sandpiles.Cycle(4)
sage: c = SandpileConfig(S, [1,2,3])
sage: c.fire_unstable()
{1: 2, 2: 1, 3: 2}
```

**fire_vertex** \((v)\)
Fire the given vertex.

**INPUT:**
\( v \) – vertex

**OUTPUT:**
SandpileConfig

**EXAMPLES:**
```python
sage: S = sandpiles.Cycle(3)
sage: c = SandpileConfig(S, [1,2])
sage: c.fire_vertex(2)
{1: 2, 2: 0}
```

**static help** \(\text{verbose=True}\)
List of SandpileConfig methods. If \( \text{verbose} \), include short descriptions.

**INPUT:**
\( \text{verbose} \) – (default: True) boolean
printed string

EXAMPLES:

```
sage: SandpileConfig.help()
Shortcuts for SandpileConfig operations:
  ~c      -- stabilize
  c & d   -- add and stabilize
  c * c   -- add and find equivalent recurrent
  c^k     -- add k times and find equivalent recurrent
              (taking inverse if k is negative)

For detailed help with any method FOO listed below,
enter "SandpileConfig.FOO?" or enter "c.FOO?" for any SandpileConfig c.

  add_random     -- Add one grain of sand to a random vertex.
  burst_size    -- The burst size of the configuration with respect to
                   the given vertex.
  deg           -- The degree of the configuration.
  dualize       -- The difference with the maximal stable
                   configuration.
  equivalent_recurrent -- The recurrent configuration equivalent to the given
                         configuration.
  equivalent_superstable -- The equivalent superstable configuration.
  fire_script   -- Fire the given script.
  fire_unstable -- Fire all unstable vertices.
  fire_vertex   -- Fire the given vertex.
  help          -- List of SandpileConfig methods.
  is_recurrent  -- Is the configuration recurrent?
  is_stable     -- Is the configuration stable?
  is_superstable -- Is the configuration superstable?
  is_symmetric  -- Is the configuration symmetric?
  order         -- The order of the equivalent recurrent element.
  sandpile      -- The configuration's underlying sandpile.
  show          -- Show the configuration.
  stabilize     -- The stabilized configuration.
  support       -- The vertices containing sand.
  unstable      -- The unstable vertices.
  values        -- The values of the configuration as a list.
```

**is_recurrent()**

Is the configuration recurrent?

**OUTPUT:**

boolean

**EXAMPLES:**

```
sage: S = sandpiles.Diamond()
sage: S.identity().is_recurrent()
True
sage: S.zero_config().is_recurrent()
False
```

**is_stable()**

Is the configuration stable?

**OUTPUT:**
boolean

EXAMPLES:

```sage
sage: S = sandpiles.Diamond()
sage: S.max_stable().is_stable()
True
sage: (2*S.max_stable()).is_stable()
False
sage: (S.max_stable() & S.max_stable()).is_stable()
True
```

`is_superstable()`

Is the configuration superstable?

OUTPUT:

boolean

EXAMPLES:

```sage
sage: S = sandpiles.Diamond()
sage: S.zero_config().is_superstable()
True
```

`is_symmetric(orbits)`

Is the configuration symmetric? Return True if the values of the configuration are constant over the vertices in each sublist of `orbits`.

INPUT:

- `orbits` – list of lists of vertices

OUTPUT:

boolean

EXAMPLES:

```sage
sage: S = Sandpile({0: {},
....: 1: {0: 1, 2: 1, 3: 1},
....: 2: {1: 1, 3: 1, 4: 1},
....: 3: {1: 1, 2: 1, 4: 1},
....: 4: {2: 1, 3: 1}})
sage: c = SandpileConfig(S, [1, 2, 2, 3])
sage: c.is_symmetric([[2, 3]])
True
```

`order()`

The order of the equivalent recurrent element.

OUTPUT:

integer

EXAMPLES:

```sage
sage: S = sandpiles.Diamond()
sage: c = SandpileConfig(S,[2,0,1])
sage: c.order()
4
sage: ~(c + c + c + c) == S.identity()
```

True
sage: c = SandpileConfig(S,[1,1,0])
sage: c.order()
1
sage: c.is_recurrent()
False
sage: c.equivalent_recurrent() == S.identity()
True

sandpile()
The configuration’s underlying sandpile.

OUTPUT:
Sandpile

EXAMPLES:

sage: S = sandpiles.Diamond()
sage: c = S.identity()
sage: c.sandpile()
Diamond sandpile graph: 4 vertices, sink = 0
sage: c.sandpile() == S
True

show (sink=True, colors=True, heights=False, directed=None, **kwds)
Show the configuration.

INPUT:

• sink – (default: True) whether to show the sink
• colors – (default: True) whether to color-code the amount of sand on each vertex
• heights – (default: False) whether to label each vertex with the amount of sand
• directed – (optional) whether to draw directed edges
• kwds – (optional) arguments passed to the show method for Graph

EXAMPLES:

sage: S = sandpiles.Diamond()
sage: c = S.identity()
sage: c.show()
sage: c.show(directed=False)
sage: c.show(sink=False,colors=False,heights=True)

stabilize (with_firing_vector=False)
The stabilized configuration. Optionally returns the corresponding firing vector.

INPUT:

with_firing_vector – (default: False) boolean

OUTPUT:
SandpileConfig or [SandpileConfig, firing_vector]

EXAMPLES:
support()  
The vertices containing sand.  

OUTPUT:  
list - support of the configuration  

EXAMPLES:  

```
sage: S = sandpiles.Diamond()
sage: c = S.identity()
sage: c
{1: 2, 2: 2, 3: 0}
sage: c.support()
[1, 2]
```

unstable()  
The unstable vertices.  

OUTPUT:  
list of vertices  

EXAMPLES:  

```
sage: S = sandpiles.Cycle(4)
sage: c = SandpileConfig(S, [1,2,3])
sage: c.unstable()
[2, 3]
```

values()  
The values of the configuration as a list.  
The list is sorted in the order of the vertices.  

OUTPUT:  
list of integers  
boolean  

EXAMPLES:  

```
sage: S = Sandpile({'a':['c','b'], 'b':['c','a'], 'c':['a']},'a')
sage: c = SandpileConfig(S, {'b':1, 'c':2})
sage: c
{'b': 1, 'c': 2}
```

(continues on next page)
class sage.sandpiles.sandpile.SandpileDivisor(S, D)

Bases: dict

Class for divisors on a sandpile.

Dcomplex()

The support-complex. (See NOTE.)

OUTPUT:

simplicial complex

EXAMPLES:

sage: S = sandpiles.House()
sage: p = SandpileDivisor(S, [1,2,1,0,0]).Dcomplex()
sage: p.homology()
{0: 0, 1: \( \mathbb{Z} \times \mathbb{Z} \), 2: 0}
sage: p.f_vector()
[1, 5, 10, 4]
sage: p.betti()
{0: 1, 1: 2, 2: 0}

Note: The “support-complex” is the simplicial complex determined by the supports of the linearly equivalent effective divisors.

add_random(distrib=None)

Add one grain of sand to a random vertex.

INPUT:

distrib – (optional) list of nonnegative numbers representing a probability distribution on the vertices

OUTPUT:

SandpileDivisor

EXAMPLES:

sage: s = sandpiles.Complete(4)
sage: D = s.zero_div()
sage: D.add_random() # random
{0: 0, 1: 0, 2: 1, 3: 0}
sage: D.add_random([0.1,0.1,0.1,0.7]) # random
{0: 0, 1: 0, 2: 0, 3: 1}

Warning: If distrib is not None, the user is responsible for assuring the sum of its entries is 1.

betti()

The Betti numbers for the support-complex. (See NOTE.)

OUTPUT:
dictionary of integers

EXAMPLES:

```
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [2,0,1])
sage: D.betti()
{0: 1, 1: 1}
```

**Note:** The “support-complex” is the simplicial complex determined by the supports of the linearly equivalent effective divisors.

**deg()**
The degree of the divisor.

**OUTPUT:**
integer

**EXAMPLES:**

```
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [1,2,3])
sage: D.deg()
6
```

**dualize()**
The difference with the maximal stable divisor.

**OUTPUT:**
SandpileDivisor

**EXAMPLES:**

```
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [1,2,3])
sage: D.dualize()
{0: 0, 1: -1, 2: -2}
sage: S.max_stable_div() - D == D.dualize()
True
```

**effective_div**(verbose=True, with_firing_vectors=False)
All linearly equivalent effective divisors. If verbose is False, the divisors are converted to lists of integers. If with_firing_vectors is True then a list of firing vectors is also given, each of which prescribes the vertices to be fired in order to obtain an effective divisor.

**INPUT:**

- verbose – (default: True) boolean
- with_firing_vectors – (default: False) boolean

**OUTPUT:**
list (of divisors)

**EXAMPLES:**
sage: s = sandpiles.Complete(4)
sage: D = SandpileDivisor(s,[4,2,0,0])
sage: sorted(D.effective_div(), key=str)
[(0: 0, 1: 2, 2: 0, 3: 4),
 (0: 0, 1: 2, 2: 4, 3: 0),
 (0: 0, 1: 6, 2: 0, 3: 0),
 (0: 1, 1: 3, 2: 1, 3: 1),
 (0: 2, 1: 0, 2: 2, 3: 2),
 (0: 4, 1: 2, 2: 0, 3: 0)]
sage: sorted(D.effective_div(False))
[[0, 2, 0, 4],
 [0, 2, 4, 0],
 [0, 6, 0, 0],
 [1, 3, 1, 1],
 [2, 0, 2, 2],
 [4, 2, 0, 0]]
sage: sorted(D.effective_div(with_firing_vectors=True), key=str)
(({0: 0, 1: 2, 2: 0, 3: 4}, (0, -1, -1, -2)),
 ({0: 0, 1: 2, 2: 4, 3: 0}, (0, -1, -2, -1)),
 ({0: 0, 1: 6, 2: 0, 3: 0}, (0, -2, -1, -1)),
 ({0: 1, 1: 3, 2: 1, 3: 1}, (0, -1, -1, -1)),
 ({0: 2, 1: 0, 2: 2, 3: 2}, (0, 0, -1, -1)),
 ({0: 4, 1: 2, 2: 0, 3: 0}, (0, 0, 0, 0))]
sage: a = _[2]
sage: a[0].values()
[0, 6, 0, 0]
sage: vector(D.values()) - s.laplacian()*a[1]
(0, 6, 0, 0)
sage: sorted(D.effective_div(False,True))
[(({0: 0, 1: 2, 2: 0, 3: 4}, (0, -1, -1, -2)),
  ({0: 0, 1: 2, 2: 4, 3: 0}, (0, -1, -2, -1)),
  ({0: 0, 1: 6, 2: 0, 3: 0}, (0, -2, -1, -1)),
  ({0: 1, 1: 3, 2: 1, 3: 1}, (0, -1, -1, -1)),
  ({0: 2, 1: 0, 2: 2, 3: 2}, (0, 0, -1, -1)),
  ({0: 4, 1: 2, 2: 0, 3: 0}, (0, 0, 0, 0))]}
sage: D = SandpileDivisor(s,[-1,0,0,0])
sage: D.effective_div(False,True)
[]

**fire_script** (sigma)

Fire the given script. In other words, fire each vertex the number of times indicated by sigma.

**INPUT:**

sigma – SandpileDivisor or (list or dict representing a SandpileDivisor)

**OUTPUT:**

SandpileDivisor

**EXAMPLES:**

```
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [1,2,3])
sage: D.unstable()
[1, 2]
sage: D.fire_script([0,1,1])
(0: 3, 1: 1, 2: 2)
sage: D.fire_script(SandpileDivisor(S,[2,0,0])) == D.fire_vertex(0).fire_vertex(0)
(continues on next page)
```
fire_unstable()
Fire all unstable vertices.

OUTPUT:
SandpileDivisor

EXAMPLES:

```python
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [1,2,3])
sage: D.fire_unstable()
{0: 3, 1: 1, 2: 2}
```

fire_vertex(v)
Fire the given vertex.

INPUT:

v – vertex

OUTPUT:
SandpileDivisor

EXAMPLES:

```python
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [1,2,3])
sage: D.fire_vertex(1)
{0: 2, 1: 0, 2: 4}
```

static help(verbosetrue)
List of SandpileDivisor methods. If verbose, include short descriptions.

INPUT:

verbose – (default: True) boolean

OUTPUT:
printed string

EXAMPLES:

```python
sage: SandpileDivisor.help()
For detailed help with any method FOO listed below, enter "SandpileDivisor.FOO?" or enter "D.FOO?" for any SandpileDivisor D.

Dcomplex -- The support-complex.
add_random -- Add one grain of sand to a random vertex.
betti -- The Betti numbers for the support-complex.
deg -- The degree of the divisor.
dualize -- The difference with the maximal stable divisor.
effective_div -- All linearly equivalent effective divisors.
fire_script -- Fire the given script.
fire_unstable -- Fire all unstable vertices.
fire_vertex -- Fire the given vertex.
help -- List of SandpileDivisor methods.
```
**is_alive** *(cycle=False)*

Is the divisor stabilizable? In other words, will the divisor stabilize under repeated firings of all unstable vertices? Optionally returns the resulting cycle.

**INPUT:**

cycle -- (default: False) boolean

**OUTPUT:**

boolean or optionally, a list of SandpileDivisors

**EXAMPLES:**

```python
sage: S = sandpiles.Complete(4)
sage: D = SandpileDivisor(S, {0: 4, 1: 3, 2: 3, 3: 2})
sage: D.is_alive()
True
sage: D.is_alive(True)
[{0: 4, 1: 3, 2: 3, 3: 2}, {0: 3, 1: 2, 2: 2, 3: 5}, {0: 1, 1: 4, 2: 4, 3: 3}]
```

**is_linearly_equivalent** *(D, with_firing_vector=False)*

Is the given divisor linearly equivalent? Optionally, returns the firing vector. (See NOTE.)

**INPUT:**

• D – SandpileDivisor or list, tuple, etc. representing a divisor

• with_firing_vector – (default: False) boolean

**OUTPUT:**

boolean or integer vector

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(3)
sage: D = SandpileDivisor(s,[2,0,0])
```
sage: D.is_linearly_equivalent([0,1,1])
True
sage: D.is_linearly_equivalent([0,1,1],True)
(1, 0, 0)
sage: v = vector(D.is_linearly_equivalent([0,1,1],True))
sage: vector(D.values()) - s.laplacian()*v
(0, 1, 1)
sage: D.is_linearly_equivalent([0,0,0])
False
sage: D.is_linearly_equivalent([0,0,0],True)
()

Note:
• If with_firing_vector is False, returns either True or False.
• If with_firing_vector is True then: (i) if self is linearly equivalent to \( D \), returns a vector \( v \) such that self - v*self.laplacian().transpose() = D. Otherwise, (ii) if self is not linearly equivalent to \( D \), the output is the empty vector, (\).

\textbf{is q\_reduced()} 
Is the divisor q-reduced? This would mean that \( self = c + kq \) where \( c \) is superstable, \( k \) is an integer, and \( q \) is the sink vertex.

**OUTPUT:**
boolean

**EXAMPLES:**

\begin{verbatim}
sage: s = sandpiles.Complete(4)
sage: D = SandpileDivisor(s, [2,-3,2,0])
sage: D.is_q_reduced()
False
sage: SandpileDivisor(s, [10,0,1,2]).is_q_reduced()
True
\end{verbatim}

For undirected or, more generally, Eulerian graphs, q-reduced divisors are linearly equivalent if and only if they are equal. The same does not hold for general directed graphs:

\begin{verbatim}
sage: s = Sandpile({0:[1],1:[1,1]})
sage: D = SandpileDivisor(s, [-1,1])
sage: Z = s.zero_div()
sage: D.is_q_reduced()
True
sage: Z.is_q_reduced()
True
sage: D == Z
False
sage: D.is_linearly_equivalent(Z)
True
\end{verbatim}

\textbf{is_symmetric(\textbf{orbits})}
Is the divisor symmetric? Return True if the values of the configuration are constant over the vertices in each sublist of \( \text{orbits} \).

**INPUT:**
orbits – list of lists of vertices

OUTPUT:
boolean

EXAMPLES:

\begin{verbatim}
sage: S = sandpiles.House()
sage: S.dict()
{0: {1: 1, 2: 1},
  1: {0: 1, 3: 1},
  2: {0: 1, 3: 1, 4: 1},
  3: {1: 1, 2: 1, 4: 1},
  4: {2: 1, 3: 1})
sage: D = SandpileDivisor(S, [0,0,1,1,3])
sage: D.is_symmetric([[2,3], [4]])
True
\end{verbatim}

\textbf{is\_weierstrass\_pt}\ (v='sink')

Is the given vertex a Weierstrass point?

INPUT:

\begin{itemize}
  \item v – (default: sink) vertex
\end{itemize}

OUTPUT:

boolean

EXAMPLES:

\begin{verbatim}
sage: s = sandpiles.House()
sage: K = s.canonical_divisor()
sage: K.weierstrass_rank_seq()
# sequence at the sink vertex, 0
(1, 0, -1)
sage: K.is_weierstrass_pt()
False
sage: K.is_weierstrass_pt(4)
(1, 0, 0, -1)
sage: K.is_weierstrass_pt(4)
True
\end{verbatim}

\textbf{Note:} The vertex \(v\) is a (generalized) Weierstrass point for divisor \(D\) if the sequence of ranks \(r(D - nv)\) for \(n = 0, 1, 2, \ldots\) is not \(r(D), r(D) - 1, \ldots, 0, -1, -1, \ldots\)

\textbf{polytope}()

The polytope determining the complete linear system.

OUTPUT:

polytope

EXAMPLES:

\begin{verbatim}
sage: s = sandpiles.Complete(4)
sage: D = SandpileDivisor(s,[4,2,0,0])
sage: p = D.polytope()
sage: p.inequalities()
(An inequality (-3, 1, 1) x + 2 >= 0,
(continues on next page)
An inequality \((1, 1, 1) x + 4 \geq 0\),
An inequality \((1, -3, 1) x + 0 \geq 0\),
An inequality \((1, 1, -3) x + 0 \geq 0\)

```python
sage: D = SandpileDivisor(s,[-1,0,0,0])
sage: D.polytope()
The empty polyhedron in QQ^3
```

**Note:** For a divisor \(D\), this is the intersection of (i) the polyhedron determined by the system of inequalities \(L^T x \leq D\) where \(L^T\) is the transpose of the Laplacian with (ii) the hyperplane \(x_{\text{sink\_vertex}} = 0\). The polytope is thought of as sitting in \((n - 1)\)-dimensional Euclidean space where \(n\) is the number of vertices.

### polytope_integer_pts()

The integer points inside divisor’s polytope. The polytope referred to here is the one determining the divisor’s complete linear system (see the documentation for `polytope`).

**OUTPUT:**

tuple of integer vectors

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(4)
sage: D = SandpileDivisor(s,[4,2,0,0])
sage: sorted(D.polytope_integer_pts())
[(-2, -1, -1),
 (-1, -2, -1),
 (-1, -1, -2),
 (-1, -1, -1),
 (0, -1, -1),
 (0, 0, 0)]
sage: D = SandpileDivisor(s,[-1,0,0,0])
sage: D.polytope_integer_pts()
()```

### q_reduced (verbose=True)

The linearly equivalent \(q\)-reduced divisor.

**INPUT:**

`verbose` – (default: True) boolean

**OUTPUT:**

SandpileDivisor or list representing SandpileDivisor

**EXAMPLES:**

```python
sage: s = sandpiles.Complete(4)
sage: D = SandpileDivisor(s,[2,-3,2,0])
sage: D.q_reduced()
{0: -2, 1: 1, 2: 2, 3: 0}
sage: D.q_reduced(False)
[-2, 1, 2, 0]
```

**Note:** The divisor \(D\) is \(q\)\textit{reduced}\(if\) where \(c\) is superstable, \(k\) is an integer, and \(q\) is the sink.
The rank of the divisor. Optionally returns an effective divisor \( E \) such that \( D - E \) is not winnable (has an empty complete linear system).

**INPUT:**

\[ \text{with\_witness} - \text{(default: False)} \text{ boolean} \]

**OUTPUT:**

integer or (integer, SandpileDivisor)

**EXAMPLES:**

```python
sage: S = sandpiles.Complete(4)
sage: D = SandpileDivisor(S,[4,2,0,0])
sage: D.rank()
3
sage: D.rank(True)
(3, {0: 3, 1: 0, 2: 1, 3: 0})
sage: E = _[1]
sage: (D - E).rank()
-1

Riemann-Roch theorem::

\[ D \text{.rank()} - (S\.canonical\_divisor()-D).\text{rank()} = D\.deg() + 1 - S\.\text{genus()} \]

\[ \text{True} \]

Riemann-Roch theorem::

\[ D \text{.rank()} - (S\.canonical\_divisor()-D).\text{rank()} = D\.deg() + 1 - S\.\text{genus()} \]

\[ \text{True} \]

sage: S = Sandpile({0:[1,1,1,1,2],1:[0,0,0,1,1,1,2,2],2:[2,2,1,1,0]},0) # multigraph with loops
sage: D = SandpileDivisor(S,[4,2,0])
sage: D.rank(True)
(2, {0: 1, 1: 1, 2: 1})
sage: S = Sandpile({0:[1,2], 1:[0,2,2], 2: [0,1]},0) # directed graph
sage: S.is_undirected()
False
sage: D = SandpileDivisor(S,[0,2,0])
sage: D.effective_div()
[[0: 0, 1: 2, 2: 0], (0: 2, 1: 0, 2: 0)]
sage: D.rank(True)
(0, {0: 0, 1: 0, 2: 1})
sage: E = D.rank(True)[1]
sage: (D - E).effective_div()
[]
```

**Note:** The rank of a divisor \( D \) is -1 if \( D \) is not linearly equivalent to an effective divisor (i.e., the dollar game represented by \( D \) is unwinnable). Otherwise, the rank of \( D \) is the largest integer \( r \) such that \( D - E \) is linearly equivalent to an effective divisor for all effective divisors \( E \) with \( \text{deg}(E) = r \).

**sandpile()**

The divisor’s underlying sandpile.
OUTPUT:
Sandpile

EXAMPLES:
```
sage: S = sandpiles.Diamond()
sage: D = SandpileDivisor(S,[1,-2,0,3])
sage: D.sandpile()
Diamond sandpile graph: 4 vertices, sink = 0
sage: D.sandpile() == S
True
```

```
show(heights=True, directed=None, **kwds)
```
Show the divisor.

INPUT:
- heights – (default: True) whether to label each vertex with the amount of sand
- directed – (optional) whether to draw directed edges
- kwds – (optional) arguments passed to the show method for Graph

EXAMPLES:
```
sage: S = sandpiles.Diamond()
sage: D = SandpileDivisor(S,[1,-2,0,2])
sage: D.show(graph_border=True,vertex_size=700,directed=False)
```

```
simulate_threshold(distrib=None)
The first unstabilizable divisor in the closed Markov chain. (See NOTE.)
```
INPUT:
- distrib – (optional) list of nonnegative numbers representing a probability distribution on the vertices

OUTPUT:
SandpileDivisor

EXAMPLES:
```
sage: s = sandpiles.Complete(4)
sage: D = s.zero_div()
sage: D.simulate_threshold()
# random
{0: 2, 1: 3, 2: 1, 3: 2}
sage: n(mean([D.simulate_threshold().deg() for _ in range(10)]) for _ in range(10)))
# random
7.10000000000000
sage: n(s.stationary_density()*s.num_verts())
6.93750000000000
```

Note: Starting at self, repeatedly choose a vertex and add a grain of sand to it. Return the first unstabilizable divisor that is reached. Also see the markov_chain method for the underlying sandpile.

```
stabilize(with_firing_vector=False)
The stabilization of the divisor. If not stabilizable, return an error.
```
INPUT:
- with_firing_vector – (default: False) boolean
EXAMPLES:

```python
sage: s = sandpiles.Complete(4)
sage: D = SandpileDivisor(s,[0,3,0,0])
sage: D.stabilize()
{0: 1, 1: 0, 2: 1, 3: 1}
sage: D.stabilize(with_firing_vector=True)
[(0: 1, 1: 0, 2: 1, 3: 1), (0: 0, 1: 1, 2: 0, 3: 0)]
```

**support()**
List of vertices at which the divisor is nonzero.

OUTPUT:
list representing the support of the divisor

EXAMPLES:

```python
sage: S = sandpiles.Cycle(4)
sage: D = SandpileDivisor(S, [0,0,1,1])
sage: D.support()
[2, 3]
sage: S.vertices()
[0, 1, 2, 3]
```

**unstable()**
The unstable vertices.

OUTPUT:
list of vertices

EXAMPLES:

```python
sage: S = sandpiles.Cycle(3)
sage: D = SandpileDivisor(S, [1,2,3])
sage: D.unstable()
[1, 2]
```

**values()**
The values of the divisor as a list.
The list is sorted in the order of the vertices.

OUTPUT:
list of integers
boolean

EXAMPLES:

```python
sage: S = Sandpile({'a':['c','b'], 'b':['c','a'], 'c':['a']},'a')
sage: D = SandpileDivisor(S, {'a':0, 'b':1, 'c':2})
sage: D
{'a': 0, 'b': 1, 'c': 2}
sage: D.values()
[0, 1, 2]
sage: S.vertices()
['a', 'b', 'c']
```
**weierstrass_div** *(verbose=True)*

The Weierstrass divisor. Its value at a vertex is the weight of that vertex as a Weierstrass point. (See `SandpileDivisor.weierstrass_gap_seq`.)

**INPUT:**

verbose – *(default: True)* boolean

**OUTPUT:**

SandpileDivisor

**EXAMPLES:**

```python
sage: s = sandpiles.Diamond()
sage: D = SandpileDivisor(s,[4,2,1,0])
sage: [D.weierstrass_rank_seq(v) for v in s]
[(5, 4, 3, 2, 1, 0, 0, -1),
 (5, 4, 3, 2, 1, 0, -1),
 (5, 4, 3, 2, 1, 0, 0, 0, -1),
 (5, 4, 3, 2, 1, 0, 0, -1)]
sage: D.weierstrass_div()
{0: 1, 1: 0, 2: 2, 3: 1}
```

**weierstrass_gap_seq** *(v='sink', weight=True)*

The Weierstrass gap sequence at the given vertex. If weight is True, then also compute the weight of each gap value.

**INPUT:**

• v – *(default: sink)* vertex
  • weight – *(default: True)* boolean

**OUTPUT:**

list or (list of list) of integers

**EXAMPLES:**

```python
sage: s = sandpiles.Cycle(4)
sage: D = SandpileDivisor(s,[2,0,0,0])
sage: [D.weierstrass_gap_seq(v,False) for v in s.vertices()]
[(1, 3), (1, 2), (1, 3), (1, 2)]
sage: D.weierstrass_gap_seq()  # gap sequence at sink vertex, 0
[(1, 3), 1]
sage: D.weierstrass_rank_seq()  # rank sequence at the sink vertex
(1, 0, 0, -1)
```

**Note:** The integer $k$ is a Weierstrass gap for the divisor $D$ at vertex $v$ if the rank of $D - (k - 1)v$ does not equal the rank of $D - kv$. Let $r$ be the rank of $D$ and let $k_i$ be the $i$-th gap at $v$. The Weierstrass weight of $v$ for $D$ is the sum of $(k_i - i)$ as $i$ ranges from 1 to $r + 1$. It measure the difference between the sequence $r, r - 1, ..., 0, -1, -1, ...$ and the rank sequence rank($D$), rank($D - v$), rank($D - 2v$),...
The Weierstrass points (vertices). Optionally, return the corresponding rank sequences.

**INPUT:**

*with_rank_seq* – (default: False) boolean

**OUTPUT:**

tuple of vertices or list of (vertex, rank sequence)

**EXAMPLES:**

```python
sage: s = sandpiles.House()
sage: K = s.canonical_divisor()
sage: K.weierstrass_pts()
(4,)
sage: K.weierstrass_pts(True)
[(4, (1, 0, 0, -1))]
```

**Note:** The vertex \(v\) is a (generalized) Weierstrass point for divisor \(D\) if the sequence of ranks \(r(D - n\cdot v)\) for \(n = 0, 1, 2, \ldots\) is not \(r(D), r(D) - 1, \ldots, 0, -1, -1, \ldots\)

The Weierstrass rank sequence at the given vertex. Computes the rank of the divisor \(D - n\cdot v\) starting with \(n = 0\) and ending when the rank is \(-1\).

**INPUT:**

\(v\) – (default: sink) vertex

**OUTPUT:**

tuple of int

**EXAMPLES:**

```python
sage: s = sandpiles.House()
sage: K = s.canonical_divisor()
sage: [K.weierstrass_rank_seq(v) for v in s.vertices()]
[(1, 0, -1), (1, 0, -1), (1, 0, -1), (1, 0, -1), (1, 0, 0, -1)]
```

The partitions of the vertices of \(S\) into \(k\) parts, each of which is connected.

**INPUT:**

\(S\) – Sandpile

\(k\) – integer

**OUTPUT:**

list of partitions

**EXAMPLES:**

```python
sage: from sage.sandpiles.sandpile import admissible_partitions
sage: from sage.sandpiles.sandpile import partition_sandpile
sage: S = sandpiles.Cycle(4)
sage: P = [admissible_partitions(S, i) for i in [2, 3, 4]]
```
sage: P
      [[[0, 2, 3], {1}],
       [{0, 3}, {1, 2}],
       [{0, 1, 3}, {2}],
       [{0}, {1, 2, 3}],
       [{0, 1}, {2, 3}],
       [{0}, {1, 2}, {3}],
       [[{0}, {1}, {2}],
        [{0}, {1}, {2}, {3}]],
       [{{0}, {1}, {2}, {3}}]]

sage: for p in P:
    ....:  sum([partition_sandpile(S, i).betti(verbose=False)[-1] for i in p])
6
8
3
sage: S.betti()
  0  1  2  3
---------------
 0: 1  -  -  -
 1: -  6  8  3
---------------
total:  1  6  8  3

sage.sandpiles.sandpile.aztec_sandpile(n)
The aztec diamond graph.

INPUT:

n – integer

OUTPUT:

dictionary for the aztec diamond graph

EXAMPLES:

sage: from sage.sandpiles.sandpile import aztec_sandpile
sage: T = aztec_sandpile(2)
sage: sorted(len(v) for u, v in T.items())
[3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 8]
sage: Sandpile(T,(0, 0)).group_order()
4542720

Note: This is the aztec diamond graph with a sink vertex added. Boundary vertices have edges to the sink so that each vertex has degree 4.

sage.sandpiles.sandpile.firing_graph(S, eff)
Creates a digraph with divisors as vertices and edges between two divisors $D$ and $E$ if firing a single vertex in $D$ gives $E$.

INPUT:

S – Sandpile

eff – list of divisors
OUTPUT:
DiGraph

EXAMPLES:

```python
sage: S = sandpiles.Cycle(6)
sage: D = SandpileDivisor(S, [1,1,1,1,2,0])
sage: eff = D.effective_div()
sage: firing_graph(S,eff).show3d(edge_size=.005,vertex_size=0.01) # long time
```

`sage.sandpiles.sandpile.glue_graphs(g, h, glue_g, glue_h)`

Glue two graphs together.

INPUT:

- `g`, `h` – dictionaries for directed multigraphs
- `glue_h`, `glue_g` – dictionaries for a vertex

OUTPUT:
dictionary for a directed multigraph

EXAMPLES:

```python
sage: from sage.sandpiles.sandpile import glue_graphs
sage: x = {0: {}, 1: {0: 1}, 2: {0: 1, 1: 1}, 3: {0: 1, 1: 1, 2: 1}}
sage: y = {0: {}, 1: {0: 2}, 2: {1: 2}, 3: {0: 1, 2: 1}}
sage: glue_x = {1: 1, 3: 2}
sage: glue_y = {0: 1, 1: 2, 3: 1}
sage: z = glue_graphs(x,y,glue_x,glue_y); z
  {'sink': {},
   'x0': {'sink': 1, 'x1': 1, 'x3': 2, 'y1': 2, 'y3': 1},
   'x1': {'x0': 1},
   'x2': {'x0': 1, 'x1': 1},
   'x3': {'x0': 1, 'x1': 1, 'x2': 1},
   'y1': {'sink': 2},
   'y2': {'y1': 2},
   'y3': {'sink': 1, 'y2': 1}}
sage: S = Sandpile(z,'sink')
sage: S.h_vector()
  [1, 6, 17, 31, 41, 41, 31, 17, 6, 1]
sage: S.resolution()
  'R^1 <-- R^7 <-- R^21 <-- R^35 <-- R^35 <-- R^21 <-- R^7 <-- R^1'
```

**Note:** This method makes a dictionary for a graph by combining those for `g` and `h`. The sink of `g` is replaced by a vertex that is connected to the vertices of `g` as specified by `glue_g` the vertices of `h` as specified in `glue_h`. The sink of the glued graph is 'sink'.

Both `glue_g` and `glue_h` are dictionaries with entries of the form `v:w` where `v` is the vertex to be connected to and `w` is the weight of the connecting edge.

`sage.sandpiles.sandpile.min_cycles(G, v)`

Minimal length cycles in the digraph `G` starting at vertex `v`.

INPUT:

- `G` – DiGraph
- `v` – vertex of `G`
OUTPUT:
list of lists of vertices

EXAMPLES:

```python
sage: from sage.sandpiles.sandpile import min_cycles, sandlib
sage: T = sandlib('gor')
sage: [min_cycles(T, i) for i in T.vertices()]
[[], [[1, 3]], [[2, 3, 1], [2, 3]], [[3, 1], [3, 2]]
```

`sage.sandpiles.sandpile.parallel_firing_graph(S, eff)`
Creates a digraph with divisors as vertices and edges between two divisors $D$ and $E$ if firing all unstable vertices in $D$ gives $E$.

INPUT:
- $S$ – Sandpile
- eff – list of divisors

OUTPUT:
DiGraph

EXAMPLES:

```python
sage: S = sandpiles.Cycle(6)
sage: D = SandpileDivisor(S, [1,1,1,1,2,0])
sage: eff = D.effective_div()
sage: parallel_firing_graph(S, eff).show3d(edge_size=.005, vertex_size=0.01)  # long
```

`sage.sandpiles.sandpile.partition_sandpile(S, p)`
Each set of vertices in $p$ is regarded as a single vertex, with an edge between $A$ and $B$ if some element of $A$ is connected by an edge to some element of $B$ in $S$.

INPUT:
- $S$ – Sandpile
- $p$ – partition of the vertices of $S$

OUTPUT:
Sandpile

EXAMPLES:

```python
sage: from sage.sandpiles.sandpile import admissible_partitions, partition_sandpile
sage: S = sandpiles.Cycle(4)
sage: P = [admissible_partitions(S, i) for i in [2,3,4]]
sage: for p in P:
....:     sum([partition_sandpile(S, i).betti(whole=False)[-1] for i in p])
6
8
3
sage: S.betti()
0 1 2 3
-------------------------------
0: 1 - - -
1: - 6 8 3
```

(continues on next page)
sage.sandpiles.sandpile.random_DAG(num_verts, p=0.5, weight_max=1)
A random directed acyclic graph with num_verts vertices. The method starts with the sink vertex and adds vertices one at a time. Each vertex is connected only to only previously defined vertices, and the probability of each possible connection is given by the argument p. The weight of an edge is a random integer between 1 and weight_max.

INPUT:
- num_verts – positive integer
- p – (default: 0.5) real number such that 0 < p ≤ 1
- weight_max – (default: 1) positive integer

OUTPUT:
a dictionary, encoding the edges of a directed acyclic graph with sink 0

EXAMPLES:

```
sage: from sage.sandpiles.sandpile import random_DAG
sage: d = DiGraph(random_DAG(5, .5)); d
doctest:...: DeprecationWarning: method random_DAG is deprecated. Please use digraphs.RandomDirectedAcyclicGraph instead.
See https://trac.sagemath.org/30479 for details.
Digraph on 5 vertices
```

sage.sandpiles.sandpile.sandlib(selector=None)
Returns the sandpile identified by selector. If no argument is given, a description of the sandpiles in the sandlib is printed.

INPUT:
- selector – (optional) identifier or None

OUTPUT:
sandpile or description

EXAMPLES:

```
sage: from sage.sandpiles.sandpile import sandlib
sage: sandlib()
Sandpiles in the sandlib:
   ci1 : complete intersection, non-DAG but equivalent to a DAG
genetic : generic digraph with 6 vertices
genus2 : Undirected graph of genus 2
gor : Gorenstein but not a complete intersection
kite : generic undirected graphs with 5 vertices
riemann-roch1 : directed graph with postulation 9 and 3 maximal weight
   superstables
   riemann-roch2 : directed graph with a superstable not majorized by a maximal
   superstable
sage: S = sandlib('gor')
sage: S.resolution()
'R^1 <-- R^5 <-- R^5 <-- R^1'
```
sage.sandpiles.sandpile.triangle_sandpile(n)
A triangular sandpile. Each nonsink vertex has out-degree six. The vertices on the boundary of the triangle are
closed to the sink.

INPUT:

n – integer

OUTPUT:

Sandpile

EXAMPLES:

```
sage: from sage.sandpiles.sandpile import triangle_sandpile
tsage: T = triangle_sandpile(5)
tsage: T.group_order()
135418115000
```

sage.sandpiles.sandpile.wilmes_algorithm(M)
Computes an integer matrix \( L \) with the same integer row span as \( M \) and such that \( L \) is the reduced Laplacian of
a directed multigraph.

INPUT:

\( M \) – square integer matrix of full rank

OUTPUT:

integer matrix (\( L \))

EXAMPLES:

```
sage: P = matrix([[2,3,-7,-3],[5,2,-5,5],[8,2,5,4],[-5,-9,6,6]])
sage: wilmes_algorithm(P)
[ 1642  -13  -1627  -1]
[  -1 1980  -1582  -397]
[   0  -1   1650  -1649]
[   0   0  -1658   1658]
```

REFERENCES:

• [PPW2013]

See also:

• sage.combinat.e_one_star
• sage.combinat.constellation
Chapter 4. Sandpiles
5.1 Generic dynamical systems on schemes

This is the generic class for dynamical systems and contains the exported constructor functions. The constructor functions can take either polynomials (or rational functions in the affine case) or morphisms from which to construct a dynamical system. If the domain is not specified, it is constructed. However, if you plan on working with points or subvarieties in the domain, it recommended to specify the domain. For products of projective spaces the domain must be specified.

The initialization checks are always performed by the constructor functions. It is possible, but not recommended, to skip these checks by calling the class initialization directly.

AUTHORS:

• Ben Hutz (July 2017): initial version

```python
class sage.dynamics.arithmetic_dynamics.generic_ds.DynamicalSystem(polys_or_rat_fncts, domain)
Bases: sage.schemes.generic.morphism.SchemeMorphism_polynomial

Base class for dynamical systems of schemes.

INPUT:

• polys_or_rat_fncts – a list of polynomials or rational functions, all of which should have the same parent
• domain – an affine or projective scheme, or product of projective schemes, on which polys defines an endomorphism. Subschemes are also ok
• names – (default: ('X', 'Y')) tuple of strings to be used as coordinate names for a projective space that is constructed

The following combinations of morphism_or_polys and domain are meaningful:

- morphism_or_polys is a SchemeMorphism; domain is ignored in this case
- morphism_or_polys is a list of homogeneous polynomials that define a rational endomorphism of domain
- morphism_or_polys is a list of homogeneous polynomials and domain is unspecified; domain is then taken to be the projective space of appropriate dimension over the common parent of the elements in morphism_or_polys
- morphism_or_polys is a single polynomial or rational function; domain is ignored and taken to be a 1-dimensional projective space over the base ring of morphism_or_polys with coordinate names given by names

EXAMPLES:
```
sage: A.<x> = AffineSpace(QQ,1)
sage: f = DynamicalSystem_affine([x^2+1])
sage: type(f)
<class 'sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine_field'>

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: type(f)
<class 'sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective_field'>

sage: P1.<x,y> = ProjectiveSpace(CC,1)
sage: H = End(P1)
sage: DynamicalSystem(H([CC.0*x^2, y^2]))
Dynamical System of Projective Space of dimension 1 over Complex Field with 53 bits of precision
  Defn: Defined on coordinates by sending (x : y) to
        (y : x)

\texttt{DynamicalSystem} defaults to projective:

sage: R.<x,y,z> = QQ[]
sage: DynamicalSystem([x^2, y^2, z^2])
Dynamical System of Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to
        (x^2 : y^2 : z^2)

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: DynamicalSystem([y, x], domain=A)
Dynamical System of Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to
        (y, x)
sage: H = End(A)
sage: DynamicalSystem(H([y, x]))
Dynamical System of Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to
        (y, x)

Note that \texttt{domain} is ignored if an endomorphism is passed in:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: P2.<x,y> = ProjectiveSpace(CC, 1)
sage: H = End(P2)
sage: f = H([CC.0*x^2, y^2])
sage: g = DynamicalSystem(f, domain=P)
sage: g.domain()
Projective Space of dimension 1 over Complex Field with 53 bits of precision

Constructing a common parent:

sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: DynamicalSystem([CC.0*x^2, 4/5*y^2])
Dynamical System of Projective Space of dimension 1 over Complex Field with 53 bits of precision
  Defn: Defined on coordinates by sending (x : y) to
        (1.00000000000000*I*x^2 : 0.800000000000000*y^2)

(continues on next page)
sage: P.<x,y> = ProjectiveSpace(GF(5), 1)
sage: K.<t> = GF(25)
sage: DynamicalSystem([GF(5)(3)*x^2, K(t)*y^2])
Dynamical System of Projective Space of dimension 1 over Finite Field in t of size 5^2
  Defn: Defined on coordinates by sending (x : y) to
        (-2*x^2 : (t)*y^2)

\textbf{as\_scheme\_morphism()}

Return this dynamical system as SchemeMorphism\_polynomial.

\textbf{OUTPUT: SchemeMorphism\_polynomial}

\textbf{EXAMPLES:}

\begin{verbatim}
sage: P.<x,y,z> = ProjectiveSpace(ZZ, 2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2])
sage: type(f.as_scheme_morphism())
<class 'sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space'>

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([x^2-y^2, y^2])
sage: type(f.as_scheme_morphism())
<class 'sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space_field'>

sage: P.<x,y> = ProjectiveSpace(GF(5), 1)
sage: f = DynamicalSystem_projective([x^2, y^2])
sage: type(f.as_scheme_morphism())
<class 'sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space_finite_field'>

sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: f = DynamicalSystem_affine([x^2-2, y^2])
sage: type(f.as_scheme_morphism())
<class 'sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space'>

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: f = DynamicalSystem_affine([x^2-2, y^2])
sage: type(f.as_scheme_morphism())
<class 'sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space_field'>

sage: A.<x,y> = AffineSpace(GF(3), 2)
sage: f = DynamicalSystem_affine([x^2-2, y^2])
sage: type(f.as_scheme_morphism())
<class 'sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space_finite_field'>
\end{verbatim}

\textbf{change\_ring}(R, check=True)

Return a new dynamical system which is this map coerced to R.

If check is True, then the initialization checks are performed.

\textbf{INPUT:}
• \( R \) – ring or morphism

OUTPUT:

A new \texttt{DynamicalSystem\_projective} that is this map coerced to \( R \).

EXAMPLES:

\begin{verbatim}
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem\_projective([3*x^2, y^2])
sage: f.change_ring(GF(5))
Dynamical System of Projective Space of dimension 1 over Finite Field of size 5

Defn: Defined on coordinates by sending (x : y) to
(-2*x^2 : y^2)
\end{verbatim}

\texttt{field\_of\_definition\_critical}(\texttt{return\_embedding=False}, \texttt{simplify\_all=False}, \texttt{names='a'})

Return smallest extension of the base field which contains the critical points

Ambient space of dynamical system must be either the affine line or projective line over a number field or finite field.

INPUT:

• \texttt{return\_embedding} – (default: False) boolean; If True, return an embedding of base field of dynamical system into the returned number field or finite field. Note that computing this embedding might be expensive.

• \texttt{simplify\_all} – (default: False) boolean; If True, simplify intermediate fields and also the resulting number field. Note that this is not implemented for finite fields and has no effect

• \texttt{names} – (optional) string to be used as generator for returned number field or finite field

OUTPUT:

If \texttt{return\_embedding} is False, the field of definition as an absolute number field or finite field. If \texttt{return\_embedding} is True, a tuple \((K, \phi)\) where \(\phi\) is an embedding of the base field in \(K\).

EXAMPLES:

Note that the number of critical points is \(2d-2\), but \((1:0)\) has multiplicity 2 in this case:

\begin{verbatim}
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([1/3*x^3 + x*y^2, y^3], domain=P)
sage: f.critical_points()
[[(1 : 0)]]
sage: N.<a> = f.field\_of\_definition\_critical(); N
Number Field in a with defining polynomial x^2 + 1
sage: g = f.change_ring(N)
sage: g.critical_points()
[(-a : 1), (a : 1), (1 : 0)]
\end{verbatim}

\begin{verbatim}
sage: A.<z> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem([z^4 + 2*z^2 + 2], domain=A)
sage: K.<a> = f.field\_of\_definition\_critical(); K
Number Field in a with defining polynomial z^2 + 1
sage: G.<a> = GF(9)
sage: R.<z> = G[]
sage: R.irreducible\_element(3, \texttt{algorithm='first\_lexicographic'})
z^3 + (a + 1)*z + a
\end{verbatim}
sage: A.<x> = AffineSpace(G,1)
sage: f = DynamicalSystem([x^4 + (2*a+2)*x^2 + a*x], domain=A)
sage: f[0].derivative(x).univariate_polynomial().is_irreducible()
True
sage: f.field_of_definition_critical(return_embedding=True, names='b')
(Finite Field in b of size 3^6, Ring morphism:
  From: Finite Field in a of size 3^2
  To:   Finite Field in b of size 3^6
  Defn: a |--> 2*b^5 + 2*b^3 + b^2 + 2*b + 2)

field_of_definition_periodic(n,  formal=False,  return_embedding=False,  sim-
plify_all=False,  names='a')

Return smallest extension of the base field which contains all fixed points of the n-th iterate

Ambient space of dynamical system must be either the affine line or projective line over a number field or finite field.

INPUT:

• n – a positive integer

• formal – (default: False) boolean; True signals to return number field or finite field over which the formal periodic points are defined, where a formal periodic point is a root of the n-th dynatomic polynomial. False specifies to find number field or finite field over which all periodic points of the n-th iterate are defined

• return_embedding – (default: False) boolean; If True, return an embedding of base field of dynamical system into the returned number field or finite field. Note that computing this embedding might be expensive.

• simplify_all – (default: False) boolean; If True, simplify intermediate fields and also the resulting number field. Note that this is not implemented for finite fields and has no effect

• names – (optional) string to be used as generator for returned number field or finite field

OUTPUT:

If return_embedding is False, the field of definition as an absolute number field or finite field. If return_embedding is True, a tuple (K, phi) where phi is an embedding of the base field in K.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([x^2, y^2], domain=P)
sage: f.periodic_points(3, minimal=False)
[(0 : 1), (1 : 0), (1 : 1)]
sage: N.<a> = f.field_of_definition_periodic(3); N
Number Field in a with defining polynomial x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: sorted(f.periodic_points(3,minimal=False, R=N), key=str)
[(-a^5 - a^4 - a^3 - a^2 - a - 1 : 1),
 (0 : 1),
 (1 : 0),
 (1 : 1),
 (a : 1),
 (a^2 : 1),
 (a^3 : 1),
 (a^4 : 1),
 (a^5 : 1)]
sage: A.<z> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem([z^2 + 1/(2*z + 1)], domain=A)
sage: K.<a> = f.field_of_definition_periodic(2); K
Number Field in a with defining polynomial z^4 + 12*z^3 + 39*z^2 + 18*z + 171
sage: F.<b> = f.field_of_definition_periodic(2, formal=True); F
Number Field in b with defining polynomial z^2 + 3*z + 6

sage: G.<a> = GF(4)
sage: A.<x> = AffineSpace(G, 1)
sage: f = DynamicalSystem([x^2 + (a+1)*x + 1], domain=A)
sage: g = f.nth_iterate_map(2)[0]

sage: (g-x).univariate_polynomial().factor()
(x + 1) * (x + a + 1) * (x^2 + a*x + 1)

sage: f.field_of_definition_periodic(2, return_embedding=True, names='b')
((Finite Field in b of size 2^4, Ring morphism:
  From: Finite Field in a of size 2^2
  To: Finite Field in b of size 2^4
  Defn: a |--> b^2 + b),
 (Finite Field in b of size 2^4, Ring morphism:
  From: Finite Field in a of size 2^2
  To: Finite Field in b of size 2^4
  Defn: a |--> b^2 + b))

field_of_definition_preimage(point, n, return_embedding=False, simplify_all=False, names='a')

Return smallest extension of the base field which contains the n-th preimages of point

Ambient space of dynamical system must be either the affine line or projective line over a number field or finite field.

INPUT:

• point – a point in this map’s domain
• n – a positive integer
• return_embedding – (default: False) boolean; If True, return an embedding of base field of dynamical system into the returned number field or finite field. Note that computing this embedding might be expensive.
• simplify_all – (default: False) boolean; If True, simplify intermediate fields and also the resulting number field. Note that this is not implemented for finite fields and has no effect

• names – (optional) string to be used as generator for returned number field or finite field

OUTPUT:

If return_embedding is False, the field of definition as an absolute number field or finite field. If return_embedding is True, a tuple (K, phi) where phi is an embedding of the base field in K.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([[1/3*x^2 + 2/3*x*y, x^2 - 2*y^2], domain=P)
sage: N.<a> = f.field_of_definition_preimage(P(1,1), 2, simplify_all=True); N
Number Field in a with defining polynomial x^8 - 4*x^7 - 128*x^6 + 398*x^5 + 3913*x^4 - 8494*x^3 - 26250*x^2 + 30564*x - 2916

sage: A.<z> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem([z^2], domain=A)
sage: K.<a> = f.field_of_definition_preimage(A(1), 3); K
Number Field in a with defining polynomial z^4 + 1
sage: G = GF(5)
sage: P.<x,y> = ProjectiveSpace(G, 1)
sage: f = DynamicalSystem([x^2 + 2*y^2, y^2], domain=P)
sage: f.field_of_definition_preimage(P(2,1), 2, return_embedding=True, names='a')
(Finite Field in a of size 5^2, Ring morphism:
  From: Finite Field of size 5
  To:  Finite Field in a of size 5^2
  Defn: 1 |--> 1)

specialization (D=None, phi=None, homset=None)

Specialization of this dynamical system.

Given a family of maps defined over a polynomial ring. A specialization is a particular member of that family. The specialization can be specified either by a dictionary or a SpecializationMorphism.

INPUT:

- D – (optional) dictionary
- phi – (optional) SpecializationMorphism
- homset – (optional) homset of specialized map

OUTPUT: DynamicalSystem

EXAMPLES:

sage: R.<c> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(R, 1)
sage: f = DynamicalSystem_projective([x^2 + c*y^2, y^2], domain=P)
sage: f.specialization({c:1})
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defined on coordinates by sending (x : y) to
  (x^2 + y^2 : y^2)

5.2 Dynamical systems on affine schemes

An endomorphism of an affine scheme or subscheme determined by polynomials or rational functions.

The main constructor function is given by DynamicalSystem_affine. The constructor function can take polynomials, rational functions, or morphisms from which to construct a dynamical system. If the domain is not specified, it is constructed. However, if you plan on working with points or subvarieties in the domain, it recommended to specify the domain.

The initialization checks are always performed by the constructor functions. It is possible, but not recommended, to skip these checks by calling the class initialization directly.

AUTHORS:

- David Kohel, William Stein
- Volker Braun (2011-08-08): Renamed classes, more documentation, misc cleanups.
- Ben Hutz (2017) relocate code and create new class

class sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine(polys_or_rat_fncts, do-main)

Bases: sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space.
An endomorphism of affine schemes determined by rational functions.

**Warning:** You should not create objects of this class directly because no type or consistency checking is performed. The preferred method to construct such dynamical systems is to use `DynamicalSystem_affine()` function.

**INPUT:**
- `morphism_or_polys` – a SchemeMorphism, a polynomial, a rational function, or a list or tuple of polynomials or rational functions
- `domain` – optional affine space or subscheme of such; the following combinations of `morphism_or_polys` and `domain` are meaningful:
  - `morphism_or_polys` is a SchemeMorphism; `domain` is ignored in this case
  - `morphism_or_polys` is a list of polynomials or rational functions that define a rational endomorphism of `domain`
  - `morphism_or_polys` is a list of polynomials or rational functions and `domain` is unspecified; `domain` is then taken to be the affine space of appropriate dimension over the common base ring, if one exists, of the elements of `morphism_or_polys`
  - `morphism_or_polys` is a single polynomial or rational function; `domain` is ignored and assumed to be the 1-dimensional affine space over the base ring of `morphism_or_polys`

**OUTPUT:** `DynamicalSystem_affine`

**EXAMPLES:**

```python
sage: A3.<x,y,z> = AffineSpace(QQ, 3)
sage: DynamicalSystem_affine([x, y, 1])
Dynamical System of Affine Space of dimension 3 over Rational Field
  Defn: Defined on coordinates by sending (x, y, z) to
        (x, y, 1)

sage: R.<x,y> = QQ[]
sage: DynamicalSystem_affine([x/y, y^2 + 1])
Dynamical System of Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to
        (x/y, y^2 + 1)

sage: R.<t> = ZZ[]
sage: DynamicalSystem_affine(t^2 - 1)
Dynamical System of Affine Space of dimension 1 over Integer Ring
  Defn: Defined on coordinates by sending (t) to
        (t^2 - 1)

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: X = A.subscheme([x-y^2])
sage: DynamicalSystem_affine([9/4*x^2, 3/2*y], domain=X)
Dynamical System of Closed subscheme of Affine Space of dimension 2 over Rational Field defined by:
- y^2 + x
  Defn: Defined on coordinates by sending (x, y) to
        (9/4*x^2, 3/2*y)
```

Chapter 5. Arithmetic Dynamical Systems
```python
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: H = End(A)
sage: f = H([x^2, y^2])
sage: DynamicalSystem_affine(f)
Dynamical System of Affine Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x, y) to (x^2, y^2)
```

Notice that \( \mathbb{Z} \) becomes \( \mathbb{Q} \) since the function is rational:

```python
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: DynamicalSystem_affine([3*x^2/(5*y), y^2/(2*x^2)])
Dynamical System of Affine Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x, y) to (3*x^2/(5*y), y^2/(2*x^2))
```

```python
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: DynamicalSystem_affine([3/2*x^2, y^2])
Dynamical System of Affine Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x, y) to (3/2*x^2, y^2)
```

If you pass in quotient ring elements, they are reduced:

```python
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: X = A.subscheme([x-y])
sage: u,v,w = X.coordinate_ring().gens()
sage: DynamicalSystem_affine([u, v, u+v], domain=X)
Dynamical System of Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
\begin{align*}
x - y
\end{align*}
Defn: Defined on coordinates by sending (x, y, z) to (y, y, 2*y)
```

```python
sage: R.<t> = PolynomialRing(QQ)
sage: A.<x,y,z> = AffineSpace(R, 3)
sage: X = A.subscheme(x^2-y^2)
sage: H = End(X)
sage: f = H([x^2/(t*y), t*y^2, x*z])
sage: DynamicalSystem_affine(f)
Dynamical System of Closed subscheme of Affine Space of dimension 3 over Univariate Polynomial Ring in t over Rational Field defined by:
\begin{align*}
x^2 - y^2
\end{align*}
Defn: Defined on coordinates by sending (x, y, z) to (x^2/(t*y), t*y^2, x*z)
```

```python
sage: x = var('x')
sage: DynamicalSystem_affine(x^2+1)
Traceback (most recent call last):
...TypeError: Symbolic Ring cannot be the base ring
```

**conjugate** \( (M) \)

Conjugate this dynamical system by \( M \), i.e. \( M^{-1} \circ f \circ M \).

If possible the new map will be defined over the same space. Otherwise, will try to coerce to the base ring of \( M \).
INPUT:

- \( M \) – a square invertible matrix

OUTPUT:

An affine dynamical system

EXAMPLES:

```sage
A.<t> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([t^2+1])
sage: f.conjugate(matrix([[1,2], [0,1]]))
Dynamical System of Affine Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (t) to
  (t^2 + 4*t + 3)
```

```sage
A.<x,y> = AffineSpace(ZZ,2)
sage: f = DynamicalSystem_affine([x^3+y^3, y^2])
sage: f.conjugate(matrix([[1,2,3], [0,1,2], [0,0,1]]))
Dynamical System of Affine Space of dimension 2 over Integer Ring
  Defn: Defined on coordinates by sending (x, y) to
  (x^3 + 6*x^2*y + 12*x*y^2 + 9*y^3 + 9*x^2 + 36*x*y + 40*y^2 + 27*x + 58*y + 28, y^2 + 4*y + 2)
```

```sage
R.<x> = PolynomialRing(QQ)
sage: K.<i> = NumberField(x^2+1)
sage: A.<x> = AffineSpace(ZZ,1)
sage: f = DynamicalSystem_affine([x^3+2*x^2+3])
sage: f.conjugate(matrix([[i,i], [0,-i]]))
Dynamical System of Affine Space of dimension 1 over Integer Ring
  Defn: Defined on coordinates by sending (x) to
  (x^3 + x^2 - x - 5)
```

dynatomic_polynomial(period)

Compute the (affine) dynatomic polynomial of a dynamical system \( f : \mathbb{A}^1 \to \mathbb{A}^1 \).

The dynatomic polynomial is the analog of the cyclotomic polynomial and its roots are the points of formal period \( n \).

ALGORITHM:

Homogenize to a map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) and compute the dynatomic polynomial there. Then, dehomogenize.

INPUT:

- \( \text{period} \) – a positive integer or a list/tuple \([m, n]\), where \( m \) is the preperiod and \( n \) is the period

OUTPUT:

If possible, a single variable polynomial in the coordinate ring of the polynomial. Otherwise a fraction field element of the coordinate ring of the polynomial.

EXAMPLES:

```sage
A.<x,y> = AffineSpace(QQ, 2)
sage: f = DynamicalSystem_affine([x^2+y^2, y^2])
sage: f.dynatomic_polynomial(2)
Traceback (most recent call last):
...
TypeError: does not make sense in dimension >1
```
Sage: A.<x> = AffineSpace(ZZ, 1)
sage: f = DynamicalSystem_affine([(x^2+1)/x])
sage: f.dynatomic_polynomial(4)
2*x^12 + 18*x^10 + 57*x^8 + 79*x^6 + 48*x^4 + 12*x^2 + 1

Sage: A.<x> = AffineSpace(CC, 1)
sage: f = DynamicalSystem_affine([(x^2+1)/(3*x)])
sage: f.dynatomic_polynomial(3)
13.0000000000000*x^6 + 117.0000000000000*x^4 + 78.0000000000000*x^2 + 1.00000000000000

Sage: A.<x> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([x^2-10/9])
sage: f.dynatomic_polynomial([2, 1])
531441*x^4 - 649539*x^2 + 1.00000000000000

Sage: A.<x> = AffineSpace(CC, 1)
sage: f = DynamicalSystem_affine([x^2+CC.0])
sage: f.dynatomic_polynomial(2)
x^2 + x + 1.00000000000000 + 1.00000000000000*I

Sage: K.<c> = FunctionField(QQ)
Sage: A.<x> = AffineSpace(K, 1)
sage: f = DynamicalSystem_affine([x^2 + c])
sage: f.dynatomic_polynomial(4)
x^12 + 6*c*x^10 + x^9 + (15*c^2 + 3*c)*x^8 + 4*c*x^7 + (20*c^3 + 12*c^2 + \rightarrow 1)*x^6 
+ (6*c^2 + 2*c)*x^5 + (15*c^4 + 18*c^3 + 3*c^2 + 4*c)*x^4 + (4*c^3 + 4*c^2 + \rightarrow 1)*x^3 
+ (6*c^5 + 12*c^4 + 6*c^3 + 5*c^2 + c)*x^2 + (c^4 + 2*c^3 + c^2 + 2*c)*x 
+ c^6 + 3*c^5 + 3*c^4 + 3*c^3 + 2*c^2 + 1

Sage: A.<z> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([z^2+3/z+1/7])
sage: f.dynatomic_polynomial([1,1]).parent()
Multivariate Polynomial Ring in z over Rational Field

Sage: R.<c> = QQ[]
Sage: A.<z> = AffineSpace(R, 1)
sage: f = DynamicalSystem_affine([z^2 + c])
sage: f.dynatomic_polynomial([1,1])
z^2 + z + c

Sage: A.<x> = AffineSpace(CC, 1)
sage: F = DynamicalSystem_affine([1/2*x^2 + CC(sqrt(3))])
sage: F.dynatomic_polynomial([1,1])
(0.125000000000000*x^4 + 0.366025403784439*x^2 + 1.50000000000000)/(0. \rightarrow 500000000000000*x^2 - x + 1.73205080756888)

**homogenize**(n)

Return the homogenization of this dynamical system.

If its domain is a subscheme, the domain of the homogenized map is the projective embedding of the domain. The domain and codomain can be homogenized at different coordinates: n[0] for the domain and n[1] for the codomain.

5.2. Dynamical systems on affine schemes 115
INPUT:

- \( n \) – a tuple of nonnegative integers. If \( n \) is an integer, then the two values of the tuple are assumed to be the same

OUTPUT: \texttt{DynamicalSystem_projective}

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: f = DynamicalSystem_affine([(x^2-2)/x^5, y^2])
sage: f.homogenize(2)
Dynamical System of Projective Space of dimension 2 over Rational Field
Defn: Defined on coordinates by sending (x0 : x1 : x2) to
     (x0^2*x2^5 - 2*x2^7 : x0^5*x1^2 : x0^5*x2^2)
```

```
sage: A.<x,y> = AffineSpace(CC, 2)
sage: f = DynamicalSystem_affine([(x^2-2)/(x*y), y^2-x])
sage: f.homogenize((2, 0))
Dynamical System of Projective Space of dimension 2 over Complex Field with ˓→53 bits of precision
Defn: Defined on coordinates by sending (x0 : x1 : x2) to
     (x0*x1*x2^2 : x0^2*x2^2 + (-2.00000000000000)*x2^4 : x0*x1^3 - x0^˓→2*x1*x2)
```

```
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: X = A.subscheme([x-y^2])
sage: f = DynamicalSystem_affine([9*y^2, 3*y], domain=X)
sage: f.homogenize(2)
Dynamical System of Closed subscheme of Projective Space of dimension 2 over Integer Ring defined by:
     x1^2 - x0*x2
Defn: Defined on coordinates by sending (x0 : x1 : x2) to
     (9*x1^2 : 3*x1*x2 : x2^2)
```

```
sage: A.<x> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([x^2-1])
sage: f.homogenize((1, 0))
Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x0 : x1) to
     (x1^2 : x0^2 - x1^2)
```

```
sage: R.<a> = PolynomialRing(QQbar)
sage: A.<x,y> = AffineSpace(R, 2)
sage: f = DynamicalSystem_affine([QQbar(sqrt(2))*x*y, a*x^2])
sage: f.homogenize(2)
Dynamical System of Projective Space of dimension 2 over Univariate Polynomial Ring in a over Algebraic Field
Defn: Defined on coordinates by sending (x0 : x1 : x2) to
     (1.414213562373095?*x0*x1 : a*x0^2 : x2^2)
```

```
sage: P.<x,y,z> = AffineSpace(QQ, 3)
sage: f = DynamicalSystem_affine([x^2 - 2*x*y + z*x, z^2 -y^2 , 5*z*y])
sage: f.homogenize(2).dehomogenize(2) == f
True
```

\texttt{multiplier}(\texttt{P}, \texttt{n}, \texttt{check=True})

Return the multiplier of the point \( P \) of period \( n \) by this dynamical system.
INPUT:

- P – a point on domain of the map
- n – a positive integer, the period of P
- check – (default: True) boolean, verify that P has period n

OUTPUT:

A square matrix of size self.codomain().dimension_relative() in the base_ring of the map.

EXAMPLES:

```sage
sage: P.<x,y> = AffineSpace(QQ, 2)
sage: f = DynamicalSystem_affine([x^2, y^2])
sage: f.multiplier(P([1, 1]), 1)
[2 0]
[0 2]
```

```sage
sage: P.<x,y,z> = AffineSpace(QQ, 3)
sage: f = DynamicalSystem_affine([x, y^2, z^2 - y])
sage: f.multiplier(P([1/2, 1, 0]), 2)
[1 0 0]
[0 4 0]
[0 0 0]
```

```sage
sage: P.<x> = AffineSpace(CC, 1)
sage: f = DynamicalSystem_affine([x^2 + 1/2])
sage: f.multiplier(P([0.5 + 0.5*I]), 1)
[1.00000000000000 + 1.00000000000000*I]
```

```sage
sage: R.<t> = PolynomialRing(CC, 1)
sage: P.<x> = AffineSpace(R, 1)
sage: f = DynamicalSystem_affine([x^2 - t^2 + t])
sage: f.multiplier(P([-t + 1]), 1)
[(-2.00000000000000)*t + 2.00000000000000]
```

```sage
sage: P.<x,y> = AffineSpace(QQ, 2)
sage: X = P.subscheme([x^2-y^2])
sage: f = DynamicalSystem_affine([x^2, y^2], domain=X)
sage: f.multiplier(X([1, 1]), 1)
[2 0]
[0 2]
```

\textbf{nth_iterate}(P, n)

Return the n-th iterate of the point P by this dynamical system.

INPUT:

- P – a point in the map’s domain
- n – a positive integer

OUTPUT: a point in the map’s codomain

EXAMPLES:
\begin{verbatim}
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: f = DynamicalSystem_affine([(x-2*y^2)/x, 3*x+y])
sage: f.nth_iterate(A(9, 3), 3)
(-104975/13123, -9566667)
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: X = A.subscheme([x-y^2])
sage: f = DynamicalSystem_affine([9*y^2, 3*y], domain=X)
sage: f.nth_iterate(X(9, 3), 4)
(59049, 243)
sage: R.<t> = PolynomialRing(QQ)
sage: A.<x,y> = AffineSpace(FractionField(R), 2)
sage: f = DynamicalSystem_affine([(x-t*y^2)/x, t*x*y])
sage: f.nth_iterate(A(1, t), 3)
((-t^16 + 3*t^13 - 3*t^10 + t^7 + t^5 + t^3 - 1)/(t^5 + t^3 - 1), -t^9 - t^7 → + t^4)
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: X = A.subscheme([x^2-y^2])
sage: f = DynamicalSystem_affine([x^2, y^2, x+y], domain=X)
sage: f.nth_iterate_map(2)
Dynamical System of Closed subscheme of Affine Space of dimension 3 over Rational Field defined by:
    x^2 - y^2
Defn: Defined on coordinates by sending (x, y, z) to
    (x^4, y^4, x^2 + y^2)
\end{verbatim}

\textbf{nth_iterate_map} \(n\)

Return the \(n\)-th iterate of \texttt{self}.

\textbf{ALGORITHM:}

Uses a form of successive squaring to reducing computations.

\textbf{Todo:} This could be improved.

\textbf{INPUT:}

- \(n\) – a positive integer

\textbf{OUTPUT:} a dynamical system of affine space

\textbf{EXAMPLES:}

\begin{verbatim}
sage: A.<x,y> = AffineSpace(ZZ, 2)
sage: f = DynamicalSystem_affine([(x^2-2)/(2*y), y^2-3*x])
sage: f.nth_iterate_map(2)
Dynamical System of Affine Space of dimension 2 over Rational Field
    Defined on coordinates by sending (x, y) to
    ((x^4 - 4*x^2 - 8*y^2 + 4)/(8*y^4 - 24*x*y^2), (2*y^5 - 12*x*y^3 + 18*x^2*y - 3*x^2 + 6)/(2*y))
sage: A.<x> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([(3*x^2-2)/(x)])
sage: f.nth_iterate_map(3)
(continues on next page)
\end{verbatim}
Dynamical System of Affine Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x) to
\((\frac{(2187x^8 - 6174x^6 + 6300x^4 - 2744x^2 + 432)}{81x^7 - 168x^5 + 112x^3 - 24x})\)

\[
\text{sage: } A.<x,y> = AffineSpace(ZZ, 2) \\
\text{sage: } X = A.subscheme([x-y^2]) \\
\text{sage: } f = DynamicalSystem_affine([9*x^2, 3*y], domain=X) \\
\text{sage: } f.nth_iterate_map(2) \\
\text{Dynamical System of Closed subscheme of Affine Space of dimension 2 over Integer Ring defined by:} \\
-\text{Defn: Defined on coordinates by sending (x, y) to} \\
(729x^4, 9y)
\]

\[
\text{sage: } A.<x,y> = AffineSpace(QQ, 2) \\
\text{sage: } f = DynamicalSystem_affine([3/5*x^2, y^2/(2*x^2)]) \\
\text{sage: } f.nth_iterate_map(2) \\
\text{Dynamical System of Affine Space of dimension 2 over Rational Field} \\
\text{Defn: Defined on coordinates by sending (x, y) to} \\
(27/125*x^4, y^4/(72/25*x^8))
\]

**orbit** \((P, n)\)
Return the orbit of \(P\) by the dynamical system.

Let \(F\) be this dynamical system. If \(n\) is an integer return \([P, F(P), \ldots, F^n(P)]\). If \(n\) is a list or tuple \(n = [m, k]\) return \([F^m(P), \ldots, F^k(P)]\).

**INPUT:**
- \(P\) – a point in the map’s domain
- \(n\) – a non-negative integer or list or tuple of two non-negative integers

**OUTPUT:** a list of points in the map’s codomain

**EXAMPLES:**

\[
\text{sage: } A.<x,y> = AffineSpace(QQ, 2) \\
\text{sage: } f = DynamicalSystem_affine([[x-2*y^2]/x, 3*x*y]) \\
\text{sage: } f.orbit(A(9, 3), 3) \\
\text{[(9, 3), (-1, 81), (13123, -243), (-104975/13123, -9566667)]}
\]

\[
\text{sage: } A.<x> = AffineSpace(QQ, 1) \\
\text{sage: } f = DynamicalSystem_affine([[x-2]/x]) \\
\text{sage: } f.orbit(A(1/2), [1, 3]) \\
\text{[(-3), (5/3), (-1/5)]}
\]

\[
\text{sage: } A.<x,y> = AffineSpace(ZZ, 2) \\
\text{sage: } X = A.subscheme([x-y^2]) \\
\text{sage: } f = DynamicalSystem_affine([9*y^2, 3*y], domain=X) \\
\text{sage: } f.orbit(X(9, 3), (0, 4)) \\
\text{[(9, 3), (81, 9), (729, 27), (6561, 81), (59049, 243)]}
\]

\[
\text{sage: } R.<t> = PolynomialRing(QQ) \\
\text{sage: } A.<x,y> = AffineSpace(FractionField(R), 2) \\
\text{[continues on next page]}
\]

### 5.2. Dynamical systems on affine schemes

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sage: f = DynamicalSystem_affine([(x-t*y^2)/x, t*x*y])
sage: f.orbit(A(1, t), 3)
[(1, t), (-t^3 + 1, t^2), ((t^5 + t^3 - 1)/(t^3 - 1), -t^6 + t^3), ((-t^16 + 3*t^13 - 3*t^10 + t^7 + t^5 + t^3 - 1)/(t^5 + t^3 - 1), -t^9 - t^7, \rightarrow t^4)]

class sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine_field(polys_or_rat_fncts, domain)

Bases: sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine
sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space_field

reduce_base_field()

Return this map defined over the field of definition of the coefficients.

The base field of the map could be strictly larger than the field where all of the coefficients are defined. This function reduces the base field to the minimal possible. This can be done when the base ring is a number field, QQbar, a finite field, or algebraic closure of a finite field. 

OUTPUT: A dynamical system

EXAMPLES:

sage: K.<t> = GF(5^2)
sage: A.<x,y> = AffineSpace(K, 2)
sage: f = DynamicalSystem_affine([x^2 + 3*y^2, 3*y^2])
sage: f.reduce_base_field()
Dynamical System of Affine Space of dimension 2 over Finite Field of size 5
  Defn: Defined on coordinates by sending (x, y) to
        (x^2 - 2*y^2, -2*y^2)

sage: A.<x,y> = AffineSpace(QQbar, 2)
sage: f = DynamicalSystem_affine([x^2 + QQbar(sqrt(3))*y^2, QQbar(sqrt(-1))*y^2])
sage: f.reduce_base_field()
Dynamical System of Affine Space of dimension 2 over Number Field in a with defining polynomial y^4 - y^2 + 1 with a = -0.866025403784439? + 0.5000000000000000?*I
  Defn: Defined on coordinates by sending (x, y) to
        (x^2 + (a^3 - 2*a)*y^2, (a^3)*y^2)

sage: K.<v> = CyclotomicField(5)
sage: A.<x,y> = AffineSpace(K, 2)
sage: f = DynamicalSystem_affine([(3*x^2 + y) / (5*x), (y^2+1) / (x+y)])
sage: f.reduce_base_field()
Dynamical System of Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y) to
        ((3*x^2 + y)/(5*x), (5*y^2 + 5)/(5*x + 5*y))

weil_restriction()

Compute the Weil restriction of this morphism over some extension field.

If the field is a finite field, then this computes the Weil restriction to the prime subfield.

A Weil restriction of scalars - denoted $Res_{L/k}$ - is a functor which, for any finite extension of fields $L/k$
and any algebraic variety $X$ over $L$, produces another corresponding variety $Res_{L/k}(X)$, defined over $k$. It is useful for reducing questions about varieties over large fields to questions about more complicated varieties over smaller fields. Since it is a functor it also applied to morphisms. In particular, the functor applied to a morphism gives the equivalent morphism from the Weil restriction of the domain to the Weil restriction of the codomain.

OUTPUT:

Scheme morphism on the Weil restrictions of the domain and codomain of the map.

EXAMPLES:

```sage
definition:
K.<v> = QuadraticField(5)
A.<x,y> = AffineSpace(K, 2)
f = DynamicalSystem_affine([x^2-y^2, y^2])
f.weil_restriction()

Dynamical System of Affine Space of dimension 4 over Rational Field
  Defn: Defined on coordinates by sending (z0, z1, z2, z3) to
        (z0^2 + 5*z1^2 - z2^2 - 5*z3^2, 2*z0*z1 - 2*z2*z3, z2^2 + 5*z3^2, -2*z2*z3)
```

```sage
definition:
K.<v> = QuadraticField(5)
PS.<x,y> = AffineSpace(K, 2)
f = DynamicalSystem_affine([x, y])
F = f.weil_restriction()
P = PS(2, 1)
Q = P.weil_restriction()
f(P).weil_restriction() == F(Q)
True
```

```class
definition:
sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine_finite_field
Bases:
sage.dynamics.arithmetic_dynamics.affine_ds.DynamicalSystem_affine_field,
sage.schemes.affine.affine_morphism.SchemeMorphism_polynomial_affine_space_finite_field
cyclegraph() Return the digraph of all orbits of this morphism mod $p$.

For subschemes, only points on the subscheme whose image are also on the subscheme are in the digraph.

OUTPUT: a digraph

EXAMPLES:

```sage
definition:
P.<x,y> = AffineSpace(GF(5), 2)
f = DynamicalSystem_affine([x^2-y, x*y+1])
f.cyclegraph()
Looped digraph on 25 vertices
```

```sage
definition:
P.<x> = AffineSpace(GF(3^3, 't'), 1)
f = DynamicalSystem_affine([x^2-1])
f.cyclegraph()
Looped digraph on 27 vertices
```

```sage
definition:
P.<x,y> = AffineSpace(GF(7), 2)
X = P.subscheme(x-y)
f = DynamicalSystem_affine([x^2, y^2], domain=X)
```
```
5.2. Dynamical systems on affine schemes

(continues on next page)
\(\text{f}.\text{cyclegraph}()\)

Looped digraph on 7 vertices

**orbit_structure**\((P)\)

Every point is preperiodic over a finite field.

This function returns the pair \([m, n]\) where \(m\) is the preperiod and \(n\) is the period of the point \(P\) by this map.

**INPUT:**

- \(P\) – a point in the map’s domain

**OUTPUT:** a list \([m, n]\) of integers

**EXAMPLES:**

```
sage: A.<x,y> = AffineSpace(GF(13), 2)
sage: f = DynamicalSystem_affine([x^2 - 1, y^2])
sage: f.orbit_structure(A(2, 3))
[1, 6]
sage: A.<x,y,z> = AffineSpace(GF(49, 't'), 3)
sage: f = DynamicalSystem_affine([x^2 - z, x - y + z, y^2 - x^2])
sage: f.orbit_structure(A(1, 1, 2))
[7, 6]
```

### 5.3 Dynamical systems on projective schemes

A dynamical system of projective schemes determined by homogeneous polynomials functions that define what the morphism does on points in the ambient projective space.

The main constructor functions are given by `DynamicalSystem` and `DynamicalSystem_projective`. The constructors function can take either polynomials or a morphism from which to construct a dynamical system. If the domain is not specified, it is constructed. However, if you plan on working with points or subvarieties in the domain, it recommended to specify the domain.

The initialization checks are always performed by the constructor functions. It is possible, but not recommended, to skip these checks by calling the class initialization directly.

**AUTHORS:**

- David Kohel, William Stein
- William Stein (2006-02-11): fixed bug where P(0,0,0) was allowed as a projective point.
- Volker Braun (2011-08-08): Renamed classes, more documentation, misc cleanups.
- Ben Hutz (2013-03) iteration functionality and new directory structure for affine/projective, height functionality
- Brian Stout, Ben Hutz (Nov 2013) - added minimal model functionality
- Dillon Rose (2014-01): Speed enhancements
- Ben Hutz (2015-11): iteration of subschemes
- Ben Hutz (2017-7): relocate code and create class
A dynamical system of projective schemes determined by homogeneous polynomials that define what the morphism does on points in the ambient projective space.

**Warning:** You should not create objects of this class directly because no type or consistency checking is performed. The preferred method to construct such dynamical systems is to use `DynamicalSystem_projective()` function.

**INPUT:**

- `morphism_or_polys` – a SchemeMorphism, a polynomial, a rational function, or a list or tuple of homogeneous polynomials.
- `domain` – optional projective space or projective subscheme.
- `names` – optional tuple of strings to be used as coordinate names for a projective space that is constructed; defaults to 'X', 'Y'.

The following combinations of `morphism_or_polys` and `domain` are meaningful:

- `morphism_or_polys` is a SchemeMorphism; `domain` is ignored in this case.
- `morphism_or_polys` is a list of homogeneous polynomials that define a rational endomorphism of `domain`.
- `morphism_or_polys` is a list of homogeneous polynomials and `domain` is unspecified; `domain` is then taken to be the projective space of appropriate dimension over the common base ring, if one exists, of the elements of `morphism_or_polys`.
- `morphism_or_polys` is a single polynomial or rational function; `domain` is ignored and taken to be a 1-dimensional projective space over the base ring of `morphism_or_polys` with coordinate names given by `names`.

**OUTPUT:** `DynamicalSystem_projective`.

**EXAMPLES:**

```python
sage: P1.<x,y> = ProjectiveSpace(QQ,1)
sage: DynamicalSystem_projective([y, 2*x])
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
        (y : 2*x)

sage: R.<t> = QQ[]
sage: DynamicalSystem_projective(t^2 - 3)
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (X : Y) to
        (X^2 - 3*Y^2 : Y^2)
sage: DynamicalSystem_projective(1/t^2)
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (X : Y) to
        (Y^2 : X^2)
```

We can define dynamical systems on $P^1$ by giving a polynomial or rational function:
Symbolic Ring elements are not allowed:

```
sage: x, y = var('x, y')
sage: DynamicalSystem_projective([x^2, y^2])
Traceback (most recent call last):
...
ValueError: [x^2, y^2] must be elements of a polynomial ring
```

When elements of the quotient ring are used, they are reduced:

```
sage: P.<x, y, z> = ProjectiveSpace(QQ, 2)
sage: X = P.subscheme([x-y])
sage: u, v, w = X.coordinate_ring().gens()
sage: DynamicalSystem_projective([u^2, v^2, w*u], domain=X)
Dynamical System of Closed subscheme of Projective Space of dimension 2 over Univariate Polynomial Ring in t over Rational Field defined by:
x
Defn: Defined on coordinates by sending (x : y : z) to
(x^2 : t*y^2 : x*z)
```

We can also compute the forward image of subschemes through elimination. In particular, let \( X = V(h_1, \ldots, h_t) \) and define the ideal \( f = (h_1, \ldots, h_t, y_0 - f_0(\bar{x}), \ldots, y_n - f_n(\bar{x})) \). Then the elimination ideal \( I_{n+1} = I \cap K[y_0, \ldots, y_n] \) is a homogeneous ideal and \( f(X) = V(I_{n+1}) \):

```
sage: P.<x, y, z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([(x-2*y)^2, (x-2*z)^2, x^2])
sage: X = P.subscheme(y-z)
sage: f(f(f(X)))
Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by:
y - z
```

(continues on next page)
```python
sage: f(P.subscheme([x,y,z]))
Closed subscheme of Projective Space of dimension 3 over Rational Field
defined by:
    w,
y,
x
```

```python
sage: T.<x,y,z,w,u> = ProductProjectiveSpaces([2, 1], QQ)
sage: DynamicalSystem_projective([x^2*u, y^2*w, z^2*u, w^2, u^2], domain=T)
Dynamical System of Product of projective spaces P^2 x P^1 over Rational Field
    Defn: Defined by sending (x : y : z , w : u) to
          (x^2*u : y^2*w : z^2*u , w^2 : u^2).
```

```python
sage: K.<v> = QuadraticField(-7)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem([x^3 + v*x*y^2, y^3])
sage: fbar = f.change_ring(QQbar)
sage: fbar.is_postcritically_finite()
False
```

```python
all_minimal_models(return_transformation=False, prime_list=None, algorithm=None, check_minimal=True)
Determine a representative in each \( SL(2, \mathbb{Z}) \)-orbit of this map.

This can be done either with the Bruin-Molnar algorithm or the Hutz-Stoll algorithm. The Hutz-Stoll
algorithm requires the map to have minimal resultant and then finds representatives in orbits with minimal
resultant. The Bruin-Molnar algorithm finds representatives with the same resultant (up to sign) of the
given map.

Bruin-Molnar does not work for polynomials and is more efficient for large primes.

INPUT:
- `return_transformation` – (default: False) boolean; this signals a return of the \( PGL_2 \) trans-
  formation to conjugate this map to the calculated models
- `prime_list` – (optional) a list of primes, in case one only wants to determine minimality at those
  specific primes
- `algorithm` – (optional) string; can be one of the following:
  - 'BM' - the Bruin-Molnar algorithm [BM2012]
  - 'HS' - for the Hutz-Stoll algorithm [HS2018]
  if not specified, properties of the map are utilized to choose
- `check_minimal` – (optional) boolean; to first check if the map is minimal and if not, compute a
  minimal model before computing for orbit representatives

OUTPUT:
A list of pairs \((F, m)\), where \( F \) is dynamical system on the projective line and \( m \) is the associated
\( PGL(2, \mathbb{Q}) \) element. Or just a list of dynamical systems if not returning the conjugation.

EXAMPLES:
```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([2*x^2, 3*y^2])
sage: f.all_minimal_models()
```
```
[Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to
(x^2 : y^2)]

```sage
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: c = 2*3^6
sage: f = DynamicalSystem([x^3 - c^2*y^3, x*y^2])
sage: len(f.all_minimal_models(algorithm='HS'))
14
sage: len(f.all_minimal_models(prime_list=[2], algorithm='HS'))
2
```

```sage
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([237568*x^3 + 1204224*x^2*y + 2032560*x*y^2
....: + 1142289*y^3, -131072*x^3 - 663552*x^2*y - 1118464*x*y^2
....: - 627664*y^3])
sage: len(f.all_minimal_models(algorithm='BM'))
2
```

REFERENCES:

- [BM2012]
- [HS2018]

```sage
automorphism_group(**kwds)
```

Calculates the subgroup of PGL2 that is the automorphism group of this dynamical system.

The automorphism group is the set of PGL(2) elements that fixes this map under conjugation.

INPUT:

keywords:

- `starting_prime` – (default: 5) the first prime to use for CRT
- `algorithm`– (optional) can be one of the following:
  - 'CRT' - Chinese Remainder Theorem
  - 'fixed_points' - fixed points algorithm
- `return_functions`– (default: False) boolean; True returns elements as linear fractional transformations and False returns elements as PGL2 matrices
- `iso_type` – (default: False) boolean; True returns the isomorphism type of the automorphism group

OUTPUT: a list of elements in the automorphism group

AUTHORS:

- Original algorithm written by Xander Faber, Michelle Manes, Bianca Viray
- Modified by Joao Alberto de Faria, Ben Hutz, Bianca Thompson

REFERENCES:

- [FMV2014]
sage: R.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2-y^2, x+y])
sage: f.automorphism_group(return_functions=True)
[x, -x]

sage: R.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + 5*x*y + 5*y^2, 5*x^2 + 5*x*y + y^2])
sage: f.automorphism_group(return_functions=True)
[x, 1/x, -x - 1, -x/(x + 1), (-x - 1)/x, -1/(x + 1)]

sage: K.<w> = CyclotomicField(3)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: D6 = DynamicalSystem_projective([y^2, x^2])
sage: sorted(D6.automorphism_group())
[-w - 1 0] [ 0 -w - 1] [w 0] [0 w] [0 1] [1 0] [0 1], [1 0], [1 0], [0 1]

canonical_height (P, **kwds)
Evaluate the (absolute) canonical height of P with respect to this dynamical system.
Must be over number field or order of a number field. Specify either the number of terms of the series to evaluate or the error bound required.

ALGORITHM:
The sum of the Green’s function at the archimedean places and the places of bad reduction.
If function is defined over \( \mathbb{Q} \) uses Wells’ Algorithm, which allows us not to have to factor the resultant.

INPUT:
- \( P \) – a projective point
kwds:
- \( \text{badprimes} \) – (optional) a list of primes of bad reduction
- \( N \) – (default: 10) positive integer. number of terms of the series to use in the local green functions
- \( \text{prec} \) – (default: 100) positive integer, float point or \( p \)-adic precision
- \( \text{error\_bound} \) – (optional) a positive real number

OUTPUT: a real number

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(ZZ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, 2*x*y]);
sage: f.canonical_height(P.point([5,4]), error_bound=0.001)
2.1970553519503404898926835324
sage: f.canonical_height(P.point([2,1]), error_bound=0.001)
1.0984430632822307984974382955
```

Notice that preperiodic points may not return exactly 0:

```python
sage: R.<X> = PolynomialRing(QQ)
sage: K.<a> = NumberField(X^2 + X - 1)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2-2*y^2, y^2])
sage: Q = P.point([a,1])
sage: f.canonical_height(Q, error_bound=0.000001)  
# Answer only within error_\longrightarrow bound of 0
5.7364919788790160119266380480e-8
sage: f.nth_iterate(Q,2) == Q  
# but it is indeed preperiodic
True
```

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: X = P.subscheme(x^2-y^2);
sage: f = DynamicalSystem_projective([x^2,y^2, 4*z^2], domain=X);
sage: Q = X([4,4,1])
sage: f.canonical_height(Q, badprimes=[2])
0.001353803087031143182555314882
```

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: X = P.subscheme(x^2-y^2);
sage: f = DynamicalSystem_projective([x^2,y^2, 30*z^2], domain=X)
sage: Q = X([4, 4, 1])
sage: f.canonical_height(Q, badprimes=[2,3,5], prec=200)
2.7054056208276961889784303469356774912979228770208655455481
```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([1000*x^2-29*y^2, 1000*y^2])
sage: Q = P(-1/4, 1)
sage: f.canonical_height(Q, error_bound=0.01)
3.7996079979254623065837411853

sage: RSA768 =
˓→123018668453011775513049495834962720772853565959533479219732245215
˓→1726400507263657518745202199786469389956474942774063845925192557326303453731548
˓→2685079170261221429134616704292143116022212404792747377940806653514195974598569
˓→02143413
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([RSA768*x^2 + y^2, x*y])
sage: Q = P(RSA768, 1)
sage: f.canonical_height(Q, error_bound=0.00000000000000001)
931.18256422718241278672729195

conjugate (M, adjugate=False, normalize=False)
Conjugate this dynamical system by M, i.e. $M^{-1} \circ f \circ M$.

If possible the new map will be defined over the same space. Otherwise, will try to coerce to the base ring of M.

INPUT:

- M – a square invertible matrix
- adjugate – (default: False) boolean, also classically called adjoint, takes a square matrix M and finds the transpose of its cofactor matrix. Used for conjugation in place of inverse when specified 'True'. Functionality is the same in projective space.
- normalize – (default: False) boolean, if normalize is 'True', then the function normalize_coordinates is called.

OUTPUT: a dynamical system

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: f.canonical_height(P(1,0))
0.00000000000000000000000000000

sage: R.<x> = PolynomialRing(QQ)
sage: K.<i> = NumberField(x^2+1)
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([[x^3+y^3, y^3]])

(continues on next page)
sage: f.conjugate(matrix([[i,0], [0,-i]]))
Dynamical System of Projective Space of dimension 1 over Integer Ring
  Defn: Defined on coordinates by sending (x : y) to
         (-x^3 + y^3 : -y^3)

sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([x^2+y^2 ,y^2, y*z])
sage: f.conjugate(matrix([[1,2,3], [0,1,2], [0,0,1]]))
Dynamical System of Projective Space of dimension 2 over Integer Ring
  Defn: Defined on coordinates by sending (x : y : z) to
         (x^2 + 4*x*y + 3*y^2 + 6*x*z + 9*y*z + 7*z^2 : y^2 + 2*y*z : y*z + 2*z^2)

sage: P.<x,y> = ProjectiveSpace(ZZ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: f.conjugate(matrix([[2,0], [0,1/2]]))
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
         (2*x^2 + 1/8*y^2 : 1/2*y^2)

Todo: Use the left and right action functionality to replace the code below with #return DynamicalSystem_projective(M.inverse()*self*M, domain=self.codomain()) once there is a function to pass to the smallest field of definition.

critical_height(**kwds)
Compute the critical height of this dynamical system.

The critical height is defined by J. Silverman as the sum of the canonical heights of the critical points. This must be dimension 1 and defined over a number field or number field order.

The computations can be done either over the algebraic closure of the base field or over the minimal extension of the base field that contains the critical points.

INPUT:

kwds:
  • badprimes – (optional) a list of primes of bad reduction
  • N – (default: 10) positive integer; number of terms of the series to use in the local green functions
  • prec – (default: 100) positive integer, float point or p-adic precision
  • error_bound – (optional) a positive real number
  • use_algebraic_closure – boolean (default: True) – If True uses the algebraic closure. If False, uses the smallest extension of the base field containing all the critical points.
OUTPUT: real number

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3+7*y^3, 11+y^3])
sage: f.critical_height()
1.1989273321156851418802151128
```

```python
sage: K.<w> = QuadraticField(2)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2+w*y^2, y^2])
sage: f.critical_height()
0.16090842452312941163719755472
```

Postcritically finite maps have critical height 0:

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3-3/4*x*y^2 + 3/4*y^3, y^3])
sage: f.critical_height(error_bound=0.0001)
0.00000000000000000000000000000
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3+3*x*y^2, y^3])
sage: f.critical_height(use_algebraic_closure=False)
0.00002347701673389711288649167991
sage: f.critical_height()
0.00002347701673389711288649167991
```

```python
critical_point_portrait(check=True, use_algebraic_closure=True)
```

This is the directed graph of iterates starting with the critical points. Must be dimension 1. If check is True, then the map is first checked to see if it is postcritically finite.

The computations can be done either over the algebraic closure of the base field or over the minimal extension of the base field that contains the critical points.

INPUT:

- check – boolean (default: True)
- use_algebraic_closure – boolean (default: True) – If True uses the algebraic closure. If False, uses the smallest extension of the base field containing all the critical points.

OUTPUT: a digraph

EXAMPLES:

```python
sage: R.<z> = QQ[]
sage: K.<v> = NumberField(z^6 + 2*z^5 + 2*z^4 + 2*z^3 + z^2 + 1)
sage: PS.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2+v*y^2, y^2])
sage: f.critical_point_portrait(check=False) # long time
Looped digraph on 6 vertices
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^5 + 5/4*x*y^4, y^5])
sage: f.critical_point_portrait(check=False)
Looped digraph on 5 vertices
```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + 2*y^2, y^2])
sage: f.critical_points()
Traceback (most recent call last):
  ...TypeError: map must be post-critically finite

sage: R.<t> = QQ[]
sage: K.<v> = NumberField(t^3 + 2*t^2 + t + 1)
sage: phi = K.embeddings(QQbar)[0]
sage: P.<x, y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem_projective([x^2 + v*y^2, y^2])
sage: f.change_ring(phi).critical_points_portrait()
Looped digraph on 4 vertices

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([8*x^4 - 8*x^2*y^2 + y^4, y^4])
sage: f.critical_point_portrait(use_algebraic_closure=False) #long time
Looped digraph on 6 vertices

sage: P.<x,y> = ProjectiveSpace(QQbar,1)
sage: f = DynamicalSystem_projective([8*x^4 - 8*x^2*y^2 + y^4, y^4])
sage: f.critical_points_portrait() #long time
Looped digraph on 6 vertices

sage: P.<x,y> = ProjectiveSpace(GF(3),1)
sage: f = DynamicalSystem_projective([x^2 + x*y - y^2, x*y])
sage: f.critical_points_portrait(use_algebraic_closure=False)
Looped digraph on 6 vertices

**critical_points** *(R=None)*

Return the critical points of this dynamical system defined over the ring \( R \) or the base ring of this map.

**INPUT:**

- \( R \) – (optional) a ring

**OUTPUT:** a list of projective space points defined over \( R \)

**EXAMPLES:**

sage: set_verbose(None)
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3-2*x*y^2 + 2*y^3, y^3])
sage: f.critical_points()
[(1 : 0)]
sage: K.<w> = QuadraticField(6)
sage: f.critical_points(K)
[(-1/3*w : 1), (1/3*w : 1), (1 : 0)]

sage: set_verbose(None)
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([2*x^2-y^2, x*y])
critical_subscheme()

Return the critical subscheme of this dynamical system.

OUTPUT: projective subscheme

EXAMPLES:

```sage
sage: set_verbose(None)
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3-2*x*y^2 + 2*y^3, y^3])
sage: f.critical_subscheme()
Closed subscheme of Projective Space of dimension 1 over Rational Field
defined by:
9*x^2*y^2 - 6*y^4
```

```sage
sage: set_verbose(None)
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([2*x^2-y^2, x*y])
sage: f.critical_subscheme()
Closed subscheme of Projective Space of dimension 1 over Rational Field
defined by:
4*x^2 + 2*y^2
```

```sage
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([2*x^2-y^2, x*y, z^2])
sage: f.critical_subscheme()
Closed subscheme of Projective Space of dimension 2 over Rational Field
defined by:
8*x^2*z + 4*y^2*z
```

```sage
sage: P.<x,y,z,w> = ProjectiveSpace(GF(81),3)
sage: g = DynamicalSystem_projective([x^3+y^3, y^3+z^3, z^3+x^3, w^3])
sage: g.critical_subscheme()
Closed subscheme of Projective Space of dimension 3 over Finite Field in
z4 of size 3^4 defined by:
0
```

```sage
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2,x*y])
sage: f.critical_subscheme()
Traceback (most recent call last):
  ...TypeError: the function is not a morphism
```

degree_sequence(iterates=2)

Return sequence of degrees of normalized iterates starting with the degree of this dynamical system.

INPUT: iterates – (default: 2) positive integer

OUTPUT: list of integers

EXAMPLES:
sage: P2.<X,Y,Z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([Z^2, X*Y, Y^2])
sage: f.degree_sequence(15)
[2, 3, 5, 8, 11, 17, 24, 31, 45, 56, 68, 91, 93, 184, 275]

sage: F.<t> = PolynomialRing(QQ)
sage: P2.<X,Y,Z> = ProjectiveSpace(F, 2)
sage: f = DynamicalSystem_projective([Y*Z, X*Y, Y^2 + t*X*Z])
sage: f.degree_sequence(5)
[2, 3, 5, 8, 13]

sage: P2.<X,Y,Z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([X^2, Y^2, Z^2])
sage: f.degree_sequence(10)
[2, 4, 8, 16, 32, 64, 128, 256, 512, 1024]

sage: P2.<X,Y,Z> = ProjectiveSpace(ZZ, 2)
sage: f = DynamicalSystem_projective([X*Y, Y*Z+Z^2, Z^2])
sage: f.degree_sequence(10)
[2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

dehomogenize(n)

Return the standard dehomogenization at the n[0] coordinate for the domain and the n[1] coordinate for the codomain.

Note that the new function is defined over the fraction field of the base ring of this map.

INPUT:

- n – a tuple of nonnegative integers; if n is an integer, then the two values of the tuple are assumed to be the same

OUTPUT:

If the dehomogenizing indices are the same for the domain and codomain, then a DynamicalSystem_affine given by dehomogenizing the source and target of self with respect to the given indices is returned. If the dehomogenizing indices for the domain and codomain are different then the resulting affine patches are different and a scheme morphism is returned.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(ZZ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: f.dehomogenize(0)
Dynamical System of Affine Space of dimension 1 over Integer Ring
  Defn: Defined on coordinates by sending (y) to
        (y^2/(y^2 + 1))

sage: f.dehomogenize((0, 1))
Scheme morphism:
  From: Affine Space of dimension 1 over Integer Ring
  To:   Affine Space of dimension 1 over Integer Ring
  Defn: Defined on coordinates by sending (y) to
        ((y^2 + 1)/y^2)

dynamical_degree(N=3, prec=53)

Return an approximation to the dynamical degree of this dynamical system. The dynamical degree is defined as $\lim_{n \to \infty} \sqrt[n]{\deg(f^n)}$.

INPUT:
• \(N\) – (default: 3) positive integer, iterate to use for approximation

• \(\text{prec}\) – (default: 53) positive integer, real precision to use when computing root

OUTPUT: real number

EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([x^2 + (x*y), y^2])
sage: f.dynamical_degree()
2.00000000000000
```

```
sage: P2.<X,Y,Z> = ProjectiveSpace(ZZ, 2)
sage: f = DynamicalSystem_projective([X*Y, Y*Z+Z^2, Z^2])
sage: f.dynamical_degree(N=5, prec=100)
1.4309690811052555010452244131
```

dynatomic_polynomial \((\text{period})\)
For a dynamical system of \(\mathbb{P}^1\) compute the dynatomic polynomial.

The dynatomic polynomial is the analog of the cyclotomic polynomial and its roots are the points of formal period \(\text{period}\). If possible the division is done in the coordinate ring of this map and a polynomial is returned. In rings where that is not possible, a \texttt{FractionField} element will be returned. In certain cases, when the conversion back to a polynomial fails, a \texttt{SymbolRing} element will be returned.

ALGORITHM:

For a positive integer \(n\), let \([F_n, G_n]\) be the coordinates of the \(n\)th iterate of \(f\). Then construct

\[
\Phi_n^*(f)(x, y) = \sum_{d|n} (yF_d(x, y) - xG_d(x, y)) \mu(n/d),
\]

where \(\mu\) is the Möbius function.

For a pair \([m, n]\), let \(f^m = [F_m, G_m]\). Compute

\[
\Phi_{m,n}^*(f)(x, y) = \Phi_n^*(f)(F_m, G_m)/\Phi_n^*(f)(F_{m-1}, G_{m-1})
\]

REFERENCES:

• [Hutz2015]

• [MoPa1994]

INPUT:

• \(\text{period}\) – a positive integer or a list/tuple \([m, n]\) where \(m\) is the preperiod and \(n\) is the period

OUTPUT:

If possible, a two variable polynomial in the coordinate ring of this map. Otherwise a fraction field element of the coordinate ring of this map. Or, a \texttt{SymbolicRing} element.

Todo:

• Do the division when the base ring is \(p\)-adic so that the output is a polynomial.

• Convert back to a polynomial when the base ring is a function field (not over \(\mathbb{Q}\) or \(F_p\)).

EXAMPLES:
```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, y^2])
sage: f.dynatomic_polynomial(2)
x^2 + x*y + 2*y^2

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, x*y])
sage: f.dynatomic_polynomial(4)
2*x^12 + 18*x^10*y^2 + 57*x^8*y^4 + 79*x^6*y^6 + 48*x^4*y^8 + 12*x^2*y^10 + y^12

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - 10/9*y^2, y^2])
sage: f.dynatomic_polynomial([2,1])
x^4*y^2 - 11/9*x^2*y^4 - 80/81*y^6

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - 29/16*y^2, y^2])
sage: f.dynatomic_polynomial([2,3])
x^12 - 95/8*x^10*y^2 + 13799/256*x^8*y^4 - 119953/1024*x^6*y^6 + 8198847/65536*x^4*y^8 - 31492431/524288*x^2*y^10 + 172692729/16777216*y^12

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2])
sage: f.dynatomic_polynomial([1,2])
x^2 - x*y

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3 - y^3, 3*x*y^2])
sage: f.dynatomic_polynomial([0,4]) == f.dynatomic_polynomial(4)
True

sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([x^2 + y^2, x*y, z^2])
sage: f.dynatomic_polynomial(2)
Traceback (most recent call last):
  ...TypeError: does not make sense in dimension >1

sage: L.<t> = PolynomialRing(QQ)
sage: P.<x,y> = ProjectiveSpace(L,1)
```

(continues on next page)

sage: f = DynamicalSystem_projective([x^2 + t*y^2, y^2])
sage: f.dynatomic_polynomial(2)
x^2 + x*y + (t + 1)*y^2

sage: K.<c> = PolynomialRing(ZZ)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2 + c*y^2, y^2])
sage: f.dynatomic_polynomial([1, 2])
x^2 - x*y + (c + 1)*y^2

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, y^2])
sage: f.dynatomic_polynomial([2, 2])
x^2 + x*y + 2*y^2

sage: R.<X> = PolynomialRing(QQ)
sage: K.<c> = NumberField(X^2 + X + 2)
sage: PP = P.change_ring(K)
sage: ff = f.change_ring(K)
sage: p = PP((c, 1))
sage: ff(ff(p)) == p
True

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, x*y])
sage: f.dynatomic_polynomial([2, 2])
x^4 + 4*x^2*y^2 + y^4

sage: R.<X> = PolynomialRing(QQ)
sage: K.<c> = NumberField(X^4 + 4*X^2 + 1)
sage: PP = P.change_ring(K)
sage: ff = f.change_ring(K)
sage: p = PP((c, 1))
sage: ff.nth_iterate(p, 4) == ff.nth_iterate(p, 2)
True

sage: P.<x,y> = ProjectiveSpace(CC, 1)
sage: f = DynamicalSystem_projective([x^2 - CC.0/3*y^2, y^2])
sage: f.dynatomic_polynomial(2)
(x^4*y + (-0.666666666666667*I)*x^2*y^3 - x*y^4 + (-0.111111111111111 - 0.
˓→333333333333333*I)*y^5)/(x^2*y - x*y^2 + (-0.333333333333333*I)*y^3)

sage: P.<x,y> = ProjectiveSpace(CC, 1)
sage: f = DynamicalSystem_projective([x^2-CC.0/5*y^2, y^2])
sage: f.dynatomic_polynomial(2)
x^2 + x*y + (1.00000000000000 - 0.200000000000000*I)*y^2

sage: L.<t> = PolynomialRing(QuadraticField(2).maximal_order())
sage: P.<x, y> = ProjectiveSpace(L.fraction_field() , 1)
sage: f = DynamicalSystem_projective([x^2 + (t^2 + 1)*y^2 , y^2])
sage: f.dynatomic_polynomial(2)
x^2 + x*y + (t^2 + 2)*y^2

sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([x^2 - 5*y^2, y^2])
sage: f.dynatomic_polynomial([3,0])

5.3. Dynamical systems on projective schemes
green_function($P, v, **kwds$)

Evaluate the local Green’s function at the place $v$ for $P$ with $N$ terms of the series or to within a given error bound.

Must be over a number field or order of a number field. Note that this is the absolute local Green’s function so is scaled by the degree of the base field.

Use $v=0$ for the archimedean place over $\mathbb{Q}$ or field embedding. Non-archimedean places are prime ideals for number fields or primes over $\mathbb{Q}$.

ALGORITHM:
See Exercise 5.29 and Figure 5.6 of [Sil2007].

INPUT:

- $P$ – a projective point
- $v$ – non-negative integer. a place, use 0 for the archimedean place

kwds:

- $N$ – (optional - default: 10) positive integer. number of terms of the series to use
- $\texttt{prec}$ – (default: 100) positive integer, float point or $p$-adic precision
- $\texttt{error\_bound}$ – (optional) a positive real number

OUTPUT: a real number

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, x*y]);
sage: Q = P(5, 1)
sage: f.green_function(Q, 0, N=30)
1.6460930159932946233759277576
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, x*y]);
sage: Q = P(5, 1)
sage: f.green_function(Q, 0, N=200, prec=200)
1.6460930160038721802875250367738355497198064992657997569827
sage: K.<w> = QuadraticField(3)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([17*x^2+1/7*y^2, 17*w*x*y])
sage: f.green_function(P.point([w, 2], False), K.places()[1])
1.7236334013785676107373093775
sage: f.green_function(P([w, 1]), K.ideal(7), N=7)
0.486477532638283262673818586
sage: f.green_function(P([w, 1]), K.ideal(17), error_bound=0.001)
-0.70813041039490996737374178059
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, x*y])
sage: f.green_function(P.point([5,2], False), 0, N=30)
1.7315451844777407992085512000
```

sage: f.green_function(P.point([2,1], False), 0, N=30)
0.8657725922318108832522609926
sage: f.green_function(P.point([1,1], False), 0, N=30)
0.43288629610862338612700146098

height_difference_bound (prec=None)

Return an upper bound on the different between the canonical height of a point with respect to
this dynamical system and the absolute height of the point.

This map must be a morphism.

ALGORITHM:
Uses a Nullstellensatz argument to compute the constant. For details: see [Hutz2015].

INPUT:
• prec – (default: RealField default) positive integer, float point precision

OUTPUT: a real number

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, x*y])
sage: f.height_difference_bound()
1.38629436111989

This function does not automatically normalize.

sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([4*x^2+100*y^2, 210*x*y, 10000*z^2])
sage: f.height_difference_bound()
11.0020998412042
sage: f.normalize_coordinates()
sage: f.height_difference_bound()
10.3089526606443

A number field example:

sage: R.<x> = QQ[]
sage: K.<c> = NumberField(x^3 - 2)
sage: P.<x,y,z> = ProjectiveSpace(K,2)
sage: f = DynamicalSystem_projective([1/(c+1)*x^2+c*y^2, 210*x*y, 10000*z^2])
sage: f.height_difference_bound()
11.0020998412042

::

sage: P.<x,y,z> = ProjectiveSpace(QQbar,2)
sage: f = DynamicalSystem_projective([x^2, QQbar(sqrt(-1))*y^2, QQbar(sqrt(3))*z^2])
sage: f.height_difference_bound()
3.43967790223022

is_PGL_minimal (prime_list=None)

Check if this dynamical system is a minimal model in its conjugacy class.

INPUT:

- prime_list – (optional) list of primes to check minimality

OUTPUT: boolean

EXAMPLES:

```python
sage: PS.<X,Y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([X^2+Y^2, X*Y])
sage: f.is_PGL_minimal()
True
```

```python
sage: PS.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([6*x^2+12*x*y+7*y^2, 12*x*y])
sage: f.is_PGL_minimal()
False
```

```python
sage: PS.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([6*x^2+12*x*y+7*y^2, y^2])
sage: f.is_PGL_minimal()
False
```

```python
sage: PS.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([X^2+Y^2, X*Y])
sage: f.is_PGL_minimal()
True
```

`is_postcritically_finite (err=0.01, use_algebraic_closure=True)`

Determine if this dynamical system is post-critically finite.

Only for endomorphisms of $\mathbb{P}^1$. It checks if each critical point is preperiodic. The optional parameter $\text{err}$ is passed into $\text{is_preperiodic()}$ as part of the preperiodic check.

The computations can be done either over the algebraic closure of the base field or over the minimal extension of the base field that contains the critical points.

INPUT:

- $\text{err}$ – (default: 0.01) positive real number
- $\text{use_algebraic_closure}$ – boolean (default: True) – If True uses the algebraic closure. If False, uses the smallest extension of the base field containing all the critical points.

OUTPUT: boolean

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2])
sage: f.is_postcritically_finite()
True
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3- y^3, y^3])
sage: f.is_postcritically_finite()
False
```

```python
sage: R.<z> = QQ[]
sage: K.<v> = NumberField(z^8 + 3*z^6 + 3*z^4 + z^2 + 1)
sage: PS.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^3+v*y^3, y^3])
sage: f.is_postcritically_finite() # long time
True
```
minimal_model

Determine if this dynamical system is minimal.

This dynamical system must be defined over the projective line over the rationals. In particular, determine if this map is affine minimal, which is enough to decide if it is minimal or not. See Proposition 2.10 in [BM2012].

INPUT:

• return_transformation – (default: False) boolean; this signals a return of the $PGL_2$ transformation to conjugate this map to the calculated minimal model
• prime_list – (optional) a list of primes, in case one only wants to determine minimality at those specific primes
• algorithm – (optional) string; can be one of the following:
  • check_primes – (optional) boolean: this signals whether to check whether each element in prime_list is a prime
    • 'BM' - the Bruin-Molnar algorithm [BM2012]
    • 'HS' - the Hutz-Stoll algorithm [HS2018]

OUTPUT:

• a dynamical system on the projective line which is a minimal model of this map
• a $PGL(2, \mathbb{Q})$ element which conjugates this map to a minimal model

EXAMPLES:
sage: PS.<X,Y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([X^2+3*Y^2, X*Y])
sage: f.minimal_model(return_transformation=True)
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (X : Y) to
  (X^2 + 3*Y^2 : X*Y),
  [1 0]
  [0 1])

sage: PS.<X,Y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([7365/2*X^4 + 6282*X^3*Y + 4023*X^2*Y^2,
  1146*X*Y^3 + 245/2*Y^4,
  -12329/2*X^4 - 10506*X^3*Y - 6723*X^2*Y^2 - 1914*X*Y^3 - 409/2*Y^4])
sage: f.minimal_model(return_transformation=True)
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (X : Y) to
  (9847*X^4 + 28088*X^3*Y + 30048*X^2*Y^2 + 14288*X*Y^3 + 2548*Y^4
  : -12329*X^4 - 35164*X^3*Y - 37614*X^2*Y^2 - 17884*X*Y^3 - 3189*Y^4),
  [2 1]
  [0 1])

sage: PS.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([6*x^2+12*x*y+7*y^2, 12*x*y])
sage: f.minimal_model()
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (x^2 + 12*x*y + 42*y^2 : 2*x*y))

sage: PS.<x,y> = ProjectiveSpace(ZZ,1)
sage: f = DynamicalSystem_projective([6*x^2+12*x*y+7*y^2, 12*x*y + 42*y^2])
sage: g,M = f.minimal_model(return_transformation=True, algorithm='BM')
sage: f.conjugate(M) == g
True

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([[2*x^2, y^2]])
sage: f.minimal_model(return_transformation=True)
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (x^2 : y^2),
  [1 0]
  [0 2])

sage: f.minimal_model(prime_list=[3])
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (continues on next page)
multiplier \((P, n, \text{check} = \text{True})\)

Return the multiplier of the point \(P\) of period \(n\) with respect to this dynamical system.

**INPUT:**

- \(P\) – a point on domain of this map
- \(n\) – a positive integer, the period of \(P\)
- \(\text{check}\) – (default: \(\text{True}\)) boolean; verify that \(P\) has period \(n\)

**OUTPUT:**

A square matrix of size \(\text{self.codomain().dimension}() \cdot \text{dimension}()\) in the \(\text{base}\) ring of this dynamical system.

**EXAMPLES:**

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([x^2, y^2, 4*z^2]);
sage: Q = P.point([4,4,1], False);
sage: f.multiplier(Q,1)
[2 0]
[0 2]
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([7*x^2 - 28*y^2, 24*x*y])
sage: f.multiplier(P(2,5), 4)
[231361/20736]
```

```python
sage: P.<x,y> = ProjectiveSpace(CC,1)
sage: f = DynamicalSystem_projective([x^3 - 25*x*y^2 + 12*y^3, 12*y^3])
sage: f.multiplier(P(1,1), 5)
[0.389017489711934]
```

```python
sage: P.<x,y> = ProjectiveSpace(RR,1)
sage: f = DynamicalSystem_projective([x^2-2*y^2, y^2])
sage: f.multiplier(P(2,1), 1)
[4.00000000000000]
```

```python
sage: P.<x,y> = ProjectiveSpace(Qp(13),1)
sage: f = DynamicalSystem_projective([x^2-29/16*y^2, y^2])
sage: f.multiplier(P(5,4), 3)
[6 + 8*13 + 13^2 + 8*13^3 + 13^4 + 8*13^5 + 13^6 + 8*13^7 + 13^8 +
8*13^9 + 13^10 + 8*13^11 + 13^12 + 8*13^13 + 13^14 + 8*13^15 + 13^16 +
8*13^17 + 13^18 + 8*13^19 + O(13^20)]
```
multiplier_spectra($n, \text{formal}=False, \text{type}'point', \text{use_algebraic_closure}=True$)

Computes the $n$ multiplier spectra of this dynamical system.

This is the set of multipliers of the periodic points of formal period $n$ included with the appropriate multiplicity. User can also specify to compute the $n$ multiplier spectra instead which includes the multipliers of all periodic points of period $n$. The map must be defined over projective space of dimension 1 over a number field or finite field.

The computations can be done either over the algebraic closure of the base field or over the minimal extension of the base field that contains the critical points.

INPUT:

- $n$ – a positive integer, the period
- $\text{formal}$ – (default: False) boolean; True specifies to find the formal $n$ multiplier spectra of this map and False specifies to find the $n$ multiplier spectra
- $\text{type}$ – (default: 'point') string; either 'point' or 'cycle' depending on whether you compute one multiplier per point or one per cycle
- $\text{use_algebraic_closure}$ – boolean (default: True) – If True uses the algebraic closure. If False, uses the smallest extension of the base field containing all the critical points.

OUTPUT: a list of field elements

EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2-y^2, y^2])
sage: f.multiplier(P(0,1), 1)
Traceback (most recent call last):
... ValueError: (0 : 1) is not periodic of period 1

sage: multiplier_spectra(1)
[-28, 1]
```

```
sage: f.multiplier_spectra(1)
[-119820502365680843999, -71981476811762560843999, -355349775155930157517261317/8192, 144 Chapter 5. Arithmetic Dynamical Systems
```

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([4608*x^10 - 2910096*x^9*y + ...
- 325988068*x^8*y^2 + 31825198932*x^7*y^3 - 4139806626613*x^6*y^4
- 44439736715486*x^5*y^5 + 2317935971590902*x^4*y^6 - ...
+ 113578270285012470*x*y^9 - 15004994023963800*y^10, 4608*y^10])
sage: sorted(f.multiplier_spectra(1))
[-119820502365680843999, -71981476811762560843999, -355349775155930157517261317/8192, 144 Chapter 5. Arithmetic Dynamical Systems
```

```
sage: set_verbose(None)
sage: z = QQ['z'].0
sage: K.<w> = NumberField(z^4 - 4*z^2 + 1, 'z')
```
sage: f = DynamicalSystem_projective([x^2 - w/4*y^2, y^2])
sage: sorted(f.multiplier_spectra(2, formal=False, type='cycle'))
[0, 0.0681483474218635 - 1.930649271699173*I, 0.0681483474218635 + 1.930649271699173*I, 5.931851652578137 + 0.2e-49*I]

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - 3/4*y^2, y^2])
sage: sorted(f.multiplier_spectra(2, formal=False, type='cycle'))
[0, 1, 1, 9]
sage: sorted(f.multiplier_spectra(2, formal=False, type='point'))
[0, 1, 1, 1, 9]
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - 7/4*y^2, y^2])
sage: f.multiplier_spectra(3, formal=True, type='cycle')
[1, 1]
sage: f.multiplier_spectra(3, formal=True, type='point')
[1, 1, 1, 1, 1]

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^4 + 3*y^4, 4*x^2*y^2])
sage: f.multiplier_spectra(1, use_algebraic_closure=False)
[0, 3*a + 3, 2*a + 1, 1, 1]
sage: f.multiplier_spectra(1)
[0, 2*z2 + 1, 3*z2 + 3, 1, 1]

sage: P.<x,y> = ProjectiveSpace(GF(5),1)
sage: f = DynamicalSystem_projective([x^4 + 2*y^4, 4*x^2*y^2])
sage: f.multiplier_spectra(1, use_algebraic_closure=False)
[0, 3*a + 3, 2*a + 1, 1, 1]
sage: f.multiplier_spectra(1)
[0, 2*z2 + 1, 3*z2 + 3, 1, 1]

sage: K = GF(3).algebraic_closure()
nth_iterate \((P, n, **kwds)\)

Return the \(n\)-th iterate of the point \(P\) by this dynamical system.

If `normalize` is `True`, then the coordinates are automatically normalized.

Todo: Is there a more efficient way to do this?

**INPUT:**

- \(P\) – a point in this map’s domain
- \(n\) – a positive integer

**kwds:**

- `normalize` – (default: `False`) boolean

**OUTPUT:** a point in this map’s codomain

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(ZZ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, 2*y^2])
sage: Q = P(1,1)
sage: f.nth_iterate(Q,4)
(32768 : 32768)
sage: P.<x,y> = ProjectiveSpace(ZZ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, 2*y^2])
sage: Q = P(1,1)
sage: f.nth_iterate(Q, 4, normalize=True)
(1 : 1)
```

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([x^2, 2*y^2, z^2-x^2])
sage: Q = P(2,7,1)
sage: f.nth_iterate(Q,2)
(-16/7 : -2744 : 1)
sage: R.<t> = PolynomialRing(QQ)
sage: P.<x,y,z> = ProjectiveSpace(R,2)
sage: f = DynamicalSystem_projective([x^2+t*y^2, (2-t)*y^2, z^2])
sage: Q = P(2+t,7,t)
sage: f.nth_iterate(Q,2)
(-29968/35 : -3329704/35 : -t^4)
```

```python
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: X = P.subscheme(x^2-y^2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2], domain=X)
```

(continues on next page)
sage: f.nth_iterate(X(2,2,3), 3)
(256 : 256 : 6561)

sage: K.<c> = FunctionField(QQ)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([[x^3 - 2*x*y^2 - c*y^3, x*y^2]])
sage: f.nth_iterate(P(c,1), 2)
((c^6 - 9*c^4 + 25*c^2 - c - 21)/(c^2 - 3) : 1)

sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([[x^2+3*y^2, 2*y^2,z^2]])
sage: f.nth_iterate(P(2, 7, 1), -2)
Traceback (most recent call last):
... 
TypeError: must be a forward orbit

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([[x^3, x*y^2], domain=P)

sage: f.nth_iterate(P(0, 1), 3, check=False)
(0 : 0)
sage: f.nth_iterate(P(0, 1), 3)
Traceback (most recent call last):
... 
ValueError: [0, 0] does not define a valid point since all entries are 0

sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([[x^3, x*y^2], domain=P)

sage: f.nth_iterate(P(2,1), 3, normalize=False)
(134217728 : 524288)
sage: f.nth_iterate(P(2,1), 3, normalize=True)
(256 : 1)

nth_iterate_map(n, normalize=False)

Return the n-th iterate of this dynamical system.

ALGORITHM:

Uses a form of successive squaring to reducing computations.

Todo: This could be improved.

INPUT:

• n – positive integer
• normalize – boolean; remove gcd’s during iteration

OUTPUT: a projective dynamical system

EXAMPLES:
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: f.nth_iterate_map(2)
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
          (x^4 + 2*x^2*y^2 + 2*y^4 : y^4)

sage: P.<x,y> = ProjectiveSpace(CC,1)
sage: f = DynamicalSystem_projective([x^2-y^2, x*y])
sage: f.nth_iterate_map(3)
Dynamical System of Projective Space of dimension 1 over Complex Field with 53 bits of precision
  Defn: Defined on coordinates by sending (x : y) to
          (x^8 + (-7.00000000000000)*x^6*y^2 + 13.00000000000000*x^4*y^4 +
           (-7.00000000000000)*x^2*y^6 + y^8 : x^7*y + (-4.00000000000000)*x^5*y^3 +
           4.000000000000000*x^3*y^5 - x*y^7)

sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([x^2-y^2, x*y, z^2+x^2])
sage: f.nth_iterate_map(2)
Dynamical System of Projective Space of dimension 2 over Integer Ring
  Defn: Defined on coordinates by sending (x : y : z) to
          (x^4 - 3*x^2*y^2 + y^4 : x^3*y - x*y^3 : 2*x^4 - 2*x^2*y^2 + y^4 + 2*x^2*z^2 + z^4)

sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: X = P.subscheme(x*z-y^2)
sage: f = DynamicalSystem_projective([x^2, x*z, z^2], domain=X)
sage: f.nth_iterate_map(2)
Dynamical System of Closed subscheme of Projective Space of dimension 2 over Rational Field defined by:
  -y^2 + x*z
  Defn: Defined on coordinates by sending (x : y : z) to
          (x^4 : x^2*z^2 : z^4)

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([y^2*z^3, y^3*z^2, x^5])
sage: f.nth_iterate_map(5, normalize=True)
Dynamical System of Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z) to
          (y^202*z^443 : x^140*y^163*z^342 : x^645)

nth_preimage_tree(Q, n, **kwds)
Return the n-th pre-image tree rooted at Q.
This map must be an endomorphism of the projective line defined over a number field, algebraic field, or finite field.

INPUT:

• Q – a point in the domain of this map

• n – a positive integer, the depth of the pre-image tree

kwds:
• return_points – (default: False) boolean; if True, return a list of lists where the index i is the level of the tree and the elements of the list at that index are the i-th preimage points as an algebraic element of the splitting field of the polynomial \( f^n - Q = 0 \)

• numerical – (default: False) boolean; calculate pre-images numerically. Note if this is set to True, preimage points are displayed as complex numbers

• prec – (default: 100) positive integer; the precision of the ComplexField if we compute the preimage points numerically

• display_labels – (default: True) boolean; whether to display vertex labels. Since labels can be very cluttered, can set display_labels to False and use return_points to get a hold of the points themselves, either as algebraic or complex numbers

• display_complex – (default: False) boolean; display vertex labels as complex numbers. Note if this option is chosen that we must choose an embedding from the splitting field field_def of the nth-preimage equation into C. We make the choice of the first embedding returned by field_def.embeddings(ComplexField())

• digits – a positive integer, the number of decimal digits to display for complex numbers. This only applies if display_complex is set to True

OUTPUT:

If return_points is False, a GraphPlot object representing the n-th pre-image tree. If return_points is True, a tuple (GP, points), where GP is a GraphPlot object, and points is a list of lists as described above under return_points.

EXAMPLES:

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, y^2])
sage: Q = P(0,1)
sage: f.nth_preimage_tree(Q, 2)
GraphPlot object for Digraph on 7 vertices
```

```python
sage: P.<x,y> = ProjectiveSpace(GF(3),1)
sage: f = DynamicalSystem_projective([x^2 + x*y + y^2, y^2])
sage: Q = P(0,1)
sage: f.nth_preimage_tree(Q, 2, return_points=True)
(GraphPlot object for Digraph on 4 vertices, 
[[(0 : 1)], [(1 : 1)], [(0 : 1), (2 : 1)]])
```

\( \text{orbit} \ (P, N, **\text{kwd}) \)

Return the orbit of the point \( P \) by this dynamical system.

Let \( F \) be this dynamical system. If \( N \) is an integer return \( [P, F(P), \ldots, F^N(P)] \). If \( N \) is a list or tuple \( N = [m, k] \) return \( [F^m(P), \ldots, F^k(P)] \). Automatically normalize the points if normalize=True. Perform the checks on point initialization if check=True.

INPUT:

• \( P \) – a point in this dynamical system’s domain

• \( n \) – a non-negative integer or list or tuple of two non-negative integers

kwds:

• check – (default: True) boolean

• normalize – (default: False) boolean
OUTPUT: a list of points in this dynamical system's codomain

EXAMPLES:

```python
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2-z^2, 2*z^2])
sage: f.orbit(P(1,2,1), 3)
[(1 : 2 : 1), (5 : 3 : 2), (34 : 5 : 8), (1181 : -39 : 128)]
```

```python
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2-z^2, 2*z^2])
sage: f.orbit(P(1,2,1), [2,4])
[(34 : 5 : 8), (1181 : -39 : 128), (1396282 : -14863 : 32768)]
```

```python
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: X = P.subscheme(x^2-y^2)
sage: f = DynamicalSystem_projective([x^2, y^2, x*z], domain=X)
sage: f.orbit(X(2,2,3), 3, normalize=True)
[(2 : 2 : 3), (2 : 2 : 3), (2 : 2 : 3), (2 : 2 : 3)]
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: f.orbit(P.point([1,2],False), 4, check=False)
[(1 : 2), (5 : 4), (41 : 16), (1937 : 256), (3817505 : 65536)]
```

```python
sage: K.<c> = FunctionField(QQ)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2+c*y^2, y^2])
sage: f.orbit(P(0,1), 3)
[(0 : 1), (c : 1), (c^2 + c : 1), (c^4 + 2*c^3 + c^2 + c : 1)]
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2,y^2], domain=P)
sage: f.orbit(P.point([1, 2], False), 4, check=False)
[(1 : 2), (5 : 4), (41 : 16), (1937 : 256), (3817505 : 65536)]
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2, 2*y^2], domain=P)
sage: f.orbit(P(0, 1), [-1, 4])
Traceback (most recent call last):
... TypeError: orbit bounds must be non-negative
```

```python
sage: f.orbit(P(2, 1), 0.1)
Traceback (most recent call last):
... TypeError: Attempt to coerce non-integral RealNumber to Integer
```

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^3, x*y^2], domain=P)
sage: f.orbit(P(0, 1), [-1, 4])
Traceback (most recent call last):
... ValueError: [0, 0] does not define a valid point since all entries are 0
```

```python
sage: f.orbit(P(0, 1), 3, check=False)
[(0 : 1), (0 : 0), (0 : 0), (0 : 0)]
```
```
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([x^3, x*y^2], domain=P)
sage: f.orbit(P(2,1), 3, normalize=False)
[(2 : 1), (8 : 2), (512 : 32), (134217728 : 524288)]
sage: f.orbit(P(2, 1), 3, normalize=True)
[(2 : 1), (4 : 1), (16 : 1), (256 : 1)]
```

```text
periodic_points (n, minimal=True, R=None, algorithm='variety', return_scheme=False)
Computes the periodic points of period n of this dynamical system defined over the ring R or the base ring of the map.

This can be done either by finding the rational points on the variety defining the points of period n, or, for finite fields, finding the cycle of appropriate length in the cyclegraph. For small cardinality fields, the cyclegraph algorithm is effective for any map and length cycle, but is slow when the cyclegraph is large. The variety algorithm is good for small period, degree, and dimension, but is slow as the defining equations of the variety get more complicated.

For rational maps, where there are potentially infinitely many periodic points of a given period, you must use the return_scheme option. Note that this scheme will include the indeterminacy locus.

**INPUT:**

- n - a positive integer
- minimal – (default: True) boolean; True specifies to find only the periodic points of minimal period n and False specifies to find all periodic points of period n
- R - a commutative ring
- algorithm – (default: 'variety') must be one of the following:
  - 'variety' - find the rational points on the appropriate variety
  - 'cyclegraph' - find the cycles from the cycle graph
- return_scheme – return a subscheme of the ambient space that defines the n th periodic points

**OUTPUT:**

A list of periodic points of this map or the subscheme defining the periodic points.

**EXAMPLES:**

```
sage: setVerbose(None)
sage: P.<x,y,z> = ProjectiveSpace(QQbar,3)
sage: f = DynamicalSystem_projective([x^2-x*y+y^2, x^2-y^2+x*y])
sage: f.periodic_points(1)
[(-0.50000000000000000? - 0.866025403784439?*I : 1),
 (-0.50000000000000000? + 0.866025403784439?*I : 1),
 (1 : 1)]
sage: P.<x,y,z> = ProjectiveSpace(QuadraticField(5,'t'),2)
sage: f = DynamicalSystem_projective([x^2 - 21/16*z^2, y^2-z^2, z^2])
sage: f.periodic_points(2)
[(-5/4 : -1 : 1), (-5/4 : -1/2*t + 1/2 : 1), (-5/4 : 0 : 1),
 (continues on next page)]
```
{-5/4 : 1/2*t + 1/2 : 1), (-3/4 : -1 : 1), (-3/4 : 0 : 1),
(1/4 : -1 : 1), (1/4 : -1/2*t + 1/2 : 1), (1/4 : 0 : 1),
(1/4 : 1/2*t + 1/2 : 1), (7/4 : -1 : 1), (7/4 : 0 : 1)}

```
sage: w = QQ['w'].0
sage: K = NumberField(w^6 - 3*w^5 + 5*w^4 - 5*w^3 + 5*w^2 - 3*w + 1,'s')
sage: P.<x,y,z> = ProjectiveSpace(K,2)
sage: f = DynamicalSystem_projective([x^2+z^2, y^2+x^2, z^2+y^2])
sage: sorted(f.periodic_points(1), key=str)
[(-2*s^5 + 4*s^4 - 5*s^3 + 3*s^2 - 4*s + 2 : -s^5 + 2*s^4 - 2*s^3 + s^2 - s :
  → 1),
(-s^5 + 3*s^4 - 5*s^3 + 4*s^2 - 3*s + 1 : s^5 - 2*s^4 + 3*s^3 - 3*s^2 + 4*s -
  → 1 : 1),
(1 : 1 : 1),
(2*s^5 - 6*s^4 + 9*s^3 - 8*s^2 + 7*s - 4 : 2*s^5 - 5*s^4 + 7*s^3 - 5*s^2 +
  → 6*s - 2 : 1),
(s^5 - 2*s^4 + 2*s^3 + s : s^5 - 3*s^4 + 4*s^3 - 3*s^2 + 2*s - 1 : l),
(s^5 - 2*s^4 + 3*s^3 - 3*s^2 + 3*s - 1 : -s^5 + 3*s^4 - 5*s^3 + 4*s^2 - 4*s -
  → 2 : 1)]
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([x^2 - 21/16*z^2, y^2-2*z^2, z^2])
sage: f.periodic_points(2, False)
(-3/4 : 2 : 1), (0 : 1 : 0), (1/4 : -1 : 1), (1/4 : 2 : 1),
(1 : 0 : 0), (1 : 1 : 0), (7/4 : -1 : 1), (7/4 : 2 : 1)]
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([x^2 - 21/16*z^2, y^2-2*z^2, z^2])
sage: f.periodic_points(2)
[(-5/4 : -1 : 1), (-5/4 : 2 : 1), (1/4 : -1 : 1), (1/4 : 2 : 1)]
```

```
sage: set_verbose(None)
sage: P.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([x^2+y^2,y^2])
sage: f.periodic_points(2, R=QQbar, minimal=False)
[(-0.5000000000000000? - 1.322875655532296?*I : 1),
(-0.5000000000000000? + 1.322875655532296?*I : 1),
(0.5000000000000000? - 0.866025403784439?*I : 1),
(0.5000000000000000? + 0.866025403784439?*I : 1),
(1 : 0)]
```

```
sage: P.<x,y> = ProjectiveSpace(GF(307), 1)
sage: f = DynamicalSystem_projective([x^10+y^10, y^10])
sage: f.periodic_points(16, minimal=True, algorithm='cyclegraph')
[(69 : 1), (185 : 1), (120 : 1), (136 : 1), (97 : 1), (183 : 1),
(170 : 1), (105 : 1), (274 : 1), (275 : 1), (154 : 1), (156 : 1),
(87 : 1), (95 : 1), (161 : 1), (128 : 1)]
```

```
sage: P.<x,y> = ProjectiveSpace(GF(13^2,'t'),1)
sage: f = DynamicalSystem_projective([x^3 + 3*y^3, x^2*y])
sage: f.periodic_points(30, minimal=True, algorithm='cyclegraph')
```
possible_periods(**kwds)

Return the set of possible periods for rational periodic points of this dynamical system.

Must be defined over \( \mathbb{Z} \) or \( \mathbb{Q} \).

ALGORITHM:
Calls `self.possible_periods()` modulo all primes of good reduction in range `prime_bound`. Return the intersection of those lists.

**INPUT:**

**kwds:**

- **prime_bound** – (default: `[1, 20]`) a list or tuple of two positive integers or an integer for the upper bound
- **bad_primes** – (optional) a list or tuple of integer primes, the primes of bad reduction
- **ncpus** – (default: all cpus) number of cpus to use in parallel

**OUTPUT:** a list of positive integers

**EXAMPLES:**

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2-29/16*y^2, y^2])
sage: f.possible_periods(ncpus=1)
[1, 3]
```

```
sage: PS.<x,y> = ProjectiveSpace(1,QQ)
sage: f = DynamicalSystem_projective([5*x^3 - 53*x*y^2 + 24*y^3, 24*y^3])
sage: f.possible_periods(ncpus=1, prime_bound=[1,10])
[1, 4, 12]
sage: f.possible_periods(ncpus=1, prime_bound=[1,20])
Traceback (most recent call last):
  ... ValueError: no primes of good reduction in that range
```

```
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([2*x^3 - 50*x*z^2 + 24*z^3, 
  ....: 5*y^3 - 53*y*z^2 + 24*z^3, 24*z^3])
sage: f.possible_periods(ncpus=10, prime_bound=10)
[1, 2, 6, 20, 42, 60, 140, 420]
sage: f.possible_periods(ncpus=20, prime_bound=20) # long time
[1, 20]
```

`preperiodic_points(m, n, **kwds)`

Computes the preperiodic points of period `m`, `n` of this dynamical system defined over the ring `R` or the base ring of the map.

This is done by finding the rational points on the variety defining the points of period `m`, `n`.

For rational maps, where there are potentially infinitely many periodic points of a given period, you must use the `return_scheme` option. Note that this scheme will include the indeterminacy locus.

**INPUT:**

- **n** - a positive integer, the period
- **m** - a non negative integer, the preperiod

**kwds:**

- **minimal** – (default: True) boolean; True specifies to find only the preperiodic points of minimal period `m`, `n` and False specifies to find all preperiodic points of period `m`, `n`
• \( R \) – (default: the base ring of the dynamical system) a commutative ring over which to find the preperiodic points

• \( \text{return\_scheme} \) – (default: False) boolean; return a subscheme of the ambient space that defines the \( m \cdot n \) th preperiodic points

OUTPUT:

A list of preperiodic points of this map or the subscheme defining the preperiodic points.

EXAMPLES:

```sage
sage: P.<x,y> = ProjectiveSpace(QQbar,1)
sage: f = DynamicalSystem_projective([x^2-y^2, y^2])
sage: f.preperiodic_points(0,1)
[(-0.618033988749895? : 1), (1 : 0), (1.618033988749895? : 1)]

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2-29/16*y^2, y^2])
sage: f.preperiodic_points(1,3)
[(-5/4 : 1), (1/4 : 1), (7/4 : 1)]

sage: P.<x,y> = ProjectiveSpace(QQbar,1)
sage: f = DynamicalSystem_projective([x^2-x*y+2*y^2, x^2-y^2])
sage: f.preperiodic_points(1,2,minimal=False)
[(-3.13185666641527? : 1),
(-1 : 1),
(-0.3478103847799310? - 1.028852254136693?*I : 1),
(-0.3478103847799310? + 1.028852254136693?*I : 1),
(0.816592833206258? - 0.6710067557437100?*I : 1),
(0.816592833206258? + 0.6710067557437100?*I : 1),
(1 : 0),
(1 : 1),
(1.695620769559862? : 1),
(3 : 1)]

sage: R.<w> = QQ[]
sage: K.<s> = NumberField(w^6 - 3*w^5 + 5*w^4 - 5*w^3 + 5*w^2 - 3*w + 1)

sage: P.<x,y,z> = ProjectiveSpace(K,2)
sage: f = DynamicalSystem_projective([x^2+z^2, y^2+x^2, z^2+y^2])
sage: sorted(f.preperiodic_points(0,1), key=str)
[(-2*s^5 + 4*s^4 - 5*s^3 + 3*s^2 - 4*s : -2*s^5 + 5*s^4 - 7*s^3 + 6*s^2 - 7*s + 3 : 1),
(-s^5 + 3*s^4 - 4*s^3 + 4*s^2 - 4*s + 2 : -s^5 + 2*s^4 - 2*s^3 + s^2 - s + 1),
(-s^5 + 3*s^4 - 5*s^3 + 4*s^2 - 3*s + 1 : s^5 - 2*s^4 + 3*s^3 - 3*s^2 + 4*s - 1 : 1),
(1 : 1 : 1),
(2*s^5 - 6*s^4 + 9*s^3 - 8*s^2 + 7*s - 4 : 2*s^5 - 5*s^4 + 7*s^3 - 5*s^2 + 6*s - 2 : 1),
(s^5 - 2*s^4 + 2*s^3 + s : s^5 - 3*s^4 + 4*s^3 - 3*s^2 + 2*s - 1 : 1),
(s^5 - 2*s^4 + 3*s^3 - 3*s^2 + 3*s - 1 : -s^5 + 3*s^4 - 5*s^3 + 4*s^2 - 4*s + 2 : 1)]
```

```sage
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: K.<v> = QuadraticField(5)
sage: phi = QQ.embeddings(K)[0]
sage: f = DynamicalSystem_projective([x^2-y^2,y^2])
```
sage: f.preperiodic_points(1,1,R=phi)
[(-1/2*v - 1/2 : 1), (1/2*v - 1/2 : 1)]

sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: X = P.subscheme(2*x-y)
sage: f = DynamicalSystem_projective([x^2-y^2, 2*(x^2-y^2), y^2-z^2],
    domain=X)
sage: f.preperiodic_points(1,1)
[(-1/2 : -1/2 : 1), (1 : 2 : 1)]

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2])
sage: sorted(f.preperiodic_points(2,1))
[(0 : 2 : 1),
 (0 : 3 : 1),
 (1 : 2 : 1),
 (1 : 3 : 1),
 (2 : 0 : 1),
 (2 : 1 : 0),
 (2 : 1 : 1),
 (2 : 2 : 1),
 (2 : 3 : 1),
 (2 : 4 : 1),
 (3 : 0 : 1),
 (3 : 1 : 0),
 (3 : 1 : 1),
 (3 : 2 : 1),
 (3 : 3 : 1),
 (3 : 4 : 1),
 (4 : 2 : 1),
 (4 : 3 : 1)]

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: f = DynamicalSystem_projective([x^2, x*y, z^2])
sage: f.preperiodic_points(2,1, return_scheme=True)
Closed subscheme of Projective Space of dimension 2 over Finite Field of size 5 defined by:
0,
x^8*z^4 - x^4*z^8,
x^7*y*z^4 - x^3*y*z^8

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: R.<z> = QQ[]
sage: K.<v> = NumberField(z^4 - z^2 - 1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2])
sage: f.preperiodic_points(2, 1, R=K)
[(v : 1), (-v : 1)]

primes_of_bad_reduction (check=True)
Determine the primes of bad reduction for this dynamical system.

Must be defined over a number field.

If check is True, each prime is verified to be of bad reduction.

ALGORITHM:
$p$ is a prime of bad reduction if and only if the defining polynomials of self have a common zero. Or stated another way, $p$ is a prime of bad reduction if and only if the radical of the ideal defined by the defining polynomials of self is not $(x_0, x_1, \ldots, x_N)$. This happens if and only if some power of each $x_i$ is not in the ideal defined by the defining polynomials of self. This last condition is what is checked. The lcm of the coefficients of the monomials $x_i$ in a Groebner basis is computed. This may return extra primes.

**INPUT:**

- check -(default: True) boolean

**OUTPUT:** a list of primes

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([1/3*x^2+1/2*y^2, y^2])
sage: f.primes_of_bad_reduction()
[2, 3]

sage: P.<x,y,z,w> = ProjectiveSpace(QQ,3)
sage: f = DynamicalSystem_projective([12*x*z-7*y^2, 31*x^2-y^2, 26*z^2, 3*w^2-→z*w])
sage: f.primes_of_bad_reduction()
[2, 3, 7, 13, 31]
```

A number field example:

```python
sage: R.<z> = QQ[]
sage: K.<a> = NumberField(z^2 - 2)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([1/3*x^2+1/a*y^2, y^2])
sage: f.primes_of_bad_reduction()
[Fractional ideal (a), Fractional ideal (3)]
```

This is an example where check = False returns extra primes:

```python
sage: P.<x,y,z> = ProjectiveSpace(ZZ,2)
sage: f = DynamicalSystem_projective([3*x*y^2 + 7*y^3 - 4*y^2*z + 5*z^3,
       -5*x^3 + x^2*y + y^3 + 2*x^2*z,
       -2*x^2*y + x*y^2 + y^3 - 4*y^2*z + x*z^→2])
sage: f.primes_of_bad_reduction(False)
[2, 5, 37, 2239, 304432717]
sage: f.primes_of_bad_reduction()
[5, 37, 2239, 304432717]
```

**ramification_type (R=None, stable=True)**

Return the ramification type of endomorphisms of $\mathbb{P}^1$.

Only branch points defined over the ring $R$ contribute to the ramification type if specified, otherwise $R$ is the ring of definition for self.

Note that branch points defined over $R$ may not be geometric points if stable not set to True.

If $R$ is specified, stable is ignored.

If stable, then this will return the ramification type over an extension which splits the Galois orbits of critical points.

**INPUT:**
• \( R \) – ring or morphism (optional)
• \( \text{split} \) – boolean (optional)

OUTPUT:
list of lists, each term being the list of ramification indices in the pre-images of one critical value

EXAMPLES:

```sage
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: F = DynamicalSystem_projective([x^4, y^4])
sage: F.ramification_type()
[[4], [4]]

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: F = DynamicalSystem_projective([x^3, 4*y^3 - 3*x^2*y])
sage: F.ramification_type()
[[2], [2], [3]]

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: F = DynamicalSystem_projective([(x + y)^4, 16*x*y*(x-y)^2])
sage: F.ramification_type()
[[2], [2, 2], [4]]

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: F = DynamicalSystem_projective([(x + y)*(x - y)^3, y*(2*x+y)^3])
sage: F.ramification_type()
[[3], [3], [3]]

sage: F = DynamicalSystem_projective([x^3-2*x*y^2 + 2*y^3, y^3])
sage: F.ramification_type()
[[2], [2], [3]]
sage: F.ramification_type(R=F.base_ring())
[[2], [3]]
```

reduced_form(**kwds)

Return reduced form of this dynamical system.

The reduced form is the \( SL(2, \mathbb{Z}) \) equivalent morphism obtained by applying the binary form reduction algorithm from Stoll and Cremona [CS2003] to the homogeneous polynomial defining the periodic points (the dynatomic polynomial). The smallest period \( n \) with enough periodic points is used and without roots of too large multiplicity.

This should also minimize the size of the coefficients, but this is not always the case. By default the coefficient minimizing algorithm in [HS2018] is applied.

See `sage.rings.polynomial.multi_polynomial.reduced_form()` for the information on binary form reduction.

Implemented by Rebecca Lauren Miller as part of GSOC 2016. Minimal height added by Ben Hutz July 2018.

INPUT:

keywords:
• \( \text{prec} \) – (default: 300) integer, desired precision
• \( \text{return_conjugation} \) – (default: True) boolean; return an element of \( SL(2, \mathbb{Z}) \)
• \( \text{error_limit} \) – (default: 0.000001) a real number, sets the error tolerance
• `smallest_coeffs` – (default: True), boolean, whether to find the model with smallest coefficients
• `dynatomic` – (default: True) boolean, to use formal periodic points
• `start_n` – (default: 1), positive integer, first period to try to find appropriate binary form
• `emb` – (optional) embedding of based field into CC
• `algorithm` – (optional) which algorithm to use to find all minimal models. Can be one of the following:
  – 'BM' – Bruin-Molnar algorithm [BM2012]
  – 'HS' – Hutz-Stoll algorithm [HS2018]
• `check_minimal` – (default: True), boolean, whether to check if this map is a minimal model
• `smallest_coeffs` – (default: True), boolean, whether to find the model with smallest coefficients

OUTPUT:
• a projective morphism
• a matrix

EXAMPLES:
```python
sage: PS.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([x^3 + x*y^2, y^3])
sage: m = matrix(QQ, 2, 2, [-201221, -1, 1, 0])
sage: f = f.conjugate(m)
sage: f.reduced_form(prec=50, smallest_coeffs=False)  # needs 2 periodic
Traceback (most recent call last):
  ... ValueError: accuracy of Newton's root not within tolerance(0.000066... > 1e-06), increase precision
sage: f.reduced_form(smallest_coeffs=False)
```

Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to
(x^3 + x*y^2 : y^3),

[ 0 -1]
[ 1 201221]
```

sage: PS.<x,y> = ProjectiveSpace(ZZ, 1)
sage: f = DynamicalSystem_projective([x^2+ x*y, y^2])  # needs 3 periodic
sage: m = matrix(QQ, 2, 2, [-221, -1, 1, 0])
sage: f = f.conjugate(m)
sage: f.reduced_form(prec=200, smallest_coeffs=False)
```

Dynamical System of Projective Space of dimension 1 over Integer Ring
Defn: Defined on coordinates by sending (x : y) to
(-x^2 + x*y - y^2) : -y^2),

[ 0 -1]
[ 1 220]
```
\texttt{sage: P.<x,y> = ProjectiveSpace(QQ, 1)}
\texttt{sage: f = DynamicalSystem_projective([x^3, y^3])}
\texttt{sage: f.reduced_form(smallest_coeffs=False)}
\begin{verbatim}
( Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (x^3 : y^3),
, [1 0]
[0 1])
\end{verbatim}
\texttt{sage: PS.<X,Y> = ProjectiveSpace(QQ,1)}
\texttt{sage: f = DynamicalSystem_projective([7365*X^4 + 12564*X^3*Y + 8046*X^2*Y^2 +,}
\quad -2292*X*Y^3 + 245*Y^4, -12329*X^4 - 21012*X^3*Y - 13446*X^2*Y^2 - 3828*X*Y^3 - 409*Y^4])
\texttt{sage: f.reduced_form(prec=30, smallest_coeffs=False)}
\begin{verbatim}
Traceback (most recent call last):
..., ValueException: accuracy of Newton's root not within tolerance(0.00008... > 1e-06), increase precision
\end{verbatim}
\texttt{sage: f.reduced_form(smallest_coeffs=False)}
\begin{verbatim}
( Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (X : Y) to
  (-7*X^4 - 12*X^3*Y - 42*X^2*Y^2 - 12*X*Y^3 - 7*Y^4 : -X^4 - 4*X^3*Y -
  -5*X^2*Y^2 - 4*X*Y^3 - Y^4),
[-1  2]
[ 2 -5])
\end{verbatim}
\texttt{sage: P.<x,y> = ProjectiveSpace(RR, 1)}
\texttt{sage: f = DynamicalSystem_projective([x^4, RR(sqrt(2))*y^4])}
\texttt{sage: m = matrix(RR, 2, 2, [1,12,0,1])}
\texttt{sage: f = f.conjugate(m)}
\texttt{sage: g, m = f.reduced_form(smallest_coeffs=False); m}
\begin{verbatim}
[ 1 -12]
[ 0  1]
\end{verbatim}
\texttt{sage: P.<x,y> = ProjectiveSpace(CC, 1)}
\texttt{sage: f = DynamicalSystem_projective([x^4, CC(sqrt(-2))*y^4])}
\texttt{sage: m = matrix(CC, 2, 2, [1,12,0,1])}
\texttt{sage: f = f.conjugate(m)}
\texttt{sage: g, m = f.reduced_form(smallest_coeffs=False); m}
\begin{verbatim}
[ 1 -12]
[ 0  1]
\end{verbatim}
\texttt{sage: K.<w> = QuadraticField(2)}
\texttt{sage: P.<x,y> = ProjectiveSpace(K, 1)}
\texttt{sage: f = DynamicalSystem_projective([x^3, w*y^3])}
\texttt{sage: m = matrix(K, 2, 2, [1,12,0,1])}
\texttt{sage: f = f.conjugate(m)}
\texttt{sage: f.reduced_form(smallest_coeffs=False)}
\begin{verbatim}
\end{verbatim}
Dynamical System of Projective Space of dimension 1 over Number Field in \(w\)
\[ \rightarrow \]
with defining polynomial \(x^2 - 2\) with \(w = 1.414213562373095?\)
Defn: Defined on coordinates by sending \((x : y)\) to
\[(x^3 : (w)*y^3)\]
\[ \rightarrow \]
\[
\begin{bmatrix}
1 & -12 \\
0 & 1 \\
\end{bmatrix}
\]

```
sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^5+x-3, embedding=(x^5+x-3).roots(ring=CC)[0][0])
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem_projective([12*x^3, 2334*w*y^3])
sage: m = matrix(K, 2, 2, [-12,1,1,0])
sage: f = f.conjugate(m)
sage: f.reduced_form(smallest_coeffs=False)
```

Dynamical System of Projective Space of dimension 1 over Number Field in \(w\)
\[ \rightarrow \]
with defining polynomial \(x^5 + x - 3\) with \(w = 1.132997565885066?\)
Defn: Defined on coordinates by sending \((x : y)\) to
\[(12*x^3 : (2334*w)*y^3)\]
\[ \rightarrow \]
\[
\begin{bmatrix}
0 & -1 \\
1 & -12 \\
\end{bmatrix}
\]

```
sage: P.<x,y> = QQ[]
sage: f = DynamicalSystem([-4*y^2, 9*x^2 - 12*x*y])
sage: f.reduced_form()
```

Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending \((x : y)\) to
\[(2*x^2 - 2*y^2 : -x^2 - 2*y^2)\]
\[
\begin{bmatrix}
2 & -2 \\
3 & 0 \\
\end{bmatrix}
\]

```
sage: P.<x,y> = QQ[]
sage: f = DynamicalSystem([-2*x^3 - 9*x^2*y - 12*x*y^2 - 6*y^3 , y^3])
sage: f.reduced_form()
```

Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending \((x : y)\) to
\[(x^3 + 3*x^2*y : 3*x*y^2 + y^3)\]
\[
\begin{bmatrix}
-1 & -2 \\
1 & 1 \\
\end{bmatrix}
\]

```
sage: P.<x,y> = QQ[]
```

(continues on next page)
\begin{verbatim}
sage: f = DynamicalSystem([4*x^2 - 7*y^2, 4*y^2])
sage: f.reduced_form(start_n=2, dynatomic=False)  # long time
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
    (x^2 - x*y - y^2 : y^2)
,)

[ 2 -1]
[ 0 2]

sage: P.<x,y> = QQ[]
sage: f = DynamicalSystem([4*x^2 + y^2, 4*y^2])
sage: f.reduced_form()  # long time
(Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
    (x^2 - x*y + y^2 : y^2)
,)

[ 2 -1]
[ 0 2]

resultant (normalize=False)
Computes the resultant of the defining polynomials of this dynamical system.

If normalize is True, then first normalize the coordinate functions with
normalize_coordinates().

INPUT:

* normalize -(default: False) boolean

OUTPUT: an element of the base ring of this map

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2+y^2, 6*y^2])
sage: f.resultant()
36

sage: R.<t> = PolynomialRing(GF(17))
sage: P.<x,y> = ProjectiveSpace(R,1)
sage: f = DynamicalSystem_projective([t*x^2+t*y^2, 6*y^2])
sage: f.resultant()
2*t^2

sage: R.<t> = PolynomialRing(GF(17))
sage: P.<x,y,z> = ProjectiveSpace(R,2)
sage: f = DynamicalSystem_projective([t*x^2+t*y^2, 6*y^2, 2*t*z^2])
sage: f.resultant()
13*t^8
\end{verbatim}
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: F = DynamicalSystem_projective([x^2+y^2,6*y^2,10*x*z+z^2+y^2])
sage: F.resultant()
1296

sage: R.<t>=PolynomialRing(QQ)
sage: s = (t^3+t+1).roots(QQbar)[0][0]
sage: P.<x,y>=ProjectiveSpace(QQbar,1)
sage: f = DynamicalSystem_projective([s*x^3-13*y^3, y^3-15*y^3])
sage: f.resultant()
871.6925062959149?

**sigma_invariants** *(n, formal=False, embedding=None, type='point')*

Computes the values of the elementary symmetric polynomials of the \( n \) multiplier spectra of this dynamical system.

Can specify to instead compute the values corresponding to the elementary symmetric polynomials of the formal \( n \) multiplier spectra. The map must be defined over projective space of dimension 1. The base ring should be a number field, number field order, or a finite field or a polynomial ring or function field over a number field, number field order, or finite field.

The parameter `type` determines if the sigma are computed from the multipliers calculated at one per cycle (with multiplicity) or one per point (with multiplicity). Note that in the cycle case, a map with a cycle which collapses into multiple smaller cycles, this is still considered one cycle. In other words, if a 4-cycle collapses into a 2-cycle with multiplicity 2, there is only one multiplier used for the doubled 2-cycle when computing \( n=4 \).

**ALGORITHM:**

We use the Poisson product of the resultant of two polynomials:

\[
\text{res}(f, g) = \prod_{f(a)=0} g(a).
\]

Letting \( f \) be the polynomial defining the periodic or formal periodic points and \( g \) the polynomial \( w - f' \) for an auxiliary variable \( w \). Note that if \( f \) is a rational function, we clear denominators for \( g \).

**INPUT:**

- \( n \) – a positive integer, the period
- `formal` – (default: False) boolean; True specifies to find the values of the elementary symmetric polynomials corresponding to the formal \( n \) multiplier spectra and False specifies to instead find the values corresponding to the \( n \) multiplier spectra, which includes the multipliers of all periodic points of period \( n \)
- `embedding` – deprecated in trac ticket #23333
- `type` – (default: 'point') string; either 'point' or 'cycle' depending on whether you compute with one multiplier per point or one per cycle

**OUTPUT:** a list of elements in the base ring

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([512*x^5 - 378128*x^4*y + 76594292*x^3*y^2 - 4570550136*x^2*y^3 - 2630045017*x*y^4 + 28193217129*y^5, 512*y^5])
sage: f.sigma_invariants(1)
```
[19575526074450617/1048576, -9078122048145044298567432325/2147483648,
-2622661149009878224381377917540931367/1099511627776,
-262266107937102104196133701280271632423/549755813888,
33852320483016116503153209450763500631714178825448006778305/
→72057594037927936, 0]

sage: set_verbose(None)
sage: z = QQ['z'].0
sage: K = NumberField(z^4 - 4*z^2 + 1, 'z')

sage: f = DynamicalSystem_projective([x^2 - 5/4*y^2, y^2])

Check that infinity as part of a longer cycle is handled correctly:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)

sage: f = DynamicalSystem_projective([y^2, x^2])

sage: f.sigma_invariants(1)
Traceback (most recent call last):
... NotImplementedError: only implemented for dimension 1

sage: K.<w> = QuadraticField(3)

sage: P.<x,y> = ProjectiveSpace(K, 1)

sage: f = DynamicalSystem_projective([x^2 - w*y^2, (1-w)*x*y])

sage: R.<c> = QQ[]

sage: Pc.<x,y> = ProjectiveSpace(R, 1)

sage: f = DynamicalSystem_projective([x^2 + c*y^2, y^2])

(continues on next page)
doubled fixed point:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([x^2 - 3/4*y^2, y^2])
sage: f.sigma_invariants(2, formal=True)
[2, 1]
```

doubled 2 cycle:

```python
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([x^2 - 5/4*y^2, y^2])
sage: f.sigma_invariants(4, formal=False)
[170, 5195, 172700, 968615, 1439066, 638125, 0]
```

class `sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective_field`(*polys*, **kwds)**

Bases: `sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective`, `sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space_field`

**all_periodic_points** (**kwds**)

Determine the set of rational periodic points for this dynamical system.

The map must be defined over \( \mathbb{Q} \) and be an endomorphism of projective space. If the map is a polynomial endomorphism of \( \mathbb{P}^1 \), i.e. has a totally ramified fixed point, then the base ring can be an absolute number field. This is done by passing to the Weil restriction.

The default parameter values are typically good choices for \( \mathbb{P}^1 \). If you are having trouble getting a particular map to finish, try first computing the possible periods, then try various different lifting_prime values.

**ALGORITHM:**

Modulo each prime of good reduction \( p \) determine the set of periodic points modulo \( p \). For each cycle modulo \( p \) compute the set of possible periods (\( m r p^e \)). Take the intersection of the list of possible periods modulo several primes of good reduction to get a possible list of minimal periods of rational periodic points. Take each point modulo \( p \) associated to each of these possible periods and try to lift it to a rational point with a combination of \( p \)-adic approximation and the LLL basis reduction algorithm.

See [Hutz2015].

**INPUT:**

**kwds:**

- `R` – (default: domain of dynamical system) the base ring over which the periodic points of the dynamical system are found
- `prime_bound` – (default: \([1,20]\)) a pair (list or tuple) of positive integers that represent the limits of primes to use in the reduction step or an integer that represents the upper bound
- `lifting_prime` – (default: 23) a prime integer; argument that specifies modulo which prime to try and perform the lifting
• `period_degree_bounds` — (default: `[4, 4]`) a pair of positive integers (max period, max degree) for which the dynatomic polynomial should be solved for
• `algorithm` — (optional) specifies which algorithm to use; current options are `dynatomic` and `lifting`; defaults to solving the dynatomic for low periods and degrees and lifts for everything else
• `periods` — (optional) a list of positive integers that is the list of possible periods
• `bad_primes` — (optional) a list or tuple of integer primes; the primes of bad reduction
• `ncpus` — (default: all cpus) number of cpus to use in parallel

OUTPUT: a list of rational points in projective space

EXAMPLES:

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2-3/4*y^2, y^2])
sage: sorted(f.all_periodic_points(prime_bound=20, lifting_prime=7)) # long time
[(-1/2 : 1), (1 : 0), (3/2 : 1)]

sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([2*x^3 - 50*x*z^2 + 24*z^3, 5*y^3 - 53*y*z^2 + 24*z^3, 24*z^3])
sage: sorted(f.all_periodic_points(prime_bound=[1,20])) # long time
[(-3 : -1 : 1), (-3 : 0 : 1), (-3 : 1 : 1), (-3 : 3 : 1), (-1 : -1 : 1),
(-1 : 0 : 1), (-1 : 1 : 1), (-1 : 3 : 1), (0 : 1 : 0), (1 : -1 : 1),
(1 : 0 : 0), (1 : 0 : 1), (1 : 1 : 1), (3 : -1 : 1),
(3 : 0 : 1), (3 : 1 : 1), (3 : 3 : 1), (5 : -1 : 1), (5 : 0 : 1),
(5 : 1 : 1), (5 : 3 : 1)]

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([-5*x^2 + 4*y^2, 4*x*y])
sage: sorted(f.all_periodic_points()) # long time
[(-2 : 1), (-2/3 : 1), (2/3 : 1), (1 : 0), (2 : 1)]

sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^2-x+1)
sage: P.<u,v> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([u^2 + v^2, v^2])
sage: sorted(f.all_periodic_points())
[(-w + 1 : 1), (w : 1), (1 : 0)]

sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^2-x+1)
sage: P.<u,v> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([u^2+v^2,u*v])
sage: f.all_periodic_points()
Traceback (most recent call last):
... NotImplementedError: rational periodic points for number fields only implemented for polynomials

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: K.<v> = QuadraticField(5)
sage: phi = QQ.embeddings(K)[0]
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2])
```
(continues on next page)
all_preperiodic_points (**kwds)

Determine the set of rational preperiodic points for this dynamical system.

The map must be defined over \( \mathbb{Q} \) and be an endomorphism of projective space. If the map is a polynomial endomorphism of \( \mathbb{P}^1 \), i.e. has a totally ramified fixed point, then the base ring can be an absolute number field. This is done by passing to the Weil restriction.

The default parameter values are typically good choices for \( \mathbb{P}^1 \). If you are having trouble getting a particular map to finish, try first computing the possible periods, then try various different values for lifting_prime.

ALGORITHM:

- Determines the list of possible periods.
- Determines the rational periodic points from the possible periods.
- Determines the rational preperiodic points from the rational periodic points by determining rational preimages.

INPUT:

kwds:

- \( R \) – (default: domain of dynamical system) the base ring over which the periodic points of the dynamical system are found
- \( \text{prime bound} \) – (default: \([1, 20]\)) a pair (list or tuple) of positive integers that represent the limits of primes to use in the reduction step or an integer that represents the upper bound
- \( \text{lifting prime} \) – (default: 23) a prime integer; specifies modulo which prime to try and perform the lifting
• **periods** – (optional) a list of positive integers that is the list of possible periods

• **bad_primes** – (optional) a list or tuple of integer primes; the primes of bad reduction

• **ncpus** – (default: all cpus) number of cpus to use in parallel

• **period_degree_bounds** – (default: \([4, 4]\)) a pair of positive integers (max period, max degree) for which the dynatomic polynomial should be solved for when in dimension 1

• **algorithm** – (optional) specifies which algorithm to use; current options are *dynatomic* and *lifting*; defaults to solving the dynatomic for low periods and degrees and lifts for everything else

**OUTPUT:** a list of rational points in projective space

**EXAMPLES:**

```python
sage: PS.<x,y> = ProjectiveSpace(1,QQ)
sage: f = DynamicalSystem_projective([x^2 -y^2, 3*x*y])
sage: sorted(f.all_preperiodic_points())
[(-2 : 1), (-1 : 1), (-1/2 : 1), (0 : 1), (1/2 : 1), (1 : 0), (1 : 1), (2 : 1)]
```

```python
sage: PS.<x,y> = ProjectiveSpace(1,QQ)
sage: f = DynamicalSystem_projective([5*x^3 - 53*x*y^2 + 24*y^3, 24*y^3])
sage: sorted(f.all_preperiodic_points(prime_bound=10))
[(-1 : 1), (0 : 1), (1 : 0), (1 : 1), (3 : 1)]
```

```python
sage: PS.<x,y,z> = ProjectiveSpace(2,QQ)
sage: f = DynamicalSystem_projective([x^2 - 21/16*z^2, y^2-2*z^2, z^2])
sage: sorted(f.all_preperiodic_points(prime_bound=[1,8], lifting_prime=7, periods=[2])) # long time
```

```python
sage: K.<w> = QuadraticField(33)
sage: PS.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2-71/48*y^2, y^2])
sage: sorted(f.all_preperiodic_points()) # long time
[(-1/12*w - 1 : 1), (-1/6*w - 1/4 : 1), (-1/12*w - 1/2 : 1), (-1/6*w + 1/4 : 1), (1/12*w - 1 : 1), (1/12*w - 1/2 : 1), (-1/12*w + 1/2 : 1), (-1/12*w + 1 : 1), (1/6*w - 1/4 : 1), (1/12*w + 1/2 : 1), (1 : 0), (1/6*w + 1/4 : 1), (1/12*w + 1 : 1)]
```

**all_rational_preimages**(points)

Given a set of rational points in the domain of this dynamical system, return all the rational preimages of those points.
In other words, all the rational points which have some iterate in the set points. This function repeatedly calls \texttt{rational_preimages}. If the degree is at least two, by Northcott, this is always a finite set. The map must be defined over number fields and be an endomorphism.

**INPUT:**
- \texttt{points} – a list of rational points in the domain of this map

**OUTPUT:** A list of rational points in the domain of this map

**EXAMPLES:**

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([16*x^2 - 29*y^2, 16*y^2])
sage: sorted(f.all_rational_preimages([P(-1,4)]))
 (5/4 : 1), (7/4 : 1)]
```

```python
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([76*x^2 - 180*x*y + 45*y^2 + 14*x*z +
 -45*y*z - 90*z^2, 67*x^2 - 180*x*y - 157*x*z + 90*y*z, -90*z^2])
sage: sorted(f.all_rational_preimages([P(-9,-4,1)]))
[(-9 : -4 : 1), (0 : -1 : 1), (0 : 0 : 1), (0 : 1 : 1), (0 : 4 : 1),
 (1 : 0 : 1), (1 : 1 : 1), (1 : 2 : 1), (1 : 3 : 1)]
```

A non-periodic example

```python
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, 2*x*y])
sage: sorted(f.all_rational_preimages([P(17,15)]))
[(1/3 : 1), (3/5 : 1), (5/3 : 1), (3 : 1)]
```

A number field example:

```python
sage: z = QQ['z'].0
sage: K.<w> = NumberField(z^3 + (z^2)/4 - (41/16)*z + 23/64);
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([16*x^2 - 29*y^2, 16*y^2])
sage: sorted(f.all_rational_preimages([P(16*w^2 - 29,16)]), key=str)
[(-w - 1/2 : 1),
 (-w : 1),
 (-w^2 + 21/16 : 1),
 (-w^2 + 29/16 : 1),
 (-w^2 - w + 25/16 : 1),
 (-w^2 - w + 33/16 : 1),
 (w + 1/2 : 1),
 (w : 1),
 (w^2 + w - 25/16 : 1),
 (w^2 + w - 33/16 : 1),
 (w^2 - 21/16 : 1),
 (w^2 - 29/16 : 1)]
```

```python
sage: K.<w> = QuadraticField(3)
sage: P.<u,v> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([u^2+v^2, v^2])
sage: f.all_rational_preimages(P(4))
[(-w : 1), (w : 1)]
```
**conjugating_set** *(other, R=None)*

Return the set of elements in PGL over the base ring that conjugates one dynamical system to the other.

Given two nonconstant rational functions of equal degree, determine if there is a rational element of PGL that conjugates one rational function to another.

The optional argument \( R \) specifies the field of definition of the PGL elements. The set is determined by taking the fixed points of one map and mapping them to all unique permutations of the fixed points of the other map. If there are not enough fixed points the function compares the mapping between rational preimages of fixed points and the rational preimages of the preimages of fixed points until there are enough points; such that there are \( n + 2 \) points with all \( n + 1 \) subsets linearly independent.

**Warning:** For degree 1 maps that are conjugate, there is a positive dimensional set of conjugations. This function returns only one such element.

**ALGORITHM:**
Implementing invariant set algorithm from the paper [FMV2014]. Given that the set of \( n \) th preimages of fixed points is invariant under conjugation find all elements of PGL that take one set to another.

**INPUT:**
- `other` – a rational function of same degree as this map
- `R` – a field or embedding

**OUTPUT:**
Set of conjugating \( n + 1 \) by \( n + 1 \) matrices.

**AUTHORS:**
- Original algorithm written by Xander Faber, Michelle Manes, Bianca Viray [FMV2014].
- Implemented by Rebecca Lauren Miller, as part of GSOC 2016.

**EXAMPLES:**

```sage
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - 2*y^2, y^2])
sage: m = matrix(QQbar, 2, 2, [-1, 3, 2, 1])
sage: g = f.conjugate(m)
sage: f.conjugating_set(g)
[[-1 3]
 [ 2 1]]
```

```sage
sage: K.<w> = QuadraticField(-1)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2 + y^2, x*y])
sage: m = matrix(K, 2, 2, [1, 1, 2, 1])
sage: g = f.conjugate(m)
sage: sorted(f.conjugating_set(g)) # long time
[[-1 -1]
 [ 2 1], [2 1]]
```
```
sage: K.<i> = QuadraticField(-1)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: D8 = DynamicalSystem_projective([y^3, x^3])
sage: sorted(D8.conjugating_set(D8)) # long time
[[-1 0] [-i 0] [ 0 -1] [ 0 -i] [ 0 i] [i 0] [ 1 0]
  [0 1], [ 0 1], [ 1 0], [ 1 0], [ 1 0], [ 0 1], [ 0 1]]
```

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: D8 = DynamicalSystem_projective([y^2, x^2])
sage: D8.conjugating_set(D8)
Traceback (most recent call last):
...
ValueError: not enough rational preimages
```

```
sage: P.<x,y> = ProjectiveSpace(GF(7),1)
sage: D6 = DynamicalSystem_projective([y^2, x^2])
sage: D6.conjugating_set(D6)
[ [1 0] [0 1] [0 2] [4 0] [2 0] [0 4]
  [0 1], [ 1 0], [ 0 1], [ 0 1], [ 1 0], [ 1 0]]
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([x^2 + x*z, y^2, z^2])
sage: f.conjugating_set(f) # long time
[ [1 0 0]
  [0 1 0]
  [0 0 1]]
```

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: R = P.coordinate_ring()
sage: f = DynamicalSystem_projective([R(3), R(4)])
sage: g = DynamicalSystem_projective([R(5), R(2)])
sage: m = f.conjugating_set(g)[0]
sage: f.conjugate(m) == g
True
```

```
sage: P.<x,y> = ProjectiveSpace(QQbar, 1)
sage: f = DynamicalSystem_projective([7*x + 12*y, 8*x])
sage: g = DynamicalSystem_projective([1645*x - 318*y, 8473*x - 1638*y])
sage: m = f.conjugating_set(g)[0]
sage: f.conjugate(m) == g
True
```

Note that only one possible conjugation is returned:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(11),2)
sage: f = DynamicalSystem_projective([2*x + 12*y, 11*y+2*z, x+z])
sage: ml = matrix(GF(11), 3, 3, [1,4,1,0,2,1,1,1,1])
sage: g = f.conjugate(ml)
sage: f.conjugating_set(g)
```

5.3. Dynamical systems on projective schemes
\begin{verbatim}

sage: L.<v> = CyclotomicField(8)
sage: P.<x,y,z> = ProjectiveSpace(L, 2)
sage: f = DynamicalSystem_projective([2*x + 12*y, 11*y+2*z, x+z])
sage: ml = matrix(L, 3, 3, [1,4,v^2,0,2,1,1,1,1])
sage: g = f.conjugate(ml)
sage: m = f.conjugating_set(g)[0]
sage: f.conjugate(m) == g
True

connected_rational_component \((P, n=0)\)

Computes the connected component of a rational preperiodic point \(P\) by this dynamical system.

Will work for non-preperiodic points if \(n\) is positive. Otherwise this will not terminate.

INPUT:

- \(P\) – a rational preperiodic point of this map
- \(n\) – (default: 0) integer; maximum distance from \(P\) to branch out; a value of 0 indicates no bound

OUTPUT:

A list of points connected to \(P\) up to the specified distance.

EXAMPLES:

\begin{verbatim}
sage: R.<x> = PolynomialRing(QQ)
sage: K.<w> = NumberField(x^3+1/4*x^2-41/16*x+23/64)
sage: PS.<x,y> = ProjectiveSpace(1,K)
sage: f = DynamicalSystem_projective([x^2 - 29/16*y^2, y^2])
sage: P = PS([w,1])
sage: sorted(f.connected_rational_component(P), key=str)
[(-w - 1/2 : 1),
 (-w : 1),
 (-w^2 + 21/16 : 1),
 (-w^2 + 29/16 : 1),
 (-w^2 - w + 25/16 : 1),
 (-w^2 - w + 33/16 : 1),
 (w + 1/2 : 1),
 (w : 1),
 (w^2 + w - 25/16 : 1),
 (w^2 + w - 33/16 : 1),
 (w^2 - 21/16 : 1),
 (w^2 - 29/16 : 1)]

sage: PS.<x,y,z> = ProjectiveSpace(2,QQ)
sage: f = DynamicalSystem_projective([x^2 - 21/16*z^2, y^2-2*z^2, z^2])
sage: P = PS([[17/16,7/4,1]])
sage: f.connected_rational_component(P,3)
[(17/16 : 7/4 : 1),
 (-47/256 : 17/16 : 1),
 (-83807/65536 : -223/256 : 1),

\end{verbatim}
\end{verbatim}
is_conjugate(other, R=None)
Return whether two dynamical systems are conjugate over their base ring (by default) or over the ring $R$ entered as an optional parameter.

ALGORITHM:
Implementing invariant set algorithm from the paper [FMV2014]. Given that the set of $n$th preimages is invariant under conjugation this function finds whether two maps are conjugate.

INPUT:
- other — a nonconstant rational function of the same degree as this map
- R — a field or embedding

OUTPUT: boolean

AUTHORS:
- Original algorithm written by Xander Faber, Michelle Manes, Bianca Viray [FMV2014].
- Implemented by Rebecca Lauren Miller as part of GSOC 2016.

EXAMPLES:

```
sage: K.<w> = CyclotomicField(3)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: D8 = DynamicalSystem_projective([y^2, x^2])
sage: D8.is_conjugate(D8)
True
```

```
sage: set_verbose(None)
sage: P.<x,y> = ProjectiveSpace(QQbar,1)
sage: f = DynamicalSystem_projective([x^2 + x*y,y^2])
sage: m = matrix(QQbar, 2, 2, [1, 1, 2, 1])
sage: g = f.conjugate(m)
sage: f.is_conjugate(g) # long time
True
```

```
sage: P.<x,y> = ProjectiveSpace(GF(5),1)
sage: f = DynamicalSystem_projective([x^3 + x*y^2,y^3])
sage: m = matrix(GF(5), 2, 2, [1, 3, 2, 9])
sage: g = f.conjugate(m)
sage: f.is_conjugate(g)
True
```
```sage
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + x*y, y^2])
sage: g = DynamicalSystem_projective([x^3 + x^2*y, y^3])
sage: f.is_conjugate(g)
False

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 + x*y, y^2])
sage: g = DynamicalSystem_projective([x^2 - 2*y^2, y^2])
sage: f.is_conjugate(g)
False

sage: P.<x,y> = ProjectiveSpace(QQbar, 1)
sage: f = DynamicalSystem_projective([7*x + 12*y, 8*x])
sage: g = DynamicalSystem_projective([1645*x - 318*y, 8473*x - 1638*y])
sage: f.is_conjugate(g)
True

conjugation is only checked over the base field by default:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([-3*y^2, 3*x^2])
sage: g = DynamicalSystem_projective([-x^2 + 2*x*y, 2*x+y + y^2])
sage: f.is_conjugate(g), f.is_conjugate(g, R=QQbar) # long time
(False, True)

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([7*x + 12*y, 8*y+2*z, x+z])
sage: m1 = matrix(QQ, 3, 3, [1,4,1,0,2,1,1,1,1])
sage: g = f.conjugate(m1)
sage: f.is_conjugate(g)
True

sage: P.<x,y,z> = ProjectiveSpace(GF(7), 2)
sage: f = DynamicalSystem_projective([2*x + 12*y, 11*y+2*z, x+z])
sage: m1 = matrix(GF(7), 3, 3, [1,4,1,0,2,1,1,1,1])
sage: g = f.conjugate(m1)
sage: f.is_conjugate(g)
True

sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem_projective([2*x^2 + 12*y*x, 11*y*x+2*y^2, x^2+z^2])
sage: m1 = matrix(QQ, 3, 3, [1,4,1,0,2,1,1,1,1])
sage: g = f.conjugate(m1)
sage: f.is_conjugate(g) # long time
True

```
OUTPUT:

A Boolean. If `return_conjugation` is `True`, then this also returns the conjugation as a matrix if `self` is Newton or `None` otherwise.

The conjugation may be defined over an extension if the map has fixed points not defined over the base field.

EXAMPLES:

```python
sage: A.<z> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([z - (z^2 + 1)/(2*z)])
sage: F = f.homogenize(1)
sage: F.is_newton(return_conjugation=True)
( True, [0 1]

sage: A.<z> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([z^2 + 1])
sage: F = f.homogenize(1)
sage: F.is_newton(return_conjugation=True)
(False, None)
```

```python
sage: PP.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([-4*x^3 - 3*x*y^2, -2*y^3])
sage: F = f.homogenize(1)
[ -a 2*a]
```

```python
sage: K.<zeta> = CyclotomicField(2*4)
sage: A.<z> = AffineSpace(K, 1)
sage: f = DynamicalSystem_affine(z-(z^3+zeta*z)/(3*z^2+zeta))
sage: F = f.homogenize(1)
sage: F.is_newton()  
True
```

```python
is_polynomial()  
Check to see if the dynamical system has a totally ramified fixed point.

The function must be defined over an absolute number field or a finite field.

OUTPUT: boolean

EXAMPLES:

```python
sage: R.<x> = QQ[]
sage: K.<w> = QuadraticField(7)
sage: P.<x,y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem_projective([x*x^2 + 2*x*y - 5*y*y^2, 2*x*y])
sage: f.is_polynomial()  
False
```

(continues on next page)
lift_to_rational_periodic(points_modp, B=None)

Given a list of points in projective space over $\mathbb{F}_p$, determine if they lift to $\mathbb{Q}$-rational periodic points. The map must be an endomorphism of projective space defined over $\mathbb{Q}$.

**ALGORITHM:**

Use Hensel lifting to find a $p$-adic approximation for that rational point. The accuracy needed is determined by the height bound $B$. Then apply the LLL algorithm to determine if the lift corresponds to a rational point.

If the point is a point of high multiplicity (multiplier 1), the procedure can be very slow.

**INPUT:**

- points_modp – a list or tuple of pairs containing a point in projective space over $\mathbb{F}_p$ and the possible period
- $B$ – (optional) a positive integer; the height bound for a rational preperiodic point

**OUTPUT:** a list of projective points

**EXAMPLES:**

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2])
sage: f.lift_to_rational_periodic([[P(0,1).change_ring(GF(7)), 4]])
[[[0 : 1], 2]]
```

There may be multiple points in the lift.

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([-5*x^2 + 4*y^2, 4*x*y])
sage: f.lift_to_rational_periodic([[P(1,0).change_ring(GF(3)), 1]]) # long...
[[[1 : 0], [2/3 : 1], [-2/3 : 1]]]
```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([16*x^2 - 29*y^2, 16*y^2])
sage: f.lift_to_rational_periodic([[P(3,1).change_ring(GF(13)), 3]])
[[(-1/4 : 1), 3]]

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem_projective([76*x^2 - 180*x*y + 45*y^2 + 14*x*z +
→ 45*y*z - 90*z^2, 67*x^2 - 180*x*y - 157*x*z + 90*y*z, -90*z^2])
sage: f.lift_to_rational_periodic([[P(14,19,1).change_ring(GF(23)), 9]])
# long time
[[(-9 : -4 : 1), 9]]

normal_form(return_conjugation=False)
Return a normal form in the moduli space of dynamical systems.
Currently implemented only for polynomials. The totally ramified fixed point is moved to infinity and the
map is conjugated to the form \( x^n + a_{n-2}x^{n-2} + \cdots + a_0 \). Note that for finite fields we can only remove
the \((n-1)\)-st term when the characteristic does not divide \( n \).

INPUT:

- \( \text{return_conjugation}\) – (default: False) boolean; if True, then return the conjugation element
  of PGL along with the embedding into the new field

OUTPUT:

- \( \text{SchemeMorphism_polynomial} \)
- (optional) an element of PGL as a matrix
- (optional) the field embedding

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem_projective([x^2 + 2*x*y - 5*x^2, 2*x*y])
sage: f.normal_form()  # long time
Traceback (most recent call last):
... NOTIMPLEMENTEDERROR: map is not a polynomial

sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^2 - 5)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2 + w*x*y, y^2])
sage: g,m,psi = f.normal_form(return_conjugation = True);m
[ 1 -1/2 *w]
[ 0 1]
sage: f.change_ring(psi).conjugate(m) == g
True

sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem_projective([[13*x^2 + 4*x*y + 3*y^2, 5*y^2]])
sage: f.normal_form()
Dynamical System of Projective Space of dimension 1 over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
  (5*x^2 + 9*y^2 : 5*y^2)

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sage: K = GF(3^3, prefix = 'w')
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^3 + 2*x^2*y + 2*x*y^2 + K.gen()·y^3, y^3])
sage: f.normal_form()
Dynamical System of Projective Space of dimension 1 over Finite Field in w3 of size 3^3
  Defn: Defined on coordinates by sending (x : y) to
         (x^3 + x^2*y + x*y^2 + (-w3)*y^3 : y^3)

sage: P.<x,y> = ProjectiveSpace(GF(3),1)
sage: f = DynamicalSystem_projective([2*x^3 + x^2*y, y^3])
sage: g,m,psi = f.normal_form(return_conjugation=True); psi
Ring morphism:
  From: Finite Field of size 3
  To:  Finite Field in z2 of size 3^2
  Defn: 1 |--> 1

potential_good_reduction(prime)
Return if this dynamical system has potential good reduction at prime.

A dynamical system has good reduction at prime if after the coefficients are reduced modulo prime
the degree remains the same. A dynamical system \( f \) has potential good reduction if there exists \( \phi \in PGL(n, K) \) such that \( \phi^{-1} \circ f \circ \phi \) has good reduction.

If this dynamical system \( f \) has potential good reduction at prime, a dynamical system \( g = \phi^{-1} \circ f \circ \phi \)
which has good reduction at prime is returned.

This dynamical system must have as its domain \( \mathbb{P}^1(K) \), where \( K \) is a number field.

INPUT:

• prime – a prime ideal of the field of definition of the fixed points of the map, or a prime number in
  \( \mathbb{Q} \) if the field of definition of the fixed points is \( \mathbb{Q} \).

OUTPUT:

• False if this dynamical system does not have good reduction

  • If this dynamical system \( f \) has potential good reduction, a dynamical system \( g \) such that \( g = \phi^{-1} \circ f \circ \phi \)
    and \( g \) has good reduction at prime.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: system = DynamicalSystem_projective([x^2-y^2, 2*x*y])
sage: prime = system.field_of_definition_periodic(1).prime_above(2)
sage: new_system = system.potential_good_reduction(prime)
sage: new_system
Dynamical System of Projective Space of dimension 1 over Number Field in a with defining polynomial x^2 + 1
  Defn: Defined on coordinates by sending (x : y) to
         ((-1/2*a)·x^2 + (-5/2*a)·y^2 : (-a)·x·y + y^2)

Note that this map has good reduction at 2:

sage: new_system.resultant()
1
rational_periodic_points(**kwds)

Determine the set of rational periodic points for this dynamical system.

The map must be defined over \(\mathbb{Q}\) and be an endomorphism of projective space. If the map is a polynomial endomorphism of \(\mathbb{P}^1\), i.e. has a totally ramified fixed point, then the base ring can be an absolute number field. This is done by passing to the Weil restriction.

The default parameter values are typically good choices for \(\mathbb{P}^1\). If you are having trouble getting a particular map to finish, try first computing the possible periods, then try various different lifting_prime values.

**ALGORITHM:**

Modulo each prime of good reduction \(p\) determine the set of periodic points modulo \(p\). For each cycle modulo \(p\) compute the set of possible periods \((m_r p^e)\). Take the intersection of the list of possible periods modulo several primes of good reduction to get a possible list of minimal periods of rational periodic points. Take each point modulo \(p\) associated to each of these possible periods and try to lift it to a rational point with a combination of \(p\)-adic approximation and the LLL basis reduction algorithm.

See [Hutz2015].

**INPUT:**

kwds:

- prime_bound – (default: \([1,20]\)) a pair (list or tuple) of positive integers that represent the limits of primes to use in the reduction step or an integer that represents the upper bound
- lifting_prime – (default: 23) a prime integer; argument that specifies modulo which prime to try and perform the lifting
- periods – (optional) a list of positive integers that is the list of possible periods
- bad_primes – (optional) a list or tuple of integer primes; the primes of bad reduction
- ncpus – (default: all cpus) number of cpus to use in parallel

**OUTPUT:** a list of rational points in projective space

**EXAMPLES:**

```python
sage: R.<x> = QQ[]
sage: K.<w> = NumberField(x^2-x+1)
sage: P.<u,v> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([u^2 + v^2, 1])
sage: sorted(f.rational_periodic_points())
doctest:warning
...[[-w + 1 : 1], (w : 1), (1 : 0)]
```
**rational_preperiodic_graph(**

**kwds**)  
Determine the directed graph of the rational preperiodic points for this dynamical system.

The map must be defined over \( \mathbb{Q} \) and be an endomorphism of projective space. If this map is a polynomial endomorphism of \( \mathbb{P}^1 \), i.e. has a totally ramified fixed point, then the base ring can be an absolute number field. This is done by passing to the Weil restriction.

**ALGORITHM:**

- Determines the list of possible periods.
- Determines the rational periodic points from the possible periods.
- Determines the rational preperiodic points from the rational periodic points by determining rational preimages.

**INPUT:**

**kwds:**

- `prime_bound` – (default: \([1, 20]\)) a pair (list or tuple) of positive integers that represent the limits of primes to use in the reduction step or an integer that represents the upper bound
- `lifting_prime` – (default: 23) a prime integer; specifies modulo which prime to try and perform the lifting
- `periods` – (optional) a list of positive integers that is the list of possible periods
- `bad_primes` – (optional) a list or tuple of integer primes; the primes of bad reduction
- `ncpus` – (default: all cpus) number of cpus to use in parallel

**OUTPUT:**

A digraph representing the orbits of the rational preperiodic points in projective space.

**EXAMPLES:**

```python
sage: PS.<x,y> = ProjectiveSpace(1,QQ)
sage: f = DynamicalSystem_projective([7*x^2 - 28*y^2, 24*x*y])
sage: f.rational_preperiodic_graph()
Looped digraph on 12 vertices

sage: PS.<x,y> = ProjectiveSpace(1,QQ)
sage: f = DynamicalSystem_projective([-3/2*x^3 +19/6*x*y^2, y^3])
sage: f.rational_preperiodic_graph(prime_bound=[1,8])
Looped digraph on 12 vertices

sage: PS.<x,y,z> = ProjectiveSpace(2,QQ)
sage: f = DynamicalSystem_projective([2*x^3 - 50*x*z^2 + 24*z^3,  .....: 5*y^3 - 53*y*z^2 + 24*z^3, 24*z^3])
sage: f.rational_preperiodic_graph(prime_bound=[1,11], lifting_prime=13) # long time
Looped digraph on 30 vertices

sage: K.<w> = QuadraticField(-3)
sage: P.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2])
sage: f.rational_preperiodic_graph() # long time
Looped digraph on 5 vertices
```
rational_preperiodic_points(**kwds)

Determine the set of rational preperiodic points for this dynamical system.

The map must be defined over $\mathbb{Q}$ and be an endomorphism of projective space. If the map is a polynomial endomorphism of $\mathbb{P}^1$, i.e. has a totally ramified fixed point, then the base ring can be an absolute number field. This is done by passing to the Weil restriction.

The default parameter values are typically good choices for $\mathbb{P}^1$. If you are having trouble getting a particular map to finish, try first computing the possible periods, then try various different values for lifting_prime.

**ALGORITHM:**

- Determines the list of possible periods.
- Determines the rational periodic points from the possible periods.
- Determines the rational preperiodic points from the rational periodic points by determining rational preimages.

**INPUT:**

kwds:

- `prime_bound` – (default: [1, 20]) a pair (list or tuple) of positive integers that represent the limits of primes to use in the reduction step or an integer that represents the upper bound
- `lifting_prime` – (default: 23) a prime integer; specifies modulo which prime to try and perform the lifting
- `periods` – (optional) a list of positive integers that is the list of possible periods
- `bad_primes` – (optional) a list or tuple of integer primes; the primes of bad reduction
- `ncpus` – (default: all cpus) number of cpus to use in parallel
- `period_degree_bounds` – (default: [4, 4]) a pair of positive integers (max period, max degree) for which the dynatomic polynomial should be solved for when in dimension 1
- `algorithm` – (optional) specifies which algorithm to use; current options are `dynatomic` and `lifting`; defaults to solving the dynatomic for low periods and degrees and lifts for everything else

**OUTPUT:** a list of rational points in projective space

**EXAMPLES:**

```python
sage: PS.<x,y> = ProjectiveSpace(1,QQ)
sage: f = DynamicalSystem_projective([x^2 -y^2, 3*x*y])
sage: sorted(f.rational_preperiodic_points())
doctest:warning...
[(-2 : 1), (-1 : 1), (-1/2 : 1), (0 : 1), (1/2 : 1), (1 : 0), (1 : 1),
 (2 : 1)]
```

reduce_base_field()

Return this map defined over the field of definition of the coefficients.

The base field of the map could be strictly larger than the field where all of the coefficients are defined. This function reduces the base field to the minimal possible. This can be done when the base ring is a number field, QQbar, a finite field, or algebraic closure of a finite field.

**OUTPUT:** A dynamical system

**EXAMPLES:**
```python
sage: K.<t> = GF(2^3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: f = DynamicalSystem_projective([x^2 + y^2, y^2, z^2+z*y])
sage: f.reduce_base_field()
Dynamical System of Projective Space of dimension 2 over Finite Field of size 2
Defn: Defined on coordinates by sending (x : y : z) to
(x^2 + y^2 : y^2 : y*z + z^2)
```

```python
sage: P.<x,y,z> = ProjectiveSpace(QQbar, 2)
sage: f = DynamicalSystem_projective([x^2 + QQbar(sqrt(3))*y^2, y^2, QQbar(sqrt(2))*z^2])
sage: f.reduce_base_field()
Dynamical System of Projective Space of dimension 2 over Number Field in a with defining polynomial y^4 - 4*y^2 + 1 with a = 1.931851652578137?
Defn: Defined on coordinates by sending (x : y : z) to
(x^2 + (a^2 - 2)*y^2 : y^2 : (a^3 - 3*a)*z^2)
```

```python
sage: R.<x> = QQ[]
sage: K.<v> = NumberField(x^3-2, embedding=(x^3-2).roots(ring=CC)[0][0])
sage: R.<x> = QQ[]
sage: L.<w> = NumberField(x^6 + 9*x^4 - 4*x^3 + 27*x^2 + 36*x + 31, embedding=(x^6 + 9*x^4 - 4*x^3 + 27*x^2 + 36*x + 31).roots(ring=CC)[0][0])
sage: P.<x,y> = ProjectiveSpace(L,1)
sage: f = DynamicalSystem([L(v)*x^2 + y^2, x*y])
sage: f.reduce_base_field().base_ring().is_isomorphic(K)
True
```

```python
sage: K.<v> = CyclotomicField(5)
sage: A.<x,y> = ProjectiveSpace(K, 1)
sage: f = DynamicalSystem_projective([3*x^2 + y^2, x*y])
sage: f.reduce_base_field()
Dynamical System of Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending (x : y) to
(3*x^2 + y^2 : x*y)
```

**class** sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective_finite_field

**Bases:**
sage.dynamics.arithmetic_dynamics.projective_ds.DynamicalSystem_projective_field,
sage.schemes.projective.projective_morphism.SchemeMorphism_polynomial_projective_space_finite_field

**all_periodic_points(**kwds)**

Return a list of all periodic points over a finite field.

**INPUT:**

**keywords:**

- **R** – (default: base ring of dynamical system) the base ring over which the periodic points of the dynamical system are found

**OUTPUT:** a list of elements which are periodic

**EXAMPLES:**
sage: P.<x,y> = ProjectiveSpace(GF(5^2),1)
sage: f = DynamicalSystem_projective([x^2+y^2, x*y])
sage: f.all_periodic_points()
[(1 : 0), (z2 + 2 : 1), (4*z2 + 3 : 1)]

sage: P.<x,y,z> = ProjectiveSpace(GF(5),2)
sage: f = DynamicalSystem_projective([x^2+y^2+z^2, x*y+x*z, z^2])
sage: f.all_periodic_points()
[(1 : 0 : 0),
 (0 : 0 : 1),
 (1 : 0 : 1),
 (2 : 1 : 1),
 (1 : 4 : 1),
 (3 : 0 : 1),
 (0 : 3 : 1)]

sage: P.<x,y>=ProjectiveSpace(GF(3),1)
sage: f = DynamicalSystem_projective([x^2-y^2, y^2])
sage: f.all_periodic_points(R=GF(3^2, 't'))
[(1 : 0), (0 : 1), (2 : 1), (t : 1), (2*t + 1 : 1)]

automorphism_group(absolute=False, iso_type=False, return_functions=False)
Return the subgroup of $PGL2$ that is the automorphism group of this dynamical system.

Only for dimension 1. The automorphism group is the set of $PGL2$ elements that fixed the map under conjugation. See [FMV2014] for the algorithm.

INPUT:

- absolute – (default: False) boolean; if True, then return the absolute automorphism group and a field of definition
- iso_type – (default: False) boolean; if True, then return the isomorphism type of the automorphism group
- return_functions – (default: False) boolean; True returns elements as linear fractional transformations and False returns elements as $PGL2$ matrices

OUTPUT: a list of elements of the automorphism group

AUTHORS:

- Original algorithm written by Xander Faber, Michelle Manes, Bianca Viray
- Modified by Joao Alberto de Faria, Ben Hutz, Bianca Thompson

EXAMPLES:

sage: R.<x,y> = ProjectiveSpace(GF(7^3,'t'),1)
sage: f = DynamicalSystem_projective([x^2-y^2, x*y])
sage: f.automorphism_group()
[ [1 0] [6 0]
 [0 1], [0 1] ]

sage: R.<x,y> = ProjectiveSpace(GF(3^2, 't'),1)
sage: f = DynamicalSystem_projective([x^3,y^3])
sage: lst, label = f.automorphism_group(return_functions=True, iso_type=True)

(continues on next page)
sage: sorted(lst, key=str), label # long time
([(2*x + 1)/(x + 1),
  (2*x + 1)/x,
  (2*x + 2)/(x + 2),
  (2*x + 2)/x,
  (x + 1)/(x + 2),
  (x + 1)/x,
  (x + 2)/(x + 1),
  (x + 2)/x,
  1/(x + 1),
  1/(x + 2),
  1/x,
  2*x,
  2*x + 1,
  2*x + 2,
  2*x/(x + 1),
  2*x/(x + 2),
  2/(x + 1),
  2/(x + 2),
  2/x,
  x,
  x + 1,
  x + 2,
  x/(x + 1),
  x/(x + 2)],
 'PGL(2,3)')

sage: R.<x,y> = ProjectiveSpace(GF(2^5,'t'),1)
sage: f = DynamicalSystem_projective([x^5,y^5])
sage: f.automorphism_group(return_functions=True, iso_type=True)
([x, 1/x], 'Cyclic of order 2')

sage: R.<x,y> = ProjectiveSpace(GF(3^4,'t'),1)
sage: f = DynamicalSystem_projective([x^2+25*x*y+y^2, x*y+3*y^2])
sage: f.automorphism_group(absolute=True)
[Univariate Polynomial Ring in w over Finite Field in b of size 3^4, 
 [1 0] 
 [0 1] ]

cyclegraph()

Return the digraph of all orbits of this dynamical system.

Over a finite field this is a finite graph. For subscheme domains, only points on the subscheme whose image are also on the subscheme are in the digraph.

OUTPUT: a digraph

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(GF(13),1)
sage: f = DynamicalSystem_projective([x^2-y^2, y^2])
sage: f.cyclegraph()
Looped digraph on 14 vertices
```python
sage: P.<x,y,z> = ProjectiveSpace(GF(3^2,'t'),2)
sage: f = DynamicalSystem_projective([x^2+y^2, y^2, z^2+y+z])
sage: f.cyclegraph()
Looped digraph on 91 vertices
```
```
sage: P.<x,y,z> = ProjectiveSpace(GF(7),2)
sage: X = P.subscheme(x^2-y^2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2], domain=X)
sage: f.cyclegraph()
Looped digraph on 15 vertices
```
```
sage: P.<x,y,z> = ProjectiveSpace(GF(3),2)
sage: f = DynamicalSystem_projective([x*z-y^2, x^2-y^2, y^2-z^2])
sage: f.cyclegraph()
Looped digraph on 13 vertices
```
```
sage: P.<x,y,z> = ProjectiveSpace(GF(3),2)
sage: X = P.subscheme([x-y])
sage: f = DynamicalSystem_projective([x^2-y^2, x^2-y^2, y^2-z^2], domain=X)
sage: f.cyclegraph()
Looped digraph on 4 vertices
```

**is_postcritically_finite(**kwds)**

Every point is postcritically finite in a finite field.

INPUT: None. kwds is to parallel the overridden function

OUTPUT: the boolean True

EXAMPLES:
```
sage: P.<x,y,z> = ProjectiveSpace(GF(5),2)
sage: f = DynamicalSystem_projective([x^2 + y^2, y^2, z^2 + y*z], domain=P)
sage: f.is_postcritically_finite()
True
```
```
sage: P.<x,y> = ProjectiveSpace(GF(13),1)
sage: f = DynamicalSystem_projective([x^4 - x^2*y^2 + y^4, y^4])
sage: f.is_postcritically_finite(use_algebraic_closure=False)
True
```

**orbit_structure(P)**

Return the pair \((m, n)\), where \(m\) is the preperiod and \(n\) is the period of the point \(P\) by this dynamical system.

Every point is preperiodic over a finite field so every point will be preperiodic.

INPUT:

- \(P\) – a point in the domain of this map

OUTPUT: a tuple \((m, n)\) of integers

EXAMPLES:
```
sage: P.<x,y,z> = ProjectiveSpace(GF(5),2)
sage: f = DynamicalSystem_projective([x^2 + y^2, y^2, z^2 + y*z], domain=P)
sage: f.orbit_structure(P(2,1,2))
(0, 6)
```
sage: P.<x,y,z> = ProjectiveSpace(GF(7),2)
sage: X = P.subscheme(x^2-y^2)
sage: f = DynamicalSystem_projective([x^2, y^2, z^2], domain=X)
sage: f.orbit_structure(X(1,1,2))
(0, 2)

sage: P.<x,y> = ProjectiveSpace(GF(13),1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2], domain=P)
sage: f.orbit_structure(P(3,4))
(2, 3)

sage: R.<t> = GF(13^3)
sage: P.<x,y> = ProjectiveSpace(R,1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2], domain=P)
sage: f.orbit_structure(P(t, 4))
(11, 6)

possible_periods(return_points=False)
Return the list of possible minimal periods of a periodic point over Q and (optionally) a point in each cycle.

ALGORITHM:
See [Hutz2009].

INPUT:
• return_points – (default: False) boolean; if True, then return the points as well as the possible periods

OUTPUT:
A list of positive integers, or a list of pairs of projective points and periods if return_points is True.

EXAMPLES:

sage: P.<x,y> = ProjectiveSpace(GF(23),1)
sage: f = DynamicalSystem_projective([x^2-2*y^2, y^2])
sage: f.possible_periods()
[1, 5, 11, 22, 110]

sage: P.<x,y> = ProjectiveSpace(GF(13),1)
sage: f = DynamicalSystem_projective([x^2 - y^2, y^2])
sage: sorted(f.possible_periods(True))
[[(0 : 1), 2], [(1 : 0), 1], [(3 : 1), 3], [(3 : 1), 36]]

Todo:
• do not return duplicate points
• improve hash to reduce memory of point-table
5.4 Dynamical systems for products of projective spaces

This class builds on the product projective space class. The main constructor functions are given by `DynamicalSystem` and `DynamicalSystem_projective`. The constructor function can take either polynomials or a morphism from which to construct a dynamical system.

The must be specified.

EXAMPLES:

```python
sage: P1xP1.<x,y,u,v> = ProductProjectiveSpaces(QQ, [1, 1])
sage: DynamicalSystem_projective([x^2*u, y^2*v, x*v^2, y*u^2], domain=P1xP1)
Dynamical System of Product of projective spaces P^1 x P^1 over Rational Field
Defn: Defined by sending (x : y , u : v) to
      (x^2*u : y^2*v , x*v^2 : y*u^2).
```

```python
class sage.dynamics.arithmetic_dynamics.product_projective_ds.DynamicalSystem_product_projective(polys, domain)

Bases: `sage.dynamics.arithmetic_dynamics.generic_ds.DynamicalSystem`, `sage.schemes.product_projective.morphism.ProductProjectiveSpaces_morphism_ring`

The class of dynamical systems on products of projective spaces.

**Warning:** You should not create objects of this class directly because no type or consistency checking is performed. The preferred method to construct such dynamical systems is to use `DynamicalSystem_projective()` function.

**INPUT:**

- `polys` - a list of \( n_1 + \cdots + n_r \) multi-homogeneous polynomials, all of which should have the same parent
- `domain` - a projective scheme embedded in \( P^{n_1-1} \times \cdots \times P^{n_r-1} \)

**EXAMPLES:**

```python
sage: T.<x,y,z,w,u> = ProductProjectiveSpaces([2, 1], QQ)
sage: DynamicalSystem_projective([x^2, y^2, z^2, w^2, u^2], domain=T)
Dynamical System of Product of projective spaces P^2 x P^1 over Rational Field
Defn: Defined by sending (x : y : z , w : u) to
      (x^2 : y^2 : z^2 , w^2 : u^2).
```

```python
nth_iterate(P, n, normalize=False)

Return the \( n \)-th iterate of \( P \) by this dynamical system.

If `normalize` is `True`, then the coordinates are automatically normalized.

**Todo:** Is there a more efficient way to do this?

**INPUT:**

- `P` - a point in `self.domain()`
- `n` - a positive integer
- `normalize` - (default: `False`) boolean
OUTPUT: A point in `self.codomain()`

EXAMPLES:

```python
sage: Z.<a,b,x,y,z> = ProductProjectiveSpaces([1, 2], QQ)
sage: f = DynamicalSystem_projective([a^3, b^3 + a*b^2, x^2, y^2 - z^2, z+y], domain=2)
sage: P = Z([1, 1, 1, 1, 1])
sage: f.nth_iterate(P, 3)
(1/1872 : 1 , 1 : 1 : 0)
```

```python
sage: Z.<a,b,x,y> = ProductProjectiveSpaces([1, 1], ZZ)
sage: f = DynamicalSystem_projective([a*b, b^2, x^3 - y^3, y^2*x], domain=2)
sage: P = Z([2, 6, 2, 4])
sage: f.nth_iterate(P, 2, normalize = True)
(1 : 3 , 407 : 112)
```

`nth_iterate_map(n)`

Return the nth iterate of this dynamical system.

ALGORITHM:

Uses a form of successive squaring to reduce computations.

Todo: This could be improved.

INPUT:

- `n` – a positive integer

OUTPUT: A dynamical system of products of projective spaces

EXAMPLES:

```python
sage: Z.<a,b,x,y,z> = ProductProjectiveSpaces([1, 2], QQ)
sage: f = DynamicalSystem_projective([a^3, b^3 + a*b^2, x^2, y^2 - z^2, z+y], domain=2)
sage: f.nth_iterate_map(3)
Dynamical System of Product of projective spaces P^1 x P^2 over Rational Field
Defn: Defined by sending (a : b , x : y : z) to
(a^27 : b^27 , x^8 : y^8 : z^8).
```

`orbit(P, N, **kwds)`

Return the orbit of `P` by this dynamical system.

Let \( F \) be this dynamical system. If `N` is an integer return \([P, F(P), \ldots, F^N(P)]\).

If `N` is a list or tuple `N = [m, k]` return \([F^m(P), \ldots, F^k(P)]\). Automatically normalize the points if `normalize` is True. Perform the checks on point initialize if `check` is True.

INPUT:

- `P` – a point in `self.domain()`
- `N` – a non-negative integer or list or tuple of two non-negative integers

kwds:

- `check` – (default: True) boolean
- `normalize` – (default: False) boolean
OUTPUT: a list of points in self.codomain()

EXAMPLES:

```
sage: Z.<a,b,x,y,z> = ProductProjectiveSpaces([1, 2], QQ)
sage: f = DynamicalSystem_projective([a^3, b^3 + a*b^2, x^2, y^2 - z^2, z+y], domain=Z)
sage: P = Z([1, 1, 1, 1, 1])
sage: f.orbit(P, 3)
[(1 : 1, 1 : 1 : 1), (1/2 : 1, 1 : 0 : 1), (1/12 : 1, -1 : 1 : 0), (1/1872 : 1)]
sage: Z.<a,b,x,y> = ProductProjectiveSpaces([1, 1], ZZ)
sage: f = DynamicalSystem_projective([a*b, b^2, x^3 - y^3, y^2*x], domain=Z)
sage: P = Z([2, 6, 2, 4])
sage: f.orbit(P, 3, normalize=True)
[(1 : 3, 1 : 2), (1 : 3, -7 : 4), (1 : 3, 407 : 112), (1 : 3, 66014215 : 5105408)]
```

class sage.dynamics.arithmetic_dynamics.product_projective_ds.DynamicalSystem_product_projective_field

Bases: sage.dynamics.arithmetic_dynamics.product_projective_ds.DynamicalSystem_product_projective

```class sage.dynamics.arithmetic_dynamics.product_projective_ds.DynamicalSystem_product_projective_field
```

cyclegraph()

Return the digraph of all orbits of this morphism mod \( p \).

OUTPUT: a digraph

EXAMPLES:

```
sage: P.<a,b,c,d> = ProductProjectiveSpaces(GF(3), [1,1])
sage: f = DynamicalSystem_projective([a^2, b^2, c^2, d^2], domain=P)
sage: f.cyclegraph()
Looped digraph on 16 vertices
```

```
sage: P.<a,b,c,d> = ProductProjectiveSpaces(GF(5), [1,1])
sage: f = DynamicalSystem_projective([a^2, b^2, c, d], domain=P)
sage: f.cyclegraph()
Looped digraph on 36 vertices
```

```
sage: P.<a,b,c,d,e> = ProductProjectiveSpaces(GF(2), [1,2])
sage: f = DynamicalSystem_projective([a^2, b^2, c, d, e], domain=P)
sage: f.cyclegraph()
Looped digraph on 21 vertices
```

Todo: Dynamical systems for subschemes of product projective spaces needs work. Thus this is not implemented for subschemes.
5.5 Wehler K3 Surfaces

AUTHORS:

- Ben Hutz (11-2012)
- Joao Alberto de Faria (10-2013)

Todo: Hasse-Weil Zeta Function
Picard Number
Number Fields

REFERENCES: [FH2015], [CS1996], [Weh1998], [Hutz2007]

sage.dynamics.arithmetic_dynamics.wehlerK3.WeherK3Surface (polys)

Defines a K3 Surface over \( \mathbb{P}^2 \times \mathbb{P}^2 \) defined as the intersection of a bilinear and biquadratic form. [Weh1998]

INPUT: Bilinear and biquadratic polynomials as a tuple or list

OUTPUT: \texttt{WehlerK3Surface\_ring}

EXAMPLES:

```
sage: PP.<x0,x1, x2, y0, y1, y2> = ProductProjectiveSpaces([2, 2],QQ)
sage: L = x0*y0 + x1*y1 - x2*y2
sage: Q = x0*x1*y1^2 + x2^2*y0*y2
sage: X = WehlerK3Surface([L, Q])
```

```
Closed subscheme of Product of projective spaces P^2 x P^2 over Rational Field defined by:
 x0*y0 + x1*y1 - x2*y2,
 x0*x1*y1^2 + x2^2*y0*y2
```

```
class sage.dynamics.arithmetic_dynamics.wehlerK3.WeherK3Surface\_field (polys)

Bases: \texttt{sage.dynamics.arithmetic_dynamics.wehlerK3.WeherK3Surface\_ring}

```
class sage.dynamics.arithmetic_dynamics.wehlerK3.WeherK3Surface\_finite\_field (polys)

Bases: \texttt{sage.dynamics.arithmetic_dynamics.wehlerK3.WeherK3Surface\_field}

```
cardinality ()

Counts the total number of points on the K3 surface.

ALGORITHM:

Enumerate points over \( \mathbb{P}^2 \), and then count the points on the fiber of each of those points.

OUTPUT: Integer - total number of points on the surface

EXAMPLES:

```
sage: PP.<x0,x1, x2, y0, y1, y2> = ProductProjectiveSpaces([2, 2], GF(7))
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \\
 3*x0*x1*y0*y1 - 2*x2^2*y0*y2 - 3*x0^2*y2^2 + 2*x1^2*y1^2 - x0*x2*y1^2 \\
- 4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 \\
+ x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: X.cardinality()
55
```
A K3 surface in \( \mathbb{P}^2 \times \mathbb{P}^2 \) defined as the intersection of a bilinear and biquadratic form. [Weh1998]

**EXAMPLES:**

```python
sage: R.<x,y,z,u,v,w> = PolynomialRing(QQ, 6)
sage: L = x*u - y*v
sage: Q = x*y*v^2 + z^2*u*w
sage: WehlerK3Surface([L, Q])
Closed subscheme of Product of projective spaces P^2 x P^2 over Rational Field defined by:
  x*u - y*v,
  x*y*v^2 + z^2*u*w
```

**Gpoly** \((\text{component}, k)\)

Return the G polynomials \(G^*_k\).

They are defined as:
\[
G^*_k = (L^*_i)^2 Q^*_k - L^*_i L^*_j Q^*_{ij} + (L^*_i)^2 Q^*_j
\]
where \(\{i, j, k\}\) is some permutation of \((0, 1, 2)\) and \(*\) is either \(x\) (Component = 1) or \(y\) (Component = 0).

**INPUT:**

- component - Integer: 0 or 1
- \(k\) - Integer: 0, 1 or 2

**OUTPUT:** polynomial in terms of either \(y\) (Component = 0) or \(x\) (Component = 1)

**EXAMPLES:**

```python
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(ZZ, 6)
sage: Y = x0*y0 + x1*y1 - x2*y2
sage: Z = x0^2*y0*y1 + x0^2*y2^2 - x0*x1*y1*y2 + x1^2*y2*y1 + x2^2*y2^2 + x2^2*y1^2 + x1^2*y2^2
sage: X = WehlerK3Surface([Z, Y])
sage: X.Gpoly(1, 0)
x0^2*x1^2 + x1^4 - x0*x1^2*x2 + x1^3*x2 + x1^2*x2^2 + x2^4
```

**Hpoly** \((\text{component}, i, j)\)

Return the H polynomials defined as \(H^*_{ij}\).

This polynomial is defined by:
\[
H^*_{ij} = 2L^*_i L^*_j Q^*_{jk} - L^*_i L^*_k Q^*_{ij} - L^*_j L^*_k Q^*_{ik} + (L^*_k)^2 Q^*_{ij}
\]
where \(\{i, j, k\}\) is some permutation of \((0, 1, 2)\) and \(*\) is either \(y\) (Component = 0) or \(x\) (Component = 1).

**INPUT:**

- component - Integer: 0 or 1
- \(i\) - Integer: 0, 1 or 2
- \(j\) - Integer: 0, 1 or 2

**OUTPUT:** polynomial in terms of either \(y\) (Component = 0) or \(x\) (Component = 1)

**EXAMPLES:**

```python
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(ZZ, 6)
sage: Y = x0*y0 + x1*y1 - x2*y2
sage: Z = x0^2*y0*y1 + x0^2*y2^2 - x0*x1*y1*y2 + x1^2*y2*y1 + x2^2*y2^2 + x2^2*y1^2 + x1^2*y2^2
sage: X = WehlerK3Surface([Z, Y])
sage: X.Hpoly(1, 0, 1)
x0^2*x1^2 + x1^4 - x0*x1^2*x2 + x1^3*x2 + x1^2*x2^2 + x2^4
```

(continues on next page)
\begin{align*}
+ x^2y^2 + x^2y^1 + x^1y^2
\end{align*}
\begin{verbatim}
sage: X = WehlerK3Surface([Z, Y])
sage: X.Hpoly(0, 1, 0)
2*y0*y1^3 + 2*y0*y1*y2^2 - y1*y2^3
\end{verbatim}

**Lxa** (a)

Function will return the L polynomial defining the fiber, given by \(L_x^a\).

This polynomial is defined as:

\[L_x^a = \{(a, y) \in \mathbb{P}^2 \times \mathbb{P}^2 : L(a, y) = 0\}\].

Notation and definition from: [CS1996]

**INPUT:** a - Point in \(\mathbb{P}^2\)

**OUTPUT:** A polynomial representing the fiber

**EXAMPLES:**

\begin{verbatim}
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1 - 2*x1^2*y0*y2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0*x2*y0*y2 + 4*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP(1, 1, 0, 1, 0, 0)
sage: X.Lxa(T[0])
y0 + y1
\end{verbatim}

**Lyb** (b)

Function will return a fiber by \(L_y^b\).

This polynomial is defined as:

\[L_y^b = \{(x, b) \in \mathbb{P}^2 \times \mathbb{P}^2 : L(x, b) = 0\}\].

Notation and definition from: [CS1996]

**INPUT:** b - Point in projective space

**OUTPUT:** A polynomial representing the fiber

**EXAMPLES:**

\begin{verbatim}
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1 - 2*x1^2*y0*y2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0*x2*y0*y2 + 4*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP(1, 1, 0, 1, 0, 0)
sage: X.Lyb(T[1])
x0
\end{verbatim}
**Qxa** *(a)*  
Function will return the Q polynomial defining a fiber given by $Q_a^x$.  
This polynomial is defined as:  
$$Q_a^x = \{(a, y) \in \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, y) = 0\}.$$  
Notation and definition from: [CS1996]  
**INPUT:** $a$ - Point in $\mathbb{P}^2$  
**OUTPUT:** A polynomial representing the fiber  
**EXAMPLES:**

```sage
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1+y0*y1...
    - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - 4*x1*x2*y1^2 
+ 5*x0*x2*y0*y2
    - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2 
    sage: Y = x0*y0 + x1*y1 + x2*y2
    sage: X = WehlerK3Surface([Z, Y])
    sage: T = PP(1, 1, 0, 1, 0, 0)
    sage: X.Qxa(T[0])
5*y0^2 + 7*y0*y1 + y1^2 + 11*y1*y2 + y2^2
```

**Qyb** *(b)*  
Function will return a fiber by $Q_b^y$.  
This polynomial is defined as:  
$$Q_b^y = \{(x, b) \in \mathbb{P}^2 \times \mathbb{P}^2 : Q(x, b) = 0\}.$$  
Notation and definition from: [CS1996]  
**INPUT:** $b$ - Point in projective space  
**OUTPUT:** A polynomial representing the fiber  
**EXAMPLES:**

```sage
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1+y0*y1...
    - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - 4*x1*x2*y1^2 
+ 5*x0*x2*y0*y2
    - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2 
    sage: Y = x0*y0 + x1*y1 + x2*y2
    sage: X = WehlerK3Surface([Z, Y])
    sage: T = PP(1, 1, 0, 1, 0, 0)
    sage: X.Qyb(T[1])
x0^2 + 3*x0*x1 + x1^2
```

**Ramification_poly** *(i)*  
Function will return the Ramification polynomial $g^*$.  
This polynomial is defined by:  
$$g^* = \frac{(H_i^*)^2-4G_i^*C_i^*}{(L_i^*)^2}.$$  
The roots of this polynomial will either be degenerate fibers or fixed points of the involutions $\sigma_x$ or $\sigma_y$ for more information, see [CS1996].
INPUT: \( i \) - Integer, either 0 (polynomial in \( y \)) or 1 (polynomial in \( x \))

OUTPUT: Polynomial in the coordinate ring of the ambient space

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1+y0*y1
     - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y1^2 + 5*x0*x2*y0*y2
     - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: X.Ramification_poly(0)
8*y0^5*y1 - 24*y0^4*y1^2 + 48*y0^2*y1^4 - 16*y0*y1^5 + y1^6 + 84*y0^3*y1^2*y2
    + 46*y0^2*y1^3*y2 - 20*y0*y1^4*y2 + 16*y1^5*y2 + 53*y0^4*y2^2 + 56*y0^3*y1*y2^2
    - 32*y0^2*y1*y2^2 - 80*y0*y1^3*y2^2 - 92*y1^4*y2^2 - 12*y0^2*y1*y2^3
    - 168*y0*y1^2*y2^3 - 122*y1^3*y2^3 + 14*y0^2*y2^4 + 8*y0*y1*y2^4 - 112*y1^2*y2^4
    + 2*y2^6
```

**Sxa** \((a)\)

Function will return fiber by \( S_a^x \).

This function is defined as:

\[
S_a^x = L_a^x \cap Q_a^x.
\]

Notation and definition from: [CS1996]

INPUT: \( a \) - Point in \( \mathbb{P}^2 \)

OUTPUT: A subscheme representing the fiber

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1
     + 3*x0*x1+y0*y1 \
     + 3*x0*x1+y0*y1 \
     - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y1^2 \
     + 5*x0*x2*y0*y2 \
     - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP(1, 1, 0, 1, 0, 0)
sage: X.Sxa(T[0])
Closed subscheme of Projective Space of dimension 2 over Rational Field
  defined by:
  y0 + y1,
  5*y0^2 + 7*y0*y1 + y1^2 + 11*y1*y2 + y2^2
```

**Syb** \((b)\)

Function will return fiber by \( S_b^y \).

This function is defined by:

\[
S_b^y = L_b^y \cap Q_b^y.
\]

Notation and definition from: [CS1996]

INPUT: \( b \) - Point in \( \mathbb{P}^2 \)
OUTPUT: A subscheme representing the fiber

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \
    3*x0*x1*y0*y1 - 2*x1^2*y0*y1 - x0^2*y1^2 + 2*x1*x2*y1^2 - x0*x2*y1^2 \ 
- 4*x1*x2*y1*y2 + 5*x0*x2*x0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 \ 
+ 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0 * y0 + x1 * y1 + x2 * y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP(1, 1, 0, 1, 0, 0)
sage: X.Syb(T[1])
Closed subscheme of Projective Space of dimension 2 over Rational Field
˓→ defined by:
    x0,
    x0^2 + 3*x0*x1 + x1^2
```

canonical_height \((P, N, \text{badprimes=None, prec=100})\)

Evaluates the canonical height for \(P\) with \(N\) terms of the series of the local heights.

ALGORITHM:
The sum of the canonical height minus and canonical height plus, for more info see section 4 of [CS1996].

INPUT:

- \(P\) – a surface point
- \(N\) – positive integer (number of terms of the series to use)
- \(\text{badprimes}\) – (optional) list of integer primes (where the surface is degenerate)
- \(\text{prec}\) – (default: 100) float point or p-adic precision

OUTPUT: A real number

EXAMPLES:

```python
sage: set_verbose(None)
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(QQ, 6)
sage: L = (-y0 - y1)*x0 + (-y0*x1 - y2*x2)
sage: Q = (-y2*y0 - y1^2)*x0^2 + ((-y0^2 - y2*y0 + (-y2*y1 - y2^2))*x1 + \
    (-y0^2 - y2*y1)*x2)*x0 + ((-y0^2 - y2*y0 - y2^2)*x1^2 + (-y2*y0 - y1^2)*x2*x1 + \
    (-y0^2 + (-y1 - y2)*y0)*x2^2)
sage: X = WehlerK3Surface([L, Q])
sage: P = X([1, 0, -1, 1, -1, 0]) # order 16
sage: X.canonical_height(P, 5) # long time
0.0000000000000000
```

Call-Silverman example:

```python
sage: set_verbose(None)
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1 + 
    3*x0*x1*y0*y1 - 2*x1^2*y0*y1 - x0^2*y1^2 + 2*x1*x2*y1^2 - x0*x2*y1^2 \ 
- 4*x1*x2*y1*y2 + 5*x0*x2*x0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 \ 
+ 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
```

(continues on next page)
sage: X = WehlerK3Surface([Z, Y])
sage: P = X(0, 1, 0, 0, 0, 1)
sage: X.canonical_height(P, 4)
0.69826458668659859569990618895

canonical_height_minus \((P, N, \text{badprimes}=None, \text{prec}=100)\)
Evaluates the canonical height minus function of Call-Silverman for \(P\) with \(N\) terms of the series of the local heights.
Must be over \(\mathbb{Z}\) or \(\mathbb{Q}\).

ALGORITHM:
Sum over the lambda minus heights (local heights) in a convergent series, for more detail see section 7 of [CS1996].

INPUT:
- \(P\) – a surface point
- \(N\) – positive integer (number of terms of the series to use)
- \(\text{badprimes}\) – (optional) list of integer primes (where the surface is degenerate)
- \(\text{prec}\) – (default: 100) float point or p-adic precision

OUTPUT: A real number

EXAMPLES:

sage: set_verbose(None)
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(QQ, 6)
sage: L = (-y0 - y1)*x0 + (-y0*x1 - y2*x2)
sage: Q = (-y2*y0 - y1^2)*x0^2 + ((-y0^2 - y2*y0 + (-y2*y1 - y2^2))*x1 + (-y0^2 - y2*y1)*x2)*x0 + ((-y0^2 - y2*y0 - y2^2)*x1^2 + (-y2*y0 - y1^2)*x2*x1 + (-y0^2 + (-y1 - y2)*y0)*x2^2)
sage: X = WehlerK3Surface([L, Q])
sage: P = X([1, 0, -1, 1, -1, 0])  #order 16
sage: X.canonical_height_minus(P, 5)  # long time
0.00000000000000000000000000000

Call-Silverman example:

sage: set_verbose(None)
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y1^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + x0*x1*y0*y1 + 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: P = X([0, 2, 0, 0, 0, 1])
sage: X.canonical_height_minus(P, 4)  # long time
0.5507370536967678815590206734

canonical_height_plus \((P, N, \text{badprimes}=None, \text{prec}=100)\)
Evaluates the canonical height plus function of Call-Silverman for \(P\) with \(N\) terms of the series of the local heights.
Must be over \( \mathbb{Z} \) or \( \mathbb{Q} \).

**ALGORITHM:**

Sum over the lambda plus heights (local heights) in a convergent series, for more detail see section 7 of [CS1996].

**INPUT:**

- \( P \) – a surface point
- \( N \) – positive integer. Number of terms of the series to use
- \( \text{badprimes} \) – (optional) list of integer primes (where the surface is degenerate)
- \( \text{prec} \) – (default: 100) float point or p-adic precision

**OUTPUT:** A real number

**EXAMPLES:**

```python
sage: set_verbose(None)
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(QQ, 6)
sage: L = (-y0 - y1)*x0 + (-y0*x1 - y2*x2)
sage: Q = (-y0^2 - y2*y0 - y1*y2)*x0 + (-y0^2 - y2*y0 - y1*y2)*x1 + (-y0^2 - y2*y0 - y1*y2)*x2*x1 + \n  (y0^2 - y2*y0 - y1*y2)*x0^2 + (y0^2 - y2*y0 - y1*y2)*x1^2 + \n  (y0^2 - y2*y0 - y1*y2)*x2^2

sage: X = WehlerK3Surface([L, Q])
sage: P = X([1, 0, -1, 1, -1, 0])  # order 16
sage: X.canonical_height_plus(P, 5)  # long time
0.00000000000000000000000000000
```

Call-Silverman Example:

```python
sage: set_verbose(None)
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \n  3*x0*x1*y0*y1 - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 \n  - 4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 \n  + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: P = X([0, 1, 0, 0, 0, 1])
```

```python
sage: X.canonical_height_plus(P, 4)  # long time
0.14752753298983071394400412161
```

**change_ring** \((R)\)

Changes the base ring on which the Wehler K3 Surface is defined.

**INPUT:** \( R \) - ring

**OUTPUT:** K3 Surface defined over input ring

**EXAMPLES:**

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], GF(3))
sage: L = x0*y0 + x1*y1 - x2*y2
sage: Q = x0*x1*y1^2 + x2^2*y0*y2
```

```python
sage: W = WehlerK3Surface([L, Q])
sage: W.base_ring()
```

Finite Field of size 3

(continues on next page)
degenerate_fibers()
Function will return the (rational) degenerate fibers of the surface defined over the base ring, or the fraction field of the base ring if it is not a field.

ALGORITHM:
The criteria for degeneracy by the common vanishing of the polynomials `self.Gpoly(1, 0), self.Gpoly(1, 1), self.Gpoly(1, 2), self.Hpoly(1, 0, 1), self.Hpoly(1, 0, 2), self.Hpoly(1, 1, 2)` (for the first component), is from Proposition 1.4 in the following article: [CS1996].

This function finds the common solution through elimination via Groebner bases by using the .variety() function on the three affine charts in each component.

OUTPUT: The output is a list of lists where the elements of lists are points in the appropriate projective space. The first list is the points whose pullback by the projection to the first component (projective space) is dimension greater than 0. The second list is points in the second component.

EXAMPLES:

```python
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(ZZ, 6)
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: Z = x0^2*y0*y1 + x0^2*y2^2 - x0*x1*y1*y2 + x1^2*y2*y1 + x2^2*y2^2 + x2^2*y1^2 + x1^2*y2^2
sage: X = WehlerK3Surface([Z, Y])
sage: X.degenerate_fibers()
[[], [(1 : 0 : 0)]]
```

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y1*y2 + x2^2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: X.degenerate_fibers()
[[], []]
```

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: R = PP.coordinate_ring()
sage: l = y0*x0 + y1*x1 + (y0 - y1)*x2
sage: q = (y1*y0 + y2^2)*x0^2 + ((y0^2 - y2*y1)*x1 + (y0^2 + (-y1^2 + y2^2))*x2)*x0 + (y2*y0 + y1^2)*x1^2 + (y0^2 + (-y1^2 + y2^2))*x2*x1
sage: X = WehlerK3Surface([l,q])
sage: X.degenerate_fibers()
[[(-1 : 1 : 1), (0 : 0 : 1)], [(-1 : -1 : 1), (0 : 0 : 1)]]
```

degenerate_primes (check=True)
Determine which primes $p$ self has degenerate fibers over $GF(p)$.

If check is False, then may return primes that do not have degenerate fibers. Raises an error if the surface is degenerate. Works only for ZZ or QQ.

INPUT: check – (default: True) boolean, whether the primes are verified
ALGORITHM:

\( p \) is a prime of bad reduction if and only if the defining polynomials of self plus the G and H polynomials have a common zero. Or stated another way, \( p \) is a prime of bad reduction if and only if the radical of the ideal defined by the defining polynomials of self plus the G and H polynomials is not \((x_0, x_1, \ldots, x_N)\). This happens if and only if some power of each \( x_i \) is not in the ideal defined by the defining polynomials of self (with G and H). This last condition is what is checked. The lcm of the coefficients of the monomials \( x_i \) in a groebner basis is computed. This may return extra primes.

OUTPUT: List of primes.

EXAMPLES:

```
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(QQ, 6)
sage: L = y0*x0 + (y1*x1 + y2*x2)
sage: Q = (2*y0^2 + y2*y0 + (2*y1^2 + y2^2))*x0^2 + ((y0^2 + y1*y0 + 
(y1^2 + 2*y2*y1))*x1 + (2*y1^2 + y2*y1 + y2^2)*x2)*x0 + ((2*y0^2\n+ (y1 + 2*y2))*y0 + (2*y1^2 + y2*y1))*x1^2 + ((2*y1 + 2*y2)*y0 + (y1^2 + 
y2*y1 + 2*y2^2))*x2*x1 + (2*y0^2 + y1*y0 + (2*y1^2 + y2^2))*x2^2)
sage: X = WehlerK3Surface([L, Q])
sage: X.degenerate_primes()
[2, 3, 5, 11, 23, 47, 48747691, 111301831]
```

```
defiber(p, component):
    Return the fibers [y (component = 1) or x (Component = 0)] of a point on a K3 Surface.

    This will work for nondegenerate fibers only.

    For algorithm, see [Hutz2007].

    INPUT:

    - p - a point in \( \mathbb{P}^2 \)

    OUTPUT: The corresponding fiber (as a list)

    EXAMPLES:

```
sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(ZZ, 6)
sage: Y = x0*y0 + x1*y1 - x2*y2
sage: Z = y0^2*x0*x1 + y0^2*x2^2 - y0*y1*x1*x2 + y1^2*x2*x1 + y2^2*x2^2 +
    y2^2*x1^2 + y1^2*x2^2
sage: X = WehlerK3Surface([Z, Y])
sage: Proj = ProjectiveSpace(QQ, 2)
sage: P = Proj([1, 0, 0])
sage: X.fiber(P, 1)
Traceback (most recent call last):
  ...)
TypeError: fiber is degenerate
```

```
sage: P.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1
        - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y1^2 + 
        5*x0*x2*y0*y2 - \n        4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + 4*x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: Proj = P[0]
sage: T = Proj([0, 0, 1])
```

(continues on next page)
sage: X.fiber(T, 1)
[(0 : 0 : 1, 0 : 1 : 0), (0 : 0 : 1, 2 : 0 : 0)]

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], GF(7))
sage: L = x0*y0 + x1*y1 - x2*y2
sage: Q=(2*x0^2 + x2*x0 + (2*x1^2 + x2^2))*y0^2 + ((x0^2 + x1*x0 + (x1^2 + 
     → x2*x1 + x2^2))*y1 + 
    (2*x1^2 + x2*x1 + x2^2)*y2)*y0 + ((2*x0^2+ (x1 + 2*x2)*x0 + (2*x1^2 + 
     → x2*x1))*y1^2 + ((2*x1 + x2)*x0 + 
    (x1^2 +x2*x1 + 2*x2^2))*y2*y1 + (2*x0^2 + x1*x0 + (2*x1^2 + x2^2))*y2^2)
sage: W = WehlerK3Surface([L, Q])
sage: W.fiber([4, 0, 1], 0)
[(0 : 1 : 0, 4 : 0 : 1), (4 : 0 : 2, 4 : 0 : 1)]

is_degenerate() Function will return True if there is a fiber (over the algebraic closure of the base ring) of dimension greater than 0 and False otherwise.

OUTPUT: boolean

EXAMPLES:

sage: R.<x0,x1,x2,y0,y1,y2> = PolynomialRing(ZZ, 6)
sage: Y = x0*y0 + x1*y1 - x2*y2
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1
     → - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y1^2 +
     → 5*x0*x2*y0*y2 + 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: X = WehlerK3Surface([Z, Y])
sage: X.is_degenerate()
True

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1
     → - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y1^2 +
     → 5*x0*x2*y0*y2 + 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: X.is_degenerate()
False

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], GF(3))
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1
     → - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - 4*x1*x2*y1^2 +
     → 5*x0*x2*y0*y2 + 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: X.is_degenerate()
True

is_isomorphic(right) Checks to see if two K3 surfaces have the same defining ideal.

INPUT:
• right - the K3 surface to compare to the original

OUTPUT: Boolean

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{PP.<}x_0,x_1,x_2,y_0,y_1,y_2\text{> = ProductProjectiveSpaces([2, 2], QQ)} \\
\text{sage: } & Z = x_0^2y_0^2 + 3x_0x_1y_0^2 + x_1^2y_0^2 + 4x_0^2y_0y_1 + \ \\
& 3x_0x_1y_0y_1 - 2x_2^2y_0y_1 - x_0^2y_1^2 + 2x_1^2y_1^2 - x_0x_2y_1^2 \ \\
& -4x_1^2x_2y_1^2 + 5x_0x_2y_0y_2 - 4x_1x_2y_0y_2 + 7x_0^2y_1y_2 + 4x_1^2y_1y_2 \ \\
& + x_0x_1y_2^2 + 3x_2^2y_2^2 \\
\text{sage: } & Y = x_0y_0 + x_1y_1 + x_2y_2 \\
\text{sage: } & X = \text{WehlerK3Surface([Z, Y])} \\
\text{sage: } & W = \text{WehlerK3Surface([Z + Y^2, Y])} \\
\text{sage: } & X.\text{is_isomorphic}(W) \\
& \text{True}
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & R.<x,y,z,u,v,w> = PolynomialRing(QQ, 6) \\
\text{sage: } & L = x*u-y*v \\
\text{sage: } & Q = x*y*v^2 + z^2*u*w \\
\text{sage: } & W1 = \text{WehlerK3Surface([L, Q])} \\
\text{sage: } & \text{PP.<}x_0,x_1,x_2,y_0,y_1,y_2\text{> = ProductProjectiveSpaces([2, 2], QQ)} \\
\text{sage: } & L = x_0y_0 + x_1y_1 + x_2y_2 \\
\text{sage: } & Q = x_1^2y_0^2 + 2x_2^2y_0y_1 + x_0^2y_1^2 -x_0x_1y_2^2 \ \\
& -4x_1x_2y_1^2 + 5x_0x_2y_0y_2 - 4x_1x_2y_0y_2 + 7x_0^2y_1y_2 + 4x_1^2y_1y_2 \ \\
& + x_0x_1y_2^2 + 3x_2^2y_2^2 \\
\text{sage: } & W2 = \text{WehlerK3Surface([L, Q])} \\
\text{sage: } & W1.\text{is_isomorphic}(W2) \\
& \text{False}
\end{align*}
\]

\text{is_smooth()} \\
Function will return the status of the smoothness of the surface.

ALGORITHM:
Checks to confirm that all of the 2x2 minors of the Jacobian generated from the biquadratic and bilinear forms have no common vanishing points.

OUTPUT: Boolean

EXAMPLES:

\[
\begin{align*}
\text{sage: } & R.<x_0,x_1,x_2,y_0,y_1,y_2> = PolynomialRing(ZZ, 6) \\
\text{sage: } & Y = x_0y_0 + x_1y_1 - x_2y_2 \\
\text{sage: } & Z = x_0^2y_0y_1 + x_0^2y_2^2 - x_0x_1y_1y_2 + x_1^2y_2y_1 + \ \\
& x_2^2y_2^2 + x_2^2y_1^2 + x_1^2y_2^2 \\
\text{sage: } & X = \text{WehlerK3Surface([Z, Y])} \\
\text{sage: } & X.\text{is_smooth()} \\
& \text{False}
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{PP.<}x_0,x_1,x_2,y_0,y_1,y_2\text{> = ProductProjectiveSpaces([2, 2], QQ)} \\
\text{sage: } & Z = x_0^2y_0^2 + 3x_0x_1y_0^2 + x_1^2y_0^2 + 4x_0^2y_0y_1 + \ \\
& 3x_0x_1y_0y_1 - 2x_2^2y_0y_1 - x_0^2y_1^2 + 2x_1^2y_1^2 - x_0x_2y_1^2 \ \\
& -4x_1^2x_2y_1^2 + 5x_0x_2y_0y_2 - 4x_1x_2y_0y_2 + 7x_0^2y_1y_2 + 4x_1^2y_1y_2 \ \\
& + x_0x_1y_2^2 + 3x_2^2y_2^2 \\
\text{sage: } & Y = x_0y_0 + x_1y_1 + x_2y_2 \\
\text{sage: } & X = \text{WehlerK3Surface([Z, Y])} \\
\text{sage: } & X.\text{is_smooth()} \\
& \text{True}
\end{align*}
\]

\text{is_symmetric_orbit}(\text{orbit}) \\
Checks to see if the orbit is symmetric (i.e. if one of the points on the orbit is fixed by \textquoteleft \text{sigma}_x\textquoteright\ or
'sigma_y').

INPUT:

- orbit - a periodic cycle of either psi or phi

OUTPUT: Boolean

EXAMPLES:

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2,2], GF(7))
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + 3*x0*x1*y0*y1 -
     -2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 -4*x1*x2*y1^2 +
     -5*x0*x2*y0*y2 -4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP([0, 0, 1, 1, 0, 0])
sage: orbit = X.orbit_psi(T, 4)
sage: X.is_symmetric_orbit(orbit)
True

lambda_minus(P, v, N, m, n, prec=100)

Evaluates the local canonical height minus function of Call-Silverman at the place v for P with N terms of the series.

Use v = 0 for the Archimedean place. Must be over Z or Q.

ALGORITHM:

Sum over local heights using convergent series, for more details, see section 4 of [CS1996].

INPUT:

- P – a projective point
- N – positive integer. number of terms of the series to use
- v – non-negative integer. a place, use v = 0 for the Archimedean place
- m, n – positive integers, We compute the local height for the divisor \( E^+_{mn} \). These must be indices of non-zero coordinates of the point P.
- prec – (default: 100) float point or p-adic precision

OUTPUT: A real number

EXAMPLES:

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2,2], QQ)
sage: L = x0*y0 + x1*y1 + x2*y2
sage: Q = x1^2*y0^2 + 2*x2^2*y0*y1 + x0^2*y1^2 - x0*x1*y2^2
sage: W = WehlerK3Surface([L, Q])
sage: T = W([-1, -1, 1, 1, 0, 1])
sage: Orb = W.orbit_phi(T, 7)
sage: W.is_symmetric_orbit(Orb)
False

(continues on next page)
\[ -4x_1x_2y_0y_2 + 7x_0^2y_1y_2 + 4x_1^2y_1y_2 + x_0x_1y_2^2 + 3x_2^2y_2^2 \]

```python
sage: Y = x_0y_0 + x_1y_1 + x_2y_2
sage: X = WehlerK3Surface([Z, Y])
```

```python
sage: P = X([0, 0, 1, 1, 0, 0])
```

```python
sage: X.lambda_minus(P, 2, 20, 2, 0, 200)
```

```
-0.1857335167204713503717280577967191488351056677474271893705
```

**lambda_plus** \((P, v, N, m, n, \text{prec}=100)\)

Evaluates the local canonical height plus function of Call-Silverman at the place \(v\) for \(P\) with \(N\) terms of the series.

Use \(v = 0\) for the archimedean place. Must be over \(\mathbb{Z}\) or \(\mathbb{Q}\).

**ALGORITHM:**

Sum over local heights using convergent series, for more details, see section 4 of [CS1996].

**INPUT:**

- \(P\) – a surface point
- \(N\) – positive integer. number of terms of the series to use
- \(v\) – non-negative integer. a place, use \(v = 0\) for the Archimedean place
- \(m, n\) – positive integers, We compute the local height for the divisor \(E_m^+\). These must be indices of non-zero coordinates of the point \(P\).
- \(\text{prec}\) – (default: 100) float point or p-adic precision

**OUTPUT:** A real number

**EXAMPLES:**

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x_0^2y_0^2 + 3x_0x_1y_0^2 + x_1^2y_0^2 + 4x_0^2y_0y_1 + 3x_0x_1y_0y_1 - 2x_2^2y_0y_1 - x_0^2y_1^2 + 2x_1^2y_1^2 - x_0x_2y_1^2 - 4x_1x_2y_1^2 + 5x_0x_2y_0y_2 - 4x_1x_2y_0y_2 + 7x_0^2y_1y_2 + 4x_1^2y_1y_2 + x_0x_1y_2^2 + 3x_2^2y_2^2
sage: Y = x_0y_0 + x_1y_1 + x_2y_2
sage: X = WehlerK3Surface([Z, Y])
sage: P = X([0, 0, 1, 1, 0, 0])
sage: X.lambda_plus(P, 0, 10, 2, 0)
```

```
0.89230705169161608922595928129
```

**nth_iterate_phi** \((P, n, **kwds)\)

Computes the nth iterate for the phi function.

**INPUT:**

- \(P\) – a point in \(\mathbb{P}^2 \times \mathbb{P}^2\)
- \(n\) – an integer

**kwds:**

- \(\text{check}\) – (default: True) boolean checks to see if point is on the surface
- \(\text{normalize}\) – (default: False) boolean normalizes the point

**OUTPUT:** The nth iterate of the point given the phi function (if \(n\) is positive), or the psi function (if \(n\) is negative)

**EXAMPLES:**

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x_0^2y_0^2 + 3x_0x_1y_0^2 + x_1^2y_0^2 + 4x_0^2y_0y_1 + 3x_0x_1y_0y_1 - 2x_2^2y_0y_1 - x_0^2y_1^2 + 2x_1^2y_1^2 - x_0x_2y_1^2 - 4x_1x_2y_1^2 + 5x_0x_2y_0y_2 - 4x_1x_2y_0y_2 + 7x_0^2y_1y_2 + 4x_1^2y_1y_2 + x_0x_1y_2^2 + 3x_2^2y_2^2
sage: Y = x_0y_0 + x_1y_1 + x_2y_2
sage: X = WehlerK3Surface([Z, Y])
sage: P = X([0, 0, 1, 1, 0, 0])
sage: X.lambda_plus(P, 0, 10, 2, 0)
```

```
0.89230705169161608922595928129
```
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: L = x0*y0 + x1*y1 + x2*y2
sage: Q = x1^2*y0^2 + 2*x2^2*y0*y1 + x0^2*y1^2 - x0*x1*y2^2
sage: W = WehlerK3Surface([L, Q])
sage: T = W([-1, -1, 1, 1, 0, 1])
sage: W.nth_iterate_phi(T, 7)
(-1 : 0 : 1 , 1 : -2 : 1)

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: L = x0*y0 + x1*y1 + x2*y2
sage: Q = x1^2*y0^2 + 2*x2^2*y0*y1 + x0^2*y1^2 - x0*x1*y2^2
sage: W = WehlerK3Surface([L, Q])
sage: T = W([-1, -1, 1, 1, 0, 1])
sage: W.nth_iterate_phi(T, -7)
(1 : 0 : 1 , -1 : 2 : 1)

sage: R.<x0,x1,x2,y0,y1,y2>=PolynomialRing(QQ, 6)
sage: L = (-y0 - y1)*x0 + (-y0*x1 - y2*x2)
sage: Q = (-y2*y0 - y1^2)*x0^2 + ((-y0^2 - y2*y0 + (-y2*y1 - y2^2))*x1 + (-y0^2 - y2*y1)*x2)*x0 + ((-y0^2 - y2*y0 + (-y2*y1)*x2)*x1 + (-y0^2 + (-y1 - y2)*y0)*x2^2)

sage: X = WehlerK3Surface([L, Q])
sage: P = X([1, 0, -1, 1, -1, 0])
sage: X.nth_iterate_psi(P, 8) == X.nth_iterate_psi(P, 8)
True

nth_iterate_psi \( (P, n, **kwds) \)
Computes the nth iterate for the psi function.

INPUT:

- \( P \) – a point in \( \mathbb{P}^2 \times \mathbb{P}^2 \)
- \( n \) – an integer

kwds:

- \( \text{check} \) – (default: True) boolean, checks to see if point is on the surface
- \( \text{normalize} \) – (default: False) boolean, normalizes the point

OUTPUT: The nth iterate of the point given the psi function (if \( n \) is positive), or the phi function (if \( n \) is negative)

EXAMPLES:

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: L = x0*y0 + x1*y1 + x2*y2
sage: Q = x1^2*y0^2 + 2*x2^2*y0*y1 + x0^2*y1^2 - x0*x1*y2^2
sage: W = WehlerK3Surface([L, Q])
sage: T = W([-1, -1, 1, 1, 0, 1])
sage: W.nth_iterate_psi(T, -7)
(-1 : 0 : 1 , 1 : -2 : 1)
**orbit PHI** \((P, N, **kwds)\)

Return the orbit of the \(\phi\) function defined by \(\phi = \sigma_y \circ \sigma_x\).

This function is defined in [CS1996].

**INPUT:**

- **P** - Point on the K3 surface
- **N** - a non-negative integer or list or tuple of two non-negative integers

**kwds:**

- **check** – (default: True) boolean, checks to see if point is on the surface
- **normalize** – (default: False) boolean, normalizes the point

**OUTPUT:** List of points in the orbit

**EXAMPLES:**

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \
7*x0*x1*y0*y1 - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 - \
4*x1*x2*y1^2 + 5*x0*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + \x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP(0, 0, 1, 1, 0, 0)
sage: X.orbit PHI(T,2, normalize = True)
\[(0 : 0 : 1 , 1 : 0 : 0), (-1 : 0 : 1 , 0 : 1 : 0), (-12816/6659 : 55413/6659 
˓→ 1 , 1 : 1/9 : 1)
```

**orbit psi** \((P, N, **kwds)\)

Return the orbit of the \(\psi\) function defined by \(\psi = \sigma_x \circ \sigma_y\).

This function is defined in [CS1996].

**INPUT:**

- **P** - a point on the K3 surface
- **N** - a non-negative integer or list or tuple of two non-negative integers

**kwds:**

- **check** - (default: True) boolean, checks to see if point is on the surface
- **normalize** – (default: False) boolean, normalizes the point
OUTPUT: a list of points in the orbit

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + 4*x0^2*y0 + y1 + 
    3*x0*x1*y0*y1 - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - 2*x0*x2*y1^2 - 
    4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + 
    x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP([0, 0, 1, 1, 0, 0])
sage: X.orbit_psi(T, 2, normalize = True)
[(0 : 0 : 1 , 1 : 0 : 0), (0 : 0 : 1 , 0 : 1 : 0), (-1 : 0 : 1 , 1 : 1/9 : 1)]
sage: X.orbit_psi(T,[2,3], normalize = True)
[(-1 : 0 : 1 , 1 : 1/9 : 1), 
  (-12816/6659 : 55413/6659 : 1 , -117756062505511/54767410965117 : -
    23134047983794359/37466994368025041 : 1)]
```

\(\phi(a, **kwds)\)

Evaluates the function \(\phi = \sigma_y \circ \sigma_x\).

ALGORITHM:

Refer to Section 6: “An algorithm to compute \(\sigma_x, \sigma_y, \phi, \text{and} \ \psi\)” in [CS1996].

For the degenerate case refer to [FH2015].

INPUT:

- \(a\) - Point in \(\mathbb{P}^2 \times \mathbb{P}^2\)

kwds:

- \(\text{check}\) - (default: True) boolean checks to see if point is on the surface
- \(\text{normalize}\) - (default: True) boolean normalizes the point

OUTPUT: A point on this surface

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + 4*x0^2*y0 + y1 + 
    3*x0*x1*y0*y1 - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - 2*x0*x2*y1^2 - 
    4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 + 
    x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP([0, 0, 1, 1, 0, 0])
sage: X.phi(T)
(-1 : 0 : 1 , 0 : 1 : 0)
```

\(\psi(a, **kwds)\)

Evaluates the function \(\psi = \sigma_x \circ \sigma_y\).

ALGORITHM:

Refer to Section 6: “An algorithm to compute \(\sigma_x, \sigma_y, \phi, \text{and} \ \psi\)” in [CS1996].

For the degenerate case refer to [FH2015].

INPUT:
• a - Point in $\mathbb{P}^2 \times \mathbb{P}^2$

kwds:

• check - (default: True) boolean checks to see if point is on the surface
• normalize - (default: True) boolean normalizes the point

OUTPUT: A point on this surface

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: c = x0^2+y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \ 
  3*x0*x1*y0*y1 - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 \ 
  - 4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 \ 
  + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([c, Y])
sage: T = PP(0, 0, 1, 1, 0, 0)
sage: X.sigmaX(T)
tuple((0 : 0 : 1 , 0 : 1 : 0))
```

`sigmaX(P, **kwds)`

Function returns the involution on the Wehler K3 surface induced by the double covers.

In particular, it fixes the projection to the first coordinate and swaps the two points in the fiber, i.e. $(x, y) \rightarrow (x, y')$. Note that in the degenerate case, while we can split fiber into pairs of points, it is not always possible to distinguish them, using this algorithm.

ALGORITHM:

Refer to Section 6: “An algorithm to compute $\sigma_x, \sigma_y, \phi$, and $\psi$” in [CS1996FH2015. For the degenerate case refer to [FH2015].

INPUT:

• P - a point in $\mathbb{P}^2 \times \mathbb{P}^2$

kwds:

• check - (default: True) boolean checks to see if point is on the surface
• normalize - (default: True) boolean normalizes the point

OUTPUT: A point on the K3 surface

EXAMPLES:

```python
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: c = x0^2+y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \ 
  3*x0*x1*y0*y1 - 2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 \ 
  - 4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 \ 
  + x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([c, Y])
sage: T = PP(0, 0, 1, 1, 0, 0)
sage: X.sigmaX(T)
tuple((0 : 0 : 1 , 0 : 1 : 0))
```

degenerate examples:
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: l = y0*x0 + y1*x1 + (y0 - y1)*x2
sage: q = (y1*y0)*x0^2 + ((y0^2)*x1 + (y0^2 + (y1^2 - y2^2))*x2)*x0 +
     (y2*y0 + y1^2)*x1^2 + (y0^2 + (-y1^2 + y2^2))*x2*x1
sage: X = WehlerK3Surface([l, q])
sage: X.sigmaX(X([1, 0, 0, 0, 1, -2]))
(1 : 0 : 0 , 0 : 1/2 : 1)
sage: X.sigmaX(X([1, 0, 0, 0, 0, 1]))
(1 : 0 : 0 , 0 : 0 : 1)
sage: X.sigmaX(X([-1, 1, 1, -1, -1, 1]))
(-1 : 1 : 1 , 2 : 2 : 1)
sage: X.sigmaX(X([0, 0, 1, 1, 1, 0]))
(0 : 0 : 1 , 1 : 1 : 0)
sage: X.sigmaX(X([0, 0, 1, 1, 1, 1]))
(0 : 0 : 1 , -1 : -1 : 1)

Case where we cannot distinguish the two points:

sage: PP.<y0,y1,y2,x0,x1,x2>=ProductProjectiveSpaces([2, 2], GF(3))
sage: l = x0*y0 + x1*y1 + x2*y2
sage: q=-3*x0^2*y0^2 + 4*x0*x1*y0^2 - 5*x0^2*y0*y1 - 190*x0*x1*y0*y1- 5*x0*x2*y0*y1 + 14*x1*x2*y0*y1 +
     5*x2^2*y0*y1 - x0^2*y1^2 - 6*x0*x1*y1^2- 2*x1^2*y1^2 + 2*x0*x2*y1^2 - 4*x2^2*y1^2 + 4*x0^2*y0*y2 - x1^2*y0*y2 + 3*x0*x2*y0*y2+ 6*x1*x2*y0*y2 -
     6*x0^2*y1*y2 - 4*x0*x1*y1*y2 - x1^2*y1*y2 + 51*x0*x2*y1*y2 - 7*x1*x2*y1*y2 - 9*x2^2*y1*y2 - x0^2*y2^2 - 4*x0*x1*y2^2 + 4*x1^2*y2^2 - x0*x2*y2^2 +
     13*x1*x2*y2^2 - x2^2*y2^2
sage: X = WehlerK3Surface([l, q])
sage: P = X([1, 0, 0, 0, 1, 1])
sage: X.sigmaX(X.sigmaX(P))
Traceback (most recent call last):
...  
ValueError: cannot distinguish points in the degenerate fiber

\textbf{sigmaY}(P, **kwds)

Function returns the involution on the Wehler K3 surfaces induced by the double covers.

In particular, it fixes the projection to the second coordinate and swaps the two points in the fiber, i.e. \((x, y) \rightarrow (x', y)\). Note that in the degenerate case, while we can split the fiber into two points, it is not always possible to distinguish them, using this algorithm.

\textbf{ALGORITHM:}

Refer to Section 6: “An algorithm to compute \(\sigma_x, \sigma_y, \phi, \text{ and } \psi\)” in [CS1996]. For the degenerate case refer to [FH2015].

\textbf{INPUT:}

- \(P\) - a point in \(\mathbb{P}^2 \times \mathbb{P}^2\)

\textbf{kwds:}

- \texttt{check} - (default: True) boolean checks to see if point is on the surface
- \texttt{normalize} - (default: True) boolean normalizes the point

\textbf{OUTPUT:} A point on the K3 surface

\textbf{EXAMPLES:}
sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: Z = x0^2*y0^2 + 3*x0*x1*y0^2 + x1^2*y0^2 + 4*x0^2*y0*y1 + \
3*x0*x1*y0*y1 -2*x2^2*y0*y1 - x0^2*y1^2 + 2*x1^2*y1^2 - x0*x2*y1^2 \
- 4*x1*x2*y1^2 + 5*x0*x2*y0*y2 - 4*x1*x2*y0*y2 + 7*x0^2*y1*y2 + 4*x1^2*y1*y2 \
+ x0*x1*y2^2 + 3*x2^2*y2^2
sage: Y = x0*y0 + x1*y1 + x2*y2
sage: X = WehlerK3Surface([Z, Y])
sage: T = PP(0, 0, 1, 1, 0, 0)
sage: X.sigmaY(T)
(0 : 0 : 1 , 1 : 0 : 0)

degenerate examples:

sage: PP.<x0,x1,x2,y0,y1,y2> = ProductProjectiveSpaces([2, 2], QQ)
sage: l = y0*x0 + y1*x1 + (y0 - y1)*x2
sage: q = (y1*y0)*x0^2 + ((y0^2)*x1 + (y0^2 + (y1^2 - y2^2))*x2)*x0 + \
(y2*y0 + y1^2)*x1^2 + (y0^2 + (-y1^2 + y2^2))*x2*x1
sage: X = WehlerK3Surface([l, q])
sage: X.sigmaY(X([1, -1, 0 ,-1, -1, 1]))
(1/10 : -1/10 : 1 , -1 : -1 : 1)
sage: X.sigmaY(X([0, 0, 1, -1, -1, 1]))
(-4 : 4 : 1 , -1 : -1 : 1)
sage: X.sigmaY(X([1, 2, 0, 0, 0, 1]))
(-3 : -3 : 1 , 0 : 0 : 1)
sage: X.sigmaY(X([1, 1, 1, 0, 0, 1]))
(1 : 0 : 0 , 0 : 0 : 1)

Case where we cannot distinguish the two points:

sage: PP.<x0,x1,x2,y0,y1,y2>=ProductProjectiveSpaces([2, 2], GF(3))
sage: l = x0*y0 + x1*y1 + x2*y2
sage: q=-3*x0^2*y0^2 + 4*x0*x1*y0^2 - 3*x0*x2*y0^2 - 5*x0^2*y0*y1 - \
-190*x0*x1*y0*y1 \
- 5*x1^2*y0*y1 + 5*x0*x2*y0*y1 + 14*x1*x2*y0*y1 + 5*x2^2*y0*y1 - x0^2*y1^2 - \
-6*x0*x1*y1^2 \
- 2*x1^2*y1^2 + 2*x0*x2*y1^2 - 4*x2^2*y1^2 + 4*x0^2*y0*y2 - x1^2*y0*y2 + \
-3*x0*x2*y0*y2 + 6*x1*x2*y0*y2 - 6*x0^2*y1*y2 - 4*x0*x1*y1*y2 - x1^2*y1*y2 + 51*x0*x2*y1*y2 - \
- 7*x1*x2*y1*y2 \
- 9*x2^2*y1*y2 - x0^2*y2^2 - 4*x0*x1*y2^2 + 4*x1^2*y2^2 - x0*x2*y2^2 + \
-13*x1*x2*y2^2 - x2^2*y2^2
sage: X = WehlerK3Surface([l, q])
sage: P = X([0, 1, 1, 0, 0])
sage: X.sigmaY(X.sigmaY(P))
Traceback (most recent call last):
...
ValueError: cannot distinguish points in the degenerate fiber

sage.dynamics.arithmetic_dynamics.wehlerK3.random_WehlerK3Surface(PP)

Produces a random K3 surface in $\mathbb{P}^2 \times \mathbb{P}^2$ defined as the intersection of a bilinear and biquadratic form.
[Weh1998]

INPUT: Projective space cartesian product

OUTPUT: WehlerK3Surface_ring

EXAMPLES:
sage: PP.<x0, x1, x2, y0, y1, y2> = ProductProjectiveSpaces([2, 2], GF(3))
sage: random_WehlerK3Surface(PP)
Closed subscheme of Product of projective spaces P^2 x P^2 over Finite Field of size 3 defined by:
x0*y0 + x1*y1 + x2*y2,
-x1^2*y0^2 - x2^2*y0^2 + x0^2*y0*y1 - x0*x1*y0*y1 - x1^2*y0*y1
+ x1*x2*y0*y1 + x0^2*y1^2 + x0*x1*y1^2 - x1^2*y1^2 + x0*x2*y1^2
- x0^2*y0*y2 - x0*x1*y0*y2 + x0*x2*y0*y2 + x1*x2*y0*y2 + x0*x1*y1*y2
- x1^2*y1*y2 - x1*x2*y1*y2 - x0^2*y2^2 + x0*x1*y2^2 - x1^2*y2^2 - x0*x2*y2^2

See also:

• sage.schemes.affine.affine_morphism
• sage.schemes.projective.projective_morphism
• sage.schemes.product_projective.morphism
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