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Index 143
1.1 Ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo $n$

EXAMPLES:

```
sage: R = Integers(97)
sage: a = R(5)
sage: a**1000000000000000000000000000000000000000000000000000000000000000061
```

This example illustrates the relation between $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_p$. In particular, there is a canonical map to $\mathbb{F}_p$, but not in the other direction.

```
sage: r = Integers(7)
sage: s = GF(7)
sage: r.has_coerce_map_from(s)
False
sage: s.has_coerce_map_from(r)
True
sage: s(1) + r(1)
2
sage: parent(s(1) + r(1))
Finite Field of size 7
sage: parent(r(1) + s(1))
Finite Field of size 7
```

We list the elements of $\mathbb{Z}/3\mathbb{Z}$:

```
sage: R = Integers(3)
sage: list(R)
[0, 1, 2]
```

AUTHORS:

- William Stein (initial code)
- David Joyner (2005-12-22): most examples
- Robert Bradshaw (2006-08-24): convert to SageX (Cython)
- Simon King (2011-04-21): allow to prescribe a category
- Simon King (2013-09): Only allow to prescribe the category of fields
class sage.rings.finite_rings.integer_mod_ring.IntegerModFactory
    Bases: sage.structure.factory.UniqueFactory

Return the quotient ring \( \mathbb{Z}/n\mathbb{Z} \).

INPUT:

- order – integer (default: 0); positive or negative
- is_field – bool (default: False); assert that the order is prime and hence the quotient ring belongs to the category of fields
- category (optional) - the category that the quotient ring belongs to.

**Note:** The optional argument `is_field` is not part of the cache key. Hence, this factory will create precisely one instance of \( \mathbb{Z}/n\mathbb{Z} \). However, if `is_field` is true, then a previously created instance of the quotient ring will be updated to be in the category of fields.

**Use with care!** Erro nephely putting \( \mathbb{Z}/n\mathbb{Z} \) into the category of fields may have consequences that can compromise a whole Sage session, so that a restart will be needed.

**EXAMPLES:**

```
sage: IntegerModRing(15)
Ring of integers modulo 15
sage: IntegerModRing(7)
Ring of integers modulo 7
sage: IntegerModRing(-100)
Ring of integers modulo 100
```

Note that you can also use `Integers`, which is a synonym for `IntegerModRing`.

```
sage: Integers(18)
Ring of integers modulo 18
sage: Integers() is Integers(0) is ZZ
True
```

**Note:** Testing whether a quotient ring \( \mathbb{Z}/n\mathbb{Z} \) is a field can of course be very costly. By default, it is not tested whether \( n \) is prime or not, in contrast to `GF()`. If the user is sure that the modulus is prime and wants to avoid a primality test, (s)he can provide `category=Fields()` when constructing the quotient ring, and then the result will behave like a field. If the category is not provided during initialisation, and it is found out later that the ring is in fact a field, then the category will be changed at runtime, having the same effect as providing `Fields()` during initialisation.

**EXAMPLES:**

```
sage: R = IntegerModRing(5)
sage: R.category()
Join of Category of finite commutative rings
    and Category of subquotients of monoids
    and Category of quotients of semigroups
    and Category of finite enumerated sets
sage: R in Fields()
True
```

(continues on next page)
sage: R.category()
Join of Category of finite enumerated fields
   and Category of subquotients of monoids
   and Category of quotients of semigroups
sage: S = IntegerModRing(5, is_field=True)
sage: S is R
True

**Warning:** if the optional argument `is_field` was used by mistake, there is currently no way to revert its impact, even though `IntegerModRing_generic.is_field()` with the optional argument `proof=True` would return the correct answer. So, prescribe `is_field=True` only if you know what you are doing!

**EXAMPLES:**

sage: R = IntegerModRing(33, is_field=True)
sage: R in Fields()
True
sage: R.is_field()
True

If the optional argument `proof = True` is provided, primality is tested and the mistaken category assignment is reported:

sage: R.is_field(proof=True)
Traceback (most recent call last):
... ValueError: THIS SAGE SESSION MIGHT BE SERIOUSLY COMPROMISED!
The order 33 is not prime, but this ring has been put into the category of fields. This may already have consequences in other parts of Sage. Either it was a mistake of the user, or a probabilistic primality test has failed. In the latter case, please inform the developers.

However, the mistaken assignment is not automatically corrected:

sage: R in Fields()
True

To avoid side-effects of this test on other tests, we clear the cache of the ring factory:

sage: IntegerModRing._cache.clear()

`create_key_and_extra_args(order=0, is_field=False, category=None)`

An integer mod ring is specified uniquely by its order.

**EXAMPLES:**

sage: Zmod.create_key_and_extra_args(7)
(7, {})
sage: Zmod.create_key_and_extra_args(7, True)
(7, {'category': Category of fields})
create_object(version, order, **kwds)

EXAMPLES:

```python
sage: R = Integers(10)
sage: TestSuite(R).run()  # indirect doctest
```

get_object(version, key, extra_args)

class sage.rings.finite_rings.integer_mod_ring.IntegerModRing_generic(order, cache=None, category=None)

Bases: sage.rings.quotient_ring.QuotientRing_generic, sage.rings.abc.IntegerModRing

The ring of integers modulo $N$.

INPUT:

- order – an integer

- category – a subcategory of CommutativeRings() (the default)

OUTPUT:

The ring of integers modulo $N$.

EXAMPLES:

First we compute with integers modulo 29.

```python
sage: FF = IntegerModRing(29)
sage: FF
Ring of integers modulo 29
sage: FF.category()
Join of Category of finite commutative rings
   and Category of subquotients of monoids
   and Category of quotients of semigroups
   and Category of finite enumerated sets
sage: FF.is_field()
True
sage: FF.characteristic()
29
sage: FF.order()
29
sage: gens = FF.unit_gens()
sage: a = gens[0]
sage: a
2
sage: a.is_square()
False
sage: def pow(i): return a**i
sage: [pow(i) for i in range(16)]
[1, 2, 4, 8, 16, 3, 6, 12, 24, 19, 9, 18, 7, 14, 28, 27]
sage: TestSuite(FF).run()
```

We have seen above that an integer mod ring is, by default, not initialised as an object in the category of fields. However, one can force it to be. Moreover, testing containment in the category of fields may re-initialise the category of the integer mod ring:
By trac ticket #15229, there is a unique instance of the integral quotient ring of a given order. Using the IntegerModRing() factory twice, and using is_field=True the second time, will update the category of the unique instance:

```
sage: F31a = IntegerModRing(31)
sage: F31a.category().is_subcategory(Fields())
False
sage: F31b = IntegerModRing(31, is_field=True)
sage: F31a is F31b
True
sage: F31a.category().is_subcategory(Fields())
True
```

Next we compute with the integers modulo 16.

```
sage: Z16 = IntegerModRing(16)
sage: Z16.category()
Join of Category of finite commutative rings
    and Category of subquotients of monoids
    and Category of quotients of semigroups
    and Category of finite enumerated sets
sage: Z16.is_field()
False
sage: Z16.order()
16
sage: Z16.characteristic()
16
sage: gens = Z16.unit_gens()
sage: gens
(15, 5)
sage: a = gens[0]
sage: b = gens[1]
sage: def powa(i):
    return a**i
sage: def powb(i):
    return b**i
sage: gp_exp = FF.unit_group_exponent()
sage: gp_exp
28
sage: [powa(i) for i in range(15)]
[1, 15, 1, 15, 1, 15, 1, 15, 1, 15, 1, 15, 1, 15, 1]
sage: [powb(i) for i in range(15)]
```

1.1. Ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo $n$
[1, 5, 9, 13, 1, 5, 9, 13, 1, 5, 9, 13, 1, 5, 9, 13, 1, 5, 9]
sage: a.multiplicative_order()
2
sage: b.multiplicative_order()
4
sage: TestSuite(Z16).run()

Saving and loading:

sage: R = Integers(100000)
sage: TestSuite(R).run()  # long time (17s on sage.math, 2011)

Testing ideals and quotients:

sage: Z10 = Integers(10)
sage: I = Z10.principal_ideal(0)
sage: Z10.quotient(I) == Z10
True
sage: I = Z10.principal_ideal(2)
sage: Z10.quotient(I) == Z10
False
sage: I.is_prime()
True

sage: R = IntegerModRing(97)
sage: a = R(5)
sage: a**(10^62)
61

cardinality()
Return the cardinality of this ring.

EXAMPLES:

sage: Zmod(87).cardinality()
87

characteristic()
EXAMPLES:

sage: R = IntegerModRing(18)
sage: FF = IntegerModRing(17)
sage: FF.characteristic()
17
sage: R.characteristic()
18

degree()
Return 1.

EXAMPLES:
sage: R = Integers(12345678900)
sage: R.degree()
1

extension(poly, name=None, names=None, **kwds)
Return an algebraic extension of self. See sage.rings.ring.CommutativeRing.extension() for more information.

EXAMPLES:

sage: R.<t> = QQ[]
sage: Integers(8).extension(t^2 - 3)
Univariate Quotient Polynomial Ring in t over Ring of integers modulo 8 with
˓→modulus t^2 + 5

factored_order()
EXAMPLES:

sage: R = IntegerModRing(18)
sage: FF = IntegerModRing(17)
sage: R.factored_order()
2 * 3^2
sage: FF.factored_order()
17

factored_unit_order()
Return a list of Factorization objects, each the factorization of the order of the units in a \( \mathbb{Z}/p^n\mathbb{Z} \) component of this group (using the Chinese Remainder Theorem).

EXAMPLES:

sage: R = Integers(8*9*25*17*29)
sage: R.factored_unit_order()
[2^2, 2 * 3, 2^2 * 5, 2^4, 2^2 * 7]

field()
If this ring is a field, return the corresponding field as a finite field, which may have extra functionality and structure. Otherwise, raise a ValueError.

EXAMPLES:

sage: R = Integers(7); R
Ring of integers modulo 7
sage: R.field()
Finite Field of size 7
sage: R = Integers(9)
sage: R.field()
Traceback (most recent call last):
  ...
ValueError: self must be a field

is_field(proof=None)
Return True precisely if the order is prime.

INPUT:
• **proof** (optional bool or None, default None): If False, then test whether the category of the quotient is a subcategory of Fields(), or do a probabilistic primality test. If None, then test the category and then do a primality test according to the global arithmetic proof settings. If True, do a deterministic primality test.

If it is found (perhaps probabilistically) that the ring is a field, then the category of the ring is refined to include the category of fields. This may change the Python class of the ring!

**EXAMPLES:**

```python
sage: R = IntegerModRing(18)
sage: R.is_field()
False
sage: FF = IntegerModRing(17)
sage: FF.is_field()
True
```

By trac ticket #15229, the category of the ring is refined, if it is found that the ring is in fact a field:

```python
sage: R = IntegerModRing(127)
sage: R.category()
Join of Category of finite commutative rings
   and Category of subquotients of monoids
   and Category of quotients of semigroups
   and Category of finite enumerated sets
sage: R.is_field()
True
sage: R.category()
Join of Category of finite enumerated fields
   and Category of subquotients of monoids
   and Category of quotients of semigroups
```

It is possible to mistakenly put \( \mathbb{Z}/n\mathbb{Z} \) into the category of fields. In this case, **is_field()** will return True without performing a primality check. However, if the optional argument `proof = True` is provided, primality is tested and the mistake is uncovered in a warning message:

```python
sage: R = IntegerModRing(21, is_field=True)
sage: R.is_field()
True
sage: R.is_field(proof=True)
Traceback (most recent call last):
  ... ValueError: THIS SAGE SESSION MIGHT BE SERIOUSLY COMPROMISED!
The order 21 is not prime, but this ring has been put into the category of fields. This may already have consequences in other parts of Sage. Either it was a mistake of the user, or a probabilistic primality test has failed.
In the latter case, please inform the developers.
```

To avoid side-effects of this test on other tests, we clear the cache of the ring factory:

```python
sage: IntegerModRing._cache.clear()
```

**is_integral_domain**(proof=None)

Return True if and only if the order of self is prime.
EXAMPLES:

```python
sage: Integers(389).is_integral_domain()
True
sage: Integers(389^2).is_integral_domain()
False
```

**is_noetherian()**
Check if `self` is a Noetherian ring.

EXAMPLES:

```python
sage: Integers(8).is_noetherian()
True
```

**is_prime_field()**
Return True if the order is prime.

EXAMPLES:

```python
sage: Zmod(7).is_prime_field()
True
sage: Zmod(8).is_prime_field()
False
```

**is_unique_factorization_domain**(proof=None)
Return True if and only if the order of `self` is prime.

EXAMPLES:

```python
sage: Integers(389).is_unique_factorization_domain()
True
sage: Integers(389^2).is_unique_factorization_domain()
False
```

**krull_dimension()**
Return the Krull dimension of `self`.

EXAMPLES:

```python
sage: Integers(18).krull_dimension()
0
```

**list_of_elements_of_multiplicative_group()**
Return a list of all invertible elements, as python ints.

EXAMPLES:

```python
sage: R = Zmod(12)
sage: L = R.list_of_elements_of_multiplicative_group(); L
[1, 5, 7, 11]
sage: type(L[0])
<... 'int'>
sage: Zmod(1).list_of_elements_of_multiplicative_group() [0]
```

**modulus()**
Return the polynomial $x - 1$ over this ring.
Note: This function exists for consistency with the finite-field modulus function.

EXAMPLES:

```
sage: R = IntegerModRing(18)
sage: R.modulus()
x + 17
sage: R = IntegerModRing(17)
sage: R.modulus()
x + 16
```

**multiplicative_generator()**

Return a generator for the multiplicative group of this ring, assuming the multiplicative group is cyclic.

Use the unit_gens function to obtain generators even in the non-cyclic case.

EXAMPLES:

```
sage: R = Integers(7); R
Ring of integers modulo 7
sage: R.multiplicative_generator()
3
sage: R = Integers(9)
sage: R.multiplicative_generator()
2
sage: Integers(8).multiplicative_generator()
Traceback (most recent call last):
  ... ValueError: multiplicative group of this ring is not cyclic
sage: Integers(4).multiplicative_generator()
3
sage: Integers(25*3).multiplicative_generator()
Traceback (most recent call last):
  ... ValueError: multiplicative group of this ring is not cyclic
sage: Integers(25*3).unit_gens()
(26, 52)
sage: Integers(162).unit_gens()
(83,)
```

**multiplicative_group_is_cyclic()**

Return True if the multiplicative group of this field is cyclic. This is the case exactly when the order is less than 8, a power of an odd prime, or twice a power of an odd prime.

EXAMPLES:

```
sage: R = Integers(7); R
Ring of integers modulo 7
sage: R.multiplicative_group_is_cyclic()
True
sage: R = Integers(9)
sage: R.multiplicative_group_is_cyclic()
True
sage: Integers(8).multiplicative_group_is_cyclic()
(continues on next page)
False

```
sage: Integers(4).multiplicative_group_is_cyclic()
True
sage: Integers(25*3).multiplicative_group_is_cyclic()
False
```

We test that trac ticket #5250 is fixed:

```
sage: Integers(162).multiplicative_group_is_cyclic()
True
```

\texttt{multiplicative_subgroups}()

Return generators for each subgroup of \((\mathbb{Z}/N\mathbb{Z})^*\).

\textbf{EXAMPLES:}

```
sage: Integers(5).multiplicative_subgroups()
((2,),)
```

```
sage: Integers(15).multiplicative_subgroups()
((11, 7), (11, 4), (2,), (11,), (14,), (7,), (4,), ())
```

```
sage: Integers(2).multiplicative_subgroups()
((,),)
```

```
sage: len(Integers(341).multiplicative_subgroups())
80
```

\texttt{order}()

Return the order of this ring.

\textbf{EXAMPLES:}

```
sage: Zmod(87).order()
87
```

\texttt{quadratic_nonresidue}()

Return a quadratic non-residue in \texttt{self}.

\textbf{EXAMPLES:}

```
sage: R = Integers(17)
sage: R.quadratic_nonresidue()
3
```

```
sage: R(3).is_square()
False
```

\texttt{random_element}(\textit{bound=\text{None}})

Return a random element of this ring.

\textbf{INPUT:}

- \texttt{bound}, a positive integer or \texttt{None} (the default). Is given, return the coercion of an integer in the interval \([-\text{bound}, \text{bound}]\) into this ring.

\textbf{EXAMPLES:}

```
sage: R = IntegerModRing(18)
sage: R.random_element().parent() is R
```

(continues on next page)
We test bound-option:

```python
sage: R.random_element(2) in [R(16), R(17), R(0), R(1), R(2)]
True
```

### square_roots_of_one()
Return all square roots of 1 in self, i.e., all solutions to $x^2 - 1 = 0$.

**OUTPUT:**
The square roots of 1 in `self` as a tuple.

**EXAMPLES:**

```python
sage: R = Integers(2^10)
sage: [x for x in R if x^2 == 1]
[1, 511, 513, 1023]
sage: R.square_roots_of_one()
(1, 511, 513, 1023)
sage: v = Integers(9*5).square_roots_of_one(); v
(1, 19, 26, 44)
sage: [x^2 for x in v]
[1, 1, 1, 1]
sage: v = Integers(9*5*8).square_roots_of_one(); v
(1, 19, 71, 89, 91, 109, 161, 179, 181, 199, 251, 269, 271, 289, 341, 359)
sage: [x^2 for x in v]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

### unit_gens(**kwds)**
Returns generators for the unit group $(\mathbb{Z}/N\mathbb{Z})^\times$.

We compute the list of generators using a deterministic algorithm, so the generators list will always be the same. For each odd prime divisor of $N$ there will be exactly one corresponding generator; if $N$ is even there will be 0, 1 or 2 generators according to whether 2 divides $N$ to order 1, 2 or $\geq 3$.

**OUTPUT:**
A tuple containing the units of `self`.

**EXAMPLES:**

```python
sage: R = IntegerModRing(18)
sage: R.unit_gens()
(11,)
sage: R = IntegerModRing(17)
sage: R.unit_gens()
(3,)
sage: IntegerModRing(next_prime(10^30)).unit_gens()
(5,)
```
The choice of generators is affected by the optional keyword `algorithm`; this can be 'sage' (default) or 'pari'. See `unit_group()` for details.

```
sage: A = Zmod(55)
sage: A.unit_gens(algorithm='sage')
(12, 46)
sage: A.unit_gens(algorithm='pari')
(2, 21)
```

**unit_group(algorithm='sage')**

Return the unit group of `self`.

**INPUT:**

- `self` – the ring \( \mathbb{Z}/n\mathbb{Z} \) for a positive integer \( n \)
- `algorithm` – either 'sage' (default) or 'pari'

**OUTPUT:**

The unit group of `self`. This is a finite Abelian group equipped with a distinguished set of generators, which is computed using a deterministic algorithm depending on the `algorithm` parameter.

- If `algorithm` == 'sage', the generators correspond to the prime factors \( p | n \) (one generator for each odd \( p \); the number of generators for \( p = 2 \) is 0, 1 or 2 depending on the order to which 2 divides \( n \)).
- If `algorithm` == 'pari', the generators are chosen such that their orders form a decreasing sequence with respect to divisibility.

**EXAMPLES:**

The output of the algorithms 'sage' and 'pari' can differ in various ways. In the following example, the same cyclic factors are computed, but in a different order:

```
sage: A = Zmod(15)
sage: G = A.unit_group(); G
Multiplicative Abelian group isomorphic to C2 x C4
sage: G.gens_values()
(11, 7)
sage: H = A.unit_group(algorithm='pari'); H
Multiplicative Abelian group isomorphic to C4 x C2
sage: H.gens_values()
(7, 11)
```

Here are two examples where the cyclic factors are isomorphic, but are ordered differently and have different generators:

```
sage: A = Zmod(40)
sage: G = A.unit_group(); G
Multiplicative Abelian group isomorphic to C2 x C2 x C4
sage: G.gens_values()
(31, 21, 17)
sage: H = A.unit_group(algorithm='pari'); H
Multiplicative Abelian group isomorphic to C4 x C2 x C2
sage: H.gens_values()
(17, 31, 21)
```

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(continued from previous page)

```
sage: A = Zmod(192)
sage: G = A.unit_group(); G
Multiplicative Abelian group isomorphic to C2 x C16 x C2
sage: G.gens_values()
(127, 133, 65)
sage: H = A.unit_group(algorithm='pari'); H
Multiplicative Abelian group isomorphic to C16 x C2 x C2
sage: H.gens_values()
(133, 127, 65)
```

In the following examples, the cyclic factors are not even isomorphic:

```
sage: A = Zmod(319)
sage: A.unit_group()
Multiplicative Abelian group isomorphic to C10 x C28
sage: A.unit_group(algorithm='pari')
Multiplicative Abelian group isomorphic to C140 x C2
sage: A = Zmod(30.factorial())
sage: A.unit_group()
Multiplicative Abelian group isomorphic to C2058 x C110 x C156 x C16 x C18 x C22 x C28
sage: A.unit_group(algorithm='pari')
Multiplicative Abelian group isomorphic to C20499647385305088000000 x C55440 x C12 x C12 x C4 x C2 x C2 x C2 x C2 x C2 x C2 x C2
```

`unit_group_exponent()`

**EXAMPLES:**

```
sage: R = IntegerModRing(17)
sage: R.unit_group_exponent()
16
sage: R = IntegerModRing(18)
sage: R.unit_group_exponent()
6
```

`unit_group_order()`

Return the order of the unit group of this residue class ring.

**EXAMPLES:**

```
sage: R = Integers(500)
sage: R.unit_group_order()
200
```

`sage.rings.finite_rings.integer_mod_ring.crt(v)`

**INPUT:**

- `v` – (list) a lift of elements of `rings.IntegerMod(n)`, for various coprime moduli `n`

**EXAMPLES:**

```
sage: from sage.rings.finite_rings.integer_mod_ring import crt
sage: crt([mod(3, 8),mod(1,19),mod(7, 15)])
1027
```
sage.rings.finite_rings.integer_mod_ring.is_IntegerModRing(x)
Return True if x is an integer modulo ring.

This function is deprecated. Use isinstance() with sage.rings.abc.IntegerModRing instead.

EXAMPLES:

```python
sage: from sage.rings.finite_rings.integer_mod_ring import is_IntegerModRing
sage: R = IntegerModRing(17)
sage: is_IntegerModRing(R)
DeprecationWarning: the function is_IntegerModRing is deprecated.
Use isinstance(..., sage.rings.abc.IntegerModRing) instead.
See https://trac.sagemath.org/32606 for details.
True
sage: is_IntegerModRing(GF(13))
True
sage: is_IntegerModRing(GF(4, 'a'))
False
sage: is_IntegerModRing(10)
False
sage: is_IntegerModRing(ZZ)
False
```

1.2 Elements of $\mathbb{Z}/n\mathbb{Z}$

An element of the integers modulo $n$.

There are three types of integer_mod classes, depending on the size of the modulus.

- **IntegerMod_int** stores its value in a int_fast32_t (typically an int); this is used if the modulus is less than $\sqrt{2^{31} - 1}$.
- **IntegerMod_int64** stores its value in a int_fast64_t (typically a long long); this is used if the modulus is less than $2^{31} - 1$. In many places, we assume that the values and the modulus actually fit inside an unsigned long.
- **IntegerMod_gmp** stores its value in a mpz_t; this can be used for an arbitrarily large modulus.

All extend IntegerMod_abstract.

For efficiency reasons, it stores the modulus (in all three forms, if possible) in a common (cdef) class NativeIntStruct rather than in the parent.

AUTHORS:

- Robert Bradshaw: most of the work
- Didier Deshommes: bit shifting
- William Stein: editing and polishing; new arith architecture
- Robert Bradshaw: implement native is_square and square_root
- William Stein: sqrt
- Maarten Derickx: moved the valuation code from the global valuation function to here

```python
class sage.rings.finite_rings.integer_mod.Int_to_IntegerMod
    Bases: sage.rings.finite_rings.integer_mod.IntegerMod_hom
```
EXAMPLES:

We make sure it works for every type.

```python
sage: from sage.rings.finite_rings.integer_mod import Int_to_IntegerMod
sage: Rs = [Integers(2**k) for k in range(1,50,10)]
```

```python
sage: [type(R(0)) for R in Rs]
[<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>, <class 'sage.rings.finite_rings.integer_mod.IntegerMod_int64'>, <class 'sage.rings.finite_rings.integer_mod.IntegerMod_gmp'>]
```

```python
sage: fs = [Int_to_IntegerMod(R) for R in Rs]
sage: [f(-1) for f in fs]
[1, 2047, 2097151, 2147483647, 2199023255551]
```

`sage.rings.finite_rings.integer_mod.IntegerMod(parent, value)`
Create an integer modulo \( n \) with the given parent.

This is mainly for internal use.

EXAMPLES:

```python
sage: from sage.rings.finite_rings.integer_mod import IntegerMod
sage: R = IntegerModRing(100)
sage: type(R._pyx_order.table)
<class 'list'>
sage: IntegerMod(R, 42)  
42
sage: IntegerMod(R, 142)  
42
sage: IntegerMod(R, 10^100 + 42)  
42
sage: IntegerMod(R, -9158)  
42
```

```python
class sage.rings.finite_rings.integer_mod.IntegerMod_abstract
Bases: sage.rings.finite_rings.element_base.FiniteRingElement

EXAMPLES:

```python
sage: a = Mod(10, 30^10); a
10
sage: type(a)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_gmp'>
sage: loads(a.dumps()) == a
True
```

`additive_order()`

Returns the additive order of self.

This is the same as `self.order()`.

EXAMPLES:

```python
sage: Integers(20)(2).additive_order()
10
sage: Integers(20)(7).additive_order()
```
20
sage: Integers(90308402384902)(2).additive_order()
45154201192451

\textbf{charpoly}(\texttt{var='x'})

Returns the characteristic polynomial of this element.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: k = GF(3)
sage: a = k.gen()
sage: a.charpoly('x')
x + 2
sage: a + 2
0
\end{verbatim}

\textbf{AUTHORS:}

• Craig Citro

\textbf{crt}(\texttt{other})

Use the Chinese Remainder Theorem to find an element of the integers modulo the product of the moduli that reduces to \texttt{self} and to \texttt{other}. The modulus of \texttt{other} must be coprime to the modulus of \texttt{self}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: a = mod(3,5)
sage: b = mod(2,7)
sage: a.crt(b)
23

sage: a = mod(37,10^8)
sage: b = mod(9,3^8)
sage: a.crt(b)
125900000037

sage: b = mod(0,1)
sage: a.crt(b) == a
True
sage: a.crt(b).modulus()
100000000
\end{verbatim}

\textbf{AUTHORS:}

• Robert Bradshaw

\textbf{divides}(\texttt{other})

Test whether \texttt{self} divides \texttt{other}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: R = Zmod(6)
sage: R(2).divides(R(4))
True
sage: R(4).divides(R(2))
\end{verbatim}
True
```
sage: R(2).divides(R(3))
False
```

**generalised_log()**

Return integers \([n_1, ..., n_d]\) such that

\[
\prod_{i=1}^{d} x_i^{n_i} = \text{self},
\]

where \(x_1, ..., x_d\) are the generators of the unit group returned by \(\text{self.parent().unit_gens()}\).

**EXAMPLES:**

```
sage: m = Mod(3, 1568)
sage: v = m.generalised_log(); v
[1, 3, 1]
sage: prod([Zmod(1568).unit_gens()[i] ** v[i] for i in [0..2]])
3
```

See also:

The method `log()`.

**Warning:** The output is given relative to the set of generators obtained by passing `algorithm='sage'` to the method `unit_gens()` of the parent (which is the default). Specifying `algorithm='pari'` usually yields a different set of generators that is incompatible with this method.

**is_nilpotent()**

Return True if \(\text{self}\) is nilpotent, i.e., some power of \(\text{self}\) is zero.

**EXAMPLES:**

```
sage: a = Integers(90384098234^3)
sage: factor(a.order())
2^3 * 191^3 * 236607587^3
sage: b = a(2*191)
sage: b.is_nilpotent()
False
sage: b = a(2*191*236607587)
sage: b.is_nilpotent()
True
```

**ALGORITHM:** Let \(m \geq \log_2(n)\), where \(n\) is the modulus. Then \(x \in \mathbb{Z}/n\mathbb{Z}\) is nilpotent if and only if \(x^m = 0\).

**PROOF:** This is clear if you reduce to the prime power case, which you can do via the Chinese Remainder Theorem.

We could alternatively factor \(n\) and check to see if the prime divisors of \(n\) all divide \(x\). This is asymptotically slower :-).

**is_one()**
**is_primitive_root()**
Determines whether this element generates the group of units modulo $n$.

This is only possible if the group of units is cyclic, which occurs if $n$ is 2, 4, a power of an odd prime or twice a power of an odd prime.

**EXAMPLES:**

```
sage: mod(1,2).is_primitive_root()
True
sage: mod(3,4).is_primitive_root()
True
sage: mod(2,7).is_primitive_root()
False
sage: mod(3,98).is_primitive_root()
True
sage: mod(11,1009^2).is_primitive_root()
True
```

**is_square()**

**EXAMPLES:**

```
sage: Mod(3,17).is_square()
False
sage: Mod(9,17).is_square()
True
sage: Mod(9,17^2*19^2).is_square()
True
sage: Mod(-1,17^30).is_square()
True
sage: Mod(1/9, next_prime(2^40)).is_square()
True
sage: Mod(1/25, next_prime(2^90)).is_square()
True
```

**ALGORITHM:** Calculate the Jacobi symbol \( (self/p) \) at each prime $p$ dividing $n$. It must be 1 or 0 for each prime, and if it is 0 mod $p$, where $p^k | n$, then $ord_p(self)$ must be even or greater than $k$.

The case $p = 2$ is handled separately.

**AUTHORS:**
- Robert Bradshaw

**is_unit()**

**lift_centered()**
Lift $self$ to a centered congruent integer.

**OUTPUT:**
The unique integer $i$ such that $-n/2 < i \leq n/2$ and $i = self \mod n$ (where $n$ denotes the modulus).

**EXAMPLES:**

```
sage: Mod(0,5).lift_centered()
0
sage: Mod(1,5).lift_centered()
1
```

(continues on next page)
sage: Mod(2,5).lift_centered()
2
sage: Mod(3,5).lift_centered()
-2
sage: Mod(4,5).lift_centered()
-1
sage: Mod(50,100).lift_centered()
50
sage: Mod(51,100).lift_centered()
-49
sage: Mod(-1,3^100).lift_centered()
-1

\textbf{log}\,(b=\text{None}, \text{logarithm\_exists}=\text{None})

Compute the discrete logarithm of this element to base \(b\), that is, return an integer \(x\) such that \(b^x = a\), where \(a\) is \texttt{self}.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{self} - unit modulo \(n\)
  \item \texttt{b} - a unit modulo \(n\). If \(b\) is not given, \texttt{R.multiplicative\_generator()} is used, where \(R\) is the parent of \texttt{self}.
\end{itemize}

\textbf{OUTPUT}: Integer \(x\) such that \(b^x = a\), if this exists; a ValueError otherwise.

\textbf{Note}: The algorithm first factors the modulus, then invokes Pari’s \texttt{znlog} function for each odd prime power in the factorization of the modulus. This method can be quite slow for large moduli.

\textbf{EXAMPLES}:

\begin{verbatim}
sage: r = Integers(125)
sage: b = r.multiplicative_generator()^3
sage: a = b^17
sage: a.log(b)
17
sage: a.log()
51
\end{verbatim}

A bigger example:

\begin{verbatim}
sage: FF = FiniteField(2^32+61)
sage: c = FF(4294967356)
sage: x = FF(2)
sage: a = c.log(x)
sage: a
2147483678
sage: x^a
4294967356
\end{verbatim}

An example with a highly composite modulus:
\begin{Verbatim}
sage: m = 2^99 * 77^7 * 123456789 * 13712923537615486607^2
sage: (Mod(5,m)^5735816763073854953388147237921).log(5)
5735816763073854953388147237921
\end{Verbatim}

Errors are generated if the logarithm doesn’t exist or the inputs are not units:

\begin{Verbatim}
sage: Mod(3, 7).log(Mod(2, 7))
Traceback (most recent call last):
  ...  
ValueError: no logarithm of 3 found to base 2 modulo 7

sage: a = Mod(16, 100); b = Mod(4, 100)
sage: a.log(b)
Traceback (most recent call last):
  ...  
ValueError: logarithm of 16 is not defined since it is not a unit modulo 100
\end{Verbatim}

AUTHORS:

- David Joyner and William Stein (2005-11)
- Simon King (2010-07-07): fix a side effect on PARI
- Lorenz Panny (2021): speedups for composite moduli

\textbf{minimal\_polynomial}(\textit{var=}'x')

Returns the minimal polynomial of this element.

\textbf{EXAMPLES:}

\begin{Verbatim}
sage: GF(241, 'a')(1).minimal\_polynomial(var = 'z')
z + 240
\end{Verbatim}

\textbf{minpoly}(\textit{var=}'x')

Returns the minimal polynomial of this element.

\textbf{EXAMPLES:}

\begin{Verbatim}
sage: GF(241, 'a')(1).minpoly()
x + 240
\end{Verbatim}

\textbf{modulus}()

\textbf{EXAMPLES:}

\begin{Verbatim}
sage: Mod(3,17).modulus()
17
\end{Verbatim}

\textbf{multiplicative\_order}()

Returns the multiplicative order of self.

\textbf{EXAMPLES:}

\begin{Verbatim}
sage: Mod(-1,5).multiplicative\_order()
2
sage: Mod(1,5).multiplicative\_order()
1
sage: Mod(0,5).multiplicative\_order()
(continues on next page)
\end{Verbatim}
norm()  
Returns the norm of this element, which is itself. (This is here for compatibility with higher order finite fields.)

EXAMPLES:

```sage
sage: k = GF(691)
sage: a = k(389)
sage: a.norm()
389
```

AUTHORS:

- Craig Citro

nth_root(n, extend=False, all=False, algorithm=None, cunningham=False)

Returns an \( n \)th root of \( \text{self} \).

INPUT:

- \( n \) - integer \( \geq 1 \)
- \( \text{extend} \) - bool (default: True); if True, return an \( n \)th root in an extension ring, if necessary. Otherwise, raise a \( \text{ValueError} \) if the root is not in the base ring. Warning: this option is not implemented!
- \( \text{all} \) - bool (default: False); if True, return all \( n \)th roots of \( \text{self} \), instead of just one.
- \( \text{algorithm} \) - string (default: None); The algorithm for the prime modulus case. CRT and p-adic log techniques are used to reduce to this case. ‘Johnston’ is the only currently supported option.
- \( \text{cunningham} \) - bool (default: False); In some cases, factorization of \( n \) is computed. If cunningham is set to True, the factorization of \( n \) is computed using trial division for all primes in the so called Cunningham table. Refer to sage.rings.factorint.factor_cunningham for more information. You need to install an optional package to use this method, this can be done with the following command line

```
sage -i cunningham_tables
```

OUTPUT:

If \( \text{self} \) has an \( n \)th root, returns one (if \( \text{all} \) is False) or a list of all of them (if \( \text{all} \) is True). Otherwise, raises a \( \text{ValueError} \) (if \( \text{extend} \) is False) or a \( \text{NotImplementedError} \) (if \( \text{extend} \) is True).

**Warning:** The ‘extend’ option is not implemented (yet).

NOTE:

- If \( n = 0 \):
  - if \( \text{all} \) = True:
    * if \( \text{self} \) = 1: all nonzero elements of the parent are returned in a list. Note that this could be very expensive for large parents.
    * otherwise: an empty list is returned
if all=False:
    * if self=1: self is returned
    * otherwise; a ValueError is raised

- If \( n < 0 \):
  - if self is invertible, the \((-n)\)th root of the inverse of self is returned
  - otherwise a ValueError is raised or empty list returned.

EXAMPLES:

```python
sage: K = GF(31)
sage: a = K(22)
sage: K(22).nth_root(7)
13
sage: K(25).nth_root(5)
5
sage: K(23).nth_root(3)
29
sage: mod(225,2^5*3^2).nth_root(4, all=True)
[225, 129, 33, 63, 255, 159, 9, 201, 105, 279, 183, 87, 81, 273, 177, 207, 111, ...
 \rightarrow 15, 153, 57, 249, 135, 39, 231]
sage: mod(275,2^5*7^4).nth_root(7, all=True)
[58235, 25307, 69211, 36283, 3355, 47259, 14331]
sage: mod(1,8).nth_root(2, all=True)
[1, 7, 5, 3]
sage: mod(4,8).nth_root(2, all=True)
[2, 6]
sage: mod(1,16).nth_root(4, all=True)
[1, 15, 13, 3, 9, 7, 5, 11]
sage: (mod(22,31)^200).nth_root(200)
5
sage: mod(3,6).nth_root(0, all=True)
[]
sage: mod(3,6).nth_root(0)
Traceback (most recent call last):
...
ValueError
sage: mod(1,6).nth_root(0, all=True)
[1, 2, 3, 4, 5]
```

ALGORITHM:

The default for prime modulus is currently an algorithm described in [Joh1999].

AUTHORS:

- David Roe (2010-02-13)

**polynomial**(\( var='x' \))

Returns a constant polynomial representing this value.

EXAMPLES:

```python
sage: k = GF(7)
sage: a = k.gen(); a
```
1
sage: a.polynomial()
1
sage: type(a.polynomial())
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>

rational_reconstruction()
Use rational reconstruction to try to find a lift of this element to the rational numbers.

EXAMPLES:

sage: R = IntegerModRing(97)
sage: a = R(2) / R(3)
sage: a
33
sage: a.rational_reconstruction()
2/3

This method is also inherited by prime finite fields elements:

sage: k = GF(97)
sage: a = k(RationalField()('2/3'))
sage: a
33
sage: a.rational_reconstruction()
2/3

sqrt(extend=True, all=False)
Return square root or square roots of self modulo \( n \).

INPUT:

- extend - bool (default: True); if True, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the square root is not in the base ring.

- all - bool (default: False); if True, return \{all\} square roots of self, instead of just one.

ALGORITHM: Calculates the square roots mod \( p \) for each of the primes \( p \) dividing the order of the ring, then lifts them \( p \)-adically and uses the CRT to find a square root mod \( n \).

See also square_root_mod_prime_power() and square_root_mod_prime() for more algorithmic details.

EXAMPLES:

sage: mod(-1, 17).sqrt()
4
sage: mod(5, 389).sqrt()
86
sage: mod(7, 18).sqrt()
5
sage: a = mod(14, 5^60).sqrt()
sage: a^a
14
sage: mod(15, 389).sqrt(extend=False)
Traceback (most recent call last):
... 
ValueError: self must be a square  
\begin{verbatim}
sage: Mod(1/9, next_prime(2^40)).sqrt()^(-2)
sage: Mod(1/25, next_prime(2^90)).sqrt()^(-2)
\end{verbatim}
9 25

\begin{verbatim}
sage: a = Mod(3,5); a
3
sage: x = Mod(-1, 360)
sage: x.sqrt(extend=False)
Traceback (most recent call last):
... 
ValueError: self must be a square
sage: y = x.sqrt(); y
sqrt359
sage: y.parent()
Univariate Quotient Polynomial Ring in sqrt359 over Ring of integers modulo 360
˓→ with modulus x^2 + 1
sage: y^2
359
\end{verbatim}

We compute all square roots in several cases:

\begin{verbatim}
sage: R = Integers(5*2^3*3^2); R
Ring of integers modulo 360
sage: R(40).sqrt(all=True)
[20, 160, 200, 340]
sage: [x for x in R if x^2 == 40]  # Brute force verification
[20, 160, 200, 340]
sage: R(1).sqrt(all=True)
[1, 19, 71, 89, 91, 109, 161, 179, 181, 199, 251, 269, 271, 289, 341, 359]
sage: R(0).sqrt(all=True)
[0, 60, 120, 180, 240, 300]
\end{verbatim}

\begin{verbatim}
sage: R = Integers(5*13^3*37); R
Ring of integers modulo 406445
sage: v = R(-1).sqrt(all=True); v
[78853, 111808, 160142, 193097, 213348, 246303, 294637, 327592]
sage: [x^2 for x in v]
[406444, 406444, 406444, 406444, 406444, 406444, 406444, 406444]
sage: v = R(169).sqrt(all=True); min(v), -max(v), len(v)
(13, 13, 104)
sage: all(x^2 == 169 for x in v)
True
\end{verbatim}

\begin{verbatim}
sage: t = FiniteField(next_prime(2^100))(4)
sage: t.sqrt(extend=False, all=True)
[2, 1267650600228229401496703205651]
sage: t = FiniteField(next_prime(2^100))(2)
sage: t.sqrt(extend=False, all=True)
[]
\end{verbatim}

1.2. Elements of $\mathbb{Z}/n\mathbb{Z}$
Modulo a power of 2:

```
sage: R = Integers(2^7); R
Ring of integers modulo 128
sage: a = R(17)
sage: a.sqrt()
23
sage: a.sqrt(all=True)
[23, 41, 87, 105]
sage: [x for x in R if x^2==17]
[23, 41, 87, 105]
```

**square_root**\(^{}(extend=True, all=False)\)

Return square root or square roots of self modulo \(n\).

**INPUT:**

- extend - bool (default: True); if True, return a square root in an extension ring, if necessary. Otherwise, raise a \(\text{ValueError}\) if the square root is not in the base ring.
- all - bool (default: False); if True, return all square roots of self, instead of just one.

**ALGORITHM:** Calculates the square roots mod \(p\) for each of the primes \(p\) dividing the order of the ring, then lifts them \(p\)-adically and uses the CRT to find a square root mod \(n\).

See also \(\text{square_root\_mod\_prime\_power()}\) and \(\text{square_root\_mod\_prime()}\) for more algorithmic details.

**EXAMPLES:**

```
sage: mod(-1, 17).sqrt()
4
sage: mod(5, 389).sqrt()
86
sage: mod(7, 18).sqrt()
5
sage: a = mod(14, 5^60).sqrt()
sage: a*a
14
sage: mod(15, 389).sqrt(extend=False)
Traceback (most recent call last):
  ...
ValueError: self must be a square
sage: Mod(1/9, next_prime(2^40)).sqrt()^(-2)
9
sage: Mod(1/25, next_prime(2^90)).sqrt()^(-2)
25
```

```
sage: a = Mod(3,5); a
3
sage: x = Mod(-1, 360)
sage: x.sqrt(extend=False)
Traceback (most recent call last):
  ...
ValueError: self must be a square
sage: y = x.sqrt(); y
```

(continues on next page)
We compute all square roots in several cases:

```
sage: R = Integers(5*2^3*3^2); R
Ring of integers modulo 360
sage: R(40).sqrt(all=True)
[20, 160, 200, 340]
sage: [x for x in R if x^2 == 40]  # Brute force verification
[20, 160, 200, 340]
sage: R(1).sqrt(all=True)
[1, 19, 71, 89, 91, 109, 161, 179, 181, 199, 251, 269, 271, 289, 341, 359]
sage: R(0).sqrt(all=True)
[0, 60, 120, 180, 240, 300]
```

```
sage: R = Integers(5*13^3*37); R
Ring of integers modulo 406445
sage: v = R(-1).sqrt(all=True); v
[78853, 111808, 160142, 193097, 213348, 246303, 294637, 327592]
sage: [x^2 for x in v]
[406444, 406444, 406444, 406444, 406444, 406444, 406444, 406444]
sage: v = R(169).sqrt(all=True); min(v), -max(v), len(v)
(13, 13, 104)
sage: all(x^2 == 169 for x in v)
True
```

```
sage: t = FiniteField(next_prime(2^100))(4)
sage: t.sqrt(extend=False, all=True)
[2, 1267650600228229401496703205651]
sage: t = FiniteField(next_prime(2^100))(2)
sage: t.sqrt(extend=False, all=True)
[]
```

Modulo a power of 2:

```
sage: R = Integers(2^7); R
Ring of integers modulo 128
sage: a = R(17)
sage: a.sqrt()
23
sage: a.sqrt(all=True)
[23, 41, 87, 105]
sage: [x for x in R if x^2==17]
[23, 41, 87, 105]
```

The `trace()` function returns the trace of this element, which is itself. (This is here for compatibility with higher order finite fields.)
EXAMPLES:

```
sage: k = GF(691)
sage: a = k(389)
sage: a.trace()
389
```

AUTHORS:

• Craig Citro

```
valuation(p)
```

The largest power \( r \) such that \( m \) is in the ideal generated by \( p^r \) or infinity if there is not a largest such power. However it is an error to take the valuation with respect to a unit.

**Note:** This is not a valuation in the mathematical sense. As shown with the examples below.

EXAMPLES:

This example shows that the \( (a*b).valuation(n) \) is not always the same as \( a.valuation(n) + b.valuation(n) \)

```
sage: R=ZZ.quo(9)
sage: a=R(3)
sage: b=R(6)
sage: a.valuation(3)
1
sage: a.valuation(3) + b.valuation(3)
2
sage: (a*b).valuation(3)
+Infinity
```

The valuation with respect to a unit is an error

```
sage: a.valuation(4)
Traceback (most recent call last):
  ...
ValueError: Valuation with respect to a unit is not defined.
```

class sage.rings.finite_rings.integer_mod.IntegerMod_gmp

Bases: sage.rings.finite_rings.integer_mod.IntegerMod_abstract

Elements of \( \mathbb{Z}/n\mathbb{Z} \) for \( n \) not small enough to be operated on in word size.

AUTHORS:

• Robert Bradshaw (2006-08-24)

```
gcd(other)
```

Greatest common divisor

Returns the “smallest” generator in \( \mathbb{Z}/N\mathbb{Z} \) of the ideal generated by \( \text{self} \) and \( \text{other} \).

INPUT:

• other – an element of the same ring as this one.

EXAMPLES:
sage: mod(2^3*3^2*5, 3^3*2^2*17^8).gcd(mod(2^4*3^17, 3^3*2^2*17^8))
12
sage: mod(0,17^8).gcd(mod(0,17^8))
0

**is_one()**

Returns True if this is 1, otherwise False.

**EXAMPLES:**

```python
sage: mod(1,5^23).is_one()
True
sage: mod(0,5^23).is_one()
False
```

**is_unit()**

Return True iff this element is a unit.

**EXAMPLES:**

```python
sage: mod(13, 5^23).is_unit()
True
sage: mod(25, 5^23).is_unit()
False
```

**lift()**

Lift an integer modulo \( n \) to the integers.

**EXAMPLES:**

```python
sage: a = Mod(8943, 2^70); type(a)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_gmp'>
sage: lift(a)
8943
sage: a.lift()
8943
```

---

**class** `sage.rings.finite_rings.integer_mod.IntegerMod_hom`

Bases: `sage.categories.morphism.Morphism`

**class** `sage.rings.finite_rings.integer_mod.IntegerMod_int`

Bases: `sage.rings.finite_rings.integer_mod.IntegerMod_abstract`

Elements of \( \mathbb{Z}/n\mathbb{Z} \) for \( n \) small enough to be operated on in 32 bits

**AUTHORS:**

• Robert Bradshaw (2006-08-24)

**EXAMPLES:**

```python
sage: a = Mod(10,30); a
10
sage: loads(a.dumps()) == a
True
```

**gcd(other)**

Greatest common divisor

---

1.2. Elements of \( \mathbb{Z}/n\mathbb{Z} \)
Returns the “smallest” generator in $\mathbb{Z}/N\mathbb{Z}$ of the ideal generated by self and other.

INPUT:

• other – an element of the same ring as this one.

EXAMPLES:

```python
sage: R = Zmod(60); S = Zmod(72)
sage: a = R(40).gcd(S(30)); a
2
sage: a.parent()
Ring of integers modulo 12
sage: b = R(17).gcd(60); b
1
sage: b.parent()
Ring of integers modulo 60
sage: mod(72*5, 3^3*2^2*17^2).gcd(mod(48*17, 3^3*2^2*17^2))
12
sage: mod(0,1).gcd(mod(0,1))
0
```

is_one()

Returns True if this is 1, otherwise False.

EXAMPLES:

```python
sage: mod(6,5).is_one()
True
sage: mod(0,5).is_one()
False
sage: mod(1, 1).is_one()
True
sage: Zmod(1).one().is_one()
True
```

is_unit()

Return True iff this element is a unit

EXAMPLES:

```python
sage: a=Mod(23,100)
sage: a.is_unit()
True
sage: a=Mod(24,100)
sage: a.is_unit()
False
```

lift()

Lift an integer modulo $n$ to the integers.

EXAMPLES:

```python
sage: a = Mod(8943, 2^10); type(a)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>
sage: lift(a)
(continues on next page)
```
sage: a.lift()
751

sqrt(extend=True, all=False)
Return square root or square roots of self modulo \( n \).

INPUT:

- extend - bool (default: True); if True, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the square root is not in the base ring.
- all - bool (default: False); if True, return all square roots of self, instead of just one.

ALGORITHM: Calculates the square roots mod \( p \) for each of the primes \( p \) dividing the order of the ring, then lifts them \( p \)-adically and uses the CRT to find a square root mod \( n \).

See also \texttt{square_root_mod_prime_power()} and \texttt{square_root_mod_prime()} for more algorithmic details.

EXAMPLES:

```sage
sage: mod(-1, 17).sqrt()
4
sage: mod(5, 389).sqrt()
86
sage: mod(7, 18).sqrt()
5
sage: a = mod(14, 5^60).sqrt()
sage: a*a
14
sage: mod(15, 389).sqrt(extend=False)
Traceback (most recent call last):
...  ValueError: self must be a square
sage: Mod(1/9, next_prime(2^40)).sqrt()^(-2)
9
sage: Mod(1/25, next_prime(2^90)).sqrt()^(-2)
25
```

```sage
sage: a = Mod(3,5); a
3
sage: x = Mod(-1, 360)
sage: x.sqrt(extend=False)
Traceback (most recent call last):
...  ValueError: self must be a square
sage: y = x.sqrt(); y
sage: y
sqrt359
sage: y.parent()
Univariate Quotient Polynomial Ring in sqrt359 over Ring of integers modulo 360 → with modulus x^2 + 1
sage: y^2
359
```

We compute all square roots in several cases:
sage: R = Integers(5*2^3*3^2); R
Ring of integers modulo 360
sage: R(40).sqrt(all=True)
[20, 160, 200, 340]
sage: [x for x in R if x^2 == 40] # Brute force verification
[20, 160, 200, 340]
sage: R(1).sqrt(all=True)
[1, 19, 71, 89, 91, 109, 161, 179, 181, 199, 251, 269, 271, 289, 341, 359]
sage: R(0).sqrt(all=True)
[0, 60, 120, 180, 240, 300]
sage: GF(107)(0).sqrt(all=True)
[0]
sage: R = Integers(5*13^3*37); R
Ring of integers modulo 406445
sage: v = R(-1).sqrt(all=True); v
[78853, 111808, 160142, 193097, 213348, 246303, 294637, 327592]
sage: [x^2 for x in v]
[406444, 406444, 406444, 406444, 406444, 406444, 406444, 406444]
sage: v = R(169).sqrt(all=True); min(v), -max(v), len(v)
(13, 13, 104)
sage: all(x^2 == 169 for x in v)
True

Modulo a power of 2:

sage: R = Integers(2^7); R
Ring of integers modulo 128
sage: a = R(17)
sage: a.sqrt()
23
sage: a.sqrt(all=True)
[23, 41, 87, 105]
sage: [x for x in R if x^2==17]
[23, 41, 87, 105]

class sage.rings.finite_rings.integer_mod.IntegerMod_int64
Bases: sage.rings.finite_rings.integer_mod.IntegerMod_abstract

Elements of \mathbb{Z}/n\mathbb{Z} for n small enough to be operated on in 64 bits

EXAMPLES:

sage: a = Mod(10,3^10); a
10
sage: type(a)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int64'>
sage: loads(a.dumps()) == a
True
sage: Mod(5, 2^31)
5

AUTHORS:

• Robert Bradshaw (2006-09-14)
**gcd**(other)

Greatest common divisor

Returns the “smallest” generator in \(\mathbb{Z}/n\mathbb{Z}\) of the ideal generated by \(\text{self}\) and \(\text{other}\).

**INPUT:**

- other – an element of the same ring as this one.

**EXAMPLES:**

```python
sage: mod(2^3*3^2*5, 3^3*2^2*17^5).gcd(mod(2^4*3*17, 3^3*2^2*17^5))
12
sage: mod(0,17^5).gcd(mod(0,17^5))
0
```

**is_one()**

Returns True if this is 1, otherwise False.

**EXAMPLES:**

```python
sage: (mod(-1,5^10)^2).is_one()  # True
True
sage: mod(0,5^10).is_one()  # False
False
```

**is_unit()**

Return True iff this element is a unit.

**EXAMPLES:**

```python
sage: mod(13, 5^10).is_unit()  # True
True
sage: mod(25, 5^10).is_unit()  # False
False
```

**lift()**

Lift an integer modulo \(n\) to the integers.

**EXAMPLES:**

```python
sage: a = Mod(8943, 2^25); type(a)
<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int64'>
sage: lift(a)
8943
sage: a.lift()
8943
```

**class** `sage.rings.finite_rings.integer_mod.IntegerMod_to_Integer`

**Bases:** `sage.categories.map.Map`

**Map to lift elements to** `Integer`.

**EXAMPLES:**

```python
sage: ZZ.convert_map_from(GF(2))
Lifting map:
  From: Finite Field of size 2
  To:   Integer Ring
```
class sage.rings.finite_rings.integer_mod.IntegerMod_to_IntegerMod
    Bases: sage.rings.finite_rings.integer_mod.IntegerMod_hom

Very fast IntegerMod to IntegerMod homomorphism.

EXAMPLES:

    sage: from sage.rings.finite_rings.integer_mod import IntegerMod_to_IntegerMod
    sage: Rs = [Integers(3**k) for k in range(1,30,5)]
    sage: [type(R(0)) for R in Rs]
    [<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>,
     <class 'sage.rings.finite_rings.integer_mod.IntegerMod_int64'>,
     <class 'sage.rings.finite_rings.integer_mod.IntegerMod_gmp'>]
    sage: fs = [IntegerMod_to_IntegerMod(S, R) for R in Rs for S in Rs
    → and S.order() > R.order()]
    sage: all(f(-1) == f.codomain()(-1) for f in fs)
    True
    sage: [f(-1) for f in fs]
    [2, 2, 2, 2, 2, 728, 728, 728, 728, 177146, 177146, 177146, 43046720, 43046720, 10460353202]

isInjective()  
    Return whether this morphism is injective.
    
    EXAMPLES:

        sage: Zmod(4).hom(Zmod(2)).is_injective()
        False

isSurjective()  
    Return whether this morphism is surjective.
    
    EXAMPLES:

        sage: Zmod(4).hom(Zmod(2)).is_surjective()
        True

class sage.rings.finite_rings.integer_mod.Integer_to_IntegerMod
    Bases: sage.rings.finite_rings.integer_mod.IntegerMod_hom

Fast $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ morphism.

EXAMPLES:

    We make sure it works for every type.

        sage: from sage.rings.finite_rings.integer_mod import Integer_to_IntegerMod
        sage: Rs = [Integers(10), Integers(10^5), Integers(10^10)]
        sage: [type(R(0)) for R in Rs]
        [<class 'sage.rings.finite_rings.integer_mod.IntegerMod_int'>,
         <class 'sage.rings.finite_rings.integer_mod.IntegerMod_int64'>,
         <class 'sage.rings.finite_rings.integer_mod.IntegerMod_gmp'>]
        sage: fs = [Integer_to_IntegerMod(R) for R in Rs]
        sage: [f(-1) for f in fs]
        [9, 99999, 9999999999]
is_injective()
Return whether this morphism is injective.

EXAMPLES:

```
sage: ZZ.hom(Zmod(2)).is_injective()
False
```

is_surjective()
Return whether this morphism is surjective.

EXAMPLES:

```
sage: ZZ.hom(Zmod(2)).is_surjective()
True
```

section()

```
sage.rings.finite_rings.integer_mod.Mod(n, m, parent=None)
```
Return the equivalence class of $n$ modulo $m$ as an element of $\mathbb{Z}/m\mathbb{Z}$.

EXAMPLES:

```
sage: x = Mod(12345678, 32098203845329048)
sage: x
12345678
sage: x^100
1017322209155072
```

You can also use the lowercase version:

```
sage: mod(12,5)
2
```

Illustrates that trac ticket #5971 is fixed. Consider $n$ modulo $m$ when $m = 0$. Then $\mathbb{Z}/0\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ so $n$ modulo 0 is equivalent to $n$ for any integer value of $n$:

```
sage: Mod(10, 0)
10
sage: a = randint(-100, 100)
sage: Mod(a, 0) == a
True
```

class sage.rings.finite_rings.integer_mod.NativeIntStruct
Bases: object

We store the various forms of the modulus here rather than in the parent for efficiency reasons.

We may also store a cached table of all elements of a given ring in this class.

inverses

```
precompute_table(parent)
```
Function to compute and cache all elements of this class.

If inverses == True, also computes and caches the inverses of the invertible elements.

EXAMPLES:
sage: from sage.rings.finite_rings.integer_mod import NativeIntStruct
sage: R = IntegerModRing(10)
sage: M = NativeIntStruct(R.order())
sage: M.precompute_table(R)
sage: M.table
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
sage: M.inverses
[None, 1, None, 7, None, None, None, 3, None, 9]

This is used by the `sage.rings.finite_rings.integer_mod_ring.IntegerModRing_generic` constructor:

sage: from sage.rings.finite_rings.integer_mod_ring import IntegerModRing-generic
sage: R = IntegerModRing_generic(39, cache=False)
sage: R(5)^-1
8
sage: R(5)^-1 is R(8)
False
sage: R = IntegerModRing_generic(39, cache=True)  # indirect doctest
sage: R(5)^-1 is R(8)
True

Check that the inverse of 0 modulo 1 works, see trac ticket #13639:

sage: R = IntegerModRing_generic(1, cache=True)  # indirect doctest
sage: R(0)^-1 is R(0)
True

table

`sage.rings.finite_rings.integer_mod.is_IntegerMod(x)`

Return True if and only if x is an integer modulo n.

EXAMPLES:

sage: from sage.rings.finite_rings.integer_mod import is_IntegerMod
sage: is_IntegerMod(5)
False
sage: is_IntegerMod(Mod(5,10))
True

`sage.rings.finite_rings.integer_mod.lucas(k, P, Q=1, n=None)`

Return \([V_k(P, Q) \mod n, Q^{[k/2]} \mod n]\) where \(V_k\) is the Lucas function defined by the recursive relation

\[V_k(P, Q) = PV_{k-1}(P, Q) - QV_{k-2}(P, Q)\]

with \(V_0 = 2, V_1 = P\).

INPUT:

- k – integer; index to compute
- P, Q – integers or modular integers; initial values
- n – integer (optional); modulus to use if P is not a modular integer

REFERENCES:
• [IIEEEP1363]

AUTHORS:
• Somindu Chaya Ramanna, Shashank Singh and Srinivas Vivek Venkatesh (2011-09-15, ECC2011 summer school)
• Robert Bradshaw

EXAMPLES:

```python
sage: [lucas(k,4,5,11)[0] for k in range(30)]
[2, 4, 6, 4, 8, 1, 8, 5, 2, 5, 10, 4, 10, 9, 8, 9, 7, 5, 7, 3, 10, 3, 6, 9, 6, 1, 7, 1, 2, 3]
sage: lucas(20,4,5,11)
[10, 1]
```

`sage.rings.finite_rings.integer_mod.lucas_q1(mm, P)`
Return $V_k(P, 1)$ where $V_k$ is the Lucas function defined by the recursive relation

$$V_k(P, Q) = PV_{k-1}(P, Q) - QV_{k-2}(P, Q)$$

with $V_0 = 2, V_1(PQ) = P$.

REFERENCES:
• [Pos1988]

AUTHORS:
• Robert Bradshaw

`sage.rings.finite_rings.integer_mod.makeNativeIntStruct`
alias of `sage.rings.finite_rings.integer_mod.NativeIntStruct`

`sage.rings.finite_rings.integer_mod.mod(n, m, parent=None)`
Return the equivalence class of $n$ modulo $m$ as an element of $\mathbb{Z}/m\mathbb{Z}$.

EXAMPLES:

```python
sage: x = Mod(12345678, 32098203845329048)
sage: x
12345678
sage: x^100
1017322209155072
```

You can also use the lowercase version:

```python
sage: mod(12, 5)
2
```

Illustrates that trac ticket #5971 is fixed. Consider $n$ modulo $m$ when $m = 0$. Then $\mathbb{Z}/0\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ so $n$ modulo 0 is equivalent to $n$ for any integer value of $n$:

```python
sage: Mod(10, 0)
10
sage: a = randint(-100, 100)
sage: Mod(a, 0) == a
True
```
sage.rings.finite_rings.integer_mod.square_root_mod_prime(a, p=None)
Calculates the square root of \( a \), where \( a \) is an integer mod \( p \); if \( a \) is not a perfect square, this returns an (incorrect) answer without checking.

ALGORITHM: Several cases based on residue class of \( p \) mod 16.

- \( p \) mod 2 = 0: \( p = 2 \) so \( \sqrt{a} = a \).
- \( p \) mod 4 = 3: \( \sqrt{a} = a^{(p+1)/4} \).
- \( p \) mod 8 = 5: \( \sqrt{a} = \zeta i a \) where \( \zeta = (2a)^{(p-5)/8}, i = \sqrt{-1} \).
- \( p \) mod 16 = 9: Similar, work in a bi-quadratic extension of \( \mathbb{F}_p \) for small \( p \), Tonelli and Shanks for large \( p \).
- \( p \) mod 16 = 1: Tonelli and Shanks.

REFERENCES:
- [Mul2004]
- [Atk1992]
- [Pos1988]

AUTHORS:
- Robert Bradshaw

sage.rings.finite_rings.integer_mod.square_root_mod_prime_power(a, p, e)
Calculates the square root of \( a \), where \( a \) is an integer mod \( p^e \).

ALGORITHM: Compute \( p \)-adically by stripping off even powers of \( p \) to get a unit and lifting \( \sqrt{\text{unit}} \) mod \( p \) via Newton’s method whenever \( p \) is odd and by a variant of Hensel lifting for \( p = 2 \).

AUTHORS:
- Robert Bradshaw
- Lorenz Panny (2022): polynomial-time algorithm for \( p = 2 \)

EXAMPLES:

```
sage: from sage.rings.finite_rings.integer_mod import square_root_mod_prime_power
sage: a = Mod(17,2^20)
sage: b = square_root_mod_prime_power(a,2,20)
sage: b^2 == a
True
```

```
sage: a = Mod(72,97^10)
sage: b = square_root_mod_prime_power(a,97,10)
sage: b^2 == a
True
```

```
sage: mod(100, 5^7).sqrt()^2
100
```
2.1 Finite Fields

Sage supports arithmetic in finite prime and extension fields. Several implementation for prime fields are implemented natively in Sage for several sizes of primes \( p \). These implementations are

- `sage.rings.finite_rings.integer_mod.IntegerMod_int`,
- `sage.rings.finite_rings.integer_mod.IntegerMod_int64`, and
- `sage.rings.finite_rings.integer_mod.IntegerMod_gmp`.

Small extension fields of cardinality \(< 2^{16}\) are implemented using tables of Zech logs via the Givaro C++ library (`sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro`). While this representation is very fast it is limited to finite fields of small cardinality. Larger finite extension fields of order \( q > 2^{16}\) are internally represented as polynomials over smaller finite prime fields. If the characteristic of such a field is 2 then NTL is used internally to represent the field (`sage.rings.finite_rings.finite_field_ntl_gf2e.FiniteField_ntl_gf2e`). In all other case the PARI C library is used (`sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt`).

However, this distinction is internal only and the user usually does not have to worry about it because consistency across all implementations is aimed for. In all extension field implementations the user may either specify a minimal polynomial or leave the choice to Sage.

For small finite fields the default choice are Conway polynomials. The Conway polynomial \( C_n \) is the lexicographically first monic irreducible, primitive polynomial of degree \( n \) over \( GF(p) \) with the property that for a root \( \alpha \) of \( C_n \) we have that \( \beta = \alpha^{(p^n-1)/(p^m-1)} \) is a root of \( C_m \) for all \( m \) dividing \( n \).

Sage contains a database of Conway polynomials which also can be queried independently of finite field construction.

A pseudo-Conway polynomial satisfies all of the conditions required of a Conway polynomial except the condition that it is lexicographically first. They are therefore not unique. If no variable name is specified for an extension field, Sage will fit the finite field into a compatible lattice of field extensions defined by pseudo-Conway polynomials. This lattice is stored in an `AlgebraicClosureFiniteField` object; different algebraic closure objects can be created by using a different prefix keyword to the finite field constructor.

Note that the computation of pseudo-Conway polynomials is expensive when the degree is large and highly composite. If a variable name is specified then a random polynomial is used instead, which will be much faster to find.

While Sage supports basic arithmetic in finite fields some more advanced features for computing with finite fields are still not implemented. For instance, Sage does not calculate embeddings of finite fields yet.

EXAMPLES:

```python
sage: k = GF(5); type(k)
<class 'sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn_with_category'>
```
sage: k = GF(5^2, 'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro_with_category'>

One can also give the cardinality \( q = p^n \) as the tuple \((p, n)\):

sage: k = GF((5, 2), 'c'); k
Finite Field in c of size 5^2
sage: k = GF(2^16, 'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_ntl_gf2e.FiniteField_ntl_gf2e_with_category'>

sage: k = GF((3, 16), 'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt_with_category'>

Finite Fields support iteration, starting with 0.

sage: k = GF(9, 'a')
sage: for i,x in enumerate(k): print("{} {}\n0 0
1 a
2 a + 1
3 2*a + 1
4 2
5 2*a
6 2*a + 2
7 a + 2
8 1
sage: for a in GF(5):
....: print(a)
0
1
2
3
4

We output the base rings of several finite fields.

sage: k = GF(3); type(k)
<class 'sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn_with_category'>
sage: k.base_ring()
Finite Field of size 3

sage: k = GF(9, 'alpha'); type(k)
<class 'sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro_with_category'>
sage: k.base_ring()
Finite Field of size 3
sage: k = GF((3, 40), 'b'); type(k)
<class 'sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt_with_category'>
sage: k.base_ring()
Finite Field of size 3

Further examples:

sage: GF(2).is_field()
True
sage: GF(next_prime(10^20)).is_field()
True
sage: GF(19^20, 'a').is_field()
True
sage: GF(8, 'a').is_field()
True

AUTHORS:

- William Stein: initial version
- Robert Bradshaw: prime field implementation
- Martin Albrecht: Givaro and ntl.GF2E implementations

class sage.rings.finite_rings.finite_field_constructor.FiniteFieldFactory(*args, **kwds)

Bases: sage.structure.factory.UniqueFactory

Return the globally unique finite field of given order with generator labeled by the given name and possibly with given modulus.

INPUT:

- order – a prime power
- name – string, optional. Note that there can be a substantial speed penalty (in creating extension fields) when omitting the variable name, since doing so triggers the computation of pseudo-Conway polynomials in order to define a coherent lattice of extensions of the prime field. The speed penalty grows with the size of extension degree and with the number of factors of the extension degree.
- modulus – (optional) either a defining polynomial for the field, or a string specifying an algorithm to use to generate such a polynomial. If modulus is a string, it is passed to irreducible_element() as the parameter algorithm; see there for the permissible values of this parameter. In particular, you can specify modulus="primitive" to get a primitive polynomial. You may not specify a modulus if you do not specify a variable name.
- impl – (optional) a string specifying the implementation of the finite field. Possible values are:
  - 'modn' – ring of integers modulo $p$ (only for prime fields).
  - 'givaro' – Givaro, which uses Zech logs (only for fields of at most 65521 elements).
  - 'ntl' – NTL using GF2X (only in characteristic 2).
  - 'pari' or 'pari_ffelt' – PARI’s FFELT type (only for extension fields).
- elem_cache – (default: order < 500) cache all elements to avoid creation time; ignored unless impl='givaro'
- repr – (default: 'poly') ignored unless impl='givaro'; controls the way elements are printed to the user:
Finite Rings, Release 9.7

- ‘log’: repr is log_repr()
- ‘int’: repr is int_repr()
- ‘poly’: repr is poly_repr()

- check_irreducible – verify that the polynomial modulus is irreducible
- proof – bool (default: True): if True, use provable primality test; otherwise only use pseudoprimality test.

ALIAS: You can also use GF instead of FiniteField – they are identical.

EXAMPLES:

```
sage: k.<a> = FiniteField(9); k
Finite Field in a of size 3^2
sage: parent(a)
Finite Field in a of size 3^2
sage: charpoly(a, 'y')
y^2 + 2*y + 2
```

We illustrate the proof flag. The following example would hang for a very long time if we didn’t use proof=False.

Note: Magma only supports proof=False for making finite fields, so falsely appears to be faster than Sage – see trac ticket #10975.

```
sage: k = FiniteField(10^1000 + 453, proof=False)
sage: k = FiniteField((10^1000 + 453)^2, 'a', proof=False) # long time -- about 5 seconds
```

```
sage: F.<x> = GF(5)[]
sage: K.<a> = GF(5**5, name='a', modulus=x^5 - x +1 )
sage: f = K.modulus(); f
x^5 + 4*x + 1
sage: type(f)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
```

By default, the given generator is not guaranteed to be primitive (a generator of the multiplicative group), use modulus="primitive" if you need this:

```
sage: K.<a> = GF(5^45)
sage: a.multiplicative_order()
7105427357601001858711242675781
sage: a.is_square()
True
sage: b.<b> = GF(5^45, modulus="primitive")
sage: b.multiplicative_order()
2842170943040400047434844970703124
```

The modulus must be irreducible:

```
sage: K.<a> = GF(5**5, name='a', modulus=x^5 - x)
Traceback (most recent call last):
```

(continues on next page)
ValueError: finite field modulus must be irreducible but it is not

You can’t accidentally fool the constructor into thinking the modulus is irreducible when it is not, since it actually tests irreducibility modulo $p$. Also, the modulus has to be of the right degree (this is always checked):

```
sage: F.<x> = QQ[]
sage: factor(x^5 + 2)
x^5 + 2
sage: K.<a> = GF(5^5, modulus=x^5 + 2)
Traceback (most recent call last):
...  
ValueError: finite field modulus must be irreducible but it is not
sage: K.<a> = GF(5^5, modulus=x^3 + 3*x + 3, check_irreducible=False)
Traceback (most recent call last):
...  
ValueError: the degree of the modulus does not equal the degree of the field
```

Any type which can be converted to the polynomial ring $\mathbb{G}F(p)[x]$ is accepted as modulus:

```
sage: K.<a> = GF(13^3, modulus=[1,0,0,2])
sage: K.<a> = GF(13^10, modulus=pari("ffinit(13,10)"))
sage: var('x')
x
sage: K.<a> = GF(13^2, modulus=x^2 - 2)
sage: K.<a> = GF(13^2, modulus=sin(x))
Traceback (most recent call last):
...  
TypeError: self must be a numeric expression
```

If you wish to live dangerously, you can tell the constructor not to test irreducibility using `check_irreducible=False`, but this can easily lead to crashes and hangs – so do not do it unless you know that the modulus really is irreducible!

```
sage: K.<a> = GF(5**2, name='a', modulus=x^2 + 2, check_irreducible=False)
```

Even for prime fields, you can specify a modulus. This will not change how Sage computes in this field, but it will change the result of the `modulus()` and `gen()` methods:

```
sage: k.<a> = GF(5, modulus="primitive")
sage: k.modulus()
x + 3
sage: a
2
```

The order of a finite field must be a prime power:

```
sage: GF(1)
Traceback (most recent call last):
...
ValueError: the order of a finite field must be at least 2
sage: GF(100)
Traceback (most recent call last):
```

(continues on next page)
Finite fields with explicit random modulus are not cached:

```python
sage: k.<a> = GF(5**10, modulus='random')
sage: n.<a> = GF(5**10, modulus='random')
sage: while k.modulus() == n.modulus():
    ....:     n.<a> = GF(5**10, modulus='random')
sage: n is k
False
sage: GF(5**10, 'a') is GF(5**10, 'a')
True
```

We check that various ways of creating the same finite field yield the same object, which is cached:

```python
sage: K = GF(7, 'a')
sage: L = GF(7, 'b')
sage: K is L          # name is ignored for prime fields
True
sage: K is GF(7, modulus=K.modulus())
True
sage: K = GF(4,'a'); K.modulus()
x^2 + x + 1
sage: L = GF(4,'a', K.modulus())
sage: K is L
True
sage: M = GF(4,'a', K.modulus().change_variable_name('y'))
sage: K is M
True
```

You may print finite field elements as integers. This currently only works if the order of field is < $2^{16}$, though:

```python
sage: k.<a> = GF(2^8, repr='int')
sage: a
2
```

The following demonstrate coercions for finite fields using Conway polynomials:

```python
sage: k = GF(5^2); a = k.gen()
sage: l = GF(5^5); b = l.gen()
sage: a + b
3*z10^5 + z10^4 + z10^2 + 3*z10 + 1
```

Note that embeddings are compatible in lattices of such finite fields:

```python
sage: m = GF(5^3); c = m.gen()
sage: (a+b)+c == a+(b+c)
True
sage: (a*b)*c == a*(b*c)
True
sage: from sage.categories.pushout import pushout
sage: n = pushout(k, l)
```

(continues on next page)
sage: o = pushout(l, m)
sage: q = pushout(n, o)
sage: q(o(b)) == q(n(b))
True

Another check that embeddings are defined properly:

sage: k = GF(3**10)
sage: l = GF(3**20)
sage: l(k.gen()**10) == l(k.gen())**10
True

Using pseudo-Conway polynomials is slow for highly composite extension degrees:

sage: k = GF(3^120)  # long time -- about 3 seconds
sage: GF(3^40).gen().minimal_polynomial()(k.gen()^((3^120-1)/(3^40-1)))  # long time
 →because of previous line
0

Before trac ticket #17569, the boolean keyword argument conway was required when creating finite fields without a variable name. This keyword argument is now removed (trac ticket #21433). You can still pass in prefix as an argument, which has the effect of changing the variable name of the algebraic closure:

sage: K = GF(3^10, prefix='w'); L = GF(3^10); K is L
False
sage: K.variable_name(), L.variable_name()
('w10', 'z10')
sage: list(K.polynomial()) == list(L.polynomial())
True

create_key_and_extra_args(order, name=None, modulus=None, names=None, impl=None, proof=None, check_prime=True, check_irreducible=True, prefix=None, repr=None, elem_cache=None, **kwds)

EXAMPLES:

sage: GF.create_key_and_extra_args(9, 'a')
((9, ('a',), x^2 + 2*x + 2, 'givaro', 3, 2, True, None, 'poly', True, True, →True), {})  

The order $q$ can also be given as a pair $(p, n)$:

sage: GF.create_key_and_extra_args((3, 2), 'a')
((9, ('a',), x^2 + 2*x + 2, 'givaro', 3, 2, True, None, 'poly', True, True, →True), {})  

We do not take invalid keyword arguments and raise a value error to better ensure uniqueness:

sage: GF.create_key_and_extra_args(9, 'a', foo='value')
Traceback (most recent call last):
...
TypeError: ...create_key_and_extra_args() got an unexpected keyword argument →'foo'

Moreover, repr and elem_cache are ignored when not using givaro:
Finite Rings, Release 9.7

```plaintext
sage: GF.create_key_and_extra_args(16, 'a', impl='ntl', repr='poly')
((16, ('a',), x^4 + x + 1, 'ntl', 2, 4, True, None, None, None, True, True), {})
sage: GF.create_key_and_extra_args(16, 'a', impl='ntl', elem_cache=False)
((16, ('a',), x^4 + x + 1, 'ntl', 2, 4, True, None, None, None, True, True), {})
sage: GF(16, impl='ntl') is GF(16, impl='ntl', repr='foo')
True
```

We handle extra arguments for the givaro finite field and create unique objects for their defaults:

```plaintext
sage: GF(25, impl='givaro') is GF(25, impl='givaro', repr='poly')
True
sage: GF(25, impl='givaro') is GF(25, impl='givaro', elem_cache=True)
True
sage: GF(625, impl='givaro') is GF(625, impl='givaro', elem_cache=False)
True
```

We explicitly take structure, implementation and prec attributes for compatibility with `AlgebraicExtensionFunctor` but we ignore them as they are not used, see trac ticket #21433:

```plaintext
create_object(version, key, **kwds)
EXAMPLES:
sage: K = GF(19)  # indirect doctest
tests = TestSuite(K).run()
```

We try to create finite fields with various implementations:

```plaintext
sage: k = GF(2, impl='modn')
sage: k = GF(2, impl='givaro')
sage: k = GF(2, impl='ntl')
Traceback (most recent call last):
... ValueError: the degree must be at least 2
sage: k = GF(2, impl='supercalifragilisticexpialidocious')
Traceback (most recent call last):
... ValueError: no such finite field implementation:
  'supercalifragilisticexpialidocious'
sage: k.<<a>> = GF(2^15, impl='modn')
Traceback (most recent call last):
... ValueError: the 'modn' implementation requires a prime order
sage: k.<<a>> = GF(2^15, impl='givaro')
sage: k.<<a>> = GF(2^15, impl='ntl')
sage: k.<<a>> = GF(2^15, impl='pari')
sage: k.<<a>> = GF(3^60, impl='modn')
Traceback (most recent call last):
```

(continues on next page)
sage.rings.finite_rings.finite_field_constructor.is_PrimeFiniteField(x)
Returns True if x is a prime finite field.

EXAMPLES:

```python
sage: from sage.rings.finite_rings.finite_field_constructor import is_PrimeFiniteField
sage: is_PrimeFiniteField(QQ)
False
sage: is_PrimeFiniteField(GF(7))
True
sage: is_PrimeFiniteField(GF(7^2, 'a'))
False
sage: is_PrimeFiniteField(GF(next_prime(10^90, proof=False)))
True
```

2.2 Base Classes for Finite Fields

AUTHORS:

- Adrien Brochard, David Roe, Jeroen Demeyer, Julian Rueth, Niles Johnson, Peter Bruin, Travis Scrimshaw, Xavier Caruso: initial version

```python
class sage.rings.finite_rings.finite_field_base.FiniteField
    Bases: sage.rings.ring.Field

Abstract base class for finite fields.

algebraic_closure(name='z', **kwds)
    Return an algebraic closure of self.

INPUT:

- name - string (default: 'z'): prefix to use for variable names of subfields

- implementation - string (optional): specifies how to construct the algebraic closure.
    
    The only value supported at the moment is 'pseudo_conway'. For more details, see
    `algebraic_closure_finite_field`.

OUTPUT:

An algebraic closure of self. Note that mathematically speaking, this is only unique up to non-unique
isomorphism. To obtain canonically defined algebraic closures, one needs an algorithm that also provides
a canonical isomorphism between any two algebraic closures constructed using the algorithm.
```
This non-uniqueness problem can in principle be solved by using *Conway polynomials*; see for example Wikipedia article *Conway_polynomial_(finite_fields)*. These have the drawback that computing them takes a long time. Therefore Sage implements a variant called *pseudo-Conway polynomials*, which are easier to compute but do not determine an algebraic closure up to unique isomorphism.

The output of this method is cached, so that within the same Sage session, calling it multiple times will return the same algebraic closure (i.e. the same Sage object). Despite this, the non-uniqueness of the current implementation means that coercion and pickling cannot work as one might expect. See below for an example.

**EXAMPLES:**

```
sage: F = GF(5).algebraic_closure()
sage: F
Algebraic closure of Finite Field of size 5
sage: F.gen(3)
z3
```

The default name is ‘z’ but you can change it through the option `name`:

```
sage: Ft = GF(5).algebraic_closure('t')
sage: Ft.gen(3)
t3
```

Because Sage currently only implements algebraic closures using a non-unique definition (see above), it is currently impossible to implement pickling in such a way that a pickled and unpickled element compares equal to the original:

```
sage: F = GF(7).algebraic_closure()
sage: x = F.gen(2)
sage: loads(dumps(x)) == x
False
```

**Note:** This is currently only implemented for prime fields.

**cardinality()**

Return the cardinality of `self`.

Same as `order()`.

**EXAMPLES:**

```
sage: GF(997).cardinality()
997
```

**construction()**

Return the construction of this finite field, as a `ConstructionFunctor` and the base field.

**EXAMPLES:**

```
sage: v = GF(3^3).construction(); v
(AlgebraicExtensionFunctor, Finite Field of size 3)
sage: v[0].polys[0]
3
sage: v = GF(2^1000, 'a').construction(); v[0].polys[0]
a^1000 + a^5 + a^4 + a^3 + 1
```
The implementation is taken into account, by trac ticket #15223:

```python
sage: k = FiniteField(9, 'a', impl='pari_ffelt')
sage: F, R = k.construction()
sage: F(R) is k
True
```

dual_basis(basis=None, check=True)

Return the dual basis of basis, or the dual basis of the power basis if no basis is supplied.

If \( e = \{e_0, e_1, ..., e_{n-1}\} \) is a basis of \( F_{p^n} \) as a vector space over \( F_p \), then the dual basis of \( e, d = \{d_0, d_1, ..., d_{n-1}\} \), is the unique basis such that \( \text{Tr}(e_i d_j) = \delta_{i,j}, 0 \leq i, j \leq n - 1 \), where \( \text{Tr} \) is the trace function.

**INPUT:**

- `basis` – (default: None): a basis of the finite field self, \( F_{p^n} \), as a vector space over the base field \( F_p \). Uses the power basis \( \{x^i : 0 \leq i \leq n - 1\} \) as input if no basis is supplied, where \( x \) is the generator of self.

- `check` – (default: True): verifies that basis is a valid basis of self.

**ALGORITHM:**

The algorithm used to calculate the dual basis comes from pages 110–111 of [McE1987].

Let \( e = \{e_0, e_1, ..., e_{n-1}\} \) be a basis of \( F_{p^n} \) as a vector space over \( F_p \) and \( d = \{d_0, d_1, ..., d_{n-1}\} \) be the dual basis of \( e \). Since \( e \) is a basis, we can rewrite any \( d_c, 0 \leq c \leq n - 1 \), as \( d_c = \beta_0 e_0 + \beta_1 e_1 + ... + \beta_{n-1} e_{n-1} \), for some \( \beta_0, \beta_1, ..., \beta_{n-1} \in F_p \). Using properties of the trace function, we can rewrite the \( n \) equations of the form \( \text{Tr}(e_i d_c) = \delta_{i,c} \) and express the result as the matrix vector product: \( A[\beta_0, \beta_1, ..., \beta_{n-1}] = i_c \), where the \( i, j \)-th element of \( A \) is \( \text{Tr}(e_i e_j) \) and \( i_c \) is the \( i \)-th column of the \( n \times n \) identity matrix. Since \( A \) is an invertible matrix, \( [\beta_0, \beta_1, ..., \beta_{n-1}] = A^{-1} i_c \), from which we can easily calculate \( d_c \).

**EXAMPLES:**

```python
sage: F.<a> = GF(2^4)
sage: F.dual_basis(basis=None, check=False)
[a^3 + 1, a^2, a, 1]
```

We can test that the dual basis returned satisfies the defining property of a dual basis: \( \text{Tr}(e_i d_j) = \delta_{i,j}, 0 \leq i, j \leq n - 1 \)

```python
sage: F.<<a> = GF(7^4)
sage: e = [4*a^3, 2*a^3 + a^2 + 3*a + 5, ....: 3*a^3 + 5*a^2 + 4*a + 2, 2*a^3 + 2*a^2 + 2]
sage: d = F.dual_basis(e, check=True); d
[3*a^3 + 4*a^2 + 6*a + 2, a^3 + 6*a^2 + 2*a + 5, 3*a^3 + 6*a^2 + 2*a + 5, 5*a^2 + 4*a + 3]
sage: vals = 
[(x * y).trace() for x in e] for y in d]
sage: matrix(vals) == matrix.identity(4)
True
```

We can test that if \( d \) is the dual basis of \( e \), then \( e \) is the dual basis of \( d \):

```python
sage: F.<<a> = GF(7^8)
sage: e = [a^0, a^1, a^2, a^3, a^4, a^5, a^6, a^7]
sage: d = F.dual_basis(e, check=False); d
[6*a^6 + 4*a^5 + 4*a^4 + a^3 + 6*a^2 + 3, 2*a^6 + 4*a^4 + a^3 + 2*a^2 + 2]
sage: matrix(vals) == matrix.identity(4)
```

(continues on next page)
\[ 6a^7 + 4a^6 + 4a^5 + 2a^4 + a^2, \\
4a^6 + 5a^5 + 5a^4 + 4a^3 + 5a^2 + a + 6, \\
5a^7 + a^6 + a^4 + 4a^3 + 4a^2 + 1, \\
2a^7 + 5a^6 + a^5 + a^3 + 5a^2 + 2a + 4, \\
a^7 + 2a^6 + 5a^5 + a^4 + 5a^2 + 4a + 4, \\
a^7 + a^6 + 5a^5 + a^3 + 5a^2 + 2a^4 + 4a^3 + 6a + 6, \\
5a^7 + a^6 + a^5 + 2a^4 + 5a^3 + 6a \]

```
sage: F.dual_basis(d)  
[1, a, a^2, a^3, a^4, a^5, a^6, a^7]  
```

We cannot calculate the dual basis if \( \text{basis} \) is not a valid basis.

```
sage: F.<a> = GF(2^3)  
sage: F.dual_basis([a], check=True)  
Traceback (most recent call last):  
...  
ValueError: basis length should be 3, not 1  
```

```
sage: F.dual_basis([a^0, a, a^0 + a], check=True)  
Traceback (most recent call last):  
...  
ValueError: value of 'basis' keyword is not a basis  
```

AUTHOR:

• Thomas Gagne (2015-06-16)

```
extension(modulus, name=None, names=None, map=False, embedding=None, latex_name=None, latex_names=None, **kwds)  
```

Return an extension of this finite field.

INPUT:

• \( \text{modulus} \) – a polynomial with coefficients in \( \text{self} \), or an integer.

• \( \text{name or names} \) – string: the name of the generator in the new extension

• \( \text{latex_name or latex_names} \) – string: latex name of the generator in the new extension

• \( \text{map} \) – boolean (default: \( \text{False} \)): if \( \text{False} \), return just the extension \( E \); if \( \text{True} \), return a pair \((E, f)\), where \( f \) is an embedding of \( \text{self} \) into \( E \).

• \( \text{embedding} \) – currently not used; for compatibility with other \text{AlgebraicExtensionFunctor} \text{calls}.

• **\text{kwds}**: further keywords, passed to the finite field constructor.

OUTPUT:

An extension of the given modulus, or pseudo-Conway of the given degree if \( \text{modulus} \) is an integer.

EXAMPLES:

```
sage: k = GF(2)  
sage: R.<x> = k[]  
sage: k.extension(x^1000 + x^5 + x^4 + x^3 + 1, 'a')  
Finite Field in a of size 2^1000  
sage: k = GF(3^4)  
sage: R.<x> = k[]  
```

(continues on next page)
sage: k.extension(3)
Finite Field in z12 of size 3^12
sage: K = k.extension(2, 'a')
sage: k.is_subring(K)
True

An example using the map argument:

sage: F = GF(5)
sage: E, f = F.extension(2, 'b', map=True)
sage: E
Finite Field in b of size 5^2
sage: f
Ring morphism:
  From: Finite Field of size 5
  To:   Finite Field in b of size 5^2
  Defn: 1 |--> 1
sage: f.parent()
Set of field embeddings from Finite Field of size 5 to Finite Field in b of size 5^2

Extensions of non-prime finite fields by polynomials are not yet supported: we fall back to generic code:

sage: k.extension(x^5 + x^2 + x - 1)
Univariate Quotient Polynomial Ring in x over Finite Field in z4 of size 3^4 with modulus x^5 + x^2 + x + 2

factored_order()
Returns the factored order of this field. For compatibility with integer_mod_ring.

EXAMPLES:

sage: GF(7^2,'a').factored_order()
7^2

factored_unit_order()
Returns the factorization of self.order()-1, as a 1-tuple.
The format is for compatibility with integer_mod_ring.

EXAMPLES:

sage: GF(7^2,'a').factored_unit_order()
(2^4 * 3,)

fetch_int(n)
Return the element of self that equals n under the condition that gen() is set to the characteristic of the finite field self.

INPUT:

• n – integer. Must not be negative, and must be less than the cardinality of self.

EXAMPLES:
sage: p = 4091
sage: F = GF(p^4, 'a')
sage: n = 100*p^3 + 37*p^2 + 12*p + 6
sage: F.fetch_int(n)
100*a^3 + 37*a^2 + 12*a + 6
sage: F.fetch_int(n) in F
True

**free_module** *(base=None, basis=None, map=None, subfield=None)*

Return the vector space over the subfield isomorphic to this finite field as a vector space, along with the isomorphisms.

**INPUT:**

- `base` – a subfield of or a morphism into this finite field. If not given, the prime subfield is assumed. A subfield means a finite field with coercion to this finite field.
- `basis` – a basis of the finite field as a vector space over the subfield. If not given, one is chosen automatically.
- `map` – boolean (default: True); if True, isomorphisms from and to the vector space are also returned.

The `basis` maps to the standard basis of the vector space by the isomorphisms.

**OUTPUT:** if `map` is False,

- vector space over the subfield or the domain of the morphism, isomorphic to this finite field.

and if `map` is True, then also

- an isomorphism from the vector space to the finite field.
- the inverse isomorphism to the vector space from the finite field.

**EXAMPLES:**

```python
sage: GF(27, 'a').vector_space(map=False)
Vector space of dimension 3 over Finite Field of size 3
sage: F = GF(8)
sage: E = GF(64)
sage: V, from_V, to_V = E.vector_space(F, map=True)
sage: V
Vector space of dimension 2 over Finite Field in z3 of size 2^3
sage: to_V(E.gen())
(0, 1)
sage: all(from_V(to_V(e)) == e for e in E)
True
sage: all(to_V(e1 + e2) == to_V(e1) + to_V(e2) for e1 in E for e2 in E)
True
sage: all(to_V(c * e) == c * to_V(e) for e in E for c in F)
True
sage: basis = [E.gen(), E.gen() + 1]
sage: W, from_W, to_W = E.vector_space(F, basis, map=True)
sage: all(from_W(to_W(e)) == e for e in E)
True
sage: all(to_W(c * e) == c * to_W(e) for e in E for c in F)
```

(continues on next page)
True
sage: all(to_W(e1 + e2) == to_W(e1) + to_W(e2) for e1 in E for e2 in E)  # long...
˓→time
True
sage: to_W(basis[0]); to_W(basis[1])
(1, 0)
(0, 1)

frobenius_endomorphism(n=1)

INPUT:

• \(n\) – an integer (default: 1)

OUTPUT:

The \(n\)-th power of the absolute arithmetic Frobenius endomorphism on this finite field.

EXAMPLES:

sage: k.<t> = GF(3^5)
sage: Frob = k.frobenius_endomorphism(); Frob
Frobenius endomorphism t |--> t^3 on Finite Field in t of size 3^5

sage: a = k.random_element()
sage: Frob(a) == a^3
True

We can specify a power:

sage: k.frobenius_endomorphism(2)
Frobenius endomorphism t |--> t^(3^2) on Finite Field in t of size 3^5

The result is simplified if possible:

sage: k.frobenius_endomorphism(6)
Frobenius endomorphism t |--> t^3 on Finite Field in t of size 3^5
sage: k.frobenius_endomorphism(5)
Identity endomorphism of Finite Field in t of size 3^5

Comparisons work:
AUTHOR:

- Xavier Caruso (2012-06-29)

galois_group()

Return the Galois group of this finite field, a cyclic group generated by Frobenius.

EXAMPLES:

```python
sage: G = GF(3^6).galois_group()
sage: G
Galois group C6 of GF(3^6)
sage: F = G.gen()
sage: F^2
Frob^2
sage: F^6
1
```

gen()

Return a generator of this field (over its prime field). As this is an abstract base class, this just raises a `NotImplementedError`.

EXAMPLES:

```python
sage: K = GF(17)
sage: sage.rings.finite_rings.finite_field_base.FiniteField.gen(K)
Traceback (most recent call last):
...  
NotImplementedError
```

is_conway()

Return `True` if self is defined by a Conway polynomial.

EXAMPLES:

```python
sage: GF(5^3, 'a').is_conway()  
True
sage: GF(5^3, 'a', modulus='adleman-lenstra').is_conway()  
False
sage: GF(next_prime(2^16, 2), 'a').is_conway()  
False
```

is_field(proof=True)

Returns whether or not the finite field is a field, i.e., always returns `True`.

EXAMPLES:

```python
sage: k.<a> = FiniteField(3^4)
sage: k.is_field()
True
```
**is_perfect()**
Return whether this field is perfect, i.e., every element has a $p$-th root. Always returns True since finite fields are perfect.

**EXAMPLES:**

```
sage: GF(2).is_perfect()
True
```

**is_prime_field()**
Return True if self is a prime field, i.e., has degree 1.

**EXAMPLES:**

```
sage: GF(3^7, 'a').is_prime_field()
False
sage: GF(3, 'a').is_prime_field()
True
```

**modulus()**
Return the minimal polynomial of the generator of self over the prime finite field.

The minimal polynomial of an element $a$ in a field is the unique monic irreducible polynomial of smallest degree with coefficients in the base field that has $a$ as a root. In finite field extensions, $\mathbb{F}_{p^n}$, the base field is $\mathbb{F}_p$.

**OUTPUT:**

- a monic polynomial over $\mathbb{F}_p$ in the variable $x$.

**EXAMPLES:**

```
sage: F.<a> = GF(7^2); F
Finite Field in a of size 7^2
sage: F.polynomial_ring()
Univariate Polynomial Ring in a over Finite Field of size 7
sage: f = F.modulus(); f
x^2 + 6*x + 3
sage: f(a)
0
```

Although $f$ is irreducible over the base field, we can double-check whether or not $f$ factors in $F$ as follows. The command $F[\text{\textbf{x}'}](f)$ coerces $f$ as a polynomial with coefficients in $F$. (Instead of a polynomial with coefficients over the base field.)

```
sage: f.factor()
x^2 + 6*x + 3
sage: F[\text{\textbf{x}'}](f).factor()
(x + a + 6) * (x + 6*a)
```

Here is an example with a degree 3 extension:

```
sage: G.<b> = GF(7^3); G
Finite Field in b of size 7^3
sage: g = G.modulus(); g
x^3 + 6*x^2 + 4
sage: g.degree(); G.degree()
```
For prime fields, this returns $x - 1$ unless a custom modulus was given when constructing this field:

```sage
sage: k = GF(199)
sage: k.modulus()
x + 198
sage: var('x')
x
sage: k = GF(199, modulus=x+1)
sage: k.modulus()
x + 1
```

The given modulus is always made monic:

```sage
sage: k.<a> = GF(7^2, modulus=2*x^2-3, impl="pari_ffelt")
sage: k.modulus()
x^2 + 2
```

**multiplicative_generator()**

Return a primitive element of this finite field, i.e. a generator of the multiplicative group.

You can use `multiplicative_generator()` or `primitive_element()`, these mean the same thing.

**Warning:** This generator might change from one version of Sage to another.

**EXAMPLES:**

```sage
sage: k = GF(997)
sage: k.multiplicative_generator()
7
sage: k.<a> = GF(11^3)
sage: k.primitive_element()
a
sage: k.<b> = GF(19^32)
sage: k.multiplicative_generator()
b + 4
```

**ngens()**

The number of generators of the finite field. Always 1.

**EXAMPLES:**

```sage
sage: k = FiniteField(3^4, 'b')
sage: k.ngens()
1
```

**order()**

Return the order of this finite field.

**EXAMPLES:**

```sage
```
polynomial(\texttt{name=None})

Return the minimal polynomial of the generator of \texttt{self} over the prime finite field.

\begin{itemize}
\item \texttt{name} – a variable name to use for the polynomial. By default, use the name given when constructing this field.
\end{itemize}

\textbf{OUTPUT}:
\begin{itemize}
\item a monic polynomial over \(\mathbb{F}_p\) in the variable \texttt{name}.
\end{itemize}

\textbf{See also:}

Except for the \texttt{name} argument, this is identical to the \texttt{modulus()} method.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: k.<a> = FiniteField(9)
sage: k.polynomial('x')
x^2 + 2*x + 2
sage: k.polynomial()
a^2 + 2*a + 2
sage: F = FiniteField(9, 'a', impl='pari_ffelt')
sage: F.polynomial()
a^2 + 2*a + 2
sage: F = FiniteField(7^20, 'a', impl='pari_ffelt')
sage: f = F.polynomial(); f
a^20 + a^10 + a^9 + a^7 + a^6 + a^5 + a^4 + a + 1
sage: f(F.gen())
0
sage: k.<a> = GF(2^20, impl='ntl')
sage: k.polynomial()
a^20 + a^10 + a^9 + a^7 + a^6 + a^5 + a^4 + a + 1
sage: k.polynomial('FOO')
FOO^20 + FOO^10 + FOO^9 + FOO^7 + FOO^6 + FOO^5 + FOO^4 + FOO + 1
sage: a^20
a^10 + a^9 + a^7 + a^6 + a^5 + a^4 + a + 1
\end{verbatim}

polynomial\_ring(\texttt{variable\_name=None})

Returns the polynomial ring over the prime subfield in the same variable as this finite field.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: k.<alpha> = FiniteField(3^4)
sage: k.polynomial_ring()
Univariate Polynomial Ring in alpha over Finite Field of size 3
\end{verbatim}

\textbf{primitive\_element()}\n
Return a primitive element of this finite field, i.e. a generator of the multiplicative group.
You can use `multiplicative_generator()` or `primitive_element()`, these mean the same thing.

**Warning:** This generator might change from one version of Sage to another.

**EXAMPLES:**

```python
sage: k = GF(997)
sage: k.multiplicative_generator()
7
sage: k.<a> = GF(11^3)
sage: k.primitive_element()
a
sage: k.<b> = GF(19^32)
sage: k.multiplicative_generator()
b + 4
```

**random_element(***args, **kwds)**

A random element of the finite field. Passes arguments to `random_element()` function of underlying vector space.

**EXAMPLES:**

```python
sage: k = GF(19^4, 'a')
sage: k.random_element().parent() is k
True
```

Passes extra positional or keyword arguments through:

```python
sage: k.random_element(prob=0)
0
```

**some_elements()**

Returns a collection of elements of this finite field for use in unit testing.

**EXAMPLES:**

```python
sage: k = GF(2^8, 'a')
sage: k.some_elements() # random output
[a^4 + a^3 + 1, a^6 + a^4 + a^3, a^5 + a^4 + a, a^2 + a]
```

**subfield**(degree, name=None, map=False)

Return the subfield of the field of degree.

The inclusion maps between these subfields will always commute, but they are only added as coercion maps if the following condition holds for the generator \( g \) of the field, where \( d \) is the degree of this field over the prime field:

The element \( g^{(p^d-1)/(p^n-1)} \) generates the subfield of degree \( n \) for all divisors \( n \) of \( d \).

**INPUT:**

- **degree** – integer; degree of the subfield
- **name** – string; name of the generator of the subfield
- **map** – boolean (default False); whether to also return the inclusion map

**EXAMPLES:**
```
sage: k = GF(2^21)
sage: k.subfield(3)
Finite Field in z3 of size 2^3
sage: k.subfield(7, 'a')
Finite Field in a of size 2^7
sage: k.coerce_map_from(_)
Ring morphism:
  From: Finite Field in a of size 2^7
  To:   Finite Field in z21 of size 2^21
  Defn: a |--> z21^20 + z21^19 + z21^17 + z21^15 + z21^14 + z21^11 + z21^9 + z21^8 + z21^6 + z21^5 + z21^4 + z21^3 + z21
sage: k.subfield(8)
Traceback (most recent call last):
  ...:
ValueError: no subfield of order 2^8
```

```
subfields(degree=0, name=None)
Return all subfields of self of the given degree, or all possible degrees if degree is 0.

The subfields are returned as absolute fields together with an embedding into self.

INPUT:

- degree – (default: 0) an integer
- name – a string, a dictionary or None:
  - If degree is nonzero, then name must be a string (or None, if this is a pseudo-Conway extension), and will be the variable name of the returned field.
  - If degree is zero, the dictionary should have keys the divisors of the degree of this field, with the desired variable name for the field of that degree as an entry.
  - As a shortcut, you can provide a string and the degree of each subfield will be appended for the variable name of that subfield.
  - If None, uses the prefix of this field.

OUTPUT:

A list of pairs (K, e), where K ranges over the subfields of this field and e gives an embedding of K into self.

EXAMPLES:

```
sage: k = GF(2^21)
sage: k.subfields()
[(Finite Field of size 2,
  Ring morphism:
    From: Finite Field of size 2
    To:   Finite Field in z21 of size 2^21
    Defn: 1 |--> 1),
  (Finite Field in z3 of size 2^3,
  Ring morphism:
    From: Finite Field in z3 of size 2^3
    To:   Finite Field in z21 of size 2^21
    Defn: z3 |--> z21^20 + z21^19 + z21^17 + z21^15 + z21^14 + z21^11 + z21^9 + z21^8 + z21^6 + z21^2),
```
```
(Finite Field in z7 of size 2^7,
Ring morphism:
   From: Finite Field in z7 of size 2^7
   To: Finite Field in z21 of size 2^21
   Defn: z7 |--> z21^20 + z21^19 + z21^17 + z21^15 + z21^14 + z21^6 + z21^4 + z21^3 + z21),
(Finite Field in z21 of size 2^21,
Identity endomorphism of Finite Field in z21 of size 2^21))

unit_group_exponent()

The exponent of the unit group of the finite field. For a finite field, this is always the order minus 1.

EXAMPLES:

```sage
sage: k = GF(2^10, 'a')
sage: k.order()
1024
sage: k.unit_group_exponent()
1023
```

zeta(n=None)

Return an element of multiplicative order n in this finite field. If there is no such element, raise ValueError.

**Warning:** In general, this returns an arbitrary element of the correct order. There are no compatibility guarantees: \( F.zeta(9)^3 \) may not be equal to \( F.zeta(3) \).

EXAMPLES:

```sage
sage: k = GF(7)
sage: k.zeta()
3
sage: k.zeta().multiplicative_order()
6
sage: k.zeta(3)
2
sage: k.zeta(3).multiplicative_order()
3
sage: k = GF(49, 'a')
sage: k.zeta().multiplicative_order()
48
sage: k.zeta(6)
3
sage: k.zeta(5)
Traceback (most recent call last):
...
ValueError: no 5th root of unity in Finite Field in a of size 7^2
```

Even more examples:

```sage
sage: GF(9,'a').zeta_order()
8
```
This works even in very large finite fields, provided that $n$ can be factored (see trac ticket #25203):

```
sage: k.<a> = GF(2^2000)
sage: p = 8877945148742945001146041439025147034098690503591013177336356694416517527310181938001
sage: z = k.zeta(p)
sage: z
a^1999 + a^1996 + a^1995 + a^1994 + ... + a^7 + a^5 + a^4 + 1
sage: z^p
1
```

### `zeta_order()`

Return the order of the distinguished root of unity in `self`.

**EXAMPLES:**

```
sage: GF(9,'a').zeta_order()
8
sage: GF(9,'a').zeta()
a
sage: GF(9,'a').zeta().multiplicative_order()
8
```

`sage.rings.finite_rings.finite_field_base.is_FiniteField(R)`

Return whether the implementation of `R` has the interface provided by the standard finite field implementation.

**EXAMPLES:**

```
sage: from sage.rings.finite_rings.finite_field_base import is_FiniteField
sage: is_FiniteField(GF(9,'a'))
True
sage: is_FiniteField(GF(next_prime(10^10)))
True
```

Note that the integers modulo $n$ are not backed by the finite field type:

```
sage: is_FiniteField(Integers(7))
False
```

`sage.rings.finite_rings.finite_field_base.unpickle_FiniteField_ext(_type, order, variable_name, modulus, kwargs)`

Used to unpickle extensions of finite fields. Now superseded (hence no doctest), but kept around for backward compatibility.

`sage.rings.finite_rings.finite_field_base.unpickle_FiniteField_prm(_type, order, variable_name, kwargs)`

Used to unpickle finite prime fields. Now superseded (hence no doctest), but kept around for backward compatibility.

### 2.2. Base Classes for Finite Fields
2.3 Base class for finite field elements

AUTHORS:

• David Roe (2010-01-14): factored out of sage.structure.element
• Sebastian Oehms (2018-07-19): added conjugate() (see trac ticket #26761)

class sage.rings.finite_rings.element_base.Cache_base
    Bases: sage.structure.sage_object.SageObject

    fetch_int(number)
    Given an integer less than \(p^n\) with base 2 representation \(a_0 + a_1 \cdot 2 + \cdots + a_k x^k\), this returns \(a_0 + a_1 x + \cdots + a_k x^k\), where \(x\) is the generator of this finite field.

    EXAMPLES:

    sage: k.<a> = GF(2^48)
    sage: k._cache.fetch_int(2^33 + 2 + 1)
a^33 + a + 1

class sage.rings.finite_rings.element_base.FinitePolyExtElement
    Bases: sage.rings.finite_rings.element_base.FiniteRingElement

    Elements represented as polynomials modulo a given ideal.

    additive_order()
    Return the additive order of this finite field element.

    EXAMPLES:

    sage: k.<a> = FiniteField(2^12, 'a')
    sage: b = a^3 + a + 1
    sage: b.additive_order()
    2
    sage: k(0).additive_order()
    1

    charpoly(var='x', algorithm='pari')
    Return the characteristic polynomial of self as a polynomial with given variable.

    INPUT:

    • var – string (default: 'x')
    • algorithm – string (default: 'pari')
      - 'pari' – use pari’s charpoly
      - 'matrix' – return the charpoly computed from the matrix of left multiplication by self

    The result is not cached.

    EXAMPLES:

    sage: from sage.rings.finite_rings.element_base import FinitePolyExtElement
    sage: k.<a> = FiniteField(19^2)
    sage: parent(a)
    Finite Field in a of size 19^2
    sage: b = a**20
The `conjugate()` method returns the result of the Frobenius morphism in the case where the field is a quadratic extension, say $GF(q^2)$, where $q = p^k$ is a prime power and $p$ the characteristic of the field.

**OUTPUT:**

Instance of this class representing the image under the Frobenius morphisms.

**EXAMPLES:**

```python
sage: F.<a> = GF(16)
sage: b = a.conjugate(); b
a + 1
sage: a == b.conjugate()
True
sage: F.<a> = GF(27)
sage: a.conjugate()
Traceback (most recent call last):
...TypeError: cardinality of the field must be a square number
```

The `frobenius()` method returns the $(p^k)^{th}$ power of self, where $p$ is the characteristic of the field.

**INPUT:**

- $k$ – integer (default: 1, must fit in C int type)

Note that if $k$ is negative, then this computes the appropriate root.

**EXAMPLES:**

```python
sage: F.<a> = GF(29^2)
sage: z = a^2 + 5*a + 1
sage: z.pth_power()
19*a + 20
sage: z.pth_power(10)
10*a + 28
sage: z.pth_power(-10) == z
True
sage: F.<b> = GF(2^12)
sage: y = b^3 + b + 1
sage: y == (y.pth_power(-3))^(2^3)
True
```
sage: y.pth_power(2)
b^7 + b^6 + b^5 + b^4 + b^3 + b

\textbf{is\_square()}

Returns True if and only if this element is a perfect square.

\textbf{EXAMPLES:}

\begin{verbatim}sage: k.<a> = FiniteField(9, impl='givaro', modulus='primitive')
sage: a.is_square()
False
sage: (a**2).is_square()
True
\end{verbatim}

\begin{verbatim}sage: k.<a> = FiniteField(4, impl='ntl', modulus='primitive')
sage: (a**2).is_square()
True
\end{verbatim}

\begin{verbatim}sage: k.<a> = FiniteField(17^5, impl='pari_ffelt', modulus='primitive')
sage: a.is_square()
False
sage: (a**2).is_square()
True
\end{verbatim}

\begin{verbatim}sage: k(0).is_square()
True
\end{verbatim}

\textbf{list()}

Return the list of coefficients (in little-endian) of this finite-field element when written as a polynomial in the generator.

Equivalent to calling list() on this element.

\textbf{EXAMPLES:}

\begin{verbatim}sage: x = polygen(GF(71))
sage: F.<u> = GF(71^7, modulus=x^7+x+1)
sage: a = 3 + u + 3*u^2 + 3*u^3 + 7*u^4
sage: a.list()
[3, 1, 3, 3, 7, 0, 0]
sage: a.list() == list(a) == [a[i] for i in range(F.degree())]
True
\end{verbatim}

The coefficients returned are those of a fully reduced representative of the finite-field element:

\begin{verbatim}sage: b = u^777
sage: b.list()
[9, 69, 4, 27, 40, 10, 56]
sage: (u.polynomial()^777).list()
[0, 0, 0, ..., 0, 1]
\end{verbatim}

\textbf{matrix(reverse=False)}

Return the matrix of left multiplication by the element on the power basis $1, x, x^2, \ldots, x^{d-1}$ for the field extension. Thus the columns of this matrix give the images of each of the $x^i$.

\textbf{INPUT:}

- reverse – if True, act on vectors in reversed order
EXAMPLES:

```python
sage: k.<a> = GF(2^4)
sage: b = k.random_element()
sage: vector(a*b) == a.matrix() * vector(b)
True
sage: (a*b).vector_(reverse=True) == a.matrix(reverse=True) * b._vector_
˓
→(reverse=True)
True
```

```python
minimal_polynomial(var='x')
```

Returns the minimal polynomial of this element (over the corresponding prime subfield).

EXAMPLES:

```python
sage: k.<a> = FiniteField(3^4)
sage: parent(a)
Finite Field in a of size 3^4
sage: b=a**20;p=charpoly(b,"y");p
y^4 + 2*y^2 + 1
sage: factor(p)
(y^2 + 1)^2
sage: b.minimal_polynomial('y')
y^2 + 1
```

```python
minpoly(var='x', algorithm='pari')
```

Returns the minimal polynomial of this element (over the corresponding prime subfield).

INPUT:

- `var` - string (default: ‘x’)
- `algorithm` - string (default: ‘pari’)
  - ‘pari’ – use pari’s minpoly
  - ‘matrix’ – return the minpoly computed from the matrix of left multiplication by self

EXAMPLES:

```python
sage: from sage.rings.finite_rings.element_base import FinitePolyExtElement
sage: k.<a> = FiniteField(19^2)
sage: parent(a)
Finite Field in a of size 19^2
sage: b=a**20
sage: p=FinitePolyExtElement.minpoly(b,"x", algorithm="pari")
sage: q=FinitePolyExtElement.minpoly(b,"x", algorithm="matrix")
sage: q == p
True
sage: p
x + 17
```

```python
multiplicative_order()
```

Returns the multiplicative order of this field element.

EXAMPLES:
sage: S.<a> = GF(5^3); S
Finite Field in a of size 5^3
dsage: a.multiplicative_order()
124
dsage: (a^8).multiplicative_order()
31
dsage: S(0).multiplicative_order()
Traceback (most recent call last):
... ArithmeticError: Multiplicative order of 0 not defined.

\textbf{norm}()  
Return the norm of self down to the prime subfield.
This is the product of the Galois conjugates of self.

\textbf{EXAMPLES:}

sage: S.<b> = GF(5^2); S
Finite Field in b of size 5^2
tsage: b.norm()
2
tsage: b.charpoly('t')
t^2 + 4*t + 2

Next we consider a cubic extension:

sage: S.<a> = GF(5^3); S
Finite Field in a of size 5^3
tsage: a.norm()
2
tsage: a.charpoly('t')
t^3 + 3*t + 3
tsage: a * a^5 * (a^25)
2

\textbf{nth\_root}(n, extend=False, all=False, algorithm=None, cunningham=False)
Returns an \(n\)th root of \(self\).

\textbf{INPUT:}

- \(n\) – integer \(\geq 1\)
- \textbf{extend} – bool (default: False); if True, return an \(n\)th root in an extension ring, if necessary. Otherwise, raise a \texttt{ValueError} if the root is not in the base ring. Warning: this option is not implemented!
- \textbf{all} – bool (default: False); if True, return all \(n\)th roots of \(self\), instead of just one.
- \textbf{algorithm} – string (default: None); ‘Johnston’ is the only currently supported option. For \texttt{IntegerMod} elements, the problem is reduced to the prime modulus case using CRT and \(p\)-adic logs, and then this algorithm used.

\textbf{OUTPUT:}

If \(self\) has an \(n\)th root, returns one (if \texttt{all} is False) or a list of all of them (if \texttt{all} is True). Otherwise, raises a \texttt{ValueError} (if \texttt{extend} is False) or a \texttt{NotImplementedError} (if \texttt{extend} is True).
Warning: The extend option is not implemented (yet).

EXAMPLES:

```python
sage: K = GF(31)
sage: a = K(22)
sage: K(22).nth_root(7)
13
sage: K(25).nth_root(5)
5
sage: K(23).nth_root(3)
29

sage: K.<a> = GF(625)
sage: (3*a^2+a+1).nth_root(13)**13
3*a^2 + a + 1

sage: k.<a> = GF(29^2)
sage: b = a^2 + 5*a + 1
sage: b.nth_root(11)
3*a + 20
sage: b.nth_root(5)
Traceback (most recent call last):
  ... ValueError: no nth root
sage: b.nth_root(5, all = True)
[]
sage: b.nth_root(3, all = True)
[14*a + 18, 10*a + 13, 5*a + 27]

sage: k.<a> = GF(29^5)
sage: b = a^2 + 5*a + 1
sage: b.nth_root(5)
19*a^4 + 2*a^3 + 2*a^2 + 16*a + 3
sage: b.nth_root(7)
Traceback (most recent call last):
  ... ValueError: no nth root
sage: b.nth_root(7, all = True)
[]
```

ALGORITHM:
The default is currently an algorithm described in [Joh1999].

AUTHOR:
• David Roe (2010-02-13)

**pth_power** *(k=1)*

Return the \((p^k)^{th}\) power of self, where \(p\) is the characteristic of the field.

**INPUT:**

• \(k\) – integer (default: 1, must fit in C int type)
Note that if $k$ is negative, then this computes the appropriate root.

**EXAMPLES:**

```
sage: F.<a> = GF(29^2)
sage: z = a^2 + 5*a + 1
sage: z.pth_power()
19*a + 20
sage: z.pth_power(10)
10*a + 28
sage: z.pth_power(-10) == z
True
sage: F.<b> = GF(2^12)
sage: y = b^3 + b + 1
sage: y == (y.pth_power(-3))^(2^3)
True
sage: y.pth_power(2)
b^7 + b^6 + b^5 + b^4 + b^3 + b
```

**pth_root** ($k=1$)

Return the $(p^k)^{th}$ root of self, where $p$ is the characteristic of the field.

**INPUT:**

- $k$ – integer (default: 1, must fit in C int type)

Note that if $k$ is negative, then this computes the appropriate power.

**EXAMPLES:**

```
sage: F.<b> = GF(2^12)
sage: y = b^3 + b + 1
sage: y == (y.pth_root(3))^(2^3)
True
sage: y.pth_root(2)
b^11 + b^10 + b^9 + b^7 + b^5 + b^4 + b^2 + b
```

**sqrt** ($extend=False$, $all=False$)

See `square_root()`.

**EXAMPLES:**

```
sage: k.<a> = GF(3^17)
sage: (a^3 - a - 1).sqrt()
a^16 + 2*a^15 + a^13 + 2*a^12 + a^10 + 2*a^9 + 2*a^8 + a^7 + a^6 + 2*a^5 + a^4 +
    2*a^2 + 2*a + 2
```

**square_root** ($extend=False$, $all=False$)

The square root function.

**INPUT:**

- `extend` – bool (default: True); if True, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the root is not in the base ring.

**Warning:** This option is not implemented!
• all – bool (default: False); if True, return all square roots of self, instead of just one.

**Warning:** The 'extend' option is not implemented (yet).

**EXAMPLES:**

```
sage: F = FiniteField(7^2, 'a')
sage: F(2).square_root()
4
sage: F(3).square_root()
2*a + 6
sage: F(3).square_root()**2
3
sage: F(4).square_root()
2
sage: K = FiniteField(7^3, 'alpha', impl='pari_ffelt')
sage: K(3).square_root()
Traceback (most recent call last):
  ... ValueError: must be a perfect square.
```

**trace()**

Return the trace of this element, which is the sum of the Galois conjugates.

**EXAMPLES:**

```
sage: S.<a> = GF(5^3); S
Finite Field in a of size 5^3
sage: a.trace()
0
sage: a.charpoly('t')
t^3 + 3*t + 3
sage: a + a^5 + a^25
0
sage: z = a^2 + a + 1
sage: z.trace()
2
sage: z.charpoly('t')
t^3 + 3*t^2 + 2*t + 2
sage: z + z^5 + z^25
2
```

***class*** `sage.rings.finite_rings.element_base.FiniteRingElement`

Bases: `sage.structure.element.CommutativeRingElement`

`sage.rings.finite_rings.element_base.is_FiniteFieldElement(x)`

Returns if x is a finite field element.

**EXAMPLES:**

```
sage: from sage.rings.finite_rings.element_base import is_FiniteFieldElement
sage: is_FiniteFieldElement(1)
False
sage: is_FiniteFieldElement(IntegerRing())
```

(continues on next page)
2.4 Homset for Finite Fields

This is the set of all field homomorphisms between two finite fields.

**EXAMPLES:**

```
sage: R.<t> = ZZ[]
sage: E.<a> = GF(25, modulus = t^2 - 2)
sage: F.<b> = GF(625)
sage: H = Hom(E, F)
sage: f = H([4*b^3 + 4*b^2 + 4*b]); f
Ring morphism:
  From: Finite Field in a of size 5^2
  To:   Finite Field in b of size 5^4
  Defn: a |--> 4*b^3 + 4*b^2 + 4*b
sage: f(2) 2
sage: f(a)
4*b^3 + 4*b^2 + 4*b
sage: len(H)
2
sage: [phi(2*a)^2 for phi in Hom(E, F)]
[3, 3]
```

We can also create endomorphisms:

```
sage: End(E)
Automorphism group of Finite Field in a of size 5^2
sage: End(GF(7))[0]
Ring endomorphism of Finite Field of size 7
  Defn: 1 |--> 1
sage: H = Hom(GF(7), GF(49, 'c'))
sage: H[0](2) 2
```

```python
class sage.rings.finite_rings.homset.FiniteFieldHomset(R, S, category=None):
    Bases: sage.rings.homset.RingHomset_generic
    Set of homomorphisms with domain a given finite field.

    index(item)
    Return the index of self.

    EXAMPLES:
    ```
sage: K.<z> = GF(1024)
sage: g = End(K)[3]
sage: End(K).index(g) == 3
    True
    ```
```
is_aut()

Check if self is an automorphism

EXAMPLES:

```python
sage: Hom(GF(4, 'a'), GF(16, 'b')).is_aut()
False
sage: Hom(GF(4, 'a'), GF(4, 'c')).is_aut()
False
sage: Hom(GF(4, 'a'), GF(4, 'a')).is_aut()
True
```

list()

Return a list of all the elements in this set of field homomorphisms.

EXAMPLES:

```python
sage: K.<a> = GF(25)
sage: End(K)
Automorphism group of Finite Field in a of size 5^2
sage: list(End(K))
[Ring endomorphism of Finite Field in a of size 5^2
  Defn: a |--> 4*a + 1,
  Ring endomorphism of Finite Field in a of size 5^2
  Defn: a |--> a]
sage: L.<z> = GF(7^6)
sage: [g for g in End(L) if (g^3)(z) == z]
[Ring endomorphism of Finite Field in z of size 7^6
  Defn: z |--> z,
  Ring endomorphism of Finite Field in z of size 7^6
  Defn: z |--> 5*z^4 + 5*z^3 + 4*z^2 + 3*z + 1,
  Ring endomorphism of Finite Field in z of size 7^6
  Defn: z |--> 3*z^5 + 5*z^4 + 5*z^2 + 2*z + 3]
```

Between isomorphic fields with different moduli:

```python
sage: k1 = GF(1009)
sage: k2 = GF(1009, modulus="primitive")
sage: Hom(k1, k2).list()
[
  Ring morphism:
    From: Finite Field of size 1009
    To: Finite Field of size 1009
    Defn: 1 |--> 1
]
sage: Hom(k2, k1).list()
[
  Ring morphism:
    From: Finite Field of size 1009
    To: Finite Field of size 1009
    Defn: 11 |--> 11
]
sage: k1.<a> = GF(1009^2, modulus="first_lexicographic")
sage: k2.<b> = GF(1009^2, modulus="conway")
```

(continues on next page)
sage: Hom(k1, k2).list()
[
  Ring morphism:
  From: Finite Field in a of size 1009^2
  To:   Finite Field in b of size 1009^2
  Defn: a |--> 290*b + 864,
  Ring morphism:
  From: Finite Field in a of size 1009^2
  To:   Finite Field in b of size 1009^2
  Defn: a |--> 719*b + 145
]

order()
Return the order of this set of field homomorphisms.

EXAMPLES:

sage: K.<a> = GF(125)
sage: End(K)
Automorphism group of Finite Field in a of size 5^3
sage: End(K).order()
3
sage: L.<b> = GF(25)
sage: Hom(L, K).order() == Hom(K, L).order() == 0
True

2.5 Finite field morphisms

This file provides several classes implementing:

- embeddings between finite fields
- Frobenius isomorphism on finite fields

EXAMPLES:

sage: from sage.rings.finite_rings.hom_finite_field import FiniteFieldHomomorphism_
  → generic

Construction of an embedding:

sage: k.<t> = GF(3^7)
sage: K.<T> = GF(3^21)
sage: f = FiniteFieldHomomorphism_generic(Hom(k, K)); f
Ring morphism:
  From: Finite Field in t of size 3^7
  To:   Finite Field in T of size 3^21
  Defn: t |--> T^20 + 2*T^18 + T^16 + 2*T^13 + T^9 + 2*T^8 + T^7 + T^6 + T^5 + T^3 + 2*T^2 + T

sage: f(t)
T^20 + 2*T^18 + T^16 + 2*T^13 + T^9 + 2*T^8 + T^7 + T^6 + T^5 + T^3 + 2*T^2 + T
The map \( f \) has a method \texttt{section} which returns a partially defined map which is the inverse of \( f \) on the image of \( f \):

\begin{verbatim}
sage: g = f.section(); g
Section of Ring morphism:
    From: Finite Field in t of size 3^7
    To:   Finite Field in T of size 3^21
    Defn: t |--> T^20 + 2*T^18 + T^16 + 2*T^13 + T^9 + 2*T^8 + T^7 + T^6 + T^5 + T^3 + 2*T^2 + T

sage: g(f(t^3+t^2+1))
t^3 + t^2 + 1

sage: g(T)
Traceback (most recent call last):
  ... ValueError: T is not in the image of Ring morphism:
    From: Finite Field in t of size 3^7
    To:   Finite Field in T of size 3^21
    Defn: t |--> T^20 + 2*T^18 + T^16 + 2*T^13 + T^9 + 2*T^8 + T^7 + T^6 + T^5 + T^3 + 2*T^2 + T

There is no embedding of \( GF(5^6) \) into \( GF(5^{11}) \):

\begin{verbatim}
sage: k.<t> = GF(5^6)
sage: K.<T> = GF(5^11)
sage: FiniteFieldHomomorphism_generic(Hom(k, K))
Traceback (most recent call last):
  ... ValueError: No embedding of Finite Field in t of size 5^6 into Finite Field in T of size 5^11
\end{verbatim}

Construction of Frobenius endomorphisms:

\begin{verbatim}
sage: k.<t> = GF(7^14)
sage: Frob = k.frobenius_endomorphism(); Frob
Frobenius endomorphism t |--> t^7 on Finite Field in t of size 7^14

sage: Frob(t)
t^7

Some basic arithmetics is supported:

\begin{verbatim}
sage: Frob^2
Frobenius endomorphism t |--> t^(7^2) on Finite Field in t of size 7^14

sage: f = k.frobenius_endomorphism(7); f
Frobenius endomorphism t |--> t^(7^7) on Finite Field in t of size 7^14

sage: f^8Frob
Frobenius endomorphism t |--> t^(7^8) on Finite Field in t of size 7^14

sage: Frob.order()
14
sage: f.order()
2
\end{verbatim}

Note that simplifications are made automatically:
Finite Rings, Release 9.7

```
sage: Frob^16
Frobenius endomorphism t |--> t^(7^2) on Finite Field in t of size 7^14
sage: Frob^28
Identity endomorphism of Finite Field in t of size 7^14
```

And that comparisons work:

```
sage: Frob == Frob^15
True
sage: Frob^14 == Hom(k, k).identity()
True
```

AUTHOR:

• Xavier Caruso (2012-06-29)

class sage.rings.finite_rings.hom_finite_field.FiniteFieldHomomorphism_generic

Bases: sage.rings.morphism.RingHomomorphism_im_gens

A class implementing embeddings between finite fields.

**is_injective()**

Return True since a embedding between finite fields is always injective.

**is_surjective()**

Return true if this embedding is surjective (and hence an isomorphism).

**section()**

Return the inverse of this embedding.

It is a partially defined map whose domain is the codomain of the embedding, but which is only defined on the image of the embedding.
```python
sage: from sage.rings.finite_rings.hom_finite_field import FiniteFieldHomomorphism_generic
sage: k.<t> = GF(3^7)
sage: K.<T> = GF(3^21)
sage: f = FiniteFieldHomomorphism_generic(Hom(k, K))
sage: g = f.section(); g
Section of Ring morphism:
    From: Finite Field in t of size 3^7
    To:   Finite Field in T of size 3^21
    Defn: t |--> T^20 + 2*T^18 + T^16 + 2*T^13 + T^9 + 2*T^8 + T^7 + T^6 + T^5 +
          T^3 + 2*T^2 + T
sage: g(f(t^3+t^2+1))
t^3 + t^2 + 1
sage: g(T)
Traceback (most recent call last):
  ...  ValueError: T is not in the image of Ring morphism:
          From: Finite Field in t of size 3^7
          To:   Finite Field in T of size 3^21
          Defn: t |--> T^20 + 2*T^18 + T^16 + 2*T^13 + T^9 + 2*T^8 + T^7 + T^6 + T^5 +
          T^3 + 2*T^2 + T
```

```python
class sage.rings.finite_rings.hom_finite_field.FrobeniusEndomorphism_finite_field
Bases: sage.rings.morphism.FrobeniusEndomorphism_generic

A class implementing Frobenius endomorphisms on finite fields.

fixed_field()
Return the fixed field of self.

OUTPUT:

* a tuple \((K, e)\), where \(K\) is the subfield of the domain consisting of elements fixed by self and \(e\) is an embedding of \(K\) into the domain.

**Note:** The name of the variable used for the subfield (if it is not a prime subfield) is suffixed by \_fixed.

EXAMPLES:

```python
sage: k.<t> = GF(5^6)
sage: f = k.frobenius_endomorphism(2)
sage: kfixed, embed = f.fixed_field()
sage: kfixed
Finite Field in t_fixed of size 5^2
sage: embed
Ring morphism:
    From: Finite Field in t_fixed of size 5^2
    To:   Finite Field in t of size 5^6
    Defn: t_fixed |--> 4*t^5 + 2*t^4 + 4*t^2 + t
sage: tfixed = kfixed.gen()
sage: embed(tfixed)
4*t^5 + 2*t^4 + 4*t^2 + t
```

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**inverse()**

Return the inverse of this Frobenius endomorphism.

EXAMPLES:

```sage
sage: k.<a> = GF(7^11)
sage: f = k.frobenius_endomorphism(5)
sage: (f.inverse() * f).is_identity()
True
```

**is_identity()**

Return true if this morphism is the identity morphism.

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: Frob.is_identity()
False
sage: (Frob^3).is_identity()
True
```

**is_injective()**

Return true since any power of the Frobenius endomorphism over a finite field is always injective.

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: Frob.is_injective()
True
```

**is_surjective()**

Return true since any power of the Frobenius endomorphism over a finite field is always surjective.

EXAMPLES:

```sage
sage: k.<t> = GF(5^3)
sage: Frob = k.frobenius_endomorphism()
sage: Frob.is_surjective()
True
```

**order()**

Return the order of this endomorphism.

EXAMPLES:

```sage
sage: k.<t> = GF(5^12)
sage: Frob = k.frobenius_endomorphism()
sage: Frob.order()
12
sage: (Frob^2).order()
6
sage: (Frob^9).order()
4
```

**power()**

Return an integer \( n \) such that this endomorphism is the \( n \)-th power of the absolute (arithmetic) Frobenius.
EXAMPLES:

```python
sage: k.<t> = GF(5^12)
sage: Frob = k.frobenius_endomorphism()
sage: Frob.power()
1
sage: (Frob^9).power()
9
sage: (Frob^13).power()
1
```

class sage.rings.finite_rings.hom_finite_field.SectionFiniteFieldHomomorphism_generic

Bases: sage.categories.map.Section

A class implementing sections of embeddings between finite fields.
3.1 Finite Prime Fields

AUTHORS:

- William Stein: initial version
- Martin Albrecht (2008-01): refactoring

```python
class sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn(p, check=True, modulus=None):
    Bases: sage.rings.finite_rings.finite_field_base.FiniteField, sage.rings.finite_rings.integer_mod_ring.IntegerModRing_generic

Finite field of order \( p \) where \( p \) is prime.
```

EXAMPLES:

```python
sage: FiniteField(3)
Finite Field of size 3

sage: FiniteField(next_prime(1000))
Finite Field of size 1009
```

`characteristic()`

Return the characteristic of \( \text{code}{self} \).  

EXAMPLES:

```python
sage: k = GF(7)
sage: k.characteristic()
7
```

`construction()`

Returns the construction of this finite field (for use by \text{code}{sage.categories.pushout}).

EXAMPLES:

```python
sage: GF(3).construction()
(QuotientFunctor, Integer Ring)
```

`degree()`

Return the degree of \( \text{self} \) over its prime field.  
This always returns 1.
EXAMPLES:

```
sage: FiniteField(3).degree()
1
```

**gen**(\(n=0\))

Return a generator of `self` over its prime field, which is a root of `self.modulus()`.

Unless a custom modulus was given when constructing this prime field, this returns 1.

**INPUT:**

- \(n\) – must be 0

**OUTPUT:**

An element \(a\) of `self` such that `self.modulus()(a) == 0`. 

**Warning:** This generator is not guaranteed to be a generator for the multiplicative group. To obtain the latter, use `multiplicative_generator()` or use the `modulus="primitive"` option when constructing the field.

EXAMPLES:

```
sage: k = GF(13)
sage: k.gen()
1
sage: k = GF(1009, modulus="primitive")
sage: k.gen()  # this gives a primitive element
11
sage: k.gen(1)
Traceback (most recent call last):
  ...
IndexError: only one generator
```

**is_prime_field()**

Return True since this is a prime field.

**EXAMPLES:**

```
sage: k.<a> = GF(3)
sage: k.is_prime_field()
True
sage: k.<a> = GF(3^2)
sage: k.is_prime_field()
False
```

**order()**

Return the order of this finite field.

**EXAMPLES:**

```
sage: k = GF(5)
sage: k.order()
5
```
polynomial(name=None)
    Returns the polynomial name.

    EXAMPLES:

    sage: k.<a> = GF(3)
sage: k.polynomial()
x

3.2 Finite field morphisms for prime fields

Special implementation for prime finite field of:

- embeddings of such field into general finite fields
- Frobenius endomorphisms (= identity with our assumptions)

See also:
sage.rings.finite_rings.hom_finite_field

AUTHOR:
- Xavier Caruso (2012-06-29)

class sage.rings.finite_rings.hom_prime_finite_field.FiniteFieldHomomorphism_prime
    Bases: sage.rings.finite_rings.hom_finite_field.FiniteFieldHomomorphism_generic

A class implementing embeddings of prime finite fields into general finite fields.

class sage.rings.finite_rings.hom_prime_finite_field.FrobeniusEndomorphism_prime
    Bases: sage.rings.finite_rings.hom_prime_finite_field.FrobeniusEndomorphism_finite_field

A class implementing Frobenius endomorphism on prime finite fields (i.e. identity map :-).

fixed_field()
    Return the fixed field of self.

    OUTPUT:
    - a tuple \((K, e)\), where \(K\) is the subfield of the domain consisting of elements fixed by self and \(e\) is an
      embedding of \(K\) into the domain.

    Note: Since here the domain is a prime field, the subfield is the same prime field and the embedding is
    necessarily the identity map.

    EXAMPLES:

    sage: k.<t> = GF(5)
sage: f = k.frobenius_endomorphism(2); f
Identity endomorphism of Finite Field of size 5
sage: kfixed, embed = f.fixed_field()
sage: kfixed == k
True
sage: [ embed(x) == x for x in kfixed ]
[True, True, True, True, True]
class sage.rings.finite_rings.hom_prime_finite_field.SectionFiniteFieldHomomorphism_prime
    Bases: sage.rings.finite_rings.hom_finite_field.SectionFiniteFieldHomomorphism_generic
4.1 Finite fields implemented via PARI’s FFELT type

AUTHORS:
• Peter Bruin (June 2013): initial version, based on finite_field_ext_pari.py by William Stein et al.

```python
class sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt(p, modulus, name=None):
    Bases: sage.rings.finite_rings.finite_field_base.FiniteField
    Finite fields whose cardinality is a prime power (not a prime), implemented using PARI’s FFELT type.
    INPUT:
    • p – prime number
    • modulus – an irreducible polynomial of degree at least 2 over the field of p elements
    • name – string: name of the distinguished generator (default: variable name of modulus)
    OUTPUT:
    A finite field of order \( q = p^n \), generated by a distinguished element with minimal polynomial \( \text{modulus} \). Elements are represented as polynomials in name of degree less than \( n \).
    
    Note: Direct construction of \texttt{FiniteField\_pari\_ffelt} objects requires specifying a characteristic and a modulus. To construct a finite field by specifying a cardinality and an algorithm for finding an irreducible polynomial, use the \texttt{FiniteField} constructor with \texttt{impl='pari\_ffelt'}. 
```

EXAMPLES:
Some computations with a finite field of order 9:

```python
sage: k = FiniteField(9, 'a', impl='pari_ffelt')
sage: k
Finite Field in a of size 3^2
sage: k.is_field()
True
sage: k.characteristic()
3
sage: a = k.gen()
sage: a
a
```
(continues on next page)
Next we compute with a finite field of order 16:

```python
sage: k16 = FiniteField(16, 'b', impl='pari_ffelt')
sage: z = k16.gen()
sage: z
b
sage: z.charpoly('x')
x^4 + x + 1
sage: k16.is_field()
True
sage: k16.characteristic()
2
sage: z.multiplicative_order()
15
```

Illustration of dumping and loading:

```python
sage: K = FiniteField(7^10, 'b', impl='pari_ffelt')
sage: loads(K.dumps()) == K
True
sage: K = FiniteField(10007^10, 'a', impl='pari_ffelt')
sage: loads(K.dumps()) == K
True
```

### Element

alias of `sage.rings.finite_rings.element_pari_ffelt.FiniteFieldElement_pari_ffelt`

#### characteristic()

Return the characteristic of self.

EXAMPLES:

```python
sage: F = FiniteField(3^4, 'a', impl='pari_ffelt')
sage: F.characteristic()
3
```

#### degree()

Returns the degree of self over its prime field.

EXAMPLES:

```python
sage: F = FiniteField(3^20, 'a', impl='pari_ffelt')
sage: F.degree()
20
```
\begin{verbatim}
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\textbf{gen}(n=0)
Return a generator of \texttt{self} over its prime field, which is a root of \texttt{self.modulus()}.

\textbf{INPUT}:
\begin{itemize}
  \item \texttt{n} – must be 0
\end{itemize}

\textbf{OUTPUT}:
An element \(a\) of \texttt{self} such that \texttt{self.modulus()}(a) == 0.

\textbf{Warning:} This generator is not guaranteed to be a generator for the multiplicative group. To obtain the latter, use \texttt{multiplicative_generator()} or use the \texttt{modulus="primitive"} option when constructing the field.

\textbf{EXAMPLES}:
\begin{verbatim}
sage: R.<x> = PolynomialRing(GF(2))
sage: FiniteField(2^4, 'b', impl='pari_ffelt').gen() b
sage: k = FiniteField(3^4, 'alpha', impl='pari_ffelt')
sage: a = k.gen()
sage: a
alpha
sage: a^4
alpha^3 + 1
\end{verbatim}
\end{verbatim}

\section*{4.2 Finite field elements implemented via PARI’s FFELT type}

\textbf{AUTHORS}:
\begin{itemize}
  \item Peter Bruin (June 2013): initial version, based on \texttt{element_ext_pari.py} by William Stein et al. and \texttt{element_ntl_gf2e.pyx} by Martin Albrecht.
\end{itemize}

\textbf{class} \texttt{sage.rings.finite_rings.element_pari_ffelt.FiniteFieldElement_pari_ffelt}
\texttt{Bases: \texttt{sage.rings.finite_rings.element_base.FinitePolyExtElement}}

An element of a finite field implemented using PARI.

\textbf{EXAMPLES}:
\begin{verbatim}
sage: K = FiniteField(10007^10, 'a', impl='pari_ffelt')
sage: a = K.gen(); a
\begin{verbatim}<class 'sage.rings.finite_rings.element_pari_ffelt.FiniteFieldElement_pari_ffelt'>
\end{verbatim}
sage: type(a)
<...>
\end{verbatim}

\textbf{charpoly}(\texttt{var='x'})
Return the characteristic polynomial of \texttt{self}.

\textbf{INPUT}:
\begin{itemize}
  \item \texttt{var} – string (default: \texttt{x}): variable name to use.
\end{itemize}

\textbf{EXAMPLES}:

### is_one()  
Return True if self equals 1.

**EXAMPLES:**

```python
sage: F.<a> = FiniteField(5^3, impl='pari_ffelt')
sage: a.is_one()
False
sage: (a/a).is_one()
True
```

### is_square()  
Return True if and only if self is a square in the finite field.

**EXAMPLES:**

```python
sage: k.<a> = FiniteField(3^2, impl='pari_ffelt')
sage: a.is_square()
False
sage: (a**2).is_square()
True
sage: k.<a> = FiniteField(2^2, impl='pari_ffelt')
sage: (a**2).is_square()
True
sage: k.<a> = FiniteField(17^5, impl='pari_ffelt')
sage: (a**2).is_square()
True
sage: a.is_square()
False
sage: k(0).is_square()
True
```

### is_unit()  
Return True if self is non-zero.

**EXAMPLES:**

```python
sage: F.<a> = FiniteField(5^3, impl='pari_ffelt')
sage: a.is_unit()
True
```

### is_zero()  
Return True if self equals 0.

**EXAMPLES:**

```python
sage: F.<a> = FiniteField(5^3, impl='pari_ffelt')
sage: a.is_zero()
```

(continues on next page)
False
sage: (a - a).is_zero()
True

**lift()**

If `self` is an element of the prime field, return a lift of this element to an integer.

**EXAMPLES:**

```python
sage: k = FiniteField(next_prime(10^10)^2, 'u', impl='pari_ffelt')
sage: a = k(17)/k(19)
sage: b = a.lift(); b
7894736858
sage: b.parent()
Integer Ring
```

**log(base)**

Return a discrete logarithm of `self` with respect to the given base.

**INPUT:**

- `base` – non-zero field element

**OUTPUT:**

An integer \( x \) such that `self` equals `base` raised to the power \( x \). If no such \( x \) exists, a `ValueError` is raised.

**EXAMPLES:**

```python
sage: F.<g> = FiniteField(2^10, impl='pari_ffelt')
sage: b = g; a = g^37
sage: a.log(b)
37
sage: b^37; a
g^8 + g^7 + g^4 + g + 1
g^8 + g^7 + g^4 + g + 1

sage: F.<a> = FiniteField(5^2, impl='pari_ffelt')
sage: F(-1).log(F(2))
2
sage: F(1).log(a)
0
```

Some cases where the logarithm is not defined or does not exist:

```python
sage: F.<a> = GF(3^10, impl='pari_ffelt')
sage: a.log(-1)
Traceback (most recent call last):
  ... ArithmeticError: element a does not lie in group generated by 2
sage: a.log(0)
Traceback (most recent call last):
  ... ArithmeticError: discrete logarithm with base 0 is not defined
sage: F(0).log(1)
```

(continues on next page)
minpoly(var='x')
Return the minimal polynomial of self.
INPUT:
• var – string (default: ‘x’): variable name to use.
EXAMPLES:

```
sage: R.<x> = PolynomialRing(FiniteField(3))
sage: F.<a> = FiniteField(3^2, modulus=x^2 + 1, impl='pari_ffelt')
sage: a.minpoly('y')
y^2 + 1
```

multiplicative_order()
Returns the order of self in the multiplicative group.
EXAMPLES:

```
sage: a = FiniteField(5^3, 'a', impl='pari_ffelt').0
sage: a.multiplicative_order()
124
sage: a**124
1
```

descriminant() (continued from previous page)

```
Traceback (most recent call last):
...  
ArithmeticError: discrete logarithm of 0 is not defined
```

polynomial(name=None)
Return the unique representative of self as a polynomial over the prime field whose degree is less than
the degree of the finite field over its prime field.
INPUT:
• name – (optional) variable name
EXAMPLES:

```
sage: k.<a> = FiniteField(3^2, impl='pari_ffelt')
sage: pol = a.polynomial()
sage: pol
a
sage: parent(pol)
Univariate Polynomial Ring in a over Finite Field of size 3
```

```
sage: k = FiniteField(3^4, 'alpha', impl='pari_ffelt')
sage: a = k.gen()
sage: a.polynomial()
alpha
sage: (a**2 + 1).polynomial('beta')
beta^2 + 1
sage: (a**2 + 1).polynomial().parent()
Univariate Polynomial Ring in alpha over Finite Field of size 3
sage: (a**2 + 1).polynomial('beta').parent()
Univariate Polynomial Ring in beta over Finite Field of size 3
```
**sqrt**(extend=False, all=False)

Return a square root of `self`, if it exists.

INPUT:

- `extend` – bool (default: False)

**Warning:** This option is not implemented.

- `all` - bool (default: False)

OUTPUT:

A square root of `self`, if it exists. If all is True, a list containing all square roots of `self` (of length zero, one or two) is returned instead.

If `extend` is True, a square root is chosen in an extension field if necessary. If `extend` is False, a ValueError is raised if the element is not a square in the base field.

**Warning:** The `extend` option is not implemented (yet).

**EXAMPLES:**

```python
sage: F = FiniteField(7^2, 'a', impl='pari_ffelt')
sage: F(2).sqrt()
4
sage: F(3).sqrt() in (2*F.gen() + 6, 5*F.gen() + 1)
True
sage: F(3).sqrt()**2
3
sage: F(4).sqrt(all=True)
[2, 5]
sage: K = FiniteField(7^3, 'alpha', impl='pari_ffelt')
sage: K(3).sqrt()  # indirect doctest
ValueError: element is not a square
```

**EXAMPLES:**

```python
sage.rings.finite_rings.element_pari_ffelt.unpickle_FiniteFieldElement_pari_ffelt(parent, elem)
```

**EXAMPLES:**

```python
sage: k.<a> = GF(2^20, impl='pari_ffelt')
sage: e = k.random_element()
sage: f = loads(dumps(e))  # indirect doctest
```

(continues on next page)
```sage
sage: e == f
True
```
Finite fields that are implemented using Zech logs and the cardinality must be less than $2^{16}$. By default, Conway polynomials are used as minimal polynomial.

```python
class sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro(q, name='a', modulus=None, repr='poly', cache=False)
```

**INPUT:**
- `q` – $p^n$ (must be prime power)
- `name` – (default: 'a') variable used for `poly_repr()`
- `modulus` – A minimal polynomial to use for reduction.
- `repr` – (default: 'poly') controls the way elements are printed to the user:
  - `log`: `repr` is `log_repr()`
  - `int`: `repr` is `int_repr()`
  - `poly`: `repr` is `poly_repr()`
- `cache` – (default: `False`) if `True` a cache of all elements of this field is created. Thus, arithmetic does not create new elements which speeds calculations up. Also, if many elements are needed during a calculation this cache reduces the memory requirement as at most `order()` elements are created.

**OUTPUT:**
Givaro finite field with characteristic $p$ and cardinality $p^n$.

**EXAMPLES:**
By default, Conway polynomials are used for extension fields:

```python
sage: k.<a> = GF(2**8)
sage: -a ^ k.degree()
a^4 + a^3 + a^2 + 1
sage: f = k.modulus(); f
x^8 + x^4 + x^3 + x^2 + 1
```

You may enforce a modulus:
sage: P.<x> = PolynomialRing(GF(2))
sage: f = x^8 + x^4 + x^3 + x + 1 # Rijndael Polynomial
sage: k.<a> = GF(2^8, modulus=f)
sage: k.modulus()
x^8 + x^4 + x^3 + x + 1
sage: a^(2^8)
a
You may enforce a random modulus:

sage: k = GF(3**5, 'a', modulus='random')
sage: k.modulus()
# random polynomial
x^5 + 2*x^4 + 2*x^3 + x^2 + 2

Three different representations are possible:

sage: FiniteField(9, 'a', impl='givaro', repr='poly').gen()
a
sage: FiniteField(9, 'a', impl='givaro', repr='int').gen()
3
sage: FiniteField(9, 'a', impl='givaro', repr='log').gen()
1

For prime fields, the default modulus is the polynomial \(x - 1\), but you can ask for a different modulus:

sage: GF(1009, impl='givaro').modulus()
x + 1008
sage: GF(1009, impl='givaro', modulus='conway').modulus()
x + 998

\(a\_times\_b\_minus\_c(a, b, c)\)
Return \(a*b - c\).

INPUT:

• \(a,b,c\) – \(\text{FiniteField\_givaroElement}\)

EXAMPLES:

sage: k.<a> = GF(3**3)
sage: k.a_times_b_minus_c(a,a,k(1))
a^2 + 2

\(a\_times\_b\_plus\_c(a, b, c)\)
Return \(a*b + c\). This is faster than multiplying \(a\) and \(b\) first and adding \(c\) to the result.

INPUT:

• \(a,b,c\) – \(\text{FiniteField\_givaroElement}\)

EXAMPLES:

sage: k.<a> = GF(2**8)
sage: k.a_times_b_plus_c(a,a,k(1))
a^2 + 1

\(c\_minus\_a\_times\_b(a, b, c)\)
Return \(c - a*b\).
INPUT:

• \(a, b, c\) – *FiniteField_givaroElement*

EXAMPLES:

```python
sage: k.<a> = GF(3^3)
sage: k.c_minus_a_times_b(a,a,k(1))
2*a^2 + 1
```

**characteristic()**

Return the characteristic of this field.

EXAMPLES:

```python
sage: p = GF(19^5, 'a').characteristic(); p
19
sage: type(p)
<class 'sage.rings.integer.Integer'>
```

**degree()**

If the cardinality of `self` is \(p^n\), then this returns \(n\).

OUTPUT:

Integer – the degree

EXAMPLES:

```python
sage: GF(3^4, 'a').degree()
4
```

**fetch_int(n)**

Given an integer \(n\) return a finite field element in `self` which equals \(n\) under the condition that `gen()` is set to `characteristic()`.

EXAMPLES:

```python
sage: k.<a> = GF(2^8)
sage: k.fetch_int(8)
a^3
sage: e = k.fetch_int(151); e
a^7 + a^4 + a^2 + a + 1
sage: 2^7 + 2^4 + 2^2 + 2 + 1
151
```

**frobenius_endomorphism(n=1)**

INPUT:

• \(n\) – an integer (default: 1)

OUTPUT:

The \(n\)-th power of the absolute arithmetic Frobenius endomorphism on this finite field.

EXAMPLES:

```python
sage: k.<t> = GF(3^5)
sage: Frob = k.frobenius_endomorphism(); Frob
Frobenius endomorphism t |--> t^3 on Finite Field in t of size 3^5
```
sage: a = k.random_element()
sage: Frob(a) == a^3
True

We can specify a power:

sage: k.frobenius_endomorphism(2)
Frobenius endomorphism $t \mapsto t^{3^2}$ on Finite Field in $t$ of size $3^5$

The result is simplified if possible:

sage: k.frobenius_endomorphism(6)
Frobenius endomorphism $t \mapsto t^3$ on Finite Field in $t$ of size $3^5$
sage: k.frobenius_endomorphism(5)
Identity endomorphism of Finite Field in $t$ of size $3^5$

Comparisons work:

sage: k.frobenius_endomorphism(6) == Frob
True

AUTHOR:

• Xavier Caruso (2012-06-29)

\texttt{gen(n=0)}

Return a generator of \texttt{self} over its prime field, which is a root of \texttt{self.modulus()}.

INPUT:

• \texttt{n} – must be 0

OUTPUT:

An element $a$ of \texttt{self} such that \texttt{self.modulus()}(a) == 0.

\textbf{Warning:} This generator is not guaranteed to be a generator for the multiplicative group. To obtain the latter, use \texttt{multiplicative_generator()} or use the modulus="primitive" option when constructing the field.

EXAMPLES:

sage: k = GF($3^4$, 'b'); k.gen()
b
sage: k.gen()
Traceback (most recent call last):
...
IndexError: only one generator
sage: F = FiniteField(31, impl='givaro')
sage: F.gen()
1
**int_to_log(n)**

Given an integer \( n \) this method returns \( i \) where \( i \) satisfies \( g^i = n \mod p \) where \( g \) is the generator and \( p \) is the characteristic of \( \text{self} \).

**INPUT:**
- \( n \) – integer representation of a finite field element

**OUTPUT:**
log representation of \( n \)

**EXAMPLES:**

```
sage: k = GF(7^3, 'a')
sage: k.int_to_log(4)
228
sage: k.int_to_log(3)
57
sage: k.gen()^57
3
```

**log_to_int(n)**

Given an integer \( n \) this method returns \( i \) where \( i \) satisfies \( g^n = i \) where \( g \) is the generator of \( \text{self} \); the result is interpreted as an integer.

**INPUT:**
- \( n \) – log representation of a finite field element

**OUTPUT:**
integer representation of a finite field element.

**EXAMPLES:**

```
sage: k = GF(2^8, 'a')
sage: k.log_to_int(4)
16
sage: k.log_to_int(20)
180
```

**order()**

Return the cardinality of this field.

**OUTPUT:**
Integer – the number of elements in \( \text{self} \).

**EXAMPLES:**

```
sage: n = GF(19^5,'a').order(); n
2476099
sage: type(n)
<class 'sage.rings.integer.Integer'>
```

**prime_subfield()**

Return the prime subfield \( F_p \) of \( \text{self} \) if \( \text{self} \) is \( F_{p^n} \).
Finite Rings, Release 9.7

sage: GF(3^4, 'b').prime_subfield()
Finite Field of size 3
sage: S.<b> = GF(5^2); S
Finite Field in b of size 5^2
sage: S.prime_subfield()
Finite Field of size 5
sage: type(S.prime_subfield())
<class 'sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn_with_category'>

random_element(*args, **kwds)
Return a random element of self.

EXAMPLES:

sage: k = GF(23**3, 'a')
sage: e = k.random_element()
sage: e.parent() is k
True
sage: type(e)
<class 'sage.rings.finite_rings.element_givaro.FiniteField_givaroElement'>
sage: P.<x> = PowerSeriesRing(GF(3^3, 'a'))
sage: P.random_element(5).parent() is P
True

5.2 Givaro Field Elements

Sage includes the Givaro finite field library, for highly optimized arithmetic in finite fields.

Note: The arithmetic is performed by the Givaro C++ library which uses Zech logs internally to represent finite field elements. This implementation is the default finite extension field implementation in Sage for the cardinality less than $2^{16}$, as it is a lot faster than the PARI implementation. Some functionality in this class however is implemented using PARI.

EXAMPLES:

sage: k = GF(5); type(k)
<class 'sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn_with_category'>
sage: k = GF(5^2, 'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro_with_category'>
sage: k = GF(2^16,'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_ntl_gf2e.FiniteField_ntl_gf2e_with_category'>
sage: k = GF(3^16, 'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt_with_category'>

(continues on next page)
sage: n = previous_prime_power(2^16 - 1)
sage: while is_prime(n):
    ....: n = previous_prime_power(n)
sage: factor(n)
251^2
sage: k = GF(n, 'c'); type(k)
<class 'sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro_with_category'>

AUTHORS:

• Martin Albrecht <malb@informatik.uni-bremen.de> (2006-06-05)
• William Stein (2006-12-07): editing, lots of docs, etc.

class sage.rings.finite_rings.element_givaro.Cache_givaro
    Bases: sage.rings.finite_rings.element_base.Cache_base

Finite Field.
These are implemented using Zech logs and the cardinality must be less than $2^{16}$. By default Conway polynomials are used as minimal polynomial.

INPUT:

• q = $p^n$ (must be prime power)
• name – variable used for poly_repr (default: 'a')
• modulus – a polynomial to use as modulus.
• repr – (default: 'poly') controls the way elements are printed to the user:
  – 'log': repr is log_repr()
  – 'int': repr is int_repr()
  – 'poly': repr is poly_repr()
• cache – (default: False) if True a cache of all elements of this field is created. Thus, arithmetic does not create new elements which speeds calculations up. Also, if many elements are needed during a calculation this cache reduces the memory requirement as at most order() elements are created.

OUTPUT:
Givaro finite field with characteristic $p$ and cardinality $p^n$.

EXAMPLES:
By default Conway polynomials are used:

```
sage: k.<a> = GF(2^8)
sage: -a ^ k.degree()
a^4 + a^3 + a^2 + 1
sage: f = k.modulus(); f
x^8 + x^4 + x^3 + x^2 + 1
```

You may enforce a modulus:
Finite Rings, Release 9.7

```python
sage: P.<x> = PolynomialRing(GF(2))
sage: f = x^8 + x^4 + x^3 + x + 1 # Rijndael polynomial
sage: k.<a> = GF(2^8, modulus=f)
sage: k.modulus()
x^8 + x^4 + x^3 + x + 1
sage: a^2^8
a
```

You may enforce a random modulus:

```python
sage: k = GF(3**5, 'a', modulus='random')
sage: k.modulus()
# random polynomial
x^5 + 2*x^4 + 2*x^3 + x^2 + 2
```

For binary fields, you may ask for a minimal weight polynomial:

```python
sage: k = GF(2**10, 'a', modulus='minimal_weight')
sage: k.modulus()
x^10 + x^3 + 1
```

**a_times_b_minus_c(a, b, c)**

Return \(a \times b - c\).

**INPUT:**

- \(a, b, c\) – `FiniteField_givaroElement`

**EXAMPLES:**

```python
sage: k.<a> = GF(3**3)
sage: k._cache.a_times_b_minus_c(a,a,k(1))
a^2 + 2
```

**a_times_b_plus_c(a, b, c)**

Return \(a \times b + c\).

This is faster than multiplying \(a\) and \(b\) first and adding \(c\) to the result.

**INPUT:**

- \(a, b, c\) – `FiniteField_givaroElement`

**EXAMPLES:**

```python
sage: k.<a> = GF(2**8)
sage: k._cache.a_times_b_plus_c(a,a,k(1))
a^2 + 1
```

**c_minus_a_times_b(a, b, c)**

Return \(c - a \times b\).

**INPUT:**

- \(a, b, c\) – `FiniteField_givaroElement`

**EXAMPLES:**
sage: k.<a> = GF(3^3)
sage: k._cache.c_minus_a_times_b(a,a,k(1))
2^a^2 + 1

**characteristic()**

Return the characteristic of this field.

**EXAMPLES:**

```python
sage: p = GF(19^3,'a')._cache.characteristic(); p
19
```

**element_from_data(e)**

Coerces several data types to `self`.

**INPUT:**

- `e` – data to coerce in.

**EXAMPLES:**

```python
sage: k = GF(3^8,'a')
sage: type(k)
<class 'sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro_with_category'>
sage: e = k.vector_space(map=False).gen(1); e
(0, 1, 0, 0, 0, 0, 0, 0)
sage: k(e) #indirect doctest
a
```

**exponent()**

Return the degree of this field over \( F_p \).

**EXAMPLES:**

```python
sage: K.<a> = GF(9); K._cache.exponent()
2
```

**fetch_int(number)**

Given an integer `n` return a finite field element in `self` which equals `n` under the condition that `gen()` is set to `characteristic()`.

**EXAMPLES:**

```python
sage: k.<a> = GF(2^8)
sage: k._cache.fetch_int(8)
a^3
sage: e = k._cache.fetch_int(151); e
a^7 + a^4 + a^2 + a + 1
sage: 2^7 + 2^4 + 2^2 + 2 + 1
151
```

**gen()**

Return a generator of the field.

**EXAMPLES:**

```python
```
sage: K.<a> = GF(625)
sage: K._cache.gen()
a

int_to_log\( (n) \)
Given an integer \( n \) this method returns \( i \) where \( i \) satisfies \( g^i = n \mod p \) where \( g \) is the generator and \( p \) is the characteristic of \( \text{self} \).

INPUT:

\[ \bullet \, n \rightarrow \text{integer representation of an finite field element} \]

OUTPUT:

log representation of \( n \)

EXAMPLES:

sage: k = GF(7**3, 'a')
sage: k._cache.int_to_log(4)
228
sage: k._cache.int_to_log(3)
57
sage: k.gen()^57
3

log_to_int\( (n) \)
Given an integer \( n \) this method returns \( i \) where \( i \) satisfies \( g^n = i \) where \( g \) is the generator of \( \text{self} \); the result is interpreted as an integer.

INPUT:

\[ \bullet \, n \rightarrow \text{log representation of a finite field element} \]

OUTPUT:

integer representation of a finite field element.

EXAMPLES:

sage: k = GF(2**8, 'a')
sage: k._cache.log_to_int(4)
16
sage: k._cache.log_to_int(20)
180

order()
Return the order of this field.

EXAMPLES:

sage: K.<a> = GF(9)
sage: K._cache.order()
9

order_c()
Return the order of this field.

EXAMPLES:
sage: K.<a> = GF(9)
sage: K._cache.order_c()
9

random_element(*args, **kwds)
Return a random element of self.

EXAMPLES:

sage: k = GF(23**3, 'a')
sage: e = k._cache.random_element()
sage: e.parent() is k
True
sage: type(e)
<class 'sage.rings.finite_rings.element_givaro.FiniteField_givaroElement'>

sage: P.<x> = PowerSeriesRing(GF(3^3, 'a'))
sage: P.random_element(5).parent() is P
True

repr

class sage.rings.finite_rings.element_givaro.FiniteField_givaroElement
Bases: sage.rings.finite_rings.element_base.FinitePolyExtElement

An element of a (Givaro) finite field.

integer_representation()
Return the integer representation of self. When self is in the prime subfield, the integer returned is equal to self.

Elements of this field are represented as integers as follows: given the element $e \in F_p[x]$ with $e = a_0 + a_1 x + a_2 x^2 + \cdots$, the integer representation is $a_0 + a_1 p + a_2 p^2 + \cdots$.

OUTPUT: A Python int.

EXAMPLES:

sage: k.<b> = GF(5^2); k
Finite Field in b of size 5^2
sage: k(4).integer_representation()
4
sage: b.integer_representation()
5
sage: type(b.integer_representation())
<... 'int'>

is_one()
Return True if self == k(1).

EXAMPLES:

sage: k.<a> = GF(3^4); k
Finite Field in a of size 3^4
sage: a.is_one()
False
sage: k(1).is_one()
True
**is_square()**
Return True if self is a square in self.parent()

ALGORITHM:
Elements are stored as powers of generators, so we simply check to see if it is an even power of a generator.

EXAMPLES:

```python
sage: k.<a> = GF(9); k
Finite Field in a of size 3^2
sage: a.is_square()
False
sage: v = set([x^2 for x in k])
sage: [x.is_square() for x in v]
[True, True, True, True, True]
sage: [x.is_square() for x in k if not x in v]
[False, False, False, False]
```

**is_unit()**
Return True if self is nonzero, so it is a unit as an element of the finite field.

EXAMPLES:

```python
sage: k.<a> = GF(3^4); k
Finite Field in a of size 3^4
sage: a.is_unit()
True
sage: k(0).is_unit()
False
```

**log**(base)
Return the log to the base $b$ of self, i.e., an integer $n$ such that $b^n = self$.

**Warning:** TODO – This is currently implemented by solving the discrete log problem – which shouldn’t be needed because of how finite field elements are represented.

EXAMPLES:

```python
sage: k.<b> = GF(5^2); k
Finite Field in b of size 5^2
sage: a = b^7
sage: a.log(b)
7
```

**multiplicative_order()**
Return the multiplicative order of this field element.

EXAMPLES:

```python
sage: S.<b> = GF(5^2); S
Finite Field in b of size 5^2
sage: b.multiplicative_order()
24
sage: (b^6).multiplicative_order()
4
```
polynomial(name=None)
Return self viewed as a polynomial over self.parent().prime_subfield().

EXAMPLES:

```python
sage: k.<b> = GF(5^2); k
Finite Field in b of size 5^2
sage: f = (b^2+1).polynomial(); f
b + 4
sage: type(f)
<class 'sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint'>
sage: parent(f)
Univariate Polynomial Ring in b over Finite Field of size 5
```

sqrt(extend=False, all=False)
Return a square root of this finite field element in its parent, if there is one. Otherwise, raise a ValueError.

INPUT:

- **extend** – bool (default: True); if True, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the root is not in the base ring.

  > **Warning:** this option is not implemented!

- **all** – bool (default: False); if True, return all square roots of self, instead of just one.

  > **Warning:** The extend option is not implemented (yet).

ALGORITHM:
sself is stored as $a^k$ for some generator $a$. Return $a^{k/2}$ for even $k$. 

EXAMPLES:

```python
sage: k.<a> = GF(7^2)
sage: k(2).sqrt()
3
sage: k(3).sqrt()
2*a + 6
sage: k(3).sqrt()**2
3
sage: k(4).sqrt()
2
sage: k.<a> = GF(7^3)
sage: k(3).sqrt()
Traceback (most recent call last):
  ... ValueError: must be a perfect square.
```

class sage.rings.finite_rings.element_givaro.FiniteField_givaro_iterator
Bases: object

Iterator over FiniteField_givaro elements. We iterate multiplicatively, as powers of a fixed internal generator.

EXAMPLES:
Finite Rings, Release 9.7

```
sage: for x in GF(2^2, 'a'): print(x)
  0
  a
  a + 1
  1
```

```
sage.rings.finite_rings.element_givaro.unpickle_Cache_givaro(parent, p, k, modulus, rep, cache)
EXAMPLES:
```
```
sage: k = GF(3**7, 'a')
sage: loads(dumps(k)) == k # indirect doctest
True
```

```
sage.rings.finite_rings.element_givaro.unpickle_FiniteField_givaroElement(parent, x)
```

5.3 Finite field morphisms using Givaro

Special implementation for givaro finite fields of:
- embeddings between finite fields
- frobenius endomorphisms

SEEALSO:
```
:smod:`sage.rings.finite_rings.hom_finite_field`
```

AUTHOR:
- Xavier Caruso (2012-06-29)

```
class sage.rings.finite_rings.hom_finite_field_givaro.FiniteFieldHomomorphism_givaro
  Bases: sage.rings.finite_rings.hom_finite_field.FiniteFieldHomomorphism_generic

class sage.rings.finite_rings.hom_finite_field_givaro.FrobeniusEndomorphism_givaro
  Bases: sage.rings.finite_rings.hom_finite_field_givaro.FiniteFieldHomomorphism_givaro
```

```
def fixed_field()
  Return the fixed field of self.
  OUTPUT:
  • a tuple \((K, \epsilon)\), where \(K\) is the subfield of the domain consisting of elements fixed by \(\text{self}\) and \(\epsilon\) is an embedding of \(K\) into the domain.

  Note: The name of the variable used for the subfield (if it is not a prime subfield) is suffixed by \(_\text{fixed}\).
```

```
EXAMPLES:
```
```
sage: k.<t> = GF(5^6)
sage: f = k.frobenius_endomorphism(2)
sage: kfixed, embed = f.fixed_field()
sage: kfixed
  Finite Field in t_fixed of size 5^2
sage: embed
```

(continues on next page)
Ring morphism:
  From: Finite Field in t_fixed of size $5^2$
  To:   Finite Field in t of size $5^6$
  Defn: $t_{\text{fixed}} \mapsto 4^*t^5 + 2^*t^4 + 4^*t^2 + t$

```
sage: tfixed = kfixed.gen()
sage: embed(tfixed)
4^*t^5 + 2^*t^4 + 4^*t^2 + t
```

class
sage.rings.finite_rings.hom_finite_field_givaro.SectionFiniteFieldHomomorphism_givaro

   Bases: sage.rings.finite_rings.hom_finite_field.SectionFiniteFieldHomomorphism_generic
CHAPTER SIX

FINITE FIELDS OF CHARACTERISTIC 2 USING NTL

6.1 Finite Fields of Characteristic 2

```python
class sage.rings.finite_rings.finite_field_ntl_gf2e.FiniteField_ntl_gf2e(q, names='a', modulus=None, repr='poly')
```

Bases: `sage.rings.finite_rings.finite_field_base.FiniteField`

Finite Field of characteristic 2 and order $2^n$.

INPUT:

- $q = 2^n$ (must be 2 power)
- `names` - variable used for `poly_repr` (default: 'a')
- `modulus` - A minimal polynomial to use for reduction.
- `repr` - controls the way elements are printed to the user: (default: 'poly')
  - 'poly': polynomial representation

OUTPUT:

Finite field with characteristic 2 and cardinality $2^n$.

EXAMPLES:

```
sage: k.<a> = GF(2^16)
sage: type(k)
<class 'sage.rings.finite_rings.finite_field_ntl_gf2e.FiniteField_ntl_gf2e_with_category'>
sage: k.<a> = GF(2^1024)
sage: k.modulus()
x^1024 + x^19 + x^6 + x + 1
sage: set_random_seed(6397)
sage: k.<a> = GF(2^17, modulus='random')
sage: k.modulus()
x^17 + x^16 + x^15 + x^10 + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1
sage: k.modulus().is_irreducible()
True
sage: k.<a> = GF(2^211, modulus='minimal_weight')
sage: k.modulus()
x^211 + x^11 + x^10 + x^8 + 1
```

(continues on next page)
sage: k.<a> = GF(2^211, modulus='conway')
sage: k.modulus()
x^211 + x^9 + x^6 + x^5 + x^3 + x + 1
sage: k.<a> = GF(2^23, modulus='conway')
sage: a.multiplicative_order() == k.order() - 1
True

characteristic()
Return the characteristic of self which is 2.

EXAMPLES:

sage: k.<a> = GF(2^16, modulus='random')
sage: k.characteristic()
2

degree()
If this field has cardinality $2^n$ this method returns $n$.

EXAMPLES:

sage: k.<a> = GF(2^64)
sage: k.degree()
64

fetch_int(number)
Given an integer $n$ less than cardinality() with base 2 representation $a_0 + 2 \cdot a_1 + \cdots + 2^k a_k$, returns $a_0 + a_1 \cdot x + \cdots + a_k x^k$, where $x$ is the generator of this finite field.

INPUT:

- number – an integer

EXAMPLES:

sage: k.<a> = GF(2^48)
sage: k.fetch_int(2^43 + 2^15 + 1)
a^43 + a^15 + 1
sage: k.fetch_int(33793)
a^15 + a^10 + 1
sage: 33793.digits(2) # little endian
[1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]

gen(n=0)
Return a generator of self over its prime field, which is a root of self.modulus().

INPUT:

- n – must be 0

OUTPUT:

An element $a$ of self such that self.modulus()(a) == 0.
**Warning:** This generator is not guaranteed to be a generator for the multiplicative group. To obtain the latter, use `multiplicative_generator()` or use the `modulus="primitive"` option when constructing the field.

```python
sage: k.<a> = GF(2^19)
sage: k.gen() == a
True
sage: a
a
```

**order()**

Return the cardinality of this field.

```python
sage: k.<a> = GF(2^64)
sage: k.order()
18446744073709551616
```

**prime_subfield()**

Return the prime subfield $F_p$ of self if self is $F_{p^n}$.

```python
sage: F.<a> = GF(2^16)
sage: F.prime_subfield()
Finite Field of size 2
```

```python
sage.rings.finite_rings.finite_field_ntl_gf2e.late_import()
Imports various modules after startup.
```

6.2 Finite Fields of characteristic 2.

This implementation uses NTL’s GF2E class to perform the arithmetic and is the standard implementation for GF($2^n$) for $n \geq 16$.

**AUTHORS:**

- Martin Albrecht <malb@informatik.uni-bremen.de> (2007-10)

```python
class sage.rings.finite_rings.finite_field_ntl_gf2e.Cache_ntl_gf2e
    Bases: sage.rings.finite_rings.element_base.Cache_base

    This class stores information for an NTL finite field in a Cython class so that elements can access it quickly.
    It’s modeled on `NativeIntStruct`, but includes many functions that were previously included in the parent (see trac ticket #12062).
```
degree()
If the field has cardinality \(2^n\) this method returns \(n\).

EXAMPLES:
```
sage: k.<a> = GF(2^64)
sage: k._cache.degree()
64
```

fetch_int(number)
Given an integer less than \(p^n\) with base 2 representation \(a_0 + a_1 \cdot 2 + \cdots + a_k \cdot 2^k\), this returns \(a_0 + a_1 x + \cdots + a_k x^k\), where \(x\) is the generator of this finite field.

INPUT:
- number – an integer, of size less than the cardinality

EXAMPLES:
```
sage: k.<a> = GF(2^48)
sage: k._cache.fetch_int(2^33 + 2 + 1)
a^33 + a + 1
```

import_data(e)
EXAMPLES:
```
sage: k.<a> = GF(2^17)
sage: V = k.vector_space(map=False)
sage: v = [1,0,0,0,0,1,0,0,1,0,0,0,0,1,0,0,0]
sage: k._cache.import_data(v)
a^13 + a^8 + a^5 + 1
```

order()
Return the cardinality of the field.

EXAMPLES:
```
sage: k.<a> = GF(2^64)
sage: k._cache.order()
18446744073709551616
```

polynomial()
Returns the list of 0's and 1's giving the defining polynomial of the field.

EXAMPLES:
```
sage: k.<a> = GF(2^20,modulus="minimal_weight")
sage: k._cache.polynomial()
[1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]
```

class sage.rings.finite_rings.element_ntl_gf2e.FiniteField_ntl_gf2eElement
Bases: sage.rings.finite_rings.element_base.FinitePolyExtElement

An element of an NTL:GF2E finite field.

charpoly(var=’x’)
Return the characteristic polynomial of self as a polynomial in var over the prime subfield.
INPUT:
  • var – string (default: 'x')

OUTPUT:
  polynomial

EXAMPLES:

```
sage: k.<a> = GF(2^8, impl="ntl")
sage: b = a^3 + a
sage: b.minpoly()
x^4 + x^3 + x^2 + x + 1
sage: b.charpoly()
x^8 + x^6 + x^4 + x^2 + 1
sage: b.charpoly().factor()
(x^4 + x^3 + x^2 + x + 1)^2
sage: b.charpoly('Z')
Z^8 + Z^6 + Z^4 + Z^2 + 1
```

integer_representation()

Return the int representation of self. When self is in the prime subfield, the integer returned is equal to self and not to log_repr.

Elements of this field are represented as ints in as follows: for \( e \in F_p[x] \) with \( e = a_0 + a_1 x + a_2 x^2 + \cdots \), \( e \) is represented as: \( n = a_0 + a_1 p + a_2 p^2 + \cdots \).

EXAMPLES:

```
sage: k.<a> = GF(2^20)
sage: a.integer_representation()
2
sage: (a^2 + 1).integer_representation()
5
sage: k.<a> = GF(2^70)
sage: (a^65 + a^64 + 1).integer_representation()
5534923223128654849
```

is_one()

Return True if self == k(1).

Equivalent to self != k(0).

EXAMPLES:

```
sage: k.<a> = GF(2^20)
sage: a.is_one()  # indirect doctest
False
sage: k(1).is_one()
True
```

is_square()

Return True as every element in \( F_{2^n} \) is a square.

EXAMPLES:
sage: k.<a> = GF(2^18)
sage: e = k.random_element()
sage: e.parent() is k
True
sage: e.is_square()
True
sage: e.sqrt()^2 == e
True

is_unit()
Return True if self is nonzero, so it is a unit as an element of the finite field.

EXAMPLES:

sage: k.<a> = GF(2^17)
sage: a.is_unit()
True
sage: k(0).is_unit()
False

log(base)
Compute an integer $x$ such that $b^x = a$, where $a$ is self and $b$ is base.

INPUT:

• base – finite-field element.

OUTPUT:

Integer $x$ such that $a^x = b$, if it exists. Raises a ValueError exception if no such $x$ exists.

ALGORITHM: pari:fflog

EXAMPLES:

sage: F = FiniteField(2^10, 'a')
sage: g = F.gen()
sage: b = g; a = g^37
sage: a.log(b)
37
sage: b^37; a
a^8 + a^7 + a^4 + a + 1
a^8 + a^7 + a^4 + a + 1

Big instances used to take a very long time before trac ticket #32842:

sage: g = GF(2^61).gen()
sage: g.log(g^7)
1976436865640309101

AUTHORS:

• David Joyner and William Stein (2005-11)
• Lorenz Panny (2021-11): use PARI’s pari:fflog instead of sage.groups.generic.discrete_log()

minpoly(var='x')
Return the minimal polynomial of self, which is the smallest degree polynomial $f \in \mathbb{F}_2[x]$ such that $f(self) = 0$.  

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INPUT:

- var – string (default: 'x')

OUTPUT:
polynomial

EXAMPLES:

```python
sage: K.<a> = GF(2^100)
sage: f = a.minpoly(); f
x^100 + x^57 + x^56 + x^55 + x^48 + x^47 + x^46 + x^45 + x^44 + x^43 + x^41 + x^37 + x^36 + x^35 + x^34 + x^31 + x^30 + x^27 + x^25 + x^24 + x^22 + x^20 + x^19 + x^16 + x^15 + x^11 + x^9 + x^8 + x^6 + x^5 + x^3 + 1
sage: f(a)
0
sage: g = K.random_element()
sage: g.minpoly()(g)
0
```

`polynomial(name=None)`

Return self viewed as a polynomial over self.parent().prime_subfield().

INPUT:

- name – (optional) variable name

EXAMPLES:

```python
sage: k.<a> = GF(2^17)
sage: e = a^15 + a^13 + a^11 + a^10 + a^9 + a^8 + a^7 + a^6 + a^4 + a + 1
sage: e.polynomial()
a^17 + a^15 + a^13 + a^11 + a^10 + a^9 + a^8 + a^7 + a^6 + a^4 + a + 1
sage: from sage.rings.polynomial.polynomial_element import is_Polynomial
sage: is_Polynomial(e.polynomial())
True
sage: e.polynomial('x')
x^17 + x^15 + x^13 + x^11 + x^10 + x^9 + x^8 + x^7 + x^6 + x^4 + x + 1
```

`sqr(t(all=False, extend=False)`

Return a square root of this finite field element in its parent.

EXAMPLES:

```python
sage: k.<a> = GF(2^20)
sage: a.is_square()
True
sage: a.sqrt()
a^19 + a^15 + a^14 + a^12 + a^9 + a^7 + a^4 + a^3 + a + 1
sage: a.sqrt()^2 == a
True
```

This failed before trac ticket #4899:
trace()
Return the trace of self.

EXAMPLES:

```python
sage: K.<a> = GF(2^25)
sage: a.trace()
0
sage: a.charpoly()
x^25 + x^8 + x^6 + x^2 + 1
sage: parent(a.trace())
Finite Field of size 2
sage: b = a+1
sage: b.trace()
1
sage: b.charpoly()[1]
1
```

weight()
Returns the number of non-zero coefficients in the polynomial representation of self.

EXAMPLES:

```python
sage: K.<a> = GF(2^21)
sage: a.weight()
1
sage: (a^5+a^2+1).weight()
3
sage: b = 1/(a+1); b
a^20 + a^19 + a^18 + a^17 + a^16 + a^15 + a^14 + a^13 + a^12 + a^11 + a^10 + a^9 + a^8 + a^7 + a^6 + a^4 + a^3 + a^2
sage: b.weight()
18
```

.. note::

    sage.rings.finite_rings.element_ntl_gf2e.unpickleFiniteField_ntl_gf2eElement(parent, elem)

EXAMPLES:

```python
sage: k.<a> = GF(2^20)
sage: e = k.random_element()
sage: f = loads(dumps(e))  # indirect doctest
sage: e == f
True
```
7.1 Finite residue fields

We can take the residue field of maximal ideals in the ring of integers of number fields. We can also take the residue field of irreducible polynomials over $GF(p)$.

**EXAMPLES:**

```sage
def K.<a> = NumberField(x^3-7)
def P = K.ideal(29).factor()[0][0]
def k = K.residue_field(P)
def k
Residue field in abar of Fractional ideal (2*a^2 + 3*a - 10)
def k.order()
841
```

We reduce mod a prime for which the ring of integers is not monogenic (i.e., 2 is an essential discriminant divisor):

```sage
def K.<a> = NumberField(x^3 + x^2 - 2*x + 8)
def F = K.factor(2); F
(Fractional ideal (1/2*a^2 - 1/2*a + 1)) * (Fractional ideal (-a^2 + 2*a - 3)) *
→ (Fractional ideal (-3/2*a^2 + 5/2*a - 4))
def F[0][0].residue_field()
Residue field of Fractional ideal (1/2*a^2 - 1/2*a + 1)
def F[1][0].residue_field()
Residue field of Fractional ideal (-a^2 + 2*a - 3)
def F[2][0].residue_field()
Residue field of Fractional ideal (-3/2*a^2 + 5/2*a - 4)
```

We can also form residue fields from $\mathbb{Z}$:

```sage
def ZZ.residue_field(17)
Residue field of Integers modulo 17
```

And for polynomial rings over finite fields:

```sage
def R.<t> = GF(5)[]
def I = R.ideal(t^2 + 2)
def k = ResidueField(I); k
Residue field in tbar of Principal ideal (t^2 + 2) of Univariate Polynomial Ring in t over Finite Field of size 5
```
AUTHORS:

- David Roe (2007-10-3): initial version
- William Stein (2007-12): bug fixes
- John Cremona (2008-9): extend reduction maps to the whole valuation ring add support for residue fields of \( \mathbb{Z} \)
- David Roe (2009-12): added support for \( \mathbb{GF}(p)(t) \) and moved to new coercion framework.

class sage.rings.finite_rings.residue_field.LiftingMap
Bases: sage.categories.map.Section

Lifting map from residue class field to number field.

EXAMPLES:

```python
sage: K.<a> = NumberField(x^3 + 2)
sage: F = K.factor(5)[0][0].residue_field()
sage: F.degree()
2
case: L = F.lift_map(); L
Lifting map:
  From: Residue field in abar of Fractional ideal (a^2 + 2*a - 1)
  To: Maximal Order in Number Field in a with defining polynomial x^3 + 2
sage: L(F.0^2)
3*a + 1
sage: L(3*a + 1) == F.0^2
True
case: R.<t> = GF(13)[[]
sage: P = R.ideal(8*t^12 + 9*t^11 + 11*t^10 + 2*t^9 + 11*t^8 + 3*t^7 + 12*t^6 + t^4 → 7*t^3 + 5*t^2 + 12*t + 1)
sage: k.<a> = P.residue_field()
sage: k.lift_map()
Lifting map:
  From: Residue field in a of Principal ideal (t^12 + 6*t^11 + 3*t^10 + 10*t^9 +→ 3*t^8 + 2*t^7 + 8*t^6 + 5*t^4 + 9*t^3 + 12*t^2 + 8*t + 5) of Univariate → Polynomial in t over Finite Field of size 13
  To: Univariate Polynomial Ring in t over Finite Field of size 13
```

class sage.rings.finite_rings.residue_field.ReductionMap
Bases: sage.categories.map.Map

A reduction map from a (subset) of a number field or function field to this residue class field.

It will be defined on those elements of the field with non-negative valuation at the specified prime.

EXAMPLES:

```python
sage: I = QQ[sqrt(17)].factor(5)[0][0]; I
Fractional ideal (5)
sage: k = I.residue_field(); k
Residue field in sqrt17bar of Fractional ideal (5)
sage: R = k.reduction_map(); R
Partially defined reduction map:
  From: Number Field in sqrt17 with defining polynomial x^2 - 17 with sqrt17 = 4. →123185625617660?
  To: Residue field in sqrt17bar of Fractional ideal (5)
```

(continues on next page)
sage: R.<t> = GF(next_prime(2^20))[]; P = R.ideal(t^2 + t + 1)
sage: k = P.residue_field()
sage: k.reduction_map()

Partially defined reduction map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of size 1048583
  To:  Residue field in tbar of Principal ideal (t^2 + t + 1) of Univariate Polynomial Ring in t over Finite Field of size 1048583

section()
Computes a section of the map, namely a map that lifts elements of the residue field to elements of the field.

EXAMPLES:

sage: K.<a> = NumberField(x^5 - 5*x + 2)
sage: P = K.ideal(47).factor()[0][0]
sage: k = K.residue_field(P)
sage: f = k.convert_map_from(K)
sage: s = f.section(); s
Lifting map:
  From: Residue field in abar of Fractional ideal (-14*a^4 + 24*a^3 + 26*a^2 - 58*a + 15)
  To:  Number Field in a with defining polynomial x^5 - 5*x + 2
sage: s(K.gen())
a
sage: L.<b> = NumberField(x^5 + 17*x + 1)
sage: P = L.factor(53)[0][0]
sage: l = L.residue_field(P)
sage: g = l.convert_map_from(L)
sage: s = g.section(); s
Lifting map:
  From: Residue field in bbar of Fractional ideal (53, b^2 + 23*b + 8)
  To:  Number Field in b with defining polynomial x^5 + 17*x + 1
sage: s(L.gen()).parent()
Number Field in b with defining polynomial x^5 + 17*x + 1

sage: R.<t> = GF(2)[]; h = t^5 + t^2 + 1
sage: k.<a> = R.residue_field(h)
sage: K = R.fraction_field()
sage: f = k.convert_map_from(K)
sage: f.section()
Lifting map:
  From: Residue field in a of Principal ideal (t^5 + t^2 + 1) of Univariate Polynomial Ring in t over Finite Field of size 2 (using GF2X)
  To:  Fraction Field of Univariate Polynomial Ring in t over Finite Field of size 2 (using GF2X)

class sage.rings.finite_rings.residue_field.ResidueFieldFactory
Bases: sage.structure.factory.UniqueFactory

A factory that returns the residue class field of a prime ideal $p$ of the ring of integers of a number field, or of a polynomial ring over a finite field.

7.1. Finite residue fields
INPUT:

- $p$ – a prime ideal of an order in a number field.
- names – the variable name for the finite field created. Defaults to the name of the number field variable but with bar placed after it.
- check – whether or not to check if $p$ is prime.

OUTPUT:

- The residue field at the prime $p$.

EXAMPLES:

```sage
sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(29).factor()[0][0]
sage: ResidueField(P)
Residue field in abar of Fractional ideal (2*a^2 + 3*a - 10)
```

The result is cached:

```sage
sage: ResidueField(P) is ResidueField(P)
True
sage: k = K.residue_field(P); k
Residue field in abar of Fractional ideal (2*a^2 + 3*a - 10)
sage: k.order()
841
```

It also works for polynomial rings:

```sage
sage: R.<t> = GF(31)[]
sage: P = R.ideal(t^5 + 2*t + 11)
sage: ResidueField(P)
Residue field in tbar of Principal ideal (t^5 + 2*t + 11) of Univariate Polynomial\n⋯→Ring in t over Finite Field of size 31
sage: ResidueField(P) is ResidueField(P)
True
sage: k = ResidueField(P); k.order()
28629151
```

An example where the generator of the number field doesn’t generate the residue class field:

```sage
sage: K.<a> = NumberField(x^3-875)
sage: P = K.ideal(5).factor()[0][0]; k = K.residue_field(P); k
Residue field in abar of Fractional ideal (5, 1/25*a^2 - 2/5*a - 1)
sage: k.polynomial()
abar^2 + 3*abar + 4
sage: k.0^3 - 875
2
```

An example where the residue class field is large but of degree 1:

```sage
sage: K.<a> = NumberField(x^3-875); P = K.ideal(2007).factor()[2][0]; k = K.residue_field(P); k
Residue field of Fractional ideal (223, 1/5*a + 11)
```

(continues on next page)
And for polynomial rings:

```python
sage: R.<t> = GF(next_prime(2^18))[

sage: P = R.ideal(t - 5)
sage: k = ResidueField(P); k
Residue field of Principal ideal (t + 262142) of Univariate Polynomial Ring in t
˓→over Finite Field of size 262147
sage: k(t)
5
```

In this example, 2 is an inessential discriminant divisor, so divides the index of \( \mathbb{Z}[a] \) in the maximal order for all \( a \):

```python
sage: K.<a> = NumberField(x^3 + x^2 - 2*x + 8); P = K.ideal(2).factor()[0][0]; P
Fractional ideal (1/2*a^2 - 1/2*a + 1)
sage: F = K.residue_field(P); F
Residue field of Fractional ideal (1/2*a^2 - 1/2*a + 1)
sage: F(a)
0
sage: B = K.maximal_order().basis(); B
[1, 1/2*a^2 + 1/2*a, a^2]
sage: F(B[1])
1
sage: F(B[2])
0
sage: F
Residue field of Fractional ideal (1/2*a^2 - 1/2*a + 1)
sage: F.degree()
1
```

create_key_and_extra_args(p, names=None, check=True, impl=None, **kwds)

Return a tuple containing the key (uniquely defining data) and any extra arguments.

EXAMPLES:

```python
sage: K.<a> = NumberField(x^3-7)
sage: ResidueField(K.ideal(29).factor()[0][0]) # indirect doctest
Residue field in abar of Fractional ideal (2*a^2 + 3*a - 10)
```

create_object(version, key, **kwds)

Create the object from the key and extra arguments. This is only called if the object was not found in the cache.

EXAMPLES:

```python
sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(29).factor()[0][0]
sage: ResidueField(P) is ResidueField(P) # indirect doctest
True
```
class sage.rings.finite_rings.residue_field.ResidueFieldHomomorphism_global
    Bases: sage.rings.morphism.RingHomomorphism

    The class representing a homomorphism from the order of a number field or function field to the residue field at a given prime.

    EXAMPLES:

    sage: K.<a> = NumberField(x^3-7)
    sage: P = K.ideal(29).factor()[0][0]
    sage: k = K.residue_field(P)
    sage: OK = K.maximal_order()
    sage: abar = k(OK.1); abar
    sage: (1+abar)^179
    24*abar + 12
    sage: phi = k.coerce_map_from(OK); phi
    Ring morphism:
        From: Maximal Order in Number Field in a with defining polynomial x^3 - 7
        To: Residue field in abar of Fractional ideal (2*a^2 + 3*a - 10)
    sage: phi in Hom(OK,k)
    True
    sage: phi(OK.1)
    abar
    sage: R.<t> = GF(19)[]; P = R.ideal(t^2 + 5)
    sage: k.<a> = P.residue_field(); f = k.coerce_map_from(R)
    sage: f.lift(a^2 + 5*a + 1)
      13*a + 5

    lift(x)
      Returns a lift of x to the Order, returning a “polynomial” in the generator with coefficients between 0 and p-1.

    EXAMPLES:

    sage: K.<a> = NumberField(x^3-7)
    sage: P = K.ideal(29).factor()[0][0]
    sage: k = K.residue_field(P)
    sage: c = OK(a)
    sage: b = k(a)
    sage: f.lift(13*b + 5)
    13*a + 5
    sage: f.lift(12821*b+918)
    3*a + 19
    sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + t^2 + 7)
    sage: k.<a> = P.residue_field(); f = k.coerce_map_from(R)
    sage: f.lift(a^2 + 5*a + 1)
\[ t^2 + 5^*t + 1 \]
\[ \text{sage: } f(f.lift(a^2 + 5^*a + 1)) == a^2 + 5^*a + 1 \]
\[ \text{True} \]

**section()**

Computes a section of the map, namely a map that lifts elements of the residue field to elements of the ring of integers.

**EXAMPLES:**

\[ \text{sage: } \]
\[ \text{K.<a> = NumberField(x^5 - 5^*x + 2)} \]
\[ \text{sage: } \]
\[ \text{P = K.ideal(47).factor()[0][0]} \]
\[ \text{sage: } \]
\[ \text{k = K.residue_field(P)} \]
\[ \text{sage: } \]
\[ \text{f = k.coerce_map_from(K.ring_of_integers())} \]
\[ \text{sage: } \]
\[ \text{s = f.section(); s} \]
\[ \text{Lifting map:} \]
\[ \text{From: Residue field in abar of Fractional ideal (-14^*a^4 + 24^*a^3 + 26^*a^2 --> 58^*a + 15)} \]
\[ \text{To: Maximal Order in Number Field in a with defining polynomial x^5 - 5^*x + 2} \]
\[ \text{sage: } \]
\[ \text{s(k.gen())} \]
\[ \text{a} \]
\[ \text{sage: } \]
\[ \text{L.<b> = NumberField(x^5 + 17^*x + 1)} \]
\[ \text{sage: } \]
\[ \text{P = L.factor(53)[0][0]} \]
\[ \text{sage: } \]
\[ \text{l = L.residue_field(P)} \]
\[ \text{sage: } \]
\[ \text{g = l.coerce_map_from(L.ring_of_integers())} \]
\[ \text{sage: } \]
\[ \text{s = g.section(); s} \]
\[ \text{Lifting map:} \]
\[ \text{From: Residue field in bbar of Fractional ideal (53, b^2 + 23*b + 8) --> 53^*b + 1} \]
\[ \text{To: Maximal Order in Number Field in b with defining polynomial x^5 + 17^*x + 1} \]
\[ \text{sage: } \]
\[ \text{s(l.gen()).parent()} \]
\[ \text{Maximal Order in Number Field in b with defining polynomial x^5 + 17^*x + 1} \]
\[ \text{sage: } \]
\[ \text{R.<t> = GF(17)[]; P = R.ideal(t^3 + t^2 + 7)} \]
\[ \text{sage: } \]
\[ \text{k.<a> = P.residue_field()} \]
\[ \text{sage: } \]
\[ \text{f = k.coerce_map_from(R)} \]
\[ \text{sage: } \]
\[ \text{f.section()} \]
\[ \text{(map internal to coercion system -- copy before use)} \]
\[ \text{Lifting map:} \]
\[ \text{From: Residue field in a of Principal ideal (t^3 + t^2 + 7) of Univariate Polynomial Ring in t over Finite Field of size 17 --> Polynomial Ring in t over Finite Field of size 17} \]
\[ \text{To: Univariate Polynomial Ring in t over Finite Field of size 17} \]

**class** `sage.rings.finite_rings.residue_field.ResidueField_generic(p)`

Bases: `sage.rings.ring.Field`

The class representing a generic residue field.

**EXAMPLES:**

\[ \text{sage: } \]
\[ \text{I = QQ[i].factor(2)[0][0]; I} \]
\[ \text{Fractional ideal (I + 1)} \]
\[ \text{sage: } \]
\[ \text{k = I.residue_field(); k} \]
Residue field of Fractional ideal (I + 1)
sage: type(k)
<class 'sage.rings.finite_rings.residue_field.ResidueFiniteField_prime_modn_with_
˓→category'>

Sage: R.<t> = GF(29)[]; P = R.ideal(t^2 + 2); k.<a> = ResidueField(P); k
Residue field in a of Principal ideal (t^2 + 2) of Univariate Polynomial Ring in t˓→over Finite Field of size 29
sage: type(k)
<class 'sage.rings.finite_rings.residue_field.ResidueFiniteField_givaro_with_
˓→category'>

construction()
Construction of this residue field.
OUTPUT:
An AlgebraicExtensionFunctor and the number field that this residue field has been obtained from.
The residue field is determined by a prime (fractional) ideal in a number field. If this ideal can be coerced
into a different number field, then the construction functor applied to this number field will return the
Corresponding residue field. See trac ticket #15223.
EXAMPLES:

sage: K.<z> = CyclotomicField(7)
sage: P = K.factor(17)[0][0]
sage: k = K.residue_field(P)
sage: k
Residue field in zbar of Fractional ideal (17)
sage: F, R = k.construction()
sage: F
AlgebraicExtensionFunctor
sage: R
Cyclotomic Field of order 7 and degree 6
sage: F(R) is k
True
sage: F(ZZ)
Residue field of Integers modulo 17
sage: F(CyclotomicField(49))
Residue field in zbar of Fractional ideal (17)

ideal()
Return the maximal ideal that this residue field is the quotient by.
EXAMPLES:

sage: K.<a> = NumberField(x^3 + x + 1)
sage: P = K.ideal(29).factor()[0][0]
sage: k = K.residue_field(P) # indirect doctest
sage: k.ideal() is P
True
sage: p = next_prime(2^40); p
1099511627791
sage: k = K.residue_field(K.prime_above(p))
sage: k.ideal().norm() == p
True

sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + t^2 + 7)
sage: k.<a> = R.residue_field(P)
sage: k.ideal()
Principal ideal (t^3 + t^2 + 7) of Univariate Polynomial Ring in t over Finite Field of size 17

lift(x)
Returns a lift of x to the Order, returning a “polynomial” in the generator with coefficients between 0 and p − 1.

EXAMPLES:

sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(29).factor()[0][0]
sage: k =K.residue_field(P)
sage: c = K.maximal_order()
sage: b = k(a)
sage: k.lift(13*b + 5)
13*a + 5
sage: k.lift(12821*b+918)
3*a + 19

sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + t^2 + 7)
sage: k.<a> = P.residue_field()
sage: k.lift(a^2 + 5)
t^2 + 5

lift_map()
Returns the standard map from this residue field up to the ring of integers lifting the canonical projection.

EXAMPLES:

sage: I = QQ[3^(1/3)].factor(5)[1][0]; I
Fractional ideal (-a + 2)
sage: k = I.residue_field(); k
Residue field of Fractional ideal (-a + 2)
sage: f = k.lift_map(); f
Lifting map:
  From: Residue field of Fractional ideal (-a + 2)
  To: Maximal Order in Number Field in a with defining polynomial x^3 - 3
      with a = 1.442249570307409?

sage: f.domain()
Residue field of Fractional ideal (-a + 2)
sage: f.codomain()
Maximal Order in Number Field in a with defining polynomial x^3 - 3 with a = 1.442249570307409?
sage: f(k.0)
1

(continues on next page)
sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + t^2 + 7)
sage: k.<a> = P.residue_field()
sage: f = k.lift_map(); f
(map internal to coercion system -- copy before use)
Lifting map:
  From: Residue field in a of Principal ideal (t^3 + t^2 + 7) of Univariate
 Polynomial Ring in t over Finite Field of size 17
  To:  Univariate Polynomial Ring in t over Finite Field of size 17
sage: f(a^2 + 5)
t^2 + 5

reduction_map()
Return the partially defined reduction map from the number field to this residue class field.

EXAMPLES:

sage: I = QQ[2^(1/3)].factor(2)[0][0]; I
Fractional ideal (a)
sage: k = I.residue_field(); k
Residue field of Fractional ideal (a)
sage: pi = k.reduction_map(); pi
Partially defined reduction map:
  From: Number Field in a with defining polynomial x^3 - 2 with a = 1.
    259921049894873?
  To:  Residue field of Fractional ideal (a)
sage: pi.domain()
Number Field in a with defining polynomial x^3 - 2 with a = 1.259921049894873?
sage: pi.codomain()
Residue field of Fractional ideal (a)

sage: K.<a> = NumberField(x^3 + x^2 - 2*x + 32)
sage: F = K.factor(2)[0][0].residue_field()
sage: F.reduction_map().domain()
Number Field in a with defining polynomial x^3 + x^2 - 2*x + 32
sage: K.<a> = NumberField(x^3 + 128)
sage: F = K.factor(2)[0][0].residue_field()
sage: F.reduction_map().codomain()
Residue field of Fractional ideal (1/4*a)

sage: R.<t> = GF(17)[]; P = R.ideal(t^3 + t^2 + 7)
sage: k.<a> = P.residue_field(); f = k.reduction_map(); f
Partially defined reduction map:
  From: Fraction Field of Univariate Polynomial Ring in t over Finite Field of
 size 17
  To:  Residue field in a of Principal ideal (t^3 + t^2 + 7) of Univariate
 Polynomial Ring in t over Finite Field of size 17
sage: f(1/t)
12*a^2 + 12*a

class sage.rings.finite_rings.residue_field.ResidueFiniteField_givaro(p, q, name, modulus,
to_vs, to_order, PB)
Bases: sage.rings.finite_rings.residue_field.ResidueField_generic, sage.rings.finite_rings.finite_field_givaro.FiniteField_givaro

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The class representing residue fields of number fields that have non-prime order strictly less than $2^{16}$.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(29).factor()[0][0]
sage: k = K.residue_field(P)
sage: k.degree()
2
sage: OK = K.maximal_order()
sage: c = OK(a)
sage: b = k(c)
sage: b*a^2
7
sage: b*a
13*a
```

```
sage: R.<t> = GF(7)[]; P = R.ideal(t^19 + t^5 + t^2 + t + 1)
sage: k.<a> = R.residue_field(P); type(k)
<class 'sage.rings.finite_rings.residue_field.ResidueFiniteField_givaro_with_category'>
sage: k(1/t)
5*a
```

```class sage.rings.finite_rings.residue_field.ResidueFiniteField_ntl_gf2e(q, name, modulus, repr, p, to_vs, to_order, PB)
```

Bases: `sage.rings.finite_rings.residue_field.ResidueField_generic`, `sage.rings.finite_rings.finite_field_ntl_gf2e.FiniteField_ntl_gf2e`

The class representing residue fields with order a power of 2.

When the order is less than $2^{16}$, givaro is used by default instead.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(29).factor()[0][0]
sage: k = K.residue_field(P)
sage: k.degree()
2
sage: OK = K.maximal_order()
sage: c = OK(a)
sage: b = k(c)
sage: b*a^2
7
sage: b*a
13*a
```

(continues on next page)
a^{18} + a^4 + a + 1
sage: k(1/t)*t
1

class sage.rings.finite_rings.residue_field.ResidueFiniteField_pari_ffelt(p, characteristic, name, modulus, to_vs, to_order, PB)

Bases: sage.rings.finite_rings.residue_field.ResidueField_generic, sage.rings.finite_rings.finite_field_pari_ffelt.FiniteField_pari_ffelt

The class representing residue fields of number fields that have non-prime order at least $2^6$.

EXAMPLES:

sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(923478923).factor()[0][0]
sage: k = K.residue_field(P)
sage: k.degree()
2
sage: OK = K.maximal_order()
sage: c = OK(a)
sage: b = k(c)
2*a
sage: b+c
664346875*a + 535606347
sage: k.base_ring()
Finite Field of size 923478923

sage: R.<t> = GF(5)[]; P = R.ideal(4*t^12 + 3*t^11 + 4*t^10 + t^9 + t^8 + 3*t^7 + 2*t^6 + 3*t^4 + t^3 + 3*t^2 + 2)
sage: k.<a> = P.residue_field()
sage: type(k)
<class 'sage.rings.finite_rings.residue_field.ResidueFiniteField_pari_ffelt_with_category'>
sage: k(1/t)
3*a^11 + a^10 + 3*a^9 + 2*a^8 + 2*a^7 + a^6 + 4*a^5 + a^3 + 2*a^2 + a

class sage.rings.finite_rings.residue_field.ResidueFiniteField_prime_modn(p, name, intp, to_vs, to_order, PB)

Bases: sage.rings.finite_rings.residue_field.ResidueField_generic, sage.rings.finite_rings.finite_field_prime_modn.FiniteField_prime_modn

The class representing residue fields of number fields that have prime order.

EXAMPLES:

sage: R.<x> = QQ[]
sage: K.<a> = NumberField(x^3-7)
sage: P = K.ideal(29).factor()[1][0]
sage: k = ResidueField(P)
sage: k
Residue field of Fractional ideal (-a^2 - 2*a - 2)
7.2 Algebraic closures of finite fields

Let $F$ be a finite field, and let $\overline{F}$ be an algebraic closure of $F$; this is unique up to (non-canonical) isomorphism. For every $n \geq 1$, there is a unique subfield $F_n$ of $\overline{F}$ such that $F \subset F_n$ and $[F_n : F] = n$.

In Sage, algebraic closures of finite fields are implemented using compatible systems of finite fields. The resulting Sage object keeps track of a finite lattice of the subfields $F_n$ and the embeddings between them. This lattice is extended as necessary.

The Sage class corresponding to $\overline{F}$ can be constructed from the finite field $F$ by using the `algebraic_closure()` method.

The Sage class for elements of $\overline{F}$ is `AlgebraicClosureFiniteFieldElement`. Such an element is represented as an element of one of the $F_n$. This means that each element $x \in F$ has infinitely many different representations, one for each $n$ such that $x$ is in $F_n$.

**Note:** Only prime finite fields are currently accepted as base fields for algebraic closures. To obtain an algebraic closure of a non-prime finite field $F$, take an algebraic closure of the prime field of $F$ and embed $F$ into this.

Algebraic closures of finite fields are currently implemented using (pseudo-)Conway polynomials; see `AlgebraicClosureFiniteField_pseudo_conway` and the module `conway_polynomials`. Other implementations may be added by creating appropriate subclasses of `AlgebraicClosureFiniteField_generic`.

In the current implementation, algebraic closures do not satisfy the unique parent condition. Moreover, there is no coercion map between different algebraic closures of the same finite field. There is a conceptual reason for this, namely that the definition of pseudo-Conway polynomials only determines an algebraic closure up to non-unique isomorphism. This means in particular that different algebraic closures, and their respective elements, never compare equal.

**AUTHORS:**
Construct an algebraic closure of a finite field.

The recommended way to use this functionality is by calling the \texttt{algebraic\_closure()} method of the finite field.

\textbf{Note:} Algebraic closures of finite fields in Sage do not have the unique representation property, because they are not determined up to unique isomorphism by their defining data.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.rings.algebraic_closure_finite_field import AlgebraicClosureFiniteField
sage: F = GF(2).algebraic_closure()
sage: F1 = AlgebraicClosureFiniteField(GF(2), 'z')
sage: F1 == F
False
\end{verbatim}

In the pseudo-Conway implementation, non-identical instances never compare equal:

\begin{verbatim}
sage: F1 == F
False
sage: loads(dumps(F)) == F
False
\end{verbatim}

This is to ensure that the result of comparing two instances cannot change with time.

\textbf{class} \texttt{sage.rings.algebraic_closure_finite_field.AlgebraicClosureFiniteFieldElement}(\textit{parent}, \textit{value})

\textbf{Bases:} \texttt{sage.structure.element.FieldElement}

Element of an algebraic closure of a finite field.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: F = GF(3).algebraic_closure()
sage: F.gen(2)
z2
sage: type(F.gen(2))
<class 'sage.rings.algebraic_closure_finite_field.AlgebraicClosureFiniteField_element_with_category.element_class'>
\end{verbatim}

\textbf{as\_finite\_field\_element}(\textit{minimal}=\texttt{False})

Return \texttt{self} as a finite field element.

\textbf{INPUT:}

\begin{itemize}
  \item \texttt{minimal} – boolean (default: \texttt{False}). If \texttt{True}, always return the smallest subfield containing \texttt{self}.
\end{itemize}

\textbf{OUTPUT:}
• a triple (field, element, morphism) where field is a finite field, element an element of field and morphism a morphism from field to self.parent().

EXAMPLES:

```
sage: F = GF(3).algebraic_closure('t')
sage: t = F.gen(5)
sage: t.as_finite_field_element()  
(Finite Field in t5 of size 3^5,  
t5,  
Ring morphism:  
  From: Finite Field in t5 of size 3^5  
  To:   Algebraic closure of Finite Field of size 3  
  Defn: t5 |--> t5)
```

By default, field is not necessarily minimal. We can force it to be minimal using the minimal option:

```
sage: s = t + 1 - t
sage: s.as_finite_field_element()[0]  
Finite Field in t5 of size 3^5
sage: s.as_finite_field_element(minimal=True)[0]  
Finite Field of size 3
```

This also works when the element has to be converted between two non-trivial finite subfields (see trac ticket #16509):

```
sage: K = GF(5).algebraic_closure()  
sage: z = K.gen(5) - K.gen(5) + K.gen(2)  
sage: z.as_finite_field_element(minimal=True)  
(Finite Field in z2 of size 5^2, z2, Ring morphism:  
  From: Finite Field in z2 of size 5^2  
  To:   Algebraic closure of Finite Field of size 5  
  Defn: z2 |--> z2)
```

There are automatic coercions between the various subfields:

```
sage: a = K.gen(2) + 1  
sage: _,b,_ = a.as_finite_field_element()  
sage: K4 = K.subfield(4)[0]  
sage: K4(b)  
z4^3 + z4^2 + z4 + 4
sage: b.minimal_polynomial() == K4(b).minimal_polynomial()  
True  
sage: K(K4(b)) == K(b)  
True
```

You can also use the inclusions that are implemented at the level of the algebraic closure:

```
sage: f = K.inclusion(2,4); f  
Ring morphism:  
  From: Finite Field in z2 of size 5^2  
  To:   Finite Field in z4 of size 5^4  
  Defn: z2 |--> z4^3 + z4^2 + z4 + 3
sage: f(b)  
z4^3 + z4^2 + z4 + 4
```
change_level\((n)\)

Return a representation of self as an element of the subfield of degree \(n\) of the parent, if possible.

EXAMPLES:

```python
sage: F = GF(3).algebraic_closure()
sage: z = F.gen(4)
sage: (z^10).change_level(6)
2*z6^5 + 2*z6^3 + z6^2 + 2*z6 + 2

sage: z.change_level(6)
Traceback (most recent call last):
... ValueError: z4 is not in the image of Ring morphism:
  From: Finite Field in z2 of size 3^2
  To: Finite Field in z4 of size 3^4
  Defn: z2 |--> 2*z4^3 + 2*z4^2 + 1

sage: a = F(1).change_level(3); a
1
sage: a.change_level(2)
1
sage: F.gen(3).change_level(1)
Traceback (most recent call last):
... ValueError: z3 is not in the image of Ring morphism:
  From: Finite Field of size 3
  To: Finite Field in z3 of size 3^3
  Defn: 1 |--> 1
```

is_square()

Return True if self is a square.

This always returns True.

EXAMPLES:

```python
sage: F = GF(3).algebraic_closure()
sage: F.gen(2).is_square()
True
```

minimal_polynomial()

Return the minimal polynomial of self over the prime field.

EXAMPLES:

```python
sage: F = GF(11).algebraic_closure()
sage: F.gen(3).minpoly()
x^3 + 2*x + 9
```

minpoly()

Return the minimal polynomial of self over the prime field.

EXAMPLES:

```python
sage: F = GF(11).algebraic_closure()
sage: F.gen(3).minpoly()
x^3 + 2*x + 9
```
**multiplicative_order()**

Return the multiplicative order of self.

EXAMPLES:

```
sage: K = GF(7).algebraic_closure()
sage: K.gen(5).multiplicative_order()
16806
sage: (K.gen(1) + K.gen(2) + K.gen(3)).multiplicative_order()
7353
```

**nth_root(n)**

Return an n-th root of self.

EXAMPLES:

```
sage: F = GF(5).algebraic_closure()
sage: t = F.gen(2) + 1
sage: s = t.nth_root(15); s
4*z6^5 + 3*z6^4 + 2*z6^3 + 2*z6^2 + 4
sage: s**15 == t
True
```

Todo: This function could probably be made faster.

**pth_power(k=1)**

Return the $p^k$-th power of self, where $p$ is the characteristic of self.parent().

EXAMPLES:

```
sage: K = GF(13).algebraic_closure('t')
sage: t3 = K.gen(3)
sage: s = 1 + t3 + t3**2
sage: s.pth_power()
10*t3^2 + 6*t3
sage: s.pth_power(2)
2*t3^2 + 6*t3 + 11
sage: s.pth_power(3)
t3^2 + t3 + 1
sage: s.pth_power(3).parent() is K
True
```

**pth_root(k=1)**

Return the unique $p^k$-th root of self, where $p$ is the characteristic of self.parent().

EXAMPLES:

```
sage: K = GF(13).algebraic_closure('t')
sage: t3 = K.gen(3)
sage: s = 1 + t3 + t3**2
sage: s.pth_root()
2*t3^2 + 6*t3 + 11
sage: s.pth_root(2)
10*t3^2 + 6*t3
```

(continues on next page)
sage: s.pth_root(3)
t3^2 + t3 + 1
sage: s.pth_root(2).parent() is K
True

sqrt()
Return a square root of self.

EXAMPLES:

```python
sage: F = GF(3).algebraic_closure()
sage: F.gen(2).sqrt()
z4^3 + z4 + 1
```

class sage.rings.algebraic_closure_finite_field.AlgebraicClosureFiniteField_generic(base_ring, name, category=None)

Bases: sage.rings.ring.Field

Algebraic closure of a finite field.

Element
alias of AlgebraicClosureFiniteFieldElement

algebraic_closure()
Return an algebraic closure of self.

This always returns self.

EXAMPLES:

```python
sage: from sage.rings.algebraic_closure_finite_field import AlgebraicClosureFiniteField
sage: F = AlgebraicClosureFiniteField(GF(5), 'z')
sage: F.algebraic_closure() is F
True
```

characteristic()
Return the characteristic of self.

EXAMPLES:

```python
sage: from sage.rings.algebraic_closure_finite_field import AlgebraicClosureFiniteField
sage: p = next_prime(1000)
sage: F = AlgebraicClosureFiniteField(GF(p), 'z')
sage: F.characteristic() == p
True
```

gen(n)
Return the $n$-th generator of self.

EXAMPLES:
sage: from sage.rings.algebraic_closure_finite_field import AlgebraicClosureFiniteField
sage: F = AlgebraicClosureFiniteField(GF(5), 'z')
sage: F.gen(2)
z2

gens()
Return a family of generators of self.

OUTPUT:
• a Family, indexed by the positive integers, whose \(n\)-th element is \(self.gen(n)\).

EXAMPLES:

sage: from sage.rings.algebraic_closure_finite_field import AlgebraicClosureFiniteField
sage: F = AlgebraicClosureFiniteField(GF(5), 'z')
sage: g = F.gens(); g
Lazy family (...(i))_{i in Positive integers}
sage: g[3]
z3

inclusion\((m, n)\)
Return the canonical inclusion map from the subfield of degree \(m\) to the subfield of degree \(n\).

EXAMPLES:

sage: F = GF(3).algebraic_closure()
sage: F.inclusion(1, 2)
Ring morphism:
  From: Finite Field of size 3
  To:  Finite Field in z2 of size 3^2
  Defn: 1 |--> 1
sage: F.inclusion(2, 4)
Ring morphism:
  From: Finite Field in z2 of size 3^2
  To:  Finite Field in z4 of size 3^4
  Defn: z2 |--> 2^4 z4^3 + 2^2 z4^2 + 1

ngens()
Return the number of generators of self, which is infinity.

EXAMPLES:

sage: from sage.rings.algebraic_closure_finite_field import AlgebraicClosureFiniteField
sage: AlgebraicClosureFiniteField(GF(5), 'z').ngens()
+Infinity

some_elements()
Return some elements of this field.

EXAMPLES:
sage: F = GF(7).algebraic_closure()
sage: F.some_elements()
(1, z2, z3 + 1)

subfield\((n)\)
Return the unique subfield of degree \(n\) of self together with its canonical embedding into self.

EXAMPLES:

sage: F = GF(3).algebraic_closure()
sage: F.subfield(1)
(Finite Field of size 3,
 Ring morphism:
  From: Finite Field of size 3
  To:   Algebraic closure of Finite Field of size 3
      Defn: 1 |---> 1)
sage: F.subfield(4)
(Finite Field in z4 of size 3^4,
 Ring morphism:
  From: Finite Field in z4 of size 3^4
  To:   Algebraic closure of Finite Field of size 3
      Defn: z4 |---> z4)

class sage.rings.algebraic_closure_finite_field.AlgebraicClosureFiniteField_pseudo_conway(base_ring, name, category=None, lattice=None, use_database=True)

Bases: sage.misc.fast_methods.WithEqualityById, 
sage.rings.algebraic_closure_finite_field.AlgebraicClosureFiniteField_generic

Algebraic closure of a finite field, constructed using pseudo-Conway polynomials.

EXAMPLES:

sage: F = GF(5).algebraic_closure(implementation='pseudo_conway')
sage: F.cardinality()
+Infinity
sage: F.algebraic_closure() is F
True
sage: x = F(3).nth_root(12); x
z4^3 + z4^2 + 4*z4
sage: x**12
3
7.3 Routines for Conway and pseudo-Conway polynomials

AUTHORS:
- David Roe
- Jean-Pierre Flori
- Peter Bruin

class sage.rings.finite_rings.conway_polynomials.PseudoConwayLattice(p, use_database=True)
   Bases: sage.misc.fast_methods.WithEqualityById, sage.structure.sage_object.SageObject

A pseudo-Conway lattice over a given finite prime field.

The Conway polynomial \( f_n \) of degree \( n \) over \( \mathbb{F}_p \) is defined by the following four conditions:
- \( f_n \) is irreducible.
- In the quotient field \( \mathbb{F}_p[x]/(f_n) \), the element \( x \mod f_n \) generates the multiplicative group.
- The minimal polynomial of \( (x \mod f_n)^{\frac{p^n-1}{p-1}} \) equals the Conway polynomial \( f_m \), for every divisor \( m \) of \( n \).
- \( f_n \) is lexicographically least among all such polynomials, under a certain ordering.

The final condition is needed only in order to make the Conway polynomial unique. We define a pseudo-Conway lattice to be any family of polynomials, indexed by the positive integers, satisfying the first three conditions.

INPUT:
- \( p \) – prime number
- \( \text{use\_database} \) – boolean. If True, use actual Conway polynomials whenever they are available in the database. If False, always compute pseudo-Conway polynomials.

EXAMPLES:

```
sage: from sage.rings.finite_rings.conway_polynomials import PseudoConwayLattice
sage: PCL = PseudoConwayLattice(2, use_database=False)
sage: PCL.polynomial(3)
x^3 + x + 1
```

```
sage: from sage.rings.finite_rings.conway_polynomials import PseudoConwayLattice
sage: PCL = PseudoConwayLattice(2, use_database=False)
sage: PCL.check_consistency(6)
sage: PCL.check_consistency(60)  # long time
```

```
sage: from sage.rings.finite_rings.conway_polynomials import PseudoConwayLattice
sage: PCL = PseudoConwayLattice(2, use_database=False)
sage: PCL.polynomial(6)
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sage: from sage.rings.finite_rings.conway_polynomials import PseudoConwayLattice
sage: PCL = PseudoConwayLattice(2, use_database=False)
sage: PCL.polynomial(6)
sage: PCL.polynomial(60)  # long time
```
ALGORITHM:

Uses an algorithm described in [HL1999], modified to find pseudo-Conway polynomials rather than Con-
way polynomials. The major difference is that we stop as soon as we find a primitive polynomial.

EXAMPLES:

```python
sage: from sage.rings.finite_rings.conway_polynomials import PseudoConwayLattice
sage: PCL = PseudoConwayLattice(2, use_database=False)
sage: PCL.polynomial(3)
x^3 + x + 1
sage: PCL.polynomial(4)
x^4 + x^3 + 1
sage: PCL.polynomial(60)
x^60 + x^59 + x^58 + x^55 + x^54 + x^53 + x^52 + x^51 + x^48 + x^46 + x^45 + x^44 + x^43 + x^42 + x^41 + x^39 + x^38 + x^37 + x^35 + x^34 + x^33 + x^32 + x^31 + x^30 + x^28 + x^24 + x^22 + x^21 + x^18 + x^17 + x^16 + x^15 + x^14 + x^10 + x^8 + x^7 + x^5 + x^3 + x^2 + x + 1
```

sage.rings.finite_rings.conway_polynomials.conway_polynomial\((p, n)\)

Return the Conway polynomial of degree \(n\) over \(\text{GF}(p)\).

If the requested polynomial is not known, this function raises a `RuntimeError` exception.

INPUT:

- \(p\) – prime number
- \(n\) – positive integer

OUTPUT:

- the Conway polynomial of degree \(n\) over the finite field \(\text{GF}(p)\), loaded from a table.

Note: The first time this function is called a table is read from disk, which takes a fraction of a second. Subse-
quent calls do not require reloading the table.

See also the `ConwayPolynomials()` object, which is the table of Conway polynomials used by this function.

EXAMPLES:

```python
sage: conway_polynomial(2,5)
x^5 + x^2 + 1
sage: conway_polynomial(101,5)
x^5 + 2^x + 99
sage: conway_polynomial(97,101)
Traceback (most recent call last):
  ...  
RuntimeError: requested Conway polynomial not in database.
```

sage.rings.finite_rings.conway_polynomials.exists_conway_polynomial\((p, n)\)

Check whether the Conway polynomial of degree \(n\) over \(\text{GF}(p)\) is known.

INPUT:

- \(p\) – prime number
- \(n\) – positive integer

OUTPUT:
• boolean: True if the Conway polynomial of degree \(n\) over GF(\(p\)) is in the database, False otherwise.

If the Conway polynomial is in the database, it can be obtained using the command `conway_polynomial(p, n)`.

**EXAMPLES:**

```plaintext
sage: exists_conway_polynomial(2, 3)
True
sage: exists_conway_polynomial(2, -1)
False
sage: exists_conway_polynomial(97, 200)
False
sage: exists_conway_polynomial(6, 6)
False
```
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