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An underlying philosophy in the development of Sage is that it should provide unified library-level access to some of the best GPL'd C/C++ libraries. Sage provides access to many libraries which are included with Sage.

The interfaces are implemented via shared libraries and data is moved between systems purely in memory. In particular, there is no interprocess interpreter parsing (e.g., pexpect), since everything is linked together and run as a single process. This is much more robust and efficient than using `pexpect`.

Each of these interfaces is used by other parts of Sage. For example, eclib is used by the elliptic curves module to compute ranks of elliptic curves and PARI is used for computation of class groups. It is thus probably not necessary for a casual user of Sage to be aware of the modules described in this chapter.
1.1 Library interface to Embeddable Common Lisp (ECL)

```python
class sage.libs.ecl.EclListIterator
    Bases: object

    Iterator object for an ECL list

    This class is used to implement the iterator protocol for EclObject. Do not instantiate this class directly but use the iterator method on an EclObject instead. It is an error if the EclObject is not a list.

    EXAMPLES:
    >>> from sage.libs.ecl import *
    >>> I = EclListIterator(EclObject('(1 2 3)'))
    >>> type(I)
    <type 'sage.libs.ecl.EclListIterator'>
    >>> [i for i in I]
    [<ECL: 1>, <ECL: 2>, <ECL: 3>]
    >>> [i for i in EclObject('(1 2 3)')]
    [<ECL: 1>, <ECL: 2>, <ECL: 3>]
    >>> EclListIterator(EclObject('1'))
    Traceback (most recent call last):
    ... 
    TypeError: ECL object is not iterable
```

```python
class sage.libs.ecl.EclObject
    Bases: object

    Python wrapper of ECL objects

    The EclObject forms a wrapper around ECL objects. The wrapper ensures that the data structure pointed to is protected from garbage collection in ECL by installing a pointer to it from a global data structure within the scope of the ECL garbage collector. This pointer is destroyed upon destruction of the EclObject.

    EclObject() takes a Python object and tries to find a representation of it in Lisp.

    EXAMPLES:

    Python lists get mapped to LISP lists. None and Boolean values to appropriate values in LISP:
    ```
    >>> from sage.libs.ecl import *
    >>> EclObject([None, true, false])
    (<ECL: (NIL T NIL)>)
    >>> EclObject(['a', 'b', 'c'])
    (<ECL: (quote (A B C))>)
    >>> EclObject({'a': 1, 'b': 'hello'})
    (<ECL: (quote (A 1 B "hello")))
    >>> EclObject(3.14159)
    (<ECL: 3.14159>)
    >>> EclObject(1+2j)
    (<ECL: 1/2 + 1/2*#C(1 0)>)
    >>> EclObject('a string')
    (<ECL: "a string">)
    >>> EclObject(None)
    (<ECL: NIL>)
    >>> EclObject(True)
    (<ECL: T>)
    >>> EclObject(False)
    (<ECL: NIL>)
    ```

    Numerical values are translated to the appropriate type in LISP:
    ```
    >>> from sage.libs.ecl import *
    >>> EclObject(123)
    (<ECL: 123>)
    >>> EclObject(-123)
    (<ECL: -123>)
    >>> EclObject(123.456)
    (<ECL: 123.456>)
    >>> EclObject(-123.456)
    (<ECL: -123.456>)
    >>> EclObject(12345678901234567890)
    (<ECL: 12345678901234567890>)
    >>> EclObject(-12345678901234567890)
    (<ECL: -12345678901234567890>)
    ```
```
Floats in Python are IEEE double, which LISP has as well. However, the printing of floating point types in LISP depends on settings:

```python
sage: a = EclObject(float(10**40))
sage: ecl_eval('(setf *read-default-float-format* 'single-float)')
<ECL: SINGLE-FLOAT>
sage: a
<ECL: 1.d40>
sage: ecl_eval('(setf *read-default-float-format* 'double-float)')
<ECL: DOUBLE-FLOAT>
sage: a
<ECL: 1.e40>
```

Tuples are translated to dotted lists:

```python
sage: EclObject((false, true))
<ECL: (NIL . T)>
sage: EclObject((1, 2, 3))
<ECL: (1 2 . 3)>
```

Strings are fed to the reader, so a string normally results in a symbol:

```python
sage: EclObject("Symbol")
<ECL: SYMBOL>
```

But with proper quotation one can construct a lisp string object too:

```python
sage: EclObject("'Symbol'")
<ECL: "Symbol">
```

Or any other object that the Lisp reader can construct:

```python
sage: EclObject('#("I am "just" a "simple" vector")')
<ECL: #("I am "just" a "simple" vector")>
```

By means of Lisp reader macros, you can include arbitrary objects:

```python
sage: EclObject([1, 2, '#.(make-hash-table :test #'equal)', 4])
<ECL: (1 2 #<hash-table ...> 4)>
```

Using an optional argument, you can control how strings are handled:

```python
sage: EclObject("String", False)
<ECL: "String">
sage: EclObject('#("I may look like a vector but I am a string")', False)
<ECL: '#(I may look like a vector but I am a string)'>
```

This also affects strings within nested lists and tuples

```python
sage: EclObject([1, 2, "String", 4], False)
<ECL: (1 2 "String" 4)>
```

EclObjects translate to themselves, so one can mix:
Calling an EclObject translates into the appropriate LISP `apply`, where the argument is transformed into an EclObject itself, so one can flexibly apply LISP functions:

```python
sage: car=EclObject("car")
sage: cdr=EclObject("cdr")
sage: car(cdr([1,2,3]))
<ECL: 2>
```

and even construct and evaluate arbitrary S-expressions:

```python
sage: eval=EclObject("eval")
sage: quote=EclObject("quote")
sage: eval([car, [cdr, [quote,[1,2,3]]]])
<ECL: 2>
```

### atomp()

Return True if self is atomic, False otherwise.

**EXAMPLES:**

```python
sage: from sage.libs.ecl import *
sage: EclObject([]).atomp()
True
sage: EclObject([[]]).atomp()
False
```

### caar()

Return the caar of self

**EXAMPLES:**

```python
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
sage: L.cdr()
<ECL: ((3 4))>
sage: L.caar()
<ECL: 1>
sage: L.cadr()
<ECL: (3 4)>
sage: L.cdar()
<ECL: (2)>
sage: L.cddr()
<ECL: NIL>
```

### cadr()

Return the cadr of self

**EXAMPLES:**

```python
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
<ECL: (1 2)>
```

(continues on next page)
car ()
Return the car of self

EXAMPLES:

```python
sage: from sage.libs.ecl import *
```
```python
sage: L=EclObject([[1,2],[3,4]])
```
```python
sage: L.car()
<ECL: (1 2)>
```
```python
sage: L.cdr()
<ECL: ((3 4))>
```
```python
sage: L.caar()
<ECL: 1>
```
```python
sage: L.cadr()
<ECL: (3 4)>
```
```python
sage: L.cdar()
<ECL: (2)>
```
```python
sage: L.cddr()
<ECL: NIL>
```

cdar ()
Return the cdar of self

EXAMPLES:

```python
sage: from sage.libs.ecl import *
```
```python
sage: L=EclObject([[1,2],[3,4]])
```
```python
sage: L.car()
<ECL: (1 2)>
```
```python
sage: L.cdr()
<ECL: ((3 4))>
```
```python
sage: L.caar()
<ECL: 1>
```
```python
sage: L.cadr()
<ECL: (3 4)>
```
```python
sage: L.cdar()
<ECL: (2)>
```
```python
sage: L.cddr()
<ECL: NIL>
```

cddr ()
Return the cddr of self

EXAMPLES:
\begin{verbatim}
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
< ECL: (1 2)>
sage: L.cdr()
< ECL: ((3 4))>
sage: L.caar()
< ECL: 1>
sage: L.cadr()
< ECL: (3 4)>
sage: L.cdar()
< ECL: (2)>
sage: L.cddr()
< ECL: NIL>
\end{verbatim}

\textbf{cdr()}

Return the cdr of self

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],[3,4]])
sage: L.car()
< ECL: (1 2)>
sage: L.cdr()
< ECL: ((3 4))>
sage: L.caar()
< ECL: 1>
sage: L.cadr()
< ECL: (3 4)>
sage: L.cdar()
< ECL: (2)>
sage: L.cddr()
< ECL: NIL>
\end{verbatim}

\textbf{characterp()}

Return True if self is a character, False otherwise

Strings are not characters

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.ecl import *
sage: EclObject('"a"').characterp()
False
\end{verbatim}

\textbf{cons(d)}

apply cons to self and argument and return the result.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.ecl import *
sage: a=EclObject(1)
sage: b=EclObject(2)
sage: a.cons(b)
< ECL: (1 . 2)>
\end{verbatim}

\textbf{consp()}

Return True if self is a cons, False otherwise. NIL is not a cons.
Examples:

```python
sage: from sage.libs.ecl import *
sage: EclObject([]).consp()
False
sage: EclObject([[]]).consp()
True
```

**eval()**
Evaluate object as an S-Expression

Examples:

```python
sage: from sage.libs.ecl import *
sage: S=EclObject("(+ 1 2)")
sage: S
<ECL: (+ 1 2)>
sage: S.eval()
<ECL: 3>
```

**fixnump()**
Return True if self is a fixnum, False otherwise

Examples:

```python
sage: from sage.libs.ecl import *
sage: EclObject(2**3).fixnump()
True
sage: EclObject(2**200).fixnump()
False
```

**listp()**
Return True if self is a list, False otherwise. NIL is a list.

Examples:

```python
sage: from sage.libs.ecl import *
sage: EclObject([]).listp()
True
sage: EclObject([[]]).listp()
True
```

**nullp()**
Return True if self is NIL, False otherwise

Examples:

```python
sage: from sage.libs.ecl import *
sage: EclObject([]).nullp()
True
sage: EclObject([[]]).nullp()
False
```

**python()**
Convert an EclObject to a python object.

Examples:
sage: from sage.libs.ecl import *
sage: L=EclObject([[1,2],("three","four")])
sage: L.python()
[1, 2, ('THREE', 'four')]

rplaca\(d\)
Destructively replace car(self) with \(d\).

EXAMPLES:

sage: from sage.libs.ecl import *
sage: L=EclObject((1,2))
sage: L
<ECL: (1 . 2)>
sage: a=EclObject(3)
sage: L.rplaca(a)
sage: L
<ECL: (3 . 2)>

rplacd\(d\)
Destructively replace cdr(self) with \(d\).

EXAMPLES:

sage: from sage.libs.ecl import *
sage: L=EclObject((1,2))
sage: L
<ECL: (1 . 2)>
sage: a=EclObject(3)
sage: L.rplacd(a)
sage: L
<ECL: (1 . 3)>

symbolp()
Return True if self is a symbol, False otherwise.

EXAMPLES:

sage: from sage.libs.ecl import *
sage: EclObject([]).symbolp()
True
sage: EclObject([[]]).symbolp()
False

sage.libs.ecl.ecl_eval\(s\)
Read and evaluate string in Lisp and return the result

EXAMPLES:

sage: from sage.libs.ecl import *
sage: ecl_eval("(defun fibo (n)(cond((= n 0) 0)((= n 1) 1)(T (+ (fibo (- n 1))(fibo (- n 2)))))")
<ECL: FIBO>
sage: ecl_eval("(mapcar 'fibo '(1 2 3 4 5 6 7))")
<ECL: (1 1 2 3 5 8 13)>

sage.libs.ecl.init_ecl()
Internal function to initialize ecl. Do not call.
This function initializes the ECL library for use within Python. This routine should only be called once and importing the ecl library interface already does that, so do not call this yourself.

EXAMPLES:

```python
sage: from sage.libs.ecl import *
```

At this point, init_ecl() has run. Explicitly executing it gives an error:

```python
sage: init_ecl()
Traceback (most recent call last):
... RuntimeWarning: ECL is already initialized
```

```
sage.libs.ecl.print_objects()
```
Print GC-protection list

Diagnostic function. ECL objects that are bound to Python objects need to be protected from being garbage collected. We do this by including them in a doubly linked list bound to the global ECL symbol SAGE-LIST-OF-OBJECTS. Only non-immediate values get included, so small integers do not get linked in. This routine prints the values currently stored.

EXAMPLES:

```
sage: a=EclObject("hello")
sage: b=EclObject(10)
sage: c=EclObject("world")
sage: print_objects() #random because previous test runs can have left objects
NIL WORLD HELLO
```

```
sage.libs.ecl.shutdown_ecl()
```
Shut down ecl. Do not call.

Given the way that ECL is used from python, it is very difficult to ensure that no ECL objects exist at a particular time. Hence, destroying ECL is a risky proposition.

EXAMPLES:

```
sage: shutdown_ecl()
```

```
sage.libs.ecl.test_ecl_options()
```
Print an overview of the ECL options

```
sage.libs.ecl.test_sigint_before_ecl_sig_on()
```
CHAPTER TWO

ECLIB

2.1 Sage interface to Cremona’s eclib library (also known as mwrank)

This is the Sage interface to John Cremona’s eclib C++ library for arithmetic on elliptic curves. The classes defined in this module give Sage interpreter-level access to some of the functionality of eclib. For most purposes, it is not necessary to directly use these classes. Instead, one can create an EllipticCurve and call methods that are implemented using this module.

Note: This interface is a direct library-level interface to eclib, including the 2-descent program mwrank.

class sage.libs.eclib.interface.mwrank_EllipticCurve(ainvs, verbose=False)

Bases: sage.structure.sage_object.SageObject

The mwrank_EllipticCurve class represents an elliptic curve using the Curvedata class from eclib, called here an ‘mwrank elliptic curve’.

Create the mwrank elliptic curve with invariants ainvs, which is a list of 5 or less integers \(a_1, a_2, a_3, a_4,\) and \(a_5\).

If strictly less than 5 invariants are given, then the first ones are set to 0, so, e.g., \([3, 4]\) means \(a_1 = a_2 = a_3 = 0\) and \(a_4 = 3, a_5 = 4\).

INPUT:

• ainvs (list or tuple) – a list of 5 or less integers, the coefficients of a nonsingular Weierstrass equation.

• verbose (bool, default False) – verbosity flag. If True, then all Selmer group computations will be verbose.

EXAMPLES:

We create the elliptic curve \(y^2 + y = x^3 + x^2 - 2x\):

```sage
e = mwrank_EllipticCurve([0, 1, 1, -2, 0])
sage: e.ainvs()
[0, 1, 1, -2, 0]
```

This example illustrates that omitted \(a\)-invariants default to 0:

```sage
e = mwrank_EllipticCurve([3, -4])
sage: e
y^2 = x^3 + 3x - 4
```
sage: e.ainvs()
[0, 0, 0, 3, -4]

The entries of the input list are coerced to \texttt{int}. If this is impossible, then an error is raised:

```python
sage: e = mwrank_EllipticCurve([3, -4.8]); e
Traceback (most recent call last):
...  
TypeError: ainvs must be a list or tuple of integers.
```

When you enter a singular model you get an exception:

```python
sage: e = mwrank_EllipticCurve([0, 0])
Traceback (most recent call last):
...  
ArithmeticError: Invariants (= 0,0,0,0,0) do not describe an elliptic curve.
```

\textbf{CPS} \_\textbf{height} \_\textbf{bound} ()

Returns the Cremona-Prickett-Siksek height bound. This is a floating point number \(B\) such that if \(P\) is a point on the curve, then the naive logarithmic height \(h(P)\) is less than \(B + \hat{h}(P)\), where \(\hat{h}(P)\) is the canonical height of \(P\).

\textbf{Warning:} We assume the model is minimal!

\textbf{EXAMPLES:}

```python
sage: E = mwrank_EllipticCurve([0, 0, 0, -1002231243161, 0])
sage: E.CPS_height_bound()
14.163198527061496
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: E.CPS_height_bound()
0.0
```

\textbf{ainvs} ()

Returns the \(a\)-invariants of this mwrank elliptic curve.

\textbf{EXAMPLES:}

```python
sage: E = mwrank_EllipticCurve([0,0,1,-1,0])
sage: E.ainvs()  
[0, 0, 1, -1, 0]
```

\textbf{certain} ()

Returns True if the last \texttt{two\_descent()} call provably correctly computed the rank. If \texttt{two\_descent()} hasn’t been called, then it is first called by \texttt{certain()} using the default parameters.

The result is True if and only if the results of the methods \texttt{rank()} and \texttt{rank\_bound()} are equal.

\textbf{EXAMPLES:}

A 2-descent does not determine \(E(\mathbb{Q})\) with certainty for the curve \(y^2 + y = x^3 - x^2 - 120x - 2183\):

```python
sage: E = mwrank_EllipticCurve([0, -1, 1, -120, -2183])
sage: E.certain()
```
E.certain()  
False  
E.rank()  
0

The previous value is only a lower bound; the upper bound is greater:

E.rank_bound()  
2

In fact the rank of $E$ is actually 0 (as one could see by computing the $L$-function), but Sha has order 4 and the 2-torsion is trivial, so mwrank cannot conclusively determine the rank in this case.

**conductor()**

Return the conductor of this curve, computed using Cremona’s implementation of Tate’s algorithm.

**Note:** This is independent of PARI’s.

**EXAMPLES:**

```
sage: E = mwrank_EllipticCurve([1, 1, 0, -6958, -224588])
sage: E.conductor()
2310
```

**gens()**

Return a list of the generators for the Mordell-Weil group.

**EXAMPLES:**

```
sage: E = mwrank_EllipticCurve([0, 0, 1, -1, 0])
sage: E.gens()
[[0, -1, 1]]
```

**isogeny_class**(verbose=False)

Returns the isogeny class of this mwrank elliptic curve.

**EXAMPLES:**

```
sage: E = mwrank_EllipticCurve([0,-1,1,0,0])
sage: E.isogeny_class()
([[0, -1, 1, 0], [0, -1, 1, -10, -20], [0, -1, 1, -7820, -263580]], [[0, 5, → 0], [5, 0, 5], [0, 5, 0]])
```

**rank()**

Returns the rank of this curve, computed using *two_descent*().

In general this may only be a lower bound for the rank; an upper bound may be obtained using the function *rank_bound*(). To test whether the value has been proved to be correct, use the method *certain*().

**EXAMPLES:**

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.rank()
0
sage: E.certain()
True
```
sage: E = mwrank_EllipticCurve([0, -1, 1, -929, -10595])
sage: E.rank()
0
sage: E.certain()
False

**rank_bound()**
Returns an upper bound for the rank of this curve, computed using `two_descent()`.

If the curve has no 2-torsion, this is equal to the 2-Selmer rank. If the curve has 2-torsion, the upper bound may be smaller than the bound obtained from the 2-Selmer rank minus the 2-rank of the torsion, since more information is gained from the 2-isogenous curve or curves.

**EXAMPLES:**
The following is the curve 960D1, which has rank 0, but Sha of order 4:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.rank_bound()
0
sage: E.rank()
0
```

In this case the rank was computed using a second descent, which is able to determine (by considering a 2-isogenous curve) that Sha is nontrivial. If we deliberately stop the second descent, the rank bound is larger:

```
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.two_descent(second_descent = False, verbose=False)
sage: E.rank_bound()
2
```

In contrast, for the curve 571A, also with rank 0 and Sha of order 4, we only obtain an upper bound of 2:

```
sage: E = mwrank_EllipticCurve([0, -1, 1, -929, -10595])
sage: E.rank_bound()
2
```

In this case the value returned by `rank()` is only a lower bound in general (though this is correct):

```
sage: E.rank()
0
sage: E.certain()
False
```

**regulator()**
Return the regulator of the saturated Mordell-Weil group.

**EXAMPLES:**

```
sage: E = mwrank_EllipticCurve([0, 0, 1, -1, 0])
sage: E.regulator()
0.05111140823996884
```

**saturate**(bound=-1)
Compute the saturation of the Mordell-Weil group at all primes up to `bound`.

**INPUT:**
• **bound**(int, default -1) – Use -1 (the default) to saturate at all primes, 0 for no saturation, or \( n \) (a positive integer) to saturate at all primes up to \( n \).

**EXAMPLES:**

Since the 2-descent automatically saturates at primes up to 20, it is not easy to come up with an example where saturation has any effect:

```python
sage: E = mwrank_EllipticCurve([0, 0, 0, -1002231243161, 0])
sage: E.gens()
[[[-1001107, -4004428, 1]]]
sage: E.saturate()
sage: E.gens()
[[[-1001107, -4004428, 1]]]
```

Check that trac ticket #18031 is fixed:

```python
sage: E = EllipticCurve([0,-1,1,-266,968])
sage: Q1 = E([-1995,3674,1])
sage: Q2 = E([157,1950,1])
sage: E.saturation([Q1,Q2])
((1 : -27 : 1), (157 : 1950 : 1)), 3, 0.801588644684981
```

**selmer_rank**()

Returns the rank of the 2-Selmer group of the curve.

**EXAMPLES:**

The following is the curve 960D1, which has rank 0, but Sha of order 4. The 2-torsion has rank 2, and the Selmer rank is 3:

```python
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.selmer_rank()
3
```

Nevertheless, we can obtain a tight upper bound on the rank since a second descent is performed which establishes the 2-rank of Sha:

```python
sage: E.rank_bound()
0
```

To show that this was resolved using a second descent, we do the computation again but turn off second_descent:

```python
sage: E = mwrank_EllipticCurve([0, -1, 0, -900, -10098])
sage: E.two_descent(second_descent = False, verbose=False)
sage: E.rank_bound()
2
```

For the curve 571A, also with rank 0 and Sha of order 4, but with no 2-torsion, the Selmer rank is strictly greater than the rank:

```python
sage: E = mwrank_EllipticCurve([0, -1, 1, -929, -10595])
sage: E.selmer_rank()
2
sage: E.rank_bound()
2
```

In cases like this with no 2-torsion, the rank upper bound is always equal to the 2-Selmer rank. If we ask for the rank, all we get is a lower bound:
set_verbose\( (\text{verbose}) \)

Set the verbosity of printing of output by the two_descent() and other functions.

**INPUT:**

- \text{verbose} (int) – if positive, print lots of output when doing 2-descent.

**EXAMPLES:**

```
\textit{sage}: \text{E} = \text{mwrank\_EllipticCurve}([0, 0, 1, -1, 0])
\textit{sage}: \text{E}.\text{saturation()} \# \text{no output}
\textit{sage}: \text{E}.\text{gens()}
[[[0, -1, 1]]]
\textit{sage}: \text{E} = \text{mwrank\_EllipticCurve}([0, 0, 1, -1, 0])
\textit{sage}: \text{E}.\text{set\_verbose}(1)
\textit{sage}: \text{E}.\text{saturation()} \# \text{tol 1e-10}
Basic pair: I=48, J=-432
disc=255744
2-adic index bound = 2
By Lemma 5.1(a), 2-adic index = 1
2-adic index = 1
One (I,J) pair
Looking for quartics with I = 48, J = -432
Looking for Type 2 quartics:
Trying positive a from 1 up to 1 (square a first...)
(1,0,-6,4,1) --trivial
Trying positive a from 1 up to 1 (...then non-square a)
Finished looking for Type 2 quartics.
Looking for Type 1 quartics:
Trying positive a from 1 up to 2 (square a first...)
(1,0,0,4,4) --nontrivial...(x:y:z) = (1 : 1 : 0)
Point = [0:0:1]
height = 0.0511114082399688402358
Rank of B=im(eps) increases to 1 (The previous point is on the egg)
Exiting search for Type 1 quartics after finding one which is globally soluble.
Mordell rank contribution from B=im(eps) = 1
Selmer rank contribution from B=im(eps) = 1
Sha rank contribution from B=im(eps) = 0
Mordell rank contribution from A=ker(eps) = 0
Selmer rank contribution from A=ker(eps) = 0
Sha rank contribution from A=ker(eps) = 0
Searching for points (bound = 8)...done:
found points which generate a subgroup of rank 1
and regulator 0.0511114082399688402358
Processing points found during 2-descent...done:
now regulator = 0.0511114082399688402358
Saturating (with bound = -1)...done:
points were already saturated.
```

\textit{silverman\_bound()}\n
Return the Silverman height bound. This is a floating point number \( B \) such that if \( P \) is a point on the curve, then the naive logarithmic height \( h(P) \) is less than \( B + \hat{h}(P) \), where \( \hat{h}(P) \) is the canonical height.
of \( P \).

**Warning:** We assume the model is minimal!

**EXAMPLES:**

```python
sage: E = mwrank_EllipticCurve([0, 0, 0, -1002231243161, 0])
sage: E.silverman_bound()
18.29545210468247
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: E.silverman_bound()
6.284833369972403
```

**two_descent** *(verbose=True, selmer_only=False, first_limit=20, second_limit=8, n_aux=-1, second_descent=True)*

Compute 2-descent data for this curve.

**INPUT:**

- `verbose` (bool, default True) – print what mwrank is doing.
- `selmer_only` (bool, default False) – selmer_only switch.
- `first_limit` (int, default 20) – bound on \(|x| + |z|\) in quartic point search.
- `second_limit` (int, default 8) – bound on \(\log \max(|x|, |z|)\), i.e. logarithmic.
- `n_aux` (int, default -1) – (only relevant for general 2-descent when 2-torsion trivial) number of primes used for quartic search. \(n_{aux}=-1\) causes default (8) to be used. Increase for curves of higher rank.
- `second_descent` (bool, default True) – (only relevant for curves with 2-torsion, where mwrank uses descent via 2-isogeny) flag determining whether or not to do second descent. Default strongly recommended.

**OUTPUT:**

Nothing – nothing is returned.

**class** `sage.libs.eclib.interface.mwrank_MordellWeil` *(curve, verbose=True, pp=1, maxr=999)*

**Bases:** `sage.structure.sage_object.SageObject`

The `mwrank_MordellWeil` class represents a subgroup of a Mordell-Weil group. Use this class to saturate a specified list of points on an `mwrank_EllipticCurve`, or to search for points up to some bound.

**INPUT:**

- `curve` (`mwrank_EllipticCurve`) – the underlying elliptic curve.
- `verbose` (bool, default False) – verbosity flag (controls amount of output produced in point searches).
- `pp` (int, default 1) – process points flag (if nonzero, the points found are processed, so that at all times only a \(\mathbb{Z}\)-basis for the subgroup generated by the points found so far is stored; if zero, no processing is done and all points found are stored).
- `maxr` (int, default 999) – maximum rank (quit point searching once the points found generate a subgroup of this rank; useful if an upper bound for the rank is already known).

**EXAMPLES:**
```sage
sage: E = mwrank_EllipticCurve([1,0,1,4,-6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ
Subgroup of Mordell-Weil group: []
sage: EQ.search(2)
P1 = [0:1:0]  is torsion point, order 1
P1 = [1:-1:1]  is torsion point, order 2
P1 = [2:2:1]  is torsion point, order 3
P1 = [9:23:1]  is torsion point, order 6

sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(2)
P1 = [0:1:0]  is torsion point, order 1
P1 = [-3:0:1]  is generator number 1
...    
P4 = [-91:804:343] = -2*P1 + 2*P2 + 1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
```

Example to illustrate the verbose parameter:

```sage
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E, verbose=False)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
sage: EQ = mwrank_MordellWeil(E, verbose=True)
sage: EQ.search(1)
P1 = [0:1:0]  is torsion point, order 1
P1 = [-3:0:1]  is generator number 1
saturating up to 20...Checking 2-saturation
Points have successfully been 2-saturated (max q used = 7)
Checking 3-saturation
Points have successfully been 3-saturated (max q used = 7)
Checking 5-saturation
Points have successfully been 5-saturated (max q used = 23)
Checking 7-saturation
Points have successfully been 7-saturated (max q used = 41)
Checking 11-saturation
Points have successfully been 11-saturated (max q used = 17)
Checking 13-saturation
Points have successfully been 13-saturated (max q used = 43)
Checking 17-saturation
Points have successfully been 17-saturated (max q used = 31)
Checking 19-saturation
Points have successfully been 19-saturated (max q used = 37)
done
P2 = [-2:3:1]  is generator number 2
saturating up to 20...Checking 2-saturation
possible kernel vector = [1,1]
This point may be in 2E(Q): [14:-52:1]
...and it is!
Replacing old generator #1 with new generator [1:-1:1]
Points have successfully been 2-saturated (max q used = 7)
Index gain = 2^1
```

(continues on next page)
Checking 3-saturation
Points have successfully been 3-saturated (max q used = 13)
Checking 5-saturation
Points have successfully been 5-saturated (max q used = 67)
Checking 7-saturation
Points have successfully been 7-saturated (max q used = 53)
Checking 11-saturation
Points have successfully been 11-saturated (max q used = 73)
Checking 13-saturation
Points have successfully been 13-saturated (max q used = 103)
Checking 17-saturation
Points have successfully been 17-saturated (max q used = 113)
Checking 19-saturation
Points have successfully been 19-saturated (max q used = 47)
done (index = 2).
Gained index 2, new generators = [ [1:-1:1] [-2:3:1] ]
P3 = [-14:25:8] is generator number 3
saturating up to 20...
Checking 2-saturation
Points have successfully been 2-saturated (max q used = 11)
Checking 3-saturation
Points have successfully been 3-saturated (max q used = 13)
Checking 5-saturation
Points have successfully been 5-saturated (max q used = 71)
Checking 7-saturation
Points have successfully been 7-saturated (max q used = 101)
Checking 11-saturation
Points have successfully been 11-saturated (max q used = 127)
Checking 13-saturation
Points have successfully been 13-saturated (max q used = 151)
Checking 17-saturation
Points have successfully been 17-saturated (max q used = 139)
Checking 19-saturation
Points have successfully been 19-saturated (max q used = 179)
done (index = 1).
P4 = [-1:3:1] = -1*P1 + -1*P2 + -1*P3 (mod torsion)
P4 = [0:2:1] = 2*P1 + 0*P2 + 1*P3 (mod torsion)
P4 = [2:13:8] = -3*P1 + 1*P2 + -1*P3 (mod torsion)
P4 = [1:0:1] = -1*P1 + 0*P2 + 0*P3 (mod torsion)
P4 = [2:0:1] = -1*P1 + 1*P2 + 0*P3 (mod torsion)
P4 = [18:7:8] = -2*P1 + -1*P2 + -1*P3 (mod torsion)
P4 = [3:3:1] = 1*P1 + 0*P2 + 1*P3 (mod torsion)
P4 = [4:6:1] = 0*P1 + -1*P2 + -1*P3 (mod torsion)
P4 = [36:69:64] = 1*P1 + -2*P2 + 0*P3 (mod torsion)
P4 = [68:-25:64] = -2*P1 + -1*P2 + -2*P3 (mod torsion)
P4 = [12:35:27] = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]

Example to illustrate the process points (pp) parameter:

```
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E, verbose=False, pp=1)
sage: EQ.search(1); EQ
# generators only
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
sage: EQ = mwrank_MordellWeil(E, verbose=False, pp=0)
sage: EQ.search(1); EQ
# all points found
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8], [-3:0:1], [-2:3:1], [-14:25:8], [-1:3:1], [0:1:1], [2:0:1], [18:7:8], [3:3:1], [4:6:1], [36:69:64], [12:35:27]]
```
points()

Return a list of the generating points in this Mordell-Weil group.

OUTPUT:

(list) A list of lists of length 3, each holding the primitive integer coordinates \([x, y, z]\) of a generating point.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(1)
P1 = [0:1:0] is torsion point, order 1
P1 = [-3:0:1] is generator number 1
...
P4 = [12:35:27] = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ.points()
[[1, -1, 1], [-2, 3, 1], [-14, 25, 8]]
```

process \((v, sat=0)\)

This function allows one to add points to a `mwrank_MordellWeil` object.

Process points in the list \(v\), with saturation at primes up to \(sat\). If \(sat\) is zero (the default), do no saturation.

INPUT:

- \(v\) (list of 3-tuples or lists of ints or Integers) – a list of triples of integers, which define points on the curve.
- \(sat\) (int, default 0) – saturate at primes up to \(sat\), or at all primes if \(sat\) is zero.

OUTPUT:

None. But note that if the `verbose` flag is set, then there will be some output as a side-effect.

EXAMPLES:

```
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: E.gens()
[[1, -1, 1], [-2, 3, 1], [-14, 25, 8]]
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.process([[1547, -2967, 343], [-1422227300, -4932830557, 12167000000]], sat=20)
P1 = [1547:-2967:343] is generator number 1
...
```

Example to illustrate the saturation parameter `sat`:

```
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.process([[1547, -2967, 343], [2707496766203306, 864581029138191, ...
-296971514023272], [-13422227300, -4932830557, 12167000000]], sat=20)
P1 = [1547:-2967:343] is generator number 1
...
```
Gained index 5, new generators = [ [-2:3:1] [-14:25:8] [1:-1:1] ]

```
sage: EQ.points()
[[-2, 3, 1], [-14, 25, 8], [1, -1, 1]]
```

Here the processing was followed by saturation at primes up to 20. Now we prevent this initial saturation:

```
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.process([[1547, -2967, 343], [2707496766203306, 864581029138191, -2969715140223272], [-13422227300, -49322830557, 12167000000]], sat=0)
P1 = [1547:-2967:343] is generator number 1
P2 = [2707496766203306:864581029138191:2969715140223272] is generator number 2
P3 = [-13422227300:-49322830557:12167000000] is generator number 3

```
sage: EQ.points()
[[1547, -2967, 343], [2707496766203306, 864581029138191, 2969715140223272], [-13422227300, -49322830557, 12167000000]]
```

```
sage: EQ.regulator()
375.4292028825455
```

```
sage: EQ.saturate(2)  # points were not 2-saturated
saturating basis...Saturation index bound = 93
WARNING: saturation at primes p > 2 will not be done;
...
Gained index 2
New regulator = 93.857...
(False, 2, ' [ ] ')

```
sage: EQ.points()
[[1547, -2967, 343], [2707496766203306, 864581029138191, 2969715140223272], [-13422227300, -49322830557, 12167000000]]
```

```
sage: EQ.regulator()
93.85730072063639
```

```
sage: EQ.saturate(3)  # points were not 3-saturated
saturating basis...Saturation index bound = 46
WARNING: saturation at primes p > 3 will not be done;
...
Gained index 3
New regulator = 10.428...
(False, 3, ' [ ] ')

```
sage: EQ.points()
[[-2, 3, 1], [-14, 25, 8], [-13422227300, -49322830557, 12167000000]]
```

```
sage: EQ.regulator()
10.4285889689596
```

```
sage: EQ.saturate(5)  # points were not 5-saturated
saturating basis...Saturation index bound = 15
WARNING: saturation at primes p > 5 will not be done;
...
Gained index 5
New regulator = 0.417...
(False, 5, ' [ ] ')

```
sage: EQ.points()
[[-2, 3, 1], [-14, 25, 8], [1, -1, 1]]
```

```
sage: EQ.regulator()
0.417143558758384
```

```
sage: EQ.saturate()  # points are now saturated
saturating basis...Saturation index bound = 3
Checking saturation at [ 2 3 ]
```

(continues on next page)
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done
(True, 1, '[]')

rank()

Return the rank of this subgroup of the Mordell-Weil group.

OUTPUT:

(int) The rank of this subgroup of the Mordell-Weil group.

EXAMPLES:

```sage
E = mwrank_EllipticCurve([0,-1,1,0,0])
sage: E.rank()
0
```

A rank 3 example:

```sage
E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.rank()
0
sage: EQ.regulator()
1.0
```

The preceding output is correct, since we have not yet tried to find any points on the curve either by searching or 2-descent:

```sage
EQ
Subgroup of Mordell-Weil group: []
```

Now we do a very small search:

```sage
EQ.search(1)
P1 = [0:1:0] is torsion point, order 1
P1 = [-3:0:1] is generator number 1
saturating up to 20...Checking 2-saturation
...
P4 = [12:35:27] = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
sage: EQ.rank()
3
sage: EQ.regulator()
0.417143558758384
```

We do in fact now have a full Mordell-Weil basis.

regulator()

Return the regulator of the points in this subgroup of the Mordell-Weil group.

**Note:** eclib can compute the regulator to arbitrary precision, but the interface currently returns the output as a float.
OUTPUT:

(float) The regulator of the points in this subgroup.

EXAMPLES:

```python
sage: E = mwrank_EllipticCurve([0,-1,1,0,0])
sage: E.regulator()
1.0
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: E.regulator()
0.417143558758384
```

`saturate (max_prime=-1, odd_primes_only=False)`

Saturate this subgroup of the Mordell-Weil group.

INPUT:

- `max_prime` (int, default -1) – saturation is performed for all primes up to `max_prime`. If -1 (the default), an upper bound is computed for the primes at which the subgroup may not be saturated, and this is used; however, if the computed bound is greater than a value set by the `eclib` library (currently 97) then no saturation will be attempted at primes above this.

- `odd_primes_only` (bool, default False) – only do saturation at odd primes. (If the points have been found via `two_descent()` they should already be 2-saturated.)

OUTPUT:

(3-tuple) `(ok, index, unsatlist)` where:

- `ok` (bool) – True if and only if the saturation was provably successful at all primes attempted. If the default was used for `max_prime` and no warning was output about the computed saturation bound being too high, then True indicates that the subgroup is saturated at all primes.

- `index` (int) – the index of the group generated by the original points in their saturation.

- `unsatlist` (list of ints) – list of primes at which saturation could not be proved or achieved. Increasing the precision should correct this, since it happens when a linear combination of the points appears to be a multiple of \( p \) but cannot be divided by \( p \). (Note that `eclib` uses floating point methods based on elliptic logarithms to divide points.)

**Note:** We emphasize that if this function returns True as the first return argument (ok), and if the default was used for the parameter `max_prime`, then the points in the basis after calling this function are saturated at all primes, i.e., saturating at the primes up to `max_prime` are sufficient to saturate at all primes. Note that the function might not have needed to saturate at all primes up to `max_prime`. It has worked out what prime you need to saturate up to, and that prime might be smaller than `max_prime`.

**Note:** Currently (May 2010), this does not remember the result of calling `search()`. So calling `search()` up to height 20 then calling `saturate()` results in another search up to height 18.

EXAMPLES:

```python
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
```

We initialise with three points which happen to be 2, 3 and 5 times the generators of this rank 3 curve. To prevent automatic saturation at this stage we set the parameter `sat` to 0 (which is in fact the default):
Now we saturate at $p = 2$, and gain index 2:

```
sage: EQ.saturate(2)  # points were not 2-saturated
saturating basis...Saturation index bound = 93
WARNING: saturation at primes $p > 2$ will not be done;
...
Gained index 2
New regulator = 93.857...
(False, 2, '[ ]')
sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1], [[-14:25:8], [-13422227300:-49322830557:12167000000]]
sage: EQ.regulator()
93.85730072063639
```

Now we saturate at $p = 3$, and gain index 3:

```
sage: EQ.saturate(3)  # points were not 3-saturated
saturating basis...Saturation index bound = 46
WARNING: saturation at primes $p > 3$ will not be done;
...
Gained index 3
New regulator = 10.428...
(False, 3, '[ ]')
sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1], [-14:25:8], [-13422227300:-49322830557:12167000000]]
sage: EQ.regulator()
10.4285889689596
```

Now we saturate at $p = 5$, and gain index 5:

```
sage: EQ.saturate(5)  # points were not 5-saturated
saturating basis...Saturation index bound = 15
WARNING: saturation at primes $p > 5$ will not be done;
...
Gained index 5
New regulator = 0.417...
(False, 5, '[ ]')
sage: EQ
Subgroup of Mordell-Weil group: [[-2:3:1], [-14:25:8], [1:-1:1]]
sage: EQ.regulator()
0.417143558758384
```
Finally we finish the saturation. The output here shows that the points are now provably saturated at all primes:

```
sage: EQ.saturate()  # points are now saturated

saturating basis...Saturation index bound = 3
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done
(True, 1, '[]')
```

Of course, the `process()` function would have done all this automatically for us:

```
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.process([\[1547, -2967, 343\], \[2707496766203306, 864581029138191, ...
\→2969715140223272\], \[-13422227300, -4932830557, 1216700000\]], sat=5)
P1 = \[1547:-2967:343\] is generator number 1
...
Gained index 5, new generators = \[ [-2:3:1] [-14:25:8] [1:-1:1] ]
sage: EQ
Subgroup of Mordell-Weil group: \[[-2:3:1], [-14:25:8], [1:-1:1]\]
sage: EQ.regulator()
0.417143558758384
```

But we would still need to use the `saturate()` function to verify that full saturation has been done:

```
sage: EQ.saturate()
saturating basis...Saturation index bound = 3
Checking saturation at [ 2 3 ]
Checking 2-saturation
Points were proved 2-saturated (max q used = 11)
Checking 3-saturation
Points were proved 3-saturated (max q used = 13)
done
(True, 1, '[]')
```

Note the output of the preceding command: it proves that the index of the points in their saturation is at most 3, then proves saturation at 2 and at 3, by reducing the points modulo all primes of good reduction up to 11, respectively 13.

`search(\texttt{height\_limit=}18, \texttt{verbose=}False)`

Search for new points, and add them to this subgroup of the Mordell-Weil group.

**INPUT:**

- height\_limit (float, default: 18) – search up to this logarithmic height.

**Note:** On 32-bit machines, this must be $< 21.48$ else $\exp(h_{\text{lim}}) > 2^{31}$ and overflows. On 64-bit machines, it must be at most 43.668. However, this bound is a logarithmic bound and increasing it by just 1 increases the running time by (roughly) $\exp(1.5) = 4.5$, so searching up to even 20 takes a very long time.

**Note:** The search is carried out with a quadratic sieve, using code adapted from a version of Michael Stoll's `ratpoints` program. It would be preferable to use a newer version of `ratpoints`. 

### 2.1. Sage interface to Cremona's `eclib` library (also known as `mwrank`)
• verbose (bool, default False) – turn verbose operation on or off.

EXAMPLES:

A rank 3 example, where a very small search is sufficient to find a Mordell-Weil basis:

```plaintext
sage: E = mwrank_EllipticCurve([0,0,1,-7,6])
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(1)
P1 = [0:1:0] is torsion point, order 1
P1 = [-3:0:1] is generator number 1
...
P4 = [12:35:27] = 1*P1 + -1*P2 + -1*P3 (mod torsion)
sage: EQ
Subgroup of Mordell-Weil group: [[1:-1:1], [-2:3:1], [-14:25:8]]
```

In the next example, a search bound of 12 is needed to find a non-torsion point:

```plaintext
sage: E = mwrank_EllipticCurve([0, -1, 0, -18392, -1186248])
#1056g4
sage: EQ = mwrank_MordellWeil(E)
sage: EQ.search(11); EQ
P1 = [0:1:0] is torsion point, order 1
P1 = [161:0:1] is torsion point, order 2
Subgroup of Mordell-Weil group: []
sage: EQ.search(12); EQ
P1 = [0:1:0] is torsion point, order 1
P1 = [161:0:1] is torsion point, order 2
P1 = [4413270:10381877:27000] is generator number 1
...
Subgroup of Mordell-Weil group: [[4413270:10381877:27000]]
```

### 2.2 Cython interface to Cremona’s eclib library (also known as mwrank)

EXAMPLES:

```plaintext
sage: from sage.libs.eclib.mwrank import _Curvedata, _mw
sage: c = _Curvedata(1,2,3,4,5)
sage: print(c)
[1,2,3,4,5]  
b2 = 9  b4 = 11  b6 = 29  b8 = 35
[1,2,3,4,5]
c4 = -183  c6 = -3429
disc = -10351  (# real components = 1)
#torsion not yet computed
sage: t = _mw(c)
sage: t.search(10)
sage: t
[[1:2:1]]
```

```plaintext
sage.libs.eclib.mwrank.get_precision()

Returns the working floating point bit precision of mwrank, which is equal to the global NTL real number precision.

OUTPUT:
```

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(int) The current precision in bits.

See also \texttt{set\_precision()}.  

\textbf{EXAMPLES:}

\begin{verbatim}
sage: mrank_get_precision()
150
\end{verbatim}

\texttt{sage.libs.eclib.mwrank.initprimes}(filename, verb=False)  
Initialises mwrank/eclib’s internal prime list.

\textbf{INPUT:}

- \texttt{filename} (string) – the name of a file of primes.
- \texttt{verb} (bool: default False) – verbose or not?

\textbf{EXAMPLES:}

\begin{verbatim}
sage: file = os.path.join(SAGE_TMP, 'PRIMES')
sage: with open(file, 'w') as fobj:
....:   _ = fobj.write(' '.join([str(p) for p in prime_range(10^7,10^7+20)]))
sage: mrank_initprimes(file, verb=True)
Computed 78519 primes, largest is 1000253
reading primes from file ...
read extra prime 10000019
finished reading primes from file ...
Extra primes in list: 10000019
sage: mrank_initprimes("x" + file, True)
Traceback (most recent call last):
... IOError: No such file or directory: ...
\end{verbatim}

\texttt{sage.libs.eclib.mwrank.set\_precision}(n)  
Sets the working floating point bit precision of mwrank, which is equal to the global NTL real number precision. 

\begin{itemize}
  \item NTL real number bit precision. This has a massive effect on the speed of mwrank calculations. The default (used if this function is not called) is \(n=150\), but it might have to be increased if a computation fails.
\end{itemize}

\textbf{INPUT:}

- \texttt{n} – a positive integer: the number of bits of precision.

\textbf{Warning:} This change is global and affects \textit{all} future calls of eclib functions by Sage.

\textbf{Note:} The minimal value to which the precision may be set is 53. Lower values will be increased to 53.

See also \texttt{get\_precision()}.  

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.eclib.mwrank import set_precision, get_precision
sage: old_prec = get_precision(); old_prec
150
sage: set_precision(50)
sage: get_precision()
50
\end{verbatim}
53

sage: set_precision(old_prec)
sage: get_precision()
150

2.3 Cremona matrices

class sage.libs.eclib.mat.Matrix
Bases: object

A Cremona Matrix.

EXAMPLES:

sage: M = CremonaModularSymbols(225)
sage: t = M.hecke_matrix(2)
sage: type(t)
<type 'sage.libs.eclib.mat.Matrix'>
sage: t
61 x 61 Cremona matrix over Rational Field

add_scalar(s)
Return new matrix obtained by adding s to each diagonal entry of self.

EXAMPLES:

sage: M = CremonaModularSymbols(23, cuspidal=True, sign=1)
sage: t = M.hecke_matrix(2); print(t.str())
[ 0 1]
[ 1 -1]
sage: w = t.add_scalar(3); print(w.str())
[3 1]
[1 2]

ccharpoly(var=’x’)
Return the characteristic polynomial of this matrix, viewed as as a matrix over the integers.

ALGORITHM:

Note that currently, this function converts this matrix into a dense matrix over the integers, then calls the
ccharpoly algorithm on that, which I think is LinBox’s.

EXAMPLES:

sage: M = CremonaModularSymbols(33, cuspidal=True, sign=1)
sage: t = M.hecke_matrix(2)
sage: t.charpoly()
x^3 + 3*x^2 - 4
sage: t.charpoly().factor()
(x - 1) * (x + 2)^2

ncols()
Return the number of columns of this matrix.

EXAMPLES:
sage: M = CremonaModularSymbols(1234, sign=1)
sage: t = M.hecke_matrix(3); t.ncols()
156
sage: M.dimension()
156

**nrows**()

Return the number of rows of this matrix.

**EXAMPLES:**

```
sage: M = CremonaModularSymbols(19, sign=1)
sage: t = M.hecke_matrix(13); t
2 x 2 Cremona matrix over Rational Field
sage: t.nrows()
2
```

**sage_matrix_over_ZZ**(sparse=True)

Return corresponding Sage matrix over the integers.

**INPUT:**

- **sparse** – (default: True) whether the return matrix has a sparse representation

**EXAMPLES:**

```
sage: M = CremonaModularSymbols(23, cuspidal=True, sign=1)
sage: t = M.hecke_matrix(2)
sage: s = t.sage_matrix_over_ZZ(); s
[0 1]
[1 -1]
sage: type(s)
<type 'sage.matrix.matrix_integer_sparse.Matrix_integer_sparse'>
sage: s = t.sage_matrix_over_ZZ(sparse=False); s
[0 1]
[1 -1]
sage: type(s)
<type 'sage.matrix.matrix_integer_dense.Matrix_integer_dense'>
```

**str**()

Return full string representation of this matrix, never in compact form.

**EXAMPLES:**

```
sage: M = CremonaModularSymbols(22, sign=1)
sage: t = M.hecke_matrix(13)
sage: t.str()
"[14 0 0 0 0]
-4 12 0 8 4
[0 -6 4 -6 0]
[4 2 0 6 -4]
[0 -10 0 0 14]"
```

class sage.libs.eclib.mat.MatrixFactory

Bases: object
2.4 Modular symbols using eclib newforms

class sage.libs.eclib.newforms.EModularSymbol
    Bases: object

Modular symbol associated with an elliptic curve, using John Cremona’s newforms class.

EXAMPLES:

```
sage: from sage.libs.eclib.newforms import EModularSymbol
sage: E = EllipticCurve('11a')
sage: M = EModularSymbol(E,1); M
Modular symbol with sign 1 over Rational Field attached to Elliptic Curve defined by y^2 + y = x^3 - x^2 - 10*x - 20 over Rational Field

By default, symbols are based at the cusp \(\infty\), i.e. we evaluate \(\{\infty, r\}\):

```
sage: [M(1/i) for i in range(1,11)]
[2/5, -8/5, -3/5, 7/5, 12/5, 12/5, 7/5, -3/5, -8/5, 2/5]
```

We can also switch the base point to the cusp 0:

```
sage: [M(1/i, base_at_infinity=False) for i in range(1,11)]
[0, -2, -1, 1, 2, 1, -1, -2, 0]
```

For the minus symbols this makes no difference since \(\{0, \infty\}\) is in the plus space. Note that to evaluate minus symbols the space must be defined with sign 0, which makes both signs available:

```
sage: M = EModularSymbol(E,0); M
Modular symbol with sign 0 over Rational Field attached to Elliptic Curve defined by y^2 + y = x^3 - x^2 - 10*x - 20 over Rational Field

sage: [M(1/i, -1) for i in range(1,11)]
[0, 0, 1, 1, 0, 0, -1, -1, 0, 0]
sage: [M(1/i, -1, base_at_infinity=False) for i in range(1,11)]
[0, 0, 1, 1, 0, 0, -1, -1, 0, 0]
```

If the EModularSymbol is created with sign 0 then as well as asking for both + and - symbols, we can also obtain both (as a tuple). However it is more work to create the full modular symbol space:

```
sage: E = EllipticCurve('11a1')
sage: M = EModularSymbol(E,0); M
Modular symbol with sign 0 over Rational Field attached to Elliptic Curve defined by y^2 + y = x^3 - x^2 - 10*x - 20 over Rational Field

sage: [M(1/i) for i in range(2,11)]
[[-8/5, 0], [-3/5, 1], [7/5, 1], [12/5, 0], [12/5, 0], [7/5, -1], [-3/5, -1], [-8/5, 0], [2/5, 0]]
```

The curve is automatically converted to its minimal model:
Non-optimal curves are handled correctly in eclib, by comparing the ratios of real and/or imaginary periods:

```python
sage: from sage.libs.eclib.newforms import ECModularSymbol
sage: E1 = EllipticCurve('11a1') # optimal
sage: E1.period_lattice().basis()
(1.26920930427955, 0.634604652139777 + 1.45881661693850*I)
sage: M1 = ECModularSymbol(E1,0)
sage: M1(0)
[2/5, 0]
sage: M1(1/3)
[-3/5, 1]
```

One non-optimal curve has real period 1/5 that of the optimal one, so plus symbols scale up by a factor of 5 while minus symbols are unchanged:

```python
sage: E2 = EllipticCurve('11a2') # not optimal
sage: E2.period_lattice().basis()
(0.253841860855911, 0.126920930427955 + 1.45881661693850*I)
sage: M2 = ECModularSymbol(E2,0)
sage: M2(0)
[2, 0]
sage: M2(1/3)
[-3, 1]
sage: all((M2(r,1)==5*M1(r,1)) for r in QQ.range_by_height(10))
True
sage: all((M2(r,-1)==M1(r,-1)) for r in QQ.range_by_height(10))
True
```

The other non-optimal curve has real period 5 times that of the optimal one, so plus symbols scale down by a factor of 5; again, minus symbols are unchanged:

```python
sage: E3 = EllipticCurve('11a3') # not optimal
sage: E3.period_lattice().basis()
(6.34604652139777, 3.17302326069888 + 1.45881661693850*I)
sage: M3 = ECModularSymbol(E3,0)
sage: M3(0)
[2/25, 0]
sage: M3(1/3)
[-3/25, 1]
sage: all((5*M3(r,1)==M1(r,1)) for r in QQ.range_by_height(10))
True
sage: all((M3(r,-1)==M1(r,-1)) for r in QQ.range_by_height(10))
True
```
2.5 Cremona modular symbols

class sage.libs.eclib.homspace.ModularSymbols

Class of Cremona Modular Symbols of given level and sign (and weight 2).

EXAMPLES:

```python
sage: M = CremonaModularSymbols(225)
sage: type(M)
<type 'sage.libs.eclib.homspace.ModularSymbols'>
```

dimension()

Return the dimension of this modular symbols space.

EXAMPLES:

```python
sage: M = CremonaModularSymbols(1234, sign=1)
sage: M.dimension()
156
```

hecke_matrix(p, dual=False, verbose=False)

Return the matrix of the $p$-th Hecke operator acting on this space of modular symbols.

The result of this command is not cached.

INPUT:

- $p$ – a prime number
- **dual** – (default: False) whether to compute the Hecke operator acting on the dual space, i.e., the transpose of the Hecke operator
- **verbose** – (default: False) print verbose output

OUTPUT:

(matrix) If $p$ divides the level, the matrix of the Atkin-Lehner involution $W_p$ at $p$; otherwise the matrix of the Hecke operator $T_p$.

EXAMPLES:

```python
sage: M = CremonaModularSymbols(37)
sage: t = M.hecke_matrix(2); t
5 x 5 Cremona matrix over Rational Field
sage: print(t.str())
[ 3 0 0 0 0]
[ -1 -1 1 1 0]
[ 0 0 -1 0 1]
[ -1 1 0 -1 -1]
[ 0 0 1 0 -1]
sage: w = M.hecke_matrix(37); w
```

(continues on next page)
5 x 5 Cremona matrix over Rational Field

sage: w.charpoly().factor()
(x - 1)^2 * (x + 1)^3

sage: sw = w.sage_matrix_over_ZZ()
sage: st = t.sage_matrix_over_ZZ()
sage: sw^2 == sw.parent()(1)
True
sage: st*sw == sw*st
True

is_cuspidal()
Return whether or not this space is cuspidal.

EXAMPLES:

sage: M = CremonaModularSymbols(1122); M.is_cuspidal()
0
sage: M = CremonaModularSymbols(1122, cuspidal=True); M.is_cuspidal()
1

level()
Return the level of this modular symbols space.

EXAMPLES:

sage: M = CremonaModularSymbols(1234, sign=1)
sage: M.level()
1234

number_of_cusps()
Return the number of cusps for \( \Gamma_0(N) \), where \( N \) is the level.

EXAMPLES:

sage: M = CremonaModularSymbols(225)
sage: M.number_of_cusps()
24

sign()
Return the sign of this Cremona modular symbols space. The sign is either 0, +1 or -1.

EXAMPLES:

sage: M = CremonaModularSymbols(1122, sign=1); M
Cremona Modular Symbols space of dimension 224 for Gamma_0(1122) of weight 2
---with sign 1
sage: M.sign()
1
sage: M = CremonaModularSymbols(1122); M
Cremona Modular Symbols space of dimension 433 for Gamma_0(1122) of weight 2
---with sign 0
sage: M.sign()
0
sage: M = CremonaModularSymbols(1122, sign=-1); M
Cremona Modular Symbols space of dimension 209 for Gamma_0(1122) of weight 2
---with sign -1
sage: M.sign()
-1

2.5. Cremona modular symbols
sparse_hecke_matrix \( (p, \text{dual}=False, \text{verbose}=False, \text{base\_ring}='ZZ') \)

Return the matrix of the \( p \)-th Hecke operator acting on this space of modular symbols as a sparse Sage matrix over \text{base\_ring}. This is more memory-efficient than creating a Cremona matrix and then applying \text{sage\_matrix\_over\_ZZ with sparse=True}.

The result of this command is not cached.

INPUT:

- \( p \) -- a prime number
- \text{dual} -- (default: False) whether to compute the Hecke operator acting on the dual space, i.e., the transpose of the Hecke operator
- \text{verbose} -- (default: False) print verbose output

OUTPUT:

(matrix) If \( p \) divides the level, the matrix of the Atkin-Lehner involution \( W_p \) at \( p \); otherwise the matrix of the Hecke operator \( T_p \).

EXAMPLES:

```
sage: M = CremonaModularSymbols(37)
sage: t = M.sparse_hecke_matrix(2); type(t)
<type 'sage.matrix.matrix_integer_sparse.Matrix_integer_sparse'>
sage: print(t)
[ 3 0 0 0 0]
[-1 -1 1 1 0]
[ 0 0 -1 0 1]
[-1 1 0 -1 -1]
[ 0 0 1 0 -1]
sage: M = CremonaModularSymbols(5001)
sage: T = M.sparse_hecke_matrix(2)
sage: U = M.hecke_matrix(2).sage_matrix_over_ZZ(sparse=True)
sage: print(T == U)
True
sage: T = M.sparse_hecke_matrix(2, dual=True)
sage: print(T == U.transpose())
True
sage: T = M.sparse_hecke_matrix(2, base\_ring=GF(7))
sage: print(T == U.change\_ring(GF(7)))
True
```

This concerns an issue reported on trac ticket #21303:

```
sage: C = CremonaModularSymbols(45, cuspidal=True, sign=-1)
sage: T2a = C.hecke_matrix(2).sage_matrix_over_ZZ()
sage: T2b = C.sparse_hecke_matrix(2)
sage: print(T2a == T2b)
True
```
2.6 Cremona modular symbols

sage.libs.eclib.constructor.CremonaModularSymbols((level, sign=0, cuspidal=False, verbose=0))

Return the space of Cremona modular symbols with given level, sign, etc.

INPUT:

- level – an integer >= 2 (at least 2, not just positive!)
- sign – an integer either 0 (the default) or 1 or -1.
- cuspidal – (default: False); if True, compute only the cuspidal subspace
- verbose – (default: False): if True, print verbose information while creating space

EXAMPLES:

```
sage: M = CremonaModularSymbols(43); M
Cremona Modular Symbols space of dimension 7 for Gamma_0(43) of weight 2 with sign 0
sage: M = CremonaModularSymbols(43, sign=1); M
Cremona Modular Symbols space of dimension 4 for Gamma_0(43) of weight 2 with sign 1
sage: M = CremonaModularSymbols(43, cuspidal=True); M
Cremona Cuspidal Modular Symbols space of dimension 6 for Gamma_0(43) of weight 2 with sign 0
sage: M = CremonaModularSymbols(43, cuspidal=True, sign=1); M
Cremona Cuspidal Modular Symbols space of dimension 3 for Gamma_0(43) of weight 2 with sign 1
```

When run interactively, the following command will display verbose output:

```
sage: M = CremonaModularSymbols(43, verbose=1)
After 2-term relations, ngens = 22
ngens = 22
maxnumrel = 32
relation matrix has = 704 entries...
Finished 3-term relations: numrel = 16 ( maxnumrel = 32)
relmat has 42 nonzero entries (density = 0.0596591)
Computing kernel...
time to compute kernel = (... seconds)
rk = 7
Number of cusps is 2
ncusps = 2
About to compute matrix of delta
delta matrix done: size 2x7.
About to compute kernel of delta
done
Finished constructing homspace.
sage: M
Cremona Modular Symbols space of dimension 7 for Gamma_0(43) of weight 2 with sign 0
```

The input must be valid or a ValueError is raised:

```
sage: M = CremonaModularSymbols(-1)
Traceback (most recent call last):
...
```

(continues on next page)
ValueError: the level (= -1) must be at least 2

```python
sage: M = CremonaModularSymbols(0)
Traceback (most recent call last):
...
ValueError: the level (= 0) must be at least 2
```

The sign can only be 0 or 1 or -1:

```python
sage: M = CremonaModularSymbols(10, sign = -2)
Traceback (most recent call last):
...
ValueError: sign (= -2) is not supported; use 0, +1 or -1
```

We do allow -1 as a sign (see trac ticket #9476):

```python
sage: CremonaModularSymbols(10, sign = -1)
Cremona Modular Symbols space of dimension 0 for Gamma_0(10) of weight 2 with
  sign -1
```
CHAPTER THREE

FLINT

3.1 Flint imports

sage.libs.flint.flint.free_flint_stack()

3.2 FLINT fmpz_poly class wrapper

AUTHORS:

• William Stein (2007-10-02) update for new flint; add arithmetic and creation of coefficients of arbitrary size.

class sage.libs.flint.fmpz_poly.Fmpz_poly
   Bases: sage.structure.sage_object.SageObject

   Construct a new fmpz_poly from a sequence, constant coefficient, or string (in the same format as it prints).

   EXAMPLES:

   sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
   sage: Fmpz_poly([1,2,3])
   3 1 2 3
   sage: Fmpz_poly(5)
   1 5
   sage: Fmpz_poly(str(Fmpz_poly([3,5,7])))
   3 3 5 7

degree()
   The degree of self.

   EXAMPLES:

   sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
   sage: f = Fmpz_poly([1,2,3]); f
   3 1 2 3
   sage: f.degree()
   2
   sage: Fmpz_poly(range(1000)).degree()
   999
   sage: Fmpz_poly([2,0]).degree()
   0
derivative()  
Return the derivative of self.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
sage: f = Fmpz_poly([1,2,6])  

sage: f.derivative().list() == [2, 12]
True
```

div_rem(other)  
Return self / other, self, % other.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
sage: f = Fmpz_poly([1,3,4,5])  

sage: g = f^23  

sage: g.div_rem(f)[1]
0  

sage: g.div_rem(f)[0] - f^22  
0

sage: f = Fmpz_poly([1..10])  

sage: g = Fmpz_poly([1,3,5])  

sage: q, r = f.div_rem(g)  

sage: q*f+r  
17 1 2 3 4 4 10 11 17 18 22 26 30 23 26 18 20

sage: g  
3 1 3 5

sage: q*g+r  
10 1 2 3 4 5 6 7 8 9 10
```

left_shift(n)  
Left shift self by n.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
sage: f = Fmpz_poly([1,2])

sage: f.left_shift(1).list() == [0,1,2]
True
```

list()  
Return self as a list of coefficients, lowest terms first.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
sage: f = Fmpz_poly([2,1,0,-1])

sage: f.list()
[2, 1, 0, -1]
```

pow_truncate(exp, n)  
Return self raised to the power of exp mod x^n.

EXAMPLES:

```
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
sage: f = Fmpz_poly([1,2])

sage: f.pow_truncate(1, 2)
[2, 1, 0, -1]
```

(continues on next page)
pseudo_div\( (\text{other}) \)

pseudo_div_rem\( (\text{other}) \)

right_shift\( (n) \)
Right shift self by \( n \).

EXAMPLES:

```python
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
g = Fmpz_poly([i, 1])
sage: g = f**10; g
11 1 10 45 120 252 210 120 45 10 1
sage: g.truncate(5)
5 1 10 45 120 210
```

truncate\( (n) \)
Return the truncation of self at degree \( n \).

EXAMPLES:

```python
sage: from sage.libs.flint.fmpz_poly import Fmpz_poly
sage: f = Fmpz_poly([1, 1])
sage: g = f**10; g
11 1 10 45 120 252 210 120 45 10 1
sage: g.truncate(5)
5 1 10 45 120 210
```

### 3.3 FLINT Arithmetic Functions

**sage.libs.flint.arith.bell_number\( (n) \)**
Return the \( n \)-th Bell number.

See Wikipedia article Bell_number.

EXAMPLES:

```python
sage: from sage.libs.flint.arith import bell_number
sage: [bell_number(i) for i in range(10)]
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
sage: bell_number(10)
115975
sage: bell_number(40)
157450588391204931289324344702531067
sage: bell_number(100)
47585391276764836587907684138720782636369686825611466663346375591144978924426226727240442177
```

**sage.libs.flint.arith.bernoulli_number\( (n) \)**
Return the \( n \)-th Bernoulli number.

See Wikipedia article Bernoulli_number.

EXAMPLES:
sage.libs.flint.arith.dedekind_sum\( p, q \)
Return the Dedekind sum \( s(p, q) \) where \( p \) and \( q \) are arbitrary integers.

See Wikipedia article Dedekind_sum.

EXAMPLES:

```
sage: from sage.libs.flint.arith import dedekind_sum
dedekind_sum(4, 5)
-1/5
```

sage.libs.flint.arith.euler_number\( n \)
Return the Euler number of index \( n \).

See Wikipedia article Euler_number.

EXAMPLES:

```
sage: from sage.libs.flint.arith import euler_number
euler_number(8)
[1, 0, -1, 0, 5, 0, -61, 0]
```

sage.libs.flint.arith.harmonic_number\( n \)
Return the harmonic number \( H_n \).

See Wikipedia article Harmonic_number.

EXAMPLES:

```
sage: from sage.libs.flint.arith import harmonic_number
n = 500 + randint(0,500)
bool( sum(1/k for k in range(1,n+1)) == harmonic_number(n) )
True
```

sage.libs.flint.arith.number_of_partitions\( n \)
Return the number of partitions of the integer \( n \).

See Wikipedia article Partition_(number_theory).

EXAMPLES:

```
sage: from sage.libs.flint.arith import number_of_partitions
number_of_partitions(3)
3
number_of_partitions(10)
42
number_of_partitions(40)
37338
```
sage: number_of_partitions(100)
190569292
sage: number_of_partitions(100000)
27493510569775696512677516320986352688173429315980054758203125984302147328114964173655050741660782345009262853831404597021307130674510624419227311238999702284408609370935531629697851569569892196108480158600569421098519
4.1 Wrappers for Giac functions

We provide a python function to compute and convert to sage a Groebner basis using the giacpy_sage module.

AUTHORS:
- Martin Albrecht (2015-07-01): initial version
- Han Frederic (2015-07-01): initial version

EXAMPLES:

```python
from sage.libs.giac import groebner_basis as gb_giac
# random
P = PolynomialRing(QQ, 6, 'x')
I = sage.rings.ideal.Cyclic(P)
B = gb_giac(I.gens())  # random
B
```

```
Polynomial Sequence with 45 Polynomials in 6 Variables
```

class sage.libs.giac.GiacSettingsDefaultContext
```
Context preserve libgiac settings.
```
sage.libs.giac.groebner_basis(gens, proba_epsilon=None, threads=None, prot=False, elim_variables=None, *args, **kwds)
Compute a Groebner Basis of an ideal using giacpy_sage. The result is automatically converted to sage.
Supported term orders of the underlying polynomial ring are lex, deglex, degrevlex and block orders with 2 degrevlex blocks.

INPUT:
- gens - an ideal (or a list) of polynomials over a prime field of characteristic 0 or p<2^31
- proba_epsilon - (default: None) majoration of the probability of a wrong answer when probabilistic algorithms are allowed.
  - if proba_epsilon is None, the value of sage.structure.proof.all.proba_epsilon is taken. If it is false then the global giacpy_sage.giacsettings.proba_epsilon is used.
  - if proba_epsilon is 0, probabilistic algorithms are disabled.
- threads - (default: None) Maximal number of threads allowed for giac. If None, the global giacpy_sage.giacsettings.threads is considered.
- prot - (default: False) if True print detailed informations
elim_variables - (default: None) a list of variables to eliminate from the ideal.

– if elim_variables is None, a Groebner basis with respect to the term ordering of the parent polynomial ring of the polynomials gens is computed.

– if elim_variables is a list of variables, a Groebner basis of the elimination ideal with respect to a degrevlex term order is computed, regardless of the term order of the polynomial ring.

OUTPUT:
Polynomial sequence of the reduced Groebner basis.

EXAMPLES:

```python
sage: from sage.libs.giac import groebner_basis as gb_giac
sage: P = PolynomialRing(GF(previous_prime(2**31)), 6, 'x')
sage: I = sage.rings.ideal.Cyclic(P)
sage: B=gb_giac(I.gens());B
// Groebner basis computation time ...
Polynomial Sequence with 45 Polynomials in 6 Variables
sage: B.is_groebner()
True
```

Elimination ideals can be computed by passing elim_variables:

```python
sage: P = PolynomialRing(GF(previous_prime(2**31)), 5, 'x')
I = sage.rings.ideal.Cyclic(P)
sage: B = gb_giac(I.gens(), elim_variables=[P.gen(0), P.gen(2)])
// Groebner basis computation time ...
sage: B.is_groebner()
True
sage: B.ideal() == I.elimination_ideal([P.gen(0), P.gen(2)])
True
```

Computations over QQ can benefit from

- a probabilistic lifting:

```python
sage: P = PolynomialRing(QQ,5, 'x')
I = ideal([P.random_element(3,7) for j in range(5)])
sage: B1 = gb_giac(I.gens(),1e-16) # long time (1s)
...Running a probabilistic check for the reconstructed Groebner basis. If successful, error probability is less than 1e-16 ...
sage: sage.structure.proof.all.polynomial(True)
sage: B2 = gb_giac(I.gens()) # long time (4s)
// Groebner basis computation time...
sage: B1 == B2 # long time
True
sage: B1.is_groebner() # long time (20s)
True
```

- multi threaded operations:

```python
sage: P = PolynomialRing(QQ, 8, 'x')
I = sage.rings.ideal.Cyclic(P)
sage: time B = gb_giac(I.gens(),1e-6,threads=2) # doctest: +SKIP
(continues on next page)```
Running a probabilistic check for the reconstructed Groebner basis...
Time: CPU 168.98 s, Wall: 94.13 s

You can get detailed information by setting prot=True

```python
sage: I = sage.rings.ideal.Katsura(P)
sage: gb_giac(I, prot=True)  # random, long time (3s)
9381383 begin computing basis modulo 535718473
9381501 begin new iteration zmod, number of pairs: 8, base size: 8
... end, basis size 74 prime number 1
G=Vector [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,...
... creating reconstruction #0
...
+++++++basis size 74
checking pairs for i=0, j=
checking pairs for i=1, j=2,6,12,17,19,24,29,34,39,42,43,48,56,61,64,69,
...
checking pairs for i=72, j=73,
checking pairs for i=73, j=
Number of critical pairs to check 373
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
Successfull check of 373 critical pairs
12380865 end final check
Polynomial Sequence with 74 Polynomials in 8 Variables
```

`sage.libs.giac.local_giacsettings(func)`
Decorator to preserve Giac’s proba_epsilon and threads settings.

**EXAMPLES:**

```python
sage: def testf(a,b):
    ....:     giacsettings.proba_epsilon = a/100
    ....:     giacsettings.threads = b+2
    ....:     return (giacsettings.proba_epsilon, giacsettings.threads)
sage: from sage.libs.giac.giac import giacsettings
sage: from sage.libs.giac import local_giacsettings
sage: gporig, gtorig = (giacsettings.proba_epsilon,giacsettings.threads)
sage: gp, gt = local_giacsettings(testf)(giacsettings.proba_epsilon,giacsettings.threads)
sage: gp<gporig, gt-gtorig
(True, 2)
```

4.1. Wrappers for Giac functions 45
5.1 The Elliptic Curve Method for Integer Factorization (ECM)

Sage includes GMP-ECM, which is a highly optimized implementation of Lenstra’s elliptic curve factorization method. See http://ecm.gforge.inria.fr/ for more about GMP-ECM. This file provides a Cython interface to the GMP-ECM library.

AUTHORS:

- Jeroen Demeyer (2012-03-29): signal handling, documentation
- Paul Zimmermann (2011-05-22) – added input/output of sigma

EXAMPLES:

```python
sage: from sage.libs.libecm import ecmfactor
sage: result = ecmfactor(999, 0.00)
sage: result[0]
True
sage: result[1] in [3, 9, 27, 37, 111, 333, 999] or result[1]
True
sage: result = ecmfactor(999, 0.00, verbose=True)
Performing one curve with B1=0
Found factor in step 1: ...
sage: result[0]
True
sage: result[1] in [3, 9, 27, 37, 111, 333, 999] or result[1]
True
sage: ecmfactor(2^128+1,1000,sigma=227140902)
(True, 5704689200685129054721, 227140902)
```

`sage.libs.libecm.ecmfactor(number, B1, verbose=False, sigma=0)`

Try to find a factor of a positive integer using ECM (Elliptic Curve Method). This function tries one elliptic curve.

**INPUT:**

- `number` – positive integer to be factored
- `B1` – bound for step 1 of ECM
- `verbose` (default: False) – print some debugging information

**OUTPUT:**

Either `(False, None)` if no factor was found, or `(True, f)` if the factor `f` was found.
EXAMPLES:

```python
sage: from sage.libs.libecm import ecmfactor

This number has a small factor which is easy to find for ECM:

```python
sage: N = 2^167 - 1
sage: factor(N)
2349023 * 7963830476685650737778616296087448490695649
sage: ecmfactor(N, 2e5)
(True, 2349023, ...)
```

If a factor was found, we can reproduce the factorization with the same sigma value:

```python
sage: N = 2^167 - 1
sage: ecmfactor(N, 2e5, sigma=1473308225)
(True, 2349023, 1473308225)
```

With a smaller B1 bound, we may or may not succeed:

```python
sage: ecmfactor(N, 1e2) # random
(False, None)
```

The following number is a Mersenne prime, so we don’t expect to find any factors (there is an extremely small chance that we get the input number back as factorization):

```python
sage: N = 2^127 - 1
sage: N.is_prime()
True
sage: ecmfactor(N, 1e3)
(False, None)
```

If we have several small prime factors, it is possible to find a product of primes as factor:

```python
sage: N = 2^179 - 1
sage: factor(N)
359 * 1433 * 14894591093600398645694019709543721664951999121
sage: ecmfactor(N, 1e3) # random
(True, 514447, 3475102204)
```

We can ask for verbose output:

```python
sage: N = 12^97 - 1
sage: factor(N)
11 * ...
```

```python
sage: ecmfactor(N, 100, verbose=True)
Performing one curve with B1=100
Found factor in step 1: 11
(True, 11, ...)
```

```python
sage: ecmfactor(N/11, 100, verbose=True)
Performing one curve with B1=100
Found no factor.
(False, None)
```
6.1 GSL arrays

class sage.libs.gsl.array.GSLDoubleArray
Bases: object

EXAMPLES:

```
sage: a = WaveletTransform(128,'daubechies',4)
sage: for i in range(1, 11):
    ....:     a[i] = 1
sage: a[:6:2]
[0.0, 1.0, 1.0]
```
7.1 Rubinstein’s lcalc library

This is a wrapper around Michael Rubinstein’s lcalc. See http://oto.math.uwaterloo.ca/~mrubinst/L_function_public/CODE/.

AUTHORS:

• Rishikesh (2010): added compute_rank() and hardy_z_function()
• Yann Laigle-Chapuy (2009): refactored
• Rishikesh (2009): initial version

class sage.libs.lcalc.lcalc_Lfunction.Lfunction
   Bases: object

Initialization of L-function objects. See derived class for details, this class is not supposed to be instantiated directly.

EXAMPLES:

```sage
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: L=Lfunction_from_character(DirichletGroup(5)[1])
L-function with complex Dirichlet coefficients
```

compute_rank()
Computes the analytic rank (the order of vanishing at the center) of of the L-function

EXAMPLES:

```sage
sage: chi=DirichletGroup(5)[2]  #This is a quadratic character
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: L=Lfunction_from_character(chi, type="int")
sage: L.compute_rank()
0
sage: E=EllipticCurve([-82,0])
sage: L=Lfunction_from_elliptic_curve(E, number_of_coeffs=40000)
sage: L.compute_rank()
3
```

find_zeros(T1, T2, stepsize)
Finds zeros on critical line between T1 and T2 using step size of stepsize. This function might miss zeros if step size is too large. This function computes the zeros of the L-function by using change in signs of areal valued function whose zeros coincide with the zeros of L-function.

Use find_zeros_via_N() for slower but more rigorous computation.
INPUT:

- \( T_1 \) – a real number giving the lower bound
- \( T_2 \) – a real number giving the upper bound
- \( \text{stepsize} \) – step size to be used for the zero search

OUTPUT:

- list – A list of the imaginary parts of the zeros which were found.

EXAMPLES:

```python
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi=DirichletGroup(5)[2]  # This is a quadratic character
sage: L=Lfunction_from_character(chi, type="int")
sage: L.find_zeros(5,15,.1)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
sage: L=Lfunction_from_character(chi, type="double")
sage: L.find_zeros(1,15,.1)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
sage: chi=DirichletGroup(5)[1]
sage: L=Lfunction_from_character(chi, type="complex")
sage: L.find_zeros(-8,8,.1)
[-4.13290370521..., 6.18357819545...]
sage: L=Lfunction_Zeta()
sage: L.find_zeros(10,29.1,.1)
[14.1347251417..., 21.0220396387..., 25.0108575801...]
```

**find_zeros_via_N**

Finds \( count \) number of zeros with positive imaginary part starting at real axis. This function also verifies that all the zeros have been found.

INPUT:

- \( count \) - number of zeros to be found
- \( \text{do_negative} \) - (default: False) False to ignore zeros below the real axis.
- \( \text{max_refine} \) - when some zeros are found to be missing, the step size used to find zeros is refined. \( \text{max_refine} \) gives an upper limit on when \( \text{lcalc} \) should give up. Use default value unless you know what you are doing.
- \( \text{rank} \) - integer (default: -1) analytic rank of the L-function. If -1 is passed, then we attempt to compute it. (Use default if in doubt)
- \( \text{test_explicit_formula} \) - integer (default: 0) If nonzero, test the explicit formula for additional confidence that all the zeros have been found and are accurate. This is still being tested, so using the default is recommended.

OUTPUT:

- list – A list of the imaginary parts of the zeros that have been found

EXAMPLES:
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: chi=DirichletGroup(5)[2]  # This is a quadratic character
sage: L=Lfunction_from_character(chi, type="int")
sage: L.find_zeros_via_N(3)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
sage: L=Lfunction_from_character(chi, type="double")
sage: L.find_zeros_via_N(3)
[6.64845334472..., 9.83144443288..., 11.9588456260...]
sage: chi=DirichletGroup(5)[1]
sage: L=Lfunction_from_character(chi, type="complex")
sage: L.find_zeros_via_N(3)
[6.18357819545..., 8.45722917442..., 12.6749464170...]
sage: L=Lfunction_Zeta()
sage: L.find_zeros_via_N(3)
[14.1347251417..., 21.0220396387..., 25.0108575801...]

hardy_z_function(s)
Computes the Hardy Z-function of the L-function at s

INPUT:

• s - a complex number with imaginary part between -0.5 and 0.5

EXAMPLES:

sage: chi = DirichletGroup(5)[2]  # Quadratic character
sage: from sage.libs.lcalc.lcalc_Lfunction import *
sage: L = Lfunction_from_character(chi, type="int")
sage: L.hardy_z_function(0)
0.231750947504...

sage: L.hardy_z_function(.5).imag()  # abs tol 1e-15
1.17253174178320e-17
sage: L.hardy_z_function(.4+.3*I)
0.2166144222685... - 0.00408187127850...*I
sage: chi = DirichletGroup(5)[1]
sage: L = Lfunction_from_character(chi, type="complex")
sage: L.hardy_z_function(0)
0.793967590477...

sage: L.hardy_z_function(.5).imag()  # abs tol 1e-15
0.000000000000000
sage: E = EllipticCurve([-82,0])
sage: L = Lfunction_from_elliptic_curve(E, number_of_coeffs=40000)
sage: L.hardy_z_function(2.1)
0.00643179176869...

sage: L.hardy_z_function(2.1).imag()  # abs tol 1e-15
-0.000000000000000

value(s, derivative=0)
Computes the value of the L-function at s

INPUT:

• s - a complex number

• derivative - integer (default: 0) the derivative to be evaluated

• rotate - (default: False) If True, this returns the value of the Hardy Z-function (sometimes called

7.1. Rubinstein’s lcalc library
the Riemann-Siegel Z-function or the Siegel Z-function).

EXAMPLES:

```python
sage: chi=DirichletGroup(5)[2]  #This is a quadratic character
sage: from sage.libs.lcalc.lcalc_Lfunction import *

sage: L=Lfunction_from_character(chi, type="int")

sage: L.value(.5)  # abs tol 3e-15
0.231750947504016 + 5.75329642226136e-18*I

sage: L.value(.2+.4*I)
0.102558603193... + 0.190840777924...*I

sage: L=Lfunction_from_character(chi, type="double")

sage: L.value(.6)  # abs tol 3e-15
0.27463335856345 + 6.59869267328199e-18*I

sage: L.value(.6+I)
0.362258705721... + 0.433888250620...*I

sage: chi=DirichletGroup(5)[1]

sage: L=Lfunction_from_character(chi, type="complex")

sage: L.value(.5)
0.763747880117... + 0.216964767518...*I

sage: L.value(.6+5*I)
0.702723260619... - 1.10178575243...*I

sage: L=Lfunction_Zeta()

sage: L.value(.5)
-1.46035450880...

sage: L.value(.4+.5*I)
-0.450728958517... - 0.780511403019...*I
```

class sage.libs.lcalc.lcalc_Lfunction.Lfunction_C

The Lfunction_C class is used to represent L-functions with complex Dirichlet Coefficients. We assume that L-functions satisfy the following functional equation.

\[
\Lambda(s) = \omega Q^s \Lambda(1 - s)
\]

where

\[
\Lambda(s) = Q^s \prod_{j=1}^{\alpha} \Gamma(\kappa_j s + \gamma_j) L(s)
\]

See (23) in arXiv math/0412181

INPUT:

- `what_type_L` - integer, this should be set to 1 if the coefficients are periodic and 0 otherwise.
- `dirichlet_coefficient` - List of dirichlet coefficients of the L-function. Only first \(M\) coefficients are needed if they are periodic.
- `period` - If the coefficients are periodic, this should be the period of the coefficients.
- `Q` - See above
- `OMEGA` - See above
- `kappa` - List of the values of \(\kappa_j\) in the functional equation
- `gamma` - List of the values of \(\gamma_j\) in the functional equation
• pole - List of the poles of L-function
• residue - List of the residues of the L-function

Note: If an L-function satisfies \( \Lambda(s) = \omega Q^s \Lambda(k - s) \), by replacing \( s \) by \( s + (k - 1)/2 \), one can get it in the form we need.

```python
class sage.libs.lcalc.lcalc_Lfunction.Lfunction_D
Bases: sage.libs.lcalc.lcalc_Lfunction.Lfunction
```

The `Lfunction_D` class is used to represent L-functions with real Dirichlet coefficients. We assume that L-functions satisfy the following functional equation.

\[
\Lambda(s) = \omega Q^s \Lambda(1 - s)
\]

where

\[
\Lambda(s) = Q^s \left( \prod_{j=1}^{a} \Gamma(\kappa_j s + \gamma_j) \right) L(s)
\]

See (23) in arXiv math/0412181

INPUT:
• what_type_L - integer, this should be set to 1 if the coefficients are periodic and 0 otherwise.
• dirichlet_coefficient - List of dirichlet coefficients of the L-function. Only first \( M \) coefficients are needed if they are periodic.
• period - If the coefficients are periodic, this should be the period of the coefficients.
• Q - See above
• OMEGA - See above
• kappa - List of the values of \( \kappa_j \) in the functional equation
• gamma - List of the values of \( \gamma_j \) in the functional equation
• pole - List of the poles of L-function
• residue - List of the residues of the L-function

Note: If an L-function satisfies \( \Lambda(s) = \omega Q^s \Lambda(k - s) \), by replacing \( s \) by \( s + (k - 1)/2 \), one can get it in the form we need.

```python
class sage.libs.lcalc.lcalc_Lfunction.Lfunction_I
Bases: sage.libs.lcalc.lcalc_Lfunction.Lfunction
```

The `Lfunction_I` class is used to represent L-functions with integer Dirichlet Coefficients. We assume that L-functions satisfy the following functional equation.

\[
\Lambda(s) = \omega Q^s \Lambda(1 - s)
\]

where

\[
\Lambda(s) = Q^s \left( \prod_{j=1}^{a} \Gamma(\kappa_j s + \gamma_j) \right) L(s)
\]
See (23) in arXiv math/0412181

**INPUT:**
- `what_type_L` - integer, this should be set to 1 if the coefficients are periodic and 0 otherwise.
- `dirichlet_coefficient` - List of dirichlet coefficients of the L-function. Only first $M$ coefficients are needed if they are periodic.
- `period` - If the coefficients are periodic, this should be the period of the coefficients.
- `Q` - See above
- `OMEGA` - See above
- `kappa` - List of the values of $\kappa_j$ in the functional equation
- `gamma` - List of the values of $\gamma_j$ in the functional equation
- `pole` - List of the poles of L-function
- `residue` - List of the residues of the L-function

**Note:** If an L-function satisfies $\Lambda(s) = \omega Q s \Lambda(k - s)$, by replacing $s$ by $s + (k - 1)/2$, one can get it in the form we need.

class sage.libs.lcalc.lcalc_Lfunction.Lfunction_Zeta
   Bases: sage.libs.lcalc.lcalc_Lfunction.Lfunction

The Lfunction_Zeta class is used to generate the Riemann zeta function.

def Lfunction_from_character(chi, type='complex')
   Given a primitive Dirichlet character, this function returns an lcalc L-function object for the L-function of the character.

**INPUT:**
- `chi` - A Dirichlet character
- `use_type` - string (default: “complex”) type used for the Dirichlet coefficients. This can be “int”, “double” or “complex”.

**OUTPUT:**
L-function object for `chi`.

**EXAMPLES:**

```
sage: from sage.libs.lcalc.lcalc_Lfunction import Lfunction_from_character
sage: Lfunction_from_character(DirichletGroup(5)[1])
L-function with complex Dirichlet coefficients
sage: Lfunction_from_character(DirichletGroup(5)[2], type="int")
L-function with integer Dirichlet coefficients
sage: Lfunction_from_character(DirichletGroup(5)[2], type="double")
L-function with real Dirichlet coefficients
sage: Lfunction_from_character(DirichletGroup(5)[1], type="int")
Traceback (most recent call last):
  ... ValueError: For non quadratic characters you must use type="complex"
```

def Lfunction_from_elliptic_curve(E, number_of_coeffs=10000)
   Given an elliptic curve `E`, return an L-function object for the function $L(s, E)$.
INPUT:
- E - An elliptic curve
- number_of_coeffs - integer (default: 10000) The number of coefficients to be used when constructing the L-function object. Right now this is fixed at object creation time, and is not automatically set intelligently.

OUTPUT:
L-function object for $L(s, E)$.

EXAMPLES:

```python
sage: from sage.libs.lcalc.lcalc_Lfunction import Lfunction_from_elliptic_curve
sage: L = Lfunction_from_elliptic_curve(EllipticCurve('37'))
sage: L
L-function with real Dirichlet coefficients
sage: L.value(0.5).abs() < 1e-15  # "noisy" zero on some platforms (see #9615)
True
sage: L.value(0.5, derivative=1)
0.305999...
```
8.1 libSingular: Functions

Sage implements a C wrapper around the Singular interpreter which allows to call any function directly from Sage without string parsing or interprocess communication overhead. Users who do not want to call Singular functions directly, usually do not have to worry about this interface, since it is handled by higher level functions in Sage.

AUTHORS:
- Michael Brickenstein (2009-07): initial implementation, overall design
- Martin Albrecht (2009-07): clean up, enhancements, etc.
- Michael Brickenstein (2009-10): extension to more Singular types
- Martin Albrecht (2010-01): clean up, support for attributes
- Simon King (2011-04): include the documentation provided by Singular as a code block.
- Burcin Erocal, Michael Brickenstein, Oleksandr Motsak, Alexander Dreyer, Simon King (2011-09) plural support

EXAMPLES:

The direct approach for loading a Singular function is to call the function `singular_function()` with the function name as parameter:

```
sage: from sage.libs.singular.function import singular_function
sage: P.<a,b,c,d> = PolynomialRing(GF(7))
sage: std = singular_function('std')
sage: I = sage.rings.ideal.Cyclic(P)
sage: std(I)
[a + b + c + d,
b^2 + 2*b*d + d^2,
b*c^2 + c^2*d - b*d^2 - d^3,
b*c*d^2 + c^2*d^2 - b*d^3 + c*d^3 - d^4 - 1,
b*d^4 + d^5 - b - d,
c^3*d^2 + c^2*d^3 - c - d,
c^2*d^4 + b*c - b*d + c*d - 2*d^2]
```

If a Singular library needs to be loaded before a certain function is available, use the `lib()` function as shown below:

```
sage: from sage.libs.singular.function import singular_function, lib as singular_lib
sage: primdecSY = singular_function('primdecSY')
Traceback (most recent call last):
...
NameError: Singular library function 'primdecSY' is not defined
```
sage: singular_lib('primdec.lib')
sage: primdecSY = singular_function('primdecSY')

There is also a short-hand notation for the above:

sage: import sage.libs.singular.function_factory
sage: primdecSY = sage.libs.singular.function_factory.ff.primdec__lib.primdecSY

The above line will load “primdec.lib” first and then load the function primdecSY.

class sage.libs.singular.function.BaseCallHandler
   Bases: object
   
   A call handler is an abstraction which hides the details of the implementation differences between kernel and library functions.

class sage.libs.singular.function.Converter
   Bases: sage.structure.sage_object.SageObject
   
   A Converter interfaces between Sage objects and Singular interpreter objects.

   ring()
       Return the ring in which the arguments of this list live.

   EXAMPLES:

sage: from sage.libs.singular.function import Converter
sage: P.<a,b,c> = PolynomialRing(GF(127))
sage: Converter([a,b,c],ring=P).ring()
Multivariate Polynomial Ring in a, b, c over Finite Field of size 127

class sage.libs.singular.function.KernelCallHandler
   Bases: sage.libs.singular.function.BaseCallHandler
   
   A call handler is an abstraction which hides the details of the implementation differences between kernel and library functions.

   This class implements calling a kernel function.

   Note: Do not construct this class directly, use singular_function() instead.

class sage.libs.singular.function.LibraryCallHandler
   Bases: sage.libs.singular.function.BaseCallHandler
   
   A call handler is an abstraction which hides the details of the implementation differences between kernel and library functions.

   This class implements calling a library function.

   Note: Do not construct this class directly, use singular_function() instead.

class sage.libs.singular.function.Resolution
   Bases: object
   
   A simple wrapper around Singular's resolutions.
class sage.libs.singular.function.RingWrap

    Bases: object

    A simple wrapper around Singular's rings.

    characteristic()

        Get characteristic.

        EXAMPLES:

        sage: from sage.libs.singular.function import singular_function
        sage: P.<x,y,z> = PolynomialRing(QQ)
        sage: ringlist = singular_function("ringlist")
        sage: l = ringlist(P)
        sage: ring = singular_function("ring")
        sage: ring(l, ring=P).characteristic()
        0

    is_commutative()

        Determine whether a given ring is commutative.

        EXAMPLES:

        sage: from sage.libs.singular.function import singular_function
        sage: P.<x,y,z> = PolynomialRing(QQ)
        sage: ringlist = singular_function("ringlist")
        sage: l = ringlist(P)
        sage: ring = singular_function("ring")
        sage: ring(l, ring=P).is_commutative()
        True

    ngens()

        Get number of generators.

        EXAMPLES:

        sage: from sage.libs.singular.function import singular_function
        sage: P.<x,y,z> = PolynomialRing(QQ)
        sage: ringlist = singular_function("ringlist")
        sage: l = ringlist(P)
        sage: ring = singular_function("ring")
        sage: ring(l, ring=P).ngens()
        3

    npars()

        Get number of parameters.

        EXAMPLES:

        sage: from sage.libs.singular.function import singular_function
        sage: P.<x,y,z> = PolynomialRing(QQ)
        sage: ringlist = singular_function("ringlist")
        sage: l = ringlist(P)
        sage: ring = singular_function("ring")
        sage: ring(l, ring=P).npars()
        0

    ordering_string()

        Get Singular string defining monomial ordering.

        EXAMPLES:
sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)

par_names()  
    Get parameter names.

EXAMPLES:

sage: from sage.libs.singular.function import singular_function
sage: P.<x,y,z> = PolynomialRing(QQ)

var_names()  
    Get names of variables.

EXAMPLES:

class sage.libs.singular.function.SingularFunction
    Bases: sage.structure.sage_object.SageObject

The base class for Singular functions either from the kernel or from the library.

class sage.libs.singular.function.SingularKernelFunction
    Bases: sage.libs.singular.function.SingularFunction

EXAMPLES:

sage: from sage.libs.singular.function import SingularKernelFunction
sage: R.<x,y> = PolynomialRing(QQ, order='lex')

class sage.libs.singular.function.SingularLibraryFunction
    Bases: sage.libs.singular.function.SingularFunction

EXAMPLES:

sage: from sage.libs.singular.function import SingularLibraryFunction
sage: R.<x,y> = PolynomialRing(QQ, order='lex')

(continues on next page)
sage: f = SingularLibraryFunction("groebner")
sage: f(I)
[1]

sage.libs.singular.function.all_singular_poly_wrapper(s)
Tests for a sequence s, whether it consists of singular polynomials.

EXAMPLES:

sage: from sage.libs.singular.function import all_singular_poly_wrapper
sage: P.<x,y,z> = QQ[

sage: all_singular_poly_wrapper([x+1, y])
True

sage: all_singular_poly_wrapper([x+1, y, 1])
False

sage.libs.singular.function.all_vectors(s)
Checks if a sequence s consists of free module elements over a singular ring.

EXAMPLES:

sage: from sage.libs.singular.function import all_vectors
sage: P.<x,y,z> = QQ[

sage: M = P**2
sage: all_vectors([x])
False

sage: all_vectors([(x,y)])
False

sage: all_vectors([M(0), M((x,y))])
True

sage: all_vectors([M(0), M((x,y)),(0,0)])
False

sage.libs.singular.function.is_sage_wrapper_for_singular_ring(ring)
Check whether wrapped ring arises from Singular or Singular/Plural.

EXAMPLES:

sage: from sage.libs.singular.function import is_sage_wrapper_for_singular_ring
sage: P.<x,y,z> = QQ[

sage: is_sage_wrapper_for_singular_ring(P)
True

sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: P = A.g_algebra(relations={y*x:-x*y}, order = 'lex')

sage: is_sage_wrapper_for_singular_ring(P)
True

sage.libs.singular.function.is_singular_poly_wrapper(p)
Checks if p is some data type corresponding to some singular poly.

EXAMPLES:

sage: from sage.libs.singular.function import is_singular_poly_wrapper
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({z*x:x*z+2*x, z*y:y*z-2*y})

sage: is_singular_poly_wrapper(x+y)
True
sage.libs.singular.function.lib(name)
Load the Singular library name.

INPUT:

- name – a Singular library name

EXAMPLES:

```python
sage: from sage.libs.singular.function import singular_function
sage: from sage.libs.singular.function import lib as singular_lib
sage: singular_lib('general.lib')
```

sage.libs.singular.function.list_of_functions(packages=False)
Return a list of all function names currently available.

INPUT:

- packages – include local functions in packages.

EXAMPLES:

```python
sage: from sage.libs.singular.function import list_of_functions
sage: 'groebner' in list_of_functions()
True
```

sage.libs.singular.function.singular_function(name)
Construct a new libSingular function object for the given name.

This function works both for interpreter and built-in functions.

INPUT:

- name – the name of the function

EXAMPLES:

```python
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: f = 3*x*y + 2*z + 1
sage: g = 2*x + 1/2
sage: I = Ideal([f,g])
sage: from sage.libs.singular.function import singular_function
sage: std = singular_function("std")
```

```python
sage: std(I)
```

```python
sage: size = singular_function("size")
```

```python
sage: size([2, 3, 3])
```

```python
sage: size("sage")
```

```python
sage: size(["hello", "sage"], 2)
```

```python
sage: factorize = singular_function("factorize")
```

```python
sage: factorize(f)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```

```python
sage: factorize(f, 1)
```
We give a wrong number of arguments:

```python
sage: factorize()
Traceback (most recent call last):
...  
RuntimeError: error in Singular function call 'factorize':
Wrong number of arguments (got 0 arguments, arity code is 303)
sage: factorize(f, 1, 2)
Traceback (most recent call last):
...  
RuntimeError: error in Singular function call 'factorize':
Wrong number of arguments (got 3 arguments, arity code is 303)
sage: factorize(f, 1, 2, 3)
Traceback (most recent call last):
...  
RuntimeError: error in Singular function call 'factorize':
Wrong number of arguments (got 4 arguments, arity code is 303)
```

The Singular function `list` can be called with any number of arguments:

```python
sage: singular_list = singular_function("list")
sage: singular_list(2, 3, 6)
[2, 3, 6]
sage: singular_list()
[]
sage: singular_list(1)
[1]
sage: singular_list(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

We try to define a non-existing function:

```python
sage: number_foobar = singular_function('number_foobar')
Traceback (most recent call last):
...  
NameError: Singular library function 'number_foobar' is not defined
```

```python
sage: from sage.libs.singular.function import lib as singular_lib
sage: singular_lib('general.lib')
sage: number_e = singular_function('number_e')
sage: number_e(10r)
67957045707/25000000000
sage: RR(number_e(10r))
2.71828182828000
```

```python
sage: singular_lib('primdec.lib')
sage: primdecGTZ = singular_function("primdecGTZ")
sage: primdecGTZ(I)
[[[y - 8/3*z - 4/3, x + 1/4], [y - 8/3*z - 4/3, x + 1/4]]]
sage: singular_list((1,2,3),3,[1,2,3], ring=P)
[(1, 2, 3), 3, [1, 2, 3]]
sage: ringlist=singular_function("ringlist")
sage: l = ringlist(P)
sage: l[3].__class__
<class 'sage.rings.polynomial.multi_polynomial_sequence.PolynomialSequence_generic'>
sage: l
```

(continues on next page)
[0, ['x', 'y', 'z'], [[dp', (1, 1, 1)], ['C', (0, )]], [0]]
sage: ring=singular_function("ring")
sage: ring(1)
<RingWrap>
sage: matrix = Matrix(P,2,2)
sage: matrix.randomize(terms=1)
sage: det = singular_function("det")
sage: det(matrix)
-3/5*x*y*z
sage: coeffs = singular_function("coeffs")
sage: coeffs(x+y+y+1,y)
[ 1]
[ x + 1]
sage: intmat = Matrix(ZZ, 2, 2, [100, 2, 3, 4])
sage: det(intmat)
394
sage: random = singular_function("random")
sage: A = random(10,2,3); A.nrows(), max(A.list()) <= 10
(2, True)
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: M=P**3
sage: leadcoef = singular_function("leadcoef")
sage: lead(v)
(0, y^3)
sage: lead(v)
(0, y^3)
sage: lead(v)
(0, y^3)
sage: jet = singular_function("jet")
sage: jet(v, 2)
(x + y, x*y, z)
sage: syz = singular_function("syz")
sage: I = P.ideal([x+y, x*y-y, y+2, x**2+1])
sage: M = syz(I)
sage: M
[(-2*y, 2, y + 1, 0), (0, -2, x - 1, 0), (x*y - y, -y + 1, 1, -y), (x^2 + 1, -x - 1, -1, -x)]
sage: singular_lib("mprimdec.lib")
sage: syz(M)
[(-x - 1, y - 1, 2*x, -2*y)]
sage: GTZmod = singular_function("GTZmod")
sage: GTZmod(M)
[[(2*y, 2, y + 1, 0), (0, -2, x - 1, 0), (x*y - y, -y + 1, 1, -y), (x^2 + 1, -x - 1, -1, -x)], [(0, 0)]]
sage: mres = singular_function("mres")
sage: resolution = mres(M, 0)
sage: resolution
<Resolution>
sage: singular_list(resolution)
[[(2*y, 2, y + 1, 0), (0, -2, x - 1, 0), (x*y - y, -y + 1, 1, -y), (x^2 + 1, -x - 1, -1, -x)], [(0, 0)]]
sage: A.<x,y> = FreeAlgebra(QQ, 2)
sage: P.<x,y> = A.g_algebra(y*x=x*y)
sage: I= Sequence([x*y,x+y], check=False, immutable=True)
sage: twostd = singular_function("twostd")
### 8.2 libSingular: Function Factory

**AUTHORS:**
- Martin Albrecht (2010-01): initial version

**class** `sage.libs.singular.function_factory.SingularFunctionFactory`

**Bases:** object

A convenient interface to libsingular functions.

**trait_names()**

**EXAMPLES:**

```python
sage: import sage.libs.singular.function_factory
sage: "groebner" in sage.libs.singular.function_factory.ff.trait_names()
True
```
8.3 libSingular: Conversion Routines and Initialisation

AUTHOR:

- Martin Albrecht <malb@informatik.uni-bremen.de>

8.4 Wrapper for Singular’s Polynomial Arithmetic

AUTHOR:

- Martin Albrecht (2009-07): refactoring

8.5 libSingular: Options

Singular uses a set of global options to determine verbosity and the behavior of certain algorithms. We provide an interface to these options in the most ‘natural’ python-ic way. Users who do not wish to deal with Singular functions directly usually do not have to worry about this interface or Singular options in general since this is taken care of by higher level functions.

We compute a Groebner basis for Cyclic-5 in two different contexts:

```
sage: P.<a,b,c,d,e> = PolynomialRing(GF(127))
sage: I = sage.rings.ideal.Cyclic(P)
sage: import sage.libs.singular.function_factory
sage: std = sage.libs.singular.function_factory.ff.std
```

By default, tail reductions are performed:

```
sage: from sage.libs.singular.option import opt, opt_ctx
sage: opt['red_tail']
True
sage: std(I)[-1]
d^2*e^6 + 28*b*c*d + ...
```

If we don’t want this, we can create an option context, which disables this:

```
sage: with opt_ctx(red_tail=False, red_sb=False):
....:     std(I)[-1]
....:     d^2*e^6 + 8*c^3 + ...
```

However, this does not affect the global state:

```
sage: opt['red_tail']
True
```

On the other hand, any assignment to an option object will immediately change the global state:

```
sage: opt['red_tail'] = False
sage: opt['red_tail']
False
sage: opt['red_tail'] = True
sage: opt['red_tail']
True
```
Assigning values within an option context, only affects this context:

```python
sage: with opt_ctx:
    ....:    opt['red_tail'] = False
sage: opt['red_tail']
True
```

Option contexts can also be safely stacked:

```python
sage: with opt_ctx:
    ....:    opt['red_tail'] = False
    ....:    print(opt)
    ....:    with opt_ctx:
    ....:        opt['red_through'] = False
    ....:        print(opt)
general options for libSingular (current value 0x00000082)  
general options for libSingular (current value 0x00000002)
sage: print(opt)
general options for libSingular (current value 0x02000082)
```

Furthermore, the integer valued options `deg_bound` and `mult_bound` can be used:

```python
sage: R.<x,y> = QQ[]
sage: I = R*[x^3+y^2,x^2*y+1]
sage: opt['deg_bound'] = 2
sage: std(I)  
[x^2*y + 1, x^3 + y^2]
sage: opt['deg_bound'] = 0
sage: std(I)  
[y^3 - x, x^2*y + 1, x^3 + y^2]
```

The same interface is available for verbosity options:

```python
sage: from sage.libs.singular.option import opt_verb
sage: opt_verb['not_warn_sb']
False
sage: opt.reset_default()  # needed to avoid side effects
sage: opt_verb.reset_default()  # needed to avoid side effects
```

**AUTHOR:**

- Martin Albrecht (2009-08): initial implementation
- Martin Albrecht (2010-01): better interface, verbosity options
- Simon King (2010-07): Python-ic option names; `deg_bound` and `mult_bound`

**class** `sage.libs.singular.option.LibSingularOptions`  
**Bases:** `sage.libs.singular.option.LibSingularOptions_abstract`  

Pythonic Interface to libSingular’s options.

Supported options are:

- `return_sb` or `returnSB` - the functions `syz`, `intersect`, `quotient`, `modulo` return a standard base instead of a generating set if `return_sb` is set. This option should not be used for `lift`.
- `fast_hc` or `fastHC` - tries to find the highest corner of the staircase (HC) as fast as possible during a standard basis computation (only used for local orderings).
• **int_strategy** or **intStrategy** - avoids division of coefficients during standard basis computations. This option is ring dependent. By default, it is set for rings with characteristic 0 and not set for all other rings.

• **lazy** - uses a more lazy approach in std computations, which was used in SINGULAR version before 2-0 (and which may lead to faster or slower computations, depending on the example).

• **length** - select shorter reducers in std computations.

• **not_regularity** or **notRegularity** - disables the regularity bound for res and mres.

• **not_sugar** or **notSugar** - disables the sugar strategy during standard basis computation.

• **not_buckets** or **notBuckets** - disables the bucket representation of polynomials during standard basis computations. This option usually decreases the memory usage but increases the computation time. It should only be set for memory-critical standard basis computations.

• **old_std** or **oldStd** - uses a more lazy approach in std computations, which was used in SINGULAR version before 2-0 (and which may lead to faster or slower computations, depending on the example).

• **prot** - shows protocol information indicating the progress during the following computations: facstd, fglm, groebner, lres, mres, minres, mstd, res, slimgb, sres, std, stdfglm, stdhilb, syz.

• **red_sb** or **redSB** - computes a reduced standard basis in any standard basis computation.

• **red_tail** or **redTail** - reduction of the tails of polynomials during standard basis computations. This option is ring dependent. By default, it is set for rings with global degree orderings and not set for all other rings.

• **red_through** or **redThrough** - for inhomogeneous input, polynomial reductions during standard basis computations are never postponed, but always finished through. This option is ring dependent. By default, it is set for rings with global degree orderings and not set for all other rings.

• **sugar_crit** or **sugarCrit** - uses criteria similar to the homogeneous case to keep more useless pairs.

• **weight_m** or **weightM** - automatically computes suitable weights for the weighted ecart and the weighted sugar method.

In addition, two integer valued parameters are supported, namely:

• **deg_bound** or **degBound** - The standard basis computation is stopped if the total (weighted) degree exceeds **deg_bound**. **deg_bound** should not be used for a global ordering with inhomogeneous input. Reset this bound by setting **deg_bound** to 0. The exact meaning of “degree” depends on the ring ordering and the command: slimgb uses always the total degree with weights 1, std does so for block orderings, only.

• **mult_bound** or **multBound** - The standard basis computation is stopped if the ideal is zero-dimensional in a ring with local ordering and its multiplicity is lower than **mult_bound**. Reset this bound by setting **mult_bound** to 0.

**EXAMPLES:**

```python
sage: from sage.libs.singular.option import LibSingularOptions
sage: libsingular_options = LibSingularOptions()
```

Here we demonstrate the intended way of using libSingular options:
The option `mult_bound` is only relevant in the local case:

```python
sage: from sage.libs.singular.option import opt
sage: Rlocal.<x,y,z> = PolynomialRing(QQ, order='ds')
sage: x^2<x
True
sage: J = [x^7+y^7+z^6, x^6+y^8+z^7, x^7+y^5+z^8, x^2+y^3+y^2+z^3+x^3+y^2+y^3+x^2+y^2+x^2+y^2]*Rlocal
sage: J.groebner_basis(mult_bound=100)
[x^3*y^2 + y^3*z^2 + x^2*z^3, x^2*y^3 + x^3*z^2 + y^2*z^3, y^5, x^6 + x*y^4*z^5, x^4*z^2 - y^4*z^2 + x^2*y*z^3 - x*y^2*z^3, z^6 - x*y^4*z^4 - x^3*y*z^5]
```

```python
sage: opt['red_tail'] = True  # the previous commands reset opt['red_tail'] to False
sage: J.groebner_basis()
[x^3*y^2 + y^3*z^2 + x^2*z^3, x^2*y^3 + x^3*z^2 + y^2*z^3, y^5, x^6, x^4*z^2 - y^4*z^2 + x^2*y*z^3 - x*y^2*z^3, z^6, y^4*z^3 - y^3*z^4 - x^2*z^5, x^3*y*z^4 - x^2*y*z^5 + y^3*z^5, x*y^3*z^5]
```

```python
reset_default()
Reset libSingular’s default options.

EXAMPLES:
```
```python
sage: from sage.libs.singular.option import opt
sage: opt
general options for libSingular (current value 0x06000082)
```

```python
class sage.libs.singular.option.LibSingularOptionsContext
Bases: object
Option context
This object localizes changes to options.

EXAMPLES:
```
```python
class sage.libs.singular.option.LibSingularOptionsContext
Bases: object
Option context
This object localizes changes to options.

EXAMPLES:
```
```python
sage: from sage.libs.singular.option import opt, opt_ctx
sage: opt
```

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```python
sage: with opt_ctx(redTail=False):
....:    print(opt)
....: with opt_ctx(redThrough=False):
....:    print(opt)
general options for libSingular (current value 0x04000082)
general options for libSingular (current value 0x04000002)
sage: print(opt)
general options for libSingular (current value 0x06000082)
```

**opt**

**class sage.libs.singular.option.LibSingularOptions_abstract**

**Bases:** object

Abstract Base Class for libSingular options.

**load**(value=None)

**EXAMPLES:**

```python
sage: from sage.libs.singular.option import opt as sopt
sage: bck = sopt.save(); hex(bck[0]), bck[1], bck[2]
('0x6000082', 0, 0)
sage: sopt['redTail'] = False
sage: hex(int(sopt))
'0x4000082'
sage: sopt.load(bck)
sage: sopt['redTail']
True
```

**save()**

Return a triple of integers that allow reconstruction of the options.

**EXAMPLES:**

```python
sage: from sage.libs.singular.option import opt
sage: opt['deg_bound']
0
sage: opt['red_tail']
True
sage: s = opt.save()
sage: opt['deg_bound'] = 2
sage: opt['red_tail'] = False
sage: opt['deg_bound']
2
sage: opt['red_tail']
False
sage: opt.load(s)
sage: opt['deg_bound']
0
sage: opt['red_tail']
True
sage: opt.reset_default()  # needed to avoid side effects
```

**class sage.libs.singular.option.LibSingularVerboseOptions**

**Bases:** sage.libs.singular.option.LibSingularOptions_abstract

Pythonic Interface to libSingular's verbosity options.

Supported options are:
• mem - shows memory usage in square brackets.
• yacc - Only available in debug version.
• redefine - warns about variable redefinitions.
• reading - shows the number of characters read from a file.
• loadLib or load_lib - shows loading of libraries.
• debugLib or debug_lib - warns about syntax errors when loading a library.
• loadProc or load_proc - shows loading of procedures from libraries.
• defRes or def_res - shows the names of the syzygy modules while converting resolution to list.
• usage - shows correct usage in error messages.
• Imap or imap - shows the mapping of variables with the fetch and imap commands.
• notWarnSB or not_warn_sb - do not warn if a basis is not a standard basis
• contentSB or content_sb - avoids to divide by the content of a polynomial in std and related algorithms. Should usually not be used.
• cancelunit - avoids to divide polynomials by non-constant units in std in the local case. Should usually not be used.

EXAMPLES:

```python
sage: from sage.libs.singular.option import LibSingularVerboseOptions
sage: libsingular_verbose = LibSingularVerboseOptions()
```

```python
sage: libsingular_verbose
verbosity options for libSingular (current value 0x00002851)
```

```python
reset_default()
Return to libSingular’s default verbosity options
```

EXAMPLES:

```python
sage: from sage.libs.singular.option import opt_verb
sage: opt_verb['not_warn_sb']
False
sage: opt_verb['not_warn_sb'] = True
sage: opt_verb['not_warn_sb']
True
sage: opt_verb.reset_default()

sage: opt_verb['not_warn_sb']
False
```

### 8.6 Wrapper for Singular’s Rings

AUTHORS:

• Martin Albrecht (2009-07): initial implementation
• Kwankyu Lee (2010-06): added matrix term order support

```python
sage.libs.singular.ringcurrRing_wrapper()
Returns a wrapper for the current ring, for use in debugging ring_refcount_dict.
```

EXAMPLES:
sage: from sage.libs.singular.ring import currRing_wrapper
sage: currRing_wrapper()
The ring pointer ...

sage.libs.singular.ring.poison_currRing(frame, event, arg)
Poison the currRing pointer.

This function sets the currRing to an illegal value. By setting it as the python debug hook, you can poison the currRing before every evaluated Python command (but not within Cython code).

INPUT:
- frame, event, arg – the standard arguments for the CPython debugger hook. They are not used.

OUTPUT:
Returns itself, which ensures that poison_currRing() will stay in the debugger hook.

EXAMPLES:
sage: previous_trace_func = sys.gettrace()  # None if no debugger running
sage: from sage.libs.singular.ring import poison_currRing
sage: sys.settrace(poison_currRing)
sage: sys.gettrace()  # built-in function poison_currRing
<sage.libs.singular.ring.poison_currRing>
sage: sys.settrace(previous_trace_func)  # switch it off again

sage.libs.singular.ring.print_currRing()
Print the currRing pointer.

EXAMPLES:
sage: from sage.libs.singular.ring import print_currRing
sage: print_currRing()  # random output
DEBUG: currRing == 0x7fc6fa6ec480

class sage.libs.singular.ring.ring_wrapper_Py
Bases: object

Python object wrapping the ring pointer.

This is useful to store ring pointers in Python containers.

You must not construct instances of this class yourself, use wrap_ring() instead.

EXAMPLES:
sage: from sage.libs.singular.ring import ring_wrapper_Py
sage: ring_wrapper_Py
<type 'sage.libs.singular.ring.ring_wrapper_Py'>
8.7 Singular’s Groebner Strategy Objects

AUTHORS:
- Martin Albrecht (2009-07): initial implementation
- Michael Brickenstein (2009-07): initial implementation
- Hans Schoenemann (2009-07): initial implementation

class sage.libs.singular.groebner_strategy.GroebnerStrategy
  Bases: sage.structure.sage_object.SageObject


  This object provides functions for normal form computations and other functions for Groebner basis computation.

ALGORITHM:
Uses Singular via libSINGULAR

ideal()
Return the ideal this strategy object is defined for.

EXAMPLES:

```sage
def from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(GF(32003))
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: strat.ideal()
Ideal (x + z, y + z) of Multivariate Polynomial Ring in x, y, z over Finite
˓ Field of size 32003
```

normal_form(p)
Compute the normal form of p with respect to the generators of this object.

EXAMPLES:

```sage
def from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(QQ)
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: strat.normal_form(x*y) # indirect doctest
z^2
sage: strat.normal_form(x + 1)
-z + 1
```

ring()
Return the ring this strategy object is defined over.

EXAMPLES:

```sage
def from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(GF(32003))
sage: I = Ideal([x + z, y + z])
sage: strat = GroebnerStrategy(I)
sage: strat.ring()
Multivariate Polynomial Ring in x, y, z over Finite Field of size 32003
```
class sage.libs.singular.groebner_strategy.NCGroebnerStrategy

Bases: sage.structure.sage_object.SageObject


This object provides functions for normal form computations and other functions for Groebner basis computation.

ALGORITHM:
Uses Singular via libSINGULAR

ideal()

Return the ideal this strategy object is defined for.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2-H.one()])
sage: strat = NCGroebnerStrategy(I)
sage: strat.ideal() == I
True
```

normal_form(p)

Compute the normal form of $p$ with respect to the generators of this object.

EXAMPLES:

```
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: JL = H.ideal([x^3, y^3, z^3 - 4*z])
sage: JT = H.ideal([x^3, y^3, z^3 - 4*z], side='twosided')
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: SL = NCGroebnerStrategy(JL.std())
sage: ST = NCGroebnerStrategy(JT.std())
sage: SL.normal_form(x*y^2)
x*y^2
sage: ST.normal_form(x*y^2)
y*z
```

ring()

Return the ring this strategy object is defined over.

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2-H.one()])
sage: strat = NCGroebnerStrategy(I)
sage: strat.ring() == H
True
```

sage.libs.singular.groebner_strategy.unpickle_GroebnerStrategy0(I)

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import GroebnerStrategy
sage: P.<x,y,z> = PolynomialRing(GF(32003))

(continues on next page)
sage: I = Ideal([x + z, y + z])

sage: strat = GroebnerStrategy(I)

sage: loads(dumps(strat)) == strat  # indirect doctest
True

sage.libs.singular.groebner_strategy.unpickle_NCGroebnerStrategy0(I)

EXAMPLES:

```
sage: from sage.libs.singular.groebner_strategy import NCGroebnerStrategy
sage: A.<x,y,z> = FreeAlgebra(QQ, 3)
sage: H.<x,y,z> = A.g_algebra({y*x:x*y-z, z*x:x*z+2*x, z*y:y*z-2*y})
sage: I = H.ideal([y^2, x^2, z^2-H.one()])
sage: strat = NCGroebnerStrategy(I)
sage: loads(dumps(strat)) == strat  # indirect doctest
True
```
9.1 Context Managers for LibGAP

This module implements a context manager for global variables. This is useful since the behavior of GAP is sometimes controlled by global variables, which you might want to switch to a different value for a computation. Here is an example how you are suppose to use it from your code. First, let us set a dummy global variable for our example:

```python
sage: libgap.set_global('FooBar', 123)
```

Then, if you want to switch the value momentarily you can write:

```python
sage: with libgap.global_context('FooBar', 'test'):
....:     print(libgap.get_global('FooBar'))
test
```

Afterward, the global variable reverts to the previous value:

```python
sage: print(libgap.get_global('FooBar'))
123
```

The value is reset even if exceptions occur:

```python
sage: with libgap.global_context('FooBar', 'test'):
....:     print(libgap.get_global('FooBar'))
....:     raise ValueError(libgap.get_global('FooBar'))
Traceback (most recent call last):
...
ValueError: test
sage: print(libgap.get_global('FooBar'))
123
```

class sage.libs.gap.context_managers.GlobalVariableContext (variable, value)
Bases: object

Context manager for GAP global variables.

It is recommended that you use the `sage.libs.gap.libgap.Gap.global_context()` method and not construct objects of this class manually.

INPUT:

- `variable` – string. The variable name.
- `value` – anything that defines a GAP object.

EXAMPLES:
9.2 Common global functions defined by GAP.

9.3 Long tests for GAP

These stress test the garbage collection inside GAP

sage.libs.gap.test_long.test_loop_1()

EXAMPLES:

sage: from sage.libs.gap.test_long import test_loop_1
sage: test_loop_1()  # long time (up to 25s on sage.math, 2013)

sage.libs.gap.test_long.test_loop_2()

EXAMPLES:

sage: from sage.libs.gap.test_long import test_loop_2
sage: test_loop_2()  # long time (10s on sage.math, 2013)

sage.libs.gap.test_long.test_loop_3()

EXAMPLES:

sage: from sage.libs.gap.test_long import test_loop_3
sage: test_loop_3()  # long time (31s on sage.math, 2013)

9.4 Utility functions for GAP

exception sage.libs.gap.util.GAPError
    Bases: ValueError
    Exceptions raised by the GAP library

class sage.libs.gap.util.ObjWrapper
    Bases: object
    Wrapper for GAP master pointers

EXAMPLES:

sage: from sage.libs.gap.util import ObjWrapper
sage: x = ObjWrapper()
sage: y = ObjWrapper()
sage: x == y
True

sage.libs.gap.util.gap_root()
    Find the location of the GAP root install which is stored in the gap startup script.
9.5 Library Interface to GAP

This module implements a fast C library interface to GAP. To use it, you simply call \texttt{libgap} (the parent of all \texttt{GapElement} instances) and use it to convert Sage objects into GAP objects.

\textbf{EXAMPLES:}

\begin{verbatim}
  sage: a = libgap(10)
  sage: a
  10
  sage: type(a)
  <type 'sage.libs.gap.element.GapElement_Integer'>
  sage: a*a
  100
  sage: timeit('a*a')  # random output
  625 loops, best of 3: 898 ns per loop

  Compared to the expect interface this is >1000 times faster:

  sage: b = gap('10')
  sage: timeit('b*b')  # random output; long time
  125 loops, best of 3: 2.05 ms per loop

  If you want to evaluate GAP commands, use the \texttt{Gap.eval()} method:

  sage: libgap.eval('List([1..10], i->i^2)')
  [ 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 ]

  not to be confused with the \texttt{libgap} call, which converts Sage objects to GAP objects, for example strings to strings:

  sage: libgap('List([1..10], i->i^2)')
  "List([1..10], i->i^2)"
  sage: type(_)
  <type 'sage.libs.gap.element.GapElement_String'>

  You can usually use the \texttt{sage()} method to convert the resulting GAP element back to its Sage equivalent:

  sage: a.sage()
  10
  sage: type(_)
  <type 'sage.rings.integer.Integer'>

  sage: libgap.eval('5/3 + 7*E(3)').sage()
  7*zeta3 + 5/3

  sage: generators = gens_of_group.sage()
\end{verbatim}

(continues on next page)
We can also specify which group in Sage the permutations should consider themselves as elements of when converted to Sage:

```
sage: A4 = groups.permutation.Alternating(4)
sage: generators = gens_of_group.sage(parent=A4); generators
[(1,2,3), (2,3,4)]
sage: all(gen.parent() == A4 for gen in generators)
True
```

So far, the following GAP data types can be directly converted to the corresponding Sage datatype:

1. GAP booleans `true`/`false` to Sage booleans `True`/`False`. The third GAP boolean value `fail` raises a `ValueError`.
2. GAP integers to Sage integers.
3. GAP rational numbers to Sage rational numbers.
4. GAP cyclotomic numbers to Sage cyclotomic numbers.
5. GAP permutations to Sage permutations.
6. The GAP containers `List` and `rec` are converted to Sage containers `list` and `dict`. Furthermore, the `sage()` method is applied recursively to the entries.

Special support is available for the GAP container classes. GAP lists can be used as follows:

```
sage: lst = libgap([1,5,7]); lst
[ 1, 5, 7 ]
sage: type(lst)
<type 'sage.libs.gap.element.GapElement_List'>
sage: len(lst)
3
sage: lst[0]
1
sage: [ x^2 for x in lst ]
[1, 25, 49]
sage: type(_[0])
<type 'sage.libs.gap.element.GapElement_Integer'>
```

Note that you can access the elements of GAP `List` objects as you would expect from Python (with indexing starting at 0), but the elements are still of type `GapElement`. The other GAP container type are records, which are similar to Python dictionaries. You can construct them directly from Python dictionaries:

```
sage: libgap({'a':123, 'b':456})
rec( a := 123, b := 456 )
```

Or get them as results of computations:

```
sage: rec = libgap.eval('rec(a:=123, b:=456, Sym3:=SymmetricGroup(3))')
sage: rec['Sym3']
```

(continues on next page)
The output is a Sage dictionary whose keys are Sage strings and whose values are instances of `GapElement()`. So, for example, `rec['a']` is not a Sage integer. To recursively convert the entries into Sage objects, you should use the `sage()` method:

```python
sage: rec.sage()
{'Sym3': NotImplementedError('cannot construct equivalent Sage object'...),
 'a': 123,
 'b': 456}
```

Now `rec['a']` is a Sage integer. We have not implemented the conversion of the GAP symmetric group to the Sage symmetric group yet, so you end up with a `NotImplementedError` exception object. The exception is returned and not raised so that you can work with the partial result.

While we don’t directly support matrices yet, you can convert them to Gap List of Lists. These lists are then easily converted into Sage using the recursive expansion of the `sage()` method:

```python
sage: M = libgap.eval('BlockMatrix([[1,1,[[1, 2],[ 3, 4]],[1,2,[[9,10],[11,12]]],[2,2,[[5, 6],[ 7, 8]]]],2,2)')
sage: M
<block matrix of dimensions (2*2)x(2*2)>
sage: M.List()  # returns a GAP List of Lists
[[1, 2, 3, 4], [11, 12, 9, 10], [0, 0, 5, 6], [0, 0, 7, 8]]
sage: M.List().sage()  # returns a Sage list of lists
[[1, 2, 3, 4], [11, 12, 9, 10], [0, 0, 5, 6], [0, 0, 7, 8]]
sage: matrix(ZZ, _)
[ 1  2  3  4]
[ 5  6  7  8]
```

### 9.5.1 Using the GAP C library from Cython

**Todo:** Expand the following text

We are using the GAP API provided by the GAP project since GAP 4.10.

**AUTHORS:**
- Volker Braun (2012-08-28, GAP/Singular workshop): update to gap-4.5.5, make it ready for public consumption.
- Dima Pasechnik (2018-09-18, GAP Days): started the port to native libgap API

```python
class sage.libs.gap.libgap.Gap
    Bases: sage.structure.parent.PARENT

    The libgap interpreter object.
```

9.5. Library Interface to GAP
Note: This object must be instantiated exactly once by the libgap. Always use the provided \texttt{libgap} instance, and never instantiate \texttt{Gap} manually.

EXAMPLES:

```python
sage: libgap.eval('SymmetricGroup(4)')
     Sym( [ 1 .. 4 ] )
```

\texttt{Element}

alias of \texttt{sage.libs.gap.element.GapElement}

\texttt{collect}()

Manually run the garbage collector

EXAMPLES:

```python
sage: a = libgap(123)
sage: del a
sage: libgap.collect()
```

\texttt{count\_GAP\_objects}()

Return the number of GAP objects that are being tracked by GAP.

OUTPUT:

An integer

EXAMPLES:

```python
sage: libgap.count\_GAP\_objects() # random output
5
```

\texttt{eval}(\texttt{gap\_command})

Evaluate a gap command and wrap the result.

INPUT:

- gap\_command – a string containing a valid gap command without the trailing semicolon.

OUTPUT:

A \texttt{GapElement}.

EXAMPLES:

```python
sage: libgap.eval('0')
0
sage: libgap.eval('"string"')
"string"
```

\texttt{function\_factory}(\texttt{function\_name})

Return a GAP function wrapper

This is almost the same as calling \texttt{libgap.eval(function\_name)}, but faster and makes it obvious in your code that you are wrapping a function.

INPUT:

- function\_name – string. The name of a GAP function.
A function wrapper `GapElement_Function` for the GAP function. Calling it from Sage is equivalent to calling the wrapped function from GAP.

**EXAMPLES:**

```python
sage: libgap.function_factory('Print')
<Gap function "Print">
```

`get_global(variable)`

Get a GAP global variable

**INPUT:**

- `variable` – string. The variable name.

**OUTPUT:**

A `GapElement` wrapping the GAP output. A `ValueError` is raised if there is no such variable in GAP.

**EXAMPLES:**

```python
sage: libgap.set_global('FooBar', 1)
sage: libgap.get_global('FooBar')
1
sage: libgap.unset_global('FooBar')
sage: libgap.get_global('FooBar')
Traceback (most recent call last):
...
GAPError: Error, VAL_GVAR: No value bound to FooBar
```

`global_context(variable, value)`

Temporarily change a global variable

**INPUT:**

- `variable` – string. The variable name.
- `value` – anything that defines a GAP object.

**OUTPUT:**

A context manager that sets/reverts the given global variable.

**EXAMPLES:**

```python
sage: libgap.set_global('FooBar', 1)
sage: with libgap.global_context('FooBar', 2):
    print(libgap.get_global('FooBar'))
2
sage: libgap.get_global('FooBar')
1
```

`load_package(pkg)`

If loading fails, raise a `RuntimeError` exception.

`one()`

Return (integer) one in GAP.

**EXAMPLES:**

```python
9.5. Library Interface to GAP  85
```
sage: libgap.one()
1
sage: parent(_)
C library interface to GAP

**set_global** *(variable, value)*
Set a GAP global variable

**INPUT:**

- **variable** – string. The variable name.
- **value** – anything that defines a GAP object.

**EXAMPLES:**

```python
sage: libgap.set_global('FooBar', 1)
sage: libgap.get_global('FooBar')
1
sage: libgap.unset_global('FooBar')
sage: libgap.get_global('FooBar')
Traceback (most recent call last):
  ... GAPError: Error, VAL_GVAR: No value bound to FooBar
```

**set_seed** *(seed=None)*
Reseed the standard GAP pseudo-random sources with the given seed.

Uses a random seed given by `current_randstate().ZZ_seed()` if `seed=None`. Otherwise the seed should be an integer.

**EXAMPLES:**

```python
sage: libgap.set_seed(0)
0
sage: [libgap.Random(1, 10) for i in range(5)]
[2, 3, 3, 4, 2]
```

**show()**
Return statistics about the GAP owned object list

This includes the total memory allocated by GAP as returned by `libgap.eval('TotalMemoryAllocated()')`, as well as garbage collection / object count statistics as returned by `libgap.eval('GasmanStatistics')`, and finally the total number of GAP objects held by Sage as `GapElement` instances.

The value `livekb + deadkb` will roughly equal the total memory allocated for GAP objects (see `libgap.eval('TotalMemoryAllocated()')`).

**Note:** Slight complication is that we want to do it without accessing libgap objects, so we don’t create new GapElements as a side effect.

**EXAMPLES:**

```python
sage: a = libgap(123)
sage: b = libgap(456)
sage: c = libgap(789)
sage: del b
```
unset_global \texttt{(variable)}

Remove a GAP global variable

\textbf{INPUT:}

- \texttt{variable} -- string. The variable name.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: libgap.set_global('FooBar', 1)
sage: libgap.get_global('FooBar')
1
sage: libgap.unset_global('FooBar')
sage: libgap.get_global('FooBar')
Traceback (most recent call last):
  ...
GAPError: Error, VAL_GVAR: No value bound to FooBar
\end{verbatim}

\textbf{zero()}

Return (integer) zero in GAP.

\textbf{OUTPUT:}

A \texttt{GapElement}.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: libgap.zero()
0
\end{verbatim}

9.6 Short tests for GAP

\texttt{sage.libs.gap.test.test_write_to_file()}

Test that libgap can write to files

See trac ticket \#16502, trac ticket \#15833.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.gap.test import test_write_to_file
sage: test_write_to_file()
\end{verbatim}
9.7 GAP element wrapper

This document describes the individual wrappers for various GAP elements. For general information about GAP, you should read the libgap module documentation.

```python
class sage.libs.gap.element.GapElement
    Bases: sage.structure.element.RingElement

    Wrapper for all Gap objects.
```

**Note:** In order to create GapElements you should use the libgap instance (the parent of all Gap elements) to convert things into GapElement. You must not create GapElement instances manually.

**EXAMPLES:**

```python
sage: libgap(0)
0
```

If Gap finds an error while evaluating, a GAPError exception is raised:

```python
sage: libgap.eval('1/0')
Traceback (most recent call last):
...  
GAPError: Error, Rational operations: <divisor> must not be zero
```

Also, a GAPError is raised if the input is not a simple expression:

```python
sage: libgap.eval('1; 2; 3')
Traceback (most recent call last):
...  
GAPError: can only evaluate a single statement
```

**deepcopy** *(mut)*

Return a deepcopy of this Gap object

Note that this is the same thing as calling StructuralCopy but much faster.

**INPUT:**

- **mut** - (boolean) wheter to return an mutable copy

**EXAMPLES:**

```python
sage: a = libgap([[0,1],[2,3]])
sage: b = a.deepcopy(1)
sage: b[0,0] = 5
sage: a
[[0, 1], [2, 3]]
sage: b
[[5, 1], [2, 3]]
sage: l = libgap([0,1])
sage: l.deepcopy(0).IsMutable()
false
sage: l.deepcopy(1).IsMutable()
true
```
**is_bool()**

Return whether the wrapped GAP object is a GAP boolean.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: libgap(True).is_bool()
True
```

**is_function()**

Return whether the wrapped GAP object is a function.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: a = libgap.eval("NormalSubgroups")
sage: a.is_function()
True
```

**is_list()**

Return whether the wrapped GAP object is a GAP List.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: libgap.eval('[1, 2, , , 5]').is_list()
True
```

**is_permutation()**

Return whether the wrapped GAP object is a GAP permutation.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: perm = libgap.PermList( libgap([1,5,2,3,4])); perm
(2,5,4,3)
sage: perm.is_permutation()
True
```

**is_record()**

Return whether the wrapped GAP object is a GAP record.

**OUTPUT:**


Boolean.

EXAMPLES:

```
sage: libgap.eval('[1, 2, 3, 4]').is_record()
False
sage: libgap.eval('rec(a:=1, b:=3)').is_record()
True
```

is_string() Function

Return whether the wrapped GAP object is a GAP string.

OUTPUT: Boolean.

EXAMPLES:

```
sage: libgap('this is a string').is_string()
True
```

sage() Function

Return the Sage equivalent of the GapElement.

EXAMPLES:

```
sage: libgap(1).sage()
1
sage: type(_)
<type 'sage.rings.integer.Integer'>

sage: libgap(3/7).sage()
3/7
sage: type(_)
<type 'sage.rings.rational.Rational'>

sage: libgap.eval('5 + 7*E(3)').sage()
7*zeta3 + 5

sage: libgap(Infinity).sage()
+Infinity
sage: libgap(-Infinity).sage()
-Infinity

sage: libgap(True).sage()
True
sage: libgap(False).sage()
False
sage: type(_)
<... 'bool'>

sage: libgap('this is a string').sage()
'this is a string'

sage: x = libgap.Integers.Indeterminate("x")

sage: p = x^2 - 2*x + 3
sage: p.sage()
```

(continues on next page)
\[ x^2 - 2x + 3 \]

```python
sage: p.sage().parent()
Univariate Polynomial Ring in x over Integer Ring
```

```python
sage: p = x^-2 + 3*x
sage: p.sage()
x^-2 + 3*x
sage: p.sage().parent()
Univariate Laurent Polynomial Ring in x over Integer Ring
```

```python
sage: p = (3 * x^2 + x) / (x^2 - 2)
sage: p.sage()
(3*x^2 + x)/(x^2 - 2)
sage: p.sage().parent()
Fraction Field of Univariate Polynomial Ring in x over Integer Ring
```

### class `sage.libs.gap.element.GapElement_Boolean`

**Bases:** `sage.libs.gap.element.GapElement`

Derived class of `GapElement` for GAP boolean values.

**EXAMPLES:**

```python
sage: b = libgap(True)
sage: type(b)
<type 'sage.libs.gap.element.GapElement_Boolean'>
```

**sage()**

Return the Sage equivalent of the `GapElement`

**OUTPUT:**

A Python boolean if the values is either true or false. GAP booleans can have the third value `Fail`, in which case a `ValueError` is raised.

**EXAMPLES:**

```python
sage: b = libgap.eval('true'); b
true
sage: type(_)
<type 'sage.libs.gap.element.GapElement_Boolean'>
sage: b.sage()
True
sage: type(_)
<... 'bool'>

sage: libgap.eval('fail')
fail
sage: _.sage()
Traceback (most recent call last):
  ... ValueError: the GAP boolean value "fail" cannot be represented in Sage
```

### class `sage.libs.gap.element.GapElement_Cyclotomic`

**Bases:** `sage.libs.gap.element.GapElement`

Derived class of `GapElement` for GAP universal cyclotomic.

**EXAMPLES:**

```python
```

9.7. GAP element wrapper 91
sage: libgap.eval('E(3)')
E(3)
sage: type(_)
<type 'sage.libs.gap.element.GapElement_Cyclotomic'>

sage (ring=None)
Return the Sage equivalent of the GapElement_Cyclotomic.

INPUT:

• ring – a Sage cyclotomic field or None (default). If not specified, a suitable minimal cyclotomic
  field will be constructed.

OUTPUT:
A Sage cyclotomic field element.

EXAMPLES:

sage: n = libgap.eval('E(3)')
sage: n.sage()
zeta3
sage: parent(_)
Cyclotomic Field of order 3 and degree 2
sage: n.sage(ring=CyclotomicField(6))
zeta6 - 1
sage: libgap.E(3).sage(ring=CyclotomicField(3))
zeta3
sage: libgap.E(3).sage(ring=CyclotomicField(6))
zeta6 - 1

class sage.libs.gap.element.GapElement_FiniteField
Bases: sage.libs.gap.element.GapElement

Derived class of GapElement for GAP finite field elements.

EXAMPLES:

sage: libgap.eval('Z(5)^2')
Z(5)^2
sage: type(_)
<type 'sage.libs.gap.element.GapElement_FiniteField'>
lift()
Return an integer lift.

OUTPUT:
The smallest positive GapElement_Integer that equals self in the prime finite field.

EXAMPLES:

sage: n = libgap.eval('Z(5)^2')
sage: n.lift()
4
sage: type(_)
<type 'sage.libs.gap.element.GapElement_Integer'>
(continues on next page)
Return the Sage equivalent of the `GapElement_FiniteField`.

INPUT:

- ring – a Sage finite field or None (default). The field to return self in. If not specified, a suitable finite field will be constructed.

OUTPUT:

An Sage finite field element. The isomorphism is chosen such that the Gap PrimitiveRoot() maps to the Sage multiplicative_generator().

EXAMPLES:

```sage
sage: n = libgap.eval('Z(25)^2')
sage: n.sage()
a + 3
sage: parent(_)
Finite Field in a of size 5^2
```

```sage
sage: n.sage(ring=GF(5))
Traceback (most recent call last):
... ValueError: the given ring is incompatible ...
```

class `sage.libs.gap.element.GapElement_Float`

Bases: `sage.libs.gap.element.GapElement`

Derived class of GapElement for GAP floating point numbers.

EXAMPLES:

```sage
sage: i = libgap(123.5)
sage: type(i)
<type 'sage.libs.gap.element.GapElement_Float'>
sage: RDF(i)
123.5
sage: float(i)
123.5
```

Return the Sage equivalent of the `GapElement_Float`.

- ring – a floating point field or None (default). If not specified, the default Sage RDF is used.

OUTPUT:

A Sage double precision floating point number

EXAMPLES:

```sage
sage: a = libgap.eval("Float(3.25)").sage()
sage: a
3.25
```
**parent (a)**

Real Double Field

---

**class**  

`sage.libs.gap.element.GapElement_Function`  

**Bases:** `sage.libs.gap.element.GapElement`  

Derived class of `GapElement` for GAP functions.

**EXAMPLES:**

```
sage: f = libgap.Cycles
sage: type(f)
<type 'sage.libs.gap.element.GapElement_Function'>
```

---

**class**  

`sage.libs.gap.element.GapElement_Integer`  

**Bases:** `sage.libs.gap.element.GapElement`  

Derived class of `GapElement` for GAP integers.

**EXAMPLES:**

```
sage: i = libgap(123)
sage: type(i)
<type 'sage.libs.gap.element.GapElement_Integer'>
sage: ZZ(i)
123
```

---

**is_C_int ()**

Return whether the wrapped GAP object is a immediate GAP integer.

An immediate integer is one that is stored as a C integer, and is subject to the usual size limits. Larger integers are stored in GAP as GMP integers.

**OUTPUT:**

Boolean.

**EXAMPLES:**

```
sage: n = libgap(1)
sage: type(n)
<type 'sage.libs.gap.element.GapElement_Integer'>
sage: n.is_C_int()
True
sage: n.IsInt()
true
sage: N = libgap(2^130)
sage: type(N)
<type 'sage.libs.gap.element.GapElement_Integer'>
sage: N.is_C_int()
False
sage: N.IsInt()
true
```

---

**sage (ring=None)**

Return the Sage equivalent of the `GapElement_Integer`

- **ring** – Integer ring or `None` (default). If not specified, the default Sage integer ring is used.
OUTPUT:
A Sage integer

EXAMPLES:

```python
sage: libgap([1, 3, 4]).sage()
[1, 3, 4]
sage: all(x in ZZ for x in _)
True
sage: libgap(132).sage(ring=IntegerModRing(13))
2
sage: parent(_)
Ring of integers modulo 13
```

class sage.libs.gap.element.GapElement_IntegerMod

Bases: sage.libs.gap.element.GapElement

Derived class of GapElement for GAP integers modulo an integer.

EXAMPLES:

```python
sage: n = IntegerModRing(123)(13)
sage: i = libgap(n)
sage: type(i)
<type 'sage.libs.gap.element.GapElement_IntegerMod'>
```

`lift()`  
Return an integer lift.

OUTPUT:
A GapElement_Integer that equals self in the integer mod ring.

EXAMPLES:

```python
sage: n = libgap.eval('One(ZmodnZ(123)) * 13')
sage: n.lift()
13
sage: type(_)
<type 'sage.libs.gap.element.GapElement_Integer'>
```

`sage (ring=None)`  
Return the Sage equivalent of the GapElement_IntegerMod

INPUT:

• ring – Sage integer mod ring or None (default). If not specified, a suitable integer mod ringa is used automatically.

OUTPUT:
A Sage integer modulo another integer.

EXAMPLES:

```python
sage: n = libgap.eval('One(ZmodnZ(123)) * 13')
sage: n.sage()
13
sage: parent(_)
Ring of integers modulo 123
```
class sage.libs.gap.element.GapElement_List

Bases: sage.libs.gap.element.GapElement

Derived class of GapElement for GAP Lists.

Note: Lists are indexed by 0..len(l) - 1, as expected from Python. This differs from the GAP convention where lists start at 1.

EXAMPLES:

```
sage: lst = libgap.SymmetricGroup(3).List(); lst
[ (), (1,3), (1,2,3), (2,3), (1,3,2), (1,2) ]
sage: type(lst)
<type 'sage.libs.gap.element.GapElement_List'>
sage: len(lst)
6
sage: lst[3]
(2,3)
```

We can easily convert a Gap List object into a Python list:

```
sage: list(lst)
[(), (1,3), (1,2,3), (2,3), (1,3,2), (1,2)]
sage: type(_)
<... 'list'>
```

Range checking is performed:

```
sage: lst[10]
Traceback (most recent call last):
... IndexError: index out of range.
```

```
matrix (ring=None)

Return the list as a matrix.

GAP does not have a special matrix data type, they are just lists of lists. This function converts a GAP list of lists to a Sage matrix.

OUTPUT:

A Sage matrix.

EXAMPLES:

```
sage: F = libgap.GF(4)
sage: a = F.PrimitiveElement()
sage: m = libgap([[a,a^0],[0*a,a^2]]); m
[ [ Z(2^2), Z(2)^0 ],
  [ 0*Z(2), Z(2^2)^2 ] ]
sage: m.IsMatrix()
true
sage: matrix(m)
[ a 1]
[ 0 a + 1]
sage: matrix(GF(4,'B'), m)
[ B 1]
[ 0 B + 1]
```

(continues on next page)
sage: M = libgap.eval('SL(2,GF(5))').GeneratorsOfGroup()[1]
sage: type(M)
<type 'sage.libs.gap.element.GapElement_List'>
sage: M[0][0]
Z(5)^2
sage: M.IsMatrix()
true
sage: M.matrix()
[4 1]
[4 0]

sage(**kwds)

Return the Sage equivalent of the GapElement

OUTPUT:
A Python list.

EXAMPLES:

sage: libgap([ 1, 3, 4 ]).sage()
[1, 3, 4]
sage: all( x in ZZ for x in _)
True

vector(ring=None)

Return the list as a vector.

GAP does not have a special vector data type, they are just lists. This function converts a GAP list to a Sage vector.

OUTPUT:
A Sage vector.

EXAMPLES:

sage: F = libgap.GF(4)
sage: a = F.PrimitiveElement()
sage: m = libgap([0*a, a, a^3, a^2]); m
[ 0*Z(2), Z(2^2), Z(2)^0, Z(2^2)^2 ]
sage: type(m)
<type 'sage.libs.gap.element.GapElement_List'>
sage: m[3]
Z(2^2)^2
sage: vector(m)
(0, a, 1, a + 1)
sage: vector(GF(4, 'B'), m)
(0, B, 1, B + 1)

class sage.libs.gap.element.GapElement_MethodProxy

Bases: sage.libs.gap.element.GapElement_Function

Derived class returned by GapElement.__getattr__.

EXAMPLES:
```python
sage: lst = libgap([])
sage: lst.Add
<Gap function "Add">
sage: type(_)
<type 'sage.libs.gap.element.GapElement_MethodProxy'>
sage: lst.Add(1)
sage: lst
[ 1 ]
```

### class sage.libs.gap.element.GapElement_Permutation

Bases: `sage.libs.gap.element.GapElement`

Derived class of `GapElement` for GAP permutations.

**Note:** Permutations in GAP act on the numbers starting with 1.

**EXAMPLES:**

```python
sage: perm = libgap.eval('(1,5,2)(4,3,8)')
sage: type(perm)
<type 'sage.libs.gap.element.GapElement_Permutation'>
```

```python
sage: perm.sage()
[5, 1, 8, 3, 2, 6, 7, 4]
sage: type(_)
<class 'sage.combinat.permutation.StandardPermutations_all_with_category.˓→element_class'>
```

```python
sage: perm.sage(PermutationGroup([(1,2),(1,2,3,4,5,6,7,8)]))
(1,5,2)(3,8,4)
sage: type(_)
<type 'sage.groups.perm_gps.permgroup_element.PermutationGroupElement'>
```

### class sage.libs.gap.element.GapElement_Rational

Bases: `sage.libs.gap.element.GapElement`

Derived class of `GapElement` for GAP rational numbers.

**EXAMPLES:**

```python
sage: r = libgap(123/456)
sage: type(r)
<type 'sage.libs.gap.element.GapElement_Rational'>
```

```
```

**Return the Sage equivalent of the `GapElement`.**

If the permutation group is given as parent, this method is *much* faster.

**EXAMPLES:**

```python
sage: perm_gap = libgap.eval('(1,5,2)(4,3,8)'); perm_gap
(1,5,2)(3,8,4)
```

```python
sage: perm_gap.sage()  # Using permutation group
[5, 1, 8, 3, 2, 6, 7, 4]
```

```python
sage: perm_gap.sage(PermutationGroup([(1,2),(1,2,3,4,5,6,7,8)]))
(1,5,2)(3,8,4)
```

```python
```
A Sage rational number.

EXAMPLES:

```
sage: r = libgap(123/456);  r
41/152
sage: type(_)
<type 'sage.libs.gap.element.GapElement_Rational'>
sage: r.sage()
41/152
sage: type(_)
<type 'sage.rings.rational.Rational'>
```

class sage.libs.gap.element.GapElement_Record
Bases: sage.libs.gap.element.GapElement

Derived class of GapElement for GAP records.

EXAMPLES:

```
sage: rec = libgap.eval('rec(a:=123, b:=456)')
sage: type(rec)
<type 'sage.libs.gap.element.GapElement_Record'>
sage: len(rec)
2
sage: rec['a']
123
```

We can easily convert a Gap rec object into a Python dict:

```
sage: dict(rec)
{'a': 123, 'b': 456}
sage: type(_)
<... 'dict'>
```

Range checking is performed:

```
sage: rec['no_such_element']
Traceback (most recent call last):
...
GAPError: Error, Record Element: '<rec>.no_such_element' must have an assigned value
```

record_name_to_index(name)

Convert string to GAP record index.

INPUT:

• py_name – a python string.

OUTPUT:

A UInt, which is a GAP hash of the string. If this is the first time the string is encountered, a new integer is returned(!)

EXAMPLES:
```python
sage: rec = libgap.eval('rec(first:=123, second:=456)')
sage: rec.record_name_to_index('first')  # random output
1812L
sage: rec.record_name_to_index('no_such_name')  # random output
3776L
```

```python
sage

Return the Sage equivalent of the \texttt{GapElement}.

**EXAMPLES:**

```python
sage: libgap.eval('rec(a:=1, b:=2)').sage()
{'a': 1, 'b': 2}
sage: all( isinstance(key,str) and val in ZZ for key,val in _.items() )
True
sage: rec = libgap.eval('rec(a:=123, b:=456, Sym3:=SymmetricGroup(3))')
sage: rec.sage()
{'Sym3': NotImplementedError('cannot construct equivalent Sage object'...),
 'a': 123,
 'b': 456}
```

```python
class sage.libs.gap.element.GapElement_RecordIterator
    Bases: object

    Iterator for \texttt{GapElement_Record}.

    Since Cython does not support generators yet, we implement the older iterator specification with this auxiliary class.

    **INPUT:**

    - \texttt{rec} – the \texttt{GapElement_Record} to iterate over.

    **EXAMPLES:**

```python
sage: rec = libgap.eval('rec(a:=123, b:=456)')
sage: sorted(rec)
[('a', 123), ('b', 456)]
sage: dict(rec)
{'a': 123, 'b': 456}
```

```python
class sage.libs.gap.element.GapElement_Ring
    Bases: sage.libs.gap.element.GapElement

    Derived class of \texttt{GapElement} for GAP rings (parents of ring elements).

    **EXAMPLES:**

```python
sage: i = libgap(ZZ)
sage: type(i)
<type 'sage.libs.gap.element.GapElement_Ring'>
```

```python
ring_cyclotomic()

Construct an integer ring.

**EXAMPLES:**

```python
sage: libgap.CyclotomicField(6).ring_cyclotomic()
Cyclotomic Field of order 3 and degree 2
```
ring_finite_field(var='a')

Construct an integer ring.

EXAMPLES:

```
sage: libgap.GF(3,2).ring_finite_field(var='A')
Finite Field in A of size 3^2
```

ring_integer()

Construct the Sage integers.

EXAMPLES:

```
sage: libgap.eval('Integers').ring_integer()
Integer Ring
```

ring_integer_mod()

Construct a Sage integer mod ring.

EXAMPLES:

```
sage: libgap.eval('ZmodnZ(15)').ring_integer_mod()
Ring of integers modulo 15
```

ring_polynomial()  

Construct a polynomial ring.

EXAMPLES:

```
sage: B = libgap(QQ['x'])
sage: B.ring_polynomial()
Univariate Polynomial Ring in x over Rational Field
sage: B = libgap(ZZ['x','y'])
sage: B.ring_polynomial()
Multivariate Polynomial Ring in x, y over Integer Ring
```

ring_rational()

Construct the Sage rationals.

EXAMPLES:

```
sage: libgap.eval('Rationals').ring_rational()
Rational Field
```

sage(**kwds)

Return the Sage equivalent of the `GapElement_Ring`.

INPUT:
• **kwds – keywords that are passed on to the ring__ method.

OUTPUT:
A Sage ring.

EXAMPLES:

```
sage: libgap.eval('Integers').sage()
Integer Ring
```
sage: libgap.eval('Rationals').sage()
Rational Field

sage: libgap.eval('ZmodnZ(15)').sage()
Ring of integers modulo 15

sage: libgap.GF(3,2).sage(var='A')
Finite Field in A of size 3^2

sage: libgap.CyclotomicField(6).sage()
Cyclotomic Field of order 3 and degree 2

sage: libgap(QQ['x','y']).sage()
Multivariate Polynomial Ring in x, y over Rational Field

class sage.libs.gap.element.GapElement_String

    Bases: sage.libs.gap.element.GapElement

    Derived class of GapElement for GAP strings.

    EXAMPLES:

    sage: s = libgap('string')
sage: type(s)
<type 'sage.libs.gap.element.GapElement_String'>
sage: s
"string"
sage: print(s)
string

sage()  

    Convert this GapElement_String to a Python string.

    OUTPUT:

    A Python string.

    EXAMPLES:

    sage: s = libgap.eval(' "string" '); s
"string"
sage: type(_)
<type 'sage.libs.gap.element.GapElement_String'>
sage: str(s)
'string'
sage: s.sage()
'string'
sage: type(_)
<type 'str'>
9.8 LibGAP Workspace Support

The single purpose of this module is to provide the location of the libgap saved workspace and a time stamp to invalidate saved workspaces.

\begin{verbatim}
sage.libs.gap.saved_workspace.timestamp()
\end{verbatim}

Return a time stamp for (lib)gap

\textbf{OUTPUT:}

Float. Unix timestamp of the most recently changed GAP/LibGAP file(s). In particular, the timestamp increases whenever a gap package is added.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.gap.saved_workspace import timestamp
sage: timestamp()
1406642467.25684
sage: type(timestamp())
<... 'float'>
\end{verbatim}

\begin{verbatim}
sage.libs.gap.saved_workspace.workspace(name='workspace')
\end{verbatim}

Return the filename of the gap workspace and whether it is up to date.

\textbf{INPUT:}

- name – string. A name that will become part of the workspace filename.

\textbf{OUTPUT:}

Pair consisting of a string and a boolean. The string is the filename of the saved libgap workspace (or that it should have if it doesn’t exist). The boolean is whether the workspace is up-to-date. You may use the workspace file only if the boolean is True.

\textbf{EXAMPLES:}

\begin{verbatim}
sage: from sage.libs.gap.saved_workspace import workspace
sage: ws, up_to_date = workspace()
sage: ws
'/.../gap/libgap-workspace-...'
sage: isinstance(up_to_date, bool)
True
\end{verbatim}
10.1 Interface between flint matrices and linbox

This module only contains C++ code (and the interface is fully C compatible). It basically contains what used to be in the LinBox source code under interfaces/sage/linbox-sage.C written by M. Albrecht and C. Pernet. The functions available are:

• void linbox_fmpz_mat_mul(fmpz_mat_t C, fmpz_mat_t A, fmpz_mat_t B): set C to be the result of the multiplication $A \times B$
• void linbox_fmpz_mat_charpoly(fmpz_poly_t cp, fmpz_mat_t A): set cp to be the characteristic polynomial of the square matrix $A$
• void linbox_fmpz_mat_minpoly(fmpz_poly_t mp, fmpz_mat_t A): set mp to be the minimal polynomial of the square matrix $A$
• size_t linbox_fmpz_mat_rank(fmpz_mat_t A): return the rank of the matrix $A$
• void linbox_fmpz_mat_det(fmpz_t det, fmpz_mat_t A): set det to the determinant of the square matrix $A$
11.1 An interface to Anders Buch’s Littlewood-Richardson Calculator lrcalc

The “Littlewood-Richardson Calculator” is a C library for fast computation of Littlewood-Richardson (LR) coefficients and products of Schubert polynomials. It handles single LR coefficients, products of and coproducts of Schur functions, skew Schur functions, and fusion products. All of the above are achieved by counting LR (skew)-tableaux (also called Yamanouchi (skew)-tableaux) of appropriate shape and content by iterating through them. Additionally, lrcalc handles products of Schubert polynomials.

The web page of lrcalc is http://sites.math.rutgers.edu/~asbuch/lrcalc/.

The following describes the Sage interface to this library.

EXAMPLES:

```sage
import sage.libs.lrcalc.lrcalc as lrcalc

sage: import sage.libs.lrcalc.lrcalc as lrcalc

Compute a single Littlewood-Richardson coefficient:

sage: lrcalc.lrcoef([3,2,1],[2,1],[2,1])
2

Compute a product of Schur functions; return the coefficients in the Schur expansion:

sage: lrcalc.mult([2,1], [2,1])
{{[2, 2, 1, 1]: 1, [2, 2, 2]: 1, [3, 1, 1, 1]: 1, [3, 2, 1]: 2, [3, 3]: 1, [4, 1, 1]: 1, [4, 2]: 1}}

Same product, but include only partitions with at most 3 rows. This corresponds to computing in the representation ring of \( \text{gl}(3) \):

sage: lrcalc.mult([2,1], [2,1], 3)
{{[2, 2, 2]: 1, [3, 2, 1]: 2, [3, 3]: 1, [4, 1, 1]: 1, [4, 2]: 1}}

We can also compute the fusion product, here for \( \text{sl}(3) \) and level 2:

sage: lrcalc.mult([3,2,1], [3,2,1], 3,2)
{{[4, 4, 4]: 1, [5, 4, 3]: 1}}
```
Compute the expansion of a skew Schur function:

```sage
lrcalc.skew([3,2,1],[2,1])
```

\{[1, 1, 1]: 1, [2, 1]: 2, [3]: 1\}

Compute the coproduct of a Schur function:

```sage
lrcalc.coprod([3,2,1])
```

\{([1, 1, 1], [2, 1]): 1, ([2, 1], [2, 1]): 2, ([2, 1], [3]): 1, ([2, 1, 1], [1, 1]): 1, ([2, 1, 1], [2]): 1, ([2, 2], [1, 1]): 1, ([2, 2], [2]): 1, ([2, 2, 1], [1]): 1, ([3, 1], [2]): 1, ([3, 1], [1, 1]): 1, ([3, 1], [2]): 1, ([3, 2], [1]): 1, ([3, 2, 1], []): 1\}

Multiply two Schubert polynomials:

```sage
lrcalc.mult_schubert([4,2,1,3], [1,4,2,5,3])
```

\{[4, 5, 1, 3, 2]: 1, [5, 3, 1, 4, 2]: 1, [5, 4, 1, 2, 3]: 1, [6, 2, 1, 4, 3, 5]: 1\}

Same product, but include only permutations of 5 elements in the result. This corresponds to computing in the cohomology ring of Fl(5):

```sage
lrcalc.mult_schubert([4,2,1,3], [1,4,2,5,3], 5)
```

\{[4, 5, 1, 3, 2]: 1, [5, 3, 1, 4, 2]: 1, [5, 4, 1, 2, 3]: 1\}

List all Littlewood-Richardson tableaux of skew shape $\mu/\nu$; in this example $\mu = [3, 2, 1]$ and $\nu = [2, 1]$. Specifying a third entry `maxrows` restricts the alphabet to $\{1, 2, \ldots, \text{maxrows}\}$:

```sage
list(lrcalc.lrskew([3,2,1],[2,1]))
```

```
[[[None, None, 1], [None, 1], [1]], [[None, None, 1], [None, 1], [2]],
 [[None, None, 1], [None, 2], [1]], [[None, None, 1], [None, 2], [3]]]
```

```sage
list(lrcalc.lrskew([3,2,1],[2,1],maxrows=2))
```

```
[[[None, None, 1], [None, 1], [1]], [[None, None, 1], [None, 1], [2]], [[None, None, 1], [None, 2], [1]]]
```

Todo: use this library in the `SymmetricFunctions` code, to make it easy to apply it to linear combinations of Schur functions.

See also:

- `lrcoef`
- `mult`
- `coprod`
• *skew()
• *lrskew()
• *mult_schubert()

Underlying algorithmic in lrcalc

Here is some additional information regarding the main low-level C-functions in *lrcalc*. Given two partitions *outer* and *inner* with *inner* contained in *outer*, the function:

```c
skewtab *st_new(vector *outer, vector *inner, vector *conts, int maxrows)
```

constructs and returns the (lexicographically) first LR skew tableau of shape *outer / inner*. Further restrictions can be imposed using *conts* and *maxrows*.

Namely, the integer *maxrows* is a bound on the integers that can be put in the tableau. The name is chosen because this will limit the partitions in the output of *skew()* or *mult()* to partitions with at most this number of rows.

The vector *conts* is the content of an empty tableau(!). More precisely, this vector is added to the usual content of a tableau whenever the content is needed. This affects which tableaux are considered LR tableaux (see *mult()* below). *conts* may also be the NULL pointer, in which case nothing is added.

The other function:

```c
int *st_next(skewtab *st)
```

computes in place the (lexicographically) next skew tableau with the same constraints, or returns 0 if *st* is the last one.

For a first example, see the *skew()* function code in the *lrcalc* source code. We want to compute a skew Schur function, so create a skew LR tableau of the appropriate shape with *st_new* (with *conts* = NULL), then iterate through all the LR tableaux with *st_next()*.

For each skew tableau, we use that *st->conts* is the content of the skew tableau, find this shape in the res hash table and add one to the value.

For a second example, see *mult(vector *sh1, vector *sh2, maxrows)*. Here we call *st_new()* with the shape *sh1 / (0)* and use *sh2* as the *conts* argument. The effect of using *sh2* in this way is that *st_next* will iterate through semistandard tableaux *T* of shape *sh1* such that the following tableau:

```
111111
222222    --- minimal tableau of shape *sh2*
333
*****
**T**
****
**
```

is a LR skew tableau, and *st->conts* contains the content of the combined tableaux.

More generally, *st_new(outer, inner, conts, maxrows)* and *st_next* can be used to compute the Schur expansion of the product $S_{outer/inner} \ast S_{conts}$, restricted to partitions with at most *maxrows* rows.

AUTHORS:

• Mike Hansen (2010): core of the interface
• Anne Schilling, Nicolas M. Thiéry, and Anders Buch (2011): fusion product, iterating through LR tableaux, finalization, documentation

11.1. An interface to Anders Buch’s Littlewood-Richardson Calculator *lrcalc*
sage.libs.lrcalc.lrcalc.coprod (part, all=0)
Compute the coproduct of a Schur function.

Return a linear combination of pairs of partitions representing the coproduct of the Schur function given by the partition part.

INPUT:
- part – a partition.
- all – an integer.

If all is non-zero then all terms are included in the result. If all is zero, then only pairs of partitions (part1, part2) for which the weight of part1 is greater than or equal to the weight of part2 are included; the rest of the coefficients are redundant because Littlewood-Richardson coefficients are symmetric.

EXAMPLES:

```python
sage: from sage.libs.lrcalc.lrcalc import coprod
sage: sorted(coprod([2,1]).items())
[(([1, 1], [1]), 1), ([2], [1]), ([2, 1], [])]
```

sage.libs.lrcalc.lrcalc.lrcoef (outer, inner1, inner2)
Compute a single Littlewood-Richardson coefficient.

Return the coefficient of outer in the product of the Schur functions indexed by inner1 and inner2.

INPUT:
- outer – a partition (weakly decreasing list of non-negative integers).
- inner1 – a partition.
- inner2 – a partition.

Note: This function converts its inputs into Partition()’s. If you don’t need these checks and your inputs are valid, then you can use lrcoef_unsafe().

EXAMPLES:

```python
sage: from sage.libs.lrcalc.lrcalc import lrcoef
sage: lrcoef([3,2,1], [2,1], [2,1])
2
sage: lrcoef([3,3], [2,1], [2,1])
1
sage: lrcoef([2,1,1,1,1], [2,1], [2,1])
0
```

sage.libs.lrcalc.lrcalc.lrcoef_unsafe (outer, inner1, inner2)
Compute a single Littlewood-Richardson coefficient.

Return the coefficient of outer in the product of the Schur functions indexed by inner1 and inner2.

INPUT:
- outer – a partition (weakly decreasing list of non-negative integers).
- inner1 – a partition.
- inner2 – a partition.
**Warning:** This function does not do any check on its input. If you want to use a safer version, use \texttt{lrcoef()}.

EXAMPLES:

```python
sage: from sage.libs.lrcalc.lrcalc import lrcoef_unsafe
sage: lrcoef_unsafe([3,2,1], [2,1], [2,1])
2
sage: lrcoef_unsafe([3,3], [2,1], [2,1])
1
sage: lrcoef_unsafe([2,1,1,1,1], [2,1], [2,1])
0
```

```python
sage.libs.lrcalc.lrcalc.lrskew(outer, inner, weight=None, maxrows=0)
```

Iterate over the skew LR tableaux of shape \texttt{outer / inner}.

INPUT:

- \texttt{outer} – a partition
- \texttt{inner} – a partition
- \texttt{weight} – a partition (optional)
- \texttt{maxrows} – an integer (optional)

OUTPUT: an iterator of \texttt{SkewTableau}

Specifying \texttt{maxrows} restricts the alphabet to \{1,2,...,\texttt{maxrows}\}.

Specifying \texttt{weight} returns only those tableaux of given content/weight.

EXAMPLES:

```python
sage: from sage.libs.lrcalc.lrcalc import lrskew
sage: for st in lrskew([3,2,1],[2]):
    ....:     st.pp()
    . . 1
    1 1
    2
    . . 1
    1 2
    2
    . . 1
    1 2
    3

sage: for st in lrskew([3,2,1],[2], maxrows=2):
    ....:     st.pp()
    . . 1
    1 1
    2
    . . 1
    1 2
    2

sage: list(lrskew([3,2,1],[2], weight=[3,1]))
[[[None, None, 1], [1, 1], [2]]]
```

11.1. An interface to Anders Buch’s Littlewood-Richardson Calculator \texttt{lrcalc}
**Sage 9.2 Reference Manual: C/C++ Library Interfaces, Release 9.2**

sage.libs.lrcalc.lrcalc.mult(part1, part2, maxrows=None, level=None, quantum=None)

Compute a product of two Schur functions.

Return the product of the Schur functions indexed by the partitions `part1` and `part2`.

**INPUT:**

- `part1` – a partition
- `part2` – a partition
- `maxrows` – (optional) an integer
- `level` – (optional) an integer
- `quantum` – (optional) an element of a ring

If `maxrows` is specified, then only partitions with at most this number of rows are included in the result.

If both `maxrows` and `level` are specified, then the function calculates the fusion product for \(\mathfrak{sl}(\text{maxrows})\) of the given level.

If `quantum` is set, then this returns the product in the quantum cohomology ring of the Grassmannian. In particular, both `maxrows` and `level` need to be specified.

**EXAMPLES:**

```python
sage: from sage.libs.lrcalc.lrcalc import mult
sage: mult([2],[])   # doctest: +NORMALIZE_WHITESPACE
{(2): 1}

sage: sorted(mult([2],[2]).items())

sage: sorted(mult([2,1],[2,1]).items())

sage: sorted(mult([2,1],[2,1],maxrows=2).items())

sage: mult([2,1],[2,1], 2, 2, quantum=q)

sage: mult([2,1],[2,1], quantum=q)
```

The quantum product:

```python
sage: q = polygen(QQ, 'q')
sage: sorted(mult([1],[2,1], 2, 2, quantum=q).items())

sage: sorted(mult([2,1],[2,1], 2, 2, quantum=q).items())

sage: sorted(mult([2,1],[2,1], quantum=q).items())

sage: mult([2,1],[2,1], quantum=q)
```

```python
Traceback (most recent call last):
...
ValueError: maxrows needs to be specified if you specify the level
```

**sage.libs.lrcalc.lrcalc.mult_schubert(w1, w2, rank=0)**

Compute a product of two Schubert polynomials.
Return a linear combination of permutations representing the product of the Schubert polynomials indexed by the permutations \( w_1 \) and \( w_2 \).

**INPUT:**
- \( w_1 \) – a permutation.
- \( w_2 \) – a permutation.
- \( \text{rank} \) – an integer.

If \( \text{rank} \) is non-zero, then only permutations from the symmetric group \( S(\text{rank}) \) are included in the result.

**EXAMPLES:**

```python
sage: from sage.libs.lrcalc.lrcalc import mult_schubert
sage: result = mult_schubert([3, 1, 5, 2, 4], [3, 5, 2, 1, 4])
sage: sorted(result.items())
[[(5, 4, 6, 1, 2, 3), 1], [(5, 6, 3, 1, 2, 4), 1],
 [(5, 7, 2, 1, 3, 4, 6), 1], [(6, 3, 5, 1, 2, 4), 1],
 [(6, 4, 3, 1, 2, 5), 1], [(6, 5, 2, 1, 3, 4), 1],
 [(7, 3, 4, 1, 2, 5, 6), 1], [(7, 4, 2, 1, 3, 5, 6), 1]]
```

**sage.libs.lrcalc.lrcalc.skew**(outer, inner, maxrows=0)

Compute the Schur expansion of a skew Schur function.

Return a linear combination of partitions representing the Schur function of the skew Young diagram \( \text{outer} / \text{inner} \), consisting of boxes in the partition \( \text{outer} \) that are not in \( \text{inner} \).

**INPUT:**
- \( \text{outer} \) – a partition.
- \( \text{inner} \) – a partition.
- \( \text{maxrows} \) – an integer or None.

If \( \text{maxrows} \) is specified, then only partitions with at most this number of rows are included in the result.

**EXAMPLES:**

```python
sage: from sage.libs.lrcalc.lrcalc import skew
sage: sorted(skew([2,1],[1]).items())
[[(1, 1), 1], [(2), 1]]
```

**sage.libs.lrcalc.lrcalc.test_iterable_to_vector**(it)

A wrapper function for the cdef function iterable_to_vector and vector_to_list, to test that they are working correctly.

**EXAMPLES:**

```python
sage: from sage.libs.lrcalc.lrcalc import test_iterable_to_vector
sage: x = test_iterable_to_vector([3,2,1]); x
[3, 2, 1]
```

**sage.libs.lrcalc.lrcalc.test_skewtab_to_SkewTableau**(outer, inner)

A wrapper function for the cdef function skewtab_to_SkewTableau for testing purposes.

It constructs the first LR skew tableau of shape \( \text{outer/inner} \) as an lrcalcskewtab, and converts it to a SkewTableau.

**EXAMPLES:**
from sage.libs.lrcalc.lrcalc import test_skewtab_to_SkewTableau

test_skewtab_to_SkewTableau([3,2,1],[])
[[1, 1, 1], [2, 2], [3]]

test_skewtab_to_SkewTableau([4,3,2,1],[1,1]).pp()
. 1 1 1
. 2 2
1 3
2
12.1 Utilities for Sage-mpmath interaction

Also patches some mpmath functions for speed

\texttt{sage.libs.mpmath.utils.bitcount(n)}

Bitcount of a Sage Integer or Python int/long.

\textbf{EXAMPLES}:

\begin{verbatim}
sage: from mpmath.libmp import bitcount
sage: bitcount(0)
0
sage: bitcount(1)
1
sage: bitcount(100)
7
sage: bitcount(-100)
7
sage: bitcount(2r)
2
sage: bitcount(2L)
2
\end{verbatim}

\texttt{sage.libs.mpmath.utils.call(func, *args, **kwargs)}

Call an mpmath function with Sage objects as inputs and convert the result back to a Sage real or complex number.

By default, a RealNumber or ComplexNumber with the current working precision of mpmath (mpmath.mp.prec) will be returned.

If \texttt{prec=n} is passed among the keyword arguments, the temporary working precision will be set to \texttt{n} and the result will also have this precision.

If \texttt{parent=P} is passed, \texttt{P.prec()} will be used as working precision and the result will be coerced to \texttt{P} (or the corresponding complex field if necessary).

Arguments should be Sage objects that can be coerced into RealField or ComplexField elements. Arguments may also be tuples, lists or dicts (which are converted recursively), or any type that mpmath understands natively (e.g. Python floats, strings for options).

\textbf{EXAMPLES}:

\begin{verbatim}
sage: import sage.libs.mpmath.all as a
sage: a.mp.prec = 53
\end{verbatim}
sage: a.call(a.erf, 3+4*I)
-120.186991395079 - 27.7503372936239*I
sage: a.call(a.polylog, 2, 1/3+4/5*I)
0.153548951541433 + 0.87511442499637*I
sage: a.call(a.barnesg, 3+4*I)
-0.000676375932234244 - 0.000042236140124728*I
sage: a.call(a.barnesg, -4)
0.000000000000000
sage: a.call(a.hyper, [2, 3], [4, 5], 1/3)
1.0703578162508
sage: a.call(a.hyper, [2, 3], [4, 2, 3], 1/3)
1.95762943509305
sage: a.call(a.quad, a.erf, [0, 1])
0.486064958112256
sage: a.call(a.gammainc, 3+4*I, 2/3, 1-pi*I, prec=100)
-274.188711307771 + 101.59521032382593402947725236*I
sage: x = (3+4*I).n(100)
sage: y = (2/3).n(100)
sage: z = (1-pi*I).n(100)
sage: a.call(a.gammainc, x, y, z, prec=100)
-274.188711307771 + 101.59521032382593402947725236*I
sage: a.call(a.erf, infinity)
1.00000000000000
sage: a.call(a.erf, -infinity)
-1.00000000000000
sage: a.call(a.gamma, infinity)
+infinity
sage: a.call(a.polylog, 2, 1/2, parent=RR)
0.582240526465012
sage: a.call(a.polylog, 2, 2, parent=RR)
2.46740110027234 - 2.17758609030360*I
sage: a.call(a.polylog, 2, 1/2, parent=RealField(100))
0.5822405264650125059026532016
sage: a.call(a.polylog, 2, 2, parent=RealField(100))
2.4674011002723396547086227500 - 2.177586090303602130506888982*I
sage: a.call(a.polylog, 2, 1/2, parent=CC)
0.582240526465012
sage: type(_)
<type 'sage.rings.complex_number.ComplexNumber'>
sage: a.call(a.polylog, 2, 1/2, parent=RDF)
0.5822405264650125
sage: type(_)
<type 'sage.rings.real_double.RealDoubleElement'>

Check that trac ticket #11885 is fixed:
sage: a.call(a.ei, 1.0r, parent=float)
1.895117816355936

Check that trac ticket #14984 is fixed:
sage: a.call(a.log, -1.0r, parent=float)
3.141592653589793j

sage.libs.mpmath.utils.from_man_exp(man, exp, prec=0, rnd='d')
Create normalized mpf value tuple from mantissa and exponent.

With prec > 0, rounds the result in the desired direction if necessary.
EXAMPLES:

```python
sage: from mpmath.libmp import from_man_exp
sage: from_man_exp(-6, -1)
(1, 3, 0, 2)
sage: from_man_exp(-6, -1, 1, 'd')
(1, 1, 1, 1)
sage: from_man_exp(-6, -1, 1, 'u')
(1, 1, 2, 1)
```

`sage.libs.mpmath.utils.isqrt(n)`

Square root (rounded to floor) of a Sage Integer or Python int/long. The result is a Sage Integer.

EXAMPLES:

```python
sage: from mpmath.libmp import isqrt
sage: isqrt(0)
0
sage: isqrt(100)
10
sage: isqrt(10)
3
sage: isqrt(10r)
3
sage: isqrt(10L)
3
```

`sage.libs.mpmath.utils.mpmath_to_sage(x, prec)`

Convert any mpmath number (mpf or mpc) to a Sage RealNumber or ComplexNumber of the given precision.

EXAMPLES:

```python
sage: import sage.libs.mpmath.all as a
sage: a.mpmath_to_sage(a.mpf('2.5'), 53)
2.50000000000000
sage: a.mpmath_to_sage(a.mpc('2.5','-3.5'), 53)
2.50000000000000 - 3.50000000000000*I
sage: a.mpmath_to_sage(a.mpf('inf'), 53)
+infinity
sage: a.mpmath_to_sage(a.mpf('-inf'), 53)
-infinity
sage: a.mpmath_to_sage(a.mpf('nan'), 53)
NaN
sage: a.mpmath_to_sage(a.mpf('0'), 53)
0.000000000000000
```

A real example:

```python
sage: RealField(100)(pi)
3.1415926535897932384626433833
sage: t = RealField(100)(pi)._mpmath_; t
mpf('3.1415926535897932384626433833')
sage: a.mpmath_to_sage(t, 100)
3.1415926535897932384626433833
```

We can ask for more precision, but the result is undefined:
 sage: a.mpmath_to_sage(t, 140) # random
3.141592653589793238462643383279333156440
 sage: ComplexField(140)(pi)
3.1415926535897932384626433832795028841972

A complex example:

 sage: ComplexField(100)([0, pi])
3.1415926535897932384626433832795028841972*I
 sage: t = ComplexField(100)([0, pi])._mpmath_(); t
mpc(real='0.0', imag='3.1415926535897932')
 sage: sage.libs.mpmath.all.mpmath_to_sage(t, 100)
3.1415926535897932384626433833*I

Again, we can ask for more precision, but the result is undefined:

 sage: sage.libs.mpmath.all.mpmath_to_sage(t, 140) # random
3.1415926535897932384626433832793333156440*I
 sage: ComplexField(140)([0, pi])
3.1415926535897932384626433832795028841972*I

sage.libs.mpmath.utils.normalize(sign, man, exp, bc, prec, rnd)
Create normalized mpf value tuple from full list of components.

EXAMPLES:

 sage: from mpmath.libmp import normalize
 sage: normalize(0, 4, 5, 3, 53, 'n')
(0, 1, 7, 1)

sage.libs.mpmath.utils.sage_to_mpmath(x, prec)
Convert any Sage number that can be coerced into a RealNumber or ComplexNumber of the given precision into an mpmath mpf or mpc. Integers are currently converted to int.

Lists, tuples and dicts passed as input are converted recursively.

EXAMPLES:

 sage: import sage.libs.mpmath.all as a
 sage: a.mp.dps = 15
 sage: print(a.sage_to_mpmath(2/3, 53))
0.666666666666667
 sage: print(a.sage_to_mpmath(2./3, 53))
0.666666666666667
 sage: print(a.sage_to_mpmath(3+4*I, 53))
(3.0 + 4.0j)
 sage: print(a.sage_to_mpmath(1+pi, 53))
4.14159265358979
 sage: a.sage_to_mpmath(infinity, 53)
mpf('+inf')
 sage: a.sage_to_mpmath(-infinity, 53)
mpf('-inf')
 sage: a.sage_to_mpmath(NaN, 53)
mpf('nan')
 sage: a.sage_to_mpmath(0, 53)
0
 sage: a.sage_to_mpmath([0.5, 1.5], 53)
[mpf('0.5'), mpf('1.5')]
sage: a.sage_to_mpmath((0.5, 1.5), 53)
(mpf('0.5'), mpf('1.5'))
sage: a.sage_to_mpmath({'n':0.5}, 53)
{'n': mpf('0.5')}
13.1 Victor Shoup’s NTL C++ Library

Sage provides an interface to Victor Shoup’s C++ library NTL. Features of this library include incredibly fast arithmetic with polynomials and asymptotically fast factorization of polynomials.
14.1 Interface between Sage and PARI

14.1.1 Guide to real precision in the PARI interface

In the PARI interface, “real precision” refers to the precision of real numbers, so it is the floating-point precision. This is a non-trivial issue, since there are various interfaces for different things.

Internal representation and conversion between Sage and PARI

Real numbers in PARI have a precision associated to them, which is always a multiple of the CPU wordsize. So, it is a multiple of 32 or 64 bits. When converting from Sage to PARI, the precision is rounded up to the nearest multiple of the wordsize:

```
sage: x = 1.0
sage: x.precision()
53
sage: pari(x)
1.00000000000000
sage: pari(x).bitprecision()
64
```

With a higher precision:

```
sage: x = RealField(100).pi()
\n\n```

When converting back to Sage, the precision from PARI is taken:

```
sage: x = RealField(100).pi()
sage: y = pari(x).sage()
sage: y
3.1415926535897932384626433832793333156
sage: parent(y)
Real Field with 128 bits of precision
```

So \texttt{pari(x).sage()} is definitely not equal to \texttt{x} since it has 28 bogus bits.

Therefore, some care must be taken when juggling reals back and forth between Sage and PARI. The correct way of avoiding this is to convert \texttt{pari(x).sage()} back into a domain with the right precision. This has to be done by
the user (or by Sage functions that use PARI library functions). For instance, if we want to use the PARI library to compute \( \sqrt{\pi} \) with a precision of 100 bits:

```python
sage: R = RealField(100)
sage: s = R(pi); s
3.1415926535897932384626433833
sage: p = pari(s).sqrt()
sage: x = p.sage(); x  # wow, more digits than I expected!
1.77245385090551602729174833410973484
sage: x.prec()  # has precision 'improved' from 100 to 128?
128
sage: x == RealField(128)(pi).sqrt()  # sadly, no!
False
sage: R(x)  # x should be brought back to precision 100
1.772453850905516027291674833
sage: R(x) == s.sqrt()  # True
True
```

### Output precision for printing

Even though PARI reals have a precision, not all significant bits are printed by default. The maximum number of digits when printing a PARI real can be set using the methods `Pari.set_real_precision_bits()` or `Pari.set_real_precision()`.

We create a very precise approximation of \( \pi \) and see how it is printed in PARI:

```python
sage: pi = pari(RealField(1000).pi())
```

The default precision is 15 digits:

```python
sage: pi
3.14159265358979
```

With a different precision:

```python
sage: _ = pari.set_real_precision(50)
sage: pi
3.1415926535897932384626433832795028841971693993751
```

Back to the default:

```python
sage: _ = pari.set_real_precision(15)
sage: pi
3.14159265358979
```

### Input precision for function calls

When we talk about precision for PARI functions, we need to distinguish three kinds of calls:

1. Using the string interface, for example `pari("\sin(1)")`.
2. Using the library interface with exact inputs, for example `pari(1).sin()`.
3. Using the library interface with inexact inputs, for example `pari(1.0).sin()`.

In the first case, the relevant precision is the one set by the methods `Pari.set_real_precision_bits()` or `Pari.set_real_precision()`:
In the second case, the precision can be given as the argument `precision` in the function call, with a default of 53 bits. The real precision set by `Pari.set_real_precision_bits()` or `Pari.set_real_precision()` is irrelevant.

In these examples, we convert to Sage to ensure that PARI’s real precision is not used when printing the numbers. As explained before, this artificially increases the precision to a multiple of the wordsize.

```
sage: s = pari(1).sin(precision=180).sage(); print(s); print(parent(s))
0.841470984807896506652502321630298999622563061
Real Field with 192 bits of precision
sage: s = pari(1).sin(precision=40).sage(); print(s); print(parent(s))
0.841470984807896507
Real Field with 64 bits of precision
sage: s = pari(1).sin().sage(); print(s); print(parent(s))
0.841470984807896507
Real Field with 64 bits of precision
```

In the third case, the precision is determined only by the inexact inputs and the `precision` argument is ignored:

```
sage: pari(1.0).sin(precision=180).sage()
0.841470984807896507
sage: pari(1.0).sin(precision=40).sage()
0.841470984807896507
sage: pari(RealField(100).one()).sin().sage()
0.84147098480789650665250232163029899962
```

**Elliptic curve functions**

An elliptic curve given with exact \( a \)-invariants is considered an exact object. Therefore, you should set the precision for each method call individually:

```
sage: e = pari([0,0,0,-82,0]).ellinit()
sage: etal = e.elleta(precision=100)[0]
sage: etal.sage()
3.6054636014326520859158205642077267748
sage: etal = e.elleta(precision=180)[0]
sage: etal.sage()
3.60546360143265208591582056420772677481026899659802474544
```
14.2 Convert PARI objects to Sage types

`sage.libs.pari.convert_sage.gen_to_sage(z, locals=None)`
Convert a PARI gen to a Sage/Python object.

**INPUT:**
- `z` – PARI gen
- `locals` – optional dictionary used in fallback cases that involve `sage_eval()`

**OUTPUT:**
One of the following depending on the PARI type of `z`:
- a `Integer` if `z` is an integer (type `t_INT`)
- a `Rational` if `z` is a rational (type `t_FRAC`)
- a `RealNumber` if `z` is a real number (type `t_REAL`). The precision will be equivalent.
- a `NumberFieldElement_quadratic` or a `ComplexNumber` if `z` is a complex number (type `t_COMPLEX`). The former is used when the real and imaginary parts are integers or rationals and the latter when they are floating point numbers. In that case The precision will be the maximal precision of the real and imaginary parts.
- a Python list if `z` is a vector or a list (type `t_VEC, t_COL`)
- a Python string if `z` is a string (type `t_STR`)
- a Python list of Python integers if `z` is a small vector (type `t_VECSMALL`)
- a matrix if `z` is a matrix (type `t_MAT`)
- a padic element (type `t_PADIC`)
- a `Infinity` if `z` is an infinity (type `t_INF`)

**EXAMPLES:**

```python
sage: from sage.libs.pari.convert_sage import gen_to_sage
```

Converting an integer:

```python
sage: z = pari('12'); z
12
sage: z.type()
't_INT'
```

```python
sage: a = gen_to_sage(z); a
12
sage: a.parent()
Integer Ring
```

```python
sage: gen_to_sage(pari('7^42'))
311973482284542371301330321821976049
```

Converting a rational number:

```python
sage: z = pari('389/17'); z
389/17
sage: z.type()
't_FRAC'
```

(continues on next page)
Converting a real number:

```python
sage: pari.set_real_precision(70)
15
sage: z = pari('1.234'); z
1.234000000000000000000000000000000000000000000000000000000000000000000000000
sage: a = gen_to_sage(z); a
1.234000000000000000000000000000000000000000000000000000000000000000000000000
sage: a.parent()
Real Field with 256 bits of precision
```

For complex numbers, the parent depends on the PARI type:

```python
sage: z = pari('(3+I)'); z
3 + I
sage: z.type()
't_COMPLEX'
sage: a = gen_to_sage(z); a
i + 3
sage: a.parent()
Number Field in i with defining polynomial x^2 + 1 with i = 1*I

sage: z = pari('(3+I)/2'); z
3/2 + 1/2*I
sage: a = gen_to_sage(z); a
1/2*i + 3/2
sage: a.parent()
Number Field in i with defining polynomial x^2 + 1 with i = 1*I

sage: z = pari('1.0 + 2.0*I'); z
1.00000000000000 + 2.00000000000000*I
sage: a = gen_to_sage(z); a
1.00000000000000 + 2.00000000000000*I
sage: a.parent()
Complex Field with 64 bits of precision
```

Converting polynomials:

```python
sage: f = pari('(2/3)*x^3 + x - 5/7 + y')
sage: f.type()
't_POL'
sage: R.<x,y> = QQ[]
```

Converting a real number:
sage: gen_to_sage(f, {'x': x, 'y': y})
2/3*x^3 + x + y - 5/7
sage: parent(gen_to_sage(f, {'x': x, 'y': y}))
Multivariate Polynomial Ring in x, y over Rational Field
sage: x, y = SR.var('x,y')
sage: gen_to_sage(f, {'x': x, 'y': y})
2/3*x^3 + x + y - 5/7
sage: parent(gen_to_sage(f, {'x': x, 'y': y}))
Symbolic Ring

traceback (most recent call last):
... NameError: name 'x' is not defined

Converting vectors:

sage: z1 = pari('[-3, 2.1, 1+I]'); z1
[-3, 2.10000000000000, 1 + I]
sage: z2 = pari('[[1.0*I, [1,2]]~]'); z2
[1.00000000000000*I, [1, 2]]~
sage: z1.type(), z2.type()
('t_VEC', 't_COL')
sage: a1 = gen_to_sage(z1)
sage: a2 = gen_to_sage(z2)
sage: type(a1), type(a2)
(<... 'list'>, <... 'list'>)
sage: [parent(b) for b in a1]
[Integer Ring,
 Number Field in i with defining polynomial x^2 + 1 with i = 1*I]
sage: [parent(b) for b in a2]
[Complex Field with 64 bits of precision, <... 'list'>]
sage: z = pari('Vecsmall([1,2,3,4])')
sage: z.type()
't_VECSMALL'
sage: a = gen_to_sage(z); a
[1, 2, 3, 4]
sage: type(a)
<... 'list'>
sage: [parent(b) for b in a]
[<... 'int'>, <... 'int'>, <... 'int'>, <... 'int'>]

Matrices:

sage: z = pari('[1,2;3,4]')
sage: z.type()
't_MAT'
sage: a = gen_to_sage(z); a
[1 2]
[3 4]
sage: a.parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring

Conversion of p-adics:
Conversion of infinities:

```python
sage: gen_to_sage(pari('oo'))
+Infinity
sage: gen_to_sage(pari('-oo'))
-Infinity
```

Conversion of strings:

```python
sage: s = pari('"foo"').sage(); s
'foo'
sage: type(s)
<type 'str'>
```

### 14.3 Ring of pari objects

**AUTHORS:**

- Simon King (2011-08-24): Use UniqueRepresentation, element_class and proper initialisation of elements.

```python
class sage.rings.pari_ring.Pari(x, parent=None):
    Bases: sage.structure.element.RingElement

    Element of Pari pseudo-ring.

class sage.rings.pari_ring.PariRing
    Bases: sage.misc.fast_methods.Singleton, sage.rings.ring.Ring

    EXAMPLES:

    sage: R = PariRing(); R
    Pseudoring of all PARI objects.
    sage: loads(R.dumps()) is R
    True

    Element
    alias of Pari

    characteristic()

    is_field (proof=True)

    random_element (x=None, y=None, distribution=None)
    Return a random integer in Pari.
```

**Note:** The given arguments are passed to \texttt{ZZ.random_element(...)}.  

**INPUT:**
• $x, y$ – optional integers, that are lower and upper bound for the result. If only $x$ is provided, then the result is between 0 and $x - 1$, inclusive. If both are provided, then the result is between $x$ and $y - 1$, inclusive.

• *distribution* – optional string, so that \( \mathbb{Z} \) can make sense of it as a probability distribution.

**EXAMPLES:**

```
sage: R = PariRing()
sage: R.random_element()  # -8
sage: R.random_element(5, 13)  # 12
sage: [R.random_element(distribution="1/n") for _ in range(10)]  # [0, 1, -1, 2, 1, -95, -1, -2, -12, 0]
```

**zeta()**

Return -1.

**EXAMPLES:**

```
sage: R = PariRing()
sage: R.zeta()  # -1
```
15.1 Hyperelliptic Curve Point Finding, via ratpoints (deprecated)

This module is deprecated, use PARI instead:

```python
sage: pari(EllipticCurve("389a1").ellratpoints(4))
[-2, 0], [-2, -1], [-1, 1], [-1, -2], [0, 0], [0, -1], [1, 0], [1, -1], [3, 5], [3, -6], [4, 8], [4, -9], [-3/4, 7/8], [-3/4, -15/8]
sage: pari("[x^3 + x^2 - 2*x, 1]").hyperellratpoints(4)
[-2, 0], [-2, -1], [-1, 1], [-1, -2], [0, 0], [0, -1], [1, 0], [1, -1], [3, 5], [3, -6], [4, 8], [4, -9], [-3/4, 7/8], [-3/4, -15/8]
```

The function `sage.libs.ratpoints.ratpoints(coeffs, H, verbose=False, max=0, min_x_denom=None, max_x_denom=None, intervals=[])` can be used to access the ratpoints library to find points on the hyperelliptic curve:

\[ y^2 = a_n x^n + \cdots + a_1 x + a_0. \]

**INPUT:**
- `coeffs` – list of integer coefficients \(a_0, a_1, \ldots, a_n\)
- `H` – the bound for the denominator and the absolute value of the numerator of the \(x\)-coordinate
- `verbose` – if True, ratpoints will print comments about its progress
- `max` – maximum number of points to find (if 0, find all of them)

**OUTPUT:**
The points output by this program are points in (1, ceil(n/2), 1)-weighted projective space. If \(n\) is even, then the associated homogeneous equation is

\[ y^2 = a_n x^2 + \cdots + a_1 x^{2n-1} + a_0 z^n \]

while if \(n\) is odd, it is

\[ y^2 = a_n x^2 + \cdots + a_1 x z^{n-1} + a_0 z^{n+1}. \]

**EXAMPLES:**

```python
sage: from sage.libs.ratpoints import ratpoints
doctest:...: DeprecationWarning: the module sage.libs.ratpoints is deprecated; use pari.ellratpoints or pari.hyperellratpoints instead
See http://trac.sagemath.org/24531 for details.
sage: for x,y,z in ratpoints([1..6], 200):
    ....:     print(-1*y^2 + 1*z^6 + 2*x*z^5 + 3*x^2*z^4 + 4*x^3*z^3 + 5*x^4*z^2 +
              6*x^5*z)
    0
    0
```

(continues on next page)
sage: for x,y,z in ratpoints([1..5], 200):
    ....: print (-1*y^2 + 1*z^4 + 2*x*z^3 + 3*x^2*z^2 + 4*x^3*z + 5*x^4)

sage: for x,y,z in ratpoints([1..200], 1000):
    ....: print("{} {} ".format(x,y,z))

1 0 0
0 1 1
0 -1 1
201 -

The denominator of $x$ can be restricted, for example to find integral points:

sage: from sage.libs.ratpoints import ratpoints
sage: coeffs = [400, -112, 0, 1]
sage: ratpoints(coeffs, 10^6, max_x_denom=1, intervals=[[[-10,0],[1000,2000]]])
[(1, 0, 0), (-8, 28, 1), (-8, -28, 1), (-7, 29, 1), (-7, -29, 1),
 (-4, 28, 1), (-4, -28, 1), (0, 20, 1), (0, -20, 1), (1368, 50596, 1),
 (1368, -50596, 1), (1624, 65444, 1), (1624, -65444, 1)]
sage: ratpoints(coeffs, 1000, min_x_denom=100, max_x_denom=200)
[(1, 0, 0),
 (-656, 426316, 121),
 (-656, -426316, 121),
 (452, 85052, 121),
 (452, -85052, 121),
 (988, 80036, 121),
 (988, -80036, 121),
 (-556, 773188, 169),
 (-556, -773188, 169),
 (264, 432068, 169),
 (264, -432068, 169)]

Finding the integral points on the compact component of an elliptic curve:

sage: E = EllipticCurve([0,1,0,-35220,-1346400])
sage: e1, e2, e3 = E.division_polynomial(2).roots(multiplicities=False)
sage: coeffs = [E.a6(),E.a4(),E.a2(),1]
sage: ratpoints(coeffs, 1000, max_x_denom=1, intervals=[[e3,e2]])
[(1, 0, 0),
 (-165, 0, 1),
 (-162, 366, 1),
 (-161798, 121224055, 169),
 (-161798, -121224055, 169)]
15.1. Hyperelliptic Curve Point Finding, via ratpoints (deprecated)

\begin{verbatim}
(-162, -366, 1),
(-120, 1080, 1),
(-120, -1080, 1),
(-90, 1050, 1),
(-90, -1050, 1),
(-85, 1020, 1),
(-85, -1020, 1),
(-42, 246, 1),
(-42, -246, 1),
(-40, 0, 1)
\end{verbatim}
16.1 Readline

This is the library behind the command line input, it takes keypresses until you hit Enter and then returns it as a string to Python. We hook into it so we can make it redraw the input area.

EXAMPLES:

```python
sage: from sage.libs.readline import *
sage: replace_line('foobar', 0)
sage: set_point(3)
sage: print('current line:', repr(copy_text(0, get_end())))
current line: 'foobar'
sage: print('cursor position:', get_point())
cursor position: 3
```

When printing with `interleaved_output` the prompt and current line is removed:

```python
sage: with interleaved_output():
    ....: print('output')
    ....: print('current line: ', repr(copy_text(0, get_end())))
    ....: print('cursor position:', get_point())
output
current line: ''
cursor position: 0
```

After the interleaved output, the line and cursor is restored to the old value:

```python
sage: print('current line:', repr(copy_text(0, get_end())))
current line: 'foobar'
sage: print('cursor position:', get_point())
cursor position: 3
```

Finally, clear the current line for the remaining doctests:

```python
sage: replace_line('', 1)
```

`sage.libs.readline.clear_signals()`
Remove the readline signal handlers

Remove all of the Readline signal handlers installed by `set_signals()`

EXAMPLES:
sage: from sage.libs.readline import clear_signals
sage: clear_signals()
0

sage.libs.readline.copy_text(pos_start, pos_end)
Return a copy of the text between start and end in the current line.

INPUT:
  • pos_start, pos_end – integer. Start and end position.

OUTPUT:
String.

EXAMPLES:

sage: from sage.libs.readline import copy_text, replace_line
sage: replace_line('foobar', 0)
sage: copy_text(1, 5)
'ooba'

sage.libs.readline.forced_update_display()
Force the line to be updated and redisplayed, whether or not Readline thinks the screen display is correct.

EXAMPLES:

sage: from sage.libs.readline import forced_update_display
sage: forced_update_display()
0

sage.libs.readline.get_end()
Get the end position of the current input

OUTPUT:
Integer

EXAMPLES:

sage: from sage.libs.readline import get_end
sage: get_end()
0

sage.libs.readline.get_point()
Get the cursor position

OUTPUT:
Integer

EXAMPLES:

sage: from sage.libs.readline import get_point, set_point
sage: get_point()
0
sage: set_point(5)
sage: get_point()
5
sage: set_point(0)
sage.libs.readline.initialize()
Initialize or re-initialize Readline’s internal state. It’s not strictly necessary to call this; readline() calls it before reading any input.

EXAMPLES:

```
sage: from sage.libs.readline import initialize
sage: initialize()
```

class sage.libs.readline.interleaved_output

Bases: object

Context manager for asynchronous output

This allows you to show output while at the readline prompt. When the block is left, the prompt is restored even if it was clobbered by the output.

EXAMPLES:

```
sage: from sage.libs.readline import interleaved_output
sage: with interleaved_output():
....:     print('output')
output
```

sage.libs.readline.print_status()

Print readline status for debug purposes

EXAMPLES:

```
sage: from sage.libs.readline import print_status
sage: print_status()
catch_signals: 1
catch_sigwinch: 1
```

sage.libs.readline.replace_line(text, clear_undo)

Replace the contents of rl_line_buffer with text.

The point and mark are preserved, if possible.

INPUT:

- **text** – the new content of the line.
- **clear_undo** – integer. If non-zero, the undo list associated with the current line is cleared.

EXAMPLES:

```
sage: from sage.libs.readline import copy_text, replace_line
sage: replace_line('foobar', 0)
sage: copy_text(1, 5)
'ooba'
```

sage.libs.readline.set_point(point)

Set the cursor position

INPUT:

- **point** – integer. The new cursor position.

EXAMPLES:
sage.libs.readline.set_signals()

Install the readline signal handlers

Install Readline’s signal handler for SIGINT, SIGQUIT, SIGTERM, SIGALRM, SIGTSTP, SIGTTIN, SIGTTOU, and SIGWINCH, depending on the values of rl_catch_signals and rl_catch_sigwinch.

EXAMPLES:

```python
sage: from sage.libs.readline import set_signals
sage: set_signals()
0
```
17.1 Symmetrica library

sage.libs.symmetrica.symmetrica.bdg_symmetrica(part, perm)
Calculates the irreducible matrix representation $D^\text{part}(\text{perm})$, whose entries are of integral numbers.


sage.libs.symmetrica.symmetrica.chartafel_symmetrica(n)
you enter the degree of the symmetric group, as INTEGER object and the result is a MATRIX object: the charactertable of the symmetric group of the given degree.

EXAMPLES:

```
sage: symmetrica.chartafel(3)
[ 1 1 1]
[-1 0 2]
[ 1 -1 1]
sage: symmetrica.chartafel(4)
[ 1 1 1 1]
[-1 0 -1 1 3]
[ 0 -1 2 0 2]
[ 1 0 -1 -1 3]
[-1 1 1 -1 1]
```

sage.libs.symmetrica.symmetrica.charvalue_symmetrica(irred, cls, table=None)
you enter a PARTITION object part, labelling the irreducible character, you enter a PARTITION object class, labeling the class or class may be a PERMUTATION object, then result becomes the value of that character on that class or permutation. Note that the table may be NULL, in which case the value is computed, or it may be taken from a precalculated charactertable.

FIXME: add table parameter

EXAMPLES:

```
sage: n = 3
sage: m = matrix([[symmetrica.charvalue(irred, cls) for cls in Partitions(n)] for \n\rightarrow irred in Partitions(n)]); m
[ 1 1 1]
[-1 0 2]
[ 1 -1 1]
sage: m == symmetrica.chartafel(n)
True
sage: n = 4
sage: m = matrix([[symmetrica.charvalue(irred, cls) for cls in Partitions(n)] for \n\rightarrow irred in Partitions(n)]
```

(continues on next page)
computes the expansion of a elementary symmetric function labeled by a INTEGER number as a POLYNOM erg. The object number may also be a PARTITION or a ELM_SYM object. The INTEGER length specifies the length of the alphabet. Both routines are the same.

EXAMPLES:

```python
sage: a = symmetrica.compute_elmsym_with_alphabet(2,2); a
x0*x1
sage: a.parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring
sage: a = symmetrica.compute_elmsym_with_alphabet([2],2); a
x0*x1
sage: symmetrica.compute_elmsym_with_alphabet(3,2)
0
sage: symmetrica.compute_elmsym_with_alphabet([3,2,1],2)
0
```

computes the expansion of a homogeneous(=complete) symmetric function labeled by a INTEGER number as a POLYNOM erg. The object number may also be a PARTITION or a HOM_SYM object. The INTEGER length specifies the length of the alphabet. Both routines are the same.

EXAMPLES:

```python
sage: symmetrica.compute_homsym_with_alphabet(3,1,'x')
x^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],1,'x')
x^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],2,'x')
x0^3 + 2*x0^2*x1 + 2*x0*x1^2 + x1^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],2,'a,b')
a^3 + 2*a^2*b + 2*a*b^2 + b^3
sage: symmetrica.compute_homsym_with_alphabet([2,1],2,'x').parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring
```

computes the expansion of a monomial symmetric function labeled by a PARTITION number as a POLYNOM erg. The INTEGER length specifies the length of the alphabet.

EXAMPLES:
sage: symmetrica.compute_monomial_with_alphabet([2,1],2,'x')
x0^2*x1 + x0*x1^2
sage: symmetrica.compute_monomial_with_alphabet([1,1],2,'x')
0
sage: symmetrica.compute_monomial_with_alphabet(2,2,'x')
x0^2 + x1^2
sage: symmetrica.compute_monomial_with_alphabet(2,2,'a,b')
a^2 + b^2
sage: symmetrica.compute_monomial_with_alphabet(2,2,'x').parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring

sage.libs.symmetrica.symmetrica.compute_powsym_with_alphabet_symmetrica(n, length, alphabet='x')
computes the expansion of a power symmetric function labeled by a INTEGER label or by a PARTITION label or a POW_SYM label as a POLYNOM erg. The INTEGER length specifies the length of the alphabet.

EXAMPLES:

sage: symmetrica.compute_powsym_with_alphabet(2,2,'x')
x0^2 + x1^2
sage: symmetrica.compute_powsym_with_alphabet(2,2,'x').parent()
Multivariate Polynomial Ring in x0, x1 over Integer Ring
sage: symmetrica.compute_powsym_with_alphabet([2],2,'x')
x0^2 + x1^2
sage: symmetrica.compute_powsym_with_alphabet([2],2,'a,b')
a^2 + b^2
sage: symmetrica.compute_powsym_with_alphabet([2,1],2,'a,b')
a^3 + a^2*b + a*b^2 + b^3

sage.libs.symmetrica.symmetrica.compute_schur_with_alphabet_det_symmetrica(part, length, alphabet='x')
Computes the expansion of a schurfunction labeled by a partition PART as a POLYNOM erg. The INTEGER length specifies the length of the alphabet.

EXAMPLES:

sage: symmetrica.compute_schur_with_alphabet_det(2,2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet_det([2],2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet_det(Partition([2]),2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet_det(Partition([2]),2,'y')
y0^2 + y0*y1 + y1^2
sage: symmetrica.compute_schur_with_alphabet_det(Partition([2]),2,'a,b')
a^2 + a*b + b^2

sage.libs.symmetrica.symmetrica.compute_schur_with_alphabet_symmetrica(part, length, alphabet='x')

Comes the expansion of a schurfunction labeled by a partition PART as a POLYNOM erg. The INTEGER length specifies the length of the alphabet.
EXAMPLES:

```python
sage: symmetrica.compute_schur_with_alphabet(2,2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet([2],2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet(Partition([2]),2)
x0^2 + x0*x1 + x1^2
sage: symmetrica.compute_schur_with_alphabet(Partition([2]),2,'y')
y0^2 + y0*y1 + y1^2
sage: symmetrica.compute_schur_with_alphabet(Partition([2]),2,'a,b')
a^2 + a*b + b^2
sage: symmetrica.compute_schur_with_alphabet([2,1],1,'x')
0
```

```python
sage.libs.symmetrica.symmetrica.dimension_schur_symmetrica(s)
you enter a SCHUR object a, and the result is the dimension of the corresponding representation of the symmetric group sn.
```

```python
sage.libs.symmetrica.symmetrica.dimension_symmetrization_symmetrica(n, part)
computes the dimension of the degree of a irreducible representation of the GL_n, n is a INTEGER object, labeled by the PARTITION object a.
```

```python
sage.libs.symmetrica.symmetrica.divdiff_perm_schubert_symmetrica(perm, a)
Returns the result of applying the divided difference operator \( \delta_i \) to a where a is either a permutation or a Schubert polynomial over QQ.

EXAMPLES:

```python
sage: symmetrica.divdiff_perm_schubert([2,3,1], [3,2,1])
X[2, 1]
sage: symmetrica.divdiff_perm_schubert([3,1,2], [3,2,1])
X[1, 3, 2]
sage: symmetrica.divdiff_perm_schubert([3,2,4,1], [3,2,1])
Traceback (most recent call last):
... 
ValueError: cannot apply \( \delta_3 \) to a (= [3, 2, 1])
```

```python
sage.libs.symmetrica.symmetrica.divdiff_schubert_symmetrica(i, a)
Returns the result of applying the divided difference operator \( \delta_i \) to a where a is either a permutation or a Schubert polynomial over QQ.

EXAMPLES:

```python
sage: symmetrica.divdiff_schubert(1, [3,2,1])
X[2, 3, 1]
sage: symmetrica.divdiff_schubert(2, [3,2,1])
X[3, 1, 2]
sage: symmetrica.divdiff_schubert(3, [3,2,1])
Traceback (most recent call last):
... 
ValueError: cannot apply \( \delta_3 \) to a (= [3, 2, 1])
```

```python
sage.libs.symmetrica.symmetrica.end()
```

```python
sage.libs.symmetrica.symmetrica.gupta_nm_symmetrica(n, m)
this routine computes the number of partitions of n with maximal part m. The result is erg. The input n,m must be INTEGER objects. The result is freed first to an empty object. The result must be a different from m and n.
```
it computes the table of the above values. The entry n,m is the result of gupta_nm. mat is freed first. max must be an INTEGER object, it is the maximum weight for the partitions. max must be different from result.

computes the so called Hall Littlewood Polynomials, i.e. a SCHUR object, whose coefficient are polynomials in one variable. The method, which is used for the computation is described in the paper: A.O. Morris The Characters of the group GL(n,q) Math Zeitschr 81, 112-123 (1963)

computes the kostkanumber, i.e. the number of tableaux of given shape, which is a PARTITION object, and of given content, which also is a PARTITION object, or a VECTOR object with INTEGER entries. The result is an INTEGER object, which is freed to an empty object at the beginning. The shape could also be a SKEWPARTITION object, then we compute the number of skewtableaux of the given shape.

computes the list of tableaux of given shape and content. shape is a PARTITION object or a SKEWPARTITION object and content is a PARTITION object or a VECTOR object with INTEGER entries, the result becomes a LIST object whose entries are the computed TABLEAUX object.

Returns the table of Kostka numbers of weight n.

EXAMPLES:

\[
\begin{align*}
\text{sage: } & \text{symmetrica.kostka_number}([2,1],[1,1,1]) \\
& 2 \\
\text{sage: } & \text{symmetrica.kostka_number}([1,1,1],[1,1,1]) \\
& 1 \\
\text{sage: } & \text{symmetrica.kostka_number}([3],[1,1,1]) \\
& 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{symmetrica.kostka_tab}([3],[1,1,1]) \\
& [[[1, 2, 3]]] \\
\text{sage: } & \text{symmetrica.kostka_tab}([2,1],[1,1,1]) \\
& [[[1, 2], [3]], [[1, 3], [2]]] \\
\text{sage: } & \text{symmetrica.kostka_tab}([1,1,1],[1,1,1]) \\
& [[[1], [2], [3]]] \\
\text{sage: } & \text{symmetrica.kostka_tab}([2,2,1],[1,1,1]) \\
& [[[None, 1], [None, 2], [3]], [[None, 1], [None, 3], [2]], [[None, 2], [None, 3], [1]]] \\
\text{sage: } & \text{symmetrica.kostka_tab}([2,2],[1,1,1]) \\
& [[[None, 1], [2, 3]], [[None, 2], [1, 3]]] \\
\end{align*}
\]

\[
\begin{align*}
\text{sage: } & \text{symmetrica.kostka_tafel}(1) \\
& [1] \\
\text{sage: } & \text{symmetrica.kostka_tafel}(2) \\
& [1 0] \\
& [1 1] \\
\text{sage: } & \text{symmetrica.kostka_tafel}(3) \\
& [1 0 0] \\
& [1 1 0] \\
\end{align*}
\]

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(continued from previous page)

\[
\begin{bmatrix}
1 & 2 & 1 \\
\end{bmatrix}
\]

\texttt{sage: symmetrica.kostka_tafel(4)}
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 \\
1 & 3 & 2 & 3 & 1 \\
\end{bmatrix}
\]

\texttt{sage: symmetrica.kostka_tafel(5)}
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 0 \\
1 & 3 & 3 & 2 & 1 & 0 \\
1 & 4 & 5 & 6 & 5 & 4 \\
\end{bmatrix}
\]

\texttt{sage.libs.symmetrica.symmetrica.kranztafel_symmetrica(a, b)}

you enter the INTEGER objects, say \(a\) and \(b\), and \(\text{res}\) becomes a MATRIX object, the charactertable of \(S_b\) wr \(S_a\), \(\text{co}\) becomes a VECTOR object of classorders and \(\text{cl}\) becomes a VECTOR object of the classlabels.

\textbf{EXAMPLES:}

\texttt{sage: \{(a,b,c) = symmetrica.kranztafel(2,2)\}}
\texttt{sage: a}
\[
\begin{bmatrix}
1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & 1 \\
0 & 0 & 2 & 0 & -2 \\
-1 & -1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
\texttt{sage: b}
\[
\begin{bmatrix}
2, 2, 1, 2, 1 \\
\end{bmatrix}
\]
\texttt{sage: for m in c: print(m)}
\[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
1 & 0 \\
0 & 2 \\
0 & 0 \\
1 & 1 \\
0 & 0 \\
2 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\texttt{sage.libs.symmetrica.symmetrica.mult_monomial_monomial_symmetrica(m1, m2)}
\texttt{sage.libs.symmetrica.symmetrica.mult_schubert_schubert_symmetrica(a, b)}

Multiplies the Schubert polynomials \(a\) and \(b\).

\textbf{EXAMPLES:}

\texttt{sage: symmetrica.mult_schubert_schubert([3,2,1], [3,2,1])}
\[
X[5, 3, 1, 2, 4]
\]

\texttt{sage.libs.symmetrica.symmetrica.mult_schubert_variable_symmetrica(a, i)}

Returns the product of \(a\) and \(x_i\). Note that indexing with \(i\) starts at 1.
EXAMPLES:

```python
sage: symmetrica.mult_schubert_variable([3,2,1], 2)
X[3, 2, 4, 1]
sage: symmetrica.mult_schubert_variable([3,2,1], 4)
X[3, 2, 1, 4, 6, 5] - X[3, 2, 1, 5, 4]
```

```python
sage.libs.symmetrica.symmetrica.mult_schur_schur_symmetrica(s1, s2)
sage.libs.symmetrica.symmetrica.ndg_symmetrica(part, perm)
sage.libs.symmetrica.symmetrica.newtrans_symmetrica(perm)
```

computes the decomposition of a schubertpolynomial labeled by the permutation `perm`, as a sum of Schurfunction.

```python
sage.libs.symmetrica.symmetrica.odd_to_strict_part_symmetrica(part)
```

implements the bijection between partitions with odd parts and strict partitions. input is a VECTOR type partition, the result is a partition of the same weight with different parts.

```python
sage.libs.symmetrica.symmetrica.odg_symmetrica(part, perm)
```

Calculates the irreducible matrix representation D^\lambda(part(perm)), which consists of real numbers.


```python
sage.libs.symmetrica.symmetrica.outerproduct_schur_symmetrica(parta, partb)
```

you enter two PARTITION objects, and the result is a SCHUR object, which is the expansion of the product of the two schurfunctions, labeled by the two PARTITION objects `parta` and `partb`. Of course this can also be interpreted as the decomposition of the outer tensor product of two irreducible representations of the symmetric group.

EXAMPLES:

```python
sage: symmetrica.outerproduct_schur([2],[2])
```

```python
sage.libs.symmetrica.symmetrica.part_part_skewschur_symmetrica(outer, inner)
```

Return the skew Schur function s_{outer/inner}.

EXAMPLES:

```python
sage: symmetrica.part_part_skewschur([3,2,1],[2,1])
s[1, 1, 1] + 2*s[2, 1] + s[3]
```

```python
sage.libs.symmetrica.symmetrica.plethysm_symmetrica(outer, inner)
sage.libs.symmetrica.symmetrica.q_core_symmetrica(part, d)
```

computes the q-core of a PARTITION object `part`. This is the remaining partition (=res) after removing of all hooks of length `d` (= INTEGER object). The result may be an empty object, if the whole partition disappears.

```python
sage.libs.symmetrica.symmetrica.random_partition_symmetrica(n)
```

Return a random partition `p` of the entered weight `w`.

`w` must be an INTEGER object, `p` becomes a PARTITION object. Type of partition is VECTOR . It uses the algorithm of Nijenhuis and Wilf, p.76

```python
sage.libs.symmetrica.symmetrica.scalarproduct_schubert_symmetrica(a, b)
```

EXAMPLES:
sage: symmetrica.scalarproduct_schubert([3,2,1], [3,2,1])
X[1, 3, 5, 2, 4]
sage: symmetrica.scalarproduct_schubert([3,2,1], [2,1,3])
X[1, 2, 4, 3]

sage.libs.symmetrica.symmetrica.scalarproduct_schur_symmetrica(s1, s2)
sage.libs.symmetrica.symmetrica.schur_schur_plet_symmetrica(outer, inner)
Calculates the irreducible matrix representation D^part(perm), which consists of rational numbers.

sage.libs.symmetrica.symmetrica.specht_dg_symmetrica(part, perm)
sage.libs.symmetrica.symmetrica.start()
sage.libs.symmetrica.symmetrica.strict_to_odd_part_symmetrica(part)
implies the bijection between strict partitions and partitions with odd parts. input is a VECTOR type partition,
the result is a partition of the same weight with only odd parts.
sage.libs.symmetrica.symmetrica.t_ELMSYM_HOMSYM_symmetrica(elmsym)
sage.libs.symmetrica.symmetrica.t_ELMSYM_MONOMIAL_symmetrica(elmsym)
sage.libs.symmetrica.symmetrica.t_ELMSYM_POWSYM_symmetrica(elmsym)
sage.libs.symmetrica.symmetrica.t_ELMSYM_SCHUR_symmetrica(elmsym)
sage.libs.symmetrica.symmetrica.t_HOMSYM_ELMSYM_symmetrica(homsym)
sage.libs.symmetrica.symmetrica.t_HOMSYM_MONOMIAL_symmetrica(homsym)
sage.libs.symmetrica.symmetrica.t_HOMSYM_POWSYM_symmetrica(homsym)
sage.libs.symmetrica.symmetrica.t_HOMSYM_SCHUR_symmetrica(homsym)
sage.libs.symmetrica.symmetrica.t_MONOMIAL_ELMSYM_symmetrica(monomial)
sage.libs.symmetrica.symmetrica.t_MONOMIAL_HOMSYM_symmetrica(monomial)
sage.libs.symmetrica.symmetrica.t_MONOMIAL_POWSYM_symmetrica(monomial)
sage.libs.symmetrica.symmetrica.t_MONOMIAL_SCHUR_symmetrica(monomial)
sage.libs.symmetrica.symmetrica.t_POLYNOM_ELMSYM_symmetrica(p)
Converts a symmetric polynomial with base ring QQ or ZZ into a symmetric function in the elementary basis.
sage.libs.symmetrica.symmetrica.t_POLYNOM_MONOMIAL_symmetrica(p)
Converts a symmetric polynomial with base ring QQ or ZZ into a symmetric function in the monomial basis.
sage.libs.symmetrica.symmetrica.t_POLYNOM_POWER_symmetrica(p)
Converts a symmetric polynomial with base ring QQ or ZZ into a symmetric function in the power sum basis.
sage.libs.symmetrica.symmetrica.t_POLYNOM_SCHUBERT_symmetrica(a)
Converts a multivariate polynomial a to a Schubert polynomial.

EXAMPLES:

sage: R.<x1,x2,x3> = QQ[]
sage: w0 = x1^2*x2
sage: symmetrica.t_POLYNOM_SCHUBERT(w0)
X[3, 2, 1]
sage.libs.symmetrica.symmetrica.t_POLYNOM_SCHUR_symmetrica($p$)
Converting a symmetric polynomial with base ring QQ or ZZ into a symmetric function in the Schur basis.

sage.libs.symmetrica.symmetrica.t_POWSYM_ELMSYM_symmetrica($powsym$)

sage.libs.symmetrica.symmetrica.t_POWSYM_HOMSYM_symmetrica($powsym$)

sage.libs.symmetrica.symmetrica.t_POWSYM_MONOMIAL_symmetrica($powsym$)

sage.libs.symmetrica.symmetrica.t_POWSYM_SCHUR_symmetrica($powsym$)

sage.libs.symmetrica.symmetrica.t_SCHUBERT_POLYNOM_symmetrica($a$)
Converting a Schubert polynomial to a ‘regular’ multivariate polynomial.

**EXAMPLES:**

```
sage: symmetrica.t_SCHUBERT_POLYNOM([3,2,1])
x0^2*x1
```

sage.libs.symmetrica.symmetrica.t_SCHUR_ELMSYM_symmetrica($schur$)

sage.libs.symmetrica.symmetrica.t_SCHUR_HOMSYM_symmetrica($schur$)

sage.libs.symmetrica.symmetrica.t_SCHUR_MONOMIAL_symmetrica($schur$)

sage.libs.symmetrica.symmetrica.t_SCHUR_POWSYM_symmetrica($schur$)

sage.libs.symmetrica.symmetrica.test_integer($x$)
Tests functionality for converting between Sage’s integers and symmetrica’s integers.

**EXAMPLES:**

```
sage: from sage.libs.symmetrica.symmetrica import test_integer
sage: test_integer(1)
1
sage: test_integer(-1)
-1
sage: test_integer(2^33)
8589934592
sage: test_integer(-2^33)
-8589934592
sage: test_integer(2^100)
1267650600228229401496703205376
sage: test_integer(-2^100)
-1267650600228229401496703205376
sage: for i in range(100):
....:     if test_integer(2^i) != 2^i:
....:         print("Failure at {}".format(i))
```

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