Hyperbolic Geometry

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CONTENTS

CHAPTER

ONE

HYPERBOLIC POINTS

This module implements points in hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

This module also implements ideal points in hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

Note that not all models of hyperbolic space are bounded, meaning that the ideal boundary is not the topological boundary of the set underlying tho model. For example, the unit disk model is bounded with boundary given by the unit sphere. The hyperboloid model is not bounded.

AUTHORS:

- Greg Laun (2013): initial version

EXAMPLES:

We can construct points in the upper half plane model, abbreviated UHP for convenience:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_point(2 + I)
Point in UHP I + 2
sage: g = UHP.get_point(3 + I)
sage: g.dist(UHP.get_point(I))
arccosh(11/2)
```

We can also construct boundary points in the upper half plane model:

```
sage: UHP.get_point(3)
Boundary point in UHP 3
```

Some more examples:

```
sage: HyperbolicPlane().UHP().get_point(0)
Boundary point in UHP 0

sage: HyperbolicPlane().PD().get_point(I/2)
Point in PD 1/2*I

sage: HyperbolicPlane().KM().get_point((0,1))
Boundary point in KM (0, 1)

sage: HyperbolicPlane().HM().get_point((0,0,1))
Point in HM (0, 0, 1)
```

class sage.geometry.hyperbolic_space.hyperbolic_point.**HyperbolicPoint**(*model*, *coordinates*, *is_boundary*, *check=True*, ***graphics_options*)

Bases: `Element`

Abstract base class for hyperbolic points. This class should never be instantiated.

INPUT:

- `model` – the model of the hyperbolic space
- `coordinates` – the coordinates of a hyperbolic point in the appropriate model
- `is_boundary` – whether the point is a boundary point
- `check` – (default: `True`) if `True`, then check to make sure the coordinates give a valid point in the model

EXAMPLES:

Comparison between different models is performed via coercion:

```
sage: UHP = HyperbolicPlane().UHP()
sage: p = UHP.get_point(.2 + .3*I); p
Point in UHP 0.200000000000000 + 0.300000000000000*I

sage: PD = HyperbolicPlane().PD()
sage: q = PD.get_point(0.2 + 0.3*I); q
Point in PD 0.200000000000000 + 0.300000000000000*I

sage: p == q
False
sage: PD(p)
Point in PD 0.231213872832370 - 0.502890173410405*I

sage: bool(p.coordinates() == q.coordinates())
True
```

Similarly for boundary points:

```
sage: p = UHP.get_point(-1); p
Boundary point in UHP -1

sage: q = PD.get_point(-1); q
Boundary point in PD -1

sage: p == q
True
sage: PD(p)
Boundary point in PD -1
```

It is an error to specify a point that does not lie in the appropriate model:

```
sage: HyperbolicPlane().UHP().get_point(0.2 - 0.3*I)
Traceback (most recent call last):
...
ValueError: 0.200000000000000 - 0.300000000000000*I is not a valid point in the␣
```

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```
↪UHP model

sage: HyperbolicPlane().PD().get_point(1.2)
Traceback (most recent call last):
...
ValueError: 1.20000000000000 is not a valid point in the PD model

sage: HyperbolicPlane().KM().get_point((1,1))
Traceback (most recent call last):
...
ValueError: (1, 1) is not a valid point in the KM model

sage: HyperbolicPlane().HM().get_point((1, 1, 1))
Traceback (most recent call last):
...
ValueError: (1, 1, 1) is not a valid point in the HM model
```

It is an error to specify an interior point of hyperbolic space as a boundary point:

```
sage: HyperbolicPlane().UHP().get_point(0.2 + 0.3*I, is_boundary=True)
Traceback (most recent call last):
...
ValueError: 0.200000000000000 + 0.300000000000000*I is not a valid boundary point␣
↪in the UHP model
```

coordinates()

Return the coordinates of the point.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(2 + I).coordinates()
I + 2

sage: HyperbolicPlane().PD().get_point(1/2 + 1/2*I).coordinates()
1/2*I + 1/2

sage: HyperbolicPlane().KM().get_point((1/3, 1/4)).coordinates()
(1/3, 1/4)

sage: HyperbolicPlane().HM().get_point((0,0,1)).coordinates()
(0, 0, 1)
```

graphics_options()

Return the graphics options of the current point.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().get_point(2 + I, color="red")
sage: p.graphics_options()
{'color': 'red'}
```

is_boundary()

Return `True` if `self` is a boundary point.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: p = PD.get_point(0.5+.2*I)
sage: p.is_boundary()
False
sage: p = PD.get_point(I)
sage: p.is_boundary()
True
```

model()

Return the model to which the `HyperbolicPoint` belongs.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(I).model()
Hyperbolic plane in the Upper Half Plane Model

sage: HyperbolicPlane().PD().get_point(0).model()
Hyperbolic plane in the Poincare Disk Model

sage: HyperbolicPlane().KM().get_point((0,0)).model()
Hyperbolic plane in the Klein Disk Model

sage: HyperbolicPlane().HM().get_point((0,0,1)).model()
Hyperbolic plane in the Hyperboloid Model
```

show(*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

```
sage: HyperbolicPlane().PD().get_point(0).show()
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().KM().get_point((0,0)).show()
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().HM().get_point((0,0,1)).show()
Graphics3d Object
```

symmetry_involution()

Return the involutory isometry fixing the given point.

EXAMPLES:

```
sage: z = HyperbolicPlane().UHP().get_point(3 + 2*I)
sage: z.symmetry_involution()
Isometry in UHP
[  3/2 -13/2]
[  1/2  -3/2]

sage: HyperbolicPlane().UHP().get_point(I).symmetry_involution()
Isometry in UHP
[ 0 -1]
[ 1  0]

sage: HyperbolicPlane().PD().get_point(0).symmetry_involution()
Isometry in PD
[-I  0]
[ 0  I]
```

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```
sage: HyperbolicPlane().KM().get_point((0, 0)).symmetry_involution()
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

sage: HyperbolicPlane().HM().get_point((0,0,1)).symmetry_involution()
Isometry in HM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

sage: p = HyperbolicPlane().UHP().random_element()
sage: A = p.symmetry_involution()
sage: p.dist(A*p)   # abs tol 1e-10
0

sage: A.preserves_orientation()
True

sage: A*A == HyperbolicPlane().UHP().get_isometry(identity_matrix(2))
True
```

to_model(*model*)

Convert `self` to the `model`.

INPUT:

- `other` – (a string representing) the image model

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: PD = HyperbolicPlane().PD()
sage: PD.get_point(1/2+I/2).to_model(UHP)
Point in UHP I + 2
sage: PD.get_point(1/2+I/2).to_model('UHP')
Point in UHP I + 2
```

update_graphics(*update=False*, ***options*)

Update the graphics options of a *HyperbolicPoint*. If `update` is `True`, update rather than overwrite.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().get_point(I); p.graphics_options()
{}

sage: p.update_graphics(color = "red"); p.graphics_options()
{'color': 'red'}

sage: p.update_graphics(color = "blue"); p.graphics_options()
{'color': 'blue'}

sage: p.update_graphics(True, size = 20); p.graphics_options()
{'color': 'blue', 'size': 20}
```

class sage.geometry.hyperbolic_space.hyperbolic_point.**HyperbolicPointUHP**(*model*, *coordinates*, *is_boundary*, *check=True*, ***graphics_options*)

Bases: *HyperbolicPoint*

A point in the UHP model.

INPUT:

- the coordinates of a point in the unit disk in the complex plane **C**

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(2*I)
Point in UHP 2*I

sage: HyperbolicPlane().UHP().get_point(1)
Boundary point in UHP 1
```

show(*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(I).show()
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().UHP().get_point(0).show()
Graphics object consisting of 2 graphics primitives
sage: HyperbolicPlane().UHP().get_point(infinity).show()
Traceback (most recent call last):
...
NotImplementedError: can...t draw the point infinity
```

symmetry_involution()

Return the involutory isometry fixing the given point.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(3 + 2*I).symmetry_involution()
Isometry in UHP
[  3/2 -13/2]
[  1/2  -3/2]
```

CHAPTER

TWO

HYPERBOLIC ISOMETRIES

This module implements the abstract base class for isometries of hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

The isometry groups of all implemented models are either matrix Lie groups or are doubly covered by matrix Lie groups. As such, the isometry constructor takes a matrix as input. However, since the isometries themselves may not be matrices, quantities like the trace and determinant are not directly accessible from this class.

AUTHORS:

- Greg Laun (2013): initial version

EXAMPLES:

We can construct isometries in the upper half plane model, abbreviated UHP for convenience:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(matrix(2,[1,2,3,4]))
Isometry in UHP
[1 2]
[3 4]
sage: A = UHP.get_isometry(matrix(2,[0,1,1,0]))
sage: A.inverse()
Isometry in UHP
[0 1]
[1 0]
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometry**(*model*, *A*, *check=True*)

Bases: `Morphism`

Abstract base class for hyperbolic isometries. This class should never be instantiated.

INPUT:

- `A` – a matrix representing a hyperbolic isometry in the appropriate model

EXAMPLES:

```
sage: HyperbolicPlane().HM().get_isometry(identity_matrix(3))
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
```

attracting_fixed_point()

For a hyperbolic isometry, return the attracting fixed point; otherwise raise a `ValueError`.

OUTPUT:

- a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = UHP.get_isometry(Matrix(2,[4,0,0,1/4]))
sage: A.attracting_fixed_point()
Boundary point in UHP +Infinity
```

axis()

For a hyperbolic isometry, return the axis of the transformation; otherwise raise a `ValueError`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2,[2,0,0,1/2]))
sage: H.axis()
Geodesic in UHP from 0 to +Infinity
```

It is an error to call this function on an isometry that is not hyperbolic:

```
sage: P = UHP.get_isometry(matrix(2,[1,4,0,1]))
sage: P.axis()
Traceback (most recent call last):
...
ValueError: the isometry is not hyperbolic: axis is undefined
```

classification()

Classify the hyperbolic isometry as elliptic, parabolic, hyperbolic or a reflection.

A hyperbolic isometry fixes two points on the boundary of hyperbolic space, a parabolic isometry fixes one point on the boundary of hyperbolic space, and an elliptic isometry fixes no points.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2,[2,0,0,1/2]))
sage: H.classification()
'hyperbolic'

sage: P = UHP.get_isometry(matrix(2,[1,1,0,1]))
sage: P.classification()
'parabolic'

sage: E = UHP.get_isometry(matrix(2,[-1,0,0,1]))
sage: E.classification()
'reflection'
```

fixed_geodesic()

If `self` is a reflection in a geodesic, return that geodesic.

EXAMPLES:

```
sage: A = HyperbolicPlane().PD().get_isometry(matrix([[0, 1], [1, 0]]))
sage: A.fixed_geodesic()
Geodesic in PD from -1 to 1
```

fixed_point_set()

Return a list containing the fixed point set of orientation-preserving isometries.

OUTPUT:

list of hyperbolic points or a hyperbolic geodesic

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: H = KM.get_isometry(matrix([[5/3,0,4/3], [0,1,0], [4/3,0,5/3]]))
sage: g = H.fixed_point_set(); g
Geodesic in KM from (1, 0) to (-1, 0)
sage: H(g.start()) == g.start()
True
sage: H(g.end()) == g.end()
True
sage: A = KM.get_isometry(matrix([[1,0,0], [0,-1,0], [0,0,1]]))
sage: A.preserves_orientation()
False
sage: A.fixed_point_set()
Geodesic in KM from (1, 0) to (-1, 0)
```

```
sage: B = KM.get_isometry(identity_matrix(3))
sage: B.fixed_point_set()
Traceback (most recent call last):
...
ValueError: the identity transformation fixes the entire hyperbolic plane
```

is_identity()

Return `True` if `self` is the identity isometry.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(matrix(2,[4,1,3,2])).is_identity()
False
sage: UHP.get_isometry(identity_matrix(2)).is_identity()
True
```

matrix()

Return the matrix of the isometry.

Note: We do not allow the `matrix` constructor to work as these may be elements of a projective group (ex. $PSL(n, \mathbf{R})$), so these isometries aren't true matrices.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(-identity_matrix(2)).matrix()
[-1  0]
[ 0 -1]
```

model()

Return the model to which `self` belongs.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_isometry(identity_matrix(2)).model()
Hyperbolic plane in the Upper Half Plane Model

sage: HyperbolicPlane().PD().get_isometry(identity_matrix(2)).model()
Hyperbolic plane in the Poincare Disk Model

sage: HyperbolicPlane().KM().get_isometry(identity_matrix(3)).model()
Hyperbolic plane in the Klein Disk Model

sage: HyperbolicPlane().HM().get_isometry(identity_matrix(3)).model()
Hyperbolic plane in the Hyperboloid Model
```

preserves_orientation()

Return `True` if `self` is orientation-preserving and `False` otherwise.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = UHP.get_isometry(identity_matrix(2))
sage: A.preserves_orientation()
True
sage: B = UHP.get_isometry(matrix(2,[0,1,1,0]))
sage: B.preserves_orientation()
False
```

repelling_fixed_point()

For a hyperbolic isometry, return the attracting fixed point; otherwise raise a `ValueError`.

OUTPUT:

- a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = UHP.get_isometry(Matrix(2,[4,0,0,1/4]))
sage: A.repelling_fixed_point()
Boundary point in UHP 0
```

to_model(*other*)

Convert the current object to image in another model.

INPUT:

- `other` – (a string representing) the image model

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: UHP = H.UHP()
sage: PD = H.PD()
sage: KM = H.KM()
sage: HM = H.HM()
```

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```
sage: A = UHP.get_isometry(identity_matrix(2))
sage: A.to_model(HM)
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
sage: A.to_model('HM')
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]

sage: A = PD.get_isometry(matrix([[I, 0], [0, -I]]))
sage: A.to_model(UHP)
Isometry in UHP
[ 0  1]
[-1  0]
sage: A.to_model(HM)
Isometry in HM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]
sage: A.to_model(KM)
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]

sage: A = HM.get_isometry(diagonal_matrix([-1, -1, 1]))
sage: A.to_model('UHP')
Isometry in UHP
[ 0 -1]
[ 1  0]
sage: A.to_model('PD')
Isometry in PD
[-I  0]
[ 0  I]
sage: A.to_model('KM')
Isometry in KM
[-1  0  0]
[ 0 -1  0]
[ 0  0  1]
```

translation_length()

For hyperbolic elements, return the translation length; otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2,[2,0,0,1/2]))
sage: H.translation_length()
2*arccosh(5/4)
```

```
sage: f_1 = UHP.get_point(-1)
sage: f_2 = UHP.get_point(1)
sage: H = UHP.isometry_from_fixed_points(f_1, f_2)
```

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```
sage: p = UHP.get_point(exp(i*7*pi/8))
sage: bool((p.dist(H*p) - H.translation_length()) < 10**-9)
True
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometryKM**(*model*, *A*, *check=True*)

Bases: *HyperbolicIsometry*

Create a hyperbolic isometry in the KM model.

INPUT:

- a matrix in $SO(2,1)$

EXAMPLES:

```
sage: HyperbolicPlane().KM().get_isometry(identity_matrix(3))
Isometry in KM
[1 0 0]
[0 1 0]
[0 0 1]
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometryPD**(*model*, *A*, *check=True*)

Bases: *HyperbolicIsometry*

Create a hyperbolic isometry in the PD model.

INPUT:

- a matrix in $PU(1,1)$

EXAMPLES:

```
sage: HyperbolicPlane().PD().get_isometry(identity_matrix(2))
Isometry in PD
[1 0]
[0 1]
```

preserves_orientation()

Return `True` if `self` preserves orientation and `False` otherwise.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.get_isometry(matrix([[-I, 0], [0, I]])).preserves_orientation()
True
sage: PD.get_isometry(matrix([[0, I], [I, 0]])).preserves_orientation()
False
```

class sage.geometry.hyperbolic_space.hyperbolic_isometry.**HyperbolicIsometryUHP**(*model*, *A*, *check=True*)

Bases: *HyperbolicIsometry*

Create a hyperbolic isometry in the UHP model.

INPUT:

- a matrix in $GL(2, \mathbf{R})$

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_isometry(identity_matrix(2))
Isometry in UHP
[1 0]
[0 1]
```

attracting_fixed_point()

Return the attracting fixed point.

Otherwise, this raises a `ValueError`.

OUTPUT:

- a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = matrix(2,[4,0,0,1/4])
sage: UHP.get_isometry(A).attracting_fixed_point()
Boundary point in UHP +Infinity
```

classification()

Classify the hyperbolic isometry as elliptic, parabolic, or hyperbolic.

A hyperbolic isometry fixes two points on the boundary of hyperbolic space, a parabolic isometry fixes one point on the boundary of hyperbolic space, and an elliptic isometry fixes no points.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(identity_matrix(2)).classification()
'identity'

sage: UHP.get_isometry(4*identity_matrix(2)).classification()
'identity'

sage: UHP.get_isometry(matrix(2,[2,0,0,1/2])).classification()
'hyperbolic'

sage: UHP.get_isometry(matrix(2, [0, 3, -1/3, 6])).classification()
'hyperbolic'

sage: UHP.get_isometry(matrix(2,[1,1,0,1])).classification()
'parabolic'

sage: UHP.get_isometry(matrix(2,[-1,0,0,1])).classification()
'reflection'
```

fixed_point_set()

Return a list or geodesic containing the fixed point set of orientation-preserving isometries.

OUTPUT:

list of hyperbolic points or a hyperbolic geodesic

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: H = UHP.get_isometry(matrix(2, [-2/3,-1/3,-1/3,-2/3]))
sage: g = H.fixed_point_set(); g
Geodesic in UHP from -1 to 1
sage: H(g.start()) == g.start()
True
sage: H(g.end()) == g.end()
True
sage: A = UHP.get_isometry(matrix(2,[0,1,1,0]))
sage: A.preserves_orientation()
False
sage: A.fixed_point_set()
Geodesic in UHP from 1 to -1
```

```
sage: B = UHP.get_isometry(identity_matrix(2))
sage: B.fixed_point_set()
Traceback (most recent call last):
...
ValueError: the identity transformation fixes the entire hyperbolic plane
```

preserves_orientation()

Return `True` if `self` is orientation-preserving and `False` otherwise.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = identity_matrix(2)
sage: UHP.get_isometry(A).preserves_orientation()
True
sage: B = matrix(2,[0,1,1,0])
sage: UHP.get_isometry(B).preserves_orientation()
False
```

repelling_fixed_point()

Return the repelling fixed point.

Otherwise, this raises a `ValueError`.

OUTPUT:

- a hyperbolic point

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = matrix(2,[4,0,0,1/4])
sage: UHP.get_isometry(A).repelling_fixed_point()
Boundary point in UHP 0
```

translation_length()

For hyperbolic elements, return the translation length; otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_isometry(matrix(2,[2,0,0,1/2])).translation_length()
2*arccosh(5/4)
```

```
sage: H = UHP.isometry_from_fixed_points(-1,1)
sage: p = UHP.get_point(exp(i*7*pi/8))
sage: Hp = H(p)
sage: bool((UHP.dist(p, Hp) - H.translation_length()) < 10**-9)
True
```

sage.geometry.hyperbolic_space.hyperbolic_isometry.**moebius_transform**(A, z)

Given a matrix `A` in $GL(2, \mathbf{C})$ and a point `z` in the complex plane return the Möbius transformation action of `A` on `z`.

INPUT:

- `A` – a 2×2 invertible matrix over the complex numbers
- `z` – a complex number or infinity

OUTPUT:

- a complex number or infinity

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_model import moebius_
↪transform
sage: moebius_transform(matrix(2,[1,2,3,4]),2 + I)
-2/109*I + 43/109
sage: y = var('y')
sage: moebius_transform(matrix(2,[1,0,0,1]),x + I*y)
x + I*y
```

The matrix must be square and 2×2:

```
sage: moebius_transform(matrix([[3,1,2],[1,2,5]]),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring

sage: moebius_transform(identity_matrix(3),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring
```

The matrix can be symbolic or can be a matrix over the real or complex numbers, but must be provably invertible:

```
sage: a,b,c,d = var('a,b,c,d')
sage: moebius_transform(matrix(2,[a,b,c,d]),I)
(I*a + b)/(I*c + d)
sage: moebius_transform(matrix(2,[1,b,c,b*c+1]),I)
(b + I)/(b*c + I*c + 1)
sage: moebius_transform(matrix(2,[0,0,0,0]),I)
Traceback (most recent call last):
...
TypeError: A must be an invertible 2x2 matrix over the complex numbers or a␣
↪symbolic ring
```

CHAPTER

THREE

HYPERBOLIC GEODESICS

This module implements the abstract base class for geodesics in hyperbolic space of arbitrary dimension. It also contains the implementations for specific models of hyperbolic geometry.

AUTHORS:

- Greg Laun (2013): initial version

EXAMPLES:

We can construct geodesics in the upper half plane model, abbreviated UHP for convenience:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2, 3)
sage: g
Geodesic in UHP from 2 to 3
```

This geodesic can be plotted using `plot()`, in this example we will show the axis.

```
sage: g.plot(axes=True)                                                    #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

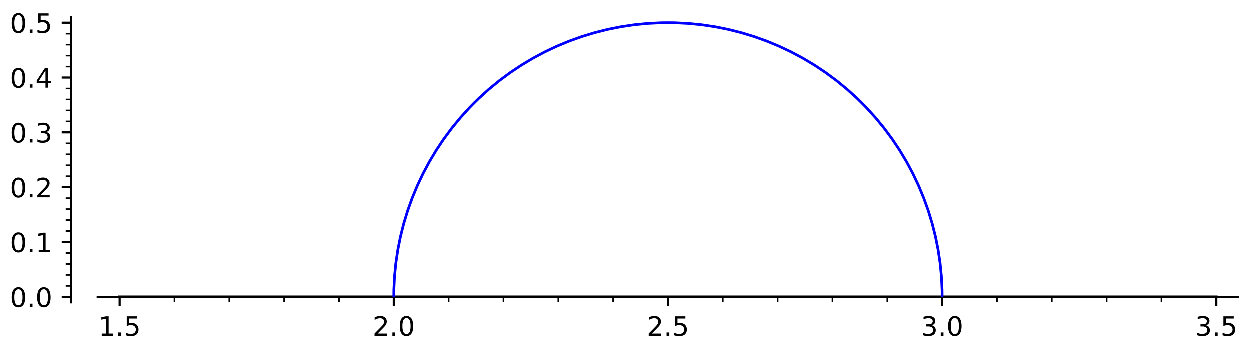
```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3 + I)
sage: g.length()
arccosh(11/2)
sage: g.plot(axes=True)                                                    #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

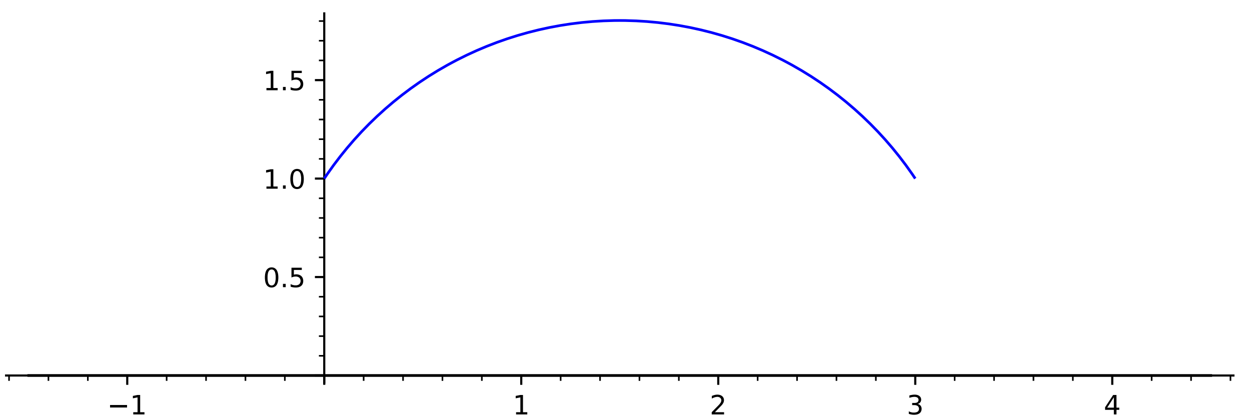
Geodesics of both types in UHP are supported:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g
Geodesic in UHP from I to 3*I
sage: g.plot()                                                             #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

Geodesics are oriented, which means that two geodesics with the same graph will only be equal if their starting and ending points are the same:

```
sage: g1 = HyperbolicPlane().UHP().get_geodesic(1,2)
sage: g2 = HyperbolicPlane().UHP().get_geodesic(2,1)
sage: g1 == g2
False
```


0.5
0.4
0.3
0.2
0.1
0.0
1.5
2.0
2.5
3.0
3.5


1.5
1.0
0.5
−1
1
2
3
4

Todo: Implement a parent for all geodesics of the hyperbolic plane? Or implement geodesics as a parent in the subobjects category?

class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesic**(*model*, *start*, *end*, ***graphics_options*)

Bases: SageObject

Abstract base class for oriented geodesics that are not necessarily complete.

INPUT:

- `start` – a HyperbolicPoint or coordinates of a point in hyperbolic space representing the start of the geodesic
- `end` – a HyperbolicPoint or coordinates of a point in hyperbolic space representing the end of the geodesic

EXAMPLES:

We can construct a hyperbolic geodesic in any model:

```
sage: HyperbolicPlane().UHP().get_geodesic(1, 0)
Geodesic in UHP from 1 to 0
sage: HyperbolicPlane().PD().get_geodesic(1, 0)
Geodesic in PD from 1 to 0
sage: HyperbolicPlane().KM().get_geodesic((0,1/2), (1/2, 0))
Geodesic in KM from (0, 1/2) to (1/2, 0)
sage: HyperbolicPlane().HM().get_geodesic((0,0,1), (0,1, sqrt(2)))
Geodesic in HM from (0, 0, 1) to (0, 1, sqrt(2))
```

angle(*other*)

Return the angle between any two given geodesics if they intersect.

INPUT:

- `other` – a hyperbolic geodesic in the same model as `self`

OUTPUT:

- the angle in radians between the two given geodesics

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: g = PD.get_geodesic(3/5*I + 4/5, 15/17*I + 8/17)
sage: h = PD.get_geodesic(4/5*I + 3/5, I)
sage: g.angle(h)
1/2*pi
```

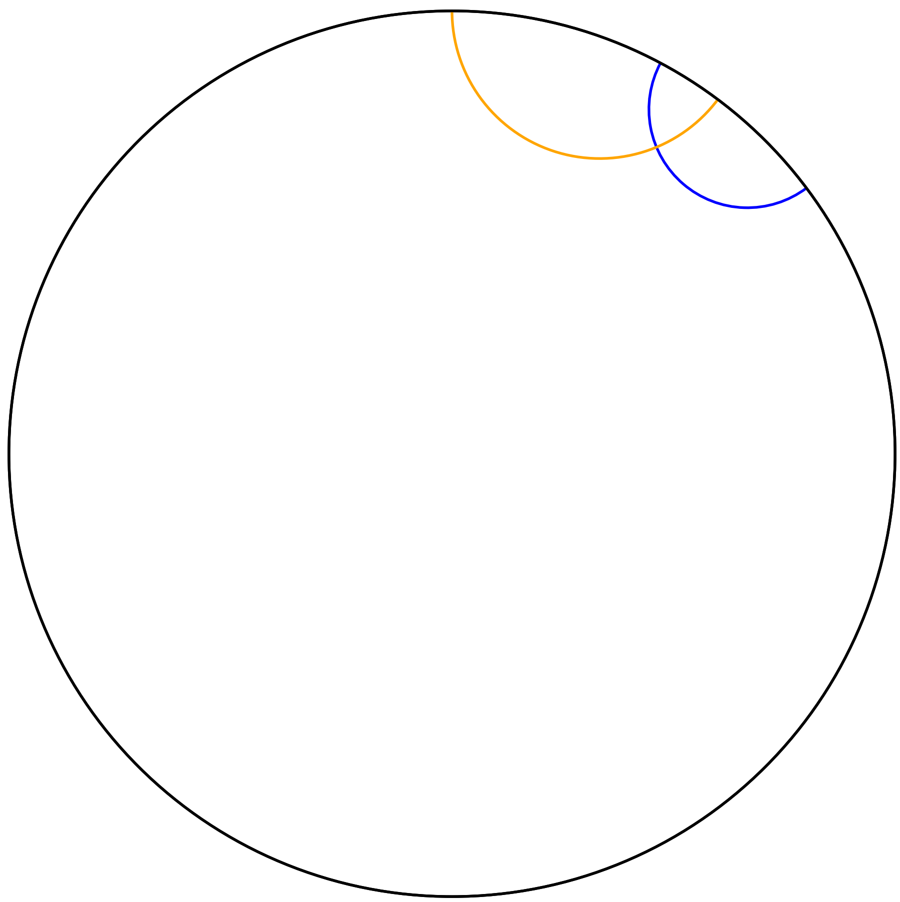
common_perpendicula(*other*)

Return the unique hyperbolic geodesic perpendicular to two given geodesics, if such a geodesic exists. If none exists, raise a ValueError.

INPUT:

- `other` – a hyperbolic geodesic in the same model as `self`

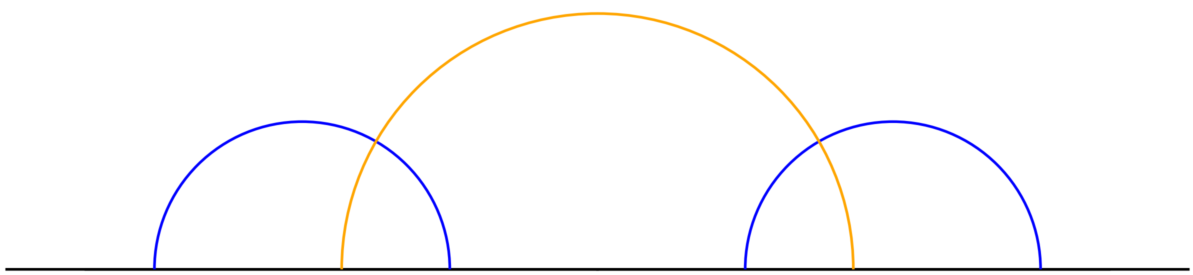
OUTPUT:



- a hyperbolic geodesic

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2,3)
sage: h = HyperbolicPlane().UHP().get_geodesic(4,5)
sage: g.common_perpendicular(h)
Geodesic in UHP from 1/2*sqrt(3) + 7/2 to -1/2*sqrt(3) + 7/2
```



It is an error to ask for the common perpendicular of two intersecting geodesics:

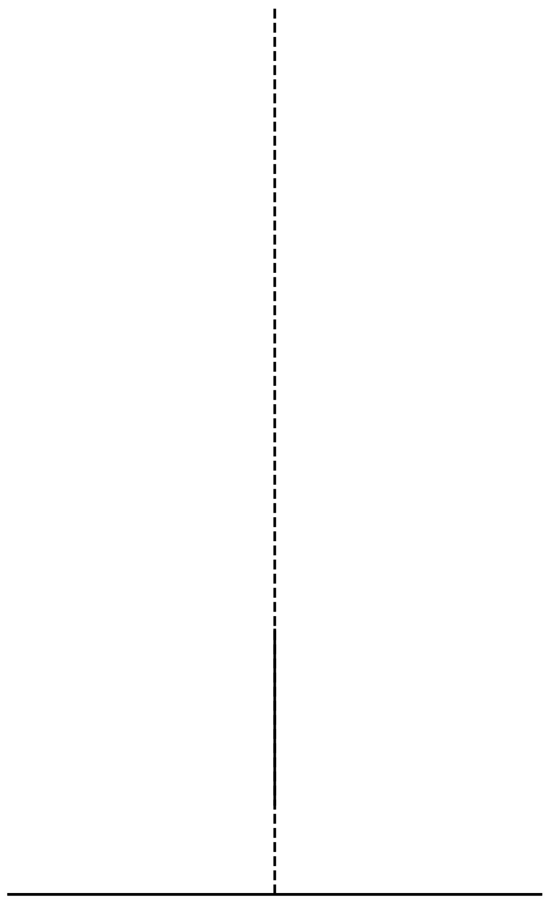
```
sage: g = HyperbolicPlane().UHP().get_geodesic(2,4)
sage: h = HyperbolicPlane().UHP().get_geodesic(3, infinity)
sage: g.common_perpendicular(h)
Traceback (most recent call last):
...
ValueError: geodesics intersect; no common perpendicular exists
```

complete()

Return the geodesic with ideal endpoints in bounded models. Raise a `NotImplementedError` in models that are not bounded. In the following examples we represent complete geodesics by a dashed line.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: UHP = H.UHP()
sage: UHP.get_geodesic(1 + I, 1 + 3*I).complete()
Geodesic in UHP from 1 to +Infinity
```

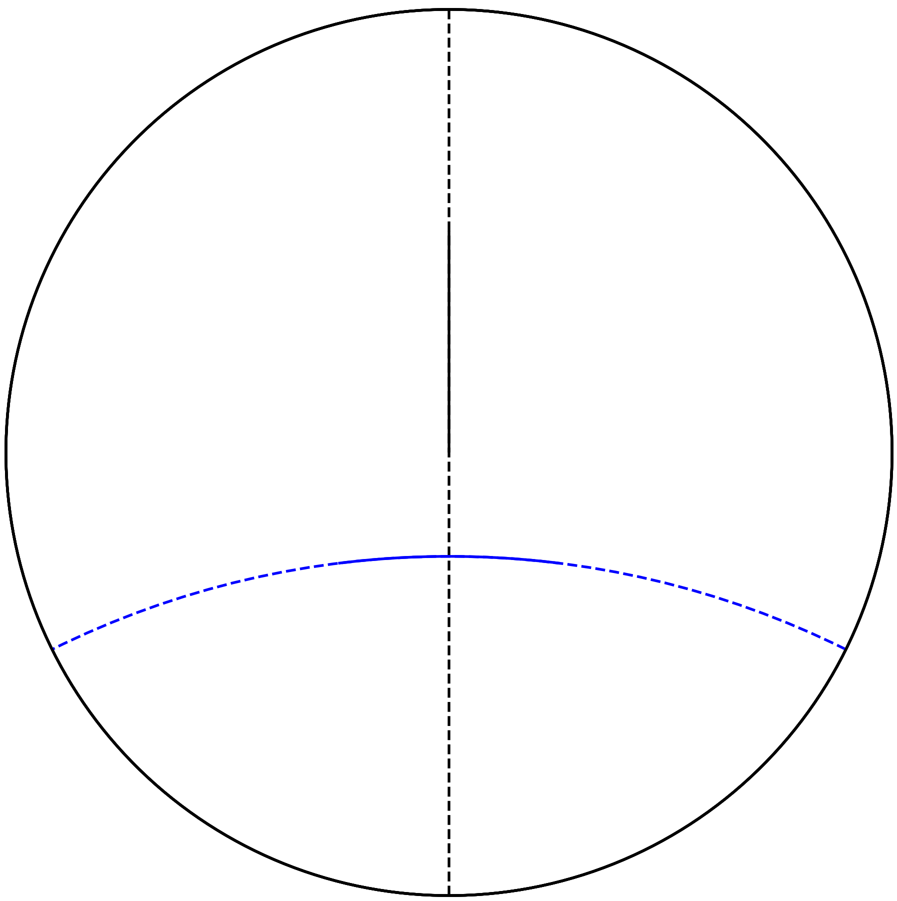


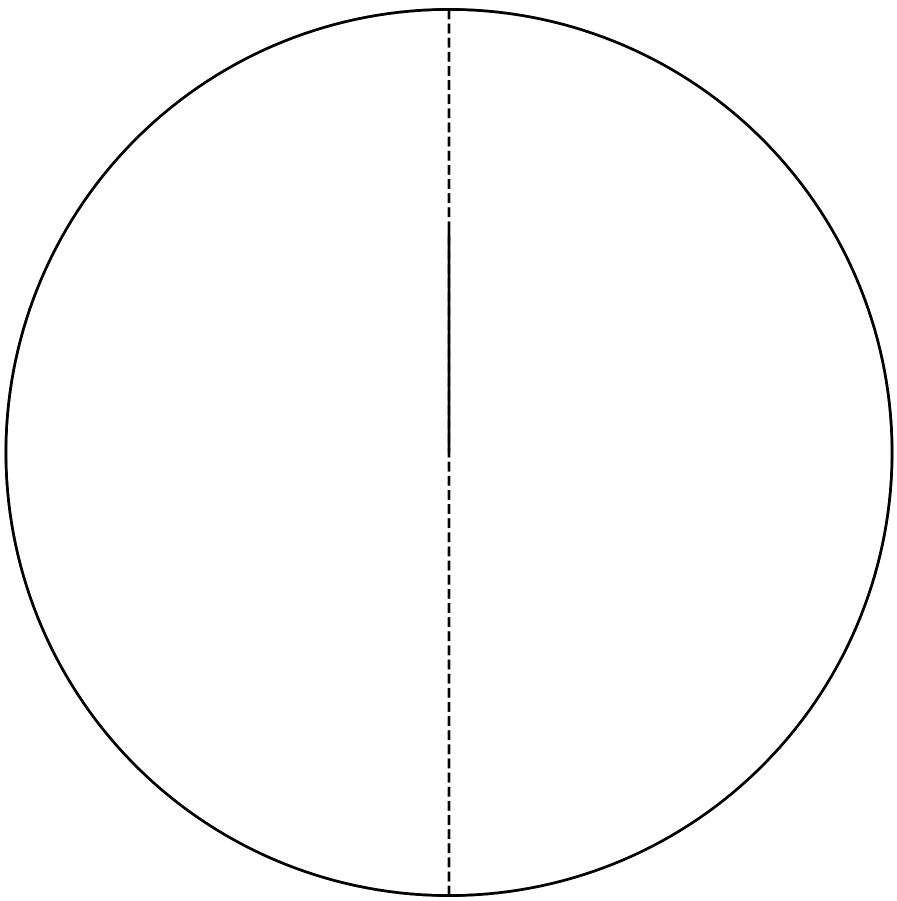
```
sage: PD = H.PD()
sage: PD.get_geodesic(0, I/2).complete()
Geodesic in PD from -I to I
sage: PD.get_geodesic(0.25*(-1-I),0.25*(1-I)).complete()
Geodesic in PD from -0.895806416477617 - 0.444444444444444*I to 0.
↪895806416477617 - 0.444444444444444*I
```

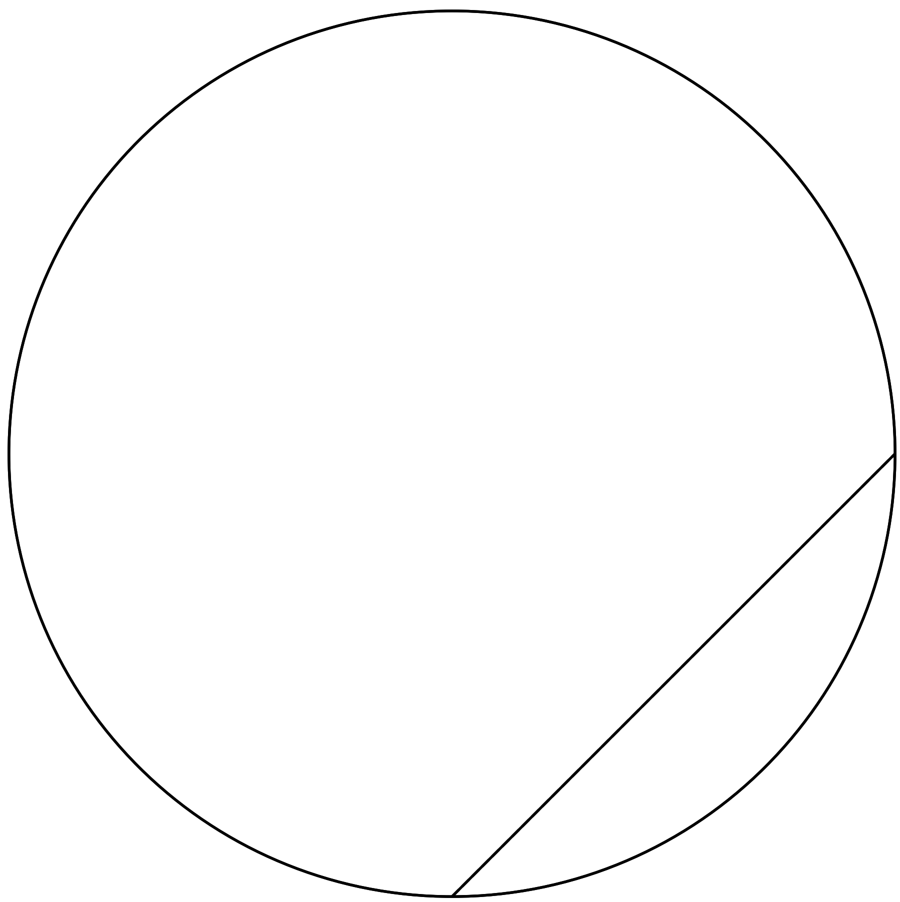
```
sage: KM = H.KM()
sage: KM.get_geodesic((0,0), (0,1/2)).complete()
Geodesic in KM from (0, -1) to (0, 1)
```

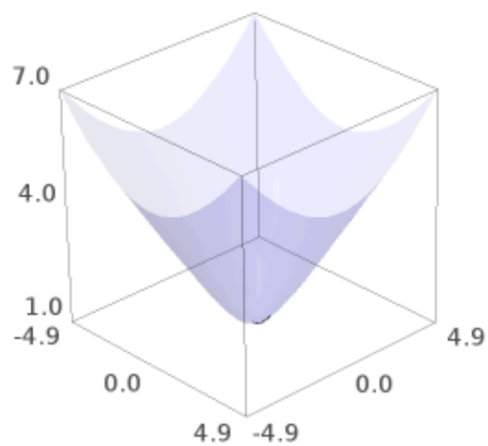
```
sage: KM.get_geodesic(-I, 1).complete()
Geodesic in KM from -I to 1
```

```
sage: HM = H.HM()
sage: HM.get_geodesic((0,0,1), (1, 0, sqrt(2))).complete()
Geodesic in HM from (0, 0, 1) to (1, 0, sqrt(2))
```








7.0
4.0
1.0
-4.9
0.0
4.9
-4.9
0.0
4.9

```
sage: g = HM.get_geodesic((0,0,1), (1, 0, sqrt(2))).complete()
sage: g.is_complete()
True
```

dist (*other*)

Return the hyperbolic distance from a given hyperbolic geodesic to another geodesic or point.

INPUT:

- `other` – a hyperbolic geodesic or hyperbolic point in the same model

OUTPUT:

- the hyperbolic distance

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2, 4.0)
sage: h = HyperbolicPlane().UHP().get_geodesic(5, 7.0)
sage: bool(abs(g.dist(h).n() - 1.92484730023841) < 10**-9)
True
```

If the second object is a geodesic ultraparallel to the first, or if it is a point on the boundary that is not one of the first object's endpoints, then return +infinity

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2, 2+I)
sage: p = HyperbolicPlane().UHP().get_point(5)
sage: g.dist(p)
+Infinity
```

end ()

Return the starting point of the geodesic.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g.end()
Point in UHP 3*I
```

endpoints ()

Return a list containing the start and endpoints.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g.endpoints()
[Point in UHP I, Point in UHP 3*I]
```

graphics_options ()

Return the graphics options of `self`.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 2*I, color="red")
sage: g.graphics_options()
{'color': 'red'}
```

ideal_endpoints()

Return the ideal endpoints in bounded models. Raise a `NotImplementedError` in models that are not bounded.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: UHP = H.UHP()
sage: UHP.get_geodesic(1 + I, 1 + 3*I).ideal_endpoints()
[Boundary point in UHP 1, Boundary point in UHP +Infinity]

sage: PD = H.PD()
sage: PD.get_geodesic(0, I/2).ideal_endpoints()
[Boundary point in PD -I, Boundary point in PD I]

sage: KM = H.KM()
sage: KM.get_geodesic((0,0), (0, 1/2)).ideal_endpoints()
[Boundary point in KM (0, -1), Boundary point in KM (0, 1)]

sage: HM = H.HM()
sage: HM.get_geodesic((0,0,1), (1, 0, sqrt(2))).ideal_endpoints()
Traceback (most recent call last):
...
NotImplementedError: boundary points are not implemented in
 the HM model
```

intersection(*other*)

Return the point of intersection of two geodesics (if such a point exists).

INPUT:

- `other` – a hyperbolic geodesic in the same model as `self`

OUTPUT:

- a hyperbolic point or geodesic

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
```

is_asymptotically_parallel(*other*)

Return `True` if `self` and `other` are asymptotically parallel and `False` otherwise.

INPUT:

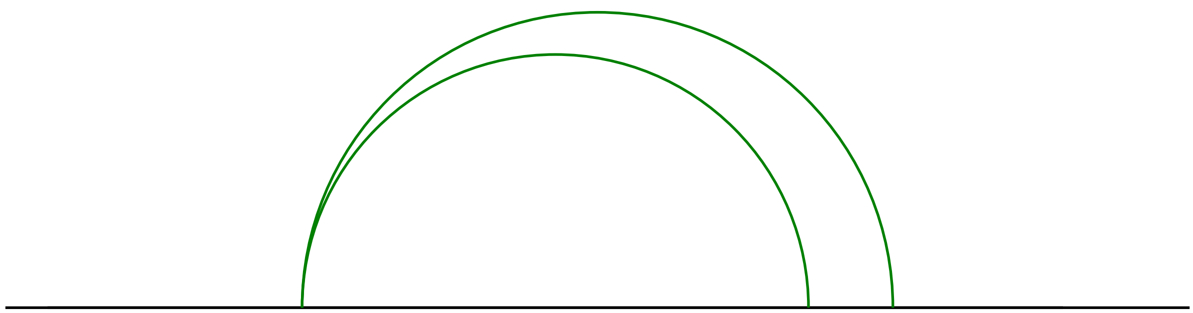
- `other` – a hyperbolic geodesic

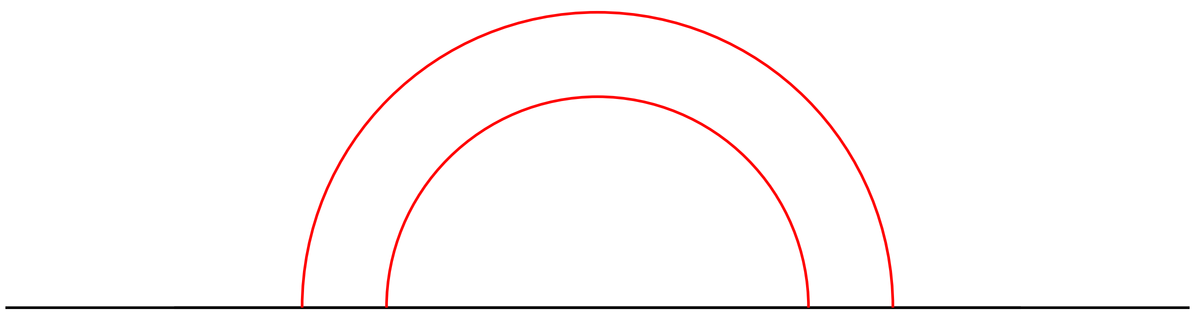
EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(-2,4)
sage: g.is_asymptotically_parallel(h)
True
```

Ultraparallel geodesics are not asymptotically parallel:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(-1,4)
sage: g.is_asymptotically_parallel(h)
False
```





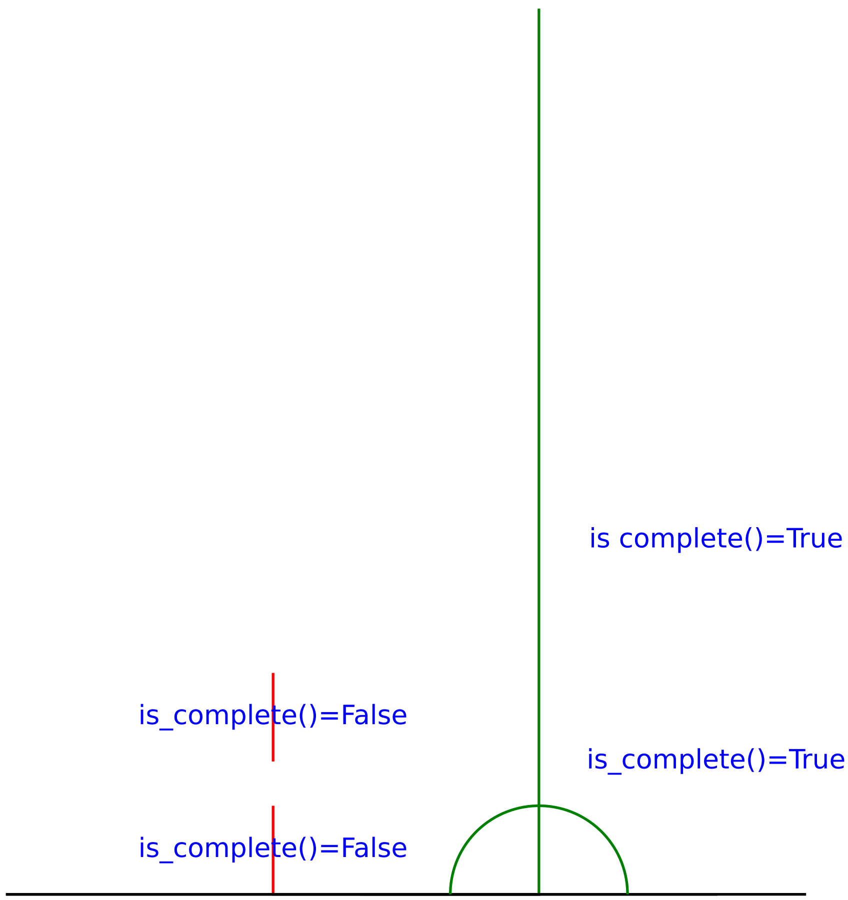
No hyperbolic geodesic is asymptotically parallel to itself:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: g.is_asymptotically_parallel(g)
False
```

is_complete()

Return `True` if `self` is a complete geodesic (that is, both endpoints are on the ideal boundary) and `False` otherwise.

If we represent complete geodesics using green color and incomplete using red colors we have the following graphic:



Notice, that there is no visual indication that the *vertical* geodesic is complete

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(1.5*I, 2.5*I).is_complete()
False
sage: UHP.get_geodesic(0, I).is_complete()
False
sage: UHP.get_geodesic(3, infinity).is_complete()
True
sage: UHP.get_geodesic(2,5).is_complete()
True
```

is_parallel(*other*)

Return `True` if the two given hyperbolic geodesics are either ultra parallel or asymptotically parallel and``False`` otherwise.

INPUT:

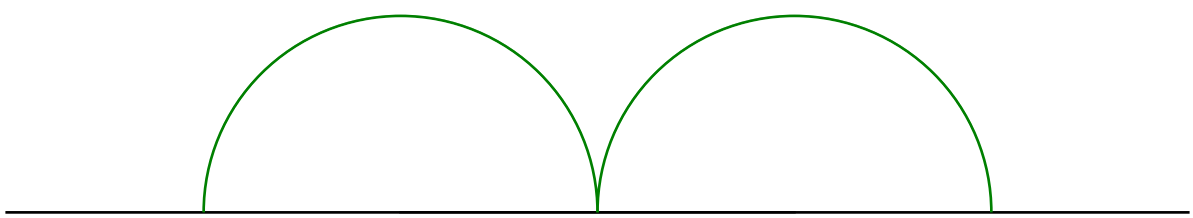
- `other` – a hyperbolic geodesic in any model

OUTPUT:

`True` if the given geodesics are either ultra parallel or asymptotically parallel, `False` if not.

EXAMPLES:

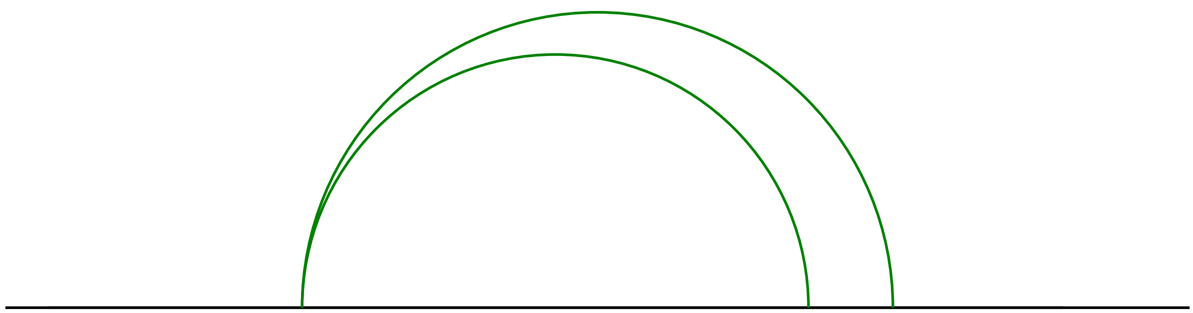
```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(5,12)
sage: g.is_parallel(h)
True
```



```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(-2,4)
sage: g.is_parallel(h)
True
```

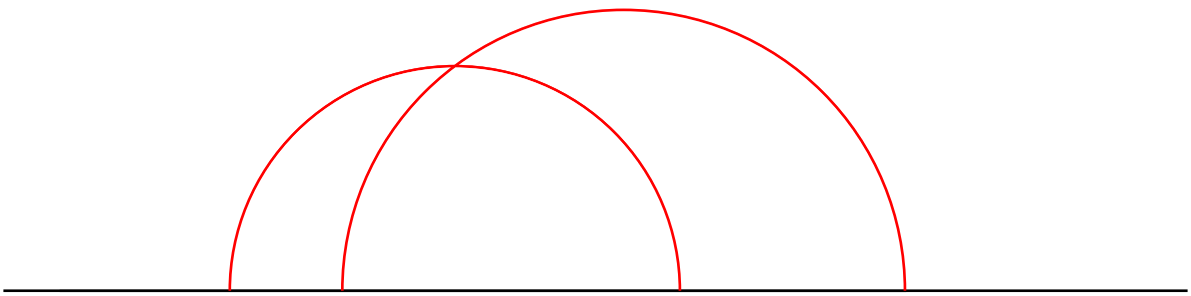
```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,2)
sage: h = HyperbolicPlane().UHP().get_geodesic(-1,4)
```

(continues on next page)



(continued from previous page)

```
sage: g.is_parallel(h)
False
```



No hyperbolic geodesic is either ultra parallel or asymptotically parallel to itself:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: g.is_parallel(g)
False
```

is_ultra_parallel(*other*)

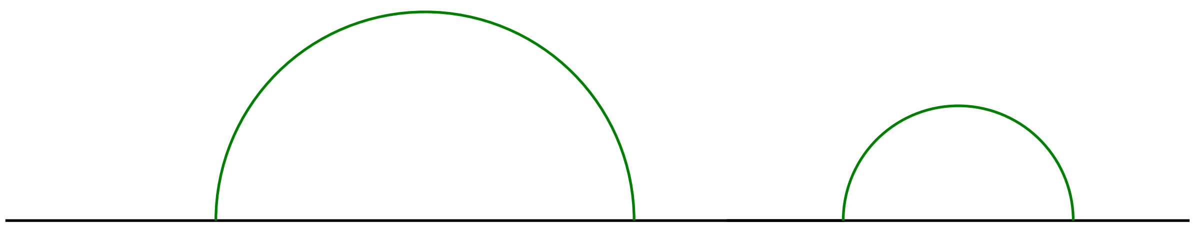
Return `True` if `self` and `other` are ultra parallel and `False` otherwise.

INPUT:

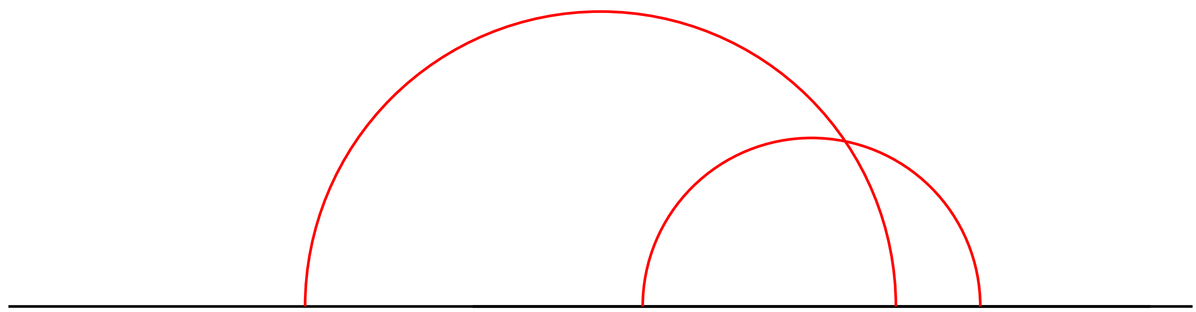
- `other` – a hyperbolic geodesic

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_geodesic \
....:   import *
sage: g = HyperbolicPlane().UHP().get_geodesic(0,1)
sage: h = HyperbolicPlane().UHP().get_geodesic(-3,-1)
sage: g.is_ultra_parallel(h)
True
```



```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: h = HyperbolicPlane().UHP().get_geodesic(2,6)
sage: g.is_ultra_parallel(h)
False
```



```
sage: g = HyperbolicPlane().UHP().get_geodesic(-2,5)
sage: g.is_ultra_parallel(g)
False
```

length()

Return the Hyperbolic length of the hyperbolic line segment.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(2 + I, 3 + I/2)
sage: g.length()
arccosh(9/4)
```

midpoint()

Return the (hyperbolic) midpoint of a hyperbolic line segment.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().random_geodesic()
sage: m = g.midpoint()
sage: end1, end2 = g.endpoints()
sage: bool(abs(m.dist(end1) - m.dist(end2)) < 10**-9)
True
```

Complete geodesics have no midpoint:

```
sage: HyperbolicPlane().UHP().get_geodesic(0,2).midpoint()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

model()

Return the model to which the *HyperbolicGeodesic* belongs.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(I, 2*I).model()
Hyperbolic plane in the Upper Half Plane Model

sage: PD = HyperbolicPlane().PD()
sage: PD.get_geodesic(0, I/2).model()
Hyperbolic plane in the Poincare Disk Model

sage: KM = HyperbolicPlane().KM()
sage: KM.get_geodesic((0, 0), (0, 1/2)).model()
Hyperbolic plane in the Klein Disk Model

sage: HM = HyperbolicPlane().HM()
sage: HM.get_geodesic((0, 0, 1), (0, 1, sqrt(2))).model()
Hyperbolic plane in the Hyperboloid Model
```

perpendicular_bisector()

Return the perpendicular bisector of `self` if `self` has finite length. Here distance is hyperbolic distance.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: g = PD.get_geodesic(-0.3+0.4*I,+0.7-0.1*I)
sage: h = g.perpendicular_bisector().complete()
sage: P = g.plot(color='blue')+h.plot(color='orange');P
Graphics object consisting of 4 graphics primitives
```

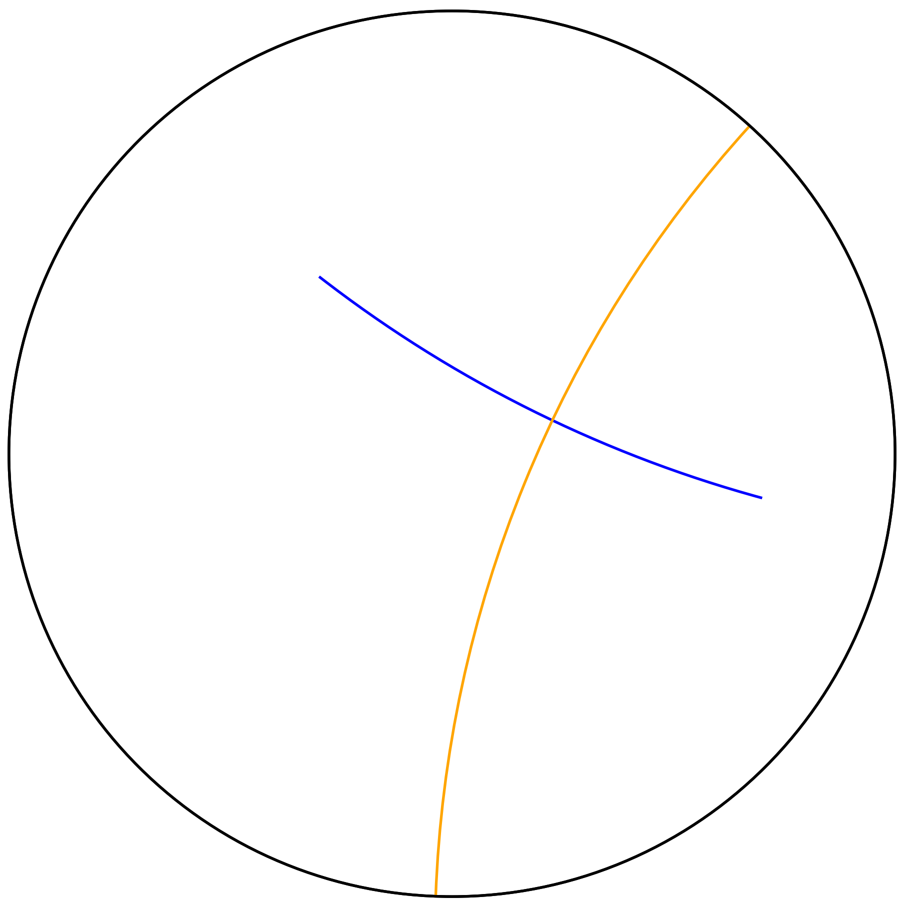
Complete geodesics cannot be bisected:

```
sage: g = HyperbolicPlane().PD().get_geodesic(0, 1)
sage: g.perpendicular_bisector()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

reflection_involution()

Return the involution fixing `self`.

EXAMPLES:



```
sage: H = HyperbolicPlane()
sage: gU = H.UHP().get_geodesic(2,4)
sage: RU = gU.reflection_involution(); RU
Isometry in UHP
[ 3 -8]
[ 1 -3]

sage: RU*gU == gU
True

sage: gP = H.PD().get_geodesic(0, I)
sage: RP = gP.reflection_involution(); RP
Isometry in PD
[ 1  0]
[ 0 -1]

sage: RP*gP == gP
True

sage: gK = H.KM().get_geodesic((0,0), (0,1))
sage: RK = gK.reflection_involution(); RK
Isometry in KM
[-1  0  0]
[ 0  1  0]
[ 0  0  1]

sage: RK*gK == gK
True

sage: HM = H.HM()
sage: g = HM.get_geodesic((0,0,1), (1,0, n(sqrt(2))))
sage: A = g.reflection_involution()
sage: B = diagonal_matrix([1, -1, 1])
sage: bool((B - A.matrix()).norm() < 10**-9)
True
```

The above tests go through the Upper Half Plane. It remains to test that the matrices in the models do what we intend.

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_isometry \
....:   import moebius_transform
sage: R = H.PD().get_geodesic(-1,1).reflection_involution()
sage: bool(moebius_transform(R.matrix(), 0) == 0)
True
```

start()

Return the starting point of the geodesic.

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 3*I)
sage: g.start()
Point in UHP I
```

to_model(*model*)

Convert the current object to image in another model.

INPUT:

- `model` – the image model

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: PD = HyperbolicPlane().PD()
sage: UHP.get_geodesic(I, 2*I).to_model(PD)
Geodesic in PD from 0 to 1/3*I
sage: UHP.get_geodesic(I, 2*I).to_model('PD')
Geodesic in PD from 0 to 1/3*I
```

update_graphics(*update=False*, ***options*)

Update the graphics options of `self`.

INPUT:

- `update` – if `True`, the original option are updated rather than overwritten

EXAMPLES:

```
sage: g = HyperbolicPlane().UHP().get_geodesic(I, 2*I)
sage: g.graphics_options()
{}

sage: g.update_graphics(color = "red"); g.graphics_options()
{'color': 'red'}

sage: g.update_graphics(color = "blue"); g.graphics_options()
{'color': 'blue'}

sage: g.update_graphics(True, size = 20); g.graphics_options()
{'color': 'blue', 'size': 20}
```

class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicHM**(*model*, *start*, *end*, ***graphics_options*)

Bases: *HyperbolicGeodesic*

A geodesic in the hyperboloid model.

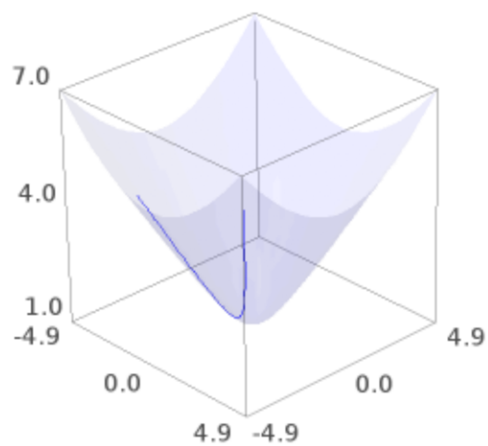
Valid points in the hyperboloid model satisfy $x^2 + y^2 - z^2 = -1$

INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: HM = HyperbolicPlane().HM()
sage: p1 = HM.get_point((4, -4, sqrt(33)))
sage: p2 = HM.get_point((-3,-3,sqrt(19)))
sage: g = HM.get_geodesic(p1, p2)
sage: g = HM.get_geodesic((4, -4, sqrt(33)), (-3, -3, sqrt(19)))
```

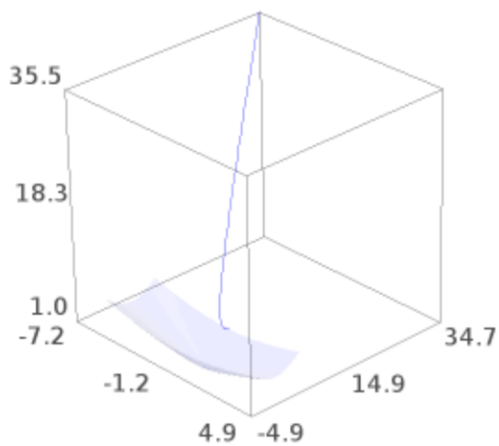

7.0
4.0
1.0
-4.9
0.0
4.9
-4.9
0.0
4.9

plot(*show_hyperboloid=True*, ***graphics_options*)

Plot `self`.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_geodesic \
....:     import *
sage: g = HyperbolicPlane().HM().random_geodesic()
sage: g.plot()                                                          #␣
↪needs sage.plot
Graphics3d Object
```



class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicKM**(*model*, *start*, *end*, ***graphics_options*)

Bases: `HyperbolicGeodesic`

A geodesic in the Klein disk model.

Geodesics are represented by the chords, straight line segments with ideal endpoints on the boundary circle.

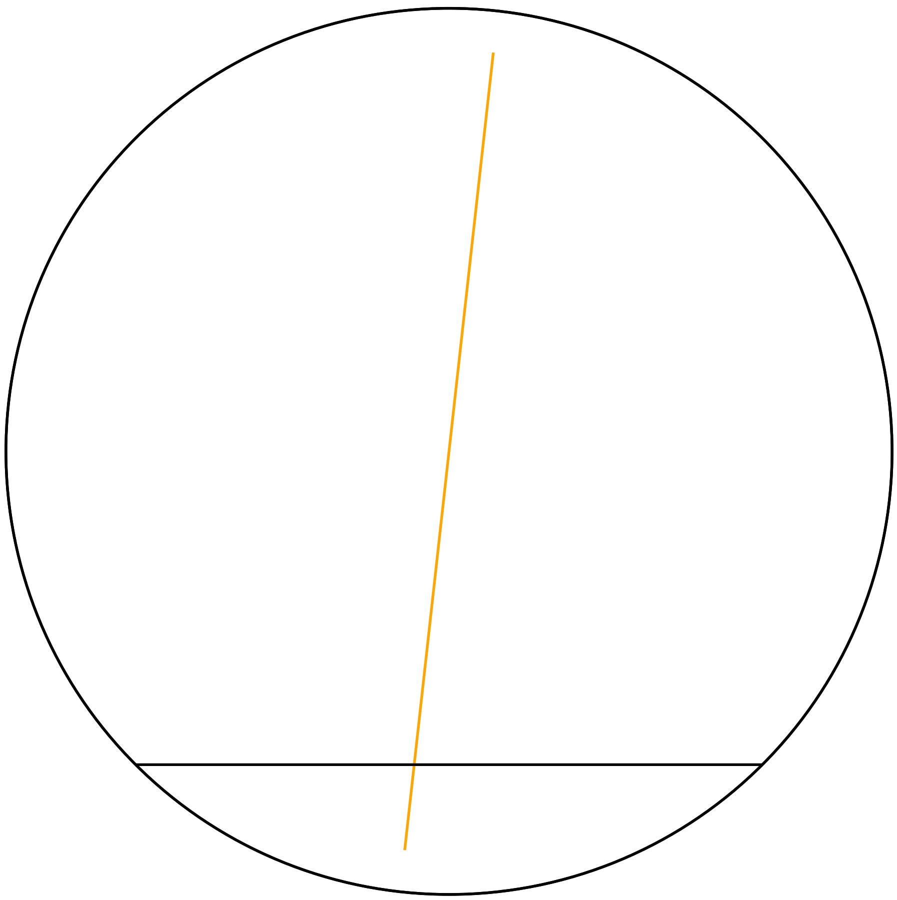
INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic

- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: g = KM.get_geodesic((0.1,0.9),(-0.1,-0.9))
sage: h = KM.get_geodesic((-0.707106781,-0.707106781),(0.707106781,-0.707106781))
sage: P = g.plot(color='orange')+h.plot(); P                                    #␣
↪needs sage.plot
Graphics object consisting of 4 graphics primitives
```



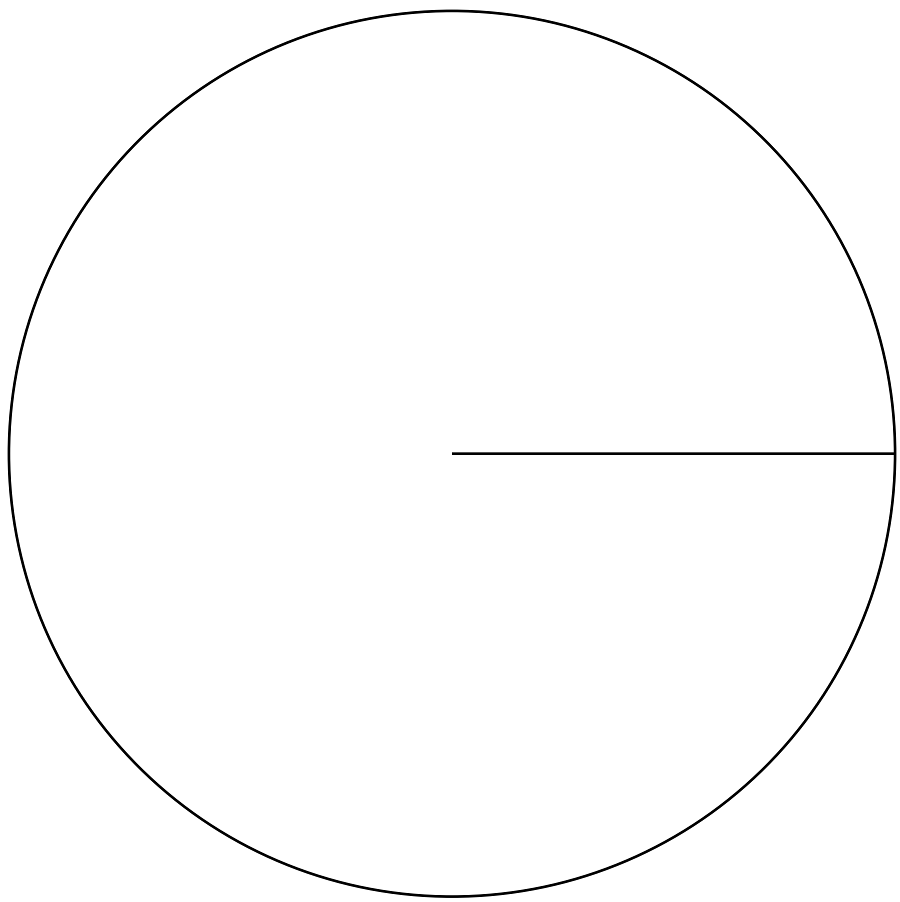
plot (*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

```
sage: HyperbolicPlane().KM().get_geodesic(0, 1).plot()                           #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicPD** (*model*, *start*, *end*, ***graphics_options*)



Bases: *HyperbolicGeodesic*

A geodesic in the Poincaré disk model.

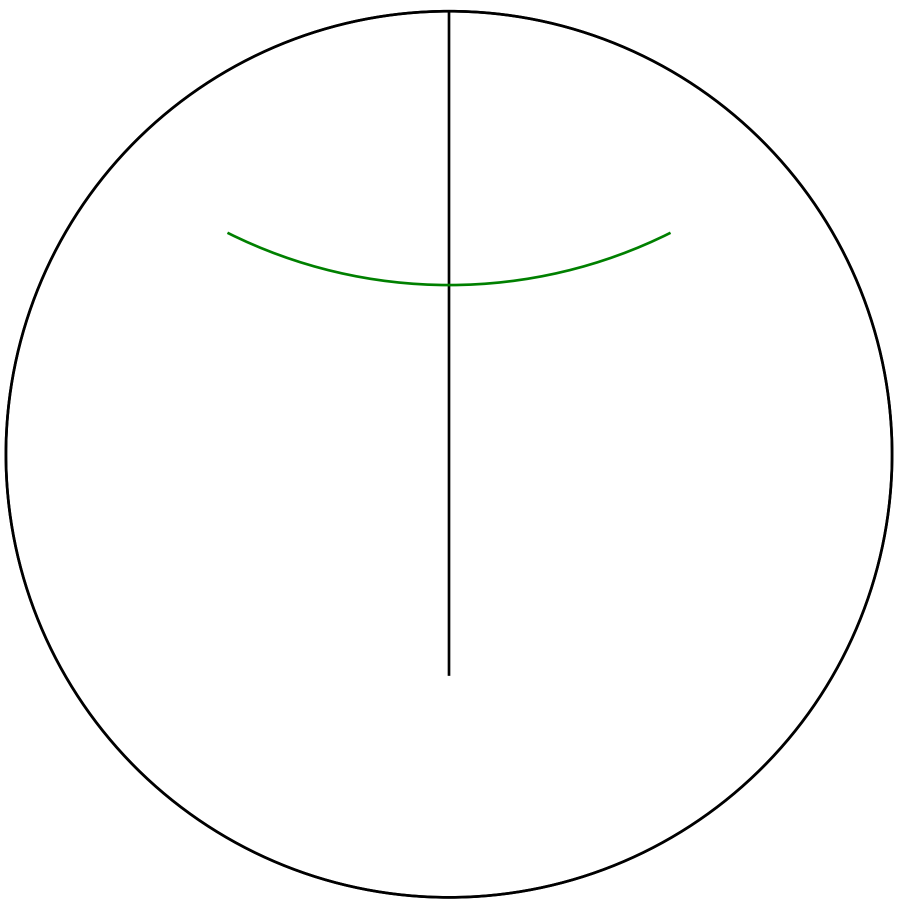
Geodesics in this model are represented by segments of circles contained within the unit disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.

INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: g = PD.get_geodesic(PD.get_point(I), PD.get_point(-I/2))
sage: g = PD.get_geodesic(I,-I/2)
sage: h = PD.get_geodesic(-1/2+I/2,1/2+I/2)
```



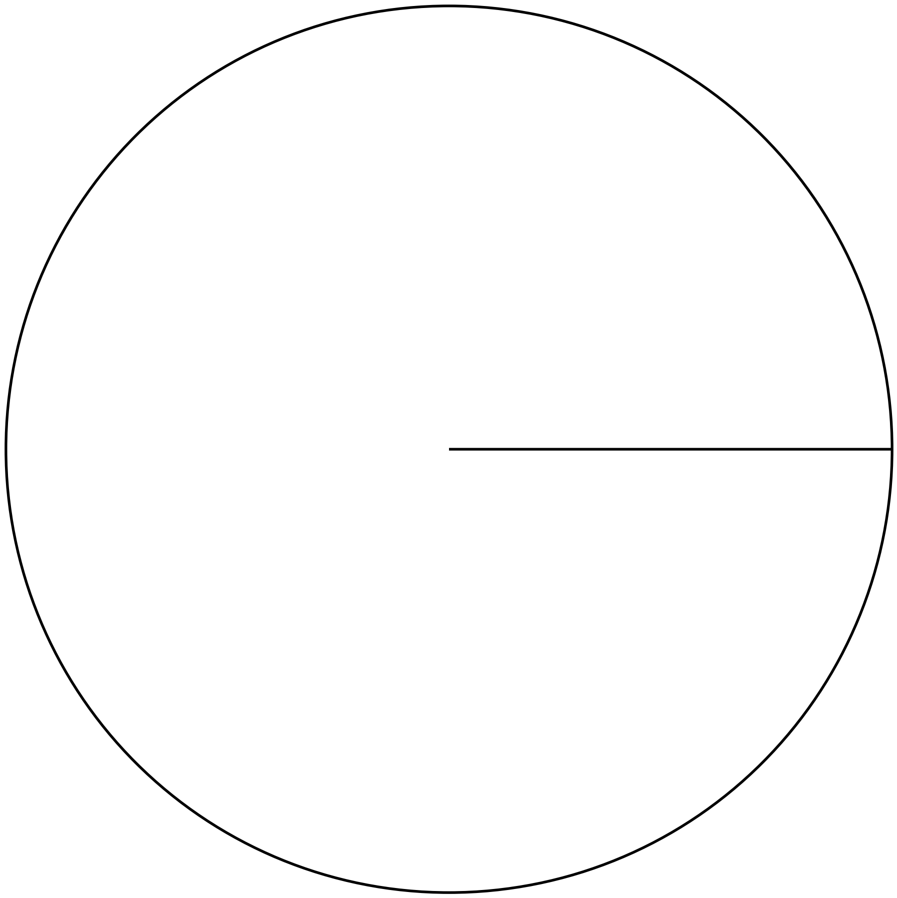
plot (*boundary=True*, ***options*)

Plot `self`.

EXAMPLES:

First some lines:

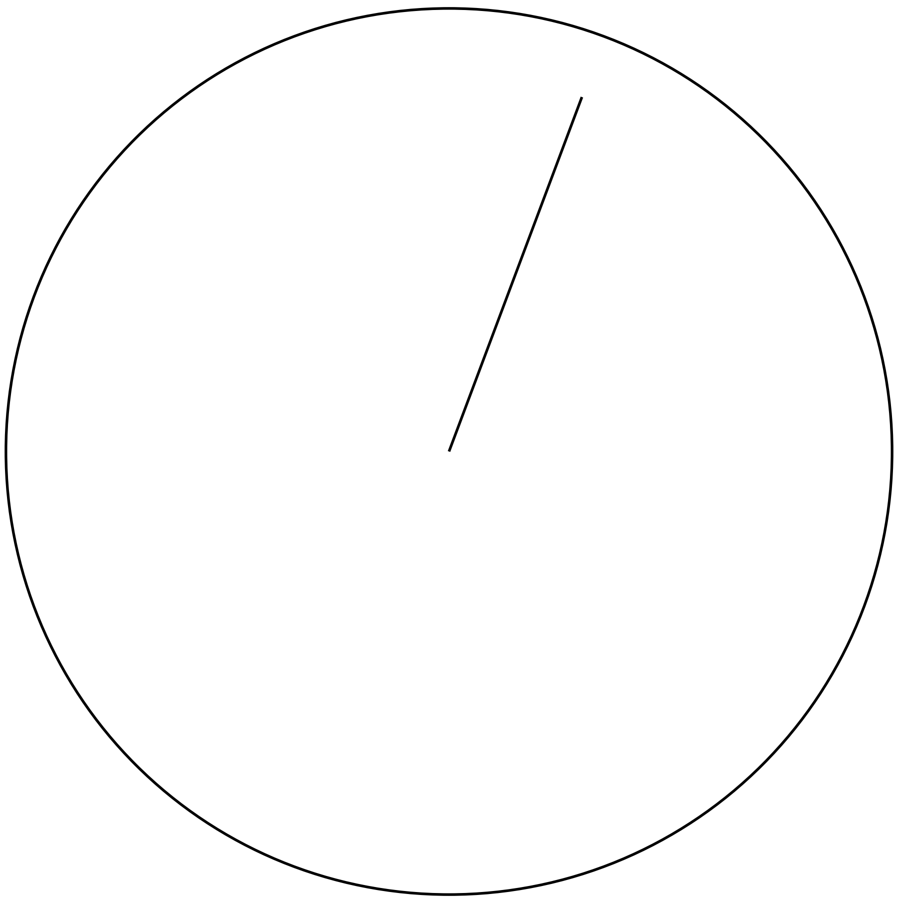
```
sage: PD = HyperbolicPlane().PD()
sage: PD.get_geodesic(0, 1).plot()                                          #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

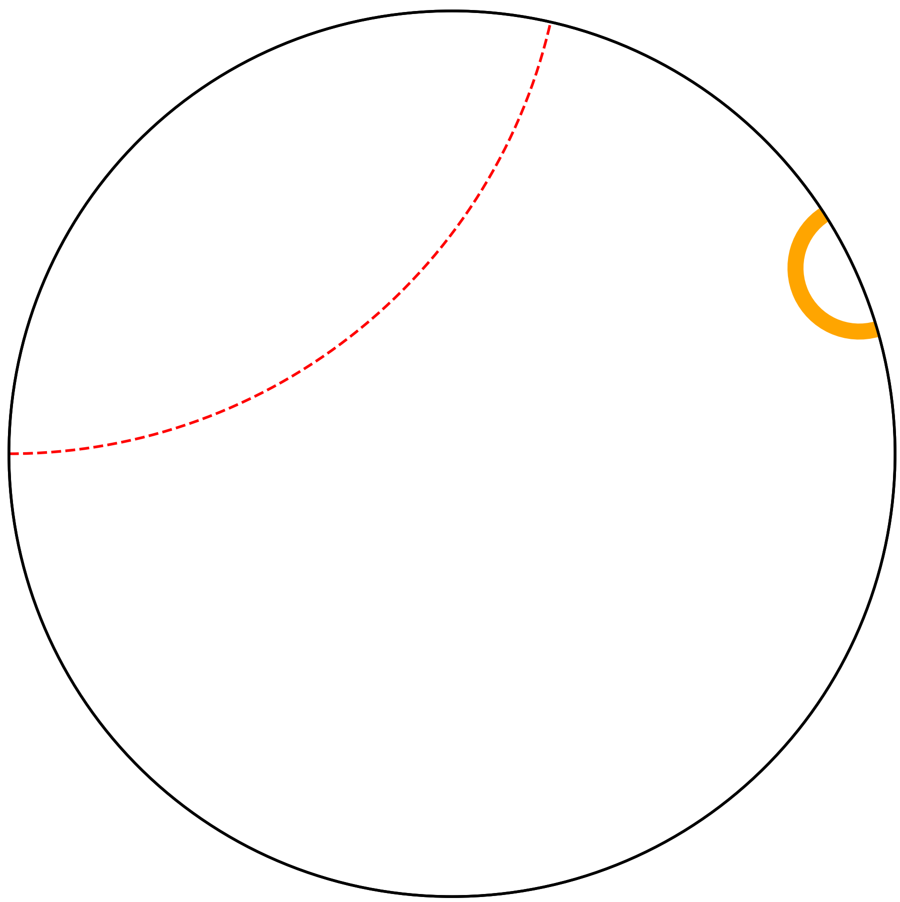


```
sage: PD.get_geodesic(0, 0.3+0.8*I).plot()                                  #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

Then some generic geodesics:

```
sage: PD.get_geodesic(-0.5, 0.3+0.4*I).plot()                               #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: g = PD.get_geodesic(-1, exp(3*I*pi/7))
sage: G = g.plot(linestyle="dashed",color="red"); G                         #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: h = PD.get_geodesic(exp(2*I*pi/11), exp(1*I*pi/11))
sage: H = h.plot(thickness=6, color="orange"); H                            #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
sage: show(G+H)                                                             #␣
↪needs sage.plot
```





class sage.geometry.hyperbolic_space.hyperbolic_geodesic.**HyperbolicGeodesicUHP**(*model*, *start*, *end*, ***graphics_options*)

Bases: *HyperbolicGeodesic*

Create a geodesic in the upper half plane model.

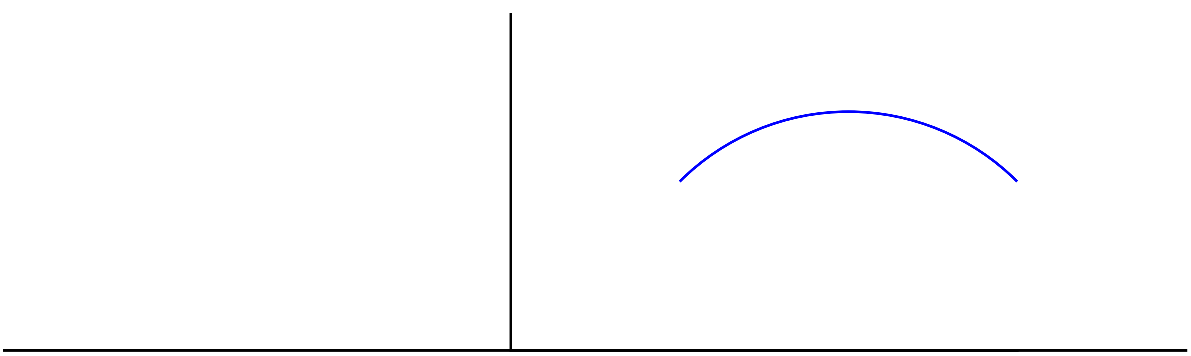
The geodesics in this model are represented by circular arcs perpendicular to the real axis (half-circles whose origin is on the real axis) and straight vertical lines ending on the real axis.

INPUT:

- `start` – a `HyperbolicPoint` in hyperbolic space representing the start of the geodesic
- `end` – a `HyperbolicPoint` in hyperbolic space representing the end of the geodesic

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(UHP.get_point(I), UHP.get_point(2 + I))
sage: g = UHP.get_geodesic(I, 2 + I)
sage: h = UHP.get_geodesic(-1, -1+2*I)
```



angle(*other*)

Return the angle between the completions of any two given geodesics if they intersect.

INPUT:

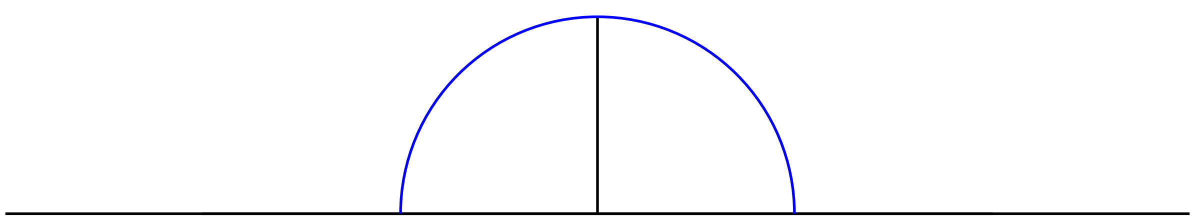
- `other` – a hyperbolic geodesic in the UHP model

OUTPUT:

- the angle in radians between the two given geodesics

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(2, 4)
sage: h = UHP.get_geodesic(3, 3 + I)
sage: g.angle(h)
1/2*pi
sage: numerical_approx(g.angle(h))
1.57079632679490
```



If the geodesics are identical, return angle 0:

```
sage: g.angle(g)
0
```

It is an error to ask for the angle of two geodesics that do not intersect:

```
sage: g = UHP.get_geodesic(2, 4)
sage: h = UHP.get_geodesic(5, 7)
sage: g.angle(h)
Traceback (most recent call last):
...
ValueError: geodesics do not intersect
```

common_perpendicular(*other*)

Return the unique hyperbolic geodesic perpendicular to `self` and `other`, if such a geodesic exists; otherwise raise a `ValueError`.

INPUT:

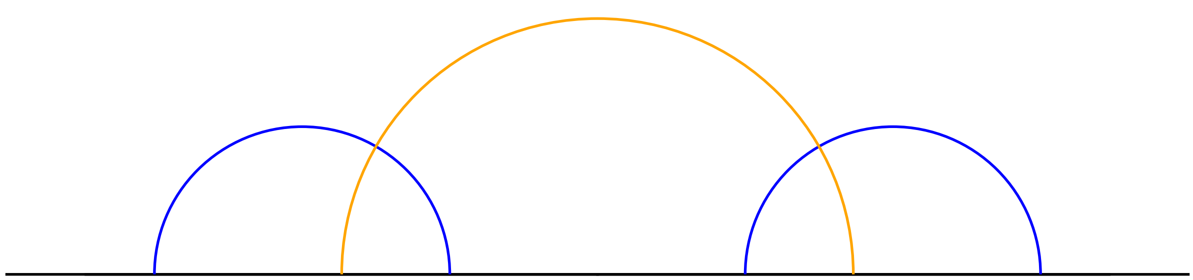
- `other` – a hyperbolic geodesic in current model

OUTPUT:

- a hyperbolic geodesic

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(2, 3)
sage: h = UHP.get_geodesic(4, 5)
sage: g.common_perpendicular(h)
Geodesic in UHP from 1/2*sqrt(3) + 7/2 to -1/2*sqrt(3) + 7/2
```



It is an error to ask for the common perpendicular of two intersecting geodesics:

```
sage: g = UHP.get_geodesic(2, 4)
sage: h = UHP.get_geodesic(3, infinity)
sage: g.common_perpendicular(h)
Traceback (most recent call last):
...
ValueError: geodesics intersect; no common perpendicular exists
```

ideal_endpoints()

Determine the ideal (boundary) endpoints of the complete hyperbolic geodesic corresponding to `self`.

OUTPUT:

- a list of 2 boundary points

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(I, 2*I).ideal_endpoints()
[Boundary point in UHP 0,
 Boundary point in UHP +Infinity]
sage: UHP.get_geodesic(1 + I, 2 + 4*I).ideal_endpoints()
[Boundary point in UHP -sqrt(65) + 9,
 Boundary point in UHP sqrt(65) + 9]
```

intersection(*other*)

Return the point of intersection of `self` and `other` (if such a point exists).

INPUT:

- `other` – a hyperbolic geodesic in the current model

OUTPUT:

- a list of hyperbolic points or a hyperbolic geodesic

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(3, 5)
sage: h = UHP.get_geodesic(4, 7)
sage: g.intersection(h)
[Point in UHP 2/3*sqrt(-2) + 13/3]
```

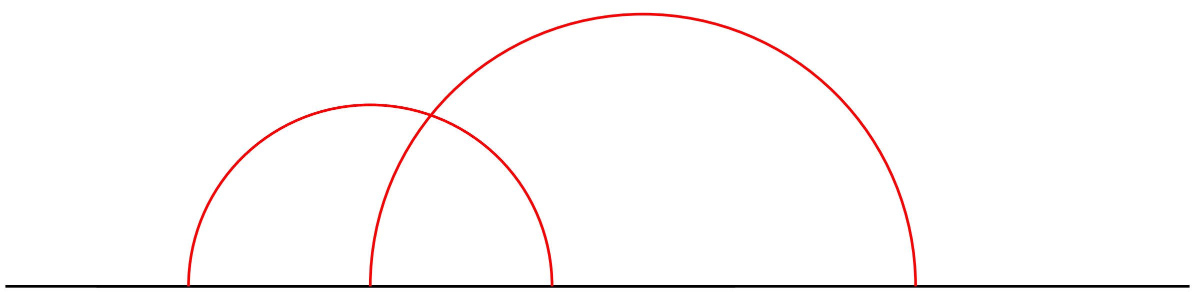
If the given geodesics do not intersect, the function returns an empty list:

```
sage: g = UHP.get_geodesic(4, 5)
sage: h = UHP.get_geodesic(6, 7)
sage: g.intersection(h)
[]
```

If the given geodesics are asymptotically parallel, the function returns the common boundary point:

```
sage: g = UHP.get_geodesic(4, 5)
sage: h = UHP.get_geodesic(5, 7)
sage: g.intersection(h)
[Boundary point in UHP 5.00000000000000]
```

If the given geodesics are identical, return that geodesic:



```
sage: g = UHP.get_geodesic(4 + I, 18*I)
sage: h = UHP.get_geodesic(4 + I, 18*I)
sage: g.intersection(h)
Geodesic in UHP from I + 4 to 18*I
```

midpoint()

Return the (hyperbolic) midpoint of `self` if it exists.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.random_geodesic()
sage: m = g.midpoint()
sage: d1 = UHP.dist(m, g.start())
sage: d2 = UHP.dist(m, g.end())
sage: bool(abs(d1 - d2) < 10**-9)
True
```

Infinite geodesics have no midpoint:

```
sage: UHP.get_geodesic(0, 2).midpoint()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

perpendicular_bisector()

Return the perpendicular bisector of the hyperbolic geodesic `self` if that geodesic has finite length.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.random_geodesic()
sage: h = g.perpendicular_bisector().complete()
sage: c = lambda x: x.coordinates()
sage: bool(c(g.intersection(h)[0]) - c(g.midpoint()) < 10**-9)
True
```

```
sage: UHP = HyperbolicPlane().UHP()
sage: g = UHP.get_geodesic(1+I,2+0.5*I)
sage: h = g.perpendicular_bisector().complete()
sage: show(g.plot(color='blue')+h.plot(color='orange'))
```

Infinite geodesics cannot be bisected:

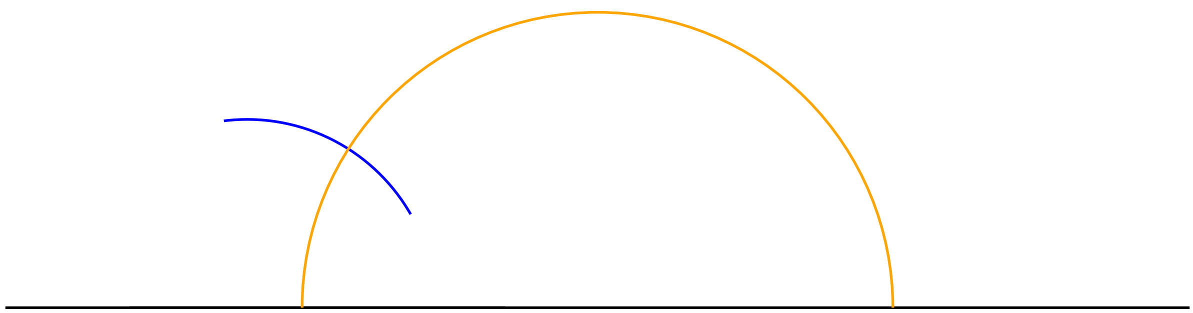
```
sage: UHP.get_geodesic(0, 1).perpendicular_bisector()
Traceback (most recent call last):
...
ValueError: the length must be finite
```

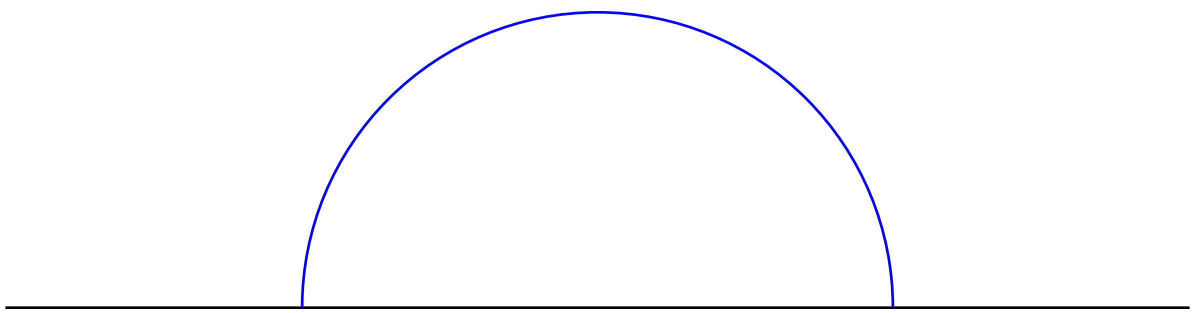
plot(*boundary=True*, ***options*)

Plot `self`.

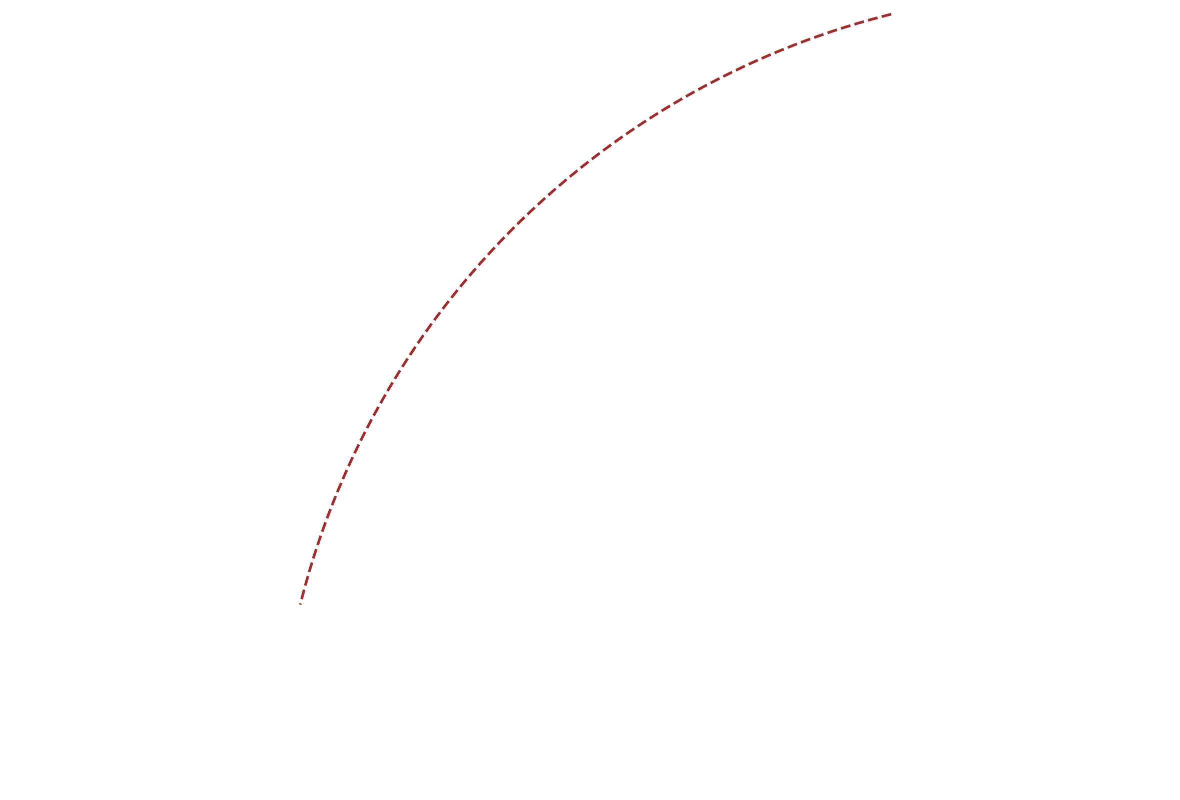
EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.get_geodesic(0, 1).plot()                                     #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```





```
sage: UHP.get_geodesic(I, 3+4*I).plot(linestyle="dashed", color="brown")     #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```



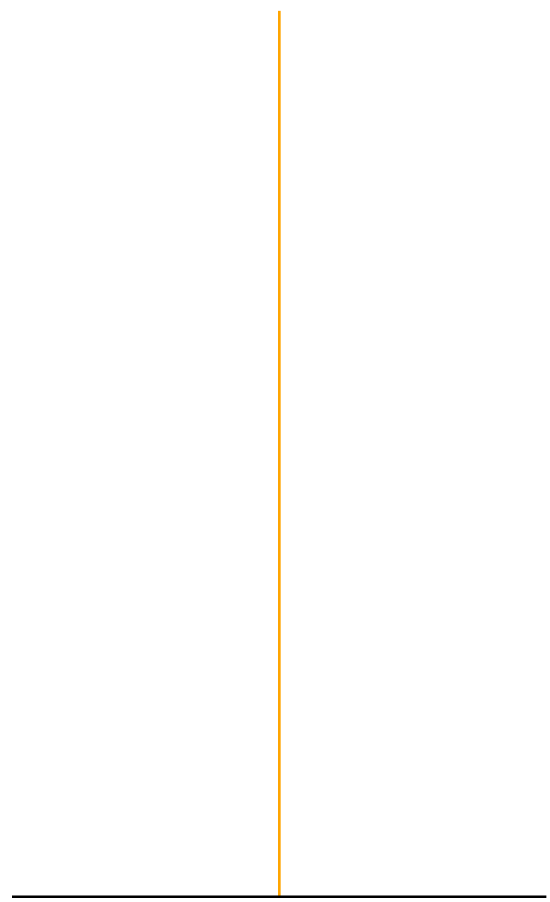
```
sage: UHP.get_geodesic(1, infinity).plot(color='orange')                     #␣
↪needs sage.plot
Graphics object consisting of 2 graphics primitives
```

reflection_involution()

Return the isometry of the involution fixing the geodesic `self`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: g1 = UHP.get_geodesic(0, 1)
sage: g1.reflection_involution()
Isometry in UHP
[ 1  0]
[ 2 -1]
sage: UHP.get_geodesic(I, 2*I).reflection_involution()
Isometry in UHP
[ 1  0]
[ 0 -1]
```



CHAPTER

FOUR

HYPERBOLIC MODELS

In this module, a hyperbolic model is a collection of data that allow the user to implement new models of hyperbolic space with minimal effort. The data include facts about the underlying set (such as whether the model is bounded), facts about the metric (such as whether the model is conformal), facts about the isometry group (such as whether it is a linear or projective group), and more. Generally speaking, any data or method that pertains to the model itself – rather than the points, geodesics, or isometries of the model – is implemented in this module.

Abstractly, a model of hyperbolic space is a connected, simply connected manifold equipped with a complete Riemannian metric of constant curvature -1. This module records information sufficient to enable computations in hyperbolic space without explicitly specifying the underlying set or its Riemannian metric. Although, see the SageManifolds project if you would like to take this approach.

This module implements the abstract base class for a model of hyperbolic space of arbitrary dimension. It also contains the implementations of specific models of hyperbolic geometry.

AUTHORS:

- Greg Laun (2013): Initial version.

EXAMPLES:

We illustrate how the classes in this module encode data by comparing the upper half plane (UHP), Poincaré disk (PD) and hyperboloid (HM) models. First we create:

```
sage: U = HyperbolicPlane().UHP()
sage: P = HyperbolicPlane().PD()
sage: H = HyperbolicPlane().HM()
```

We note that the UHP and PD models are bounded while the HM model is not:

```
sage: U.is_bounded() and P.is_bounded()
True
sage: H.is_bounded()
False
```

The isometry groups of UHP and PD are projective, while that of HM is linear:

```
sage: U.is_isometry_group_projective()
True
sage: H.is_isometry_group_projective()
False
```

The models are responsible for determining if the coordinates of points and the matrix of linear maps are appropriate for constructing points and isometries in hyperbolic space:

```
sage: U.point_in_model(2 + I)
True
sage: U.point_in_model(2 - I)
False
sage: U.point_in_model(2)
False
sage: U.boundary_point_in_model(2)
True
```

class `sage.geometry.hyperbolic_space.hyperbolic_model.`**`HyperbolicModel`**(*space*, *name*, *short_name*, *bounded*, *conformal*, *dimension*, *isometry_group*, *isometry_group_is_projective*)

Bases: `Parent`, `UniqueRepresentation`, `BindableClass`

Abstract base class for hyperbolic models.

`Element`

alias of *`HyperbolicPoint`*

`bdry_point_test`(*p*)

Test whether a point is in the model. If the point is in the model, do nothing; otherwise raise a `ValueError`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().bdry_point_test(2)
sage: HyperbolicPlane().UHP().bdry_point_test(1 + I)
Traceback (most recent call last):
...
ValueError: I + 1 is not a valid boundary point in the UHP model
```

`boundary_point_in_model`(*p*)

Return `True` if the point is on the ideal boundary of hyperbolic space and `False` otherwise.

INPUT:

- any object that can converted into a complex number

OUTPUT:

- boolean

EXAMPLES:

```
sage: HyperbolicPlane().UHP().boundary_point_in_model(I)
False
```

`dist`(*a*, *b*)

Calculate the hyperbolic distance between `a` and `b`.

INPUT:

- a, b – a point or geodesic

OUTPUT:

- the hyperbolic distance

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1.0 + I)
sage: UHP.dist(p1, p2)
2.23230104635820

sage: PD = HyperbolicPlane().PD()
sage: p1 = PD.get_point(0)
sage: p2 = PD.get_point(I/2)
sage: PD.dist(p1, p2)
arccosh(5/3)

sage: UHP(p1).dist(UHP(p2))
arccosh(5/3)

sage: KM = HyperbolicPlane().KM()
sage: p1 = KM.get_point((0, 0))
sage: p2 = KM.get_point((1/2, 1/2))
sage: numerical_approx(KM.dist(p1, p2))
0.881373587019543

sage: HM = HyperbolicPlane().HM()
sage: p1 = HM.get_point((0,0,1))
sage: p2 = HM.get_point((1,0,sqrt(2)))
sage: numerical_approx(HM.dist(p1, p2))
0.881373587019543
```

Distance between a point and itself is 0:

```
sage: p = UHP.get_point(47 + I)
sage: UHP.dist(p, p)
0
```

Points on the boundary are infinitely far from interior points:

```
sage: UHP.get_point(3).dist(UHP.get_point(I))
+Infinity
```

get_geodesic(*start*, *end=None*, ***graphics_options*)

Return a geodesic in the appropriate model.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_geodesic(I, 2*I)
Geodesic in UHP from I to 2*I

sage: HyperbolicPlane().PD().get_geodesic(0, I/2)
Geodesic in PD from 0 to 1/2*I

sage: HyperbolicPlane().KM().get_geodesic((1/2, 1/2), (0,0))
Geodesic in KM from (1/2, 1/2) to (0, 0)
```

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```
sage: HyperbolicPlane().HM().get_geodesic((0,0,1), (1,0, sqrt(2)))
Geodesic in HM from (0, 0, 1) to (1, 0, sqrt(2))
```

get_isometry(*A*)

Return an isometry in `self` from the matrix `A` in the isometry group of `self`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_isometry(identity_matrix(2))
Isometry in UHP
[1 0]
[0 1]

sage: HyperbolicPlane().PD().get_isometry(identity_matrix(2))
Isometry in PD
[1 0]
[0 1]

sage: HyperbolicPlane().KM().get_isometry(identity_matrix(3))
Isometry in KM
[1 0 0]
[0 1 0]
[0 0 1]

sage: HyperbolicPlane().HM().get_isometry(identity_matrix(3))
Isometry in HM
[1 0 0]
[0 1 0]
[0 0 1]
```

get_point(*coordinates*, *is_boundary=None*, ***graphics_options*)

Return a point in `self`.

Automatically determine the type of point to return given either:

1. the coordinates of a point in the interior or ideal boundary of hyperbolic space, or
2. a *HyperbolicPoint* object.

INPUT:

- a point in hyperbolic space or on the ideal boundary

OUTPUT:

- a *HyperbolicPoint*

EXAMPLES:

We can create an interior point via the coordinates:

```
sage: HyperbolicPlane().UHP().get_point(2*I)
Point in UHP 2*I
```

Or we can create a boundary point via the coordinates:

```
sage: HyperbolicPlane().UHP().get_point(23)
Boundary point in UHP 23
```

However we cannot create points outside of our model:

```
sage: HyperbolicPlane().UHP().get_point(12 - I)
Traceback (most recent call last):
...
ValueError: -I + 12 is not a valid point in the UHP model
```

```
sage: HyperbolicPlane().UHP().get_point(2 + 3*I)
Point in UHP 3*I + 2

sage: HyperbolicPlane().PD().get_point(0)
Point in PD 0

sage: HyperbolicPlane().KM().get_point((0,0))
Point in KM (0, 0)

sage: HyperbolicPlane().HM().get_point((0,0,1))
Point in HM (0, 0, 1)

sage: p = HyperbolicPlane().UHP().get_point(I, color="red")
sage: p.graphics_options()
{'color': 'red'}
```

```
sage: HyperbolicPlane().UHP().get_point(12)
Boundary point in UHP 12

sage: HyperbolicPlane().UHP().get_point(infinity)
Boundary point in UHP +Infinity

sage: HyperbolicPlane().PD().get_point(I)
Boundary point in PD I

sage: HyperbolicPlane().KM().get_point((0,-1))
Boundary point in KM (0, -1)
```

is_bounded()

Return `True` if `self` is a bounded model.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().is_bounded()
True
sage: HyperbolicPlane().PD().is_bounded()
True
sage: HyperbolicPlane().KM().is_bounded()
True
sage: HyperbolicPlane().HM().is_bounded()
False
```

is_conformal()

Return `True` if `self` is a conformal model.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.is_conformal()
True
```

is_isometry_group_projective()

Return `True` if the isometry group of `self` is projective.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.is_isometry_group_projective()
True
```

isometry_from_fixed_points(*repel*, *attract*)

Given two fixed points `repel` and `attract` as hyperbolic points return a hyperbolic isometry with `repel` as repelling fixed point and `attract` as attracting fixed point.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: PD = HyperbolicPlane().PD()
sage: PD.isometry_from_fixed_points(-i, i)
Isometry in PD
[   3/4  1/4*I]
[-1/4*I    3/4]
```

```
sage: p, q = PD.get_point(1/2 + I/2), PD.get_point(6/13 + 9/13*I)
sage: PD.isometry_from_fixed_points(p, q)
Traceback (most recent call last):
...
ValueError: fixed points of hyperbolic elements must be ideal

sage: p, q = PD.get_point(4/5 + 3/5*I), PD.get_point(-I)
sage: PD.isometry_from_fixed_points(p, q)
Isometry in PD
[ 1/6*I - 2/3 -1/3*I - 1/6]
[ 1/3*I - 1/6 -1/6*I - 2/3]
```

isometry_in_model(*A*)

Return `True` if the input matrix represents an isometry of the given model and `False` otherwise.

INPUT:

- a matrix that represents an isometry in the appropriate model

OUTPUT:

- boolean

EXAMPLES:

```
sage: HyperbolicPlane().UHP().isometry_in_model(identity_matrix(2))
True

sage: HyperbolicPlane().UHP().isometry_in_model(identity_matrix(3))
False
```

isometry_test(*A*)

Test whether an isometry `A` is in the model.

If the isometry is in the model, do nothing. Otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().isometry_test(identity_matrix(2))
sage: HyperbolicPlane().UHP().isometry_test(matrix(2, [I,1,2,1]))
Traceback (most recent call last):
...
ValueError:
[I 1]
[2 1] is not a valid isometry in the UHP model
```

name()

Return the name of this model.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.name()
'Upper Half Plane Model'
```

point_in_model(*p*)

Return `True` if the point `p` is in the interior of the given model and `False` otherwise.

INPUT:

- any object that can converted into a complex number

OUTPUT:

- boolean

EXAMPLES:

```
sage: HyperbolicPlane().UHP().point_in_model(I)
True
sage: HyperbolicPlane().UHP().point_in_model(-I)
False
```

point_test(*p*)

Test whether a point is in the model. If the point is in the model, do nothing. Otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_model import␣
↪HyperbolicModelUHP
sage: HyperbolicPlane().UHP().point_test(2 + I)
sage: HyperbolicPlane().UHP().point_test(2 - I)
Traceback (most recent call last):
...
ValueError: -I + 2 is not a valid point in the UHP model
```

random_element(***kwargs*)

Return a random point in `self`.

The points are uniformly distributed over the rectangle $[-10, 10] \times [0, 10i]$ in the upper half plane model.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().random_element()
sage: bool((p.coordinates().imag()) > 0)
True
```

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```
sage: p = HyperbolicPlane().PD().random_element()
sage: HyperbolicPlane().PD().point_in_model(p.coordinates())
True

sage: p = HyperbolicPlane().KM().random_element()
sage: HyperbolicPlane().KM().point_in_model(p.coordinates())
True

sage: p = HyperbolicPlane().HM().random_element().coordinates()
sage: bool((p[0]**2 + p[1]**2 - p[2]**2 - 1) < 10**-8)
True
```

random_geodesic(***kwargs*)

Return a random hyperbolic geodesic.

Return the geodesic between two random points.

EXAMPLES:

```
sage: h = HyperbolicPlane().PD().random_geodesic()
sage: all( e.coordinates().abs() <= 1 for e in h.endpoints() )
True
```

random_isometry(*preserve_orientation=True*, ***kwargs*)

Return a random isometry in the model of `self`.

INPUT:

- `preserve_orientation` – if `True` return an orientation-preserving isometry

OUTPUT:

- a hyperbolic isometry

EXAMPLES:

```
sage: A = HyperbolicPlane().PD().random_isometry()
sage: A.preserves_orientation()
True
sage: B = HyperbolicPlane().PD().random_isometry(preserve_orientation=False)
sage: B.preserves_orientation()
False
```

random_point(***kwargs*)

Return a random point of `self`.

The points are uniformly distributed over the rectangle $[-10, 10] \times [0, 10i]$ in the upper half plane model.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().random_point()
sage: bool((p.coordinates().imag()) > 0)
True

sage: PD = HyperbolicPlane().PD()
sage: p = PD.random_point()
sage: PD.point_in_model(p.coordinates())
True
```

short_name()

Return the short name of this model.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.short_name()
'UHP'
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelHM**(*space*)

Bases: *HyperbolicModel*

Hyperboloid Model.

boundary_point_in_model(*p*)

Return False since the Hyperboloid model has no boundary points.

EXAMPLES:

```
sage: HM = HyperbolicPlane().HM()
sage: HM.boundary_point_in_model((0,0,1))
False
sage: HM.boundary_point_in_model((1,0,sqrt(2)))
False
sage: HM.boundary_point_in_model((1,2,1))
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize. For the hyperboloid model, the background object is the hyperboloid itself.

EXAMPLES:

```
sage: H = HyperbolicPlane().HM().get_background_graphic()
```

isometry_in_model(*A*)

Test that the matrix A is in the group $SO(2,1)^+$.

EXAMPLES:

```
sage: A = diagonal_matrix([1,1,-1])
sage: HyperbolicPlane().HM().isometry_in_model(A)
True
```

point_in_model(*p*)

Check whether a complex number lies in the hyperboloid.

EXAMPLES:

```
sage: HM = HyperbolicPlane().HM()
sage: HM.point_in_model((0,0,1))
True
sage: HM.point_in_model((1,0,sqrt(2)))
True
sage: HM.point_in_model((1,2,1))
False
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelKM**(*space*)

Bases: *HyperbolicModel*

Klein Model.

boundary_point_in_model(*p*)

Check whether a point lies in the unit circle, which corresponds to the ideal boundary of the hyperbolic plane in the Klein model.

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: KM.boundary_point_in_model((1, 0))
True
sage: KM.boundary_point_in_model((1/2, 1/2))
False
sage: KM.boundary_point_in_model((1, .2))
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize.

For the Klein model, the background object is the ideal boundary.

EXAMPLES:

```
sage: circ = HyperbolicPlane().KM().get_background_graphic()
```

isometry_in_model(*A*)

Check if the given matrix `A` is in the group $SO(2, 1)$.

EXAMPLES:

```
sage: A = matrix(3, [[1, 0, 0], [0, 17/8, 15/8], [0, 15/8, 17/8]])
sage: HyperbolicPlane().KM().isometry_in_model(A)
True
```

point_in_model(*p*)

Check whether a point lies in the open unit disk.

EXAMPLES:

```
sage: KM = HyperbolicPlane().KM()
sage: KM.point_in_model((1, 0))
False
sage: KM.point_in_model((1/2, 1/2))
True
sage: KM.point_in_model((1, .2))
False
```

class sage.geometry.hyperbolic_space.hyperbolic_model.**HyperbolicModelPD**(*space*)

Bases: *HyperbolicModel*

Poincaré Disk Model.

boundary_point_in_model(*p*)

Check whether a complex number lies in the open unit disk.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.boundary_point_in_model(1.00)
True
sage: PD.boundary_point_in_model(1/2 + I/2)
False
sage: PD.boundary_point_in_model(1 + .2*I)
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize.

For the Poincaré disk, the background object is the ideal boundary.

EXAMPLES:

```
sage: circ = HyperbolicPlane().PD().get_background_graphic()
```

isometry_in_model(*A*)

Check if the given matrix `A` is in the group $U(1,1)$.

EXAMPLES:

```
sage: z = [CC.random_element() for k in range(2)]; z.sort(key=abs)
sage: A = matrix(2,[z[1], z[0],z[0].conjugate(),z[1].conjugate()])
sage: HyperbolicPlane().PD().isometry_in_model(A)
True
```

point_in_model(*p*)

Check whether a complex number lies in the open unit disk.

EXAMPLES:

```
sage: PD = HyperbolicPlane().PD()
sage: PD.point_in_model(1.00)
False
sage: PD.point_in_model(1/2 + I/2)
True
sage: PD.point_in_model(1 + .2*I)
False
```

class `sage.geometry.hyperbolic_space.hyperbolic_model.`**HyperbolicModelUHP**(*space*)

Bases: *HyperbolicModel*

Upper Half Plane model.

Element

alias of *HyperbolicPointUHP*

boundary_point_in_model(*p*)

Check whether a complex number is a real number or `\infty`. In the `UHP.model_name_name`, this is the ideal boundary of hyperbolic space.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.boundary_point_in_model(1 + I)
False
sage: UHP.boundary_point_in_model(infinity)
```

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```
True
sage: UHP.boundary_point_in_model(CC(infinity))
True
sage: UHP.boundary_point_in_model(RR(infinity))
True
sage: UHP.boundary_point_in_model(1)
True
sage: UHP.boundary_point_in_model(12)
True
sage: UHP.boundary_point_in_model(1 - I)
False
sage: UHP.boundary_point_in_model(-2*I)
False
sage: UHP.boundary_point_in_model(0)
True
sage: UHP.boundary_point_in_model(I)
False
```

get_background_graphic(***bdry_options*)

Return a graphic object that makes the model easier to visualize. For the upper half space, the background object is the ideal boundary.

EXAMPLES:

```
sage: hp = HyperbolicPlane().UHP().get_background_graphic()
```

isometry_from_fixed_points(*repel*, *attract*)

Given two fixed points `repel` and `attract` as complex numbers return a hyperbolic isometry with `repel` as repelling fixed point and `attract` as attracting fixed point.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.isometry_from_fixed_points(2 + I, 3 + I)
Traceback (most recent call last):
...
ValueError: fixed points of hyperbolic elements must be ideal

sage: UHP.isometry_from_fixed_points(2, 0)
Isometry in UHP
[  -1    0]
[-1/3 -1/3]
```

isometry_in_model(*A*)

Check that `A` acts as an isometry on the upper half plane. That is, `A` must be an invertible 2×2 matrix with real entries.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: A = matrix(2,[1,2,3,4])
sage: UHP.isometry_in_model(A)
True
sage: B = matrix(2,[I,2,4,1])
sage: UHP.isometry_in_model(B)
False
```

An example of a matrix A such that $\det(A) \neq 1$, but the A acts isometrically:

```
sage: C = matrix(2,[10,0,0,10])
sage: UHP.isometry_in_model(C)
True
```

point_in_model (*p*)

Check whether a complex number lies in the open upper half plane.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: UHP.point_in_model(1 + I)
True
sage: UHP.point_in_model(infinity)
False
sage: UHP.point_in_model(CC(infinity))
False
sage: UHP.point_in_model(RR(infinity))
False
sage: UHP.point_in_model(1)
False
sage: UHP.point_in_model(12)
False
sage: UHP.point_in_model(1 - I)
False
sage: UHP.point_in_model(-2*I)
False
sage: UHP.point_in_model(I)
True
sage: UHP.point_in_model(0) # Not interior point
False
```

random_isometry (*preserve_orientation=True*, ***kwargs*)

Return a random isometry in the Upper Half Plane model.

INPUT:

- `preserve_orientation` – if `True` return an orientation-preserving isometry

OUTPUT:

- a hyperbolic isometry

EXAMPLES:

```
sage: A = HyperbolicPlane().UHP().random_isometry()
sage: B = HyperbolicPlane().UHP().random_isometry(preserve_orientation=False)
sage: B.preserves_orientation()
False
```

random_point (***kwargs*)

Return a random point in the upper half plane. The points are uniformly distributed over the rectangle $[-10, 10] \times [0, 10i]$.

EXAMPLES:

```
sage: p = HyperbolicPlane().UHP().random_point().coordinates()
sage: bool((p.imag()) > 0)
True
```

CHAPTER

FIVE

INTERFACE TO HYPERBOLIC MODELS

This module provides a convenient interface for interacting with models of hyperbolic space as well as their points, geodesics, and isometries.

The primary point of this module is to allow the code that implements hyperbolic space to be sufficiently decoupled while still providing a convenient user experience.

The interfaces are by default given abbreviated names. For example, UHP (upper half plane model), PD (Poincaré disk model), KM (Klein disk model), and HM (hyperboloid model).

Note: All of the current models of 2 dimensional hyperbolic space use the upper half plane model for their computations. This can lead to some problems, such as long coordinate strings for symbolic points. For example, the vector `(1, 0, sqrt(2))` defines a point in the hyperboloid model. Performing mapping this point to the upper half plane and performing computations there may return with vector whose components are unsimplified strings have several `sqrt(2)`'s. Presently, this drawback is outweighted by the rapidity with which new models can be implemented.

AUTHORS:

- Greg Laun (2013): Initial version.
- Rania Amer, Jean-Philippe Burelle, Bill Goldman, Zach Groton, Jeremy Lent, Leila Vaden, Derrick Wigglesworth (2011): many of the methods spread across the files.

EXAMPLES:

```
sage: HyperbolicPlane().UHP().get_point(2 + I)
Point in UHP I + 2

sage: HyperbolicPlane().PD().get_point(1/2 + I/2)
Point in PD 1/2*I + 1/2
```

class sage.geometry.hyperbolic_space.hyperbolic_interface.**HyperbolicModels**(*base*)

Bases: `Category_realization_of_parent`

The category of hyperbolic models of hyperbolic space.

class ParentMethods

Bases: `object`

super_categories()

The super categories of `self`.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_interface import␣
↪HyperbolicModels
sage: H = HyperbolicPlane()
sage: models = HyperbolicModels(H)
sage: models.super_categories()
[Category of metric spaces,
 Category of realizations of Hyperbolic plane]
```

class sage.geometry.hyperbolic_space.hyperbolic_interface.**HyperbolicPlane**

Bases: Parent, UniqueRepresentation

The hyperbolic plane $\mathbb{H}^2$.

Here are the models currently implemented:

- UHP – upper half plane
- PD – Poincaré disk
- KM – Klein disk
- HM – hyperboloid model

HM

alias of *HyperbolicModelHM*

Hyperboloid

alias of *HyperbolicModelHM*

KM

alias of *HyperbolicModelKM*

KleinDisk

alias of *HyperbolicModelKM*

PD

alias of *HyperbolicModelPD*

PoincareDisk

alias of *HyperbolicModelPD*

UHP

alias of *HyperbolicModelUHP*

UpperHalfPlane

alias of *HyperbolicModelUHP*

a_realization()

Return a realization of self.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: H.a_realization()
Hyperbolic plane in the Upper Half Plane Model
```

sage.geometry.hyperbolic_space.hyperbolic_interface.**HyperbolicSpace**(*n*)

Return n dimensional hyperbolic space.

EXAMPLES:

```
sage: from sage.geometry.hyperbolic_space.hyperbolic_interface import␣
↪HyperbolicSpace
sage: HyperbolicSpace(2)
Hyperbolic plane
```

CHAPTER

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